Hybrid Inverse Problems

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Abstract

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Inverse problems arise in different disciplines including exploration geophysics, medical imaging and nondestructive evaluation. In some settings, a single modality displays either high contrast or high resolution but not both. In favorable situations, physical effects couple one high-contrast modality with another high-resolution modality. Hybrid inverse problems, also called coupled-physics inverse problems or multi-wave inverse problems, are motivated to study these coupling mechanisms to display both high contrast and high resolution. In photo-acoustic tomography(PAT) and thermo-acoustic tomography(TAT), acoustic waves couple with optical radiations, while in electro-seismic(ES) effect, seismic waves couple with electrical fields.

The solution strategies of hybrid methods typically consist of two steps. Normally in the first step, a high-resolution-low-contrast modality is considered to reconstruct some internal data. In PAT and TAT, we invert a wave equation and reconstruct the initial wave pressure from available boundary measurements. In ES conversion, we invert Biot's system to recover the internal source. In our work, we assume that this step has been performed.

The second step consists of the quantitative reconstruction of coefficients of interest by applying a high-contrast modality on the high-resolution internal data obtained during the first step. In the second step of PAT, our main objective is to recover diffusion and absorption coefficients from the internal data. The second step of ES conversion works with Maxwell's equation to reconstruct the conductivity and the coupling coefficient from

internal data.

This thesis mainly focuses on the second steps of PAT and ES conversion. Indeed, assuming the internal measurements are obtained already, we mainly prove the uniqueness and the stability of the constructions of the coefficients of interest in PAT and ES conversion. Precisely, We show that knowledge of two internal data based on well-chosen boundary conditions uniquely determine the coefficients of interest. Moreover, Lipschitz type stability results are proved based on the same sets of well-chosen boundary conditions.

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NOTATION LIST

Ω: Domain of interest, open and bounded subset in $\mathbb{R}^n$.

Ω_0:   Open set in $\mathbb{R}^n$ and $\Omega \subset \Omega_0$.

Ω_1:   Open subset of Ω with neighborhood of tangent points with respect to vector field removed.

$\tilde{\Omega}$:   Open subset of Ω, defined in (2.204).

$\partial\Omega$:   Boundary of Ω. $\partial\Omega$ is of class of C^2.

$\partial\Omega_{\pm}$:   The front and back sides of boundary, defined in (2.79).

P_0: $P_0 = -h^2\Delta = \sum(hD_{x_k})^2$, while $D_{x_k} = -i\partial_{x_k}$.

P: $P = P_0 + h^2q = -h^2\Delta + h^2q$.

P_φ: $P_\varphi = e^{-\frac{\varphi}{h}}Pe^{\frac{\varphi}{h}}$.

P_φ^*:   The adjoint of P_φ, i.e., $P_\varphi^* = e^{\frac{\varphi}{h}}(-h^2\Delta + h^2\bar{q})e^{-\frac{\varphi}{h}}$.

x: Point in Ω, i.e., $x \in \Omega$.

x_0:   Point on $\partial\Omega$, i.e., $x_0 \in \partial\Omega$.

σ_a: Attenuation coefficient.

D: Diffusion coefficient.

μ: $\mu = \frac{\sigma_a}{\sqrt{D}}$, see (2.34).

q: Potential function in Schrödinger equation.

u_j: Solutions to Schrödinger equation with boundary condition g_j.

g_j: Boundary illuminations.

d_j: Measured internal data, $d_j = \mu u_j$.

β_d: Vector field defined with d by (2.62).

β: Scaled imaginary part of β_d, $\beta = \frac{h}{2}\Im\beta_d$.

$\check{u}_j$:   CGO solutions, $\check{u}_j = e^{\rho \cdot x}(a + r)$ or $\check{u} = e^{\frac{1}{h}(\varphi + i\psi)}(a + r)$.

$\check{d}$:   Internal data constructed by CGO solutions, $\check{d} = \mu \check{u}$.

$\check{\beta}_d$:   Vector filed defined with $\check{d}$ by (2.62).

$\check{\beta}$:   Scaled imaginary part of $\check{\beta}_d$, $\check{\beta} = \frac{h}{2}\Im\check{\beta}_d$, defined in (2.64).

ρ: Phase parameter of CGO solutions, $e^{\rho \cdot x}(a + r)$.

$a(x)$:   Amplitude function in CGO solutions.

$r(x)$:   Correction term in CGO solutions.

$\mathfrak{k}$:   Vector in $\mathbb{R}^n$ such that $\rho = \frac{1}{h}(\mathfrak{k} + i\mathfrak{k}^{\perp})$, while $|\rho| = \frac{1}{h}$.

$\mathfrak{k}^{\perp}$:   Vector in $\mathbb{R}^n$ that is perpendicular to $\mathfrak{k}$, i.e., $\mathfrak{k} \cdot \mathfrak{k}^{\perp} = 0$.

φ: Limiting Carleman weight.

$\varphi + i\psi$: Phase function of CGO solutions, $e^{\frac{1}{h}(\varphi+i\psi)}(a+r)$.

h: Small positive parameter.

n: Special dimension.

j: Index of data, $j = 1, \ldots, J$. $J \in \mathbb{N}^*$ is the total number of measurements.

ε: Small positive parameter, applied in $H^{\frac{n}{2}+k+\varepsilon}$.

k: Positive integer, applied in $H^{\frac{n}{2}+k+\varepsilon}$.

s: The order of Sobolev spaces, $s = \frac{n}{2} + k + \varepsilon$.

ϵ: Sufficiently small positive parameter, applied in (2.68).

α: Real parameter $\alpha > \frac{1}{2}$, applied in $C^{k,\alpha}(\partial\Omega)$

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DEDICATION

to my parents Maixiu Wang and Genxiao Chen.

Chapter 1

INTRODUCTION

Inverse problems arise in different disciplines including exploration geophysics, medical imaging and nondestructive evaluation. In some settings, a single modality displays either high contrast or high resolution but not both. In favorable situations, physical effects couple one high-contrast modality with another high-resolution modality. Hybrid inverse problems, also called coupled-physics inverse problems or multi-wave inverse problems, are motivated to study these coupling mechanisms to display both high contrast and high resolution.

Several medical imaging modalities succeed to display high resolution. Typical examples are computerized tomography(CT), magnetic resonance imaging(MRI) and ultrasound imaging(UI). In some situation, these modalities fail to exhibit a sufficient contrast between different types of tissues. On the other hand, some modalities, based on optical, elastic, or electrical properties of tissues, do display such high contrast, but involve a highly smoothing measurement operator and are thus typically low-resolution. Such examples are optical tomography(OP), electrical impedance tomography(EIT) and elasto-graphic imaging(EI).

To produce both high resolution and high contrast, one strategy is introduced to fuse data acquired independently from two or more imaging modalities. Such method is referred as multi-modality imaging. Another strategy, referred as hybrid inverse problems, also called coupled-physics inverse problems or multi-wave inverse problems, considers the combination of high resolution modalities with high contrast modalities. By combination, we mean the existence of a physical mechanism that couples these two modalities. Our work focus on hybrid inverse problems. Typical physical coupling and hybrid inverse problems include: optics or electromagnetism with ultrasound in Photo-Acoustic Tomography (PAT)[7, 10], Thermo-Acoustic Tomography (TAT)[6] and Ultrasound Modulated Optical Tomography (UMOT)[43, 48, 49], also called Acousto-Optic Tomography (AOT); electrical currents with ultrasound in Ultrasound Modulated Electrical Impedance Tomography (UMEIT)[50], also

called Electro-Acoustic Tomography (EAT); electrical currents with magnetic resonance in Magnetic Resonance EIT (MREIT)[34, 35, 32] or Current Density Impedance Imaging (CDII); and elasticity with ultrasound in Transient Elastography (TE)[31]. In geophysics, Electro-seismic or Seismo-electrical(ES/SE)[29, 30, 11] conversion considers the coupling of electromagnetic fields with mechanical waves through the phenomenon of electro-kinetics. Some hybrid methods are extensively studied and explored experimentally whereas other ones are less well understood. Readers are referred to the recent books [1, 33, 44] and their references for general information about practical and theoretical aspects of medical imaging. We are mainly interested in PAT, TAT and ES conversion. The coupling mechanisms are explained as follows.

In PAT and TAT, when a body is exposed to short pulse radiation, it absorbs energy and expands thermo-elastically. The expansion emits acoustic pulses, which travel to the boundary of the domain of interest where they are measured. What distinguishes these two modalities is that in PAT, radiation is high-frequency radiation (near-infra-red with sub-μm wavelength), while in thermo-acoustic, radiation is low-frequency radiation (microwave with wavelengths comparable to 1m). The inverse problems are to reconstruct diffusion and absorption coefficients from the measurements of the boundary illuminations and the acoustic waves observed on body boundary.

Electro-seismic(ES) and Seismo-electrical(SE) conversion concern electro-kinetic phenomenon in porous medium. When a porous rock is saturated with an electrolyte, an electric double layer is formed at the interface of solid and fluid. One side of the solid-fluid interface is negatively charged and the other side is positively charged. An electric field or electromagnetic waves acting on the electric double layer will move charges in different directions, creating relative movement of fluid and solid. This is named electro-seismic conversion. Conversely, a mechanical wave moving fluid and solid will generate electromagnetic fields. This is named seismo-electric conversion.

Solution strategies of hybrid methods typically consist of two steps. Normally in the first step, a high-resolution-low-contrast modality is considered to reconstruct some internal measurements, which, therefore, are of high resolution. In PAT and TAT, we invert a wave equation and reconstruct the initial wave pressure from available boundary measurements.

In ES conversion, we invert Biot's system to recover the internal source. In our work, we assume that this step has been performed.

The second step of the hybrid methods consists of the quantitative reconstruction of the coefficients of interest by applying a high-contrast modality on the high-resolution internal data obtained during the first step. Normally, the internal data are functionals of the coefficients of interest. Precisely, if D is the internal data, σ is the coefficient of interest and u is the solution to the partial differential equation involving σ. The obtained internal measurements are of the form $D(x) = \sigma(x)u$ or $D(x) = \sigma(x)|u|^2$. The second step in PAT and TAT is referred as quantitative photo-acoustic tomography(QPAT) and quantitative thermo-acoustic tomography(QTAT), respectively. Our main objective is to recover diffusion and absorption coefficients from the internal measurements. Due to different physical properties, a diffusion equation and internal data of the form $D = \sigma u$ are considered in QPAT, while Maxwell's equation and internal data of the form $D = \sigma|u|^2$ are applied in QTAT. The second step of ES conversion also works with Maxwell's equation to reconstruct the conductivity and the coupling coefficient L from the internal data of the form $D = LE$.

We are more interested in the second step. Indeed, assuming the internal measurements are obtained already, we mainly study the uniqueness and the stability of the construction of the coefficients of interest in the second step. In many hybrid inverse problems, the uniqueness and the stability properties are not guaranteed in general. However, if forward solutions satisfy certain prescribed boundary conditions, we can achieve a unique and stable reconstruction. Although reconstruction algorithms depend on different physical models of interest, there are important common features shared by most hybrid methods.

The most important feature of hybrid inverse problems is to display both high resolution and high contrast. When a high-contrast modality in the second step is applied to high-resolution data obtained from the first step, the high quality in both contrast and resolution are expected. Mathematically, the internal data provides local, point-wise information about coefficients of interest, i.e., singularities of these coefficients are well preserved. We thus expect resolution of hybrid modalities to be significantly improved compared to the stand-alone high-contrast-low-resolution modalities.

Another feature shared by many hybrid inverse problems is that their solution strategies

often require that the forward solution u satisfies certain qualitative properties. Complex geometric optics(CGO) solutions are then constructed explicitly to characterize these qualitative properties. When a solution u falls in a neighborhood of a CGO solution, we expect it also characterizes the qualitative requirements. By the elliptic regularity, the perturbation normally can be controlled on the domain boundary. Thus the neighborhood of CGO solutions defines a restricted class of boundary conditions with which solutions to the hybrid inverse problems are shown to be uniquely and stably determined by the internal measurements.

The rest of this thesis is organized as follows. Chapter 2 is devoted to PAT and TAT. The physical modeling is introduced in Section 2.1. It is followed by review of known results about the first step of PAT and TAT in Section 2.2. QPAT with full boundary data is reviewed in Section 2.3 and our new results of QPAT with partial boundary data are presented in Section 2.4. QTAT with full boundary data is also reviewed in Section 2.5. In Chapter 3, we study the hybrid inverse problem in ES conversion. The physical modeling is introduced at the beginning. The first step of the inverse problem, i.e., the inversion of Biot's system, is studied in Section 3.1. Related future research problem is also proposed. Section 3.2 presents main results of our research about the uniqueness and stability of the reconstruction in the second step of the inverse problem. Chapter 4 contains a list of proposed problems for future research.

Chapter 2

PHOTO-ACOUSTIC TOMOGRAPHY (PAT) AND THERMO-ACOUSTIC TOMOGRAPHY(TAT)

2.1 Introduction

Photo-acoustic tomography(PAT) and thermo-acoustic tomography(TAT) are recent hybrid methods based on the photo-acoustic effect which couples optical and ultrasonic waves. When a body is exposed to short pulse radiation, it absorbs energy and expands thermo-elastically. The expansion emits acoustic pulses, which travel to the boundary of the domain of interest where they are measured. The physical coupling between the absorbed radiation and the emitted sound is called the photo-acoustic effect.

What distinguishes PAT and TAT is the radiation applied. In PAT, near-infra-red photons, with wavelengths between $600nm$ and $900nm$, are used. Radiations in this frequency window are considered because they are not significantly absorbed by water molecules and thus can propagate relatively deep into tissues. In TAT, radiations are low-frequency microwave, with wavelengths comparable to $1m$, due to that they are less absorbed than optical frequencies and thus propagate into deeper tissues.

The first step of the inverse problems in both PAT and TAT is the reconstruction of the absorbed radiation or deposited energy from time-dependent boundary measurements of acoustic signals. Acoustic signals propagate in fairly homogeneous domains as the sound speed is assumed to be known. Notice that the light speed is much faster than the sound speed. When a short pulse of radiation is emitted into the medium, we may assume that it propagates at a time scale that is much shorter than that of ultrasound. Therefore, we can model that the initial ultrasound pressure $p|_{t=0}$ is proportional to the energy absorption $D(x)$, i.e.,

$$p|_{t=0} = \mathfrak{G}(x)D(x), \qquad (2.1)$$

where $\mathfrak{G}(x)$ is the Grüneisen coefficient assumed to be constant and known. The acoustic

wave is modeled by

$$
\begin{cases}
\dfrac{1}{c^2}\dfrac{\partial^2 p}{\partial t^2} - \Delta p &= 0 \qquad \text{in } (0,T) \times \mathbb{R}^n \\[2mm]
p|_{t=0} &= f \qquad \text{in } \mathbb{R}^n \\[2mm]
\partial_t p|_{t=0} &= 0 \qquad \text{in } \mathbb{R}^n,
\end{cases}
\tag{2.2}
$$

where c is the speed of the acoustic wave, $T > 0$ is fixed, and f is the unknown initial pressure. The pressure $p(t,x)$ is then measured on $[0,T] \times \partial\Omega$.

The inverse problem in this step is to recover the initial pressure $f(x)$ from boundary measurements of $p(t,x)$ on $[0,T] \times \partial\Omega$, and thus the energy absorption. This step has been studied extensively in the mathematical literature, see, e.g. [2, 13, 14, 15, 18, 19, 20, 22, 23, 24, 27, 28, 36, 46, 47]. Main reconstruction results for the cases with continuous or discontinuous sound speed are reviewed in Section 2.2.

The second steps in PAT and TAT consist of quantitative reconstruction of attenuation and diffusion coefficients from knowledge of internal energy absorption. They are referred as quantitative photo-acoustic tomography(QPAT) and quantitative thermo-acoustics tomography(QTAT), respectively. Due to radiation properties, different physical models are applied in QPAT and QTAT. Radiation is typically modeled by transport or diffusion equations in the former case and Maxwell's equations in the latter case.

In [7], G. Bal and G. Uhlmann studied QPAT with whole boundary illuminations and showed the uniqueness and stability of the reconstruction of attenuation and diffusion coefficients. Our work extends Bal and Uhlmann's results to partial boundary illumination conditions and prove the uniqueness and stability of the reconstruction. Particularly, we show that two coefficients in the diffusion equation are uniquely determined by four well-chosen illuminations at part of the domain boundary. The stability of the reconstruction is established from either four internal data under geometric condition of strict convexity on the domain of interest, or from $4n$ well-chosen partial boundary conditions, where n is the spatial dimension.

Mathematically, by the standard Liouville change of variables, the diffusion equation is replaced by a Schrödinger equation with unknown potential and with internal and partial boundary measurements. By adapting the complex geometrical optics(CGO) solutions

proposed in [26], we are able to obtain uniqueness and stability results for the inverse Schrödinger problem. The inverse Liouville change of variables concludes the uniqueness and stability of the the diffusive regime.

This chapter is organized as follows: In Section 2.2, we review some current results on the first step of PAT and TAT. In Section 2.3, we review G. Bal and G. Uhlmann's [7] results of QPAT with whole boundary data. We present our main results of the uniqueness and the stability of the reconstruction in QPAT with partial boundary data in Section 2.4. We also review some results in QTAT[6] in Section 2.5.

2.2 The first step of PAT and TAT

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected domain with smooth boundary $\partial\Omega$. Let $p(t,x)$ solve the wave equation

$$
\begin{cases}
\dfrac{1}{c^2}\dfrac{\partial^2 p}{\partial t^2} - \Delta p &= 0 \quad \text{in } (0,T) \times \mathbb{R}^n \\
p|_{t=0} &= f \quad \text{in } \mathbb{R}^n \\
\partial_t p|_{t=0} &= 0 \quad \text{in } \mathbb{R}^n,
\end{cases}
\tag{2.3}
$$

where $T > 0$ is fixed.

Assume that f is supported in $\bar{\Omega}$. The measurements are modeled by the operator Λ

$$
\mathfrak{h} = \Lambda f := u|_{[0,T]\times\partial\Omega}.
\tag{2.4}
$$

The problem now is to reconstruct the unknown f from measurement $\mathfrak{h}$, i.e., to invert Λ.

In the case when $T = \infty$, we can solve a problem with Cauchy data 0 at $t = \infty$ and boundary data $\mathfrak{h} = \Lambda f$. The zero Cauchy data are justified by the energy decay that holds for non-trapping geometry. Then solving the resulting problem backwards recovers f. We will study the case when $T < \infty$ is fixed.

We define the energy of a function $u(t,x)$ in Ω by

$$
E_\Omega(t,u) = \int_\Omega \left(|\nabla u|^2 + c^{-2}|u_t|^2 \right) dx.
\tag{2.5}
$$

We also define the space $H_D(\Omega)$ to be the completion of $\mathbb{C}_0^\infty(\Omega)$ under the Dirichlet norm

$$
\|f\|_{H_D}^2 = \int_\Omega |\nabla u|^2 dx.
\tag{2.6}
$$

Assume for now the speed $c > 0$ is smooth. The speed c defines a Riemannian metric $c^{-2}dx^2$. For any piecewise smooth curve $[a, b] \to \gamma \in \mathbb{R}^n$, the length of the curve in that metric is given by

$$\text{length}(c) = \int_a^b \frac{\dot{\gamma}(t)}{c(\gamma(t))}dt. \tag{2.7}$$

The so defined length is independent of the parametrization of γ. The distance function $\text{dist}(x, y)$ is then defined as the infimum of the lengths of all such curves connecting x and y.

We define

$$T_0 := \max\{\text{dist}(x, \partial\Omega) : x \in \bar{\Omega}\}, \tag{2.8}$$

and define T_1 to be the supremum of the lengths of all maximal geodesics lying in $\bar{\Omega}$. When $T_0 < \infty$, one can prove that

$$T_0 \leq \frac{T_1}{2}. \tag{2.9}$$

If $T_1 < \infty$, the sound speed c is called non-trapping. Let $\mathcal{K} \subset \Omega$ be a fixed compact subset. We define $T_1(\mathcal{K})$ as the supremum of the lengths of all geodesics passing through $\mathcal{K}$.

2.2.1 Smooth speed and full boundary data

If $T \gg 1$, Λf recovers f uniquely. We have the following sharp result based on the unique continuation theorem by Tataru[40].

Theorem 2.2.1 (Stefanov and Uhlmann[36]). *Let $\Lambda f = 0$. Then $f(x) = 0$ for $\text{dist}(x, \partial\Omega) \leq T$. Moreover, $f(x)$ can be arbitrary in the set $\text{dist}(x, \partial\Omega) > T$, if the latter set is non-empty.*

Corollary 2.2.2. *Λ is injective on $H_D(\Omega)$ if and only if $T \geq T_0$.*

The proof is mainly based on the reconstruction method in Theorem 2.2.3 as follows. The idea is to solve (2.3) approximately.

One method to get an approximation solution of f is the following time reversal method.

Given $\mathfrak{h}$, let v_0 solve

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 v_0}{\partial t^2} - \Delta v_0 & = & 0 \qquad \text{in } (0,T) \times \mathbb{R}^n \\ v_0|_{[0,T]\times\partial\Omega} & = & \mathfrak{h} \\ \partial_t v_0|_{t=T} & = & 0 \\ v_0|_{t=T} & = & 0. \end{cases} \qquad (2.10)$$

The we define the following approximate inverse",

$$A_0\mathfrak{h} := v_0(0,\cdot) \quad \text{in } \bar{\Omega}. \qquad (2.11)$$

Then $A_0\Lambda f$ is viewed as an approximation to f. This is actually true asymptoticly as $T \to \infty$. Since $\mathfrak{h}$ may not vanish on $\{T\} \times \partial\Omega$, the mixed problem above has boundary data with a possible jump type singularity at $\{T\} \times \partial\Omega$. That singularity will propagate back to $t = 0$ and will affect v_0, and then v_0 may not be in the energy space. For this reason, $\mathfrak{h}$ is usually cut off smoothly near $t = T > T_0$, i.e., $\mathfrak{h}$ is replaced by $\chi(t)\mathfrak{h}$, where $\chi \in C^\infty(\mathbb{R})$, $\chi = 0$ for $t = T$, and $\chi = 1$ in a neighborhood of $(-\infty, T_0]$.

Another method will modify this approach in a way that would make the error operator a contraction. We proceed as follows. Given $\mathfrak{h}$, let $v(t,x)$ solve

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 v_0}{\partial t^2} - \Delta v_0 & = & 0 \qquad \text{in } (0,T) \times \mathbb{R}^n \\ v_0|_{[0,T]\times\partial\Omega} & = & \mathfrak{h} \\ \partial_t v_0|_{t=T} & = & 0 \\ v_0|_{t=T} & = & \phi, \end{cases} \qquad (2.12)$$

where ϕ solves the elliptic boundary value problem

$$\Delta\phi = 0, \quad \phi|_{\partial\Omega} = \mathfrak{h}(T,\cdot). \qquad (2.13)$$

We introduce notation P_Ω for the Poisson operator of harmonic extension: $P_\Omega(\mathfrak{h}(T,\cdot)) := \phi$. Since 0 is not a Dirichlet eigenvalue of Δ in Ω, (2.13) is uniquely solvable. Note that the initial data at $t = T$ satisfy compatibility conditions of first order. Then we define the following pseudo-inverse

$$A\mathfrak{h} := v(0,\cdot) \quad \text{in } \bar{\Omega}. \qquad (2.14)$$

The operator A maps continuously the closed subspace of $H^1([0,T] \times \partial\Omega)$ consisting of functions that vanish at $t = T$ to $H^1(\Omega)$. It also sends the range of Λ to $H_0^1(\Omega) \approx H_D(\Omega)$.

Theorem 2.2.3 (Stefanov and Uhlmann[36])**.** *Let $(\Omega, c^{-2}dx^2)$ be non-trapping and let $T > T_0$. Then $A\Lambda = Id - K$, where K is compact in $H_D(\Omega)$ and $\|K\|_{H_D(\Omega)} < 1$. In particular, $Id - K$ is invertible on $H_D(\Omega)$, and the inverse problem in (2.13) has an explicit solution of the form*

$$f = \sum_{m=0}^{\infty} K^m A\mathfrak{h}, \quad \mathfrak{h} := \Lambda f. \tag{2.15}$$

Theorem 2.2.3 provides an estimate of the error in the reconstruction.

Corollary 2.2.4. *Under that same hypothesis in Theorem 2.2.3, the error of the reconstruction is controlled by*

$$\|f - A\Lambda f\|_{H_D(\Omega)} \leq \left(\frac{E_\Omega(u, T)}{E_\Omega(u, 0)} \right)^{\frac{1}{2}} \|f\|_{H_D(\Omega)}, \quad \forall f \in H_D(\Omega), f \neq 0, \tag{2.16}$$

where u is the solution of (2.3).

The stability results are given as:

Theorem 2.2.5 (Stefanov and Uhlmann[36])**.** *Let $\mathcal{K} \subset \Omega$ be compact.*

(a) Let $T > T_1(\mathcal{K})/2$. Then there exists a constant $C > 0$ so that

$$\|f\|_{H_D(\Omega)} \leq C\|\Lambda f\|_{H^1([0,T] \times \partial\Omega)}. \tag{2.17}$$

(b) Let $T < T_1(\mathcal{K})/2$. Then for any s_1, s_2 and $C > 0$, there exists $f \in C^\infty$ supported in $\mathcal{K}$ so that

$$\|f\|_{H^{s_1}} \geq C\|\Lambda f\|_{H^{s_2}([0,T] \times \partial\Omega)}. \tag{2.18}$$

2.2.2 Smooth speed and partial boundary data

The case of partial boundary measurements has been discussed in the literature as well. One of the motivations is that in breast imaging, in which case, measurements are possible only on part of the domain boundary.

Let $\Gamma \subset \partial\Omega$ be a relatively open subset of $\partial\Omega$. The measurements are taken on $[0, T] \times \Gamma$ and denoted as

$$\Lambda f|_{[0,T]\times\Gamma}. \tag{2.19}$$

Then the uniqueness result is given as follows.

Theorem 2.2.6 (Stefanov and Uhlmann[36])**.** *Let $T > T_0$ and $\partial\Omega$ be strictly convex. If $\Lambda f = 0$ on $[0, T] \times \Gamma$ for $f \in H_D(\Omega)$ with $suppf \subset \mathcal{K}$, then $f = 0$.*

For any $(x, \xi) \in \Omega \times S^{n-1}$ we denote by $\gamma_{x,\xi}$ the unit speed (i.e., $|\dot\gamma| = c(\gamma)$) geodesics issued at x in the direction ξ. Define $t_\pm(x, \xi)$ by the condition

$$t_\pm(x, \xi) = \max(t \geq 0 : \gamma_{x,\xi}(t_\pm) \in \bar\Omega). \tag{2.20}$$

We assume that $\mathcal{K}$ satisfies the following condition

$$\forall (x, \xi) \in \mathcal{K} \times S^{n-1}, \quad \gamma_{x,\xi}(t_\sigma(x, \xi)) \in \Gamma \quad \text{for } \sigma = + \text{ or } -, \tag{2.21}$$

i.e., every geodesic initialed in $\mathcal{K}$ exits the domain on Γ. Then can define that

$$T_0(\mathcal{K}, \Gamma) := \max\{t_\sigma(x, \xi) : \forall (x, \xi) \in \mathcal{K} \times S^{n-1}, \gamma_{x,\xi}(t_\sigma(x, \xi)) \in \Gamma, \sigma = + \text{ or } -\}. \tag{2.22}$$

The stability result then follows.

Theorem 2.2.7 (Stefanov and Uhlmann[36])**.** *Let $\mathcal{K} \subset \Omega$ be compact satisfying (2.21).*

(a) Let $T > T_0(\mathcal{K}, \Gamma)$. Then there exists a constant $C > 0$ so that

$$\|f\|_{H_D(\Omega)} \leq C\|\Lambda f\|_{H^1([0,T]\times\Gamma)}. \tag{2.23}$$

(b) Let $T < T_0(\mathcal{K}, \Gamma)$. Then for any s_1, s_2 and $C > 0$, there exists $f \in C^\infty$ supported in $\mathcal{K}$ so that

$$\|f\|_{H^{s_1}} \geq C\|\Lambda f\|_{H^{s_2}([0,T]\times\Gamma)}. \tag{2.24}$$

2.2.3 Discontinuous sound speed and Brain imaging

Let Ω be a bounded domain with smooth boundary. Let $\mathcal{S} \subset \Omega$ be a smooth, closed and orientable surface. Let sound speed $c(x) > 0$ be piecewise smooth with a nonzero jump across $\mathcal{S}$. For $x \in \mathcal{S}$, let $c_-(x)$ be the limit of c from inside, and let $c_+(x)$ be that from outside. Our assumption then is that those limits are positive and $c_-(x) \neq c_+(x)$, for $x \in \mathcal{S}$.

