IMPROVING INSTRUCTION IN MECHANICS THROUGH IDENTIFICATION AND ELICITATION OF PIVOTAL CASES IN STUDENT REASONING

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This is to certify that I have examined this copy of a doctoral dissertation by

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and have found that it is complete and satisfactory in all respects,
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Abstract

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Chair of the Supervisory Committee:
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This dissertation reports efforts to develop instructional materials on linear momentum, angular momentum, and forces on rigid bodies in standard introductory physics course. Many common errors in understanding of physics concepts are described. Some of these have been previously reported. Other cases represent new instances of common patterns of student reasoning, notably the conflation of related concepts. In cases where persistent student errors were already well known, tutorial instructional materials existed at the start of this project. These materials had been developed with specific student difficulties in mind, but in some cases, were less effective than was hoped. Through further exploration into student thinking by analysis of responses to written and online questions, individual student interviews, and classroom interactions, we improved our understanding of students' conceptual errors. We also identified several specific cases on which students appeared to hinge naturally their understanding of a concept or principle. For some students in some situations, invocation of one of these cases as support for a principle appears to be relatively spontaneous; however, in most cases, students may think more productively when guided to consider the case and assign it a pivotal role. Thus, in the final design, the instructional strategies of the tutorials are intended to promote naturally productive paths of student thinking. In this dissertation, we report on the observations and reflections that drove the instructional development and demonstrate the increased effectiveness of the instruction.

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Innumerable people made vital contributions to this project. I am pleased to have been given the task of synthesizing and recording the results of their hard work.

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DEDICATION

To Eleanor and all you orcadillos

CHAPTER 1. INTRODUCTION

This dissertation describes research and curriculum development that are part of an overall effort by the Physics Education Group to improve student learning in undergraduate physics. As part of this effort the group has been developing tutorials that supplement lecture and laboratory instruction in the standard introductory course. Tutorial sessions replace the traditional weekly recitation session and are intended to help students develop a functional understanding of important physical concepts and lines of reasoning. The tutorials are written on the basis of research on student learning in general and of specific physics topics, and they typically result in significant improvements in student performance on selected conceptual tasks.

The initial goals of the project described here were to refine published tutorials on linear momentum and rigid-body dynamics, and to develop a tutorial on conservation of angular momentum. As we progressed toward these goals we came to recognize that student understanding of specific types of forces, especially static friction, had an impact on their ability to apply general principles. Thus the scope of the project expanded to include this topic as well. This dissertation documents the patterns in student reasoning that we identified, describes the iterative process of curriculum development, and provides evidence of improved student learning with the current versions of the tutorials.

This dissertation also has another purpose: to provide explicit and detailed examples of the process whereby the development of curriculum is informed not only by knowledge of student difficulties, but by knowledge of spontaneous (or easily stimulated) patterns of reasoning and specific physical situations upon which students seem to hinge their understanding of abstract physical principles and definitions. In some cases, we have found that students tend to justify an incorrect general interpretation of a principle by spontaneously appealing to an incorrect interpretation of a particular case. When we have observed that a particular case has been given such authority in student understanding, we have called it a *pivotal case*. We have found that centering tutorial instruction on a pivotal case and teaching students how to interpret it can help students understand and apply the corresponding abstract principle in a variety of problem contexts. More generally, we understand pivotal cases as a particular type of students' productive thinking. Thus, in general, we have sought through this research to identify patterns and tendencies in student reasoning that may be useful to elicit in instruction, not necessarily because they need correction, but because they may be directed in a way that is helpful for students.

A. INSTRUCTIONAL CONTEXTS

The data presented in this dissertation were taken in a variety of courses at the University of Washington and elsewhere. This section describes each of the courses in which either data were gathered or instruction was administered.

1. Calculus-based introductory mechanics course at the University of Washington (UW 121)

Most of the data presented in this dissertation come from this population of students. UW 121 is a one-quarter (10 week) course that covers basic mechanics, including kinematics, Newton's laws, conservation of momentum, conservation of energy, rotational kinematics and dynamics, and conservation of angular momentum. The enrollment of each section is typically between 90 and 180 students, with an average of roughly 130. Students meet for three 50-minute lectures, one 3-hour laboratory session, and one 50-minute tutorial session each week. Students are required to attend some minimum number of lab sessions in order to pass Physics 121, and they are given some credit for participating in tutorial sessions. In some cases, students also receive credit for participating in "votes" during lecture sessions using personal response systems. Individual tutorial sessions usually have enrollments of between 22 and 24 students. In most cases, 2 tutorial instructors are assigned to each section. Tutorial instructors prepare for tutorials by meeting weekly to work through the tutorial and discuss issues related to teaching the tutorial. (This meeting is described in more detail in section 7 below.) The textbook used for this course (during the study reported in this dissertation) was *Physics for Scientists and Engineers* by Giancoli,[1] or *Physics for Scientists and Engineers* by Knight.[2] In the tutorial sessions, students use *Tutorials in Introductory Physics*. However, in general, if student understanding of a particular physics topic is receiving serious research attention, then the associated tutorial(s) that students use is often not the version found in *TiIP*, but a revised version, which is printed and given to the students as a handout.

2. Algebra-based introductory mechanics course at the University of Washington (UW 114)

The algebra-based introductory physics course at the University of Washington has not included a formal tutorial component. Typically, students in UW 114 meet for a large lecture 4 times per week, with no small section meeting. The course has an associated standard laboratory course that is recommended but not required. However, in winter 2005, each of two sections of UW 114 that were offered that quarter had atypical instructional conditions. The instructor of UW 114A was an experienced physics education researcher and curriculum developer, and based many of his lectures on instructional strategies found in *Tutorials in Introductory Physics*. The instructor of UW 114B had arranged to reduce the number of lectures per week by 1 (to 3) and add small (22-24) weekly tutorial sections. The tutorial system used in UW 114B was very similar to that for UW 121.

3. Calculus-based introductory mechanics course at Purdue University (PRD 152)

The calculus-based introductory mechanics course at Purdue University is one-semester (16 week) course. Because the course is longer than that at UW by about 6 weeks, the course covers more topics. Topics covered in PRD 152 that are not covered in UW 121 include: hydrostatics and hydrodynamics, wave motion, and sound. Students meet for two 50-minute lectures, two 50-minute recitations, and one 100-minute laboratory session each week. One recitation session each week is devoted to working through a tutorial from a custom edition of *Tutorials in Introductory Physics*. The tutorials and homework that appear in the custom edition are the same as those found in the standard first edition. Tutorial sessions at Purdue are similar to those at UW in most respects; one notable difference is that the typical student enrollment for a recitation section at Purdue is ~50 students, compared with ~20 students at UW. Students work in groups of 4 in a large room with 2 TAs. The TAs prepare for each tutorial in a weekly meeting similar to that at UW (see UW 501, below). Students in PRD 152 tend to perform very similarly to students in UW 121 with similar instruction.

4. Calculus-based introductory mechanics course at the University of Colorado (COL 1110)

The calculus-based introductory mechanics course at the University of Colorado is a semester course similar to PRD 152. COL 1110 covers the same material covered in UW 121 along with an introduction to thermodynamics. Students in COL 1110 meet each week for three 1-hour lectures and one 1-hour tutorial section, in which students work through a tutorial from the first edition of *Tutorials in Introductory Physics*. COL 1110 does not have a laboratory component; students take a separate laboratory course after completing COL 1110. Tutorial sections at the University of Colorado have approximately the same number of students and TAs as at the University of Washington. Tutorial TAs prepare for each week's tutorial with a meeting similar to that at UW (see UW 501, below). Students in COL 1110 tend to perform very similarly to students in UW 121 with similar instruction.

5. Calculus-based introductory mechanics course at the University of Cincinnati (CIN 201)

The calculus-based introductory mechanics course at the University of Cincinnati is a one-quarter (10-week) course similar to that at UW. The sequence of topics in CIN 201 is very similar to that in UW 121. Unlike UW 121, students in CIN 201 are not required to take the introductory laboratory course concurrently with CIN 201, but it is strongly recommended. Lectures meet three times each week, and tutorial sessions meet once each week. Tutorials have a student-instructor ratio similar to that at UW, and tutorial instructors prepare each week in a meeting similar to that of UW 501 (described below).

6. Advanced classical mechanics course at the University of Washington (UW 424)

This course is a senior-level one-quarter course on classical mechanics, and only the first mechanics course that a physics major at the University of Washington would take after Physics 121. In Physics 424, students are introduced to Lagrangian and Hamiltonian formalisms and apply them to complex mechanical systems. We did not have the opportunity to develop or administer any instructional interventions for the students in this course, but we were able to ask the students some written questions, before the instructor for that course began to teach the related material.

7. Graduate teaching seminar at the University of Washington (UW 501)

In Physics 501 at the University of Washington, graduate student TAs prepare to teach each week's tutorial for Physics 121. The enrollment in UW 501 is usually ~25 students. In each weekly meeting of UW 501, TAs take a pretest that is either identical or similar to that taken by students in UW 121. After working through the pretest, TAs work through the tutorial in groups of 5-6, with the guidance of one experienced TA, who sits with them (instead of roaming around the room, as in UW 121). As the TAs work through the tutorial, discussion tends to wander through a variety of perspectives: engaging the ideas in the tutorial on a basic level, exploring the relationship of the basic ideas to more advanced ideas that are clearly beyond the scope of Physics 121, seeking and offering issue-specific teaching strategies, getting acquainted with typical lecture instruction and student background, placing the tutorial in the context of a sequence of ideas spanning several tutorials, *etc*. After TAs have completed the tutorial, the instructor for UW 501 distributes pretest responses from current UW 121 students to the TAs. TAs inspect the responses, looking for trends in reasoning, and offer their own observations in a large class discussion. In some cases, the instructor presents some numerical evidence that the tutorial is both appropriately targeted to the students' initial understanding and effective in helping the students perform better on some non-trivial tasks.

We also present some data from this course in this dissertation. We compare performance by the TAs on the pretest, before working through the tutorial, to that by UW 121 students, after working through the tutorial, as some measure of the effectiveness of the tutorial instruction.

B. RESEARCH METHODS

This section summarizes how we gathered, interpret, and report data. We discuss the different modes by which we collect student responses, general principles of task design and interpretation of individual student responses, some statistical considerations for combining aggregate data, and our approach to interpreting aggregate data.

1. Data gathering

Here we survey the various means by which we collected student responses to intellectual tasks.

a. Interviews

Individual demonstration interviews are a crucial component of research on student understanding. For every round of interviews described in this dissertation, a single researcher conducted the interviews with one student at a time. Each interview lasted about 1 hour and was audio-recorded. We solicited volunteers for the interviews by making an announcement in several tutorial sections of Physics 122 (Introductory Electricity and Magnetism). Thus, all students who participated had completed Physics 121 at the University of Washington. In exchange for participating, we offered students one hour of tutoring, to be redeemed at the student's convenience. Each round of interviews attracted 5-8 participants. Students who volunteer for interviews tend to earn better grades in physics than the average physics student.

For each round of interviews, we designed a rough interview protocol and followed it loosely, so that students might be encouraged to share ideas that naturally came to their minds, without feeling like they were "off topic." Students were told that the interviewer was interested in how they think, that they should try to express all of their thoughts out loud, and that the interviewer would try not to reveal whether a student's responses were correct or incorrect. Some rounds of interviews involved a demonstration, but all followed the basic pattern: intellectual task, discussion (repeat). In other words, the interviews were always based on accomplishing some relatively concrete task, like making a prediction, or working a conceptual pencil-and-paper problem. Students were welcome to alter the nature of the discussion toward considering the meaning of a term, another experiment, or the interpretation of a general, abstract principle, and in many cases, were invited to do so by the interviewer.

We have used the interview data mainly as a source of inspiration and insight, rather than for any quantitative claims. For this reason, we were not concerned that the questions asked in the interviews were done under very similar, restricted conditions.

b. Web-based pretests

In UW 121, COL 1110, PRD 152, and CIN 201, students take a web-based pretest each week. The pretests are constructed by researchers in the Physics Education Group at UW, using the program *WebQ*.[3] The pretest is typically open for submissions after lecture is over on Friday and before lecture begins on Monday. Students may log in to the pretest at any time while the pretest is available. After logging in, each student is given 15:00 minutes to submit. If a student does not click submit before the time runs out, the responses are automatically submitted. Each 'question' on a pretest is usually split into two parts, a pull-down menu, with which students select an option, such as "the final speed of cart A is greater than the final speed of cart B," and a separate free-response part in which students explain their reasoning for the previous part. In order to prevent systematic bias toward the first option in a list, each pull-down menu begins with the option 'unanswered.' This is the default option that is submitted if a student logs in and submits responses right away. Such responses typically constitute a very small fraction of all responses. Though using radio buttons instead of pull-down menus would obviate the need for an 'unanswered' option, pull-down menus make the pretest as a whole appear shorter, keeping the figures closer to each other and to the corresponding questions. We believe this is more convenient for the students.

Though pretests are not graded for correctness, responses tend to appear fairly considered and honest. (In other topic areas we have seen that student responses to questions on the ungraded pretests tend to be similar to responses to the same questions when used in course examinations after traditional instruction.) Most students write about one sentence for each explanation. We adjust the length of the entire pretest so that the average student spends about 8-12 minutes; the result is that most pretests have two major contexts, about which we ask ~ 3 (two-part) questions.

Students in UW 121 are sent an email reminder to take the pretest each week. At the beginning of a typical UW 121 course, around 90% of the students take the pretest each week, and this fraction steadily drops to around 70% as the quarter comes to a close, perhaps because students find it increasingly difficult to stay on top of the many obligations for each of their courses. We find no obvious relationship between whether students take the pretest and how students perform in the course. Thus, we consider the fraction of students who take a given pretest to be representative.

c. Paper pretests

Two courses (UW 114, and UW 501) described above used paper pretests instead of web-based pretests. This difference was not motivated by research design but by practicality. In general, we have found that students perform similarly on paper and web-based pretests. For particular topics and particular questions, we have seen student responses differ between web-based and paper versions when the options shown on the web-based version are unusually revealing or instructive (perhaps by virtue of not being exhaustive). However, the types of tasks discussed in this dissertation are those for which students tend to perform very similarly with either format, so we do not emphasize this variable when discussing the data.

d. Free-response exam questions

For each course in which tutorials were part of the formal instruction (UW 121, COL 1110, PRD 152, UW 114, CIN 201), students were informed that part (~20%) of each midterm exam would test their understanding of material covered in the tutorial sessions and the corresponding homework. Students in UW 121, COL 1110, UW 114, and CIN 201 (not PRD 152) were given a multi-page exam, of which about one page asked free-response tutorial-style questions. The questions on these pages are written by researchers in the Physics Education Group, and are reviewed by the lecture instructors for the course. The final exam for each course also included at least one free-response page based on material covered in tutorial. Student performance on any given question does not appear to depend on whether the question was asked on a midterm exam or the final exam.

Students' free response exam papers were collected and scanned electronically before being given to the exam grader. In every case, the author of the tutorial questions then met with the exam grader to discuss the solution and give suggestions for how to give partial credit, *etc.* We then use the electronic files of the scanned papers to tally student responses and study their reasoning on each question.

e. Multiple-choice exam questions

Due to limited grading resources, it was necessary sometimes in UW 121, UW 114, and PRD 152 to ask the students multiple-choice questions on the exams instead of (or in addition to) free-response questions. In some cases, transforming a question to multiple-choice was trivial, as in "Is X greater than, less than, or equal to Y?" In other cases, it was necessary to construct a small set of non-exhaustive options on the basis of some knowledge of how students tend to answer that question (in free-response form) or reason about similar questions. Only in special cases do we suspect that the format (free-response or multiple-choice) of an exam question affects student performance, so we treat them here as equally valid measurements of aggregate student understanding.

f. In-class votes

In UW 121 and UW 114, many instructors began using infrared voting systems in lecture, beginning around the year 2004. A number of such products are currently available; at the University of Washington, the physics department has used the H-ITT Classroom Response System.[4] A similar system was used in COL 1110. The system can be used in a variety of ways: some instructors give reading quizzes, ask conceptual questions to test comprehension of a lecture presentation or as a means of stimulating student discussion, or as a way to measure students' ability to teach each other.[5] In order to encourage students to respond, and to respond correctly, some instructors give each student one point for each vote in which the student participates, and two points for each correct answer.

g. Informal classroom observations

Members of the Physics Education Group at the University of Washington maintain a high level of interaction with the students in the introductory course – each graduate student in the group usually serves as an instructor for 5-6 sections of the tutorials each week, and for 1-2 hours each week in the physics study center. (Students go to the study center for help on homework and to ask questions about any part of the introductory course.) Such involvement allows for many opportunities to reflect on student interactions, questions, and confusions about physics topics in the tutorials, related topics, or the tutorial materials themselves. We have drawn continually upon these observations as a source for reflection on student learning throughout the research and curriculum development process. We also draw on them in this dissertation when we describe each piece of tutorial curriculum in detail.

2. Data analysis and task design

The issues of how to extract meaningful information from student responses to intellectual tasks and how to construct tasks in way that maximizes meaningful responses are intimately linked. In the research described here, we have taken the following approach. First, we compose a question or task that we believe requires students to use the concept or principle in which we are interested, and that has an unambiguously correct answer (within the context of a particular theory). Because students' responses tend to contain much information (elements of explanation, word choices, logical progression, *etc.*) we first distill the response into some basic form, like "greater than," if students are asked whether quantity A is greater than, less than or equal to quantity B; or "up and to the right," if students are asked to draw an arrow to indicate the direction of some vector quantity. Whenever possible, we ask students to explain their reasoning in arriving at an answer. In general, the sorts of distinct reasoning that students offer for support of a particular basic response tend to be small in number. As we examine the responses to a particular task, it may become apparent that a certain response corresponds to more than one sort of distinct reasoning. Instead of recording these kinds of reasoning with the same precision as we do with the basic response, we make notes of the general trends in reasoning. We then construct a new task, either by asking another question about the same physical context, or by changing the physical context itself, so that we can improve the one-to-one correspondence between clear basic responses and trends in reasoning. In some cases, we find it sufficient to achieve a one-to-one correspondence between major trends in reasoning and a *combination* of basic responses. Ideally, then, we gradually construct a task for which students do not choose the correct response by using incorrect reasoning. With this approach, we aim to eliminate the subjectivity that comes with attempting to distinguish "correct reasoning" from "incomplete" or "incorrect reasoning." By reducing this subjectivity in data analysis, we hope to improve the reliability and reproducibility of our measurements, as well as the ability of researchers in this field to communicate results.

3. Data reporting

The research on student learning that is reported in this dissertation was conducted primarily for the purpose of improving instruction in physics. For each physics topic discussed here, we present some tendencies in student reasoning, one or more versions of short curricular interventions (tutorials) to help student develop and transform their ideas, and evidence of the positive effects on student reasoning about that topic. In almost all cases, this evidence consists of comparisons of the fraction of students answering a question in a particular way at various stages of instruction, or after different versions of the tutorial instruction. An example is: "90% of Class A answered Question X correctly after Instruction P, while 65% of Class B answered Question X correctly after Instruction R." Another example is: "45% of Classes A and B answered Question Y correctly, after similar instruction."

Presenting numerical data in this manner raises a few questions: What should we count as "similar instruction?" How much do the data vary when the same question is asked of multiple sections of students after similar instruction? What precision do we require if we want to use numbers to compare Instruction P to Instruction R?

First, we explain what in this dissertation we call "similar instruction." We acknowledge that many different physics lecturers have many different ideas for explanations, demonstrations, examples, discussion topics, *etc.* For the most part, we did not track such variations in lecture instruction from section to section. The main reason for this is that most differences in lecture instruction tend to make no difference to student performance *on the kinds of questions we have asked.*[6] The lack of impact of differences in lecture instruction makes large-scale research like that reported here possible. In order to make the best sense of student performance on different tasks under different conditions, we selectively ignore much of the possibly relevant (but most often irrelevant) information. However, in some cases, a group of students performs anomalously. For example: The success rate on a particular task is usually between 35% and 45% after all lecture instruction, but, in one class, 70% of the students answer correctly. In a case like this, we aim to construct a *plausible* explanation of the anomalous performance by examining the instruction of the anomalous class. We may discover that the lecturer of that section discussed a point or worked an example that was very similar to a question we asked the students, thus, in a sense, revealing the answer to our question shortly before we asked it. Most large (> ~15%) variations can be explained in this way. By searching for differences in lecture instruction *after* detecting differences in performance, we miss many opportunities to observe that this or that specific instructional element does *not* make a difference to student performance on our conceptual tasks, but after all, *most* lecture instruction does not have this effect. Instead, we are able to detect here and there that some comment or demonstration in lecture probably *did* make a difference to student performance on some task. In these cases, we present the data separately and describe how the instruction was different. Thus, when we report that students in many lecture sections had "similar instruction," we mean that students in these sections enjoyed a variety of different lecture presentations, none of which strongly affected student performance on the task on which we are reporting.

Table 1-1: Typical variations among different lecture sections on the same task after similar instruction (before relevant tutorial instruction). The data are what we would call "similar," and are combined in the second row from the bottom. The bottom row shows how we report the data in this dissertation. (Data are taken from the *N2R* question asked in the "Block and Spool" context.) Statistical uncertainties are calculated by assuming that data are drawn from a Poisson parent distribution.

Section	Number of students	Response P	Response Q (correct)	Response R	Response S
UW034A	112	56 ± 7%	4 ± 2%	13 ± 3%	22 ± 4%
UW034C	132	56 ± 7%	5 ± 2%	6 ± 2%	27 ± 5%
UW041A	115	68 ± 8%	4 ± 2%	8 ± 3%	13 ± 3%
UW041B	95	65 ± 8%	3 ± 2%	9 ± 3%	12 ± 4%
UW042A	97	55 ± 8%	8 ± 3%	5 ± 2%	23 ± 5%
UW042B	126	63 ± 7%	9 ± 3%	6 ± 2%	19 ± 4%
UW044D	71	59 ± 9%	6 ± 3%	6 ± 3%	21 ± 5%
Mean as in equation 1-5		60%	6%	8%	20%
Mean and uncertainty as in equations 1-1 and 1-2		60 ± 3%	5 ± 1%	7 ± 1%	18 ± 2%
We report:	748	60%	5%	5%	20%

How much do the data vary? This can depend slightly on the physics topic and on the task. In general, though, for a given section of ~100 students, the frequency of any particular response (chosen from a short list) on a given task tends to lie within 5-10% of the weighted average over different sections. When a section varies more than ~15% from this average, then we start searching for an explanation for that section's different performance. Table 1-1 shows a set of real data, separated into different lecture sections, then combined, and then rounded to the nearest 5%, as we would report it in this dissertation. Here we have calculated statistical uncertainties for each sort of response in each section, assuming that the data are drawn from a large-N Poisson parent distribution. For each section i and response-type j, the statistical uncertainty is calculated from:

$$\sigma_{ij} = \frac{\sqrt{n_{ij}}}{N_i} = \sqrt{\frac{x_{ij}}{N_i}} \tag{1-1}$$

where n_{ij} is the number of students in section i who gave response j, N_i is the number of students in section i who gave some response ($N_i = \sum_j n_{ij}$), and x_{ij} is the fraction of students in section i who gave response j ($x_{ij} = \frac{n_{ij}}{N_i}$). In the second row from the bottom, we show the mean value of x_j, averaged over all sections, and weighted according to the statistical uncertainty σ_{ij}.

$$x_j = \frac{\sum_i x_{ij}\left(\frac{1}{\sigma_{ij}^2}\right)}{\sum_i \frac{1}{\sigma_{ij}^2}} = \frac{\sum_i N_i}{\sum_i \frac{N_i^2}{n_{ij}}} \tag{1-2}$$

This mean has a statistical uncertainty σ_j, which is calculated according to:

$$\frac{1}{\sigma_j^2} = \sum_i \frac{1}{\sigma_{ij}^2} \tag{1-3}$$

Because usually $N_i \approx 100$ (and n_{ij} is always less than N_i), a conservative short-cut estimate for the uncertainty of any x_{ij} is $\sigma_{ij} \approx 10\%$. Assuming that each section i has σ_{ij} around 10%, then the total uncertainty for all sections should be around

$$\sigma \approx \frac{\sigma_{ij}}{\sqrt{m}} \tag{1-4}$$

where m is the number of sections. For the data in Table 1-1, this shortcut yields a conservative estimate for σ of around 4%. If the number of sections is smaller, say, $m = 2$, then we would estimate $\sigma \approx 7\%$.

Another statistical shortcut is shown in the third row from the bottom in Table 1-1. This row shows means calculated according to:

$$x_j = \frac{\sum_i x_{ij} N_i}{\sum_i N_i} = \frac{\sum_i n_{ij}}{\sum_i N_i} \tag{1-5}$$

In this approach, individual measurements are, in a sense, ignored, and data are treated as if they had been gathered in a single measurement. Using the data in Table 1-1 as an example, it is as if we had a single class of 748 students, with 451 students giving response P. Though it is true that 451 of 748 students *did* give response P, this occurred over seven separate measurements. In the naïve approach shown in Equation 1-5, each measurement n_{ij} is given equal weight in the sum, but in the Poisson approach of Equation 1-2, each measurement n_{ij} is dilated by the factor N_i/n_{ij}, thus, from one perspective, giving greater weight to sections with lower x_{ij}. As shown, these figures tend to be very similar because the x_{ij}'s are closely grouped and class sizes (N_i) are approximately the same. If both of these conditions were not satisfied, then the two methods would give means that are more different than those shown. Because these means are approximately equal for our data, we believe that the naïve mean, which is more easily calculated, is sufficient for our purposes.

In some cases, data may appear more similar before rounding to the nearest 5% than after. Therefore, we pay little attention to apparent differences in reported data if they are less than 10%. Furthermore, remembering that this research has been done in the name of improving instruction, if we were to observe differences of 10, 20, or 30% between classes, we ask whether such a difference would change the *instruction* that we would offer to that class. In most cases, our judgment of the best short (tutorial) instruction on a topic does not depend on whether 20% or 50% of the class answers a relevant question correctly before instruction; in either case, the issues that the students (as a group) need to address are the same. Thus, data such as those presented in Table 1-1 help give us a sense of what ideas and skills students have, and roughly how prevalent they are. Though the data could be used to zero in on some parent distribution, such precise predictability is not a high research priority for us. By overestimating the statistical uncertainty, we weaken the apparent precision of our measurements to a level appropriate to our main purpose: improving physics instruction.

Finally, how do we use the data to compare different manners or contents of instruction? As teachers, we do not believe that the observation "90% of Class A answered Question X correctly, while 65% of Class B answered Question X correctly" strictly demonstrates that Class A's instruction is better than Class B's. Even if such statistics held for Classes A and B over all sorts of questions and tasks, we would not be satisfied to conclude that Class A had experienced, considered, thought, or learned all that they ought to have in their course. However, if the tasks on which student performance differed in Classes A and B were relevant to the definitions of and relations between fundamental concepts of the course, we expect that students in Class A would be more prepared to consider (and more likely to raise) subtler issues in class discussion, and more able to think productively about difficult and unfamiliar problems. These and other intangible outcomes are what we consider parts of the undefined concept *learning*. In other words, we do not equate improvements on selected tasks with learning, nor do we assume that such improvements necessarily lead to future learning. But, in this dissertation, we do treat such improvements as *standing for* learning, because we believe that facility with difficult conceptual tasks is one factor that enables learning.

C. ORGANIZATION OF THE DISSERTATION

In Chapter 2 we describe previous research on student understanding of linear momentum, rigid-body dynamics, static friction forces, and angular momentum (and the related mathematical formalism). We also review some research done by cognitive psychologists on student understanding of the material conditional from classical logic. The concept of the pragmatic reasoning schema, which the psychologists introduce to the context-dependence of students' treatment of conditionals, strongly influenced the research and curriculum development reported in this dissertation.

In chapters 3 and 4, we detail an investigation of student learning of linear momentum conservation in one dimension and of forces exerted on rigid (extended) bodies, respectively. The research described in each of these chapters began by examining the effect of published tutorial curricula on student understanding. These curricula were written on the basis of some understanding of how students learn about linear momentum conservation and rigid-body dynamics, but were less effective than we had hoped. Each chapter describes the process by which we improved our understanding of student learning of that physics topic and demonstrates that the new version of the curriculum results in better student performance than before on selected tasks.

Chapter 5 describes an investigation that was an extension of that reported in chapter 4. As part of the research on student understanding of Newton's 2^{nd} law in the context of rotating, rigid bodies, we explored how students think about constraint forces on objects that roll without slipping. Student responses in this context suggested that many students treated forces according to their type (*e.g.*, friction, tension), rather than according to some understanding of the relationship between motion and forces in general. Thus, chapter 5 explores student understanding of different types of forces, first by comparing and contrasting student understanding of friction forces with that of tension forces after traditional instruction, and then by describing an instructional intervention with which we attempted to help students connect their understanding of static friction to that of (static) tension.

Chapters 6 and 7 report on student learning of angular momentum and its conservation. Because angular momentum is a complex topic, we have divided our presentation of the research and curriculum development into two chapters. In chapter 6, we describe student learning of the angular momentum associated with "spinning" motion, that is, of bodies that rotate around some central axis such that relevant centers of mass are exactly or approximately at rest. In chapter 7, we focus on student learning of the more abstract concept of the angular momentum of a (classical) particle, especially in the case of constant-velocity motion. This chapter also illustrates how student understanding of angular momentum conservation is explicitly coupled to that of linear momentum conservation. Though the research and curriculum development is presented here in two chapters, we developed a single tutorial and corresponding homework for the topic of angular momentum and its conservation. Discussion of the curriculum and its effect on student understanding is divided into two parts: Part 1 of the tutorial and homework is discussed in chapter 6, and Part 2 is discussed in chapter 7.

In chapter 8, we summarize the findings from our research and draw some general conclusions, including some implications for instruction and assessment.

NOTES TO CHAPTER 1

[1] Giancoli, D. C., *Physics for Scientists and Engineers with Modern Physics*, Third Edition (Prentice-Hall, Upper Saddle River, NJ, 2000).

[2] Knight, R. D., *Physics for Scientists and Engineers: A Strategic Approach with Modern Physics* (Pearson/Addison-Wesley, San Francisco, 2004).

[3] Information about *WebQ* can be found at <catalyst.washington.edu>. *WebQ* is a product of the Center for Teaching, Learning, and Technology at the University of Washington.

[4] Information about this product can be found at <www.h-itt.com>

[5] Mazur, E., *Peer Instruction: A User's Manual,* (Prentice Hall, Upper Saddle River, NJ, 1997).

[6] For an example of the tendency of student performance on conceptual tasks to be instructor-independent, see McDermott, L. C. and P. S. Shaffer, "Research as a guide for curriculum development: An example from introductory electricity. Part I: Investigation of student understanding," *Am. J. Phys.* **60**, 994-1002 (1992).

CHAPTER 2. REVIEW OF PREVIOUS RESEARCH

This chapter reviews previous research on student understanding of linear momentum, rigid-body dynamics, static friction, angular momentum, and the vector cross product.

Much of the relevant previous research on student understanding of specific topics in physics was done by former graduate students in the Physics Education Group at the University of Washington and is reported in their Ph.D. dissertations. All dissertation research done at UW was conducted in a manner very similar to the research reported in this dissertation (with very similar student populations and research methods).

We also describe in this chapter some research done by cognitive psychologists. We used this research as a guide for improving our understanding of the idea of productive *natural* thinking among students.

A. RESEARCH ON STUDENT UNDERSTANDING OF LINEAR MOMENTUM

This section reviews the small number of studies on student understanding of linear momentum of which we are aware.

Singh and Rosengrant[1] developed a 25-item multiple-choice test of energy and momentum concepts as part of an exploratory investigation. The test was administered to over 3000 students in introductory physics courses at different colleges and universities, both at the beginning of the course and at the end. The authors chose a variety of contexts and questions that would correspond to typical expected learning outcomes of the introductory calculus-based physics course. These questions (and corresponding distractors) were designed to measure the prevalence of various tendencies in reasoning that the authors had identified in individual student interviews. Because the test was intended to be comprehensive, the study did not focus on specific patterns in reasoning, but instead summarized them, giving examples of illustrative student responses from the interviews. Of relevance to the research in this dissertation are the following conclusions: "many students did not realize that work and energy are scalar quantities and momentum is a vector quantity;" students tended to explain phenomena in a context according to ideas other than the formal principles of energy or momentum conservation; students often failed to recognize whether momentum or energy was the more appropriate concept for a given context and question.

Graham and Berry[2] developed a "momentum hierarchy questionnaire," which was designed "to identify stages through which students' understanding of momentum develops." The authors also sought to develop a general method for determining how student understanding of a particular scientific concept develops. By identifying a set of levels of understanding of momentum through which students progress, the authors hoped to model the development of the momentum concept. The questionnaire included 20 problems that tested different aspects of momentum, with varied level of complexity. The test was administered to almost 600 17-18 year olds who had studied momentum, impulse, and conservation of momentum. The analysis of student responses revealed three major levels in student understanding of momentum. To summarize briefly: students at level 1 recognize the importance of the product of mass and speed and can compare the momentum of two objects moving in the same direction; students at level 2 can perform most tasks involving momentum in one *direction* (that is, for situations in which no object "turns" *or* "rebounds"), including application of the conservation principle and the impulse-momentum theorem; and students at level 3 can perform tasks with momentum in two dimensions. The authors conclude that students evolve from a speed-dominant understanding of momentum to a scalar understanding and then to a vector understanding. Furthermore, the authors suggest that instruction that focuses on momentum in one dimension may reinforce students' treatment of momentum as a scalar, and they recommend that students work with momentum in two dimensions as early as possible in momentum instruction.

In her Ph.D. dissertation (at the University of Washington), O'Brien Pride[3] investigated student understanding of momentum conservation in one and two dimensions. She documented many contexts in which students treated the concepts *energy* and *momentum* as essentially indistinguishable. Many conservation-of-momentum tasks described in this dissertation derive from tasks developed in O'Brien Pride's work, especially the "qualitative two collision comparison" (see O'Brien Pride, p. 126, and Figure 3-1 in Chapter 3 of this dissertation). Perhaps the most striking result of her research is the fact that student ability to apply conservation of momentum (as a vector quantity) in one dimension appears *not* to depend on whether students worked through an additional tutorial on momentum conservation in two dimensions (O'Brien Pride, p. 130). This result appears to conflict with the common view (see previous paragraph) that student understanding of the conservation of momentum (as a vector) in one dimension should improve after students have practiced momentum conservation in two dimensions, because, as some have said, "that's where you see momentum is really a vector."

Kautz[4] also reported in his Ph.D. dissertation (at the University of Washington) that many students at both the introductory level and the sophomore level (in a thermal physics course) have difficulty with the vector nature of velocity and momentum, and with determining the vector quantity *change in momentum.* Because Kautz's research focused on student understanding of the ideal gas law, including the microscopic model of an ideal gas, he was concerned with students' ability to relate the pressure exerted on a piston by the gas in terms of the mechanical impulse delivered to the piston in each collision between a gas molecule and the piston. Students were given "colliding-particle tasks," (Kautz, p. 169) in which they were asked to determine the change in momentum of a particle that collides with an extremely massive wall. The particle moves initially in a direction normal to the wall, and rebounds with the same speed. Many students stated incorrectly that the change in momentum of the particle was zero, supporting their answers with statements like "the momentum of the particle doesn't change, only its direction" or by adding the initial and final velocity vectors. In some cases, students related this answer to an incorrect scalar understanding of momentum conservation.

Boudreaux's[5] Ph.D. dissertation research (at the University of Washington) reported on student understanding of Galilean relativity. His study included an investigation of student difficulties with momentum in the context of multiple frames of reference. In addition to observing yet more contexts in which students treat momentum as a scalar (Boudreaux, p. 483), Boudreaux also observed that students tend to transform momentum vectors from frame to frame by using a momentum vector (incorrect), rather than the appropriate velocity vector (correct) (Boudreaux, p. 507). This tendency in reasoning may be understood as a particular instance of the general tendency of students to conflate related quantities. Furthermore, some students treated momentum as frame-independent (Boudreaux, p. 498), sometimes supporting this result with the statement "momentum is conserved in all reference frames," thus confusing conservation with frame-invariance (Boudreaux, p. 502).

Lawson and McDermott[6] studied students' ability to apply the impulse-momentum and work-energy theorems in one dimension. This research was conducted primarily with individual student interviews that included a demonstration in which two pucks of very different mass were on an air table and pushed with identical constant forces across a fixed distance. Students were asked to compare the final kinetic energies of the two pucks, as well as their final momenta. Since the pucks were pushed by the same force (F) through the same distance (Δx), the work done on them ($W=F\Delta x$) was the same; therefore their final kinetic energies were the same ($W=\Delta K$). The heavier puck took a longer time (Δt) to traverse the distance, and thus spent more time under the influence of the force and received a larger impulse ($J=F\Delta t$); therefore, the heavier puck had final momentum of greater magnitude than the lighter puck ($J=\Delta p$). This study found that students tended to think about kinetic energy and momentum primarily in terms of their mathematical definitions ($K=\frac{1}{2}mv^2$ and $\vec{p}=m\vec{v}$) rather than as quantities that change according to work done on a system or impulse delivered to a system, respectively. Of special interest is the authors' observation of the "compensation argument," in which students reasoned primarily on the basis of the algebraic relationship $p=mv$, stating for example, that the pucks (probably) had equal momentum (mv) because one had a greater mass (m), while the other had a greater velocity (v). We present more examples of this type of reasoning in this dissertation.

B. Research on Student Understanding of Newton's 2$^{\text{nd}}$ Law for Rotating Objects

We know of no published research on student ability to apply Newton's 2$^{\text{nd}}$ law to objects undergoing combined translational and rotational motion. The research presented on this topic in this dissertation is an extension of that reported by Ortiz in her Ph.D. dissertation[7] at the University of Washington. In that research, Ortiz characterized student reasoning as *location-based force reasoning,* according to which students "assume that the motion of the center of mass of a rigid body is determined by the sum of only the external forces that *act at the center of mass* rather than by the sum of all external forces acting on the entire body" (Ortiz, p. 35). Ortiz found that students often tended to break an individual applied force into orthogonal components, such that one component was directed at the center of mass. This component of the force was what some students thought related most directly to the acceleration of the center of mass of the object. A closely related tendency in student reasoning is to treat the relationship between forces and center-of-mass acceleration as though "the net force is the cause of *both* rotational motion and translational motion (of the center of mass) as if 'any force given to the block not used up in rotational acceleration will be given to translational acceleration'" (Ortiz, p. 44). Students sometimes gave similar explanations, referring to *energy* instead of *force*, but Ortiz concludes these two approaches probably represent a more primitive, common reasoning difficulty than two distinct conceptual errors.

The most important finding from Ortiz' research regarding addressing these difficulties with combined translation and rotational motion is that they are not easily addressed. That is, the typical success rate (fraction of students answering correctly) for tasks that required students to apply the idea that a force applied "off-center" has the same effect on the acceleration of the center of mass as a force applied "on-center" was about 35%. A majority of students, therefore, continued to use location-based force reasoning, even after all tutorial instruction. This result was the primary motivating factor for the present research – that, though student difficulties had been identified (with some understanding of the relationships between them) and curriculum had been written with these student difficulties specifically in mind, many students apparently did not change how they thought about forces in combined translational and rotational motion.

C. RESEARCH ON STUDENT UNDERSTANDING OF STATIC FRICTION

Besson and Viennot[8] describe an instructional intervention in which university students were taught to visualize static friction as resulting from contact between saw-tooth shaped surfaces. The authors show that student performance on complicated tasks involving static friction (*e.g.*, finding the direction of the static friction force on the drive and non-drive wheels of a car that is speeding up) improves dramatically when students have been taught this model for static friction forces. They conclude that students had a strong need for the *mechanism* of static friction, which instruction on Newton's laws could not provide. In this dissertation, we describe an attempt to help students transfer Newtonian reasoning about tension forces to situations with static friction forces, without explicitly instructing students on a mechanism for friction forces. Our choice not to include instruction on a mesoscopic model for friction was not because of any objection to the work by Besson and Viennot, but rather to explore students' ability to apply whatever productive ideas (including a mechanism, perhaps) that students had about tension to problems with friction.

D. RESEARCH ON STUDENT UNDERSTANDING OF ANGULAR MOMENTUM

In an effort primarily directed toward exploring the problem-solving strategies of experts in unfamiliar situations, Singh[9] studied how introductory students and physics professors responded to a particular problem requiring (according to one solution) the principle of angular momentum conservation and understanding of both the "spinning" and "particle" forms of angular momentum. In the problem, a rigid wheel spinning around a horizontal axis with angular speed ω_0 is dropped onto the floor, slips for some time, and then rolls without slipping. Subjects were asked to determine how the final speed of the center of mass of the wheel depended on the friction coefficient μ. Most professors interviewed had not solved the problem before, but were able to reason productively toward a correct solution. Though about half of the professors entertained the possibility of solving the problem using angular momentum conservation, none of them successfully applied the principle in their solutions. The introductory students, in contrast, did not identify the problem as one to which angular momentum conservation could be applied. This study helped us by identifying the problem as one that is probably not of appropriate difficulty for introductory students.

Ortiz' Ph.D. dissertation[7] also reported on students' ability to use the vector cross product, which is necessary for some tasks involving the definition of angular momentum for a particle: $\vec{L} = \vec{r} \times \vec{p}$. The task from that study that is most relevant to the research presented in this dissertation is a torque ranking in the context Ortiz called "Version A: Four hexagonal blocks" (Ortiz, p. 65). In that context, forces of equal magnitude and direction are applied at different points on four identical hexagonal blocks. Three of the four forces are exerted on the same line of action (*i.e.*, with the same moment arm) and thus result in equal torques (even though the forces are exerted at different points along that line of action). This problem is similar to tasks described in Chapter 7 of this dissertation, in which students compare the angular momentum of an object moving at constant velocity at different instants. Ortiz found that students tended to neglect the angle between the position and force vectors, focusing only on the magnitudes of each. Students also tended to treat torque as maximized when the position and force vectors are perpendicular, thus overemphasizing the role of the angle between the vectors and underemphasizing the role of the magnitude of the position vector. These patterns in reasoning are similar to those Ortiz observed in cross-product tasks without a physical context (Ortiz, p. 213) and to those we have observed in students' treatment of the cross product of position and linear momentum.

E. Psychological Research on Natural Reasoning

Another field of research that has influenced the research presented in this dissertation is psychologists' investigations of how people reason deductively. Perhaps the most well-known study (among cognitive psychologists) of college students' deductive reasoning ability was reported by Wason[10], in which students were asked to select cards to check whether they conformed to a certain logical rule. Students were told that they would be shown cards that have numbers on one side and letters on the other, and should turn over all and only those cards necessary to test violations of a rule like, "If a card has a vowel on one side, then it has an even number on the other." Students were presented with four cards, showing an "A," a "B," a "4," and a "7." (To check the rule given above, the correct response would be to turn over the cards with the "A" and the "7.") Students generally perform poorly (about 10% answer correctly) on such tasks, both before and after a one-semester introductory college course on formal logic, but are known to perform much more successfully in some cases for which the selection task is presented in a more "realistic" context.

In order to explain these and other similar results, Cheng and Holyoak[11] introduced the notion of the *pragmatic reasoning schema* as part of an alternative view of deductive reasoning, in contrast with two dominant views: that people reason using syntactic domain-independent rules of logic (a view usually associated with Piaget), and that people primarily use domain-specific knowledge without reasoning abstractly. The authors describe a pragmatic reasoning schema as an abstract knowledge structure that people induce from ordinary life experiences, and that consists of generalized, context-sensitive rules that are based on classes of goals and relationships of circumstances to these goals. Thus, a pragmatic reasoning system is sensitive to inferences that are *useful* to the realization of goals, unlike a purely syntactic system, which only distinguishes between valid and invalid inferences. An example of a pragmatic reasoning schema is the "permission schema," in which a conditional's consequent is understood as an action (or condition) that grants permission for a subsequent action (the antecedent of the conditional) to be taken. (For example, in order to enter a theater, a person should first obtain a ticket. *If* a person enters the theater, *then* the person must have a ticket. If a person has a ticket, the person need not enter the theater. But, *if* the person does *not* have a ticket, *then* the person may *not* (does not) enter the theater. And finally, if the person does not enter the theater, then the person (obviously) need not have a ticket.) Cheng and Holyoak present evidence that students are much more successful on selection tasks when they rationalize the task in terms of a pragmatic reasoning schema.

In a separate paper[12] that followed shortly afterwards, Cheng and her colleagues showed that instruction targeted at refining students' use of pragmatic reasoning schemas was more effective than instruction centered on helping students learn classical logical syntax, even when this instruction used a combination of "abstract training" and "examples training." The authors explain clearly that, though students' improved understanding of how to use pragmatic reasoning schemas to solve logical problems may look like an improved understanding of classical logical syntax, the systems do not function identically. Thus, in a sense, perfect facility with pragmatic reasoning does not strictly substitute for perfect facility with classical logic. However, they advocate teaching logic through pragmatic reasoning because such instruction is likely to be more effective and (by definition) more useful to most students.

The research and curriculum development reported in this dissertation have been strongly influenced by the work described above. In our tentative, working understanding we have associated "classical logic," which students do not naturally understand well, and in which instruction tends to be very ineffective, to correspond to the formal principles, definitions, and typical forms of persuasion (*e.g.*, derivations) of physics. We have associated "pragmatic reasoning schema" with various sorts of natural tendencies in reasoning in physics that are potentially useful to us in designing instruction. As in these psychological studies, we have sought to improve student performance on intellectual tasks by shifting the cognitive load onto students' knowledge structures, thus gaining some instructional (rather than mechanical) advantage.

NOTES TO CHAPTER 2

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[3] O'Brien Pride, T., "An investigation of student difficulties with two dimensions, two-body systems, and relativity in introductory mechanics," Ph.D. dissertation, Department of Physics, University of Washington, 1997 (unpublished).

[4] Kautz, C. H., "Investigation of student understanding of microscopic and macroscopic behavior of the ideal gas," Ph.D. dissertation, Department of Physics, University of Washington, 1999 (unpublished).

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CHAPTER 3. STUDENT LEARNING OF LINEAR MOMENTUM IN ONE DIMENSION

When this investigation began, part of the regular curriculum of the introductory calculus-based course at the University of Washington was the published version[1] of the tutorial *Conservation of Momentum in One Dimension.* This version of the tutorial was designed on the basis of research[2,3] on student understanding of momentum that had been conducted previously by the Physics Education Group at UW. We began the investigation by examining student performance after tutorial instruction on a set of questions about momentum conservation in one-dimensional collisions. These questions consisted of comparisons of the outcomes of collisions with similar initial conditions, differing only in elasticity. The task was for students to rank the final speeds of a set of initially stationary target objects, using information about how the incident objects moved after the collisions. In studying responses to these questions, we had two reactions. First, we were disappointed at the typical success rate. It seemed to us that students ought to be able to do as well on this sort of task as was typical with other related tasks with momentum vectors in the same contexts. Second, our attention was drawn to a particular type of incorrect response that had not been previously described in the known research. Namely, some students gave final speed rankings that were not consistent with the tendency of students to neglect the directional aspect of momentum or confuse momentum with energy. We modified the published tutorial to try to address the error directly, but our initial efforts were ineffective. We pursued a deeper understanding of the error through a series of one-on-one student interviews and new pre- and post-tests. As we followed the path of new tasks and students' responses to them, we understood that students found a particular special case important – a case on which they appeared to base a significant aspect of their understanding of linear momentum. The case was an elastic collision with mass ratio very far from unity. Students apparently used their interpretation of momentum in this case as a logical reference point for considering other, more mundane cases. Returning to the tutorial instruction, we modified the tutorial so that it centered logically on this special case. Results from post-tests suggest that students have a better understanding of momentum after they are guided to analyze this important case than they do after deriving momentum relationships from Newton's laws, or after having their own conceptual errors about momentum presented plainly to them as such.

In part A of this chapter, we describe the published version of the tutorial, student performance on some linear momentum tasks after working through the published tutorial, and some particular tendencies in student reasoning that inspired further development of the tutorial instruction. In part B, we characterize student understanding of linear momentum after standard instruction and before tutorial instruction, especially on some tasks not previously described in the known research. In part C, we describe the current tutorial and homework, detailing the motivations for and intended functions of each part of the curriculum. In part D, we present evidence that student understanding of linear momentum after the current version of the tutorial sequence is better than that after the published version of the tutorial sequence, as measured on a variety of tasks.

A. STUDENT UNDERSTANDING OF MOMENTUM IN COLLISIONS AFTER INITIAL (PUBLISHED) VERSION OF TUTORIAL *CONSERVATION OF MOMENTUM IN ONE DIMENSION*

In her Ph.D. dissertation research[4], T. O'Brien Pride described some common tendencies among student approaches to the concept of linear momentum. These tendencies included: a failure to distinguish between momentum and energy in collisions; a failure to recognize that the outcome of a collision is not uniquely predicted by applying momentum conservation only; a failure to treat momentum as a vector quantity; the common prediction of "complete transfer" of some sort of motion in collision demonstrations. In an ongoing and iterative response to these research results, O'Brien Pride and other members of the Physics Education Group designed curriculum to address these issues. Instructional strategies that were tried included: an emphasis on relating momentum conservation to Newton's laws; having students analyze an inelastic collision and calculate the change in mechanical energy; an additional tutorial on momentum conservation in two dimensions; frequent use of vector diagrams to represent momentum. The following section describes the published version of the tutorial, the authoring of which (*c.* autumn 1997) was the culmination of a period of focused research and curriculum development on momentum.

1. Description of published version of tutorial

The beginning of the tutorial has students consider a context that we now call "2 collisions, equal target mass, with stop." In this context, glider A is incident on glider M in Experiment 1, and incident on glider N in Experiment 2, as shown in Figure 3-1.

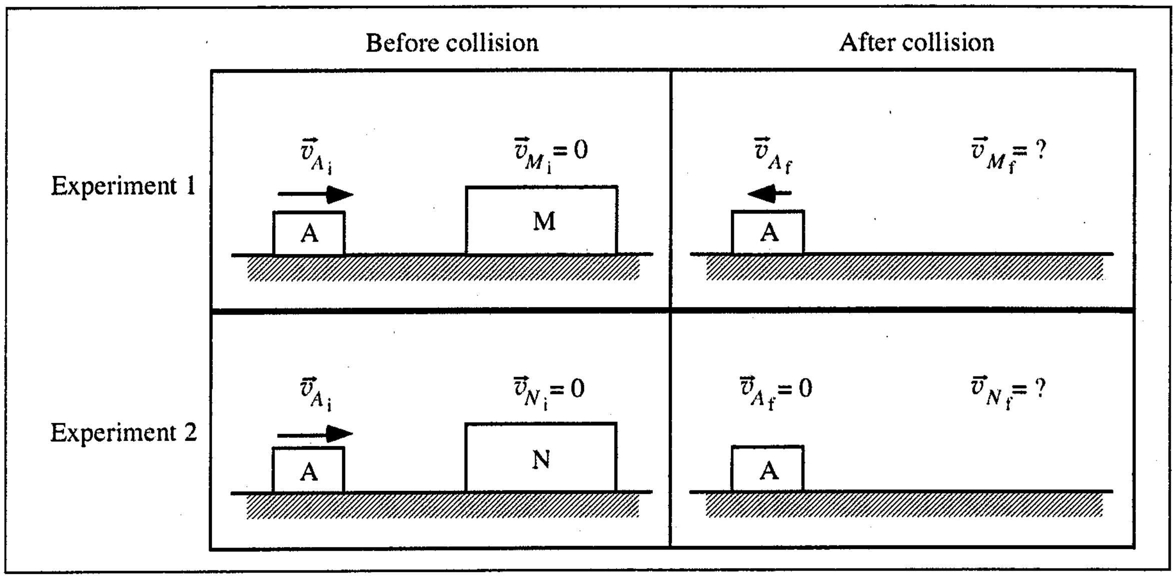

Figure 3-1: Collision context from the published version of the tutorial *Conservation of Momentum in One Dimension.*

Glider A has the same initial velocity in each experiment. Gliders M and N have the same mass, which is five times the mass of glider A. After Experiment 1, glider A is moving backwards, and after Experiment 2, glider A is at rest. (Though students are not asked at this point in the tutorial whether they think the final speed of glider M will be greater than, less than, or equal to the final speed of glider N, in principle they would have answered that question on the pretest, a few days before working through the tutorial.) Students draw free-body diagrams of both gliders A and M for some arbitrary instant while they are in contact during Experiment 1. They are expected to use Newton's third law to say that the net force on glider A has necessarily the same magnitude as the net force on glider M, since the only horizontal force on either glider is that by the other glider. Next, students manipulate Newton's second law into a form that relates the impulse on an object to its change in momentum. Students compare the net impulse on glider A to that on glider M and conclude that the gliders must have equal and opposite changes in momentum. Continuing in a single line of logic, students show that the change in momentum of glider A in Experiment 1 is greater than that in Experiment 2, since glider A reversed direction instead of merely coming to rest. Putting the two previous pieces together, students conclude that the change in momentum of glider M must therefore be greater than the change in momentum of glider N. At the end of this chain of logic is the result that glider M must have a greater final speed than glider N. After concluding this, students read a hypothetical comment by a student who claims that the final speed of glider N must be *greater than* that of glider M: "In experiment 2, glider A transfers all of its momentum to glider N, whereas in experiment 1, glider A still has some momentum left so glider M does not get as much." (This type of comment was common on pretests dealing with these collisions.) Students discuss whether they agree or disagree with this statement, and then they discuss their answers with a tutorial instructor.

In the next section of the tutorial, students consider the free-body diagrams for each individual glider and the system of both gliders during a collision. They observe that the forces of the gliders' mutual interaction are internal to the system of both gliders, and therefore do not appear in its free-body diagram. Next, students use the definition of the momentum of a system of multiple objects to derive a relation between the change in momentum of that system and the changes in momentum of the individual objects. Students then observe that the momentum of each glider changes during the collision, but the momentum of the combination remains the same. Students can reach this conclusion either by noting that the net force on the combined system is zero, or by using the previous result, that the change in momentum of a system equals the sum of the changes in momentum of its parts, along with the earlier result that these changes in momentum are equal and opposite.

The final section of the tutorial is a variation on the previous section; in this case, the second glider is fixed to the table when the first glider collides with it. Again, students draw free-body diagrams for the gliders during the collision, establishing that the net force on the stationary glider must be zero, while the net force on the first glider and that on the system of both gliders must each be non-zero. Students infer (from the description of motion) that the momentum of the first glider must change, the momentum of the second glider remains the same, and, therefore, that of the system also must change. Students are then guided to generalize the correlation between the free-body diagram of an object and how its momentum changes. Specifically, they note that it is not the presence of external forces *per se* that change the momentum of a system, but the presence of a non-zero net force.

The published homework for *Conservation of Momentum in One Dimension* included exercises designed to help students: distinguish the relationship between colliding objects' changes in momentum from the relationship between their changes in velocity; relate the change in momentum of an object or system of multiple objects to the net force on the object or system; and understand that the Galilean transformations of both velocity vectors and momentum vectors for objects in a collision must be performed through the addition of velocity vectors rather than momentum vectors.

2. Description of contexts in which questions about linear momentum were asked

This section describes various collision contexts within which we asked students written questions to test their understanding of various aspects of linear momentum, both after traditional instruction and the tutorial instruction described above. The questions themselves were asked in many contexts and are described separately in the next section.

Some of the characteristics of the question contexts described in this chapter are common to all of the contexts; other characteristics are varied from context to context. Unless otherwise noted, all of the contexts involve an incident object moving to the right across a level, frictionless surface toward an initially stationary target object. If more than one collision is described in a context, then the incident objects all have the same mass and move with the same initial velocity.

Because there are a number of different combinations of contexts and questions described in this chapter, we have developed a notation to describe them concisely. First, objects in a collision are given standard names. If a context involves two collisions, then in the first collision ("Experiment 1"), the incident object "A" collides with the target object "X;" in the second collision ("Experiment 2"), the incident object "B" collides with the target object "Y." If the context includes a third collision, then object "C" collides with object "Z."

Contexts are also named according to a condensed naming scheme. For example, the context name "3.ETM.S" indicates that the context describes three collisions, with equal target masses, including one collision in which the incident object comes to a stop. The first field of the context name indicates the number of collisions and has possible values of 1, 2, or 3. The second field indicates whether the target masses are all equal (ETM), all different (DTM), or all "giant" (GTM). (In "GTM" collisions, the ratio of the mass of the target glider to the mass of the incident glider is much greater than unity. If a GTM context involves more than one collision, then the target masses are of equal giant mass.) The third field will contain either "S" for "stop" or "NS" for "no stop." "S" indicates that one incident object comes to a stop; "NS" indicates that no incident objects come to a stop. An optional fourth field indicates which, if any, objects are known to have equal final speeds. If no objects have equal final speed, then the field will be empty. If the incident (B) and target (Y) gliders in the second collision have equal final speed, then the field will contain "BYEFS." If the incident objects in the second and third collision have equal final speed, then the field will contain "BCEFS." An example of this usage would be "2.DTM.S.BYEFS," to describe a context involving two collisions with different target masses, in which one incident object stops after the collision, while in the other collision, the incident and target objects move away from each other at equal final speeds.

The following three contexts are those that we used to assess student understanding after the published version of the tutorial. More contexts are described in this chapter's later sections, in which we characterize student understanding of linear momentum before tutorial and after the current version of the tutorial.

a. 3.ETM.S

This context involves three collisions, with equal target mass, in which one of the incident objects comes to a stop (see Figure 3-2). Objects A, B, and C move to the right with speed v_0 and collide with objects X, Y, and Z, respectively. The incident objects each have mass m, which is less than M, the mass of each of the target objects. Objects X, Y, and Z are initially at rest. After the collisions, object A is moving to the left with speed 1/3 v_0, object B is at rest, and object C is moving to the right with speed 1/6 v_0. The final speeds of objects X, Y, and Z are not known. The objects are pucks, and are shown from a top-view.

b. 3.ETM.S backwards

This context is a variation on 3.ETM.S. Instead of describing how particular incident objects move after their collisions, we tell students about the final speeds of the target objects: the final speed of object X is greater than that of object Y, which is greater than that of object Z (see Figure 3-2). All target objects move to the right after the collisions. Students are also told that, in one of the collisions, the incident object has zero final velocity; in another, the incident object moves with final velocity to the left; in another, the incident object moves with final velocity to the right. Students are not told in which collision the incident glider has any particular final motion. The objects are gliders and are shown from a side-view.

c. 3.ETM.NS.ABCEFS 2-D

In this context, objects A, B, and C all have equal final speed, which is less than their initial speed. The objects are pucks, and are shown in a top-view in Figure 3-2. The initial velocity of objects A and B are to the right, while the initial velocity of object C is down and to the right. After the collisions, object A moves to the left, object B moves to the right, and object C moves down and to the left. The final speeds of objects X, Y, and Z are not known. The incident object mass is less than the target object mass.

In one slight variation on this context, students were told that the elasticity of each collision may not be the same. In another variation, students were given an extra reminder: "Recall that targets X, Y, and Z have equal masses."

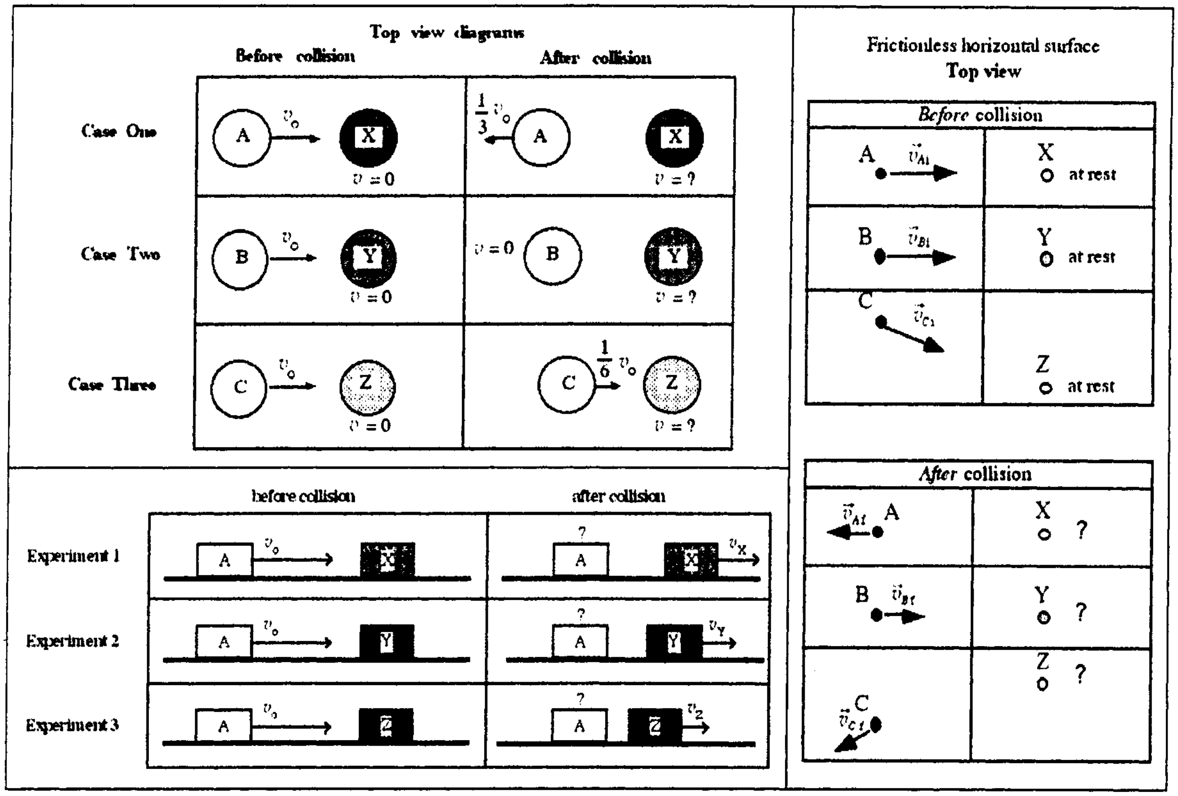

Figure 3-2: The "3.ETM.S" context (upper left), the "3.ETM.S backwards" context (lower left), and the "3.ETM.NS.ABCEFS 2-D" context (right).

3. Description of questions about linear momentum

This section describes questions that were used to assess student understanding of linear momentum, both after traditional instruction only and after all instruction on the topic, including the published tutorial and homework. The questions described below were all asked within one of the contexts described in the previous section.

a. The *Momentum transfer* question

In the *Momentum transfer* question, students are asked to compare the magnitudes of the change in momentum vectors for two objects in a single collision. In most cases, the question was asked about a collision in which the incident object rebounded after colliding with the initially stationary target object. The purpose of this question was to determine the frequency with which students treated the quantity "change in momentum" as representing an exchange of a conserved quantity of motion between interacting objects. To answer the question correctly, it is not necessary to realize that momentum is a *vector* quantity of motion. For the contexts described above, the question appears as "Is the magnitude of the change in momentum of object A greater than, less than, or equal to the magnitude of the change in momentum of object X?"

In some cases, the question was phrased in terms of impulse instead of change in momentum: "Is the magnitude of the impulse imparted to object A greater than, less than, or equal to the magnitude of the impulse imparted to object X?" This version was asked in only one section.

b. The *Incident* Δv question

The *Incident* Δv question asks students either to compare the magnitudes of the change in velocity vectors for a pair of incident objects, or to rank the magnitudes of the change in velocity vectors for three incident objects. In the published version of the tutorial, students constructed the change in momentum vectors for the incident objects and used the fact that the change in momentum vector for any incident object is equal in magnitude to that of its target object to compare the final momenta (or speeds) of the target objects. Therefore, the *Incident* Δv question was intended as a "helper" question on post-tests – an opportunity for students to be reminded to look at changes in momentum through the process of vector subtraction. The question also helped us understand where the conceptual failure occurred, in those cases in which students were unsuccessful at comparing the final momenta of target objects. That is, if students were very successful on the *Incident* Δv question, then we would know that a failure to compare the final momenta of the targets was generally not due to a failure of students' understanding of how to construct changes in vector quantities.

c. The *Final speed* question

In the *Final speed* question, students are asked to compare the final speeds of two equal mass targets, or to rank the final speeds of three equal mass targets. In the context "3.ETM.S backwards," the *Final speed* question is reformed into the task of identifying in which collision the incident glider moves in which direction (or comes to a stop).

The *Final speed* question has been asked only in ETM contexts. If it were known that one target mass was greater than another (as in DTM contexts), it could still be inferred how their final momenta compare in magnitude, but more information would be needed in order to determine how their final speeds compare. For example, if it were known that both collisions were elastic, then it could be determined that the target mass with the greater final momentum would actually have the lower final speed. In GTM contexts, the final speed of the target is necessarily very small in comparison with other speeds in the problem. Therefore, in order to ask students to perform a task in DTM and GTM contexts that is roughly equivalent to the *Final speed* question in ETM contexts, we ask students the *Final momentum* question (described later in this section). The *Final momentum* question serves a similar but not identical set of research functions as the *Final speed* question.

4. Correct answers to questions about linear momentum

a. The *Momentum transfer* question

In every context described above, the magnitude of the change in momentum of object A must be equal to the magnitude of the change in momentum of object X. Since the net force on the system (of objects A and X) is zero during the collision, the change in momentum of the system must be zero. The change in momentum of the system is the vector sum of the changes in momentum of the parts of the system. In order for the changes in momentum of the parts to add to zero, they must be of equal magnitude and in opposite direction. Note that a scalar understanding of momentum can also lead to the correct answer, according to the logic: the system does not lose or gain any momentum; whatever amount of momentum is lost by object A is gained by object X.

Alternatively, it can be reasoned in terms of impulse (perhaps more suitable for the "impulse" version of the question), that at every instant, the force exerted on object A by object X is equal in magnitude and opposite in direction to the force exerted on object X by object A. The time integral of each of these forces over the duration of the collision results in the same magnitude of impulse.

b. The *Incident Δv* question

Using the context 3.ETM.S as an example, in which object A rebounds to the left, object B comes to rest, and object C continues to the right, the magnitude of the change in velocity of object A is greater than that of object B, which is greater than that of object C. A leftward change in velocity vector must be added to object A's initial rightward velocity to result in a final velocity to the left. A *shorter* change in velocity vector must be added to object B's initial rightward velocity to make a final velocity of zero. An even shorter change in velocity vector must be added to C's initial velocity to the right to make a shorter final velocity that points to the right.

c. The *Final speed* question

Using the context 3.ETM.S as an example, the final speed ranking of targets X, Y, and Z is $v_X > v_Y > v_Z$. According to the reasoning encouraged by the published tutorial, in each collision, the change in momentum of the incident object is equal in magnitude to that of the target object. The change in momentum of object A is greater than that of object B, which is greater than that of object C, by using the change in velocity ranking above with the fact that A, B, and C have equal mass. Therefore, if object A has the greatest change in momentum of the incident objects, object X must have the greatest change in momentum of the target objects, and so on. The ranking of X, Y, and Z according to magnitude of the change in momentum is the same as the ranking according to final speed, since they all start from rest and have equal mass.

5. Administration of questions

All of the following results are from questions asked on midterm exams after all instruction on linear momentum conservation, including the published tutorial and homework.

The *Momentum transfer* question (in the context 3.ETM.NS.ABCEFS 2-D) was asked in three sections of UW 121. In one of the sections, the "impulse" version of the question was asked. Results were similar[5] and have been combined.

The *Incident Δv* question was asked in three sections of UW 121; two sections answered the question in the context 3.ETM.NS.ABCEFS 2-D, and one section answered it in the context 3.ETM.S. Student performance on this question did not appear to depend on which of these two contexts was used, so the results have been combined.

The *Final speed* question was asked in four sections of UW 121 and one section of CIN 201. Two sections of UW 121 answered the question in the 3.ETM.NS.ABCEFS 2-D context, one section answered it in the 3.ETM.S context, and one section answered it in the 3.ETM.S backwards context. The section of CIN 201 also answered the question in the 3.ETM.S backwards context. Because students in CIN 201 often tend to perform somewhat differently from students in UW 121 with similar instruction, these results are reported separately.

6. Overview of student performance

a. The *Momentum transfer* question

After all instruction, approximately 70% of students answered the *Momentum transfer* question correctly, as shown in Table 3-1. Considering that the tutorial strongly emphasized the point that the changes in momentum of two objects in a collision are equal and opposite, and that the further conclusions about momentum conservation are based on this rule, it might seem unsatisfactory for 70% of the class to understand this key point. In fact, because these data report the number of students choosing certain answers, without regard to reasoning, a student response that merely stated that "changes in momentum are always equal and opposite" would be tallied in the "correct" category. If we look at the correct answer as an easily memorized nugget, 70% correct might seem significantly disappointing. What might explain the low success rate then, is some inhibiting factor – some (incorrect) way of understanding momentum that might be more compelling to the students than a rule they have derived or been told.

Table 3-1: Student performance on the *Momentum transfer* question in the context 3.ETM.NS.ABCEFS 2-D after traditional instruction and the published version of tutorial *Conservation of momentum in one dimension*. Success did not depend on whether students were asked to compare "change in momentum" or "impulse."

A rebounds after colliding with X.	UW 121, N=436 3 sections
$\lvert\Delta\vec{p}_A\rvert = \lvert\Delta\vec{p}_X\rvert$ (correct)	70%
$\lvert\Delta\vec{p}_A\rvert > \lvert\Delta\vec{p}_X\rvert$	20%
$\lvert\Delta\vec{p}_A\rvert < \lvert\Delta\vec{p}_X\rvert$	10%

b. The *Incident Δv* question

As shown in Table 3-2, student performance on the *Incident Δv* question also appears to be good (80% correct), especially considering how much trouble students tend to have with vector subtraction at the beginning of the course. Because the question was asked in contexts with three collisions, students ranked three change-in-velocity vectors according to magnitude. The sorts of incorrect rankings students gave are consistent with those reported by recent research[6] on student understanding of vector operations. There are two main purposes in presenting these data here. First, students are capable of a fairly high success rate on at least one task involving momentum vectors, which we interpret as meaning that the students are capable of greater success on the *Final speed* question than they demonstrated after working through the published tutorial. (Data from the *Final speed* question are presented in the next section.) Second, in order to evaluate revisions of the curriculum, we want to compare student performance after different instruction on as many tasks as possible.

Table 3-2: Student performance on the *Incident* Δv question in various 3.ETM contexts after the published version of the tutorial *Conservation of Momentum in One Dimension*. Results from different contexts were similar and are combined.

	UW 121, N=392 3 sections
Correct ranking	80%

c. The *Final speed* question

As shown in Table 3-3, Student success on the *Final speed* question appears not to depend strongly on the context in which the question is asked. Furthermore, certain sorts of errors stand out, across contexts, and at different universities. Our examination of student responses to the *Final speed* question was what sparked the notion that we could make significant progress in our understanding of student treatment of momentum. That is, we were motivated both by the dissatisfying typical success rate (~55%) on the *Final speed* question and by the consistent tendency of some students (~10%) to respond with a "reverse ranking" (in which students give a ranking that is the correct ranking in reverse order). We were also motivated by our frustration with our limited understanding of the known, yet persistent, student tendency to treat momentum as a scalar quantity (~20%).

Table 3-3: Student performance on the *Final speed* question in various 3.ETM contexts after the published version of the tutorial *Conservation of momentum in one dimension*.

Example rankings for 3.ETM.S	UW 121, N=289 2 sections 3.ETM.NS. ABCEFS	UW 121, N=109 1 section 3.ETM.S	UW 121, N=125 1 section 3.ETM.S backwards	CIN 201, N=293 1 section 3.ETM.S backwards
$v_X > v_Y > v_Z$ (correct)	60%	55%	55%	50%
$v_Y > v_X > v_Z$ (scalar)	10%	20%	25%	25%
$v_Z > v_Y > v_X$ (reverse)	10%	15%	10%	10%

7. Discussion of patterns in student reasoning

a. Tendency to treat momentum as a scalar quantity

The tendency of students to treat momentum as a scalar quantity has been reported previously.[7] It was with this tendency in mind that the published tutorial and tasks like the *Final speed* question were designed. The signature characteristic of "scalar momentum" reasoning is the combination of emphasizing the final speed with which various incident objects move, while deemphasizing the *direction* in which they move. In some cases, students use the word *momentum* to refer to the scalar quantity of motion; in other cases, they refer to a transfer of *kinetic energy* or *speed.* Our identification of which responses exemplify "scalar momentum" reasoning has not depended on which word students used, since it is generally well known that students often do not distinguish between different quantities of motion.

The following three typical responses are taken from the *Final speed* question of the context 3.ETM.S, by students who gave the "scalar" ranking $v_Y > v_Z > v_X$:

> "(Puck) Y gets all of the p and v from B. Z gets almost all of the velocity, and X gets the least. Since they are all the same mass they can be compared using $p=mv$ and $\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2'$."
>
> "I would say Y would be the fastest because all energy from B has been transferred to it. Next would be Z because it received 5/6 of C's energy. And last would be X, because A rebounded."
>
> "Y has the largest final speed because upon collision, B has a velocity of 0, and Y moves, since momentum is conserved, and B has a velocity of 0, all final momentum will be converted to Y. (Good chance that happened because collision was elastic – pucks didn't stick.) Z is second because 5/6 v_0 can go to Z, making it greater than X."

Besides exhibiting scalar momentum reasoning, the above responses also show a common tendency to exchange the words *energy, momentum,* and *velocity* freely. The first response above shows that a student can recognize that momentum ought to be symbolized as a "p" with a vector symbol over it, without understanding what implications that symbol might have for the treatment of momentum.

In the context 3.ETM.S backwards, "scalar momentum" reasoning looks like this:

> "A transfers its energy to X, so it stops while X moves. That's why X is moving the fastest."
>
> "Conservation of momentum, X block takes up the most velocity transfer. Since final speeds are X > Y > Z, this means that all of the momentum is transferred in X, only some in Y, and none into Z, since it is stationary after the collision."

It is not possible to answer the *Final speed* question in the 3.ETM.S backwards context completely according to scalar momentum reasoning; since students are asked to determine the direction of final velocity for each of the incident gliders, the best a student can do with scalar momentum reasoning alone is to determine the collision in which the incident glider stops. Then, in order to match leftward and rightward final velocities of incident gliders with faster and slower target speeds, students must appeal to some other idea about how collisions work. Therefore, no complete response to the *Final speed* question in this context can be considered "purely scalar," but either response that states that A stops (B moves right, C moves left; or B moves left, C moves right) would be considered "partially scalar."

b. Tendency to treat collisions with differing elasticity and equal target masses as elastic collisions with different target masses

In our examination of reverse ranking responses to the *Final speed* question, we found that a significant minority of students seemed to understand the set of collisions in 3.ETM.S as something different. Namely, the collisions in 3.ETM.S all have equal target mass and differ in outcome because of differences in elasticity; however, some responses described a set of collisions that were (or may have been) all elastic, and differed in outcome because of differing target masses. The following two responses are typical, and show the varying levels of explicitness:

> "Puck Z is the lightest among the pucks, therefore it bounces off C. Puck Y is the second lightest, and puck X is the heaviest."
>
> "C keeps moving in the same direction so Z didn't resist moving to the right very much, so it must have moved the fastest. ... X was strongly resisting movement so it won't move very fast."

This set of responses could be interpreted as resulting from students' having difficulty understanding the context being described in a tense exam situation. In order to make sure that the information that the target masses were equal was not hard for the students to find among all of the other information (though it appeared twice already - in a sentence and in an equation) we tried adding a special reminder of this fact, immediately adjacent to the *Final speed* question. The reminder did not affect the incidence of reverse rankings. This fact suggested to us that students might have a strong visual image of such collisions, and that it might be helpful in instruction to focus on helping students understand those collisions.

c. Tendency to associate a large change in motion for one object with a small change in motion for the other object

Other reverse ranking responses to the *Final speed* question used a kind of reasoning distinct from those that focused on varying the target mass. Some students seemed to be applying an idea (that we call "shared delta-v") that there is a roughly inverse relationship between the sizes of the changes in velocity (or momentum) of the two objects.

> "Z > Y > X. We know this because momentum is conserved. So, the object that had the smallest change in velocity vector passes the greater momentum to the object it collides with."

> "$v_Z > v_Y > v_X$ because $\Delta v_A > \Delta v_B > \Delta v_C$. The greater change in speed of the left puck (A, B, C), the less change in speed right puck (X, Y, Z)."
>
> "Z > Y > X. Because of the law of conservation of momentum, we can see that the smaller the change in momentum of puck with mass *m*, the greater the final velocity will be of pucks X, Y, or Z."

It is clear from these responses that these students did not answer the *Final speed* question incorrectly because they answered the *Incident Δv* question incorrectly; the students used their correct answers to the *Incident Δv* question in an incorrect way. Though the second student refers to "change in speed," the ranking she quotes from her answer to the *Incident Δv* question is correct, suggesting that her understanding of the words "change in speed" is more like our understanding of "change in velocity." Many responses, like the first and third ones above, suggested that some students associated the "shared delta-v" idea with conservation of momentum, perhaps thinking that the idea they were using was equivalent to applying the principle of conservation of momentum.

Some students combined the "shared delta-v" idea with the idea of varying the target mass:

> "Z > Y > X. Because the less ABC changes, the lighter M must be so the more it should change."

Such responses suggested to us that the two ideas were perhaps related. The next section describes our attempt to make sense of this assortment of persistent conceptual errors.

8. Unification of apparently distinct errors

This section shows that the scalar momentum error and the reverse ranking error may be understood as closely related. That is, under certain typical conditions for collisions, these errors are indistinguishable, linked logically by the more basic error of conflating velocity and momentum.

An interesting fact about all elastic collisions in one dimension is that the relative speed with which the objects approach each other is the same as that with which they recede from each other. (The direction of the relative velocity is of course reversed as a result of the collision.) This can be seen most easily by analyzing the collision in the center-of-mass frame, ensuring that the initial and final kinetic energies are equal, and noting that relative velocities are frame-independent. For a collision between objects 1 and 2, this can be written as:

$$\vec{v}_{1i} - \vec{v}_{2i} = -(\vec{v}_{1f} - \vec{v}_{2f}) \tag{3-1}$$

Taking the magnitude of each side, we have:

$$\left|\vec{v}_{1i} - \vec{v}_{2i}\right| = \left|\vec{v}_{1f} - \vec{v}_{2f}\right| \tag{3-2}$$

As long as the objects do not move in the same direction either before or after the collision (a condition that is satisfied in the familiar situation where $\vec{v}_{2i} = 0$ and $m_2 > m_1$), then we can write:

$$\left|\vec{v}_{1i}\right| + \left|\vec{v}_{2i}\right| = \left|\vec{v}_{1f}\right| + \left|\vec{v}_{2f}\right| \tag{3-3}$$

This statement looks like a *conservation of speed* relationship, if such a thing were possible. We do not suggest that any but perhaps a tiny minority of students may have worked through this logic or have even seen the above equations. However, we do suspect that many students have roughly perceived this principle when viewing elastic collisions in which the target mass is initially at rest and bigger than the incident mass. That is, one can easily observe that the faster the incident object is moving after it rebounds, the slower the target object moves. If one is even more perceptive, then it can be seen that the sum of these speeds (the relative speed at which they separate) is roughly equal to the incident object's initial speed. If a student had observed such a pattern, it might be easy to mistake it for conservation of momentum (to which it is not logically equivalent) and infer something like:

$$\left|\vec{p}_{1i}\right| + \left|\vec{p}_{2i}\right| = \left|\vec{p}_{1f}\right| + \left|\vec{p}_{2f}\right| \qquad \text{“Scalar momentum conservation” (false)} \tag{3-4}$$

Equation (3-4) is not something that we believe many students ever wrote down; rather, it represents an interpretation students may have of formally correct statements of momentum conservation. In the way described above, we may understand scalar momentum reasoning to have an empirical basis in a certain set of elastic collisions.

Similarly, "shared delta-v" reasoning may share this empirical basis with scalar momentum reasoning. Starting again from Equation (1), and adding the left-hand side to each side, we obtain:

$$2(\vec{v}_{1i} - \vec{v}_{2i}) = -(\vec{v}_{1f} - \vec{v}_{2f}) + (\vec{v}_{1i} - \vec{v}_{2i}) \tag{3-5}$$

Rearranging terms and taking the magnitude of each side:

$$\left|2(\vec{v}_{1i} - \vec{v}_{2i})\right| = \left|-\Delta\vec{v}_1 + \Delta\vec{v}_2\right| \tag{3-6}$$

Because $\Delta\vec{v}_1$ and $\Delta\vec{v}_2$ are always in opposite directions, the right-hand side can be changed:

$$\left|2(\vec{v}_{1i} - \vec{v}_{2i})\right| = \left|\Delta\vec{v}_1\right| + \left|\Delta\vec{v}_2\right| \tag{3-7}$$

If we consider a set of collisions (as students do when answering our momentum questions) for which $\vec{v}_{1i}$ and $\vec{v}_{2i}$ are each the same over the collisions, then a result is that:

$$constant = \left|\Delta\vec{v}_1\right| + \left|\Delta\vec{v}_2\right| \tag{3-8}$$

Just as equation (3-3) is true for elastic collisions in which the two objects do not move in the same direction before or after the collision, equation (3-8) is *true*, for a set of elastic collisions that have the same pair of initial velocities. Equation (3-8) leads logically to the (perhaps) easily observable empirical principle: The more one object's velocity changes, the less the other object's velocity changes. If students had observed such a pattern and incorrectly generalized to all collisions in which momentum is supposed to be conserved, they might answer the *Final speed* question with a reverse ranking. Many students did this; some stated the principle as an empirical fact (which it is, in a different set of collisions), some described the rule as equivalent to momentum conservation (which it is not), and some (correctly) described how the target masses would have to compare in order for this relationship between changes in velocity to hold.

In constructing this sort of perspective on two important and persistent errors students made, we understood the role of elastic collisions in instruction differently than before. We had originally thought of partially or completely inelastic collisions as more pedagogically useful for teaching momentum conservation than elastic ones. This is because, in such collisions, the momentum of the system is constant while the kinetic energy is not, which demonstrates that momentum and kinetic energy are distinct concepts. As a result of the above reflections on student errors and reasoning, we understood an instructional context of elastic collisions as offering an opportunity to help students distinguish between momentum and *velocity*. It is with this approach in mind that the following interview tasks, tutorial curriculum, and assessments of student learning were designed.

9. Interviews: round 1

In this first round of interviews, we hoped to find some students who would reveal more about why they might use something like a "shared delta-v" idea in a context with equal target masses. All of the students in this set of interviews had recently completed Physics 121 at the University of Washington.

a. Description of interview tasks

The interviews centered on a demonstration of two collisions, each involving two carts, on parallel tracks. The demonstration was designed to be as similar as possible to the context 2.ETM.S. In order for collisions to involve the same masses and yet have different outcomes, the collisions must differ in the elasticity of the interaction between the carts. To achieve this, one set of carts (A and X) interacted through the (mostly) elastic plunger, which was initially in its relaxed, extended position; in the other collision, the plunger was also initially extended, but wrapped with just the right amount of tissue paper and tape so that the incident cart (B) stopped as a result of the collision. The carts were released from the same height on inclined tracks, which connected to level tracks, on which the collisions occurred.

In the interviews, the differences and similarities between the setup of the collisions was described to the students. Following this, the students were asked to predict which target cart would move faster after the collision. The interviewer then performed the demonstration, and invited the student to reflect on the relationship between the prediction and the outcome, accounting for any differences.

b. Discussion of results

The most important result from this round of interviews was that, when students observed an outcome that differed from their predictions, they tended to question the assumption that each system had constant momentum rather than questioning their approach to the concept of momentum. The following excerpt illustrates this process. Before observing the demonstration, the student recognizes, with some conflict, that he was taught that momentum was constant even if the collision was not elastic.

> S: I'm pretty sure momentum is only conserved in an elastic collision. But then in lab we tested inelastic collisions, and momentum still worked. Now I'm confused!

After observing the demonstration (in which the target (X) hit by the rebounding incident glider (A) moved faster than the target glider (Y) hit by the stopping incident glider(B)), the student remarked:

> S: No, momentum isn't conserved here [points to the partially inelastic collision involving B and Y]. I mean, how can it be? There's a greater momentum here (A) and a greater momentum there (X)!

Such responses were typical; students sooner concluded that momentum was not constant in the "deader" collision than concluded that they were thinking about momentum incorrectly. If students are to confront their understanding of what "conservation of momentum" means, then we thought the situation in which they should experience some cognitive conflict ought to be one for which they would not find the conclusion "momentum is not conserved" acceptable.

10. Description of intermediate modifications to tutorial instruction

This section describes several modifications to the tutorial instruction, none of which changed student performance on various versions of the *Final speed* question after tutorial instruction.

In one version of the tutorial, we introduced a demonstration that corresponded to the 2.ETM.S context at the beginning of the tutorial. The purpose of the demonstration was to emphasize that the target objects ("M and N") did have equal mass, since we knew that students had a tendency to infer that the target masses must be different if the outcomes of the collisions are different.

In another modification, we added a student dialogue, in which one of the hypothetical students explained that the more the incident object's velocity changes, the less the final speed of the target object. For the most part, students seemed to have trouble understanding why someone would say something like that. Another hypothetical student explained that the masses of the target gliders must be different, since the incident glider bounced back in one collision and not the other. To this, most students responded by observing flatly that the problem stated that the target gliders had the same mass. Instead of being stimulated to affirm cautiously why one might *expect* the masses to be different, students simply labeled the statement as incorrect, apparently reluctant to sympathize with reasoning that to them was clearly wrong.

In the last ineffective modification, we added a context between the second and third pages of the published tutorial, intending the new context as an opportunity for students to apply the change-in-momentum procedure they had just learned to a pair of collisions with different target mass. In this context, one incident object (A) stopped and one (B) rebounded, but the one (X) hit by the stopping object moved faster than the one (Y) hit by the rebounding object. Students were asked to compare the magnitudes of final momentum of the target objects. The correct treatment would be to ignore the final speeds of the targets and focus on the behavior of the incident objects. Students did not appear to connect their thinking about the previous problems to the present problem, usually not even attempting to apply the change-in-momentum procedure that they had ostensibly just learned.

11. Interviews: round 2

On the written questions described earlier, when we asked students to consider collisions with equal target mass and varying elasticity, some students instead considered collisions with different target mass and (possibly) the same elasticity. The purpose of the second round of interviews was to explore how students responded to questions about this context involving only elastic collisions, and about which we had previously not been asking any questions.

The students and the experimental setup involved in the demonstration were similar to those described in round 1, except that this time, in both collisions, the carts interacted through repelling magnets, and the target carts were visibly different in mass. Glider X would be "hit" by glider A, and was less massive than glider Y, which would be hit by glider B.

a. Description of interview tasks

First, we showed students the collision between gliders A and X, and asked students to predict the outcome of the second collision. In particular, students were asked how the final speed of glider B would compare to that of glider A, and how the final speed of glider Y would compare to that of glider X. These comparisons can be determined using momentum and energy conservation either with much algebra or with a frame-transformation procedure, neither of which we expected students to be comfortable doing. The purpose of this question was to observe how intuitive students would find the outcomes of the collisions. Therefore, we asked students to predict and explain briefly, without requiring very much logical support for the students' answers.

After showing students the collisions, in which they observed that gliders A and B both rebounded, A traveled slower than B, and X traveled faster than Y, we asked students to explain why they thought the outcomes were different. Next we asked them to compare gliders X and Y according to final momentum. The interviewer then tried (through unscripted conversation) to explore the reasoning students used to make the comparison.

b. Discussion of results

First, these interviews taught us that students were concerned with a certain limiting case of the progression that they were being shown. This in itself was very encouraging, since we consider the spontaneous concern with limits to be one of the signature qualities of "expert" physics thinking. Second, this limiting case was serving a logical function for the students; the limit had instructive value for how to treat momentum. In particular, students seemed to understand the limit as support for the idea that momentum should be treated as a scalar.

