Student-Generated Representations by Middle-Schoolers with MLD: An Exploration of Accessibility within an Equal Sharing Fractions Intervention

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Abstract

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An exploratory qualitative case study of three middle-school students with MLD was conducted to explore the accessibility of a Cognitively Guided Instruction Equal-Sharing intervention. Here I employ a difference perspective to extend Vygotsky's sociocultural theory of mediational tools in relation to the theoretical constructs of student-generated representations (SGRs) to consider issues of accessibility. SGRs are appropriate for this inquiry as they are both evidence of students' understandings and of patterns of reasoning during mathematical problem-solving. Student-generated representations have not been typically used to think about issues of accessibility, however they are well-suited to. Through 3 case studies with 5-8 hour long session each, I explored the ways in which students, who demonstrated unique patterns of reasoning around fraction content, engaged with the Equal-Sharing intervention, used and created SGRs. In a detailed retrospective analysis of understandings in conjunction with tool use and problem-solving, it was revealed that some aspects of SGRs and the action-oriented environment of the intervention enhanced accessibility, but that some issues of inaccessibility remained. Differentiated and scaffolded content was still necessary to alleviate tensions created between inaccessible SGRs and the students' understandings. Implications for research and practice are provided.

Keywords: mathematical learning disability, dyscalculia, fractions, intervention, Equal-Sharing, representations, modeling, atypical fraction understandings, Cognitively Guided Instruction, student generated representations

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Researchers have argued that the ability to think representationally is a necessary skill for success in academics (Jitendra et al., 2016), and has marked the difference between novice and expert (van Garderen, 2006). This sentiment has been echoed in the mathematical domain of fractions; where proficiency is equated with increased success in higher level mathematics (National Center for Educational Statistics, 2009). However, the use of traditional pedagogical tools and content is not equally effective for all students; this is especially true for those whose cognitive profiles differ from the intended population of most curricula. These barriers have been experienced by learners with a math learning disability (MLD; dyscalculia) (Lewis, 2014).

Unfortunately, research has not explored how issues of inaccessibility emerge for students with MLD when they learn about fractions. Research has only briefly examined issues of inaccessibility, that is, how common pedagogical tools for teaching fractions do not support learners with the unique patterns of reasoning inherent to learners with MLD (Lewis, 2014). Therefore, it remains an open question in the field of MLD research how to address this incongruence. What has been shown, more generally, is that Constructivist oriented mathematics curricula, such as Cognitively Guided Instruction (CGI; Carpenter, Fennema, Franke, Levi, & Empson, 2015), where sense-making and reasoning about quantities is encouraged, is an effective approach for typically achieving students, and there is some research to suggest that this may be true for students with disabilities (e.g., Hunt & Empson, 2015). Still it is unclear if CGI would also be effective for students with dyscalculia or MLD, which results in qualitatively different ways of reasoning and results in issues of access (Lewis, 2014). Therefore, my research aims to clarify this ambiguity by investigating the ways a highly touted and promising pedagogical approach that elicits fraction knowledge (i.e., CGI Equal-Sharing intervention; henceforth, ES intervention) can be used productively to create accessible pathways for engagement for learners with MLD.

This inquiry is broadly situated within Vygotsky's (1929/1993) sociocultural theoretical framing of disability. "*Defectologia"* or "defectology" (Vygotsky, 1929/1993), a now outdated term for the study of individuals with sensory, physical, and cognitive disabilities, provided the foundation for his theories of typical and atypical human development. Here I view disability as a cognitive difference (Lewis, 2014). To unearth points of incongruence, and possible elements of inaccessibility, I propose the use of mathematical research on modeling and representations as a method for analyzing understandings and problem-solving strategies that would otherwise be invisible. Model generating activities (Lesh, Carmona, & Post, 2002), such as those embedded in the CGI ES intervention, make it possible to use student mathematical representations as a unit of analysis. Student representations are evidence of student understandings – internalized processes that are otherwise difficult to capture—but these representations also serve as student specific evidence of problem-solving strategies and behaviors that learners with MLD bring to solving fractions problems.

The use of representation theory as it relates to mathematical understanding and problem-solving skills is not a new avenue of research (see for example, Schoenfeld, 2016; Rellensmann, Schukajilow, & Leopold, 2016) but the same cannot be said for its application to research within the field of MLD where it has only been minimally applied (e.g., Hunt & Empson, 2015). My research aims to fill the gap of using representation theory as a method for understanding the curricular accessibility for learners with MLD, that is, to deepen our understanding of the alignment of students' understanding of fractions within this specific ES context. To that end, in this paper, I present a case-study of three middle school students with an MLD as they embark on a CGI Equal Sharing fractions intervention paying particular attention to student-generated representations.

Literature Review and Conceptual Framing

MLD as a Cognitive Difference

Mathematical learning disability (MLD; i.e., dyscalculia) is a brain-based learning difference that affects the ability of learners to engage with mathematical content (Butterworth, 2010), and is differentiated from the broadly defined category of mathematical *difficulties* which can be attributed to any number of social or environmental factors (e.g., socioeconomic factors; Lewis & Fisher, 2016). Due to these biological differences, learners with MLD commonly experience chronic challenges engaging with the canonical representations of mathematical concepts and pedagogical resources encountered in the classroom (Lewis, 2014). Understanding these differences is paramount for designing and implementing appropriate interventions.

Members of the research community have approached the study of learners with MLDs from different perspectives to explain and understand academic challenges. In this section I briefly describe and critique the dominant deficit perspective and then describe an alternative model grounded in a sociocultural theory. The dominant deficit perspective for understanding MLD research is one that unfortunately pathologizes learning disabilities (Reid & Valle, 2004), which can result in interpreting difficulties learning math content as individual deficiencies.

Research from this perspective typically takes the form of a quantitative analysis of errors over a nuanced depiction of student problem solving (Lambert, Tan, Hunt, & Candela, 2018).This interpretation leads to research that defines MLD in terms of its area of cognitive deficits (such as limited numeric processing or deficient mental representations of the number line (e.g., Butterworth, Varma, & Laurillard, 2011; Geary, Hoard, Nugent, & Byrd-Craven, 2008)). Any difficulties are attributed to an internal characteristic (read: deficit) within the learner.

To broaden disability inquiry and to combat a deficit perspective to disability research, some researchers have begun to adopt a difference perspective on MLD. For example, Mazzocco and colleagues have identified qualitative differences in students with MLD. An error analysis demonstrated that learners with MLD do not simply underperform in mathematics compared to typically achieving learners; their errors, which have the potential to lead to underperformance in mathematics, are *qualitatively* different (Mazzocco et al., 2013). Mazzocco, Devlin, and McKenney (2008) demonstrated that individuals with MLD make qualitatively different kinds of errors compared to low-achieving and typically achieving peers on timed math facts assessment, and furthermore, these cognitive differences in understanding of quantity extend to the mathematical domain of fractions (Mazzocco & Devlin 2008; Mazzocco et al. 2013). Others have gone further in rejecting the deficit model and have taken a social model of disability, which conceptualizes learning disability as a cognitive difference, differences that are viewed as natural, normal human variations and not evidence of disorders of deficiencies (Miller, 1993 see for example Lambert & Tan, 2017; Lambert, Tan, Hunt, & Candela, 2018; Dudley-Marling & Burns, 2014).

I draw on multiple theoretical constructs to make sense of the interaction between the ES intervention and the conceptualization of learners with MLD, all while attempting to broadly situate this inquiry within an anti-deficit framing of disability. In this article, I use the terms *concepts* and *conceptualization* to mean "the shared public aspects of some idea" and "the way students and others think about a concept" (Roth & Thom, 2009, p. 176), respectively. First, this research leverages Vygotsky's (1929/1993) sociocultural theoretical framing of disability, Defectology, which is a useful framework for further elaborating on his assumptions about mediational tools and scaffolding for learners with disabilities. This broad framework positions the research inquiry from a place of difference and allowed me to focus on diverse patterns of conceptualizing and problem-solving to consider issues of accessibility ascribed to cognitive differences.

Vygotsky's Sociocultural Theory

In seeking a theoretical grounding for this research, I draw upon the work of Vygotsky (1929/1993) for several complementary reasons. A central tenet of his sociocultural perspective is that humans are innately equipped to use artifacts (physical, symbolic, or material) to mediate activities. Artifacts are historically created within a culture and are developed to meet the specific needs of the practice, in this context, mathematical ones. Furthermore, in Vygotsky's view, all mental functioning that characterizes human thought is mediated, and tools facilitate mental processes. From this perspective, mathematics learning can be described as the "appropriation of practices associated with the sets of artefacts that have historically come to represent the body of knowledge we call mathematics" (Healy & Fernandes, 2011, p. 157). In any given circumstance mediational tools take diverse forms. Take for instance the specific context of teaching fractions in an educational setting, commonly used mediational tools for communicating concepts and relationships are area models, fraction tiles, and Unifix cubes. However, what will be established is that these mediational tools are not equally useful for all learners.

Vygotsky's Defectology (1929/1993) research emphasized the educational needs and development on blind learners, deaf learners, and learners with different cognitive disabilities. Vygotsky asserted that rather than focusing on quantitative differences in achievements between those with and without LD, a qualitative stance would be a more fruitful avenue for understanding the educational and developmental needs of these individuals. For example, in the case of the blind or deaf learner, he argued that some traditional instructional tools were inaccessible to learners without the sensory resources of vision and speech, and that other mediational tools or artifacts could act as a substitute for their function (Vygotsky, 1929/1993) in their absence. In other words, the pedagogical utility of the written word for a blind learner is negligible, and therefore, should be substituted for a more suitable mediational tool such as braille. Vygotsky advocated for the need for educators to seek the appropriate mediational tools for streamlining learning, understanding, and engagement for different learners; however, he did not broach how subtle differences in cognition could be attended to from this perspective.

Vygotsky and neovygotskian math education researchers (see for example, Swanson & Williams, 2014) argue that all mathematical activity is mediated. As outlined above, Vygotsky considered accessibility for individuals with physical differences and the need to develop more accessible tools more aligned with these learners' perceptual abilities. However, Vygotsky did not consider cognitive differences such as those seen in MLD. The present inquiry centers accessibility of mediational tools and an ES intervention for learners with MLD by drawing a parallel to Vygotsky's example of inaccessibility for blind learners. This is motivated by Lewis' (2014) assertion, and call to action, that issues of access exist between standard mathematical mediational and the unique cognitive profiles of learners with MLD. Mediational tools do not function or support as intended for these learners. There is a need to ascertain what mediational tools are useful, accessible, and are conducive of productive fraction problem-solving (see for example, Lewis' (2017) use of a 'scale' intervention to increase accessibility). In order to evaluate and assess cognitive accessibility of an ES intervention for students with MLD, I drew upon a theoretical construct that permitted an analysis of student understandings.

Atypical Fraction Understandings

Taking a difference-not-deficit perspective to MLD research, and similarly grounding her research in a Vygotskian framing, Lewis (2014) articulated a rationale for the common fraction difficulties experienced by learners with MLD that leads to recognizing the limitations and inaccessibility of common classroom materials in communicating math content to these learners. To make this claim, Lewis (2014) "focused on how and what each student understood about the mathematics" in her exploration and referred to those as *understandings* (p. 352). Here, too, I use *understandings* in the same way. She "identified the understandings the student relied upon and the ways in which these understandings persisted and were incompatible with standard instructional approaches" (p. 352). From there, she found that learners with MLD behaved in a way consistent with two persistent understandings—that lead to an incongruence between student understandings of fraction quantities and canonical representations of fractions, such as area models.

These very same persistent understandings became the foundation for creating a pen and paper assessment for screening the presence of these persistent understandings (Lewis, Thompson, & Arvey, 2020) as a method for possibly identifying individuals with MLD. In Lewis, Thomson, and Arvey (2020), the term persistent understandings are referred to as *atypical understandings*, and the present inquiry will employ this terminology throughout.

The first atypical understanding, which is what Lewis named "halving," and was operationalized as the learner interpreting the quantity ½ as a splitting action rather than the quantity ½ (2014). For example, a learner might draw ½ by splitting a shape in two and emphasizing the process of cutting into the two pieces, rather than 1 of the two pieces. These models tend to resemble a circle with a line down the middle, with the understanding that the line itself indicates half. In many instances, the individual with MLD focused on balance as a focal element in identifying one-half. The atypical understanding, *fractional complement,* was operationalized as interpreting area models or other fraction representations based on the fractional complement (e.g., the non-shaded area of an area model). For example, the learner might interpret an area model of 2/3 as 1/3 based on the non-shaded area (Lewis, 2014). Fractional complement demonstrated that learners with MLD can find ambiguity when assigning meaning to shading (or the act of shading) in area model representations, commonly fluctuating between canonical and atypical interpretations within discussions. The operationalization of these AUs helped to establish a clearer cognitive profile of learners with MLD and has implications for designing interventions that tap into the conceptualizations of these learners and for creating accessible pathways to math content. This study builds on Lewis's (2014, 2017) argument that alternative methods or models are necessary for learners with MLDs to successfully engage with fractions.

Representation Theory: Model Eliciting Activities and Student Generated Drawing

It has already been demonstrated in prior research on AUs and standard mediational tools for fractions create barriers of access for students with MLD. In order to understand and potentially circumscribe these barriers, I employ theoretical research on modeling and student-generated representations. Without such a theoretical construct it is challenging to make claims about students' internalized models of mathematics, a sentiment that is reified in Lesh, Carmona, and Post's (2002) assertion that:

> it is impossible for anyone to know exactly what's inside a students' mind. But, when students are asked to develop a model (which is expressed in some kind of representational media), many inferences can be made about the nature of their mathematical knowledge and its development (p. 89)

These externalized representations are a means for researchers and teachers to view student thinking and conceptualizations. To unearth the areas of inaccessibility that require additional attention and modification – SGRs can help teachers and researchers interpret and gain access to student thinking.

