Realization spaces of polytopes and matroids

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Abstract

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Chapter 1 describes several models for the realization space of a polytope. These models include the classical model, a model representing realizations in the Grassmannian, a new model which represents realizations by slack matrices, and a model which represents polytopes by their Gale transforms. We explore the connections between these models, and show how they can be exploited to obtain useful parametrizations of the slack realization space.

Chapter 2 introduces a natural model for the realization space of a polytope up to projective equivalence which we call the slack realization space of the polytope. The model arises from the positive part of an algebraic variety determined by the slack ideal of the polytope. This is a saturated determinantal ideal that encodes the combinatorics of the polytope. The slack ideal offers an effective computational framework for several classical questions about polytopes such as rational realizability, non-prescribability of faces, and realizability of combinatorial polytopes.

Chapter 3 studies the simplest possible slack ideals, which are toric, and explores their connections to projectively unique polytopes. We prove that if a projectively unique polytope has a toric slack ideal, then it is the toric ideal of the bipartite graph of vertex-facet non-incidences of the polytope. The slack ideal of a polytope is contained in this toric ideal if and only if the polytope is morally 2-level, a generalization of the 2-level property in polytopes. We show that polytopes that do not admit rational realizations cannot have toric slack ideals.

A classical example of a projectively unique polytope with no rational realizations is due to Perles. We prove that the slack ideal of the Perles polytope is reducible, providing the first example of a slack ideal that is not prime.

Chapter 4 studies a certain collection of polytopal operations which preserve projective uniqueness of polytopes. We look at their effect on slack matrices and use this to classify all "McMullen-type" projectively unique polytopes in dimension 5. From this we identify one of the smallest known projectively unique polytopes not obtainable from McMullen's constructions.

Chapter 5 extends the slack realization space model to the setting of matroids. We show how to use this model to certify non-realizability of matroids, and describe an explicit relationship to the standard Grassmann-Plücker realization space model. We also exhibit a way of detecting projectively unique matroids via their slack ideals by introducing a toric ideal that can be associated to any matroid.

Chapter 6 addresses some of the computational aspects of working with slack ideals. We develop a `Macaulay2` [27] package for computing and manipulating slack ideals. In particular, we explore the dehomogenizing and rehomogenizing of slack ideals, both from a computational and theoretical perspective.

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DEDICATION

To mom and dad.

Chapter 1

INTRODUCTION

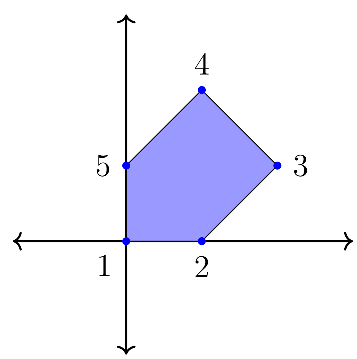
1.1 Realization space models

Let P be an abstract labeled d-polytope with v vertices and f facets. A realization of P is an assignment of coordinates to each vertex label, $i \mapsto \boldsymbol{q}_i \in \mathbb{R}^k$ so that the polytope $Q = \operatorname{conv}\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_v\}$ is combinatorially equivalent to P; that is, it has the same face lattice as P. We think of realizations as geometric instances of a particular combinatorial structure.

Example 1.1.1. *If P is a pentagon with 5 vertices organized into facets $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{4, 5\}$, and $\{5, 1\}$, then*

$$\boldsymbol{q}_1 = (0, 0), \qquad \boldsymbol{q}_2 = (1, 0),$$
$$\boldsymbol{q}_3 = (2, 1), \qquad \boldsymbol{q}_4 = (1, 2),$$
$$\boldsymbol{q}_5 = (0, 1) \in \mathbb{R}^2$$

gives a realization $Q = \operatorname{conv}\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_5\}$ of P.



The *realization space* of P is, essentially, the set of all polytopes Q which are realizations of P, or equivalently, the set of all "geometrically distinct" polytopes which are combinatorially equivalent to P. In this chapter, we will see four models for this realization space, relaxations of these models and the relationships between them, as depicted in Figure 1.1. Each portion of the diagram will be explained in the upcoming sections. We begin by introducing three of the models before we compare the information which is encoded by each model.