The following excerpt is taken from a discussion between the interviewer (I) and "Student 1" (S1), and shows the student in the process of comparing the final momentum of gliders X and Y by considering a limit:

S1: Well, if you had enough mass on that cart (target) when A collides with it, it would come in and bounce back with the same speed.

I: According to your coordinate system, after the collision, the total momentum is positive, negative, or zero for A and X?

S1: positive.

I: B and Y?

S1: positive.

I: What about this other situation that you just described…

S1: Absolutely.

I: …in which (the target) is so heavily loaded that A pretty much has the same speed afterwards as it did before?

S1: Um… now I'm not quite sure. A would still have momentum that would have changed direction but it would be the same value. The direction shouldn't matter.

I: You're remembering that…?

S1: No, it's just kind of popping into my head.

I: Do you remember from your course somebody saying direction was not important or that it was important?

S1: No, honestly I do not recall. But that's what it seems like to me, direction shouldn't matter.

In another interview, with "Student 2," a similar phenomenon occurred as the student considered the final momenta of X and Y:

S2: I guess if you imagine another cart that has an unlimited amount of weight, the incident cart bounces off of it. I would think that that cart would bounce off with the same velocity that it went in with. The incident cart would get faster until it reached that velocity.

I: And the momentum of the target object, as you add the mass, what happens to the momentum of the target object?

S2: It decreases, because the vel… [laughs]. I guess because the velocity is getting lower, even though the mass is getting higher. At the end, having an unlimited mass, I'm guessing the velocity will be zero. So the momentum would be zero.

As each of these conversations continued, the interviewer invited the students to consider the directional aspect of momentum. After this mild intervention, Students 1 and 2 each concluded (correctly) that the final momentum of the more massive glider (Y) should be greater than that of less massive glider (X). The interviewer then reminded each student of the giant-target limit s/he had introduced:

I: How is it possible that the final momentum of Y could be greater than X when it's slower?

S1: The speed could be less and the mass could be more so that the momentum is more.

I: Is it possible that the momentum could actually get larger as you go down the line?

S1: I don't know see how that would be possible, but I don't really see how I can explain what I said before (correct comparison). It seems false to me. Because if the momentum just keeps getting larger, then that falsifies a cart coming in and rebounding at the same speed.

Student 1 still found the giant-target limit so compelling that it seemed to nullify his previous correct result, without explaining otherwise how or why his previous reasoning was flawed. The conversation with Student 2 also continued:

I: Now let's go back to (the giant target).

S2: Ohhhh! That doesn't work!

I: Why not?

S2: Because the magnitude of the momentum is just getting bigger and bigger. It's not approaching zero.

I: Is that possible, for the speed to be getting smaller, as you increase the mass of the target, and for the momentum to be getting bigger?

S2: Um… yeah, I guess the rate of the mass getting bigger has to be, well… the velocity is still approaching zero, though, so as it approaches zero, the momentum still approaches zero, I think.

Briefly summarizing each student's path of reasoning, Student 1 was initially disposed to ignore the direction when considering momentum, because of the giant-target limit. He later correctly compared the momenta of X and Y, but recanted when he remembered the giant-target limit. Student 2 initially ignored directions of motion, remembering later that "velocity has direction" and correctly compared the momenta. She worried that the giant-target limit would conflict with her previous result, as shown above, but ultimately relied on her confidence in the notion that "velocity has direction."

We concluded from these interviews that the giant-target limit plays an important role in how students think about momentum. We also supposed that incorporating a proper treatment of this particular extreme case in tutorial instruction on momentum conservation might improve student understanding of intermediate cases.

B. STUDENT UNDERSTANDING OF MOMENTUM IN COLLISIONS BEFORE TUTORIAL *CONSERVATION OF MOMENTUM IN ONE DIMENSION*

1. Description of additional question contexts

a. 2.DTM.S.BYEFS

As shown in Figure 3-3, glider A collides with glider X in Experiment 1. In Experiment 2, glider B collides with glider Y. Before the collision, gliders X and Y are both at rest. Gliders A and B each move to the right with initial speed v_0. After the collisions: glider A is at rest, while glider B moves to the left with speed v_{Bf}; glider X moves with final speed v_{Xf}, which is greater than the final speed of glider Y, v_{Yf}. The mass of glider Y is greater than that of glider X, which is greater than that of glider B, which is equal to that of glider A ($m_Y > m_X > m_B = m_A$).

In this context, students were not told that gliders B and Y had equal final speed. However, the arrows we drew to represent the final velocities of gliders B and Y appeared to be approximately equal in length. Some students referred to this feature in their explanations.

b. 2.DTM.S

This context is very similar to 2.DTM.S.BYEFS, except that the arrows that we drew to represent the final velocities of gliders B and Y were clearly different in length. The arrows suggested that the final speed of glider B was significantly greater than the final speed of glider Y.

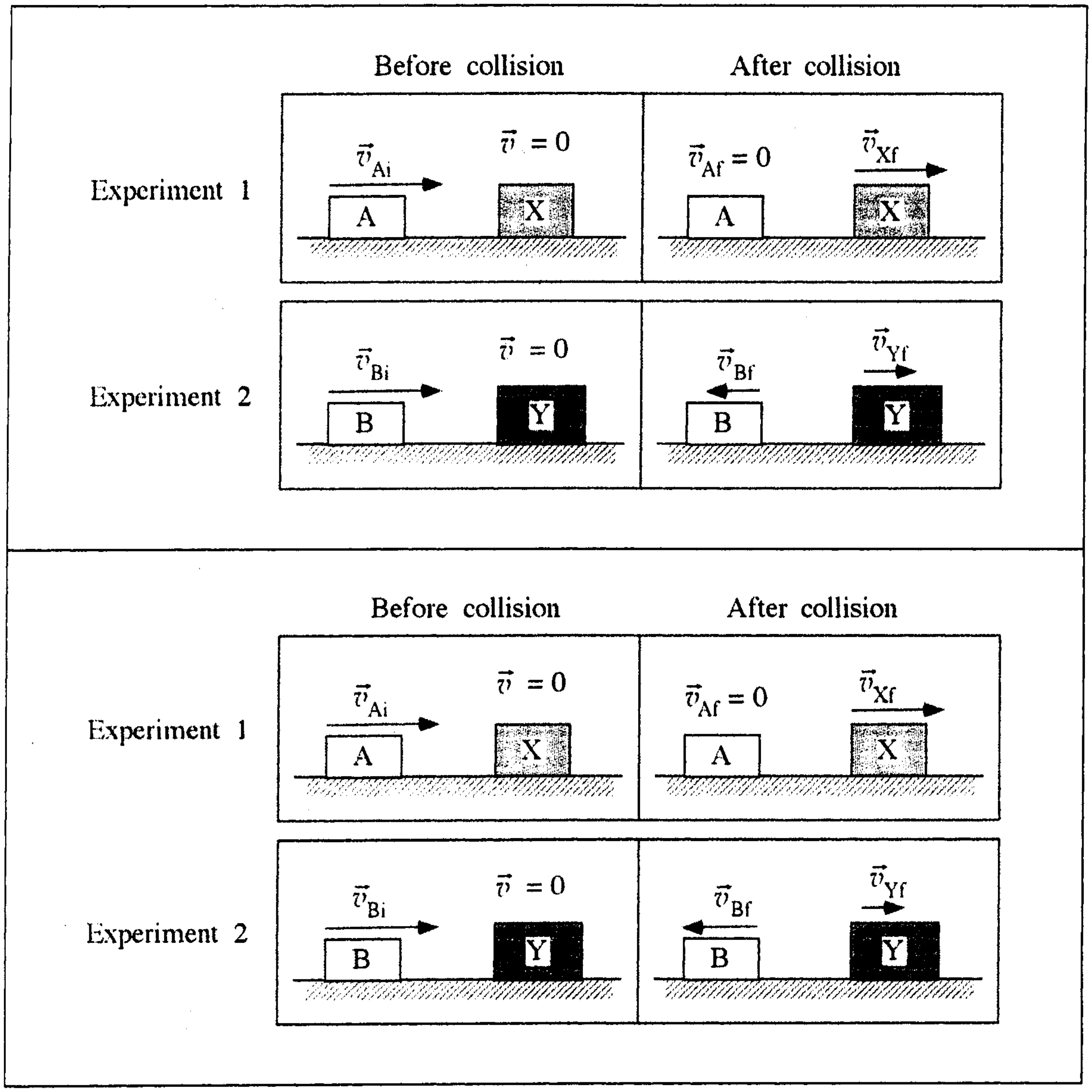

Figure 3-3: The "2.DTM.S.BYEFS" context (top) and the "2.DTM.S" context (bottom).

c. 1.GTM

In this context, there is one collision, between an incident object and a target object that is initially at rest (see Figure 3-4). The incident object moves toward the target object and rebounds with a velocity that is less in magnitude than its initial velocity, and in the opposite direction. The mass of the target object is much greater than the mass of the incident object. After the collision, the target object moves with a speed that is "very close to" zero (or in one variation, "much smaller than 1 m/s"). In each version of the context, enough information is given about the initial and final motion of the incident object that the magnitude of the final momentum of the target object can be calculated.

i. level

In the level version, the two objects are gliders on a level, frictionless track. The mass of the incident object is 1 kg. The initial speed of the incident glider is 10 m/s, and its final speed is 7 m/s. Because we were concerned that "very close to zero" may not be a well-defined mathematical concept, we also asked a variation (only in the level version) in which the final speed of the target was described as "much smaller than 1 m/s" (see the lower left cell of Figure 3-4).

ii. angled

In the angled version, an astronaut of mass m moves toward a space station of mass 10,000 m and then pushes off of it. The magnitudes of the initial and final momentum of the astronaut are 20 kg•m/s and 13 kg•m/s, respectively. The space station does not spin.

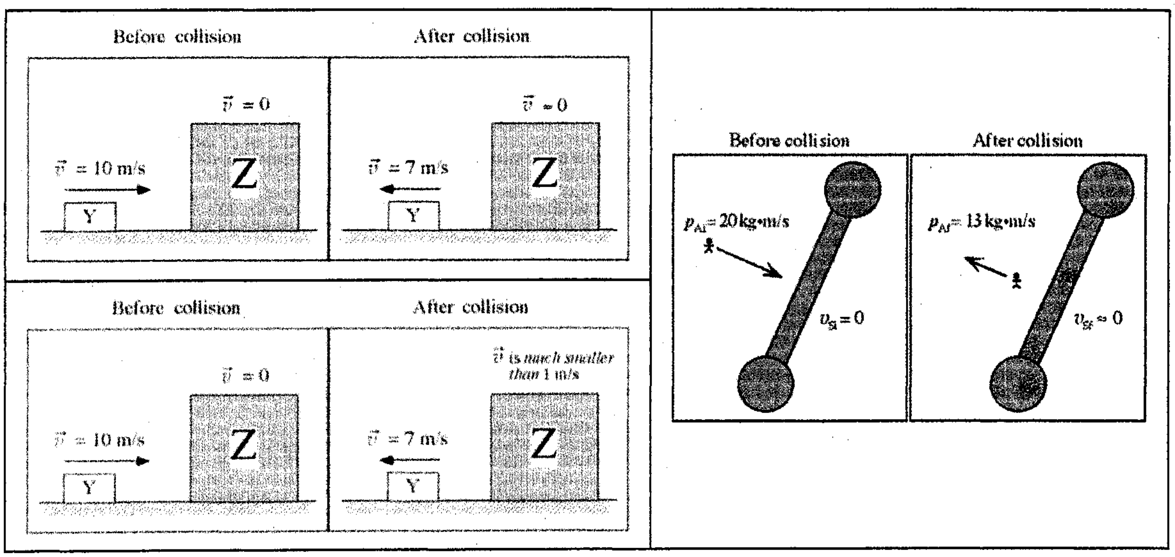

Figure 3-4: "Level" (left) and "angled" (right) versions of the 1.GTM context.

d. 3.ETM.S.BCEFS

This context is very similar to 3.ETM.S, except that students were told that both objects B and C had final speed w_0. The problem also stated explicitly that objects B and C had the same final speed, and that only their directions of motion were different. Additionally, this context used the "finish line" device, in which students were not asked to compare the final speeds of objects X, Y, and Z, but instead to state the order in which these objects crossed a finish line, after starting at rest from a line marked "start." The intention of this strategy was to frame the outcomes of the collisions in terms that were maximally concrete. We thought perhaps that asking the *Final speed* question in more concrete terms might cue students to think more about what they would expect to observe if they were watching the collisions, rather than encouraging students to use abstract ideas and rules that they thought they were being taught but did not believe were accurate descriptions of reality. We have informally observed that what students say they believe they are being taught is distinct from and less accurate than what we are trying to teach.

e. 2.ETM.S book

Another attempt to construct a question context that encouraged students to employ a physical intuition was 2.ETM.S book. As a lecture demonstration, this context is common, with the lecturer often posing a question that is roughly equivalent to "Would a rubber ball or a clay ball (of the same mass) be more effective in knocking over a book?" We considered this question to be a variant on the *Final speed* question. In 2.ETM.S book, a book is standing on end, as shown in Figure 3-5. A ball (either rubber or clay, of equal mass) is suspended from a string and released so that it swings and strikes the book. We found it difficult to pose a clear multiple-choice question that probed the same idea as the lecture question described above. The following two variants were attempts to pose clear multiple-choice questions, to which we hoped students could give clear responses.

i. One-part version

When the *Final speed* question was asked as a single question, the problem states, "A student wants to knock over a book by hitting it with either a clay ball or a rubber ball, of equal mass. The rubber and clay are each attached to a string and released from rest from the same position. When the clay ball hits the book, it stops; when the rubber ball hits the book, it bounces back." The question read, "Which of the following options describes an *impossible* set of outcomes?" Students were shown four options: "The clay ball knocks the book over, but the rubber ball does not," "The rubber ball knocks the book over, but the clay ball does not," "The book is knocked over in both cases," and "The book is not knocked over in either case."

ii. Four-part version

We constructed the four-part version because we suspected that some students may have thought more than one option was impossible in the one-part version. The situation became more complicated when we expanded it, but we thought student responses would be easier to interpret. In this version, a group of four students wants to knock over a book by hitting it with either a clay ball or a rubber ball, of equal mass. Each of the students (1-4) will release both balls from the same position, but different students may choose different positions for releasing the balls. In the first question, students read, "Student 1 starts with the rubber ball, which knocks the book over. Student 1 releases the clay ball from the same position. Will the clay ball knock the book over?" Students were then given three options: "Yes. If the rubber ball knocks the book over, the clay ball will definitely knock the book over," "No. If the rubber ball knocks the book over, the clay ball will definitely not knock the book over," and "Can't tell on the basis of the information given." The second, third, and fourth questions were very similar to the first one; as a group of four questions, they covered all of the combinations of rubber vs. clay, and knocking the book over vs. not knocking the book over. The options on these three questions were parallel (not identical) in content to those in the first question.

f. 2.ETM.S TV cart

This context is very similar to 2.ETM.S book, but balls hit a small cart instead of a book (see Figure 3-5). The change is not superficial – we were concerned that the physics involved in knocking over an object (or not knocking it over, as the case may be) was not strictly limited to linear momentum conservation. Even if a book is approximated as a rigid, rectangular solid, the static friction force on the bottom of the book and the various torques on the book (about the point around which it rotates as it falls over) are all factors that ought to be accounted for when predicting the motion of the book. Therefore, the TV cart version was constructed to eliminate these potential problems. In this version, "You are watching TV when you hear your roommate come home. In order to look busy, you want to turn off the TV as quickly as possible. A wheeled cart is in front of the TV switch. You have two balls of equal mass that you could throw at the cart. If you throw the rubber ball, it will bounce back after it hits the cart. If you throw the soft 'squeeze' ball, it will fall straight to the ground after it hits the cart. (The cart is blocking the TV switch, so you can't throw the ball directly at the switch.)" Students were then asked which ball they should throw at the switch and were given three options: "the rubber ball," "the 'squeeze' ball," and "It doesn't matter which ball you throw."

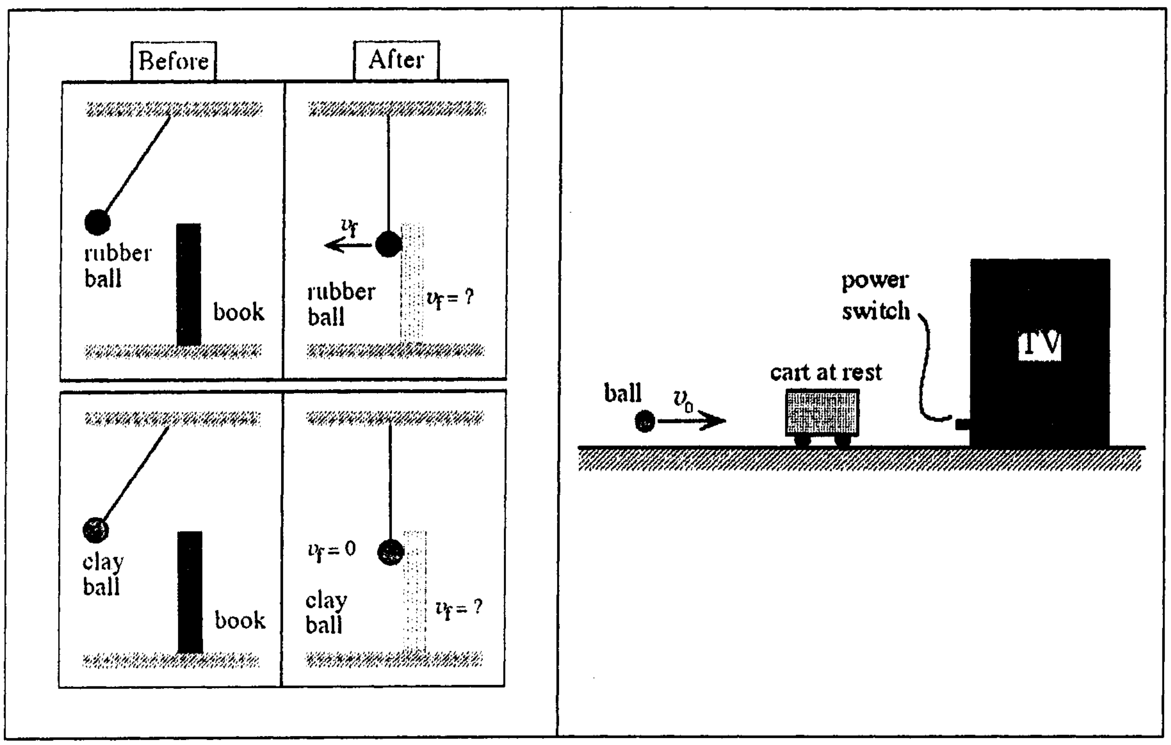

Figure 3-5: The "2.ETM.S book" (left) and "2.ETM.S TV cart" (right) contexts.

2. Description of additional questions

a. The *Incident Δp* question

The *Incident Δp* question is very similar to the *Incident Δv* question, described earlier in this section. The *Incident Δp* question was designed to determine the degree to which students treated momentum as a vector when finding the change in momentum for a single object. That is, student responses to this question are consistent with a vector subtraction procedure, a scalar subtraction procedure, or neither, but not both. This question would appear as "Is the magnitude of the change in momentum of Glider A greater than, less than, or equal to the magnitude of the change in momentum of Glider B?" in the context 2.DTM.S. To answer this question correctly, it is not necessary to realize that momentum is a *conserved* quantity of motion, just that it is a vector quantity of motion.

b. The *Final momentum* question

The *Final momentum* question is similar to the *Final speed* question described earlier in this section. In ETM contexts, these questions are functionally identical. However, in DTM contexts, students are told how the target masses compare (*e.g.*, $m_X < m_Y$) and how the final speeds compare (*e.g.*, $v_{Xf} > v_{Yf}$). Strictly speaking, these pieces of information are superfluous (and useless) if one wants to compare the final momenta of objects X and Y correctly. Similarly, in GTM contexts, some information is given about the mass and final speed of the target object, though a correct treatment of the target object's final momentum should ignore such information. However, because many students treat these kinds of information with importance, the *Final momentum* question (when asked in DTM or GTM contexts) should be considered non-trivially distinct from the *Final speed* question.

The *Final momentum* question has served numerous functions throughout this research project, but its primary function is to determine the extent to which students understand momentum not just as a conserved quantity of motion, nor just as a vector quantity of motion, but as a *conserved vector* quantity of motion. Any deviation from this understanding of momentum tends to lead students to an incorrect answer to this question. In its most common instantiation, the question reads: "Is the magnitude of the final momentum of Glider X greater than, less than, or equal to the magnitude of the final momentum of Glider Y?"

3. Correct answers to additional questions

a. The *Incident* Δp question

The correct reasoning for the *Incident* Δp question is the same as that for the *Incident* Δv question, with the word *momentum* replacing the word *velocity*. Note that it is not necessary to understand momentum as a conserved quantity to answer this question correctly – a student may understand how the momentum of Gliders A and B change without understanding that the momentum must be transferred to another object.

b. The *Final momentum* question

The magnitude of the final momentum of Glider X is less than the magnitude of the final momentum of Glider Y. If a student has answered both of the previous questions correctly, then he or she can infer that the momentum transferred by Glider B to Glider Y must be greater in magnitude than the momentum transferred from Glider A to Glider X. Alternatively, focusing on the momentum of the system rather than changes in momentum of the objects: Glider X has a final momentum equal to that of the system, since Glider A is at rest; Glider Y, however, has final momentum of greater magnitude than that of its system, since Glider B moves backwards after the collision. For either logical path, it is necessary to use the idea of conservation and either addition or subtraction of vectors.

If the *Final momentum* question is posed numerically, as in the 1.GTM contexts, then the final momentum of the target must be equal to the sum of the magnitudes of the initial and final momenta of the incident object, in order that the system's momentum keeps the same magnitude and direction. In the level version of the 1.GTM context, the magnitude of the final momentum of the target object must be 17 kg•m/s; in the angled version, it must be 33 kg•m/s.

c. Additional versions of the *Final speed* question

For the 2.ETM.S (book, 1 part) version of the *Final speed* question, the only impossible outcome is that the clay ball knocks the book over but the rubber ball does not. If the balls are released from a very low height, the force of interaction between either ball and the book should be so small that book will stay upright. Similarly, we can imagine the balls being released from too great a height, so that they each knock the book over.

For the 4-part version of the same question, a "correct" response would be to select correct answers to all four questions. These would be: if the clay ball knocks the book over, then the rubber ball also will knock it over; if the clay ball does not knock the book over, then we cannot tell whether the rubber ball will knock it over; if the rubber ball does knock the book over, then we cannot tell whether the clay ball will knock it over; and if the rubber ball does not knock the book over, then the clay ball will not knock it over. A scalar response would be a set of answers identical to those above, exchanging the words *rubber* and *clay* for each other. An "equal-effects" response would be, in each of the four cases, stating that the second ball would have the same effect as the first ball, never choosing the "can't tell" answer.

4. Administration of questions

The *Momentum transfer, Incident* Δp, and *Final momentum* questions have been administered to thousands of students in the introductory calculus-based course at the University of Washington, at various stages of lecture and laboratory instruction on linear momentum conservation.[8] In those cases for which the questions were asked before tutorial instruction, we do not describe the different stages of instruction or lecture treatments of the topic here, because student performance on these questions appears not to depend on whether any lecture or laboratory instruction had been completed at the time we asked students the questions. In one case, we describe a bit of lecture instruction that appeared to have an effect on student performance.

For most of the data that are presented here, the *Momentum transfer, Incident* Δp, and *Final momentum* questions were asked in succession, in that order. In a small fraction of cases, either the *Momentum transfer* question or the *Incident* Δp question was absent from the sequence; this change did not appear to affect performance on the remaining questions.

Whenever questions were asked in 1.GTM or 2.ETM.S contexts, they were asked as the second major part of the pretest, following the usual trio of questions in the 2.DTM.S context.

5. Overview of student performance

a. The *Momentum transfer* question

Among UW students, success in answering this question appears to depend somewhat on the whether the population comes from the calculus-based course (45% correct) or the algebra-based course (30% correct). We discovered after administering the question to many students that a certain version of the question had perhaps been giving us an overestimate of the number of students who understood the question to be about an exchange of some conserved quantity of motion (whether understood as a vector or scalar). Among those student responses that correctly stated that the changes in momentum were of equal magnitude, a small number were accompanied by a particular type of incorrect explanation that focused on comparing the final velocities of gliders B and Y. In that version of the question context, these velocities had been represented with arrows of roughly equal size. We expect that some students who chose "equal" would not have done so if these two final velocities had been shown as different in size. For this reason, we changed the context slightly to show clearly that the final speed of glider B is greater than the final speed of glider Y. In Table 3-4, results from the version in which $v_{Bf} = v_{Yf}$ are shown under the heading 2.DTM.S.BYEFS, while results from the version in which $v_{Bf} > v_{Yf}$ are shown under the heading 2.DTM.S. As it turns out, student performance was similar on these two versions, so it would appear that the tendency to base changes in momentum on final momentum vectors is very uncommon. However, this estimate could depend on many factors, perhaps including whether the change that students are thinking about ought to be found by applying the idea of "final minus initial," the conservation concept, or some other logic.

Table 3-4: Student performance on the *Momentum transfer* question in 2.DTM.S contexts before working through the tutorial *Conservation of Momentum in One Dimension.*

	2.DTM.S	2.DTM.S.BYEFS			
	UW 121		COL 1110	CIN 201	UW 114
B rebounds after colliding with Y.	N=217 2 sections	N≈1400 13 sections	N=445 2 sections	N=88 1 section	N=217 2 sections
$\lvert\Delta\vec{p}_B\rvert = \lvert\Delta\vec{p}_Y\rvert$ (correct)	50%	45%	40%	40%	30%
$\lvert\Delta\vec{p}_B\rvert > \lvert\Delta\vec{p}_Y\rvert$	40%	40%	50%	45%	55%
$\lvert\Delta\vec{p}_B\rvert < \lvert\Delta\vec{p}_Y\rvert$	10%	10%	10%	15%	15%

b. The *Incident Δp* question

Students are generally more successful on the *Incident Δp* question (~60%) than they are on the *Momentum transfer* question (~45%) (see Table 3-5). This comparison suggests that teaching students to understand momentum as a conserved quantity (and therefore distinct from velocity) is at least as urgent a need as teaching students to treat momentum as a vector quantity.

Table 3-5: Student performance on the *Incident Δp* question before working through the tutorial *Conservation of Momentum in One Dimension*. In the group UW 121, student performances on the question in the contexts 2.DTM.S and 3.ETM.S were similar and have been combined.

A comes to rest, and B rebounds.	UW 121 N≈1800 16 sections	COL 1110 N=445 2 sections	CIN 201 N=88 1 section	UW 114 N=217 2 sections
$\lvert\Delta\vec{p}_A\rvert < \lvert\Delta\vec{p}_B\rvert$ (correct)	60%	65%	55%	55%
$\lvert\Delta\vec{p}_A\rvert > \lvert\Delta\vec{p}_B\rvert$	20%	15%	25%	25%
$\lvert\Delta\vec{p}_A\rvert = \lvert\Delta\vec{p}_B\rvert$	10%	15%	20%	20%

c. The *Final momentum* question

Several hundred students have answered the 2.DTM.S version of the *Final momentum* question. Among these many lecture sections, the stage of instruction prior to answering the question varies essentially across all possibilities. Since the results were very similar, they have been combined. In particular, performance on the *Final momentum* question seems not to depend on whether the students have had any lecture instruction on momentum, a lab focusing on elastic collisions, or a lab comparing collisions of varying elasticity (see Table 3-6).

In Section B of UW 114, the lecturer worked an example that was very similar to the *Final momentum* question, soon before the students in that section took the pretest that included the question. The emphasis of the example was that the magnitude of the final momentum of the target is greater than the initial momentum of the incident object, if the incident object rebounds. The fraction of students in Section B that answered the *Final momentum* question correctly (35%) was higher than in Section A (10%), where such an example was not worked. Though working the example apparently had an effect on student performance, the effect may not be as large as what any teacher might hope for.

Correct answers to the *Final momentum* are, for the most part, accompanied by correct explanations of some sort. Some students refer to their (correct) answers to the *Momentum transfer* and *Incident Δp* questions; others emphasize the "negative" final momentum of glider B, for which the final momentum of Glider Y must compensate. Occasionally, students explain simply that the final momentum of Glider Y is greater than that of Glider X because the mass of Y is greater than that of X. However, responses that suggest a mass-dominant concept of momentum do not constitute a significant fraction of the total responses.

Table 3-6: Student performance on the *Final momentum* question in the context 2.DTM.S, before working through the tutorial *Conservation of Momentum in One Dimension.* The lecturer of UW 114 Section B worked a very similar problem two days before students answered the *Final momentum* question on a pretest.

	UW 121 N≈1800 16 sections	COL 1110 N=445 2 sections	CIN 201 N=88 1 section	UW 114 N=150 Section A	UW 114 N=67 Section B
$\lvert\vec{p}_{Xf}\rvert < \lvert\vec{p}_{Yf}\rvert$ (correct)	20%	15%	15%	10%	35%
$\lvert\vec{p}_{Xf}\rvert > \lvert\vec{p}_{Yf}\rvert$	50%	55%	60%	55%	35%
$\lvert\vec{p}_{Xf}\rvert = \lvert\vec{p}_{Yf}\rvert$	30%	30%	25%	25%	25%

The GTM versions of the *Final momentum* question give us a sense of how strongly students base their thinking about momentum on the velocity of the object. As shown in Table 3-7, the velocity-dominant momentum answer is more popular (40-45%) than the scalar momentum answer (25-30%), when the final speed of the giant target is described as "very close to zero." When the final speed of the target is described as "much smaller than 1 m/s," the velocity-dominant response becomes slightly less popular, but it is still given by many students (30%).

Table 3-7: Student performance on the GTM versions of the *Final momentum* question before working through the tutorial *Conservation of Momentum in One Dimension*. Results from the level and angled versions were similar and have been combined. Results from the "much smaller than 1 m/s" variation are presented separately.

	"$v_f << 1$ m/s"	"$v_f \sim 0$"	
	UW 121		COL 1110
	N=217 2 sections	N=346 3 sections	N=388 2 sections
Correct (*e.g.*, 17 kg•m/s)	30%	20%	15%
Scalar (*e.g.*, 3 kg•m/s)	35%	30%	25%
Velocity-dominant (~0 kg•m/s)	30%	40%	45%

d. Additional versions of the *Final speed* question

The lecturer of section B of UW 114 (described in Table 3-6) was the lecturer for all three sections in the first and second columns of Table 3-8. The lecturer worked the example described previously in all three sections. Apparently, having students consider the *Final momentum* question before the pretest also has a noticeable effect on how they answer the *Final speed* question in the four-part version of the 2.ETM.S book context.

Table 3-8: Student performance on the *Final speed* question in the contexts 2.ETM.S book and 2.ETM.S TV cart before working through the tutorial *Conservation of Momentum in One Dimension*. The three sections in the first and second columns all had the same lecturer, who worked an example problem similar to the *Final momentum* question prior to the pretest.

	UW 114 N=67 1 section TV cart	UW 121 N=220 2 sections book, 4-part	UW 121 N=560 5 sections book, 4-part	COL 1110 N=445 2 sections book, 1-part
Correct	35%	35%	15%	20%
Scalar	45%	25%	30%	55%
Equal effects	10%	15%	20%	N/A

6. Discussion of patterns in student reasoning

The most common answer to the 2.DTM.S version of the *Final momentum* question (that the final momentum of Glider X is greater than that of Glider Y) generally comes with two varieties of explanations, which we call "scalar momentum"[4] (discussed here) and "velocity-dominant" (discussed in the next section).

a. Tendency to treat momentum as a scalar quantity

The following examples illustrate scalar momentum reasoning:

> "Glider X carries all of the momentum that was in glider A, while glider Y carries only part of the momentum from glider B."
>
> "All the force in experiment one is transferred to glider X, but only half the force is transferred to glider Y."

The defining characteristic of "scalar momentum" reasoning is that the direction of motion of glider B after the collision is ignored; instead, the focus is on the fact that glider B is moving. Furthermore, the word *momentum* appears not to be an essential component of scalar momentum reasoning; students may substitute *force, velocity, etc.* for *momentum* and still use the same logic, namely, that there is some total scalar amount of motion that is shared by the objects.

Each version of the *Final momentum* question that has been discussed here has some canonical "scalar response," which we would expect if students were to apply scalar momentum reasoning to that question. The data above show that in every case, this response is quite popular.

b. Tendency to conflate velocity and momentum

In both versions of the *Momentum transfer* question, a very popular incorrect answer is that the magnitude of the change in momentum of glider B is greater than that of glider Y. We have interpreted students' tendency to answer this way as a consequence of conflating momentum and velocity. Even if students have seen both momentum and velocity vectors, they may not realize that a set of velocity vectors and a set of momentum vectors that describe the same experiment may look significantly different. Some example student responses are shown below:

> "(Glider) B goes from a high velocity to a velocity in the opposite direction, while (glider) Y goes from rest up to a low velocity."
>
> "Glider B started with a direction of motion opposite the final direction of motion while glider Y started with no direction of motion to having a small direction of motion vector."
>
> "Since momentum is mass times velocity, the change in momentum must be the mass times the change in velocity. Thus, by looking at the diagram, it can be seen that the magnitude of the change in velocity is greater in glider B than in glider Y. From this, it can be inferred that the magnitude change in momentum of glider B is greater than the magnitude of the change in momentum of glider Y."

The responses above show some typical patterns. The first student reads a question about momentum and answers in terms of velocity only, as if they were synonyms. The second student does not use either *velocity* or *momentum*, but instead refers to *motion* as though it were as well defined as velocity and momentum. This usage also seems consistent with failing to see the difference between the velocity and momentum concepts. Finally, the last student response above implies that momentum and velocity are not identical (by citing $p=mv$)

but treats them as functionally equivalent for this question. This response suggests that the student has not recognized that the relative sizes of a set of velocity vectors will generally be different from the relative sizes of a set of momentum vectors, when objects of different mass are involved.

Some responses to the 2.DTM.S version of the *Final momentum* question appear to use "velocity-dominant" reasoning. In these responses, students tend not to comment on the motion of glider B at all. These responses support the conclusion that glider X has greater momentum by noting that its speed is greater:

> "The final speed of glider X is higher than glider Y."
>
> "Glider X has more speed so it has more momentum."
>
> "X's vector is longer and in the same direction."

In some cases, students mentioned the fact that glider X is less massive than glider Y, apparently as some sort of support for concluding that the final momentum is greater for glider X:

> "Glider X is hit with a greater force, being as its mass is smaller than glider Y, which sends it forward with more momentum."
>
> "(Glider X) has less mass and can more easily absorb the velocity from A."

We believe students intend, by drawing attention to the mass comparison, not to apply the information to a concept of the product of mass and velocity, but to present support for the comparison of the final speeds of gliders X and Y. Ironically, after students have explained (without our request) why glider X moves faster than glider Y, they fail to see the important role of mass in distinguishing the concepts of velocity and momentum.

In the GTM contexts, in which the final velocity of the giant target is indistinguishable from zero, many students again base their answer to the *Final momentum* question on their knowledge of the object's velocity:

> "Since v is a component of momentum, and for (the target), v is almost zero, then its momentum must be very small (unless it is made, say, out of neutron star material), and therefore approximately 0 kg*m/s."
>
> "If it is barely moving then its momentum is just above 0 kg*m/s."

> "Since the velocity is almost 0, the mass multiplied by this number is still very small, even if the mass of (the target) is huge."

We had been concerned that students might not appreciate the difference between a velocity that is "exactly zero" and one that is "approximately zero," but most of the responses, like the ones above, show that students recognized that the velocity was not exactly zero. The third response above shows that if a student states that the momentum of the object is approximately zero, that student did not necessarily miss the information about the "huge" mass of the target. We have already presented other evidence that students' thinking about momentum is often dominated by velocity, but these responses suggest that there is a more general very-small-number-dominant concept of multiplication, regardless of what the quantities being multiplied represent. This question could be investigated in any context in which students are reasoning about a product, or in contexts where two effects might potentially compete (like a very large velocity multiplied by a very small mass).

c. Tendency to use "compensation reasoning"

Explanations that focus on the concept of momentum *as the product of mass and velocity* tend to be given by students who state that gliders X and Y have equal final momenta. This perspective that students tend to take toward quantities in physics, and toward momentum in particular, has been described in previous research by Lawson and McDermott.[9] Such reasoning has been called "compensation reasoning," since some students express the idea that changes in quantities may tend to cancel each other out when combining mathematically, as in a product. The following responses to the *Final momentum* question exhibit compensation reasoning:

> "A larger slower moving object can have the same momentum as a smaller faster object, as intuited through the p=mv formula."
>
> "Although we don't have quantitative data, since X has less mass and greater velocity, and Y has a greater mass and less velocity, it is likely that their change in momenta are the same."
>
> "It is possible that they are equal because glider X weighs less but is traveling faster and glider Y weighs more but is traveling slower. It is difficult to tell."

Students hardly ever state baldly that the product *will* or *must* be equal under these circumstances. Generally, students tend to soften the conclusion that the momenta are equal by using weaker words like *can*, *likely*, and *possible*. Alternatively, students may support the conclusion of equal final momenta by focusing on some aspect of the collisions that is the same. In the case of the research done by Lawson and McDermott, the common factor students found was that the air hose used in the experiments they described exerted the same force. In these collision questions, some students focus on how the initial motion of gliders A and B are the same. For example:

> "They were struck by A and B at the same velocity, the momentum imparted on them should be the same."
>
> "While glider X has a higher change in velocity, its mass is less. These two factors end up being equal because the same glider is being acted on them."

d. Tendency to conflate force and momentum

Some students focus on the identical initial velocities of the incident objects and associate them explicitly with equal forces. The following examples are taken from 2.ETM.S book (four-part version):

> "The same force is being applied to the books, so they should react the same in each case."
>
> "The amount of force in each case is equal for both balls since the mass of each is the same and for each case, (1-4) the releasing point is the same. So from kinematics the velocity at which each ball hits the book in each case is the same. The material out of which the balls are made does not matter."
>
> "If the masses are the same, and the balls are released with the same angle, then the momentum should consequently be the same. If the momentum is the same, the force applied to the book should be as well, regardless of the elasticity of the ball."
>
> "No matter what the material is made up of, it will have the same momentum or force if it has the same mass."

Many of these responses state that the material out of which the balls are made does not matter. It is true that the material does not matter, if we look at the velocity, momentum, or kinetic energy of the ball before it hits the book. However, the mistake these students make is to associate the concept (or word) force with a quantity of motion, either instead of, or in addition to, an interaction between the ball and the book.

7. Discussion of student performance on combinations of questions

In addition to studying how groups of students answer a particular single question at various stages of instruction, we can also study how students perform on various combinations of questions at a single stage of instruction. Observations of how students perform on a series of questions offer opportunities for unique insights that are important for instruction as well as assessment of student understanding, whether for the purposes of research or assigning grades to individual students in a course.

In particular, we have observed two important trends in student performance on series of questions about linear momentum. First, for similar questions in different contexts, student performance on the two questions is often very weakly correlated. In other words, knowing how a given student answered a single question often does not improve one's ability to predict how that student will answer another question that most of us physicists would say tests the same idea. Second, for a series of related questions in a single context, strings of answers to these questions may correspond to some coherent understanding of a concept. This coherent understanding may be correct, or it may be "alternative." Specifically, if a student answers every question about momentum according to a coherent[10] understanding of momentum as a conserved scalar quantity, they should answer the *Momentum transfer, Incident Δp,* and *Final momentum* questions with the responses $|\Delta\vec{p}_A| = |\Delta\vec{p}_X|$ (correct), $|\Delta\vec{p}_A| > |\Delta\vec{p}_B|$ (incorrect), and $|\vec{p}_{Xf}| > |\vec{p}_{Yf}|$ (incorrect). If a student bases all answers to these questions on a velocity-dominant understanding of momentum, then the student should answer $|\Delta\vec{p}_A| > |\Delta\vec{p}_X|$ (incorrect), $|\Delta\vec{p}_A| < |\Delta\vec{p}_B|$ (correct), and $|\vec{p}_{Xf}| > |\vec{p}_{Yf}|$ (incorrect).

Table 3-9: Percentages of students that gave sets of responses to the *Momentum transfer, Incident Δp,* and *Final Momentum* questions in the 2.DTM.S context corresponding to consistent application of various concepts of momentum.

"Coherent" responses	UW 121, N=757 7 sections	UW 114, N=217 2 sections	COL 1110, N=445 2 sections
Correct	5%	<5%	5%
Conserved scalar	10%	5%	5%
Velocity-dominant	20%	20%	25%

In Table 3-9 above, only about 35% of the students give a set of three answers that faithfully follows the logic of any of the patterns of thinking that we have identified. In other words, most students give combinations of answers that are mutually logically incoherent. Furthermore, comparing the rows of the table to each other, it is much more common for students to be consistent in treating momentum like velocity than it is for them to be consistent in treating momentum as a conserved scalar quantity. We do want to teach students that momentum is not a conserved scalar quantity, but we want to be sure that we are focusing on the central issue. It is with these observations in mind that the tutorial described in the following section was developed.

C. DESCRIPTION OF CURRENT VERSION OF TUTORIAL SEQUENCE *CONSERVATION OF MOMENTUM IN ONE DIMENSION*

The primary conceptual goal of the tutorial sequence *Conservation of Momentum in One Dimension* is for students to understand the meaning and implications of the concept of momentum as a conserved vector quantity. We consider student ability to recognize that a collision in which an object rebounds involves greater momentum transfer for both colliding objects than if the first object were to come to rest to be an indicator of strong understanding of linear momentum conservation. A copy of the current version of the tutorial and homework can be found in Appendix A of this dissertation.

1. Detailed description of tutorial

a. The concept of the momentum vector

Section I of the tutorial emphasizes the mathematical procedures of (1) determining the momentum of a single object from its mass and velocity and (2) determining the momentum vector of a system of multiple objects that are moving at different velocities. Students are first told the definition of momentum of an object ($\vec{p} = m\vec{v}$) and of a system of multiple objects (p). ($\vec{p}_{sys} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ...$). (It is assumed that the tutorial follows lecture instruction on momentum; the definitions are presented here so that tutorial students and instructors can refer to them easily.) In part A, students find the momentum vector for each of two objects that are moving with the same velocity but have different mass (see Figure 3-6). Then the students determine the momentum of the system that comprises those two objects. In this case, the addition of the momentum vectors is as simple as adding two positive numbers. In part B, the same objects are now moving with velocities that are different in magnitude and direction. Students are asked to determine the momentum vector for the same system of two objects in this new case. Students must realize that the addition of the momentum vectors in this case involves an arithmetical subtraction of magnitudes. The goals of this section are (1) to give students practice adding vectors in opposite directions, (2) to give students an opportunity to observe that the momentum vector for a system may be much smaller in magnitude than any of the momentum vectors of the objects in the system (in particular, when the momentum vectors of the individual objects are in opposite directions) and (3) to illustrate that the magnitude of the momentum of a system of multiple objects will generally not have a special value; in particular, it is not generally zero. Hardly any students have any substantial trouble with this section of the tutorial. Minor issues include questions about the units of momentum and about how to express the direction of momentum. Instructors guide students to describe the direction of momentum as closely as possible in directional terms stated in the questions (*viz.*, east and west). Despite a lack of significant conceptual struggle in this section, however, students make some errors in the following sections that are inconsistent with correct answers they give in this section. Thus, instructors can use section I as a resource, by inviting students to check for consistency between their answers to a present question and (these) previous questions.

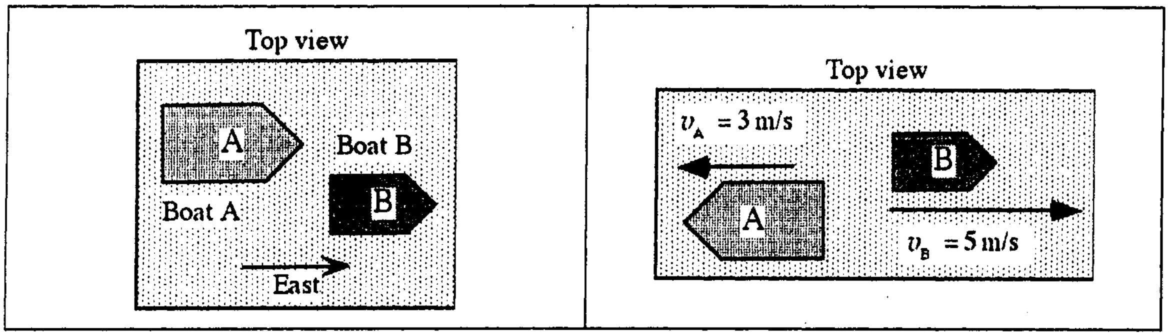

Figure 3-6: Part A (left) and part B (right) of the context for Section I of the tutorial *Conservation of Momentum in One Dimension*. Boat A has mass 10 kg, and boat B has mass 5 kg.

b. Momentum of objects in a collision

Section II of the tutorial leads students to understand the general story of momentum vectors in a collision of two objects in one dimension, in which the collision is perfectly elastic, the target object (glider B) is initially at rest, and the incident object (glider A) has mass less than that of the target (see Figure 3-7). These conditions lead to a rebound of the incident object, thereby leading students to deal with a situation that includes constant momentum for the system, "negative momentum," and some consequences of combining these concepts.

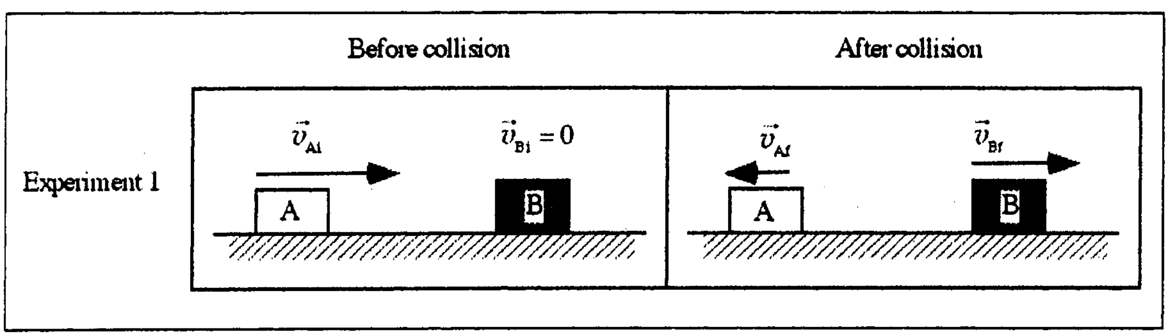

Figure 3-7: "Experiment 1" from section II of *Conservation of Momentum in One Dimension.*

The condition of perfect elasticity is not stated as such, because introductory students often misunderstand the term "elastic." Students may erroneously take the phrase "the collision is elastic" to be equivalent to a statement that the momentum of the system is constant, or that the incident object stops as a result of the collision, regardless of the ratio of the incident and target masses. However, we wanted students to deal with collisions that are elastic (even if we do not call them elastic), because we have observed that students fail to distinguish between velocity and momentum, and elastic collisions offer an opportunity to make this distinction. That is, with elastic collisions, it is easy to construct a situation for which a slower target object has final momentum of greater magnitude than a faster target object. We have also observed that some students, when trying to understand a pair of collisions that differ only in elasticity, will state that the momentum of the system decreases in the partially inelastic case. These students seem to understand inelastic collisions as excepted from the rule of momentum conservation rather than as a subset of the many cases that follow the rule. Practically speaking, when students read in the tutorial that the carts in a collision interact via repelling magnets, we observe that students do not explicitly question whether the momentum of a system is constant. By initially limiting the scope to elastic collisions, we postpone the discussion of whether and under what conditions the momentum of a system is constant until after students have successfully applied momentum conservation and discussed some of its implications.

After the collision is described to the students (in terms of relative qualitative comparisons of masses and all velocities), students are shown an arrow that represents the momentum vector of the incident glider. Students are then asked to complete a table, drawing arrows (roughly to scale) to represent the initial and final momentum of each object and the system (see Figure 3-8). Students are not told in the tutorial that the phrase "momentum is conserved" means that (in the case of zero net force on the system) the initial and final momentum vectors for the system will be identical. Students arrive at this correct interpretation through discussion with each other and with tutorial instructors. This discussion often includes the following errors: (1) Students first draw arrows to represent the initial and final momenta of gliders A and B; these arrows are drawn according to some estimation of the product of mass and velocity for each object. Students then properly add the momentum vectors to obtain the initial and final momentum vectors for the system. According to this procedure, the system momentum vectors will generally not be identical. Some students who draw arrows like these may not be thinking of any quantity staying constant during the collision. Others may have carefully drawn the arrows for gliders A and B so that the sum of the magnitudes of the momentum vectors of the two objects stays unchanged. Meanwhile, their final momentum vector of the system is shorter than the initial; they say that this shortening shows that the momentum vectors of gliders A and B are now in opposite directions. (2) Alternatively, students may first draw correct arrows to represent the initial and final momentum vectors of the system, by correctly adding the initial momentum vectors of A and B. However, these students also aim to make the sum of the magnitudes p_A and p_B stay constant from before to after, while the momentum vector for the system stays constant. The result is that the vector sum of the final momenta of A and B does not equal the final momentum vector for the system.

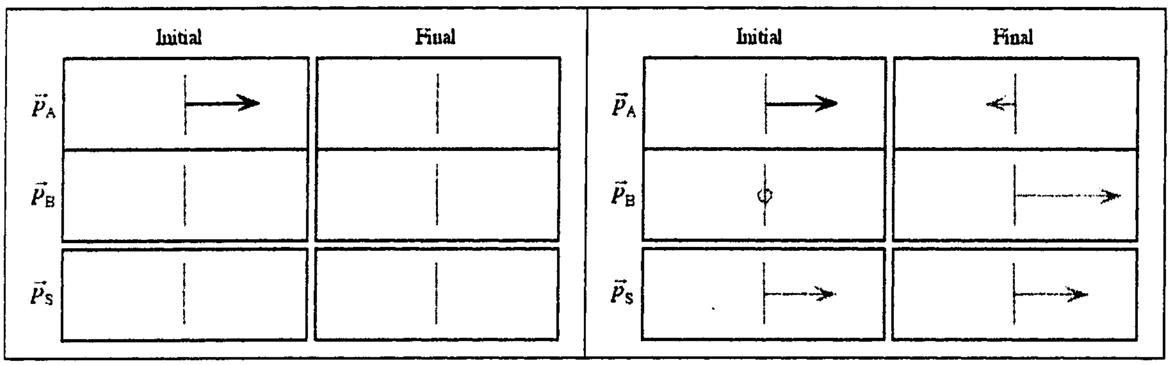

Figure 3-8: Table of momentum vectors for Experiment 1 as printed in the tutorial (left) and correctly completed in gray (right).

Though they often do not know operationally what the phrase "momentum is conserved" means, most students are familiar with the phrase from lecture and lab. We have seen that students can be easily led to accept that "momentum conservation" means (in this case) that the initial and final momentum vectors for the system are identical. After confirming the operational meaning of momentum conservation, it helps some students to look back at their vector addition from section I. After some discussion, students accept (often with uneasiness) that the only solution that allows them to fill in the table in a logically consistent way is this: the final momentum vector of glider B is longer than the final momentum vector of the system. Students are explicitly asked to compare these two vectors in magnitude. Then students are asked whether their answer would change if glider A moved to the right after the collision. Here we expect the students to recognize that the comparison of the magnitude of the final momentum of B to the final momentum vector of the system depends on the direction in which glider A moves after the collision. Next, students read and critique a hypothetical Student statement:

> "Glider A is still moving after the collision. Glider A keeps some of the momentum that it had initially, giving B only a portion of it. There is a limited amount of momentum in the system, and the most any one object can have is all of the system's momentum. Therefore, p_{Bf} must be less than p_{Sf}."

The aim of this exercise is for students to recognize that the reasoning of the statement neglects the direction of the final motion of the incident glider. The next item in the tutorial is an instruction that students should check that their answers are consistent with the law of conservation of momentum. Few students seem to need this instruction by this point in the tutorial.

Though it may seem that the directional nature of momentum would be secure in the minds of students after working through the exercises described above, we had observed that, in interviews, students were strongly influenced by their misunderstanding of a certain limiting case – what we call the "giant target" limit. That is, students who demonstrated a correct understanding of the vector nature of momentum in one context later questioned that correct analysis when they considered the limit in which the ratio of the mass of the target object to the mass of the incident object goes to infinity. Specifically, they are reluctant to believe that a mass, no matter how large, has significant momentum its speed is close to zero. Therefore, the next part of the tutorial leads students to extend the momentum vector analysis of the first collision through a discrete spectrum of collisions in which the mass of glider B increases, and ultimately, goes to infinity. Two new experiments are introduced, "Experiment 2" and "Experiment 3" (see Figure 3-9). (The original experiment is called "Experiment 1.") Students are told that the only difference in setup of the three experiments is the mass of the target object – as the experiment number gets larger, the mass of the target increases. The mass of the incident object and its initial velocity are the same in all three experiments. The following results of this change in target mass are also given to the students: the final speed of glider B decreases with experiment number, and the final (rebound) speed of glider A increases with experiment number. The students are asked plainly to rank the magnitudes of the final momentum vectors of gliders B1, B2, and B3. Many students who correctly analyzed the first collision now struggle to apply the same procedure to this spectrum of collisions. First, before any struggles are observed, most students do not spontaneously use the momentum vector approach that they used to analyze Experiment 1. Though all students, by this point, have stated that the final momentum of B is greater than the final momentum of the system, some are strongly inclined to have the final momentum of B decrease as its final speed decreases, even though glider A rebounds more strongly in later collisions. Other students focus on momentum as the product of mass and velocity, saying that as the mass of B increases and its final speed decreases, its momentum should stay constant. After students have written and explained their ranking, they are given a handout, on which they draw a set of momentum vector diagrams like the one they drew for the first experiment. Of course, students often use the tables as a place to represent the incorrect ranking that they have already given, disregarding the rules they employed previously in the first table. Through discussion, however, they eventually establish that the momentum vector of the glider B is larger (1) where the rebound speed of glider A is faster and (2) where the final speed of glider B is less. Thus, students are guided to view velocity and momentum as distinct quantities.

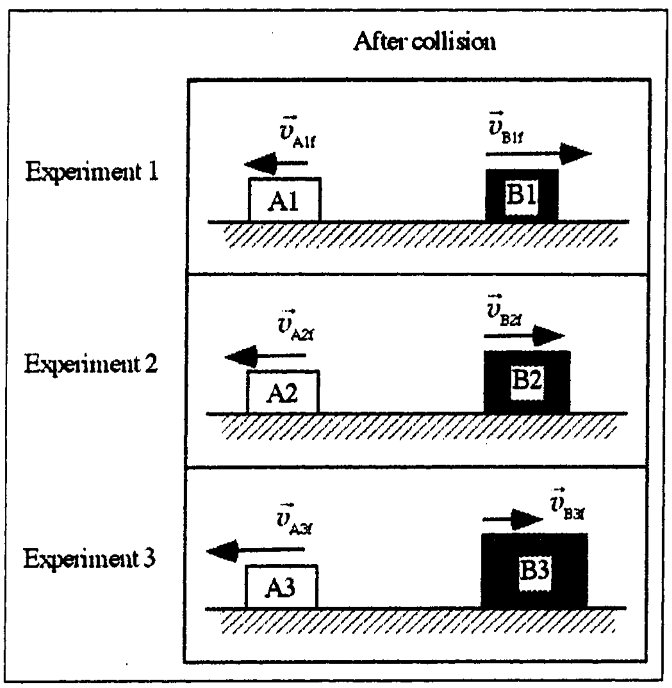

Figure 3-9: Experiments 1, 2, and 3 of section II of *Conservation of Momentum in One Dimension.* The initial velocities of the gliders in Experiments 2 and 3 are the same as in Experiment 1.

After analyzing the first three experiments in the spectrum, students go on to think about the limiting case ("Experiment 100"), for which students are told that the rebound speed of glider A is almost the same as its initial speed, and that the final speed of glider B is very nearly zero. Students are asked to compare the magnitude of the initial momentum of glider A to the final momentum of glider B. This question is similar to one asked at the beginning of this section; there students were asked to compare the final momentum of B to the final momentum of the system. The content of the new question is similar to the old one so that students can reconsider the "scalar momentum error" through a hypothetical Student discussion. The form of the new question is different in order to emphasize the fact that the sum of the magnitudes of the momentum vectors of the objects after the collision may be larger than the magnitude of the momentum "brought in" by the incident object. In the printed Student discussion, three Students offer their perspectives on the final momentum of this giant target.

Student 1: The final speed of A100 is almost the same as its initial speed. This means that A kept almost all of the momentum, giving B almost no momentum.

Student 2: Right. You can also see that the momentum of B100 has to be nearly zero because momentum is mass times velocity. If B's velocity is very small, then its momentum also must be very small.

Student 3: From each experiment to the next, the mass of B goes up, while its final speed goes down. Therefore, the final momentum of B most likely stays the same.

Instructors expect students not just to identify whether each statement is correct or incorrect but to explain what error they think is being made in each case. Finally, in this section, students work a numerical version of the giant target problem. They are told that the magnitude of the initial momentum of glider A100 is 3 kg*m/s, and glider B100 is initially at rest. Afterwards, glider A has momentum of magnitude 3 kg*m/s, in the opposite direction. Students solve for the magnitude of the final momentum of glider B100. We intend for this numerical question to offer students a view of the concrete implications of their momentum analysis of the giant target limit, perhaps offering them a final opportunity to object. However, by this point, most students treat the numerical result as fairly obvious.

c. Changes in momentum of a system of multiple objects

The purpose of this section is to help students understand the conditions under which the momentum of a system is constant. In particular, we aim to help students replace a more specific version of this condition - that there are no external forces on the system - with a more general statement - that the *net* force on the system is zero. It is a common expression among physics instructors to refer to this condition as "no net external force" (sometimes unfortunately shortened to "no external force" - unfortunate because such systems are not readily available in introductory physics laboratories). In the phrase "no net external force," the word "external" is unnecessary as long as it is understood that only external forces contribute to the net force on a system. The idea that the net force on a system does not depend on forces internal to the system is introduced in the tutorial *Newton's second and third laws.*

Students consider a collision experiment involving two gliders (C and D) of equal mass. Glider C moves to the right toward glider D, which is fixed to the table, as shown in Figure 3-10. After the collision, glider C rebounds back to the left, moving at the same speed it had initially. Students draw free-body diagrams for each glider and the system of both gliders for some instant while the gliders are in contact. Correct diagrams for C and D show vertical normal and gravitational forces on each glider; a force exerted on C by D, pointing to the left, and its Newton's-third-law partner (a force on D by C), pointing to the right; and a force to the left on D by whatever is holding it in place. A correct diagram for the system shows a normal force by the table, a gravitational force by Earth, and a leftward force on the system by the object holding D in place. Common errors include: a force on C to the right instead of to the left, perhaps representing the force that accelerated C from rest, or "the force of C's motion;" neglecting the additional force on D required to balance the force on D by C so that D does not accelerate; and thinking that the one leftward force on the system is the leftward force on C by D rather than on D by the object holding D in place. Students usually ask the tutorial instructors what object is holding glider D in place, after which instructors may invite students to use their imaginations. Glue, nail, gum, and "mystery object" are some common answers. After the free-body diagrams is a reminder to check that the free-body diagrams are consistent with the fact that D is fixed in place; some students seem to need this reminder.

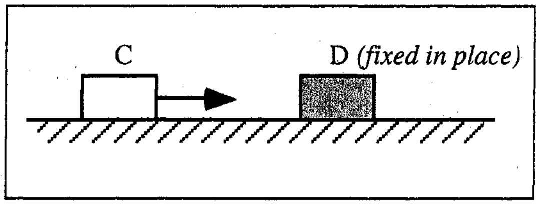

Figure 3-10: Context of section III of *Conservation of Momentum in One Dimension.*

Students then state for each of glider C, glider D, and the system, whether the momentum of that object (or system) changed as a result of the collision. In the case of glider C, we want students to say "yes, it changed;" however, students often want to make a softer statement, such as "it changed only in direction." Some instructors urge students to answer the strict "yes or no" question. The tutorial then asks students to note any correlation between whether an object's momentum changes and (1) whether there are any external forces on that object and (2) whether there is a net force on that object. It is very common for students to state that there are no external forces on all of the objects for which the momentum did not change, an assertion that is false for glider D (and the system of gliders A and B, in the earlier parts of the tutorial). When the question is rephrased by the instructor to "are there any forces on glider D that are exerted by objects outside of glider D?" students are usually able to answer correctly. When asked by an instructor whether one can tell if the momentum of an object is changing if it is known simply that there are external forces on the object, students again often answer affirmatively (and incorrectly). Of course, more discussion follows in this case. However, it seems that a non-negligible fraction of the students do not see the importance of the distinction being made, and therefore are not motivated to understand the distinction itself. Students are then asked to write a description of how to judge whether an object's momentum is changing by inspecting the free-body diagram. The intended correct answer is "the momentum of an object or system will remain constant if and only if there is zero net force on the system." We follow up on this issue in the homework, in which students work a problem about a firecracker that explodes while falling and thus has changing momentum.

Finally, students consider another printed student discussion, in which both students connect the experiment with gliders C and D with Experiment 100. In both experiments, the initial and final momenta of the incident object are equal in magnitude and opposite in direction. We want students to understand the similarities and differences between the two experiments, physically and analytically.

Student 1: This experiment is just like Experiment 100. The momentum of glider C is the same before and after the collision - only the direction of motion is different.

Student 2: Right. Glider D has no momentum afterwards, just like glider B100. So the momentum of the system is unchanged.

We expect students to note that since the net force on the system of C and D is non-zero, the momentum of that system will change, unlike the momentum of the system in Experiment 100. To our pleasant surprise, some students have spontaneously made the connection that the experiment with gliders C and D could be considered to be like Experiment 100, if it is imagined that glider D is rigidly attached to a body like glider B100, and then the system is expanded to include that object, resulting in a system on which the net force is zero.

2. Detailed description of homework

The homework for the tutorial *Conservation of Momentum in One Dimension* consists of four problems. Problems 1 and 2 are intended to be variations on the theme of describing the momentum vectors of two objects and their combined system, before and after a single collision. Problem 3 extends this theme by first emphasizing the difference between momentum and kinetic energy conservation in a set of collisions, and then goes on to the topic of the "change in momentum" vectors in collisions, and their relationship to Newton's laws. Finally, problem 4 challenges the ideas that the momentum of the system is always unchanged (and that changes in momentum in a system are always equal and opposite) by having students relate the change in momentum of a system to the non-zero net impulse on it.

Problem 1 involves the following familiar context: Glider A moves to the right on a level, frictionless surface toward glider B, which is initially at rest (see Figure 3-11). After the collision, glider B moves to the right, and glider A moves to the left. The final speed of glider A is greater than the final speed of glider B. The mass of glider A is less than the mass of glider B. Students are asked to compare the magnitude of the final momentum of glider A to that of glider B, and to support their answers by drawing a momentum vector diagram. The problem is set up so that the comparison cannot be made if the student looks at momentum simply as the product of mass and velocity. (Since the mass of glider A is less than that of glider B, while its final speed is greater, the product of mass and final speed cannot be determined to be greater than, less than, or equal to the same product for glider B.) Similarly, if a student uses the concept of momentum as a positive scalar quantity that is distributed among the objects in a collision, then the problem is again unworkable. (This idea by itself is not sufficient to determine whether glider A "keeps" more than half of its initial momentum, less than half, or exactly half.) For the most part, students do not seem to find this problem very challenging. Students draw a momentum vector diagram that shows the familiar result that the final momentum of B must be larger in magnitude than that of the system, since glider A moves backwards. Furthermore, the final momentum of glider B must be larger than the final momentum of glider A, since the vector sum of these two must be to the right.

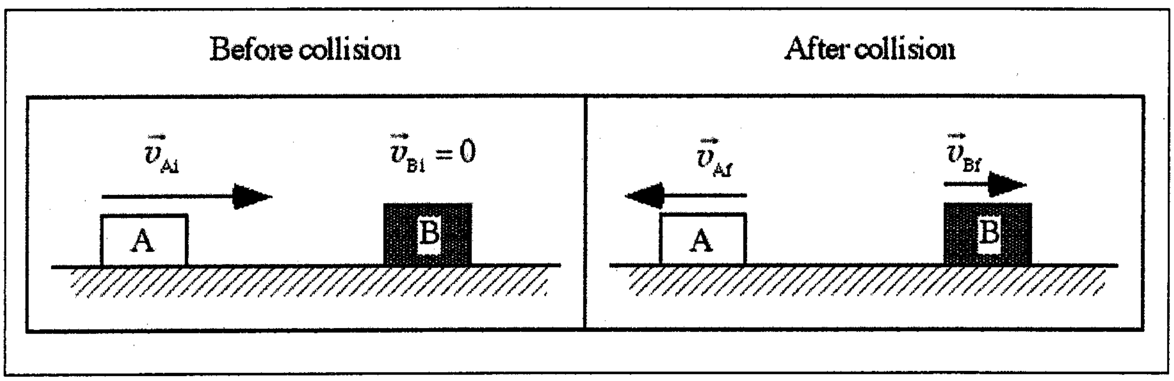

Figure 3-11: Context for problem 1 of the homework for *Conservation of Momentum in One Dimension.*

Problem 2 is similar to problem 1 in design. In problem 2, two gliders, C and D, move toward each other on a level, frictionless surface, as shown in Figure 3-12. C moves to the right, while D moves to the left. The mass of glider C is less than that of glider D, and the initial speed of glider C is greater than that of glider D. After the collision, glider C moves back to the left, and glider D moves to the right. The final speed of glider C is the same as that of glider D. Students are asked to compare the magnitudes of the initial momenta of glider C and glider D. Again, the answer cannot be found by looking at the product of mass and velocity, since there is not sufficient quantitative information. Also, the idea of momentum as a conserved positive scalar quantity cannot determine how the scalar momentum would get redistributed in the collision. Most students draw a momentum vector diagram and work the problem correctly: The final momentum of glider D is to the right, and the final momentum of glider C is to the left. Since C and D have the same final speed, but D has more mass, then the final momentum of D is greater than that of C. Thus, the system momentum is to the right. Initially, the momentum of C is to the right, while the momentum of D is to the left. In order for the system momentum to be to the right, both before and after the collision, the initial momentum of C must be greater in magnitude than that of D.

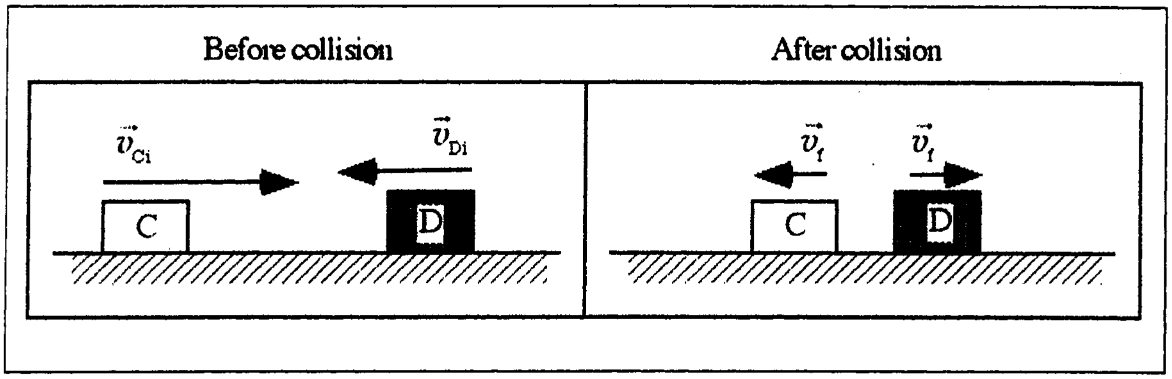

Figure 3-12: Context for problem 2 of the homework for *Conservation of Momentum in One Dimension.*

Problem 3 reintroduces a context that had been a central point of the tutorial instruction in earlier versions: a comparison of collisions that differ only in elasticity. (Since the objects that collide in this problem are animate, what makes the collisions different is not properly "elasticity," but a difference in how hard the "objects" decide to push on each other. However, the momentum and kinetic energy in the problem function as though the difference were one of elasticity, since the different collisions have different final total kinetic energy.) Currently, the tutorial itself focuses on comparisons of collisions that are all elastic, differing only in target mass. We want students to practice analyzing both types of sets of collisions, and we also want students to understand that momentum conservation and kinetic energy "conservation" are not identical concepts. This problem is meant to serve both purposes. For this problem, two astronauts perform three collision experiments in a weightless, frictionless environment, as shown in Figure 3-13. The motion of the astronauts is observed from a point of view that is at rest with respect to a nearby space station. In all three experiments, astronaut A moves to the right initially with momentum of magnitude 20 kg*m/s, toward astronaut B, who is at rest. The astronauts push on each other in different ways in the three experiments so that the outcome of each experiment is different. After the collision in experiment 1, astronaut A has stopped. In experiment 2, astronaut A's final momentum is 5 kg*m/s, to the left. In experiment 3, astronaut A's final momentum is 5 kg*m/s, to the right.

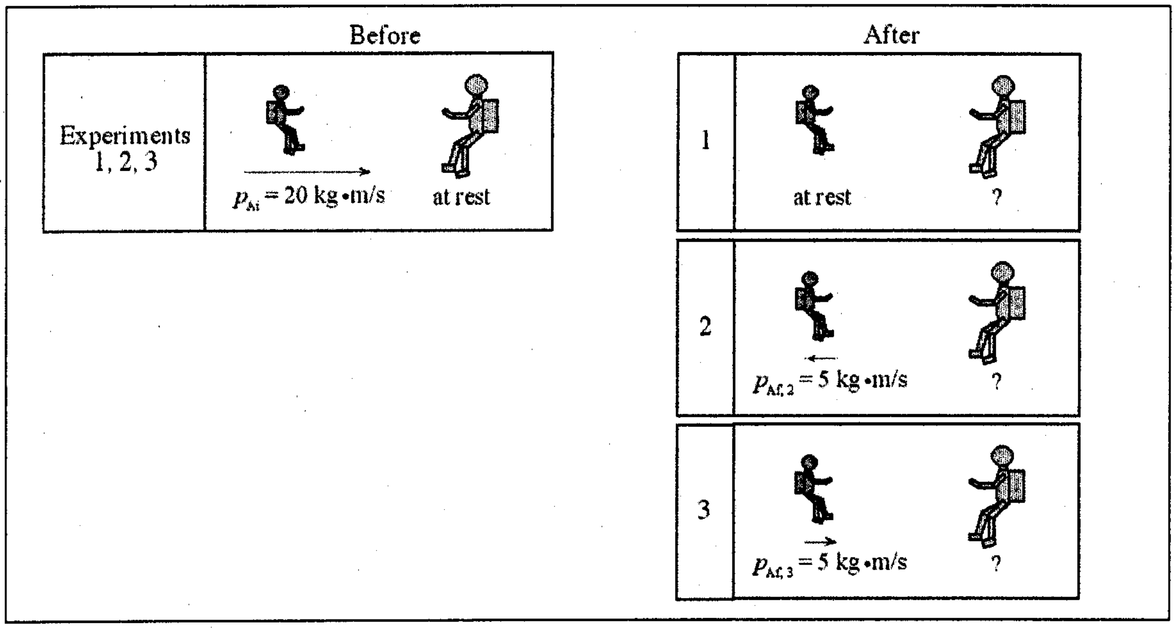

Figure 3-13: Context for problem 3 of the homework for *Conservation of Momentum in One Dimension.*

First students are asked to give the magnitude of the final momentum of astronaut B in each of the three experiments. Then, students rank the final kinetic energy of astronaut B in the three experiments. Here, we expect them to realize that the ranking of astronaut B's kinetic energies will be the same as if one were ranking the final speeds or the magnitudes of final momentum of astronaut B. The previous ranking is meant as a resource for answering the next question: Is the total final kinetic energy of the astronauts in experiment 2 greater than, less than, or equal to the total final kinetic energy of the astronauts in experiment 3? We want students to realize that astronaut B has greater final kinetic energy in experiment 2 than in experiment 3, while astronaut A has the same final kinetic energy in experiments 2 and 3. Therefore, together, there must be more kinetic energy after the collision in experiment 2. Most students do this correctly; however, for some students, the idea of equal total kinetic energy is so strong that it can override other correct ideas. For some, the momentum analysis may be disregarded completely, with an apparently blind application of a principle of conservation of kinetic energy; for others, they may re-conceptualize kinetic energy to be negative when the object moves "backwards," thereby ostensibly saving the principle of conservation of kinetic energy. These tendencies of students can probably be helped with a generally improved understanding of energy, particularly with regard to its myriad forms.

The students then consider a hypothetical student statement, composed of two sentences:

> "The momentum of the system is conserved in each experiment because there is no net force on the system. If momentum is conserved, then kinetic energy must also be conserved, because both momentum and kinetic energy are made up of mass and velocity."

Students are told that one of the sentences is correct and that they should identify the correct sentence and explain the error in reasoning in the other sentence. Here we are looking for students to affirm the content of the first sentence, and, in critiquing the second, to comment on the directional quality of momentum (in contrast to kinetic energy) and reflect briefly on what effect the functional forms mv and $(1/2)mv^2$ may have when calculating the momentum or kinetic energy of a system.

Problem 3 continues into a second phase, using the same context, but with a different conceptual focus. In this part of the problem, students draw arrows to represent the initial momentum, *change in momentum,* and final momentum of both astronauts in all three experiments. Change in momentum, as a single vector object, had been the central concept of analyzing collisions in a previous version of the tutorial. There were many reasons for removing the concept from that role, first among which was the tendency of the students to prefer the concept of system momentum to that of the change in momentum (of individual objects) in interview tasks. Having removed the concept from its previous central role, we thought the students still ought to understand its properties in a collision and its relationship to Newton's laws. Therefore, in drawing these vector diagrams for the three collisions between the two astronauts, a pattern emerges: the change in momentum of one object is equal in magnitude and opposite in direction to that of the other object. Students form the rule on the basis of this pattern and then explain it in terms of the impulse-momentum theorem as it is applied to each astronaut considered separately and considered together as a combined system.

Taken separately, the net force on each astronaut is the single force exerted on him by the other astronaut. These forces are equal in magnitude and opposite in direction, by Newton's third law. The time over which these forces are exerted must also be the same, since Newton's third law holds instant by instant. Therefore, the astronauts must have equal and opposite changes in momentum. Taking the astronauts together, the net force on the system is zero, so the change in momentum of the system is zero, which means that the changes in momentum within the system must add to zero.

Problem 4 has the students describe the momentum vectors of two parts of a firecracker as it falls and explodes. The goal of this problem is help students understand that even when the momentum of a system is not constant in time, there are various regular relationships between the many momentum vectors that describe the motion of different parts at different times. Namely: the momentum vector of the system at any instant is the sum of the momentum vectors of the parts at that instant; the change in momentum vector of a system over a specified interval of time is the sum of the change in momentum vectors of the parts over that same interval of time; the change in momentum vector of any object is the vector that must be added to the initial momentum vector to equal the final momentum vector; and the change in momentum vector of any object over a specified time interval is equal to the net impulse on that object over the same time interval. These rules are more general than the rule stating that the initial and final momentum vectors of the system are the same, as long as the net force on the system is zero. Even if this condition of zero net force does not hold, the principle of momentum conservation, which relates the initial and final momentum of a system to its net impulse, is still true. For some physicists, the phrase "momentum is conserved" is used to indicate that the initial and final momenta of a system are equal; others are careful to use it only to mean that that the momentum of a system changes not nonsensically, but according to the net impulse on it. However, it is not our goal to get introductory students to use the phrase "momentum is conserved" correctly – we are satisfied that students might be able to understand and apply the principle of momentum conservation in its more general form.

Returning to the specific content of problem 4, a pyrotechnician drops a firecracker of mass 3 kg by releasing it from rest at $t = 0.0$ s (see Figure 3-14). A short time later ($t_i = 0.4$ s), the firecracker is moving downward with speed 4 m/s. At this same instant, the firecracker begins to explode into two pieces, "top" and "bottom," with masses m_{top} = 1 kg and m_{bottom} = 2 kg. At the end of the explosion (t_f = 0.6 s), the top piece is moving upward with speed 6 m/s. The mass of the explosive substance is negligible in comparison to the mass of the two pieces.

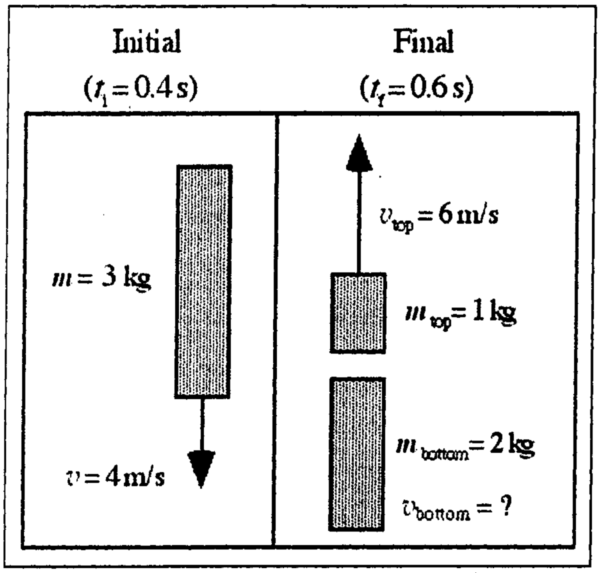

Figure 3-14: Context for problem 4 of the homework for *Conservation of Momentum in One Dimension.*

Students see a three-by-three table (shown in Figure 3-15) for entering various momentum vectors, and are advised to answer the following questions before completing the vector diagram. (The three rows are for "top," "bottom," and "system," while the three columns are for "initial," "change," and "final.") First students determine the magnitude and direction of the net force on the firecracker just before the explosion ((m_{top} + m_{bottom})*g = 30 N, down). Next students determine the magnitude and direction of the net force on the firecracker at an instant during the explosion. (Here a hint is given: Does the net force on a system depend on forces that are internal to that system? Without this hint, some students had attempted to find the net force on the system by finding the sum of the net forces on the parts, by determining the product of mass and acceleration of each part during that interval. There is not enough information given to do this, since the final velocity of the bottom part is not known.) Students then determine the magnitude and direction of the net impulse on the firecracker system during the explosion ((30 N, down)*(0.2 s) = 6 N*s, down). Next, using the impulse-momentum theorem, students determine the magnitude and direction of the change in momentum of the firecracker system during the explosion (6 N*s, down). Students are instructed to enter this vector in the bottom-center cell of the table. The next instruction is to complete the table. The instructions described above may seem unnecessarily explicit. However, even with these instructions, and after correctly calculating the net impulse on the system, a significant minority of students remains compelled to enter a zero for the change in momentum of the system, and equal vectors (12 N*s, down) for the initial and final momentum vectors of the system. Most students turn in one of two complete diagrams: a completely correct version or the version in which the non-zero net impulse on the system is ignored.

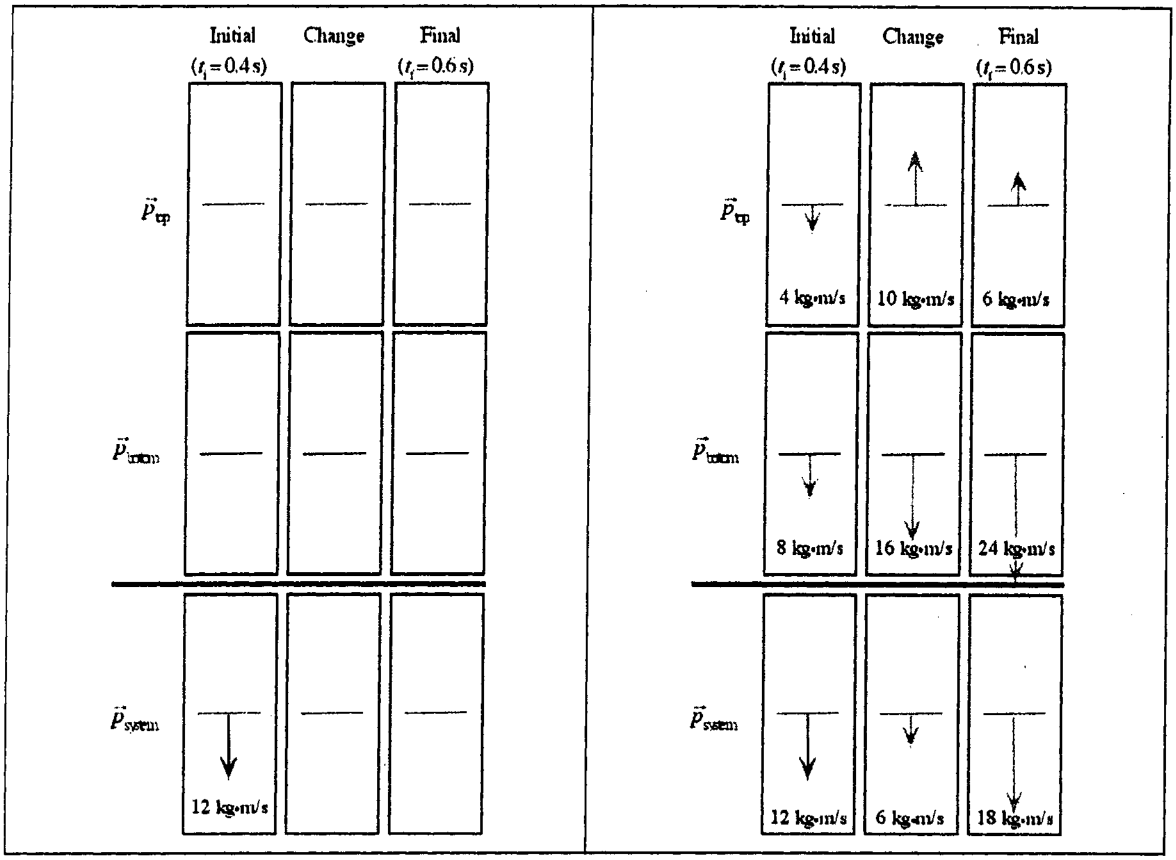

Figure 3-15: Table of momentum vectors for problem 4 of the homework for *Conservation of Momentum in One Dimension* shown as printed in the homework (left) and correctly completed in gray (right).

D. Student understanding of momentum in collisions after current version of tutorial sequence *Conservation of Momentum in One Dimension*

Two major changes from the published version of the tutorial to the current version were: changing the logical focus from the change in momentum vector to the momentum of the system; and changing the collisions of interest from those with equal target mass and varying elasticity to elastic collisions with varying target mass. Therefore, we wanted to observe whether student performance on tasks concerning either change in momentum or ETM contexts (or both) was at least as good as after the published version of the tutorial. In general, we thought that we should be satisfied with the revised tutorial only if there were no tasks on which students performed less well with the new instruction. The following sections present evidence that the current version of the tutorial helps students perform significantly better on most tasks and at least as well on other tasks.

1. Description of additional question contexts

In order to evaluate the effect of the tutorial instruction, we used questions and contexts described previously in this chapter and also some variations of them, which are described below.

a. 2.DTM.NS

This context is very similar to 2.DTM.S (see Figure 3-3), except that neither incident object comes to a stop. Instead, both incident objects rebound, with the incident object (B) that hit the larger target (Y) moving faster after the collision than the incident object (A) that hit the lighter target object (X).

b. 2.GTM.S

This context may be thought of as a hybrid of 2.DTM.S and 1.GTM level. It is different from 2.DTM.S in the following ways: the target object is the same large-mass object in both collisions, instead of the target masses being different; the final speed of the target object in each collision is given as approximately zero; the incident object that "stops" sticks to its target, thus also moving with a final speed that is approximately zero rather than exactly zero. See Figure 3-16.

c. 2.GTM.NS

This context is identical to 2.GTM.S, except that the incident object rebounds in each collision, rather than in only one. The incident objects have different final speeds.

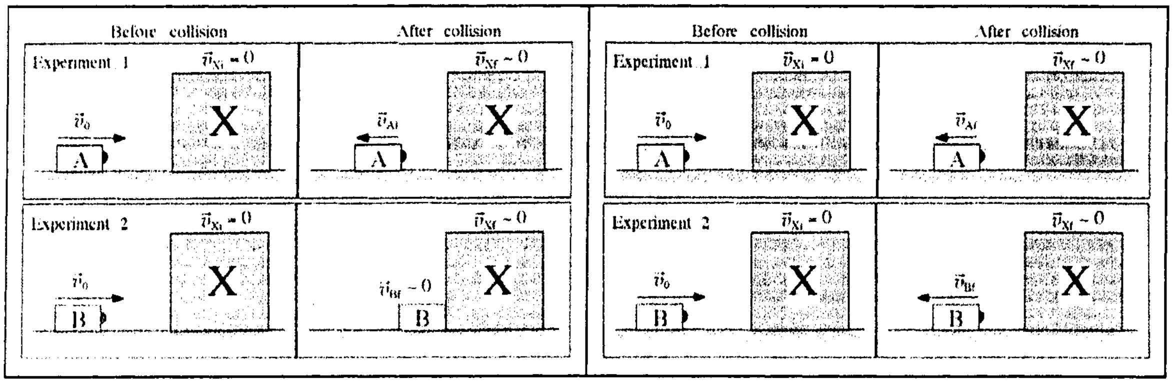

Figure 3-16: The "2.GTM.S" (left) and "2.GTM.NS" (right) contexts.

2. Administration of questions after tutorial sequence

We asked the *Momentum transfer, Incident Δp,* and *Final speed,* and *Final momentum* questions in a variety of contexts, after tutorial instruction, at various phases during the development of the tutorial curriculum.

Of particular interest are data from the introductory calculus-based physics course at Purdue University, where we asked students the *Momentum transfer, Incident Δp,* and *Final momentum* questions in the 2.DTM.NS context after they had completed the published versions of the tutorials *Changes in Energy and Momentum* and *Conservation of Momentum in One Dimension.*[11] This data set is interesting because we were able to ask Purdue students questions that had been developed after we had started modifying the tutorial *Conservation of Momentum in One Dimension* at the University of Washington. This situation enabled us to observe how students would perform on some newly constructed assessments after having used the published version of the tutorial. In previous research[12] done by members of the Physics Education Group at UW, it has been well established that Purdue students and UW students perform similarly with similar instruction.

3. Overview of student performance after tutorial sequence

a. The *Momentum transfer* question

As a reminder, in every case that the *Momentum transfer* question was asked, students compared the changes in momentum of two objects in a collision in which the incident object rebounded. Student performance on the *Momentum transfer* question in ETM contexts (Table

3-10) is separated from that in DTM contexts (Table 3-11), because in ETM contexts, no arrow is shown representing the final velocity of the target object, which some students seem to use when answering the question. Student performance in GTM contexts (Table 3-12) was similar to that in DTM contexts.

In both ETM and DTM contexts, performance was similar with both versions of the tutorial. This result suggests that the current version of the tutorial's de-emphasis on change in momentum did not have a negative effect on student learning of the rule of equal and opposite momentum transfers in a collision.

Table 3-10: Comparison of student performance on ETM versions of the *Momentum transfer* question after the published version of the tutorial and after the current version of the tutorial.

	Published tutorial	Current tutorial		
	UW 121		COL 1110	
B rebounds after colliding with Y.	N=436 3 sections 3.ETM.NS 2-D	N=204 2 sections 3.ETM.S backwards	N=241 2 sections 3.ETM.S backwards	N=240 2 sections 3.ETM.S
$\lvert\Delta\vec{p}_B\rvert = \lvert\Delta\vec{p}_Y\rvert$ (correct)	70%	65%	70%	70%
$\lvert\Delta\vec{p}_B\rvert > \lvert\Delta\vec{p}_Y\rvert$	20%	25%	25%	15%
$\lvert\Delta\vec{p}_B\rvert < \lvert\Delta\vec{p}_Y\rvert$	10%	10%	5%	15%

Table 3-11: Comparison of student performance on DTM versions of the *Momentum transfer* question after the published version of the tutorial and after the current version of the tutorial.