Prior to discussing issues of representations and their potential for conceptual inaccessibility it is first necessary to impart some definitions relevant to modeling and representation theory. In many ways, modeling (Lesh, Carmona, & Post, 2002) and student-generated representations (Rellensmann et. al., 2016) are similar but not synonymous constructs. Models are generally an artifact of a model-eliciting activity, which are activities designed to provoke student thinking around real-world problems. Model eliciting activities support mathematical conceptual development by encouraging students to develop mathematical models using "representational media" (p. 90). Representations refer to any configuration of characters, images, or concrete objects, that symbolizes an abstract idea (Goldin & Kaput, 1996) and may include manipulative materials (physical objects), pictures, or diagrams, real-life situations, spoken language, or written symbols (Lesh, Post & Behr, 1987). These models can be subjected to further revisions and refinement as students alter and improve their conceptual thinking.

A more nuanced form of representations is known as student-generated representations (SGRs). These were defined by Rellensmann, Schukajilow, and Leopold (2016) as "the process and the product of generating an illustration that corresponds to the objects and relations described in a task" (p.54). From these definitions, representations can be viewed on a spectrum, with modeling signifying the more sophisticated and developed end. SGRs are an artifact of student problem-solving, and modeling can be described as a more holistic act of representing, refining, and describing internalized mathematical constructs. Both are evidence of student conceptualizations, and both will be used to discuss issues of cognitive accessibility.

The act of constructing representations has been identified by researchers as an effective mathematical problem-solving strategy (Csíkos, Szitányi, & Kelemen, 2012; De Bock et al., 2003). Much research has been devoted to demonstrating how student-generated representations and mathematical modeling enhance mathematical problem-solving (see Schoenfeld, 2016; Larkin & Simon, 1987; Van Meter & Garner, 2005). Extant research has demonstrated that the creation and use of representations does not necessarily lead to correct solutions, but that it helps to create an environment conducive of mathematical thinking. In sum, representations are considered a helpful and successful cognitive tool in mathematical problem solving (De Bock et al., 2003).

In line with constructivist ideals of effective mathematical learning (De Corte, 2000), the successful construction and communication of SGRs are more positively correlated to increased mathematical thinking including developing skills related to mathematical dialogue. SGRs are useful for a multitude of reasons. Per De Bock et. al. (2003), these SGRs "stimulate an even more deep-level and mindful approach of the task...[and]… increases the chance that a problem will be conceptualised correctly" (p. 446). Furthermore, the act of drawing helps students to organize elements of the given tasks, it develops inferential skills, increases information processing (Rellensmann, Schukajlow, & Leopold, 2016, p. 54).

Mathematical dialogue that includes, "students' descriptions, explanations, and constructions… reveal how they are interpreting the mathematical situations they encounter by disclosing how these situations are quantified, organized, coordinated, and interpreted" (p. 90). This articulation process can be pushed further through the use of guided discourse with instructors or other students (Alexander, 2008) to reveal points of contentions between mediational tools and student thought processes. Student dialogue, can best be elicited through the use of scaffolding (Wood, Bruner, & Ross, 1976), which is described as "… processes that enable the child or novice to solve a problem, or carry out a task or achieve a goal which would be beyond his unassisted efforts" (p. 90). In the original formulation of scaffolding, a method for adapting teaching to unique students, places the onus of decision making on the educator – intervening, supporting, altering content or approach—to improve problem-solving circumstances (Wood, 2001). Throughout the present inquiry, scaffolding in the form of altered content and explicit instruction, was used to track student thinking around fraction representations and problem-solving. In other words, we found that meaning-making around the SGRs was more fully realized in the presence of dialogue and scaffolded instruction.

In this inquiry, I am positioning student generated representations as a type of mediational tool that facilitates the communication of mathematical reasoning and knowledge. I consider them active living elements of the problem-solving process which are more fully realized in the presence of dialogue and scaffolded instruction. Since standard representations presented issues of inaccessibility for leaners with an MLD, this inquiry focuses on SGRs and the ES Intervention that elicits them has not been conducted in the field of MLD as a means for understanding accessibility. Representation theory will be used to describe students' thinking throughout this inquiry. The role of scaffolding will be discussed throughout, both as a method for eliciting student thinking, but also as a means for distilling which elements of the intervention were perhaps inaccessible and best circumvented or altered.

Cognitively Guided Instruction (CGI)

The Equal Sharing intervention, as will be explicated, is a prototypical example of a model eliciting activity and is one of the few theorized and evidence-based strategies for approaching learners with MLD that offers numerous opportunities for differentiation and the creation of SGRs. The ES Intervention was developed based on Cognitively Guided Instruction (CGI; Carpenter et. al., 2015), which is not a specific curriculum, rather it is a flexible teaching philosophy that positions all learners as competent regardless of ability status by leveraging strengths, experiences, and interests. Per Carpenter et. al (2017), CGI has been described as universally well-suited to any type of learner, because it positions them as experienced, knowledgeable, and active participants in mathematical meaning making; anthesis to an empty receptacle passively amassing formal mathematical convention. CGI is also a sequenced way of using problem solving task (types) to help students reason and make sense of math by building on intuitions and not relying on explicit instruction.

Correspondingly, Empson and Levi (2011) articulate that the varied experiences that children possess are an entry point into teaching mathematics: teachers can draw upon these experiences and "intuitive understandings of mathematics and the world" (p. 5) to teach math content. Furthermore, Carpenter et al. (2017) have argued that by "[f]ocusing on what children already know and can do mathematically allows [the teacher] to position them as competent and to support them to communicate their understandings" (p. 3). To that end, Empson and Levi (2011) designed an Equal Sharing curriculum.

The foundation of the ES curriculum are word problems that attempt to leverage experiences with partitioning and sharing to support an understanding of fractional quantities (Empson & Levi, 2011). The general makeup of ES problems are as follows; it begins with "a total number of items to be distributed to a given number of …people. This type of problem requires that children give each person the same-sized share and use up (or exhaust) all the sharing material" (p.8) One such example is the problem of equally sharing 10 brownies amongst four children. In order to solve these types of problems, the learner is typically given access to various mediational tools to represent their thinking.

CGI posits that most, if not all, learners have prior experiences that potentially enable them to understand fraction concepts (Empson & Levi, 2011), and in particular "equal sharing situations seem to trigger students' informal ways of reasoning…." (Hunt, Welch-Ptak, & Silve, 2016, p. 214) using familiar scenarios such as sharing food with friends or siblings to support problem solving and reasoning (Empson & Levi, 2011, p. 6). Using this theory as an access point for learners with disabilities, Hunt and Empson (2015) conducted an exploratory study of informal problem-solving strategies employed by learners labeled as having a learning disability (LD), specifically engaging with ten 3rd through 5th grade students labeled with an LD and having IEP goals in mathematics.

Their analysis demonstrated three categories of problem-solving approaches: no-coordination, non-anticipatory, and emergent anticipatory, each indicative of varying levels of sophistication aligned with an understanding of the relationship (coordination) between partitioning and resulting fraction (Empson et al., 2005; Olive & Steffe, 2002; Pothier & Sawada, 1983; Steffe & Olive, 2009; Tzur, 1999). Their study focused on errors and student assumptions made in the process of creating SGRs but did not articulate any perceived areas of inaccessibility between student and content. In the end, their study demonstrated a tendency for learners with LD to rely on a condensed set of problem-solving approaches tantamount to trial and error and evaluated the ES intervention as a productive context for learners with LD.

Although, Hunt and Empson (2015) were able to make empirical claims about the beneficial outcomes of an ES intervention for learners labeled as having a learning disability, I argue that their participant selection varied significantly enough from the present study to create a gap in making conclusions regarding the efficacy and accessibility of the intervention for those very *specific* learners who exhibit AUs around fraction content (i.e., have an MLD). Therefore, the aim of the present study is to extend upon what Hunt and Empson (2015) and Empson and

Levi (2011) have suggested for engaging learners with disabilities in productive fraction problem-solving to address inaccessibility experienced by learners with MLD.

Research questions guiding this qualitative case-study are as follows:

1. In what ways do SGRs diminish inaccessibility of standard fraction representations as they relate to atypical understandings?
2. How do students with MLD use these models to make sense of fractions considering their atypical understandings?
3. How do SGRs change, or influence, the students' relationship with canonical representations?

To answer these questions, I have conducted an exploratory, qualitative microanalysis of weekly tutoring sessions of two researchers – Katherine Lewis and myself—working with three students with an MLD. Each student was met with approximately 6 times, for a total of 19 hours of tutoring data. We utilized a qualitative case study methodology to describe the impact of the students' atypical understandings on and within an ES intervention. This specific context can be used or extended upon on a macrolevel or future studies (Stake, 2010) involving the purposeful, strategic attention to student conceptualizations and accessibility to math content in the classroom.

Methods

Atypical Understanding Screener

In prior research, Lewis, Thompson, and Arvey (2020) established the utility of a 13-question group-administered pencil-and-paper screener in capturing atypical fraction understandings (halving and fractional complement) within the general population. A complete list of questions and scoring are provided in Appendix A. The Atypicality Screener was administered to multiple samples of students to ascertain the effectiveness of the screener in identifying AUs in the general population. It was first administered to a group of 390 middle school students, and then later to a sample of 90 students that the present study recruited from.

We scored this screener by analyzing each problem for evidence of characteristic atypical understandings. One atypical understanding point was received for each instance of an atypical understanding. The criteria for "high atypicality" was set at four or more atypical points, which is equivalent to demonstrating an atypical answer on more than 30% of problems (Lewis, Thompson, & Arvey, 2020). The study demonstrated that those students who exhibited high levels of atypical understandings also met the clinical criteria for a math learning disability; thus, demonstrating the ability of a screener to isolate patterns of reasoning that correspond to the numerical processing abilities of students with MLD. In essence, this Atypicality Screener demonstrated it possible to identify learners with MLD based on cognitive characteristics and did not rely on using low performance as a proxy for an MLD (Lewis & Thompson, 2015).

Recruitment Procedures

The entire middle-school population (n=80) at a private school in the Pacific Northwest for learners with reading-based learning disabilities were assessed. Students were recruited from this cohort to take part in an investigative exploration of fraction problem-solving for learners with MLD.

In order to explore the conceptualizations and problem-solving characteristics that students with MLD bring to solving fraction problems we employed a case study. There are multiple definitions and interpretations of case study research. I have chosen to align with Merriam's (1998) assertion that: "a qualitative case study is an intensive, holistic description and analysis of single instance, phenomenon, or social unit" (p. 21); with particular care taken to "delimiting the object of the study, the case" (p. 27). "Case studies are also targeted at information-rich sources for in-depth understanding and can also be used to inform policies or to uncover contributing reasons for cause-and-effect relationships" (Bhattacharya, 2017, p.109). I agree with Bhattacharya that hopes for a holistic account are untenable in qualitative case study research: researchers can hope to describe, explain, and attempt to understand in the moment phenomena. That is precisely what is meant to transpire in the following pages.

Participants

The Atypicality Screener was administered by the students' mathematics teacher. To maintain anonymity, the cover page and first page of the assessment both contained the student's test ID, but the teacher was tasked with removing the cover sheet which contained the student's name. The teacher was asked to retain the cover sheet for subsequent student recruitment. To recruit students for the current study, each math teacher was given a list of test IDs associated with the 7 students (9%) who had Atypicality scores of at least 4 points and were asked to distribute consent materials to these students. Individuals who were interested were asked to return to the research team via the U.S. Postal Service or otherwise, asked to discard consent materials if disinterested.

Of the seven recruitment offers made, three students accepted, Ryan, Lily, and Maddie (all student names are pseudonyms) and returned consent forms. Ryan, Lily, and Maddie were 13 years old and in the 8^{th} grade. All identified as white and were native English speakers. Results from their Atypicality Screener can be found in Table 1.

Table 1

Atypicality Screener Results

	Halving	Fractional Complement	Total Atypical Understandings
Ryan	1	3	4
Lily	1	3	4
Maddie	0	4	4

Data Collection

Data took multiple complex forms throughout this research project. I conducted a microgenetic analysis that looked at problem solving process, gestures, inscriptions, and explanations to discern meaning embedded in the students' representations and explanations.

In session 1 we administered the following: 1) background history interview, 2) atypicality interview, 3) standardized assessment of mathematics. Session 1 contained the pre-Atypicality Interview, the background interview, and the Woodcock Johnson IV. Sessions 2 through the penultimate were devoted to ES problems, while the final session was a cumulative assessment that began with a re-administration of the Atypicality Interview (subsequently referred to as the post-Atypicality Interview) and a survey of the ES problems the students encountered.

Background History Interview

The background interview was administered, and audio and video recorded in the first session to gain insight into the student as a learner and as a person with lived relevant experiences. It included about ten minutes of questions regarding the students' relationship to mathematics and school, hobbies, academic experiences and resources, participant cultural background, and English proficiency status. This participant interview was transcribed and acted as a measure to help tease out any possible confounding environmental or social factors that may explain for low mathematical performance other than an MLD.

Atypicality Interview

The Atypicality Interview (Appendix B) was administered and video and audio recorded during the first and last session and. Questions from the Atypicality screener were presented verbally in a clinical interview style with printouts. The reasoning for this verbal presentation was two-fold: (1) verbal presentation allowed researchers to potentially eliminate any reading-related difficulties that could complicate the problem-solving process, and (2) the clinical interview process provides further opportunity to probe the student's understanding of the mathematical concept in question that would otherwise be invisible if given solely in written form (Ginsburg, 1997), and (3) this interview was also used to corroborate that the written responses on the screener were characteristic of the atypical understandings. The Atypicality interview was transcribed, all student written artifacts were collected, and scanned into PDFs.