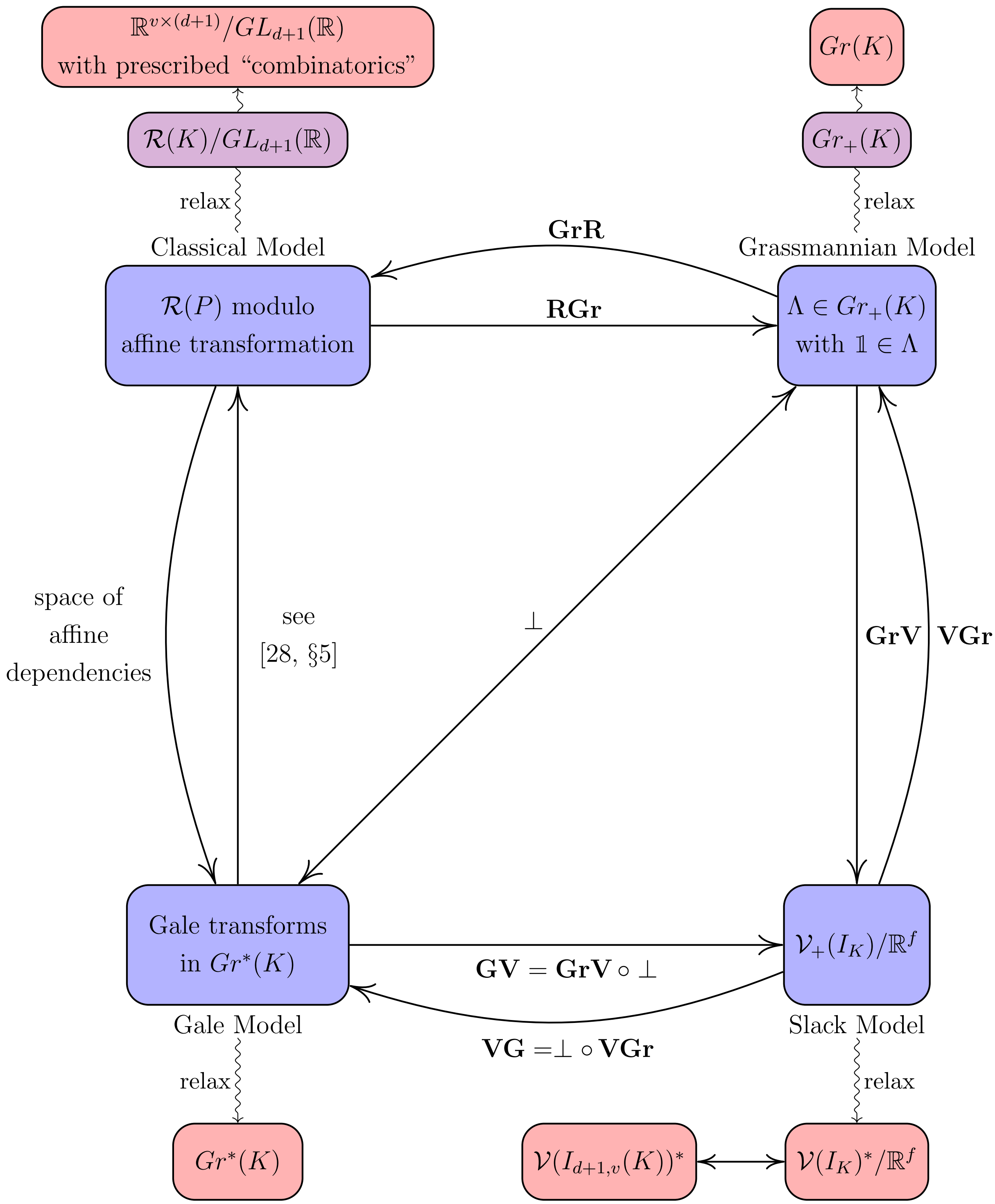


Figure 1.1: Realization space models for polytope P with $K = P^h$ and the relationships between them.