	Published tutorial	Current tutorial
B rebounds after colliding with Y.	PRD 152, N=443 2 sections 2.DTM.NS	UW 121, N=216 2 sections 2.DTM.S
$\lvert\Delta\vec{p}_B\rvert = \lvert\Delta\vec{p}_Y\rvert$ (correct)	55%	55%
$\lvert\Delta\vec{p}_B\rvert > \lvert\Delta\vec{p}_Y\rvert$	20%	25%
$\lvert\Delta\vec{p}_B\rvert < \lvert\Delta\vec{p}_Y\rvert$	20%	15%

Table 3-12: Student performance on GTM versions of the *Momentum transfer* question after the current version of the tutorial *Conservation of momentum in one dimension*.

	Current tutorial	
B rebounds after colliding with giant target Y.	UW 121, N=178 1 section 2.GTM.S	UW 121, N=129 1 section 2.GTM.NS
$\lvert\Delta\vec{p}_B\rvert = \lvert\Delta\vec{p}_Y\rvert$ (correct)	55%	50%
$\lvert\Delta\vec{p}_B\rvert > \lvert\Delta\vec{p}_Y\rvert$	20%	20%
$\lvert\Delta\vec{p}_B\rvert < \lvert\Delta\vec{p}_Y\rvert$	20%	30%

b. The *Incident Δp* question

Student performance on the *Incident Δp* question after the current version of the tutorial appears to be just as good as, if not better than, after the published version of the tutorial (see Table 3-13). Like the results in the previous section on the *Momentum transfer* question, this result suggests that student understanding of change in momentum vectors is not harmed by the current tutorial's emphasis on system momentum vectors instead of change in momentum vectors.

Table 3-13: Comparison of student performance on the *Incident* Δp question after the published version of the tutorial and after the current version of the tutorial. Performance did not depend on whether students were asked the *Incident* Δp question or the *Incident* Δv question, so these results are shown together.

	Published tutorial	Current tutorial		
	UW 121		COL 1110	
	N=392 3 sections 3.ETM (mixed)	N=134 1 section 3.ETM.S	N=240 2 sections 3.ETM.S	N=241 2 sections 3.ETM.S backwards
Correct response	80%	80%	85%	90%
Scalar response	<5%	5%	15%	5%

c. The *Final speed* question

Student performance on the *Final speed* question (in ETM contexts), after the current version of the tutorial, was better than that after the published version, even though the attention given to ETM collisions in the tutorial was significantly reduced (see Table 3-14).

Table 3-14: Comparison of student performance on 3.ETM versions of the *Final speed* question after the published version of the tutorial and after the current version of the tutorial.

	Published tutorial	Current tutorial			
	UW 121			COL 1110	
	N=234 2 sections 3.ETM (mixed)	N=134 1 section 3.ETM.S	N=158 1 section 3.ETM.S backwards	N=240 2 sections 3.ETM.S	N=241 2 sections 3.ETM.S backwards
Correct response	60%	85%	80%	75%	75%
Scalar response	20%	5%	10%	10%	15%
Reverse response	10%	5%	$<5\%$	5%	10%

d. The *Final momentum* question

The success rate on the *Final momentum* question after the published tutorial (25%) is very similar to the typical success rate on the question before any tutorial instruction on momentum (20%) (see Table 3-15). In fact, PRD 152 had also worked through the published *Changes in Energy and Momentum* tutorial before working through the published version of *Conservation of Momentum in One Dimension.* More students are successful on the *Final momentum* question after working through the current version of the tutorial (~50%), but they are not nearly as successful as they are on the *Final speed* question in ETM contexts, even though the tutorial instruction focused on a DTM rather than an ETM context. These considerations suggest that employing a correct understanding of momentum conservation may be especially difficult when students are faced with the temptation to determine an object's momentum from $p=mv$ (or $p \propto v$), instead of considering the object's role in a system.

In two instances, students could select a fourth option as an answer to the *Final momentum* question – something to the effect of "can't tell on the basis of the information given." This response was chosen by 10% of the class in each case, less than the fraction who stated that the final momenta of X and Y were equal. The fact that so many students selected "equal" when they could have chosen "can't tell" suggests some level of confidence among those students who selected "equal."

Table 3-15: Comparison of student performance on various versions of the *Final momentum* question before tutorial instruction, after the published version of the tutorial, and after the current version of the tutorial.

	Before tutorial	Published tutorials*	Current tutorial		
	UW 121	PRD 152	UW 114	UW 121	
	N≈1800 16 sections 2.DTM.S	N=443 2 sections 2.DTM.NS	N=51 1 section 2.DTM.S	N=124 1 section 2.DTM.NS	N=163 1 section 2.DTM.S
$\lvert\vec{p}_{Xf}\rvert < \lvert\vec{p}_{Yf}\rvert$ (correct)	20%	25%	45%	55%	50%
$\lvert\vec{p}_{Xf}\rvert > \lvert\vec{p}_{Yf}\rvert$	50%	20%	25%	15%	25%
$\lvert\vec{p}_{Xf}\rvert = \lvert\vec{p}_{Yf}\rvert$	30%	40%	15%	30%	20%
"Can't tell"	N/A	10%	10%	N/A	N/A

* Students in PRD 152 had completed both the tutorials *Changes in Energy and Momentum* and *Conservation of Momentum in One Dimension.*

Student performance on the *Final momentum* question in GTM contexts was also significantly improved from before tutorial instruction (from 20% correct to 60% correct) (see Table 3-16 and Table 3-17). In 2.GTM contexts, unlike in DTM contexts, we can label one response as the "scalar" response, since scalar momentum reasoning and velocity-dominant reasoning would lead to distinct answers.

In 1.GTM contexts, it appears that almost all students recognize, after working through the current version of the tutorial, that an object with nearly zero velocity need not have nearly zero momentum (see the bottom row of Table 3-17). However, it appears that roughly 10% more students give the scalar response on the post-test than on the pretest. This suggests that some students' thinking may evolve from treating momentum as depending primarily on (or even indistinct from) velocity to treating it as a conserved scalar quantity, as a result of working through the tutorial.

Table 3-16: Student performance on 2.GTM versions of the *Final momentum* question after the current version of the tutorial *Conservation of Momentum in One Dimension.*

	Current tutorial	
	UW 121, N=178 1 section 2.GTM.S	UW 121, N=129 1 section 2.GTM.NS
$\lvert\vec{p}_{Xf}\rvert < \lvert\vec{p}_{Yf}\rvert$ (correct)	65%	65%
$\lvert\vec{p}_{Xf}\rvert > \lvert\vec{p}_{Yf}\rvert$ (scalar)	15%	20%
$\lvert\vec{p}_{Xf}\rvert = \lvert\vec{p}_{Yf}\rvert$	20%	10%

Table 3-17: Comparison of student performance on 1.GTM versions of the *Final momentum* question before and after the current version of the tutorial *Conservation of Momentum in One Dimension.*

	Before tutorial		Current tutorial	
	UW 121	COL 1110		UW 121
	N=346 3 sections	N=388 2 sections	N=515 2 sections	N=72 1 section
Correct response	20%	15%	60%	55%
Scalar response	30%	25%	35%	40%
Velocity-dominant response	40%	45%	5%	5%

That student thinking about momentum may evolve from velocity-dominant to "conserved scalar" is also supported by the following summary of combinations of responses given by one class before and after the current version of the tutorial (see Table 3-18).

Table 3-18: Comparison of combinations of responses to the *Momentum transfer, Incident Δp,* and *Final momentum* questions, before and after the current version of the tutorial *Conservation of Momentum in One Dimension.* In the first column, one of the two sections is the section in the second column.

"Coherent" responses	UW 114, N=217 2 sections	UW 114, N=51 1 section
Correct	<5%	25%
Conserved scalar	5%	10%
Velocity-dominant	20%	<5%

E. SUMMARY

We have presented evidence that students' tendency to conflate velocity and momentum is as least as prevalent as the tendency to treat momentum as a conserved scalar quantity. These tendencies should not be treated as independent conceptual errors, but as two incorrect generalizations of a misinterpretation of velocities in elastic collisions. When considering elastic collisions in interviews, some students spontaneously introduced a limit of large target mass as support for an incorrect treatment of momentum. Revising the tutorial curriculum so that it centered on helping students construct an understanding of momentum in this "giant target" limit has increased student success on tasks relating to multiple kinds of collision sets. We understand this increase in student performance to be a product of helping students to connect their vector skills with a pivotal case they naturally consider (in some form) and usually associate with support for treatment of momentum as a scalar (and virtually indistinguishable from velocity). By expanding the set of tasks with which we evaluate student understanding, we observed that the published tutorial had very little effect on improving student understanding of elastic collisions. Furthermore, reducing the emphasis in the tutorial on issues relating to change in momentum did not diminished students' ability to succeed on tasks related to change in momentum. This result suggests that the concept *change in momentum* did not function as a useful tool for many students.

NOTES TO CHAPTER 3

[1] McDermott, L. C., P. S. Shaffer, and the Physics Education Group at the University of Washington, *Tutorials in Introductory Physics*, First Edition, (Prentice-Hall, Upper Saddle River, NJ, 2002).

[2] "Student understanding of the work-energy and impulse-momentum theorems," R. A. Lawson and L. C. McDermott, Am. J. Phys. **55**, 811-817 (1987).

[3] "The challenge of matching learning assessments to teaching goals: An example from the work-energy and impulse-momentum theorems," T. O'Brien Pride, S. Vokos, and L. C. McDermott, Am. J. Phys. **66**, 147-156 (1998).

[4] O'Brien Pride, T., "An investigation of student difficulties with two dimensions, two-body systems, and relativity in introductory mechanics," Ph.D. dissertation, Department of Physics, University of Washington, 1997 (unpublished).

[5] For a discussion of what we mean by "similar instruction," see Chapter 1.

[6] Nguyen, N.-L., and D. E. Meltzer, "Initial understanding of vector concepts among students in introductory physics courses," *Am. J. Phys.* **71**, 630-638 (2003).

[7] Graham, T., and J. Berry, "A hierarchical model of the development of student understanding of momentum," *Int. J. Sci. Educ.* **18**, 75-89 (1996).

[8] It may be interesting to note that some of the UW sections used a different textbook, in which momentum is treated before energy, rather than the (more common) other way round. This change in the order of topics appeared to have no effect on student success on the *Momentum transfer, Incident* Δp, and *Final momentum comparison* questions.

[9] "Student understanding of the work-energy and impulse-momentum theorems," R. A. Lawson and L. C. McDermott, Am. J. Phys. **55**, 811-817 (1987).

[10] It may be argued that the only coherent understanding of any physics concept is the correct one. We do not mean that a student's alternative understanding would withstand all logical challenges. Rather, we mean that the student's experience of answering the present questions is that his/her understanding is not challenged or shown to be flawed.

[11] McDermott, L. C., P. S. Shaffer, and the Physics Education Group at the University of Washington, *Tutorials in Introductory Physics*, First Edition, (Prentice-Hall, Upper Saddle River, NJ, 2002).

[12] Ortiz, L. G., "Identifying and addressing student difficulties with rotational dynamics," Ph.D. dissertation, Department of Physics, University of Washington, 2001 (unpublished).

CHAPTER 4. STUDENT LEARNING OF NEWTON'S 2ND LAW FOR ROTATING RIGID BODIES

In her PhD dissertation[1], Luanna G. Ortiz described an investigation of students' ability to apply Newton's second law to extended rigid bodies. To summarize briefly, the research showed that most students do not use Newton's second law correctly; instead, it is far more common for students to use what Ortiz called "location-based force reasoning," in which the effect of a force on the acceleration of the center of mass of a body depends on where the force is applied to the body. She also found that students are highly resistant to changing how they think about forces on rigid bodies – that is, only about a third of them successfully applied Newton's 2nd law in questions about the center-of-mass accelerations of rigid bodies after tutorial instruction. This research is summarized in Chapter 2. This chapter reports on an extension of Ortiz' investigation, in which our efforts were directed somewhat toward characterizing students' errors and their relationships to problem contexts, but primarily toward improving students' success rate on tasks related to applying Newton's 2nd law to rotating rigid bodies.

The main concern of this research is student understanding of this principle: The effect of a force on an object's center-of-mass acceleration does not depend on how that force is affecting the rotational motion of the object. For brevity, in this dissertation we will call this consequence of Newton's 2nd law "N2R." What basis do we have for supposing that introductory students might be significantly more successful on N2R tasks after tutorial instruction? First, an informal survey of introductory physics textbooks reveals that little attention is given in these texts to maintaining the truth of Newton's 2nd law, even for rotating rigid bodies. In most cases, Newton's 2nd law is said to have a "rotational analogue," which may imply to students that Newton's 2nd law itself is not appropriate for rotating bodies. Furthermore, most textbook explanations of phenomena like rolling with slipping emphasize energy relations, and how the rotational motion of an object influences its translational motion, rather than using the Newtonian concept that translational motion is influenced only by the net force. Textbooks sometimes have worked examples in which N2R is implied. For example, when determining the force of constraint on a falling yo-yo or a wheel rolling down an incline, it is necessary to use N2R. However, we have not seen any textbook make the point explicit that the reader might be surprised to learn that Newton's 2nd law is still true, and that this might have some surprising consequences. Given these observations, we became less convinced that N2R was necessarily very difficult, thinking instead that perhaps students had very few opportunities to reflect on whether it was true. Second, the principle we want students to be able to apply is easy to state: "The effect of a force on an object's translational motion does not depend on how that force affects the object's rotational motion." Our experience with introductory students and their ability to reason about concepts more abstract than this leads us to believe that a correct understanding and ability to apply N2R is within reach of most introductory students.

We have incorporated an experiment into tutorial instruction that can be used to demonstrate the validity of the N2R principle. That is, the experiment involves two objects on which equal forces are exerted at different locations, so that one object rotates while the other does not. It can be observed that the objects undergo identical center-of-mass accelerations. However, this experiment, though a *testable* case, does *not* appear to function as a pivotal case in student reasoning. That is, the outcome of that case did not affect student thinking about N2R as much as we had hoped. In interviews, we observed that students remembered the results of the experiment but did not understand its meaning or importance. By identifying a different case with which almost all students associated their understanding of N2R (in whatever correct or incorrect form), and guiding students to link their reasoning about the pivotal case with the testable case, we were able to increase the student success rate on N2R tasks more than with instruction involving the testable case alone.

In part A of this chapter, we describe the published version of *Dynamics of Rigid Bodies* and summarize the effectiveness of the published tutorial. In part B, we describe our additional research on student understanding after traditional instruction. In part C, we briefly describe intermediate attempts to improve the tutorial *Dynamics of Rigid Bodies* and the reasons why we found the attempts unsatisfactory. In part D, we describe in detail the current version of the tutorial sequence for *Dynamics of Rigid Bodies*. In part E, we present evidence that student understanding of N2R and related issues is much better after the current version of the tutorial than after any previous version. We also introduce the notion that student success on N2R tasks is significantly affected by student understanding of specific types of forces. This force-type dependence of student understanding of forces is the subject of the next chapter.

A. STUDENT UNDERSTANDING OF NEWTON'S 2ND LAW FOR ROTATING RIGID BODIES AFTER INITIAL (PUBLISHED) VERSION OF TUTORIAL SEQUENCE *DYNAMICS OF RIGID BODIES*

1. Description of published version of tutorial

The first section of the tutorial concerns an experiment that (behind the scenes) we called "the magic ruler" (see Figure 4-1). Students are told that a ruler is placed on a pivot and held at an angle; the pivot passes through the center of the ruler. Students predict the motion of the ruler after it is released from rest. After recording their predictions, students observe the demonstration, in which the ruler remains at rest in the orientation from which it is released. Students then consider (on the basis of this observation) whether the angular acceleration of the ruler is clockwise, counter-clockwise, or zero. After noting that the angular acceleration of the at-rest ruler is zero, students infer that the net torque on the ruler (about the pivot) is also zero (according to $\vec{\tau}_{\text{net}} = I\vec{\alpha}$). Next, students answer analogous questions about the center of mass of the ruler; after noting that the center of mass has zero acceleration, students infer that the net force on the ruler must be zero (according to $\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}}$). The tutorial then introduces the notion of an *extended free-body diagram*, which shows the forces exerted on an object and the locations at which they are exerted. Students then check their free-body diagram for consistency with their earlier statements about the net torque and net force on the ruler.

In the second and final section of the tutorial, students consider what we now call the "unconnected spools experiment." In this experiment, two identical spools (A and B) are held the same height above the floor. The thread attached to spool A is tied to a support, while spool B is not connected to a support.

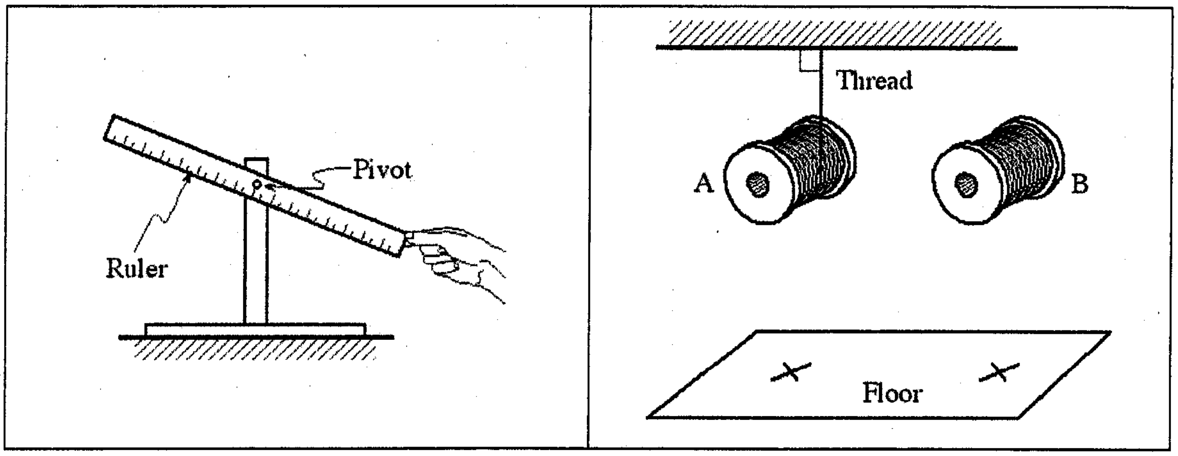

Figure 4-1: Contexts from the published version of the tutorial *Dynamics of Rigid Bodies:* the "magic ruler" (left) and the "unconnected spools" (right).

Both spools are released from rest at the same instant. Students draw extended free-body diagrams for each spool for some instant after the spools are released, and determine the direction of the net torque on each spool about its center of mass. Then students predict which spool will reach the floor first and explain how their predictions are consistent with their free-body diagrams. The tutorial then explicitly asks students to describe how the net force on an object is related to the individual forces on a free-body diagram when the forces are exerted at different points on the object. Students perform the experiment and check their predictions. (Spool B hits the floor before spool A.) On the basis of their observations, students compare the center-of-mass accelerations of spools A and B in direction (they both point down) and magnitude (it is greater for B). Students then consider the following hypothetical discussion among three students:

Student 1: The string exerts a force that is tangent to the rim of spool A. This force has no component that points toward the center of the spool, so this force does not affect the acceleration of the center of mass.

Student 2: I disagree. The acceleration of the center of mass of the spool is affected by the string. Any of the force not used up in rotational acceleration will be given to translational acceleration. This is why the acceleration of the center of mass of spool A is less than *g*.

Student 3: The net force on spool A is the gravitational force minus the tension force. By Newton's second law, the acceleration of the center of mass is the net force divided by the mass. A force will have the same effect on the motion of the center of mass regardless of whether the force causes rotational motion or not.

Students discuss whether they agree with any of the above statements and whether they should revise their earlier description of the relationship between the net force and the individual forces on an object. A special note indicates that students should discuss their answers with tutorial instructors at this point. Finally, students write down (a) Newton's 2^{nd} law for each spool, in terms of individual forces, and (b) the rotational analogue of Newton's 2^{nd} law for each spool, expressing the net torque in terms of individual forces and distances.

The published version of the homework deals with two physical situations. In problem 1, a ruler is initially at rest on a frictionless pivot (as in Figure 4-1) and horizontal when a student briefly exerts a downward force on the right end. Students describe the motion of the ruler while the hand is pushing and after the hand has stopped pushing. They also draw extended free-body diagrams for both instants and rank the magnitudes of all forces acting on the ruler over both instants. In problem 2, three identical rectangular blocks are at rest on a horizontal, frictionless ice rink, as shown in Figure 4-2. Forces of equal magnitude and direction are exerted on each of the three blocks. Each force is exerted at a different point on the block (block 1, to the left of the center of mass; block 2, directed at the center of mass; block 3, far to the right of the center of mass). Students first describe the direction and magnitudes of the angular accelerations of the three blocks, and then they describe the directions and magnitudes of the center-of-mass accelerations of the three blocks.

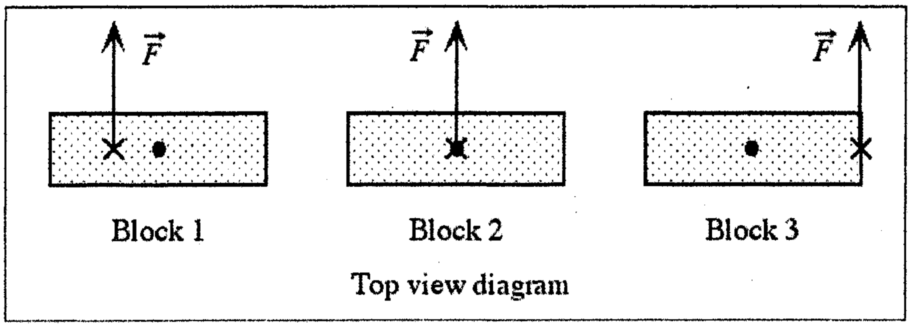

Figure 4-2: The "Three blocks" problem.

Our current view of the published materials described above is that they provided sufficient opportunity for the question "Does a force that is causing rotation affect the motion of the center of mass in the same way as a force that does not cause rotation?" to arise in discussion. Our classroom experience with this version of the tutorial and homework recalls many discussions with students in which the N2R issue was made clear, and in which tutorial instructors testified unambiguously to the truth of N2R. Homework problem 2, in which students rank the center-of-mass accelerations, was often the graded problem. Graded homework papers (with clear indications from the grader that all three blocks indeed had the *same* center-of-mass acceleration) were, in most cases, returned to the students in time for them to reflect on the relationship between forces and acceleration of the center of mass of a rigid body before taking the next exam. In short, to whatever degree the published version of the tutorial sequence *Dynamics of Rigid Bodies* is *ineffective* is not (we believe) because students were not told or otherwise informed that N2R is true.

B. STUDENT UNDERSTANDING OF NEWTON'S 2ND LAW FOR ROTATING RIGID BODIES BEFORE TUTORIAL *DYNAMICS OF RIGID BODIES*

1. Prior instruction

In essentially all cases at this stage of the introductory course, students have had much instruction on Newton's second law as it applied to non-rotating objects, and little instruction emphasizing that the law can be applied to rotating objects as well, in (almost) its original form (if the original form did not specify that the acceleration is that of the center of mass).

2. Description of contexts in which questions about Newton's second law for rotating rigid bodies were asked

This section describes contexts that were designed to probe student understanding of N2R and related concepts. In most of the contexts used by Ortiz to assess student understanding of N2R, the forces involved are (1) depicted by arrows and are exerted by unidentified external agents, or (2) shown as exerted by hands pushing normally on the object. Although all questions that Ortiz asked about resulting center of mass acceleration (for which the applied forces were known) were about "the instant shown," student responses often showed interest in significantly later instants. For instance, when a hand pushes normally to the surface of a block, after the block starts to turn, does the force by the hand stay normal to the surface, or does it maintain the same direction as before? Does the magnitude of the force stay the same, or does it diminish, since the hand may be pushing in a direction that is "less normal" to the surface after the block has turned? In order to address these concerns (though not without possibly introducing new concerns), we developed tasks for which no exact distinction between the instant shown and some later instant is necessary. In other words, in the new tasks, all forces and center-of-mass accelerations involved are of constant magnitude and direction over a finite period of time.

Another motivation for this shift in the problem contexts was to match assessments of student understanding with the character of the experiments contained in the published tutorial. Though this experiment was modified from the published version, both the original and the modified versions of the experiment involve constant forces and center-of-mass accelerations. An advantage of this feature of the experiments is that discussion of magnitudes of forces and center-of-mass accelerations is easier if the time interval of interest does not have to be very precisely defined. Namely, it is easier for students and instructors to discuss "while the spools are falling" than it is to discuss "*just after* the finger starts pushing." Furthermore, to focus assessments on these sorts of situations seemed more realistic; it is easier to keep a tension force or a static friction force constant (in magnitude and direction) over time than it is to ensure that a finger exerts a constant normal force.

Though this extension of Ortiz' research has used newly designed assessments of a theme that contrasts with that of the original research, we believe that the research on student understanding of N2R has benefited from continued use of assessments of both themes, that is, with varied level of abstraction. Considering student responses to these different assessments has helped us understand how students interpret certain terms (like *force*) in this domain of physics, and has also helped reveal that student understanding of forces on rigid bodies truly diverges not just from conventional terminology, but also from the results of realizable experiments. Furthermore, we believe that formal instruction in physics should aim to help students (1) perform tasks that lie on a wide spectrum of abstraction and (2) understand the relationship between statements and questions of varied level of abstraction.

a. Circular pucks

This context is similar to many described in Ortiz' PhD dissertation.[1] Both this context and the next (Four square blocks) are posed in the most formal terms of *applied forces* ("*F*"), exerted on the object(s) of interest by other unknown objects and in an unknown manner, and *acceleration of the center of mass* for some instant shown.

In this context, some number (2 or 3) of identical circular pucks is initially at rest on a flat, frictionless ice rink (see Figure 4-3). The pucks are shown in top view. An arrow near each puck shows the magnitude, direction, and point of application of a horizontal force that is exerted on the rim of each puck. The center of mass of each puck is indicated in with an 'X.' Students are told that the forces are exerted for a short time interval Δt. An adjacent figure shows the four cardinal directions, with north directed toward the top of the page.

i. 3 pucks

Forces of equal magnitude (F) and direction (north) are exerted on pucks 1 and 2. The force on puck 2 is applied at the southernmost point on its rim such that the force is directed at the center of the puck. The force on puck 1 is applied at the south-easternmost point on the rim of the puck. The force on puck 3 is of magnitude $2F$, directed to the south, and is applied at the easternmost point on rim of the puck.

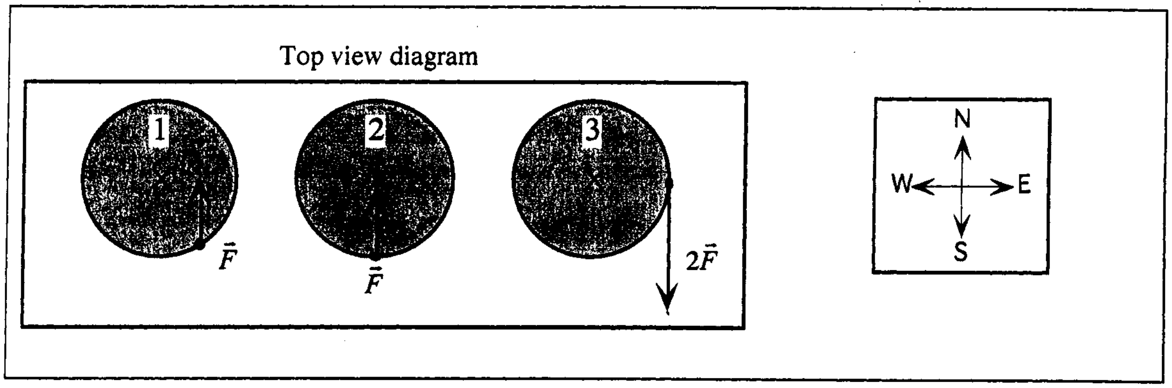

Figure 4-3: The "Circular pucks" context.

ii. 2 pucks

This context is the same as "3 pucks," except that puck 3 is omitted.

b. 4 square blocks

This context was also described in Ortiz' dissertation[1] (page 133). Four identical square blocks are on a frictionless surface. Two horizontal forces are exerted on each block, at various locations and at an angle θ either to the south or the north of the east-west line, as shown in Figure 4-4. All forces are represented abstractly by arrows and are of the same magnitude F.

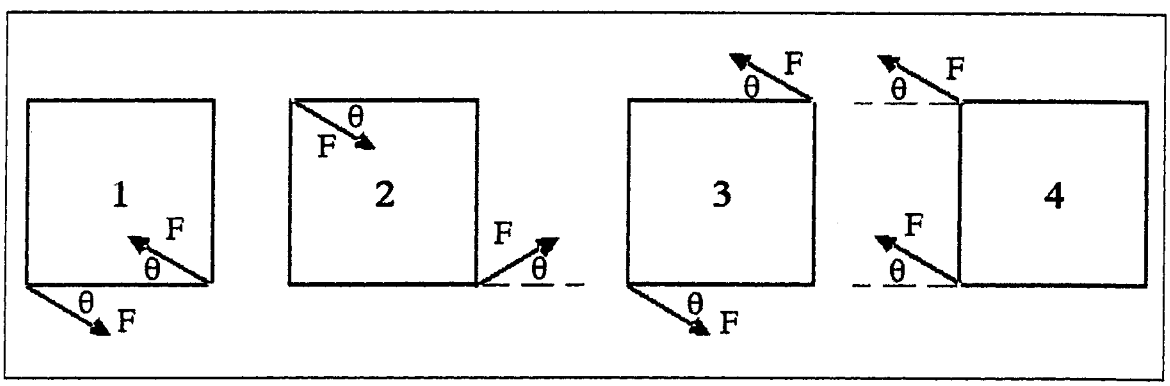

Figure 4-4: The "4 square blocks" context.

c. Block and spool

This context was initially inspired by the difficulties brought by the abstract representation of a force as an arrow. We aimed to design a context in which students would be confident that all forces exerted were of constant magnitude and direction. Furthermore, we had observed that some students tended to misinterpret the meaning of the term *acceleration of the center of mass*, sometimes mistaking it for an angular acceleration *about* the center of mass, or possibly some combination of these two accelerations. In order to avoid these problems when assessing student understanding via responses to written questions, we introduced two devices: (1) the "off-center" force is exerted as the tension in a thread that is wrapped around the object, and (2) the object traverses a fixed distance, and we ask students about the time the object takes to cross that distance. Even though we sought to minimize the level of abstraction in the "Block and spool" context, we suspected that some students were interpreting the word *tension* in a non-conventional way. Thus, we created variations of "Block and spool" that explored student understanding of the abstraction *tension*, by replacing it or juxtaposing it with other various elements.

i. Standard version

In "Block and spool," a block and a spool, of equal mass, are on a level, frictionless surface (see Figure 4-5). The objects are initially at rest next to a line marked "start." A thread is attached to the center of the front face of the block. Another thread is wrapped many times around the spool, which sits on end, and may unwind as it is pulled. The threads start pulling at the same time, with the same constant tension. The threads have negligible mass.

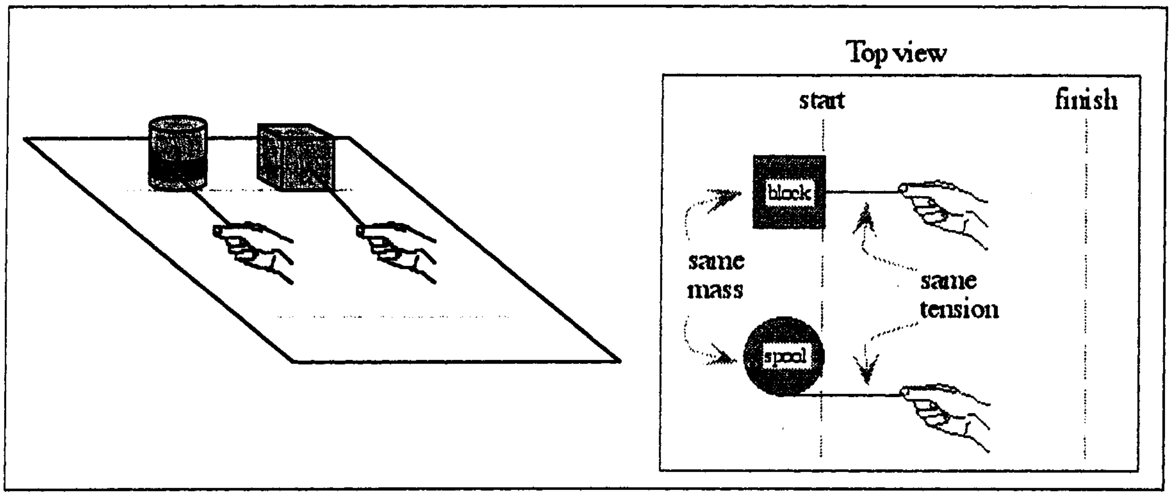

Figure 4-5: Perspective-view (left) and top-view (right) diagrams of the "Standard" version of the "Block and spool" context.

ii. Standard plus pictures of spool at later instant

In this variation, students are also shown three pictures of how the spool, its thread, and the hand pulling the thread might look at an instant some time after the hand starts pulling on the thread (see Figure 4-6). In all three pictures, the hand pulling the thread has moved the same distance. In the first picture, the thread has been extended, and the center of the spool has remained in place. In the second picture, the thread has been extended, and the spool has moved forward, but not as far as the hand has moved. In the third picture, the spool has moved forward the same distance that the hand has moved, and the thread has not been extended. We thought perhaps that asking the students to visualize a later instant in this process might call forth a kinesthetic sense that an experimenter might find it difficult to maintain a specified tension in the thread as it extends. We thought some students might connect this sense with an idea that equal tensions in the two threads would correspond with equal translational motions of the block and spool rather than equal motions of the hands. In other words, we thought that student performance on the questions in the "Block and spool" context might improve with these cues, thereby helping us to avoid underestimating student understanding of N2R before tutorial instruction.

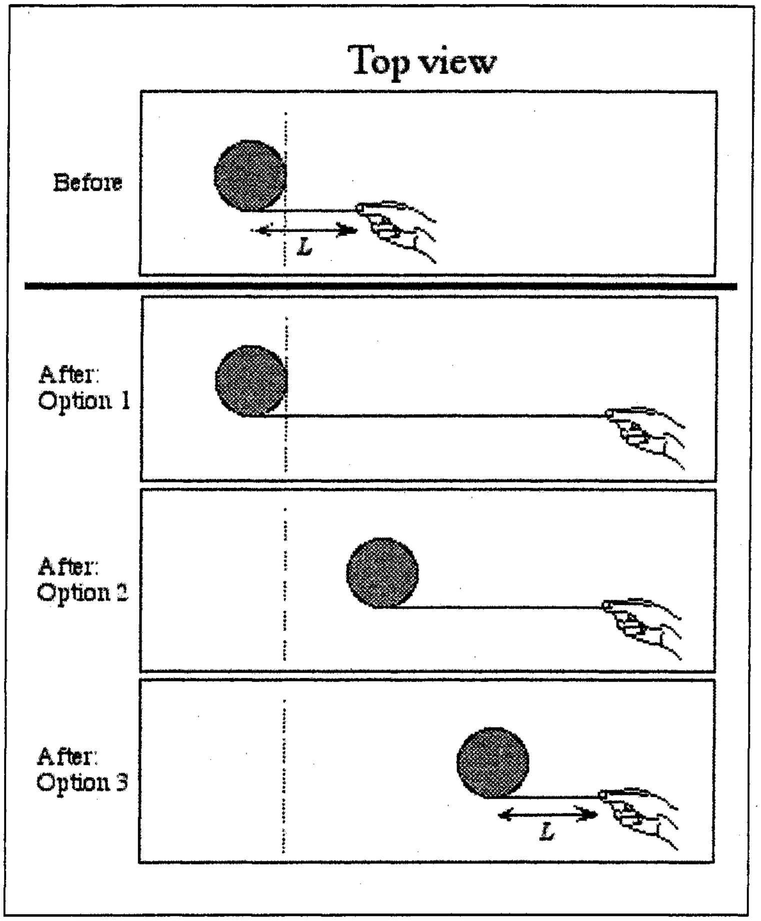

Figure 4-6: Diagram showing options for how the spool and hand might look at a later instant in the "Standard plus pictures" version of the "Block and spool" context.

iii. Standard plus hands moving together

Here we describe two experiments involving the block and spool (see Figure 4-7). In Experiment A, the experimenters pull on the threads so that the tension in each thread is the same constant value; thus, Experiment A is the same experiment that is described in the standard version of "block and spool." In Experiment B, the experimenters pull so that their hands are side-by-side at each instant. Students are told that the hands may or may not remain side-by-side in Experiment A. Our intent in juxtaposing these experiments was to allow students the opportunity to distinguish the experiments. In other words, we thought presenting Experiments A and B together might affect how some students interpret the phrase *same constant tension,* as used in the description of Experiment A.

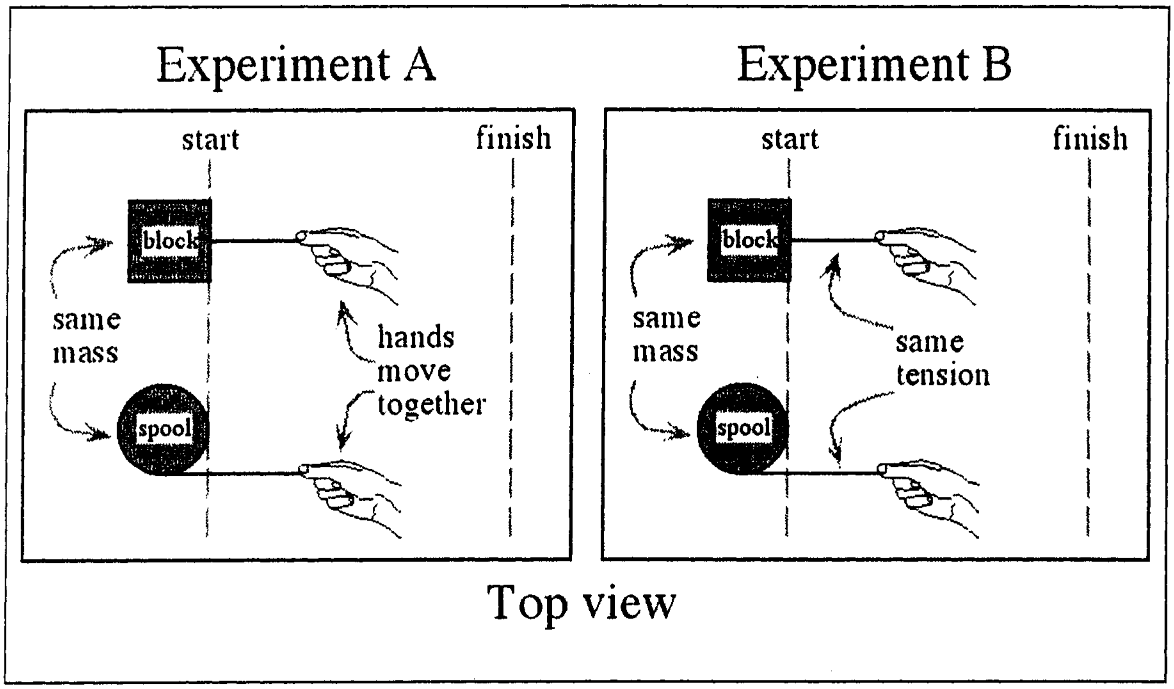

Figure 4-7: The "Standard plus hands" version of the "Block and spool" context.

iv. With springs only

In another effort to explore how students interpret the phrase *same constant tension,* we designed a version of the "block and spool" context that eliminated the abstraction *tension* from the problem description. Here, the threads attached to the objects have been cut, and identical springs have been inserted and reattached to the threads, as shown in Figure 4-8. The threads are pulled so that the springs are equally stretched throughout the motion. The figures from the standard version are modified to show that the inserted springs are the same length. An additional picture (see Figure 4-9) shows the spring at its natural, unstretched length and at an extended length, as in the experiment. Though we expected that some students would not realize that the above description constitutes the operational meaning of *same tension,* and thus might find the problem more challenging, our intention was to make the context more (not less) understandable, by making it more concrete. Though this more concrete version may seem contrived, it yields the advantage of revealing the extent to which student performance on the standard version depends on student understanding of the more abstract term *tension*.

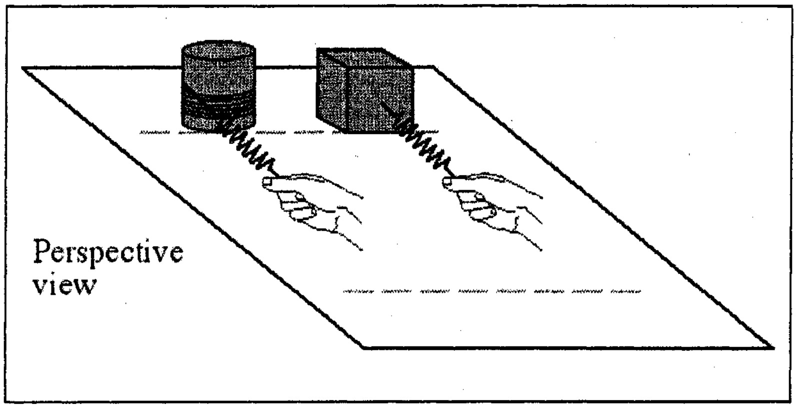

Figure 4-8: Perspective-view diagram of the experiment in the "With springs only" version of the "Block and spool" context.

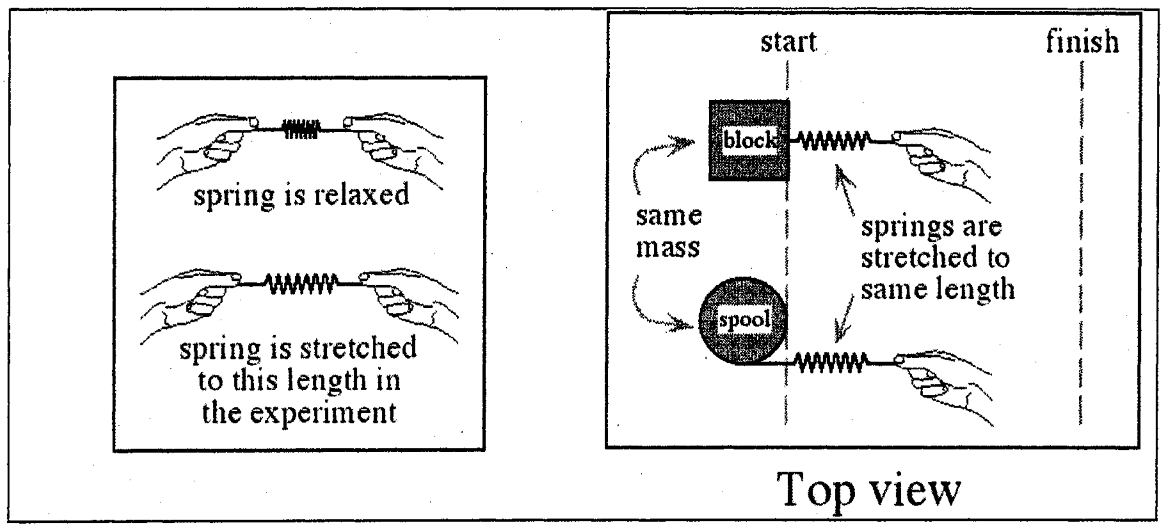

Figure 4-9: Diagram comparing natural length of spring to that in the experiment (left) and Top-view diagram of the experiment (right) in the "With springs only" version of the "Block and spool" context.

v. Standard plus springs

This context combines the standard version of the "Block and spool" context (which includes the phrase *same tension*) with the variation described immediately above. Here the two versions of the experiment are presented together as Experiment A and Experiment B, as shown in Figure 4-10. Each experiment is described as indicated above. Just as in the combined variation "standard plus hands," our intent was to observe whether the presence of the "springs" version would affect how students responded to the standard version.

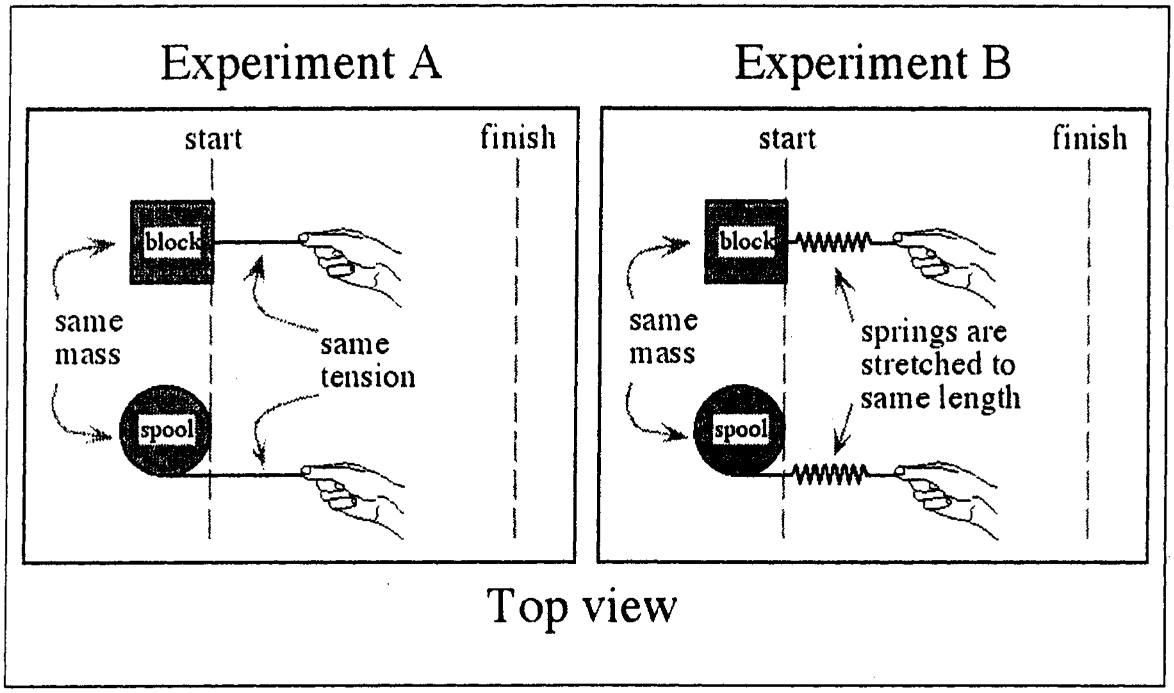

Figure 4-10: The "Standard plus springs" version of the "Block and spool" context.

d. Connected spools

This problem context is meant to correspond exactly with the experiment that students perform in the current version of the tutorial. In this experiment, two identical spools (A and B) are connected by a long massless thread over an ideal pulley, as shown in Figure 4-11. The thread is wrapped many times around spool A, passes over the pulley, and then is attached to a fixed point on spool B. The spools are released from the same height above the floor. We wanted to ask students about this context for at least three reasons: so that students may prepare for tutorial by beginning to think about the experiment, so that we might predict how students will respond to the experiment when it is presented in tutorial, and so that we can observe how frequently students relate the experiment to some (correct or incorrect) form of N2R.

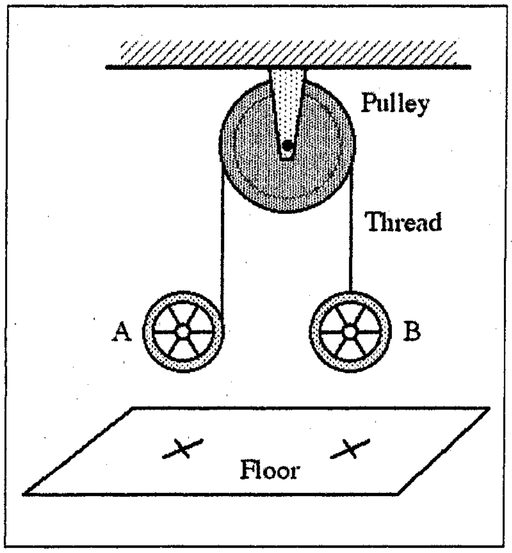

Figure 4-11: The "Connected spools" context.

e. Unconnected spools

This context is described at length in Ortiz' dissertation (*p.* 45-46), and corresponds to the experiment described (and performed by students) in the second half of the published tutorial (see Figure 4-1). The context is described here because we asked an additional question (not described by Ortiz) in "Unconnected spools," and because we wanted to observe relationships between student responses to questions in this context and some of the new contexts when both were asked on the same pretest. In "Unconnected spools," two identical spools are released from the same height above the ground. Spool A's thread is attached to a fixed support above the spool, while Spool B's thread is not connected to any other objects.

f. 3 descending objects

This pair of contexts provides students with an opportunity to apply N2R in situations that students usually associate with an energy analysis, rather than one involving forces and center-of-mass acceleration. There are two versions of "3 descending objects." In each version, three objects (A, B, and C) of equal mass start from rest and move with some one-dimensional center-of-mass acceleration through the same fixed distance. Objects A and B are both round, with the same radius, and accelerate in both a rotational and a translational sense. Students are told explicitly that the mass inside objects A and B are distributed differently, so they have different moments of inertia. Object C is a block and does not rotate. In both versions: All objects begin accelerating at the same instant, from the same height; object A reaches the end of the fixed distance first, followed by objects B and C which reach the end at the same instant.

i. Incline *f*

In the Incline *f* version of "3 descending objects," the objects start at the top of three identical ramps (see Figure 4-12). Objects A and B roll down their ramps without slipping. One figure shows the objects beginning at the tops of their ramps, and another figure shows object A arriving at the bottom of its ramp, with objects B and C trailing A by the same distance. Students were told not to assume that the coefficients of friction between any object and its ramp were the same as for any other object. The '*f*' denotes that the forces of interest are friction forces.

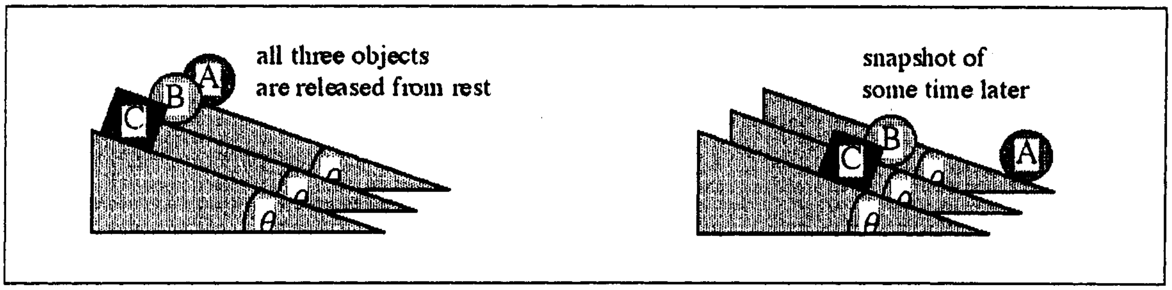

Figure 4-12: The Incline *f* version of the "3 descending objects" context.

ii. Vertical *T*

In the Vertical *T* version of "3 descending objects," objects A and B are each wrapped many times with a light thread, each of which is attached to a fixed point on the ceiling above the object (see Figure 4-13). A hand holds the thread that is attached to the top of Block C; the hand moves down so that the vertical position of block C is the same as the vertical position of object B at every instant. One figure shows the objects all starting from the same height (marked "start"), and a second figure shows the objects at the instant that object A is crossing a lower "finish" line. At this later instant, objects B and C are at the same height, above object A. The '*T*' denotes that the forces of interest are tension forces.

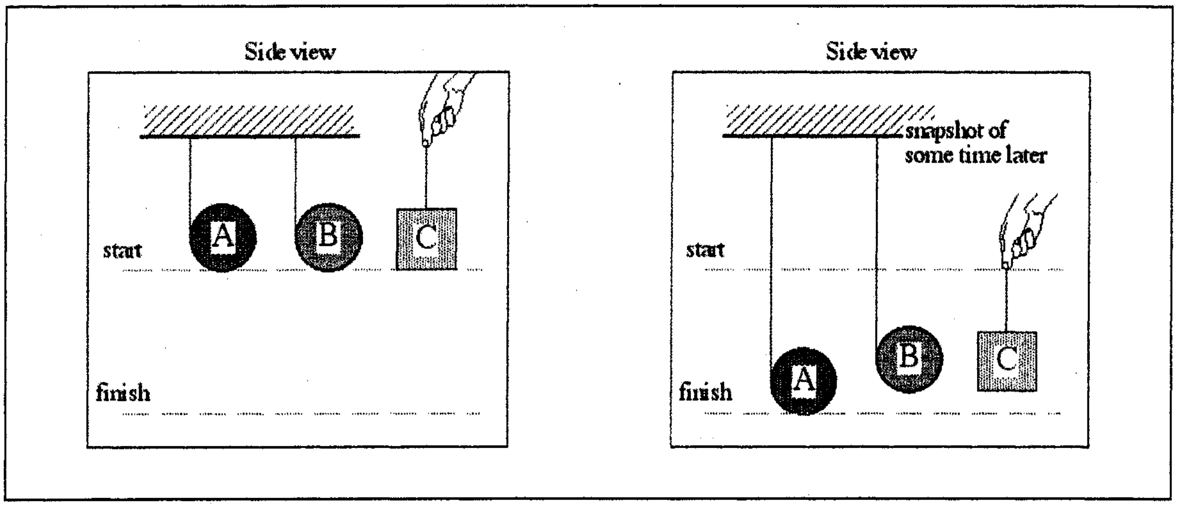

Figure 4-13: The Vertical *T* version of the "3 descending objects" context.

3. Description of questions about Newton's second law for rotating rigid bodies

a. The *Uniform tension* question

The *Uniform tension* question was asked only in the "Connected spools" context. In order to interpret student responses to the *N2R* question (described below), in which students relate forces to center-of-mass accelerations, it is necessary to find out how students think the forces involved compare to each other. The question is: Is the tension in the thread just above spool A was greater than, less than, or equal to the tension in the thread just above spool B? Furthermore, in the current version of the tutorial (described in part C of this chapter), students are not guided to determine the answer to this question, though it is necessary for them to have the correct comparison in order to have a meaningful discussion about N2R. Thus, in the tutorial design, we assumed that students did not need guidance to answer this question. Asking this question in written form on a pretest can help indicate whether that assumption is appropriate.

b. The *Thread length* question

This question was asked in only one context: the "Standard plus pictures of spool at later instant" variation of "Block and spool." This question asks students to choose which of the three figures (described above) best depicts how the spool, thread, and hand would look at some instant after the hand starts pulling on the thread. Students could also choose "none of the above."

c. The *N2R* question

The *N2R* question is the central question of this research project, and it was asked in each context in some form. An *N2R* question in any context is a question that asks students to compare the forces on different objects by comparing their center-of-mass accelerations, or vice versa. As discussed above, however, in most of the more recently designed tasks, we have asked students to relate forces to travel time over a fixed distance. (In one case, we asked students to relate travel time to what might be considered a concept even further removed - the extension of a spring.) Thus strictly speaking, these versions of the question involve more steps in logic than the ones connecting forces to center-of-mass acceleration. We might state more generally, then, that *N2R* questions ask students to relate forces to some aspect of the translational or "forward" motion of the object, "as a whole."

In the "Circular pucks" context, students are asked to rank the center-of-mass accelerations of the three pucks (or in multiple-choice format, to compare them pairwise). In "Four square blocks," students are asked to compare the center-of-mass accelerations for two of the blocks (blocks 1 and 3) that have the same two forces exerted on them, except at different locations.

In all of the "Block and spool" contexts, the *N2R* question asks students to state whether the spool crosses the finish line before, after, or at the same instant as the block, or whether the center of the spool stays in place or moves backwards, away from the finish line. Since one purpose of the question is to observe how students think about the relationship between rotational and translational motion, it was necessary for us to ask an additional question: "Will the spool begin to rotate?" We consider the combination of the "travel time" question and the question about rotation to constitute the "*N2R* question" in this context, since it is only with both questions that we can observe how students treat the effect of a force on translational motion when rotation is also involved.

In "Connected spools," students are asked whether spool A hits the ground before, after, or at the same instant as spool B. In the incline version of "3 rolling objects," students are asked to rank the friction forces on the objects as they move down their ramps; in the vertical version, students rank the tension forces in the threads that pull up on the objects as they move downward.

In "3 descending objects," students are asked to rank either the friction forces on the objects by their ramps (Incline *f* version), or the tension forces on the objects by their threads (Vertical *T* version).

d. The *Kinematics* question

In any case where students are asked explicitly to relate travel times of objects to their center-of-mass accelerations, we call the task the *Kinematics* question. Student responses to this question help us understand the ways that students interpret the term *acceleration of the center of mass* for objects that are both rotating and translating. Responses are also useful in affording us an opportunity to look for consistency between a student's force rankings and center-of-mass acceleration rankings, so that we might identify cases in which students correctly relate forces and center-of-mass acceleration, but incorrectly relate such accelerations to objects' travel times.

In cases for which students (should) start from information about the forces on a set of objects (as in "Connected spools"), they were asked the *Kinematics* question after the *N2R* question. In cases for which students start from information about the travel times of the objects (as in "3 descending objects"), they were asked the *Kinematics* question before being asked the *N2R* question.

e. The *Experiment identity* question

We asked this question in the "Standard plus hands" and "Standard plus springs" variations on the "Block and spool" context. In each of these contexts, two experiments with the block and spool are described; the question essentially asks students whether the two experiments described are the *same* experiment or not. However, in order to avoid responses like "The experiments are not the same experiment because they are not described the same way," we posed the question in terms of what could be inferred by a by-standing student who was familiar with both experiments. In the "Standard plus hands" context, the question read: "Supposed a classmate walks up just as the experimenters are performing one of the two experiments. The classmate is familiar with both experiments, but the experimenters do not tell their classmate *which* experiment they are performing. Would the classmate be able to identify which experiment was being performed, just by watching the motions of the block and spool?" In the "Standard plus springs" context, the question was phrased similarly, except we thought that a student might explain that a bystander would be able to distinguish the experiments by looking for the presence of springs. For this reason, we added the sentence "Also the classmate cannot see whether springs have been attached or not" to that version of the question. In both instances, students could choose one of two options: "Yes. The classmate would be able to tell, because the experiments have different outcomes" and "No. The classmate would not be able to tell, because the experiments have the same outcome."

4. Correct answers to questions about Newton's second law for rotating rigid bodies

a. The *Uniform tension* question

The tension in the thread just above spool A is equal to the tension in the thread just above spool B. If the mass of the thread is very small when compared with that of the spools, the net force required to accelerate any finite length of thread is negligible compared to that required to give the spool the same acceleration. The net force on any very light piece of thread is the difference between the tension forces exerted on either side of it. If this net force is zero, the two tension forces (which observation of the straight thread suggests are oppositely directed) are of equal magnitude. The above argument concludes that the tension in the thread is the same everywhere along a straight piece of the thread. To determine that the tensions are equal on either side of the pulley requires that we think about the rotational dynamics of the pulley. The net torque on a simple (frictionless) pulley about its own center is the difference in the two tensions multiplied by the radius of the pulley. If the moment of inertia of the pulley is $I=\beta MR^2$, let us call βM the *effective mass* of the pulley. The difference in the tensions on either side will be small (when compared with either individual tension) if the effective mass of the pulley is small when compared with the mass of either spool. Since the problem states that the pulley is massless, and the axle of the pulley is frictionless, the difference in the tensions on either side is zero.

By asking the *Uniform tension* question, we were not looking to see if students could reproduce any version of the above solution. Instead, we were interested to see how many students *assumed* that the tensions were equal, so that we might not have to help students understand why they are equal in a tutorial that is targeted at a more basic issue: Newton's second law (not its rotational analogue) for rotating bodies with non-negligible mass (unlike the thread or pulley) and unchanging shape (unlike the thread).

b. The *Thread length* question

The correct choice is the picture that satisfies both of these conditions: (1) The spool moves forward, because there is a net force exerted on it, and (2) the distance between the hand and the spool increases, because the thread unwinds as the spool rotates, which is because there is a net torque on the spool about its center of mass. There is a net torque on the spool because there is a non-zero lever arm extending from the spool's center of mass to the line of action of the tension force on the spool by the thread.

c. The *N2R* question

For any version of the *N2R* question, the correct answer is the one that corresponds to the following principle: the center-of-mass acceleration of an object depends only on the magnitudes and directions of the forces exerted on it, without any regard to the location at which those forces are applied. In most of the questions we have asked, two objects are known to have the same forces exerted on them at different locations, or they are known to move through the same distance in the same time. For example, in the "Circular pucks" contexts, pucks 1 and 2 have the same center-of-mass acceleration, because they have the same forces exerted on them. In "Block and spool," since the tension forces exerted on the objects are equal, the objects will have exactly the same translational motion, thus crossing the finish line at the same time. (This result does not depend on the fact that the spool will begin to rotate, which is because the thread exerts a non-zero net torque on the spool, about its center of mass.) In "Connected spools," the spools have the same weight, and the same tension force is exerted on each, so they move with the same acceleration and hit the ground at the same time.

In the "Standard plus hands" version of "Block and spool," in the experiment in which the hands move together, there is a net torque on the spool, so the spool will unwind. If the thread extends, and the hands move together, then the spool must lag behind the block. Thus, a correct answer to this question can be found without using the N2R principle at all. However, one could go on to conclude that the tension exerted on the spool must be less than that exerted on the block, since it lags behind. (Some students did this (correctly), thus answering the "true" *N2R* question hidden within.) Because the form of the question and context in which it was asked is so similar to a true *N2R* question, we have named it that way, for the sake of easier organization.

In the "Standard plus springs" and "With springs only" versions of "Block and spool," in the experiment in which the springs are inserted, the identical extension of identical springs results in equal tensions in the threads. Thus, the correct answer to these versions is the same as that for the standard version: the spool rotates and crosses the finish line after the block.

In "3 descending objects," a complete ranking of the friction (or tension) forces on objects A, B, and C can be found by comparing the motion of object A to that of object B and that of object B to that of object C. Object A has a greater center-of-mass acceleration than object B; thus the retarding friction (or tension) force must be less on object A than on object B. Objects B and C have the same center-of-mass acceleration, so the retarding forces on objects B and C must be the same magnitude (even though one friction force is static and the other is kinetic).

d. The *Kinematics* question

For all instantiations of this question, a greater center-of-mass acceleration corresponds to a lesser time to travel the fixed distance, and vice versa. Formally, the travel time Δt, the acceleration of the center of mass a_{cm}, and the fixed distance traveled Δx are related through $\Delta x=(1/2)a_{cm}(\Delta t)^2$, since all objects in every context start from rest.

e. The *Experiment identity* question

In the "Standard plus hands" version of "Block and spool," the two experiments described do not result in the same outcomes; in the experiment with equal tensions, the block and spool cross at the same time, while in the experiment in which the hands have the same motion, the spool lags behind the block. In the "Standard plus springs" version, the two experiments described do result in the same outcomes. In one experiment, the tension forces are stated to be the same; in the other experiment, the equal extension of the springs indicates that the tension forces in the threads are the same.

5. Administration of questions

All of the questions described above have been asked to students in UW 121, in the form of an online, multiple-choice pretest, in which students were explained their choices. Some of the questions were also asked to students in COL 1110, in the same format. In all cases, the pretest followed basic instruction on Newton's second law, as it applies to non-rotating objects. In most cases, the pretest also followed a lab in which students measured the angular acceleration of a rigid object on a fixed axis, and all lecture instruction on rotational dynamics and its relationship to translational dynamics. However, student performance in different lecture sections was virtually indistinguishable, so we do not describe the lecture instruction, nor do we separate results from different lecture sections from the same university.

6. Overview of student performance

In this section we present results from individual questions. However, in many cases, it is useful to consider how students responded to multiple questions on a single pretest. Results from student responses to multiple questions are presented in the next section.

a. The *Uniform tension* question

One section of the current version of the tutorial *Dynamics of Rigid Bodies* was written with the assumption that students would not need much guidance to recognize that the tension in a thread that passes over an ideal pulley is the same on both sides of the thread. In the practice of interacting with students when teaching the tutorial, the assumption seems to be safe; that is, virtually all students appear to assume that the tensions are equal (at least before they try on the notion of a lesser "effective" tension, which many students do). Table 4-1 shows how students performed on the *Uniform tension* question before the tutorial. A majority of students (60%) state that the tension in the thread above spool A is equal to that above spool B. The more common (30%) incorrect response is to state that the tension above spool A is less than that above spool B. This response is consistent with the notion that the tension (or some 'effective tension') above spool A is reduced as a result of spool A's rotational motion. In tutorial, students often say that the thread above spool A does not "hold back" spool A as much as the thread above spool B holds back spool B.

The published tutorial *Tension* was reintroduced into the tutorial sequence of UW 121 in winter quarter 2005, so we are able to compare student performance on the *Uniform tension* question with and without the influence of the *Tension* tutorial. We think of *Tension* as a relatively demanding review of Newton's laws, as they are applied to the motion of a massive rope. Students consider how the forces on the rope differ in different parts of the rope, and how they change as the rope is replaced by a lighter string (keeping the acceleration of the system fixed in this limit). The tutorial guides students to conclude that the tension in different parts of a very light string is approximately uniform, regardless of its acceleration. The students then apply this rule to situations in which the string passes over an ideal pulley. As shown in Table 4-1, student performance on the *Uniform tension* question apparently does not depend on whether the students have worked through *Tension*. One interpretation of these data is that it is not necessary for students to have worked through *Tension* before working through *Dynamics of Rigid Bodies*. However, we still believe that *Tension* can be very instructive for Newton's laws and the concept of idealization in physics. It may be, also, that how students approach issues with tension in the tutorial *Dynamics of Rigid Bodies* may be improved as a result of having worked through *Tension*.

Table 4-1: Student performance on the *Uniform tension* question (asked in the "Connected spools" context) after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies*, with and without having previously worked through the published tutorial *Tension*.

Tension above spool X: "T(X)" *Spool A unwinds,* *and Spool B does not.*	UW 121, N=360 3 sections without *Tension*	UW 121, N=377 4 sections with *Tension*
$T(A) = T(B)$ (correct)	60%	60%
$T(A) > T(B)$	10%	10%
$T(A) < T(B)$	30%	30%

b. The *Thread length* question

Student performance on the *Thread length* question (see Table 4-2) was not itself of great research interest; instead, we were more interested to observe any effect that the *Thread length* question might have on the subsequent *N2R* question. However, the results on this question do strengthen our interpretation of some responses to the *N2R* question in the same context. For each of these questions in the "Block and spool" context, there is a tendency of students to state that the spool rotates but does not move forward, and a tendency to state that the spool moves forward without rotating. The *Thread length* question does not, however, help us distinguish between students who think that the center of mass of the spool is accelerated by the "full" tension force or some fraction of it. Students who are using either of these ideas tend to give the correct response to the *Thread length* question.

Table 4-2: Student performance on the *Thread length* question (asked in the "Standard plus pictures" version of the "Block and spool" context) after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies*.

	UW 121, N=324 3 sections
The spool and hand both move to the right, and the distance between them increases (correct).	75%
The spool stays in place, and the hand moves to the right.	15%
The spool and the hand both move to the right, and the distance between them remains the same.	5%

c. The *N2R* question

i. Student performance on the *N2R* question in "Block and spool" contexts

This task, when asked in the standard version of the "Block and spool" contexts, has the lowest success rate (5%) of any of the *N2R* questions we have asked (see Table 4-3). Also, informally, we have noticed that students tend to have high confidence in their answers to this question. For these reasons, we see it as a useful tool for eliciting students' ideas about the relationship between forces and center-of-mass motion.

The most common response (60%) is that the spool moves forward and crosses after the block. This response represents the tendency of students to treat a force that causes an object to rotate as having a diminished (but non-zero) effect on the center-of-mass motion. The next most common response (20%) is that the spool rotates in place, which we associate with the tendency (similar to the first) of students to treat a force that causes rotation to have zero effect on the center of mass motion if it acts "at the very edge" of the object. The categories "S rotates and crosses after B" and "S rotates and stays in place" are not partitioned as much as the other responses - some students choose the former response and explain that the spool will rotate in place until the thread "runs out" and yanks the spool across the finish line. Thus, it may be appropriate for us to take the sum of those frequencies (80%) more seriously than the individual frequencies.

It is also important that about 35% of students state either that the spool does not rotate or that it does not translate. It is for this reason that, in the current version of the tutorial, we thought it was necessary first to establish that both motions would occur in some non-zero amount before directing the students to discuss in what amount each motion occurs.

Finally, notice that virtually no students answered that the spool crosses after the block but does not rotate. If student reasoning about forces and center-of-mass acceleration were essentially "location-based" as Ortiz described, then we should expect some of the students who said that the spool would not rotate would also state that the spool moves more slowly than the block, because the force is applied off-center. However, because all students who state that the spool does not rotate also state that the spool crosses the finish line at the same instant as the block, we do believe that student reasoning about forces on rigid bodies is not properly "location-based," but "rotation-based." Of course, rotation is itself "location-based," but differently located forces do not necessarily lead to different rotational motions.

Table 4-3: Student performance on the *N2R* question asked in the "Standard" version of the "Block and spool" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies*.

Block (B) and spool (S) are pulled with the same tension	UW 121, N=748 7 sections	PRD 152, N=316 2 sections
S rotates and crosses at the same time as B (correct)	5%	10%
S rotates and crosses after B	60%	55%
S rotates and stays in place	20%	20%
S does not rotate, and crosses at the same time as B	10%	5%
Total of "S does not rotate"	15%	10%

The "Standard plus hands" version of the "Block and spool" context is interesting because performance on the "standard" part of the context appears to be affected by the presence of the "hands" part, as shown in Table 4-4. That is, when students are given the opportunity to distinguish between the threads pulling with the same tension and the hands moving together, the success rate on the standard part improves from the usual 5% to 25%. However, even with this cue, the most common answer (35%) to the standard part is that the spool crosses after the block. Thus, though many students correctly recognize what effect changing the experimental conditions will have, most students do not. Furthermore, and not surprisingly, the overall tendency of students to state that the spool either rotates in place or moves forward without turning does not appear to depend on whether the experiment is described as "equal tensions" or "hands move together."

Table 4-4: Student performance on the *N2R* question asked in the "Standard plus hands" version of the "Block and spool" context after relevant lecture instruction and before tutorial *Dynamics of Rigid Bodies*.

Block (B) and spool (S) are pulled with the same tension	UW 121, N=122 1 section		*Block (B) and spool (S) are pulled so that hands move together*
S rotates and crosses at the same time as B (correct)	25%	5%	S rotates and crosses at the same time as B
S rotates and crosses after B	35%	55%	S rotates and crosses after B (correct)
S rotates and stays in place	20%	20%	S rotates and stays in place
S does not rotate, and crosses at the same time as B	10%	10%	S does not rotate, and crosses at the same time as B
Total of "S does not rotate"	15%	15%	Total of "S does not rotate"

Table 4-5 shows that asking students the *Thread length* question did have an effect on their answers to the subsequent *N2R* question, though not the effect we had expected. In teaching the tutorial *Dynamics of Rigid Bodies*, we had seen some students find some resolution to the "Block and spool" problem by observing that "the hand pulling the spool has to pull faster in order to pull with the same tension," after they had been guided to conclude that the block and spool cross the finish line at the same time. By asking the *Thread length* question, we had hoped to stimulate this idea of how the hand would move relative to the spool, thus improving students' success at recognizing that the spool crosses at the same time as the block. Instead, we see that the tendency to state that the spool crosses after the block is augmented in this case. The number of correct responses appears to be about the same, and the number of "no translation" and "no rotation" responses each appears to be slightly reduced. Thus, one interpretation of these results is that the *Thread length* question helps students realize that the spool both moves forward and rotates, but it does not help students think correctly about the effect of the tension force on the center of mass motion of the spool.

Table 4-5: Student performance on the *N2R* question (preceded immediately by the *Thread length* question), asked in the "Standard plus pictures" version of the "Block and spool" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies*.

Block (B) and spool (S) are pulled with the same tension	UW 121, N=324 3 sections
S rotates and crosses at the same time as B (correct)	5%
S rotates and crosses after B	75%
S rotates and stays in place	10%
S does not rotate, and crosses at the same time as B	5%
Total of "S does not rotate"	5%
Students who gave responses that were consistent with those to the *Thread length* question	85%

Yet another effort to stimulate correct treatment of the *N2R* question in the "Block and spool" context is the "Standard plus springs" version. Here we expected that removing the abstract notion of *force* might help students answer the question correctly, if the idea *force* was associated with an idea of a limited ability to produce motion. As shown in Table 4-6, student performance on the "standard" part (10%) of this context is similar to what we see (5%) on the standard version of the context. Students were slightly more successful (20%) on the "springs" part than on the standard part. An unexpected result is that many students (20%) stated that the spool does not rotate in the "springs" experiment but does rotate in the standard "same tension" experiment. Thus, for some students, the presence of the extended spring attached to the spool did not connect with a notion of "constant pull," but rather as something that requires and "consumes" some or all of the force applied by the hand.

Table 4-6: Student performance on the *N2R* question asked in the "Standard plus springs" version of the "Block and spool" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies*.

Block (B) and spool (S) are pulled with the same tension	UW 121, N=71 1 section		*Block (B) and spool (S) are pulled by identical springs stretched by the same length*
S rotates and crosses at the same time as B (correct)	10%	20%	S rotates and crosses at the same time as B (correct)
S rotates and crosses after B	70%	40%	S rotates and crosses after B
S rotates and stays in place	20%	15%	S rotates and stays in place
S does not rotate, crosses at the same time as B	~ 0%	10%	S does not rotate, crosses at the same time as B
Total of "S does not rotate"	~ 0%	20%	Total of "S does not rotate"

In contrast to the results in Table 4-6, Table 4-7 shows responses from students who worked the *N2R* question in "Springs only" version of the "Block and spool" context. Student performance on this version more closely resembles that on the standard version than it does that of the same context when juxtaposed with the standard version. That is, when the "springs" version is presented by itself, students seem to treat it as they would the standard version. Only when students have the opportunity to contrast the "springs" version with the standard version do they treat the springs as behaving differently from tense threads.

Table 4-7: Student performance on the *N2R* question asked in the "Springs only" version of the "Block and spool" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies*.

Block (B) and spool (S) are pulled by identical springs stretched by the same length	UW 121, N=123 1 section
S rotates and crosses at the same time as B (correct)	5%
S rotates and crosses after B	65%
S rotates and stays in place	20%
S does not rotate, and crosses at the same time as B	~ 0%
Total of "S does not rotate"	5%

ii. Student performance on the *N2R* question in other contexts

Table 4-8 shows student performance on the *N2R* question in the "Connected spools" context after traditional instruction and before the tutorial *Dynamics of Rigid Bodies*. More significant than the number of students who answered correctly (40%), we think, is the number of students who gave answers that were consistent with Newton's 2nd law and their answers to the *Uniform tension* question (45%). For example, some students gave "consistent" answers by stating that the tension above spool A was less than that above spool B and that the center-of-mass acceleration of spool A was greater than that of spool B (since the tension acts in a direction opposite that of the acceleration). Thus most students (55%) did not correctly relate the individual forces on an object to the acceleration of its center of mass.

Table 4-8: Student performance on the *N2R* question (preceded immediately by the *Uniform tension* question) in the "Connected spools" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies.*

Spool A unwinds, and Spool B does not.	UW 121, N=832 8 sections
a_{cm}(A) = a_{cm}(B) (correct)	40%
a_{cm}(A) > a_{cm}(B)	40%
a_{cm}(A) < a_{cm}(B)	20%
Responses that gave a_{cm} comparison that was consistent (via Newton's laws) with tension comparison in the *Uniform tension* question	45%

Table 4-9 shows student performance on the *N2R* question in the "3 Circular pucks" context. As expected, most students treat a force as a having a greater effect on the object's translational motion the more it is directed at the object's center of mass.

Table 4-9: Student performance on the *N2R* question in the "3 circular pucks" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies.*

Forces of magnitude 'F' are applied to Pucks 1 and 2 at different locations; force of magnitude '2F' is applied to puck 3.	UW 121, N=102 1 section	COL 1110, N=812 4 sections
$a_{cm}(3) > a_{cm}(1) = a_{cm}(2)$ (correct)	10%	15%
$a_{cm}(2) > a_{cm}(1) > a_{cm}(3)$	50%	45%
Total of $a_{cm}(1) = a_{cm}(2)$	20%	25%
Total of $a_{cm}(1) < a_{cm}(2)$	65%	65%

Table 4-10 shows student performance on the *N2R* question in the "Incline *f*" version of the "3 descending objects" context after traditional instruction and before the tutorial *Dynamics of Rigid Bodies.* The success rate on this task was similar in two sections (20% and 25%), but the sections have been separated because the number of students giving different incorrect responses appears to be somewhat different. We did not identify any factors that might account for these differences in performance. One unexpected result was the relatively high tendency of students to state that the friction force exerted on object B is *less than* the friction force exerted on object C. (Since object B rotates, while object C does not, we expected that most students would see the friction force on object B as responsible for affecting the rotation *and* translation of object B, while the friction force on object C merely affected the translation of C, thus (incorrectly) requiring a greater friction force on object B.) In fact, many students gave responses that suggested they thought the friction forces on objects A and B were almost unnecessary (*i.e.*, very small).

Table 4-10: Student performance on the *N2R* question in the "3 descending objects" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies*. A ranking is presented here if at least 5% of students in either section chose it.

Object A arrives at the bottom of its ramp first, followed by objects B and C, which arrive at the bottom at the same time (B rotates, C does not).	UW 121, N=71 1 section (D044)	UW 121, N=126 1 section (B042)
$f(B) = f(C) > f(A) > 0$ (correct)	20%	25%
$f(C) > f(B) > f(A) > 0$	30%	15%
$f(C) > f(A) = f(B) > 0$	10%	5%
$f(C) > f(A) = f(B) = 0$	0%	5%
$f(B) > f(C) > f(A) > 0$	5%	10%
Total of $f(B) = f(C)$	30%	35%
Total of $f(B) > f(C)$	15%	30%
Total of $f(B) < f(C)$	55%	35%

d. The *Kinematics* question

As shown in Table 4-11, a majority of students (70%) can successfully rank the center-of-mass accelerations of objects (using information about the time each object takes to travel a fixed distance) before working through *Dynamics of Rigid Bodies*. The most common error (15%) appears to be to misinterpret the term "acceleration of the center of mass" in a way that makes the quantity greater when the object also has non-zero angular acceleration. Some responses suggested that students were somehow adding the "ordinary" acceleration (which, to physicists, is the "acceleration of the center of mass") to the angular acceleration to obtain a "total" acceleration, with which they associated the term "acceleration of the center of mass."

Table 4-11: Student performance on the *Kinematics* question in the incline version of the "3 descending objects" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies.*

Object A arrives at the bottom of its ramp first, followed by objects B and C, which arrive at the bottom at the same time (B rotates, C does not).	UW 121, N=197 2 sections
$a_{cm}(A) > a_{cm}(B) = a_{cm}(C)$ (correct)	70%
$a_{cm}(A) > a_{cm}(B) > a_{cm}(C)$	10%
Total of $a_{cm}(B) = a_{cm}(C)$	75%
Total of $a_{cm}(B) > a_{cm}(C)$	15%

Table 4-12 shows student performance on the *Kinematics* question in the "Connected spools" context. As with the *N2R* question in this context, it is more useful to observe how many students give answers that are consistent (65%) with previous answers than how many students answer correctly (40%). For example, it would be consistent with the definition of "acceleration of the center of mass" if a student stated (incorrectly) that spool A hit the ground before spool B and also that spool A had a greater center-of-mass acceleration. The "consistency" rate (65%) for the "Connected spools" context is about the same size as the success rate (70%) on the *Kinematics* question in the "3 descending objects" context. The fact that only 65-70% of students give answers that are consistent with the relationship between an object's center-of-mass acceleration and its travel time indicates that many students need practice associating kinematical quantities "of the center of mass" with the motion of the object "as a whole."

Table 4-12: Student performance on the *Kinematics* question (preceded immediately by the *N2R* question) in the "Connected spools" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies*.

Spool A unwinds, and Spool B does not.	UW 121, N=638 6 sections
Spools A and B hit the ground at the same instant. (correct)	40%
Spool A hits the ground before Spool B.	35%
Spool A hits the ground after Spool B.	25%
Responses that gave travel time comparison that was consistent (via kinematics) with a_{cm} comparison in the *N2R* question	65%

e. The *Experiment identity* question

Student performance on the *Experiment Identity* question in both the "Standard plus hands" and "Standard plus springs" contexts is shown in Table 4-13. Perhaps more informative than the fraction of students who answered this question correctly is the fraction who gave an answer that was consistent with their previous answers. That is, before answering the *Experiment identity* question, students described the outcome of each experiment by answering the *N2R* question for each experiment. For example, if a student said that the block and spool cross the finish line at the same time in one experiment but at different times in the other experiment, then that student "ought" to answer the *Experiment identity* question by stating that the experiments have different outcomes. In that sense of consistency, about 70-75% of students gave consistent answers. "Inconsistent" responses seem to suggest that some students have trouble focusing their attention on "only the motions of the block and the spool" and ignoring all other cues for distinguishing the experiments, as the question requires.

Table 4-13: Student performance on the *Experiment identity* question asked in two different versions of the "Block and spool" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies.*

In Exp. A, the threads are pulled with the same tension; in Exp. B, the hands move together.	UW 121, N=122 1 section "Standard plus hands"	UW 121, N=71 1 section "Standard plus springs"	*In Exp. A, the threads are pulled with the same tension; in Exp. B, the springs are stretched the same length.*
The experiments have the same outcome.	35%	50%	The experiments have the same outcome. (correct)
The experiments have different outcomes. (correct)	60%	45%	The experiments have different outcomes.
Responses that were consistent with answers to *N2R* questions	70%	75%	Responses that were consistent with answers to *N2R* questions

7. Discussion of student performance on combinations of questions

Table 4-14 shows student performance on the *Uniform tension, N2R,* and *Kinematics* questions in the "Connected spools" context after traditional instruction and before working through *Dynamics of Rigid Bodies.* About 25% of students answer all three questions correctly. Even though analysis of the "Connected spools" context may seem more complicated than that of the "Block and spool" context, students are more successful with questions about the former than the latter. We interpret this trend as an indication that the "Block and spool" context more strongly elicits the common idea that a force affects translational motion less when it is affecting rotational motion more and that it is therefore a better context for instruction.

Table 4-14: Student performance on the *Uniform tension, N2R,* and *Kinematics* questions, asked successively in the "Connected spools" context after relevant lecture instruction and before the tutorial *Dynamics of Rigid Bodies.*

Percentage of students answering question correctly	UW 121, N=737 7 sections
Uniform tension	60%
N2R	40%
Kinematics	35%
Correct responses to all three questions	25%

8. Discussion of patterns in student reasoning

a. Tendency to treat force as partially or completely "used up" in rotational motion

This tendency in reasoning is extremely common among students at the introductory level (and not uncommon among students at more advanced levels) and can be observed in essentially any context that involves forces exerted on objects that rotate and translate. The clearest examples come from responses to the "Block and spool" context:

> "Some of the tension is used to rotate the spool while some is used to pull the spool towards the finish line. The block is using all the tension to move it, so the block will finish first."
>
> "Because some of the force is going into spinning the spool and some of the force is going into pulling the spool, whereas all of the force is going into pulling the block."

Thus, students tend to treat an applied force as partially "used up" in the rotational motion of the object. This specific tendency, which we interpret as an instance of a more abstract tendency to associate a cause of fixed "magnitude" with a limited amount of "total effect," may be related to the tendency of students to treat electric current as "used up" in a circuit.[2]

In some cases, students give similar reasoning in terms of "energy" rather than in terms of "force," though we agree with Ortiz[3] that this variation in word choice probably does not represent a distinct conceptual error. The following examples illustrate this variation:

> "The spool will lose some of the energy given to it from the string to rotational energy, so it will cross the finish line after the block."
>
> "The energy on the spool will split into translational motion and rotational motion. Since there are two types of motion and both object are applied with equal energy, the block should reach the finish line faster."

Yet another variation on this tendency in reasoning is the idea that a force can be exerted on an object in such a way that it has zero effect (as opposed to a reduced, though still non-zero, effect) on the translation motion of the object.

> "The force of the string will cause torque... There is no force applied directly to the spool to make it go forward."
>
> "The force is like a tangential force on the outside edge, which all gets turned into rotational motion."

This particular variation in reasoning is consistent with Ortiz' observation[4] that students tend to break a force into orthogonal components, with one component directed at the center of mass of the object and which alone affects the object's translational motion. In the "Block and spool" context, because the force is already perpendicular to the line connecting the point of application to the center of mass, the component of the force directed at the center of mass is zero.

b. Augmented tendency to distinguish between pulls of equal force and pulls of equal speed when both cases are presented

As shown above in Table 4-4, some students recognized that pulls of equal speed are not necessarily pulls of equal force. That is, when students were given the opportunity to contrast these conditions in the "Standard plus hands" version of the "Block and spool" context, more students associated *equal tension* with equal translational motions of the block and spool than when the contrast was not presented (compare 25% correct to 5% correct). The following responses illustrate how some students saw this contrast:

> "If the tensions are equal it will make up for the difference in speed caused by the spool letting out its slack."
>
> "If they are being pulled with the same force, they should get there in the same amount of time. The hand pulling the spool would have to pull much harder though to compensate for the unwinding."
>
> "To maintain the same tensions, the spool will have to be pulled faster than the block because it will rotate as the string unwinds. They will both cross at the same time because of equal tensions."

These students express some interest in making sense of the relationship between the tension and the relative motions of the hand and the spool, but they do not clearly distinguish between the speed of the hand and the speed of the spool, or between pulling *harder* and pulling *faster*. The fact that most students did not distinguish clearly between pulls of equal speed and pulls of equal force connects, in a sense, with students' tendency to treat a force as "used up" in rotation through the more general idea that students treat *force* as a quantity of motion (rather than as a reciprocal interaction that influences motion). This interpretation of these observations agrees with the well-known tendency of students to connect force "too directly" to an object's velocity.[5] Because the "Block and spool" context (and many other problems involving combined translational and rotational motion) does offer a unique opportunity to distinguish concretely between equal speeds and equal forces, it may be that introducing the context earlier in the introductory mechanics course can help students understand what is meant by the term *force*, and in particular, how it is not a synonym for the velocity (or speed) of an object.

c. Tendency to recognize that identical extended springs signify equal forces

As shown above in Table 4-6, students were given the "Standard plus springs" version of the "Block and spool" context were more successful on the "springs" part than on the "standard" part. This result suggests that, though some students may not understand how to interpret the phrase "equal tension" operationally, they may understand some more concrete version of Newton's 2nd law, by treating the equal extensions of the springs as equal "causes," which result in equal translational motions (equal effects).

d. Tendency to treat force as "used up" in the extension of a spring

Another feature of the data shown in Table 4-6 is that many students (~ 20%) stated that the spool would rotate in the "equal tension" experiment but not in the "springs" case. The following responses illustrate the contrast that students saw between the experiments:

> "Since the tension of the string is constant and there is no spring to relieve tension as it is pulled, then the spool will begin to unwind. In (experiment) B since there is a spring, the tension is reduced and will not unwind."
>
> "The spool in experiment A will begin to rotate even with the same amount of tension since the tension applied is fixed to a place that has the ability to move the object. In experiment B the spool will not move since the spring absorbs the tension."
>
> "The spring in B will take the tension that made it spin in A."
>
> "The spool will begin to rotate in experiment A because it is on a frictionless surface, so there is not resistant force to stop it from spinning around when the string is pulled. The spool in experiment 2 will not rotate because the spring will absorb the force from the hand pulling it."
>
> "The spring will only extend to its fullest and not pull the spool anywhere."