Standardized Assessment of Mathematics

During the first session the Woodcock Johnson IV Test of Achievement (Schrank, Mather, & McGrew, 2014) was administered. The subtests included, Applied Problems, Calculation, and Math Facts Fluency (Table 2) were administered to students as a measure of mathematics achievement during the first and second sessions due to time constraints. It was used to provide norm referenced mathematics achievement data to help establish that the classification of students as having MLD was warranted. In order to confirm a MLD status, students were expected to meet a cutoff score of the tenth percentile on the mathematics composite (Lewis & Fisher, 2016); Ryan and Lily did. Maddie met the slightly higher threshold of less than the twenty-fifth percentile.

Table 2

Woodcock-Johnson Test of Achievement IV composite and subtest percentile scores for the case study students.

	Ryan	Lily	Maddie
Mathematics Composite	1	0.2	24
Broad Mathematics	<0.1	<0.1	8
Math Calculation Skills	<0.1	<0.1	7
Applied Problems	7	2	29
Calculation	1	0.1	25
Math Facts Fluency	<0.1	0.2	2

Cognitively Guided Equal Sharing Intervention Sessions

Setting and Frequency

Due to scheduling limitations, tutoring sessions were conducted in October 2015, 8 months after the original screener was administered. Each student was seen approximately once a week for one hour each, Ryan for 8 sessions, Lily for 6, and Maddie for 5. All weekly tutoring sessions took place from October 2015 to December 2015 and were conducted in a small open cubicle located in the school's learning resource center by either Katherine Lewis or in an office space located in the College of Education with first author.

Overview of CGI ES Intervention

We employed and adapted a specific variation of CGI to focus on equal-sharing experiences meant to elicit fractional understandings as seen in Empson and Hunt (2011) book *Extending Children's Mathematics*: *Innovations in Cognitively Guided Instruction*. Each of the weekly lessons was based on a single protocol that was then tailored specifically to each students' sense-making and current level of understanding to create and sequence lessons. Decisions were made on a weekly basis and will be discussed in the Researcher Debriefs section. The underlying goal always being to ascertain which elements of the curriculum were accessible.

Each ES Intervention session the students were asked four to eight problems. The students were provided a written copy of the problem, one page with a single question. These questions were also verbally presented. The research team decided to read each problem to the students to for the same reason that we chose to read the Atypicality Interview to the students. The range of four to eight problems per sessions was dependent on student speed and challenges. Understandably, the students' speeds through the intervention were unique; therefore, the administration of problems differed for each student. Individual student trajectory through the protocol is included in Appendix D. An example of an ES problem: *4 children want to share 21 donut holes so that everyone gets the same amount. How much can each child have?*

Prior to asking any questions the students were supplied an assortment of mediational tools, Unifix cubes, markers, paper, fraction tiles, paper strips, and so forth. To start the problem, students were given free agency to engage with the content in any way they saw fit. In this decision, was the acknowledgement that learners with MLD are likely more knowledgeable about the aspects of math learning that are meaningful and helpful for them than two researchers without an MLD. After the student was given an opportunity to free problem-solve, the researcher may request that the student use an alternative method. The rationale for this will be discussed in the next section. The tutoring session ended by having the student document in a journal entry an experience or idea from the tutoring session that felt particularly salient for them. These journal entries were mutually reviewed between researcher and student at the beginning of each session.

Instructional Decisions and Design

Table 3 shows the problem types that were employed in the ES intervention. We began the protocol with problems that elicited solutions larger than 1, as the research indicates that these are a more comfortable starting place for novice learners to think about fractions (Empson & Levi, 2011). Throughout the intervention there were global goals to be targeted, but with the underlying decision to make amendments to the protocol as necessary for each student.

Taking a step back to examine the larger structure of the intervention. Ryan was the first student to begin the ES intervention, and his experience was used to guide the creation of the initial sequencing of the protocol. After reviewing each of the other students' first sessions and presence of AUs, it was felt that the CGI ES intervention would align with the way they interacted with whole numbers, and was still felt to be a viable model for encouraging productive fraction engagement for learners with MLD (Researcher Debrief, 10/27/15).

Table 3

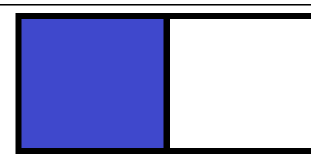
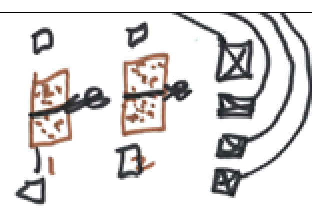
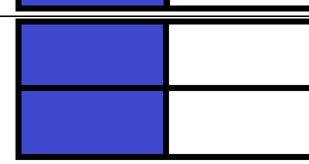
Equal Sharing Problem Type and Sequence

Problem Type	# of people sharing	Elicited solution	Example
1	2	Whole	John and Kelly share 8 jellybeans
2	3 or 4	Whole	3 students share 15 baseball cards
3	2	Whole + ½	Talin and Eddie share 9 oranges.
4	4	Whole + ½	4 students share 10 submarine sandwiches
5	4	Whole + ¼	4 students share 13 brownies
5p	Varies	Unit fraction Prompt for Equivalent Fractions	4 children share a package of Play-doh
6	4	Whole + ¾	4 share 15 slices of cheese
6p	3 or 8	Non-unit fraction < 1	3 share 2 candy bars
7	4	Whole + ½ or 2/4, Equivalent Fraction	4 share 14 biscuits; equate to equivalent model
8	4, 6, or 8	Fractions < 1 and equal to ½ Prompt for Equivalent Fractions	8 children share 4 Snickers; equate to equivalent model
9	6 or 8	Fractions < 1 and not equal to 1/2 Prompt for Equivalent Fractions	8 share 2 pepperoni pizzas; equate to equivalent model
10	Varies	Comparison between unit fractions with same numerators	Who gets more? 3 children sharing a medium pizza or 4 children sharing a medium pizza
11	Varies	Comparison between non-unit fractions with same numerator	Who gets more? 3 sharing 2 watermelon or 5 sharing 2 watermelon
12	Varies	Comparison between non-unit fractions with same denominator	Who gets more? 8 children sharing 3 bananas or 8 sharing 5 bananas

In order to get the students to think about the meaning of fraction names and connect their SGRs to canonical models, we had them begin thinking about representing what the solution would look like for just one of the individuals in the equal sharing situation. In the earlier sessions we wanted to encourage diverse forms of correct responses, but as the students progressed, we encouraged interaction with canonical representations and symbolic forms. Depending on the session and previous success with naming fractions, this may have taken the form of written expression of the naming quantity (e.g., “one-half”), symbolic (i.e., ½), and then ideally representationally. This sequence is represented in Table 4. To get the students to think about naming without explicit instruction, we would scaffold by asking questions such as, “how many of these parts can fit in a whole?” to connect to the meaning of the denominator, and “how many of these parts does each kid get?” to connect to the meaning of the numerator.

Table 4

Fraction Forms in the ES Intervention

	Naming Quantity	**Symbolic Form**	**Student-generated Representation**	**Canonical Representation**
Standard	One-half	½		
Equivalent Fractions	Two-fourths	2/4		

As the student solved, the researchers probed the students’ thinking and actions, trying to uncover conceptualizations and tensions. Once the student solved and articulated their process, the researcher would request that the student represent using an alternative mediational tool, if they had not already done so. The reason for this was to transition the students' thinking from concrete models to abstract models. This transition would ease future attempts to have them engage with canonical models if that felt appropriate for the students' trajectory. For example, if the student used Unifix cubes in their first attempt, then the researcher might request that the student solve by using markers and paper. This encouraged an examination of the accessibility and conceptualizations of these various mediational tools, as well as the transferability of their thought processes and conceptualizations from one medium to the next. These types of questions pushed student thinking from their SGRs to canonical representations.

Once students demonstrated a certain level of proficiency with naming fractions and modeling, we ideally wanted them to engage with problem types that have been shown to be historically challenging for learners with AUs, specifically, equivalent fractions and comparison problems (Lewis, 2014, 2016). These types of problems encourage more sophisticated thinking about fractional quantities that are necessary skills for future success in higher level mathematic (NMAP, 2008).

We very intentionally designed this ES intervention around student sense-making, and therefore, avoided explicit instruction as much as possible. However, there were times when students required *highly* targeted instruction to address a specific difficulty. We did not want to overly influence student sense-making beyond these targeted instances. For example, for Lily, naming conventions presented a challenge that were due in part to an atypical understanding, but also in part to an inadequate understanding of how to name fractions. We could not expect her to figure out naming fraction conventions on her own and felt it our obligation to help her develop this skill. Therefore, the researchers devoted eight minutes to conduct a mini lesson to discuss how to name fraction quantities. This episode will be further unpacked in Lily's case-study findings.

To address errors throughout the intervention, researchers scaffolded content to bring students to a more comfortable place of understanding (Lynch, Hunt, & Lewis, 2018). Again, we were not interested in disrupting student sensemaking, so explicit instruction was not the default strategy for resolving errors. We attempted to scaffold in a variety of ways, typically using minimal and subtle methods, such as a simple reframing of a question to familiar contexts, or back-tracking to a previous problem or an earlier step in the present problem. Sometimes we simply suggested that they use an alternative representational tool to accomplish a task that was ill served by a particular tool (Lynch, Hunt, & Lewis, 2018).

The problems for each session were decided beforehand in researcher debrief meetings, with some flexibility for in-the-moment tutoring decisions embedded in the design structure. Each session began by administering a problem type that the researchers felt the students would benefit from additional review, and to begin from a place of minimal to moderate challenge. We did not want to start the session with a problem that was too challenging. From there, the students' proficiency dictated the administration of more challenging problems within a session. Although the protocol itself had instructions to administer an entire set of problems, due to time constraints, we made a research team decision from the onset that permitted us to make judgement calls in the moment to forego certain problems. The criteria for moving onto the next problem set was based on the students' experience. If the student solved the previous problem correctly with relative ease, and it was assumed that the student would likely employ the same strategy to solve the next problem with minimal to no difficulty, then the researcher would move on to the next problem type.

Each of the weekly tutoring sessions was video recorded, with the camera facing toward the researcher and student in order to capture student speech, tool use, researcher-student interactions, and students' worked examples, and gestures. Students' worked examples were created with various mediational tools – including markers, paper, and concrete manipulatives (e.g., Unifix cubes, strips of paper). Each of these 19 sessions was transcribed to preserve elements of speech, gesture, and interactions. Audio backup recordings were used as a secondary source in situations of low video quality.

Transcripts were parsed into individual problem-solving segments. Each segment began with the posing of a question, and each ended with the student's answer or explanation. A single problem could be parsed into multiple segments.

Researcher Debrief Sessions

Each tutoring session was followed by weekly researcher debrief which took the form of iterative cycles of refinement, similar to Design Based Research (Deaton & Malloy, 2017; Cobb et. al., 2003). Each researcher independently watched a video recording of the tutoring session from the prior week, and documented observations to discuss as a team. Then during the debrief itself, we discussed interpretations of the students' meaning making process, made conjectures about the students thinking and learning trajectory, and made mutual decisions about how to proceed with each student's equal-sharing protocol. We even outlined decisions about what should be contained in each lesson, down to specifying the shared items and the solutions type (e.g., mixed, unit). Documentation of notes and decisions were saved via a shared Google Doc and were audio recorded.

As an additional benefit, this collaborative decision-making process allowed us to triangulate our independent analyses, a necessary element for ensuring validity of our assumptions and conclusions (Patton, 2015, p. 665). These analysis and debrief sessions allowed for reflection on how to continue to engage with each of the learners from a place of difference rather than deficit. They ensured that the tutoring techniques, clinical interview questions, and my own assumptions about student thinking were in line with an anti-deficit theoretical framing.

Analysis

We engaged in many analytic passes through the data to makes sense of the students' stories. Therefore, although SGRs take a focal place in the current inquiry, it was only through an evolving coding scheme that this transpired. SGRs significance was more fully realized in the second phase of coding. Phase one coding, focused on more superficial aspects of student problem-solving and conceptualizations.

Phase One Coding

For each problem segment the following was coded: (a) the correctness of the student answer, (b) whether the student's answers or gestures were consistent with a fractional complement or halving understanding, (c) mediational tool use.

Correctness. The pre- and post- Atypicality Interview problems were coded for correctness based on whether the student's answer aligned with the canonical explanation. I coded an answer as incorrect if that answer was inconsistent with convention. Problem segments in which the student did not provide an answer were coded as unanswered, and problem segments in which the tutor provided guidance were coded as tutor guided.

Atypical Understandings. This coding was used to isolate students' Atypical Understanding in all screeners, interviews, and intervention data. This coding was conducted based on prior research, codebook can be found in Appendix A. Students received an Atypicality point for each instance that the student exhibited either a halving or fractional complement understanding based on prior research definitions. Coding these conceptualizations in the initial session allowed for researchers to establish a baseline understanding of the types and frequency of AUs the students experienced. Coding throughout the intervention for these AUs allowed for a thorough picture of the robustness of such understandings, relative frequency of each, and how their appearance interacted with elements in the intervention.

Engagement with Material Resources and SGRs. This coding was an open-ended format of researcher notes that documented observed mediational tool use. Open coding of transcripts and video data was conducted to outline the preferences that students had for material resources. The difficulties and affordances associated with certain types of representation were noted. Here is where the significance of SGRs was first recognized, that the manner in which these learners used and created SGRs was a meaningful tool for thinking about sense-making during engagement with the ES intervention.

Phase Two Coding

Phase Two coding focused attention on SGRs, and the manner in which students created and used them. To focus my analysis, and to narrow my findings, I began to examine the trends of correctness and AUs on the pre- and post- Atypicality Interviews. This helped me to see each of the students' AU use and any transformations from beginning to end. From these trends, I went back to the interview data, and focused my SGR analysis on instances that contained elements of AUs. In some of these instances I re-evaluated whether the instance contained an AU based on their operational definitions. Four instances were removed from analysis due to this secondary coding since there was insufficient data to code as an Atypical Understanding or they were attributed to an alternative conceptualization or difficulty. From there, in each instance of AU in the intervention I examined the students' problem-solving strategies, their use of mediational tools, and their SGRs to analyze themes of engagement and accessibility.