1.1.1 The classical model

The construction of the classical model for the realization space of P is perhaps the most intuitive (see [37] for more details). In this model, we identify each realization of P with the set of its vertices in d-dimensional space, i.e.,

$$\mathcal{R}(P) := \{\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_1^\top \\ \vdots \\ \boldsymbol{q}_v^\top \end{bmatrix} \in \mathbb{R}^{v \times d} : Q = \operatorname{conv}\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_v\} \text{ is a realization of } P\}.$$

One often wishes to consider realizations equivalent if they are related via some simple transformation.

Definition 1.1.2. *Let Q, Q' be two realizations of polytope P. They are called* affinely equivalent *if they are related via an affine transformation; that is, $Q' = \boldsymbol{A}Q + \boldsymbol{b}$, for some invertible matrix $\boldsymbol{A} \in \mathbb{R}^{d \times d}$ and $\boldsymbol{b} \in \mathbb{R}^d$. They are called* projectively equivalent *if they are related via a projective transformation; that is, for all $\boldsymbol{q}_i, \boldsymbol{q}'_i$*

$$\boldsymbol{q}'_i = \frac{\boldsymbol{A}\boldsymbol{q}_i + \boldsymbol{b}}{\boldsymbol{c}^\top \boldsymbol{q}_i + \gamma},$$

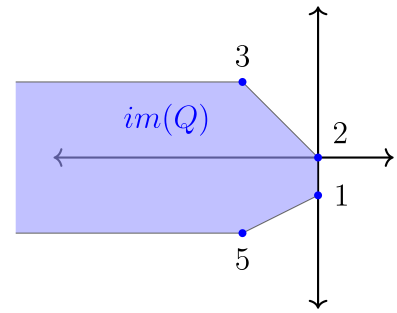
for some invertible matrix $\boldsymbol{A} \in \mathbb{R}^{d \times d}$, $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$ with $\det \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^\top & \gamma \end{bmatrix} \neq 0.$

Definition 1.1.3. *A polytope P is called* projectively unique *if all of its realizations are projectively equivalent.*

Remark 1.1.4. It is important to notice that not all projective transformations are "admissible" for a realization Q. In particular, if $\boldsymbol{c}^\top \boldsymbol{q}_i + \gamma = 0$ for some $\boldsymbol{q}_i$, the result of applying such a projective transformation will not be another polytope Q'.

Example 1.1.5. *Consider the projective transformation given by $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\boldsymbol{b} = \boldsymbol{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\gamma = -2$. Applying this transformation to Q of Example 1.1.1, we get*

$$\boldsymbol{q}_1 = (0,0) \mapsto (0,-1/2)$$
$$\boldsymbol{q}_2 = (1,0) \mapsto (0,0)$$
$$\boldsymbol{q}_3 = (2,1) \mapsto (-1,1)$$
$$\boldsymbol{q}_4 = (1,2) \mapsto \frac{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{(0,1)\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2} = \frac{(2,0)}{0}$$
$$\boldsymbol{q}_5 = (0,1) \mapsto (-1,-1)$$

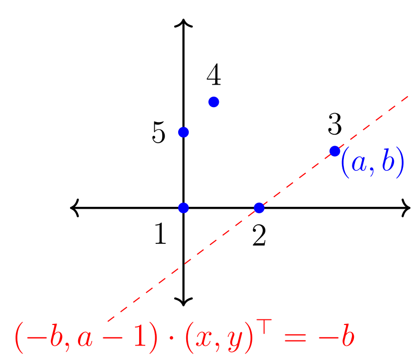


Usually (see [37]) one would also fix an *affine basis*, that is, $d+1$ necessarily affinely independent vertices of P, whose coordinates would remain fixed in all realizations in $\mathcal{R}(P)$, thus modding out affine transformations. This gives an explicit way to model the set of affine equivalence classes of realizations of P. However, no explicit version of this model exists for projective equivalence classes of realizations, due to the difficulty pointed out in Remark 1.1.4.