These students appear to be using a variation on the idea that force is "used up in rotation;" here, the force is "used up in extension." Ironically, these responses suggest that students associate a greater extension of the spring with more of the force "from the hand" being consumed by the spring, rather than a greater extension resulting in a greater pull on the spool. Here, again, students seem to treat *force* not as an interaction but as a substance that is transferred between or contained in objects, as motion or deformation. These and previously discussed patterns in reasoning are further examples of how students mix the concept force and quantities of motion (or latent potential for motion), especially energy.

e. Tendency to treat friction as a requirement for rotational motion

In the "Block and spool" context, some students tended to associate the absence of friction between the spool and the table with the impossibility of the spool's rotation. The following explanations were given by students who stated that the spool would not rotate:

> "The surface is frictionless, it will have nothing to rotate against."
>
> "I believe you would need some friction in the ground to make it start to rotate."

> "If the surface is frictionless then the spool can't rotate because it can't get a grip."
>
> "No friction, so the spool won't twirl, as rotating means that the spool has friction and tries to stay in the original position."

Informal observations in the classroom also suggest that students tend to associate the term "frictionless" with an idealization that does not correspond with any realizable set of circumstances. For instance, when asked by a tutorial instructor if a pen lying on a table will rotate when flicked near its end by a finger, students who have expressed ideas like those above usually predict (correctly) that the pen *will* rotate. They explain that friction plays a critical role in this or any similar realizable experiment, thus dissociating the approximation of a surface as "frictionless" from reality. This general tendency of students to resist allowing a correspondence between real situations and idealizations through approximations is also described briefly by Hammer.[6]

f. Tendency to treat translational/rotational partition of kinetic energy as having direct causal influence on translational motion

Though a correct solution of the *N2R* question in the "3 descending objects" context requires very similar reasoning to that in the "Block and spool" context, students tend to reason quite differently about them. As shown in Table 4-10, many students stated that, for two objects with the same translational motion, the friction force on the object that rotated (object B) was less than the friction force on the object that did not rotate (object C). Whereas in the "Block and spool" context, rotational motion counts as an "extra" effect or "responsibility" of the applied force, in the "3 descending objects" context, rotational motion seems to serve more as something that *relieves* responsibility of a force. That is, students tend to think of the rotational motion in rolling without slipping as *natural* (not requiring friction as its cause) and having a direct, inhibiting influence on translational motion. The following examples illustrate variations on this idea:

> "C will have a greater friction force on it in order to arrive at the same time (as B). This is because if there was no friction, C would reach the bottom first, because all its KE is translational, whereas B also has rotational which slows it down."

> "One reason that B is accelerating slower than A is rotational inertia, and C doesn't have that. So C must have a larger friction force. I assumed A had less of a friction force because it accelerated at a faster rate."
>
> "The masses are equal but the rotational acceleration of B takes out some of the need for a frictional force."
>
> "(The friction force on) C is greater than (the friction force on) B because they took the same amount of time to reach the bottom, but B's shape slows the acceleration."

In the first response, the student states that the rotational kinetic energy of object B "slows it down," as if friction, which also helps to "slow the object down" (or, more accurately, to reduce the rate at which the object speeds up) need not be as great in magnitude as it would be in other circumstances in which it did not have extra help. As in the third response, the rotational motion reduces the *need* for a frictional force. The common feature of these responses is that they all admit non-forces into the domain of *things that causally influence motion,* which in the Newtonian view, belongs strictly to forces alone. We believe students' tendency to treat non-forces in this way is probably a result of the fact that instruction on rolling without slipping tends to lean away from Newtonian explanations and toward energy explanations. In fact, introductory mechanics textbooks often use causal language when discussing the division of energy for objects that roll without slipping. For example:

> "The reason the rolling objects... move down the slope more slowly than if they were sliding ... is *because* some of the gravitational potential energy is converted to rotational kinetic energy, leaving less for the translational energy."[7] (emphasis added)

It is not incorrect (and not inappropriate for the introductory mechanics course) to use the word *because* in the manner above, but it is also not Newtonian. Because the *N2R* question is essentially a Newtonian task, while contexts like "3 descending objects" usually taught in terms of the energy concept (which is not Newtonian), we believe that students, in trying to make sense of the two perspectives, unwittingly form a hybrid sense of causality from two systems of thought with two subtly different perspectives on causality.

We also understand this particular tendency in student reasoning about causes in rolling without slipping as perhaps an instantiation of how people attribute causes to phenomena in general. Psychological research on how people attribute causes to social behavior[8] shows a similar trend: when multiple causes could account for behavior, people tend to change their estimates of the "size" of one cause in response to knowledge about the "size" of another cause. For example, in a professional context, a person who is agreeable and in a formally subordinate role is usually judged to be less friendly than another person who is equally agreeable but in a formally senior role. Thus, when equal effects are observed (equal agreeability or equal center-of-mass accelerations) the magnitude of a cause is diminished (friendliness or friction force) when another cause (subordinate role or rotational motion) that combines constructively with the first is of increased magnitude.

g. Tendency to treat rolling without slipping as a "low-friction" process

Some students who responded that the friction force on object B was less than that on object C stated (without much support) that friction is low when objects roll without slipping.

> "A and B are balls, so friction is minimal for both, but equal because they are same in composition/mass. C is a block, so it will have a higher magnitude of friction."
>
> "Um, the friction on a rotating object is less than an object that slides."
>
> "The block has no rotational energy, therefore must have the greatest frictional force to result in the slowest acceleration. The (other) objects are rolling and therefore don't experience a frictional force."

This tendency may be related to that discussed in the previous section, in which students expressed the idea that less friction was *needed* to retard the objects.

h. Tendency to treat static friction force as influenced by contact area

In yet another variation on explanations of why the friction force on object C is greater than that on object B, some students focused on the contact area between the object and its ramp:

> "C has more friction because it has more surface touching the ramp whereas A and B have the same, but A gets to the bottom faster so it has less friction acting on it."
>
> "Since C is a block and not a circular shape like the other two, it should have the largest magnitude of the friction force because more surface is touching the ramp. And since B and A have the same mass, yet A reaches the bottom

> first, I am assuming that the frictional force on B must be greater than frictional force on A."

An interesting feature of these responses is that students tend to use contact-area reasoning when comparing the friction on objects B and C while using (loosely) Newtonian reasoning to compare the friction forces on objects A and B. This pattern is one example of a common trend in student responses to ranking tasks: many students tend to piece together an explanation of a ranking by using different lines of reasoning for different parts of the ranking, often including an idea or conceptual procedure that, if applied to the *whole* ranking, would yield a correct answer.

In a minor variation on contact-area reasoning, some students justify their statement that the friction force was *greater* on object B by thinking about contact area:

> "There must be more friction on B than C because they arrive at the same time and C is dealing with friction on a very large area whereas B is dealing with it on a small surface (point)."

The tendency of students to focus on contact area when thinking about friction forces in rolling without slipping is one example of students' general tendency to base their thinking on the *type* of force involved. This observation led us to extend our investigation by studying how students think about different types of forces on point-like objects in static equilibrium. This extension is described in the next chapter.

9. Description of intermediate versions of tutorial sequence *Dynamics of Rigid Bodies*

All revisions of the tutorial sequence *Dynamics of Rigid Bodies* were designed to target the issue of Newton's second law for rotating rigid bodies more directly than in the published version. This section describes two intermediate versions, A and B, of the tutorial instruction, and interviews that led to the current version.

Some results for student understanding of forces on rotating rigid bodies after these intermediate versions of the tutorial are presented later in this chapter, with the results associated with the current version of the tutorial.

a. Intermediate version A of *Dynamics of Rigid Bodies*

In the first major revision of the tutorial from the published version, we rewrote the first two pages of the tutorial, leaving the last two pages intact (as described in part A of this chapter). The first two pages of the published tutorial guided students to predict and explain the motion of a symmetric ruler placed on a smooth pivot through its center in terms of the net force and net torque on the ruler (about its center of mass). The last two pages guided students to analyze the "Unconnected spools" experiment (see Figure 4-1) in terms of the net force and net torque on each spool (about its center of mass). We decided that the ruler context did not do much to stimulate students to think about the question "Does the effect of a force on the motion of the center of mass of an object depend on how the force affects the rotational motion of the object?" (The ruler context also appeared to reinforce incorrect ideas about the location of the center of mass for objects less symmetrical than a ruler.) In order to help students begin to think about this question, we introduced the "hanging dowel" context (see Figure 4-14). In this situation, a string is attached to the center of a dowel. The top of the string is attached to a fixed support, and the dowel hangs above an 'X' marked on the table below it. Students were asked to think first about how the dowel would move if a finger quickly struck the dowel near its end, both in a translational sense (Will the center of the dowel remain above the 'X'? If not, describe the direction that it will move.) and a rotational sense (Will the center of the dowel start to rotate clockwise, counterclockwise, or not at all?). This section was meant mainly to establish that a force applied at the edge of an object would still tend to accelerate the center of mass of the object.

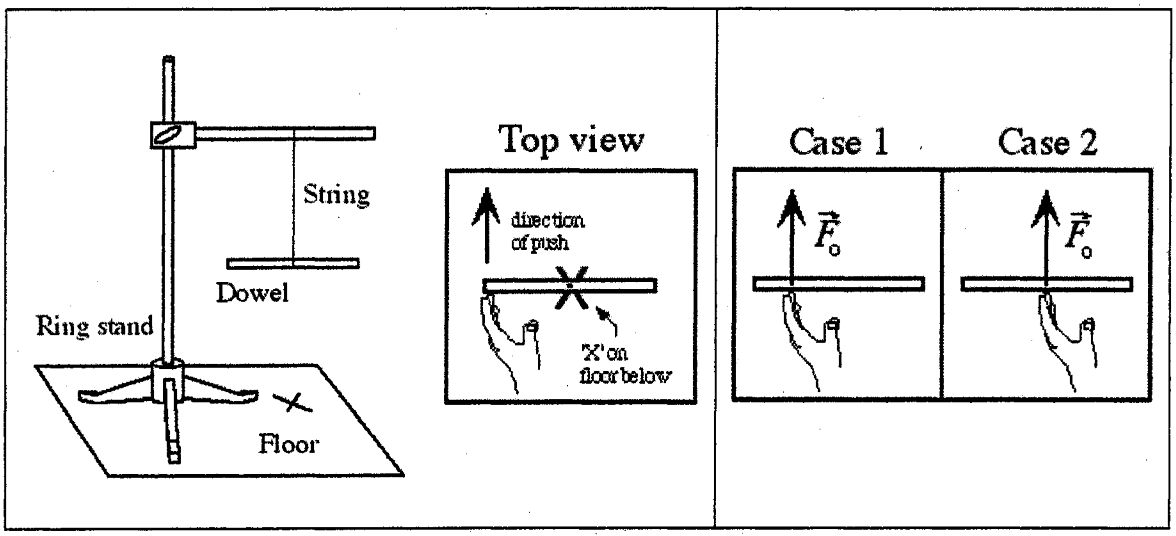

Figure 4-14: The "Hanging dowel" context.

The next section aimed at helping students distinguish between the effects of net force and net torque, asking students to draw arrows on top-view diagrams of the dowel indicating how they could strike the dowel to achieve zero net force and non-zero net torque, non-zero net force and zero net torque, and so on.

Finally, before moving on to the "Unconnected spools" experiment, we prompted students to think about how the center of mass of the dowel would accelerate when the same force is applied (1) near the edge of the dowel or (2) near the center of the dowel. Students considered a printed student dialogue, in which one hypothetical student gave a Newtonian statement that the center of mass of the dowel would accelerate equally in both cases, and the other explained that the force would have a greater effect on the center of mass when the force is applied near the center. Students had no way of checking which approach was correct outside of discussions with each other and tutorial instructors. In general, tutorial instructors were encouraged to ask students to consider what is meant when the problem states that the "same force was exerted," and how that might be achieved (or not achieved). The issue could be brought up again in the context of the unconnected spools, but again, there were no empirical means by which the Newtonian approach could be shown to be correct. This problem of experimental "proof" inspired the "Connected spools" experiment, which appeared in the tutorial for the first time in intermediate version B.

b. Intermediate version B of *Dynamics of Rigid Bodies*

The main change between versions A and B was the replacement of the "Unconnected spools" experiment with the "Connected spools" experiment (see Figure 4-11). The first half of the tutorial remained essentially the same as in version A. With this version of the tutorial, we supposed that students would be able to determine that the Newtonian approach to forces on rotating bodies was the correct approach, because the experiment involved objects on which equal forces were exerted at different locations, resulting in different rotational motions and identical translational motions of the objects. We have come to understand that this "proof" is, for many students, insufficient by itself for demonstrating what it was intended to demonstrate.

10. Interviews

We observed that, after working through version B of *Dynamics of Rigid Bodies*, about half of the students persisted in treating the effect of a force on the center-of-mass motion of an object as dependent on the object's rotational motion. The interviews described here were designed to explore how students who had completed the tutorial thought about the "Connected spools" experiment. In particular, we were interested to find out whether students correctly remembered the results of the experiment, and what they thought the purpose of the experiment was (if they did not understand it to be demonstrating the correctness of a Newtonian approach to forces on rotating objects). Furthermore, we also wanted to understand better the tendency of students to treat retarding (or constraint) forces as diminished by the presence of rotational motion (if translational motion is not varied), and any possible relationships between this tendency and student understanding of the "Connected spools" experiment.

a. Description of interview tasks

We asked students to answer the *Kinematics* and *N2R* questions in the "Incline *f*" version of the "3 descending objects" context in written form, discussing their thinking out loud as they worked through the problems. After the students found some solution to these questions, we showed them a paper with a discussion between three students:

Student 1: The friction force on B is greater than the friction force on C. The friction force on C makes C arrive after A. But the friction force on B is doing two things: making B slower than A and causing B to rotate. Therefore B requires more friction in order arrive at the same time as C.

Student2: The friction force on B is the equal to the friction force on C. B and C have the same center-of-mass acceleration and the same mass, so the net force on them is equal. In this situation, that means the friction forces on them are the same.

Student 3: The friction force on B is less than the friction force on C. If there were no friction at all, then C would win, because all of its energy would go into translation. B doesn't require as much friction as C in order to arrive after A because, in addition to the effect of friction, it also has some of its potential energy going into its rotation.

We asked students to respond to this discussion, stating with whom (if any) they agreed and explaining why. We then showed students the "Connected spools" apparatus and asked students if they remembered the experiment and if the experiment could be used to determine which of the statements in the printed discussion was correct. Students were allowed to perform the experiment again if they wished.

b. Discussion of results

The most important results of these interviews were, first, that students correctly remembered the results of the experiment, and, second, that few students correctly interpreted these results. Some students were unsure how to interpret the experiment, and other students were sure of an interpretation that was not what we intended. The following excerpt illustrates these trends:

I: Could you use the results of this experiment to help you decide how to work this other problem?

[Student performs experiment]

S: Yeah, they fall down at the same time. But I think in this case it's kind of different, because they are connected.

I: OK. So this experiment doesn't tell you ... anything?

S: No, it tells me… I just don't know how to… I'm just trying to apply it to here.

Thus, these interviews told us that, though the experiment was working well, students did not have the guidance they needed to interpret the experiment as having import for Newton's second law for rotating objects.

C. DESCRIPTION OF CURRENT VERSION OF TUTORIAL SEQUENCE *DYNAMICS OF RIGID BODIES*

The main conceptual objective of the tutorial *Dynamics of Rigid Bodies* is to help students understand that, when a force is applied to a rigid object, the effect of that force on the acceleration of the center of mass of that object does not depend on how the force is affecting the rotational motion of the object, or where on the object the force is applied. A copy of the tutorial and homework can be found in Appendix B of this dissertation.

1. Detailed description of tutorial

a. Spool on a variety of surfaces

The primary purpose of the first section of the tutorial is to establish that a spool pulled across a frictionless surface (as in the "block and spool problem" described above) will both rotate and move in the direction that the thread is pulling it. The following experiment is described: A spool sits on a piece of sandpaper that is fixed to the table; a hand pulls on a thread that is wrapped around the bottom of the spool; the hand moves in a straight horizontal line, away from the spool (see Figure 4-15). Students are asked to predict whether the spool will start to rotate, whether the spool will move across the sandpaper, and if so, to state the direction that it will move. Before checking the results of this experiment, students predict what would happen if the experiment were performed on a smooth tabletop instead of on the sandpaper. They are asked to explain their reasoning for their predictions, but instructors do not try to help students make correct predictions or use correct reasoning at this point. The instructor here should encourage students to articulate and record whatever reasoning they use, facilitating discussion of the predictions among the students, perhaps by asking students to take note of any differences in reasoning or predictions among the members of their group.

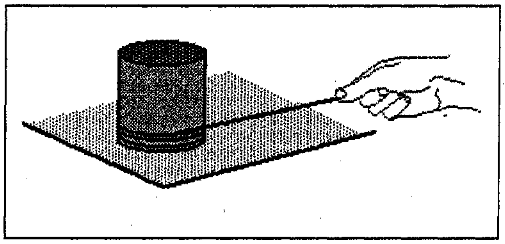

Figure 4-15: Diagram in section I, "Spool on a variety of surfaces," in the current version of the tutorial *Dynamics of Rigid Bodies*.

The students then check their predictions by performing the experiments. The results of the experiments seem to be fairly clear to the students: in both cases, the spool rotates, and it moves across the surface, in the direction that the thread is pulling. Students are asked to record the results, but not to explain them. The emphasis we want at this point is not on a conventionally correct explanation of the phenomena; rather, we intend to set a tone of the *authority of experiment* by having the students describe and generalize from phenomena. After students have recorded the results, they are asked to imagine performing the experiment again, on an even smoother surface. Then, the students think about how the motion of the spool on the new surface would differ from the motion they observed in the previous two experiments. Most students realize spontaneously that the rough observations they made do not support the idea that the motion of the spool would be significantly different if the surface were frictionless. Some students think they may have observed that some "ratio" of rotational motion to translational motion is different for the two surfaces; here an instructor can ask the students to repeat the experiments and to look carefully for this feature. Students who watch together for this effect usually observe that there is no clear difference between the two cases. If students are still concerned that there is some difference, an instructor can invite them to use a ruler to measure the distance that the hand moves when the spool moves a given distance. Changing the surface, the distance can be measured again, as the spool travels the same fixed distance. Because the spool tends to walk a little bit to each side more when it is pulled across the sandpaper than when it is pulled across the table, the results of this experiment can vary slightly. However, this difference in the path of the spool can be pointed out and discussed. This in-depth discussion is usually not necessary, if the goal is to convince students that making the surface smoother and smoother does not result in a trend toward either zero rotation or zero translation.

Summarizing, in addition to the primary goal of having students develop a (roughly) correct visualization of the motion of a spool pulled across a frictionless surface, we also have the goal of establishing a tone of the authority of experiment. There is also a third, less fundamental goal, which is for students to recognize that the distance between the hand and the spool increases as the thread is pulled. This observation is needed for a problem on the homework, and it appears to help some students resolve the block and spool problem later.

b. The "Block and spool" problem

The "Block and spool" problem is both pedagogically and literally the central problem of the tutorial. This problem has been used as an assessment tool so often because we believe that it is the clearest indicator of how students are thinking about forces on a rigid body. Because students are almost always very confident in their answer to this problem, we decided to use it as a point of discussion in the tutorial. That is, we believe using this problem in the tutorial is the clearest way to bring the more abstract conceptual issue to the front of the students' attention. After students read the description of the setup, they are asked about the order in which the block and spool cross the finish line. Although, on the pretest, a significant minority of students stated that the spool would stay in place as it rotates, no students seem the least bit inclined to say that after having made the observations from the previous section. Immediately after making their own claim regarding the order of crossing the finish line, students read three statements by hypothetical students and say with whom they agree, and why:

Student 1: The spool will rotate, and it will cross the finish line at the same time as the block. They have the same mass and the same net force, so their centers of mass will have the same acceleration. It doesn't matter whether the spool starts rotating - the tension force will still have the same effect on the spool's translational motion.

Student 2: The spool will rotate, and it will cross after the block. This is because some of the tension force on the spool is being used to rotate the spool. When a force causes an object to rotate, it has less of an effect on the object's translational motion.

Student 3: The spool will rotate, and it will cross after the block. I was thinking about energy. The spool and block must each have the same total kinetic energy when they get to the finish line. Since the spool will have some rotational kinetic energy, it must have less translational kinetic energy than the block. Therefore, it will be slower, and it will arrive at the finish line later.

Usually, students start by quickly dismissing the (correct) reasoning of Student 1 as obviously wrong, and go from there. Students tend to view the reasoning of Students 2 and 3 as similar. Most students choose allegiance with the second or third student, but some students do not want to commit to one or the other. Instructors may then encourage students to "pick a side," though the "sides" are not yet clearly differentiated. At this point, students tend to favor the reasoning of Student 3, because it sounds more "scientific." In the next section, students have a better opportunity to distinguish the two types of reasoning.

It cannot at this point be overemphasized that instructors must *not* persuade the students that Student 1's analysis of the block and spool problem is the correct one. At the bottom of this second page of the tutorial is a note that, in the next section, students will use the results of an experiment to determine which of the students gave correct reasoning about the block and spool problem. If students are persuaded by an instructor to analyze the problem in terms of Newton's second law, the purpose of the experiment *as a test of ideas* is defeated. Though an instructor may think that asking questions (such as "How do the net forces on the two objects compare?" *etc.*) is innocent in comparison to giving away the answer, the instructor is, in a sense, giving away the answer by directing the students' attention to the proper considerations. Asking certain directed questions can tell students what is (or what the instructor regards as) relevant or not. We believe that, if students are persuaded to think of their nagging doubts as irrelevant, those doubts will come back to bite them when they are solving problems later, often without help. Therefore, we intend that instructors will merely draw students' attention to the note mentioned above, and encourage students to think of the issue as, for the moment, unresolved. (We have also seen what we think is students' increased sense of the *meaning* of an experiment as time to resolution and length of discussion increase. This increased sense of meaning can be seen in improved student ability to apply the abstract principle demonstrated by the experiment, and in comments like "I'm dying to see this experiment!" or students' animated reactions to seeing the experiment performed.)

c. Testing ideas about combined translation and rotation with an experiment

In this section, students perform an experiment to test which of the three solutions to the block and spool problem is correct. We decided that performing the block and spool experiment would be too difficult, practically speaking. Instead, we have the students predict the outcome of a similar experiment that is much easier to set up and perform. Though the experiment is easier to perform, it is more difficult to analyze conceptually. Aside from the benefit of having an experiment that can actually be performed easily, we believe that this increased conceptual load on the students, used wisely, can serve as an opportunity for the students to *increase* their clarity on the general, abstract issue of how to deal with forces on a rigid body.

The experiment students consider is that described in the "Connected spools" context. First, students predict the direction of motion of each spool after it is released and the order in which the spools will hit the floor (if a student thinks one spool will move down only after the other hits the floor, s/he is encouraged to say that, though it is not a common prediction). Students then draw an extended free-body diagram (in which forces on the object are shown at the location where they act) for each spool. Then they draw point free-body diagrams (in which forces on the object are shown as if the object were located at a single point). It is necessary at this point to realize that the tension force on each spool is the same (or, to a student, *would* be the same if neither spool could rotate) since the tension in a light thread must be the same everywhere (if there is no friction between the thread and pulley, or if the pulley is massless and there is no friction in the axle of the pulley). Explaining why the tension is the same everywhere is a difficult problem; fortunately, for the purposes of this tutorial, students assume either at the outset or soon after that the tension forces on the two spools are the same. Students typically draw both pairs of free-body diagrams correctly (with one exception that we will discuss later), though often students' diagrams benefit in clarity and precision with some questions from the instructor.

Given the comments earlier about wanting to avoid prematurely directing students' attention toward the correct treatment of the problem, it may seem that having the students draw free-body diagrams of the spools without motivation or explanation could have a derailing effect on student thinking. However, we have observed that having the students draw the diagrams does not seem to direct any students away from their own reasoning and toward the correct treatment before they are ready; rather, students use the diagrams as a resource for the following discussion, which is their intended purpose.

This next activity is a crucial point of the tutorial. Students are asked to reconsider the three Students' solutions to the "Block and spool" problem. Here, our students are asked to discuss and record the predictions that they think each of the three Students would make if they were to use the same reasoning in this new experiment that they used before. We have multiple goals for this part of the tutorial. First, we want students to see that there is some abstract relationship between the two problems, namely that both problems involve two objects with the same forces on them, and the only difference between the two objects in each problem is where the forces are applied (and consequently, whether the object rotates). In interviews with students who had worked through a previous version of the tutorial (in which the connection between the "Connected spools" experiment and any other problem was much less explicit), it was clear that, though they remembered the experiment from tutorial several weeks before, many students did not see the point of it, or the similarities between it and the block and spool problem. To understand the purpose of the experiment is to understand the general principle that is being tested, which we see as a prerequisite for improving student ability to apply the principle to new situations. Second, as mentioned before, we want to emphasize the function of experiments as a way to test ideas. Third, we believe it is valuable for the student to be able to use ideas that do not belong to him/her, and to work in a logical space that operates on assumptions or rules with which the student does not agree. Furthermore, we believe this entire situation accurately models the process of theory testing that is an important part of scientific knowledge, as developed by people.

When real students make predictions on behalf of the three hypothetical Students, they are encouraged to agree with the other members of their group in how the different arguments are applied. Almost all students agree that Student 1 (though, to most students, clearly wrong about the block and spool problem) will predict that the spools will hit the floor at the same time. Virtually all students agree that Student 3 would predict that the non-rotating spool will hit first, since all of its kinetic energy will be translational, rather than having to split up the same amount of kinetic energy between rotation and translation. (Again, the argument relies on an unsupported assumption about the objects' total kinetic energies.) Student 2's reasoning seems to be the most challenging to apply– that Student talked about a force getting diminished because it causes rotation. Which force on the rotating spool (the tension or the gravitational force) exerts a non-zero torque on the spool depends on the reference point chosen. However, most students do not yet understand this subtlety of torque well enough to worry about this point. It is more common for students to think of the tension force causing the rotation; thus the tension force shown in the point free-body diagram for the rotating spool would get diminished (according to Student 2), resulting in a *larger* net force on that spool, causing it to hit first. This conclusion is the one we prefer, since students see that now, all three Students make different predictions according to their different arguments, in contrast to the block and spool problem. When students do not give three distinct predictions, it is not because of some alternative way of thinking about which force causes the spool to rotate – it is usually because students do not look carefully enough at differences in the reasoning given by Students 2 and 3. It seems to have been productive for these students when the instructor has challenged them to come up with a way of applying the reasoning of Student 2 so that Student 2 gives a prediction different from Student 3. That is, students usually are able to come up with the reasoning described above, in which the tension force is diminished. Working out these various predictions requires much thought, and the tutorial has been designed so that students will spend 20-30 minutes on this activity.

After the students have discussed their predictions with each other and with a tutorial instructor, they are asked if the reasoning they used in their prediction is the same as that of any of the three Students. This is another opportunity for students to check for consistency. We have seen students take these cues to make sure that if they agree with Student X in the block and spool problem, then they also agree with Student X in the falling spools experiment. Finally, students are asked to explain how they will use the results of the spools experiment to decide which of the Students was right about the "Block and spool" problem. The question also invites students to explain why the experiment will not decide who is right, if they think that it will not. It is rare for a student to take this option at this point, unlike in interviews with students who had worked through a previous version of the tutorial.

It seems, at this point in the tutorial, that students have a clear sense of why they are dropping spools. The students perform the experiment, observe that the spools hit the floor at the same time, and then record how the accelerations compare in direction and magnitude. (In a minority of cases they do not hit the floor at the same time, and this can usually be explained by lack of attention to the proper configuration of the spools before they are dropped. A group may need to perform the experiment a small number of times, before they get it right, but usually three tries is more than enough. By this time, there is general agreement among the students that the spools do hit the floor at the same time.) Students then state which of the solutions to the "Block and spool" problem is supported (or not refuted) by the results of this experiment. Then they answer some general questions about forces exerted on a rigid body: In order to know how a force affects the motion of the center of mass of an object, is it necessary to know (a) where on the object the force is exerted? (b) how the force is affecting the rotational motion of the object? Finally, students are advised that they have analyzed both the block and spool problem and the falling spools experiment in terms of forces, and they will be guided to describe the energy in these situations in the homework.

2. Detailed description of homework

The homework for the tutorial *Dynamics of Rigid Bodies* is intended to reinforce the main moral from the tutorial: the effect of a force on the acceleration of the center of mass of an object is independent of where the force is applied on the object. In this homework, students explore issues of energy in the contexts studied in tutorial, and practice using Newton's 2^{nd} law in new contexts. The homework consists of six problems, all of which were usually assigned.

Problem 1 returns to the context of the "Block and spool" problem that students considered in tutorial. Here students are guided to give a correct description of work and energy for the block and spool. The problem description is specified slightly more than it was in tutorial; the distance from start to finish is labeled *d,* and the tension in each thread is called *T.* All questions concern the time interval during which the spool moves from the start to the finish line. The first question asks students whether the distance traveled by the hand that is pulling the spool's thread is greater than, less than, or equal to *d.* In tutorial, students pulled a spool across a table in this manner, but they were not asked to make this exact observation at that time. We expect students to remember that, as the spool unwinds and moves forward, the thread pulling the spool gets longer, thereby increasing the distance between the hand and the spool. Thus, the hand travels a distance that is greater than the distance traveled by the center of the spool, which is d. The next question asks whether the work done by the hand pulling the spool's thread is greater than, less than, or equal to the product Td. In calculating the work done by the hand, the magnitude of force that should be used is the tension T, and the appropriate distance is the distance traveled by the hand, which is greater than d. Thus, the work done by the hand is greater than the product Td. Then, students are asked how the works done by the two hands compare to each other during that time interval. Since the work done by the hand pulling the block's thread is Td, the hand pulling the spool's thread does more work. Most students are able to answer these three questions correctly; the most common mistake is to use the distance d in finding the work done by the hand pulling the spool. Finally, students read a statement by a hypothetical Student, who explains:

> "It's not possible for the spool to rotate, AND cross the finish line at the same time as the block. If that were true, then the spool would have more total kinetic energy than the block. This is because they would have the same translational kinetic energy, and the spool would have additional rotational kinetic energy. But they must have the same total kinetic energy because the work done on them is the same."

Students are then instructed to explain, using the work-energy principle, why the spool has more total kinetic energy than the block when they cross the finish line together. Having answered the previous questions, it would not be necessary for the student to say more than this: the spool has more total kinetic energy because the net work done on an object is equal to its change in energy, and there was more work done on the spool. (Strictly speaking, it had been established only that one hand did more work than the other; that this work in each case results in a change in energy of the block/spool only follows from the masslessness and (tacitly assumed) inextensiblility of the thread. No student has ever expressed doubt of either of these premises.) The purpose of this problem is to reinforce the idea that the same force must result in the same center-of-mass acceleration by affirming to the student, that although the objects have the same a_{cm}, there is an important difference in the formal description of their motions. In interviews, students had expressed the conviction that the fact that only one object rotates must make *some* difference.

Problem 2 concerns the "Connected spools" experiment that students performed in tutorial. As in Problem 1, students have already analyzed the problem correctly in terms of forces. Neither experiment's outcome can be predicted from an energy perspective, but having observed (or decided) the outcome of each experiment, we may then give an energy description of each experiment. We want to encourage students to look at problems from many possibly useful perspectives and make judgments about which perspectives can help solve the problem, *without* sending the message that some of the perspectives are completely inappropriate.

In the first question, students are asked to recall their observation from tutorial of the relative order in which the spools hit the floor. Next a third spool, C, is introduced. Spool C is dropped from the same height at the same time as the other two spools, as shown in Figure 4-16. Spool C falls under the influence of the gravitational force only, and is brought in to help the students distinguish the motion of spool B from free-fall motion. In previous versions of the tutorial homework, many students asserted that spool B had a kinetic energy (just before landing) equal to its starting gravitational potential energy. This assertion neglects the (negative) work done on spool B by the thread pulling up on it. We intend for students to realize (by answering the following questions) that spool C and spool B must not have the same motion and thus will not have the same kinetic energy just before landing. After spool C is introduced, students rank the three spools according to center-of-mass acceleration. Students have already deduced that A and B have the same a_{cm}. Spool C has the same gravitational force on it as on A and B, but it has no retarding force on it, as A and B do. Therefore the ranking by a_{cm} is C > A = B. Next, students rank the spools according to the *translational* kinetic energy each spool has just before hitting the ground. (Students are reminded of the definition of translational kinetic energy: $K_{trans} = (1/2)mv_{cm}^2$.) Since all spools fall the same distance, under constant acceleration, the greater acceleration corresponds to the greater final speed and the greater final translational kinetic energy. Therefore the ranking by K_{trans} is the same: C > A = B.

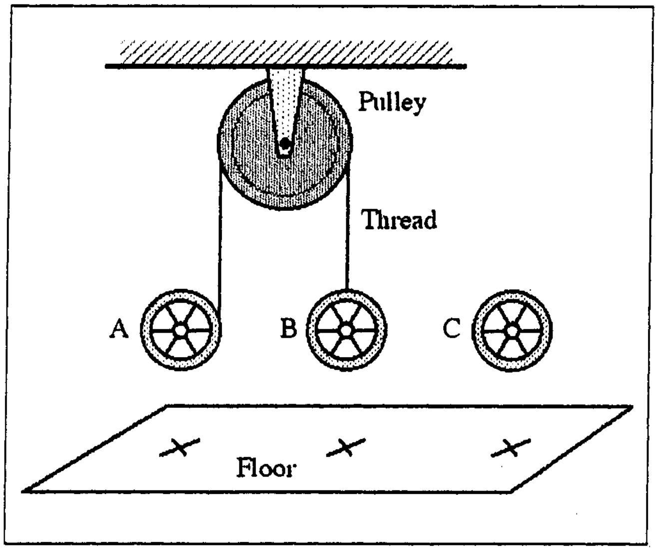

Figure 4-16: The "Connected spools," with a third spool in free-fall, as in problem 2 of the homework for *Dynamics of Rigid Bodies.*

The next question involves more sophisticated reasoning, some of which we have decided to shortcut for the students. Students are asked to consider the system consisting of spools A and B, the connecting thread, pulley, and Earth. We tell the students that there is zero net work on this system as the spools fall, and thus the sum of all of the translational and rotational kinetic energies, and gravitational potential energies remains constant ($K_{trans,A}$+ $K_{trans,B}$ + $K_{rot,A}$ + $K_{rot,B}$ + $U_{grav,A}$ + $U_{grav,B}$ = constant). (The only significant external forces on this system are the upward force on the pulley by the ceiling and the downward normal force on the Earth by whatever supports the ceiling. Neither force does any work, since the displacement of each point on the system where a force is applied is zero.) Each gravitational potential energy starts at a value of 9 J. Just before spools A and B land ($U_{grav,A}$ = $U_{grav,B}$ = 0 J), the translational kinetic energy of A ($K_{trans,A}$) is 4 J. Students are asked to find the rotational kinetic energy of spool A just before it hits the ground. The answer we expect, and which is given by a majority of students, is that the final value of $K_{trot,A}$ is 10 J. If the final value of $K_{trans,A}$ is 4 J, then so is $K_{trans,B}$. The total energy in the system is 18 J, so 10 J remain. Of course, the most common incorrect answer is that the final value of $K_{rot,A}$ is 9 J – 4 J = 5 J. These students assume that there is no energy transfer from one spool to the other via the thread. Ideally, a description of energy transfer in mechanics is accompanied by an account of work done, which is the mechanism of energy transfer in mechanics. To look at this energy transfer in terms of work done on each spool by the thread is an exercise that is probably far too advanced for most introductory physics students. In order to look at the work done, we must look at an infinitesimal displacement of the point of application of the tension force on each spool, and find the work done by taking the dot product of the force with that displacement. If this is done, we find that the point where the thread pulls on spool A moves upward as the center of the spool moves down. Since this displacement is in the direction of the force, the thread does positive work on the spool. This positive work corresponds with an increase in energy of the system of spool A and the Earth; this system began with a potential energy of 9 J, and ended with total kinetic energy of 10 J + 4 J = 14 J. Similarly, the total final kinetic energy of spool B must be less than 9 J, since the thread does negative work on it. We believe it is reasonable to expect introductory students to be able to describe the redistribution of energy among parts of a system with constant total energy, without necessarily describing the mechanism of each energy transfer in terms of work done. Finally, students are asked to rank the three spools according to the total kinetic energy each has just before it hits the ground. We are satisfied for a student to say something like: "A > C > B; $K_{\text{total,A}}$ = 14 J, $K_{\text{total,B}}$ = 4 J, $K_{\text{total,C}}$ = 9 J; B gave some of its energy to A."

With a problem as tricky as this one, it may be useful to explain here why we consider it worth the trouble. First, the falling spools experiment is the simplest instance we could find of a real situation for which applied forces are the same and rotational motions are different; these constraints are necessary for demonstrating the truth of Newton's 2$^{\text{nd}}$ law for rotating objects (with a qualitative observation). Second, we have observed that students generally prefer arguments in terms of mechanical energy to arguments in terms of forces when considering combined rotational and translational motion. The energy analysis of these problems, though somewhat complicated, is meant as a kind of mental relief for the student who feels generally more comfortable making sense of things in terms of energy. This student should not be encouraged that an energy approach will always work first, but in many cases, an energy description may be given afterward.

Problem 3 is called the "Three blocks" problem (see Figure 4-2). This problem provides yet another context for students to practice applying Newton's 2$^{\text{nd}}$ law to rotating rigid bodies. In this context, three blocks are at rest on an ice rink; an identical horizontal force is applied to each block. The force on block 1 is applied on the left side, about halfway between the middle and the far left end, the force applied to block 2 is directed right at its center of mass, and the force on block 3 is near the far right end of the block. Students draw arrows to indicate the direction of the center-of-mass acceleration of each block at the instant shown in the figure, and they rank the three accelerations, according to magnitude. Since all forces are the same in magnitude and direction, the three center-of-mass accelerations must all be the same. Students are encouraged in a hint to draw a point free-body diagram for each block before ranking the accelerations. We cannot tell how helpful the hint is to those students who give a correct ranking; only a minority of students draw the diagrams, and some students who draw the diagrams draw them such that they support an incorrect ranking. The most common incorrect ranking is 2>1>3, for which the a_{cm} increases as the force is applied closer to the center of mass. Some students who give this ranking draw point free-body diagrams that show three single arrows, with sizes matching the incorrect ranking. Many students who give this ranking state explicitly that the effect of the force on the acceleration of the center of mass depends on where the force is applied. It is possible that these students are thinking about the results from tutorial and how they are consistent with this ranking, but we believe it is much more likely that the student has not tried to make sure they are consistent. Therefore, such a student would probably be helped on this question by the suggestion that the two parts ought to be consistent.

Problem 4 is meant to demonstrate that the net force and net torque on an object are not necessarily zero or non-zero under the same circumstances. A rigid rod sits on a frictionless surface. The dowel is depicted four times, as in the top part of Figure 4-17, and students are asked to draw arrows on each picture to show how one could push on the dowel to produce all four combinations of zero and non-zero, net force and net torque. For example, on the second picture, students are asked to show where one could push on the dowel to produce a zero net force and a non-zero net torque. If a combination is not possible, students should say explicitly that it is not possible. However, all combinations are possible, and most students can come up with all four combinations successfully. In the case described above, one could draw an arrow pushing down on the left side and another pushing up on the right side. Both pushes make a counter-clockwise torque, but if the forces are equal in magnitude and opposite in direction, they result in zero net force.

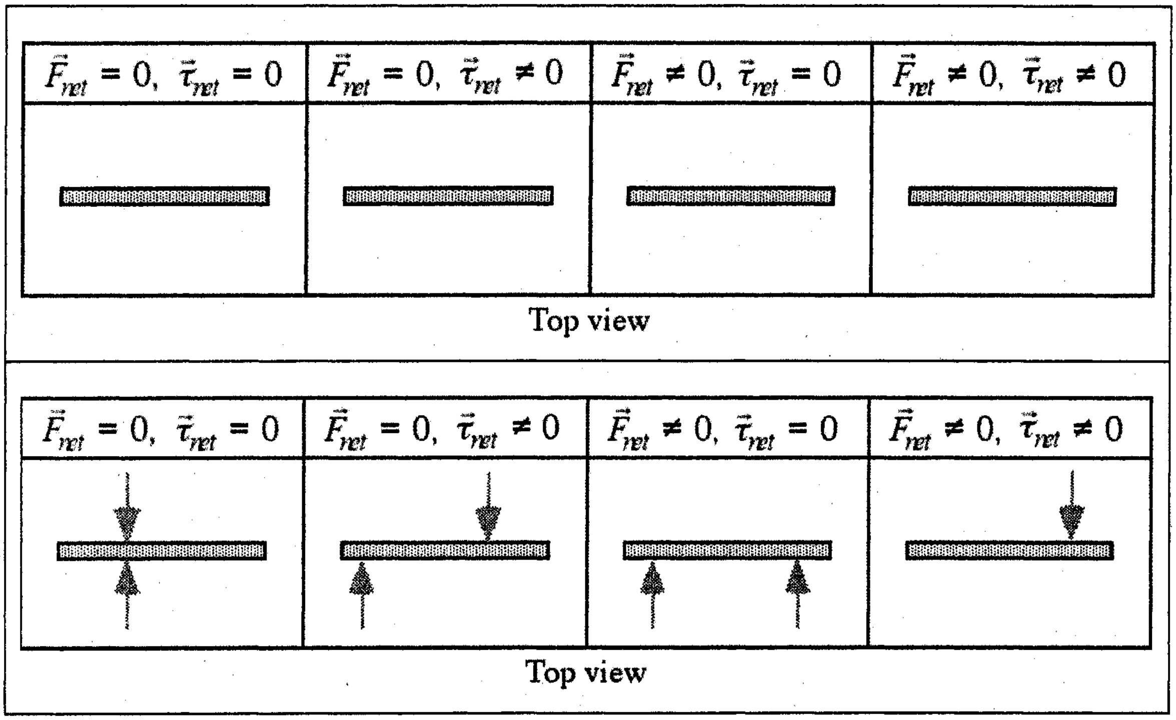

Figure 4-17: Diagram for problem 4 of the homework for *Dynamics of Rigid Bodies*, as printed in the homework (top), and correctly completed in gray (bottom).

In Problem 5, students summarize their results from the tutorial and homework. Students answer whether, when considering the effect of a force on the motion of the center of mass of an object, they must account for or ignore the rotational motion of the object. Then they answer whether, when considering the effect of a force on the total energy of an object, they must account for or ignore the rotational motion of the object. The expected answers are "ignore" and "account for," respectively. These questions provide students with an easy way to start discussing their ideas with each other; also, instructors can check quickly how students are answering. We have seen, however, that it is not uncommon for a student to give correct answers to these questions without incorrect answers on other homework or exam questions on this topic.

Finally, Problem 6 is a variation of the "3 descending objects" context, which we have used to evaluate student understanding on pretests and exams. We hoped that some students would find it useful to refer to the summary in Problem 5 when working the more challenging Problem 6. In Problem 6, three objects, A, B, and C, of equal mass, are released from rest at the same time from the same height on identical ramps, as shown in Figure 4-18. Objects A and B are both blocks, and they slide down the ramps without rotating. Object C rolls down the ramp without slipping. Its moment of inertia is unknown. Objects A, B, and C are made of different materials. (*i.e.,* It should not be assumed that the coefficient of friction between any object and its ramp is the same as that for any other object.) Object A reaches the bottom of its ramp first, followed by objects B and C, which reach the bottom at the same time. First, students rank the three objects according to their center-of-mass accelerations. This ranking can be done according to the travel time of the objects down their ramps. Since A reaches the bottom first, it had the greatest acceleration; B and C took the same time, so they had the same acceleration. The correct ranking of accelerations is then $A > B = C$. The next two questions are "helper" questions that students are not asked on exams or pretests. We intend for the students to learn by doing this problem what steps would be helpful to think about if a similar situation were to appear on an exam. The first of the helper questions asks students to rank the objects according to net force. Using Newton's second law, the ranking of net forces is the same as the ranking of accelerations: $A > B = C$. In the second helper question, students draw *point* free-body diagrams for the three objects. To do this, students should think about what forces are acting on each object and in what direction each force acts. The gravitational and normal forces must be the same for all three objects, while the difference in net force must be due entirely to a difference in frictional force on each object by its ramp. The next question, for which the previous two were helpers, is to rank the three friction forces acting on the objects by their ramps. Since for each object, the friction force is directed against the direction of the acceleration, the ranking of the friction forces must be opposite that of the net forces: $B = C > A$.

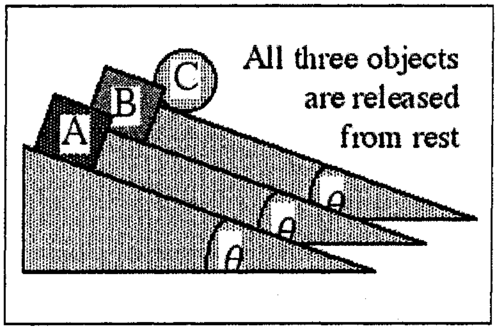

Figure 4-18: Variation of the "3 descending objects" context in problem 5 of the homework for *Dynamics of Rigid Bodies*.

The purpose of Problem 6 is to guide students to conclude that the validity of Newton's 2nd law for rotating objects is independent of the type of force that is affecting both the rotational and translational motion of the body. In particular, though there are expressions relating friction forces to normal forces, a friction force is still a force, which must always behave agreeably with Newton's 2nd law. We have seen that students often start thinking about friction forces by mis-remembering the relationship between friction and normal forces; namely, students often assume that friction and normal force are proportional, even in the case of static friction. This incorrect assumption would lead one to an incorrect ranking of the friction forces, even if we chose to assume that the coefficient of friction were the same between each object and its ramp.

D. STUDENT UNDERSTANDING OF NEWTON'S 2ND LAW FOR ROTATING RIGID BODIES AFTER CURRENT VERSION OF TUTORIAL SEQUENCE *DYNAMICS OF RIGID BODIES*

1. Description of additional question contexts

a. "Block and 2 spools"

This context is very similar to the "Standard plus hands" version of "Block and spool," except that the two experiments from that context are combined here into one experiment with a total of three objects, as shown in Figure 4-19. The experiment involves a block and two spools, all of equal mass. The tension in the thread attached to the block (Block B) is the same as the tension in the thread attached to one of the spools (Spool A). The hand pulling the other spool (Spool C) moves so that it has the same lateral position as the hand pulling the block. All three objects start at rest at the "start" line. (If we consider just Objects A and B, we have what might be called the "Standard portion" of "Block and 2 spools.") When the context was used in a lecture setting, students were shown the top two pictures before being asked the *N2R* question. After student discussion, the lecturer presented the bottom picture as a representation of the correct answer. When the context was used in an exam setting, students saw only the upper left picture.

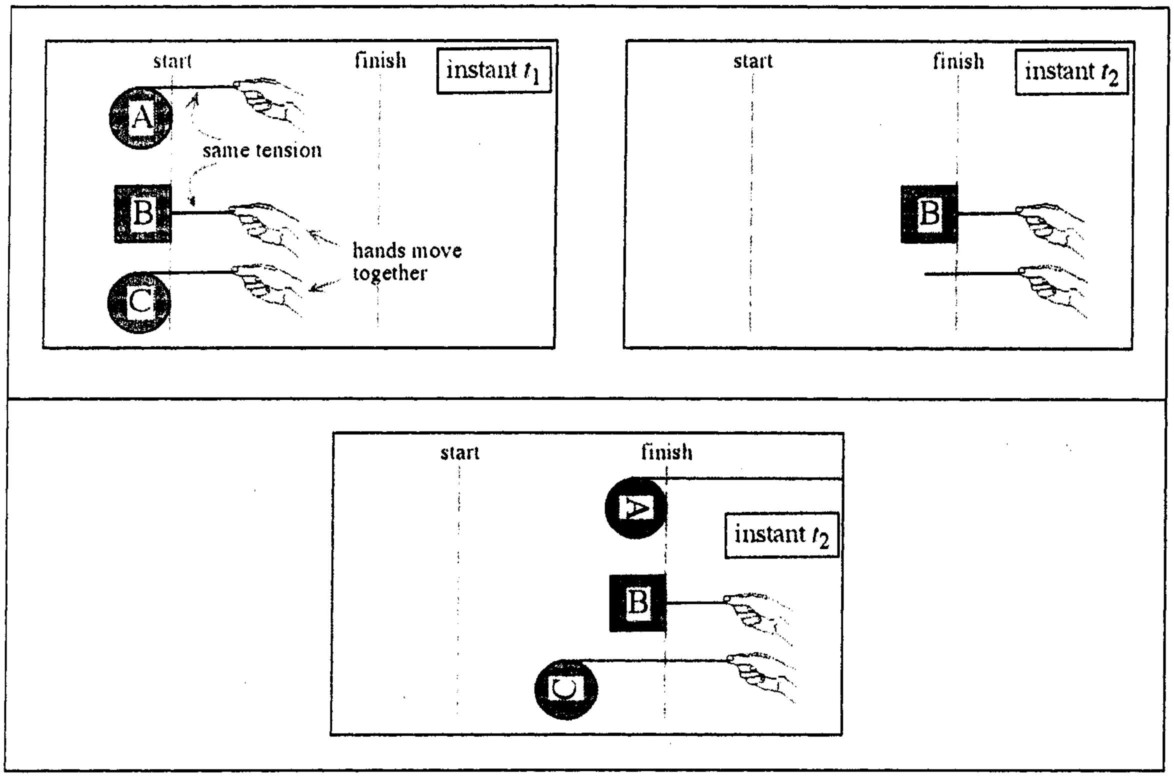

Figure 4-19: The "Block and 2 spools" context. In a lecture setting, students were shown the top two pictures before being asked the *N2R* question. After student discussion, the lecturer presented the bottom picture as a representation of the correct answer. In an exam setting, students saw only the upper left picture.

2. Description of additional questions

a. The *Work* question

The *Work* question asks students to compare the work done on each of the three objects described in the "Block and 2 spools" context by its thread, between the instant at which the threads start pulling and the instant at which the block crosses the finish line. As a multiple-choice question, students were shown five options: $W_A > W_B = W_C$; $W_A > W_B > W_C$; $W_A = W_B = W_C$; $W_A = W_B > W_C$; and $W_B > W_A > W_C$.

3. Correct answers to additional questions

a. The *Work* question

The ranking of the work done on objects A, B, and C is $W_A > W_B > W_C$. Objects A and B have identical translational motions and equal masses, so they have the same translational kinetic energy at the final instant. However, A also has some rotational energy. Using the principle of energy conservation, the change in the total energy of a system is equal to the net work done on it (in the absence of heat transfer). Thus, since the final energy of A is greater than that of B, the net work done on it must be greater. (The net work done on each object is equal to the work done on it by its thread, since no other forces do work on any of the objects.) Alternatively, the work done by the hand pulling A is more than that for B because the hands pull with the same tension, yet the hand pulling A moved a farther distance. The work each hand did on its thread must correspond with energy put completely into the spool/block, since the thread itself is massless, inextensible, and bearing a finite tension. According to this same line of reasoning, the work done on C is less than that done on B, since the distance each hand pulled is the same, but the tension must have been less on object C if it lagged behind object B (which it must do if it unwinds, as the hands move together.)

4. Administration of questions

The *Kinematics* question (see part B of this chapter) was asked in four sections of UW 121, after students had worked through either version B or the current version of *Dynamics of Rigid Bodies*. One of the four sections worked through the current version of the tutorial, and answered the question in multiple-choice form, by first comparing the center-of-mass accelerations of A and B, then those of B and C.

The *Work* question was asked in two sections under different conditions: one section answered the question during lecture, as a silent in-class (multiple-choice) vote; the other section answered it as a multiple-choice question on their final exam. However, in both cases, the *Work* question followed the *N2R* question. For the in-class vote condition, the *N2R* question was asked in the standard form. After the students voted on the *N2R* question, the correct solution was discussed before the class moved on to the *Work* question, and the expanded context "Block and 2 spools." Students were encouraged to discuss the *Work* question with each other before and during the voting period. For the final exam condition, students answered the *N2R* question in the "standard portion" of the "Block and 2 spools" context.

5. Overview of student performance

a. The *Kinematics* question

As shown in Table 4-15, student performance on this question does not appear to depend on whether students are considering the "incline" or "vertical" version of the "3 descending objects" context. Students did perform slightly more successfully when the information about the finishing times of objects B and C was depicted, as well as described verbally. Students also performed slightly better still on the multiple-choice version of the problem; asking students to compare the accelerations pair-wise may have helped some students organize their thinking. Since the online pretest also asked students to compare the accelerations pair-wise, this comparison may be most appropriate: before tutorial, 70% of students compared the accelerations correctly (see Table 4-11), and after tutorial 90% gave correct comparisons.

Table 4-15: Student performance on the *Kinematics* question in various versions of the "Three descending objects" context after different versions of the tutorial *Dynamics of Rigid Bodies*, including the current version.

A arrives at the bottom of its path first, followed by B and C, which arrive at the bottom at the same time (B rotates, C does not).	UW 121				
	Intermediate Version B			Current version	
	Incline*	Incline	Vertical		
	N=132 1 section	N=120 1 section	N=147 1 section	N=158 1 section Multiple-choice	N=170 1 section
$a_{cm}(A) > a_{cm}(B) = a_{cm}(C)$ (correct)	70%	80%	80%	90%	80%
$a_{cm}(A) > a_{cm}(B) > a_{cm}(C)$	20%	15%	10%	<5%	5%

*In this version of the "incline" context, there was no second figure showing the positions of A, B, and C at the instant when A reaches the bottom of its ramp.

b. The *N2R* question

Table 4-16 shows student performance on the *N2R* question in the "Circular pucks" context. Looking at student understanding of N2R from this perspective, it appears that the tendency of students to take a Newtonian approach to abstract problems about forces on rotating rigid bodies did not increase by increasing the instructional attention to the issue (as in version A), nor by introducing experimental proof that the Newtonian approach is correct (as in version B), but only by providing the proof and assisting students in connected the results of the experiment to a prototypical example (as in the current version). Student success on this task was basically the same after both intermediate versions as with the published version (~45%). Only with the current version did the success rate increase (by ~20%).

Table 4-16: Student performance on the *N2R* question asked in the "Circular pucks" context after different versions of the tutorial *Dynamics of Rigid Bodies*, including the current version.

	Published version	Intermediate version A	Intermediate version B	Current version	
Something about pucks	PRD 152 N=427 2 sections	UW 121 N=158 1 section	UW 121 N=149 1 section	UW 121 N=155 1 section	UW 114 N=51 1 section
$a_{cm}(1) = a_{cm}(2)$ (correct)	40%	45%	40%	60%	65%
$a_{cm}(1) < a_{cm}(2)$	50%	50%	50%	35%	30%
$a_{cm}(3)$ is zero	55%	30%	20%	25%	N/A*

*Students in this section worked a version of this question that did not include puck 3.

Table 4-17 shows that student performance on the *N2R* question in the "Block and spool" context did not increase when the attention paid to the issue of N2R was increased in the tutorial (as in version A). The success rate did improve, however, merely by exchanging the "unconnected spools" experiment for the "connected spools" experiment (as in version B). Yet, greater success on this task was achieved by teaching students how to relate the results of the experiment to the "Block and spool" context.

It may occur to the reader that student success with the "Block and spool" context after the current version of the tutorial may be so high because "we told them the answer." Much of the research in physics education (and of learning in general) indicates that teaching often constitutes the process of preparing students' minds to find a particular idea acceptable. Without such preparation, the idea that is being taught is often either misinterpreted or overtly rejected by the learner. The far left column of Table 4-17 shows the results of an in-class vote, in which students thought about the block and spool after working through the published version of the tutorial. The lecturer of these sections reported that the class discussion of the task was long and difficult; one student explained that the spool lags behind the block because: "That's what we saw in tutorial yesterday. The spool with a string attached fell slower than the one without the string, because it started spinning and used up some of the force." Later that day, the instructor received a "long, agitated email" that stated, "Everyone knows that if the quarterback gets undercut by a flying tackle, he spins but doesn't displace. We've all seen this on TV many times." After this discussion, on the final exam, the students answered a question similar to the *N2R* question in "Circular pucks," except that the objects were rods. About 50% of approximately 1000 students in COL 1110 stated that the rods would have equal center-of-mass accelerations. If we compare this success rate with that for "Circular pucks," the success rate is better than that for the published or intermediate tutorials (compare 50% to 40-45%), but not as good as that for the current version of the tutorial (compare 50% to 60-65%). These results suggest that student discussion and resolution of the "Block and spool" context is important for instruction on N2R. Another instructor (with many years of experience in teaching and curriculum development) at the University of Washington reported an experience of near mutiny when he tried to convince his students that the block and spool would cross the finish line at the same time. These anecdotes (along with increased student performance on other N2R tasks) suggest that improved student performance on *N2R* questions in the "Block and spool" context signifies meaningful learning of N2R, as a general principle.

Table 4-17: Comparison of student performance on the *N2R* question asked in the "Standard" version of the "Block and spool" context after different versions of the tutorial *Dynamics of Rigid Bodies*, including the current version.

	Published version	Intermediate version A	Intermediate version B	Current version
Block (B) and spool (S) are pulled with the same tension	COL 1110, N=385 2 sections In-class vote with discussion	UW 121, N=135 1 section Final exam	UW 121, N=130 1 section Final Exam	UW 121, N=66 1 section Silent in-class vote
S rotates and crosses at the same time as B. (correct)	30%	30%	50%	70%
S rotates and crosses after B.	60%	55%	40%	25%
S rotates and stays in place.	5%	5%	5%	0%
S does not rotate, and crosses at the same time as B.	5%	5%	5%	5%

We had observed that, before working through *Dynamics of Rigid Bodies,* student performance on the *N2R* question in the "Block and spool" context was augmented when students had the opportunity to distinguish between pulls of equal force and pulls of equal speed, as in the "Standard plus hands" version of the context. This factor may account for the small difference in success rates in Table 4-18 (compare 70% to 80%). Students who worked the *N2R* question in the "Standard portion" of "Block and 2 spools" may have been helped by the presence of the second spool, whose hand moved with the hand that pulled the block.

Table 4-18: Student performance on the *N2R* question in different versions of the "Block and spool" context after working through the current version of the tutorial *Dynamics of Rigid Bodies*.

	Current version of tutorial	
Spool A is pulled with the same tension as Block B. The hands pulling Block B and Spool C move together.	UW 121, N=149 1 section Final exam "Standard portion" of "Block and 2 spools"	UW 121, N=66 1 section Silent in-class vote "Standard" version
A rotates and crosses at the same time as B. (correct)	80%	70%
A rotates and crosses after B.	20%	25%
A rotates and stays in place.	<5%	<5%
A does not rotate, and crosses at the same time as B.	<5%	<5%

Table 4-19 compares student performance on all questions in the "Connected spools" context before tutorial instruction to that after tutorial instruction. It appears that almost all (95%) students recognized that the spools would (or did) hit the ground at the same time. This result is important because we were concerned that a significant minority of students might get a different impression from seeing the experiment performed. That is, to perform the experiment requires some skill; when the spools are released with a slight push, or from slightly different heights, a difference in arrival times is easily detected. Thus, we are encouraged that virtually all of the students correctly identified, as they often say, "what was supposed to happen." It is also encouraging that almost all students (90%) interpreted this experimental result correctly in terms of center-of-mass acceleration, a term that many students misinterpreted on the pretest. Student performance on the *Uniform tension* question also improved significantly (from 60% to 80%).

Table 4-19: Comparison of student success on the *Uniform tension, N2R,* and *Kinematics* questions, asked in the "Connected spools" context before and after the current version of the tutorial *Dynamics of Rigid Bodies.*

	Before tutorial	After current tutorial
Percentage of students answering question correctly	UW 121, N=737 7 sections	UW 121, N=100 1 section
Uniform tension	60%	80%
N2R	40%	90%
Kinematics	35%	95%
Correct responses to all three questions	25%	75%

Table 4-20 shows student performance on another relatively abstract task, the *N2R* question in the "4 square blocks" context. Student performance improved (from 35% to 55%) with Intermediate version A. Student performance on this question after the current version of the tutorial seems more difficult to interpret. First, these students saw only a version of the "4 square blocks" with blocks 2 and 4 removed. Thus, in the original version of the context, students were asked to rank the center-of-mass accelerations of all four blocks, while in the later version of the context, students were asked only to compare the center-of-mass accelerations of blocks 1 and 3 (seen by the students as blocks "1" and "2"). On one hand, the original task seems more difficult because students would have to think about more blocks; but, on the other hand, it may be that (with other factors being equal) students would tend toward using a correct version of N2R the greater the number of blocks, since applying a modified version of the N2R principle to many blocks requires making many judgments about how effective various forces are in accelerating the blocks' centers-of-mass. In other words, if applying a modified version of N2R becomes too complicated, some students may shift to a simpler solution, which in this case is the correct one.

Another interesting feature of these data is that the students in the far right column of Table 4-20 are the same students in the far right column of Table 4-19. That is, the students who performed very well on the questions relating to the "Unconnected spools" context performed relatively poorly on the *N2R* question in the "4 square blocks" context. The two right-most columns of Table 4-20 represent our attempt to observe the effect of the "Connected spools" context on the "4 square blocks" context. We thought it possible that students who were "reminded" of the experiment in tutorial just before answering a more abstract *N2R* question might perform better than students who did not receive the "reminder." The data appear to show that the cueing we intended to provide instead had a negative effect. We are confident in concluding that the "Connected spools" context does not have a positive cueing effect, especially since students generally do not see the same relationships between physical situations that physicists do. We are not, however, as prepared to conclude that reminding students of the "Connected spools" experiment causes them to perform more poorly, since we do not currently see a way to understand such an influence.

Table 4-20: Student performance on the *N2R* question in the "4 square blocks" context after different versions of the tutorial *Dynamics of Rigid Bodies*, including the current version.

The same pair of forces are exerted on blocks 1 and 3, at different locations	Students saw all 4 blocks		Students saw only block 1 and block 3 from "4 square blocks"	
	Published version	Intermediate version A	Current version	
	UW 121, N=260 2 sections	UW 121, N=165 1 section	UW 121, N=169 1 section	UW 121, N=100 1 section‡
$a_{cm}(1) = a_{cm}(3)$ (correct)	35%*	55%	50%	30%

*As reported in Ortiz' dissertation

‡These students answered the question just after reconsidering the "Connected spools" context.

The main purpose of Table 4-21 is to illustrate that student responses to the *N2R* question in the "3 descending objects" context depend on whether the force of interest is a friction force or a tension force. When we first administered the Incline *f* version of the context, we did not expect students to assume that the friction coefficients between the objects and their ramps were all equal. About 20% of the students made this assumption and used it to rank the friction forces (incorrectly). In later administrations of the context, we added the following note: "Objects A, B, and C are made of different materials (*i.e.*, Do not assume that the coefficient of friction between any object and its ramp is the same as that for any other object.)" Student success on this task improved somewhat with the note (from 10% to 25%). Students were much more likely to say that the friction force on B was less than that on C (in the Incline *f* context) than they were to say that the tension force on B was less than that on C (in the Level *T* context) (bottom row of Table 4-21). Students were also much more likely to say that the forces on B and C were equal if they were tension forces than if they were friction forces. These results motivated the research described in the next chapter of this dissertation, in which we investigated differences and similarities in student understanding of tension and friction forces.

Table 4-21: Student performance on the *N2R* question asked in various versions of the "3 descending objects" context after working through the current version of the tutorial *Dynamics of Rigid Bodies*. ('*F*' stands for the magnitude of the friction force or tension force on the object, depending on the context version.)

A arrives at the bottom of its path first, followed by B and C, which arrive at the bottom at the same time (B rotates, C does not).	UW 121			
	Vertical *T*		Incline *f*	
	N=158 1 section multiple-choice	N=147 1 section free response	N=161 2 sections free response (with note*)	N=132 1 section free response (without note*)
F(B) = *F*(C) > *F*(A) (correct)	30%	40%	25%	10%
F(A) = *F*(B) = *F*(C)	30%	10%	5%	20%
F(C) > *F*(B) > *F*(A)	5%	20%	35%	35%
F(C) > *F*(A) = *F*(B)	5%	5%	10%	20%
Total of *F*(B) = *F*(C)	75%	60%	35%	30%
Total of *F*(B) > *F*(C)	5%	10%	15%	15%
Total of *F*(B) < *F*(C)	15%	30%	50%	55%

*In one section, students read a note that advised them not to assume that the friction coefficients between the objects and the ramps were equal.

c. The *Work* question

Table 4-22 shows student performance on the *Work* question. For each administration of the question, the most popular answer (35-40%) was "$W_A = W_B > W_C$." Since this ranking (A = B > C) is also the ranking of the accelerations of the objects, or the distances traveled by the objects after any fixed time interval, we might think of this answer as the "object dominant" answer. Similarly, because the ranking "$W_A > W_B = W_C$" (given by 20-30% of students) corresponds to the correct ranking of the distances traveled by the hands after any fixed time interval, we might call this answer the "agent dominant" answer. Since the work in each case is the product of the force on the object and the distance traveled by the agent, the correct answer can be understood as a "product" of the "agent dominant" and "object dominant" answers. That is to say, these results are consistent with previous reports of students disproportionately emphasizing the role of either the force or the displacement in more basic questions about work.[9]

Table 4-22: Student performance on the *Work* question after working through the tutorial *Dynamics of Rigid Bodies*.

Spool A is pulled with the same tension as Block B. The hands pulling Block B and Spool C move together.	UW 121, N=149 1 section Final exam	UW 121, N=66 1 section In-class vote with discussion
$W_A > W_B > W_C$ (correct)	20%	30%
$W_A > W_B = W_C$	20%	30%
$W_A = W_B = W_C$	20%	<5%
$W_A = W_B > W_C$	35%	40%
$W_B > W_A = W_C$	5%	0%

6. Performance of advanced students on questions about forces on rotating rigid bodies

As shown in Table 4-23, students in the introductory course perform better on the *N2R* question in the "Block and spool" context after the current version of the tutorial than either students in an advanced undergraduate classical mechanics course or graduate student TAs, before working through the tutorial. Though the introductory students are probably less well prepared to apply the N2R principle to new situations than are the advanced students, we believe that success on this task is a significant step toward a deeper understanding of the principle and its consequences.

Table 4-23: Comparison of performance of introductory students, graduate student TAs, and advanced undergraduates on the *N2R* question in the "Block and spool" context.

	After current tutorial	Before tutorial	Without tutorial
Block (B) and spool (S) are pulled with the same tension.	Introductory students UW 121, N=66	Graduate teaching seminar UW 501, N=22	Advanced undergrads UW 424, N=22
S rotates and crosses at the same time as B. (correct)	70%	40%	25%
S rotates and crosses after B.	25%	55%	35%
S rotates and stays in place.	<5%	0%	25%
S does not rotate, and crosses at the same time as B.	<5%	5%	5%

Table 4-24 shows performance by graduate student TAs on the *Uniform tension* and *N2R* questions in the "Connected spools" context. Almost all graduate students in UW 501 recognize that the tension in the thread is the same everywhere, but many of them incorrectly relate the individual forces on each spool to its center-of-mass acceleration. The most common error (25%) was to state that the center-of-mass acceleration of spool A would be greater than that of spool B, even though the tension forces on them were equal. To quote some responses:

> "a_{cm}(A) > a_{cm}(B) because the tension acts on the leftmost point of spool A, which doesn't 'support' spool A as well as spool B is 'supported.'"
>
> "The net force on A is just W_A, because T_A exerts torque, not force. For B, the force W_B is reduced by T_B, so its acceleration is less."

We conclude from these data (and those in Table 4-23) that advanced coursework in physics does not necessarily help students understand the N2R principle.

Table 4-24: Performance of graduate student TAs on the *Uniform tension* and *N2R* questions, asked in the "Connected spools" context before working through *Dynamics of Rigid Bodies*.

Spool A unwinds, and Spool B does not.	Graduate teaching seminar UW 501, N=22		
	$a_{cm}(A) = a_{cm}(B)$ (correct)	$a_{cm}(A) > a_{cm}(B)$	$a_{cm}(A) < a_{cm}(B)$
$T(A) = T(B)$ (correct)	55%	25%	10%
$T(A) > T(B)$	0%	0%	0%
$T(A) < T(B)$	0%	5%	5%

Finally, in Table 4-25, we compare success on all questions in the "Connected spools" context by students in the introductory course, both before and after the current version of the tutorial, to that of the graduate students before working through the tutorial. These data show that the ability of most of the introductory students to remember the results of the "connected spools" experiment and relate it correctly to the abstract concepts *acceleration of the center of mass,* and *tension in the thread* was better than graduate students' ability to predict the outcome of the experiment on the basis of these concepts. We believe that a solid understanding of the "Connected spools" experiment and its meaning is probably not itself sufficient (or strictly necessary) to for a student to apply the N2R principle in a variety of new situations, but we believe it (with an understanding of the "Block and spool" context) is a good foundation for introductory instruction on the general topic of N2R.

Table 4-25: Comparison of success of students in introductory course (before and after *Dynamics of Rigid Bodies*) to that of graduate students in a teaching seminar for the "Connected spools" version of the *Uniform tension, N2R,* and *Kinematics* questions.

	Introductory students		Graduate teaching seminar
	Before tutorial	After current tutorial	Before tutorial
Percentage of students answering question correctly	UW 121, N=737 7 sections	UW 121, N=100 1 section	UW 501, N=22 1 section
Uniform tension	60%	80%	85%
N2R	40%	90%	60%
Kinematics	35%	95%	50%
Correct responses to all three questions	25%	75%	50%

E. SUMMARY

In this chapter, we have shown that most students can be guided to change how they think a force affects the acceleration of the center of mass of a rigid body. We identified a particular case (the "Block and spool" context) in which previously identified conceptual errors are very strongly elicited, and with which students strongly associated their general ideas about the relationship of forces to translational motion. This pivotal case was not an experiment that was easily testable, and many students were not convinced by instructors' claims about its outcome. We also identified another case (the "Connected spools" experiment), which though testable, did not play a pivotal role in student understanding of forces in combined rotational and translational motion. That is, students remembered its outcome but did not appreciate its significance. We developed tutorial curriculum that guides students to connect the pivotal case with the testable case with various (correct and incorrect) theoretical perspectives on the relationship between forces and combined rotation and translation. As a result, students better understood the meaning of experimental results that demonstrate the validity of Newton's 2^{nd} law and were able to apply this understanding to concrete and more abstract contexts.

Further investigation of student ability to apply Newton's 2^{nd} law in contexts requiring a high degree of transfer led to an improved understanding of students' perspectives on the relationship between energy conservation and Newtonian dynamics. Specifically, we observed that students tend to combine energy and force analyses into one (when they are properly performed in parallel), sometimes giving an energy relation the power of causal influence that belongs only to forces (and torques). This investigation also led us to understand that student treatment of forces may depend strongly on the type of force (friction, tension, normal) involved. An extension of the investigation described in this chapter is detailed in the next chapter, in which we explored student understanding of static friction and static tension in more basic contexts, and the relationship between student understanding of these types of forces.

Notes to Chapter 4

[1] Ortiz, L. G., "Identifying and addressing student difficulties with rotational dynamics," Ph.D. dissertation, Department of Physics, University of Washington, 2001 (unpublished).

[2] McDermott, L. C. and P. S. Shaffer, "Research as a guide for curriculum development: An example from introductory electricity. Part I: Investigation of student understanding," *Am. J. Phys.* **60**, 994-1002 (1992).

[3] Ortiz, op. cit., p. 45

[4] Ibid., p. 42

[5] For a careful discussion of different interpretations of students' tendency to connect "force" and "velocity," see Viennot, L., "Analyzing students' reasoning: tendencies in interpretation," *Am. J. Phys.* **53**, 432-436 (1985).

[6] Hammer, D., "More than misconceptions: multiple perspectives on student knowledge and reasoning, and an appropriate role for education research," *Am. J. Phys.* **64**, 1316-1325 (1996).

[7] Giancoli, D. C., *Physics for Scientists and Engineers with Modern Physics*, Third Edition (Prentice-Hall, Upper Saddle River, NJ, 2000), p. 263.

[8] Kelley, H. H., "The processes of causal attribution," *American Psychologist* **28**, 107-128 (1973).

[9] Loverude, M. E., C. H. Kautz, and P. R. L. Heron, "Student understanding of the first law of thermodynamics: relating work to the adiabatic compression of an ideal gas," *Am. J. Phys.* **70**, 137-148 (2002).

CHAPTER 5. STUDENT LEARNING OF STATIC FRICTION AND TENSION FORCES ON POINT-LIKE OBJECTS

In our research on student learning of forces exerted on rotating rigid bodies, we investigated students' treatments of the constraint force of static friction in the case of rolling without slipping. After examining student reasoning in this context, we suspected that students may have certain ideas specific to static friction forces that conflict with the correct treatment of the static friction force *qua* force (*i.e.*, subject to consistency with Newton's laws). In order to observe these ideas more clearly, we developed questions about static friction that did not involve rotation, and that we could ask students in the earlier part of the course, before and after basic instruction on Newton's laws and friction forces. We had also examined students' treatment of combined rotation and translation with tension as a constraint force, instead of static friction. We observed differences in student performance that seemed to be due to the difference in the type of constraint. We therefore also sought to observe the extent to which students differed in their treatment of static friction and tension forces both before and after basic instruction on friction and tension. We present some evidence here that students are much more successful in their treatment of tension than of static friction. We also present evidence that suggests that students may be able to use their correct understanding of tension as a tool for learning to treat static friction correctly.

In part A of this chapter, we analyze students' written responses to questions about static friction forces on point-like objects in static equilibrium. In part B, we examine student responses to analogous questions about tension. We also discuss the positive effect that questions about tension have on student performance on subsequent questions about friction. In part C, we describe the initial version of an instructional intervention (in the form of an interactive lecture) that was an attempt to capitalize on students' apparent ability to relate friction and tension forces. In part D, we show the effects that this intervention had on student performance on exam questions about static friction. In part E, we describe how the instructional intervention was revised on the basis of classroom experience and student understanding after the initial version. In part F, we compare student understanding after the revised version of the intervention with that after the initial version.

A. STUDENT UNDERSTANDING OF STATIC FRICTION ON POINT-LIKE OBJECTS IN STATIC EQUILIBRIUM BEFORE AND AFTER THE TUTORIAL *FORCES*

1. Description of contexts in which questions about static friction forces were asked

This section describes various contexts within which we asked students written questions to test their ability to apply Newton's laws to determine the magnitude and direction of static friction forces. These contexts are depicted in Figure 5-1. The questions themselves were asked in several contexts and are described separately in the next section.

a. Incline *f*

There are two variations of the Incline *f* context: "Single object" and "Stacked objects." We used "Single object" much more often than "Stacked objects," so unless otherwise noted, the phrase "Incline *f*" refers to the "Single object" form.

i. Single object

In this context, a book sits at rest on an incline. As different questions are asked in this context, the setup can be changed slightly: the book can be rotated so that a different surface of the book makes contact with the plane; a piece of sandpaper can be fixed to the plane, so that the book rests on the sandpaper, without touching the plane itself; or the end of the incline can be raised, increasing the angle that the incline makes with the horizontal. In all cases, the book remains at rest.

ii. Stacked objects

In this variation on the Incline *f* context, one object sits on top of a second object, which sits on an incline. Both objects remain at rest. No changes are made to the setup; the purpose was simply to provide a context in which we could probe student understanding of the frictional interaction between two objects.

b. Level *f*

There are two variations of the Level *f* context: "Single object" and "Stacked objects." We used "Single object" much more often than "Stacked objects," so unless otherwise noted, the phrase "Level *f*" refers to the "Single object" form.

i. Single object

In this version of the Level *f* context, a block sits on a level surface. A string is attached to the block and passes horizontally over a frictionless peg and then vertically downward, where it is attached to a bucket that contains some rocks. In this context, the setup is changed in ways similar to those in "Incline *f*." The block can turned so that a different surface of the block makes contact with the level surface; a piece of sandpaper can be fixed to the surface, so that the block rests on the sandpaper, without touching the surface itself; or rocks can be added to the bucket. In all cases, the block remains at rest.

ii. Stacked objects

In this variation on the Level *f* context, one block sits on top of a second block, which sits on a level surface. A string is attached to the top block and connected to the peg-and-bucket apparatus as in the "Single object" version. No changes are made to the setup; the configuration of blocks allows an opportunity to ask students about the frictional interaction between the blocks.

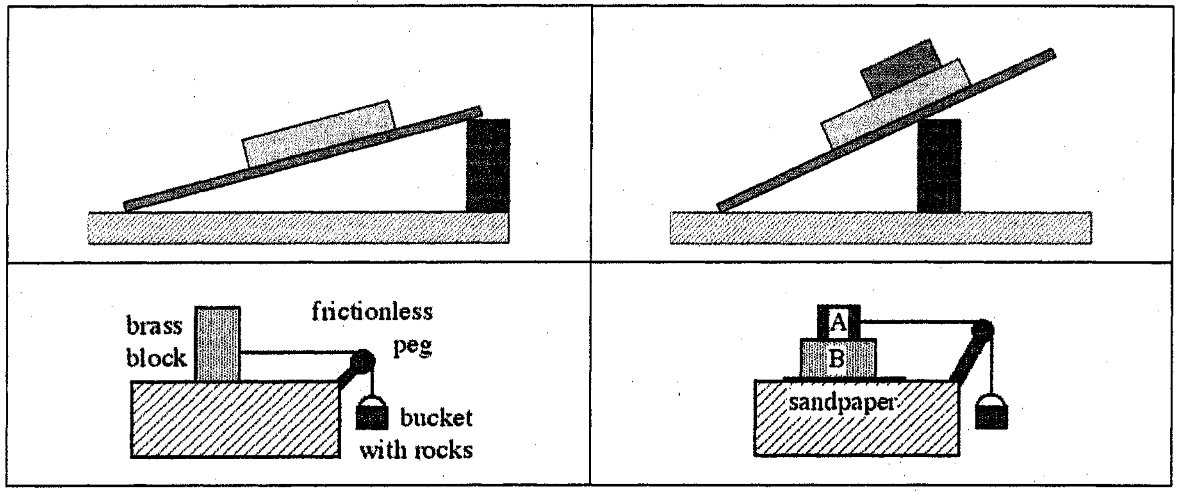

Figure 5-1: The "Incline *f*" context with a single object (upper left) or stacked objects (upper right), and the "Level *f*" context with a single object (lower left) or stacked objects (lower right).

2. Description of questions about static friction forces

In every case in which we asked students questions about static friction forces on point-like objects in equilibrium, the questions were arranged in a trio: the *Contact area* question, the *Greater maximum force* question, and the *New equilibrium* question (in that order). We asked this trio of questions in both the Incline *f* and Level *f* contexts. In each version, a student performs four experiments (Experiments 1, 2, 3, and 4), each time making a single change to the setup. In each experiment, the book (or block) remains at rest. After each change is made, students are asked to compare the friction force on the book in the new experiment to that in the immediately previous experiment. These changes (and the corresponding questions) are summarized in Figure 5-2.

Later in this chapter, we introduce contexts in which we asked the *Greater maximum force* and *New equilibrium* questions that involve tension forces instead of friction forces, so we refer here to the "Incline *f*" and "Level *f*" contexts, to be contrasted with "Incline *T*" and "Level *T*" contexts.

a. The *Contact area* question

In each version of the *Contact area* question, the orientation of the object is changed (from Experiment 1 to Experiment 2) so that the contact area between the object and whatever is exerting the friction force on the object is increased. Students are asked whether the friction force on the object in Experiment 2 is greater than, less than, or equal to that in Experiment 1. The purpose of this question is to infer to what extent students' contact-area-dependent concept of friction dominates other concepts (including the Newtonian concept). Of course, a correct answer to this question does not necessarily correspond to correct treatment of static friction in all contexts.

b. The *Greater maximum force* question

In Experiment 3, sandpaper is fixed to the original supporting surface, and the object is placed on top of the sandpaper (in the same orientation as in Experiment 2). Students are asked to compare the friction force on the object in Experiment 3 to that in Experiment 2. The phrase "coefficient of friction" is not used in the description of the problem; nor is the symbol μ. We expected that students would comfortably assume that sandpaper is rougher than the incline itself. The purpose of this question was to determine to what extent students' thinking about friction is dominated by the relationship $f=\mu N$ (rather than $F_{net}=ma$).

c. The *New equilibrium* question

For Experiment 4, a change is made to the setup that changes some of the other (non-friction) forces on the object. The purpose of this question is similar to those of the *Contact area* and *Greater maximum force* questions: to determine to what extent students use a Newtonian concept of balanced forces on an object in equilibrium. However, in both of the previous questions, the Newtonian concept of friction corresponded to no change in the friction force; in the *New equilibrium* question, the friction force changes according to the Newtonian concept. If a student were inclined for some reason to state that the friction force remained the same in all three questions, then the *New equilibrium* question would allow us to detect this departure from the Newtonian concept without relying too strongly on student explanations, which can often be difficult to interpret.

In the Incline f version, Experiment 4 involves raising the angle of the incline. In the Level version, rocks are added to the bucket. In both cases, students compare the friction force on the object in Experiment 4 to that in Experiment 3.

d. The *Third law* question

This question was asked in the "Stacked objects" version of each of the Incline *f* and Level *f* contexts. In the Incline version of the question, we asked students to draw an arrow to indicate the direction of the friction force exerted on the bottom object by the top object. Students were told that if they thought there were no such force, they should state that explicitly. In the Level version, students drew arrows to indicate the directions of three friction forces: that exerted on the top object (A) by the bottom object (B), that exerted on the bottom object by the top object, and that exerted on the bottom object by the sandpaper.

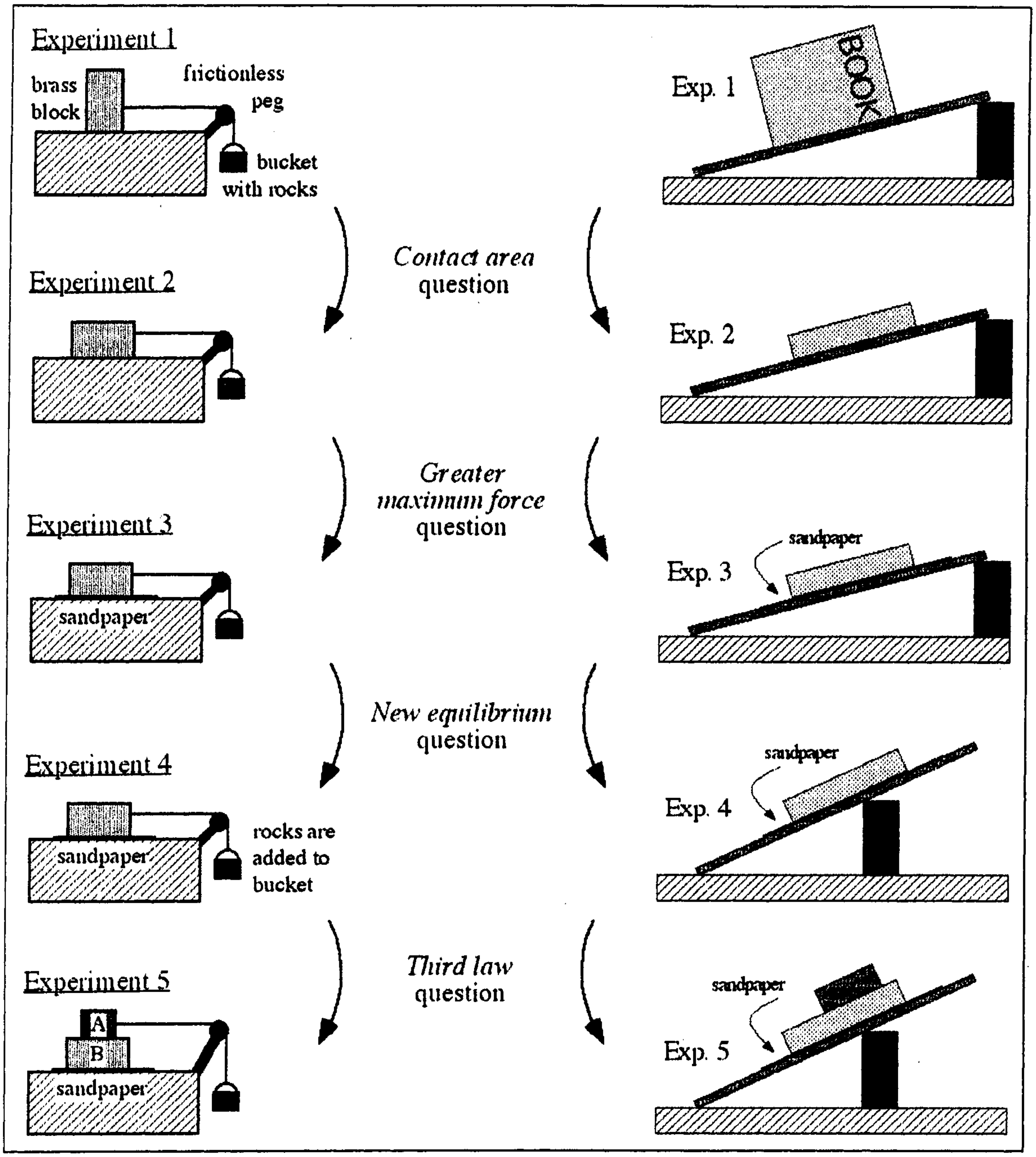

Figure 5-2: Summary of changes made to the system and the corresponding questions about static friction for the "Incline *f*" and "Level *f*" contexts.

3. Correct answers to questions about static friction forces

a. The *Contact area* and *Greater maximum force* questions

The correct answer and reasoning to both versions of both of the first two questions are the same. The object remains at rest when either the contact area or coefficient of friction is changed. Since the object remains at rest, the vector sum of the forces on the object must remain zero. If we draw and examine a free-body diagram for the object, it can be determined that each of the other forces on the object must remain the same. The static friction force must exactly cancel out the sum of the other forces if the sum of the forces is to remain zero. Therefore, like all of the other forces on the object, the friction force must remain the same in magnitude and direction.

b. The *New equilibrium* question

As with the *Contact area* and *Greater maximum force* questions, the correct answer involves realizing that the forces on the object must be balanced. When the incline is made steeper in the Incline *f* version of the *New equilibrium* question, the component of the gravitational force on the book that is parallel to the incline increases. Since the friction force is exactly balancing this force, the friction force must also increase. In the Level *f* version, rocks are added to the bucket, which effectively increases the weight of the object supported by the string. The tension in the string must increase to match this weight. Since the tension in the string increases everywhere, the tension force exerted on the block increases. The friction force on the block must match this tension force, and so it also increases. Therefore, in both versions of the *New equilibrium* question, the friction force is greater in Experiment 4 than in Experiment 3.

c. The *Third law* question

In the Incline version: The top object remains at rest on the bottom object, so in addition to a downward gravitational force on the object and the normal force up and the right, there must also be a static friction force exerted on the top object by the bottom object that points to the left and up. By Newton's third law, the friction force exerted on the bottom object by the top object must point in the opposite direction: to the right and down.

In the Level version: The top object remains at rest on the bottom object, so in addition to a rightward tension force on the object by the string, there must also be a static friction force exerted on the top object by the bottom object that points to the left. By Newton's third law, the friction force on the bottom object by the top object must point in the opposite direction: to the right. Since the bottom object also does not accelerate, there must be a static friction force exerted on the bottom object by the sandpaper that is directed to the left.