Three main themes occurred and are the focus of the findings section: 1) The student's sharing of material items in their distribution of cookies, brownies, etc in SGRs creates an accessible environment for interpreting fractions, typically complicated by a Fractional Complement understanding, 2) The action-oriented nature of the intervention allows for SGRs to incorporate and address the students' conceptualizations of one-half associated with a halving understanding, and 3) Scaffolded content is still necessary to increase the accessibility of fraction representations.

Data Quality

Validity and Credibility

Issues of validity and credibility during data collection were addressed by the weekly debriefs. As described above, each researcher reviewed video-recorded sessions of student sensemaking and used these weekly reflections to make informed decisions about the shape and direction of subsequent lessons; in other words, we attempted to triangulate data by using both multiple researcher perspectives as well as with multiple data sources or instances as proposed by Lincoln and Guba (1985). Outside data collection, and during analysis – guidance through mentored feedback was sought to continue to establish and adhere to anti-deficit perspectives, contributing to the trustworthiness of the analysis.

Limitations

As with any tutoring inquiry that utilizes multiple researchers to engage with students as tutors there is invariably different levels of expertise in teaching math content, in teaching math content specifically to learners with MLD, and to conducting and collecting clinical interview data. In an effort to mitigate the effects of multiple researchers, we attempted to equalize the tutoring experiences for each of the students by co-creating a master protocol of ES questions coupled with standardized expectations for each session. These expectations outlined that there was no set number of problems that the students were meant to finish by the end of each session, therefore, we did not need to influence the number of problems solved by the student. In each problem instance, we were interested with the students' understandings and were not tutoring to attain correctness, rather we aimed to uncover patterns of reasoning. From these reasonings, we subtly directed student learning to align with canonical models.

Findings

Recall, that this inquiry was interested in investigating the accessibility of an ES intervention for learners with MLD and whether it changed the students' relationship with canonical representations. The primary lens for evaluating this interaction was through student-generated representations (SGRs). After each ES problem was presented the students created these SGRs as part and parcel to the problem-solving process, creating representations ranging from concrete to abstract. Throughout each problem-solving episode the students were probed to explain their actions, intentions, intuitions, and conceptualizations of the problem-solving process. These explanations served to disambiguate any confusion or lack of clarity of SGRs, as these depictions could be highly tailored and perhaps only relevant to the creator.

Through an analysis of the problem-solving instances that contained an atypical understanding, it is suggested that the highly flexible and differentiable aspects of the CGI ES intervention can be leveraged to support productive problem-solving for learners with MLD. The intervention was effective in aligning with the students' action-oriented interpretations of fractions and their creation. It was observed that the act of distributing inside of the ES contexts was an invaluable environment for retaining relevant fractional quantities, as it diminished the difficulties associated with recalling the meaning of shading in representations and offered a placeholder for fractional quantities previously perceived (in error) as inconsequential.

However, the lack of substantial improvement in accuracy over the course of the intervention for one of the students, Lily, indicates that the ES intervention may not address all issues of inaccessibility. In the findings, I will show how each of the students experienced the ES intervention in unique ways, including varying degrees of improvement, exhibition of Atypical

Understandings, and the characteristics of SGRs over the course of the intervention. From these students' experiences, it is suggested that that the ES intervention in conjunction with SGRs offers some answers on how to improve the accessibility to fraction content for learners with an MLD, but is by no means a universal measure.

To make this clear, I will sequence the findings as follows: I will first provide a clearer picture of productive engagement with the ES intervention as it relates to each of the atypical understandings associated with an MLD more broadly. Then I will turn to each of the case study participants, Ryan, Maddie, and Lily, to I present the different experiences and interactions with the intervention.

Overview: Interaction of the Atypical Understandings and the ES Intervention

Fractional Complement and the ES Intervention

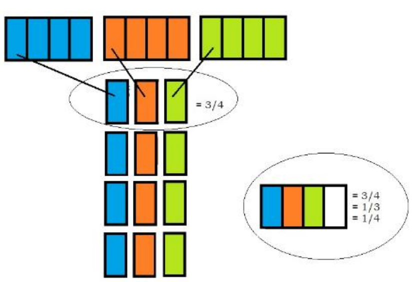
Recall that a fractional complement understanding is one that finds ambiguity around the meaning of shading in area models, and that there is a tendency to associate shading with the removal of pieces, resulting in the unshaded pieces as being understood as "left or remaining" (Lewis, 2017, p. 336). Conceptualizing shading as the act of removing (taken-away or "gone" (p.336)) complicated students' ability to ascertain which fractional quantity was focal, the non-shaded or shaded.

During the inception of the ES intervention, the research team was unsure of the ES problems' ability to create an accessible environment for learners with MLD who experience a FC understanding. What the data uncovered was that there was some instances of alignment and compatibility. The ES Intervention problems required that the students think about distributing items (and fractions of items) to a specific number of individuals. This act of distributing was observed to be similar to the act of removing as seen in a FC Understanding. The similarity between the act of distributing and shading, suggests that the ES intervention may provide a scenario where students can create a placeholder for items that were previously conceptualized as taken away or gone. Therefore, now what was once conceptualized as inconsequential has been correctly interpreted as focal.

All three learners similarly interpreted the ES problems as opportunities to distribute items to imaginary individuals and demonstrated with various mediational tools that the context provided a stable location to distribute and store items. In Figure 1, I have created a generic model to indicate this process. In this model, 4 individuals are sharing 3 items. The three items populate the top portion of the figure, have been cut into four pieces each, and then distributed to the imaginary four people (as represented by the column of four matching sets of the three colored fraction pieces). Based on this section of the SGR alone, the students were frequently able to identify the fraction value associated with the three distributed pieces, ¾.

Figure 1

Generic Model of the influence of ES problems on a FC understanding in an example of 4 individuals sharing 3 items.



a. **b.**

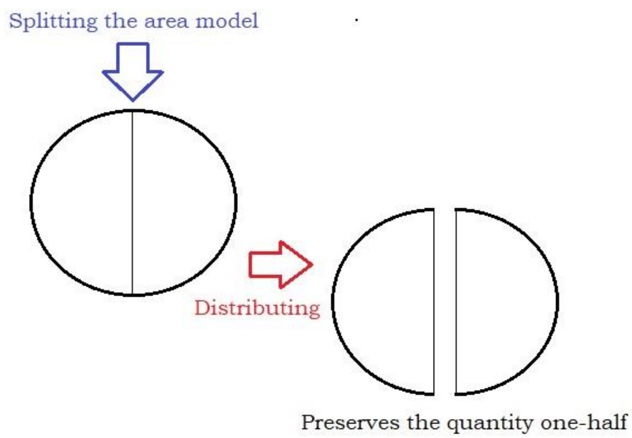
However, problems did arise when students were asked to isolate or represent the fractional amount that each person in the problem would receive. This typically translated to the student creating an SGR akin to a canonical area model (Figure 1b) of the solution, reintroducing the typical challenges associated with interpretation tasks and these types of mediational tools. There are indeed many beneficial aspects in the intervention, but the fact remains, that the tendency for the SGRs to evolve and approximate inaccessible mediational tools suggests that these are not enough to resolve the difficulties associated with engaging with fraction content of this form for all learners. For each of the students I will highlight the ways in which this action-oriented approach to solving problems is taken up by each in their respective case-study profiles.

Halving and the ES Intervention

As a reminder, a halving understanding is the conflation of the act of cutting something in half to create two separate halves as the quantity one-half. The ES intervention and the various mediational tools that the students were provided elicited many renditions of halving by all learners, but most prominently by Ryan. The ES environment created opportunities to harness this understanding to encourage successful problem solving by allowing the student to continue to envision and enact ‘cutting’ as part of fraction creation. From here, the quantity of one-half was preserved by the act of distributing pieces to each of the imaginary individuals in the sharing scenarios (Figure 2). This placeholder quality was similarly demonstrated in the ES Intervention’s interaction with a Fractional Complement understanding.

Figure 2

The Influence of ES problems on a Halving Understanding



Now, I will present student specific data in each of their respective case-study profiles. Here, I will provide an overall profile of their atypical understandings, correctness, and general interaction with the intervention. I will relate the action-oriented nature of the intervention and the aforementioned relationships to Halving and a Fractional Complement to outline the ways in which the ES Intervention and corresponding SGRs created or permitted accessible interactions with fraction content.

Case-study Students

Table 5

Atypical Understandings Frequency and Overall Correctness on Assessments

	Assessment	Total Halving	Total Fractional Complement	Total Other*	Total AUs	Total Correct (out of 16)	Percent Correct
Ryan	Pre	3	1	1	5	5	31
	Post	1	0	0	1	13	81
Lily	Pre	1	7	0	8	6	38
	Post	0	6	0	6	6	38
Maddie	Pre	0	5	0	5	7	44
	Post	2	5	0	7	10	63

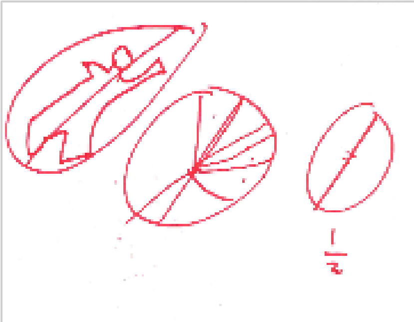
Ryan— Increase in Accessibility and Decrease in Problematic Atypical Understandings

Of all the three students, Ryan derived the most benefit from the ES intervention. By the end of the intervention there was an overall increase in correctness, and a decrease in problematic atypical understanding. His demonstration of AUs over time decreased, with only one blatant instance on the post-Atypicality Interview. In terms of correctness, he demonstrated the most improvement from pre- to post- Atypicality Interview, answering 31% and 81% correctly, respectively.

Ryan had unique ways of thinking of one-half, that developed over the course of the intervention. A focus on the pre-Atypicality Interview for now demonstrates that Ryan understood the quantity of one-half as an act of partitioning into two pieces, focusing on the balance of the two pieces, and less on any resultant quantity (Figure 3, pie depicted as an unshaded area model). For example, in his explanation of the Pedestrian Sign in Figure 3, he stated: "sometimes, like when they're walking….So, half of a person can go through but the other can't." This idea of balance is reiterated in his explanation of the other representations. He was asked how his pictures represented one-half, and he remarked that all his representations had "a line right down the middle" and emphasized balance in his picture of pie by simultaneously tapping either side of the circle and remarking on their equal sizes. These combine to suggest that Ryan, had at the start of the intervention, an understanding of half that was consistent with a halving understanding commonly attributed to having an MLD.

Figure 3

Ryan's Representation of ½. From left to right, a pedestrian sign, a pizza, a pie



Aside from gestures and the use of shading (or crossing out) to depict the removal of pieces, any obvious presence of AUs throughout the intervention for Ryan became increasingly less so. This either suggests that he was relying less heavily on this conceptualization or that the ES intervention aligned with a halving understanding. To explore this possibility, I present some data excerpts from the intervention to show how useful Ryan's SGRs were for him in thinking about the quantities in more meaningful and accessible ways for someone who has historically demonstrated problematic atypical understandings. Furthermore, I will show how the ES intervention allowed Ryan to access canonical models to productively solve fraction problems and how the researchers facilitated this transition.

Ryan used Unifix cubes in complex and action-oriented ways to create fractions and corresponding representations. Ryan routinely used Unifix cubes to enact the process of splitting elements in half (or smaller fractions sizes) to then distribute to individuals in the ES word problems. He also used Unifix cubes to represent stacks of items designated for each individual in the problem. Unfortunately, he tended to lose track of the total number of items, and of the number of cubes corresponding to the whole. For example, in the problem of 4 students sharing 14 biscuits, Ryan set out 14 cubes to be distributed, and this required considerable attention to count them out:

Katie (researcher): How are you going to go about solving this one?

Ryan: Um... [[reconnects long red cube section]] 14 biscuits. Well, that's 19. [[breaks off cube]] Take away one. [[counts backward, tapping finger on cubes]] 19, 18, 17, 16, 15... 14. [[breaks off extra red cube]] Okay. [Actually] –

K: You want to count them again to make sure you have 14?

R: Yeah. [[counts with fingers on cubes]] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. [[retrieves one more cube]] 14, right on the dot, first try. [[sets out 4 individual black cubes]] Okay. 4 children. [[checks question on paper]] 14 biscuits.

After a few moments of counting the cubes out, and then distributing it was found that he did not in fact have 14 cubes after all.

K: Okay, then how many do they have? So... [[gesturing to R's stacks]], let's count the number of biscuits we have. You have 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. You're supposed to have 14.

Although the Unifix cubes were a useful concrete mediational tool, they were not sufficient to prevent miscounts.

Another issue Ryan experienced with concrete manipulative like the Unifix cubes was the tracking of the whole. Ryan used Unifix cubes in a single representation with varying fractional or whole number values, sometimes associated with a specific color. This caused difficulty in tracking the whole versus fractional quantities.

At this point in the biscuit problem, Ryan had distributed all of the wholes (three to each) and was left with two biscuits. He added an additional cube to each of the two extra biscuits, so that he could replicate the action of splitting by snapping the cubes apart and distributing halves. What was problematic, was the fact that his two-cube stacks and the single cubes previously distributed both indicated a whole (similar to Figure 4:). To have him reidentify his whole, Katie attempted to reorient his focus, to only have him lose track of the whole again:

K: Okay, where are the whole biscuits, if there are any?

R: [[points to one red cube section stack]] That's 3 biscuits…they would get 5 biscuits.

To ameliorate this difficulty, Katie suggested using a different color to track the two-cube stacks that represented a whole. This scaffolding technique relied on reincorporating a strategy that he had created and employed successfully in a prior session. See Figure 4 for this excerpt.

K: Can we make these a different color so that we don't get confused?

R: Okay. Black.

K: Black? Okay [[removes black cube section from box]], let's just make sure we're on the same page, here. [[counting cubes in stacks]] We have 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, [[then referring to last two separate blocks]], 13, 14. Is that the right number?