Example 1.1.6. *Fixing an affine basis for the pentagon, say* $\boldsymbol{q}_1 = (0,0)$, $\boldsymbol{q}_2 = (1,0)$, *and* $\boldsymbol{q}_5 = (0,1)$, *the classical realization space becomes*

$$\mathcal{R}(P) = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ a & b \\ s & t \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{5\times 2} : \begin{array}{lll} b > 0, & a+b > 1, & a(t-1)+s(1-b) > 0, \\ s > 0, & s+t > 1, & t(a-1)+b(1-s) > 0 \end{array} \right\},$$

where the conditions on a, b, s and t come from the (labeled) convexity constraints on P. For example, vertices 2 and 3 define the facet hyperplane $(-b, a-1)\cdot(x,y)^\top = -b$. *The remainder of the vertices need to lie on the same side of this hyperplane, so that since* $(-b,a-1)\cdot(0,0)^\top = 0 > -b$, *we must also have* $(-b,a-1)\cdot(0,1)^\top = a-1 > -b$.



This realization space model naturally generalizes to polyhedral cones. Let K be an abstract labeled $(d+1)$-polyhedral cone with v extreme rays and f facets. A realization of K is an assignment of coordinates to each extreme ray, $i \mapsto \boldsymbol{r}_i \in \mathbb{R}^{d+1}$. Recording a cone by the generators of its extreme rays, we get the following realization space model

$$\mathcal{R}(K) = \{\boldsymbol{R} = \begin{bmatrix} \boldsymbol{r}_1^\top \\ \vdots \\ \boldsymbol{r}_v^\top \end{bmatrix} \in \mathbb{R}^{v\times(d+1)} : K' = \text{cone}\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_v\} \text{ is a realization of } K\}.$$

Remark 1.1.7. Unlike for polytopes, it is possible for different elements of $\mathcal{R}(K)$ to represent the same realization. In particular

$$\text{cone}\{\lambda_1\boldsymbol{r}_1, \ldots, \lambda_v\boldsymbol{r}_v\} = \text{cone}\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_v\}$$

for any choice of $\lambda_1, \ldots, \lambda_v \in \mathbb{R}_{>0}$.

If K has the same combinatorial type as polytope P, write $K = P^h$. One way to obtain a realization of $K = P^h$ is to take $K' = Q^h$, the homogenization cone of a realization Q of P,

$$\boldsymbol{R} = \begin{bmatrix} \mathbb{1} & \boldsymbol{Q} \end{bmatrix}, \quad \text{for some } \boldsymbol{Q} \in \mathcal{R}(P).$$

Definition 1.1.8. *Let K_1, K_2 be two realizations of a cone K. They are called* linearly equivalent *if they are related via a linear transformation; that is, $K_2 = \boldsymbol{A}K_1$, for some invertible matrix $\boldsymbol{A} \in \mathbb{R}^{(d+1)\times(d+1)}$.*

Realization spaces are important in the study of polytopes. A well-known result in polytope theory, Steinitz's theorem, characterizes which face lattices correspond to (realizable) 3-dimensional polytopes, and its proof can also be used to show that the realization space of a 3-polytope is *contractible* (a trivial set topologically) [37]. Steinitz's theorem was proved in 1922, and it was not until 1986 that Mnëv proved that realization spaces can be more complicated, in fact, arbitrarily so. Mnëv's Universality theorem, and the subsequent result for 4-polytopes by Richter-Gebert in 1996 are the following.

Theorem 1.1.9 ([33], [37])**.** *For every primary basic semi-algebraic set V defined over $\mathbb{Z}$ there is a rank 3 oriented matroid (4-polytope) whose realization space is stably equivalent to V.*

Theorem 1.1.9 has many powerful implications: for example, that determining realizability for rank 3 oriented matroids/4-polytopes is polynomially equivalent to the "Existential Theory of the Reals," so in particular, it is NP-hard; and that all algebraic numbers are needed to coordinatize rank 3 oriented matroids/4-polytopes. [37, Section 1.3]

Many foundational questions in the study of polytopes, including the following, can be phrased in terms of realization spaces: Is a combinatorial sphere (or arbitrary abstract polytope) realizable as a convex polytope? Realizable with rational coordinates? Can its faces be freely prescribed? Is it projectively unique?