4. Description of treatment of static friction in the tutorial *Forces*

The tutorial *Forces* focuses on developing certain skills with drawing free-body diagrams for objects that are in static equilibrium – primarily, identifying the forces that belong on a free-body diagram for an object, and balancing these forces. One of the situations in the tutorial includes an object with a static friction force exerted on it. Students are not told that there is friction; they infer that, in order for the block not to move when two people try to move it, there must be a friction force. Furthermore, most students realize that this friction force must be equal in magnitude to the sum of the forces exerted by the people. However, students are not asked to think about changing the contact area between the block and the ground, changing the coefficient of friction, or changing the magnitudes of the forces the people are exerting. The tutorial homework includes a problem in which a hand changes how it is pushing on a block that is at rest on an incline. This problem includes a question about how the friction force on the block changes when the hand changes how it is pushing.

5. Administration of questions

The trio of friction questions was administered to different classes at different stages of instruction: before any reading assignments or lectures on forces, Newton's laws, or friction; after a reading assignment that introduced friction but before students attended any lectures or worked through the tutorial *Forces*; or after both lecture and tutorial instruction. No student was given the same version of the questions twice. In all cases for which a student worked two versions of the questions, one version was worked as an online, un-graded quiz after reading but before lectures and tutorial, and the other version was worked after the lectures and tutorial, on a midterm exam.

The Incline *f* version of the *Third law* question was given to one section of UW 121 as part of a midterm exam, after lecture and tutorial instruction. The question immediately followed the Incline *f* versions of the *Contact area, Greater maximum force*, and *New equilibrium* questions.

6. Overview of student performance

a. The *Contact area* question

Student performance on the *Contact area* question (see Table 5-1) was similar before and after the reading assignment that introduced friction, so these sections have been grouped and labeled "pretest." Performance appeared not to depend on which version (Level *f* or Incline *f*) of the question students worked, so these have also been grouped.

Student performance after instruction (lecture and tutorial) has been separated into two columns. Performance was very similar in three of four sections. After careful review of student responses and lecture notes from all sections, we suspect that the difference between these three and the fourth section was whether the lecturer explained friction microscopically. In the three sections that performed similarly (and better than the fourth), the lecturers avoided a microscopic picture of friction.

In the section that performed differently (labeled "section M," for microscopic) the lecturer emphasized that friction is related to the formation of bonds between molecules from the different objects. Lecture slides in this section showed several magnified views of springs (bonds) connecting the balls (atoms or molecules) in two materials. One slide showed that a car tire deforms when it touches the road. Next to the picture of the tire were two captions: "molecular bonds break as the wheel rolls forward," and "the wheel flattens where it touches the surface, giving a contact area rather than a point of contact." Another slide, entitled "The causes of friction," stated that "at the molecular level, friction is the repeated bonding and breaking of molecular bonds between the two materials." We do not present these statements in order to support or refute them, but rather to suggest that students may have misunderstood them. We also do not infer from these data that the treatment of a microscopic model of friction is inappropriate for an introductory course; it may be that such a model could help students be more successful on some basic questions about friction. For example, Laurence Viennot claims that students are more successful at applying Newton's laws to frictional forces when they have been taught to use a mesoscopic model of the mechanism of friction forces.[1]

Table 5-1: Student performance on the *Contact area* question. Student responses to both versions of the questions were similar and are grouped here.

Comparison of friction forces for lesser (1) and greater (2) contact area	Before lectures and *Forces*	After lectures and *Forces*	
	UW 121, N=580 4 sections	UW 121, N=318 3 sections	UW 121, N=157 Section M*
$f(2) = f(1)$ (correct)	30%	85%	70%
$f(2) > f(1)$	60%	10%	25%
$f(2) < f(1)$	10%	5%	5%

*Lectures in this section emphasized a microscopic model of friction.

The combination of standard instruction (that does not emphasize a microscopic model of friction as in section M) with the tutorial *Forces* appears to be strongly effective at helping students realize that static friction forces do not, in general, depend on the contact area between the objects.

b. The *Greater maximum force* question

As shown in Table 5-2, performance on the *Greater maximum force* question appears to depend on whether students had been assigned textbook reading that introduced friction. Namely, students who had not yet been assigned this reading performed slightly better than students who had been assigned the reading. In all but one case, students in different lecture sections performed similarly at similar stages of instruction. One section (section M) did not perform as well as the others on the post-test.

Table 5-2: Student performance on the *Greater maximum force* question. Student responses on both versions of the question were similar and are grouped here. In the columns marked "Before reading" and "After reading," the word "reading" refers to a reading assignment that included introductory remarks about friction, but did not include further clarifying remarks.

Static friction forces for lesser (1) and greater (2) μ	Before reading UW 121, N=232 2 sections	After reading UW 121, N=348 3 sections	After lectures and *Forces*	
			UW 121, N=318 3 sections	UW 121, N=157 Section M*
$f(2) = f(1)$ (correct)	30%	15%	45%	25%
$f(2) > f(1)$	65%	80%	50%	65%
$f(2) < f(1)$	5%	5%	5%	10%

*Lectures in this section emphasized a microscopic model of friction.

The reading assignment on friction that appears to have affected student performance introduced the idea of a microscopic picture of friction. The reading contained the following passage:

> "Friction, like the normal force, is exerted by a surface. On a microscopic level, friction arises as atoms from the object and atoms on the surface run into each other. The rougher the surface is, the more these atoms are forced into close proximity and, as a result, the larger the friction force."

If a student were to take the last sentence of the above passage at face value, and directly apply it to the *Greater maximum force* question, that student would likely answer it incorrectly. In contrast, an experienced physicist would realize that such a statement should be understood only in ways that agree with Newton's laws. In the next chapter of the textbook, the author offers the following refining comments:

> "(The) figure shows a person pushing (to the right) on a box with horizontal force F_{push}. If the box remains at rest, "stuck" to the floor, it must be because of a static friction force pushing back to the left. The box is in static equilibrium, so the static friction must exactly balance the pushing force... The harder the person pushes, the harder the floor pushes back. Reduce the pushing force, and the static friction force will automatically be reduced to match. Static friction acts in *response* to an applied force."

If a student were to take this passage and attempt to apply it directly to the *Greater maximum force* question, the student would probably respond correctly. However, student responses in the column labeled "After reading" in Table 5-2 were assigned the second, clarifying passage two class meetings after answering the question. Again, our purpose in presenting these passages is not to agree or disagree with the statements contained therein, but only to suggest that students probably misinterpreted some of them.

Student responses in the columns marked "After lectures and *Forces*" had been assigned all relevant reading on friction when they answered the *Greater maximum force* question the second time. The combination of lectures, textbook reading, and the tutorial *Forces* does not appear to be sufficient for most students to realize that a rougher surface does not necessarily exert a larger friction force than a smoother one.

c. The *New equilibrium* question

As shown in Table 5-3, student performance on the *New equilibrium* question appears to depend on whether students had been assigned reading introducing friction (50% correct without reading, 35% correct with reading). Student success after the reading assignment also appears to be independent of the version of the question. However, there are significant differences between the two versions in the relative popularities of the incorrect choices. Specifically, 40% of students responded that the friction force would remain the same when the incline was raised in the Incline *f* version, while 60% of students said that the friction force would remain the same when rocks were added to the bucket in the Level *f* version. Also, 25% of students stated that the friction force would decrease when the incline was raised, and only 5% said that the friction force would decrease when rocks were added to the bucket. This difference can be explained by the tendency of students to treat a static friction force and its corresponding normal force as strictly proportional. Because the normal force on the book decreases in the Incline version, and the normal force on the block remains the same in the Level version, we ought to expect more students to state that the friction force decreases in the Incline version than in the Level version, and more students to state that the friction force remains the same in the Level version than in the Incline version.

Table 5-3: Student performance on both friction versions of the *New equilibrium* question, before lectures and the tutorial *Forces*. In the columns marked "Before reading" and "After reading," the word "reading" refers to a reading assignment that included introductory remarks about friction, but did not include further clarifying remarks.

Static friction forces for lesser (1) and greater (2) opposing forces	Incline *f* version		Level *f* version
	Before reading UW 121, N=232 2 sections	After reading UW 121, N=213 2 sections	After reading UW 121, N=135 1 section
$f(2) > f(1)$ (correct)	50%	35%	35%
$f(2) = f(1)$	30%	40%	60%
$f(2) < f(1)$	20%	25%	5%

After lectures and the tutorial *Forces*, student performance appears to be very similar on both versions of the question (see Table 5-4). Students in other sections slightly outperformed students in section M, in which lectures emphasized a microscopic model of friction. The combination of lectures and the tutorial *Forces* appears to be very effective at helping students realize that the static friction force on an object must increase when another force that is opposing the friction is increased, and the object remains at rest.

Table 5-4: Student performance on both friction versions of the *New equilibrium* question after lectures and the tutorial *Forces*.

Static friction forces for lesser (1) and greater (2) opposing forces	Incline *f* version		Level *f* version
	UW 121, N=103 1 section	UW 121, N=157 Section M*	UW 121, N=215 2 sections
$f(2) > f(1)$ (correct)	80%	70%	80%
$f(2) = f(1)$	10%	20%	10%
$f(2) < f(1)$	5%	10%	<5%

*Lectures in this section emphasized a microscopic model of friction.

d. The *Third law* question

On first inspection, students appeared to perform quite poorly on the *Third law* question; only about 20% of the students correctly described the direction of the friction force on the bottom object by the top object (down and to the right). 45% of students described a direction opposite the correct one (up and to the left). We do not take this comparison very seriously, however, since it appeared that many students did not take (or have) the time to read the question carefully. That is, many students seemed to be trying to describe the direction of the friction force on the top object by the bottom object. Of course it is not clear exactly which of these two questions each student intended to answer. What cannot be explained similarly, and is therefore definitely of research interest, is the tendency of students (35%) to state that there was no friction force on the bottom object by the top object.

Table 5-5: Student performance on the *Third law* question after lectures and the tutorial *Forces*.

Direction of friction force on bottom object by top object	UW 121, N=260 2 sections
Down and to the right (correct)	20%
Up and to the left	45%
No such force	35%

7. Discussion of patterns in student reasoning

a. Tendency to relate friction force directly to contact area

Of those students who stated that the friction force would increase with greater area of contact, most did not explain why they saw a direct relationship between contact area and friction. The presence of student responses that do not explain in detail is, in general, quite common. However, in this case, almost all students spoke about the relationship in similar terms: friction force is directly related to contact (or "surface") area. The following three examples illustrate this trend:

> "There is more surface area to create friction."
>
> "The friction force in experiment 2 is greater than in experiment 1 because the block on its side has a greater surface area, so the greater area causes more area for friction on the surface, because more of the block is touching the surface."
>
> "It is greater because in experiment 2, the block is laid on its side so there is more surface area rubbing on the other surface. More surface contact means more friction."

We have also observed that, in a lecture setting, asking the class to explain *why* they think more surface area corresponds to a greater friction force tends to evoke silence. We believe the reasoning many students are using is as simple as "contact is required for friction, so more contact (what is required) leads to more friction (result)."

b. Tendency to explain friction microscopically

Some students relate the roughness of a surface to a microscopic view of surface area, either by focusing on either a greater or lesser effective surface area for a rougher surface, as illustrated in the following three responses:

> "With the sandpaper, the surface area is greater compared to the flat surface creating a greater frictional force."
>
> "Once again there is more surface area."
>
> "There is less surface area being touched by the same amount of box, so the pressure per unit of area will increase with the same amount of block, making it harder to slide (because) each unit of box area has more pressure against the sandpaper than the other surface."

Other students relate the friction force to the idea of atoms:

> "Sandpaper is a rougher surface, which means there are more atoms forced into a larger friction force."
>
> "The friction force is greater because the rougher the surface is, the more these atoms that work against each other between the object and the surface are forced into close proximity, which results in larger friction force."

Students' microscopic explanation of friction forces is reminiscent of students' microscopic explanations of the temperature increase of an ideal gas during adiabatic compression,[2] in which molecules that are pushed together collide more often and "create heat."

c. Tendency to treat static friction forces as dependent on the coefficient of friction and normal force only

Many students seem to think of the relationship $f=\mu N$ as a *definition* of friction force, rather than as a guide for thinking about a particular type of force, which follows the same general rules as other types of forces.

> "If we take the equation Force of Friction = mu * Normal Force, then in this case, mu would be much greater while the Normal Force is equal to what it was in experiment 2. This change would give a higher friction force than the previous experiment."
>
> "By using the equation f= Mu*(Fn), the roughness of the surface (Mu) acts proportionally towards the friction. Since the sandpaper has higher Mu, the friction is greater than the previous experiment."

d. Tendency to treat friction as a "one-way" interaction

When answering the *Third law* question, many students give responses that violate Newton's third law. In some of these responses, it is apparent that friction, as understood by some students, is not a *mutual* interaction between objects, but more like a one-way influence that can be exerted *on* this object by that object, or vice versa, but not both. Students often talk about "friction" as something that one object gives to another, in a way that excludes the "giving back" of friction from the second object to the first.

> "There is no such force because the bottom book is the one exerting the friction force on the top book."

> "There is no frictional force on the bottom book by the top book because any frictional force between the books would be on the top book by the surface of the bottom book."

> "The top book cannot exert a friction force on the bottom book. Only the object on the bottom exerts a friction force on the object on top, not the other way around."

It may be that students treat friction this way because of the idea that static friction's "purpose" is to prevent motion, combined with the idea that "preventing motion" is not a symmetric relationship that objects share. Since the bottom book prevents the top book from sliding down, but the top book does not prevent the bottom book from sliding down, students may not see a need for a corresponding friction force (though, from a Newtonian point of view, the top book "prevents" the bottom book from sliding up the incline). For example:

> "If the question is asking if there is a friction force exerted by the top book onto the bottom book that inhibits the lower book from traveling down the ramp, then there is not."

8. Comparison of performance of introductory students and graduate student TAs

a. Administration of questions to graduate students

The Incline *f* version of the three questions was administered in sequence to a group of graduate students in a seminar on the learning and teaching of physics. The questions were given as a pretest to the grad students just before working through the tutorial *Forces*.

Table 5-6: Comparison of success of students in introductory course to that of graduate students in a teaching seminar for the Incline *f* version of the *Contact area, Greater maximum force,* and *New equilibrium* questions.

Percentage of students answering question correctly	Introductory students UW 121, N=318 After lectures and *Forces*	Graduate teaching seminar UW 501, N=22 Before *Forces*
Contact area	85%	95%
Greater maximum force	45%	75%
New equilibrium	80%	75%

As shown in Table 5-6, performance of the introductory students on the *Contact area* and *New equilibrium* questions compares favorably with that of the graduate students. However, the graduate students performed much better than the introductory students on the *Greater maximum force* question. In the next section, we describe different assessments, with which student understanding of static friction appears better than it does here, after the same instruction.

B. STUDENT UNDERSTANDING OF TENSION ON POINT-LIKE OBJECTS IN STATIC EQUILIBRIUM AND ITS EFFECT ON QUESTIONS ABOUT STATIC FRICTION

1. Description of contexts in which questions about static tension were asked

In order to compare student treatment of tension forces with that of friction forces, we constructed "*T*" versions of the *Greater maximum force* and *New equilibrium* questions. (We found no natural analog for the notion that static friction forces increase with contact area.) Each *T* version (Incline or Level) of each question is meant to correspond very closely to the *f* version (Incline or Level) of that question. These contexts are depicted in Figure 5-3.

a. Incline *T*

The Incline *T* context is meant to be very similar to the Incline *f* context, except replacing the friction force in question with a tension force. In this context, a cart with "very good wheels" is on an incline. A thread is tied to the cart and to a post at the top of the incline. When the thread is taut, it is parallel to the incline. Two changes are made to this setup, in sequence: first the thread is replaced by very strong (yet like the thread, low-mass) fishing line; second, the higher end of the incline is raised, increasing the angle between the incline and the horizontal. In all cases, the cart remains at rest.

b. Level *T*

In the Level *T* context, a block, string (string A), peg, bucket, and rocks are arranged as in the Level *f* context. However, in this context, the bottom of the block has been coated with very slippery grease. Instead of a friction force holding the block in place, a second string (string B) is tied to the left side of the block, extending back to a wall. Two changes are made serially to this setup: string B is replaced by a stronger string (string C), and rocks are added to the bucket. Students are told that the maximum tension that the "stronger string" can have without breaking is greater than that of the weaker string. In all cases the block remains at rest.

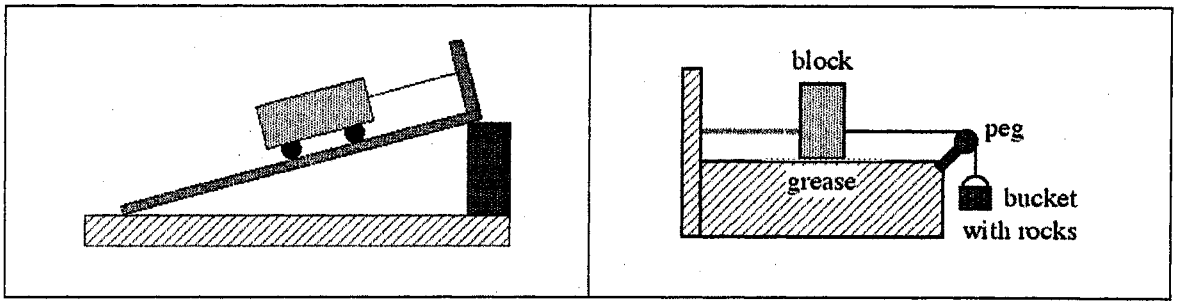

Figure 5-3: The "Incline *T*" (left) and the "Level *T*" (right) contexts.

2. Description of questions about static tension

The following questions (and the corresponding experimental changes) are depicted in Figure 5-4.

a. The *Greater maximum force* question

Both the Incline *T* and Level *T* versions of the *Greater maximum force* question are based on the notion that the maximum tension that a string (or similar object) can exert without breaking and the maximum friction force that two surfaces can exert on each other before they slip are similar concepts. That is, the tension in a string cannot exceed a certain maximum amount, but otherwise, its value will be whatever it must be in order to be consistent with Newton's laws. This reasoning is also how we would like students to think about static friction. Because we had observed in the "3 descending objects" context (described in the previous chapter) that students tended slightly more toward Newtonian reasoning when the constraint force was tension than when it was friction, we expected that students would more readily apply this "responsive force" reasoning to tension forces than to friction forces. We sought to observe the extent of this difference in equilibrium contexts. Also, if students were to work a friction version of a question immediately after working a tension version of that same question, we expected that students would be more likely to answer the friction version of the question correctly, since some students would recognize the opportunity to re-apply "responsive force" reasoning. Therefore, we also wanted to observe how many students would take this cue.

In the Incline *T* version of the *Greater maximum force* question, students compare the tension force on the cart by the light thread to that by the strong fishing line. In the Level *T* version, students compare the tension in string B to that in string C.

b. The *New equilibrium* question

In the Incline *T* version of the *New equilibrium* question, the incline is raised to a higher angle, and students compare the tension force on the cart after the change to that before. In the Level *T* version, rocks are added to the bucket, and students compare the tension force on the block after the change to that before.

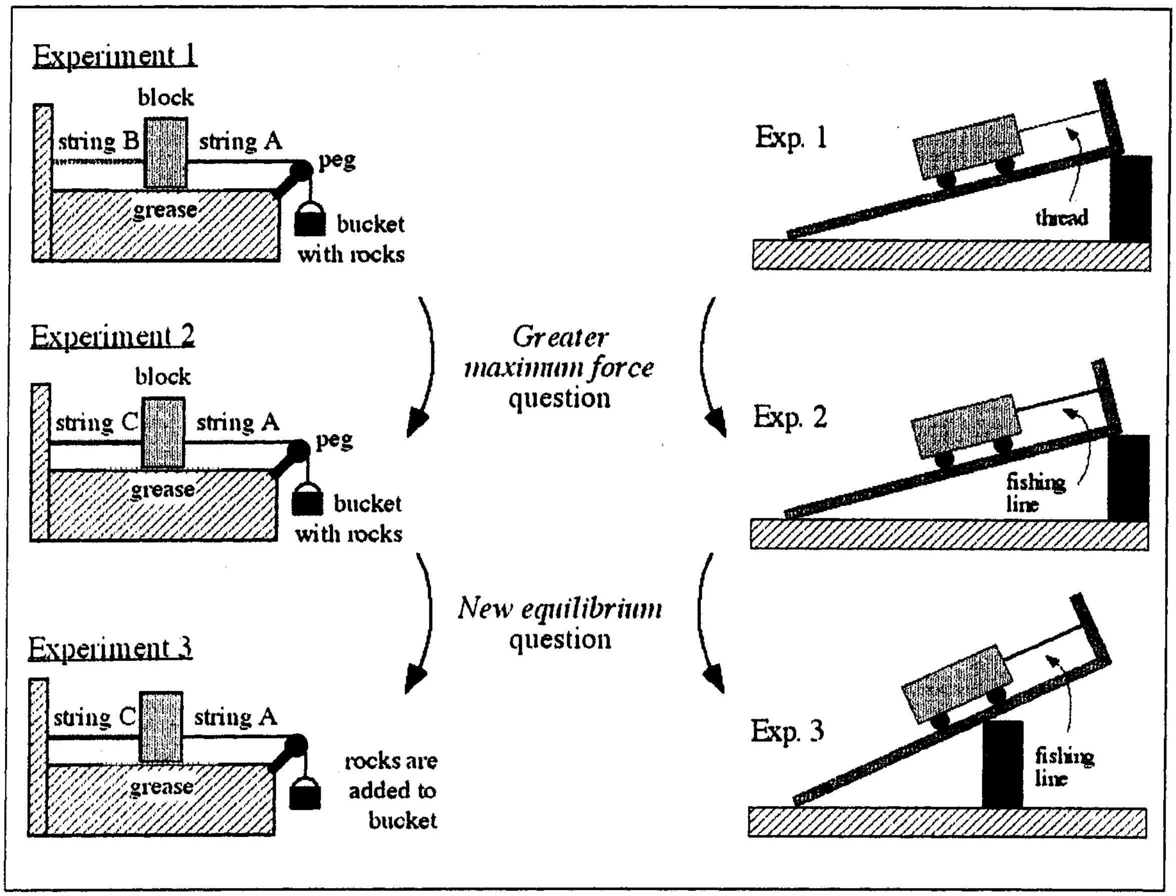

Figure 5-4: Summary of changes made to the system and the corresponding questions about static tension for the "Incline *T*" and "Level *T*" contexts.

3. Correct answers to questions about static tension

In both tension versions of the *Greater maximum force* question, the fact that the string *could* exert a greater tension force does not mean that it does in any particular case. The other forces on the object do not change in any way, so the tension force also does not change. In both tension versions of the *New equilibrium* question, the tension force must increase, just as the friction force increased in both friction versions of question.

4. Administration of questions

The Level *T* versions of the *Greater maximum force* and *New equilibrium* questions were given to three sections of students in Physics 121 at UW, before any lectures or tutorial instruction on forces. Performance on the questions did not depend on whether the students had been assigned reading that introduced tension and modeled a rope as molecular balls, connected by springs.

After students in one ("121D044") of the three sections answered both of the above questions, they were immediately asked the Level *f* version of the *New equilibrium* question. Students in the other two sections ("121A052" and "121B052") were immediately asked the Level *f* version of the *Greater maximum force* question. All of the questions described above were asked as part of an online, un-graded quiz. One week later, in the next online quiz, students in sections 121A052 and 121B052 then answered the Incline *T* versions of the *Greater maximum force* and *New equilibrium* questions, followed immediately by the Incline *f* version of the *Greater maximum force* question. During that week, the students had three lectures on forces, and had worked through the tutorial *Forces*. Student performance did not appear to depend on whether students had been assigned reading on forces, so these results are combined.

5. Overview of student performance

a. The *Greater maximum force* question

First, in Table 5-7, we compare student performance on tension questions to results presented earlier from friction questions, without the influence of tension on friction. Students were much more successful on the Level T version of the *Greater maximum force* question (75% correct) than on the Level f version (15% correct). In fact, if we compare performance on the tension version before *any* instruction on forces (75% correct) to student performance on the friction version after much instruction on forces (45% correct, see Table 5-2), students still are much more successful on the tension version. This fact suggested to us that students' "natural" understanding of tension in static situations might in some manner be a more valuable resource for learning about friction forces than formal instruction on friction *per se*.

Table 5-7: Comparison of student performance on Level T and Level f versions of the *Greater maximum force* question at identical stages of instruction (introductory reading assignment on friction and tension, before lectures or tutorial *Forces*). Students who answered friction questions were not asked tension questions on the same pretest.

Static tension forces for lesser (1) and greater (2) maximum force	Level T version UW 121, N=335 3 sections	Level f version UW 121, N=348 3 sections	*Static friction forces for lesser (1) and greater (2) maximum force*
$T(2) = T(1)$ (correct)	75%	15%	$f(2) = f(1)$ (correct)
$T(2) > T(1)$	10%	80%	$f(2) > f(1)$
$T(2) < T(1)$	10%	5%	$f(2) < f(1)$

Table 5-8 shows the effect of instruction on student performance on questions about tension. Assuming student performance on T versions of the *Greater maximum force* question does not depend on which version is used, performance appears to improve slightly (from 75% correct to 85% correct) after lectures and the tutorial *Forces*.

Table 5-8: Comparison of student performance on the Level *T* (before lectures and tutorial *Forces*) and Incline *T* (after lectures and tutorial *Forces*) versions of the *Greater maximum force* question.

Static tension forces for lesser (1) and greater (2) maximum force	Level *T* version Before *Forces* UW 121, N=335 3 sections	Incline *T* version After *Forces* UW 121, N=245 2 sections	*Static tension forces for lesser (1) and greater (2) maximum force*
T(2) = *T*(1) (correct)	75%	85%	*T*(2) = *T*(1) (correct)
T(2) > *T*(1)	10%	5%	*T*(2) > *T*(1)
T(2) < *T*(1)	10%	5%	*T*(2) < *T*(1)

As shown in Table 5-9, asking tension versions of the *Greater maximum force* question appears to affect how students treat static friction. With no instruction on forces, the effect is weak (compare 30% correct without *T* (see Table 5-2) to 40% with *T*), but the effect is slightly stronger after lectures and *Forces* (45% correct without *T* (see Table 5-2), 65% correct with *T*). Performance is best with instruction and the tension questions (65% correct), but still not as good as performance on the analogous tension question with *no* instruction (75% correct (see Table 5-8)).

Table 5-9: Comparison of student performance on the Level *f* (before lectures and tutorial *Forces*) and Incline *f* (after lectures and tutorial *Forces*) versions of the *Greater maximum force* question immediately after answering corresponding *T* versions of the question.

Static friction forces for lesser (1) and greater (2) maximum force	Level *f* version Before *Forces* UW 121, N=241 2 sections	Incline *f* version After *Forces* UW 121, N=245 2 sections	*Static friction forces for lesser (1) and greater (2) maximum force*
$f(2) = f(1)$ (correct)	40%	65%	$f(2) = f(1)$ (correct)
$f(2) > f(1)$	50%	35%	$f(2) > f(1)$
$f(2) < f(1)$	5%	<5%	$f(2) < f(1)$

b. The *New equilibrium* question

As shown in Table 5-10, student performance on the Level *T* version of the *New equilibrium* question was very good, (90% correct without lectures and *Forces*, with or without reading) – much better than performance on the *f* versions of the question at similar stages of instruction (35% correct with reading, 50% correct without reading (see Table 5-3)).

Table 5-10: Comparison of student performance on Level *T* and Level *f* versions of the *New equilibrium* question at identical stages of instruction (introductory reading assignment on friction and tension, before lectures or tutorial *Forces*). Students who answered friction questions were not asked tension questions on the same pretest.

Static tension forces for lesser (1) and greater (2) opposing forces	Level *T* version UW 121, N=335 3 sections	Level *f* version UW 121, N=135 1 sections	*Static friction forces for lesser (1) and greater (2) opposing forces*
$T(2) > T(1)$ (correct)	90%	35%	$f(2) > f(1)$ (correct)
$T(2) = T(1)$	5%	60%	$f(2) = f(1)$
$T(2) < T(1)$	<5%	5%	$f(2) < f(1)$

As shown in Table 5-11, student performance on *T* versions of the *New equilibrium* question does not appear to be strongly affected by lecture and tutorial instruction (90% correct without, 95% correct with).

Table 5-11: Comparison of student performance on the Level *T* (before lectures and tutorial *Forces*) and Incline *T* (after lectures and tutorial *Forces*) versions of the *New equilibrium* question.

Static tension forces for lesser (1) and greater (2) opposing forces	Level *T* version Before *Forces* UW 121, N=335 3 sections	Incline *T* version After *Forces* UW 121, N=245 2 sections	*Static tension forces for lesser (1) and greater (2) opposing forces*
$T(2) > T(1)$ (correct)	90%	95%	$T(2) > T(1)$ (correct)
$T(2) = T(1)$	5%	5%	$T(2) = T(1)$
$T(2) < T(1)$	<5%	<5%	$T(2) < T(1)$

As shown in Table 5-12, student performance on the Level *f* version of the *New equilibrium* question (70% correct on *f*, with *T*) also appears to have been affected by the presence of the tension version that preceded it (50% correct on *f*, without *T* (also in Table 5-3)). However, the effect was not so great that it was better than that of lectures and tutorials (80% correct on *f*, with lectures and tutorials (see Table 5-4)).

Table 5-12: Comparison of student performance on the Level *f* version (just after answering Level *T* version) and the Incline *f* version (without *T* version) of the *New equilibrium* question, before lectures, reading, or the tutorial *Forces*.

	Level *f* after Level *T*	Incline *f* without *T*
Static friction forces for lesser (1) and greater (2) opposing forces	UW 121, N=94 1 section	UW 121, N=232 2 sections
$f(2) > f(1)$ (correct)	70%	50%
$f(2) = f(1)$	25%	30%
$f(2) < f(1)$	<5%	20%

6. Discussion of patterns in student reasoning

a. Strong tendency to treat magnitude of tension force as determined primarily by Newton's 2nd law before any instruction

As shown in many of the tables in the previous section, a large majority of students are successful on questions about static tension forces, even before any instruction in Physics (UW) 121. When students explain their correct responses, they tend to express their reasoning as if it were common sense, rather than an application of an abstract formal principle. For example, these are responses from three students to the Level *T* version of the *Greater maximum force* question:

> "String C has a greater resistance to snapping but does not exert more force on the block by itself."
>
> "The mass of the bucket has not changed between experiments, so the forces remain the same. Just because string C can take more force before breaking doesn't mean that simply putting it into the experiment will create more force."

> "Even though one string is much stronger than the other, the block in both pictures is doing the same thing: remaining at rest. So in both pictures the tension force has to be the same, it is a force equal to the force by string A but in the opposite direction."

Similarly, when students responded to the *New equilibrium* question:

> "The heavier bucket in experiment 3 causes the force on string C to be greater because the bucket is pulling harder on the block because it is heavier and so the block is pulling harder on string C."

These examples show that students are able to use Newtonian reasoning (or something very similar) in basic equilibrium situations with confidence and ease. In parts C and D of this chapter, we describe curriculum that we constructed as an attempt to elicit not only the skills exhibited above, but also the attitude that to use such skills is sensible.

b. Tendency to transfer correct reasoning from tension context to friction context

On both the *Greater maximum force* and *New equilibrium* questions, more students answered the friction version correctly after being asked the tension version than did without the tension version. In most cases, students who answered correctly did not explain that they answered by thinking about the tension version. Thus for any single correct response, it cannot be demonstrated that a basic Newtonian explanation was given *because* of the presence of a similar question about tension forces. However, in some cases, students explicitly related the cases. The following two examples are taken from the *Greater maximum force* question.

> "This is similar to (tension forces in) experiments 1 and 2. The friction forces are equal because they balance the same force that the rocks create."
>
> "The force of friction is changed by changing the mass of rocks in the bucket. If this is unchanging then the friction force is equal in experiment 4 and 5. This is much like the difference between string B and string C. The block can now take more force without moving, a higher potential of sorts. Though this is independent of the amount of force on the block due to friction."

Similarly, in responding to the *New equilibrium* question, some students explained:

> "The friction force is like the tension force on string C. ...The force increases when you add rocks until the string breaks or the friction of the sand paper breaks loose."

> "Much like in (the previous question), when rocks were added it increased the force that the string pulls on the block with. In order for the block to stay at rest in each occasion there must be a force in the opposite direction from the string and equal in magnitude. So because there is more force from the string in experiment 5, then simply because the object remains at rest it must be because the friction force in the opposite direction is greater."

Other students did not refer explicitly to the previous tension version of the question, but used language that suggests that their thinking may have been influenced by the ideas of "potential" or "maximum." These responses come from the *Greater maximum force* question.

> "Since the weight in the bucket did not change, the force exerted on the block by the weight of the rocks is the same. Since the block did not move in experiment 4 with a plain surface, then the addition of sand paper only gives a great potential for friction force but doesn't increase the actual force itself."
>
> "(The block and the surface) are being pulled apart by the same tension so they should have the same frictional force, while the block on the sandpaper would be able to withstand a higher frictional force."
>
> "They are the same because the block is not moving in either experiment, though experiment 5 has a higher maximum amount of friction that it can handle."

If we think of those students who worked both versions of the question as being "instructed" to treat friction like tension, the effect of this instruction lasted long enough (~5 minutes) to show in the performance on the friction version of the question on that same online quiz, but not so long that it showed on their midterm (12-14 days later). That is, one section of students worked a pretest in which they were cued to treat friction and tension similarly; they later worked Level *f* versions of the *Contact area, Greater maximum force,* and *New equilibrium* questions (without cueing by tension questions) on a midterm exam and performed typically, as in Table 5-6. Still, though, we suppose that student performance on isolated friction questions might be more affected if we sent a stronger signal to the students that encouraged them explicitly to make the connection between friction and tension. The next section describes an attempt at doing so.

C. DESCRIPTION OF INITIAL VERSION OF INTERACTIVE TUTORIAL LECTURE *FRICTION AND TENSION*

This section describes a specific example of a general type of instructional document called an "interactive tutorial lecture" (ITL). Typically, the aim of an ITL is to simulate the tutorial experience for students in a large classroom environment. Students are given a worksheet, similar in appearance to a tutorial, and work in groups by discussing their answers with their neighboring students. The role of the lecturer in this case is to explain briefly the purpose of the worksheet as a whole (or of particular parts), to direct students' attention to sections of the tutorial ("Now work on parts A and B, and talk to your neighbors"), to have discussions with small groups about their ideas, to engage the class in a large-group discussion at the conclusion of each major section (possibly reviewing the answers to each section), and to direct in-class votes if the instructor chooses, or if the ITL is designed to include votes.

1. Overview of instructional sequence

This section describes the motivation behind the design of the initial version of the interactive tutorial lecture worksheet *Friction and Tension.*

Friction and Tension (*F&T*) was designed to take advantage of students' apparently spontaneous use of a Newtonian concept of static tension to encourage them to take a similar approach to static friction. *F&T* makes frequent use of an in-class voting system, in which students use infrared transmitters to cast individual anonymous[3] votes on the answers to multiple-choice questions posed by the instructor. For each vote, students simultaneously start discussing the question with their neighbors and casting votes. Under these conditions, students may influence each other, and each student may change his/her vote up to 2 times, until the voting is closed. Immediately after students finish voting, both the students and the instructor view the distribution of choices made by the class. Not only does the initial version of *F&T* have students vote on many (seven) questions, it also includes tables in which students record the results of each vote. Our intent in encouraging the students to record the results was that the students would have some record that they (as a group) changed how they were thinking about a problem. Because students view the results of their votes, we consider these data to be part of the instruction, and they are presented in the next section.

Another component we considered important in the design of *F&T* was the instructional value of student explanations offered vocally to the entire class. After each vote takes place, the instructor calls on the class to offer reasoning for their choices. Just after the results of the vote are shown, and students have recorded these results, the instructor asks something like "Would somebody who said 'greater than' like to say why they chose that?" After a student answers, the instructor may ask for a representative from a different choice; or if the instructor is aware that students may give distinct lines of reasoning for a particular option, the instructor may ask "Did anyone have a *different* reason for choosing 'greater than?'" The instructor may choose to ask the class if any person would like to respond to any particular explanation, either to add to it, or to argue against it, *etc*. (These student explanations tended to be very similar to students' responses to written questions, which are characterized in earlier parts of this chapter.)

F&T contains two re-votes, for which students vote a second time on a friction question, after having considered an analogous tension question. Therefore, after students vote the first time on such a question, the instructor does not identify or explain the correct answer. (To do so would be to undermine the central principle of the design of *F&T*, which is that many students are capable of applying reasoning about tension to questions about friction, when the problems are juxtaposed, without additional help.) After students have reconsidered the friction question and re-voted, it would then be appropriate to explain the correct answer to the students, if the instructor chooses. The students should be able to see a large shift in the distribution of responses toward the correct answer; thus, the re-vote, combined with volunteered student explanations, will speak for themselves, for many students. For the remaining students, it may be helpful for the instructor to confirm the correct approach explicitly.

All of the questions in the first three sections of *F&T* use the incline contexts: the friction questions are the Incline *f* versions of the *Contact area, Greater maximum force,* and *New equilibrium* questions; the tension questions are the Incline *T* versions of the *Greater maximum force* and *New equilibrium* questions. Two fictional students perform the experiments: "Sean" performs the friction experiments, and "Mila" performs the tension experiments. Students alternately consider the friction experiments and the tension experiments, so that they can contrast how they think about a friction experiment, before and after having thought about an analogous tension experiment.

2. Detailed description of instructional sequence and administration

This section describes in more detail the initial version of the interactive tutorial lecture worksheet *Friction and Tension,* how it was administered to one section of UW 114, and the aggregate student responses to in-class votes during that administration.

Students in this section did not take a pretest for *F&T*. Before working through *F&T,* they had worked through the tutorial *Forces* in standard tutorial sections (~20 students in each section).

a. Relationship between static friction forces and contact area

F&T began by having the students think about the Incline *f* version of the *Contact area* question. Students discussed the answer with each other, recorded their own answers and explanations, and voted. After the vote, the instructor asked for students to volunteer explanations. Table 5-13 shows the results of the vote.

Table 5-13: Results of an in-class vote on the "Incline *f*" version of the *Contact area* question during the interactive tutorial lecture *Friction and Tension.* Percentages are presented here (rounded to the nearest 1%) as they appeared to the students.

Comparison of friction forces for less (1) and greater (2) contact area	UW 114, N~75 1 section
$f(2) = f(1)$ (correct)	60%
$f(2) > f(1)$	38%
$f(2) < f(1)$	1%

b. Relationship between static friction forces and coefficient of static friction

This section of *F&T* began by having students consider the Incline *f* version of the *Greater maximum force* question. After the students voted on and discussed the *f* version of the *Greater maximum force* question, they turned the page and began thinking about the *T* version of the same question (in which light thread is replaced with stronger fishing line). Students discussed the question and voted. Continuing through *F&T*, students read that the package of fishing line was marked with the label "20 lbs. test strength." Students discussed whether the number 20 indicated how much the fishing line was pulling in the experiment they considered. Students were then asked to reconsider the *f* version of the *Greater maximum force* question. They discussed with each other again, and re-voted. Table 5-14 shows the results of all three votes.

Table 5-14: Results of in-class votes on the "Incline *f*" version of the *Greater maximum force* question, followed by the "Incline *T*" version, which was followed by a re-vote on the "Incline *f*" version.

Comparison of tension/friction forces for greater (2) or (1) less maximum possible force	UW 114, N~75 1 section		
	Incline *f* version (first vote)	Incline *T* version	Incline *f* version (second vote)
F(2) = *F*(1) (correct)	42%	96%	88%
F(2) > *F*(1)	55%	0%	11%
F(2) < *F*(1)	3%	4%	1%

After voting the second time on the Incline *f* version of the *Greater maximum force* question, students discussed the following question: "Suppose that the coefficient of static friction between the book and the sandpaper is 0.75. Suppose also that the normal force on the book by the incline is 8N. How would you interpret the number 6 in this instance?" The instructor suggested the number 6 should be understood as the maximum friction force (in Newtons) that the sandpaper could exert on the book for the given normal force and friction coefficient.

c. Relationship between static friction and changes in other forces on an object

The structure of this section of *F&T* is very similar to the previous section: students discussed and voted on the *f* version of the *New equilibrium* question, then the *T* version of the *New equilibrium* question, and then re-voted on the *f* version. Table 5-15 shows the results of the votes.

Table 5-15: Results of in-class votes on the "Incline *f*" version of the *New equilibrium* question, followed by the "Incline *T*" version, which was followed by a re-vote on the "Incline *f*" version.

Comparison of tension/friction forces for steeper (2) and (1) less steep incline	UW 114, N~75 1 section		
	Incline *f* version (first vote)	Incline *T* version	Incline *f* version (second vote)
$F(2) > F(1)$ (correct)	73%	96%	89%
$F(2) = F(1)$	16%	4%	8%
$F(2) < F(1)$	11%	0%	3%

d. Determining the existence and direction of friction forces

The final section of *F&T* was intended to give students practice identifying friction forces and their directions in more complex situations, using Newton's second and third laws. Students were shown a series of pictures, all involving block A sitting on block B. In the first four pictures, A is not slipping on B. First, a string pulls to the right on block A, and both blocks remain at rest. Second, the string pulls to the right on block B, and the blocks remain at rest. Third, the string pulls to the right on block B, and both blocks move to the right with the same constant speed. Fourth, the string pulls to the right on block B, and both blocks move to the right and slow down. Last, the string pulls to the right on block B, both blocks move to the right and speed up, and block A slips on block B.

Students were asked to determine the directions of these three forces in each case: the friction force exerted on A by B, the friction force exerted on B by A, and the friction force exerted on B by the table. They were also asked to indicate whether the friction force was static or kinetic (if non-zero).

This section was not finished in lecture, and was assigned to the students to finish outside of class.

3. Discussion of results of in-class votes

The student success rate on each tension question and on each re-vote of a friction question was very high. We expected this outcome, and we do not interpret it as an indicator of long-lasting student learning. That is, we would not predict on the basis of the voting results that success on similar questions on an exam would be higher than usual. Therefore, at this point, we also refrained from judging the success (or failure) of the instruction, until we could examine student responses to questions on an exam.

D. STUDENT UNDERSTANDING OF STATIC FRICTION AFTER INITIAL VERSION OF THE INTERACTIVE TUTORIAL LECTURE *FRICTION AND TENSION*

1. Administration of questions

The Level *f* versions of the *Contact area, Greater maximum force, New equilibrium,* and *Third law* questions were asked on a midterm exam to one section of UW 114, after lectures (including *Friction and Tension*) and the tutorial *Forces* (with associated tutorial homework).

2. Overview of student performance

The effect of the initial version of *Friction and Tension* on student understanding of static friction appears to be good, though we cannot directly compare performance by students in UW 114 to students in UW 121. UW 114 students typically perform at or slightly below the level of the UW 121 population at similar stages of instruction. The fact that the UW 114 students outperformed the UW 121 students on the most difficult question is encouraging. We predict that, if UW 121 students were to work through *Friction and Tension,* their post-test performance on each question would be better than both their own performance without *F&T* and that of UW 114 students with *F&T*. That is, we would expect UW 121 students to have success rates better than 90%/65%/80% (the higher number from each row in Table 5-16) on the typical trio of questions.

Table 5-16: Comparison of performance of UW 121 students after lectures and the tutorial *Forces* with that of UW 114 students after lectures, *Forces*, and the initial version of the interactive tutorial lecture *Friction and Tension*.

Percentage of students answering question correctly	UW 121, N=318 After lectures and *Forces*	UW 114, N=93 After lectures, *Forces*, and the initial version of *Friction and Tension*
Contact area	85%	90%
Greater maximum force	45%	65%
New equilibrium	80%	65%

Student performance on the *Third law* question is shown in Table 5-17. Only about 35% of students answered all three parts of the question correctly. Most students recognized that the friction force on A by B points to the left, but less than half correctly stated that the friction force on B by A points to the right. Almost all students who stated that the friction force on B by A was zero also stated correctly that the friction force on A by B was to the left. The number of (UW 114) students who stated that the friction on B by A was zero (20%) is noticeably different from the number of (UW 121) students who stated that the friction on the bottom book by the top book was zero (35%). This difference may be due to the difference in context (Incline or Level) in which the question was asked, whether students were asked to think about other friction forces (as they were in the Level version but not in the Incline version), or whether students worked through *Friction and Tension* (as those who worked the Level version did, while those who worked the Incline version did not). In any case, we conclude that most students do not understand how to apply Newton's third law to friction forces in static situations after working through *Forces* and *Friction and Tension*.

Table 5-17: Student performance on the *Third law* question asked in the Level *f* context after lectures, *Forces,* and the initial version of the interactive tutorial lecture *Friction and Tension.*

Block A is on top of block B. The string attached to block A is pulling to the right. Both blocks remain at rest. *Rows below show directions of three friction forces: on A by B, on B by A, and on B by the sandpaper.*	UW 114, N=93 After lectures, *Forces,* and the initial version of *Friction and Tension*
Left, right, left (correct)	35%
Left, right, zero	10%
Left, zero, zero	5%
Left, left, left	10%
Left, zero, left	10%
Total of "*f* on B by A points to the right." (correct)	45%
Total of "*f* on B by A is zero."	20%

After further reflection on the initial version of *F&T,* we decided that it was somewhat unrefined. In particular, at no point in the initial version of *F&T* were students explicitly asked to state in their own words how to answer a question about static friction by reflecting on (and using) the reasoning they would use if the friction force were instead a tension force. Nor were they invited to apply such a generalized procedure to a new problem. Instead, the initial version assumed that students would learn this procedure by implication. We expected that a revised version of *F&T* that made these steps explicit would be more successful than the initial version. The next two sections describe a revised version of *F&T,* with which we attempted to connect friction to tension more explicitly, and student understanding of static friction after the revised version.

E. DESCRIPTION OF REVISED VERSION OF INTERACTIVE TUTORIAL LECTURE *FRICTION AND TENSION*

This section describes a revised version of the interactive tutorial lecture *Friction and Tension,* and the aggregate student responses to in-class votes in one particular administration (to another section of UW 114) of the revised *F&T.* A copy of this revised version of *F&T* can be found in Appendix D of this dissertation.

1. Overview of modifications to instructional sequence

After using the initial version of *F&T* with one lecture section of students in UW 114, we decided that the "moral" (that students should treat static friction forces and static tension forces with similar logic) should be made more explicit. We also thought students could use more guidance in connecting a correct understanding of static friction to the familiar relation $f=\mu N$. In order to make room in *F&T* for these additions, both re-votes were cut. Thus, the total number of votes was reduced from 7 to 5. The final section "Determining the existence and direction of friction forces" was unchanged.

2. Detailed description of modified instructional sequence and administration

Students who worked through the modified version of *Friction and Tension* had worked through the tutorial *Forces* in a manner similar to that in standard tutorial sections, except that students were in a large lecture hall, with 3 tutorial instructors. In contrast to the previous administration of *F&T,* this group of students worked a paper pretest for *F&T* at the beginning of the same lecture period in which they worked through *F&T.*

The basic structure and problem contexts of the revised version are the same as that of the initial version. The revised version begins with a vote on the Incline *f* version of the Contact *area* question. The results of the vote are shown in Table 5-18, along with the students' performance on the same question, asked on a paper pretest that students worked at the beginning of class.

Instead of moving directly to the Incline *f* version of the *Greater maximum force* question (as we did in the initial version), we asked students to draw a free-body diagram for the book when it was in its initial orientation. At this point the instructor elicited suggestions from the students about how to draw the free-body diagram: how many forces there were, their directions, their magnitudes, *etc.* The instructor chose to spend much time (in pieces) on the details of this diagram, returning to it and refining it in small steps as students offered comments throughout the progression of ideas and questions in *F&T*. Much of this discussion took place directly after the vote on the *Contact area* question. Students discussed how they knew how large to draw the arrow representing the friction force, and whether (or how) the free-body diagram would be different when the book is lying on a different side. Students were then asked explicitly to reflect on whether they would change their answers to the *Contact area* question after having thought about the free-body diagram for the book.

Table 5-18: Comparison of student performance on the "Incline *f*" version of the *Contact area* question on the paper pretest with that on the in-class vote, taken minutes later.

Comparison of friction forces for less (1) and greater (2) contact area	UW 114, 1 section	
	N=60 Paper pretest	N=52 In-class vote*
$f(2) = f(1)$ (correct)	40%	54%
$f(2) > f(1)$	50%	35%
$f(2) < f(1)$	10%	12%

*Percentages in this column rounded to nearest 1% (instead of the usual 5%).

Students were then asked to vote on the "Incline *f*" version of the *Greater maximum force* question. As discussion continued, the instructor often referred students back to the free-body diagram drawn on the board, asking individual students how (and if) they had been thinking about the diagram and what changes, if any, they might make to it when the sandpaper was added. Table 5-19 shows how students responded to this question, on the paper pretest and on the vote. After asking students to explain their reasoning for the vote, the instructor continued directly to the next vote: the "Incline *T*" version of the *Greater maximum force* question, the results of which are shown in the far right column of Table 5-19. (Students reacted audibly with some combination of dismay and amusement when the instructor proceeded to the tension question without confirming the correct approach to the friction question. At this point, he assured the students, "I promise, the answer will be made clear before we are finished.") Student thinking about the *Greater maximum force* question does appear to have been influenced (20% correct on the pretest, ~60% correct on the vote) by the preceding discussion, which focused on the free-body diagram. (The instructor had not yet stated that the free-body diagram was the proper tool for answering these questions, but he had expressed high interest in students' ideas about it.)

Table 5-19: Comparison of student performance on the "Incline *f*" version of the *Greater maximum force* question on the paper pretest with that on the in-class vote, taken later that hour. Also shown are the results of the in-class vote on the "Incline *T*" version of the *Greater maximum force* question, taken shortly after the vote on the "Incline *f*" version.

Comparison of tension/friction forces for greater (2) or (1) less maximum possible force	UW 114, 1 section		
	N=60 Paper pretest Incline f	N=53 In-class vote* Incline f	N=51 In-class vote* Incline T
$F(2) = F(1)$ (correct)	20%	62%	71%
$F(2) > F(1)$	65%	38%	8%
$F(2) < F(1)$	10%	0%	22%

*Percentages in this column rounded to nearest 1% (instead of the usual 5%).

Students then discussed whether the number 20 (as on the label of fishing line, "20 lbs. test strength") indicates how hard the fishing line is pulling in any particular experiment. After this, *F&T* asks students to explain how the reasoning required to answer the friction question is similar to that for the tension question. Then, students were told to suppose that the normal force on the book by the incline (or sandpaper) is 8N, while the coefficient of static friction (μ_S) is 0.75. They then read the following statement:

> "The friction force is given by $f = \mu_S N$, so the friction force in Experiment 3S is $0.75 \times 8\text{N} = 6\text{N}$."

Students discussed whether they agreed or disagreed with it. (About half of them appeared to agree.) After the instructor confirmed that the statement was incorrect, students discussed how to interpret the number 6 in the calculation. Some students explained correctly that the number 6 in this experiment should be understood similarly to the number 20 that was previously discussed. The instructor affirmed this interpretation.

Finally, students voted on the "Incline *f*" version of the *New equilibrium* question. The results of this vote are shown in Table 5-20, along with student performance on the same question on the paper pretest, which they had taken earlier that hour. There was not enough time in the period for a vote on the "Incline *T*" version.

Table 5-20: Comparison of student performance on the "Incline *f*" version of the *New equilibrium* question on the paper pretest with that on the in-class vote, taken later that hour.

Comparison of friction forces for steeper (2) and (1) less steep incline	UW 114, 1 section	
	N=60 Paper pretest	N=52 In-class vote*
$f(2) > f(1)$ (correct)	40%	73%
$f(2) = f(1)$	30%	12%
$f(2) < f(1)$	20%	15%

*Percentages in this column rounded to nearest 1% (instead of the usual 5%).

With about one minute remaining in the lecture period, the instructor attempted to summarize the lecture with the following questions, which the students answered responsively: "Does $f=\mu N$ tell you how big a static friction force actually is?" ("No!") "What tells you how big the friction force actually is?" ("The free-body diagram!")

Students were advised to finish the worksheet outside of class, and were invited to ask the instructor any questions they had about the lecture.

F. STUDENT UNDERSTANDING OF STATIC FRICTION AFTER CURRENT VERSION OF THE INTERACTIVE TUTORIAL LECTURE *FRICTION AND TENSION*

1. Administration of questions

Students who worked through the modified version of *F&T* answered the same questions as the students who had worked through the initial version (the Level *f* versions of the *Contact area, Greater maximum force, New equilibrium,* and *Third law* questions). Unlike the previous group of students, these students had worked through the tutorial *Forces* in a large lecture hall rather than in smaller (standard) tutorial sections, and had not worked through the homework associated with those tutorials.

2. Overview of student performance

As shown in Table 5-21, student performance on the *Contact area, Greater maximum force,* and *New equilibrium* questions after the modified version of *F&T* was about the same as that after the initial version. If we assume that the data are identical, we prefer the modified version to the initial version, for the following reasons. In the modified version, the connection between the questions about friction and free-body diagrams is more explicit; some of the questions are phrased more clearly, with correspondingly clearer answers; and less time is spent on voting on a question a second time, allowing more time for discussion. Furthermore, it is possible that the modified version would result in clearly higher student success, if the students who had worked through the modified version had worked through *Forces* in tutorial sections of standard size (~20 students) with more opportunity for in-depth discussion with tutorial instructors (as did the students who worked through the initial version of *F&T*).

Table 5-21: Comparison of performance of UW 121 students after lectures and the tutorial *Forces* with that of UW 114 students after lectures, *Forces,* and the both the initial and modified versions of *Friction and Tension.*

Percentage of students answering question correctly	UW 121, N=318 After lectures and *Forces*	UW 114, N=93 After lectures, *Forces,* and the initial version of *Friction and Tension*	UW 114, N=72 After lectures, *Forces,** and the modified version of *Friction and Tension*
Contact area	85%	90%	95%
Greater maximum force	45%	65%	65%
New equilibrium	80%	65%	70%

*These students worked through *Forces* as a large group in the lecture hall.

Student performance on the *Third law* question is shown in Table 5-22. Since, in both administrations of *F&T,* the lecture period expired before students reached the last page, which deals with the application of Newton's third law to friction forces (and which was the same in both versions of *F&T*), we do not believe *F&T* helped many students improve their understanding of Newton's third law. Student performance on this task would probably be more successful if *F&T* were used in its entirety, perhaps over more than one lecture period.

Table 5-22: Student performance on the *Third law* question asked in the "Level *f*" context after lectures, *Forces*, and either the initial version or the modified version of the interactive tutorial lecture *Friction and Tension*.

Block A is on top of block B. The string attached to block A is pulling to the right. Both blocks remain at rest. *Rows below show directions of three friction forces: on A by B, on B by A, and on B by the sandpaper.*	UW 114, N=93 After lectures, *Forces*, and the initial version of *Friction and Tension*	UW 114, N=72 After lectures, *Forces*,* and the modified version of *Friction and Tension*
Left, right, left (correct)	35%	20%
Left, right, zero	10%	15%
Left, zero, zero	5%	20%
Left, left, left	10%	5%
Left, zero, left	10%	5%
Total of "*f* on B by A points to the right." (correct)	45%	40%
Total of "*f* on B by A is zero."	20%	35%

*These students worked through *Forces* as a large group in the lecture hall.

G. Summary

Student understanding of static friction forces before tutorial instruction on forces (in general) tends to be dominated by microscopic ideas and the relationship $f=\mu N$. In contrast, student understanding of static tension forces before any instruction on forces tends to be Newtonian. Furthermore, many students are able to transfer a Newtonian understanding of tension forces to problems involving friction forces, if analogous problems are juxtaposed. Tutorial instruction on forces (in general) helps students treat friction forces as subject to Newton's laws, but with much room for improvement. We developed an interactive tutorial lecture that encouraged students to apply logic they naturally use for tension forces to problems involving friction forces. Student understanding of static friction appears to be further improved by relating tension and friction forces in this way.

NOTES TO CHAPTER 5

[1] Viennot, L., U. Besson, "Using models at the mesoscopic scale in teaching physics: two experimental interventions in solid friction and fluid statics," *Int. J. Sci. Educ.* **26**, 1083-1110 (2004).

[2] Loverude, M. E., C. H. Kautz, and P. R. L. Heron, "Student understanding of the first law of thermodynamics: relating work to the adiabatic compression of an ideal gas," *Am. J. Phys.* **70**, 137-148 (2002).

[3] When individual votes appear as boxes on the screen, students can recognize their own transmitter IDs, so that they know their vote has been cast. Contents of the votes do not appear to the students. The instructor receives an electronic report of the votes, in which each student and his or her choice is identified. In this sense, students' votes are "partially anonymous."

CHAPTER 6. STUDENT LEARNING OF ANGULAR MOMENTUM OF ROTATING RIGID BODIES

The initial motivation for this sub-project was to study relationships between patterns in student reasoning about linear momentum and patterns in student reasoning about angular momentum. We thought perhaps that studying student reasoning about angular momentum might help clarify our understanding of student reasoning about linear momentum. As it turns out, one of the major errors in student treatment of angular momentum is as we expected; angular quantities of motion are just as subject to being facets of an undifferentiated quantity of motion as are linear quantities of motion. Apart from this result, however, specific errors that students made when answering our questions about angular momentum tended not to have corresponding elements in the catalog of errors for linear momentum. Most notably, student responses to written questions about a common demonstration (a spinning bicycle wheel and rotatable stool) were not consistent with student thinking about linear momentum (as we understood it). Because of these results, we thought it might not be appropriate to assume students could easily transfer their skills with linear momentum to situations with angular momentum. We therefore expanded the scope of our research to include the questions: "What ideas and problems do students have when thinking about the bicycle-wheel demonstration?" and "How might instruction on angular momentum (in general) through the bicycle-wheel context (in particular) be improved through a better understanding of students' ideas about that context?"

In part A of this chapter, we describe student understanding of the angular momentum of rigid bodies that rotate with a (roughly) stationary center of mass after instruction, including initial versions of a tutorial on angular momentum conservation. In part B, we describe student understanding after all lecture instruction, but before any tutorial instruction. In part C, we describe the current version of Part 1 (which deals primarily with the angular momentum associated with "spinning" motion) of the tutorial (and homework) *Conservation of Angular Momentum.* In part D, we present evidence that the current version of

Conservation of Angular Momentum helps students understand angular momentum in the bicycle-wheel context and more abstract contexts.

A. STUDENT UNDERSTANDING OF ANGULAR MOMENTUM OF ROTATING RIGID BODIES AFTER PREVIOUS VERSIONS OF TUTORIAL *CONSERVATION OF ANGULAR MOMENTUM*

1. Description of previous versions of tutorial

a. Initial version

The initial version of the tutorial focused on helping students understand the relationships $\vec{l} = \vec{r} \times \vec{p}$ and $\Delta\vec{l} = \vec{\tau}_{net}\Delta t$ in the "spinning barbell" context (see Figure 6-1).

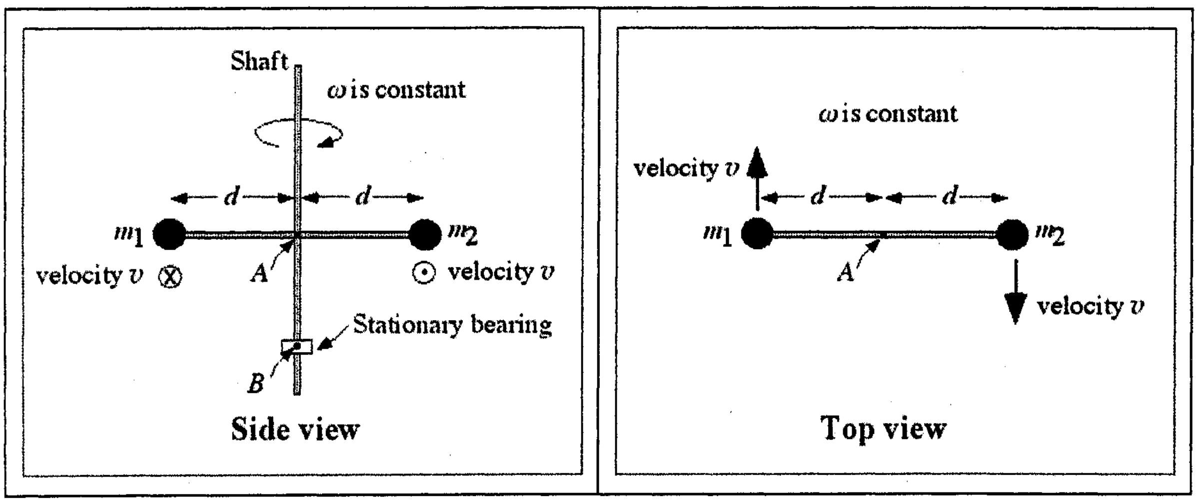

Figure 6-1: The "spinning barbell" context, as in the initial version of the tutorial *Conservation of Angular Momentum.*

In this context, two balls (of equal mass) are attached to a rigid rod, which spins with constant angular velocity around a shaft. Students are guided to determine the direction of the angular momentum of each ball with respect to point A, as seen from both a top-view and a side-view perspective. They combine these angular momentum vectors into one and determine that this vector does not change in any respect as the balls move around the circle.

In the next section, students drew a top-view free-body diagram for each ball, and determined that, though there was a non-zero net force on each ball, the net torque on each ball with respect to point A was zero. Students then found that their results were consistent the relationship $\Delta\vec{l} = \vec{\tau}_{net}\Delta t$ (specifically, 0 = 0).

The tutorial then directed students to consider point B as the reference point, instead of point A. With this reference point, the angular momentum of each ball remains the same in magnitude, but changes direction as the ball moves, tracing out a cone. Referring back to the free-body diagrams, students determine that the net torque on each ball about point B is not zero. They compare this result with the fact that the angular momentum of each ball is changing and find that it is again consistent with $\Delta\vec{l} = \vec{\tau}_{net}\Delta t$. Thus, students conclude that, though the angular momentum may be constant with respect to one reference point but not to another, the validity of the relation $\Delta\vec{l} = \vec{\tau}_{net}\Delta t$ is not reference-point dependent.

The last page of the tutorial deals with a more standard conservation problem (that is, a system with constant angular momentum, but for which the parts transfer angular momentum). As shown in the left cell of Figure 6-2, two systems, A and B, are rotating around the same axle with angular velocities of the same magnitude (ω_1) but opposite direction, when system B is dropped onto system A, allowing both systems to come to a common final angular velocity.

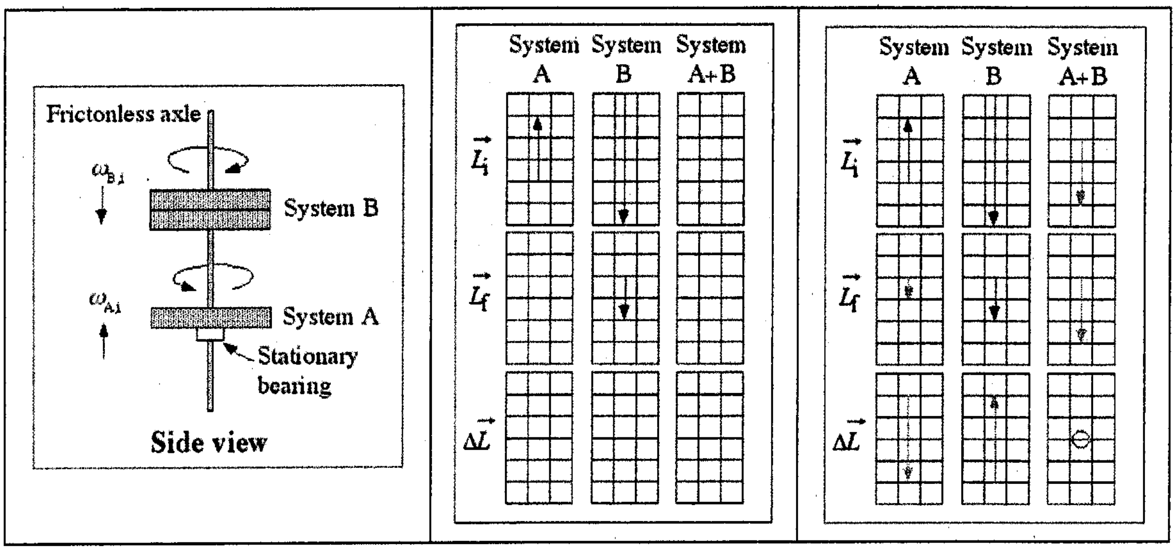

Figure 6-2: The context for the last section of the initial version of the tutorial *Conservation of Angular Momentum.* Students were guided to analyze the collision depicted at left by completing the vector diagram shown in the middle cell. The diagram is correctly completed in gray at right.

A stationary bearing prevents system A from sliding down the axle, but does not affect the rotational motion of system A. Students think about the net torque on the entire system (A+B) about the axle (it is zero) and relate it to the angular momentum of the system (it is constant) via $\Delta\vec{l} = \vec{\tau}_{net}\Delta t$. Students then conclude that the change in angular momentum of system A must be equal in magnitude and opposite in direction to that of system B, and also, therefore, that the torque exerted on system A by B is equal in magnitude and opposite in direction to that exerted on B by A. Finally, students complete the vector diagram shown in Figure 6-2 and determine the final angular speed of the system in terms of ω_i.

The homework guided students to extend their work on the "spinning barbell" context by having them consider a similar system for which the barbell is rigidly fixed to the axle, but not perpendicular to it. Thus, each ball moves in a different horizontal circle around the vertical axle. Students find that the angular momentum of each ball with respect to the midpoint of the barbell is not constant and also that the angular momentum of the two-ball system is not constant. Thus, they conclude that such motion would require a net torque about the barbell's midpoint.

b. Intermediate version A0

The initial version of the tutorial had not been based on research directed specifically at student understanding of angular momentum. When we began to do such research, we decided to modify the tutorial as part of an exploratory investigation.

The first half of Intermediate version A0 of the tutorial was very similar to that described in detail in the next chapter, under the name "Version A of Part 2 of the tutorial *Conservation of Angular Momentum*." Essentially this section sought to teach students some properties of the angular momentum of a point object moving with constant velocity, specifically, that this angular momentum is constant, generally non-zero, and generally reference-point dependent in both magnitude and direction. The following section guided students to combine this "orbital" angular momentum of an extended body with its own "spinning" angular momentum, yielding the *total* angular momentum of the body with respect to a particular point. Students found that the total angular momentum was also reference-point dependent, and that a reference point could be chosen such that the total angular momentum was zero, even if the object was spinning (if the two contributions to the total angular momentum of the body mutually cancel). Finally, students applied the conservation principle to the total angular momentum in a collision between a clay ball and a rod. In this exercise, students were guided to choose a reference point for which the initial angular momentum of the incident clay ball was zero. After the collision, when the clay ball had stuck to the rod, the system moved forward and spun, so that the total angular momentum of the system with respect to the chosen point remained zero, indicating that the "orbital" and "spin" contributions to the total angular momentum were equal in magnitude and opposite in direction.

Experience teaching the initial version and Intermediate version A0 of the tutorial suggested that both pieces of curriculum assumed a basic understanding and facility with angular momentum that many students did not have. Thus, even before we understood what students' issues were, we had some sense that the treatment of angular momentum in both versions of the tutorial was too abstract and unmotivated. In later versions of the tutorial, we sought to relate the concept of angular momentum to more concrete experiences, focusing on the question "Why do we need the angular momentum concept?"

The following sections describe contexts and questions with which we assessed student understanding after the initial version and Intermediate version A0 of the tutorial.

2. Description of contexts in which questions about angular momentum were asked

a. Dropped disk

In this context, a disk (Disk A) is spinning on a vertical frictionless axle with an initial angular velocity that is unknown in both magnitude and direction, as shown in Figure 6-3. A second disk (disk B) rotates about the same axis. Disk B is spinning with initial angular velocity of specified magnitude and directed upward (*i.e.*, it is spinning counter-clockwise, when viewed from above) before it is dropped onto disk A. (Disk A is prevented from sliding down the axle by a frictionless bearing.) After the collision, the disks rotate together with the same final angular velocity of specified magnitude, directed downward (clockwise, when viewed from above). In different versions of the context, the numerical values provided were different; however, the values were chosen so that both the correct treatment of angular momentum as a vector and the incorrect treatment of angular momentum as a scalar would yield distinct (whole-number) answers to each question for which the treatments diverged.

In both versions of this context: the rotational inertia of disk A was 1 $kg*m^2$, the initial angular velocity of disk B was 2 rad/s, directed upward (counter-clockwise, from above), and the final angular velocity of disks A and B was 2 rad/s, directed downward (clockwise, from above). The rotational inertia of disk B was 2 $kg*m^2$ in one version and 3 $kg*m^2$ in the other version.

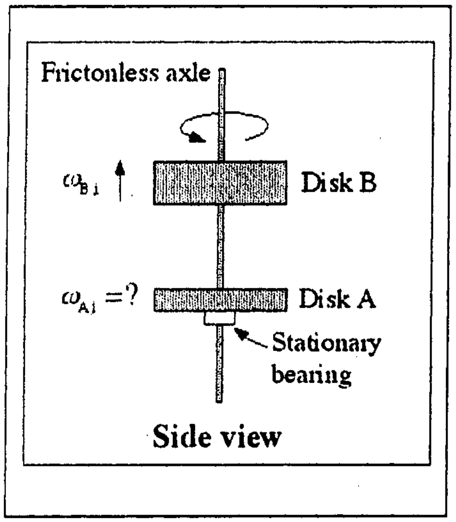

Figure 6-3: The "Dropped disk" context

b. Bicycle wheel

This context is meant to remind students of a real situation that they may have seen in lecture or elsewhere. A student is on some kind of low-friction, rotatable platform (either seated or standing). The student is handed a bicycle wheel that is already spinning counter-clockwise (when viewed from above) around a vertical axis. The wheel spins without friction.

i. 3.ETI.S

As an extension of the context described above, the student can make various changes to the motion of the wheel after it is handed to her. In "3.ETI.S," the student sits on a stool with a rotatable seat and is handed the spinning wheel (see Figure 6-4). In Experiment 1, she places her hand against the wheel, slowing the wheel to one half its initial speed. In Experiment 2, she places her hand against the wheel, bringing the wheel to a stop. In Experiment 3, she quickly flips the wheel over, so that it is spinning clockwise (when viewed from above). This context was created so that we might study similarities in student treatment of this context with the 3.ETM.S contexts from our research on student understanding of linear momentum. The name "3.ETI.S" refers to the fact that there are three experiments, or "collisions," with similar initial conditions; in each of the three experiments, the "target" object (the student and stool-seat) has approximately equal rotational inertia; and in one of the experiments (Experiment 2), the "incident" object (the wheel) "stops." Strictly speaking, Experiment 2 is more like a collision in which the objects stick together than one in which the incident object stops while the target object moves forward without it. Constructing a situation in which the final angular momentum of the wheel is exactly zero (for an arbitrary reference point) while it does not spin about its own axle would require the wheel to be at rest, rather than rotating around the stool with the student. For the most part, students treat Experiment 2 as one of "complete transfer" of motion; that is, it appears to function as we intended, as a "stop" case.

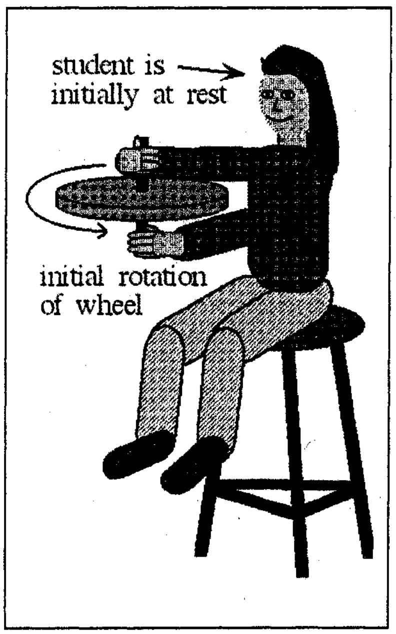

Figure 6-4: The initial state for experiments in the "3.ETI.S" context.

ii. 2.DTI.S

This context is a variation on "Bicycle wheel: 3.ETI.S" and is intended to be analogous to the context 2.DTM.S from our research on student understanding of linear momentum. In this context, two students each stand on a platform and hold a spinning wheel, as shown in Figure 6-5. The wheels are identical and spin with the same initial upward angular velocity. Each platform can rotate on its axle without friction. In Experiment 1, student 1 places her hand against the side of the wheel, bringing it to a stop. In Experiment 2, student 2 quickly flips the wheel over, so that its final angular velocity is downward. The rotational inertia of the combination of the student and platform in Experiment 1 is less than that in Experiment 2.

The final angular speed of the student and platform in Experiment 1 is greater than that in Experiment 2.

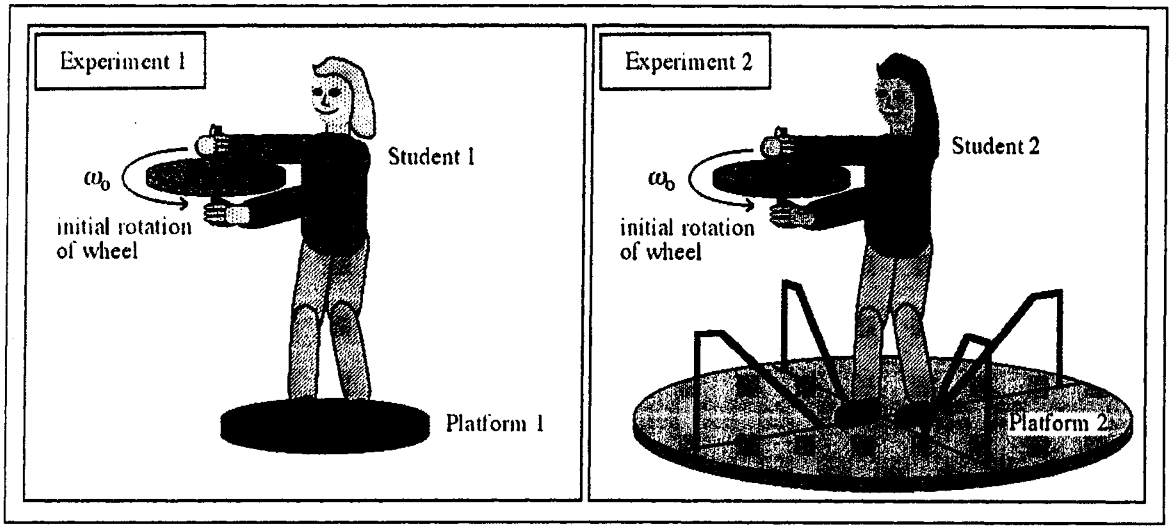

Figure 6-5: The initial state for experiment in the "2.DTI.S" context.

3. Description of questions about angular momentum

The questions described below were intended, for the most part, to correspond with similar questions from our research on student understanding of linear momentum conservation (described in Chapter 3 of this dissertation). One particular issue in constructing this correspondence stands out. Our research on student understanding of linear momentum centered on situations in which at least one object changed its direction of motion because, in such situations, treatments of linear momentum as a vector (correct) or as a scalar (incorrect) mutually diverge. We found it much less natural to pose physical situations in which one object changed the direction (or sense) in which it was *rotating*, while keeping its rotation on a single axis at every instant. The contexts described above and the questions described below represent our attempts to reach a compromise between the two goals of asking questions about (1) (relatively) realistic situations and (2) situations conceptually parallel to those involving linear momentum.

a. The *Angular momentum transfer* question

This question was asked in both the "3.ETI.S" and "2.DTI.S" versions of the "Bicycle wheel" context. When asked in the "3.ETI.S" version, students were asked to consider

"Experiment 1," in which the student slows the wheel to half its initial angular speed; in the "2.DTI.S" version, students were asked to consider "Experiment 1," in which the student brings the spinning wheel to a stop. In both cases, we asked the question "Is the magnitude of the change in angular momentum of the wheel greater than, less than, or equal to the change in angular momentum of the *combination* of the student and platform (or seat)?" We intended the *Angular momentum transfer* question to be as similar as possible to the *Momentum transfer* question (see Chapter 3).

b. The *Initial angular speed* question

Both the *Initial angular speed* and *Final angular speed* questions are intended to test whether students can treat angular momentum as a directed (rather than scalar) quantity, just as we used the *Final speed* question to assess student understanding of linear momentum. The *Initial angular speed* question was asked in the "Dropped disk" context. Because disks that spin on the same axis and rub against each other interact primarily inelastically (tending toward the same state of rotational motion), the disks do not "rebound" off each other, in a rotational sense. Thus, "rebounding" was not available as a means to test student understanding of angular momentum as a directed quantity. Therefore, we chose to make the initial angular velocity of one of the disks unknown to the students, so that "scalar" and "vector" approaches to solving for this unknown resulted in distinct (whole-number) answers. The question itself was "Determine the magnitude and direction of the initial angular velocity of disk A. Show your work and explain your reasoning."

c. The *Final angular speed* question

This question was asked only in the "3.ETI.S" version of the "Bicycle wheel" context. Students were asked to rank Experiments 1, 2, and 3 according to the final angular speed of the student. We intended this question to be very similar to the *Final speed* question, especially as asked in the "3.ETM.S" context (see Figure 3-2).

d. The *Final angular momentum* question

This question was asked only in the "2.DTI.S" version of the "Bicycle wheel" context. Students were asked "Is the magnitude of the final angular momentum of student 1 and platform 1 greater than, less than, or equal to the magnitude of the final angular momentum of student 2 and platform 2?" We intended the *Final angular momentum* question to be very similar to the *Final momentum* question, especially as asked in the "2.DTM.S" context (see Figure 3-3).

4. Correct answers to questions about angular momentum

a. The *Angular momentum transfer* question

In each case, the gravitational force on the system (of wheel, student, and platform) and the net upward normal force on the system are collinear, so that each force makes zero torque on the system about its center of mass. If the net torque on a system about some point is zero, then the angular momentum of the system about that same point remains constant. When the angular momentum of a system does not change, and we think of the system as consisting of two parts, then the changes in angular momentum of those two parts must be equal in magnitude and opposite in direction. Thus, the answer to the *Angular momentum transfer* question is that the magnitude of the change in angular momentum of the wheel is equal to that of the student and platform (or seat).

b. The *Initial angular speed* question

If the rotational inertia of disk B is 2 $kg*m^2$, then the final (and initial) angular momentum of the system is (1 $kg*m^2$ + 2 $kg*m^2$)×(2 rad/s, down) = 6 $kg*m^2/s$, down. Since the initial angular momentum of disk B is (2 $kg*m^2$)×(2 rad/s, up) = 4 $kg*m^2/s$, up, then the initial angular momentum of disk A must be 10 $kg*m^2/s$, down (that which, when added to the initial angular momentum of disk B, equals the initial angular momentum of the system). Since the rotational inertia of disk A is 1 $kg*m^2$, the initial angular velocity of disk is 10 rad/s, down (clockwise, from above).

Similarly, if the rotational inertia of disk B is 3 $kg*m^2$, the initial angular velocity of disk A is 14 rad/s, directed down (clockwise, from above).

c. The *Final angular speed* question

The final angular speed of the student is greatest in Experiment 3 because the angular momentum transfer between the wheel and the student is greatest. This is because the *change in angular momentum* vector of the wheel is greatest in Experiment 3 (one unit downward (final) – one unit upward (initial) = two units upward (change)). By similar reasoning, the change in angular momentum of the wheel in Experiment 2 is one unit downward, and that in Experiment 1 is one-half unit downward, so the correct ranking of the experiments by final angular speed of the student is 3 > 2 > 1. Alternatively, the total angular momentum of the system (initial and final) for all three experiments is one unit upward. In order for the system to maintain this angular momentum vector, the student's (and seat's) angular momentum must be greater in the upward direction the more downward (or the less upward) the final angular momentum of the wheel is. When the final angular momentum of the wheel is one unit downward, the student and seat must have angular momentum two units upward so that the system maintains a total angular momentum of one unit upward.

d. The *Final angular momentum* question

The reasoning required here is similar to that for the *Final angular speed* question. Regardless of the rotational inertia or final angular speed of the student and platform, flipping the wheel over corresponds to greater angular momentum transfer than does stopping the wheel. Thus, if the wheel has initial angular momentum of one unit upward, the student and platform in Experiment 1 have angular momentum one unit upward, while the student and platform in Experiment 2 have angular momentum two units upward. Therefore the magnitude of the angular momentum of the student platform combination in Experiment 1 is less than that in Experiment 2.

5. Administration of questions

Each of the following questions was administered to students on a final exam, after all instruction, including either the initial version or Intermediate version A0 of the tutorial *Conservation of Angular Momentum*.

6. Overview of student performance

As shown in Table 6-1, student performance on the *Angular momentum transfer* question appears not to depend on whether the wheel is slowed to half its speed or brought to a stop. In either case, the more common incorrect response was that the change in angular momentum of the wheel was greater than that of the rest of the system. This response is consistent with a confusion between angular momentum and angular velocity, since, assuming that the student (and platform) has greater rotational inertia than the wheel, the magnitude of the change in angular velocity of the wheel will be greater than that of the student (and platform).

Table 6-1: Student performance on the *Angular momentum transfer* question asked in both the 3.ETI.S and 2.DTI.S versions of the "bicycle wheel" context after all instruction, including Intermediate version A0 of *Conservation of Angular Momentum.*

Wheel is slowed either to half its initial speed (3.ETI.S) or to a stop (2.DTI.S).	UW 121, N=139 1 section 3.ETI.S	UW 121, N=133 1 section 2.DTI.S
$\left\|\Delta\vec{L}_{wheel}\right\| = \left\|\Delta\vec{L}_{student,etc.}\right\|$ (correct)	70%	70%
$\left\|\Delta\vec{L}_{wheel}\right\| > \left\|\Delta\vec{L}_{student,etc.}\right\|$	25%	20%
$\left\|\Delta\vec{L}_{wheel}\right\| < \left\|\Delta\vec{L}_{student,etc.}\right\|$	5%	<5%

As shown in Table 6-2, less than half (40%) of the students treated angular momentum as a directed quantity when working a fairly straightforward conservation problem (that is, with only "spinning" angular momentum). About one quarter of the class gave a particular numerical answer to the *Initial angular speed* question that corresponded to treating angular momentum like a scalar quantity. A large minority (35%) gave other answers, with no single answer being given by more than ~5% of the students. We believe this means that many students do not have any conceptual understanding of angular momentum, either correct or incorrect (as with those who think of it as a scalar). This result motivated us to help students relate the abstract concept of angular momentum to more concrete experiences.

Table 6-2: Student performance on the *Initial angular speed* question in the "Dropped disk" context after all instruction, including the initial version of the tutorial *Conservation of Angular Momentum.*

Initial angular velocity of disk A is unknown in magnitude and direction.	UW 121, N=185 1 section
Vector response (correct)	40%
Scalar response	25%
All other responses	35%

About the same number who gave the (correct) vector response to the *Initial angular speed* (see Table 6-2) question gave the vector response to the *Final angular speed* question in the "Bicycle wheel" context (see Table 6-3). Similarly, about one quarter gave a scalar response to both questions. Student responses to the *Final angular speed* question gave us one of our first clues that students' issues with angular momentum tended to be distinct from those with linear momentum; the next most common incorrect response was not a "reverse" response (as described in Chapter 3), but a "matching" response. ("Matching" responses were not at all common in questions about linear momentum.) These students gave a ranking for the final angular speed of the student that "matched" the ranking of the final angular speed of the wheel. In some of these responses, students seemed to be either misreading the question (as asking for a ranking of the wheel's final angular speeds) or perhaps ranking the wheel's speeds because they did not know how else to approach the problem. However, in other cases, students appeared to believe that the wheel's angular velocity at any instant *determined* the student's angular velocity at that same instant (or shortly thereafter). In some responses, the student appeared to be thinking the angular velocities of the wheel and student were opposite in direction; in others, they were the same direction. These responses inspired the construction of the *Handoff* question, described in the next section.

Table 6-3: Student performance on the *Final angular speed* question asked in the "3.ETI.S" version of the "Bicycle wheel" context after all instruction, including Intermediate version A0 of the tutorial *Conservation of Angular Momentum.*

Student slows wheel to ½ its initial angular speed in Exp.1, stops the wheel in Exp. 2, and quickly flips the wheel over in Exp. 3. ($\omega(X)$ is the final angular speed of the student in Exp. X.)	UW 121, N=305 2 sections
$\omega(3) > \omega(2) > \omega(1)$ (correct)	45%
$\omega(2) > \omega(1) > \omega(3)$ ("scalar" response)	20%
$\omega(3) > \omega(1) > \omega(2)$ ("matching" response)	15%

Student performance on the *Final angular momentum* question is shown in Table 6-4. Frequencies of correct and scalar responses are similar to those with other questions about angular momentum, described above. However, the "compensation" response is slightly more popular in this context (40%) than in linear momentum contexts (~20-30%, see Table 3-15). This result suggested to us that, after Intermediate version A0 of *Conservation of Angular Momentum,* more students understood angular momentum primarily in terms of the formula $L=I\omega$ than understood linear momentum primarily as $p=mv$ after the current version of *Conservation of Momentum in One Dimension.* In this sense, then, we believed that the linear momentum tutorial was more effective than Intermediate version A0 of the angular momentum tutorial in helping students understand the purpose and uses of the (linear or angular) momentum concept.

Table 6-4: Student performance on the *Final angular momentum* question asked in the "2.DTI.S" context after all instruction, including Intermediate version A0 of the tutorial *Conservation of Angular Momentum.*

Student 1 stops the wheel in Exp. 1. Student 2 quickly flips the wheel over in Exp. 2. *(L(X) is the final angular momentum of the student and platform in Exp. X.)*	UW 121, N=133 1 section
$L(2) > L(1)$ (correct)	45%
$L(2) < L(1)$ ("scalar" response)	15%
$L(2) = L(1)$ ("compensation" response)	40%

7. Discussion of patterns in student reasoning

a. Tendency to treat angular momentum as a scalar quantity

When students treat angular momentum as a scalar quantity, their reasoning tends to look very similar to reasoning that we have seen in linear momentum contexts. The following three examples are taken from the *Final angular speed* question in the 3.ETI.S version of the "Bicycle wheel" context.

> "Case 2 is the greatest since all angular momentum is transferred from the wheel to the student. In case 1, only half the angular momentum is transferred to the student. Case 3 is zero since angular momentum just did a 180. Even though it is upside down, it has no change in L."

> "When the student stops the wheel, the momentum is transferred to the student. When the student only slows down the wheel, it only transfers some of the momentum. When the student flips the wheel, barely any momentum is transferred."

> "Momentum is conserved in this case, so when she stops the wheel from rotating, the momentum transferred to her, making her go faster than case 1 or 3. Case 3 is the lowest because she didn't take any momentum from the wheel, just changed its rotation."

Occasionally, however, a student seems to use reasoning that he/she would probably not use when thinking about linear momentum:

> "(The student does not rotate in Experiment 3) because no angular velocity has been lost, since ω is not a true vector."

We do believe that it is appropriate for instructors to talk about the "artificial" vector nature of angular quantities at the introductory level. It is also probably wise to expect that some students may come to false conclusions when thinking about such an abstract concept.

b. Tendency to conflate angular velocity and angular momentum

We presented evidence in Chapter 3 that many students treat linear momentum as if it were linear velocity. Since angular quantities are more abstract than their (analogous) linear counterparts, we expected that many students would also have difficulty separating the concepts of angular velocity and angular momentum. The following three examples are taken from the *Angular momentum transfer* question, asked in the "Bicycle wheel" context.