This problem first arises in brain imaging due to the jump of sound speed in brain and skull[37]. Another motivation is to model the classical case of smooth speed in a patient's body but account for a possible speed jump when acoustic waves leave the body and enter the liquid surrounding it.

The formulation of the problem is still the same as the classical case, while the limits of the solution $u(t,x)$ and its normal derivative match as x approaches $\mathcal{S}$ from either side. Let $u(t,x)$ solve the problem

$$
\begin{cases}
\left(\dfrac{1}{c^2} \dfrac{\partial^2}{\partial t^2} - \Delta \right) u &= 0 \quad \text{in } (0,T) \times \mathbb{R}^n \\
u|_{\mathcal{S}_-} &= u|_{\mathcal{S}_+} \\
\frac{\partial u}{\partial \nu}|_{\mathcal{S}_-} &= \frac{\partial u}{\partial \nu}|_{\mathcal{S}_+} \\
u|_{t=0} &= f \\
\partial_t u|_{t=0} &= 0,
\end{cases}
\tag{2.25}
$$

where $T > 0$ is fixed, $u|_{\mathcal{S}_\pm}$ are the limits of u on $\mathcal{S}$ from outside and from inside. We similarly define the outside/inside normal derivatives, $\frac{\partial u}{\partial \nu}|_{\mathcal{S}_\pm}$, with ν the exterior unit normal to $\mathcal{S}$.

Assume that f is supported in $\bar{\Omega}$. The measurements are modeled by the operator

$$
\Lambda f = u|_{[0,T] \times \partial \Omega}.
\tag{2.26}
$$

The problem is to reconstruct the unknown f from Λf.

In the case when $c_- > c_+$, this ray splits into two parts when hitting $\mathcal{S}$ from inside. One of them reflects according to the usual reflection laws. Another one is the transmitted (refracted) ray at an angle satisfying Snell's law:

$$
\frac{\sin \alpha_-}{\sin \alpha_+} = \frac{c_-}{c_+}.
\tag{2.27}
$$

In the opposite case, i.e., $c_- < c_+$, there is a critical angle given by $\alpha_0 = \sin^{-1}(c_-/c_+)$. If $\alpha_- > \alpha_0$, there are still a reflected and a transmitted ray as above satisfying Snell's law. If $\alpha_- > \alpha_0$, then there is no transmitted ray, while the reflected ray still exists. This is known as a full internal reflection.

The uniqueness results in Theorem 2.2.1 and Corollary 2.2.2 still hold in this case.

The stability results requires the analysis of the propagation of singularities. However, in the case of full internal reflection, there might be singularities that never reach $\partial\Omega$. Such singularities will be invisible, no matter what T we choose. Therefore, the singularities that are certain to be visible up to time T consist of the following set:

$$\mathcal{U} = \{(x,\xi) \in (\Omega\backslash\mathcal{S}) \times S^{n-1} : \text{ there is a path of the "geodesic" issued from either}$$

$$(x,\xi) \text{ or } (x,-\xi) \text{ at } t = 0 \text{ never tangent to } \mathcal{S}, \text{ that is outside } \bar{\Omega} \text{ at time } t = T\}. \quad (2.28)$$

If f is supported in some compact subset $\mathcal{K} \subset \Omega\backslash\mathcal{S}$, then the stability condition can be formulated as follows:

$$\mathcal{U} = \{(x,\xi) \in \mathcal{K} \times S^{n-1} : \text{ there is a path of the "geodesic" issued from either}$$

$$(x,\xi) \text{ or } (x,-\xi) \text{ at } t = 0 \text{ never tangent to } \mathcal{S}, \text{ that is outside } \bar{\Omega} \text{ at time } t = T\}. \quad (2.29)$$

Let $\Pi_\mathcal{K} : H_D(\Omega) \to H_D(\mathcal{K})$ be the orthogonal projection. It turns out that the orthogonal projection is given by $\Pi_\mathcal{K}(f) = f|_\mathcal{K} - P_\mathcal{K}(f|_{\partial\mathcal{K}})$, where $P_\mathcal{K}$ is the Poisson operator of harmonic extension from $\partial\mathcal{K}$ to $\mathcal{K}$ defined in (2.13).

The reconstruction formula of f is given by the following theorem.

Theorem 2.2.8 (Stefanov and Uhlmann[37]). *Let $\mathcal{K}$ satisfy (2.29). Then $\Pi_\mathcal{K} A\Lambda = Id - K$ in $H_D(\mathcal{K})$, with $\|K\|_{H_D(\mathcal{K})} < 1$. In particular, $Id - K$ is invertible on $H_D(\mathcal{K})$, and Λ restricted to $H_D(\mathcal{K})$ has an explicit left inverse of the form*

$$f = \sum_{m=0}^{\infty} K^m \Pi_\mathcal{K} A\mathfrak{h}, \qquad \mathfrak{h} = \Lambda f. \quad (2.30)$$

The reconstruction formula in Theorem 2.2.8 also shows that, under the condition (2.29), we have the similar stability result as in Theorem 2.2.5.

2.3 Quantitative photo-acoustic tomography(QPAT)

2.3.1 Reduction to Schrödinger equation

Let Ω be an open, bounded, connected domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the diffusion equation with unknown diffusion coefficient D and unknown attenuation coefficient σ_a:

$$-\nabla \cdot D\nabla v + \sigma_a v = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{2.31}$$

Using the standard Liouville change of variables, $u = \sqrt{D}v$ solves

$$\Delta u + qu = 0, \tag{2.32}$$

with

$$q = -\frac{\Delta\sqrt{D}}{\sqrt{D}} - \frac{\sigma_a}{D}. \tag{2.33}$$

The internal data in photo-acoustics is given by

$$d = \sigma_a v = \frac{\sigma_a}{\sqrt{D}}u = \mu u, \quad \mu := \frac{\sigma_a}{\sqrt{D}}, \tag{2.34}$$

while the new boundary condition is given by $\sqrt{D}g$ on $\partial\Omega$, assuming D is known on $\partial\Omega$. We also find that

$$-\Delta\sqrt{D} - q\sqrt{D} = \mu, \tag{2.35}$$

so that we can solve for $\sqrt{D}$ and get $\sigma_a = \mu\sqrt{D}$.

We now consider Schrödinger equation of the form

$$\Delta u + qu = 0 \quad \text{in } \Omega \qquad u = g \quad \text{on } \partial\Omega, \tag{2.36}$$

where q is an unknown potential. We assume that the homogeneous problem with $g = 0$ admits the unique solution $u \equiv 0$ so that $\lambda = 0$ is not in the spectrum of $\Delta + q$. We assume that q on Ω is the restriction to Ω of a function $\tilde{q}$ compactly supported on $\mathbb{R}^n$ and such that $\tilde{q} \in H^{n/2+k+\varepsilon}(\mathbb{R}^n)$ with $\varepsilon > 0$ for $k \geq 1$. Moreover, we assume that the extension is chosen so that

$$\|q\|_{H^{n/2+k+\varepsilon}(\mathbb{R}^n)} \leq C\|\tilde{q}\|_{H^{n/2+k+\varepsilon}(\Omega)}, \tag{2.37}$$

for some constant $C = C(\Omega, k, n)$ independent of q. That such a constant exists may be found e.g. in [38], Chapter VI, Theorem 5.

We assume that $g \in C^{k,\alpha}(\partial\Omega)$ with $\alpha > \frac{1}{2}$ and $\partial\Omega$ is of class C^{k+1} so that (2.36) admits a unique solution $u \in C^{k+1}(\Omega)$, see [21], Theorem 6.19. The internal data are of the form

$$d(x) = \mu(x)u(x) \quad \text{in } \Omega, \tag{2.38}$$

Here $\mu \in C^{k+1}(\bar{\Omega})$ is bounded below and above by some positive constants.

The *inverse Schrödinger problem with internal data* (ISID) consists of reconstructing (q, μ) in Ω from knowledge of $d = (d_j)_{1 \leq j \leq J} \in (C^{k+1}(\Omega))^J$ and illuminations $g = (g_j)_{1 \leq j \leq J} \in (C^{k,\alpha}(\partial\Omega))^J$, where $J \in \mathbb{N}^*$ is the number of illuminations.

2.3.2   Complex geometrical optics solutions

In this section, we will construct *complex geometrical optics* (CGO) solutions to the Schrödinger equation

$$(\Delta + q)u = 0 \quad \text{in } \Omega. \tag{2.39}$$

The potential q is assumed to be in $L^\infty(\Omega)$. When $q = 0$, CGOs are harmonic solutions of the form $e^{\rho \cdot x}$ for $\rho \in \mathbb{C}^n$ such that $\rho \cdot \rho = 0$. When $q \neq 0$, the function $u(x) = e^{\rho \cdot x}$ is not an exact solution of (2.39) any more, but we can find solutions which resemble complex exponentials. These are the CGO solutions of the form

$$u(x) = e^{\rho \cdot x}(1 + r(x)). \tag{2.40}$$

Here $r(x)$ is a correction term which is needed to convert the approximate solution $e^{\rho \cdot x}$ to an exact solution. We are interested in the *asymptotic limit* as $|\rho| \to \infty$.

There are two ways to prove the existence of CGOs as in (2.40). Here we construct CGOs by proving the resolvability of $r(x)$. In Section 2.4.1, we present another proof using Carleman estimate.

We note that (2.40) is a solution of (2.39) if and only if that

$$e^{-\rho \cdot x}(\Delta + q)e^{\rho \cdot x}(1 + r) = 0. \tag{2.41}$$

Thus r is a weak solution of

$$(\Delta + 2\rho \cdot \nabla + q)r = -q. \tag{2.42}$$

The resolvability of (2.42), see [41], is the most important step in the construction of CGO solutions. We first consider the free case in which there is no potential on the left hand side of (2.42).

Theorem 2.3.1. *There is a constant C_0 depending only on Ω and n, such that for any $\rho \in \mathbb{C}^n$ satisfying $\rho \cdot \rho = 0$ and $|\rho| > 1$, and for any $f \in L^2(\Omega)$, the equation*

$$(-\Delta + 2\rho \cdot \nabla)r = f \quad in \ \Omega \tag{2.43}$$

has a unique solution $r \in H^1(\Omega)$ satisfying

$$\|r\|_{L^2(\Omega)} \ \leq \ \frac{C_0}{|\rho|}\|f\|_{L^2(\Omega)}, \tag{2.44}$$

$$\|\nabla r\|_{L^2(\Omega)} \ \leq \ C_0\|f\|_{L^2(\Omega)}. \tag{2.45}$$

The idea of the proof is that (2.43) is a linear equation with constant coefficients, so one can try to solve it by the Fourier transform. Since $(D_{x_k}u)^\wedge(\xi) = \xi_k \hat{u}(\xi)$, the Fourier transform equation is

$$(\xi^2 + 2\rho \cdot \xi)\hat{r}(\xi) = \hat{f}(\xi). \tag{2.46}$$

We would like to divide by $\xi^2 + 2\rho \cdot \xi$ and use the inverse Fourier transform to get a solutioin r. However, the symbol $\xi^2 + 2\rho \cdot \xi$ vanishes form some $\xi \in \mathbb{R}^n$, and the division cannot be done directly.

It turns out that we can divide by the symbol if we use Fourier series in a large cube instead of the Fourier transform, and moreover take the Fourier coefficients in a shifted lattice instead of the usual integer coordinate lattice.

Proof of Theorem 2.3.1. Write $\rho = \frac{1}{h}(\mathfrak{k} + i\mathfrak{k}^\perp)$, where $h = \sqrt{2}/|\rho|$, $|\mathfrak{k}| = |\mathfrak{k}^\perp| = 1$ and $\mathfrak{k} \cdot \mathfrak{k}^\perp = 0$. By rotating coordinates in a suitable way, we can assume that $\mathfrak{k} = e_1$ and $\mathfrak{k}^\perp = e_2$ (the first and second coordinate vectors). Thus we need to solve the equation

$$(D^2 + \frac{2}{h}(D_1 + iD_2))r = f. \tag{2.47}$$

We assume for simplicity that Ω is contained in the cube $Q = [-\pi, \pi]^n$. Extend f by zero outside Ω into Q, which gives a function in $L^2(Q)$ also denoted by f. We need to solve

$$(D^2 + \frac{2}{h}(D_1 + iD_2))r = f \quad \text{in } Q. \tag{2.48}$$

Let $w_k(x) = e^{i(k+\frac{1}{2}e_2)\cdot x}$ for $k \in \mathbb{Z}^n$. That is, we consider Fourier series in the lattice $\mathbb{Z}^n + \frac{1}{2}e_2$. Writing

$$(u, v) = \frac{1}{(2\pi)^n} \int_Q u\bar{v}dx, \quad u, v \in L^2(Q), \tag{2.49}$$

we see that $(w_k, w_l) = 0$ if $k \neq l$ and $(w_k, w_k) = 1$, so $\{w_k\}$ is an orthonormal set in $L^2(Q)$. It is also complete: if $v \in L^2(Q)$ and $(v, w_k) = 0$ for all $k \in \mathbb{Z}^n$ then $(ve^{-\frac{1}{2}ix_2}, w_k) = 0$ for all $k \in \mathbb{Z}^n$, which implies $v = 0$.

Hilbert space theory gives that f can be written as the series $f = \sum_{k \in \mathbb{Z}^n} f_k w_k$, where $f_k = (f, w_k)$ and $\|f\|_{L^2(Q)}^2 = f = \sum_{k \in \mathbb{Z}^n} |f_k|^2$. Seeking also r in the form $r = \sum_{k \in \mathbb{Z}^n} r_k w_k$, and using that

$$Dw_k = (k + \frac{1}{2}e_2)w_k,$$

the equation (2.48) results in

$$p_k r_k = f_k, \quad k \in \mathbb{Z}^n,$$

where

$$p_k := (k + \frac{1}{2}e_2)^2 + \frac{2}{h}(k_1 + i(k_2 + \frac{1}{2})).$$

Note that $\Im p_k = \frac{2}{h}(k_2 + \frac{1}{2})$ is never zero, which was the reason for considering the shifted lattice. We define

$$r_k := \frac{f_k}{p_k}$$

and

$$r = \sum_{k \in \mathbb{Z}^n} r_k w_k.$$

The last series converges in $L^2(Q)$ to a function $r \in L^2(Q)$ since

$$|r_k| = \frac{|f_k|}{|p_k|} \leq \frac{h}{|2(k_2 + \frac{1}{2})|}|f_k| \leq h|f_k|,$$

and then

$$\|r\|_{L^2(Q)} = \left(\sum_k |r_k|^2\right)^{1/2} \leq h\left(\sum_k |f_k|^2\right)^{1/2} = h\|f\|_{L^2(Q)}.$$

This shows the desired estimate in $L^2(Q)$.

It remains to show that $Dr \in L^2(Q)$ with correct bounds. We have

$$Dr = \sum_{k \in \mathbb{Z}^n} (k + \frac{1}{2}e_2)r_k w_k.$$

The derivative is justified since this is a convergent series in $L^2(Q)$: we claim

$$|(k + \frac{1}{2}e_2)r_k| \leq 4|f_k|, \quad k \in \mathbb{Z}^n, \tag{2.50}$$

which implies that $\|Dr\|_{L^2(Q)} \leq 4\|f\|_{L^2(Q)}$. To show (2.50) we consider two cases: if $|k + \frac{1}{2}e_2| \leq \frac{4}{h}$ we have

$$|(k + \frac{1}{2}e_2)r_k| \leq \frac{4h}{2h|k_2 + \frac{1}{2}|}f_k \leq 4|f_k|,$$

and if $|k + \frac{1}{2}e_2| > \frac{4}{h}$ then

$$||k + \frac{1}{2}e_2|^2 + 2\frac{k_1}{h}| \geq |k + \frac{1}{2}e_2|^2 - \frac{2}{h}|k + \frac{1}{2}e_2| \geq \frac{1}{2}|k + \frac{1}{2}e_2|^2,$$

which implies

$$|(k + \frac{1}{2}e_2)r_k| \leq \frac{|k + \frac{1}{2}e_2|}{\frac{1}{2}|k + \frac{1}{2}e_2|^2} \leq \frac{h}{2}|f_k|.$$

The statement is proved. $\qquad\square$

For further analysis on CGO solutions, we introduce the space L^2_δ for $\delta \in \mathbb{R}$ as the completion of $C^\infty_0(\mathbb{R}^2)$ with respect to the norm $\|\cdot\|_{L^2_\delta}$ given as

$$\|u\|_{L^2_\delta} = \left(\int_{\mathbb{R}^n} \langle x \rangle^{2\delta}|u|^2 dx\right)^{\frac{1}{2}}, \quad \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}. \tag{2.51}$$

We also introduce the spaces H^s_δ for $s > 0$ as the completion of $C^\infty_0(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{H^s_\delta}$ defined as

$$\|u\|_{H^s_\delta} = \left(\int_{\mathbb{R}^n} \langle x \rangle^{2\delta}|(I - \Delta)^{\frac{s}{2}}u|^2 dx\right)^{\frac{1}{2}}. \tag{2.52}$$

Here $(I - \Delta)^{\frac{s}{2}}$ is defined as the inverse Fourier transform of $\langle \xi \rangle^s \hat{u}(\xi)$, where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$.

We next consider the solution of (2.42) with nonzero potential.

Theorem 2.3.2. *Let* $-1 < \delta < 0$ *and* $k \in \mathbb{N}^*$. *Let* $q \in H_1^{\frac{n}{2}+k+\epsilon}$ *and hence in* $H_{\delta+1}^{\frac{n}{2}+k+\epsilon}$, *for* $\epsilon > 0$, *and* ρ *be such that*

$$\|q\|_{H_1^{\frac{n}{2}+k+\epsilon}} + 1 \leq C_0|\rho|. \tag{2.53}$$

Then r *the unique solution to* (2.42) *belongs to* $H_\delta^{\frac{n}{2}+k+\epsilon}$ *and*

$$|\rho|\|r\|_{H_\delta^{\frac{n}{2}+k+\epsilon}} \leq C\|q\|_{H_{\delta+1}^{\frac{n}{2}+k+\epsilon}}, \tag{2.54}$$

for a constant C *that depends on* δ *and* C_0.

Proof. From Theorem 2.3.1, we see that for $|\rho|$ large and $f \in L^2_{\delta+1}$ with $-1 < \delta < 0$, then equation (2.43) admits a unique weak solution $r \in L^2_\delta$ with

$$\|r\|_{L^2_\delta} \leq \frac{C}{|\rho|}\|f\|_{H^2_{\delta+1}}.$$

Now sine $-\Delta + 2\rho \cdot \nabla$ and $(I - \Delta)^s$ are constant coefficient operators and hence commute, we deduce that when $f \in H^s_{\delta+1}$ for any $s > 0$, then

$$\|r\|_{H^s_\delta} \leq \frac{C}{|\rho|}\|f\|_{H^s_{\delta+1}}. \tag{2.55}$$

The solution to (2.42) is known to admit the decomposition

$$r = \sum_{j=0}^{\infty} r_j, \quad -\Delta r_j + 2\rho \cdot \nabla r_j = -q r_{j-1},$$

with $r_{-1} = 1$. Let $s = \frac{n}{2} + k + \varepsilon$. Assume $q r_{j-1} \in H^s_{\delta+1}$ which is true for $j = 0$ by assumption on q. Then $r_j \in H^s_\delta$. Since H^s is algebra, we want to prove that

$$\|q r_j\|_{H^s_{\delta+1}} \leq \|q\|_{H^s_1}\|r_j\|_{H^s_\delta}. \tag{2.56}$$

Indeed, decompose $\mathbb{R}^n$ into cubes. On each cube Q, $\langle x \rangle^{2s}$ is more or less constant up to a $C^{\pm 2s}$. Now $H^s(Q)$ is an algebra so that $\|qu\|_{H^s(Q)} \leq \|q\|_{H^s(Q)}\|u\|_{H^s(Q)}$. Since $\langle x \rangle$ is more or less constant and equal to $\langle x_Q \rangle$,

$$\langle x_Q \rangle^{2\delta+2}\|qu\|_{H^s(Q)} \leq C\|\langle x \rangle(I - \Delta)^{\frac{s}{2}}q\|^2_{L^2(Q)}\|\langle x \rangle^\delta(I - \Delta)^{\frac{s}{2}}u\|^2_{L^2(Q)}.$$

It remains to sum over all the cubes Q to get the result. When the size of cubes tends to 0, the constant C tends to 1, which yield (2.56). This and (2.55) show that

$$\|r_j\|_{H^s_\delta} \leq \frac{C}{|\rho|}\|q\|_{H^s_1}\|r_{j-1}\|_{H^s_\delta}.$$

By selecting C_0 such that $C|\rho|^{-1}\|q\|_{H_1^s} \leq \frac{1}{2}$, we obtain

$$\|r_j\|_{H_\delta^s} \leq \frac{C}{2^j|\rho|}\|q\|_{H_{\delta+1}^s}.$$

Summation of the geometric series implies the result. $\qquad\square$

When restricted to Ω, we obtain the following estimate.

Corollary 2.3.3. *Let us assume the regularity hypotheses of the previous proposition. Then we find that*

$$|\rho|\|r\|_{H^{\frac{n}{2}+k+\epsilon}(\Omega)} + \|r\|_{H^{\frac{n}{2}+k+1+\epsilon}(\Omega)} \leq C\|q\|_{H^{\frac{n}{2}+k+\epsilon}(\Omega)}. \tag{2.57}$$

Proof. On the bounded domain Ω, $\langle x \rangle$ is bounded above and below by positive constants. Since q is compactly supported on $\mathbb{R}^n$ and (2.37), we obtain that

$$|\rho|\|r\|_{H^{\frac{n}{2}+k+\epsilon}(\Omega)} \leq C\|q\|_{H^{\frac{n}{2}+k+\epsilon}(\mathbb{R}^n)} \leq C(\Omega)\|q\|_{H^{\frac{n}{2}+k+\epsilon}(\Omega)}. \tag{2.58}$$

Now we have

$$-\Delta r = -2\rho \cdot \nabla r - q(1+r).$$

By elliptic regularity, with Ω_0 a smooth domain in $\mathbb{R}^n$ such that $\bar{\Omega} \subset \Omega_0$, we find for all $s = \frac{n}{2} + k + \varepsilon$, that

$$\|r\|_{H^{s+1}(\Omega)} \leq C\|2\rho \cdot \nabla r - q(1+r)\|_{H^{s-1}(\Omega_0)} + \|r\|_{H^s(\Omega_0)}.$$

The latter is bounded by $|\rho|\|r\|_{H^s(\Omega_0)} + \|q\|_{H^{s-1}(\Omega_0)}\|r\|_{H^{s-1}(\Omega_0)}$ since $s - 1 > \frac{n}{2}$ so that $H^{s-1}(\Omega_0)$ is a Banach algebra. By using the above bound on $\|r\|_{H^s(\Omega_0)}$, with $C(\Omega)$ replace by the larger $C(\Omega_0)$, we get the result. $\qquad\square$

By Sobolev embedding, we have that

Proposition 2.3.4. *Under the hypotheses of Corollary 2.3.3, the restriction to Ω of the CGO solution verifies that*

$$|\rho|\|r\|_{C^k(\bar{\Omega})} + \|r\|_{C^{k+1}(\bar{\Omega})} \leq C\|q\|_{H^{\frac{n}{2}+k+\epsilon}(\Omega)}. \tag{2.59}$$

2.3.3 Uniqueness

For this section, we choose $j = 1, 2$. We assume that we can impose the complex-valued illuminations $g_j \in C^{k,\alpha}(\partial\Omega; \mathbb{C})$ on $\partial\Omega$ and observe the complex-valued internal data d_j. Note that, to make up complex-valued g_j and d_j, we need two real observations, since $g_1 = \bar{g}_2$. Let d_j be of the form $d_j = \mu u_j$ in Ω, where u_j is the solutions of

$$\Delta u_j + q u_j = 0 \ \text{ in } \Omega \qquad u_j = g_j \ \text{ on } \partial\Omega. \tag{2.60}$$

Direct calculation gives us that

$$u_1 \Delta u_2 - u_2 \Delta u_1 = 0.$$

We assume that $\mu \in C^{k+1}(\bar{\Omega})$ is bounded above and below by positive constants. By substituting $u_j = d_j/\mu$, we obtain that

$$2(d_1 \nabla d_2 - d_2 \nabla d_1) \cdot \nabla \mu - (d_1 \Delta d_2 - d_2 \Delta d_1)\mu = 0,$$

or equivalently,

$$\beta_d \cdot \nabla \mu + \gamma_d \mu = 0, \tag{2.61}$$

where

$$\begin{aligned}
\beta_d &:= \chi(x)(d_1 \nabla d_2 - d_2 \nabla d_1) \\
\gamma_d &:= -\frac{1}{2}\chi(x)(d_1 \Delta d_2 - d_2 \Delta d_1) = \frac{-\beta \cdot \nabla \mu}{\mu}.
\end{aligned} \tag{2.62}$$

Here, $\chi(x)$ is any smooth known complex-valued function with $|\chi(x)|$ uniformly bounded below by a positive constant on $\bar{\Omega}$. Note that by assumption on μ, we have that $\beta_d \in (C^k(\bar{\Omega}; \mathbb{C}))^n$ and $\gamma \in C^k(\bar{\Omega}; \mathbb{C})$.

The methodology for the reconstruction of (μ, q) is therefore as follows: we first reconstruct μ using the real part or the imaginary part of (2.61) and the boundary condition $\mu = d/g$ on $\partial\Omega$. Then we may recover $u_j = d_j/\mu$, explicitly and therefore q from the Schrödinger equation (2.60).

The reconstruction of μ is unique as long as the integral curves of (the real part or the imaginary part of) β_d join any interior point $x \in \Omega$ to a point on $\partial\Omega$, denoted as $x_0(x)$. This property of β is not guaranteed in general, unless the boundary conditions g_j are properly

chosen. CGO solutions are then employed to construct a family of boundary conditions g_j, which could produce well-performed vector fields β_d.

To make a distinction from the observed data, we denote the CGO solutions by $\check{u}_1$ and $\check{u}_2$, and denote the internal data and the vector field constructed from CGO solutions by $\check{d}$ and $\check{\beta}_d$, respectively.

Let us consider two CGOs $\check{u}_j$ with parameters ρ_j. Let $\check{d}_j$ be the complex-valued corresponding internal data. Let us choose $\chi(x) = e^{-(\rho_1+\rho_2)\cdot x}$ in (2.62). Then we find after some algebra that $\check{\beta}_d$ in (2.62) is given by

$$\check{\beta}_d = \mu^2 \left((\rho_1 - \rho_2)(1 + r_1)(1 + r_2) + \nabla r_2(1 + r_1) - \nabla r_1(1 + r_2)\right). \tag{2.63}$$

We may then define

$$\check{\beta} := \frac{1}{2|\rho_1|}\Im\check{\beta}_d = \mu^2 \frac{\Im(\rho_1 - \rho_2)}{2|\rho_1|} + \mu^2\Im\varpi, \quad \check{\gamma} = \frac{1}{2|\rho_1|}\Im\check{\gamma}_d, \tag{2.64}$$

where $\Im\check{\beta}_d$ is the imaginary part of $\check{\beta}_d$ and

$$\varpi = \frac{\rho_1 - \rho_2}{2|\rho_1|}(r_1 + r_2 + r_1 r_2) + \frac{\nabla r_2(1 + r_1) - \nabla r_1(1 + r_2)}{2|\rho_1|}. \tag{2.65}$$

We know from Proposition 2.3.4 that $|\rho_j||r_j|$ and ∇r_j are bounded in $C^k(\bar{\Omega})$. Thus,

$$\left\|\check{\beta} - \mu^2 \frac{\Im(\rho_1 - \rho_2)}{2|\rho_1|}\right\|_{C^k(\bar{\Omega})} = \|\mu^2\Im\varpi\|_{C^k(\bar{\Omega})} \leq \frac{C_0}{|\rho_1|}, \tag{2.66}$$

for some constant C_0 independent of ρ_j.