1.1.2 The Grassmannian model

The next realization space model lives in the Grassmannian, $Gr(d+1, v)$, which is the set of all $(d+1)$-dimensional linear subspaces of a fixed v-dimensional vector space. In particular, if we consider subspaces of $\mathbb{R}^v$, then a point of the Grassmannian can be described as the column space of a $v \times (d+1)$ matrix $\boldsymbol{X}$; that is, there is a surjective map

$$\begin{aligned} \rho : \mathbb{R}^{v\times(d+1)} &\to Gr(d+1, v) \\ \boldsymbol{X} &\mapsto \text{column space}(\boldsymbol{X}). \end{aligned}$$

However, two matrices $\boldsymbol{X}, \boldsymbol{Y}$ can have the same column space, which happens exactly if they differ by an element of $GL_{d+1}(\mathbb{R})$,

$$\rho(\boldsymbol{X}) = \rho(\boldsymbol{Y}) \quad \Leftrightarrow \quad \boldsymbol{Y} = \boldsymbol{X}\boldsymbol{A}, \text{ for some } \boldsymbol{A} \in GL_{d+1}(\mathbb{R}). \tag{1.1}$$

Instead of recording points of the Grassmannian by some (non-unique choice of) matrix of basis elements, we will record each subspace by its Plücker vector which is defined as follows.

For a matrix $\boldsymbol{X} \in \mathbb{R}^{v\times(d+1)}$, denote by $\boldsymbol{X}_J$ the submatrix of $\boldsymbol{X}$ formed by taking the rows indexed by $J \subset [v] = \{1, 2, \ldots v\}$.

Definition 1.1.10. *For a $(d+1)$-subset $J = \{j_0, \ldots, j_d\} \subset [v]$, the* Plücker coordinate *of $\boldsymbol{X}$ indexed by J is* $\det(\boldsymbol{X}_J)$. *Notice if J contains repeated elements, the Plücker coordinate is necessarily zero, and Plücker coordinates given by rearranging elements of J are related by the sign of the appropriate permutation. Hence we record only the Plücker coordinates indexed by sets $J = \{j_0 < j_1 < \ldots < j_d\} \in \binom{[v]}{d+1}$ in the* Plücker vector *of $\boldsymbol{X}$,*

$$pl(\boldsymbol{X}) := \left(\det[\boldsymbol{X}_J]\right)_{J\subset\binom{[v]}{d+1}}.$$

Now if $\boldsymbol{X} \in \mathbb{R}^{v\times(d+1)}$ is any matrix with $\rho(\boldsymbol{X}) = \Lambda$, then the following map is a well-known embedding of the Grassmannian into projective space

$$\begin{aligned} pl : Gr(d+1, v) &\to \mathbb{RP}^{\binom{v}{d+1}-1} \\ \Lambda &\mapsto pl(\boldsymbol{X}). \end{aligned}$$

We call $pl(\boldsymbol{X})$ *the Plücker vector* of Λ. Notice that this is a well-defined by (1.1), since $pl(\boldsymbol{X}\boldsymbol{A}) = pl(\boldsymbol{X}) \cdot \det(\boldsymbol{A})$ so that matrices related by elements of $GL_{d+1}(\mathbb{R})$ map to the same point in projective space under pl.

The Grassmannian $Gr(d+1, v)$ is cut out as a subvariety of projective space by the *Plücker ideal*, $I_{d+1,v}$ (See, for instance [31, Section 2.2]). This ideal lives in the polynomial ring in *Plücker variables*, $I_{d+1,v} \subset \mathbb{R}[\boldsymbol{p}] := \mathbb{R}[p_{i_0\cdots i_d} : 1 \leq i_0 < \cdots < i_d \leq v]$, where we assume the variables are listed in colexicographic[1] order unless otherwise stated. This ideal consists of all polynomials which vanish on every vector of $(d+1)$-minors coming from an arbitrary $v \times (d+1)$ matrix.

Example 1.1.11. *The column space of the following matrix $\boldsymbol{X}$ is a point in $Gr(3, 5)$. One*

[1] This order is defined by $i_0 \cdots i_d <_{\text{colex}} j_0 \cdots j_d$ if $i_k < j_k$ for the largest index k with $i_k \neq j_k$.

can check that its Plücker vector satisfies the generators of the Plücker ideal $I_{3,5}$.