> "($\left|\Delta\vec{L}_{wheel}\right| > \left|\Delta\vec{L}_{student,etc.}\right|$) because the wheel was spinning so much faster, when it stops, it must reduce angular velocity by much greater."
>
> "($\left|\Delta\vec{L}_{wheel}\right| > \left|\Delta\vec{L}_{student,etc.}\right|$) You would think that they are equal because the system conserved angular velocity, but the inertia of the wheel must be taken into account when calculating the angular momentum of the student and platform."
>
> "The change in angular momentum of the wheel would be greater than the change in magnitude of the student because the wheel goes from spinning to rest, while the student will only begin to rotate a very small amount."

In the second response above, the student appears to treat the words *velocity* and *momentum* as distinct, but exchanges them for each other. In the last quotation, the student refers to the "amount" or "magnitude" of the motion, without specifying whether *angular velocity* or *angular momentum* is the quantity meant, probably because the student did not understand the difference between them.

c. Tendency to use "compensation reasoning"

Like the 2.DTM.S context described in Chapter 3, the 2.DTI.S context was designed to measure students' tendency to use compensation reasoning, and thus, in a sense, the degree to which students thought of angular momentum primarily in terms of a formula: $L=I\omega$. The following three responses are taken from the *Final angular momentum* question.

> "They are equal to each other, since $L=I\omega$, $I_1 < I_2$ and $\omega_1 > \omega_2$, momentum is conserved, differences are proportional."
>
> "They are equal because of the conservation of angular momentum. If both wheels start out with the same ω_0, and if there is no net force, the final angular momentum is the same. In this case $I_1 < I_2$ but $\omega_1 > \omega_2$, so these two values in each case compensate for the different platforms. So the final value of angular momentum remains the same."
>
> "Equal. $L=I\omega$, Since the $I_1 < I_2$ but $\omega_1 > \omega_2$, it should cancel out and the final L should be equal. The bigger the I, the ω will get smaller because it gets harder to rotate."

The first and second responses above use the idea (or phrase) of momentum conservation as support for the claim that the products $I_1\omega_1$ and $I_2\omega_2$ are equal. The second student seems to have some significant understanding of momentum conservation and its relationship to net force (or net torque), but did not pay sufficient attention to each system's interacting parts. The third student seems to have a good intuitive understanding of what is represented by the symbols I and ω, but again, fails to see the system as composed of interacting parts.

All of these responses may have been encouraged by the common equation $I_1\omega_1 = I_2\omega_2$, which usually accompanies the demonstration (or textbook example) in which a person spinning on a swivel seat changes his/her moment of inertia by holding dumbbells at arm's length and pulling them in closer. That is, $I_1\omega_1 = I_2\omega_2$ is not an expression of angular momentum conservation for the system in 2.DTI.S, but it is a correct expression of angular momentum conservation for the "dumbbell" system. Some students may have taken the relation to be equivalent to angular momentum conservation in every case.

8. Discussion of student performance on combinations of questions

We were interested to observe any relationships between how the same students performed on questions about linear momentum and "analogous" questions about angular momentum. For instance, it is conceivable that, since some of the errors students make are common to both contexts, we might be able to predict how an individual student will perform on an angular momentum task if we know how that student performed on the linear momentum task. We administered two pairs of "analogous" questions to a total of two sections of UW 121. Each section worked the linear momentum task on their third midterm exam (after the current version of *Conservation of Momentum in One Dimension)* and the angular momentum task on the final exam (after Intermediate version A0 of *Conservation of Angular Momentum)*. One section worked the *Final speed* question in the 3.ETM.S context and the *Final angular speed* question in the 3.ETI.S context (see Table 6-5). The other section worked the *Final momentum* question in the 2.DTM.S context and the *Final angular momentum* question in the 2.DTI.S context (see Table 6-6).

Table 6-5: Performance of students in one section on the *Final speed* question asked in the 3.ETM.S context after the current version of the tutorial *Conservation of Momentum in One Dimension,* with that of the same students on the *Final angular speed* question asked in the 3.ETI.S context after Intermediate version A0 of the tutorial *Conservation of Angular Momentum.*

UW 121 N=134	*"L" question*				
"p" question	Vector	Scalar	Reverse	Matching	Total
Vector	32%	14%	-	14%	82%
Scalar	5%	-	-	-	6%
Reverse	-	-	-	-	6%
Matching	-	-	-	-	-
Total	42%	16%	-	18%	

Percentages less than 5% are not shown.

Table 6-6: Performance of students in one section on the *Final momentum* question asked in the 2.DTM.S context after the current version of the tutorial *Conservation of Momentum in One Dimension,* with that of the same students on the *Final angular momentum* question asked in the 2.DTI.S context after Intermediate version A0 of the tutorial *Conservation of Angular Momentum.*

UW 121 N=117	*"L" question*			
"p" question	Vector	Scalar	Compensation	Total
Vector	28%	11%	17%	55%
Scalar	8%	-	5%	15%
Compensation	10%	5%	15%	30%
Total	46%	18%	37%	

Percentages less than 5% are not shown.

We conclude that, in general, individual student performance cannot be predicted from these data any more accurately than if the midterm and final exams had been taken by two separate groups of students. That is, supposing that we are confident that 46% of students will give the "vector" response to the *Final angular momentum* question, to what extent would it help to know how that student responded to the *Final momentum* question? Without knowing how that student performed on the "p" question, we would say that a random student has 46% chance of answering the "L" question correctly. Knowing that a particular student answered the "p" question correctly, we would say that the student has conditional probability (28%÷55%) 51% of answering the "L" question correctly. We do not consider 51% a significant improvement in certainty from 46%. From this we conclude that students who are successful on linear momentum tasks do not understand angular momentum any better than the class as a whole.

Other patterns (of smaller magnitude) are interesting to note. Students who gave a scalar response to either "p" question tended not to give a second scalar response when answering the corresponding "L" question. Perhaps most students who made this error took note when they received their graded midterms. In contrast, most students who give a scalar response to an "L" question answered the "p" question correctly. It may be that it is to most students' advantage to make the "scalar mistake" in a memorable way at some point. Also curious is that students who gave the compensation response to the "p" question seem not have been strongly discouraged from giving a similar response to the "L" question.

From a broader perspective, we conclude the more obvious: many students do not easily transfer skills learned in linear momentum contexts to angular momentum contexts. We believe that most students probably can be taught to treat linear momentum and angular momentum with similar logic, but probably not with instruction as implicit as that in Intermediate version A0 of *Conservation of Angular Momentum.*

B. STUDENT UNDERSTANDING OF ANGULAR MOMENTUM OF ROTATING RIGID BODIES BEFORE TUTORIAL *CONSERVATION OF ANGULAR MOMENTUM*

1. Prior instruction

At the time that students take the pretest for the tutorial *Conservation of Angular Momentum,* they have often been introduced to the angular momentum concept in lecture and seen demonstrations of (1) a person holding dumbbells and changing his/her moment of inertia, while sitting on a rotating seat, (2) a person rapidly turning the handles of a spinning bicycle wheel while sitting on a rotating seat, thereby changing his/her rotational motion back and forth, and/or (3) the precession of a spinning bicycle wheel, when supported from one handle only. In those cases in which we believe such demonstrations had a significant effect on how students answered the pretest questions, we describe what happened in the demonstration and why we think it had an effect.

2. Description of additional question contexts

a. Bicycle wheel

Student responses to the *Final angular speed* question in the Bicycle wheel 3.ETI.S context suggested to us that students may think of a wheel's spinning motion as itself having potential causal influence on the spinning motion of other objects. In order to investigate this idea, we designed the "Handoff" contexts described below.

A student sits at rest on a stool. The seat of the stool can rotate without friction. A wheel that is already spinning is handed to the student, who touches only the handles of the wheel. The wheel spins without friction.

i. Handoff (V)

In Case A, when the spinning wheel is handed to the student, the axle of the wheel is vertical and spins counter-clockwise (when viewed from above).

ii. Handoff (H)

In Case B, the axle of the wheel is horizontal, and the top of the wheel is moving away from the student.

Handoff (H) never appeared to the students without being accompanied by Handoff (V), which, in some cases, appeared without Handoff (H), because it was designed first. In those cases where students thought about both vertical and horizontal initial orientations of the wheel, it was emphasized to the students that, regardless of what they thought might happen in Case A, the student was again initially at rest in Case B, before being handed the wheel in the new orientation.

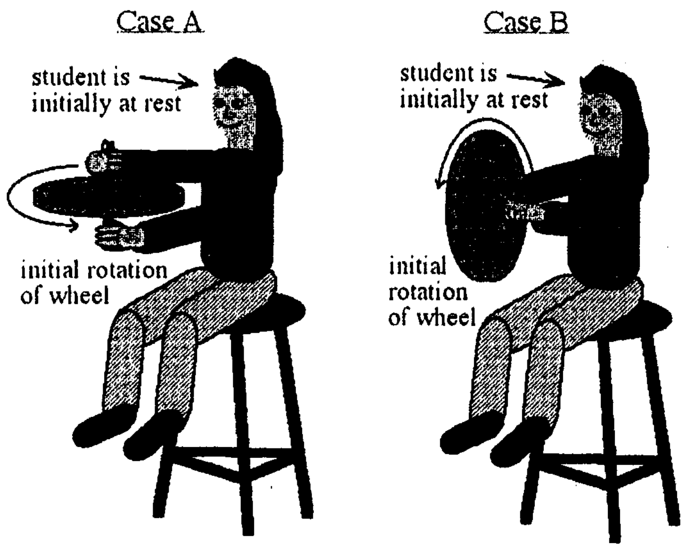

Figure 6-6: Handoff (V) ("Case A") and Handoff (H) ("Case B") from the "Handoff" context.

b. Pendulum

This context was inspired by the trouble some astronomy instructors report with teaching students that the axis of rotation of the earth keeps (approximately) the same orientation in space throughout one year. This fact is a consequence of the conservation of angular momentum, and we sought to observe to what extent students understood this in a less celestial context. A similar phenomenon can be observed in the context described below.

A thin wheel is suspended from its center by a string, which is attached to the ceiling. The wheel is first made to spin in a horizontal plane, around its axis of symmetry. The string is then lifted to the side and released so that the wheel swings like a pendulum.

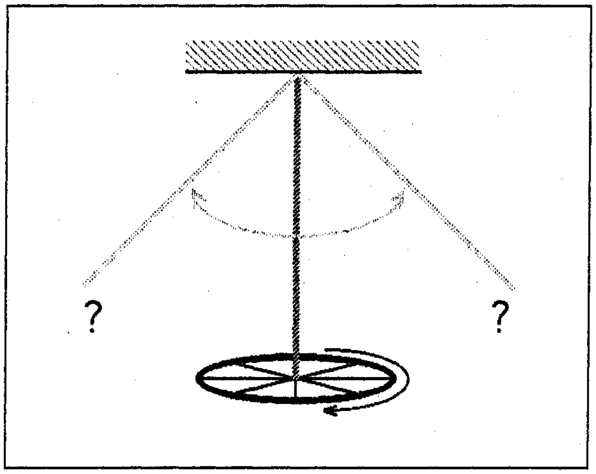

Figure 6-7: The "Pendulum" context.

3. Description of additional questions

a. The *Handoff* question

This question was asked only in the "Handoff" context. By asking students this question, we intended to observe student tendency to treat the motion of a spinning wheel (as distinct from *changes* in the wheel's motion) as having its own causal influence on the motion of the person holding the wheel. We asked students to state whether the student will begin to rotate counter-clockwise, rotate clockwise, or remain at rest when she holds the spinning wheel. In most cases, the question was asked twice, once for the vertical orientation of the wheel's axis, and once for the horizontal orientation. In a minority of cases, a single multiple-choice question was asked. This version listed the following (or equivalent) options: "A: stay at rest; B: counter-clockwise;" "A: counter-clockwise; B: stay at rest;" "A: clockwise; B: stay at rest;" "A: stay at rest; B: stay at rest;" or "none of these."

b. The *Non-local conservation* question

This question was asked in both the "Handoff" contexts and the "Pendulum" context. In the "Handoff" contexts, we asked the students whether the angular speed of the spinning wheel would increase, decrease, or remain the same while the student held the wheel in the orientation described. Because some students tended to state that the student would start to spin when she held the wheel in one orientation or the other, we wanted to see whether students thought that an increase in the motion of the student should be accompanied by a decrease in the motion of the wheel. When these questions were introduced, we added an extra remark, which explained that the phrase "the wheel spins without friction" means that when the wheel is spinning and not interacting with other objects, it does not slow down on its own. This remark was added because we were concerned that students might state that the wheel slowed down because any real wheel would slow down eventually, due to friction in its axle. The name *Non-local conservation* refers to the tendency among students to require motion to decrease in one place if they think it is increasing in another place, even if there is no means (linear impulse, work, or angular impulse) by which the quantity of motion can be transferred.

In the "Pendulum" context, the *Non-local conservation* question asks students whether the angular speed of the wheel when it is at a higher point of its swinging path (point A) is greater than, less than, or equal to the angular speed of the wheel when it is at its lowest point (point B). Here the idea is that students may want to say that the spinning of the wheel should slow down when it reaches a higher place, since rotational kinetic energy is converted to gravitational potential energy, even though there is no mechanism (work) for this particular energy transfer.

c. The *Orientation of spin* question

This question was asked only in the "Pendulum" context. Two diagrams were drawn for the students; each diagram showed the possible orientation of the spinning wheel at different points on the path of its pendulum motion. As shown in Figure 6-8, one diagram ("Figure 1") showed the wheel to have the same horizontal orientation in space during the pendulum motion. In the other diagram, ("Figure 2"), the wheel spun in a plane that changed orientation throughout the motion so that it was always perpendicular to the taut string that supported the wheel. Students were asked which diagram better represented how the spinning wheel would appear during the swinging motion. Students could also state that neither of the diagrams was accurate.

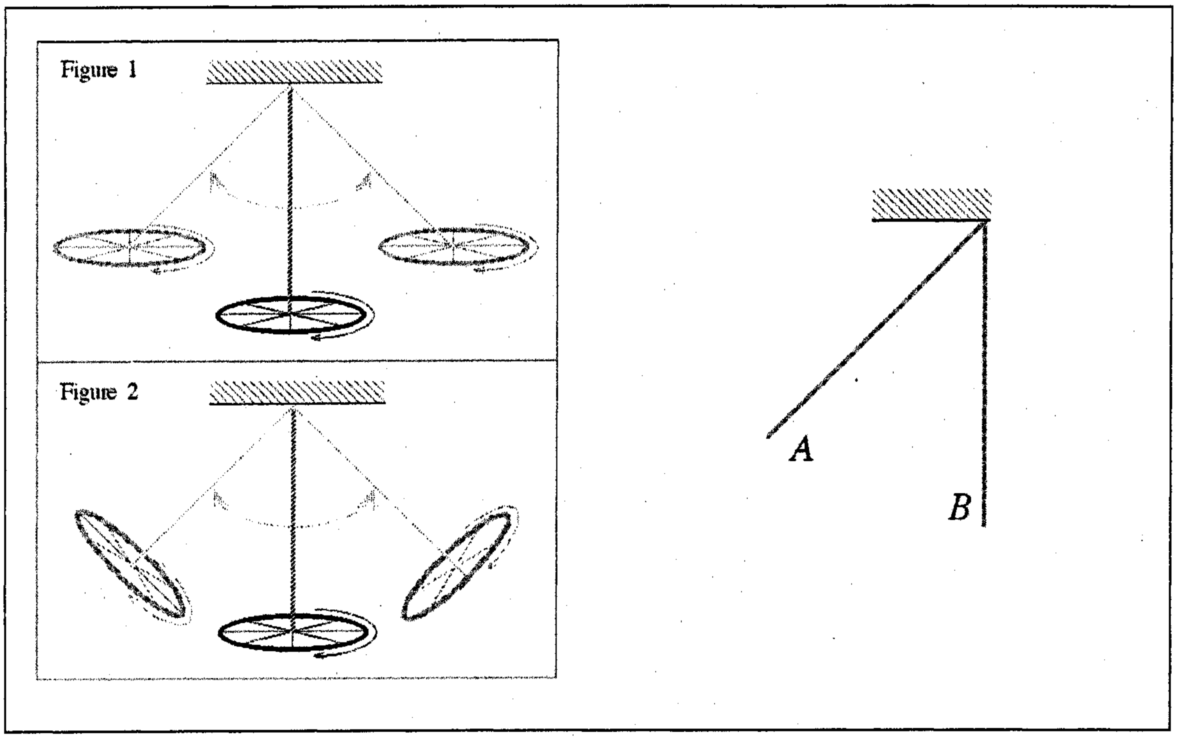

Figure 6-8: Diagrams that accompanied the *Orientation of spin* question (left) and the *Non-local conservation* question (right) in the "Pendulum" context.

4. Correct answers to additional questions

a. The *Handoff* question

The student remains at rest, regardless of the orientation of the wheel handed to her. If the axle of the wheel is not changed in direction, and if the speed of the wheel is not altered, then the angular momentum of the wheel remains the same. If the angular momentum of the wheel remains the same, then there is no angular momentum transfer from the wheel to the student.

b. The *Non-local conservation* question

In the "Handoff" contexts, there is no mechanism that transfers the angular momentum (or rotational kinetic energy) of the wheel to the student, thus there is no way for the angular speed of the wheel to change. It is true that a real spinning wheel would slow down only in so far as it has friction in its axle, meaning that the correct answer is that the angular speed of the wheel remains the same when such friction is negligible. However, our main interest in asking the question was not to test whether students could answer correctly, but to observe whether students were concerned with applying a rough conservation of motion principle. We would not consider it more (or less) correct if a response stated both that the angular speed of the wheel remains the same and that the student starts to spin than if a response stated that the angular speed had to decrease because the student started to spin.

In the "Pendulum" context, the angular speed of the wheel remains the same as the wheel moves from a low point of the swing to a high point. To change the angular momentum of the disk (about its own center of mass) in either magnitude or direction would require a net torque on the disk (about its center of mass). The only forces applied to the wheel are a gravitational force (applied as if at the center of mass), a tension force (applied at the center of mass), and, since we are analyzing the motion of the wheel in an accelerated reference frame (the center-of-mass frame), an "*-ma*" pseudo-force (applied at the center of mass). None of these forces exert non-zero torques on the wheel, so the wheel spins with the same angular momentum of constant magnitude and direction.

c. The *Orientation of spin* question

The orientation of spin must remain the same (as in "Figure 1" of Figure 6-8), since all torques on the disk about its center of mass are zero, as explained above.

5. Administration of questions

All of the questions described in this section were administered to students as part of an online pretest. Each student took the pretest at some time during the weekend before the last (tenth) week of the quarter. In some cases, the instructor had lectured on angular momentum, in other cases, not. We describe the lecture instruction in those cases for which we think lectures affected student performance significantly.

6. Overview of student performance

Table 6-7 shows student performance on the *Handoff* question after lecture instruction but before tutorial instruction. The data in the first column of Table 6-7 are what we consider typical; most of the sections to which we administered the *Handoff* question are shown here. The instructor of the 2 sections in the second (middle) column of the table was aware that students had a tendency to state that a person would begin to rotate in Case A, and he performed the relevant demonstration in lecture before the pretest. Notice that students in these sections had a greater tendency to state that the student remains at rest in Case A (65% for that column, compared to the more typical 35%), but were slightly less successful with case B (60% for that column, but typically 70%). This result suggests that students may be more likely to predict motion in Case B after they have observed that there is no motion in Case A, perhaps because they know that "something does happen," though they may not understand what conditions are necessary for motion to occur.

Students in the section in the far right column of Table 6-7 were significantly less successful with Case A but more successful than usual with Case B. After we observed this result, we determined that the instructor of that section had performed this demonstration prior to the pretest. The instructor began with the spinning wheel as in Case B, was observed not to spin, rotated the axis of the wheel himself to a vertical orientation, and was observed to begin rotating in the direction opposite the rotation of the wheel. If students remembered the demonstration when they were taking the pretest, we would expect them to associate the first phase of the demonstration with Case B (correctly) and the second phase with Case A (incorrectly). Notice that the tendency of students in that section to state that the person would rotate in a direction opposite that of the wheel in Case A is augmented (50%, compared to the typical 25%).

Table 6-7: Student performance on the *Handoff* question after lecture instruction but before the tutorial *Conservation of Angular Momentum.*

Axle of the wheel is vertical in Case A (L points up) and horizontal in Case B (L points to the student's left).	UW 121, N=733 8 sections	UW 121, N=204 2 sections*	UW 121, N=112 1 section‡
A: rest, B: rest (correct)	15%	30%	5%
A: counter-clockwise, B: rest	30%	10%	25%
A: clockwise, B: rest	25%	20%	50%
A: rest, B: counter-clockwise	15%	20%	5%
A: rest, B: clockwise	5%	15%	5%
Total of A: rest (correct)	35%	65%	15%
Total of B: rest (correct)	70%	60%	80%

*Both of these sections had the same instructor, who demonstrated "Case A" of the *Handoff* question in lecture before the pretest.

‡The instructor in this section performed a demonstration that many students mistook for the experiments described in the pretest.

Student performance on the *Non-local conservation* question in the "Bicycle wheel" context is shown in Table 6-8. We may think of students who said that the angular speed of the wheel would decrease in both cases (10%) as failing to notice or take seriously the idealization that "the wheel does not slow down on its own." What cannot be explained similarly is that 30% of students said that the wheel would slow down in only one case. That this is the most common incorrect response is consistent with the fact that most students respond that the student will begin to rotate in either Case A or Case B, but not both. The bottom two rows of Table 6-8 show a relationship between student responses to this question and to the *Handoff* question. For both Case A and Case B (considered individually), about 45% of those students who said that the student would rotate also said that the wheel would slow down, while only 15% of those who said that the student would remain at rest said that the wheel would slow down. (Thus, it may be that 15% of all students are neglecting the idealization of the frictionless axle.) Therefore, some students probably see a connection between the angular speed of the wheel and the motion of the student (regardless of how the wheel is oriented).

Table 6-8: Student performance on the *Non-local conservation* question asked in the "Bicycle wheel" context after lecture instruction but before the tutorial *Conservation of Angular Momentum.*

'ω' is the angular speed of the wheel about its own center.	UW 121, N=542 6 sections
ω remains the same in both cases. (correct)	40%
ω decreases in one case and remains the same in the other case.	30%
ω decreases in both cases.	10%
ω increases in one or both cases.	15%
Fraction of those who said student will rotate who also said ω decreases.*	45%
Fraction of those who said student will remain at rest who also said ω decreases.*	15%

*These percentages were similar for Cases A and B, so they are combined here.

Table 6-9 shows student performance on the *Final angular speed* question, asked in the 3.ETI.S version of the "Bicycle wheel" context. The instructor for the two sections in the left-hand column performed a demonstration very similar to Case A of the *Handoff* question. While students in that section performed much better on the *Handoff* question, they did not perform any better on the *Final angular speed* question than students who had not seen the demonstration. We conclude from this that, though the observation that a person does not start to rotate in either Case A or Case B may be important for students to make, it is probably not sufficient for helping students understand how to apply a conservation concept to that system.

Another important feature of the data in Table 6-9 is that many students (35-40%) gave responses that do not fall into the three major categories shown. From this we infer that, though many students may have strong incorrect ideas coming into the tutorial, many others have either widely diverging ideas or do not know how to begin thinking about angular momentum.

Table 6-9: Student performance on the *Final angular speed* question asked in the "3.ETI.S" version of the "Bicycle wheel" context after lecture instruction but before the tutorial *Conservation of Angular Momentum.*

Student slows wheel to ½ its initial angular speed in Exp.1, stops the wheel in Exp. 2, and quickly flips the wheel over in Exp. 3. ($\omega(X)$ is the final angular speed of the student in Exp. X.)	UW 121, N=249 2 sections*	UW 121, N=110 1 section
$\omega(3) > \omega(2) > \omega(1)$ (correct)	10%	20%
$\omega(2) > \omega(1) > \omega(3)$ ("scalar" response)	25%	10%
$\omega(3) > \omega(1) > \omega(2)$ ("matching" response)	25%	35%

*Both of these sections had the same instructor, who demonstrated "Case A" of the *Handoff* question in lecture before the pretest.

Student performance on the *Orientation of spin* question is shown in Table 6-10. Roughly equal numbers of students chose each of the two options presented. Most students did not appear to relate this question to angular momentum conservation. Instead most explained their answers with "There is no reason for it to change orientation," or "There is no reason for it not to remain perpendicular to the rope." These results indicate that the "Pendulum" context may be useful as a lecture demonstration (with predictions, voting, and discussion), since students would probably be equally split when predicting the wheel's orientation, and the results of the experiment would be clear.

Table 6-10: Student performance on the *Orientation of spin* question asked in the "Pendulum" context after lecture instruction but before the tutorial *Conservation of Angular Momentum.*

The spinning wheel swings like a pendulum.	UW 121, N=567 5 sections
The wheel remains horizontal. (correct)	50%
The wheel remains perpendicular to the rope.	45%

Most students (65%) realize that the rotation of the wheel would not slow down as it moved to higher points on its path, as shown in Table 6-11. As expected, the most common (25%) incorrect answer was that the wheel would spin slower when it was higher. Only about 35% of students recognized that the angular velocity of the wheel would remain the same in both magnitude and direction.

Table 6-11: Student performance on the *Non-local conservation* question asked in the "Pendulum" context after lecture instruction but before the tutorial *Conservation of Angular Momentum.*

The spinning wheel swings like a pendulum. Point A is higher on the path than point B. ($\omega(X)$ is the angular speed of the wheel when its center is at point X.)	UW 121, N=359 3 sections
$\omega(A) = \omega(B)$ (correct)	65%
$\omega(A) < \omega(B)$	25%
$\omega(A) > \omega(B)$	10%
$\omega(A) = \omega(B)$ and the wheel remains horizontal.	35%

7. Discussion of patterns in student reasoning

a. Tendency to conflate torque and angular momentum

This particular pattern in student reasoning may be understood as analogous to the well-known[1] tendency of students to associate an object's state of translational motion (velocity/momentum) with the object's influence (force) on other objects' states of motion. (For example, many students talk about the "force of an object's motion," whereby how hard an object pushes on another is determined by the force "stored" in the first object in the form of its motion.) Students' tendency to apply this reasoning to rotational motion is most apparent in responses to Case A of the *Handoff* question.

> "Because there is torque that is causing the wheel to rotate in the counterclockwise direction, therefore she will also begin rotating ccw slightly."
>
> "I've seen this done before in my high school physics class. She moves whichever way the wheel is spinning because of rotational kinetic energy."
>
> "The rotational motion of the wheel will cause the student to rotate to her left."
>
> "Because there is an angular acceleration of the wheel, this will cause the student to start to spin in the same direction as the wheel."
>
> "Counter clockwise because the tangential force resulting from the counter clockwise motion of the spinning wheel will have the tendency to move the student in the same direction as rotation."
>
> "The torque will be transferred."
>
> "She will go in the direction of the wheel because the momentum will want to stay in that direction."
>
> "The force of the spinning bicycle wheel will act upon her."

All of these responses associate some kind of influence (properly associated with torque) with the wheel's state of motion (properly associated with angular velocity, angular momentum, or kinetic energy). In reality, there could be an influence on the person in the direction in which the wheel spins, if there were some mechanism for the angular momentum transfer; it is possible that some students were thinking about such an experiment. However, hardly any students who talked about an influence also explained how the influence was working. That is, the very common *absence* of mechanistic explanation in student responses suggests that most of these students thought the motion of the wheel itself had an unmediated influence on the motion of the person.

b. Tendency to apply conservation of angular momentum to a system of changing membership

When answering the *Handoff* question, many students attempted to apply angular momentum conservation to a system that is initially just the person but is then enlarged to include the spinning wheel. These students identified "the system" as having zero initial angular momentum and then applied "conserved vector quantity" reasoning to that system, as shown in the three example responses below.

> "As long as she rotates without friction, then in order to conserve momentum she must counter the added angular momentum given by the wheel."
>
> "I think that her motion will oppose the motion of the wheel since it will try to cancel out the wheel's motion."
>
> "The momentum of the student before the wheel is handed to her is zero. In order to conserve momentum, the momentum of the student after the wheel is handed to her is equal and opposite to that of the wheel."

Students who attempt to apply angular momentum conservation to this improper "system" tend to do so "correctly" – that is, they appear understand that two angular momentum vectors can sum to zero. However, these students tend not to try to account for how the angular momentum is transferred; moreover, they generally do not explain that there is transfer of momentum at all. In this way, students treat conservation as a mathematical procedure, rather than a physical process, in which some quantity of motion is transferred between bodies, according to some mechanism.

c. Tendency to misinterpret the direction of the angular momentum vector

As shown in the leftmost column of Table 6-7, most students who said that the person would begin to rotate in Case B (vertical) of the *Handoff* question said that the person would rotate to her left, rather than to her right. The following three student responses illustrate that this preference is related to the right-hand rule.

> "By the right hand rule the angular momentum of the wheel is in the direction she is moving."

> "Using the right hand rule, we see the torque is pointing to her left (positive) so that's the direction she'll go."
>
> "She should rotate to her left. Because of the right hand rule, the momentum of the wheel is directed straight out the left hand side of the wheel. This will cause her to rotate too."

These students have not understood the abstract idea that the direction of the angular momentum vector is conventional, and have mistaken the direction of the vector for the direction of a push or the direction in which some real object moves. Occasionally, a student justifies the opposite answer (person rotates clockwise) with similar reasoning:

> "From the right hand rule, the direction of the angular momentum of the wheel is pointing to the left of the student, which will push the student to the right."

Such responses may be a result of students combining a misinterpretation of the angular momentum vector with the tendency to "cancel the wheel's motion," as described in the previous section.

d. Tendency to apply non-local conservation of motion

The sections above have presented evidence that students tend to apply variations on the conservation concept with little concern for the mechanism by which parts of the system exchange the conserved quantity. Since many students do not require mechanism for transfer, we would expect many students to state plainly that an increase in motion in one place should accompany a decrease in motion in another place. The following three responses are examples of such reasoning in the "Pendulum" context:

> "The angular speed of A must be less than that of B because the energy remains constant and at A all of the energy is potential and at B all of the energy is kinetic, so the speed must be changing to account for the change of energy."
>
> "Since some of the energy is used for other purposes at point A it is less."
>
> "The law of conservation of energy: entirely kinetic energy at point B and entirely potential energy at A."

These responses treat conservation as a mathematical rule that systems must follow, rather than as a logical outcome of a system's dynamic evolution. In the "Bicycle wheel" context students wrote:

> "The momentum of the system must be conserved and since the girl is gaining momentum the momentum of the wheel must decrease."
>
> "Because the angular speed of the student increases, the angular speed of the wheel must decrease to conserve angular momentum."

In defense of these student responses, it should be noted that we physicists value conservation principles in part because when using them, we need not be concerned with mechanism. However, we are concerned that there exists some plausible mechanism, and are only sometimes unconcerned with the quantitative details of that mechanism. These tendencies in student reasoning suggest that, as instructors, we might emphasize that, for every process in which some quantity is conserved overall, it should always be possible to think about the process in terms of local mechanisms.

C. DESCRIPTION OF PART 1 OF CURRENT VERSION OF TUTORIAL SEQUENCE *CONSERVATION OF ANGULAR MOMENTUM*

There are two main sections of the tutorial "Conservation of Angular Momentum." The primary conceptual goal for the first section is for students to understand that the angular momentum of spinning objects can be treated as a conserved vector quantity, following the same principles that the students have learned for dealing with linear momentum. The content, development, and evaluation of the first section, which we will call *Part 1,* are discussed in this chapter. The second section of the tutorial focuses on applying angular momentum in a more abstract sense, as the moment of momentum. In particular, students are guided to describe the angular momentum of a point-like object that moves with constant velocity. The second section of the tutorial is discussed in the next chapter. A copy of the entire tutorial and homework can be found in Appendix C of this dissertation.

1. Detailed description of Part 1 of tutorial

a. Angular momentum of a spinning wheel

The tutorial begins by having students think about an apparatus that is common in lecture demonstrations: a person sits at rest on a low-friction, rotatable seat; a spinning bicycle wheel is handed to the person, who then may make various changes to the motion of the wheel. The general aim of the first two pages of the tutorial is for students to explain processes in this situation in terms of angular momentum as a conserved vector quantity.

Initially, students are told that a person sits on a frictionless, rotatable seat. In separate experiments, the person is handed a spinning bicycle wheel (with a frictionless axle) in one of two orientations, as shown in Figure 6-6. In Case A, the axis of the wheel is vertical, and the wheel is spinning counter-clockwise, when viewed from above; in Case B, the axis of the wheel is horizontal, and the wheel is spinning counter-clockwise, when viewed from the person's left side. First, students are asked to predict whether the person would begin to rotate if she holds the handles of the spinning wheel in each of the orientations described (without changing the angular speed or orientation of the wheel). The prediction and explanations that we would like students to give are that the person will not begin to rotate in either case – if the spinning wheel is held in place, the motion of the wheel is not altered at all. If the wheel's motion is unchanged, the angular momentum associated with the motion remains the same, in which case there is no opportunity for angular momentum transfer between the wheel and the person.

We do not expect realistically that any students will give this explanation at this point. Most students erroneously predict that the person will begin to rotate in one of the two cases; hardly any students predict that the person will begin to rotate in both cases. Among students who predict that the person will rotate in Case A, there is a roughly equal split between students predicting that the person will rotate counter-clockwise (with the wheel) and that the person will rotate clockwise (against the wheel).

After making predictions, students observe the demonstration. An instructor reminds the students that they are not being asked to explain the result; they should simply write down their observations. (We think that students are better equipped to explain these results after they have thought about situations in which the person does begin to rotate, and the accompanying explanations in terms of angular momentum transfer.) After students record the results of the demonstration, they read text that describes how to determine the direction of the angular momentum vector for a spinning object, according to the right-hand rule. Then, on the pictures of the person holding the spinning wheel in each orientation, students draw an arrow to represent the angular momentum vector of the wheel. Students then discuss whether there is any part of the wheel that is moving in the direction of the angular momentum vector, and whether there appears to be any force on the person (or supporting object) in the direction of the wheel's angular momentum. By asking these questions of the students, we intend to acknowledge and affirm any surprise on their part. Also, we want students to participate in a community decision about how to (or how not to) interpret a concept. For example, the students collectively agree that the angular momentum vector does not represent the direction of motion of the center of the wheel, though what is does represent may not yet be clear.

b. Angular momentum as a conserved vector quantity

In this section of the tutorial, students develop an understanding of the conservation of angular momentum in the context of the person and spinning wheel described above. The person is initially at rest and holds the spinning bicycle wheel as in case A (with angular momentum directed upward.) Students are asked to consider three experiments. In Experiment 1, the person holds her hand against the side of the wheel, slowing it to one-half its initial speed; in Experiment 2, the person holds her hand against the side of the wheel, bringing to a stop; in Experiment 3, the person quickly flips the wheel over (so that it is spinning clockwise when viewed from above, with the *same* angular speed it had initially). Students are asked to predict the direction of spinning of the person after the wheel's motion has changed in each experiment; they are also asked to predict a ranking of the final angular speeds of the person in the three experiments. A correct analysis, which most students are not yet using, is to treat the angular momentum vector of the system of wheel + person as a constant. (In experiment 3, the angular momentum vector of this system must change while the wheel is flipping. However, the angular impulse on the system can be minimized by reducing the time taken to flip the wheel.) This analysis would result in the predictions: (1) the direction the person spins is counter-clockwise (from top) in all three experiments, and (2) the ranking of the final angular speed of the person is 3>2>1. Students usually predict the direction of spinning correctly in all three. However, students often rank the final angular speeds of the person as 3>1>2 or 2>1>3. Students who give the ranking 3>1>2 often describe the final angular speed of the person as resulting from the final angular speed of the wheel only, rather than as a result of the change in angular motion of the wheel. For these responses, instructors may guide students to consider the demonstration considered in section I, in which the motion of the wheel itself did not cause the person to rotate. Students who give the ranking 2>1>3 are often correctly thinking about the motion of the person correlating with a *change* in motion of the wheel, but the motion they are considering is a sort of scalar angular momentum – such students often use the terms angular momentum and energy (and others) interchangeably. For example, students may note that the wheel still has the same rotational kinetic energy afterwards in Experiment 3; therefore, the person must not have received any energy from the wheel, and thus will not rotate. (We consider the issue of energy transfer in these examples to be important for students to resolve, and we have chosen to address them in the tutorial homework.) Students then complete a momentum vector diagram, similar to the ones they used in the tutorial *Conservation of Momentum in One Dimension*. The connection between the method used here and that from the linear momentum tutorial is not made explicit, but of course it is our intent that students understand that the methods are the same. The fact that students generally have less trouble completing the diagrams correctly in this context than when they were introduced in the earlier tutorial suggests that many students are applying an already practiced skill here. When completing the angular momentum vector diagrams for these three experiments with the bicycle wheel, most students complete them correctly and revise their ranking from the previous question so that the two are consistent. After completing the table, students are asked whether their predictions would change if they were to use a "left-hand rule" instead of the right-hand rule. Here we want students to see that the arrows in their table would change, while the ranking and directions of spinning would not. More generally, we want the students to understand a distinction between a change in an abstract representation, and a change in physical observables. We hope that this exercise will introduce students (implicitly, at least) to the logical species "convention," which is a key element in the logical structure of scientific theories. After students have completed the table of angular momentum vectors and finalized their predictions, they are advised that the experiments will be performed as a demonstration at the end of class. The students then proceed to the next section of the tutorial.

2. Detailed description of Part 1 of the homework

The homework for the tutorial *Conservation of Angular Momentum* consists of two problems. Problem 1 is based on the three experiments performed with the spinning wheel described on the second page of the tutorial. The problem reviews angular momentum conservation and also aims to help students think about an energy description of these three experiments in order that they may further differentiate angular momentum from energy. Problem 2 deals with simultaneous linear and angular momentum conservation in the context of collisions between balls and a stick in a weightless, frictionless environment. Problem 2 is discussed in the next chapter.

With problem 1, we aim to highlight two differences between angular momentum and energy: (1) energy is a scalar quantity, unchanged under rotations or parity transformations; (2) energy has other forms that are not associated with motion at all, while angular momentum does not. First, the three experiments described on the second page of the tutorial are described again. Students are reminded that they observed (at the end of tutorial) that the final angular speed of the student was greater when she flipped the wheel over than when she brought the wheel to a stop. In the first question, students review angular momentum by explaining this result. In the second question, students rank the final rotational kinetic energy of the wheel in the three experiments. A note reminds students that the kinetic energy of an object is never negative. We observed in interviews (with students who had completed an earlier version of the tutorial) that there was a tendency to explain the student's greater final angular speed after flipping the wheel in terms of a principle of conservation of kinetic energy, in which the kinetic energy of the wheel would be negative when it was flipped over. This ranking of the final kinetic energies of the wheel is very easy for students, as long as they are confident that the kinetic energy of the wheel is always non-negative. Next the students rank the final rotational kinetic energy of the student in the three experiments. Again, most students find this ranking easy. Occasionally, a student will rank the final kinetic energies of the student, not according to the observed final angular speeds of the student, but according to an overriding principle of conservation of kinetic energy, in which the student's final kinetic energy is whatever kinetic energy the wheel does not retain after its motion is changed.

Finally, students compare four quantities: the initial kinetic energy of the wheel (K_{initial}), and the final kinetic energy of the wheel + student in each of the three experiments (K_1, K_2, K_3). (A hint is given that it may be useful to think about changes in non-mechanical energy.) The reasoning we expect here is as follows: K_3 is greater than K_{initial}, because the wheel, after being flipped over, has the same kinetic energy that it had initially. The quantity K_3 is then numerically equal to the sum of the initial kinetic energy of the wheel and the final kinetic energy of the student, which is not zero. K_2 is less than K_{initial} (even though the kinetic energy of the wheel decreased while that of the student increased) because some energy was converted to thermal energy in the rubbing process that brought the wheel and student to have eventually the same angular speed. Similarly, K_1 is greater than K_2, but less than K_{initial}, since there is less rubbing in experiment 1 than in experiment 2. (Alternatively, that K_2 is less than K_{initial} can also be shown mathematically. In experiment 2, the final angular momentum (L) of the student is equal to the initial angular momentum of the wheel, and it can be easily observed that the final angular speed (ω) of the student is much less than the initial angular speed of the wheel. The kinetic energy $K = (1/2)I\omega^2 = (1/2)L\omega$ will be less when L is the same but ω is less.)

Students generally have significant trouble giving the reasoning described above. The main problem appears to be their understanding of the principle of energy conservation. In particular, students generally do not appear to be comfortable with the concepts of thermal energy and biological energy (as stored chemically in the human body, for example). The combination of (1) a strong tendency of the students to believe in some kind of energy conservation concept with (2) a strong tendency to disregard non-mechanical forms of energy leads to students to a principle of mechanical energy conservation, in which all energy that is not kinetic is some kind of mechanical potential energy. We found through informal discussions with students that, at this point, they usually follow one of two strategies in understanding the energy changes in these experiments. They may either give rankings of total kinetic energy that are consistent with the previous kinetic energy rankings of the wheel and the student (associating any changes in total kinetic with opposite changes in a vaguely identified "potential energy"), or they may insist on a principle of conservation of kinetic energy, destroying the overall consistency between the three rankings and their observations of the final angular speed of the student in the three experiments. In the first case, a complete ranking cannot be determined, since, for example, the final kinetic energy of the wheel in experiment 1 is greater than that in experiment 2, while the reverse is true for the student. When these students try to account for a change in kinetic energy by proposing a corresponding change in "potential energy" (in a situation for which a physicist would not introduce a mechanical potential energy) they usually fail to describe where or how the potential energy is stored. If asked to describe the potential energy, students sometimes appear surprised that they are being asked for more detail, as if they are not accustomed to such a question. Alternatively, when students stick to a notion of conservation of kinetic energy, students state that all four quantities are equal, sometimes supporting the ranking by explaining that "no external forces are doing work on the system." If an instructor draws their attention to the global inconsistency between their answer and observations, they usually cannot resolve it. In discussions with students about where the "extra" kinetic energy comes from in experiment three, students are often completely at a loss. (Sometimes they try to assert that the "extra" is not really extra, because the student took a small amount of kinetic energy from the wheel, which has a final kinetic energy just a little bit less than its initial kinetic energy.) When the instructor suggests the possibility of the energy being converted from biological energy to kinetic energy, students very often laugh nervously, which suggests they think such an account is inappropriate, perhaps because it is too closely related to their experience of reality to be real physics.

All of the preceding remarks are meant as a rough sketch of how learning of conservation of angular momentum is related to student understanding of energy conservation. Given the conceptual difficulties described above, we believe that improved learning of energy conservation will improve learning of angular momentum conservation, since distinguishing angular momentum and energy is necessary for understanding both.

D. STUDENT UNDERSTANDING OF ANGULAR MOMENTUM OF ROTATING RIGID BODIES AFTER CURRENT VERSION OF PART 1 OF TUTORIAL SEQUENCE *CONSERVATION OF ANGULAR MOMENTUM*

1. Description of additional question contexts

a. The "4 steps" version of the "Bicycle wheel" context

In this version of the "Bicycle wheel" context, a student is sitting at rest on a frictionless, rotatable seat. The student has been handed a bicycle wheel that is spinning counterclockwise around a vertical axis (when viewed from above). The student changes the motion of the wheel in two steps, A1 and A2. In step A1, the student places her hand against the side of the wheel, slowing it to half its initial angular speed. In step A2, the student again places her hand against the side of the wheel, bringing it to a stop.

In an alternative to steps A1 and A2, the student instead changes the motion of the wheel by performing steps B1 and B2. In step B1, the student flips the wheel over, so that it is spinning clockwise, with the angular speed unchanged. In step B2, the student places her hand against the side of the wheel, bringing it to a stop.

b. The "Shaded handles" version of the "Bicycle wheel" context

We constructed the "Shaded handles" version of the "Bicycle wheel" context as a way to ask questions about different orientations of a spinning wheel without using much space for large figures on an exam. As shown in Figure 6-9, the handles on the bicycle wheel have been shaded so that orientation A is visually distinct from orientation C, without any representation of the spinning motion of the wheel. Students are asked to imagine sitting on a frictionless, rotatable seat and receiving a spinning wheel that has one bright handle and one dark handle and that is initially spinning counterclockwise (when viewed from above) when the handles are in orientation A.

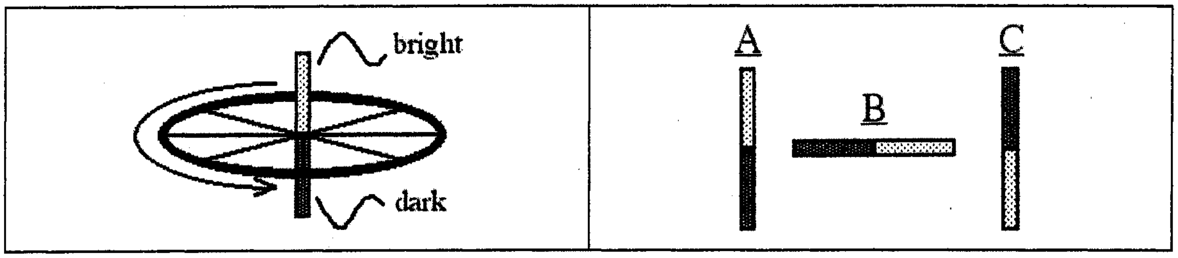

Figure 6-9: The "Shaded handles" version of the "Bicycle wheel" context.

2. Description of additional questions

a. The *Final angular speed* question

In the "4 steps" version of the "Bicycle wheel" context, the *Final angular speed* question was asked in two parts: "Does the student's angular speed during step A2 increase, decrease, or remain the same?" and "Does the student's angular speed during step B2 increase, decrease, or remain the same? We used students' responses to both questions combined as the primary indicator of understanding of the directed nature of angular momentum. If a student applies a vector understanding of angular momentum, the student should give different answers to the two parts of the question; if a student applies a scalar understanding, the student should give the same answer.

b. The *Final angular momentum* question

In the "Shaded handles" version of the "Bicycle wheel" context, students were told to imagine quickly flipping the wheel from orientation A to orientation C, without changing the angular speed of the wheel. They were then asked, "Would the magnitude of the final angular momentum of you (and the seat) be greater than, less than, or equal to the magnitude of the initial angular momentum of the wheel?"

c. The *Direction of spin* question

After imagining flipping the spinning wheel from orientation A to orientation C, students were asked, "What would be your resulting motion?" and were given the choices: "spin clockwise (to your right)," "spin counter-clockwise (to your left)," "stay at rest," and "not enough information."

d. The *Final kinetic energy* question

After imagining flipping the spinning wheel from orientation A to orientation C, students were asked, "Would the kinetic energy of the system consisting of you (and the seat) and the wheel increase, decrease, or remain the same?"

3. Correct answers to questions in additional contexts

a. The *Final angular speed* question

In the "4 steps" version of the "Bicycle wheel" context, the student's angular speed will increase during step A2 and decrease during step B2. When the student has slowed the wheel to half its initial angular speed in step A1, half of the upward angular momentum of the system ($+L_0$) has been transferred to the student; during step A2, in which the wheel is brought to a stop, the remaining half is transferred. Thus, the student's angular speed increases as she takes on this additional upward angular momentum. In step B1, the angular momentum of the wheel is switched from upward ($+L_0$) to downward ($-L_0$), so an upward angular momentum of magnitude double that of the wheel initially ($+2L_0$) is transferred to the student. When the wheel is slowed to a stop in step B2, the wheel's angular momentum changes from $-L_0$ to 0, which corresponds to a change in the angular momentum of the student from $+2L_0$ to $+L_0$. Thus, the student slows down during step B2.

b. The *Final angular momentum* question

If the wheel begins with initial upward angular momentum $+L_0$ and is flipped over, its final angular momentum will be $-L_0$. Thus, in order for the system to maintain total angular momentum $+L_0$, the student (and seat) will have final angular momentum $+2L_0$, which is greater than the initial angular momentum of the wheel.

c. The *Direction of spin* question

Since the student will have upward angular momentum after the interaction, she will be rotating counter-clockwise (when viewed from above), according to the right-hand rule.

d. The *Final kinetic energy* question

The kinetic energy associated with the wheel spinning around its own axis remains the same, since the wheel's angular speed does not change. Since the student begins at rest and is rotating after the interaction, the kinetic energy of the combination of student (and seat) and wheel must increase.

4. Administration of questions

All data presented in the next section are from questions asked on final examinations for Physics 121 at the University of Washington. Questions in the "Dropped disk" context and the "4 steps" version of the "Bicycle wheel" context were asked in free-response form; all other questions were asked in multiple-choice form.

5. Overview of student performance

As shown in Table 6-12, student performance on the *Initial angular speed* question is better after the current version of the tutorial than after the initial version. We believe these data indicate that more students have a better understanding of what it means for angular momentum to be a directed (rather than scalar) quantity. The bottom row of Table 6-12 is expressed as "all other responses" because no single response (except the "vector" and "scalar" responses) was more frequent than ~5% in any administration of the "Dropped disk" context. We believe this means that a large minority of students still has trouble bringing a coherent approach to basic problems involving angular momentum, after either version of the tutorial.

Table 6-12: Comparison of student performance on the *Initial angular speed* question in the "Dropped disk" context after the initial and current versions of the tutorial *Conservation of Angular Momentum.*

Initial angular velocity of disk A is unknown in magnitude and direction.	Initial version	Current version
	UW 121, N=185 1 section	UW 121, N=441 3 sections
Vector response (correct)	40%	55%
Scalar response	25%	15%
All other responses	35%	30%

Table 6-13 shows student performance on the *Final angular speed* question in the "4 steps" context after the current version of the tutorial. Because the "4 steps" context required students to think about experiments in the "Bicycle wheel" context that they had not seen, we believe that success on this question is a good indicator of understanding of what it means (in that context) for angular momentum to be a directed quantity.

Table 6-13: Student performance on the *Final angular* speed question asked in the "4 steps" context after all instruction, including the current version of the tutorial *Conservation of Angular Momentum.*

	UW 121, N=69 1 section
Vector response (increase/decrease) (correct)	60%
Scalar response (increase/increase)	20%
All other responses	20%

Student performance on the *Handoff* question after the current version of the tutorial is shown in Table 6-14. Almost all (85%) students recognize (or remember) that the student remains at rest in both cases. We believe that success on this task is an important basic piece for understanding angular momentum as a quantity that is conserved and *transferred* within the system in the "Bicycle wheel" context.

Table 6-14: Student performance on the *Handoff* question after all instruction, including the current version of the tutorial *Conservation of Angular Momentum.*

Axle of the wheel is vertical in Case A (L points up) and horizontal in Case B (L points to the student's left).	UW 121, N=227 2 sections
A: rest, B: rest (correct)	85%
A: counter-clockwise, B: rest	5%
A: clockwise, B: rest	5%
A: rest, B: counter-clockwise	5%

As shown in Table 6-15, about half (45%) answer the *Final angular momentum* question (asked in the "Shaded handles" version of the "Bicycle wheel" context) correctly. These results are similar to those in Table 6-4, in which about 40% of students stated that the total angular momentum of a person and platform after flipping a spinning wheel is equal to the angular momentum of a person and platform after stopping the spinning wheel. In both cases, we suspect that many of the students giving this response have trouble thinking about the angular momentum of the person (and seat or platform) alone, when the person is holding the wheel. That is, in linear momentum contexts, objects tend to separate spatially after their momentum transfer; in the "Bicycle wheel" contexts, we are asking students to think about a system's parts separately, even though the parts are not physically separated (much). Some student responses that chose "equal to" seemed to compare the final angular momentum of each entire system, rather than the angular momentum of the student and platform *part* of the system. Thus, it may be that more than 45% of students would answer the *Final angular momentum* question correctly, if they had more help separating the parts of the system. It may help to show both a realistic figure, in which the parts are adjacent, and a schematic figure, in which, the parts are separated both before and after the interaction. The fact that typical figures in linear momentum contexts are both somewhat realistic *and* organized may account for some of the difference between student success on linear momentum and angular momentum tasks.

Table 6-15: Student performance on the *Final angular momentum* question asked in the "Shaded handles" version of the "Bicycle wheel" context after all instruction, including the current version of the tutorial *Conservation of Angular Momentum.*

Wheel (w) is flipped from orientation A (L points up) to orientation C (L points down). Angular momentum of person and seat: $\vec{L}_{(p+s)}$	UW 121, N=97 1 section
$\lvert\vec{L}_{(p+s),f}\rvert > \lvert\vec{L}_{w,i}\rvert$ (correct)	45%
$\lvert\vec{L}_{(p+s),f}\rvert = \lvert\vec{L}_{w,i}\rvert$	35%
$\lvert\vec{L}_{(p+s),f}\rvert < \lvert\vec{L}_{w,i}\rvert$	15%

Table 6-16 shows the results of the *Direction of spin* question. Like the *Handoff* question, this question asks students about an experiment they saw in tutorial. However, the success rates are different (85% *vs.* 65%). We might ask why these numbers are different when students could answer each question by remembering outcomes of experiments they had seen. It might be because answering the *Handoff* question correctly does not rely at all on distinguishing between clockwise and counter-clockwise. Answering the *Direction of spin* question requires students to reason spatially (or remember spatial information), and offers students a greater opportunity to apply the right-hand rule, which can lead some to make a mistake. Another possibility is that students may be having trouble distinguishing between two different angular velocities of the wheel: though after the flip the wheel spins clockwise around its own center, its center rotates counter-clockwise around the person's axis of rotation. Simplifying the motion of the wheel by removing it from the person's hands after the interaction (and again, showing separate parts of the system as spatially separated) may improve student performance on this task. Making such a change would make the experiment itself appear less natural, and therefore probably more complicated, but it might help students think more clearly about angular momentum transfer in the experiment.

Table 6-16: Student performance on the *Direction of spin* question asked in the "Shaded handles" version of the "Bicycle wheel" context after all instruction, including the current version of the tutorial *Conservation of Angular Momentum.*

Wheel is flipped from orientation A (L points up) to orientation C (L points down).	UW 121, N=227 2 sections
Person spins counter-clockwise. (correct)	65%
Person spins clockwise.	20%
Person remains at rest.	10%

Perhaps the most difficult task we have asked in the "Bicycle wheel" context is the *Final kinetic energy* question. As shown in Table 6-17, only about 40% of students responded that the kinetic energy of the student/wheel system increases as a result of flipping the wheel over. Most students said that the kinetic energy of the system remains the same during this process. We included a very similar question on the tutorial homework and have observed (when helping students with their homework) that they find the prospect of increasing mechanical energy disturbing, or at least confusing. Students tend to treat (correct or incorrect) ideas about energy and its conservation with higher authority than they do their ideas about momentum or forces. Thus, students often "overrule" results of correct reasoning about momentum or forces with incorrect ideas about energy. We expect that improving students' comfort with identifying changes in non-mechanical energy (especially increases in thermal energy and decreases in biological energy) will improve student performance on tasks in contexts like "Bicycle wheel."

Table 6-17: Student performance on the *Final kinetic energy* question asked in the "Shaded handles" version of the "Bicycle wheel" context after all instruction, including the current version of the tutorial *Conservation of Angular Momentum.*

Wheel is flipped from orientation A (L points up) to orientation C (L points down).	UW 121, N=227 2 sections
Kinetic energy of system increases. (correct)	40%
Kinetic energy of system remains constant.	55%
Kinetic energy of system decreases.	5%

6. Comparison of performance of introductory students and graduate student TAs

Table 6-18 compares performance on the *Handoff* question by students in the introductory course to that of graduate student TAs. In general, graduate student TAs tend to be very successful on tasks that require the use of "conserved vector quantity" reasoning. Therefore, comparisons between introductory students and graduate students tend not to be useful for evaluating instruction on "straightforward" linear momentum or angular momentum conservation. However, the *Handoff* question appears to be a relatively difficult task for the TAs. Only ~35% of the TAs correctly predicted that the student would remain at rest in both Cases A and B. The most common incorrect response (30%) was that the student would begin to rotate clockwise in Case A. Most of these responses explained that the total angular momentum began as zero and so must remain zero. Thus, like the introductory students, many of the graduate students did not correctly identify a proper system to which angular momentum conservation could be applied and also failed to realize that objects with different angular velocities do not necessarily transfer angular momentum. We conclude then, that though graduate students may be very facile with the procedure for applying "conserved vector quantity" reasoning, they sometimes do so uncritically, without considering whether there is a mechanism for transfer, or whether they have defined an appropriate system.

Table 6-18: Comparison of performance of introductory students on the *Handoff* question after the current version of the tutorial *Conservation of Angular Momentum* to that of graduate students in a teaching seminar.

Axle of the wheel is vertical in Case A (L points up) and horizontal in Case B (L points to the student's left).	After current tutorial Introductory students UW 121, N=227 2 sections	Before tutorial Graduate teaching seminar UW 501, N=25
A: rest, B: rest (correct)	85%	35%
A: counter-clockwise, B: rest	5%	10%
A: clockwise, B: rest	5%	30%
A: rest, B: counter-clockwise	5%	10%

E. SUMMARY

In this chapter, we have presented evidence that many students have trouble understanding a common lecture demonstration (a spinning bicycle wheel and person on a rotating seat) in terms of conservation and transfer of angular momentum. Tutorial instruction on angular momentum that focuses on helping students understand angular momentum in this context appears to help students perform better on tasks in this context and in other, more abstract contexts than with an initial version of the tutorial. However, even after students have been guided to think about the bicycle-wheel context, many students have difficulty performing basic tasks set in the same context. We believe instruction on angular momentum of spinning objects in general can be further improved by helping students distinguish relevant parts of the system in the bicycle-wheel context and by improving instruction on non-mechanical energy and energy conservation, in general.

NOTES TO CHAPTER 6

[1] Viennot, L., "Analyzing students' reasoning: tendencies in interpretation," *Am. J. Phys.* **53**, 432-436 (1985).

CHAPTER 7. STUDENT LEARNING OF ANGULAR MOMENTUM OF A POINT-LIKE OBJECT MOVING WITH CONSTANT VELOCITY

This chapter describes an investigation of student learning of the relationship $\vec{L} = \vec{r} \times \vec{p}$, especially as it applies to a point-like object moving with constant velocity. (In this dissertation, we will refer to the angular momentum of an object moving with constant velocity with the abbreviation CVL, for "constant-velocity angular momentum.") This investigation was motivated partly by the fact that the concept of CVL is included in the standard syllabus for the introductory calculus-based mechanics course at the University of Washington.[1] As the number of topics covered in the standard introductory course tends to grow (while the length of the course stays the same), research on advanced topics in physics may be useful in helping to determine which advanced topics are of an appropriate difficulty for students in the introductory course. Also of possible value is what insight research on student understanding of the relation $\vec{L} = \vec{r} \times \vec{p}$ might afford to our understanding of how students think about other vector cross products in introductory physics, like $\vec{\tau} = \vec{r} \times \vec{F}$ and $\vec{F} = q\vec{v} \times \vec{B}$, which correspond to topics that most instructors of introductory physics would probably agree *should* be included.

What has been revealed most clearly by the research conducted so far is that student treatment of CVL appears to depend strongly on whether the question context involves collisions. In a context with collisions, many students justified treating CVL as either identically zero or identical to linear momentum by treating linear momentum and angular momentum as if they were jointly (rather than separately) conserved. These tendencies were far less common in contexts without collisions. When we modified the tutorial instruction to help students conclude that linear and angular momentum must be separately conserved, students made the above errors less often, but still far more often than in non-collision contexts. We have interpreted this context dependence as students' sensitivity to the necessity (or utility, perhaps) of the concept of CVL. This finding has highlighted our sense of the possibility that, in some cases across all research on student understanding in physics, we may grossly overestimate how well students appreciate the purpose or function of a physical concept.

In part A of this chapter, we characterize student understanding of particle angular momentum after standard instruction and before tutorial instruction. In part B we describe version A of Part 2 of the tutorial sequence *Conservation of Angular Momentum.* In part C we evaluate student understanding of CVL after version A of the tutorial sequence and summarize the observations and reflections that led us to revise the instruction. Part D describes version B of Part 2 of the tutorial sequence, while part E describes version C. In part F of this chapter, we compare the effects of versions A, B, and C on student understanding of CVL. Part G summarizes our findings and offers suggestions for how instruction on angular momentum might eventually become more effective.

A. STUDENT UNDERSTANDING OF CONSTANT VELOCITY ANGULAR MOMENTUM BEFORE TUTORIAL *CONSERVATION OF ANGULAR MOMENTUM*

1. Prior instruction

Over the course of this research, the pretest for the tutorial *Conservation of Angular Momentum* for UW 121 has tended to follow the textbook reading assignment on the definition of particle angular momentum $\vec{L} = \vec{r} \times \vec{p}$. In some cases, lecture instructors assigned homework problems that required students to apply the definition of particle angular momentum. Lecture instruction itself on the topic was generally minimal; lecture instructors' attention to particle angular momentum has tended to focus on the abstract mathematical properties of the vector cross product, rather than on the purpose or function of the definition of angular momentum as a physical concept.

2. Description of contexts in which questions about angular momentum of an object with constant velocity were asked

a. Free small ball

Through this family of contexts, we investigate student understanding of the angular momentum of an object moving with constant velocity, in the absence of any collisions or other interactions. A small ball moves to the right (across the page) with a constant velocity in a weightless, frictionless environment (see Figure 7-1). The ball does not spin. Fixed locations in space are marked with 'X's and labeled with letters. Students are asked various questions about the angular momentum of the ball with respect to different marked locations. (No reference frame is explicitly specified in these contexts. It is implied that the perspective from which the ball moves and the locations are fixed is a single perspective.) All reference points are in the plane of the page.

As we describe the particular contexts here, we indicate whether reference points had the same or different "x" and "y" coordinates, where the "x" and "y" axes are parallel and perpendicular to the top of the page, respectively. However, the positions of reference points were not described to students with coordinates.

i. 2 points, same y

In this context, two points (A and B) are marked. The line containing points A and B is parallel to the line on which the ball moves and is in the plane of the page. A figure shows a particular instant, at which: the ball has passed its closest approach to point A and has not yet passed its closest approach to point B; also, the ball is clearly closer to point A than it is to point B.

ii. 2 points, different x and y

Here, two points (A and B) are marked. Point A lies on the line on which the ball moves. Point B is above point A and to the left. A figure shows a particular instant, at which: the ball is clearly closer to point B than to point A, and has not yet passed its closest approach to either point.

iii. 3 points

This context is similar to "2 points, same y," except that a third point is added. Point C has the same x-coordinate as point B, but is closer to the ball's path and is on the opposite side of the path.

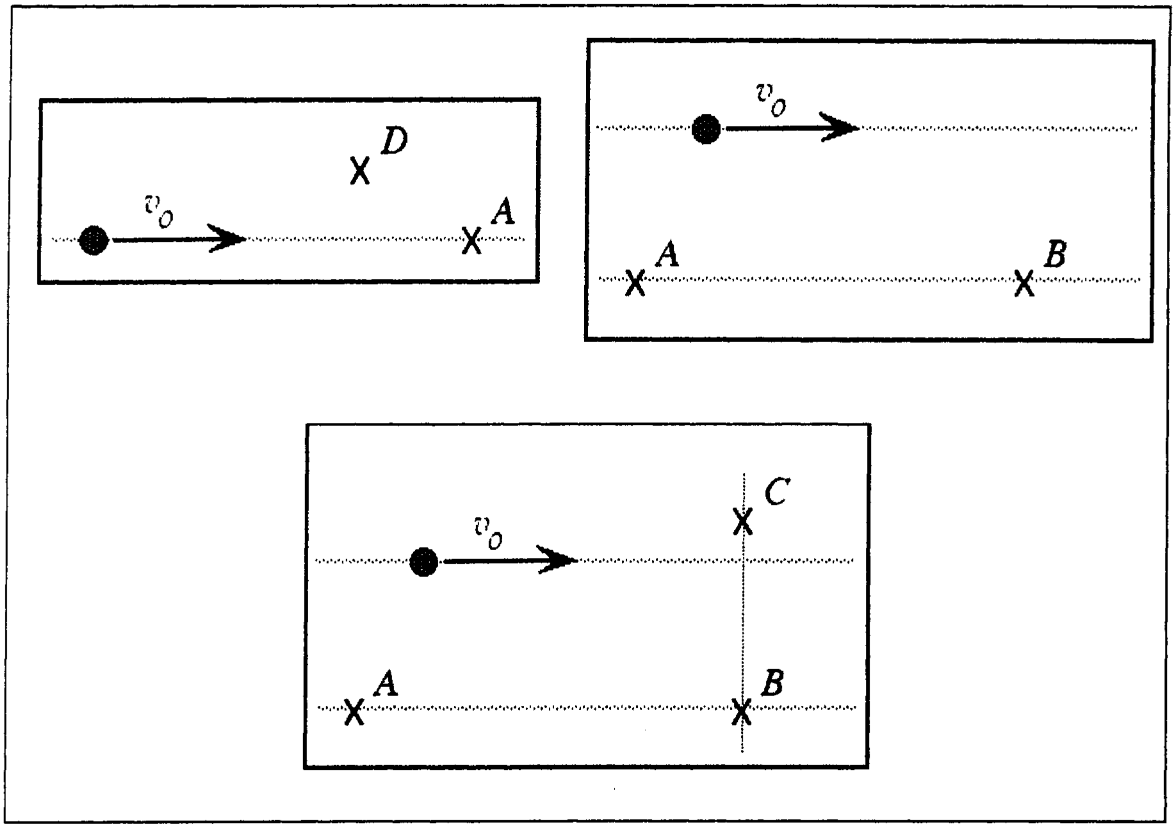

Figure 7-1: Variations on the "Free small ball" context: "2 points, different x and y" (upper left), "2 points, same y" (upper right), and "3 points" (bottom).

iv. 2 points, same y, with collisions

We thought perhaps students might be more inclined to attribute angular momentum to an object moving in a straight line if the object were involved in a collision in which the target object seemed likely to rotate. Therefore, we designed a context based on "2 points, same y" in which a target object starts with its center either point A or point B (from the original context). In order to make the collision purely planar, we changed the free small ball to a small puck moving across the surface of an air table. The target object is a large, flat disk, which though much larger than the puck, is about the same mass. (We did not want students to imagine that the disk was so massive that its motion after the collision would be negligible.) In Experiment A, the disk starts with its center above point A, and in Experiment B, the disk starts with its center above point B (see Figure 7-2). The puck has the same initial path and velocity in both experiments, and in each experiment, the puck sticks to the disk.

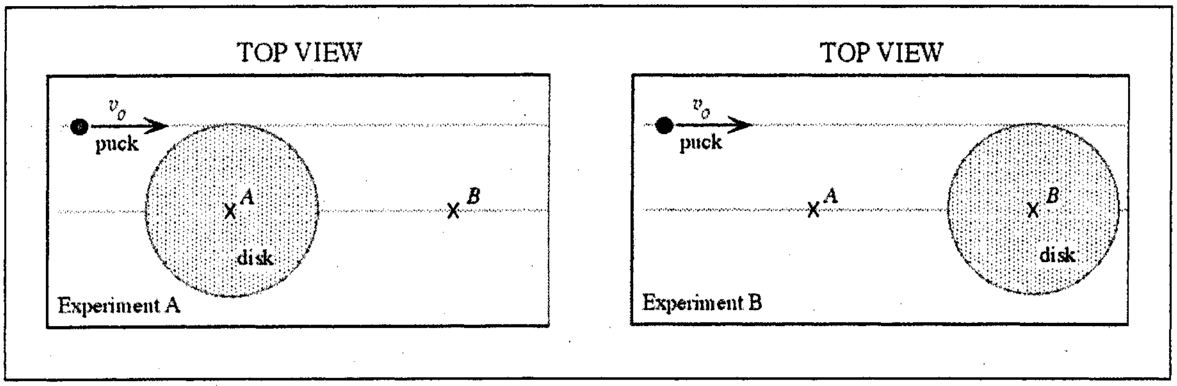

Figure 7-2: Experiment A (left) and Experiment B (right) of the "2 points, same y, with collisions" version of the "Free small ball" context.

3. Description of questions about angular momentum of an object with constant velocity

a. The *Point dependence* question

This question asks students to compare the magnitudes of the angular momentum of the ball with respect to various reference points for the instant shown. For example, in the "2 points, same y" context, the question was "Is the magnitude of the angular momentum of the ball with respect to point A greater than, less than, or equal to the magnitude of the angular momentum with respect to point B?"

In some cases, to reduce the space required to ask a series of questions, we defined these quantities symbolically, and then asked the questions using the condensed notation. For example, "the magnitude of the angular momentum of the ball with respect to point A" was shortened to "$\left|\vec{L}_{ball,A}\right|$." Also, when this question was asked in multiple-choice format on a final exam, there were often five options: "greater than," "less than," "equal to and both are not zero," "equal to and both are equal to zero," and "not enough information."

b. The *Time dependence* question

In its more basic form, this question is, "Is the angular momentum of the ball with respect to point B increasing, decreasing, or remaining constant?" However, as a multiple-choice question on a final exam, we asked students about angular momentum with respect to two different points in a single question. In this form, students read the question, "Which of the following options best describes how the magnitudes of $\vec{L}_{ball,A}$ and $\vec{L}_{ball,B}$ are changing at the instant shown?" Students were then given the following options: "They are both constant," "They are both increasing," "They are both decreasing," "$\left|\vec{L}_{ball,A}\right|$ is increasing, and $\left|\vec{L}_{ball,B}\right|$ is decreasing," and "$\left|\vec{L}_{ball,A}\right|$ is decreasing, and $\left|\vec{L}_{ball,B}\right|$ is increasing."

c. The *Direction of CVL* question

As a free-response question, students are asked to describe the direction of the angular momentum with respect to some specified point. In one multiple-choice form, students chose from a text menu of directions the one that best described the direction of the angular momentum of the object with respect to a certain point. The menu included "up (toward the top of the page)," "up and to the right," "to the right," "down and to the right," "down (toward the bottom of the page)," "down and to the left," "to the left," "up and to the left," "out of the page," "into the page," and "zero." In another multiple-choice form, asked in the "3 points" context, students were asked, "Which of the following best describes the directions of $\vec{L}_{ball,A}$, $\vec{L}_{ball,B}$, and $\vec{L}_{ball,C}$?" and given five options: "These vectors all point in the same direction," "Each of these vectors points in a different direction," "Two of these vectors point out of the page, and one of them points into the page," "Two of these vectors point into the page, and one of them points out of the page," and "not enough information."

d. The *Translated effects* questions

The *Translated effects* questions were asked in only in the "2 points, same y, with collisions" context. We asked students two questions: whether the final speed of the center of mass of the puck-disk system in Experiment A was greater than, less than, or equal to that in Experiment B; and whether the final angular speed of the puck-disk system about its own center of mass in Experiment A was greater than, less than, or equal to that in Experiment B. Since we were aware that students sometimes have difficulty interpreting the term "speed of the center of mass," we told them that they could associate this term with the speed at which the puck-disk system as a whole moves across the table. We had two purposes in asking these questions. First, we wanted to observe to what extent students realized that Experiments A and B were essentially the same experiment, related by a meaningless spatial shift across the table. Second, and primarily, since we expected students to be fairly successful on the *Translated effects* questions, we were interested in how many students would take their own answers to these questions as a hint that the initial angular momentum of the puck with respect to each point was the same non-zero constant.

4. Correct answers to questions about angular momentum of an object with constant velocity

The correct answers to all of the above questions can be found by using the definition of angular momentum for a particle: $\vec{L}_{ball,A} = \vec{r}_{ball,A} \times \vec{p}_{ball} = \vec{r}_{\perp} \times \vec{p}_{ball}$, where $\vec{r}_{ball,A}$ is the vector that begins at the reference point (*A*) and ends at the instantaneous position of the ball, $\vec{p}_{ball}$ is the linear momentum of the ball, and $\vec{r}_{\perp}$ is the component of $\vec{r}_{ball,A}$ that is perpendicular to $\vec{p}_{ball}$.

In order to answer the *Point dependence* question, it is necessary only to compare the magnitudes of $\vec{r}_{\perp}$ for each point (since the linear momentum is the same with respect to different points that do not move with respect to each other), which are the distances from each reference point to the straight-line path of the ball. In "2 points, same y," this distance is the same, so the angular momentum of the ball with respect to each point is the same. In "2 points, different x and y," point A is on the ball's path, and point D is off the path; thus, the distance from point D to the path is greater, and therefore the angular momentum of the ball is greater with respect to point D than it is to point A.

To answer the *Time dependence* question, one can observe that the magnitude of the angular momentum of the ball depends only on the perpendicular distance from the reference point to the ball's path and the linear momentum of the ball, neither of which changes. Thus, in all instantiations of the question, the correct answer is that the angular momentum of the ball remains constant.

The *Direction of CVL* question requires use of the right-hand rule to determine a vector direction for angular momentum from the cross product of position with linear momentum. There are a variety of forms of the right-hand rule, but they all should give the result that "down" × "right" = "out of the page," and "up" × "right" = "into the page." In other words, when the ball moves to the right reference point is above the ball's path, then the angular momentum of the ball with respect to that point is directed out of the page, and when the ball moves to the right and the point is below the path, the angular momentum is into the page.

One solution to the *Translated effects* questions is found by observing that Experiments A and B are indistinguishable if all arbitrary markers of space and time (which have no physical significance) are removed. Thus, all physical quantities (including the final speed of the system's center of mass and its final angular speed about the center of mass) have the same values in Experiments A and B.

5. Administration of questions

In all cases, students answered the above questions about CVL after they had been assigned reading from the textbook[2] that included an introduction to the vector cross product, the right-hand rule, the definition of particle angular momentum, and its relationship to the net torque on a particle. Lecture coverage on this topic varied from section to section at the time students took the pretest for *Conservation of Angular Momentum,* but student performance does not appear to depend strongly on whether particle angular momentum was discussed in lecture.

All sections of students were given the *Point dependence, Time dependence,* and *Direction of CVL* questions. One section of students answered the *Translated effects* before answering the other three questions.

6. Overview of student performance

a. The *Point dependence* question

Table 7-1 shows student performance on the "2 points, different x and y" version of the *Point dependence* question. Because point A is noticeably farther from the ball than point D at the instant of interest, students who over-emphasized the role of distance in the relation $\vec{L} = \vec{r} \times \vec{p}$ tended to choose "$\left|\vec{L}_{ball,A}\right| > \left|\vec{L}_{ball,D}\right|$" (25%). (We will call this the *r*-dominant concept of CVL.) Some students (15%) stated that the angular momentums would be equal but not equal to zero; this number may be understood as an upper limit on the number of students who answered the question by thinking only about the linear momentum of the ball (call this the p-dominant concept of CVL). We were surprised to see such a small number (10%) state that the angular momentum of the ball was zero in this context. We suspect that directing students attention to the contrast between a reference point on the path and another off the path cued many students to attribute *something* angular to the point off the path.

Table 7-1: Student performance on the *Point dependence* question asked in the "2 points, different x and y" version of the "Free small ball" context before working through the tutorial *Conservation of Angular Momentum.*

Point A is on the path of the ball; point D is not.	UW 121, N=303 3 sections
$\lvert\vec{L}_{ball,A}\rvert < \lvert\vec{L}_{ball,D}\rvert$ (correct)	50%
(r-dominant) $\lvert\vec{L}_{ball,A}\rvert > \lvert\vec{L}_{ball,D}\rvert$	25%
$\lvert\vec{L}_{ball,A}\rvert = \lvert\vec{L}_{ball,D}\rvert = 0$	10%
(p-dominant) $\lvert\vec{L}_{ball,A}\rvert = \lvert\vec{L}_{ball,D}\rvert \neq 0$	15%

In contrast to the results just discussed, Table 7-2 shows student performance on the *Point dependence* question in the "2 points, same y" and "2 points, same y, with collisions" versions of the "Free small ball" context. First, focusing on the version without collisions, there are several interesting differences between student performance in this context and that in the previous context. Students are much less successful on this version (10%) than on the previous version (50%). Inspection of responses to the "2 points, different x and y" version revealed that some students were supporting the correct answer with a common incorrect reason. (We had initially identified the reasoning as "basically" correct, but we later understood that it would lead to incorrect answers in other contexts, which was part of the motivation for the design of the "2 points, same y" version.) These students answered primarily on the basis of the instantaneous angular speed with which the ball moves past the reference point. (We will call this the ω-dominant concept of CVL.) In the "2 points, same y" version, students using ω-dominant reasoning tended to choose "$\left|\vec{L}_{ball,A}\right| > \left|\vec{L}_{ball,B}\right|$" (25%). (Because the ball is closer to point A at the instant shown, the angular speed of the ball with respect to point A is, in fact, greater than that with respect to point B). While the correct answer in "2 points, different x and y" included one of the major errors, the correct answer in "2 points, same y" also coincides with the p-dominant error (since the linear momentum is the same with respect to each point and not zero). Thus, in both contexts, the number of students who answer the *Point dependence* question correctly is an overestimate of the number of students who understand that angular momentum is not identical to either angular velocity or linear momentum. The number who gave an r-dominant answer in this context (20%) is about the same as in the previous context (25%). Also, the number who stated that the angular momentum was identically zero was much greater in this context (45%) than in the previous context. As stated above, we suspect that students see more contrast between the reference points in "2 points, different x and y" than in "2 points, same y" and thus may be more open to considering an angular description of the motion. This difference suggests that instruction on angular momentum may be improved by including reference points with which students naturally find some contrast in an angular sense.

In the second column of Table 7-2, we show the effect of cueing students to think about collisions (with the *Translated effects* question) before answering the *Point dependence* question. Student performance appears to be very similar, except that the number of students who stated that the angular momentum was zero was reduced (from 45% to 30%). This difference (if reproducible) may indicate that it is useful for students to consider the concept of CVL in the context of collisions.

Table 7-2: Student performance on the *Point dependence* question asked in the "2 points, same y" and "2 points, same y, with collisions" versions of the "Free small ball" context before working through the tutorial *Conservation of Angular Momentum.*

A small ball moves with constant velocity on a line parallel to the line containing points A and B; the ball has passed point A, but not point B; the ball is closer to point A.	Without collisions	with collisions
	UW 121, N=57 1 section	UW 121, N=112 1 section
$\lvert\vec{L}_{ball,A}\rvert = \lvert\vec{L}_{ball,B}\rvert \neq 0$ (correct)	10%	15%
$\lvert\vec{L}_{ball,A}\rvert = \lvert\vec{L}_{ball,B}\rvert = 0$	45%	30%
(ω-dominant) $\lvert\vec{L}_{ball,A}\rvert > \lvert\vec{L}_{ball,B}\rvert$	25%	25%
(r-dominant) $\lvert\vec{L}_{ball,A}\rvert < \lvert\vec{L}_{ball,B}\rvert$	20%	25%

b. The *Time dependence* question

As shown in Table 7-3, most students (70%) tend to think that the angular momentum of an object changes as it passes by a reference point. As the object approaches the reference point, the distance between them decreases; if a student focuses on how the distance is changing, they tend to state that the angular momentum also decreases (25%). (These students tend to be the same ones that used the r-dominant approach to the *Point dependence* question.) Many students observed that the angular speed of the object with respect to the reference point increases as the object moves closer. Students who focused on this fact tended to state that the angular momentum increases as well (40%). (Again, these students tended to be the be the same as those who gave an ω -dominant approach to the *Point dependence* question.) In the bottom three rows of Table 7-3, we have divided students' correct answers to the *Time dependence* question according to their answers to the *Direction of CVL* question, in which students stated the direction of the angular momentum. Thus, the number of students (20%) in the bottom row represents those who were thinking correctly *or* attributing some direction to the angular momentum other than zero or the direction of the object's linear momentum.

Table 7-3: Student performance on the *Time dependence* question asked in the "2 points, different x and y" version of the "Free small ball" context before working through the tutorial *Conservation of Angular Momentum.*

Point A is on the path of the ball; point D is not.	UW 121, N=303 3 sections
$\vert\vec{L}_{ball,D}\vert$ is constant (correct).	30%
(ω-dominant) $\vert\vec{L}_{ball,D}\vert$ is increasing.	40%
(r-dominant) $\vert\vec{L}_{ball,D}\vert$ is decreasing.	25%
(p-dominant) $\vec{L}_{ball,D}$ is constant and points to the right.	5%
$\vec{L}_{ball,D}$ remains equal to zero.	10%
Other "$\vert\vec{L}_{ball,D}\vert$ is constant."	20%

The bottom group of rows shows correct responses filtered according to students' responses to the *Direction of CVL* question.

Table 7-4 shows student performance on the *Time dependence* question in the "2 points, same y" and "2 points, same y, with collisions" versions of the "Free small ball" context. As in Table 7-2, having students think about collisions before answering the question may reduce tendency to treat CVL as identically zero. Otherwise, performance appears not to depend on whether the students have thought about the collisions involved in answering the *Translated effects* questions. Again, here the correct answer to the *Time dependence* question includes the major errors of treating CVL like linear momentum or as identically zero. These errors are separated in the bottom three rows of Table 7-4 according to students' responses to the *Direction of CVL* question. It is interesting that most students who think that the magnitudes of the angular momentums change in time think that they change in opposite ways; very few students think that both angular momentums are increasing or that both are decreasing. We can understand "$|\vec{L}_{ball,A}|$ is increasing, $|\vec{L}_{ball,B}|$ is decreasing" as the r-dominant response and "$|\vec{L}_{ball,A}|$ is decreasing, $|\vec{L}_{ball,B}|$ is increasing" as the ω-dominant response. As before, students tended to be consistent in taking one of these approaches to both the *Point dependence* and *Time dependence* questions.

Table 7-4: Student performance on the *Time dependence* question asked in the "2 points, same y" and "2 points, same y, with collisions" versions of the "Free small ball" context before working through the tutorial *Conservation of Angular Momentum.*

A small ball moves with constant velocity on a line parallel to the line containing points A and B; the ball has passed point A, but not point B; the ball is closer to point A.	Without collisions UW 121, N=57 1 section	with collisions UW 121, N=112 1 section
$\lvert\vec{L}_{ball,A}\rvert$ and $\lvert\vec{L}_{ball,B}\rvert$ are both constant (correct).	50%	45%
(r) $\lvert\vec{L}_{ball,A}\rvert$ is increasing, $\lvert\vec{L}_{ball,B}\rvert$ is decreasing.	20%	20%
(ω) $\lvert\vec{L}_{ball,A}\rvert$ is decreasing, $\lvert\vec{L}_{ball,B}\rvert$ is increasing.	25%	25%
Both $\lvert\vec{L}_{ball,A}\rvert$ and $\lvert\vec{L}_{ball,B}\rvert$ are increasing.	5%	5%
Both $\lvert\vec{L}_{ball,A}\rvert$ and $\lvert\vec{L}_{ball,B}\rvert$ are decreasing.	5%	5%
(p) $\vec{L}_{ball,A}$ and $\vec{L}_{ball,B}$ are constant and point to the right.	10%	10%
$\vec{L}_{ball,A}$ and $\vec{L}_{ball,B}$ remain equal to zero.	35%	20%
Other "$\lvert\vec{L}_{ball,A}\rvert$ and $\lvert\vec{L}_{ball,B}\rvert$ remain constant."	5%	15%

The bottom group of rows shows correct responses filtered according to students' responses to the *Direction of CVL* question.

c. The *Direction of CVL* question

As shown in Table 7-5 and Table 7-6, students have a variety of ideas about the direction of CVL. Only about half of the available answers were chosen with any significant frequency. Students tend to state that the direction of CVL is zero, in the direction of linear momentum, directed toward the reference point (as if taking the radial component of the linear momentum), or directed perpendicular to the line connecting the object to the reference point (is if taking the "tangential" component of the linear momentum). In each context, about 10% of students chose either "out of the page" or "into the page." (We consider choosing the direction opposite the correct one to be a very mild error, which may result simply from using the left hand instead of the right hand.) Though the two answers to this question that seem to involve taking components of the linear momentum would seem to correspond well with either r-dominant or ω-dominant reasoning, responses to the *Direction of CVL* question tend to correlate only very weakly with corresponding responses to either the *Point dependence* or *Time dependence* questions.

Table 7-5: Student performance on the *Direction of CVL* question asked in the "2 points, different x and y" version of the "Free small ball" context before working through the tutorial *Conservation of Angular Momentum.*

At the instant of interest, the ball is below and to the left of point D.	UW 121, N=303 3 sections
$\vec{L}_{ball,D}$ points out of the page (correct).	10%
$\vec{L}_{ball,D}$ points into the page.	0%
$\vec{L}_{ball,D}$ is zero.	10%
(p-dominant) $\vec{L}_{ball,D}$ points to the right.	15%
$\vec{L}_{ball,D}$ points up and to the right (toward point D).	25%
$\vec{L}_{ball,D}$ points down and to the right.	10%

Table 7-6: Student performance on the *Direction of CVL* question asked in the "points, same y" and "2 points, same y, with collisions" versions of the "Free small ball" context before working through the tutorial *Conservation of Angular Momentum.*

	without collisions	with collisions
At the instant of interest, the ball is above and to the right of point B.	UW 121, N=57 1 section	UW 121, N=112 1 section
$\vec{L}_{ball,B}$ points into the page (correct).	< 5%	5%
$\vec{L}_{ball,B}$ points out of the page.	5%	5%
$\vec{L}_{ball,B}$ is zero.	35%	20%
(p-dominant) $\vec{L}_{ball,B}$ points to the right.	20%	25%
$\vec{L}_{ball,B}$ points down and to the right (toward point B).	20%	10%
$\vec{L}_{ball,B}$ points up and to the right.	10%	15%

d. The *Translated effects* questions

As expected, most students are successful at recognizing that Experiments A and B in the *Translated effects* questions are essentially identical (75% correct on both parts). Students who state incorrectly that the final angular speeds are different in the two experiments seem mostly to be misinterpreting the question, instead focusing on the angular speed of the puck with respect to point A or B before the collision. The fact that students are very successful on this task means that, if the task were incorporated into instruction in the future, it could be used as part of the foundation for some line of reasoning, without much (if any) instruction required to establish it. However, whether it would be *useful* to students to build on such ideas is another question. The data we have gathered so far do not suggest that the *Translated effects* questions strongly affect how students think about CVL on a pretest (that is, without guidance from some form of explicit instruction).

Table 7-7: Student performance on the *Translated effects* questions, asked in the "2 points, same y, with collisions" version of the "Free small ball" context before working through the tutorial *Conservation of Angular Momentum.*

$v_{cm}(X)$ *is the final speed of the center of mass of the puck-disk system in Experiment X, and* $\omega(X)$ *is the final angular speed of the puck-disk system about its own center of mass.*	UW 121, N=112 1 section
$v_{cm}(A) = v_{cm}(B)$ (correct)	85%
$v_{cm}(A) > v_{cm}(B)$	5%
$v_{cm}(A) < v_{cm}(B)$	5%
$\omega(A) = \omega(B)$ (correct)	75%
$\omega(A) > \omega(B)$	10%
$\omega(A) < \omega(B)$	15%
Both parts correct	75%

7. Discussion of patterns in student reasoning

a. Tendency to treat CVL as identically zero

As data presented above have suggested, many students appear to think that an object moving with constant velocity (without spinning) has zero angular momentum. Two example responses to the *Direction of CVL* question in the "Free small ball context" illustrate this trend:

> " $\vec{L}_{ball,B}$ is zero. Since there is no rotation, the angular momentum is zero."
>
> " $\vec{L}_{ball,B}$ is zero. The ball is not spinning so it doesn't have any angular momentum."