R: Yeah.

K: Okay, so now we're swapping out these 2 [[red]] for black biscuits.

R: Yes.

K: So, each of these [[black two-cubes]] black ones is worth one of these [[red cubes]], right?

From here we see that Ryan was able to create an SGR that contained the necessary three whole biscuits, and fractional quantities of the remaining two biscuits with minimal scaffolding (Figure 4).

Figure 4:

Ryan's Unifix cube SGR of 4 sharing 14 biscuits.

Note: Single black cubes on bottom represent the 4 people. The 3 red cubes represent the whole biscuits. The two black cubes in each represent 2/4.

Ryan: So then it would be... [[pointing with marker]] 3... How did I get to 1, to 5, to 3? [[crosses out 5, pauses, studies stacks, writes 3 2/4]] 3 and two-fourths? Is that right?

K: Cool. Yeah, that's awesome.

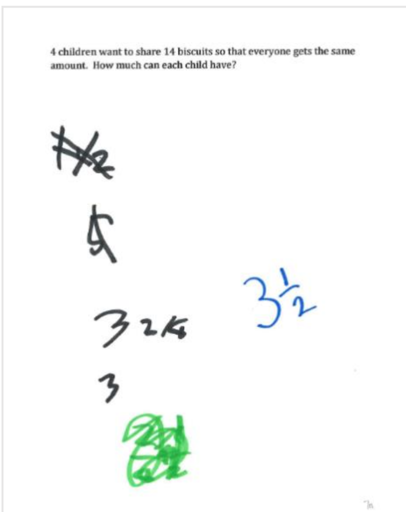
After he has documented his solution of three and two-fourths, Katie then had him think about his representation as it related to the equivalent solution created by Katie of three and one-half (Figure 5). He easily acknowledged that their solutions were the same, even writing out that 2/4 = ½ before scribbling it out with green marker (Figure 6).

Figure 5

Katie's (researcher) created solution of 3 and 1/2

Figure 6

Ryan's various symbolic solutions for 4 share 14 biscuits



Then in a move to get Ryan to think more abstractly about his concrete representation Katie asked Ryan (as we eventually did for all the students) to use drawing to represent the solution. In asking the student to draw "what one person would have" we were attempting to get students to a place where they could conduct comparisons. Here, to compare what one person would get in each of Ryan and Katie's solutions would translate to thinking about comparing the quantities three and two-fourths to three and one-half.

The task of moving from a concrete to abstract SGR caused some confusion for Ryan, and translated to him constructing a model of Katie's solution and not his own. He represented the solution as three and one half, with three green and brown circles representing the wholes and then using an unshaded area model to represent one-half (Figure 7), indicating the presence of a halving understanding. This moment represents the not always seamless transition from concrete to representational mediational tools. Furthermore, it demonstrates that the use of abstract models offers the opportunity for the reintroduction of problematic instances of atypical understandings and canonical models in problem-solving. However, it was still important to our research questions to see if we could continue to push student thinking about abstract representations of solutions.

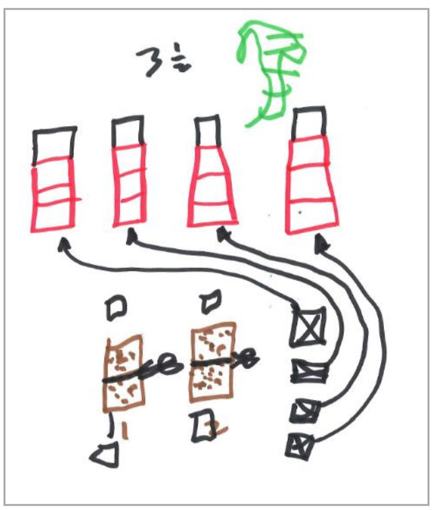
Figure 7

Ryan's solution of 4 sharing 14 biscuits. 3 green circles at the top are the wholes. Unshaded circular area model is 1/2.

It should be noted that the halving understanding that was exemplified in Figure 7 was not addressed by researcher scaffolding, and that the focus was on reorienting Ryan to focus on his own representation of three and two-fourths. We were interested in the students' depiction and transition from concrete to abstract. As a scaffolding tactic, Katie rephrased the representation task to one that emphasized the entire solution (and process), rather than the single quantity. This rephrasing did not alter Ryan's focus on representing the researcher's equivalent fraction of three and one-half. In his abstract representation he created a hybrid model that incorporated the physical elements of creating alongside the real world "biscuit." In this example (Figure 8), Ryan cuts the biscuit in half by depicting scissors along the square's middle, and then represented each of the resulting half pieces in black along the side. He crossed out boxes (black halves) as he distributed them to each of the black cubes that represented people along the top of the page:

Figure 8

Ryan represents cutting biscuits with scissors in an equivalent fraction problem of 4 students sharing 14 biscuits.



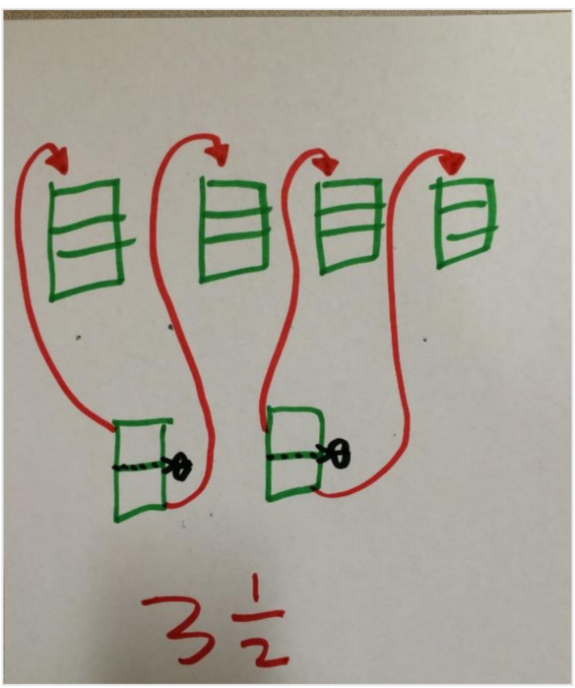
Ryan: And then this one [[drawing connecting lines to each black block to each red rectangle]] goes to that, he gets it, and then that one's dead, [[Xs through one box]], and this one gets one [[adds block to rectangle drawn above, repeating process]], and this one gets one, that's dead. Then he gets one, that's dead, this guy's dead, he gets one.

This exhaustive process both aligns with how learners with MLD interpret shading, as being given away, and is typical for these ES problems.

To get students to capture elements of the ES intervention, we asked that they draw or write down something from the present session that felt particularly salient for them. From this episode, Ryan captured the utility of representing the solution of three and one-half. He continued to preserve the action-oriented nature of the ES intervention problems in his representations, with the use of scissors and arrows to indicate partitioning and distribution, respectively.

Figure 9

Ryan's Journal Activity. He documented that the problem and model in 4 share 14 biscuits was helpful.

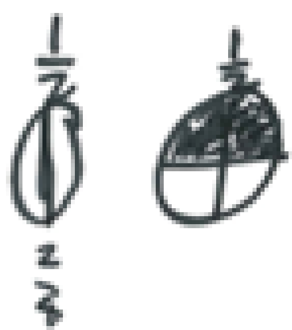


To reiterate this episode, Ryan relied on an action-oriented approach to solving ES problems. His SGRs demonstrated that even though Ryan frequently focused on the mechanics of splitting and chopping during the ES intervention, he was able to productively orient to accurate fractional quantities so successfully that he could even use them to think about equivalent fractions. Thus, suggesting that student-generated representations that start from the whole and emphasize the actions used to create fractions are a useful experience for learners who exhibit a halving understanding.

For Ryan, based on (Table 5) the results and trends indicated from the three assessments (increase in correctness) and the overall frequency of AUs over the course of the ES intervention (decrease in presence of AUs) suggests that the ES intervention provided an accessible environment to experience a halving understanding, at the very least.

Figure 10

Ryan's representation from post-Atypicality Interview of 1/2



To make this claim clearer, I turn to data from Ryan’s pre- and post- Atypicality Interviews to draw comparisons. His understanding and representations of one-half, as demonstrated in (**Figure 10**), showed that he still relied on a halving understanding, but that one of his SGRs evolved to one more canonical in nature. He drew two black area models of one-half. One representing a halving interpretation and the other a canonical representation of 2/4. These two representations suggest that the halving understanding remained present, but upon completing the intervention, he was able to draw and correctly interpret a canonical model of one-half in the more advanced form of an equivalent fraction. This supports the claim that the ES intervention increased Ryan's ability to engage with canonical fraction models, suggesting that the ES intervention increased the accessibility of these models for him, at least temporarily.

This approximation of canonical models is replicated in his representations of 3/5. Ryan employed some intriguing thinking when he developed and created his SGRs for 3/5 on the pre-Atypicality Interview that transformed drastically by the end of the intervention. Ryan's two representations of 3/5 in the pre-Atypicality Screener are found in Figure 11 and 12.

Here he represented the fraction 3/5 by drawing five ice cream cones and indicating the numerator (typically defined with shading) by adding three small dots to three of the cones. Clearly, this is not a canonical representation, as this would not universally be understood as 3/5. In another effort to represent 3/5 on the pre-Atypicality Interview (Figure 12) he attempted to approximate a canonical area model, but represented the three in the numerator separate from the denominator, and only used shading to indicate the removal of a piece made in error. These two examples demonstrate that prior to the intervention, Ryan did not always represent fractions in canonical ways, which makes sense given their history of inaccessibility. By the end of the intervention, Ryan was able to represent 3/5 by drawing a green area model and shading in the appropriate three pieces (Figure 13)

Figure 11

Ryan's representation from pre-Interview of 3/5. He drew five ice cream cones with three dots

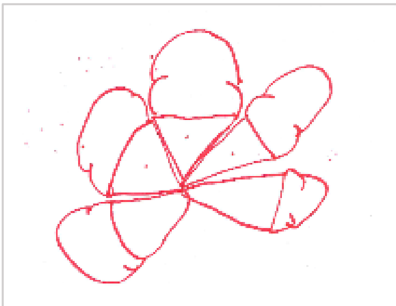


Figure 12

Ryan's representation from pre-Interview of 3/5. Represented 3 and 5 in two separate area models

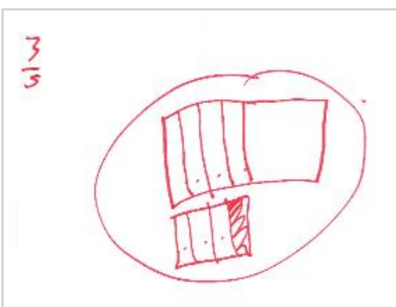
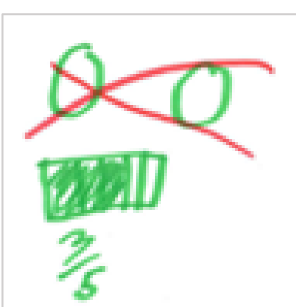


Figure 13

Ryan's representation from post-Interview of 3/5.



A closer approximation of and correct interpretation of canonical representations became a trend for Ryan on the post-Atypicality Screener, positioning him upon exiting the intervention as more able to engage with standard mediational tools. The key takeaway for Ryan, was that the ES intervention seemed to serve him well. It allowed him to create SGRs in a way that maintained an action-oriented approach to fraction problem-solving that both preserved quantity and aligned with his halving and fractional complement understanding. As an additional benefit, the ES intervention managed to increase his ability to use canonical models as targeted by the researchers.

Maddie—Increase in Accessibility through Compensation

Maddie represented an interesting case of increased exhibition of AUs from pre- to post-Atypicality Interview yet with an overall increase in correctness, from 44% to 63% (see Table 5). This is suggestive that Maddie was able to (at least sometimes) successfully draw on compensatory strategies to resolve challenges associated with incompatible canonical representations and her AUs. This possibility will be further explored in the Discussion section, for now, I will outline the trends in her conceptualizations and the character of her problem-solving and SGRs.

A closer look at Maddie's trend in conceptualizations demonstrated that she infrequently exhibited AUs outside of the Atypicality Screener, and the pre-and post-Atypicality Interview (Table 5). This was somewhat surprising since she exemplified high atypicality on all screeners and interviews. She rarely, only once, demonstrated a halving understanding during the intervention, and unexpectedly demonstrated four instances of a fractional complement understanding on the final session. This may have perhaps been influenced by initiating the session with the Atypicality Interview but remains an open question.

Table 6

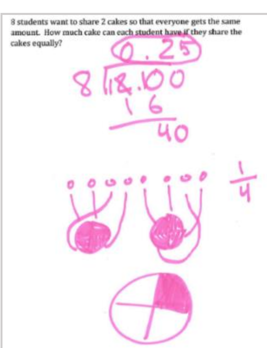
Frequency and Percentage of Long Division Use by Maddie by Session

	Session Number				
	1	**2**	**3**	**4**	**5**
Long-Division Use	0	4	3	3	0
Percentage of Problems	0%	57%	43%	43%	0%

Maddie similarly solved problems like Ryan that leveraged the distributive possibilities of the ES problems, using that characteristic to track items previously conceived as taken away. In Figure 14,she used the act of shading to indicate the removal of pieces, and signified the distribution and assignment to each of the people (the row of eight dots). Otherwise, Maddie took a slightly different approach than either Ryan or Lily in solving ES problems. Her reliance on the long-division algorithm was pronounced (Table 6) and seemed to help her navigate these types of problems. She generally began each problem instance, as seen in the same figure, by employing the long-division algorithm.