By choosing $\rho = \rho_1 = \bar{\rho}_2 = \frac{1}{h}(\mathfrak{k} + i\mathfrak{k}^\perp)$ with $h = \sqrt{2}/|\rho_1|$, $|\mathfrak{k}| = |\mathfrak{k}^\perp| = 1$ and $\mathfrak{k} \cdot \mathfrak{k}^\perp = 0$, we have $\overline{e^{\rho\cdot x}} = e^{\bar{\rho}\cdot x}$ and $\bar{r}_1 = r_2$ by uniqueness of the solution to the equation satisfied by r_j. Then (2.66) is simplified as

$$\left\|\check{\beta} - \mu^2\mathfrak{k}^\perp\right\|_{C^k(\bar{\Omega})} \leq hC_0. \tag{2.67}$$

When h is sufficiently small, we obtain that every point in $x \in \Omega$ is connected to a point on $\partial\Omega$ by an integral curve of $\check{\beta}$. Thus, Eq. (2.61) admits a unique solution.

We next define boundary conditions $g_j \in C^{k,\alpha}(\partial\Omega;\mathbb{C})$, $\alpha > 1/2$, such that

$$\|g_j - \check{u}_j|_{\partial\Omega}\|_{C^{k,\alpha}(\partial\Omega;\mathbb{C})} \leq \epsilon, \tag{2.68}$$

for some $\epsilon > 0$ sufficiently small. Let u_j be the solution of (2.60) with boundary illumination g_j as in (2.68). By elliptic regularity, we thus have

$$\|u_j - \check{u}_j\|_{C^{k+1}(\bar{\Omega};\mathbb{C})} \leq C_0 \epsilon, \tag{2.69}$$

form some positive constant C_0. Define the complex-valued internal data $d_j = \mu u_j$. Since $\mu \in C^{k+1}(\bar{\Omega})$, we thus have

$$\|d_j - \check{d}_j\|_{C^{k+1}(\bar{\Omega};\mathbb{C})} \leq C_1 \epsilon, \tag{2.70}$$

for $C_1 > 0$. Define the vector field β as in (2.62) and (2.64). We then deduce from (2.67) and (2.70) that

$$\left\| \beta - \mu^2 \mathfrak{k}^\perp \right\|_{C^k(\bar{\Omega})} \leq C(1 + \epsilon)h, \tag{2.71}$$

for some $C > 0$. As a consequence, as soon as h and ϵ are sufficiently small, we obtain that any point $x \in \Omega$ is mapped to a point on $\partial\Omega$ in a finite time by an integral curve of β.

Moreover, we have the equation with real-valued coefficients:

$$\beta \cdot \nabla\mu + \gamma\mu = 0. \tag{2.72}$$

Since $\mu = d/g$ is known on $\partial\Omega$, this equation provides a unique reconstruction for μ.

Let us define the set of parameters

$$\mathcal{P} = \Big\{ (\mu, q) \in C^{k+1}(\bar{\Omega}) \times H^{\frac{n}{2}+k+\varepsilon}(\Omega) : \ 0 \text{ is not an eigenvalue of} \tag{2.73}$$

$$\Delta + q, \ \|u\|_{C^{k+1}(\bar{\Omega})} + \|q\|_{H^{\frac{n}{2}+k+\varepsilon}} \leq P < \infty \Big\}. \tag{2.74}$$

The above construction of the vector field allows us to obtain the following uniqueness result.

Theorem 2.3.5. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2. Let (μ, q) and $(\tilde{\mu}, \tilde{q})$ be two elements in $\mathcal{P}$. Let d and $\tilde{d}$ be the internal data for the coefficients (μ, q) and $(\tilde{\mu}, \tilde{q})$, respectively and with boundary conditions $g_j, j = 1, 2$. Then there is an open set of illuminations $g \in (C^{k,\alpha}(\partial\Omega))^2$ for some $\alpha > \frac{q}{2}$ such that $d = \tilde{d}$ implies that $(\mu, q) = (\tilde{\mu}, \tilde{q})$.*

Proof. Since the two measurements $d = \tilde{d}$, we have that μ and $\tilde{\mu}$ solve the same equation (2.72). Since $\mu = \tilde{\mu} = d/g$ on $\partial\Omega$, we deduce that $\mu = \tilde{\mu}$ since the integral curves of β map any point $x \in \Omega$ to the boundary $\partial\Omega$. More precisely, consider the flow $\theta_x(t)$ associated to β, i.e., the solution to

$$\dot{\theta}_x(t) = \beta(\theta_x(t)), \quad \theta_x(0) = x \in \bar{\Omega}. \tag{2.75}$$

By the Picard-lindelöf theorem, the above equations admit unique solutions since β is of class C^1. And by (2.71) for h sufficiently small, any point $x \in \bar{\Omega}$ is mapped to two points on $\partial\Omega$ by the flow $\theta_x(t)$ in finite time. For $x \in \bar{\Omega}$, let us define $x_\pm(x) \in \partial\Omega$ and $\pm t_\pm(x) > 0$ such that

$$\theta_x(t_\pm(x)) = x_\pm(x) \in \partial\Omega. \tag{2.76}$$

Then by the method of characteristics, the solution of (2.72) is given by

$$\mu(x) = \mu_0(x_\pm(x))e^{-\int_0^{t_\pm(x)} \gamma(\theta_x(s)ds)}. \tag{2.77}$$

The solution $\tilde{\mu}(x)$ is given by the same formula since $\theta_x(t) = \tilde{\theta}_x(t)$ so that $\mu = \tilde{\mu}$. This implies that $u = \tilde{u}$ since $d = \tilde{d}$. Moreover, since CGO solution $\check{u} \neq 0$ and boundary illumination g is properly chosen, we have $u \neq 0$ by (2.69). Then q is uniquely determined in $\bar{\Omega}$. $\square$

Applying the inversion of Liouville change of variable, we can prove the uniqueness result of the *inverse diffusion problem with internal data*(IDID).

We define the set of coefficients (D, σ_a) as

$$\mathcal{M} = \left\{ (D, \sigma_a) : (\sqrt{D}, \sigma_a) \in Y \times C^{k+1}(\bar{\Omega}), \|\sqrt{D}\|_Y + \|\sigma_a\|_{C^{k+1}(\bar{\Omega})} \leq M \right\}, \tag{2.78}$$

where $Y = H^{\frac{n}{2}+k+2+\varepsilon}(\bar{\Omega})$ and M is fixed.

Theorem 2.3.6. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D|_{\partial\Omega} = \tilde{D}|_{\partial\Omega}$. Let d and $\tilde{d}$ be the internal data for the coefficients (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$, respectively and with boundary conditions $g_j, j = 1, 2$. Then there is an open set of illuminations $g \in (C^{k,\alpha}(\partial\Omega))^2$ for some $\alpha > 1/2$ such that if $d = \tilde{d}$, then $(D, \sigma_a) = (\tilde{D}, \tilde{\sigma}_a)$.*

Proof. From Theorem 2.3.5, we can uniquely construct (μ, q). When we assume that we know $\sqrt{D}$ on $\partial\Omega$, (2.35) admits a unique solution for $\sqrt{D}$ and thus a unique $\sigma_a = \mu\sqrt{D}$. $\square$

2.3.4 Stability result for 2 real observations

We next consider the stability of the proposed reconstruction method. We divide the proof into two cases. When two complex-valued data are measured, strict convexity is assumed on the domain of interest. In another case, $2n$ complexed-valued data are measured and the stability result follows without any geometric constrain on the domain.

We can define the front and back side of the boundary by

$$\partial\Omega_\pm = \{x_0 \in \partial\Omega : \pm\nu(x_0) \cdot \mathfrak{k}^\perp > 0\}, \tag{2.79}$$

where $\nu(x_0)$ is the exterior unit norm of $\partial\Omega$ at x_0. Let $\theta_x(t)$ be the flow associated to vector fields β. Thus, $\theta_x(t)$ will map every internal point $x \in \Omega$ to two boundary points, denoted as $x_\pm(x) = \theta_x(t_\pm(x)) \in \partial\Omega_\pm$, while $t_\pm(x)$ are the time parameters. We can define $\tilde\theta_x(t)$, $\tilde x_\pm$ and $\tilde t_\pm(x)$ according to $\tilde\beta$ in the similar way.

From the equality

$$\theta_x(t) - \tilde\theta_x(t) = \int_0^t [\beta(\theta_x(s)) - \tilde\beta(\tilde\theta_x(s))]ds, \tag{2.80}$$

and using the Lipschitz continuity of β and Gronwall's lemma, we deduce the existence of a constant C such that

$$|\theta_x(t) - \tilde\theta_x(t)| \le Ct\|\beta - \tilde\beta\|_{C^0(\bar\Omega)}, \tag{2.81}$$

when $\theta_x(t)$ and $\tilde\theta_x(t)$ are in $\bar\Omega$. The inequality (3.99) is uniform in t as all characteristics exit $\bar\Omega$ in finite time.

To see higher order estimates, we define $W := D_x\theta_x(t)$, which solves the equation, $\dot W = D_x\beta(\theta_x)W$, with $W(0) = I$. Define $\tilde W$ similarly. By using Gronwall's lemma again, we deduce that

$$|W - \tilde W| \le Ct\|D_x\beta - D_x\tilde\beta\|_{C^0(\bar\Omega)} \tag{2.82}$$

when $\theta_x(t)$ and $\tilde\theta_x(t)$ are in $\bar\Omega$. Since β and $\tilde\beta$ are of class $C^k(\bar\Omega)$, then we obtain iteratively that

$$|D_x^{k-1}\theta_x(t) - D_x^{k-1}\tilde\theta_x(t)| \le Ct\|D_x\beta - D_x\tilde\beta\|_{C^{k-1}(\bar\Omega)}, \tag{2.83}$$

when $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$.

We define that $\Omega_1 \subset \Omega$ by removing a neighborhood of each tangent point of $\partial\Omega$ with respect to $\mathfrak{k}^\perp$. Precisely, for some small $\eta > 0$,

$$\Omega_1 = \Omega \backslash \{x \in \Omega : |\nu(x_\pm(x)) \cdot \mathfrak{k}^\perp| < \eta\}, \tag{2.84}$$

where $\nu(x_\pm(x))$ is the exterior norm vector of $\partial\Omega$ at $x_\pm(x)$.

We first need to consider the following lemma.

Lemma 2.3.7. *Let Ω be an open bounded and convex subset in $\mathbb{R}^n$ with C^k boundary and Ω_1 be defined as in (2.84). Let $k \geq 2$ and assume β and $\tilde{\beta}$ are $C^k(\bar{\Omega})$ vector fields which satisfy (2.91). Let $x \in \Omega_1$, we have that*

$$\|x_-(x) - \tilde{x}_-(x)\|_{C^{k-1}(\Omega_1)} + \|t_-(x) - \tilde{t}_-(x)\|_{C^{k-1}(\Omega_1)} \leq C\|\beta - \tilde{\beta}\|_{C^{k-1}(\Omega_1)}, \tag{2.85}$$

where C is a constant depending on Ω.

Proof. (**Chen and Yang[11]**) Without loss of generality, we assume $t_+(x) \leq \tilde{t}_+(x)$. We also denote that $A := \theta_x(t_+(x)) \in \partial\Omega$, $B := \tilde{\theta}_x(\tilde{t}_+(x)) \in \partial\Omega$ and $C := \tilde{\theta}_x(t_+(x)) \in \Omega$.

We first want to show that the angle $\angle AxB$ is controlled by

$$\angle AxB \leq C_1\|\beta - \tilde{\beta}\|_{C^k(\Omega_1)} + C_2 h, \tag{2.86}$$

for some C_1, C_2. Indeed, by applying (2.81) and sine theorem, we can see that $\angle AxC$ is bounded by $C_1\|\beta - \tilde{\beta}\|_{C^k(\Omega_1)}$. Also notice that $\tilde{\beta}$ satisfies (2.91). Therefore, similar argument shows that, for any t_1, t_2, the angle between the vector from x to $\tilde{\theta}_x(t_1)$ and the vector from x to $\tilde{\theta}_x(t_2)$ is bounded by $C_2 h$. Thus $\angle CxB \leq C_2 h$. This proves (2.86).

By the definition of Ω_1, a neighborhood of the boundary point at which the tangent plane of $\partial\Omega$ is parallel to ζ_0 is removed. therefore, there exists a constant value $\eta > 0$ depending only on Ω_1 such that, for any $x \in \Omega_1$, $\eta_1 \geq \eta$, where η_1 is the angle between the vector $\overline{xA}$ and the tangle plane of $\partial\Omega$ at A, as in Fig. 2.1. Then by (2.86), when $\|\beta - \tilde{\beta}\|_{C^k(\Omega_1)}$ and h are so small that $\eta_0 := \eta - C_1\|\beta - \tilde{\beta}\|_{C^k(\Omega_1)} - C_2 h > 0$, the extension of $\overline{xB}$ will intersect the tangent plane of $\partial\Omega$ at A, with intersection point D. Then it is easy to check that

$$\angle ABC > \angle ADx = \eta_1 - \angle AxB > \eta_0 > 0. \tag{2.87}$$
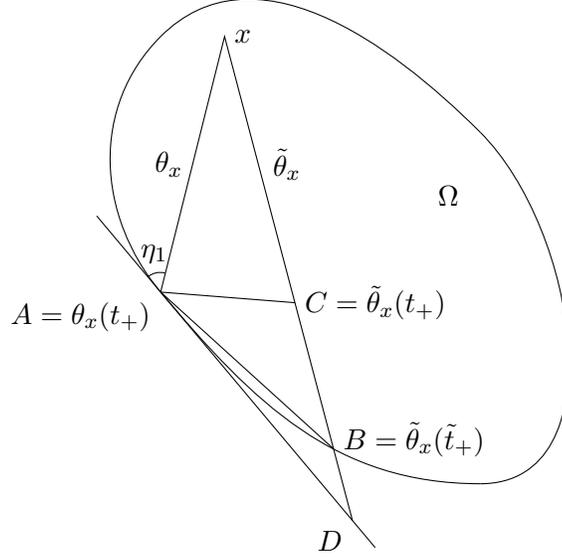
Figure 2.1: Vector fields β and $\tilde{\beta}$

The sine theorem gives that

$$|AB| = \frac{|AC|}{\sin(\angle ABC)} \sin(\angle ACB). \tag{2.88}$$

(2.81) directly implies $|AB| = |x_+ - \tilde{x}_+| \le C' \|\beta - \tilde{\beta}\|_{C^k(\Omega_1)}$.

Since $\beta, \tilde{\beta} \in C^k(\Omega)$ and $\partial\Omega$ is of class C^k, it is clear that $\angle ABC$ and $\angle ACB$ are C^k functions with respect to $x \in \Omega$. By differentiating (2.88) and applying (3.101), we get higher order estimates

$$\|x_+ - \tilde{x}_+\|_{C^{k-1}(\Omega_1)} \le C'' \|\beta - \tilde{\beta}\|_{C^k(\Omega_1)}. \tag{2.89}$$

To see the second part in (2.85), we have that

$$|CB| = \int_{t_+(x)}^{\tilde{t}_+(x)} \tilde{\beta}(\tilde{\theta}_x(d)) ds = |\tilde{\beta}(\tilde{\theta}_x(\tau))|(\tilde{t}_+(x) - t_+(x)), \tag{2.90}$$

for $t_+(x) \le \tau \le \tilde{t}_+(x)$. Similar argument shows the estimate of $t_+ - \tilde{t}_+$ in (2.85). $\qquad\square$

Proposition 2.3.8. *Let Ω be an open bounded and convex subset in $\mathbb{R}^n$ with C^k boundary and Ω_1 be defined as in (2.84). Let $k \ge 1$. Let μ and $\tilde{\mu}$ be solution of (2.61) corresponding*

to coefficients (β, γ) and $\tilde{\beta}, \tilde{\gamma}$, respectively, where

$$\left\|\beta - \mu^2 \mathfrak{k}^\perp\right\|_{C^k(\bar{\Omega})} \leq Ch \quad \text{and} \quad \left\|\tilde{\beta} - \mu^2 \mathfrak{k}^\perp\right\|_{C^k(\bar{\Omega})} \leq Ch. \tag{2.91}$$

Let us assume that $\mu|_{\partial\Omega} = \mu_0$ and $\tilde{\mu}|_{\partial\Omega} = \tilde{\mu}_0$ for $\mu_0, \tilde{\mu}_0 \in C^k(\partial\Omega)$. We also assume that h is sufficiently small. Then there is a constant C_1 such that

$$\|\mu - \tilde{\mu}\|_{C^{k-1}(\bar{\Omega})} \leq C_1 \|\mu_0\|_{C^k(\partial\Omega)} (\|\beta - \tilde{\beta}\|_{C^{k-1}(\Omega_1)} + \|\gamma - \tilde{\gamma}\|_{C^{k-1}(\Omega_1)}) \tag{2.92}$$
$$+ C_1 \|\mu_0 - \tilde{\mu}_0\|_{C^k(\partial\Omega)}.$$

Proof. We recall that $\theta_x(t)$ is the flow defined in (2.75) and that $x_\pm(x)$ and $t_\pm(x)$ are defined in (2.76). By the method of characteristics, $\mu(x)$ is determined explicitly as

$$\mu(x) = \mu_0(x_\pm(x))e^{-\int_0^{t_\pm(x)} \gamma(\theta_x(s)ds)}. \tag{2.93}$$

The solution $\tilde{\mu}(x)$ has a similar expression.

$$|\mu(x) - \tilde{\mu}(x)| \leq \left|(\mu_0(x_+(x)) - \tilde{\mu}_0(\tilde{x}_+(x)))e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds}\right| +$$
$$\left|\tilde{\mu}_0(\tilde{x}_+(x)) \left(e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds} - e^{-\int_0^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\theta}_x(s))ds}\right)\right|$$

Applying Lemma 2.3.7, we deduce that

$$|D_x^{k-1}[\mu_0(x_+(x)) - \tilde{\mu}_0(\tilde{x}_+(x))]| \leq \|\mu_0 - \tilde{\mu}_0\|_{C^{k-1}(\partial\Omega)}$$
$$+ C\|\mu_0\|_{C^{k-1}(\partial\Omega_+)} \|\beta - \tilde{\beta}\|_{C^{k-1}(\tilde{\Omega})}.$$

This proves the $\mu_0(x_+(x))$ is stable. To consider the second term, by the Leibniz rule it is sufficient to prove the stability result for $\int_0^{t_+(x)} \gamma(\theta_x(s))ds$.

Assume without loss of generality that $t_+(x) < \tilde{t}_+(x)$. Then we have, applying (2.81),

$$\int_0^{t_+(x)} [\gamma(\theta_x(s)) - \tilde{\gamma}(\tilde{\theta}_x(s))]ds = \int_0^{t_+(x)} [(\gamma(\theta_x(s)) - \gamma(\tilde{\theta}_x(s))) + (\gamma - \tilde{\gamma})(\tilde{\theta}_x(s))]ds$$
$$\leq C\|\gamma\|_{C^0(\tilde{\Omega})} \|\beta - \tilde{\beta}\|_{C^0(\tilde{\Omega})} + C\|\gamma - \tilde{\gamma}\|_{C^0(\tilde{\Omega})}.$$

Derivatives of order $k-1$ of the above expression are uniformly bounded since $t_+(x) \in C^{k-1}(\tilde{\Omega})$, γ has C^k derivatives bounded on $\tilde{\Omega}$ and $\theta_x(t)$ is stable as in (2.83).

It remains to handle the term

$$v(x) = \int_{t_+(x)}^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\theta}_x(s))ds.$$

$\tilde{\beta}$ and $\tilde{\gamma}$ are of class $C^k(\Omega)$, then so is the function $x \to \tilde{\gamma}(\tilde{\theta}(s))$. Derivatives of order $k-1$ of $v(x)$ involve terms of size $\tilde{t}_+(x) - t_+(x)$ and terms of form

$$D_x^m\big(\tilde{t}_+ D_x^{k-1-m}\tilde{\gamma}(\tilde{\theta}_x(\tilde{t}_+)) - t_+ D_x^{k-1-m}\tilde{\gamma}(\tilde{\theta}_x(t_+))\big), \quad 0 \le m \le k-1.$$

Since the function has $k-1$ derivatives that are Lipschitz continuous, we thus have

$$|D_x^{k-1}v(x)| \le C\|\tilde{t}_+ - t_+\|_{C^{k-1}(\Omega)}.$$

The rest of the proof follows Lemma 2.71. $\qquad\square$

We are now in a position to state our main stability result.

Theorem 2.3.9. *Let us assume that (μ, q) and $\tilde{\mu}, \tilde{q}$ are elements in $\mathcal{P}$ and that $\|g - \check{u}|_{\partial\Omega}\|_{C^0(\partial\Omega)}$ is sufficiently small so that u does not vanish in Ω. Under the assumption of Proposition 2.3.8 and assuming h is sufficiently small, we have that*

$$\|\mu - \tilde{\mu}\|_{C^{k-1}(\Omega_1)} \le C\|d - \tilde{d}\|_{(C^k(\Omega_1))^2}. \tag{2.94}$$

Moreover, we have the following stability result provided that $k \ge 3$:

$$\|q - \tilde{q}\|_{C^{k-3}(\Omega_1)} \le C\|d - \tilde{d}\|_{(C^k(\Omega_1))^2}. \tag{2.95}$$

Proof. Let us define $\mu_0 = d|_{\partial\Omega}/g$ and $\tilde{\mu}_0 = d|_{\partial\Omega}/g$. By assumption, g does not vanish on $\partial\Omega$. We thus deduce that $\|\mu_0 - \tilde{\mu}_0\|_{C^k(\partial\Omega)}$ is controlled by $\|d - \tilde{d}\|_{(C^k(\Omega_1))^2}$. The rest of the proof is direct consequence of Theorem 2.3.8. $\qquad\square$

By applying inverse Liouville change of variable, we can get the stability result for ISID.

Theorem 2.3.10. *Let $k \ge 3$. Let Ω be a bounded, open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^{k+1}. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D|_{\partial\Omega} = \tilde{D}|_{\partial\Omega}$. Let d and $\tilde{d}$ be the internal data for the coefficients (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$, respectively and with boundary conditions $g_j, j = 1, 2$. Then there is an open set of illuminations $g \in (C^{k,\alpha}(\partial\Omega))^2$ for some $\alpha > 1/2$ and a constant C such that*

$$\|D - \tilde{D}\|_{C^{k-1}(\Omega_1)} + \sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(\Omega_1)} \le C\|d - \tilde{d}\|_{(C^k(\Omega_1))^2}. \tag{2.96}$$

Proof. The main result consists of getting the stability on D mentioned above. Since $k \geq 3$, we have stability of the reconstruction of $q \in C^{k-3}(\tilde{\Omega})$ and $\mu \in C^{k-1}(\tilde{\Omega})$ provided that the boundary conditions are well-chosen. The we have

$$-(\Delta + q)(\sqrt{D} - \sqrt{\tilde{D}}) = \mu - \tilde{\mu} + (q - \tilde{q})\sqrt{\tilde{D}}.$$

By elliptic regularity, we deduce that $(\sqrt{D} - \sqrt{\tilde{D}})$ is bounded in $C^{k-1}(\tilde{\Omega})$, and hence the result. $\qquad\square$

2.3.5 *Stability result for 2n real observations*

Let us now consider the setting in which we can access $2n$ real-valued internal data viewed as n complex-valued internal data (since the measurements are linear in u, we can measure the real and imaginary parts separately).

Let us define the complex vectors

$$\rho_1 = \frac{1}{h}(-e_1 - ie_2) = -\rho_2, \quad \text{and} \quad \rho_j = \frac{1}{h}(e_1 + ie_j), \quad 2 \leq j \neq n \tag{2.97}$$

Let $\check{u}_j$ be the corresponding CGOs. We choose boundary conditions g_j such that

$$\|g - \check{u}_j|_{\partial\Omega}\|_{C^{k,\alpha}(\partial\Omega)} \leq \epsilon, \tag{2.98}$$

for ϵ sufficiently small. We define u_j as the solution to (2.60) with boundary conditions g_j. These are n complex-valued solutions whose real and imaginary parts consist of $2n$ real-valued solutions. For $1 \leq j \leq n$, we define $d_j = \mu u_j$. We now construct the n vector field β_j. For $2 \leq j \leq n$, the real-valued vector fields and scalr terms are constructed as in the preceding section; for $j = 1$, the vector field is constructed by using $\rho_2 = -\rho - 1$:

$$\beta_1 = \frac{h}{2}\Re(d_2\nabla d_1 - d_1\nabla d_2), \qquad \gamma_1 = \frac{h}{4}\Re(d_2\Delta d_1 - d_1\Delta d_2),$$

$$\beta_j = \frac{he^{-2e_1\cdot x/h}}{2}\Im(d_j\nabla\bar{d}_j - \bar{d}_j\nabla d_j), \quad \gamma_j = \frac{he^{-2e_1\cdot x/h}}{4}\Im(d_j\Delta\bar{d}_j - \bar{d}_j\Delta d_j), \tag{2.99}$$

for $2 \leq j \leq n$. As in the preceding section, we verify that

$$\|\beta_j - \mu^2 e_j\|_{C^k(\bar{\Omega})} \leq Ch(1 + \epsilon). \tag{2.100}$$

For h sufficiently small and assuming that μ is bounded below and above by positive constants, we obtain that as each point $x \in \Omega$, the vector $\beta_j(x)$ form a basis. Moreover, the matrix a_{ij} such that $\beta_j = \sum a_{jk} e_k$ is an invertible matrix with inverse of class $C^k(\bar{\Omega})$. In other words, we have constructed a vector-valued function $\Lambda(x) \in (C^k(\bar{\Omega}))^n$ such that (2.60) may be recast as

$$\nabla \mu + \Lambda(x)\mu = 0. \tag{2.101}$$

Finally, the construction of Λ is stable under small perturbation in data d_j. Indeed, invertibility of a_{jk} is ensured for vector fields close to β_j. Let Λ and $\tilde{\Lambda}$ be two vector fields constructed from knowledge of two sets of internal data $d = \{d_j, 1 \leq j \leq n\}$ and $\tilde{d} = \{\tilde{d}_j, 1 \leq j \leq n\}$. Then we find that

$$\|\Lambda - \tilde{\Lambda}\|_{(C^k(\bar{\Omega}))^n} \leq C\|d - \tilde{d}\|_{(C^{k+1}(\bar{\Omega};\mathbb{C}))^n}, \tag{2.102}$$

provided the right-hand side is sufficiently small.

Let us now assume that Ω is connected and $\mu = \mu_0 = d/g$ is known form some point $x_0 \in \partial\Omega$. In other words, we want to solve the over-determined problem

$$\nabla \mu + \Lambda(x)\mu = 0 \quad \text{in } \Omega, \qquad \mu(x_0) = \mu_0(x_0) \quad x_0 \in \partial\Omega. \tag{2.103}$$

Let $x \in \Omega$ be an arbitrary point and assume that Ω is bounded and connected and $\partial\Omega$ is smooth. Then we find a smooth curve that links x to the point $x_0 \in \partial\Omega$. Restricted to this curve, (2.103) becomes a stable ordinary differential equation. The solution of the ordinary differential equation is then stable with respect to modification in Λ (the curve between x and x_0 is kept constant). The solution μ then clearly inherits the smoothness of $\Lambda(x)$ directly from (2.103). Moreover, since $\mu_0(x_0) - \tilde{\mu}_0(x_0)$ (with $\tilde{\mu}_0 = \tilde{d}/\tilde{g}$ on $\partial\Omega$) is small and equation (2.102) is stable with respect to changes in the value of $\mu_0(x_0)$, we deduce that the reconstruction of μ is stable with respect to perturbations in d.