Thus, students who treat CVL as identically zero appear to associate angular momentum exclusively with the "spinning" motion of objects only. In defense of these "incorrect" responses, we note that these responses exhibit a kind of conceptual minimalism that is appropriate in science. That is, it is *inappropriate* in science to believe in concepts for which there is no evidence or that have no function in theory. If students have not seen any evidence that suggests angular momentum ought to be attributed to an object moving with constant velocity, then, in that sense, it is "correct" to refrain from adding angular momentum to the descriptions of motion of the ball. (In a similar way, even though physicists now believe neutrinos "exist," we think it is correct that physicists did not believe in (or had not imagined) the existence of the neutrino before it had been observed that electrons in beta-decay have a continuous spectrum of energy. When Pauli observed this fact, he proposed the neutrino concept as a means of saving energy conservation.) Without this sense of frugality with concepts, students might be inclined to accept without question an endless list of different quantities of motion such as mv, $mv^2/2$, mvr, $m^3v^4/6$, $m^5v^6/15$, *etc.* We believe most physics instructors would prefer that their students require some justification for introducing yet another quantity of motion. Thus, though treatment of CVL as identically zero is in one sense incorrect, it indicates a valuable pattern of thinking among students and suggests perhaps that students might more easily treat CVL correctly if they understood the evidence for and purpose of the concept.

b. Linear momentum (*p*) dominant concept of CVL

A pattern in student reasoning that we believe is closely related to the one described above is the tendency to treat CVL as if it were identical to linear momentum. The following two responses are drawn from the *Direction of CVL* question in the "Free small ball" context:

> "$\vec{L}_{ball,B}$ points to the right. Since the ball is moving to the right, the center of mass of the ball must have some momentum to the right."
>
> "$\vec{L}_{ball,B}$ points to the right. The momentum will be carried the same direction as the velocity vector."

These responses could be interpreted as yet another instance of the common tendency of students to conflate closely related concepts. We believe it is more optimistic (though perhaps not more correct) to interpret this tendency in reasoning as another example (like that in the previous section) of students' sense of frugality with concepts. That is, to treat angular momentum as if it were linear momentum in this context may not be necessarily to confuse them, but instead perhaps to see value in using a small number of concepts to describe a simple motion.

c. Angular velocity (ω) dominant concept of CVL

Many students treat CVL as though it is identical to the angular velocity with which the line connecting the object to the reference point moves. We say that these students use an "ω-dominant" concept of CVL. The following two explanations were given by students who answered the *Time dependence* question (in the "Free small ball" context) by stating that the angular momentum of the ball with respect to point B is increasing:

> "Since as the ball moves along the path, the change in the angle increases, increasing the angular velocity and thus increasing the angular momentum."
>
> "The closer the ball is to the point, the more degrees it changes per unit time."

Most students who answer the *Time dependence* question this way use similar reasoning when answering the *Point dependence* question in the same context. The two students who gave the following explanations stated that the angular momentum of the ball was greater with respect to point A than point B:

> "Just a guess, probably because it is closer to point A and it therefore it would seem faster ... Like looking outside a moving car really close ... It seems to be moving really fast."
>
> "The angle is changing for A at a greater rate than it is for B at that instant."

d. Radius (r) dominant concept of CVL

Another common tendency in student reasoning about angular momentum is to focus on the distance (or magnitude of the position vector) between the object and the reference point. We call this approach the "r-dominant" concept of CVL. These two responses to the *Time dependence* question illustrate the pattern:

> "Angular momentum = mvd, and if the distance between the ball and point B is decreasing so must angular momentum."
>
> "The angular momentum is decreasing because the ball is getting closer to point B."

Just as students who used the ω-dominant concept of CVL tended to use similar reasoning to answer both the *Time dependence* and *Point dependence* questions, so did students who used the r-dominant concept. Two students who stated that the angular momentum is less with respect to point A than point B gave the following explanations:

> "L = Pr. r is greater at this instant from B."
>
> "Angular momentum is less because the ball is closer to A (the vectors are shorter), so both crosses will yield a larger angular momentum."

The first explanation above treats the (cross) product of position and linear momentum as if it were a simple product of scalars. In the second explanation, the student appears to be aware that angular momentum is a cross product of vectors, but appears to conclude something about the magnitude of the cross product from a judgment about the magnitudes of the vectors in the product. This tendency in reasoning about cross products in abstract contexts was described by Ortiz.[3]

B. DESCRIPTION OF VERSION A OF PART 2 OF TUTORIAL SEQUENCE *CONSERVATION OF ANGULAR MOMENTUM*

As discussed in the previous chapter, the tutorial *Conservation of Angular Momentum* can be thought of as having two main parts, *Part 1* and *Part 2*. The previous chapter discusses Part 1 of the tutorial and homework, and related student performance on pre- and post-tests. This chapter discusses the corresponding elements for Part 2. The following sections of this chapter describe three versions of Part 2, "version A," "version B," and "version C." (A copy of each version can be found in Appendix C of this dissertation.) Version A is not strictly an "initial" version, since the tutorial was modified many times, with modifications of varied magnitude, before it arrived at version A. However, the contrast between versions A, B, and C, and the intermediate improvements of our understanding of student learning that led to the development of later versions are what we expect will be most useful to the reader.

There is only one version of Part 2 of the homework, which is also described in this section.

1. Detailed description of version A of Part 2 of tutorial

The angular momentum of a point object may be thought of (and has been called) the "moment of momentum." This definition of angular momentum as a vector product (or outer product) of position with linear momentum, though very abstract, may be understood as logically prior to the attribution of angular momentum to a more familiar object like a spinning wheel. (That is, we physicists formally define the angular momentum of a spinning object as a summation of the infinitesimal angular momenta of its constituent parts.) Also, some homework problems assigned to students of the introductory calculus-based physics course involve angular momentum transfer between a point-like object and a rigid, extended object. Therefore, in order to help students understand these issues in the textbook and standard homework, we focused Part 2 of the tutorial on the introduction and properties of the angular momentum of a point-like object moving with constant velocity. As they worked through version A of Part 2, students explored properties such as: constant velocity results in constant angular momentum, which is generally not zero; and this angular momentum is reference-point dependent, both in magnitude and direction.

Version A of Part 2 began by describing a point-like object moving with constant velocity past two specified stationary points. The example was an astronaut floating through a large doorway in a space station, as shown in the left cell of Figure 7-3. At the edges of the doorway were two points, C and D, which were different distances away from and on opposite sides of the straight-line path of the astronaut. The line connecting points C and D intersected the path of the astronaut at a right angle. Students were shown two figures; the first showed an instant when the astronaut is at this point of intersection, "directly beneath point C," and the second figure showed the astronaut at some later time, after passing through the doorway, as in the upper right cell of Figure 7-3. Students were asked to draw, on each figure, an arrow representing the position vector of the astronaut with respect to point C ($\vec{r}_{a,C}$), and an arrow representing the linear momentum of the astronaut ($\vec{p}_a$). After doing this, on the second figure, students drew the vector components of the position vector that are perpendicular ($\vec{r}_{\perp}$) and parallel ($\vec{r}_{\parallel}$) to the astronaut's linear momentum (see the lower right cell of Figure 7-3).

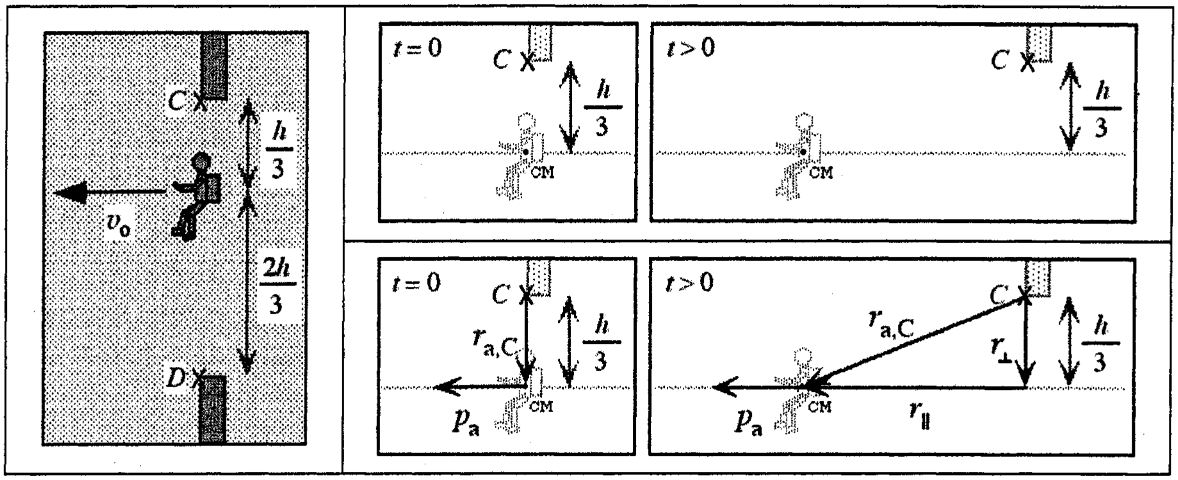

Figure 7-3: Context for version A of Part 2 of the tutorial *Conservation of Angular Momentum*. The diagram at upper right is shown as printed in the tutorial and is correctly completed at lower right.

Some students seemed to need help determining the directions of these component vectors, or sometimes even a reminder that the components, not just the original vectors, also require arrowheads. Instructors sometimes asked a student to describe the direction of $\vec{r}_{a,C}$ in terms of up. down, left, and right. When the student answered "down and to the left," then they often quickly saw that $\vec{r}_\perp$ was down, and $\vec{r}_\parallel$ was to the left. The students were then instructed to write a *vector* equation relating $\vec{r}_{a,C}$, $\vec{r}_\perp$, and $\vec{r}_\parallel$. The equation we wanted students to write stated that the position vector was the vector sum of the two component vectors ($\vec{r}_{a,C} = \vec{r}_\perp + \vec{r}_\parallel$). However, we found that with this instruction, most students wrote a relation according to the Pythagorean theorem, since the vectors form a right triangle. When students wrote such an equation, the tutorial instructors then had to make sure to get to each table to help students arrive at a vector equation. When we inserted into the tutorial the note "The Pythagorean theorem is not a vector equation," the effect was positive – more students were able to have productive discussions without requiring TA intervention. The next instruction was to use the vector equation to replace $\vec{r}_{a,C}$ in the definition of the angular momentum of the astronaut with respect to point C: $\vec{l}_{a,C} = \vec{r}_{a,C} \times \vec{p}_a$. Here students substituted the vector sum of the components of $\vec{r}_{a,C}$, and distributed the cross product over the sum. Then they used the fact (that a sufficient number of our students seem to know off-hand) that the cross product of parallel vectors is zero. Therefore, the term $\vec{r}_\parallel \times \vec{p}_a$ was zero, leaving

$\vec{l}_{a,C} = \vec{r}_\perp \times \vec{p}_a$. We then asked students to determine the direction of $\vec{l}_{a,C}$. Here we expected either that at least one student at a table of four would know how to determine the direction of the cross product of perpendicular vectors, or that a tutorial instructor would quickly describe a procedure for doing it. One goal of the preceding exercises was first to establish that the definition of angular momentum gave us a non-zero angular momentum for the astronaut (with respect to point C), even though the astronaut was not spinning or moving in a curved path. In the next question, we tried to help students get an intuitive sense for why it is not absurd that the angular momentum is non-zero.

We changed the context from an imaginary setting with an astronaut to an activity set in the classroom. Each student visualized (or performed, according to preference) the motion of a friend walking around him/her, first in a semi-circular path, with the subject at the center of the circle, and then in a straight-line path, passing by the subject. Students were told to stand in place and make sure to keep facing the friend as the friend moved. Then they were asked to describe how their *own* motion is the same in each case. Here we wanted students to recognize that, in each case, in order to follow the motion of the friend, they had to turn their heads (or bodies) as the friend moved. Students then used the results of this activity to explain why it is reasonable that the angular momentum of the astronaut was not zero, even though it did not spin or move in a curved path.

Students then considered the time-dependence of the angular momentum of the astronaut with respect to point C. They observed that the expression that they had derived showed the magnitude of $\vec{l}_{a,C}$ as equal to the product $r_\perp p_a$. Since neither of these quantities changed with time, the angular momentum was also constant. Some students, perhaps wondering if this result depended on some special feature of the chosen path of the astronaut, spontaneously explored the possibility of a "diagonal" path past point C, attempting to draw a straight-line path for which the perpendicular component of the position vector would not be constant. These students then usually realized that such a construction is not possible. Students then compared the angular momentum of the astronaut with respect to point C ($\vec{l}_{a,C}$) to that with respect to point D ($\vec{l}_{a,D}$), in both magnitude and direction. Since point D was farther from the astronaut's path than point C, $l_{a,D}$ was greater than $l_{a,C}$. Also, since points C and D were on opposite sides of the path, $\vec{l}_{a,C}$ and $\vec{l}_{a,D}$ pointed in opposite directions. Then students were asked whether their comparisons would change if they were to determine the direction of angular momentum with the left hand instead of the right hand. We observed a few times that one student started to answer affirmatively, that the comparison of directions *would* change when the opposite hand is used, and then another student pointed out that, since each angular momentum is has switched directions, they would still be opposite to each other. (Here, again, we hoped to teach students implicitly something about the nature of conventional knowledge in physics – that when a different convention is chosen, some comparisons may change and others may remain unchanged. In particular, the "conventional" wisdom in physics is that an even number of applications of the right-hand rule will result in an opportunity to make a statement that does not depend on the right-hand rule. We did not expect that many students would take these lessons away from this part of the tutorial, but we wanted to create the space in which such discussions could happen, if instructional time and interests were right.)

Students then considered a printed Student dialogue, in which different ideas about the angular momentum of the astronaut were stated:

Student 1: The angular momentum of a system does not change when there is zero net torque on the system. There is clearly no torque at all on the astronaut, so the angular momentum must stay constant as the astronaut floats by.

Student 2: The angular momentum of the astronaut is not conserved because the angular momentum with respect to point C is not the same as the angular momentum with respect to point D.

This alternative (correct) reasoning given by Student 1 was present so that students might see another way to answer the previous question about time-dependence, while acknowledging an issue not emphasized in the tutorial – the relationship between angular momentum and net torque. Most students recognized the statement as correct, though they often asked for confirmation from a TA, perhaps because they expected "Student" statements to be incorrect in some way. The second statement demonstrated confusion between reference-point invariance and conservation (similar to the confusion between reference-*frame* invariance and conservation described by Boudreaux[4]). Most students correctly identified the error. (It was encouraging to see students handle this statement well, considering how commonly physics graduate students miss the distinction between frame invariance and conservation.)

We then asked students to find a stationary reference point with respect to which the angular momentum was zero. Students had little trouble finding a point on the path of the astronaut, though often they identified the single point on that path that is collinear with points C and D. We encouraged students to consider whether there were more points that satisfied the specification, and to describe where they were.

Finally, students summarized their results in a general form by stating whether a point-like object moving with constant velocity has constant angular momentum (it does) and whether the angular momentum depends on the reference point (it does).

2. Detailed description of Part 2 of the homework

The homework for the tutorial *Conservation of Angular Momentum* consists of two problems. The first problem deals with angular momentum of a spinning wheel, and is discussed in the previous chapter. Problem 2 concerns the simultaneous conservation of both linear and angular momentum, in the context of "off-center" collisions in systems under the influence of no external forces. Four "candidate" collisions of two balls and a stick are depicted; students determine whether each collision violates linear momentum conservation, angular momentum conservation, or both. On this basis, students judge whether each collision is possible or impossible. Of course, most students, unlike physicists, do not prefer to judge the possibility of an event by finding whether it is consistent with the laws of physics. We have observed that students, when working this problem, tend to judge the possibility of the event (before considering the relevant laws of physics) by comparing the proposed outcome with some intuitively expected outcome; if the two do not match, the student concludes that the event is not possible. We hope that the exercise helps students, if only in some cases incrementally and gradually, learn to explain the impossibility of various events by employing laws of physics. Therefore, in order to encourage students to relate the possibility of an event to the laws of physics, students are *first* asked to draw arrows (or symbols representing out of or into the page) to represent the linear momentum vector of the system both before and after the collision and the angular momentum vector (with respect to a specified point) both before and after the collision. If either the linear momentum or the angular momentum is not constant in direction, then students should infer that the conservation of that quantity has been violated, and the event is not possible.

All four collisions (shown in Figure 7-4) occur in a weightless, frictionless environment. In the first collision, a ball moves to the right toward the top end of the stick. Another ball is sitting at rest next to the bottom of the stick, on the right side (see figure). After the collision, the first ball is at rest next to the stick (on the top left), and the second ball moves away from the stick with the same velocity that the first ball had initially. In this case, the linear momentum of the system is the same before and after the collision. The angular momentum, however, is initially into the page with respect to the center of the stick, and after the collision, it is out of the page with respect to the center of the stick. Since the angular momentum has flipped directions, it cannot be constant. Therefore, the first collision is not possible. The most common incorrect assessment of this collision is that neither conservation law is violated. It is common for students to associate zero angular momentum with an object that moves at constant velocity, regardless of the reference point with respect to which the angular momentum is taken. Thus, according to this thinking, the angular momentum of the system is zero before and after the collision, thereby not violating angular momentum conservation. (However, students who think this way are still likely to say that the collision is not possible, because it does not look right.)

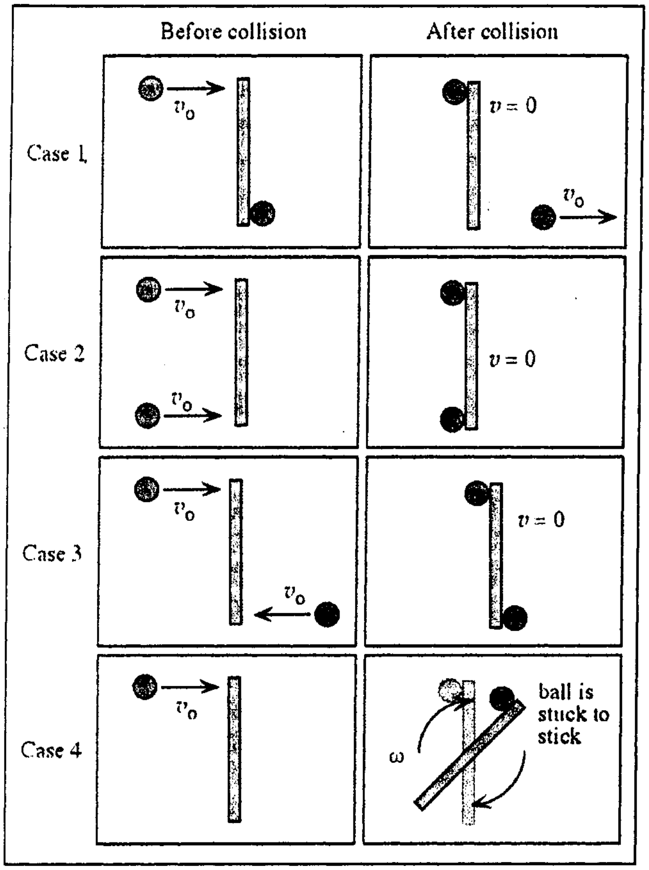

Figure 7-4: Context for problem 2 of the homework for *Conservation of Angular Momentum.*

In the second collision, two balls move to the right, one ball approaching the top end and the other the bottom end of the stick. After the collision, the balls are next to the stick, at rest. In this case, linear momentum conservation is violated, since initially the system momentum was to the right, and after the collision it is zero. Angular momentum conservation, however, is not violated; the top ball has angular momentum into the page, and the bottom ball has angular momentum out of the page of equal magnitude. Thus, the angular momentum both before and after the collision is zero. Of course, it is possible here to give the correct answer with incorrect reasoning, by associating zero initial angular momentum with each ball.

In the third collision, a ball moves right, approaching the top left side of the stick, and another ball moves left, toward the bottom right side of the stick. In this case, the initial angular momentum of the system is out of the page, and the final angular momentum of the system is zero. The linear momentum of the system is zero both before and after the collision.

If a student does not recognize the non-zero angular momentum of each ball, then they will conclude that angular momentum conservation is not violated. However, even though both conservation laws would be satisfied in that analysis, students expect that the stick "would start rotating." We hope any such conflict would encourage students to reconsider the angular momentum analysis. (Contrariwise, it is possible that students might find the depicted outcome palatable, thinking, as we have observed students do sometimes, that frictionlessness can cause an absence of rotation where one might otherwise expect it.)

Finally, the fourth collision shows a single ball moving to the right toward the top left end of the stick. Afterwards, the ball is attached to the stick, which rotates about its own center, without translating. Before and after the collision, the angular momentum of the system is into the page; there is not enough information given to determine whether it is the same magnitude. Linear momentum, however, is not conserved here, since the linear momentum vector, after the collision, would decrease in magnitude and perpetually change direction while the ball moves in a circle around the center of the stick (whose linear momentum is zero).

The goal of having students think about this last impossible collision, in particular, is to help them understand that it is because of linear momentum conservation that the (center of mass of the) stick-and-ball system *must* move in the original direction of motion of the ball. Furthermore, the final linear momentum of the system, and thus the final velocity of the center of mass of the system, does not depend on the impact parameter of the ball. This point is made more explicit in version B of Part 2 of the tutorial, but in version A, this homework problem was the only place in the tutorial sequence that we implied that linear momentum and angular momentum were separately conserved. At the time we were using version A, we were interested in observing the degree to which students went beyond the questions asked in the homework to reach the conclusion that linear momentum and angular momentum are separately conserved. The next section includes the description of a post-test on which we measured students' tendency to recognize that the final velocity of the center of mass is independent of the impact parameter of the ball.

C. STUDENT UNDERSTANDING OF CONSTANT VELOCITY ANGULAR MOMENTUM AFTER VERSION A OF TUTORIAL SEQUENCE *CONSERVATION OF ANGULAR MOMENTUM*

1. Description of additional question contexts

a. Ball and rod

In this context, a ball collides with a rod in two separate experiments (1 and 2) in a weightless, frictionless environment (see Figure 7-5). The ball and rod are not in contact with any other objects. In both experiments the ball moves with the same initial speed v_0 toward the rod and sticks to it. In Experiment 1, the ball hits the rod at its center; in Experiment 2, the ball hits the rod near its end. Before the ball hits the rod, the center of the rod is at point A, which is fixed in space (*i.e.*, if the rod begins to move, its center may leave point A).

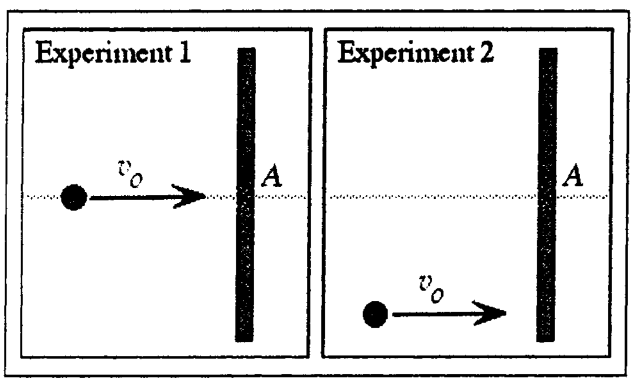

Figure 7-5: The "Ball and rod" context.

2. Description of additional questions

a. The *p vectors* question

In the *p vectors* question, we ask students to draw arrows to represent the magnitude and direction of the total linear momentum of the ball/rod system both before and after each experiment. Students draw their arrows in the table shown in Figure 7-6 and explain their reasoning briefly.

Linear momentum vectors	Experiment 1	Experiment 2
Before collision		
After collision		

Figure 7-6: Table presented to students in the *p vectors* question.

b. The *L vectors* question

In the *L vectors* question, students are asked to indicate the direction (not magnitude) of the angular momentum of the ball/rod system with respect to point A before and after the collision in each experiment. Students are reminded to use the symbols ⊗ and ⊙ to represent the directions "into the page" and "out of the page," respectively. They draw the symbols in the table shown in Figure 7-7 and explain their reasoning briefly.

Angular momentum vectors	Experiment 1	Experiment 2
Before collision		
After collision		

Figure 7-7: Table presented to students in the *L vectors* question.

c. The *Final center-of-mass speed* question

After working the *p vectors* and *L vectors* questions, students are asked, "If the final speed of the center of mass of the ball/rod system in Experiment 1 greater than, less than, or equal to that in Experiment 2?"

3. Correct answers to additional questions

The initial momentum vectors of the ball/rod system in Experiments 1 and 2 are equal because, in each experiment, the ball moves with the same initial velocity and the rod is at rest. Each final momentum vector is equal (in direction *and* magnitude) to the initial momentum vectors because the net force on each system is zero. A correctly completed diagram is shown in Figure 7-8.

Linear momentum vectors	Experiment 1	Experiment 2
Before collision	→	→
After collision	→	→

Figure 7-8: A correctly completed table for the *p vectors* question.

The initial angular momentum vector of the ball/rod system with respect to point A is zero in Experiment 1, because the radius vector pointing from point A to the ball is anti-parallel to its linear momentum vector. The angular momentum of the ball (and the system), which is the cross product of these vectors, is zero because they are anti-parallel. The angular momentum of the ball with respect to point A in Experiment 2 is directed out of the page, because the radius vector in this case has a component directed down the page, and according to the right-hand rule, down × right = out of the page. (Alternatively, we would also consider it correct to refer to this angular momentum as counter-clockwise in direction, or sense.) The angular momentum vector of the system in each experiment is unchanged from before the collision to after, because the net torque on each system is zero. Figure 7-9 shows a correctly completed table for the *L vectors* question.

Angular momentum vectors	Experiment 1	Experiment 2
Before collision	zero	⊙ or ↺
After collision	zero	⊙ or ↺

Figure 7-9: A correctly completed table for the *L vectors* question.

The final speed of the center of mass is the same in the two experiments. "Velocity of the center of mass" is defined to be the total linear momentum of the system divided by its total mass. Since both systems have the same linear momentum and the same mass, the speed of the center of mass is also the same.

4. Administration of questions

Students enrolled in UW 121 in autumn quarter 2004 worked through version A of the tutorial sequence *Conservation of Angular Momentum.* Two of these sections were given a series of questions about CVL on their final exam. In one section, students answered (in free-response form) the *p vectors, L vectors,* and *Final center-of-mass speed* question in the "Ball and rod" context. In the other section, students were given (in multiple-choice form) the *Point dependence, Time dependence,* and *Direction of CVL* questions in the "3 points" version of the "Free small ball" context. Because it was a final exam, all instruction had been completed.

5. Overview of student performance

a. Questions asked in the "Free small ball" context

As shown in Table 7-8, there is a sharp decrease (from pretest to post-test) in the number of students who say that the angular momentum of the ball with respect to point B is zero (compare 30-45% to less than 5%). However, many students still used *r*-dominant and ω-dominant concepts of CVL after tutorial instruction. In fact, before the tutorial, about 50% used one of those two lines of reasoning to answer the *Point dependence* question, and after version A of the tutorial, still, about 50% gave *r*-dominant or ω-dominant answers. However, we would hesitate to say that 50% of students did not change how they think; there may still be a positive evolution in student thinking such that there is a net zero flux into (and out of) this combination of categories.

Table 7-8: Student performance on the *Point dependence* question asked in the "3 points" version of the "Free small ball" context after version A of the tutorial *Conservation of Angular Momentum.*

A small ball moves with constant velocity on a line parallel to the line containing points A and B; the ball has passed point A, but not point B; the ball is closer to point A.	UW 121, N=97 1 section Multiple-choice
$\left\|\vec{L}_{ball,A}\right\| = \left\|\vec{L}_{ball,B}\right\| \neq 0$ (correct)	40%
$\left\|\vec{L}_{ball,A}\right\| = \left\|\vec{L}_{ball,B}\right\| = 0$	<5%
(ω-dominant) $\left\|\vec{L}_{ball,A}\right\| > \left\|\vec{L}_{ball,B}\right\|$	20%
(r-dominant) $\left\|\vec{L}_{ball,A}\right\| < \left\|\vec{L}_{ball,B}\right\|$	35%

Table 7-9 shows similar trends, from another perspective. After version A of the tutorial, about 40% of students recognized that the angular momentum of the ball is constant with respect to both points A and B. As with the *Point dependence* question, around 50-60% of students gave r-dominant or ω-dominant answers. Also, as students did before tutorial, they tended to take a consistent approach: about 25% gave r-dominant answers to both the *Point dependence* and *Time dependence* questions, while 15% gave ω-dominant answers to both.

Table 7-9: Student performance on the *Time dependence* question asked in the "3 points" version of the "Free small ball" context after version A of the tutorial *Conservation of Angular Momentum.*

A small ball moves with constant velocity on a line parallel to the line containing points A and B; the ball has passed point A, but not point B; the ball is closer to point A.	UW 121, N=97 1 section Multiple-choice
$\lvert\vec{L}_{ball,A}\rvert$ and $\lvert\vec{L}_{ball,B}\rvert$ are both constant (correct).	40%
(r-dominant) $\lvert\vec{L}_{ball,A}\rvert$ is increasing, $\lvert\vec{L}_{ball,B}\rvert$ is decreasing.	35%
(ω-dominant) $\lvert\vec{L}_{ball,A}\rvert$ is decreasing, $\lvert\vec{L}_{ball,B}\rvert$ is increasing.	25%
Both $\lvert\vec{L}_{ball,A}\rvert$ and $\lvert\vec{L}_{ball,B}\rvert$ are increasing.	<5%
Both $\lvert\vec{L}_{ball,A}\rvert$ and $\lvert\vec{L}_{ball,B}\rvert$ are decreasing.	<5%

Student performance on the *Direction of CVL* question after version A of the tutorial is shown in Table 7-10. Here about 60% of students correctly determined the directions of the angular momentum vectors of the ball with respect to points A, B, and C, while 25% gave a response that is correct except for a "sign" or "convention" error. Only 5% stated that these angular momentum vectors all point in different directions, as we would expect a student who found various components of the linear momentum to conclude. Finally, 10% chose the option that these vectors all point in the same direction, as we might expect if a student were using a p-dominant concept of CVL. However, because two of the four options seemed to emphasize the directions "into the page" and "out of the page," students were probably drawn to choose one of those. That is, we would not expect as many as 60% of students to determine the directions of the three angular momentum vectors correctly if we were to ask the same question in free-response form.

Table 7-10: Student performance on the *Direction of CVL* question asked in the "3 points" version of the "Free small ball" context after version A of the tutorial *Conservation of Angular Momentum.*

A small ball moves with constant velocity on a line parallel to the line containing points A and B; point C is on the opposite side of the ball's path from points A and B. Students compared the directions of the angular momentum vectors of the ball with respect to each point.	UW 121, N=97 1 section Multiple-choice
Two vectors point into the page, and one points out of the page (correct).	60%
Two vectors point out of the page, and one points into the page.	25%
Each of these vectors points in a different direction.	5%
These vectors all point in the same direction.	10%

b. Questions asked in the "Ball and rod" context

Most students gave one of three responses to the *p vectors* question, as shown in Table 7-11. About 40% stated correctly that the momentum vectors were all equal. A total of about 35% gave responses that included the feature that the final linear momentum vector was reduced in Experiment 2 but not Experiment 1, suggesting that the "introduction" of rotational motion of the ball and rod in Experiment 2 ought to correspond with some kind of reduction of linear motion. Some students (10%) even stated that the final linear momentum would be zero in Experiment 2, as if "all" of the linear momentum were converted somehow into rotational motion.

Table 7-11: Student performance on the *p vectors* question (in the "Ball and rod" context) after version A of the tutorial *Conservation of Angular Momentum.*

A ball collides with and sticks to a rod near its center in Experiment 1, and near its end in Experiment 2.	UW 121, N=156 1 section
All p vectors point to the right, and are equal in magnitude (correct).	40%
All p vectors point to the right, and the final momentum in Exp. 2 is reduced.	20%
All p vectors point to the right, except the final momentum in Exp. 2, which is zero.	10%
p will decrease in Experiment 2, but not Experiment 1.	35%

In Table 7-12, we see student performance on the *L vectors* question, in which students treated CVL much differently than they did in the "Free small ball" context. The most common response (35%) was to state that the angular momentum of the ball/rod system is zero before and after the collision in Experiment 1 (correct), that the angular momentum of the ball/rod system after the collision in Experiment 2 is out of the page (correct), but that the angular momentum before the collision in Experiment 2 is zero (incorrect). In fact, 60% of students stated that this initial angular momentum was zero. This number can be compared with that in the second row of Table 7-8, which is the number of students (< 5%) who said $|\vec{L}_{ball,A}|$ and $|\vec{L}_{ball,B}|$ are each zero in the "Free small ball" context. Similarly, the number of students who gave the correct answer to the *L vectors* question was much smaller (15%) than the number who gave a correct answer to the *Direction of CVL* question in the "Free small ball" context (60%). Meanwhile, 20% of students answering the *L vectors* question gave a *p*-dominant response by stating that the final angular momentum of the ball/rod system points to the right. In contrast, less than 5% of students answering questions in the "Free small ball" context stated that the angular momentum for the ball was the same for all three points and not zero. Our interpretation of these results is that students' tendency to treat CVL as identically zero as identical to linear momentum is strongly augmented by the presence of a collision, in which, according to student thinking, the final angular momentum of the ball/rod system is *not present* before the collision, but rather supplied by and converted from the linear momentum, in a way that either does (L_{2f} points to the right) or does not (L_{2f} points out of the page) preserve the direction of the original linear momentum.

Table 7-12: Student performance on the *L vectors* question (in the "Ball and rod" context) after version A of the tutorial *Conservation of Angular Momentum.*

A ball collides with and sticks to a rod near its center in Experiment 1, and near its end in Experiment 2.	UW 121, N=156 1 section
Both *L* vectors in Exp. 1 are zero, both *L* vectors in Exp. 2 are out of the page (correct).	15%
All *L* vectors are zero, except the final *L* in Exp. 2, which is out of the page.	35%
All *L* vectors are zero, except the final *L* in Exp. 2, which points to the right.	10%
Total of "final *L* in Exp. 2 points to the right"	20%
Total of "initial *L* in Exp. 2 is zero"	60%

Finally, student performance on the *Final center-of-mass speed* question after version A of the tutorial is shown in Table 7-13. Most students (55%) stated incorrectly that the final speed of the center of mass of the ball/rod system in Experiment 2 was less than that in Experiment 1. Only 25% of students correctly stated that these speeds were equal, drew four linear momentum vectors of equal magnitude, and also confirmed that the ball/rod system does rotate in Experiment 2. (A few students appeared to think that the rod will not rotate if there is no friction.) Many students justified their (incorrect) answers to this question by writing false conservation-of-energy equations, in which they equated the initial kinetic energy of the ball to the total final kinetic energy of the ball/rod system. (A correct description of energy in this situation would require a thermal energy term. However, such an equation would not be useful unless the relevant speeds had already been found using momentum conservation, or the thermal energy had been independently calculated according to the thermal properties of the objects.) We understand both the tendency of students to prefer to think in terms of energy conservation and the tendency to treat linear and angular momentum as jointly conserved as different forms of a more basic, primitive idea that physical processes happen in a way that keeps some visible, scalar quantity of motion fixed.

Table 7-13: Student performance on the *Final center-of-mass speed* question (in the "Ball and rod" context) after version A of the tutorial *Conservation of Angular Momentum.*

A ball collides with and sticks to a rod near its center in Experiment 1, and near its end in Experiment 2.	UW 121, N=156 1 section
$v_{cm}(1) = v_{cm}(2)$ (correct)	40%
$v_{cm}(1) > v_{cm}(2)$	55%
$v_{cm}(1) < v_{cm}(2)$	5%
Total of "$v_{cm}(1) = v_{cm}(2)$," "all *p* vectors are equal magnitude," and "system rotates in Exp. 2"	25%

6. Discussion of patterns in student reasoning

a. Tendency to treat linear momentum and angular momentum as jointly conserved

A very common theme in student reasoning in the "Ball and rod" context was the idea that linear momentum and angular momentum are conserved jointly, rather than separately. Students who answered the *p vectors* question by stating (correctly) that the linear momentum in Experiment 1 remained the same during the collision but that the linear momentum in Experiment 2 was reduced (either to zero or to some lesser non-zero amount) gave the following explanations:

> "Linear momentum after collision in experiment 2 is zero since it becomes angular momentum."

"In experiment 2, the clay ball will hit at one end, causes the rod to rotate, thus although the total momentum may be conserved, the linear momentum will be zero."

"Exp. 2: The system will rotate about its center of mass when the ball hits so there will be some linear momentum conserved but it will also take form as angular momentum."

"In Exp. 2, the ball strikes away from center. Since $L = \vec{r} \times \vec{p}$, and r is nonzero, some linear momentum is converted to angular momentum."

"In Exp. 2, linear momentum decreases because angular momentum increases."

"Exp. 2: Because there are no external forces, momentum is conserved, but some goes into angular momentum."

"Exp. 2, ball hits near end of the rod and produces torque, so $\vec{p}$ is turned into $\vec{L}$, so $\vec{p}_f < \vec{p}_i$."

"In Exp. 2, however, all of the ball's initial linear $\vec{p}$ is converted into angular $\vec{p}$ after the collision. That's why its final linear $\vec{p}$ is zero."

"In experiment 2 some momentum is converted to rotational momentum."

As the above responses show, there are many small variations on this idea. One student refers to "angular $\vec{p}$;" another calls it "rotational momentum." These different ways of referring to angular momentum are not strictly incorrect, though they are unconventional. What we believe these observations show is that, when thinking about momentum, students tend to use procedures (and language) that they have been taught are appropriate to use when thinking about *energy*. Some students gave answers similar to those above but explained in *terms* of energy but used the same logic, which is essentially as follows: because the system in Experiment 2 rotates, but has the same available initial motion, it must not translate as much (or at all). In fact, some students explained their answers in those more concrete terms, without appealing to any abstract terms like *momentum* or *energy*. Thus, we believe the kind of thinking represented by the responses shown above is probably better described as a common idea about "motion," rather than about linear momentum or angular momentum, *per se*.

b. Tendency to conflate force and linear momentum

Some students who answered the *Final center-of-mass speed* question correctly did so by applying a result they had learned in the tutorial *Dynamics of Rigid Bodies:*

> "Equal to. Because momentum must be conserved, the speeds must be the same. Also, in one of the tutorials we did a spool and string experiment, and both fell at the same speed. It didn't matter where the string was attached to the spool." [Response includes sketch of "Connected spools" experiment.]
>
> "Final speed of CM of system in Ex. 1 is same as the final speed of CM of system in Ex. 2 because the motion of an object is not considered when determining how the force affects the motion of the center of mass."
>
> "The final speed of the center of mass of the system in experiment 1 is equal to the final speed of the center of mass of the system in experiment 2, because the forces exerted on them have the same magnitude and *if you want to determine how a force affects the motion of the center of mass of an object, it doesn't matter where on the object the force is exerted and it doesn't matter how the force affects the rotational motion of the object.*" [Italics added to indicate quotation from *Dynamics of Rigid Bodies*]

In the third response, the student somehow quoted verbatim a long excerpt from the tutorial *Dynamics of Rigid Bodies*. (Students in UW 121 may not use any written resources during exams except a sheet of handwritten notes. Thus, assuming this student did not cheat, it is likely that she found the excerpt important enough that she either memorized it or copied it onto her allowed sheet of notes.) What all of the above responses have in common is that they found the conclusions drawn from *Dynamics of Rigid Bodies* relevant to this context. To apply the N2R principle (the above excerpt) to the "Ball and rod" context is not strictly appropriate because the ball does not exert forces of equal magnitude and direction on the rod in both experiments. In fact, even after the collision is "over" in Experiment 2, the ball and rod never stop exerting forces (of perpetually changing direction) on each other as they mutually revolve around one other.

Because an analysis of forces in any collision tends to complicated (in which lies part of the value of conservation principles), and because these students have applied a correct version of a difficult idea about forces and center-of-mass acceleration *and* arrived at the correct answer to the *Final center-of-mass speed* question, we would not emphasize to students in the introductory course that the argument presented in the above student responses is incorrect. However, as we have seen earlier in this dissertation, the tendency to treat linear momentum as a force can lead students to incorrect answers in more basic contexts than this (see Chapter 3, page 72). Therefore, even though we would not stress the inapplicability of the N2R principle to the "Ball and rod" context, we believe that it is appropriate to expect introductory students to have a basic appreciation for the difference between *momentum* and *force*.

7. Interviews

Our primary motivation for this set of interviews was to investigate the strong context dependence of student treatment of CVL after version A of the tutorial.

a. Description of interview tasks

In the interviews, we asked students to answer all of the questions that we had administered after version A of the tutorial *Conservation of Angular Momentum* (*i.e.*, the *Point dependence*, *Time dependence*, and *Direction of CVL* questions in the "3 points" version of the "Free small ball" context, and the *p vectors*, *L vectors*, and *Final center-of-mass speed* question in the "Ball and rod" context) in written form, discussing their thinking out loud as they worked through the problems. Students first answered all questions in one context and then moved on to the other context. We expected that students would answer some combinations of questions in a mutually inconsistent way and that the bulk of the interview time would be spent discussing such inconsistencies.

b. Discussion of results of interviews

We observed a variety of patterns in student thinking during these interviews that guided the development of version B of the tutorial (described in the next section). In this section, we present some excerpts that we found especially illuminating.

After "Student 1" (S1) answered the *p vectors* question correctly and the *Final center-of-mass speed* question incorrectly, he changed his answer to the *p vectors* question so that it was consistent with his answer to the *Final center-of-mass speed* question (but incorrect). The interviewer (I) then inquired about how the student decided to fix the inconsistency:

> I: You had an inconsistency there. They didn't match together, the way you saw it, and then you chose to fix one, but not the other one. How did you make that choice?

S1: I like to think in terms of energies. And because, I looked at the energies, the energy going in, and the momentum going in, I guess, and since I just I keep going back and saying it has less linear velocity, which is what we said down here in C [the *Final center-of-mass speed* question], since momentum is determined by 'mv,' since v is smaller after the collision, it would have to be, yeah, it would be a smaller momentum than the system before the collision, which is just a ball moving.

I: And that's based on the energy argument, and the reason that that wins, is because you prefer to think about energy?

S1: Yeah, I do. Momentum seems abstract to me. I mean, it works. I've seen it. Momentum is conserved, but energy is a tangible... It does work, and momentum is just this quantity that we can create and you can find with math and it's conserved. Energy makes sense, something so high has this much energy. You can see it transferred. I just think better that way.

I: You feel like you know more what it is? Is that – I don't want to put words in your mouth -

S1: No, I know - yeah, I think it's, yeah, I do, I guess I do understand a little bit better what, yeah, what it is. It's something that can very easily do work, or takes work to do other, to put energy into potential energy, and momentum is just kind of this 'you take mass, you multiply it by velocity, and it has a momentum.' It's a definition.

I: So it's kind of mathematical?

S1: Yeah, but that's not really my problem with it, it's not hard math.

I: Oh right, not that the operation is hard, but...

S1: But just that it's not – it's almost like – to me it seems like an easy way of defining energy conservation: momentum conservation. You just multiply this by this, and look afterwards, it's going to be conserved. And that's kind of the, oh what's the word I'm looking for, it's kind of *because* of energy conservation. So I like to just go right to the energy conservation, so I can see what's happening.

Thus, Student 1 overruled a correct result he found using momentum conservation with incorrect ideas about energy, thinking that they ought to yield the same results. To him, an energy approach was preferable because it was more intuitive, unlike momentum, which to him seemed like an arbitrary definition. The interviewer then asked the student to apply momentum conservation to Experiment 1 to determine the final speed of the ball/rod system. After the student correctly determined the answer, the interviewer suggested applying a similar approach using kinetic energy. The student did this, and found "correctly" that the final speed given by this approach was not the same as the one given by momentum conservation. The student was puzzled, because he had strongly expected the two methods to yield the same result. When he saw that they were not the same, he searched (in vain) for a calculation error for some time. The interviewer then asked if they could change the subject.

Later, Student 1 was determining the direction of the angular momentum of the ball with respect to each of points B and C in the "Free small ball" context (see Figure 7-1).

> S1: This one points into the page, this one points out of the page [correct]. And I define that by the right-hand rule.
>
> I: OK. How did you do that?
>
> S1: I pictured them as, this is stupid, but I pictured them as disks in my mind. As the ball hits them, that's the direction they would spin had that been perfect, like I flicked it almost.
>
> I: OK. Why did you say that was stupid?
>
> S1: It's not mathematical. It's just in my own brain.

In this excerpt, Student 1 pictured collisions between the ball and a stationary disk centered at different points to help him determine the direction of the angular momentum of the ball. We thought that this idea might be productive for students if the tutorial could somehow encourage them to use it. (As an "epistemological" aside, it is interesting to see Student 1's attitude toward the formal/mathematical shift here; in a previous excerpt, he preferred energy to momentum allegedly because he preferred his intuition to formal definitions, while here the intuition that comes "from his own brain" and is "not mathematical" is regarded as "stupid.")

When Student 1 thought about the *Time dependence* question, he determined that the angular momentum of the ball is constant with respect to both points A and B but based his answer in part on the fact that the path of the ball is parallel to the line containing points A and B.

> S1: How I defined it earlier is how it's independent of where the ball is, (so) I'd have to go with they are both constant as the ball's traveling not at an angle, as long as the points B and A are parallel they're going to be constant but if you change that angle, you'd have to start worrying about where the ball is to...

I: I'm sorry ... I missed that part.

S1: If this line [the path of the ball]... were like that [not parallel to the line containing A and B], then it would matter where the ball is to calculate that v, but at all points, as long as they're parallel, at all points, the angular momentum will be the same, so I said that these two func..., they're constant functions, there is no function. If this were tilted, there would be a function to describe the place of the ball.

I: Can you draw a picture of that, like maybe down there?

S1: [Drawing] This is where points A and B are, and if the line the ball is on is say, tilted, like that. If that were tilted, you would have to describe, you would create a function of this ball's position vs. time, and then you could define the angular momentum at either of these points, as a function of that position vs. time. But seeing as how they're parallel, and at all points, even if you, you could describe a function, even of that, but it'd be real boring, it'd be a constant, and so I think the angular momentums would not be changing, they're both constant.

I: So like in this case [lines not parallel], if you were to look at like the angular momentum with respect to point B, would that be changing?

S1: Um... depending on where the ball is? Yeah.

This discussion above provided some more detail on a pattern we had observed in the tutorial classroom with version A. When students considered whether the angular momentum of the astronaut with respect to point C was changing in time, they often failed to recognize changing the path of the astronaut would also affect the identification of the component of the position vector that is perpendicular to the linear momentum vector. In some cases, they seemed to think that "r-perp" would remain in the same orientation with respect *to the paper* when the object's path was changed, so that r-perp would change in magnitude (and "flip" in direction, if necessary) according to the position of the astronaut along the axis defined by r-perp.

The interviewer then asked Student 1 about his conclusion that the angular momentum would depend on time if the lines were not parallel, seeking to observe what question asked of Student 1 might help him change his mind:

I: If I ask you a question that doesn't refer to point A, like how the angular momentum of the ball is changing with respect to point B, you know, is it increasing, decreasing, or remaining constant, you would still want to think about the line that A and B are sharing?

S1: You don't have to... There's no requirement for a line in angular momentum, you just need a point to calculate it around. The line isn't really necessary. So, if the line is not necessary [pause] huh. [pause] Then that's wrong. All right. Because if you remove the line, and don't take into consideration these lines,

I: Mm-hmm.

S1: Then you're always parallel. I guess, well you're not. You have no dimension near a point, so it should be constant along that.

I: Mm-hmm.

S1: So... this still plays a part from whatever point you're speaking of, so I'm not 100% sure that this [correct answer to the *Time dependence* question] is wrong, but, [rereads question, inaudibly], but I think my reasoning down here was bad, or wrong. But now, yeah, this [correct answer to *Time dependence* question] is pretty much no diff. Because any point you define in space can be said to be this point up here. It's, you know, 90 degrees to there.

I: You can draw that perpendicular?

S1: Yeah, anywhere, you can draw it totally perpendicular.

Thus, Student 1 concluded that the perpendicular component of the position vector can always be drawn, regardless of the orientation of the path, by noting that no lines (except for the path of the object, perhaps) are necessary.

In the following excerpt, "Student 2" (S2) has just considered the *Final center-of-mass speed* question and concluded that the center of mass of the ball/rod system in Experiment 2 must move more slowly than that in Experiment 1. The interviewer then chose to explore how students might interpret "proof" that linear and angular momentum are separately conserved:

I: What if I could show you photos that looked like the rate at which the (rod), when the (ball) hits the center [Experiment 1], goes forward is the same as the rate at which the other one [Experiment 2] goes forward, what would you think of that?

S2: That would be hard for me to figure out in my head because it seems like you're not having a conservation type of situation. Seems like this one has more energy going into it than this one.

I: How's that?

S2: Because that's moving in more dimensions, therefore more is going on.

I: More is going on, OK. You mentioned energy. Does it violate momentum conservation?

S2: Yes, I think it would because what I've been saying before is that I think angular momentum plus the translational momentum would have to equal some sort of constant.

I: What's that based on?

S2: What's that based on? Empirical knowledge. I don't know. Just what I've seen in the real world.

Student 2 expressed the idea that such photographs would violate his understanding that linear and angular momentum were jointly conserved. Discussions like this suggested to us that students would readily interpret photographs of collisions like those in the "Ball and rod" context as having consequences for how to treat linear and angular momentum conservation. At the time of these interviews, we did not have any photographic evidence that the centers of mass would move forward at the same speed. If we had been able to present them to Student 2 at this time, we would have. However, since we did not have such evidence, the interviewer attempted to simulate the experience of looking at real photographs by appealing to Student 2's ability to imagine actually being in his tutorial section:

I: And then (the tutorial) goes, "ask a tutorial instructor to provide you with the photographs so that you can (*etc.*)." So, you know, you sit down, and you lean over a sheet with some photographs on it, and the photographs show them moving at the same forward rate. What would you do, how would you make sense of that?

S2: If I saw that, I would say, WELL, I have to change my opinion about the world. Thank you for bestowing this wonderful knowledge, that's why I'm here.

I: OK.

S2: And I learned something.

I: And what would that be?

S2: That angular momentum and translational momentum are separate things completely, and I have to treat them as such.

I: OK. What about energy? Isn't there still the energy factor?

S2: Yeah, I wouldn't... I really don't know how to figure that out because [long pause]. I mean, like I said, it seems like more is going on, therefore more energy is being used, or something, rather than just translational motion. And so it couldn't equal... like, I wouldn't believe your pictures. They were doctored.

I: OK. Good to know.

Thus, with little guidance (except for the discussion leading up to the "presentation" of evidence, which may not have been insignificant), Student 2 readily concluded that linear momentum and angular momentum are "separate things." Then, when the interviewer tested his confidence by referring back to his preference to look at the situation in terms of energy, he abandoned his (correct) conclusion that linear and angular momentum are separately conserved. We do not think that Student 2's conclusion that the pictures "were doctored" should be taken completely seriously; after all, the interviewer and student were playfully pretending to look at photographs that did not exist. However, we have had various similar classroom experiences in which students appeared to feel like they were being taught something that was patently incorrect, as if the instructor were simply *wrong*. We believe such situations probably cannot be completely avoided and that it may help to have some expectation of which situations are more likely to arouse such responses. In any case, this excerpt (like many others in this dissertation) illustrates the common tendency of students to prefer to think in terms of energy, and often according to an incorrect understanding of energy.

The next two sections describe how we applied insights from these interviews and student performance on exams after version A to revising part 2 of the tutorial.

D. DESCRIPTION OF VERSION B OF PART 2 OF TUTORIAL SEQUENCE *CONSERVATION OF ANGULAR MOMENTUM*

This section describes changes made to Part 2 of the tutorial (from version A to version B) and the motivations for these changes. Part 1 of the tutorial and both parts of the homework were essentially unchanged.

1. Detailed description of version B of Part 2 of tutorial

In the interest of evaluating the effect of version A of the tutorial, we explored student understanding of angular momentum of an object moving with constant velocity (or "CVL," for "constant velocity angular momentum") by asking a variety of exam questions. Student performance on new post-tests revealed that student understanding of CVL was not as significantly improved as we previously thought. In some cases, we observed that students' correct treatment of CVL was highly context-sensitive; in other cases, we realized that there were sorts of incorrect reasoning of which we had not been very aware, and in the course of applying such incorrect reasoning, some students would answer some multiple choice questions correctly. After taking these revelations into account, we decided that the tutorial had not actually been very effective.

In order to learn more about why the tutorial was not very effective, we conducted individual student interviews with students who had recently completed Physics 121. To give a short summary of the results of those interviews: students barely remembered the notion of CVL, they expressed little understanding of why we would attribute angular momentum to an object moving with constant velocity (in particular, that the attribution is motivated by a commitment to conservation), and they also expressed the idea of the dispensability of CVL, on the basis of some kind of joint conservation between linear and angular momentum. That is, according to many students, angular momentum need not be present before an isolated interaction if it is present afterwards, as long as "momentum" can be converted from linear to angular form.

Part 2 of the tutorial was revised, in order to motivate students properly, first, to believe in the separate conservation of linear momentum and angular momentum, second, to accept that angular momentum can be attributed to an object moving with constant velocity, and third, to understand how to determine the direction and (coarse quantitative) magnitude of such an angular momentum. We expected that this sequence would make the notion of CVL seem somewhat sensible to students, rather than arbitrary, as we suspect many students found it in version A of the tutorial.

a. On-center and off-center collisions

As described above, because of the tendency of students to treat linear momentum and angular momentum as jointly conserved, we decided it would be necessary first to "prove" that they are separately conserved. In order to demonstrate this abstract notion, we had students consider a context (as shown in Figure 7-10) that we knew (from previous experience) would strongly elicit the idea of joint conservation. Students read: Two collision experiments are performed with a puck and a rod on a level, low-friction air table. In each experiment: the puck moves with initial speed v_0, and does not spin; the rod is initially at rest; the puck sticks to the rod. The puck and rod have approximately equal mass. In Experiment 1, the puck moves toward the center of the rod. In Experiment 2, the puck moves toward a point near the end of the rod.

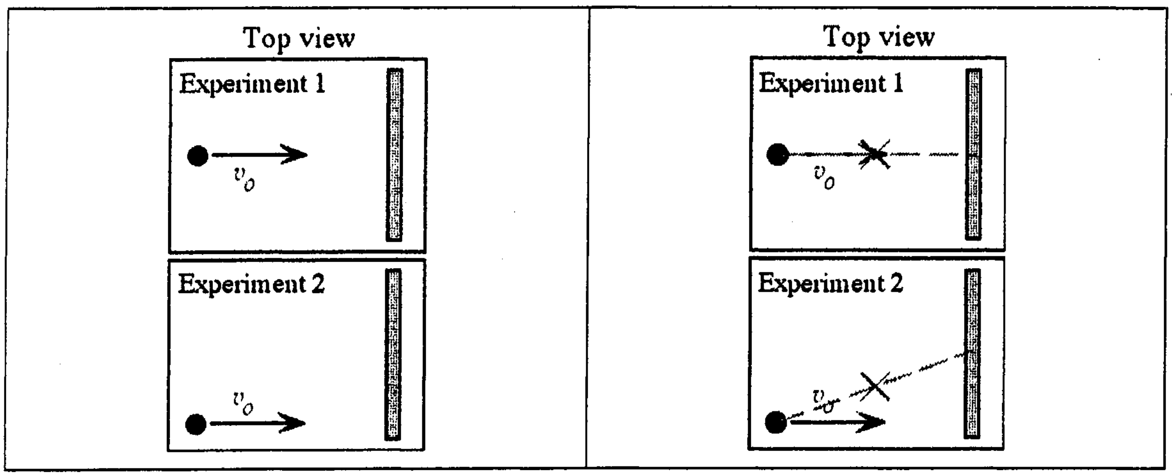

Figure 7-10: The "puck and rod" context in Section III (the first section in Part 2) of version B of the tutorial *Conservation of Angular Momentum*, as printed (left), and with correct location of the center of mass of each system in gray (right).

In these experiments, the translational motion of the center of mass of the puck-rod system relates directly to the total linear momentum of that system. Therefore, in order to draw students' attention to this point, they are asked to draw the center of mass of the system on the figures that depict the puck and rod in each experiment at some instant before the puck hits the rod (see Figure 7-10). To do so, one should first determine the "present" positions of the center of mass of each object (puck or rod) separately. Once this has been done, the combined center of mass lies on the line connecting the two individual centers of mass. To determine where on this line it lies, we consider the mass ratio of the objects, since the center of mass of any system may be conceived as the "weighted" average position of the mass of the system. Since the puck and rod have equal mass, their combined center of mass lies halfway in between them. For some students, this may be a non-trivial exercise.

Next, students are asked to predict whether the final speed of the center of mass of the puck-rod system in Experiment 1 will be greater than, less than, or equal to that in Experiment 2. The rough intent of the question is to ask students to compare the "forward progress" of the two systems; however, some students do not necessarily relate the motion of the center of mass to the forward progress of the system, as a whole. That is, some students may say that the center of mass of each system has the same final speed, while the forward progress is different. To some students, these answers may seem consistent: the center of mass of the puck-rod system in Experiment 2 is "below" the center of the rod. If the center of the rod were the special point around which rotational motion occurs and which marks forward progress, the center of mass could move at the same speed in both experiments, which is greater than the speed of the center of the rod in Experiment 2, since the system rotates in a way that (initially) apparently augments the speed of the center of mass. Alternatively, some students seem to interpret the phrase "speed of the center of mass" as referring to some sort of summation of linear speed and angular speed. This may not seem so odd when we consider that the phrase "angular speed" is often followed by the words "about the center of mass." Student may not take notice of the different prepositions "of" and "about" and what each might be intended to signify. These students may therefore also agree that the speeds of the centers of mass are the same, even though they imagine one system moving forward more quickly than the other because the "total speed" is the same for both. A third possibility is that some students may be thinking that there is no single point whose motion indicates the forward progress of the system. In any case, it is necessary that most students already understand this special property of the motion of the center of mass, or that the question is translated to the students as referring to forward progress.

After students have predicted how the final speeds of the centers of mass compare, they consider a printed Student dialogue:

Student 1: The center of mass will move faster in Experiment 1, because the momentum is purely linear. In Experiment 2, some of the linear momentum is converted into angular momentum when the puck and rod start spinning.

Student 2: Both systems will move at the same speed because the linear momentum is the same in each experiment. Linear momentum can't be turned into angular momentum – each kind of momentum is conserved separately.

Some students anticipate the conceptual development of the tutorial by asking, "Then where does the angular momentum come from?" after reading Student 2's statement. Instructors may affirm the importance of that question and assure students that an answer to it is coming.

Tutorial instructors then bring a sequence of photographs of the two collisions. Frames 1-10 are shown for each collision, for a total of twenty photographs, as shown in Figure 7-11. The photographs are still frames taken from videos of the collisions (which were performed successively). In the first two frames of each experiment, the puck has not yet hit the rod. With these photographs, students can easily confirm that the puck had the same initial velocity in each experiment. Then, in order to determine whether the center of mass has the same final speed in each experiment, it is necessary only to look at the tenth (and last) frame of each film. In this pair of frames, the students can see that the total distance traveled by the center of mass of the first system is indistinguishable from that of the second system. The intermediate frames can serve to show that the second puck-rod system does have a significant angular speed around its own center of mass.

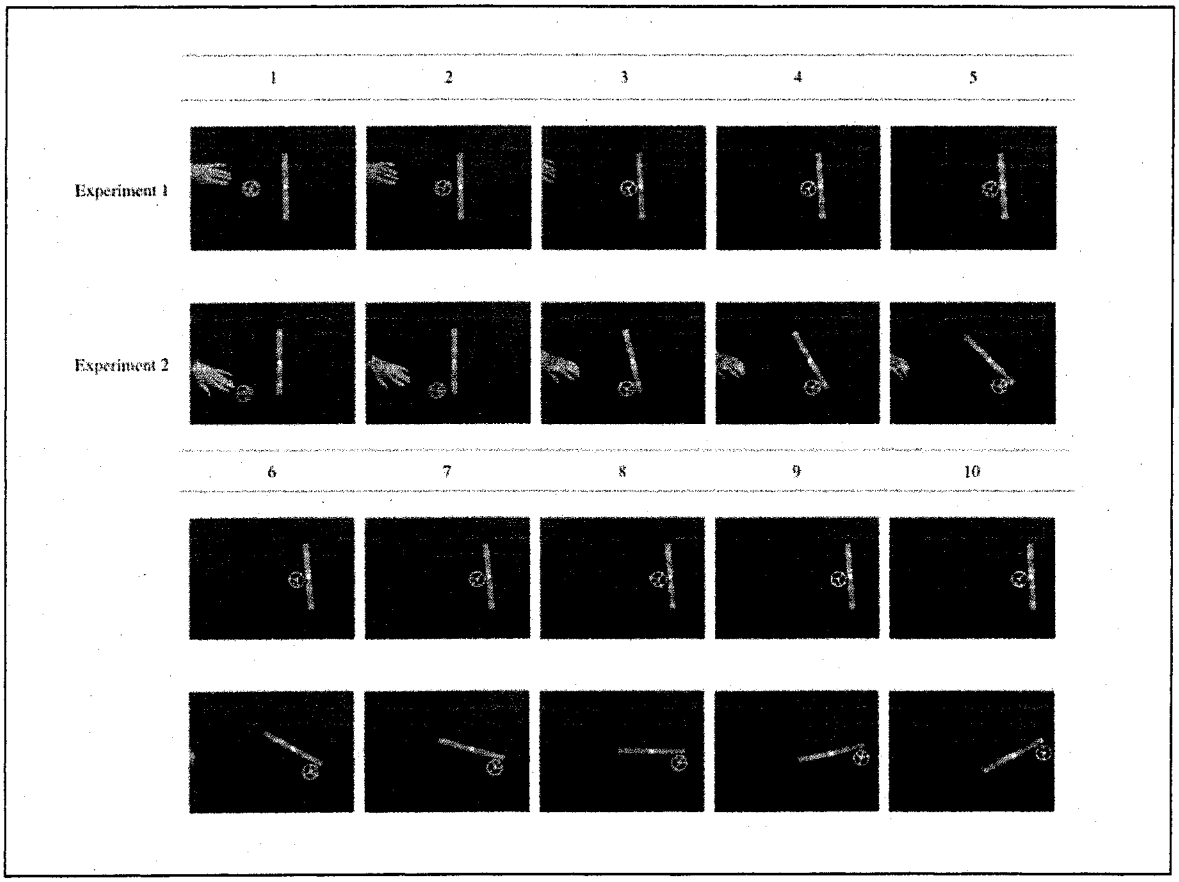

Figure 7-11: Photographs depicting the "On-center and off-center collisions." Students are given a handout that show these photographs.

For better or worse, many students connect this pair of experiments with the "Connected spools" experiment from the tutorial *Dynamics of Rigid Bodies*. In that experiment, the purpose was to show that the acceleration of the center of mass of a rigid body depends on the forces that are applied to it, but not on where those forces are applied. For each spool, the forces were identical, except for the location at which the tension forces were applied. In these collision experiments, many students seem to think that an incoming object with some linear momentum will exert the same force(s) on the target object, wherever or however they collide. This is incorrect; the net force on the puck after the collision of Experiment 2 is perpetually centripetal (and non-zero), while the net force on the puck in Experiment 1 is zero after the collision. However, we are not eager to point out this disparity to most students. When a student to makes the connection between these two situations, it seems to be primarily productive for him/her. For example, when answering exam questions about on-center and off-center collisions, students who write that they see the situation as equivalent to the "Connected spools" experiment state correctly, virtually without exception, that the final speed of the center of mass of the puck-rod system does not depend on where the puck hits (if it sticks). As one student remarked to her classmates in tutorial, after seeing the collision photographs: "This is just like the spools. We should stop being tricked by this."

This section of the tutorial concludes by asking the students whether linear momentum can be "turned into" angular momentum. Students seem to agree that the photographs show clearly that it cannot. The issue of "where the angular momentum came from" is resolved in the next section.

b. Angular momentum of an object moving with constant velocity

After it has been established that the presence of angular motion in Experiment 2 does diminish its translational motion, we introduce the idea that the angular momentum that is apparent *after* the collision is also present *before* the collision. In order to help students see the attribution of angular momentum to straight-line motion as sensible, we chose to try to elicit skills that we expected many students would be comfortable using. These skills are (1) facility with the general conservation concept: "amount of something" at the beginning of a process is the same as the amount afterwards, and (2) facility with associating varying degrees of rotational motion with varied impact parameter (or moment arm). Essentially, then, this section guides students to think about the angular momentum of a puck with respect to any point by imagining the effect that puck would have on the rotational motion of a disk centered at that point. (We chose to use a disk instead of a rod, because unlike a rod, a disk is always "properly" oriented, regardless of the direction from which the colliding puck approaches it.)

In three different experiments, the center of the disk is placed at different starting locations: points A, B, and C, which are marked on the table. (Students may imagine that the disk in Figure 7-12 is translucent.) In all of these experiments, the puck has the same initial velocity and moves on the path shown in Figure 7-12. The puck sticks to the disk in all cases.

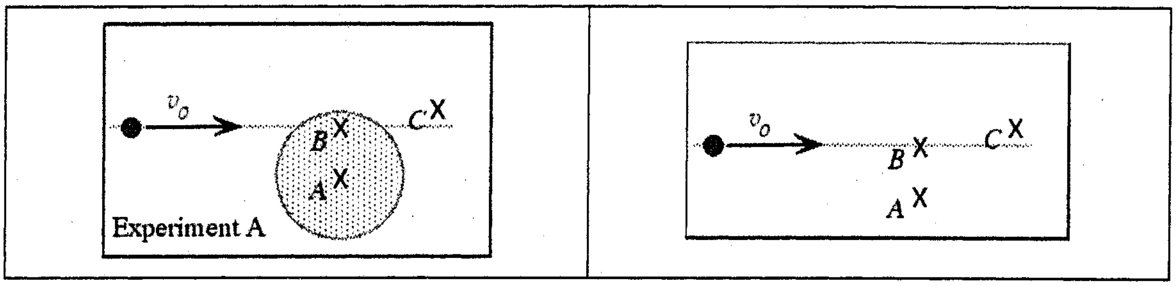

Figure 7-12: The "puck and disk" context in Section IV of version B of the tutorial *Conservation of Angular Momentum.* Experiment A is shown at left. Students later use the figure at right to compare Experiments A, B, and C.

First, restricting our attention to a single experiment, we have students consider applying the conservation concept to the angular momentum of the puck/disk system by reflecting on the following student dialogue:

Student 1: The system must somehow have angular momentum before the collision. After the collision, the puck and disk will spin, so there is clearly some angular momentum. The angular momentum had to come from somewhere.

Student 2: That makes sense, but how there could be initial angular momentum? There's nothing angular about an object moving in a straight line.

Students are often at a loss for a solution to the dilemma presented above. If instructors feel that students understand the conflict expressed in the dialogue, they may suggest students continue through the tutorial, which offers a solution. Students read that they can account for the rotation motion of the system in a way that is consistent with angular momentum conservation if they make the following assumptions: The puck somehow does have angular momentum before the collision, and the manner in which the system spins *after* the collision can be used as an indicator of both the magnitude and direction of the angular momentum of the puck *before* the collision. The tutorial then guides students to understand the standard point-referential language of angular momentum, by stating that the construction "the angular momentum of the puck with respect to point A is different from the angular momentum of the puck with respect to point B" is a way of saying that the puck/disk system rotates differently if the disk starts at point A or point B.

Then using these assumptions, students compare the angular momentum of the puck with respect to points A, B, and C (according to the diagram in the right cell of Figure 7-12) in direction and magnitude. Most students do not have trouble with this exercise.

In order to show that the angular momentum of an object with respect to a particular point depends on both the point and the path of the object, we guide students to think about the angular momentum of the puck with respect to the same points as it moves on a different path, as in Figure 7-13. Here students repeat the previous exercise, comparing the angular momentum of the puck with respect to points A, B, and C in direction and magnitude.

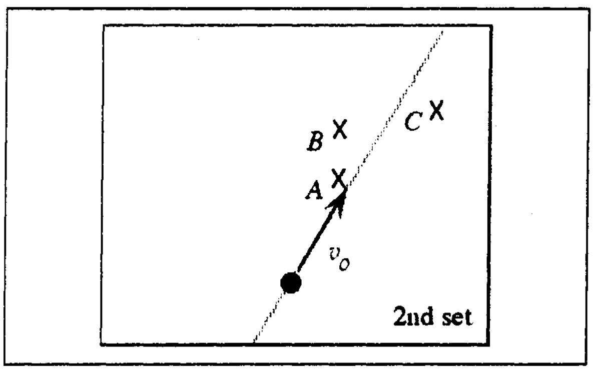

Figure 7-13: Continuation of the "puck and disk" context. Points A, B, and C are the same as before, but the puck moves on a new path.

Students then construct these general results from their work: the angular momentum of an object moving in a straight line (without spinning) depends on the reference point and on the path on which the object moves.

At this point, after students have some comfort with comparing different angular momentums, the tutorial introduces a formal definition for the magnitude of angular momentum: $|\vec{l}| = r_{\perp} p$, where p is the magnitude of the object's linear momentum, and $r_{\perp}$ is the shortest distance from the reference point to the object's straight-line path. Students then draw $r_{\perp}$ for each point on both diagrams they have considered. Finally, students are asked to consider whether an object's angular momentum changes or remains constant as the object approaches a reference point. Instructors may encourage students to answer in two ways - using the formal definition of angular momentum and by reflecting on the original motivation for "inventing" such angular momentum: commitment to angular momentum *conservation*.

E. DESCRIPTION OF VERSION C OF PART 2 OF TUTORIAL SEQUENCE *CONSERVATION OF ANGULAR MOMENTUM*

This section describes changes we made to version B of the tutorial (resulting in version C). Some of the changes were made because we observed that students misinterpreted or were confused by some word choices or sentence structures in version B. Other changes were motivated by limited student success on some tasks after working through version B. These post-test results are described in the next section, along with those for version C.

The first revision of version B of the tutorial was in the first Student dialogue. We observed that some students had trouble interpreting Student 1's original comment about the speed of the center of mass. Some students did not realize that the speed of the center of mass was the speed at which the system "as a whole, moves across the table." That is, they agreed with the Student 1's original statement (that the center-of-mass speeds would be equal) but also predicted that the systems would move forward at different rates. In order to clarify the issue being discussed, we eliminated the term *center of mass* from Student 1's comment.

Student 1: The system in Experiment 1 will move forward across the table at a faster rate, because the final momentum will be purely linear. In Experiment 2, some of the initial linear momentum is converted into angular momentum when the puck and rod start spinning.

Student 2: Both systems will move at the same speed because the linear momentum is the same in each experiment. Linear momentum can't be turned into angular momentum – each kind of momentum is conserved separately.

We also changed the Student 1's statement in the second Student dialogue. In version B, Student 1's reasoning did not specify a system to which he/she was applying angular momentum conservation. We saw some students explain that the angular momentum that "had to come from somewhere" came from the torque applied to the disk by the puck. Thus, in order to emphasize that in the *closed* system of the puck and disk, angular momentum must be the same before and after, the dialogue was revised as shown below:

Student 1: After the collision, the puck and disk will spin, so there must be some final angular momentum. If the system has angular momentum after the collision, it must have it before the collision, since angular momentum is conserved.

Student 2: That makes sense, but how could there be initial angular momentum? There's nothing angular about an object moving in a straight line.

Finally, we thought that, with version B, students were not guided to think about the conditions under which the angular momentum of a moving object was the same with respect to two points. That is, in the original continuation of the "puck and disk" context, the angular momentum of the puck was not the same with respect to any pair of labeled points. Since we valued this issue in our assessment of student understanding, we chose to target it more directly in the instruction. We also thought students would have a stronger impression of the concept if they were more active in constructing a path for the puck and points with respect to which the angular momentum of the puck was the same when it moved on that path. As shown in Figure 7-14, the path of the puck is not shown in the second set of experiments in version C.

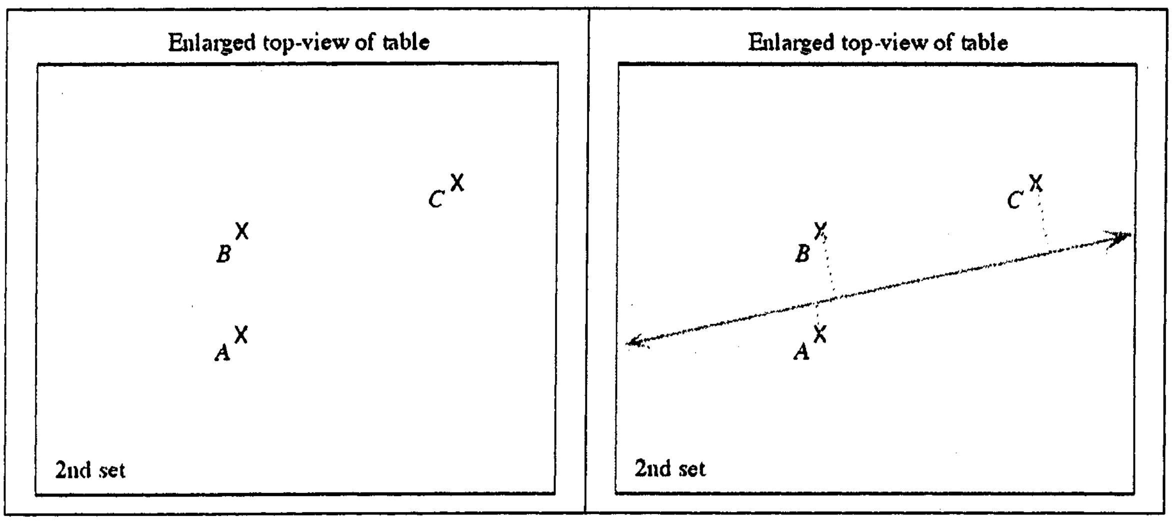

Figure 7-14: Revised version of the continuation of the "puck and disk" context, as printed in the tutorial (left), and correctly completed in gray (right).

Students are told that, in the second set of experiments, the puck moves on a path such that: the angular momentum of the puck with respect to points B and C is the same in direction and magnitude; and the angular momentum of the puck with respect to point A is smaller in magnitude than that with respect to point B, and points in the opposite direction. Students draw a path on which the path could move such that the above conditions are satisfied. One solution is shown in Figure 7-14. (The puck moves in either direction along a line that is parallel to the line containing points B and C and closer to point A than it is to B and C. The perpendicular distance from each point to the line is also indicated.) After drawing the path of the puck, students mark two more points for which the angular momentum of the puck is the same as with respect to point A. (Any two points on the line through A and parallel to the path of the puck are solutions.)

F. STUDENT UNDERSTANDING OF CONSTANT VELOCITY ANGULAR MOMENTUM AFTER VERSIONS B AND C OF TUTORIAL SEQUENCE *CONSERVATION OF ANGULAR MOMENTUM*

1. Administration of questions

Students enrolled in UW 121 in winter quarter 2005 (two lecture sections) worked through version B of the tutorial sequence *Conservation of Angular Momentum*, while students in UW 121 in spring quarter 2005 (two lecture sections) worked through version C. In each quarter, one section was given the *p vectors, L vectors,* and *Final center-of-mass speed* questions in the "Ball and rod" context. In winter 2005, the second section was given the *Point dependence* and *Time dependence* questions (in multiple-choice form) in the "2 points, same y" version of the "Free small ball" context. In spring 2005, the second section was given the *Point dependence* and *Time dependence* questions (in free-response form) in the "2 points, same y" version of the "Free small ball" context.

Though particle angular momentum is officially included in the standard syllabus of UW 121 (and therefore is covered in the standard reading assignments), the section that was given the "Ball and rod" context in winter 2005 had no lectures that addressed the topic of particle angular momentum (or any form of the relation $\vec{L} = \vec{r} \times \vec{p}$).

Four sections (two in autumn 2004, and two in winter 2005) were assigned online homework that required students to think about CVL. The two sections from spring 2005 were not assigned this particular (final) homework.

2. Overview of student performance

a. Questions asked in the "Ball and rod" context

As shown in Table 7-14, student performance on the *p vectors* question improved with both versions B and C of the tutorial. With both versions, very few (< 5%) students stated that the final linear momentum in Experiment 2 would be zero, and the number of students who stated that the linear momentum would decrease in Experiment 2 (but not Experiment 1) decreased from 35% (after version A) to 15%.

Table 7-14: Comparison of student performance on the *p vectors* question (in the "Ball and rod" context) after versions A, B, and C of the tutorial *Conservation of Angular Momentum.*

	UW 121		
A ball collides with and sticks to a rod near its center in Experiment 1, and near its end in Experiment 2.	Version A N=156 1 section	Version B N=120 1 section	Version C N=100 1 section
All *p* vectors point to the right, and are equal in magnitude (correct)	40%	70%	55%
All *p* vectors point to the right, and the final momentum in Exp. 2 is reduced	20%	10%	15%
All *p* vectors point to the right, except the final momentum in Exp. 2, which is zero	10%	<5%	<5%
p will decrease in Experiment 2, but not Experiment 1.	35%	15%	15%

Table 7-15 shows student performance on the *L vectors* question after versions A, B, and C of the tutorial. With version B, the number of students who stated that the initial angular momentum of the system in Experiment 2 is zero decreased from 60% (after version A) to 35%. Even with this improvement in performance, many more students treated CVL as identically zero in the "Ball and rod" context after version B (35%) than in the "Free small ball" context after version A (< 5%). The number of students who treated CVL like linear momentum also dropped from 20% (after version A) to 5%. After version B, many more students (45%) correctly identified the directions of all four angular momentum vectors than after version A (15%). Thus, even though (in our opinion) CVL was clearly presented in version A of the tutorial, more students treated CVL correctly after version B, in which the purpose of the concept of CVL played a more central role.

Student performance after version C appears to be more similar to that after version A than after version B. However, as discussed in the previous sections, version C itself is far more similar to version B than to version A. This result has led us to suspect that the online homework problem that required students to think about CVL in a context outside of tutorial may have played an important role in instruction. In the homework problem, a ball moves toward the end of a rod, which is fixed at its center to a pivot; students are asked to determine a numerical value for the angular momentum of the ball with respect to the pivot before it hits and sticks to the rod. When working on this problem, some students probably entered either zero or a number corresponding to the linear momentum of the ball. This particular online homework system immediately returns the message "NO" with a red "X" if the student enters an incorrect answer. (In UW 121, students can submit an unlimited number of responses until the computer tells the student "OK," with a green check.) Though it does not seem plausible that students gained much positive understanding of CVL from this experience, it may be that some students took away the lesson "angular momentum (of an object moving with constant velocity) is not zero, and it is not the same as linear momentum." As mentioned earlier, students who worked through version C of the tutorial did not have the opportunity to check their understanding of CVL with a computer. In future quarters at the University of Washington, there will be sections of UW 121 that will work through version C and online homework problems about CVL. Until we can test the understanding of those students, version B and C cannot be directly compared.

Table 7-15: Comparison of student performance on the *L vectors* question (in the "Ball and rod" context) after versions A, B, and C of the tutorial *Conservation of Angular Momentum.*

	UW 121		
A ball collides with and sticks to a rod near its center in Experiment 1, and near its end in Experiment 2.	Version A N=156 1 section	Version B N=120 1 section	Version C N=100 1 section
Both *L* vectors in Exp. 1 are zero, both *L* vectors in Exp. 2 are out of the page (correct)	15%	45%	20%
All *L* vectors are zero, except the final *L* in Exp. 2, which is out of the page	35%	25%	40%
All *L* vectors are zero, except the final *L* in Exp. 2, which points to the right	10%	~0%	10%
Total of "final *L* in Exp. 2 points to the right"	20%	5%	15%
Total of "initial *L* in Exp. 2 is zero"	60%	35%	60%

Student performance on the *Final center-of-mass speed* question after versions A, B, and C of the tutorial is shown in Table 7-16. Students who had either version B or C (70-75%) performed much better than students who had version A (40%). Even though this question is very similar to ones that students considered in versions B and C, many students (20-25%) still tended to state that the system in Experiment 1 moves with a greater center-of-mass speed than the system in Experiment 2.

Table 7-16: Comparison of student performance on the *Final center-of-mass speed* question (in the "Ball and rod" context) after versions A, B, and C of the tutorial *Conservation of Angular Momentum.*

A ball collides with and sticks to a rod near its center in Experiment 1, and near its end in Experiment 2.	UW 121		
	Version A N=156 1 section	Version B N=120 1 section	Version C N=100 1 section
$v_{cm}(1) = v_{cm}(2)$ (correct)	40%	70%	75%
$v_{cm}(1) > v_{cm}(2)$	55%	25%	20%
$v_{cm}(1) < v_{cm}(2)$	5%	5%	5%
Total of "$v_{cm}(1) = v_{cm}(2)$," "all p vectors are equal magnitude," and "system rotates in Exp. 2"	25%	60%	50%

b. Questions asked in "Free small ball" contexts

Student performance on the *Point dependence* question after versions A, B, and C is shown in Table 7-17. After version B, students performed about as well (30%) as after version A (40%). After both versions B and C, the number of students who treated CVL as identically zero is slightly higher (10-15%) than after version A (< 5%). It may be that, with the increased emphasis on collisions in the treatment of CVL, a few students incorrectly concluded that CVL is zero when the object is not involved in a collision, since no angular momentum is "necessary" to account for the rotational motion of anything. (We have not yet observed this line of reasoning among students.) Although both versions B and C gradually bring the treatment of CVL away from actual collisions to the abstract definition of particle angular momentum, they are both slightly longer than version A, and we have observed that many students did not reach the final sections of versions B and C. It may also be that the presence of point C in the "3 points" version of the "Free small ball" context tends to help students think about angular momentum as non-zero, as the contrast between points A and D in the "2 points, different x and y" version appeared to do.

Another important feature of the data in Table 7-17 is that both r-dominant and ω-dominant approaches to CVL remain very common after any version of the tutorial so far.

Table 7-17: Comparison of student performance on the *Point dependence* question asked in different versions of the "Free small ball" context after versions A, B, and C of the tutorial *Conservation of Angular Momentum.*

	UW 121		
A small ball moves with constant velocity on a line parallel to the line containing points A and B; the ball has passed point A, but not point B; the ball is closer to point A.	"3 points"	"2 points, same y"	
	Version A N=97 1 section Multiple-choice	Version B N=149 1 section Multiple-choice	Version C N=170 1 section Free-response
$\lvert\vec{L}_{ball,A}\rvert = \lvert\vec{L}_{ball,B}\rvert \neq 0$ (correct)	40%	30%	15%
$\lvert\vec{L}_{ball,A}\rvert = \lvert\vec{L}_{ball,B}\rvert = 0$	<5%	10%	15%
(ω-dominant) $\lvert\vec{L}_{ball,A}\rvert > \lvert\vec{L}_{ball,B}\rvert$	20%	25%	40%
(r-dominant) $\lvert\vec{L}_{ball,A}\rvert < \lvert\vec{L}_{ball,B}\rvert$	35%	35%	30%

Student performance on the *Time dependence* question appears to be about the same after each version of the tutorial (see Table 7-18). As we saw with student performance on the *Point dependence* question, students' treatment of CVL according to either an r-dominant (30-35%) or ω-dominant (25-40%) concept remains very common.

Table 7-18: Comparison of student performance on the *Time dependence* question (in the "Free small ball" context) after versions A, B, and C of the tutorial *Conservation of Angular Momentum.*

	UW 121		
	"3 points"	"2 points, same y"	
A small ball moves with constant velocity on a line parallel to the line containing points A and B; the ball has passed point A, but not point B; the ball is closer to point A.	Version A N=97 1 section Multiple-choice	Version B N=149 1 section Multiple-choice	Version C N=170 1 section Free-response
$\vert\vec{L}_{ball,A}\vert$ and $\vert\vec{L}_{ball,B}\vert$ are both constant (correct)	40%	40%	30%
(r) $\vert\vec{L}_{ball,A}\vert$ is increasing, $\vert\vec{L}_{ball,B}\vert$ is decreasing	35%	35%	30%
(ω) $\vert\vec{L}_{ball,A}\vert$ is decreasing, $\vert\vec{L}_{ball,B}\vert$ is increasing	25%	25%	40%
Both $\vert\vec{L}_{ball,A}\vert$ and $\vert\vec{L}_{ball,B}\vert$ are increasing	<5%	<5%	<5%
Both $\vert\vec{L}_{ball,A}\vert$ and $\vert\vec{L}_{ball,B}\vert$ are decreasing	<5%	<5%	<5%

G. SUMMARY

In this chapter we have described various errors that students make when thinking about the angular momentum of a point-like object moving with constant velocity (CVL). Among these errors is the idea that CVL is identically zero. We found that students hinge their understanding that CVL is identically zero on the notion that linear momentum and angular momentum are jointly, rather than separately, conserved. That is, more students recognized that CVL is not identically zero after they had worked through a tutorial that guided them to think about evidence that linear momentum and angular momentum are separately conserved, as well as some of the consequences of this conclusion. In this sense, a particular context ("Puck and rod") served as a pivotal case in student understanding of the formal definition of angular momentum. However, even after tutorial instruction, many students fail to understand what it means for angular momentum to be conserved in the case of an object moving with constant velocity.

We also observed that an incorrect understanding of energy conservation (especially with regard to non-mechanical forms of energy) often takes a dominant role in student reasoning, overruling correct ideas about linear and angular momentum conservation. We believe that student performance on some questions in angular momentum will improve as research-based curriculum on student understanding of energy conservation improves.

NOTES TO CHAPTER 7

[1] The course content and reading assignments for UW 121 are chosen by committee each quarter. However, each instructor may emphasize or deemphasize various elements of the course in his/her choice of homework and exam problems.

[2] See, for example, pages 396-401 of Knight, R. D., *Physics for Scientists and Engineers: A Strategic Approach with Modern Physics* (Pearson/Addison-Wesley, San Francisco, 2004).

[3] See pages 213-215 of Ortiz, L. G., "Identifying and addressing student difficulties with rotational dynamics," Ph.D. dissertation, Department of Physics, University of Washington, 2001 (unpublished).

[4] See page 502 of Boudreaux, A. Q., "An investigation of student understanding of Galilean relativity," Ph.D. dissertation, Department of Physics, University of Washington, 2002 (unpublished).

CHAPTER 8. CONCLUSION

This dissertation describes efforts to improve instruction at the introductory level on a variety of topics in mechanics: linear momentum conservation in one dimension, Newton's 2^{nd} law for forces that affect both the translational and rotational motion of an object, static friction forces on objects in equilibrium, "one-dimensional" angular momentum conservation for systems of spinning objects, and the angular momentum of a point-like object moving with constant velocity. For each physics topic, we have described student understanding of that topic after lecture instruction, the specific process through which we learned more about how students learn about that topic, the current version of the tutorial curriculum and its rationale, and evidence that students understand the topic better after the current version of the tutorial than after either the published or any intermediate revised version. Thus, the first purpose of this dissertation has been to demonstrate that instruction on each of these physics topics has improved.

Broadly, when we are working to improve instruction, we intentionally follow an iterative cycle. We often begin by exploring student responses to a variety of tasks that relate to a particular concept or principle. We characterize the major errors (and the prevalence of each) that students make when performing these tasks. We design curriculum that addresses the most common errors directly; we examine student responses to tasks after they have worked through the curriculum, noting any broad changes in how students perform. We revise the curriculum in light of which instructional strategies appear to have been effective or ineffective, and we observe how a new group of students performs after working through the revised curriculum.

However, this iterative approach to curriculum development, though it may reveal the major problems students have when learning specific topics, is not guaranteed to produce effective curriculum. That is, it will not necessarily result in high student success on the most relevant fundamental tasks. In many cases, we were well aware of the specific problems students had in learning a topic; we had incorporated our understanding of these problems into the curriculum; and the curriculum appeared to help fewer students than we had hoped. Thus, it does not, in general, suffice to compose curriculum simply on the basis of an awareness of students' problems. What, then, makes effective curriculum effective? How do we take our understanding of student difficulties in a particular topic and fashion effective instructional materials?