Figure 14

Maddie's representation of 8 share 2 cakes



In her SGRs, (e.g., Figure 14) she used shading and crossing out of fraction pieces in the same figure in a manner that aligned with a Fractional Complement understanding. This shading to indicate the removal and distribution of pieces in her SGRs proved to be a beneficial aspect of this mediational tool for problem-solving, but more importantly, for identifying fraction values for someone who has a Fractional Complement understanding. Even though she found ambiguity around the meaning of shading in canonical models on the pre- and post- Atypicality Interviews, she was able to correctly represent and interpret solutions in the ES intervention. That is, when prompted to depict "what each person would have" as a single solution she competently represented using canonical representations.

Figure 15

Maddie drawing and writing one-half from the post-Atypicality Interview

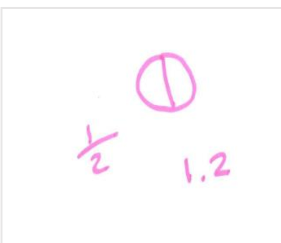
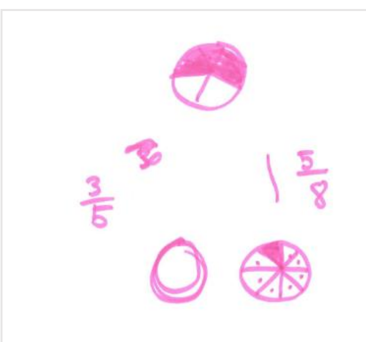


Figure 16

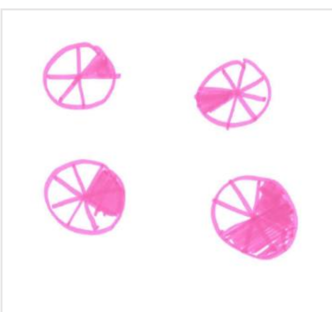
Maddie's representation of 3/5 and 1 5/8 from the post-Atypicality Interview



Maddie demonstrated remarkable proficiency with the ES problems. This progress did not seem to affect the results on the Atypicality Interviews. She demonstrated an increase presence of AUs in the post Atypicality Interview compared to the pre. This increase in AUs was matched by an increase in correctness. By the end of the intervention, her examples of one-half and one and five-eighths, which were depicted as canonical representations, remained inaccessible mediational tools outside the ES intervention. When asked to draw or write one-half, she indicated an unshaded area model and the numeral ½ as her responses (Figure 15). She managed to represent and interpret 3/5 correctly (Error! Reference source not found.), but on the s ame page, she encountered issues with one and five-eighths, where shading was omitted. In each of these, aside from the decimal, 1.2, the numerals or symbolic form of the fraction seemed more meaningful to her.

Figure 17

Maddie's representation of the comparison problems of 1/6 compared to 1/8 and 2/8 compared to and 5/8.



The inaccessibility of canonical area models is really brought home in her explanation of her SGR (Figure 17) in the comparison problem, *which is larger two-eighths or five-eighths?* When asked which is larger or are they equal, she stated that two-eighths was larger, “because, um, since there's less circles colored in two-eighths, more people can have it. Or... yeah…. because,… I wasn't sure if you…were measuring how much were colored in, or how much was left." Maddie demonstrated profound awareness of her challenges understanding the meaning of shading in this remark; even still, her final answer is that two-eighths is larger since there were "only two pieces colored in."

Perhaps the secret to Maddie's successful navigation of the ES intervention was her adherence to a small range of SGRs that allowed her to interact with fractions. SGRs that allowed her to distribute and track items compatible with a FC understanding, and others that were purely algorithmic, allowing her to translate to fraction decimal form. The fact that it remains unclear how much, if any, she connected decimal fractions to her representations. This unknown factor may have contributed to her success.

Moreover, a lack of AUs throughout the intervention could suggest a couple of things. One, it is possible that the intervention was ideal and highly accessible for her conceptualizations. That is, that the ability to connect to decimal fractions, and to models that enacted the process of shading, removing, and setting in a placeholder was an invaluable process for her Fractional Complement understanding. Or two, that she was able to compensate through other means of solving. The most important aspect of Maddie's problem solving seemed to be her reliance on long-division to begin each ES problem. Maddie seemed to truly characterize a student who had learned to compensate for challenges associated with incompatible mediational tools.

Lily—Inaccessibility Unchanged by the ES Intervention

Table 7

Lily's Atypical Understandings by Session

	Session #				
	2	**3**	**4**	**5**	**6[a]**
Halving	**2***	**0***	**1**	**1**	**0**
Fractional Complement	**2**	**7**	**7**	**3**	**4**

a. Final session totals only included ES problems
*Chopping gestures used, but at least one H "instance" not coded due to insufficient verbal data to confirm

Here I provide perhaps a fuller account of Lily's experiences with the ES intervention compared to that of Ryan or Maddie. The reason being, is her experience contained more difficulties associated with inaccessible SGRs. A presentation of Lily's experiences combined with Ryan and Maddie's to creates a complete picture of the affordances of the intervention and SGRs in effectively communicating fraction content. Overall, from Table 5 it is seen that Lily tended to rely most heavily on a FC understanding throughout all assessments, and the frequency remained relatively static. Her correctness on the pre- and post- Atypicality Interview remained unchanged at 38%. In Table 7 the presence of a fractional complement understandings remained quite high throughout the intervention.

Figure 18

Lily's representation of 1/6 (top) and 1/8 (bottom) in the comparison problem asking which is larger? From her pre-Atypicality Interview

Figure 19

A model of ¾ that Lily interpreted as 1/4, 1/3, or ½ from the cumulative ES problems in the final session

To fully capture her Fractional Complement understanding, take the example from Lily's pre-Atypicality Interview, which was mirrored in her post-Atypicality Interview, where she was asked to ascertain which fraction was larger, one-sixth or one-eighth (Figure 18). After drawing canonical circular area models, she remarked that one-eighth is larger "because there's more of not shaded in on this side [[indicating 1/8]] than there is that side [[indicating 1/6]]." For Lily, comparison problems or interpretation problems that required that she make decisions about the significance of shading, proved problematic, and even extended to problems in the ES intervention unlike Maddie.

Another example of naming difficulty is in her interpretation of a researcher made area model of ¾ (Figure 19) in which she rapidly vacillates from ¼, 1/3, and ½ in her responses. In this example I constructed my own area model to get Lily to connect to the meaning of shading and to her own SGR. I created an area model, cut it into fourths, and then had her add up the pieces with me as I colored in three of them, one at a time.

LL: One-fourth.

G: One-fourth. So that's one right there... [[pointing to LL drawing of cubes, begins shading in another section of square area model]] Now what is this? I colored in another one-fourth.

LL: That is two-fourths.

G: Correct. [[starts to shade in another section of square]]

LL: Three-fourths.

G: Yeah!

She then makes a sophisticated connection between her SGR and the canonical one I had just depicted.

LL: [[reaching to point at her drawing of cubes]] One-fourth, two-fourths, three-fourths.

G: Yup, you got it. So it's three-fourths.

LL: Three-fourths! 4 wholes and three-fourths –

And then, only seconds later, she is no longer focused on the canonical meaning of the shading in the representation, relying on previous fractional complement and the default one-half to name the fractional quantity.

LL: and one... half... One-third, one-third, it was one-third.

G: So, what did we say this was?

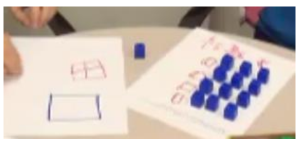
LL: One-fourth, one-fourth, one-third, and one-half.

To reorient her back to her previous thinking, Lily and I reviewed the sequence of steps that originally led us to three-fourths. Therefore, this episode did eventually culminate with her identifying the canonical area model as three-fourths.

The presence of a Fractional Complement understanding remained consistent for Lily over the course of the ES Intervention including pre-and post- Interviews. This trend demonstrates the robustness of this conceptualization for Lily regardless of the context, and the incremental increase in correctness suggests that the ES intervention was tenuous in supporting her. That said, there were numerous opportunities to scaffold and deviate from the ES intervention to address Lily's considerable challenges in interpreting and naming fractions. This quality of the intervention was invaluable for creating a more accessible teaching environment for Lily.

Figure 20

Lily's representation of 4 share 13 brownies



Much like Ryan, Lily similarly benefitted from the action-oriented nature of the ES intervention in that it aligned with her conceptualization of the removal of pieces in creating fractions. In the end, her most profound difficulty was interpreting and naming fractional quantities. To start, the ability to account for items that had been distributed proved beneficial for Lily and was exemplified repeatedly throughout the intervention. In the face of her robust fractional complement understanding, Lily was able to represent and dialogue sharing situations productively, increasing accessibility of fractional content up to a point, but not ensuring it. Take for instance the example of Lily sharing thirteen brownies with four individuals (Figure 20). Lily first used Unifix cubes to represent the distribution of three whole brownies to each of the four individuals and concluded that there was a single remaining brownie to be dealt with. She 'cut' the (brown) brownie into four pieces, and then bodily simulated the distribution of these fourth-sized chunks of brownie and then drew them next to her blue wholes. In the end, she was correctly able to create an SGR that depicted 3 ¼ as the final solution, and even was able to write and verbally assign the same correct value. This enactment demonstrates the utility of the intervention in allowing the student to depict sharing as an act of removing and distributing; to keep track of the relevant aspects of the problem.

For Lily, the act of distributing in representational form leveraged her FC understanding, creating a more accessible environment to think about fractional problem-solving. Even so, the ability to distribute was insufficient in eliminating errors associated with naming fractions complicated and usually attributed to inaccessible mediational tools and a FC understanding. In terms of naming fractions, Lily relied on two strategies or conceptualizations for naming fractions. Inherent to a Fractional Complement understanding is finding ambiguity of the meaning of shading in area models, and that translates to difficulties naming fractions. For instance, naming conventions of fractions dictate that the shaded pieces are assigned to the numerator value, and the total number of pieces are assigned to the denominator. These naming conventions were challenging for Lily. If she did not rely on a Fractional Complement interpretation of the area model, focusing on the non-shaded pieces, then she tended to rely on the common default value of "half." Research shows that novice learners are familiar with the quantity of one-half, and then default to naming all fractions as half (Empson & Levi, 2011). She frequently fluctuated in a single problem instance through a series of fractional values in her process to identify the fraction name associated with a representation, which was exemplified in Figure 19.

In this next excerpt, Lily is far less successful in identifying the fraction associated with the solution (i.e., 2 2/4) compared to the brownie example. Here, Lily has been asked to share 10 submarine sandwiches with 4 people. She has so far successfully distributed the 2 whole sandwiches using Unifix cubes and is now focused on the two remaining that must be split to be distributed equally.

While creating an SGR of this portion of the problem (Figure21a), she partitioned the two remaining submarine sandwiches in a coordinated manner between the number of individuals and the number of fraction pieces in the sharing scenario (Hunt & Empson, 2015). From this, the solution would be that each of the four individuals (Figure 21b)in the problem would receive an additional two-fourths. In the following dialogue between Lily and myself, she assigned the resultant fraction a variety of values, typically influenced by either a FC understanding ("one-third") or a typical error associated with novice learners ("one-half", a placeholder fraction name). The following passage demonstrates her challenges in interpreting her own SGR.

> Lily: There's 2 left over [[holding up 2 cubes]], so one of the people don't feel left out, that's why you divide the sub sandwiches. I was thinking you could divide them into fourths, because then each student will get enough of the sub sandwiches…. [[leans forward to look closely at her drawing, sits back]] *One-third.* Because it's *one-third* of the sandwich.
>
> Grace: Can you tell me where you see one-third?
>
> L: Well, you take the one-half, and then there's 2 left over, so it's one-third.

Here she is referring to a single piece in one of the fourths models as a "half", and then focuses on its removal of a portion of the model to identify the value left as one-third. She is then asked where she sees one-half in the SGR, and she reorients her interpretations. She responds with, L: [[tapping picture]] One-third, actually, one-third. Because there's 1 left over, and then there's 3 here, so 1. You're taking away the 1, and then the third. 3 left over. Um, I don't know. Like, like a half, [[gesturing with marker on sandwich drawing]], the four of half, because... it's not really like a half, I guess, it's kind of four-half, I guess…

In this last portion of her explanation, she relies heavily on a Fractional Complement understanding, focusing on the removal of pieces, and emphasizing the remaining pieces as relevant to naming conventions. Then quickly reverts to the default half value. A conjecture that comes from her last fraction name of ‘four of half’, or ‘four-half’ is that she have meant 2/4 but was unable to articulate that knowledge according to fraction naming convention standards. This problem demonstrates that she was able to successfully model the process of distributing items to people, partitioning remaining items in intelligent ways, and then completing the rest of distributing process.

Figure 21

*A portion of Lily's depiction of sharing 10 submarine sandwiches with 4 people. She is now focusing on the two remaining sandwiches which require fractions***.**

a) **b)**

Note: a) She partitioned the two remaining into fourths, b) SGR of four people sharing one of the sandwiches

In general, her SGR based on this series of activities associated with solving an ES problem were visually correct, suggesting that accessibility to fraction content is increased in the ES problem, at least representationally. However, the fact remained that Lily had naming challenges attached to these SGRs. The ES intervention left untouched Lily's naming challenges. We employed scaffolding in the form of explicit instruction to address this missing piece of naming fractions.

Scaffolding for Lily

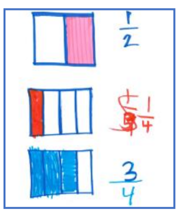
First of all, I recognize that explicit instruction represents a real deviation from the ES intervention but was necessary for supporting Lily in areas of inaccessibility. Explicit instruction only took the form of very brief mini lessons on content about mathematical convention (e.g., naming, conventional notation; Saxe et al. (2005)) that could not be addressed by the ES intervention alone and was viewed as a tool to attempt to increase the accessibility of fraction content. We wanted Lily to have a deeper understanding of fraction conventions. Furthermore, we understood that some students require additional metacognitive support to become more proficient mathematical problem-solvers (Schoenfeld, 2016). It was felt that Lily would benefit from this form of scaffolded support.