We may thus state the main result of this section:

Theorem 2.3.11. *Let $k \geq 2$. We assume that we have access to n well-chosen complex-valued measurements and the (μ, q) and $(\tilde{\mu}, \tilde{q})$ are elements in $\mathcal{P}$. Under the hypotheses above, and provided that $\|g_j - \check{u}_j\|_{C^{k,\alpha}(\bar{\Omega};\mathbb{C})}$ is sufficiently small, then we have the following*

stability result:

$$\|\mu - \tilde{\mu}\|_{C^k(\bar{\Omega})} + \|q - \tilde{q}\|_{C^{k-2}(\bar{\Omega})} \leq C\|d - \tilde{d}\|_{(C^{k+1}(\bar{\Omega}))^{2n}}. \qquad (2.104)$$

Proof. The inequality for $\mu - \tilde{\mu}$ is a direct consequence of the results proved above. This provides a stability result for μ^{-1} and for $u_j = d_j/\mu$ from the data d_j. We thus have a stability result for $\Delta u_j = -u_j q$ and hence the above stability result for $u_j(q - \tilde{q})$ since $(u_j - \tilde{u}_j \tilde{q})$ is small.

Now, $\check{u} = e^{\rho \cdot x}(1 + r)$ does not vanish on Ω when $|\rho|$ is sufficiently large since $|\rho|r$ is bounded. When the boundary condition $g_j - \check{u}_j|\partial\omega$ is small, then by the maximum principle, u_j does not vanish on Ω either. This means that either its real part or its imaginary part does not vanish everywhere in Ω. This provides control of $q - \tilde{q}$ in Ω as given in (2.104). $\qquad \square$

Theorem 2.3.12. *Let $k \geq 3$ and assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D|_{\partial\Omega} = \tilde{D}|_{\partial\Omega}$. Then there is an open set of $2n$ real-valued boundary conditions $g \in C^{k,\alpha}(\partial\Omega)$ for $\alpha > 1/2$ such that, we have the stability estimate in Ω_1 as*

$$\|D - \tilde{D}\|_{C^{k-1}(\Omega_1)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(\Omega_1)} \leq C\|d - \tilde{d}\|_{(C^k(\Omega_1))^2}. \qquad (2.105)$$

Proof. The proof is the same as that of Theorem 2.3.10. $\qquad \square$

2.4 Quantitative photo-acoustic tomography with partial data

We now consider QPAT with illuminations only on part of the domain boundary. Recall that in Section 2.3.1, diffusion equation (2.31) is transformed to Schrödinger equation (2.36) by the standard Liouville change of variable.

We assume that the boundary illumination $g \in C^{k,\alpha}(\partial\Omega)$ with $\alpha > \frac{1}{2}$ and $\partial\Omega$ is of class C^{k+1} so that (2.36) admits a unique solution $u \in C^{k+1}(\Omega)$ The internal data are of the form

$$d(x) = \mu(x)u(x) \quad \text{in } \Omega.$$

Here $\mu \in C^{k+1}(\bar{\Omega})$ is bounded below and above by some positive constants.

The *inverse Schrödinger problem with internal data* (ISID) consists of reconstructing (q, μ) in Ω from knowledge of $d = (d_j)_{1 \leq j \leq J} \in (C^{k+1}(\Omega))^J$ and illuminations $g = (g_j)_{1 \leq j \leq J} \in (C^{k,\alpha}(\partial\Omega))^J$, where $J \in \mathbb{N}^*$ is the number of illuminations.

Our reconstruction strategy, similar to that in Section 2.3, is mainly based on explicitly constructed CGO solutions to the Schrödinger equation. For the partial data case, we will require the trace of CGOs is supported only on the part of the domain boundary. The construction of CGOs in Section 2.3.2 fails to satisfy this boundary requirement. Instead, we introduce Carleman estimate to construct CGOs.

2.4.1 Carleman estimate and complex geometrical optics solutions

In this section, we study Carleman estimate which gives another way to construct CGO solutions. We consider the Schrödinger equation $(-\Delta + q)u = 0$ in Ω, where $q \in L^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary.

Recall that Theorem 2.3.2 states the unique resolvability of r for

$$(D^2 + 2\rho \cdot D + q)r = f \quad \text{in } \Omega,$$

where $\rho \in \mathbb{C}^n$ satisfying $\rho \cdot \rho = 0$. Equivalently, r solves

$$e^{-i\rho \cdot x}(-\Delta + q)e^{i\rho \cdot x}r = f \quad \text{in } \Omega.$$

We also have the estimate

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\rho|}\|f\|_{L^2(\Omega)}.$$

If we write $\rho = \frac{1}{h}(\mathfrak{k} + i\mathfrak{k}^\perp)$, the estimate for r may be rewritten as

$$\|r\|_{L^2(\Omega)} \leq C_0 h \|e^{-i\rho \cdot x}(-\Delta + q)e^{i\rho \cdot x}r\|_{L^2(\Omega)}.$$

This estimate can also be viewed as a uniqueness result. It turns out the above estimate can be proved directly without Fourier analysis, and this is sufficient to imply also existence of a solution.

We introduce some notation first. If $u, v \in L^2(\Omega)$, we write

$$(u|v) = \int_\Omega u\bar{v}dx,$$

$$\|u\| = (u|u)^{1/2} = \|u\|_{L^2(\Omega)}.$$

Consider the semiclassical Laplacian

$$P_0 = -h^2\Delta = \sum (hD_{x_k})^2, \tag{2.106}$$

and corresponding Schrödinger operator

$$P = h^2(-\Delta + q) = P_0 + h^2 q. \tag{2.107}$$

The operators conjugated with exponential weights will be denoted by

$$P_{0,\varphi} = e^{\varphi/h} P_0 e^{-\varphi/h}, \tag{2.108}$$

$$P_\varphi = e^{\varphi/h} P e^{-\varphi/h} = P_{0,\varphi} + h^2 q.$$

Here h is a small parameter, $D_{x_k} = -i\partial_{x_k}$ and $\varphi \in C^\infty(\Omega; \mathbb{R})$, with $\nabla\varphi \neq 0$ everywhere. Consider the conjugated operator

$$P_{0,\varphi} = \sum_{k=1}^n (hD_{x_k} + i\partial_{x_k}\varphi)^2 = \mathcal{A} + i\mathcal{B}, \tag{2.109}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the formally selfadjoint operators,

$$\begin{aligned} \mathcal{A} &= (hD)^2 - (\nabla\varphi)^2, \\ \mathcal{B} &= \nabla\varphi \cdot hD + hD \cdot \nabla\varphi, \end{aligned} \tag{2.110}$$

having the principal symbols

$$\mathfrak{a} = \xi^2 - (\nabla\varphi)^2, \qquad \mathfrak{b} = 2\nabla\varphi \cdot \xi. \tag{2.111}$$

We want the conjugated operator $e^{\varphi/h} \circ P_0 \circ e^{-\varphi/h}$ to be locally solvable in a semi-classical sense, which means its principal symbol satisfies Hörmander's condition

$$\{\mathfrak{a}, \mathfrak{b}\} = 0, \qquad \text{when } \mathfrak{a} = \mathfrak{b} = 0, \tag{2.112}$$

where $\{\mathfrak{a}, \mathfrak{b}\} = \sum_k (\partial_{\xi_k}\mathfrak{a}\partial_{x_k}\mathfrak{b} - \partial_{x_k}\mathfrak{a}\partial_{\xi_k}\mathfrak{b})$ is the Poisson bracket.

Definition 2.4.1. *A real smooth function φ on an open set Ω is said to be a limiting Carleman weight if it has non-vanishing gradient on Ω and if the symbols (2.111) satisfy the condition (2.112). This is equivalent to say that*

$$\langle \varphi''\nabla\varphi, \nabla\varphi \rangle + \langle \varphi''\xi, \xi \rangle = 0 \quad \text{when } \xi^2 = (\nabla\varphi)^2 \text{ and } \nabla\varphi \cdot \xi = 0, \tag{2.113}$$

where φ'' is the Hessian matrix of φ.

We begin with the simplest Carleman estimate while the limiting Carleman weight is a linear function of x, i.e., $\varphi(x) = \rho \cdot x$ for $\rho \in \mathbb{R}^n$. The following estimate is valid for test functions and does not involve boundary terms.

Theorem 2.4.2 (Carleman estimate). *Let $q \in L^\infty(\Omega)$, let $\rho \in \mathbb{R}^n$ be a unit vector, and let $\varphi(x) = \rho \cdot x$. There exist constants $C > 0$ and $h_0 > 0$ such that whenever $0 < h < h_0$ and $u \in C_c^\infty(\Omega)$, we have*

$$h\|u\|_{L^2(\Omega)} \leq C\|P_\varphi u\|_{L^2(\Omega)}. \tag{2.114}$$

Proof. We first consider the case $q = 0$, that is, the estimate

$$h\|u\| \leq C\|P_{0,\varphi}u\|, \quad u \in C_0^\infty(\Omega). \tag{2.115}$$

Since $\varphi(x) = \rho \cdot x$ where ρ is a unit vector, we obtain

$$P_{0,\varphi} = \sum_{k=1}^n (hD_{x_k} + i\rho_k) = \mathcal{A} + i\mathcal{B}, \tag{2.116}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the formally selfadjoint operators,

$$\mathcal{A} = (hD)^2 - 1,$$
$$\mathcal{B} = 2\rho \cdot hD.$$

Now we have

$$\begin{aligned}
\|P_{0,\varphi}u\|^2 &= (P_{0,\varphi}u|P_{0,\varphi}u) = ((\mathcal{A} + i\mathcal{B})u|(\mathcal{A} + i\mathcal{B})u) \\
&= (\mathcal{A}u|\mathcal{A}u) + (\mathcal{B}u|\mathcal{B}u) + i(\mathcal{B}u|\mathcal{A}u) - i(\mathcal{A}u|\mathcal{B}u) \\
&= \|\mathcal{A}u\| + \|\lfloor u\| + (i[\mathcal{A}, \mathcal{B}]u|u),
\end{aligned}$$

where $[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ is the commutator of $\mathcal{A}$ and $\mathcal{B}$. This argument used integration by parts and the fact that $\mathcal{A}* = \mathcal{A}$ and $\mathcal{B}^* = \mathcal{B}$. There are no boundary terms since $u \in C_c^\infty(\Omega)$.

Also notice that $\mathcal{A}$ and $\mathcal{B}$ are constant coefficient differential operators and satisfy

$$[\mathcal{A}, \mathcal{B}] = 0. \tag{2.117}$$

Therefore,

$$\|P_{0,\varphi}u\|^2 = \|\mathcal{A}u\|^2 + \|\mathcal{B}\|^2. \tag{2.118}$$

By the Poincaré inequality

$$\|\mathcal{B}u\| = 2h\|\rho \cdot Du\| \geq Ch\|u\|, \tag{2.119}$$

where C depends on Ω. This shows that for any $h > 0$, one has

$$h\|u\| \leq C\|P_{0,\varphi}u\|, \quad u \in C_0^\infty(\Omega). \tag{2.120}$$

Finally, consider the case where q may be nonzero. The last estimate implies that for $u \in C_0^\infty(\Omega)$, one has

$$\begin{aligned} h\|u\| &\leq C\|P_{0,\varphi}u\| \leq C\|(P_{0,\varphi} + h^2 q)u\| + C\|h^2 qu\| \\ &\leq C\|P_\varphi u\| + Ch^2\|q\|_{L^\infty(\Omega)}\|u\|. \end{aligned}$$

Choose h_0 so that $C\|q\|_{L_0^\infty(\Omega)}h_0 = 1/2$, that is

$$h_0 = \frac{1}{2C\|q\|_{L_0^\infty(\Omega)}}$$

Then, if $0 < h \leq h_0$,

$$h\|u\| \leq C\|P_\varphi u\| + \frac{1}{2}h\|u\|.$$

The last term may be absorbed in the left hand side, which completes the proof. $\qquad\square$

When $\varphi \in C_c^\infty(\Omega)$ is a nonlinear limiting Carleman weight, we could have the same Carleman estimate, but with a more complicated proof.

Proposition 2.4.3 (Carleman estimate)**.** *Let Ω_0 be a open set and φ be a limiting Carleman weight. Let $\Omega \subset\subset \Omega_0$ be open and let $q \in L^\infty(\Omega)$. Then if $u \in C_c^\infty(\Omega)$, we have*

$$h(\|e^{\varphi/h}u\| + \|hDe^{\varphi/h}u\|) \leq C\|e^{\varphi/h}(-h^2\Delta + h^2 q)u\|, \tag{2.121}$$

where C depends on Ω, and $h > 0$ is small enough so that $Ch\|q\|_{L^\infty(\Omega)} \leq 1/2$.

Proof. Recall that $\mathcal{A}, \mathcal{B}, \mathfrak{a}$ and $\mathfrak{b}$ are defined in (2.109) and (2.111), satisfying (2.112). The Poisson bracket

$$\{\mathfrak{a}, \mathfrak{b}\} = 4\langle\varphi''_{xx}(x)|\xi \otimes \xi + \nabla\varphi \otimes \nabla\varphi\rangle \tag{2.122}$$

is a quadratic polynomial in ξ, which vanishes when $\xi^2 - (\nabla\varphi)^2 = 0$ and $\nabla\varphi \cdot \xi = 0$. Then we find $c(x)$ and $l(x,\xi)$, while $l(x,\xi)$ is linear in ξ, such that

$$\{\mathfrak{a}, \mathfrak{b}\} = c(x)\mathfrak{a}(x,\xi) + l(x,\xi)\mathfrak{b}(x,\xi). \tag{2.123}$$

The commutator $[\mathcal{A}, \mathcal{B}]$ can be computed directly as

$$\begin{aligned}
[\mathcal{A}, \mathcal{B}] = \frac{h}{i}\Big(&\sum_{j,k} \big[(hD_{x_j} \circ \varphi''_{x_j x_k} + \varphi''_{x_j x_k} \circ hD_{x_j})hD_{x_k} \\
&+ hD_{x_k}(\sum_{j,k} \big[(hD_{x_j} \circ \varphi''_{x_j x_k} + \varphi''_{x_j x_k} \circ hD_{x_j})\big] \\
&+ 4\langle \varphi''_x x, \nabla\varphi(x) \otimes \nabla\varphi(x)\rangle\Big).
\end{aligned} \tag{2.124}$$

The Weyl symbol of $[\mathcal{A}, \mathcal{B}]$ as a semi-classical operator is

$$\frac{h}{i}\{\mathfrak{a}, \mathfrak{b}\} + h^3 p_0(x). \tag{2.125}$$

Combining with (2.123), we get with a new p_0:

$$i[\mathcal{A}, \mathcal{B}] = h\left(\frac{1}{2}(c(x) \circ \mathcal{A} + \mathcal{A} \circ c(x)) + \frac{1}{2}(L\mathcal{B} + \mathcal{B}L) + h^2 p_0(x)\right), \tag{2.126}$$

where L denotes the Wyel quantization of l.

We now derive the Carleman estimate for $u \in C_c^\infty(\Omega)$, $\Omega \subset\subset \Omega_0$. Start from $P_0 u = v$ and let $\tilde{u} = e^{\varphi/h}u$, $\tilde{v} = e^{\varphi/h}$, so that

$$(\mathcal{A} + i\mathcal{B})\tilde{u} = \tilde{v}. \tag{2.127}$$

Using the formal self-adjointness of $\mathcal{A}, \mathcal{B}$, we get

$$\|\tilde{v}\|^2 = ((\mathcal{A} - i\mathcal{B})(\mathcal{A} + i\mathcal{B})\tilde{u}|\tilde{u}) = \|\mathcal{A}\tilde{u}\|^2 + \|\mathcal{B}\|^2 + (i[\mathcal{A}, \mathcal{B}]\tilde{u}|\tilde{u}). \tag{2.128}$$

Using (2.126), we get for $u \in C_0^\infty(\Omega)$ that

$$\begin{aligned}
\|\tilde{v}\|^2 &\geq \|\mathcal{A}\tilde{u}\| + \|\mathcal{B}\tilde{u}\| - \mathcal{O}(h)(\|\mathcal{A}\tilde{u}\|\|\tilde{u}\| + \|L\tilde{u}\|\|\mathcal{B}\tilde{u}\|) - \mathcal{O}(h^3)\|\tilde{u}\|^2 \\
&\geq \frac{2}{3}\|\mathcal{A}\tilde{u}\|^2 + \frac{1}{2}\|\mathcal{B}\tilde{u}\|^2 - \mathcal{O}(h^2)(\|\tilde{u}\|^2 + \|L\tilde{u}\|^2). \\
&\geq \frac{1}{2}(\|\mathcal{A}\tilde{u}\|^2 + \|\mathcal{B}\tilde{u}\|^2) - \mathcal{O}(h^2)\|\tilde{u}\|^2,
\end{aligned} \tag{2.129}$$

where in the last step we used a *priori* estimate

$$\|h\nabla\tilde{u}\|^2 \le \mathcal{O}(1)(\|\mathcal{A}\tilde{u}\|^2 + \|\tilde{u}\|^2), \tag{2.130}$$

which follows from the classical ellipticity of $\mathcal{A}$.

Now we could try to use that $\mathcal{B}$ is associated to a non-vanishing gradient field (and hence without any closed or even trapped trajectories in Ω_0), to obtain the Poincaré estimate:

$$h\|\tilde{u}\| \le \mathcal{O}(1)\|\mathcal{B}\tilde{u}\|. \tag{2.131}$$

To absorb the last term in (2.129), we make a light modification of φ by introducing

$$\varphi_\varepsilon = f \circ \varphi, \quad \text{with } f = f_\varepsilon \tag{2.132}$$

to be chosen below, and write $\mathfrak{a}_\varepsilon + i\mathfrak{b}_\varepsilon$ for the conjugated symbol. The Poisson bracket $\{\mathfrak{a}_\varepsilon, \mathfrak{b}_\varepsilon\}$ becomes with φ equal to the original weight:

$$\{\mathfrak{a}_\varepsilon, \mathfrak{b}_\varepsilon\}(x, f'(\varphi)\eta) = f'(\varphi)^3 \left(\mathfrak{a}, \mathfrak{b}(x,\eta) + \frac{4f''(\varphi)}{f'(\varphi)}\|\nabla\varphi\|^4\right), \tag{2.133}$$

when $\mathfrak{a}(x,\eta) = \mathfrak{b}(x,\eta) = 0$.

By substituting $\xi \to f'(\varphi)\eta$, we deduce that if $\mathfrak{a}(x,\eta) = \mathfrak{b}(x,\eta) = 0$, then $\mathfrak{a}_\varepsilon(x, f'(\varphi)\eta) = \mathfrak{b}_\varphi(x, f'(\varphi)\eta) = 0$. Now let

$$f_\varepsilon(\lambda) = \lambda + \varepsilon\lambda^2/2, \tag{2.134}$$

with $0 < \varepsilon \ll 1$, so that

$$\frac{4f''(\varphi)}{f'(\varphi)} = \frac{4\varepsilon}{1 + \varepsilon\varphi} = 4\varepsilon + \mathcal{O}(\varepsilon^2). \tag{2.135}$$

In view of (2.133) and (2.112), we get

$$\{\mathfrak{a}_\varepsilon, \mathfrak{b}_\varepsilon\} = 4f''_\varepsilon(\varphi)(f'_\varepsilon(\varphi))^2\|\nabla\varphi\|^4 + c_\varepsilon(x)\mathfrak{a}_\varepsilon(x,\xi) + l_\varepsilon(x,\xi)\mathfrak{b}_\varepsilon(x,\xi), \tag{2.136}$$

with $l_\varepsilon(x,\xi)$ linear in ξ.

Instead of (2.129), we get with $\hat{u}e^{\varphi_\varepsilon/h}u$ and $\hat{v} = e^{\varphi_\varepsilon/h}v$ when $P_0 u = v$:

$$\|\hat{v}\|^2 \ge h(t\varepsilon + \mathcal{O}(\varepsilon^2)) \int \|\nabla\varphi\|^4|\hat{u}(x)|^2 dx + \frac{1}{2}\|\mathcal{A}_\varepsilon\hat{u}\|^2 \tag{2.137}$$

$$\frac{1}{2}\|\mathcal{B}_\varepsilon\hat{u}\|^2 - \mathcal{O}(h^2)\|\hat{u}\|^2,$$

while the analogue of (2.131) remains uniformly valid when ε is small:

$$h\|\hat{u}\| \leq \mathcal{O}(1)\|\mathcal{B}_\varepsilon \hat{u}\|, \tag{2.138}$$

even though we will not use this estimate.

Choose $h \ll \varepsilon \ll 1$, so that (2.136) gives

$$\|\hat{v}\|^2 \geq \varepsilon\|\hat{u}\|^2 + \frac{1}{2}\|\mathcal{B}_\varepsilon \hat{u}\|^2. \tag{2.139}$$

We want to transform this into an estimate for $\tilde{u}, \tilde{v}$. From the special form of $\mathcal{A}_\varepsilon$, we see that

$$\|hD\hat{u}\|^2 \leq (\mathcal{A}_\varepsilon \hat{u}|\hat{u}) + \mathcal{O}(1)\|\hat{u}\|^2, \tag{2.140}$$

leading to

$$\|hD\hat{u}\|^2 \leq \frac{1}{2}\|\mathcal{A})_\varepsilon \hat{u}\|^2 + \mathcal{O}(1)\|\hat{u}\|. \tag{2.141}$$

Combing this with (2.139), we get

$$\|\hat{v}\|^2 \geq \frac{\varepsilon h}{C_0}(\|\hat{u}\|^2 + \|hD\hat{u}\|^2) + \left(\frac{1}{2} - \mathcal{O}(\varepsilon h)\right)\|\mathcal{A}_\varepsilon \hat{u}\|^2 \frac{1}{2}\|\mathcal{B}_\varepsilon \hat{u}\|^2. \tag{2.142}$$

Write $\varphi_\varepsilon = \varphi + \varepsilon\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{g}_\varepsilon$ is $\mathcal{O}(1)$ with all its derivatives. We have

$$\hat{u}e^{\varepsilon\mathfrak{g}/h}\tilde{u}, \quad \hat{v} = e^{\varepsilon\mathfrak{g}/h}\tilde{v}, \tag{2.143}$$

so

$$hD\hat{u}e^{\varepsilon\mathfrak{g}/h}(hD\tilde{u} + \frac{\varepsilon}{i}\mathfrak{g}'\tilde{u}) = e^{\frac{\varepsilon}{h}\mathfrak{g}}(hD\tilde{u} + \mathcal{O}(\varepsilon)\tilde{u}), \tag{2.144}$$

and

$$\|\hat{u}\|^2 + \|hD\hat{u}\|^2 \geq \|e^{\varepsilon\mathfrak{g}/h}\tilde{u}\|^2 + \|e^{\varepsilon\mathfrak{g}/h}hD\tilde{u}\|^2$$
$$-C\varepsilon\|e^{\varepsilon\mathfrak{g}/h}\tilde{u}\|\|e^{\varepsilon\mathfrak{g}/h}hD\tilde{u}\| - C\varepsilon\|e^{\varepsilon\mathfrak{g}/h}\tilde{u}\|^2 \tag{2.145}$$
$$\geq (1 - C\varepsilon)(\|e^{\varepsilon\mathfrak{g}/h}\tilde{u}\|^2 + \|e^{\varepsilon\mathfrak{g}/h}hD\tilde{u}\|^2). \tag{2.146}$$

So from (2.142), we obtain after increasing C_0 by a factor $(1 + \mathcal{O})$:

$$\|e^{\varepsilon\mathfrak{g}/h}\tilde{v}\|^2 \geq \frac{\varepsilon h}{C_0}(\|e^{\varepsilon\mathfrak{g}/h}\tilde{u}\|^2 + \|e^{\varepsilon\mathfrak{g}/h}hD\tilde{u}\|^2). \tag{2.147}$$

If we take $\varepsilon = Ch$ with $C \gg 1$ but fixed, then $\varepsilon \mathfrak{g}/h$ is uniformly bounded in Ω and we get the Carleman estimate

$$h^2(\|\tilde{u}\|^2 + \|hD\tilde{u}\|^2) \leq C_1 \|\tilde{v}\|^2. \tag{2.148}$$

This clear extends to solution of the equation

$$(-h^2\Delta + h^2 q)u = v, \tag{2.149}$$

if $q \in L^\infty$ is fixed, since we can start by apply (2.148) with $\tilde{v}$ replace by $\tilde{v} - h^2 q\tilde{u}$. This finishes the proof. $\qquad\square$

In previous theorems, we choose u is compactly supported in Ω, i.e., $u \in C_0^\infty(\Omega)$. We now consider Carleman estimate with boundary terms. Let ν denote the exterior unit normal to $\partial\Omega$ and define

$$\partial\Omega_\pm = \{x_0 \in \partial\Omega | \pm \nabla\varphi \cdot \nu \geq 0\}. \tag{2.150}$$

Proposition 2.4.4. *Let Ω_0, φ be as in Proposition 2.4.3. Let $\Omega \subset\subset \Omega_0$ be an open set with C^∞ boundary and let $q \in L^\infty(\Omega)$. Let ν denote the exterior unit normal to $\partial\Omega$ and define $\partial\Omega_\pm$ as in (2.150). Then there exists a constant $C_0 > 0$, such that for every $u \in C^\infty(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$, we have for $0 < h \ll 1$:*

$$-\frac{h^3}{C_0}((\nabla\varphi_x \cdot \nu)e^{\varphi/h}\partial_\nu u | e^{\varphi/h}\partial_\nu u)_{\partial\Omega_-} + \frac{h^2}{C_0}(\|e^{\varphi/h}u\|^2 + \|e^{\varphi/h}h\nabla u\|^2) \tag{2.151}$$

$$\leq \|e^{\varphi/h}(-h^2\Delta + h^2 q)u\|^2 + C_0 h^3((\nabla\varphi_x \cdot \nu)e^{\varphi/h}\partial_\nu u | e^{\varphi/h}\partial_\nu u)_{\partial\Omega_+}.$$

Proof. Let $P_0 u = v$, $u \in C^\infty(\Omega)$, $u|_{\partial\Omega} = 0$ and $\Omega \subset\subset \Omega_0$ is a domain with C^∞ boundary. Let $\hat{u} = e^{\varphi/h}u$, $\hat{v} = e^{\varphi}/h$, with $\varphi = \varphi_\varepsilon$, $0 \leq \varepsilon \ll 1$. With $\mathcal{A} = \mathcal{A}_\varepsilon$, $\mathcal{B} = \mathcal{B}_\varepsilon$, we have

$$(\mathcal{A} + i\mathcal{B})\hat{u} = \hat{v}, \tag{2.152}$$

and

$$\|\hat{v}\|^2 = ((\mathcal{A} + i\mathcal{B})\hat{u}|(\mathcal{A} + i\mathcal{B})\hat{u}) \tag{2.153}$$

$$= \|\mathcal{A}\hat{u}\|^2 + \|\mathcal{B}\hat{u}\|^2 + i((\mathcal{B}\hat{u}|\mathcal{A}\hat{u}) - (\mathcal{A}\hat{u}|\mathcal{B}\hat{u})).$$

Using that $\mathcal{B}$ is a first order differential operator and that $\hat{u}|_{\partial\Omega} = 0$, we see that

$$(\mathcal{A}\hat{u}|\mathcal{B}\hat{u}) = (\mathcal{B}\mathcal{A}\hat{u}|\hat{u}). \tag{2.154}$$

Similarly, we have

$$(\mathcal{B}\hat{u}|(\nabla\varphi)^2\hat{u}) = ((\nabla\varphi)^2\mathcal{B}\hat{u}|\hat{u}). \tag{2.155}$$

Finally, we use Green's formula, with ν denoting the exterior unit normal, to transform

$$(\mathcal{B}\hat{u}| - h^2\Delta\hat{u})_\Omega = -h^2(\mathcal{B}\hat{u}|\partial_\nu\hat{u})_{\partial\omega} + (-h^2\Delta\mathcal{B}\hat{u}|\hat{u})_\Omega, \tag{2.156}$$

where we also used that $\hat{u}|_{\partial\Omega} = 0$.