With the research presented here, we hope to contribute to a growing common understanding of how to "connect the dots" of student difficulties and student learning. Toward this goal, we claim here that a curriculum is successful to the extent that students experience it as meaningful. Furthermore, we believe that 'meaning,' though difficult to define, may be conceived as a subject of research on student learning. Operationally, we have associated what we thought was 'meaningful,' in student experience, with the 'spontaneous' (or easily stimulated), in student behavior. Thus, we have sought to observe the perspectives, issues, problems, and skills that students appear to employ with little intervention and the contexts in which they appear. In whatever way these semi-spontaneous ideas seemed useful to us in instruction, we tried to evoke them and direct them to the students' advantage. In some cases, this process was relatively straightforward. In other cases, ideas that were relatively spontaneous for students in one context tended not to be spontaneous in another relevant (and sometimes quite important) context. In these cases, we tried to elicit ideas that were easily stimulated in the first context so that we could encourage students to apply them to the second context.

As we developed the curriculum described in this dissertation, we understood two basic forms for this research on meaning in student understanding. First, we sought to identify students' comfortable skills and preferred representations. For example, though they had been taught to analyze collisions by using the concept of the *change in momentum* vector, students in interviews preferred to work with the momentum vector for the system of objects without invoking the change in momentum vector at all. Our interpretation of this behavior was that the system momentum vector was more meaningful to most students than the change in momentum vector, and that any conclusions students were guided to draw using the change in momentum vector would be less memorable or trustworthy.

Second, when students articulated an incorrect understanding of a general principle (*e.g.*, that linear momentum is a conserved *scalar* quantity), we asked students (directly and indirectly) to explain why they understood the principle that way. We found that students often tended to hinge their understanding of a general principle on a single, common case (what we have called here a *pivotal case*). By centering each piece of curriculum on the appropriate pivotal case (there may not be such a case for every issue), we have attempted to relate the general, abstract principle at hand to the case to which students *naturally* relate it, and to guide students to a correct understanding of that case. Because we appealed to students' natural thinking, we hoped that what we had tried to teach them might amount to a lighter cognitive load than "less natural" ideas, thus improving the ease with which students later applied the ideas or communicated them to their peers. In other words, we expected that a correct understanding of a case that was naturally meaningful to students, though not by itself an understanding of an abstract principle, might instead *represent* the abstract principle well enough to help students achieve a functional understanding of the principle that they could apply to various unfamiliar situations.

In this way, we believe that instruction in introductory mechanics has improved because we have better understood how to make the curriculum meaningful to students. We have also shown many of the limitations of both standard physics instruction and the curriculum described in this dissertation. We believe these issues will eventually be resolved as researchers and teachers better understand how to identify and elicit that which students naturally find meaningful.

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APPENDIX A: TUTORIAL SEQUENCE *CONSERVATION OF MOMENTUM IN ONE DIMENSION*

- Pretest
- Tutorial
- Handout
- Homework

PRETEST: CONSERVATION OF MOMENTUM Name:____________________

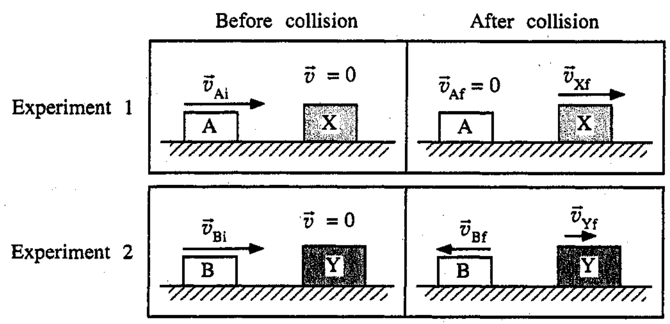
1. Two experiments are conducted with gliders on a level, frictionless track:

   Experiment 1: Glider A is launched toward a stationary target, glider X. After the collision, glider A is at rest.

   Experiment 2: Glider B is launched toward a stationary target, glider Y. Glider B has the same mass and initial velocity as glider A in Experiment 1. After the collision, glider B has reversed direction.

   The final speed of glider X is greater than the final speed of glider Y (*i.e.*, $v_{Xf} > v_{Yf}$).

   The mass of glider X is less than the mass of glider Y (*i.e.*, $m_X < m_Y$).



   a. In Experiment 2, is the magnitude of the <u>change in momentum</u> of glider B *greater than, less than,* or *equal to* the magnitude of the <u>change in momentum</u> of glider Y? Explain.

   b. Is the magnitude of the change in momentum of glider A *greater than, less than,* or *equal to* the magnitude of the change in momentum of glider B? Explain.

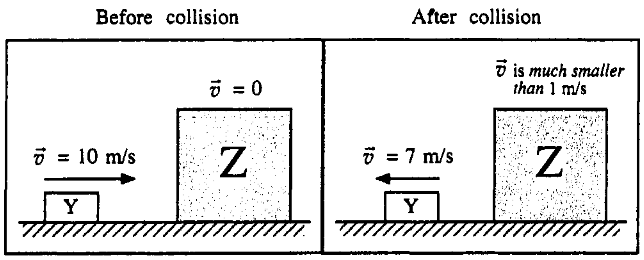
   c. Is the magnitude of the final momentum of glider X *greater than, less than,* or *equal to* the magnitude of the final momentum of glider Y? Explain.

Pretest for tutorial sequence *Conservation of Momentum in One Dimension* (page 2 of 2)

2. A collision experiment is performed with two gliders, Y and Z, on a level, frictionless track.

   The mass of glider Y is 1 kg. Glider Z is *much* more massive than glider Y. Despite the very large mass of glider Z, it is still free to move without friction on the track.

   Before the collision, glider Y moves to the right with speed 10 m/s, toward glider Z, which is initially at rest. After the collision, glider Y moves to the left with final speed 7 m/s. Glider Z moves with a final speed *much smaller than* 1 m/s.



Determine the magnitude of the final momentum of glider Z. Explain your reasoning.

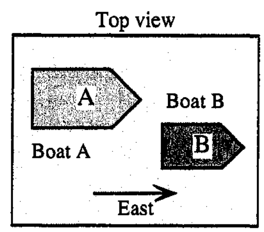
Current version of tutorial *Conservation of Momentum in One Dimension* (page 1 of 4)

CONSERVATION OF MOMENTUM IN ONE DIMENSION

I. The concept of the momentum vector

The momentum vector, $\vec{p}$, of an object is defined as the product of its mass and its velocity ($\vec{p} = m\vec{v}$). The momentum vector of a system of multiple objects is defined as the sum of the momentum vectors of the individual objects ($\vec{p}_{system} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ...$).

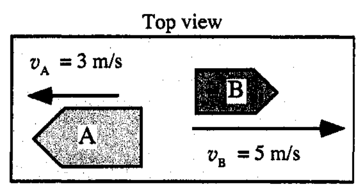
A. Two small boats, boat A and boat B, are in a lake. An observer on the shore sees the boats moving with the same velocity, 1 m/s, due east. The mass of boat A is 10 kg, and the mass of boat B is 5 kg.



1. Find the magnitude and direction of the momentum vector of each boat.

2. Consider the system of boats A and B together. Call this system C. Determine the magnitude and direction of the momentum vector of system C. Explain.

B. Much later, an observer on the shore sees the boats moving with different velocities. Boat A moves with velocity 3 m/s, due west, and boat B moves with velocity 5 m/s, due east.



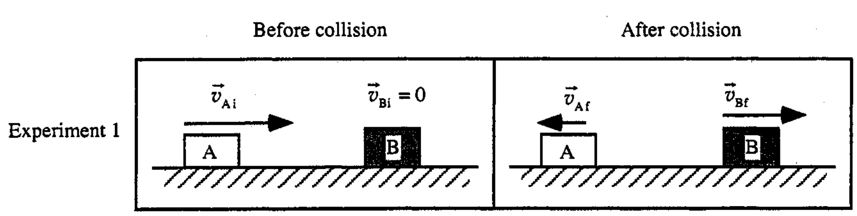
1. Find the magnitude and direction of the momentum vector of each boat.

2. Determine the magnitude and direction of the momentum vector of system C. Explain.

II. Momentum of objects in a collision

A. Experiment 1 is conducted with two gliders on a level, *frictionless* track. In Experiment 1, glider A moves toward glider B, which is initially at rest. After the collision, glider A has reversed direction.

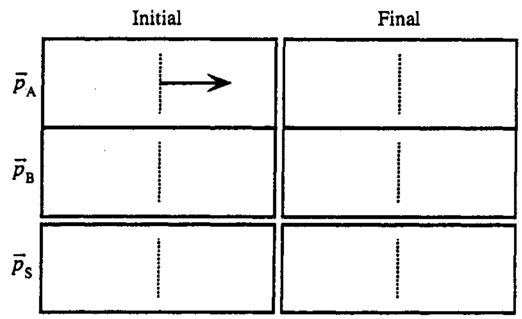
The mass of glider B is greater than the mass of glider A ($m_B > m_A$). Magnets are affixed to the gliders so that the gliders repel each other without touching.



Current version of tutorial *Conservation of Momentum in One Dimension* (page 2 of 4)

Conservation of momentum in one dimension

1. In the boxes at right, draw vectors to represent the momentum of glider A, glider B, and system S (the system of both gliders) both before and after the collision.



2. Is the magnitude of the final momentum of glider B ($|\vec{p}_{Bf}|$) *greater than*, *less than*, or *equal to* the magnitude of the final momentum of the system ($|\vec{p}_{Sf}|$)? Explain.

   Would your answer change if glider A had moved to the right after the collision? Explain.

3. Consider the following statement about Experiment 1:

   "Glider A is still moving after the collision. Glider A keeps some of the momentum that it had initially, giving B only a portion of it. There is a limited amount of momentum in the system, and the most any one object can have is all of the system's momentum. Therefore, p_{Bf} must be less than p_{Sf}."

   Do you agree or disagree with the statement? Explain.

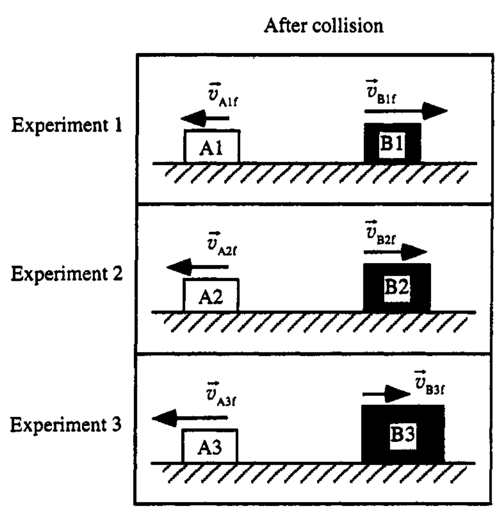
4. Check that your answer to question 2 is consistent with your answer to question 3 and with the law of conservation of momentum.

B. Two new experiments, 2 and 3, are performed. They are identical to Experiment 1, except that mass has been added to the target gliders B2 and B3 so that $m_{B3} > m_{B2} > m_{B1}$. Gliders A1, A2, and A3 are identical and have the same initial velocity.

   Repelling magnets are used in all three experiments.

   It is *observed* that, after the collisions:

   - gliders A1, A2, and A3 move to the left with $v_{A3f} > v_{A2f} > v_{A1f}$.
   - gliders B1, B2, and B3 move to the right with $v_{B1f} > v_{B2f} > v_{B3f}$.



Current version of tutorial *Conservation of Momentum in One Dimension* (page 3 of 4)

1. Rank $|\vec{p}_{B1f}|$, $|\vec{p}_{B2f}|$, and $|\vec{p}_{B3f}|$ from greatest to least, where $|\vec{p}_{B1f}|$ is the magnitude of the final momentum of glider B1, *etc*. Explain the reasoning you used to determine the ranking.

2. A tutorial instructor will provide you with a handout that you can use to draw momentum vector diagrams for experiments 1, 2, and 3. Copy your group's diagrams in the space below after discussion.

   Are your vector diagrams consistent with your ranking in question 1?

C. Experiments 4, 5, 6,...100, *etc.*, are set up in the same pattern as the first three experiments. The mass of glider B100 is *much larger* than the mass of glider A100.

It is *observed* in Experiment 100, that:

- the final speed of glider B100 is very nearly zero.
- glider A100 moves to the left with a final speed very nearly equal to its initial speed.

1. Is the magnitude of the final momentum of glider B100 *greater than, less than*, or *equal to* the magnitude of the initial momentum of glider A100? Explain.

2. Three students discuss question 1, about the final momentum of glider B100:

   Student 1: *"The final speed of A100 is almost the same as its initial speed. This means that A kept almost all of the momentum, giving B almost no momentum."*

   Student 2: *"Right. You can also see that the momentum of B100 has to be nearly zero because momentum is mass times velocity. If B's velocity is very small, then its momentum also must be very small."*

   Student 3: *"From each experiment to the next, the mass of B goes up, while its final speed goes down. Therefore, the final momentum of B most likely stays the same."*

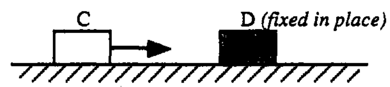
   With which student, if any, do you agree? For any statements that are incorrect, discuss the errors in reasoning in those statements.

D. Suppose that glider A100 has initial momentum 3 kg•m/s, to the right. After the collision, glider A100 has final momentum very nearly equal to 3 kg•m/s, to the left. What is the magnitude and direction of the final momentum of glider B100? Show your work.

Current version of tutorial *Conservation of Momentum in One Dimension* (page 4 of 4)

III. Changes in momentum of a system of multiple objects

A. A new experiment is performed with two gliders, C and D, of *equal mass*. Glider D is fixed in place. Glider C is launched toward glider D and rebounds with the same speed that it had initially.



1. In the spaces provided, draw separate free-body diagrams for each glider and for the system of the two gliders at an instant during the collision in this new experiment.

Free-body diagram for glider C	Free-body diagram for glider D	Free-body diagram for system S

Explain how the fact that glider D is fixed in place is reflected in your free-body diagrams.

2. Does the momentum of each of the following change during the collision? Explain.

- glider C
- glider D
- system S

B. Consider the experiments described in parts II and III. When the momentum of an object or system of objects did not change:

- were *external* forces exerted on the object (or system)?
- was there a *net* force on the object (or system)?

C. On the basis of your results above, describe how you can tell whether the momentum of an object or system is constant by inspecting the free-body diagram for that object or system.

D. Consider the following student discussion about the above experiment.

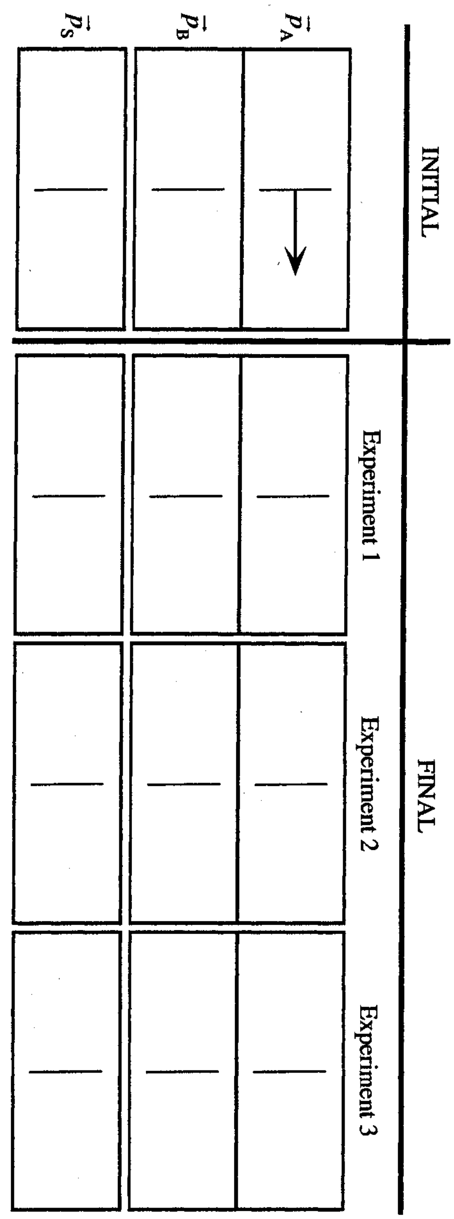
Student 1: *"This experiment is just like Experiment 100. The momentum of glider C is the same before and after the collision - only the direction of motion is different."*

Student 2: *"Right. Glider D has no momentum afterwards, just like glider B100. So the momentum of the system is unchanged."*

Describe the *error* in each statement.

Handout for tutorial *Conservation of Momentum in One Dimension*

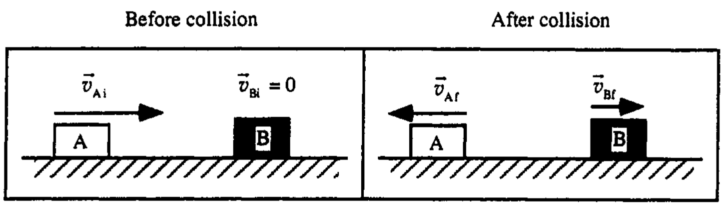
CONSERVATION OF MOMENTUM: HANDOUT



HOMEWORK:
CONSERVATION OF MOMENTUM Name________________

1. Two gliders, A and B, collide on a level, frictionless track, as shown below.

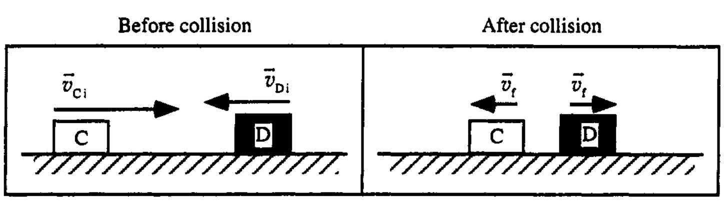
   The mass of glider A is less than the mass of glider B (*i.e.*, $m_A < m_B$). The final speed of glider A is greater than the final speed of glider B (*i.e.*, $v_{Af} > v_{Bf}$).



   Is the magnitude of the final momentum of glider A (p_{Af}) *greater than, less than,* or *equal to* the magnitude of the final momentum of glider B (p_{Bf})? Draw a momentum vector diagram to support your answer. (An example of a momentum vector diagram can be found on the second page of the tutorial.)

2. Two gliders, C and D, collide on a level, frictionless track, as shown below.

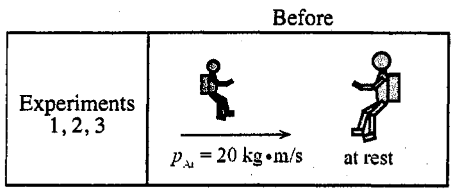
   The mass of glider C is less than the mass of glider D (*i.e.*, $m_C < m_D$). The initial speed of glider C is greater than the initial speed of glider D (*i.e.*, $v_{Ci} > v_{Di}$). After the collision, Gliders C and D move in opposite directions with the same final speed, v_f.

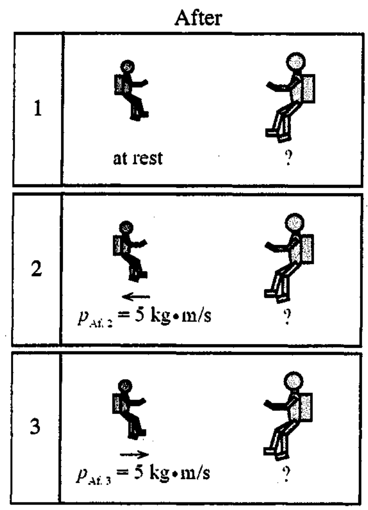


   Is the magnitude of the initial momentum of glider C (p_{Ci}) *greater than, less than,* or *equal to* the magnitude of the initial momentum of glider D (p_{Di})? Draw a momentum vector diagram to support your answer.

CONSERVATION OF MOMENTUM

3. Two astronauts, A and B, participate in three collision experiments in a weightless, frictionless environment. In each experiment, astronaut B is initially at rest, and astronaut A has initial momentum p_{Ai} = 20 kg•m/s to the right. (The velocities of the astronauts are measured with respect to a nearby space station.)





The astronauts push on each other in different ways so that the outcome of each experiment is different. As shown in the figure at right, astronaut A has a different final momentum in each experiment.

a. Determine the magnitude of the final momentum of astronaut B in each experiment. Explain.

b. Rank the final kinetic energy of astronaut B in the three experiments. Explain.

c. Is the *total* kinetic energy after the collision in Experiment 2 *greater than, less than,* or *equal to* the *total* kinetic energy after the collision in Experiment 3? ("Total kinetic energy" means the sum of the kinetic energies of the two astronauts.) Explain.

d. Consider the following statement:

"The momentum of the system is conserved in each experiment because there is no net force on the system. If momentum is conserved, then kinetic energy must also be conserved, because both momentum and kinetic energy are made up of mass and velocity."

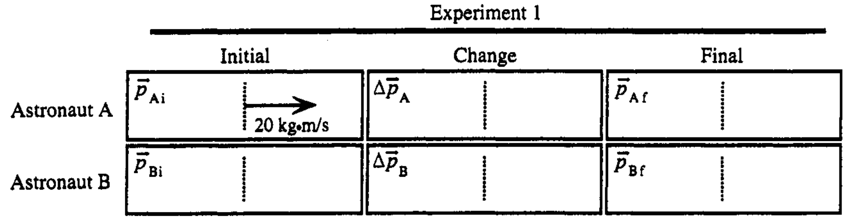
One of the sentences above is completely correct. Discuss the error(s) in reasoning in the other sentence.

Tutorial homework for *Conservation of Momentum in One Dimension* (page 3 of 4)

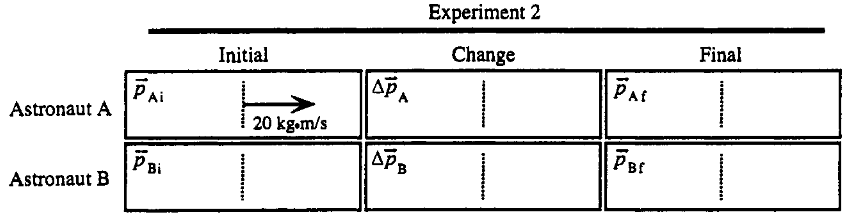
CONSERVATION OF MOMENTUM Name____________________

e. In the boxes below, draw the initial momentum, *change in momentum,* and final momentum vectors for each astronaut in the three experiments.

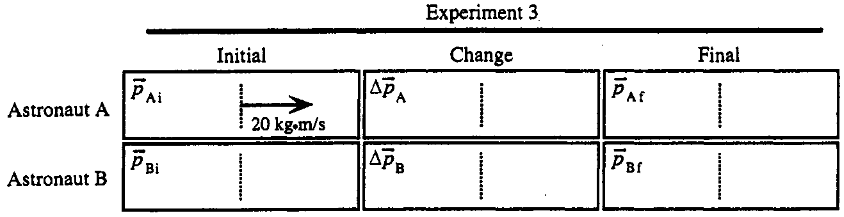
Experiment 1



Experiment 2



Experiment 3



f. Generalize from the diagrams you have drawn to answer the following questions:

If the *net* force on a system of two colliding objects is zero, how does the change in momentum of one object compare to the change in momentum of the other object:

- in magnitude?
- in direction?

g. Explain how your answers to part f are consistent with Newton's third law and the impulse-momentum theorem ($\vec{F}_{net}\Delta t = \Delta\vec{p}$) for:

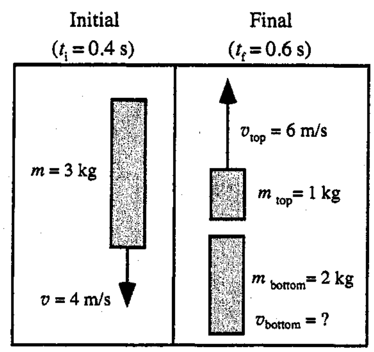
- each astronaut considered separately
- for the system of both astronauts together

CONSERVATION OF MOMENTUM

4. A pyrotechnician drops a firecracker of mass 3 kg by releasing it from rest at $t = 0.0$ s. A short time later ($t_i = 0.4$ s), the firecracker is moving downward with speed 4 m/s. At this same instant, the firecracker begins to explode into two pieces, "top" and "bottom," with masses $m_{top} = 1$ kg and $m_{bottom} = 2$ kg. At the end of the explosion ($t_f = 0.6$ s), the top piece is moving upward with speed 6 m/s.

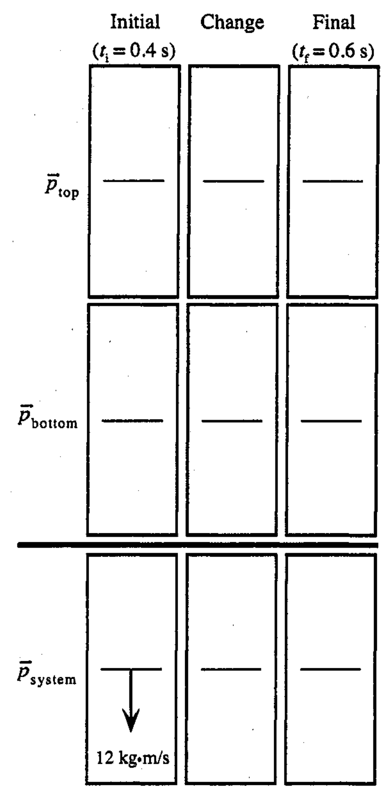
The mass of the explosive substance is negligible in comparison to the mass of the two pieces.

Before completing the momentum vector diagram below, consider the following:



a. Determine the magnitude of the net force on the firecracker system *before* the explosion. (Use $g = 10$ m/s^2.) Explain.

b. Determine the magnitude of the net force on the firecracker system at an instant *during* the explosion. (*Hint:* Does the net force on a system depend on forces that are internal to that system?)



c. Determine the magnitude and direction of the net *impulse* ($\vec{F}_{net}\Delta t$) on the firecracker system for the time interval starting at $t_i = 0.4$ s and ending at $t_f = 0.6$ s. Explain.

d. Use the impulse-momentum theorem to determine the magnitude and direction of the *change in momentum* of the firecracker system during the explosion. Enter this vector in the bottom-center cell of the table.

e. Determine the final momentum of the firecracker system and enter it in the bottom-right cell of the table. (Is it the same as the initial momentum?)

f. Complete the vector diagram.

APPENDIX B: TUTORIAL SEQUENCE *DYNAMICS OF RIGID BODIES*

- Pretest
- Tutorial
- Homework

Pretest for tutorial sequence *Dynamics of Rigid Bodies* (page 1 of 2)

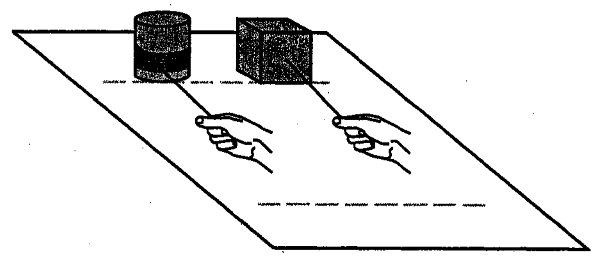
PRETEST: DYNAMICS OF RIGID BODIES Name:_______________

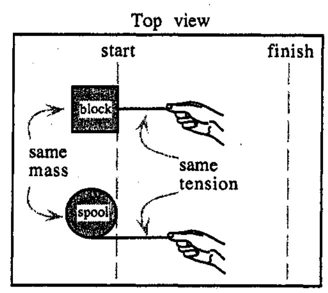
1. Two objects, a block and a spool, are each pulled across a level, frictionless surface by a string.

   The string pulling the block is tied to a small hook at the center of the front face of the block.

   The block and the spool have the same mass. The strings are pulled with the same constant tension and start pulling at the same time.

   (Make the approximation that the strings and the hook are massless.)





   a. Will the spool begin to rotate? Explain.

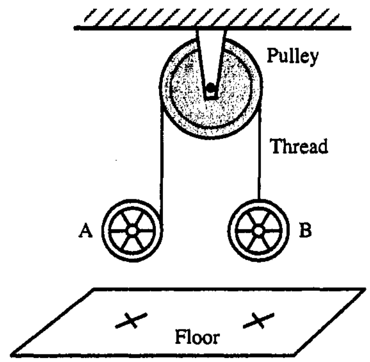
   b. Will the spool cross the finish line *before*, *after*, or *at the same instant as* the block? If the center of the spool will not cross the finish line at all, state that explicitly. Explain.

Pretest for tutorial sequence *Dynamics of Rigid Bodies* (page 2 of 2)

2. Two identical spools are held the same height above the floor. A thread is wrapped many times around spool A. The same thread passes over a pulley, and is attached to a fixed point on spool B, so that spool B will not rotate. An "X" is marked on the floor directly below each spool.

   Both spools are released from rest at the same instant.

   (Assume that the pulley and thread are massless and that the axle of the pulley is frictionless.)



   a. Is the tension in the part of the thread just above spool A *greater than, less than,* or *equal to* the tension in the part of the thread just above spool B (after the spools are released but before either spool hits the floor)? Explain.

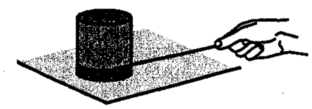
   b. Is the magnitude of the acceleration of the center of mass of spool A *greater than, less than,* or *equal to* the magnitude of the acceleration of the center of mass of spool B? Explain.

   c. Will spool A hit the floor *before, after,* or *at the same instant as* spool B? Explain.

DYNAMICS OF RIGID BODIES

I. Spool on a variety of surfaces

A. A spool sits on a piece of sandpaper. Thread has been wrapped many times around the bottom of the spool. A hand pulls on the thread, moving horizontally, in a straight line away from the spool. (The sandpaper is fixed in place.)



Predict:

- Whether the spool will rotate.
- Whether the center of the spool will move across the sandpaper. If it will move, describe the direction it will move.

Explain the reasoning you used to make each prediction.

B. Imagine repeating the above experiment *without* the sandpaper, on a smooth tabletop.

Predict:

- Whether the spool will rotate.
- Whether the center of the spool will move across the table. If it will move, describe the direction it will move.

Explain the reasoning you used to make each prediction.

C. Ask a tutorial instructor to provide you with the equipment you need to check your predictions. (Make sure that the thread is wrapped *very close to the bottom of the spool*, so that the spool does not start to tip over when the thread is pulled.)

1. When the spool is <u>on the sandpaper</u>, and you pull the thread:

    a. Does the spool rotate?

    b. Does the center of the spool move? If so, in what general direction does it move?

2. When the spool is <u>on the table</u>, and you pull the thread:

    a. Does the spool rotate?

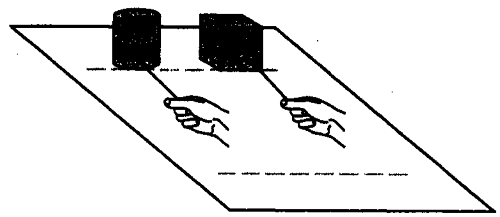
    b. Does the center of the spool move? If so, in what general direction does it move?

D. Suppose that you could repeat the experiment again, using an even smoother surface. Describe how (or whether) you think the resulting motion of the spool would be different from the motion you observed in the experiments above.

➪Discuss your answers with a tutorial instructor before continuing.

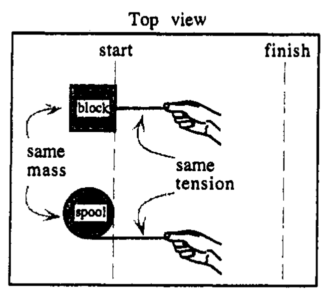
Current version of tutorial *Dynamics of Rigid Bodies* (page 2 of 4)

II. The "block and spool" problem



Two objects, a block and a spool, are each pulled across a level, frictionless surface by a string.

The string pulling the block is tied to a small hook at the center of the front face of the block. The string pulling the spool is wrapped many times around the spool and may unwind as it is pulled.



The block and the spool have the same mass. The strings are pulled with the same constant tension and start pulling at the same time.

(Make the approximation that the strings and the hook are massless.)

A. Will the spool cross the finish line *before*, *after*, or *at the same instant that* the block crosses the finish line? Explain.

B. Three students discuss the block and spool problem:

Student 1: *"The spool will rotate, and it will cross the finish line at the same time as the block. They have the same mass and the same net force, so their centers of mass will have the same acceleration. It doesn't matter whether the spool starts rotating - the tension force will still have the same effect on the spool's translational motion."*

Student 2: *"The spool will rotate, and it will cross after the block. This is because some of the tension force on the spool is being used to rotate the spool. When a force causes an object to rotate, it has less of an effect on the object's translational motion."*

Student 3: *"The spool will rotate, and it will cross after the block. I was thinking about energy. The spool and block must each have the same total kinetic energy when they get to the finish line. Since the spool will have some rotational kinetic energy, it must have less translational kinetic energy than the block. Therefore, it will be slower, and it will arrive at the finish line later."*

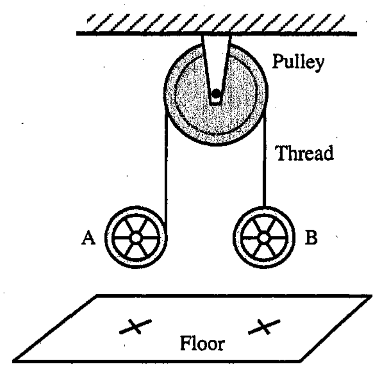
With which student(s), if any, do you agree? Explain your reasoning.

In the next section, you will consider a way to test which of the above ideas is a correct solution to the "block and spool" problem.

Current version of tutorial *Dynamics of Rigid Bodies* (page 3 of 4)

III. Testing ideas about combined translation and rotation with an experiment

Two identical spools are held the same height above the floor. A thread is wrapped many times around spool A. The same thread passes over a pulley, and is attached to a fixed point on spool B, so that spool B will not rotate. An "×" is marked on the floor directly below each spool.



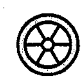
Both spools are released from rest at the same instant. (Assume that the pulley and thread are massless and that the axle of the pulley is frictionless.)

A. *Predict* whether spool A will hit the floor *before*, *after*, or *at the same instant as* spool B. If either spool starts to move only after the other spool hits the floor, state that explicitly. Explain.

B. Draw an *extended* free-body diagram for each spool after they are released. (An extended free-body diagram shows where on the object each force is exerted.)

C. Draw a *point* free-body diagram for each spool after they are released. (A point free-body diagram shows the forces on an object as if the object were located at a single point.)

Extended free-body diagram for spool A



Extended free-body diagram for spool B

Point free-body diagram for spool A

Point free-body diagram for spool B

D. Recall the student discussion from the previous page. For each of the three students, state the prediction you think that student would make if s/he were to use the same reasoning in this experiment that s/he used when thinking about the "block and spool" problem. (Try to agree with your partners about what you think each of the students *would* predict, and why.)

Student 1:

Student 2:

Student 3:

Current version of tutorial *Dynamics of Rigid Bodies* (page 4 of 4)

1. Which student's reasoning, if any, is the same as the reasoning you used to make your prediction?

2. Can you use the outcome of this experiment to decide which reasoning about the "block and spool" problem is correct? If so, explain how. If not, explain why not.

➪Discuss your answers with a tutorial instructor before continuing.

E. Ask a tutorial instructor to provide you with the equipment you need to check your predictions. (Ignore *small* differences in the motion of the spools.)

   How does the acceleration of the center of mass of spool A compare to that of spool B:

   - in magnitude?
   - in direction?

   If necessary, describe how you would revise your free-body diagrams to be consistent with your observations.

F. Do the results of the experiment support any of the ideas presented by Student 1, Student 2, or Student 3? Explain.

G. Generalize from your observations to answer the following questions:

   If you want to determine how a force affects the motion of the center of mass of an object, should you consider:

   - where on the object the force is exerted?

   - how the force is affecting the rotational motion of the object?

H. Energy analysis

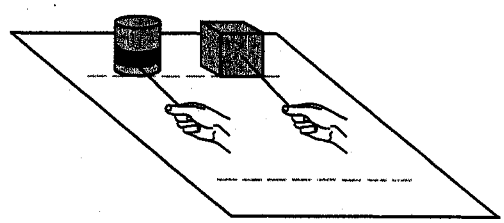
   You have analyzed the "block and spool" problem and the falling spools experiment in terms of forces. You will be guided to describe the energy in these situations in the homework.

HOMEWORK:
DYNAMICS OF RIGID BODIES Name_______________

1. In tutorial, you considered the situation described below:

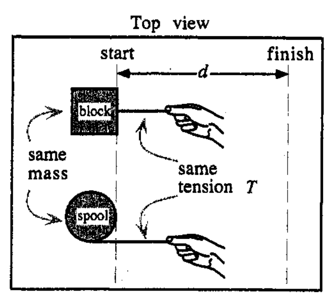
Two objects, a block and a spool, are each pulled across a level, frictionless surface by a string.

The string pulling the block is tied to a small hook at the center of the front face of the block. The string pulling the spool is wrapped many times around the spool and may unwind as it is pulled.



The block and the spool have the same mass. The strings are pulled with the same constant tension T and start pulling at the same time. (Make the approximation that the strings and the hook are massless.)

The distance between the start and finish lines is d.



<u>As the spool moves from the start to the finish line:</u>

a. Is the distance traveled by the hand that is pulling the spool's thread *greater than, less than,* or *equal to d?* Explain.

b. Is the work done by the hand that is pulling the spool's thread *greater than, less than,* or *equal to* the product Td? Explain.

c. Is the work done by the hand that is pulling the spool's thread *greater than, less than,* or *equal to* the work done by the hand that is pulling the block's thread? Explain.

d. Consider the following statement:

"It's not possible for the spool to rotate, AND cross the finish line at the same time as the block. If that were true, then the spool would have more total kinetic energy than the block. This is because they would have the same translational kinetic energy, and the spool would have additional rotational kinetic energy. But they must have the same total kinetic energy because the work done on them is the same."

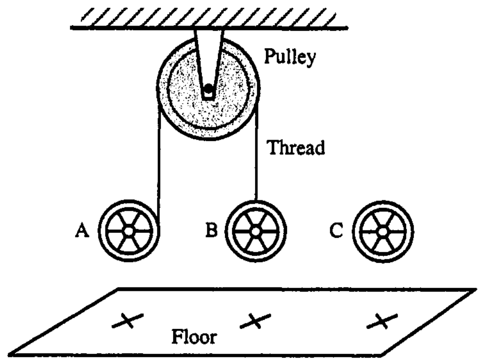
Explain, <u>using the work-energy principle</u>, why the spool has more total kinetic energy than the block when they cross the finish line together.

DYNAMICS OF RIGID BODIES

2. In tutorial, you considered the situation described below:

   Two identical spools are held the same height above the floor. A thread is wrapped many times around spool A. The same thread passes over a pulley, and is attached to a fixed point on spool B, so that spool B will not rotate. An "x" is marked on the floor directly below each spool.

   Both spools are released from rest at the same instant. (Assume that the pulley and thread are massless and that the axle of the pulley is frictionless.)



   a. You observed the motion of spools A and B after they were released. Describe the motion of each spool. (*i.e.*, In which direction did each spool move? In what order did the spools hit the floor?) Ignore small differences.

A third identical spool, C, is added to the experiment. All three spools are released from the same height at the same time. Spool C is not in contact with any other objects as it falls.

   b. Rank spools A, B, and C according to the magnitude of the acceleration of the center of mass of each spool. Explain.

   c. Rank spools A, B, and C according to the *translational* kinetic energy that each spool has just before it hits the floor. Explain. (Use the definition $K_{trans} = \frac{1}{2}mv_{cm}^2$.)

Consider the system consisting of all of these objects: spool A, spool B, thread, pulley, and Earth. Since the net work done on this system is zero, the sum $U_{grav,A} + U_{grav,B} + K_{trans,A} + K_{trans,B} + K_{rot,A} + K_{rot,B}$ is constant as spools A and B fall.

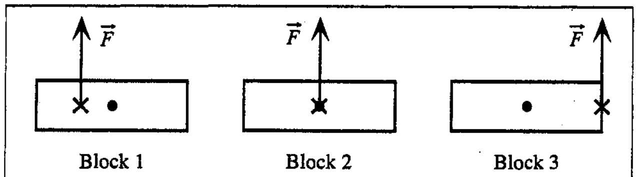
   d. Suppose that this system starts with $U_{grav,A} = U_{grav,B} = 9$ J. Just before the spools hit the ground (*i.e.*, when $U_{grav,A} = U_{grav,B} = 0$ J), spool A has translational kinetic energy $K_{trans,A} = 4$ J. Determine the value of the rotational kinetic energy of spool A. Show your work.

   e. Rank spools A, B, and C according to the *total* kinetic energy that each spool has just before it hits the floor. Explain.

HOMEWORK:
DYNAMICS OF RIGID BODIES Name________________________

3. Three identical rectangular blocks are at rest on a horizontal, frictionless ice rink. Forces of equal magnitude and direction are exerted on each of the three blocks. Each force is exerted at a different point on the block (indicated by the symbol "×"), as shown in the top view diagram below. (Each small circle in the diagram indicates the location of the block's center of mass.)

For the instant shown in the diagram below:



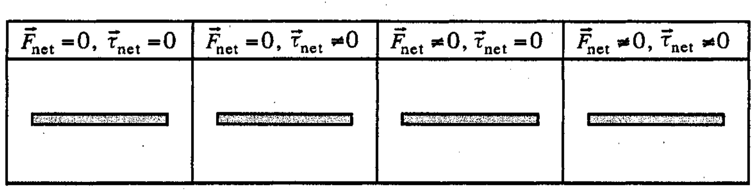
Top view diagram

a. On the diagram above, sketch a vector on each block to indicate the direction of the acceleration of the center of mass ($\vec{a}_{cm}$) of that block. If for any block $\vec{a}_{cm} = 0$, state that explicitly. Explain.

b. Rank the magnitudes of the accelerations of the centers of mass of the blocks ($a_{cm,1}$, $a_{cm,2}$, $a_{cm,3}$). Support your ranking by drawing a *point* free-body diagram for each block.

4. A rigid rod sits on a frictionless surface. The boxes below indicate four different combinations of net force on the rod and net torque on the rod about its own center. In each box, draw arrows that show where you can strike the rod (in one place or in two places at the same time) in order to achieve each combination. If any combination is not possible, write that explicitly.

(*Example:* In the second box, show where you could strike the rod so that the net force on it is zero, and the net torque on it is *not* zero, <u>while you are striking it</u>.)



Top view

DYNAMICS OF RIGID BODIES

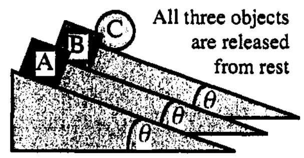
5. Summarize your results thus far by answering the questions below:

   a. If you want to determine how a force affects the *motion of the center of mass* of an object, should you *ignore* or *account for* the rotational motion of the object?

   b. If you want to determine how a force affects the *total energy* of the object, should you *ignore* or *account for* the rotational motion of the object?

6. Three objects, A, B, and C, of equal mass, are released from rest at the same time from the same height on identical ramps.

   Objects A and B are both blocks, and they slide down the ramps without rotating. Object C rolls down the ramp without slipping. Its moment of inertia is unknown.



   Objects A, B, and C are made of different materials. (*i.e.*, Do not assume that the coefficient of friction between any object and its ramp is the same as that for any other object.)

   Object A reaches the bottom of its ramp first, followed by objects B and C, which reach the bottom at the same time.

   a. Rank the objects according to the magnitude of the acceleration of the center of mass of each object. Explain.

   b. Rank the objects according to the magnitude of the *net force* on each object. Explain.

   c. In the boxes below, draw and label a *point* free-body diagram for each object.

	Object A	Object B	Object C
Point free-body diagrams	•	•	•

   d. Rank the objects according to the magnitude of the frictional force exerted on each object by its ramp. Explain your reasoning.

   Make sure that your ranking is consistent with your answer to part a. of question 5.

APPENDIX C: TUTORIAL SEQUENCE *CONSERVATION OF ANGULAR MOMENTUM*

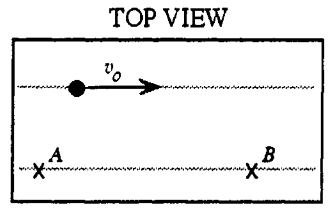
- Pretest
- Part 1 of tutorial
- Part 2 of tutorial (version A)
- Part 2 of tutorial (version B)
- Part 2 of tutorial (version C)
- Handout
- Homework

PRETEST: CONSERVATION OF ANGULAR MOMENTUM

Name:________________

1. A small puck is moving across the surface of a large level air table. The puck moves with constant velocity $\vec{v}_0$ to the right without spinning.

   Two locations are marked on the table: point A and point B. The line containing points A and B is parallel to the path of the puck.



<u>All questions below refer to the instant shown in the figure at right.</u>

a. Is the magnitude of the angular momentum of the puck with respect to point A ($|\vec{L}_{\text{puck,A}}|$) *greater than, less than,* or *equal to* the magnitude of the angular momentum of the puck with respect to point B ($|\vec{L}_{\text{puck,B}}|$)? Explain.

b. Describe the direction of the angular momentum of the puck with respect to point B ($\vec{L}_{\text{puck,B}}$). If this quantity is zero, state that explicitly. Explain.

c. State whether each of the following quantities is *increasing, decreasing,* or *remaining constant*. Explain.

- $|\vec{L}_{\text{puck,A}}|$
- $|\vec{L}_{\text{puck,B}}|$

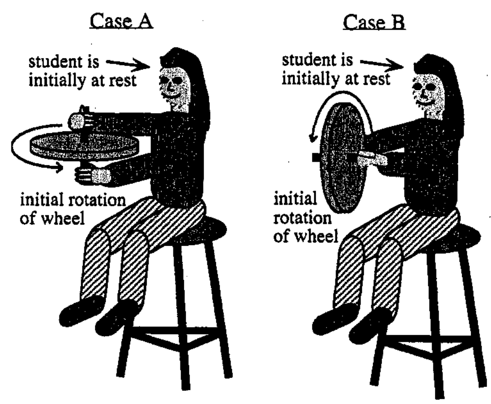
Pretest for tutorial sequence *Conservation of Angular Momentum* (page 2 of 2)

2. A student sits at rest on a stool. The seat of the stool can rotate without friction. The student is handed a bicycle wheel that is *already spinning*. The wheel spins *without friction*.

   In Case A, the axle of the wheel is vertical, and the spins counter-clockwise (when viewed from above).

   In Case B, the axle of the wheel is horizontal, and the top of the wheel is moving away from the student.

   In both cases, the student touches only the handles of the wheel.



a. After the student is handed the wheel in Case A, will she begin to rotate *clockwise* (when viewed from above), *counterclockwise,* or *stay at rest?* Explain your reasoning.

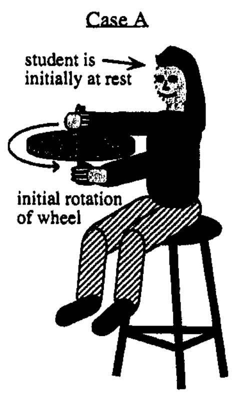
b. After the student is handed the wheel in Case B, will she begin to rotate *clockwise* (when viewed from above), *counterclockwise,* or *stay at rest?* Explain your reasoning.

Current version of Part 1 of tutorial *Conservation of Angular Momentum* (page 1 of 2)

CONSERVATION OF ANGULAR MOMENTUM

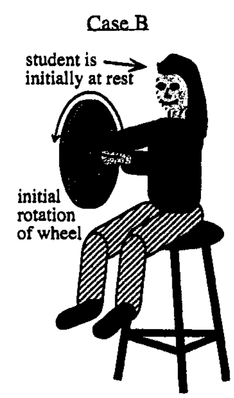
I. Angular momentum of a spinning wheel

A student sits at rest on a stool that can rotate without friction. In each of the following separate cases, the student is handed a bicycle wheel that is *already spinning*. The axle of the wheel is also low-friction.



A. For each case, *predict* whether the student will begin to rotate *counterclockwise* (when viewed from above), *clockwise*, or *stay at rest*. Explain your reasoning in each case.

Case A:
The center of the wheel is stationary. The axle of the wheel is vertical. The wheel spins counterclockwise (when viewed from above).



Case B:
The center of the wheel is stationary. The axle of the wheel is horizontal. The wheel spins so that the top of the wheel moves away from the student.

Check your predictions by observing the demonstration. What was the resulting motion of the student in each case?

- Case A
- Case B

Just as an object moving through space has linear momentum $\vec{p}$, we say that a spinning object has *angular* momentum $\vec{L}$. It is customary to determine the direction of $\vec{L}$ for a spinning object by using the *right hand rule*. When the fingers of the right hand curl in the direction of the spinning motion, the direction of $\vec{L}$ is the same as the direction in which the thumb points.

B. For each case described above, draw a vector next to the figure to represent the angular momentum of the wheel at the instant shown.

C. On the basis of your observations thus far:

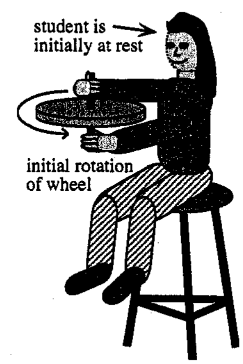
- Does a wheel that is spinning and held in place appear to exert a force on its support in the direction of its angular momentum?

- Does any part of the wheel move in the direction of the angular momentum?

Current version of Part 1 of tutorial *Conservation of Angular Momentum* (page 2 of 2)

II. Angular momentum as a conserved vector quantity

A student sits at rest on a stool and holds a spinning bicycle wheel, as in Case A of section I. Consider the following three experiments:



Experiment 1: The student places her hand against the side of the wheel, slowing it to 1/2 its initial angular speed.

Experiment 2: The student places her hand against the side of the wheel, bringing it to a stop.

Experiment 3: The student quickly flips the wheel over (so that it is spinning clockwise when viewed from above, with the *same* angular speed it had initially).

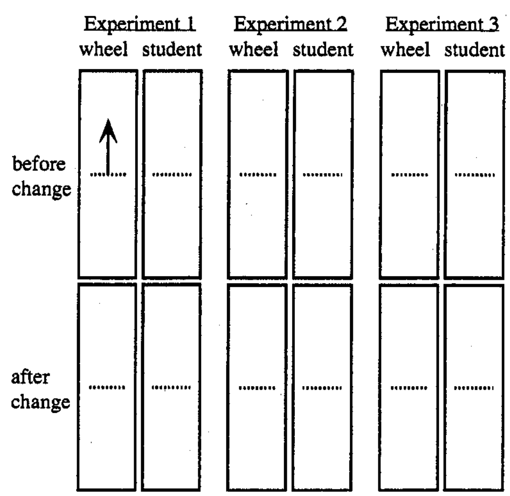
A. *Predict*:

- whether the student will rotate *clockwise, counterclockwise*, or *not at all* at the end of each experiment
- the ranking of the three experiments according to the final angular speed of the student

Explain.

B. Complete the angular momentum vector diagrams at right.

Are your predictions above consistent with the results of your diagrams? Explain.



C. How, if at all, would your predictions in part A change if the convention were to determine the direction of angular momentum by using the left hand instead of the right hand? Explain.

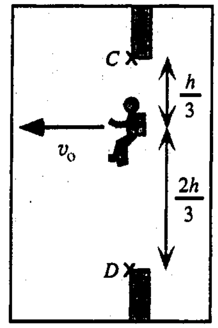
➪ Continue working through the following pages of the tutorial. Before you leave class, return to part D below.

D. Check your predictions from part A by observing the demonstration. Record your observations.

Is the ranking you observed consistent with the assumption that the total angular momentum of the student-wheel system is constant?

Version A of Part 2 of tutorial *Conservation of Angular Momentum* (page 1 of 2)

Note: For the following section, use the convention that ⊗ indicates a vector pointing *into* the page and ⊙ indicates a vector pointing *out of* the page.



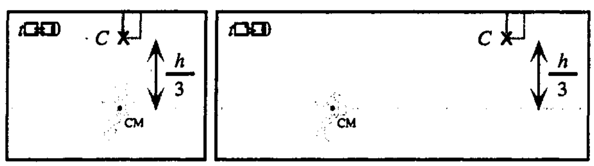
III. Angular momentum of an object moving with constant velocity

An astronaut moves to the left with constant velocity v_o, through a large doorway in a space station, as shown at right. Points *C* and *D* mark the top and bottom of the doorway, respectively, and are a distance *h* apart. At time $t = 0$, the center of mass of the astronaut passes directly below point C, at a distance $h/3$. The astronaut does not spin.

A. The diagram below shows the astronaut at two instants, $t = 0$, and some time later. A gray line shows the path of the center of mass of the astronaut.

1. In both boxes below, draw:
    - $\vec{r}_{a,C}$ (the position vector of the astronaut with respect to point *C*)
    - $\vec{p}$ (the linear momentum vector of the astronaut)

2. For $t > 0$, draw $\vec{r}_{\perp}$ and $\vec{r}_{\parallel}$, the vector components of $\vec{r}_{a,C}$ that are perpendicular and parallel to the linear momentum vector, $\vec{p}$, respectively.



3. Write a vector equation relating $\vec{r}_{a,C}$, $\vec{r}_{\parallel}$, and $\vec{r}_{\perp}$. (The Pythagorean theorem is *not* a vector equation.)

4. The angular momentum of the astronaut with respect to point C is defined by $\vec{l}_{a,C} = \vec{r}_{a,C} \times \vec{p}$. Replace $\vec{r}_{a,C}$ in this definition by using the relationship from question 3, and explain why the definition can be simplified to $\vec{l}_{a,C} = \vec{r}_{\perp} \times \vec{p}$.

5. What is the direction of the angular momentum of the astronaut with respect to point C? Explain.

B. Imagine that you are standing in one place, watching a friend walk by. You make sure that as your friend moves, you are always directly facing him/her. First, your friend walks in a path shaped like a semi-circle, with you at the center. Then, from the same starting place, your friend walks by with a constant velocity (in a straight line).

1. Describe how *your* motion in watching your friend is similar in the two cases.

Version A of Part 2 of tutorial *Conservation of Angular Momentum* (page 2 of 2)

Conservation of angular momentum

2. Use your answer to the previous question to explain why it makes sense that $\vec{l}_{a,C}$ is *not* zero (even though the astronaut is neither spinning nor moving in a curved path).

C. Does the angular momentum of the astronaut with respect to point C ($\vec{l}_{a,C}$) change with time? Explain.

D. How does the angular momentum of the astronaut about point D ($\vec{l}_{a,D}$) compare to the angular momentum of the astronaut about point C ($\vec{l}_{a,C}$):

- in direction?
- in magnitude?

Would your answers to the above questions change if you were to determine the direction of angular momentum with your left hand instead of your right hand? Explain.

(Keep in mind that the convention is to determine the direction of angular momentum by using the right hand.)

E. Two students discuss the angular momentum of the astronaut:

Student 1: *"The angular momentum of a system does not change when there is zero net torque on the system. There is clearly no torque at all on the astronaut, so the angular momentum must stay constant as the astronaut floats by."*

Student 2: *"The angular momentum of the astronaut is not conserved because the angular momentum with respect to point C is not the same as the angular momentum with respect to point D."*

With which student(s), if either, do you agree? Explain.

F. Find a reference point that is stationary with respect to the space station and for which the angular momentum of the astronaut is zero. Explain the reasoning you used to find the point.

Does the angular momentum of the astronaut with respect to this point change over time?

G. Generalize from your answers in section III to answer the following questions. For an object that is moving with *constant velocity* (and not spinning):

- Do the magnitude and direction of the angular momentum depend on the reference point?
- Does an object moving with constant velocity have constant angular momentum?

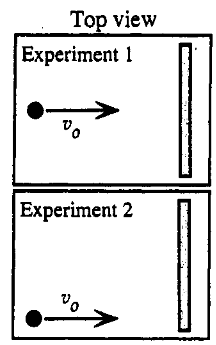
Version B of Part 2 of tutorial *Conservation of Angular Momentum* (page 1 of 3)

III. On-center and off-center collisions

Two collision experiments are performed with a puck and a rod on a level, low-friction air table. In each experiment: the puck moves with initial speed v_o and does not spin; the rod is initially at rest; the puck sticks to the rod. The puck and rod have approximately equal mass.

In Experiment 1, the puck moves toward the center of the rod.

In Experiment 2, the puck moves toward a point near the end of the rod.



A. On each of the figures at right, mark the approximate location of the center of mass of the puck-rod system at the instant shown.

B. *Predict* whether the final speed of the center of mass of the puck-rod system in Experiment 1 will be *greater than, less than,* or *equal to* that in Experiment 2. Explain your reasoning.

C. Two students discuss their predictions before performing the experiments:

Student 1: *"The center of mass will move faster in Experiment 1, because the momentum is purely linear. In Experiment 2, some of the linear momentum is converted into angular momentum when the puck and rod start spinning."*

Student 2: *"Both systems will move at the same speed because the linear momentum is the same in each experiment. Linear momentum can't be turned into angular momentum – each kind of momentum is conserved separately."*

With which student, if either, do you agree? Explain.

D. A tutorial instructor will provide you with photographs of the collisions described above.

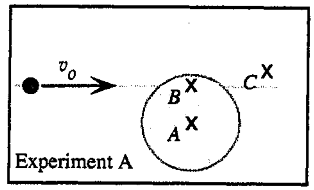
1. How does the speed of the center of mass of the puck-rod system in Experiment 1 compare to that in Experiment 2? Explain how you can tell from the photographs.

2. What do the photographs suggest about whether linear momentum is "turned into" angular momentum in Experiment 2?

Version B of Part 2 of tutorial *Conservation of Angular Momentum* (page 2 of 3)

IV. Angular momentum of an object moving with constant velocity

The rod in the previous experiments is replaced by a disk. In three different experiments, the center of the disk is placed at different starting locations: points A, B, and C, which are marked *on the table*. In all of these experiments, the puck has the same initial velocity and moves on the path shown.



The puck sticks to the disk in all cases.

A. In the first experiment (Experiment A), the disk is placed with its center on top of point A, as shown.

Consider the following discussion between two students:

Student 1: *"The system must somehow have angular momentum before the collision. After the collision, the puck and disk will spin, so there is clearly some angular momentum. The angular momentum had to come from somewhere."*

Student 2: *"That makes sense, but how there could be initial angular momentum? There's nothing angular about an object moving in a straight line."*

With which student, if either, do you agree? Explain.

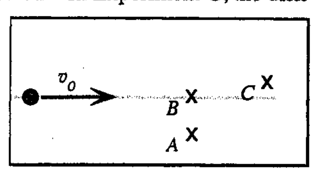
In order to account for the rotational motion of the system after the collision in a way that is consistent with angular momentum conservation, we will make the following assumptions:

- The puck somehow *does* have angular momentum before the collision.
- The manner in which the system spins *after* the collision can be used as an indicator of both the magnitude and direction of the angular momentum of the puck *before* the collision.

For instance, if the manner in which the system spins depends on whether the disk starts at point A or point B, we say that the angular momentum of the puck *with respect to* point A is different from the angular momentum of the puck *with respect to* point B.

B. In Experiment B, the center of the disk is placed on top of point B. In Experiment C, the disk is placed on top of point C.

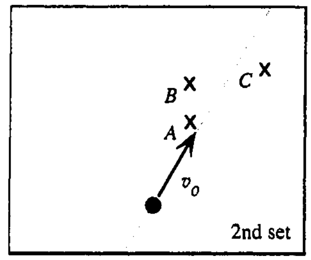
Use the assumptions listed above to compare the angular momentum of the puck with respect to points A, B, and C:



- in direction
- in magnitude

Version B of Part 2 of tutorial *Conservation of Angular Momentum* (page 3 of 3)

C. In a 2nd set of experiments, the puck is given an initial velocity in a new direction. In each experiment, the center of the disk is placed at a different point. Points A, B, and C are at the same locations as in the previous experiments.



Compare the angular momentum of the puck with respect to points A, B, and C:

- in direction

- in magnitude

D. According to your results so far, do the magnitude and direction of the angular momentum of an object that is moving in a straight line (without spinning) appear to depend on:

- The reference point (*i.e.* the point with respect to which the angular momentum is taken)? Give a specific example from the cases above.

- The path on which the object moves? Give a specific example from the cases above.

E. The magnitude of the angular momentum of an object q with respect to any point X $|\vec{l}_{qX}|$ is defined as $|\vec{l}_{qX}| = r_\perp p$, where $r_\perp$ ("r-perp") is the shortest distance from the reference point to the object's straight-line path, and p is the magnitude of the linear momentum of the object.

1. Draw $r_\perp$ for each labeled point on both figures on the previous page.
2. Explain how both of your angular momentum rankings above are consistent with the formal definition of angular momentum.

3. Does the angular momentum of an object *change* or *remain constant* as the object approaches a reference point?

   Explain how your answer is consistent with the definition of angular momentum.

Version C of Part 2 of tutorial *Conservation of Angular Momentum* (page 1 of 3)

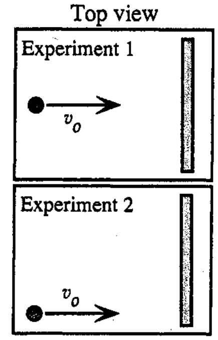
III. On-center and off-center collisions

Two collision experiments are performed with a puck and a rod on a level, low-friction air table. In each experiment: the puck moves with initial speed v_o and does not spin; the rod is initially at rest; the puck sticks to the rod. The puck and rod have approximately equal mass.

In Experiment 1, the puck moves toward the center of the rod.

In Experiment 2, the puck moves toward a point near the end of the rod.

Top view



A. On each of the figures at right, mark the approximate location of the center of mass of the puck-rod system at the instant shown.

B. *Predict* whether the final speed of the center of mass of the puck-rod system in Experiment 1 will be *greater than, less than,* or *equal to* that in Experiment 2. Explain your reasoning.

C. Two students discuss their predictions before performing the experiments:

Student 1: *"The system in Experiment 1 will move forward across the table at a faster rate, because the final momentum will be purely linear. In Experiment 2, some of the initial linear momentum is converted into angular momentum when the puck and rod start spinning."*

Student 2: *"Both systems will move at the same speed because the linear momentum is the same in each experiment. Linear momentum can't be turned into angular momentum – each kind of momentum is conserved separately."*

With which student, if either, do you agree? Explain.

D. A tutorial instructor will provide you with photographs of the collisions described above.

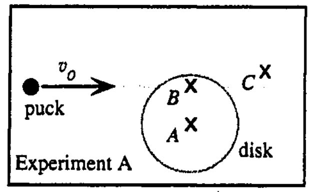
1. How does the speed of the center of mass of the puck-rod system in Experiment 1 compare to that in Experiment 2? Explain how you can tell from the photographs.

2. Do the photographs support either of the ideas presented by Student 1 or Student 2? Explain.

Version C of Part 2 of tutorial *Conservation of Angular Momentum* (page 2 of 3)

IV. Angular momentum of an object moving with constant velocity

The rod in the previous experiments is replaced by a disk. In three different experiments, the center of the disk is placed at different starting locations: points A, B, and C, which are marked *on the table*. In all of these experiments, the puck has the same initial velocity and moves on the path shown.



The puck sticks to the disk in all cases.

A. In the first experiment (Experiment A), the disk is placed with its center on top of point A, as shown.

Consider the following discussion between two students:

Student 1: *"After the collision, the puck and disk will spin, so there must be some final angular momentum. If the system has angular momentum after the collision, it must have it before the collision, since angular momentum is conserved."*

Student 2: *"That makes sense, but how could there be initial angular momentum? There's nothing angular about an object moving in a straight line."*

With which student, if either, do you agree? Explain.

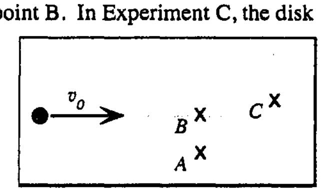
In order to account for the rotational motion of the system after the collision in a way that is consistent with angular momentum conservation, we will make the following assumptions:

- The puck somehow *does* have constant angular momentum before the collision.
- The manner in which the system spins *after* the collision can be used as an indicator of both the magnitude and direction of the angular momentum of the puck *before* the collision.

For instance, if the manner in which the puck/disk system spins depends on whether the disk starts at point A or point B, we say that the angular momentum of the puck *with respect to* point A is different from the angular momentum of the puck *with respect to* point B.

B. In Experiment B, the center of the disk is placed on top of point B. In Experiment C, the disk is placed on top of point C.

Use the assumptions listed above to compare the angular momentum of the puck with respect to points A, B, and C:



- in direction
- in magnitude

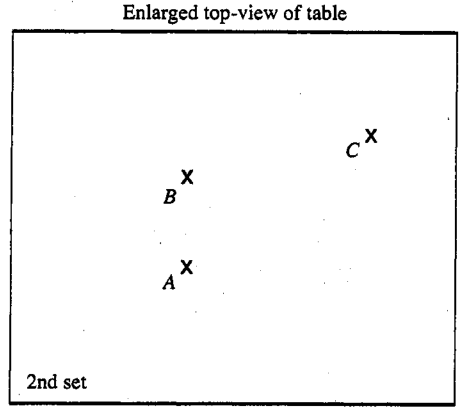
Version C of Part 2 of tutorial *Conservation of Angular Momentum* (page 3 of 3)

C. In the 2nd set of experiments, the center of the disk is again placed at points A, B, and C. (Points A, B, and C are at the same locations as in the previous experiments.)

In each of these experiments, the puck has the same initial velocity and moves on the same path, such that:

- The angular momentum of the puck is the same (in magnitude and direction) with respect to points B and C.
- The angular momentum of the puck with respect to point A is smaller in magnitude than the angular momentum with respect to point B, and points in the opposite direction.

Enlarged top-view of table



1. On the enlarged figure, sketch a path for the puck that satisfies the above conditions.
2. Mark at least two new points with respect to which the angular momentum of the puck is the same (in magnitude and direction) as that with respect to point A.
3. Compare your sketches with those drawn by other members of your group. Discuss the similarities and differences in your sketches.

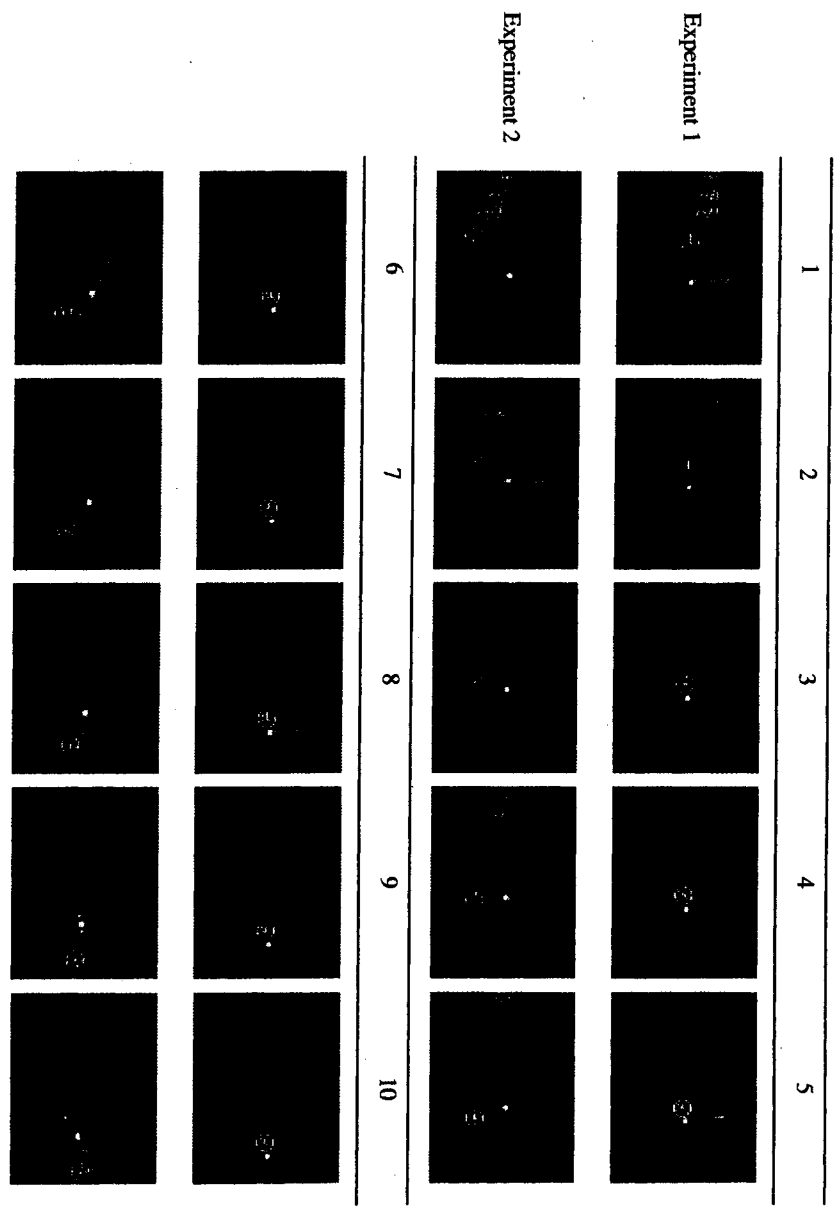
   If necessary, revise your sketch.

D. The magnitude of the angular momentum of an object q with respect to any point X $\left|\vec{l}_{q,X}\right|$ is defined as $\left|\vec{l}_{q,X}\right| = r_{\perp}p$, where $r_{\perp}$ ("r-perp") is the shortest distance from the reference point to the object's straight-line path, and p is the magnitude of the linear momentum of the object.

1. Draw $r_{\perp}$ for each labeled point on the two previous figures.
2. Explain how your answers to parts B and C are consistent with the *formal* definition of angular momentum.
3. Does the angular momentum of an object *change* or *remain constant* as the object approaches a reference point?

   Explain how your answer is consistent with the definition of angular momentum.

CONSERVATION OF ANGULAR MOMENTUM: HANDOUT

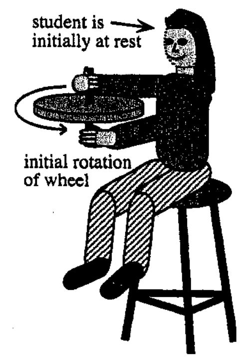


Tutorial homework for *Conservation of Angular Momentum* (page 1 of 2)

HOMEWORK: CONSERVATION OF ANGULAR MOMENTUM

Name________________________

1. In tutorial, you considered the following situation:



A student sits at rest on a stool and holds a bicycle wheel that is spinning counterclockwise, when viewed from above. The wheel spins without friction. The stool has a seat that can rotate without friction. Consider the following three experiments:

<u>Experiment 1:</u> The student places her hand against the side of the wheel, slowing it to 1/2 its initial angular speed.

<u>Experiment 2:</u> The student places her hand against the side of the wheel, bringing it to a stop.

<u>Experiment 3:</u> The student quickly flips the wheel over (so that it is spinning clockwise when viewed from above, with the *same* angular speed it had initially).

a. You observed that the final angular speed of the <u>student</u> in Experiment 3 is *greater than* that in Experiment 2. Explain this result.

b. Rank the final kinetic energy of the wheel in Experiments 1, 2, and 3. (Recall that the kinetic energy of an object is *never* negative.) Explain.

c. Rank the final kinetic energy of the student in Experiments 1, 2, and 3. Explain.

d. Rank the following <u>four</u> quantities: the initial kinetic energy of the wheel ($K_{wheel,i}$) and the final kinetic energy of the student-wheel system in Experiments 1, 2 , and 3 ($K_{system,f,1}$, *etc.*). Explain. (*Hint:* It may be helpful to think about changes in *non-mechanical* energy.)

CONSERVATION OF ANGULAR MOMENTUM

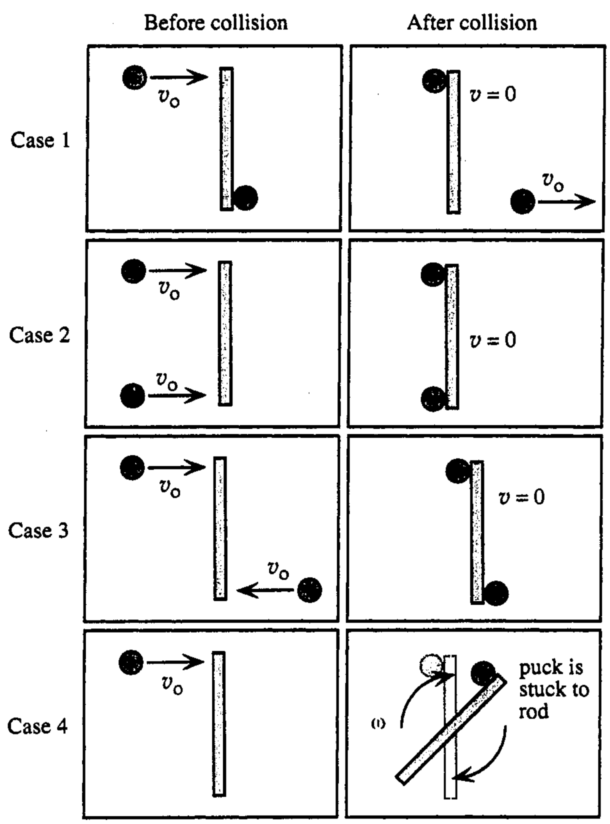
2. The four cases depicted below involve collision experiments between one or two identical pucks and a rod on a level, low-friction air table. The collisions are shown from a top-view perspective. (If any linear velocity or angular velocity is not labeled, assume it is zero. If distances appear to be equal, assume they are equal.)

   For each case:

   a. Indicate the direction of the total angular momentum of the pucks-and-rod system, with respect to the center of the rod, both before and after the collision.

   b. Indicate the direction of the total linear momentum of the pucks-and-rod system, both before and after the collision.

   c. On the basis of your answers to parts a and b, state whether the proposed process *could occur* or *could not occur*. If the process could not occur, state whether it violates (1) the principle of linear momentum conservation, (2) the principle of angular momentum conservation, or both.



Top view

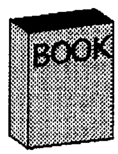
APPENDIX D: INTERACTIVE TUTORIAL LECTURE *FRICTION AND TENSION*

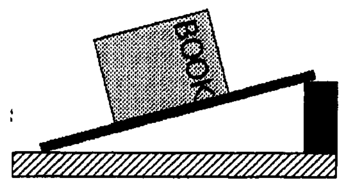
- Pretest
- Interactive tutorial lecture worksheet

Pretest for interactive tutorial lecture *Friction and Tension*

PRETEST: FRICTION AND TENSION Name:________________

A student performs a few experiments with a book and a wooden incline.

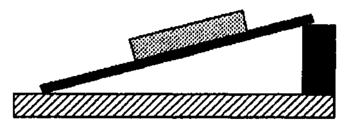




In the first experiment, the student places the book on the incline as shown above right. The book remains at rest.

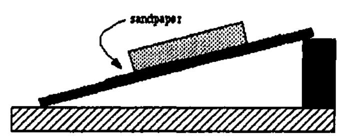
Next, the student performs a second experiment (shown below right) by making a single change to the previous setup. The book is placed on its back, on the same spot on the incline. (You are looking at the binding of the book.) The book remains at rest.

a. Is the friction force on the book by the incline in the second experiment *greater than, less than,* or *equal to* the friction force on the book in the first experiment? Explain your reasoning.



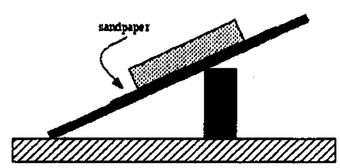
In a third experiment, the student makes another single change from the previous setup. This time, the book is placed on its back on top of a piece of sandpaper. The book remains at rest.

b. Is the friction force on the book by the incline in the third experiment *greater than, less than,* or *equal to* the friction force on the book in the second experiment? Explain your reasoning.



In a fourth experiment, the student makes another single change from the previous setup. The block that supports the incline on the far right is moved to the left, so that the incline makes a higher angle with the horizontal. The incline and the book remain at rest when the block is in the new position.

c. Is the friction force on the book by the incline in the fourth experiment *greater than, less than,* or *equal to* the friction force on the book in the third experiment? Explain your reasoning.



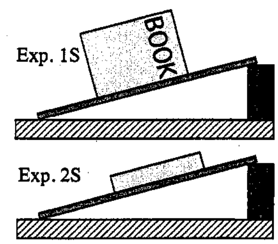
FRICTION AND TENSION

I. Surface Area

A. A student (named Sean) performs a few experiments with a book and a wooden incline. The book is shown in perspective at right.

In Experiments 1S and 2S, Sean places the book on the incline in two different orientations, as shown. In each case, the book remains at rest.

1. Is the magnitude of the friction force on the book in Experiment 2S *greater than, less than,* or *equal to* the magnitude of the friction force on the book in Experiment 1S? Explain your reasoning.



VOTE.

2. Record the results of the class vote here.

Greater than	Less than	Equal to
%	%	%

Friction forces can be categorized into two types: *static* friction and *kinetic* friction (or *sliding*) friction. A static friction force is one for which the two interacting objects do not move relative to one another.

B. In the space below, sketch a free-body diagram for the book in Experiment 1S.

Explain how you knew how large to draw the arrow that represents the friction force on the book.

How, if at all, would the free-body diagram for the book be different in Experiment 2S?

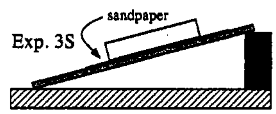
C. Reconsider your answer to part A above. Is it consistent with your results from part B? If not, how could you change your answer to part A to make it consistent with part B?

Current version of interactive tutorial lecture worksheet *Friction and Tension* (page 2 of 4)

II. Coefficient of static friction

A. Sean now performs Experiment 3S. Experiment 3S is exactly the same as 2S, except that, in 3S, Sean has attached sandpaper to the part of the incline directly beneath the book.

1. Is the magnitude of the friction force on the book in Experiment 3S *greater than, less than,* or *equal to* the magnitude of the friction force in Experiment 2S? Explain your reasoning.

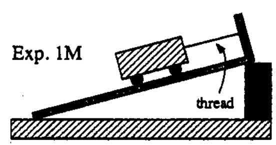


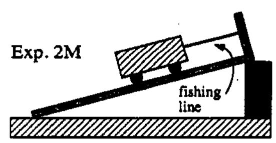
VOTE.

Greater than	Less than	Equal to
%	%	%

B. Another student, Mila, performs two experiments, 1M and 2M. (Mila and Sean are using different sets of equipment.) In Experiment 1M, Mila has put a cart with very good wheels on the incline. The cart does not roll down the incline, because she has tied a thread to the cart and to the top of the incline. Experiment 2M is exactly the same as Experiment 1M, except that the thread is replaced by much stronger fishing line.

1. Is the magnitude of the tension force on the cart in Experiment 2M *greater than, less than,* or *equal to* the magnitude of the tension force on the cart in Experiment 1M? Explain your reasoning.





VOTE.

Greater than	Less than	Equal to
%	%	%

C. The label on the package of fishing line says "20 lbs. test strength." Does the number 20 tell you how much the fishing line is pulling in Experiment 2M? Explain.

D. Reconsider Experiments 2S and 3S. Given your answer for Experiments 1M and 2M, would you change in any way your answer for Experiments 2S and 3S?

Explain what is similar in the reasoning needed to answer parts A and B.

Current version of interactive tutorial lecture worksheet *Friction and Tension* (page 3 of 4)

E. Suppose that the coefficient of static friction (μ_s) between the book and the sandpaper is 0.75 and that the normal force on the book by the incline in Experiment 3S is 8N.

1. Consider the following statement:

   "The friction force is given by $f = \mu_s N$, so the friction force in Experiment 3S is $0.75 \times 8N = 6N$."

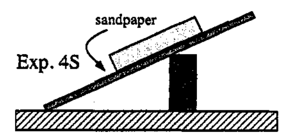
   Do you agree or disagree with this statement? Explain.

2. How would you interpret the number 6 in this instance? Explain.

III. Increasing the angle of the incline

A. Sean now performs Experiment 4S. In Experiment 4S, he moves the block that supports the incline so that the incline makes a larger angle with the horizontal. The book remains at rest on the incline.

1. Is the magnitude of the friction force on the book in Experiment 4S *greater than, less than,* or *equal to* the magnitude of the friction force on the book in Experiment 3S? Explain your reasoning.

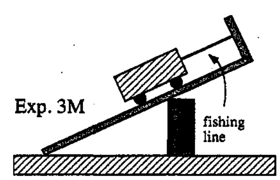


VOTE.

Greater than	Less than	Equal to
%	%	%

B. Mila performs Experiment 3M. In Experiment 3M, Mila moves the block that supports the incline so that the incline makes a larger angle with the horizontal.

1. Is the magnitude of the tension force on the cart in Experiment 3M *greater than, less than,* or *equal to* the magnitude of the tension force on the cart in Experiment 2M? Explain your reasoning.



VOTE.

Greater than	Less than	Equal to
%	%	%

C. Reconsider Experiments 3S and 4S. Given your answer for Experiments 2M and 3M, would you change in any way your answer for Experiments 3S and 4S?

D. Consider the following statement:

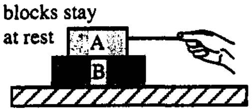
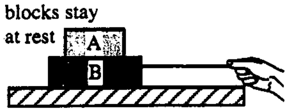
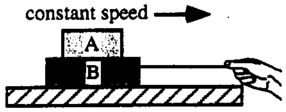
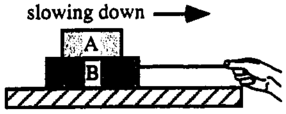
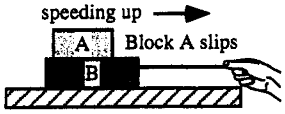
"Since the incline was raised from Experiment 3S to Experiment 4S, the normal force on the book decreased. This is because $N=mg\cos\theta$, and θ increased. Therefore the friction force also decreased, because $f=\mu_S N$."

Indicate which parts of the statement you think are correct or incorrect.

Current version of interactive tutorial lecture worksheet *Friction and Tension* (page 4 of 4)

IV. Determining the existence and direction of friction forces

Block A is on top of block B, which is on a table. In each of the five cases depicted below, a person is pulling a string that is attached to one of the blocks. In the first four cases, block A does not slip on block B. Draw an arrow to represent the direction of the friction force described in each column of the table, and indicate whether it is *static* or *kinetic* friction. If any friction force is zero, write that explicitly.

	Directions of friction forces:		
	on A by B	on B by A	on B by table
blocks stay at rest 			
blocks stay at rest 			
constant speed → 			
slowing down → 			
speeding up → 			

It is often said that "the friction force always opposes the direction of motion." Can you find any counterexamples to this statement in the cases above?

APPENDIX E: OPINION: "HOW SHOULD WE USE THE WORD CONSERVE?"

Most introductory mechanics textbooks, instructors, and students use the word "conserved" as a synonym of the word "constant." This paper advocates a different usage of "conserve," by which a conserved quantity is one that changes according to certain rules. The "conserved = constant" usage seems more common for quantities in mechanical contexts, while the usage advocated here seems more common for quantities in other contexts (*e.g.*, charge, baryon number, energy of heat engines). If we use the word "conserve" as a description of the special manner in which some quantities change, conserved quantities will not necessarily be constant, and not all constant quantities will be said to be conserved. Further, while the description of the value of a quantity or how it is changing requires specification of a particular object or system in context, statements about conserved quantities should not be so restricted, but should rather be stated without qualification: "(momentum, energy, charge, *etc.*) is conserved."

Consider first two closed cardboard boxes: one containing pennies and the other containing bunnies. If we carefully count the number of pennies in the box at different times, it is conceivable that the counts will be different. If the number of pennies in the box increases, the most reasonable conclusion (under ordinary circumstances) is that someone put some pennies in the box. Pennies (that used to be somewhere else) must have passed through the bounding surface area of the box. A large number of tiny guards, restlessly patrolling the boundary of the box, could give reports of pennies passing in or out of the box that would be perfectly consistent with the net change in the number of pennies inside the box (even though the guards were not watching the pennies in the deep interior regions of the box). Under ordinary household circumstances, pennies cannot be created or destroyed. Contrariwise, if the number of bunnies inside the other box were to increase in a way that was inconsistent with reports given by the border guards, we should not conclude that box security had been breached. Bunnies can be created and (lamentably) destroyed. To apply the word "conserve" to this story: pennies are conserved, and bunnies are not conserved. (Of course, in the final picture, pennies are not conserved – I destroyed one myself in high school with a Bunsen burner.) To say that pennies are conserved is not to say that the number of pennies in the box does not change; it is to say that the number of pennies in the box changes exactly according to a "penny current." Just as "pennies are neither created nor destroyed" is not qualified by referring to a particular set of pennies in a particular place, "Pennies are conserved" says something about the nature of the pennies in general. Though there can be bunny currents, bunnies are not conserved, because there are bunny sources and sinks.

It is common to hear physicists describe energy and momentum in one-dimensional collisions like "In an inelastic collision, momentum is conserved, but kinetic energy is not," or "In an elastic collision, momentum and kinetic energy are both conserved." First, to demonstrate the usage suggested above, momentum is conserved, and kinetic energy is not. The momentum of an object or system changes according to momentum currents (impulses), without being created or destroyed. On the other hand, kinetic energy can be created and destroyed. When two clay balls of equal mass move with equal and opposite velocities toward each other along a line and stick together, the kinetic energy in that system decreases (from some positive value to zero). This kinetic energy decreases without a corresponding current of kinetic energy passing out of the system and kinetic energy appearing somewhere else. Instead, the total energy of the system remains the same, and the kinetic energy is converted to thermal energy. It is because energy did not pass into or out of the system that the total energy of the system remains the same. Energy is conserved. (By the way, even in collisions for which kinetic energy is allegedly conserved, it does not remain constant. Though the initial and final values may be the same, there is a dip in the kinetic energy while the colliding objects are transferring momentum.)

We should also not say a quantity is conserved just because it is staying constant. In the case of bunnies, because they are able to change in number without currents, we should still refrain from saying bunnies are conserved, even if the number of bunnies in the box is unchanged. Also consider intensive quantities. If the temperature in an office building is held constant, should we say that the temperature of the building is conserved? If an opera singer sings a long note of constant pitch, is the frequency of the vocal cord vibrations conserved? Most physicists would not think to use "conserve" in these contexts. If we use "is conserved" to mean "is constant," then the act of remaining constant is sometimes called conserved and other times not. How do students learn to tell the difference? If "conserved" means "stays constant," what do we gain by using the word, aside from the pleasure of saying a technical term? We know that our students use words synonymously that we do not: velocity, acceleration, force, momentum, energy, *etc*. To many students, all of these words are physics terms that stand for one concept: motion. If we use technical terms in place of common terms (read "conserved" in place of "stays the same"), then we encourage students to believe that part of learning science is using fancy but fundamentally unnecessary language.

Constancy is a special case of conservation. This case obtains when there is a net zero current of the conserved quantity from the system to its surroundings. When there is zero net impulse on a system, the momentum of the system remains constant. When there is zero net work done on (and zero net heat transfer to) a system, then the total energy of the system remains constant. The concept "universe" is defined as that system that has no surroundings with which to interact. Thus the currents of conserved quantities are zero for the universe also.

When a conserved quantity changes for a particular system, it is usually said not to be conserved. We often hear "the momentum of the system was not conserved because there was a net external force on the system." Such a statement makes it sound like the principle of momentum conservation does not apply when there is a net force on the system. We might naively infer that, after a process for which a system experiences a net force, its momentum could be just about *anything,* since "momentum is not conserved." If we do not imply falsely that conservation laws are *true* only in the zero-net-current case described above, then we certainly imply that this case is the only one for which conservation laws are useful. However, conservation laws are useful *for solving problems* when we know something certain about the currents of conserved quantities, whether or not those currents are zero. They may be useful *for instruction* in even more cases. For example, it may benefit students to practice describing the transfers of multiple conserved quantities during a process, in contexts for which certain conserved quantities would normally not get any attention.

To summarize, we should speak unconditionally about the conservation of a fundamentally conserved quantity: "charge is conserved," "linear momentum is conserved," *etc.* We should strictly reserve the word "conserve" for reference to fundamentally conserved quantities only. Finally, we should teach students that conservation laws are always true by teaching them to employ conservation laws in cases of non-zero net currents of conserved quantities.

The opinions expressed here were strongly influenced by *Teaching Introductory Physics*, by A. B. Arons, especially Chapter 5 of Part I.

VITA

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