Fraction Naming

In the above Figure 22, Lily and I co-created a sheet of common fractions. This scaffolded exercise was intended to create a 'cheat sheet' of common fractions and canonical area models that Lily had challenges with due to the inaccessibility of area models. I chose to use canonical models, since the SGRs that tended to confuse Lily the most were the ones she created that approximated a canonical one. She was permitted and encouraged to use the sheet in subsequent sessions. In this sheet, it can be seen that the creation of one-half was relatively straightforward, whereas one-fourth proved less so. Lily colored and labeled the fraction one-fourth then assigned the fractional value of the 1/3. To reorient Lily to the meaning of shading, I related it to the one-half area model. She relabeled the fraction as ¼. This sheet proved useful in subsequent problems, where she interpreted the fraction ¼ as 1/3, and similar errors.

Figure 22

Fraction Naming Convention Sheet



Planning as Part of Mathematical Problem Solving

Lily could best be described as an impulsive yet persistent problem-solver, consistently eager to engage with mediational tools to solve ES intervention problems. This alacrity later led to research decisions on how to help to increase Lily's problem-solving planning skills, as her tenacity to start problems frequently increased the difficulty of the task and as a result created errors that she was reluctant to resolve. For example, when she encountered an error in solving an ES problem, and the researcher asked if she might like to try an alternative strategy, her response was a pithy "nah." Variations on this resistance, or perhaps indifference, were viewed throughout.

This pre-planning development was the simple use of a breadbox to store all the mediational tools while Lily and I wrote out the steps necessary to complete the problem. This allowed her to pause and think through her process, she would eventually be given access to the various mediational tools to begin problem-solving. In instances where she was off track or unclear on the next steps, we successfully referenced the planning sheet to guide her work.

Figure 23

Breadbox Problem Solving Toolkit

To begin this problem (Figure 24) we read the problem together: *15 slices of cheese for 4 people. How many slices of cheese can each person have?* Then I instructed Lily to tell me specifics about the problem that we need to consider before solving, she mentioned the number of people and slices of cheese, of which I wrote in Figure 24a. She then bodily gestured that we should distribute the cheese until everyone gets the same amount. When probed about whether she had a gut intuition about whether it would divide evenly, without any fractions, she was unsure. At that point, since she was unsure what cheese may be remaining, I had her begin the process without completing the planning phase with the clause that we would pause later once we had firmer grounding on next steps. With access to the mediational tools in the breadbox she created a SGR using Unifix cubes that allowed her to ascertain the number of leftover cheese to convert to fractions for sharing. At this point she decided to cut the remaining cheese "in half," but her actions and subsequent SGR did not align with this declaration. She represented three cheese slices cut into fourths. This shows that although the planning process can be useful in outlining steps, it too, can contain errors. To be noted, her SGR was closer to a more sophisticated coordinated strategy (Hunt & Empson, 2015). In combination with the Fraction Naming Convention Sheet (Figure 22), her SGR (Figure 24b) enabled her to eventually correctly conclude that each person in the problem would receive the sum of 3 and ¼+ ¼ + ¼ (Figure 24c), but did not name as three and three-fourths.

Figure 24

SGR and Planning Sheet from the Problem 4 share 15 slices of cheese

a) **b)** **c)**

Note: a) Planning instructions for 4 sharing 15 slices of cheese. b) Lily's SGR for 4 share 15 c) Lily's solution for 4 share 15

For Lily as opposed to the other two students, the ES intervention was less useful in addressing the inaccessibly of representations made so by the incompatibility of mediational tools and her conceptualizations. Even her SGRs were challenging for her to interact with throughout the ES intervention. By the post-Atypicality Interview, the intervention resulted in a reduction of AUS by only two instances still indicating a high level of AUs. This is not meant to be presented as a negative result, simply that it contrasts with Ryan’s experience. Furthermore, her level of correctness remained unchanged from pre- to post- Atypicality Interview. Assessment results aside, specific anecdotes from these interviews suggest that some useful transformations in problem-solving were made over the course of the intervention, and by extension, are indicative that accessibility concerns were modestly addressed.

Figure 25

Lily's attempts to draw 3/5 in the post-Atypicality Interview

a) **b)** **c)**

Prior to Lily's engagement with the ES intervention she was unable to successfully draw or write 3/5. When asked to do so on the pre-Atypicality Interview she drew an area model of 5/6. Some aspects of her problem-solving and modeling improved by the end of the intervention. When asked to draw 3/5 on the post-Atypicality Interview, she demonstrated a tenacity to persevere through uncertainty around how to precisely construct fifths by attempting to construct multiple models with incorrect partitioning (Figure 25a and b). She also competently arrived at a viable representation of 3/5 (Figure 25c). Furthermore, in her explanation on the post-Interview, she employs thinking used during the intervention to draw an analogy to her SGR and an ES context to allow her to productively engage with her SGR.

> LL: Well, as you see I'm trying to figure out a shape that worked, and I found one….So, you take the whole shape, you divide it up into 5 pieces, you color 3, like it's a chocolate bar, because doesn't it look like a chocolate bar? And then you take out the chunks that... [[waving hand at shaded sections]] Because people eat these…

This excerpt further demonstrates her continued reliance on a Fractional Complement understanding, focusing on the removal ("because people eat these…") to help her focus on the relevant pieces. Although she is relying on a Fractional Complement understanding here, she still correctly identifies the relevant fraction name. It remains unclear whether the content of the intervention or the differentiated instruction from session four and five that addressed fraction naming conventions, and planning as part of the problem-solving process, respectively, were responsible for diminishing the power of inaccessible representation in the previous example. The chocolate bar analogy that she employed and the fact that she seemed more comfortable making errors as part of solving, makes me think it might be a joint effect.

These findings of the various interactions with the intervention demonstrated the deeply complex relationship that learners with MLD experience while engaging with fractions, particularly as they are presented in an ES environment. Many of the SGRs were beneficial for pushing and supporting student mathematical thinking, these limitations and affordances have been shown. Progress for Lily was stunted by the fact that many of these student-generated representations eventually evolved to reflect canonical representations of fractions, whereas, the ES Intervention may have acted as a form of bridging intervention for Maddie and Ryan.

Discussion

These findings suggest that the complex relationship that learners have with engaging with fractions continues to pervade even in the face of an effective ES intervention. Of interest were the sometimes unique and often shared conceptualizations for engaging in mathematical problem-solving of Ryan, Lily, and Maddie. By tracking the instances of the atypical understandings associated with having an MLD during the intervention it was ascertained that the ES environment created opportunities to productively leverage aspects of these conceptualizations.

The difficulties associated with inaccessible mediational tools and a halving understanding were ameliorated through the creation of SGRs inspired by ES problem contexts. These models allowed learners to maintain an action-oriented perspective creating one-half, but importantly, also preserved the quantity one-half for successful interpretation. Additionally, for those who experienced an FC understanding, the inaccessibility of mediational tools was decreased in an ES context since the context enabled students to distribute and track items previously assumed as gone. However, when requested to interpret solutions based on canonical models of their own devising, students continued to experience inaccessibility. Thus, demonstrating that the ES intervention is useful, but insufficient in entirely address all inaccessibility concerns.

In thinking of Ryan's progress, his overall increase in correctness, and single instance of any AUs by the end of the intervention, it is also possible that the intervention acted as a sort of bridging intervention as seen in similar fraction intervention research for individuals with MLD (Lewis, 2017). In this research, Lewis was able to create a scale model that allowed a learner with MLD to both engage with fractions starting from the whole and contained a receptacle for placing removed pieces. This latter aspect aligned nicely with a fractional complement understanding. In both Lewis (2017) and this present inquiry it was seen that it is possible to increase the accessibility of canonical representations.

Cognitively Guided Instruction was chosen for its ability to respond to the individual needs of each of the students regardless of their ability status. The specific Equal Sharing intervention was selected for it being touted as an intervention that leveraged lived experiences of students in the form of equal sharing encounters (Empson & Levi, 2011). Per DeBock (2003) rich, contextualized knowledge that has been activated by experiences is helpful in understanding and solving mathematical problems. Therefore, it was assumed that SGRs were created based on meaningful contexts and non-abstract associations. With this in mind, we were conscientious in creating word-problems that were accessible, yet in the face of inaccessible and impersonal mathematical content, scaffolded instruction was mandated to alleviate the strain of a purely abstract mathematical context (Boaler, 1993; DeBock 2003). An example of this for Ryan occurred in a sharing context that used a measurement of string in yards which we substituted for a more relatable context of candy bars.

One method that we used to hopefully create a more accessible playing field for learners with MLD was to give students problem solving freedom. Each student was given agency to decide what tools to use and what direction to take to solve at least initially. This decision is supported by claims made by Boaler (1993), that asserts that open ended investigations are able to “connect with a student's meaning and allow the attainment of personal goals” (p. 17). In the findings, I was able to articulate the conceptualizations and some problem-solving habits of each of the students as they engaged with these open-ended ES problems and created SGRs to expose these interactions Lesh, Carmona, & Post, 2002). Attention was drawn to the ways in which these resources helped or hindered student’s’ successful navigation of ES problems, and how the use of dialogue and student-generated representations pushed both student and researchers thinking about the ways the intervention remained inaccessible in respect to AUs.

Limitations

An obvious limitation of this study is the duration of the ES intervention. It is possible that the years of standard representation confusion could not be easily resolved in such a short timeframe. It is furthermore possible, that the ES intervention's lack of universal usefulness in increasing accessibility for all learners was in fact researcher fallibility. The researchers may not have done a sufficient job in connecting standard representations to Ryan, Lily, and Maddie, and that there remains a better way to do so.

Implications for Research

It was shown that student-generated representations were useful in student problem-solving, for pushing thinking about the ES context. However, the downside was the influence of internalized models of canonical representations on SGRs. These canonical representations of fractions are known to be inaccessible for learners with MLD (Lewis, 2014), even still, all three students attempted to create abstract representations that resembled these inaccessible tools and were met with difficulties. Representation theory has documented that the level of abstraction of student representations is correlated with problem solving performance (Hegarty & Kozhevnikov, 1999; Van Garderen & Montague, 2003), and that when students use drawings that are "structured and relevant to the mathematical model needed, the performance of students improves. However, when they make drawings that are merely pictorial or not relevant, their performance is worse, as is to be expected" (Hoogland et. al., 2018, p. 124). Canonical models, in many ways, are not relevant for learners with an MLD, although this research made an effort to alter this relationship. From this information and my assertion, future studies may wish to investigate what forms abstract SGRs may take, without the influence of prior schooling of canonical models, to see if student derived depictions may be more beneficial for learners with MLD.

Additionally, both Ryan and Maddie would be interesting candidates for investigating the prevalence and use of compensatory strategies that have been documented in students with MLD that has led to successful engagement with mathematics (Lewis & Lynn, 2018). It was shown that Maddie was able to circumvent difficulties with representations by employing the long-division algorithm to solve ES problems. Subsequent research may find it fruitful to investigate the ways in which learners with MLD, such as Maddie, have developed their own accessible pathways for navigating mathematical coursework. The teachability of these habits and skills to other students would be interesting avenue of research to pursue.

Implications for Practice

This research also has implications for educational settings. The ES intervention was likely beneficial in part because it allowed the learners to retain an action-oriented approach to thinking about fractions. A feature that has been successful in the past for one adult learner with a MLD, who exhibited reliance on a fractional complement understanding (Lewis, 2017). Furthermore, it has been observed that an impediment to fraction mastery is the depiction of fractions as part-whole relationships (NMAP, 2008), such as pieces of pizza or pie. Therefore, in both Lewis (2017) and the present inquiry, students were productively positioned to think about fractions starting from a whole. This has been shown to be tremendously valuable in conceptualizing fractions as “continuous and infinitely divisible” (Misquitta, 2011; Behr, Lesh, Post, & Silver, 1983; Hiebert & Tonnessen, 1978). Therefore, in thinking about instructing individuals who have atypical understandings, it is even more paramount to start with the whole, and to encourage the act of constructing representations to align with their general preference for action.

Conclusion

An exploratory qualitative case study of three middle-school students with MLD was conducted to explore the accessibility of a Cognitively Guided Instruction Equal-Sharing intervention. I employed a difference perspective to extend Vygotsky’s sociocultural theory of mediational tools in relation to the theoretical constructs of student-generated representations to consider issues of accessibility. Each student had their own unique set and presentation of AUs throughout the intervention. This study showed that some aspects of the intervention did align with the students’ unique conceptualizations, but that it was insufficient to address issues of inaccessibility. The ability to distribute was helpful in light of an FC understanding and the ability to represent an action-oriented version of one-half was useful in the face of a halving understanding. Differentiated and scaffolded content was still necessary to alleviate tensions created between inaccessible SGRs and the students’ conceptualizations, particularly when these SGRs took the form of canonical inaccessible mediational tools.