On $\partial\Omega$, we have

$$\mathcal{B} = 2(\nabla\varphi \cdot \nu)\frac{h}{i}\partial_\nu + \mathcal{B}', \tag{2.157}$$

where $\mathcal{B}'$ acts along the boundary, so using again the Dirichlet condition, we get

$$(\mathcal{B}\hat{u}|\partial_\nu\hat{u})|_{\partial\Omega} = \frac{2h}{i}((\nabla\varphi \cdot \nu)\partial_\nu\hat{u}|\partial_\nu\hat{u})_{\partial\Omega}. \tag{2.158}$$

Putting together the calculations and using (2.110) for $\mathcal{A}$, we get

$$\|\hat{v}\|^2 = \|\mathcal{A}\hat{u}\|^2 + \|\mathcal{B}\hat{u}\|^2 + i([\mathcal{A}, \mathcal{B}]\hat{u}|\hat{u}) - 2h^3((\nabla\varphi \cdot \nu)\partial_\nu\hat{u}|\hat{u})_{\partial\Omega}. \tag{2.159}$$

Define the front and back sides of the boundary with respect to $\nabla\varphi$ by

$$\partial\Omega_\pm = \{x \in \partial\Omega : \pm\nabla\varphi \cdot \nu \geq 0\}. \tag{2.160}$$

Notice that $\partial\Omega_\pm$ are independent of ε. We rewrite (2.158) as

$$-2h^3((\nabla\varphi \cdot \nu)\partial_\nu\hat{u}|\partial_\nu\hat{u})_{\partial\Omega_-} + i([\mathcal{A}, \mathcal{B}]\hat{u}|\hat{u}) + \|\mathcal{A}\hat{u}\|^2 + \|\mathcal{B}\hat{u}\|^2 \tag{2.161}$$

$$= \|\hat{v}\|^2 + 2h^3((\nabla\varphi \cdot \nu)\partial_\nu\hat{u}|\partial_\nu\hat{u})_{\partial\Omega_+}. \tag{2.162}$$

This is analogous to (2.128) and the extra boundary terms can be added in the discussion leading from (2.137) to (2.142) and we get

$$-2h^3((\nabla\varphi \cdot \nu)\partial_\nu\hat{u}|\partial_\nu\hat{u})_{\partial\Omega_-} + \frac{\varepsilon h}{C_0}(\|\hat{u}\|^2 + \|hD\hat{u}\|^2)$$

$$+ (\frac{1}{2} - \mathcal{O}(\varepsilon h))\|\mathcal{A}_\varepsilon\hat{u}\|^2 + \frac{1}{2}\|\mathcal{B}\hat{u}\|^2 \tag{2.163}$$

$$\leq \|\hat{v}\|^2 + 2h^3((\nabla\varphi \cdot \nu)\partial_\nu\hat{u}|\partial_\nu\hat{u})_{\partial\Omega_+}, \tag{2.164}$$

with $\varphi = \varphi_\varepsilon$, provided $\varepsilon \gg h$. Fixing $\varepsilon = Ch$ for $C \gg 1$, we get with $\varphi = \varphi_{\varepsilon_0}$ from some $C_0 > 0$:

$$-\frac{h^3}{C_0}((\nabla\varphi \cdot \nu)\partial_\nu\tilde{u}|\partial_\nu\tilde{u})_{\partial\Omega_-} + \frac{h^2}{C_0}(\|\tilde{u}\|^2 + \|hD\tilde{u}\|^2)$$

$$\leq \|\tilde{v}\|^2 + C_0 h^3((\nabla\varphi \cdot \nu)\partial_\nu\tilde{u}|\partial_\nu\tilde{u})_{\partial\Omega_+}. \quad (2.165)$$

Here $\tilde{u} = e^{\varphi/h}u$ and $\tilde{v} = e^{\varphi/h}v$. This finishes the proof. $\qquad\square$

Let $H^s(\Omega)$ be the Sobolev space. We denote by $H^1_{scl}(\Omega)$ the semi-classical Sobolev space of order 1 on Ω with associated norm

$$\|u\|^2_{H^1_{scl}(\Omega)} = \|h\nabla u\|^2 + \|u\|^2 \quad (2.166)$$

and by $H^s_{scl}(\mathbb{R}^n)$ the semi-classical Sobolev space of order s on $\mathbb{R}^n$ with associated norm

$$\|u\|^2_{H^s_{scl}(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|^2_{L^2(\mathbb{R}^n)} = \int (1 + h^2\xi^2)^s |\hat{u}(\xi)|^2 d\xi. \quad (2.167)$$

Recall that $P_0 = -h^2\Delta$ and $P = P_0 + h^2q$. Carleman estimates in Proposition 2.4.3 and the Hahn-Banach theorem give the following resolvability result.

Proposition 2.4.5. *Let φ be a limiting Carleman weight and $q \in L^\infty(\Omega)$. Let $0 \leq s \leq 1$. There exists $h_0 > 0$ such that for $0 < h \leq h_0$ and for every $f \in H^{s-1}_{scl}(\Omega)$, there exists $r \in H^s_{scl}(\Omega)$ such that*

$$e^{\frac{\varphi}{h}}Pe^{-\frac{\varphi}{h}}r = f, \quad h\|r\|_{H^s_{scl}(\Omega)} \leq C\|f\|_{H^{s-1}_{scl}(\Omega)}.$$

Proof. (2.148) can be written as

$$h\|r\|_{H^1} \leq C\|e^{\varphi/h}P_0e^{-\varphi/h}r\|, \quad u \in C^\infty_c(\Omega). \quad (2.168)$$

Here $\Omega \subset\subset \Omega_0$ has smooth boundary. Recall that $P_{0,\varphi} = e^{\varphi/h}P_0e^{-\varphi/h}$ has te semi-classical Weyl symbol $\xi^2 - (\nabla\varphi)^2 + 2i\nabla\varphi \cdot \xi = \mathfrak{a} + i\mathfrak{b}$, which is elliptic in the region $|\xi| \geq 2|\nabla\varphi|$. It is therefore clear that (2.168) can be extended to

$$h\|r\|_{H^{-s+1}} \leq C_{s,\Omega}\|e^{\varphi/h}P_0e^{-\varphi/h}r\|_{H^{-s}}, \quad r \in C^\infty_c(\Omega), \quad (2.169)$$

for every fixed $s \in \mathbb{R}$. If $0 \leq s \leq 1$, we have

$$\|qr\|_{H^{-s}} \leq \|qr\| \leq \|q\|_{L^\infty}\|r\| \leq \|q\|_{L^\infty}\|r\|_{H^{-s+1}}, \tag{2.170}$$

and for $h > 0$ small enough, we get from (2.169) that

$$h\|r\|_{H^{-s+1}} \leq C_{s,\Omega}\|e^{\varphi/h}P_0 e^{-\varphi/h}r\|_{H^{-s}}, \quad r \in C_c^\infty(\Omega). \tag{2.171}$$

Let $P_\varphi^* = P_{0,-\varphi} + h^2\bar{q}$ is the conjugate operator of P_φ. (2.171) also gives

$$h\|r\|_{H^{-s+1}} \leq C_{s,\Omega}\|P_\varphi^* r\|_{H^{-s}}, \quad u \in C_c^\infty(\Omega). \tag{2.172}$$

Let $\mathcal{D} = P_\varphi^* C_c^\infty(\Omega)$ be a subspace of $H^{-s}(\Omega)$. Consider the linear functional

$$\mathcal{L} : \mathcal{D} \to \mathbb{C}, \quad \mathcal{L}(P_\varphi^* r) = (r|f), \quad \text{for } r \in C_c^\infty(\Omega). \tag{2.173}$$

This is well defined since any element of $\mathcal{D}$ has a unique representation as $P_\varphi^* u$ with $u \in C_c^\infty(\Omega)$, by the Carleman estimate. Also, the Carleman estimate implies

$$|\mathcal{L}(P_\varphi^* r)| \leq \|r\|_{H^{-s+1}}\|f\|_{H^{s-1}} \leq \frac{C}{h}\|P_\varphi^* r\|\|f\|_{H^{s-1}}. \tag{2.174}$$

Thus $\mathcal{L}$ is a bounded linear functional on $\mathcal{D}$.

The Hahn-Banach theorem ensures that there is a bounded linear functional $\hat{\mathcal{L}} : H^{-s} \to \mathbb{C}$ satisfying $\hat{\mathcal{L}}|_\mathcal{D} = \mathcal{L}$ and $\hat{\mathcal{L}} \leq ch^{-1}\|v\|_{H^{s-1}}$. By the Riesz representation theorem, there is $r \in H^s(\Omega)$ such that

$$\hat{\mathcal{L}}w = (w|r), \quad w \in H^{-s}(\Omega), \tag{2.175}$$

and $\|r\|_{H^s(\Omega)} \leq Ch^{-1}\|f\|_{H^{s-1}}$.

Therefore, for $\phi \in C_c^\infty(\Omega)$, by the definition of weak derivatives, we have

$$(\phi|P_\varphi r) = (P_\varphi^* \phi|r) = \hat{\mathcal{L}}(P_\varphi^* \phi) = \mathcal{L}(P_\varphi^* \phi) = (\phi|f), \tag{2.176}$$

which shows that $P_\varphi r = f$ in the weak sense and $\|r\|_{H^s(\Omega)} \leq Ch^{-1}\|f\|_{H^{s-1}}$. $\qquad\square$

Now we start to construct CGO solutions by choosing $\varphi \in C^\infty(\Omega)$ to be a limiting Carleman weight and $\psi \in C^\infty(\Omega)$ such that

$$(\nabla\psi)^2 = (\nabla\varphi)^2, \quad \nabla\psi \cdot \nabla\varphi = 0. \tag{2.177}$$

Then ψ is a local solution to the Hamilton-Jacobi problem

$$\mathfrak{a}(x, \nabla\psi) = \mathfrak{b}(x, \nabla\psi) = 0. \tag{2.178}$$

Therefore,

$$
\begin{aligned}
e^{-\frac{1}{h}(-\varphi+i\psi)} P_0 e^{\frac{1}{h}(-\varphi+i\psi)} a &= \big[((hD + \nabla\psi)^2 - \nabla\varphi^2) + i\big(\nabla\varphi(hD + \nabla\psi) \\
&\quad + (hD + \nabla\psi)\nabla\varphi\big)\big]a \\
&= (h\mathfrak{L} - h^2\Delta)a, \tag{2.179}
\end{aligned}
$$

where $\mathfrak{L}$ is the transport operator given by

$$\mathfrak{L} = \nabla\psi D + D\nabla\psi + i(\nabla\varphi D + D\nabla\varphi). \tag{2.180}$$

There exists a non-vanishing smooth function $a \in C^\infty(\Omega)$, see [16, 26], such that

$$\mathfrak{L}a = 0. \tag{2.181}$$

Assume that $q \in L^\infty(\Omega)$ and recall $P = h^2(-\Delta + q) = P_0 + h^2 q$. Then (2.179) implies that

$$Pe^{\frac{1}{h}(-\varphi+i\psi)} a = e^{-\varphi/h} h^2 \kappa,$$

with $\kappa = \mathcal{O}(1)$ in L^∞ and hence in L^2. Now Proposition 2.4.5 implies there exists $r(x, h) \in H^1_{scl}(\Omega)$ such that

$$
\begin{aligned}
e^{\varphi/h} Pe^{\frac{1}{h}(-\varphi+i\psi)} r &= -h^2 \kappa, \quad \text{and} \\
\|r\|_{H^1_{scl}(\Omega)} = \|h\nabla r\| + \|r\| &\leq Ch, \quad \text{for some } C.
\end{aligned} \tag{2.182}
$$

Hence,

$$P(e^{\frac{1}{h}(-\varphi+i\psi)}(a + r)) = 0,$$

i.e., we constructed a solution of the Schrödinger equation of the form

$$e^{\frac{1}{h}(-\varphi+i\psi)}(a + r). \tag{2.183}$$

In our partial data model, illumination is supported on part of the boundary. But the CGO solution in (2.183) is non-vanishing everywhere. Fortunately, one can use the Carleman estimate (2.151) and the Hahn-Banach theorem to construct CGO solutions for the conjugate operator $P_\varphi^* = (e^{\frac{\varphi}{h}} Pe^{-\frac{\varphi}{h}})$ where $*$ denotes the adjoint. Notice that P_φ^* has the same form as P_φ except that q is replaced by $\bar{q}$ and φ by $-\varphi$.

Proposition 2.4.6. *Let φ be as in (2.151). Let $f \in H_{scl}^{-1}(\Omega)$,*

$$f_- \in L^2(\partial\Omega_-; (-\nabla\varphi \cdot \nu)dS). \tag{2.184}$$

Then $\exists r \in H_{scl}^0(\Omega)$ such that

$$P_\varphi^* r = f, \quad r|_{\partial\Omega_-} = f_-. \tag{2.185}$$

Moreover,

$$\|r\|_{H_{scl}^0} + \sqrt{h}\|(\nabla\varphi \cdot \nu)^{-\frac{1}{2}}r\|_{\partial\Omega_+} \leq C\left(\frac{1}{h}\|f\|_{H_{scl}^{-1}} + \sqrt{h}\|(-\nabla\varphi \cdot \nu)^{-\frac{1}{2}}f_-\|_{\partial\Omega_-}\right) \tag{2.186}$$

Proof. We use the Carleman estimate (2.151). Let f as in the proposition. For $w \in (H_0^1 \cap H^2)(\Omega)$ we have

$$|(w|f)_\Omega + (h\partial_\nu w|f)_{\partial\Omega_-}| \leq \|w\|_{H^1}\|f\|_{H^{-1}} + \left((-\nabla\varphi \cdot \nu)^{\frac{1}{2}}h\partial_\nu w|(-\nabla\varphi \cdot \nu)^{-\frac{1}{2}}f_-\right). \tag{2.187}$$

Therefore,

$$|(w,f)_\Omega + (h\partial_\nu w|f)_{\partial\Omega_-}|$$
$$\leq C\left(\frac{1}{h}\|f\|_{H^{-1}h\|w\|h^1} + \frac{1}{\sqrt{h}}\|(-\nabla\varphi \cdot \nu)^{-\frac{1}{2}}f_-\|_{\partial\Omega_-}\sqrt{h}\|(-\nabla\varphi \cdot \nu)^{\frac{1}{2}}h\partial_\nu w\|_{\partial\Omega_-}\right). \tag{2.188}$$

Now by using (2.151), we get

$$|(w,f)_\Omega + (h\partial_\nu w|f)_{\partial\Omega_-}|$$
$$\leq C\left(\frac{1}{h}\|f\|_{H^{-1}} + \frac{1}{\sqrt{h}}\|(-\nabla\varphi \cdot \nu)^{-\frac{1}{2}}f_-\|_{\partial\Omega_-}\right)\left(\|\|P_\varphi w\| + \sqrt{h}(-\nabla\varphi \cdot \nu)^{\frac{1}{2}}h\partial_\nu w\|_{\partial\Omega_+}\right). \tag{2.189}$$

By the Hahn-Banach theorem, $\exists r \in H^0(\Omega)$, $r_+ \in L^2(\partial\Omega_+, (\nabla\varphi \cdot \nu)^{-\frac{1}{2}}dS)$, r_+ on $\partial\Omega_+$ such that

$$(w,f)_\Omega + (h\partial_\nu w|f)_{\partial\Omega_-} = (P_\varphi w|r) + (h\partial_v w|u_+)_{\partial\Omega_+}, \tag{2.190}$$

for $\forall w \in (H_0^1 \cap H^2)(\Omega)$ with

$$\|r\|_{H^0} + \frac{1}{\sqrt{h}}\|(\nabla\varphi \cdot \nu)^{-\frac{1}{2}}\|_{\partial\Omega_+}$$
$$\leq C\left(\frac{1}{h}\|f\|_{H^{-1}} + \frac{1}{\sqrt{h}}\|(-\nabla\varphi \cdot \nu)^{-\frac{1}{2}}f_-\|_{\partial\Omega_-}\right). \tag{2.191}$$

Since $P_\varphi = -h^2\Delta$ is a first order operator and $w|_{\partial\Omega} = 0$, we have $(P_\varphi w|r) = (w|P_\varphi^* r) - h^2(\partial_\nu w|r)_{\partial\Omega}$.

Using (2.190) we obtain

$$0 = (w|f - P_\varphi^* r) + h\big((\partial_\nu w|1_{\partial\Omega_-} f_-)_{\partial\Omega} - (\partial_\nu w|1_{\partial\Omega_+} r_+)_{\partial\Omega} + (\partial_\nu w|hu)_{\partial\Omega}\big),$$

where $1|_{\partial\Omega_\pm}$ denotes the indicator function of $\partial\Omega_\pm$.

By varying w in $(H_0^1 \cap H^2)(\Omega)$, we get

$$P_\varphi^* r = f, \quad hr|_{\partial\Omega} = -1_{\partial\Omega_-} f_- + 1_{\partial\Omega_+} r_+,$$

which implies the proposition after replacing f_- above by $-hf_-$. $\qquad\square$

Let Γ_- be a strict open subset of $\partial\Omega_-$.

Proposition 2.4.7. *We can construct a solution of*

$$Pu = 0, \quad u\big|_{\Gamma_-} = 0 \tag{2.192}$$

of the form

$$u = e^{\frac{1}{h}(\varphi + i\psi)}(a + r) + z \tag{2.193}$$

where φ, ψ and a are chosen as above and $z = e^{i\frac{l}{h}}b(x; h)$ with b a symbol of order zero in h and

$$\operatorname{Im} l(x) = -\varphi(x) + k(x) \tag{2.194}$$

where $k(x) \sim \operatorname{dist}(x, \partial\Omega_-)$ in a neighborhood of $\partial\Omega_-$ and b has its support in that neighborhood. Moreover, $\|r\|_{H^0} = \mathcal{O}(h)$, $r|_{\partial\Omega_-} = 0$, $\|(\nabla\varphi \cdot \nu)^{\frac{1}{2}} r\|_{\partial\Omega_+} = \mathcal{O}(h^{\frac{1}{2}})$.

Proof. We start by constructing a WKB solution u in Ω of

$$-h^2\Delta u = 0, \quad u|_{\partial\Omega_-} = e^{\frac{1}{h}(\varphi + i\psi)}(\chi a)|_{\partial\Omega_-} \tag{2.195}$$

where $\chi \in C_c^\infty(\partial\Omega_-)$, $\chi = 1$ on Γ_-.

We try $u = e^{\frac{i}{h}l(x)}z(x; h)$. The eikonal equation for l is

$$\begin{aligned}
(l')^2 &= 0 \text{ to infinite order at } \partial\Omega \\
l|_{\partial\Omega_-} &= \psi - i\varphi.
\end{aligned} \tag{2.196}$$

Of course $g := \psi - i\varphi$ is a solution but we look for the second solution, corresponding to having u equal to a "reflected wave". We decompose on $\partial\Omega_-$

$$g' = g'_t + g'_\nu,$$

where t denotes the tangential part and ν the normal part.

the in order to stisfy the eikonal equation we need

$$0 = (g'_t)^2 + (g'_\nu)^2.$$

Therefore, we can solve (2.196) to ∞-order on $\partial\Omega_-$ with l satisfying

$$l|_{\partial\Omega_-} = g|_{\partial\Omega_-}, \quad \partial_\nu l|_{\partial\Omega_-} = -\partial_\nu g|_{\partial\Omega_-}.$$

By the definition of $\partial\Omega_-$ we have

$$\partial_\nu \Im g = -\partial_\nu \varphi > 0 \text{on } \partial\Omega_-.$$

Since ν is the unit exterior normal we have that (2.194) is satisfied.

solving also the transport equation to ∞-order, at the boundary we get a symbol $\mathfrak{z}$ of order 0 with support arbitrarily close to $\mathrm{supp}\chi$, such that

$$\begin{cases} -h^2 \Delta(e^{\frac{il}{h}} z(x;h)) &= e^{\frac{il}{h}} \mathcal{O}((\mathrm{dist}(x, \partial\Omega))^\infty + h^\infty) \\ e^{\frac{il}{h}}|_{\partial\Omega} &= e^{\frac{il}{h}} \chi a|_{\partial\Omega}. \end{cases}$$

our new WKB input to u_2 will be

$$(e^{\frac{ig}{h}} a - e^{\frac{il}{h}} z).$$

We now have

$$P(e^{\frac{ig}{h}} a - e^{\frac{il}{h}} z) = e^{\frac{\varphi}{h}} h^2 d, \tag{2.197}$$

where $d = \mathcal{O}(1)$ in $L^2(\Omega)$.

using Proposition 2.4.6 we can solve

$$e^{-\frac{\varphi}{h}} P e^{\frac{\varphi}{h}} (e^{i\frac{\psi}{h}} \tilde{r}_2) = -h^2 d \tag{2.198}$$

$$\tilde{r}_2|_{\partial\Omega_-} = 0$$

with

$$\|\tilde{r}_2\|_{H^0} + \sqrt{h}\|(\nabla\varphi \cdot \nu)^{-\frac{1}{2}}\tilde{r}_2\|_{\partial\Omega_+} \le \frac{C}{h}\|h^2 d\|_{H^{-1}} = \mathcal{O}(h). \tag{2.199}$$

Thus

$$\|\tilde{r}_2\| = \mathcal{O}(h), \quad \|(\nabla\varphi \cdot \nu)^{-\frac{1}{2}}\tilde{r}_2\|_{\partial\Omega_+} = \mathcal{O}(\sqrt{h}). \tag{2.200}$$

Now we take

$$u_2 = e^{\frac{1}{h}(\varphi+i\psi)}(a + \tilde{r}_2) - e^{\frac{il}{h}}z. \tag{2.201}$$

Clearly $Pu_2 = 0$, $u_2|_{\partial\Omega} = 0$ in γ_-. $\qquad\square$

Remark: According to the proof in [26], supp(z) is arbitrarily close to $\partial\Omega_-$ in Ω.

Notice that φ is a limiting Carleman weight, so is $-\varphi$. We construct the CGO solutions of the form

$$\begin{aligned}
\check{u}_1 &= e^{\frac{1}{h}(\varphi+i\psi)}(a_1 + r_1) + z_1 \\
\check{u}_2 &= e^{\frac{1}{h}(-\varphi+i\psi)}(a_2 + r_2) + z_2.
\end{aligned} \tag{2.202}$$

In particular, we choose

$$\varphi(x) = \log|x - x_0| \quad \text{and} \quad \psi(x) = d_{S^{n-1}}\left(\frac{x - x_0}{|x - x_0|}, \omega\right), \tag{2.203}$$

where $x_0 \in \mathbb{R}^n \backslash \overline{\text{ch}(\Omega)}$ and $\omega \in S^{n-1}$.

Note that z_j, $j = 1, 2$, are supported only in an arbitrarily neighborhood of $\partial\Omega_-$. For the rest of this paper, we will mainly consider the uniqueness and the stability of the solutions on the subregion defined by

$$\tilde{\Omega} = \Omega\backslash(\text{supp}(z_1) \cup \text{supp}(z_2)). \tag{2.204}$$

Remark that a neighborhood of the point $y \in \{y \in \partial\Omega | (x_0 - y) \cdot \nu(y) = 0\}$ is excluded from $\tilde{\Omega}$. Then, there exists a fixed constant $\eta > 0$, such that $\forall x \in \tilde{\Omega}$, we have $(x_0 - y) \cdot \nu(y) \ge \eta$. In the following proof, we mainly focus on $\tilde{\Omega}$, where z_1 and z_2 vanish.

2.4.2 Uniqueness

The methodology to prove the uniqueness of the reconstruction will be similar to that in Section 2.3.3. Let $j = 1, 2$. We assume that we can impose the complex-valued illuminations

$g_j \in C^{k,\alpha}(\partial\Omega; \mathbb{C})$ on $\partial\Omega$ and observe the complex-valued internal data d_j. For partial data case, to make up two complex-valued g_j and d_j, we need to four real observations, since CGO solutions $\check{u}_j$ constructed in (2.202) are not conjugate of each other and the trick used in Section 2.3.3 fails. Let d_j be of the form $d_j = \mu u_j$ in Ω, where u_j is the solutions of

$$\Delta u_j + q u_j = 0 \text{ in } \Omega \qquad u_j = g_j \text{ on } \partial\Omega. \tag{2.205}$$

Direct calculation gives us that

$$u_1 \Delta u_2 - u_2 \Delta u_1 = 0.$$

We assume that $\mu \in C^{k+1}(\bar{\Omega})$ is bounded above and below by positive constants. By substituting $u_j = d_j/\mu$, we obtain that

$$2(d_1 \nabla d_2 - d_2 \nabla d_1) \cdot \nabla \mu - (d_1 \Delta d_2 - d_2 \Delta d_1)\mu = 0,$$

or equivalently,

$$\beta_d \cdot \nabla \mu + \gamma_d \mu = 0, \tag{2.206}$$

where

$$\begin{aligned}
\beta_d &:= \chi(x)(d_1 \nabla d_2 - d_2 \nabla d_1) \\
\gamma_d &:= -\frac{1}{2}\chi(x)(d_1 \Delta d_2 - d_2 \Delta d_1) = \frac{-\beta \cdot \nabla \mu}{\mu}.
\end{aligned} \tag{2.207}$$

Here, $\chi(x)$ is any smooth known complex-valued function with $|\chi(x)|$ uniformly bounded below by a positive constant on $\bar{\Omega}$. Note that by assumption on μ, we have that $\beta_d \in (C^k(\bar{\Omega}; \mathbb{C}))^n$ and $\gamma_d \in C^k(\bar{\Omega}; \mathbb{C})$.

Proposition 2.4.8. *Let β be the normalized vector field, defined by*

$$\beta = \frac{h}{2}\beta_d. \tag{2.208}$$

There is a open set of g in $C^{k,\alpha}(\partial\Omega)$, with $\text{supp}(g) \subseteq \Gamma$, such that, for any small ϵ,

$$\left\| \beta(x) - \mu^2 \Theta \frac{x_0 - x}{|x_0 - x|^2} \right\|_{C^k(\tilde{\Omega}, \mathbb{C})} \leq C h(1 + \epsilon) \quad \text{on } \tilde{\Omega}, \tag{2.209}$$

where $x_0 \in \mathbb{R}^n \backslash \overline{\text{ch}(\Omega)}$ and Θ is a function of class $C^k(\Omega)$. Therefore, (2.206) admits a unique solution μ on $\tilde{\Omega}$.

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Proof. Let $\check{u}_1, \check{u}_2$ be CGO solutions in (2.202). By substituting $\check{d}_j = \mu \check{u}_j$ and $\chi(x) = e^{-\frac{2i}{h}\psi}$ in (2.207), we find $\check{\beta}_d$, restricted to $\tilde{\Omega}$, is given by

$$\check{\beta}_d = \mu^2 \Big(-\frac{2\nabla\varphi}{h}(a_1+r_1)(a_2+r_2) + (a_1+r_1)(\nabla a_2 + \nabla r_2)$$
$$- (a_2+r_2)(\nabla a_1 + \nabla r_1) \Big).$$

We may then define, on $\tilde{\Omega}$,

$$\check{\beta} = \frac{h}{2}\check{\beta}_d = -\mu^2 \Theta \nabla\varphi + \mu^2 \beth, \tag{2.210}$$

where $\Theta = (a_1+r_1)(a_2+r_2)$ and $\beth = \frac{h}{2}((a_1+r_1)(\nabla a_2 + \nabla r_2) - (a_2+r_2)(\nabla a_1 + \nabla r_1)) \leq C_0 h$ for some constant C_0. Also note that, from (2.203), $\nabla\phi = \frac{x_0 - x}{|x_0 - x|^2}$.

We will next choose appropriate boundary conditions g_j, $j = 1, 2$, on $\partial\Omega$, which could lead to small $\|\beta_d - \check{\beta}_d\|$ in Ω. In particular, recall that $\check{u}_j = 0$ on $\partial\Omega_-$, for some $\epsilon > 0$, we choose $g_j \in C^{k,\alpha}(\partial\Omega)$ and $g_j = 0$ on $\partial\Omega_-$, such that,

$$\|g_j - \check{u}_j\|_{C^{k,\alpha}(\partial\Omega)} \leq \epsilon \quad \text{on } \partial\Omega, \quad j = 1, 2. \tag{2.211}$$

Let u_j be the solutions of (2.60) with boundary conditions g_j from (2.211). By elliptic regularity, we have that

$$\|u_j - \check{u}_j\|_{C^{k+1}(\bar{\Omega}, \mathbb{C})} \leq C\epsilon \quad \text{on } \bar{\Omega}, \quad j = 1, 2, \tag{2.212}$$

for some positive constant C. Notice that $d_j = \mu u_j$ and $\mu \in C^{k+1}(\bar{\Omega})$, we deduce that

$$\|d_j - \check{d}_j\|_{C^{k+1}(\bar{\Omega}, \mathbb{C})} \leq C_0 \epsilon \quad \text{on } \bar{\Omega}, \quad j = 1, 2. \tag{2.213}$$

Thus, restricting to $\tilde{\Omega}$, (2.210) and (2.213) induce (2.209), which indicates, when h, ϵ are sufficiently small, β is close to a non-vanishing vector $-\mu^2 \Theta \nabla\varphi = \mu^2 \Theta \frac{x_0 - x}{|x_0 - x|^2}$. Approximately, the integral curves of β are rays from any $x \in \tilde{\Omega}$ to $x_0 \in \mathbb{R}^n \backslash \overline{\text{ch}(\Omega)}$, intersecting $\partial\Omega_+$ at a point $x_+(x)$. Therefore, with $\mu = d_1/g_2 = d_2/g_2$ known on $\partial\Omega_+$,

$$\beta \cdot \nabla\mu + \gamma\mu = 0, \quad \gamma = \frac{h}{2}\gamma_d \tag{2.214}$$

provides a unique reconstruction for μ, so does (2.206). More precisely, consider the flow $\theta_x(t)$ associated to β, i.e., the solution to

$$\dot{\theta}_x(t) = \beta(\theta_x(t)), \quad \theta_x(0) = x \in \tilde{\Omega}.$$

By the Picard-Lindelöf theorem, the above equations admit unique solution θ_x while β is of class C^1. Since β is non-vanishing, θ_x reaches $x_+(x) \in \partial\Omega_+$ in a finite time, denoted as $t_+(x)$, i.e.,

$$\theta_x(t_+(x)) = x_+(x).$$

Then by the method of characteristics, $\mu(x)$ is uniquely determined by

$$\mu(x) = \mu_0(x_+(x))e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds}, \tag{2.215}$$

where $\mu_0 = d/g$ on $\partial\Omega_+$. This finishes the proof. $\qquad\square$

Let us define the set of parameters

$$\mathcal{P} = \Big\{(\mu, q) \in C^{k+1}(\Omega) \times H^{\frac{n}{2}+k+\epsilon}(\Omega); \ 0 \text{ is not an eigenvalue of } \Delta + q,$$
$$\|\mu\|_{C^{k+1}(\Omega)} + \|q\|_{H^{\frac{n}{2}+k+\epsilon}(\Omega)} \le P < \infty\Big\}. \tag{2.216}$$

The above construction of the vector field allows us to obtain the following uniqueness result.

Theorem 2.4.9. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with boundary of class C^{k+1}. Let (μ, q) and $(\tilde{\mu}, \tilde{q})$ be two elements in $\mathcal{P}$ and $\tilde{\Omega}$ be defined in (2.204). When h and ϵ are sufficiently small, for $j = 1, 2$, let g_j be constructed according to (2.211) with the CGO solutions $\check{u}_j$. d_j and $\tilde{d}_j$ are two sets of observations of the internal data on Ω.*

Restricting to $\tilde{\Omega}$, $d_j = \tilde{d}_j$ implies that $(\mu, q) = (\tilde{\mu}, \tilde{q})$.

Proof. We have proved that, for $j = 1, 2$, when g_j is properly chosen, μ is uniquely reconstructed on $\tilde{\Omega}$, i.e., $\mu = \tilde{\mu}$. Directly, $d_j = \tilde{d}_j$ also implies $u := u_j = \tilde{u}_j$. By unique continuation, u cannot vanish on an open set in $\tilde{\Omega}$ different from the empty set. Otherwise u vanishes everywhere and this is impossible to satisfy the boundary conditions. Therefore, the set $F = \{|u| > 0\} \cap \tilde{\Omega}$ is open and $\bar{F} = \bar{\tilde{\Omega}}$ since the complement of $\bar{F}$ has to be empty. By continuity, this shows that $q = \tilde{q}$ on $\tilde{\Omega}$. $\qquad\square$

Since the coefficient q in the Schrödinger equation is unknown, the CGO solutions $\check{u}_j$, $j = 1, 2$, cannot be explicitly determined. Therefore, although we know that g_j can be chosen from a open set close to $\check{u}_j$, a more explicit characterization of the open set is lacking.

Also notice that the parameters h and ϵ need to be small to make β flat enough, while cannot be too small, otherwise g_1 will be so large that the imposed illuminations become physically infeasible and g_2 will be so small that the imposed illuminations become physically undetectable.

By applying the inverse Liouville change of variables, we get the uniqueness results for the inversion of the diffusion problem.

Theorem 2.4.10. *Let Ω be an open, bounded, connected domain with C^2 boundary $\partial\Omega$. Let Γ and $\tilde{\Omega}$ be defined as above. Assume that $(D(x), \sigma_a(x))$ and $(\tilde{D}(x), \tilde{\sigma}_a(x))$ are in $\mathcal{M}$ with $D|_\Gamma = \tilde{D}|_\Gamma$. Let $d = (d_j)$ and $\tilde{d} = (\tilde{d}_j)$, $j = 1, \ldots, 4$, be the internal data for coefficients $(D(x), \sigma_a(x))$ and $(\tilde{D}(x), \tilde{\sigma}_a(x))$, respectively and with boundary conditions $g = (g_j)$, $j = 1, \ldots, 4$. Then there is a set of illuminations $g \in (C^{k,\alpha}(\partial\Omega))^4$, $\mathrm{supp}(g) \subseteq \Gamma$, for some $\alpha > 1/2$, such that if $d = \tilde{d}$, then $(D(x), \sigma_a(x)) = (\tilde{D}(x), \tilde{\sigma}_a(x))$ in $\tilde{\Omega}$.*

Proof. The proof is the same as that of Theorem 2.3.6. □

2.4.3 Stability result for 2 complex observations

In this section, we consider the stability of the proposed reconstruction method. We divide the proof into two cases. When two complex-valued data are measured, strict convexity is assumed on the domain of interest. In another case, $2n$ complexed-valued data are measured and the stability result follows without any geometric condition on the domain.

Recall that $\theta_x(t)$ is the flow associated to β and $\theta_x(t)$ reaches the boundary at x_+ and at time t_+, i.e., $\theta_x(t_+) = x_+ \in \partial\Omega$. Similar notations are use for $\tilde{\beta}$.

Lemma 2.4.11. *Let $k \geq 1$ and assume that β and $\tilde{\beta}$ are $C^k(\bar{\Omega})$ vector fields that are sufficiently flat, i.e., h is sufficiently small. Then, restricting to $\tilde{\Omega}$, we have that*

$$\|x_+ - \tilde{x}_+\|_{C^k(\tilde{\Omega})} + \|t_+ - \tilde{t}_+\|_{C^k(\tilde{\Omega})} \leq C\|\beta - \tilde{\beta}\|_{C^k(\tilde{\Omega})}, \tag{2.217}$$

where C is a constant depending on h and R.

Proof. By the definition of $\tilde{\Omega}$ in (2.204). Let $x \in \tilde{\Omega}$ and $y \in \partial\Omega_+ \cap \partial\tilde{\Omega}$, then $\beta(x) \cdot \nu(y) > \eta$, for some constant $\eta > 0$, where β satisfies (2.209) and $\nu(y)$ is the unit outer normal. Then, (2.217) follows directly from Lemma 2.3.7. □

Proposition 2.4.12. *Let $k \geq 1$. Let μ and $\tilde{\mu}$ be solutions to (2.214) corresponding to coefficients (β, γ) and $(\tilde{\beta}, \tilde{\gamma})$, respectively, where (2.209) holds for both β and $\tilde{\beta}$.*

Let us define $\mu_0 = \mu|_{\partial\Omega}$ and $\tilde{\mu}_0 = \tilde{\mu}|_{\partial\Omega}$, thus $\mu_0, \tilde{\mu}_0 \in C^k(\partial\Omega)$. Then there is a constant C such that

$$\|\mu - \tilde{\mu}\|_{C^{k-1}(\tilde{\Omega})} \leq C\|\mu_0\|_{C^k(\partial\Omega_+)}(\|\beta - \tilde{\beta}\|_{C^k(\tilde{\Omega})} + \|\gamma - \tilde{\gamma}\|_{C^{k-1}(\tilde{\Omega})})$$
$$+ C\|\mu_0 - \tilde{\mu}_0\|_{C^k(\partial\Omega_+)}. \quad (2.218)$$

Proof. By the method of characteristics, $\mu(x)$ is determined explicitly in (2.77), while $\tilde{\mu}(x)$ has a similar expression.

$$|\mu(x) - \tilde{\mu}(x)| \leq \left|(\mu_0(x_+(x)) - \tilde{\mu}_0(\tilde{x}_+(x)))e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds}\right| +$$
$$\left|\tilde{\mu}_0(\tilde{x}_+(x))\left(e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds} - e^{-\int_0^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\theta}_x(s))ds}\right)\right|$$

Applying Lemma 2.4.11, we deduce that

$$|D_x^{k-1}[\mu_0(x_+(x)) - \tilde{\mu}_0(\tilde{x}_+(x))]| \leq \|\mu_0 - \tilde{\mu}_0\|_{C^{k-1}(\partial\Omega)}$$
$$+ C\|\mu_0\|_{C^{k-1}(\partial\Omega_+)}\|\beta - \tilde{\beta}\|_{C^{k-1}(\tilde{\Omega})}.$$

This proves the $\mu_0(x_+(x))$ is stable. To consider the second term, by the Leibniz rule it is sufficient to prove the stability result for $\int_0^{t_+(x)} \gamma(\theta_x(s))ds$.

Assume without loss of generality that $t_+(x) < \tilde{t}_+(x)$. Then we have, applying (2.81),

$$\int_0^{t_+(x)} [\gamma(\theta_x(s)) - \tilde{\gamma}(\tilde{\theta}_x(s))]ds = \int_0^{t_+(x)} [(\gamma(\theta_x(s)) - \gamma(\tilde{\theta}_x(s))) + (\gamma - \tilde{\gamma})(\tilde{\theta}_x(s))]ds$$
$$\leq C\|\gamma\|_{C^0(\tilde{\Omega})}\|\beta - \tilde{\beta}\|_{C^0(\tilde{\Omega})} + C\|\gamma - \tilde{\gamma}\|_{C^0(\tilde{\Omega})}.$$

Derivatives of order $k - 1$ of the above expression are uniformly bounded since $t_+(x) \in C^{k-1}(\tilde{\Omega})$, γ has C^k derivatives bounded on $\tilde{\Omega}$ and $\theta_x(t)$ is stable as in (2.81).

It remains to handle the term

$$v(x) = \int_{t_+(x)}^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\theta}_x(s))ds.$$

$\tilde{\beta}$ and $\tilde{\gamma}$ are of class $C^k(\Omega)$, then so is the function $x \to \tilde{\gamma}(\tilde{\theta}(s))$. Derivatives of order $k - 1$ of $v(x)$ involve terms of size $\tilde{t}_+(x) - t_+(x)$ and terms of form

$$D_x^m\left(\tilde{t}_+ D_x^{k-1-m}\tilde{\gamma}(\tilde{\theta}_x(\tilde{t}_+)) - t_+ D_x^{k-1-m}\tilde{\gamma}(\tilde{\theta}_x(t_+))\right), \quad 0 \leq m \leq k - 1.$$

Since the function has $k - 1$ derivatives that are Lipschitz continuous, we thus have

$$|D_x^{k-1} v(x)| \leq C \|\tilde{t}_+ - t_+\|_{C^{k-1}(\Omega)}.$$

The rest of the proof follows Lemma 2.4.11. $\qquad\square$

With all prepared, we are ready to prove the following stability result.

Theorem 2.4.13. *Let $k \geq 3$. Assume that (μ, q) and $(\tilde{\mu}, \tilde{q})$ are in $\mathcal{P}$ and that $|g_j - \check{u}_j|_{\partial\Omega}$, $j = 1, 2$, are sufficiently small. Let $\mu, \tilde{\mu}$ be solutions of (2.214) with $(\beta, \gamma), (\tilde{\beta}, \tilde{\gamma})$ and h sufficiently small. Assume on the boundary, $\mu_0 = \tilde{\mu}_0$. Then, restricting to $\tilde{\Omega}$, we have that*

$$\|\mu - \tilde{\mu}\|_{C^{k-1}(\tilde{\Omega})} + \|q - \tilde{q}\|_{C^{k-3}(\tilde{\Omega})} \leq C \|d - \tilde{d}\|_{(C^k(\tilde{\Omega}))^2}. \tag{2.219}$$

Proof. The first part follows directly from (2.207) and Proposition 2.4.12. This provides a stability result for $\nu = 1/\mu$ and thus for $u_j = \nu d_j$. Notice $\check{u}_j$ is non-vanishing on $\tilde{\Omega}$. So when choosing $|g_j - \check{u}_j|_{\partial\Omega}$ sufficiently small, the arguments in (2.211) and (2.212) show that u_j is non-vanishing in $\tilde{\Omega}$. Thus $\Delta u_j + u_j q = 0$ gives the stability control of q. $\qquad\square$

Theorem 2.4.14. *Let $k \geq 3$. Let (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ be in $\mathcal{M}$. We assume that $D|_{\partial\Omega} = \tilde{D}_{\partial\Omega}$. Then there is an open set of 2 real-valued boundary conditions $g \in C^{k,\alpha}(\partial\Omega)$ for $\alpha > 1/2$ such that, restricting to $\tilde{\Omega}$, we have the stability estimate*

$$\|D - \tilde{D}\|_{C^{k-1}(\tilde{\Omega})} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(\tilde{\Omega})} \leq C \|d - \tilde{d}\|_{(C^k(\tilde{\Omega}))^2}. \tag{2.220}$$

Proof. The main result consists of getting the stability on D mentioned above. Since $k \geq 3$, we have stability of the reconstruction of $q \in C^{k-3}(\tilde{\Omega})$ and $\mu \in C^{k-1}(\tilde{\Omega})$ provided that the boundary conditions are well-chosen. The we have

$$-(\Delta + q)(\sqrt{D} - \sqrt{\tilde{D}}) = \mu - \tilde{\mu} + (q - \tilde{q})\sqrt{\tilde{D}}.$$

By elliptic regularity, we deduce that $(\sqrt{D} - \sqrt{\tilde{D}})$ is bounded in $C^{k-1}(\tilde{\Omega})$, and hence the result. $\qquad\square$

2.4.4   Stability Result for $2n$ complex observations

We now consider the case that $4n$ real-valued observations are taken, with which we can construct $2n$ sets of complex-valued boundary and internal data, denoted as $g_{1,2}^j = \{g_1^j, g_2^j\}$ and $d_{1,2}^j = \{d_1^j, d_2^j\}$. For the rest of this section, we choose $j = 1, \ldots, n$.

Let H be a hyperplane in $\mathbb{R}^n \backslash \overline{\text{ch}(\Omega)}$. Choose $x^j \in H$, such that $\{x^j - x^1\}$ form a basis of H, thus span$\{x^j - x^1\}$ has dimension $n - 1$. Then for $\forall x \in \Omega$, $\{x^j - x\}$ form a basis of $\mathbb{R}^n$. In fact, since $x \notin H$,

$$\text{span}\{x^j - x\} = \text{span}\{x^j - x^1, x^1 - x\}$$

has dimension m.

Define

$$\varphi^j = \log |x - x^j|, \qquad \psi = \text{dist}\left(\frac{x - x^j}{|x - x^j|}, \omega\right), \qquad j = 1, \ldots, n.$$

Then the matrix $B_\varphi := (\nabla \varphi^j)$ is invertible.

Corresponding to x^j, φ^j and ψ^j, we can define the front and back sides of the boundary, $\partial \Omega_+^j$ and $\partial \Omega_-^j$, by (2.79), and define the CGO solutions, $\check{u}_{1,2}^j$, by (2.202).

Let us now choose boundary conditions $g_{1,2}^j$ close to $\check{u}_{1,2}^j$ on $\partial \Omega$, precisely, by (2.211). With internal data $d_{1,2}^j$, we can define β^j by (2.207) and (2.208).

Proposition 2.4.8 shows that the matrix $B = (\beta^j)$ is close to an invertible matrix B_φ on a subregion $\tilde{\Omega}$. Therefore, we can make B invertible on $\tilde{\Omega}$ with inverse of class $C^k(\tilde{\Omega})$ by choosing h sufficiently small. Consequently, (2.214) can be rewrite

$$\nabla \mu + \Upsilon \mu = 0, \tag{2.221}$$

where Υ is a vector-valued function in $(C^k(\tilde{\Omega}))^n$. Finally, the construction of Υ is stable under small perturbations in the data d^j. Indeed, let β and $\tilde{\beta}$ be two vector fields constructed from the internal measurements $d_{1,2}^j$ and $\tilde{d}_{1,2}^j$, respectively. Then when h is sufficiently small,

$$\|\beta - \tilde{\beta}\|_{(C^k(\tilde{\Omega}))^n} \leq \|d - \tilde{d}\|_{(C^{k+1}(\tilde{\Omega}))^n}. \tag{2.222}$$

Now we consider equation (2.221) with boundary condition $\mu = \mu_0$. Assume Ω is bounded and connected and $\partial \Omega$ is smooth. Let $x \in \tilde{\Omega}$. Find a smooth curve from x

to a point on the boundary. Restricted to this curve, (2.221) is a stable ordinary differential equation. Keep the curve fixed. Let $\mu, \tilde{\mu}$ be solutions to (2.221) with respect to $\beta, \tilde{\beta}$, respectively. Assume $\mu_0 = \tilde{\mu}_0$ on $\partial\Omega$. By solving the equation explicitly and applying (2.222), we find that

$$\|\mu - \tilde{\mu}\|_{C^k(\tilde{\Omega})} \leq \|d - \tilde{d}\|_{C^{k+1}(\tilde{\Omega})}. \tag{2.223}$$

We can now state the main theorem of this section.

Theorem 2.4.15. *Let $k \geq 2$. Assume that (μ, q) and $(\tilde{\mu}, \tilde{q})$ are in $\mathcal{P}$ and that we have $2n$ well-chosen complex valued measurement, such that $|g_{1,2}^j - \check{u}_{1,2}^j|_{\partial\Omega}$, $j = 1, \ldots, n$, are sufficiently small. Let $\mu, \tilde{\mu}$ be solutions of (2.221) with $\beta, \tilde{\beta}$ and h sufficiently small. Assume on the boundary, $\mu_0 = \tilde{\mu}_0$. Then we have that*

$$\|\mu - \tilde{\mu}\|_{C^k(\tilde{\Omega})} + \|q - \tilde{q}\|_{C^{k-2}(\tilde{\Omega})} \leq C\|d - \tilde{d}\|_{(C^{k+1}(\tilde{\Omega}))^2}. \tag{2.224}$$

Proof. The first result is directly from (2.223). The proof of the stability of q is exactly that same as in the proof of Theorem 2.4.13. □

Theorem 2.4.16. *Let $k \geq 3$ and assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D|_{\partial\Omega} = \tilde{D}|_{\partial\Omega}$. Then there is an open set of $2n$ real-valued boundary conditions $g \in C^{k,\alpha}(\partial\Omega)$ for $\alpha > 1/2$ such that, we have the stability estimate in $\tilde{\Omega}$ as*

$$\|D - \tilde{D}\|_{C^{k-1}(\tilde{\Omega})} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(\tilde{\Omega})} \leq C\|d - \tilde{d}\|_{(C^k(\tilde{\Omega}))^2}. \tag{2.225}$$

Proof. The proof is the same as that of Theorem 2.4.14. □

2.5 Quantitative thermo-acoustic tomography(QTAT)

In thermo-acoustic tomography, low-frequency radiation (microwave with wavelengths comparable to 1m) is used. The propagation of such electromagnetic radiation is modeled by

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E + \sigma\mu\frac{\partial}{\partial t}E + \nabla \times \nabla \times E = S(t,x), \tag{2.226}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$. Here, $c^2 = (\epsilon\mu)^{-1}$ is the light speed in the domain of interest, ϵ the permittivity, μ the permeability, and $\sigma = \sigma(x)$ the unknown conductivity.

Let us consider the scalar approximation to the above problem:

$$\frac{\partial^2}{\partial t^2}u + \sigma\mu\frac{\partial}{\partial t}u - \Delta u = S(t,x) \tag{2.227}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ for $n \geq 2$, an arbitrary dimension. We assume that $S(t,x)$ is a narrow-band pulse with central frequency ω_c of the form $S(t,x) = -e^{i\omega_c t}\phi(t)S(x)$, where $\phi(t)$ is the envelope of the pulse and $S(x)$ is a superposition of plane waves with wavenumber k such that $\omega_c = ck$. By taking the Fourier transform of (2.227), we get that

$$u(t,x) \sim \phi(t)u(x), \tag{2.228}$$

where $u(x)$ is given by

$$\Delta u + (k^2 + ikc\mu\sigma)u = S(x). \tag{2.229}$$

The amount of absorbed radiation by internal tissue as the electromagnetic waves propagate is given by

$$d(t,x) = \sigma(x)|u(t,x)|^2 \sim \phi(t)^2\sigma(x)|u(x)|^2. \tag{2.230}$$

We assume the recover of the internal measurement $d(x)$ is performed in the first step of the hybrid method. Therefore, QTAT can be formulated as

$$\begin{cases} \Delta u + q(x)u = 0, & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.231}$$

where g is the boundary illumination of u and $q(x) = k^2 + ikc\mu\sigma$. The internal data are then of the form

$$d(x) = \sigma|u|^2(x). \tag{2.232}$$

The inverse problem in QTAT is to recover σ from measurement $(d(x), g(x))$.

We define the spaces H_δ^s as in (2.52) and $\mathcal{M}$ as the space of functions in $H^p(\Omega)$, $p \geq \frac{n}{2}$, with norm bounded by a fixed $M > 0$. The uniqueness and stability results of the reconstruction are obtained by G. Bal, etc [6]. An explicit reconstruction algorithm is also given.

Theorem 2.5.1. *Let $\rho\mathbb{C}^n$ be such that $|\rho|$ is sufficiently large and $\rho \cdot \rho = 0$. Let σ and $\tilde{\sigma}$ be function in $\mathcal{M}$.*

Let $g \in H^{p-\frac{1}{2}}(\partial\Omega)$ be a given illumination and $H(x)$ be the internal measurement of the form in (2.232) for u solution of (2.231). Let $\tilde{H}(x)$ be the measurement constructed by replacing σ by $\tilde{\sigma}$ in (2.231) and (2.232).

Then there is an open set of illuminations $g \in H^{p-\frac{1}{2}}(\partial\Omega)$ such that $H(x) = \tilde{H}(x)$ in $H^p(\Omega)$ implies that $\sigma(x) = \tilde{\sigma}(x)$ in $H^p(\Omega)$. Moreover, there exists a constant C independent of σ and $\tilde{\sigma}$ such that

$$\|\sigma - \tilde{\sigma}\|_{H^p(\Omega)} \leq C\|H - \tilde{H}\|_{H^p(\Omega)}. \tag{2.233}$$

More precisely, we can write the reconstruction of σ as finding the unique fixed point to the equation

$$\sigma(x) = e^{(\rho+\bar{\rho})\cdot x}H(x) - \mathcal{H}_g[\sigma](x), \quad \text{in } H^p(\Omega), \tag{2.234}$$

The functional $\mathcal{H}_g[\sigma]$ defined as

$$\mathcal{H}_g[\sigma](x) = \sigma(x)(\psi_g(x) + \overline{\psi}_g(x) + \psi_g(x)\overline{\psi}_g(x)), \tag{2.235}$$

is a contraction map for g in the open set described above, where ψ_g is define as the solution to

$$(\Delta + 2\rho \cdot \nabla)\psi_g = -q(1 + \psi_g), \quad \text{in } \Omega, \qquad \psi_g = e^{-\rho\cdot x}g - 1 \quad \text{on } \partial\Omega. \tag{2.236}$$

We thus deduce the reconstruction algorithm

$$\sigma = \lim_{m \to \infty}, \tag{2.237}$$

where

$$\sigma_0 = 0, \quad \sigma_m(x) = e^{-(\rho+\bar{\rho})\cdot x}H(x) - \mathcal{H}_g[\sigma_{m-1}](x), \quad m \geq 1. \tag{2.238}$$

Chapter 3

ELECTRO-SEISMIC CONVERSION

When a porous rock is saturated with an electrolyte, an electric double layer is formed at the interface of the solid and the fluid. One side of the interface is negatively charged and the other side is positively charged. Such electric double layer(EDL) system is also called Debye layer. Due to the EDL system, electromagnetic(EM) fields and mechanical waves are coupled through the phenomenon of electro-kinetics. Precisely, electrical fields or EM waves acting on the EDL will move the charges, creating relative movement of fluid and solid. This is called electro-seismic conversion. Conversely, mechanical waves moving fluid and solid will generate EM fields. This is called seismo-electric conversion. Thompson and Gist [42] have made field measurement clearly demonstrating seismo-electric conversion in saturated sediments. Zhu et al. [51, 52, 53] made laboratory experiments and observed the seismo-electric conversion in model wells, and their experimental results confirm that seismo-electric logging could be a new bore-hole logging technique.

The investigation of wave propagation in fluid-saturated porous media was early developed by Biot [8, 9]. The governing equations of the electro-seismic conversion was derived by Pride [29] as following.

$$\nabla \times E = i\omega\mu H, \tag{3.1}$$

$$\nabla \times H = (\sigma - i\epsilon\omega)E + L(-\nabla p + \omega^2\rho_f u) + J_s, \tag{3.2}$$

$$-\omega^2(\rho u + \rho_f w) = \nabla \cdot \tau, \tag{3.3}$$

$$-i\omega w = LE + \frac{\kappa}{\eta}(-\nabla p + \omega^2\rho_f u), \tag{3.4}$$

$$\tau = (\lambda\nabla \cdot u + C\nabla \cdot w)I + G(\nabla u + \nabla u^T), \tag{3.5}$$

$$-p = C\nabla \cdot u + M\nabla \cdot w, \tag{3.6}$$

where the first two are Maxwell's equations, the remaining are Biot's equations. The notations are as follows:

E electric field,

H magnetizing field or magnetic field intensity,

ω seismic wave frequency,

σ conductivity,

ϵ dielectric constant or relative permittivity,

μ magnetic permeability,

J_s source current,

p pore pressure,

ρ_f density of pore fluid,

L electro-kinetic mobility parameter,

κ fluid flow permeability,

u the displacement of the solid frame,

w a vector function defined by $w = \beta(u - u_f)$, where u_f is the fluid displacement and β is the porosity fo the medium,

τ bulk stress tensor,

η viscosity of pore fluid,

λ, G Lamé parameters of elasticity,

C, M Biot moduli parameters.

Pride and Haartsen [30] also analyzed the basic properties of seismo-electric waves.

Notice that the coupling is non-linear, namely electro-seismic and seismo-electric conversions happen simultaneously. Under the assumptions that the coupling is so weak that multiple coupling is neglectable, we can linearize the forward system in two steps. Particularly, we focus on the electro-seismic conversion and ignore the seismo-electric conversion. The first step in the forward system is modeled by Maxwell equations without the effect of the seismic waves, i.e., $L = 0$ in (3.2). While the electro-seismic conversion happens, the seismic waves are generated and modeled by Biot's equations with potential LE in (3.4).

In the present work, we mainly focus on the inverse problem of the linearized electro-seismic conversion, which is a hybrid problem and consists of two steps. The first step of the inverse problem is the inversion of Biot's equations to recover the source potential LE in (3.4) from any measurements observed on the domain boundary. Williams [45] presented an approximation to Biot's equations, which reduce Biot's system to the inhomogeneous Helmholtz equation.

Assuming the first step is implemented successfully, the second step of the inverse problem is to invert Maxwell's equations, which consists of reconstructing the conductivity σ and the electro-kinetic mobility parameter or the coupling coefficient L from boundary measurements of the electrical fields and the internal data LE obtained in the first step.

The problem of interest in our work is the second step of the inverse problem. We study the reconstruction of the conductivity σ and the coupling coefficient L and prove uniqueness and stability results of the reconstructions. Particularly, we show that σ, L are uniquely determined by 2 well-chosen electrical fields at the domain boundary. The explicit reconstruction procedure is presented. The stability of the reconstruction is established from either 2 measurements under geometrical conditions or from 6 well-chosen boundary conditions.