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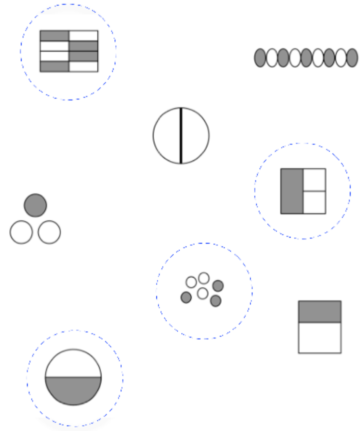
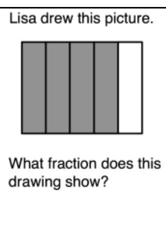
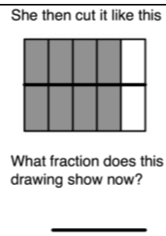
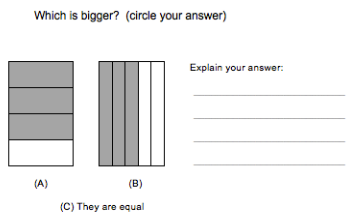
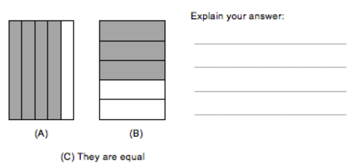
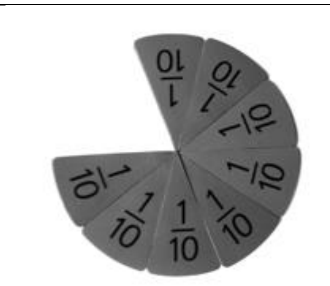
Appendix A: Questions and Scoring Criteria for the Atypical Understandings Screener

#	Problem	Correctness	Atypical Points	Not Atypical
1	Draw a picture of $\frac{1}{2}$	1 point – a canonical drawing or representation of ½ (i.e., area model, number line, decimal, percent, or semicircle)	1 atypical point – a drawing of a shape partitioned into two parts but without shading or labeling of either part.	A drawing of a shape partitioned into two with one of the parts labeled "½"
2	Draw another way to show $\frac{1}{2}$	(same as above)	(same as above)	(same as above)
3	Draw a picture of $\frac{3}{5}$	1 point – a canonical representation of 3/5 (e.g., area model, number line, or discrete set).	1 atypical point – an area model or discrete set drawing of 2/5 (i.e., 3 out of 5 parts unshaded).	Partitioning issues, because students have difficulty accurately partitioning into fifths. (note: to receive 1 point for a correct answer, the area model drawings must be between 1/2 and 3/4.)
4	Draw a picture of $1\frac{5}{8}$	1 point – a canonical representation of 1 5/8 (e.g., area model, number line, or discrete set).	1 atypical point – a representation where the whole is not shaded or labeled.	A drawing where the wholes are different sizes.
5	Circle all the pictures that you think show $\frac{1}{2}$? (*correct answers circled below*)	1 point for each correctly circled canonical representation of 1/2 (see	1 atypical point – for circling the halved circle with no shading.	

	 (adapted from Ni, 2001)	circled answers) -1 point for each incorrect answer.		
6	Which is more $\frac{1}{6}$ or $\frac{1}{8}$? (you can draw pictures to help you) Explain your answer:	1 point for correct answer (1/6) (explanations are used to disambiguate student answer, not required)	1 atypical point for incorrect answer (1/8) *with* an explanation and/or drawing that focuses on the non-shaded amount (e.g., more left in the 1/8 drawing).	A written answer that states that 1/8 is bigger than 1/6 because 8 is bigger than 6.
7	Which is more $\frac{2}{8}$ or $\frac{5}{8}$? Explain your answer:	1 point for correct answer (5/8) (explanations are used to disambiguate student answer, not required)	1 atypical point for an incorrect answer (2/8) *with* an explanation and/or drawing that focuses on the fractional complement (e.g., 6/8 unshaded, 3/8 unshaded).	Answer of 2/8 with no explanation or drawing.
8		1 point for correct answer (4/5).	1 atypical point for answers where the numerator is determined by number of parts not shaded (e.g., 1/5 or 1/4).	Answers where student has miscounted the number of pieces, (e.g., 5/6)

9		1 point for correct answer (8/10 or 4/5).	1 atypical point for answers where the numerator is determined by number of parts not shaded (e.g., 2/10, 2/8, 1/5 or 1/4).	Answers where student has miscounted the number of pieces, (e.g., 10/12).
10	 (adapted from Armstrong & Larson, 1995)	1 point for correct answer (A) (explanations are used to disambiguate student answer, not required)	1 atypical point for selecting B and an explanation focusing on the number “left” or non-shaded amount.	An incorrect answer (B or C) with either no explanation or an explanation that suggests miscounting, (e.g., “C because 3/5=3/5”)
11	 (adapted from Armstrong & Larson, 1995)	1 point for correct answer (A) (explanations are used to disambiguate student answer, not required)	1 atypical point for selecting B and an explanation focusing on the number “left” or non-shaded amount.	An incorrect answer (B or C) with either no explanation or an explanation that suggests miscounting, (e.g., “C because 3/5=3/5”)
12	Solve the problem $\frac{1}{2} + \frac{1}{4} =$ using pictures.	1 point for correct answer (3/4). Student not required to draw pictures.	1 atypical point for (a) answers that include a drawing of ½ without shading or (b) an answer of 2/4 (unshaded/shaded) with canonical area models of ½ and ¼.	An incorrect answer of 1/6 or 2/6 are not considered atypical by themselves.
13	What fraction does this picture show?	1 point for correct answer (e.g., 8/10 or 4/5).	1 atypical point for an answers that determine the numerator based on the missing pieces (e.g., 2/10 or 1/5, or 1/10).	An incorrect answer in which the student has miscounted (e.g., 7/10 or 9/10).

<table>
<tr><td></td><td>
</td><td></td><td></td><td></td></tr>
<tr><td colspan="5">Global coding: Any time the student interpreted a representation of as the fractional complement (e.g., interpreting 2/3 as 1/3) the student got an atypical point.</td></tr>
</table>

Appendix B: Atypicality Interview

Question 1 & 2:

- How would you draw a picture of 1/2?
- Can you think of another way of drawing a picture of 1/2?
- Can you explain how your picture is the same thing as 1/2?

Question 3:

- How would you draw a picture of 3/5?
- Can you explain how your picture is the same thing as 3/5?

Question 4:

- How would you draw a picture of one and five-eighths (1 5/8)?
- Can you explain how your picture is the same thing as 1 5/8?

Question 5:

- Circle all the pictures you think are the same as 1/2. Why did you pick this one [*point to each circled answer in turn*]? Why didn't you pick this one [*point to non-circled*]?

Question 6:

- Which is more 1/6 or 1/8?
- How do you know? / How would you explain it to someone who didn't understand fractions?

Question 7:

- Which is more 2/8 or 5/8?
- How do you know? How would you explain it to someone who didn't understand fractions?

Question 8 & 9:

- One student I was working with drew a picture like this (draw 4/5) – what fraction did she draw?
- Another student came along and cut it in half like that. What fraction does this drawing show now?

Question 10:

- Which is larger, or are they equal? [*page with area models of* 3/4 or 3/5]

Question 11:

- Which is larger, or are they equal? [*pare with area models of* 4/5 or 3/5]

Question 12:

- How would you solve this problem [write problem "½ + ¼ ="]?
- Can you explain how you solved it?

Question 13:

- What fraction does the picture show [*present page with eight 1/10 pieces*]? How did you figure that out?

Standard Follow-up Probes:

- Can you tell me how you got that answer?
- How do you know?
- How did you figure that out?

Can you explain your answer?

Appendix C: Equal-sharing Master Protocol

1. Evenly divided with 2 people (**admin 1st one, if correct, skip to next set**) – make sure he draws or uses manipulatives to demonstrate his thinking. (both manipulatives and drawing for each level of question)	
a) John and Kelly have 8 jellybeans. If they share the jellybeans equally, how many jellybeans would each person get?	8/2 = 4
b) Simone and Raphael have 10 candy corn. If they share the candy corn equally, how many candies would each person get?	10/2 = 5
c) Robert and Jill have 18 brownies. If they share the brownies equally, how many brownies would each person get?	18/2 = 9
d) Jeremy and Allison have 12 apples. If they share the apples equally, how many apples would each person get?	12/2 = 6
2. Evenly divided with 3 or 4 people **(admin all)**	
a) 3 students want to share 15 baseball cards. If they share the cards equally, how many cards would each student get?	15/3=5
b) 4 students want to share 12 pencils. If they share the pencils equally, how many pencils would each student get?	12/4=3
c) 5 students want to share 20 Oreos. If they share the Oreos equally, how many Oreos would each student get?	20/5=4
3. Equally divide in half (**make judgment**)	
a) Karl and Chloe have 7 cookies. If they share the cookies equally, how many cookies would each person get?	7/2=3.5
b) Kara and Samuel have 13 cupcakes. If they share the cupcakes equally, how many cupcakes would each person get?	13/2=7 1/2
c) Lawrence and Lily have 21 brownies. If they share the brownies equally, how many brownies would each person get?	17/2 = 10.5
d) Talin and Eddie have 9 oranges. If they share the oranges equally, how many oranges would each person get?	9/2= 4 1/2

e) Rebecca and Leslie have 11 strawberries. If they share the strawberries equally, how many strawberries would each person get?	11/2 = 5 ½
4. Fraction is half, but dividing with more people (**if correct, STOP**)	
a) 4 students want to share 10 submarine sandwiches so that everyone gets the same amount. How much can each student have?	10/4 = 2.5
b) 4 children want to share 14 mini pizza so that everyone gets the same amount. How much can each child have?	14/4 = 3.5
c) 4 students want to share 6 bagels so that everyone gets the same amount. How much can each student have? (Note that RW was administered 6 mini pizzas)	6/4=1.5
d) 4 students want to share 10 burritos so that everyone gets the same amount. How much can each student have?	10/4=2 1/2
e) e) 4 students want to share 18 pieces of sushi so that everyone gets the same amount. How much sushi can each child have?	18/4= 4 1/2
5. Fraction ends as 1/4	
a) 4 students want to share 13 brownies so that everyone gets the same amount. How much can each student have?	13/4 = 3.25
b) 4 students want to share 17 cupcakes so that everyone gets the same amount. How much can each student have?	17/4 = 4.25
c) 4 children want to share 9 chocolate bars so that everyone gets the same amount. How much can each child have?	9/4=2.25
d) 4 children want to share 21 donut holes so that everyone gets the same amount. How much can each child have?	21/4=2.25
5p. Proper fractions: unit fractions (when appropriate, prompt for EF)	
a) 4 children want to share a package of Play-doh so that everyone gets the same amount. How much Play-doh can each child have?	1/4
b) 3 students want to share a loaf of banana bread so that everyone gets the same amount. How much banana bread can each child have?	1/3

c) 8 children want to share a large pizza so that everyone gets the same amount. How much pizza can each child have?	1/8
6. Non unit fraction amount (3/4)	
Note 1: original question, later amended "There are 11 yards of ribbon for 4 people to share, how yards of ribbon can each person get if they share the ribbon equally?") Note 2: RW was administered: 11 fun-sized Twix for 4 students to share. a) There are 11 donuts for 4 people to share. How many donuts does each person get if they share the donuts equally?	11/4=2.75
b) There are 15 slices of cheese for 4 people to share. How many slices of cheese can each person have if they share the cheese equally?	15/4=3.75
c) 4 children want to share 19 chocolate bars so that everyone gets the same amount. How much can each child have?	19/4=4.75
6p. Proper fractions: non-unit fractions	
a) 3 children want to share 2 candy bars so that everyone gets the same amount. How many candy bars can each person get if they share them equally?	2/3
b) 8 students want to share 3 cakes so that everyone gets the same amount. How much cake can each student have if they share the cakes equally?	3/8
c) 8 students want to share 5 submarine sandwiches so that everyone gets the same amount. How many sandwiches can each student have if they share them equally?	5/8
7. Equivalent fractions (½ with 2/4) (**Prompt** student to think about equivalent fraction: e.g. "I saw another student who split the last 2 parts into 4 (rather than 2) so everyone got 3 2/4…")	
a) 4 children want to share 14 biscuits so that everyone gets the same amount. How much can each child have?	14/4 = 3 ½ or 3 2/4
b) 4 students want to share 22 jumbo marshmallows so that everyone gets the same amount. How much can each student have?	22/4 = 5 ½ or 5 2/4
c) There are 10 muffins for 4 people to share. How muffins can each person get if they share the muffins equally?	10/4 = 2 ½ or 2 2/4

8. Proper fractions: (non-unit fractions prompt for EF – all equal to 1/2) *[should we try to connect to canonical area model?]*	
a) 6 students want to share a 3 pumpkin pies. How much pie would each person get if they share the pie equally? (Note: amended for RW to 3 watermelons rather than pie)	3/6 = 1/2 or 3/6
b) 8 children want to share 4 packages of clay so that everyone gets the same amount. How much clay can each child have? (For RW: "8 children who want to share 4 Snickers so that everyone gets the same amount. How much Snickers can each child have?")	4/8 = 1/2 or 2/4
c) 4 students want to share 2 cartons of ice cream so that everyone gets the same amount. How much ice cream can each child have?	2/4 = 1/2 or 2/4
9. Proper fractions: (non-unit fractions, prompt EF – NOT equal to 1/2)	
a) 6 children want to share 2 candy bars so that everyone gets the same amount. How many candy bars can each person get if they share them equally?	2/6 = 1/3 or 2/6
b) 8 students want to share 2 cakes so that everyone gets the same amount. How much cake can each student have if they share the cakes equally? (for RW: 8 students wants to share two pepperoni pizzas so that everyone gets the same amount. How much pepperoni pizza can each student have if they share the pizzas equally?"	2/8 = 1/4 or 2/8
c) 6 students want to share 4 submarine sandwiches so that everyone gets the same amount. How many sandwiches can each student have if they share them equally?	4/6 = 2/3 or 4/6

Appendix D: ES Problem Presentation by Student and Session

	Participant		
Session #	Ryan	Lily	Maddie
1	Atypicality Interview, History Interview, & WJIV		
2	1A 1B 2A 3A 3B 4A	1A 1B 2A 2B 2C 3A 3B 3C 4A	½+1/4 from screener 1A 2A 3A 3B 4A 5A
3	1C 2B 2C 3C 4B 4C 5A	1D 3D (REMEDIATION: fraction name conventions) 4D 5A 5B 5C 6A	4D 5B 6A 6B (REMEDIATION: decimal fraction equivalence) 7A 8A 8B
4	1D 3D 5B	4E 5p.A 5p.B	8C 9A 9B

	5C 6A (Differentiated) 6C	5p.C 6p.A	10A 10B 11A 12A (REMEDIATION: long division)
5	5D 6B 6C 7A	4C 5D 6B 10A	SCREENER 10C 11B 12C Long division
6	7B 8A 9A 9B	SCREENER 3E 7B 5E 6C	N/A
7	8B 10A 10B 11A	N/A	N/A
8	SCREENER 10C 8C	N/A	N/A