Mathematically, our proof relies on explicit solutions to Maxwell's equations, namely Complex Geometrical Optics (CGO) solutions, constructed by Colton and Päivärinta[12]. In our reconstruction procedure, the coupling coefficient L satisfies a transport equation with vector field β. With CGO solutions, we can prove the integral curves of the vector field β are close to straight lines and exit the domain in finite time. Therefore, L can be

uniquely and explicitly solved by the characteristic method. Stability follows the analysis of the method of characteristic.

The rest of the chapter is structured as follows. The inversion of Biot's equations is analyzed in Section 3.1. We focus on the second step of the inverse problem in Section 3.2. Section 3.2.1 presents our main uniqueness and stability results of the reconstructions. The CGO solutions are introduced in section 3.2.2. The inverse Maxwell's equations and an explicit reconstruction algorithm are addressed in the rest of section 3.2, while section 3.2.3 focusing on the proof of the uniqueness result and section 3.2.4 and 3.2.5 focusing on the stability proof.

3.1 Biot's system and its inversion

The wave propagation in a porous medium saturated with fluid is mainly modeled by Biot's system. Considering the electro-seismic effect, we have an internal potential $D = LE$ in (3.8) as follows

$$-\omega^2(\rho u + \rho_f w) = \nabla \cdot \tau, \tag{3.7}$$

$$-i\omega w = D + \tfrac{\kappa}{\eta}(-\nabla p + \omega^2 \rho_f u), \tag{3.8}$$

$$\tau = (\lambda \nabla \cdot u + C \nabla \cdot w)I + G(\nabla u + \nabla u^T), \tag{3.9}$$

$$-p = C \nabla \cdot u + M \nabla \cdot w. \tag{3.10}$$

The inverse problem we proposed is to recover the internal potential from any possible boundary measurements of the acoustic waves.

The direct inversion of Biot's system is very challenging due to the facts that (1) the seismic wave generated by electro-seismic conversion is weak, and (2) the Biot slow wave is a diffusive wave, which decays rapidly to zero with propagation distance and is therefore difficult to observe.

Williams [45] presented the effective density fluid model(EDFM), which is an very accurate approximation to Biot's equations and could simplify the formulation.

Let $u = u_s$ be the displacement of the solid frame and u_f be the displacement of the saturated fluid. Define $w = \beta(u_s - u_f)$ with β the porosity of the medium. Stoll [39]

introduced potentials $(\Phi_s, \Phi_f, \Psi_s, \Psi_f)$ defined by

$$u = \nabla \Phi_s + \nabla \times \Psi_s, \tag{3.11}$$

$$w = \nabla \Phi_f + \nabla \times \Psi_f.$$

Assume $D = 0$ for now. By substituting (3.11) into (3.7)-(3.10), we have

$$H \Delta^2 \Phi_s + C \Delta^2 \Phi_f = -\omega \rho \Delta \Phi_s - \omega^2 \rho_f \Delta \Phi_f, \tag{3.12}$$

$$C \Delta^2 \Phi_s + M \Delta^2 \Phi_f = -\omega^2 \rho_f \Delta \Phi_s - \frac{i\omega\eta}{\kappa} \Delta \Phi_f. \tag{3.13}$$

We have notations as the following

$$
\begin{aligned}
\rho &= \beta \rho_f + (1 - \beta) \rho_s, \\
H &= \lambda + 2G = [(K_r - K_b)^2/(\Xi - K_b)] + K_b + 4\mu/3, \\
C &= K_r(K_r - K_b)/(\Xi - K_b), \\
M &= K_r^2/(\Xi - K_b), \\
\Xi &= \dot{K}_r[1 + \beta(K_r/K_f - 1)],
\end{aligned}
\tag{3.14}
$$

where ρ is the total mass density, K_r is the bulk modulus of individual sediment grains, and K_f is the bulk modulus of the pore fluid.

By the fact that for sand sediments the frame and shear moduli are much lower than other moduli, we take $K_b = \mu = 0$, which also implies that the slow wave and shear wave are neglected in this model. We then have

$$H = C = M = \left(\frac{1 - \beta}{K_r} + \frac{\beta}{K_f} \right)^{-1}. \tag{3.15}$$

Let $U = \nabla^2 \Phi_s$ and $W = \nabla^2 \Phi_f$. Taking the Fourier transform of (3.12) and (3.13) gives that

$$H k^2 \hat{U} + H k^2 \hat{W} = \omega \rho \hat{U} + \omega^2 \rho_f \hat{W}, \tag{3.16}$$

$$H k^2 \hat{U} + H k^2 \hat{W} = \omega^2 \rho_f \hat{U} + \frac{i\omega\eta}{\kappa} \hat{W}. \tag{3.17}$$

Williams [45] introduce the effective mass density ρ_{eff} in such a way that

$$k^2 = \frac{\omega^2 \rho_{\text{eff}}(\omega)}{H} \tag{3.18}$$

or precisely

$$\rho_{\text{eff}}(\omega) = \rho_f \left(\frac{\omega^2 \rho_f^2 + \rho \frac{i\eta}{\kappa}}{2\omega^2 \rho_f - \omega\rho + \frac{i\omega\eta}{\kappa}} \right). \tag{3.19}$$

Using (3.17) and (3.18), $\hat{W}$ can be found in terms of $\hat{U}$,

$$\hat{W} = \left(\frac{\rho_f - \rho_{\text{eff}}(\omega)}{\rho_{\text{eff}}(\omega) - \frac{i\eta}{\omega\kappa}} \right) \hat{U}. \tag{3.20}$$

Then using (3.19), (3.20) and (3.16), we can derive that

$$k^2 H(\hat{U} + \hat{W}) = \omega^2 \rho_{\text{eff}}(\omega)(\hat{U} + \hat{W}). \tag{3.21}$$

The inverse Fourier transform gives that

$$H\nabla^2 \Phi + \omega^2 \rho_{\text{eff}}(\omega)\Phi = 0, \tag{3.22}$$

where $\Phi = U + W$.

In the case when $D \neq 0$, we also define $\rho_{\text{eff}}(\omega)$ as in (3.19). We will then get

$$H\nabla^2 \Phi + \omega^2 \rho_{\text{eff}}(\omega)\Phi = \nabla \cdot D. \tag{3.23}$$

The inverse problem now becomes the inversion of the inhomogeneous Helmholtz equation to recover $\nabla \cdot D$, with which we can calculate D.

3.2 Inversion of Maxwell's equations with internal data

In this section, we study the second step of the linearized electro-seismic conversion. Particularly, we invert the Maxwell's equations and recover the coupling coefficient L and medium conductivity σ from the boundary measurement of the electrical field E and the internal data $D = LE$ from the first step. We prove the reconstruction is unique and stable with some boundary conditions of E prescribed by CGO solutions.

Section 3.2.1 presents our main uniqueness and stability results. Section 3.2.2 is devoted to CGO solutions to Maxwell's equations as a prerequisite section. We study the reconstruction procedure and its uniqueness in Section 3.2.3. The stable results are proved in Section 3.2.4 and 3.2.5.

3.2.1   Main results

Let Ω be an open, bounded and connected domain in $\mathbb{R}^3$ with C^2 boundary $\partial\Omega$. In the second step of the electro-seismic conversion, the propagation of the electrical fields is modeled by Maxwell's equations in Ω,

$$
\begin{cases}
\nabla \times E & = & i\omega\mu H, \\
\nabla \times H & = & (\sigma - i\epsilon\omega)E + J_s.
\end{cases}
\tag{3.24}
$$

In the case when $\mu \equiv \mu_0$ is constant and $J_s = 0$ in Ω, we can rewrite the system in (3.24) as

$$
\nabla \times \nabla \times E - k^2 nE = 0,
\tag{3.25}
$$

and

$$
\nabla \cdot nE = 0
\tag{3.26}
$$

where the wave number $k > 0$ and the refractive index n are given by

$$
k = \omega\sqrt{\epsilon_0\mu_0}, \quad n = \frac{1}{\epsilon_0}\left(\epsilon + i\frac{\sigma}{\omega}\right).
\tag{3.27}
$$

The measurements available for the inverse problem include the internal data from the first step

$$
D := LE, \quad \text{in } \Omega
\tag{3.28}
$$

and the boundary illumination, or precisely, the tangential boundary measurement of the electrical field

$$
G := tE, \quad \text{on } \partial\Omega.
\tag{3.29}
$$

Define the operator

$$
\Lambda(L, \sigma) := (J_s, D, G).
\tag{3.30}
$$

The problem now is to invert the operator Λ_M, or namely, to reconstruct (L, σ) from some measurements (J_s, D, G), assuming μ and ϵ are given.

We define the set of coefficients $(L, \sigma) \in \mathcal{M}$ as

$$
\mathcal{M} = \{(L, \sigma) \in C^{d+1} \times H^{\frac{3}{2}+3+d+\iota} :
\tag{3.31}
$$

$$
\text{and } 0 \text{ is not an eigenvalue of } \nabla \times \nabla \times \cdot - k^2 n\},
$$

where the wave number $k > 0$, $\iota > 0$ is small and the refractive index n are given by (3.27).

The main results are as follows, where the measurements G and D are complex-valued.

Theorem 3.2.1. *Let Ω be an open, bounded subset of $\mathbb{R}^3$ with boundary $\partial\Omega$ of class C^k. Let (L,σ) and $(\tilde{L},\tilde{\sigma})$ be two elements in $\mathcal{M}$. Let $D := (D_1, D_2)$ and $\tilde{D} := (\tilde{D}_1, \tilde{D}_2)$, be two sets of internal data on Ω for the coefficients (L,σ), $(\tilde{L},\tilde{\sigma})$, respectively and with boundary illuminations $G := (G_1, G_2)$.*

Then there is a subset of $G \in (C^{d+4}(\partial\Omega))^2$, such that if $D_j = \tilde{D}_j$, $j = 1,2$, we have $(L,\sigma) = (\tilde{L}, \tilde{\sigma})$.

Here and in the following, we shall abuse the notation and use $C^d(\overline{\Omega})$ to denote either set of complex-valued functions or set of vector-valued functions whose elements have up to d order continuous derivatives. It should be clear from the context which one it is. The function space $(C^{d+4}(\partial\Omega))^2$ is an abbreviation of the product space $C^{d+4}(\partial\Omega) \times C^{d+4}(\partial\Omega)$.

To consider the stability of the reconstruction, we need to restrict to a subset of Ω. Let ζ_0 be a constant unit vector. Let $x_0 \in \partial\Omega$ be the tangent point of $\partial\Omega$ with respect to ζ_0, i.e., $\nu(x_0) \cdot \zeta_0 = 0$, where $\nu(x_0)$ is the exterior norm vector of $\partial\Omega$ at x_0. Define Ω_1 to be the subset of Ω by removing a neighborhood of each tangent point $x_0 \in \partial\Omega$.

Theorem 3.2.2. *let $k \geq 3$. Let Ω be convex with C^k boundary $\partial\Omega$ and Ω_1 is defined as above. Assume that (L,σ) and $(\tilde{L},\tilde{\sigma})$ are two elements in $\mathcal{M}$. Let $D = (D_j)$ and $\tilde{D} = (\tilde{D}_j)$, $j = 1,2$, be the internal data for coefficients (L,σ) and $(\tilde{L},\tilde{\sigma})$, respectively, with boundary conditions $G = (G_j)$, $j = 1,2$.*

Then there is a set of illuminations $G \in (C^{d+4}(\partial\Omega))^2$ such that restricting to Ω_1, we have

$$\|L - \tilde{L}\|_{C^{k-1}(\Omega_1)} + \|\sigma - \tilde{\sigma}\|_{C^{k-3}(\Omega_1)} \leq C\|D - \tilde{D}\|_{(C^k(\Omega_1))^2}. \tag{3.32}$$

The geometric conditions can be removed when more measurements are available. In particular, when 6 complex measurements are provided, we have the following stability result.

Theorem 3.2.3. *let $k \geq 3$. Let Ω be convex. Assume that (L,σ) and $(\tilde{L},\tilde{\sigma})$ are two elements in $\mathcal{M}$. Let $D = (D_1^j, D_2^j)$ and $\tilde{D} = (\tilde{D}_1^j, \tilde{D}_2^j)$, $j = 1,2,3$, be the internal data for coefficients (L,σ) and $(\tilde{L},\tilde{\sigma})$, respectively, with boundary conditions $G = (G_1^j, G_2^j)$, $j = 1,2,3$.*

Then there is a set of illuminations $G \in (C^{d+4}(\partial\Omega))^6$, such that

$$\|L - \tilde{L}\|_{C^{k-1}(\Omega)} + \|\sigma - \tilde{\sigma}\|_{C^{k-3}(\Omega)} \leq C\|D - \tilde{D}\|_{(C^k(\Omega))^6}. \tag{3.33}$$

Note that the above measurements are all complex-valued. We will need two real measurements to make up one complex data.

3.2.2 Complex Geometrical Optics solutions

We would like to consider the equations (3.25) and (3.26) in the whole $\mathbb{R}^3$. For this purpose, we extend $n \in H^{\frac{3}{2}+3+d+\iota}(\Omega)$ to be a function defined in the whole $\mathbb{R}^3$ in such a way that n is positive, and $1 - n \in H^{\frac{3}{2}+3+d+\iota}(\mathbb{R}^n)$ is compactly supported. We denote this extension still by n.

Colton and Päivärinta[12] constructed explicit solutions, namely *Complex Geometrical Optics* solutions(CGOs), to the Maxwell's equation (3.25) and (3.26). CGOs will be the main technique we will use to solve the inverse Maxwell problem. We follow the construction of CGOs in [12] and extend their properties from L^2 space to higher order sobolev spaces. CGOs are of the form

$$E(x) = e^{i\zeta \cdot x}(\eta + R_\zeta(x)), \tag{3.34}$$

where $\zeta \in \mathbb{C}^3\backslash\mathbb{R}^3$, $\eta \in \mathbb{C}^3$, are constant vectors satisfying

$$\zeta \cdot \zeta = k^2, \ \zeta \cdot \eta = 0. \tag{3.35}$$

Substituting (3.34) into (3.25) and (3.26) gives

$$\tilde{\nabla} \times \tilde{\nabla} \times R_\zeta \ = \ k^2(n-1)\eta + k^2 n R_\zeta, \tag{3.36}$$

$$\tilde{\nabla} \cdot R_\zeta \ = \ -\alpha \cdot (\eta + R_\zeta) \tag{3.37}$$

where $\tilde{\nabla} := \nabla + i\zeta$ and $\alpha := \nabla n(x)/n(x)$. We further define $\tilde{\Delta} := \Delta + 2i\zeta \cdot \nabla - k^2$. By substituting the formula $\tilde{\nabla} \times \tilde{\nabla} \times R_\zeta = -\tilde{\Delta}R_\zeta + \tilde{\nabla}\tilde{\nabla} \cdot R_\zeta$ into (3.36) and (3.37), we see that R_ζ is a solution to

$$(\Delta + 2i\zeta \cdot \nabla)R_\zeta = -\tilde{\nabla}(\alpha \cdot (\eta + R_\zeta)) + k^2(1-n)(\eta + R_\zeta). \tag{3.38}$$

It was proved in [12] the existence of R_ζ to (3.38) as a $C^2(\mathbb{R}^3)$ functions. For our analysis, we need to extend the results of CGOs in [12] to smoother function spaces.

Recall that the spaces L^2_δ and H^s_δ are defined to be the completion of $C^\infty_0(\mathbb{R}^3)$ with respect to norms in (2.51) and (2.52), respectively.

We recall [41] for $|\zeta|$ large and $v \in L^2_{\delta+1}$ with $-1 < \delta < 0$, the equation

$$(\Delta + 2i\zeta \cdot \nabla)u = v \tag{3.39}$$

admits a unique week solution $u \in L^2_\delta$ with

$$\|u\|_{L^2_\delta} \leq C(\delta, c) \frac{\|v\|_{L^2_{\delta+1}}}{|\zeta|}. \tag{3.40}$$

Since $(\Delta + 2i\zeta \cdot \nabla)$ and $(I - \Delta)^s$ are constant coefficient operators and hence commute, we deduce that when $v \in H^s_{\delta+1}$, for $s \geq 0$, then

$$\|u\|_{H^s_\delta} \leq C(\delta, c) \frac{\|v\|_{H^s_{\delta+1}}}{|\zeta|}. \tag{3.41}$$

We define the integral operator $G_\zeta : H^s_{\delta+1}(\mathbb{R}^3) \to H^s_\delta(\mathbb{R}^3)$ by

$$G_\zeta(v) := \mathcal{F}^{-1}\left(\frac{\hat{v}}{\xi^2 + 2\zeta \cdot \xi}\right), \tag{3.42}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. We see that G_ζ is bounded and there exists a positive constant $C(\delta)$ such that

$$\|G_\zeta\| \leq \frac{C}{|\zeta|}. \tag{3.43}$$

Before we can prove the existence of a unique solution to (3.38), we first prove the following lemma. The Lemma 3.1 in [12] proves for the case when $s = 0$. We study any $s = \frac{5}{2} + d + \iota$ here.

Lemma 3.2.4. *For any $v \in H^s_{\delta+1}(\mathbb{R}^3)$ and $|\zeta|$ sufficiently large, the equation*

$$(\Delta + 2i\zeta \cdot \nabla + \alpha \cdot \tilde{\nabla})u = v \tag{3.44}$$

has a unique solution $u \in H^s_\delta(\mathbb{R}^3)$ satisfying

$$\|u + n^{-1/2}G_\zeta(n^{1/2}v)\|_{H^s_\delta} \leq \frac{C}{|\zeta|^2}, \tag{3.45}$$

for some positive constant C independent of ζ.

Proof. From the identity

$$n^{-1/2}(\Delta + 2i\zeta \cdot \nabla)(n^{1/2}u) = (\Delta + 2i\zeta \cdot \nabla + \alpha \cdot \tilde{\nabla})u + qu, \tag{3.46}$$

where $q := \Delta n^{1/2}/n^{1/2} \in H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$, we can rewrite (3.44) as

$$(\Delta + 2i\zeta \cdot \nabla - q)f = g, \tag{3.47}$$

where $f := n^{1/2}u$ and $g := n^{1/2}v$. The assumption on n ensures that $1 - n^{1/2} \in H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ and is compactly supported, so $1 - n^{1/2} \in H^{\frac{5}{2}+d+\iota}_{\delta+1}(\mathbb{R}^3)$. Therefore,

$$g = n^{1/2}v = v - (1 - n^{1/2})v \in H^{\frac{5}{2}+d+\iota}_{\delta+1}(\mathbb{R}^3).$$

Applying the integral operator $-G_\zeta$ gives

$$f + G_\zeta(qf) = -G_\zeta(g). \tag{3.48}$$

Since $q \in H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ is compactly supported, multiplication by q is a bounded operator mapping $H^{\frac{5}{2}+d+\iota}_\delta(\mathbb{R}^3)$ into $H^{\frac{5}{2}+d+\iota}_{\delta+1}(\mathbb{R}^3)$, so $I + G_\zeta(q\cdot)$ is invertible on $H^{\frac{5}{2}+d+\iota}_\delta(\mathbb{R}^3)$ for $|\zeta|$ sufficiently large. This shows that (3.48) has a unique solution f in $H^{\frac{5}{2}+d+\iota}_\delta(\mathbb{R}^3)$, correspondingly $u = n^{-1/2}f \in H^{\frac{5}{2}+d+\iota}_\delta(\mathbb{R}^3)$ is the unique solution of (3.44). Eq. (3.43) also gives

$$\|f + G_\zeta(g)\|_{H^s_\delta} = \|G_\zeta(q(G_\zeta(qf) + G_\zeta(g)))\|_{H^{\frac{5}{2}+d+\iota}_\delta} \leq \frac{C}{|\zeta|^2}, \tag{3.49}$$

for some positive constant C independent of ζ. This proves the lemma. $\qquad\square$

Proposition 3.2.5. *Let $s = \frac{5}{2} + d + \iota$. For $|\zeta|$ sufficiently large, there is a unique solution $R_\zeta \in H^s_\delta(\mathbb{R}^3)$ to (3.38). Thus, the CGO solution E defined by (3.34) satisfies (3.25) and (3.26). Moreover, R_ζ satisfies*

$$\|R - in^{-1/2}G_\zeta(n^{1/2}\alpha \cdot \eta)\zeta\|_{H^s_\delta} = O\left(\frac{1}{|\zeta|}\right). \tag{3.50}$$

Proof. By applying the vector identity

$$\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A, \tag{3.51}$$

we see that

$$\tilde{\nabla}(\alpha \cdot (\eta + R_\zeta)) = \alpha \times (\tilde{\nabla} \times R_\zeta) + (\alpha \cdot \tilde{\nabla})R_\zeta + (R_\zeta \cdot \nabla)\alpha + \tilde{\nabla}(\alpha \cdot \eta). \tag{3.52}$$

The terms which are potentially troublesome are $\alpha \times (\tilde{\nabla} \times R_\zeta)$ and $(\alpha \cdot \tilde{\nabla})R_\zeta$. The latter can be dealt with using Lemma 3.2.4, so we need only consider the term $\alpha \times (\tilde{\nabla} \times R_\zeta)$. Denote $Q := \tilde{\nabla} \times R_\zeta$, by (3.36) we have that

$$\tilde{\nabla} \times Q = k^2(n-1)\eta + k^2 n R_\zeta \tag{3.53}$$

and hence

$$\tilde{\nabla} \times \tilde{\nabla} \times Q = k^2 \nabla n \times (\eta + R_\zeta) + k^2(n-1)\zeta \times \eta + k^2 n Q. \tag{3.54}$$

Since $\tilde{\nabla} \cdot Q = 0$, we now have

$$\Delta Q + 2i\zeta \cdot \nabla Q = k^2 \nabla m \times (\eta + R_\zeta) + k^2(1-n)(i\zeta \times \eta + Q). \tag{3.55}$$

Rearrange the terms to get

$$(\Delta + 2i\zeta \cdot \nabla - k^2(1-n))Q = k^2 \nabla(1-n) \times (\eta + R_\zeta) + k^2(1-n)(i\zeta \times \eta). \tag{3.56}$$

Applying $-G_\zeta$ to this identity yields

$$Q = -(I + k^2 G_\zeta(1-n))^{-1} G_\zeta(k^2 \nabla(1-n) \times (\eta + R_\zeta) + k^2(1-n)(i\zeta \times \eta)) \tag{3.57}$$

for large $|\zeta|$, since the operator $I + k^2 G_\zeta(1-n)$ is invertible on $H_\delta^s(\mathbb{R}^3)$ for large $|\zeta|$. On the other hand, applying $-G_\zeta$ to (3.38) and using (3.52) gives

$$R_\zeta = G_\zeta[\alpha \times Q] + G_\zeta[(\alpha \cdot \tilde{\nabla})R_\zeta] + G_\zeta[(R_\zeta \cdot \nabla)\alpha] + G_\zeta[\tilde{\nabla}\alpha \cdot \eta] - k^2 G_\zeta[(1-n)(\eta + R_\zeta)] \tag{3.58}$$

where Q is given by (3.57). The integral equation (3.57), (3.58) has a unique solution in $H_\delta^s(\mathbb{R}^3)$ due to (3.43) and Lemma 3.2.4.

Finally, we use the unique solvability of (3.57) and (3.58) to deduce the unique solvability of (3.38). To do this, let $B \subset \mathbb{R}^3$ be a ball containing Ω. Applying G_ζ to (3.38) yields

$$R_\zeta = G_\zeta[\tilde{\nabla}(\alpha \cdot (\eta + R_\zeta))] - k^2 G_\zeta[(1-n)(\eta + R_\zeta)]. \tag{3.59}$$

This integral equation is of Fredholm type in $H_\delta^s(B)$ as both $G_\zeta(1-n)$ and $R_\zeta \mapsto G_\zeta[\tilde{\nabla}(\alpha \cdot R_\zeta)]$ are smoothing operators. Now, suppose R_ζ^h is a solution of the following homogeneous equation in $H_\delta^s(B)$,

$$R_\zeta^h = G_\zeta[\tilde{\nabla}(\alpha \cdot R_\zeta^h)] - k^2 G_\zeta[(1-n)R_\zeta^h]. \tag{3.60}$$

Then R_ζ^h also satisfies the homogeneous equation corresponding to (3.57), (3.58). Since these equations are uniquely solvable, we conclude that $R_\zeta^h = 0$. Therefore, by Fredholm alternative, we conclude that (3.59) admits a unique solution in $H^s(B)$. Defining $R_\zeta(x)$ for $x \in \mathbb{R}^3$ by the right-hand side of (3.59) and recalling $\Omega \subset B$ yields a solution of (3.59), which is defined in $\mathbb{R}^3$. This shows that (3.38) has a unique solution in $H_\delta^s(\mathbb{R}^3)$.

Furthermore, by Lemma 3.2.4, we see that

$$\|R + n^{-1/2}G_\zeta(n^{1/2}[-\alpha \times (\tilde{\nabla} \times R_\zeta) - (R_\zeta \cdot \nabla)\alpha \tag{3.61}$$
$$- \tilde{\nabla}(\alpha \cdot \eta) + k^2(1-n)(\eta + R_\zeta)])\|_{H_\delta^s} = O\left(\frac{1}{|\zeta|^2}\right).$$

Substituting $\tilde{\nabla}(\alpha \cdot \eta) = (\nabla + i\zeta)(\alpha \cdot \eta)$, (3.61) implies that

$$\|R + in^{-1/2}G_\zeta(n^{1/2}\alpha \cdot \eta)\zeta\|_{H_\delta^s} \le \|n^{-1/2}G_\zeta(n^{1/2}[-\alpha \times (\tilde{\nabla} \times R_\zeta) \tag{3.62}$$
$$- (R_\zeta \cdot \nabla)\alpha - \nabla\alpha \cdot \eta) + k^2(1-n)(\eta + R_\zeta)])\|_{H_\delta^s} + O\left(\frac{1}{|\zeta|^2}\right).$$
$$= O\left(\frac{1}{|\zeta|}\right) \tag{3.63}$$

This complete the proof.

$\square$

By applying Sobolev embedding theorem to on a bounded domain, we have the estimate in $C^{d+1}(\overline{\Omega})$.

Corollary 3.2.6. *Let Ω be an open, bounded domain. With same hypotheses as the previous proposition, we then have*

$$\|R_\zeta - in^{-1/2}G_\zeta(n^{1/2}\alpha \cdot \eta)\zeta\|_{H^s(\Omega)} \le \frac{C}{|\zeta|}, \tag{3.64}$$

for some positive constant C independent of ζ. Moreover, when $s = \frac{5}{2} + d + \iota$, we also have

$$\|R_\zeta - in^{-1/2}G_\zeta(n^{1/2}\alpha \cdot \eta)\zeta\|_{C^{d+1}(\Omega)} \le \frac{C}{|\zeta|}, \tag{3.65}$$

for some positive constant C.

Proposition 3.2.7. *Let $s = \frac{5}{2} + d + \iota$. Suppose $\zeta \in \mathbb{C}^3 \backslash \mathbb{R}^3$, $\eta \in \mathbb{C}^3$, satisfy $\zeta \cdot \zeta = k^2$ and $\zeta \cdot \eta = 0$ such that as $|\zeta| \to \infty$ the limits $\zeta/|\zeta|$ and η exist and,*

$$\left| \zeta/|\zeta| - \zeta_0 \right| = O\left(\frac{1}{|\zeta|}\right), \quad |\eta - \eta_0| = O\left(\frac{1}{|\zeta|}\right). \tag{3.66}$$

$R_\zeta \in H_\delta^s(\mathbb{R}^3)$ is the unique solution to (3.38) in Proposition 3.2.5. For $|\zeta|$ large,

$$\| R_\zeta - i|\zeta| n^{-1/2} G_\zeta (n^{1/2} \alpha \cdot \eta_0) \zeta_0 \|_{C^{d+1}(\Omega)} = O\left(\frac{1}{|\zeta|}\right). \tag{3.67}$$

Proof. The proof follows directly by substituting (3.66) into (3.65). $\qquad\square$

To sum up, we can construct CGO solutions in the form of (3.34). We choose the specific sets of ζ, η as in [12]. Precisely, Let h be a small real parameter and choose arbitrary $a \in \mathbb{R}$. We define ζ_1, ζ_2 and η_1, η_2 by

$$
\begin{aligned}
\zeta_1 &= (a/2, i\sqrt{1/h^2 + a^2/4 - k^2}, 1/h), \\
\zeta_2 &= (a/2, -i\sqrt{1/h^2 + a^2/4 - k^2}, -1/h), \\
\eta_1 &= \frac{1}{\sqrt{1/h^2 + a^2}}(1/h, 0, -a/2), \\
\eta_2 &= \frac{1}{\sqrt{1/h^2 + a^2}}(1/h, 0, a/2),
\end{aligned}
\tag{3.68}
$$

and note that

$$
\begin{aligned}
\lim_{c \to \infty} \eta_j = \eta_0 &:= (1, 0, 0), \quad j = 1, 2, \\
\lim_{c \to \infty} \zeta_1/|\zeta_1| &= \zeta_0 := \frac{1}{\sqrt{2}}(0, i, 1), \\
\lim_{c \to \infty} \zeta_2/|\zeta_2| &= -\zeta_0,
\end{aligned}
\tag{3.69}
$$

and

$$\zeta_1 + \zeta_2 = (a, 0, 0), \quad \zeta_0 \cdot \zeta_0 = 0, \quad \eta_0 \cdot \zeta_0 = 0. \tag{3.70}$$

Proposition 3.2.7 implies that

$$(\eta_1 + R_{\zeta_1}) \cdot (\eta_2 + R_{\zeta_2}) = 1 + o(1) \tag{3.71}$$

in the C^k norm over bounded domain Ω.

3.2.3 Construction of vector fields and uniqueness result

Let us now consider the reconstruction of (L, σ). Assume E_j, $j = 1, 2$, be two complex solutions to

$$\nabla \times \nabla \times E_j - k^2 n E_j = 0 \text{ in } \Omega, \tag{3.72}$$

with the tangential boundary conditions

$$tE_j = G_j \text{ on } \partial\Omega, \tag{3.73}$$

with G_j well-chosen boundary values and $j = 1, 2$. We will see that

$$\nabla \times \nabla \times E_1 \cdot E_2 - \nabla \times \nabla \times E_2 \cdot E_1 = 0. \tag{3.74}$$

Let $D_j = LE_j$, $j = 1, 2$, be the internal complex-valued measurements. Assume $L \in C^{k+1}(\Omega)$ is non-vanishing. By substituting $E_j = D_j/L$ in (3.74), we have, after some algebraical calculation,

$$\beta \cdot \nabla L + \gamma L = 0, \tag{3.75}$$

where

$$\begin{aligned}
\beta &= \chi(x)\{[(\nabla D_1)D_2 - (\nabla D_2)D_1] + [(\nabla \cdot D_1)D_2 - (\nabla \cdot D_2)D_1] & \text{(3.76)} \\
&\quad -2[(\nabla D_1)^T D_2 - (\nabla D_2)D_1]\}, \\
\gamma &= \chi(x)\{[\nabla(\nabla \cdot D_1) \cdot D_2 - \nabla(\nabla \cdot D_2) \cdot D_1] - [\nabla^2 D_1 \cdot D_2 - \nabla^2 D_2 \cdot D_1]\}. & \text{(3.77)}
\end{aligned}$$

Here, $\chi(x)$ is a smooth known complex-valued function with $|\chi(x)|$ uniformly bounded from below by a positive constant on $\bar{\Omega}$.

To show the transport equation (3.75) has a unique solution, it suffices to prove that the direction of the vector field β is close to a constant and thus the integral curves of β connects every internal point to two boundary points.

Let $\check{E}_1, \check{E}_2$ be two CGOs with parameters ζ_1, ζ_2 and η_1, η_2 defined in (3.68), i.e.,

$$\check{E}_j = e^{i\zeta_j \cdot x}(\eta_j + R_{\zeta_j}), \quad j = 1, 2. \tag{3.78}$$

Let $\check{D}_j = L\check{E}_j$, $j = 1, 2$, be the corresponding internal data. By choosing $\chi(x) = -e^{-i(\zeta_1 + \zeta_2) \cdot x} \frac{h}{4\sqrt{2}}$ and substituting $\check{D}_j$ into (3.76), we can analyze the asymptotic behavior of the vector field

$\check{\beta}$ as $|\zeta_j| \to \infty$, or equivalently, $h \to 0$. Indeed, we have

$$\nabla \check{D}_j = e^{i\zeta_j \cdot x}[(\eta_j + R_{\zeta_j})(\nabla L)^T + i\mu(\eta_j + R_{\zeta_j})\zeta_j^T + \mu\nabla(\eta_j + R_{\zeta_j})], \quad j = 1, 2, \qquad (3.79)$$

and

$$
\begin{aligned}
& \chi(x)(\nabla \check{D}_1)\check{D}_2 \\
= \ & -\frac{Lh}{4\sqrt{2}}[(\eta_1 + R_{\zeta_1})(\nabla L)^T + iL(\eta_1 + R_{\zeta_1})\zeta_1^T + \mu\nabla(\eta_1 + R_{\zeta_1})](\eta_2 + R_{\zeta_2}) \\
= \ & -\frac{Lh}{4\sqrt{2}}[(\eta_1 + R_{\zeta_1})[\nabla L^T(\eta_2 + R_{\zeta_2}) + iL\zeta_1^T(\eta_2 + R_{\zeta_2})] \\
& +\mu\nabla(\eta_1 + R_{\zeta_1})(\eta_2 + R_{\zeta_2})] \\
= \ & -i\frac{L^2 h}{4\sqrt{2}}(\eta_1 + R_{\zeta_1})[\zeta_1 \cdot (\eta_2 + R_{\zeta_2})] + O(h).
\end{aligned}
\qquad (3.80)
$$

Therefore, by Proposition 3.2.7, $\chi(x)(\nabla \check{D}_1)\check{D}_2 \to 0$ in $C^k(\Omega)$ norm as $h \to 0$ on a bounded domain Ω. Similarly,

$$
\begin{aligned}
& \chi(x)(\nabla \check{D}_1)^T \check{D}_2 \\
= \ & -\frac{Lh}{4\sqrt{2}}[\nabla L(\eta_1 + R_{\zeta_1})^T(\eta_2 + R_{\zeta_2}) + iL\zeta_1(\eta_1 + R_{\zeta_1})^T(\eta_2 + R_{\zeta_2}) \\
& +L\nabla(\eta_1 + R_{\zeta_1})^T(\eta_2 + R_{\zeta_2})] \\
= \ & -iL^2\frac{\zeta_1 h}{4\sqrt{2}}(\eta_1 + R_{\zeta_1}) \cdot (\eta_2 + R_{\zeta_2}) + O(h) \\
\to \ & -\frac{iL\zeta_0}{4} \quad \text{in } C^k(\Omega) \text{ as } h \to 0.
\end{aligned}
\qquad (3.81)
$$

More calculation gives that

$$
\begin{aligned}
& \chi(x)(\nabla \cdot \check{D}_1)\check{D}_2 \\
= \ & -\frac{L}{4\sqrt{2}c}[(\nabla\mu \cdot (\eta_1 + R_{\zeta_1}))(\eta_2 + R_{\zeta_2}) + iL(\zeta_1 \cdot (\eta_1 + R_{\zeta_1}))(\eta_2 + R_{\zeta_2}) \\
& +L\nabla \cdot (\eta_1 + R_{\zeta_1})(\eta_2 + R_{\zeta_2})] \\
\to \ & 0 \quad \text{in } C^k(\Omega) \text{ as } h \to 0.
\end{aligned}
\qquad (3.82)
$$

By substituting (3.80), (3.81) and (3.82) into (3.76), we have

$$\lim_{h \to 0} \|\check{\beta} - L^2\zeta_0\|_{C^k(\Omega)} = 0, \qquad (3.83)$$

i.e., the vector fields have approximately constant directions for small h and their integral curves connect every internal point to two boundary points. Thus, the transport equation (3.75) admits a unique solution.

To see the dependence of vector fields on the boundary conditions, we need to introduce a regularity theorem of Maxwell's equations. Let tE be the tangential boundary condition of E. Define the Div-spaces as

$$H_{\mathrm{Div}}^s(\Omega) = \{u \in H^s\Omega^1(\Omega) : \mathrm{Div}(tu) \in H^{s-1/2}(\partial\Omega)\}, \tag{3.84}$$

$$TH_{\mathrm{Div}}^s(\partial\Omega) = \{g \in H^s\Omega^1(\partial\Omega) : \mathrm{Div}(g) \in H^s(\partial\Omega)\}, \tag{3.85}$$

where $H^s\Omega^1(\Omega)$ is a space of vector functions of which each component is in $H^s(\Omega)$. These are Hilbert spaces with norms

$$\|u\|_{H_{\mathrm{Div}}^s(\Omega)} = \|u\|_{H^s(\Omega)} + \|\mathrm{Div}(tu)\|_{H^{s-1/2}(\partial\Omega)}, \tag{3.86}$$

$$\|g\|_{TH_{\mathrm{Div}}^s(\partial\Omega)} = \|g\|_{H^s(\partial\Omega)} + \|\mathrm{Div}(g)\|_{H^s(\partial\Omega)} \tag{3.87}$$

It is clear that $t(H_{\mathrm{Div}}^s(\Omega)) = TH_{\mathrm{Div}}^{s-1/2}(\partial\Omega)$.

Theorem 3.2.8 (Kenig-Salo-Uhlmann[25]). *Let $\epsilon, \mu \in C^s$, $s > 2$, be positive functions. There is a discrete subset $\Sigma \subset \mathbb{C}$ such that if ω is outside this set, then one has a unique solution $E \in H_{\mathrm{Div}}^d$ to (3.25) given any tangential boundary condition $G \in TH_{\mathrm{Div}}^{s-1/2}(\partial\Omega)$. The solution satisfies*

$$\|E\|_{H_{\mathrm{Div}}^s(\Omega)} \leq C\|G\|_{TH_{\mathrm{Div}}^{s-1/2}(\partial\Omega)} \tag{3.88}$$

with C independent of G.

Note that when the tangential boundary condition is prescribed by CGOs, i.e., $\check{G}_j = t\check{E}_j$, $j = 1, 2$. By Theorem 3.2.8, E_j is the unique solution to (3.72) and (3.73). Then the corresponding vector field $\check{\beta}$ defined in (3.76) satisfies (3.83), which implies that the direction of $\check{\beta}$ is close to constant direction and thus its integral curves connect every internal point to two boundary points. Therefore, (3.75) admits a unique solution.

Furthermore, Theorem 3.2.8 also allows one to relax the boundary condition $\check{G}_j = t\check{E}_j$ and still to get the uniqueness of the solution to (3.75).

Proposition 3.2.9. *Under the assumption of Theorem 3.2.8, when G_j is in a neighborhood of $\check{G} = t\check{E}$ in $C^{k+3}(\partial\Omega)$, $j = 1, 2$, the corresponding vector field β defined in (3.76) satisfies*

$$\|\beta - iL^2\zeta_0\|_{C^k(\Omega)} = O(h), \tag{3.89}$$

for small h.

Proof. By definition $\|E\|_{H^s(\Omega)} \le \|E\|_{H^s_{\check{D}iv}(\Omega)}$ and $\|G\|_{TH^s_{\check{D}iv}(\partial\Omega)} \le \|G\|_{H^{s+1}(\partial\Omega)}$. In particular, when $s = \frac{5}{2} + d + \iota$, from Sobolev embedding theorem and Proposition 3.2.8 we have that

$$\|E\|_{C^{d+1}(\overline{\Omega})} \le C\|E\|_{H^{\frac{5}{2}+d+\iota}(\Omega)} \le C\|E\|_{H^{\frac{5}{2}+d+\iota}_{\check{D}iv}(\Omega)}$$

$$\le \quad C\|G\|_{TH^{d+2+\iota}_{\check{D}iv}(\partial\Omega)} \le C\|G\|_{H^{d+3+\iota}(\partial\Omega)} \le C\|G\|_{C^{d+4}(\partial\Omega)}, \tag{3.90}$$

where various constants are all named "C". Hence

$$\|E\|_{C^{d+1}(\Omega)} \le C\|G\|_{C^{d+4}(\partial\Omega)}. \tag{3.91}$$

Let us now define boundary conditions $G_j \in C^{d+4}(\partial\Omega)$, $j = 1, 2$, such that

$$\|G_j - t\check{E}_j\|_{C^{d+4}(\partial\Omega)} \le \varepsilon, \tag{3.92}$$

for some $\varepsilon > 0$ sufficiently small. let E_j be the solution to (3.25) and (3.26) with $tE_j = G_j$. By (3.91), we thus have

$$\|E_j - \check{E}_j\|_{C^{d+1}(\Omega)} \le C\varepsilon, \tag{3.93}$$

for some positive constant C. Define the complex valued internal data $D_j = LE_j$. We deduce that

$$\|D_j - \check{D}_j\|_{C^{d+1}(\Omega)} \le C_0\varepsilon, \tag{3.94}$$

for $C_0 > 0$. Define β by (3.76). We can easily deduce (3.89) from (3.83) and (3.94). This finishes the proof. $\qquad\square$

Recall $\mathcal{M}$ is the parameter space of (L, σ) defined in (3.31) and h is the parameter in (3.68). We are in the place to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. By Proposition 3.2.9, we choose the set of illuminations as a neighborhood of $(\check{G}_j) = (t\check{E}_j)$ in $(C^{d+4}(\partial\Omega))^2$. Since the measurements $D = \tilde{D}$, we have that L

and $\tilde{L}$ solve the same transport equation (3.75) while $L = \tilde{L} = D/G$ on $\partial\Omega$. As β satisfies (3.89), we deduce that $L = \tilde{L}$ since the integral curves of β map any $x \in \Omega$ to the boundary $\partial\Omega$. More precisely, consider the flow $\theta_x(t)$ associated to β, i.e., the solution to

$$\dot{\theta}_x(t) = \beta(\theta_x(t)), \quad \theta(0) = x \in \bar{\Omega}. \tag{3.95}$$

By the Picard-Lindelöf theorem, (3.95) admits a unique solution since β is of class $C^1(\Omega)$. For $x \in \Omega$, let $x_\pm(x) \in \partial\Omega$ and $t_\pm(x) > 0$ such that

$$\theta_x(t_\pm(x)) = x_\pm(x) \in \partial\Omega. \tag{3.96}$$

By the method of characteristics, the solution L to the transport equation (3.75) is given by

$$L(x) = L_0(x_\pm(x))e^{-\int_0^{t_\pm(x)} \gamma(\theta_x(s))ds}, \tag{3.97}$$

where $L_0 := L|_{\partial\Omega}$ is the restriction of L on the boundary. The solution $\tilde{L}$ is given by the same formula since $\theta_x(t) = \tilde{\theta}_x(t)$. This implies $E_j = \check{E}_j = D_j/L$, $j = 1, 2$. By the choice of illuminations, we have $|E_j| \neq 0$ due to (3.93) and $|\check{E}_j| \neq 0$. Under the assumption that $D_j = \tilde{D}_j$, we have $E_j = \tilde{E}_j$, $j = 1, 2$. Therefore, $k^2 n = k^2 \tilde{n}$ and thus $\sigma = \tilde{\sigma}$. $\qquad\square$

3.2.4 Vector fields and stability result with 2 complex internal measurements

Recall that $\theta_x(t)$ is the flow associated with β defined in (3.95). From the equality

$$\theta_x(t) - \tilde{\theta}_x(t) = \int_0^t [\beta(\theta_x(s)) - \tilde{\beta}(\tilde{\theta}_x(s))]ds, \tag{3.98}$$

and using the Lipschitz continuity of β and Gronwall's lemma, we deduce the existence of a constant C such that

$$|\theta_x(t) - \tilde{\theta}_x(t)| \leq Ct\|\beta - \tilde{\beta}\|_{C^0(\bar{\Omega})}, \tag{3.99}$$

when $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$. The inequality (3.99) is uniform in t as all characteristics exit $\bar{\Omega}$ in finite time.

To see higher order estimates, we define $W := D_x\theta_x(t)$, which solves the equation, $\dot{W} = D_x\beta(\theta_x)W$, with $W(0) = I$. Define $\tilde{W}$ similarly. By using Gronwall's lemma again, we deduce that

$$|W - \tilde{W}| \leq Ct\|D_x\beta - D_x\tilde{\beta}\|_{C^0(\bar{\Omega})} \tag{3.100}$$

when $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$. Since β and $\tilde{\beta}$ are of class $C^d(\bar{\Omega})$, then we obtain iteratively that

$$|D_x^{d-1}\theta_x(t) - D_x^{d-1}\tilde{\theta}_x(t)| \le Ct\|D_x\beta - D_x\tilde{\beta}\|_{C^{d-1}(\bar{\Omega})}, \tag{3.101}$$

when $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$.

Recall that Ω_1 is defined to be the subset of Ω by removing a neighborhood of each tangent point of $\partial\Omega$ with respect to ζ_0.

Lemma 3.2.10. *Let Ω be an open bounded and convex subset in $\mathbb{R}^3$ with C^d boundary. Let $d \ge 2$ and assume β and $\tilde{\beta}$ are $C^d(\bar{\Omega})$ vector fields which satisfy (3.89). Restricting to Ω_1, we have that*

$$\|x_+ - \tilde{x}_+\|_{C^{d-1}(\Omega_1)} + \|t_+ - \tilde{t}_+\|_{C^{d-1}(\Omega_1)} \le C\|\beta_+ - \tilde{\beta}_+\|_{C^{d-1}(\Omega_1)}, \tag{3.102}$$

where C is a constant depending on Ω.

The proof is the same as that of Lemma 2.3.7.

Proposition 3.2.11. *Let $d \ge 1$. Let L and $\tilde{L}$ be solutions to (3.75) corresponding to coefficients (β, γ) and $(\tilde{\beta}, \tilde{\gamma})$, respectively, where (3.89) holds for both β and $\tilde{\beta}$.*

Let $L_0 = L|_{\partial\Omega}$ and $\tilde{L}_0 = \tilde{L}_{\partial\Omega}$, thus $L_0, \tilde{L}_0 \in C^d(\partial\Omega)$. We also assume h is sufficiently small and Ω is convex. Then there is a constant C such that restricting to Ω_1

$$\begin{aligned}
\|L - \tilde{L}\|_{C^{d-1}(\bar{\Omega}_1)} \le &C\|L_0\|_{C^{d-1}(\partial\Omega_1)}[\|\beta - \tilde{\beta}\|_{C^{d-1}(\Omega_1)} \\
&+ \|\gamma - \tilde{\gamma}\|_{C^{d-1}(\Omega_1)}] + C\|L_0 - \tilde{L}_0\|_{C^{d-1}(\partial\Omega_1)}
\end{aligned} \tag{3.103}$$

The proposition and the proof follows Proposition 2.3.8.

Proof. By the method of characteristics, $L(x)$ is determined explicitly in (3.97), while $\tilde{L}$ has a similar expression. We thus have

$$\begin{aligned}
|L(x) - \tilde{L}(x)| \le &|(L_0(x_+(x)) - \tilde{L}_0(\tilde{x}_+(x)))e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds}| \\
&+ |\tilde{L}_0(\tilde{x}_+(x))(e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds} - e^{-\int_0^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\theta}_x(s))ds})|.
\end{aligned} \tag{3.104}$$

Applying Lemma 3.2.10, we deduce that

$$|D_x^{d-1}[L_0(x_+(x)) - \tilde{L}_0(\tilde{x}_+(x))]| \le \|L_0 - \tilde{L}_0\|_{C^{d-1}(\partial\Omega)} + C\|L_0\|_{C^{d-1}(\partial\Omega)}\|\beta - \tilde{\beta}\|_{C^{d-1}(\Omega_1)}.$$
(3.105)

This proves $L_0(x_+(x))$ is stable. To consider the second term, by the Leibniz rule, it is sufficient to prove the stability result for $\int_0^{t_+(x)} \gamma(\theta_x(s))ds$.

Assume without loss of generality that $t_+(x) < \tilde{t}_+(x)$. Then by applying (3.101), we have

$$\int_0^{t_+(x)} [\gamma(\theta_x(s)) - \tilde{\gamma}(\tilde{\theta}_x(s))]ds = \int_0^{t_+(x)} [\gamma(\theta_x(s)) - \gamma(\tilde{\theta}_x(s)) + (\gamma - \tilde{\gamma})\tilde{\theta}_x(s)]ds$$

$$\le C\|\gamma\|_{C^0(\Omega_1)}\|\beta - \tilde{\beta}\|_{C^0(\Omega_1)} + C\|\gamma - \tilde{\gamma}\|_{C^0(\Omega_1)}.$$

Derivatives of order $d-1$ of the above expression are uniformly bounded since $t_+(x) \in C^{d-1}(\Omega_1)$, γ has C^d derivatives bounded on Ω and $\theta_x(t)$ is stable as in (3.101).

It remains to handle the term

$$v(x) := \int_{t_+(x)}^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\theta}_x(s))ds.$$

$\tilde{\beta}$ and $\tilde{\gamma}$ are of class $C^d(\Omega_1)$, so is the function $x \to \tilde{\gamma}(\tilde{\theta}_x(s))$. Derivatives of order $d-1$ of $v(x)$ involve terms of size $t_+(x) - \tilde{t}_+(x)$ and terms of form

$$D_x^m\left(\tilde{t}_+ D_x^{d-1-m}\tilde{\gamma}(\tilde{\theta}_x(\tilde{t}_+)) - t_+ D_x^{d-1-m}\tilde{\gamma}(\tilde{\theta}_x(t_+))\right), \quad 0 \le m \le d-1.$$

Since the function has $d-1$ derivatives that are Lipschitz continuous, we thus have

$$|D_x^{d-1}v(x)| \le C\|t_+ - \tilde{t}_+\|_{C^{d-1}(\Omega_1)}.$$

The rest of the proof follows Lemma 3.2.10. □

Now we can prove the main stability theorem.

Proof of Theorem 3.2.2. From (3.76) (3.77) it is easy to check that

$$\|\beta - \tilde{\beta}\|_{C^{d-1}(\overline{\Omega}_1)} \le C\|D - \tilde{D}\|_{C^d(\overline{\Omega}_1)} \qquad \|\gamma - \tilde{\gamma}\|_{C^{d-1}(\overline{\Omega}_1)} \le C\|D - \tilde{D}\|_{C^{d+1}(\overline{\Omega}_1)} \qquad (3.106)$$

where $C > 0$ is a positive constant. The first part follows directly from (3.76) and Proposition 3.2.11. This also provides a stability result for $E_j = D_j/L$ as L is non-vanishing. By

choosing the boundary illuminations close to the boundary conditions of CGO solutions, (3.92) and (3.93) imply that E_j is non-vanishing since the CGO solutions are non-vanishing. Thus (3.25) gives the stability control of $k^2 n$ and thus σ. □

3.2.5 Stability with 6 complex internal data

Rather than applying the characteristics method to (3.75), we can rewrite (3.75) into matrix form by introducing more internal measurements. We first construct proper CGO solutions. Let $j = 1, 2, 3$ in this section. We can choose unit vectors ζ_0^j and η_0^j, such that $\zeta_0^j \cdot \zeta_0^j = 0$, $\zeta_0^j \cdot \eta_0^j = 0$ and $\{\zeta_0^j\}$ are linearly independent. Also, choose (ζ_1^j, ζ_2^j) and (η_1^j, η_2^j) such that $|\zeta| := |\zeta_1^j| = |\zeta_2^j|$,

$$\lim_{|\zeta| \to \infty} \frac{\zeta_1^j}{|\zeta|} = \lim_{|\zeta| \to \infty} \frac{\zeta_2^j}{|\zeta|} = \zeta_0^j, \quad \text{and} \quad \lim_{|\zeta| \to \infty} \eta^j = \eta_0^j. \tag{3.107}$$

We construct CGO solutions $\check{E}_1^j, \check{E}_2^j$ corresponding to (ζ_1^j, η_1^j) and (ζ_2^j, η_2^j). Let the boundary illuminations G_1^j, G_2^j be chosen according to (3.92) for ε small enough. The measured internal data are then given by D_1^j, D_2^j. Proposition 3.2.8 shows that the vector field defined by (3.76) satisfies that

$$\|\beta^j - iL^2 \zeta_0^j\|_{C^d(\Omega)} \leq \frac{C}{|\zeta|}. \tag{3.108}$$

While $|\zeta|$ is sufficiently large and $L \neq 0$ on $\bar{\Omega}$, we obtain that the vector $\{\beta^j(x)\}$ are linear independent at every $x \in \Omega$. Thus matrix $(\beta^j(x))$ is invertible with inverse of class $C^d(\Omega)$. By constructing vector-valued function $\Upsilon(x) \in (C^d(\bar{\Omega}))^3$, the transport equation (3.75) now becomes the matrix equation

$$\nabla L + \Upsilon(x) L = 0. \tag{3.109}$$

Notice that $\Upsilon(x)$ is stable under small perturbations in the data $D := (D_1^j, D_2^j) \in (C^d(\bar{\Omega}))^6$, i.e.,

$$\|\Upsilon - \tilde{\Upsilon}\|_{(C^{d-1}(\Omega))^3} \leq C\|D - \tilde{D}\|_{(C^d(\Omega))^6}. \tag{3.110}$$

Assume Ω is connected and $L_0 = L|_{\partial\Omega}$ is known. Choose a smooth curve from $x \in \Omega$ to a point on the boundary. Restricting to the curve, (3.109) is a stable ordinary differential

equation. Keep the curve fixed. Let L and $\tilde{L}$ be solutions to (3.109) with respect to Υ and $\tilde{\Upsilon}$, respectively. By solving the equation explicitly and (3.110), we find that

$$\|L - \tilde{L}\|_{C^{d-1}(\Omega)} \leq C\|D - \tilde{D}\|_{(C^d(\Omega))^6}. \tag{3.111}$$

Proof of Theorem 3.2.3. The first result in (3.33) is directly from (3.111). The proof of the stability of σ is exactly the same as in the proof of theorem 3.2.2. □

Chapter 4

FUTURE PROBLEMS

Due to my interest in both theoretical research and applications of inverse problems, my proposed questions will include the following.

1. An immediate research problem is the quantitative thermo-acoustic tomography(QTAT) with partial boundary data. QTAT studies diffuse electromagnetic inverse problem with internal data. The objective is to reconstruct the diffusion and attenuation coefficients from boundary illuminations and internal energy attenuation. QTAT with full boundary data is studied in [6]. By applying CGO solutions with nonlinear limiting Carleman weights, one could expect similar reconstruction results for partial boundary data case.

2. Another potential research project is the diffuse electromagnetic inverse problem with boundary conditions. The objective will be the reconstruction of the attenuation coefficient from the controlled source perspective or magnetotelluric. The leading topics would be conditional Lipschitz-type stability estimates of the reconstruction. The numerical implementation and the integration with seismic will also be very interesting.

3. One future research problem will be the first step of the inverse problem of Electro-seismic, which is modeled by Biot's equations. The inversion of Biot's equations is to reconstruct the internal potential due to electro-seismic conversion from any boundary measurements. This step is challenging due to the facts that (1) the seismic wave generated by electro-seismic conversion is weak, and (2) the Biot slow wave is a diffusive wave, which decays rapidly to zero with propagation distance and is therefore difficult to observe.

Williams [45] presented the effective density fluid model(EDFM), which is an very accurate approximation to Biot's equations. In EDFM, Biot slow wave and shear wave are neglected, and Biot's equations are reduced to the form of Helmholtz equation with an effective density depending on frequency. By employing the idea of effective density, we can reform the inversion of Biot's equations as the inversion of Helmholtz equation to find the internal potential.

4. It will be very interesting to consider the nonlinear inverse problem of electro-seismic conversion due to the nonlinear coupling mechanism. A possible idea is an analog to inverse scattering theory. One can analyze the forward iterative coupling and develop methods to remove the multiple coupling.

5. It is practically important to study the inverse problem of electro-seismic conversion with partial data. It is confirmed that seismo-electric conversion could be a new bore-hole logging technique. Thus partial data inversion will be the main application in field. Imitating the QPAT with partial data, a possible approach will be the CGO solutions with nonlinear limiting Carleman weights for Maxwell's equations. The coupling coefficient L is solved by a transport equation with vector field β. By applying CGO solutions with nonlinear Carleman weights, we can have control on β so that the integral curves of β exit the domain of interest only on part of the boundary, in which case bore-hole measurements will be enough for the inversion. The difficult will be the construction of such CGO solutions for Maxwell's equations.

6. The final goal is the numerical implementation of the methods I have developed and applications with real data.

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