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \begin{array}{rcl} {\scriptstyle i_0 i_1 i_2} & & \begin{array}{cccccccccc} {\scriptstyle 123} & {\scriptstyle 124} & {\scriptstyle 134} & {\scriptstyle 234} & {\scriptstyle 125} & {\scriptstyle 135} & {\scriptstyle 235} & {\scriptstyle 145} & {\scriptstyle 245} & {\scriptstyle 345} \end{array} \\ pl(\boldsymbol{X}) & = & [1 : \; 2 : \; 3 : \; 2 : \; 1 : \; 2 : \; 2 : \; 1 : \; 2 : \; 2] \\ I_{3,5} & = & \left\langle \begin{array}{ll} p_{235}p_{145} - p_{135}p_{245} + p_{125}p_{345}, & p_{234}p_{145} - p_{134}p_{245} + p_{124}p_{345}, \\ p_{234}p_{135} - p_{134}p_{235} + p_{123}p_{345}, & p_{234}p_{125} - p_{124}p_{235} + p_{123}p_{245}, \\ p_{134}p_{125} - p_{124}p_{135} + p_{123}p_{145} & \end{array} \right\rangle . \end{array}$$

Notice that the rows of the matrix in the above example generate a cone over a pentagon. In general, a matrix whose v rows generate the cone over a d-polytope will have rank $d+1$ and hence define a point in $Gr(d+1, v)$. Thus if K is a $(d+1)$-dimensional polyhedral cone with v (labeled) extreme rays, we can identify each matrix of generators $\boldsymbol{R} \in \mathcal{R}(K)$ with the corresponding point in the Grassmannian. We now consider the structure of this resulting realization space of (linear equivalence classes of) a polyhedral cone inside $Gr(d+1, v)$ by looking at how the combinatorics of K impose conditions on the rays of possible realizations.

As always, it suffices to consider the extreme ray-facet incidences of K. Furthermore, we can determine these incidences by considering only sets of extreme rays of size $d+1$. To see this, notice that we need d linearly independent extreme rays to determine a d-dimensional subspace (and thus a potential facet of K). Then, by convexity, these d rays determine a facet of K if and only if every other extreme ray is contained in the same (closed) half-space defined by the hyperplane containing the original d rays. That is, if the d independent rays are indexed by J, the set J defines a facet if and only if for some normal $\boldsymbol{\alpha}_J$ to the hyperplane containing rays J, we have $\langle \boldsymbol{\alpha}_J, \boldsymbol{r}_j \rangle \geq 0$ for all extreme rays $j \in [v]$. Furthermore, the extreme rays contained in that facet are exactly those j for which $\langle \boldsymbol{\alpha}_J, \boldsymbol{r}_j \rangle = 0$.

To determine $\boldsymbol{\alpha}_J$ from the original d rays, we use the following basic linear algebra fact.

Fact 1.1.12. *Given d linearly independent vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_d$ in $\mathbb{R}^{d+1}$, the linear functional defined by*

$$\langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle := \det[\, \boldsymbol{y}_1 \,|\, \cdots \,|\, \boldsymbol{y}_d \,|\, \boldsymbol{x} \,], \quad \boldsymbol{x} \in \mathbb{R}^{d+1}$$

vanishes exactly on the d-dimensional subspace spanned by $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_d$*. In other words, we can calculate* $\boldsymbol{\alpha}$*, a normal to the hyperplane spanned by* $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_d$*, via the above determinant.*

This fact tells us that the above conditions defining facets of K depend only on determinants of sets of $d+1$ extreme rays. Thus we can encode the combinatorial information about K as conditions on the Plücker vector of a point in the Grassmannian.

Definition 1.1.13. *Call a set* $J \subset [v]$ *a* facet extension *if it contains* d *elements which span a facet of* K *and a single element not on that facet. Denote the set of facet extensions of* K *by* $\overline{\mathcal{F}}(K) \subset \binom{[v]}{d+1}$*.*

Then the elements of the Grassmannian which correspond to realizations of K are

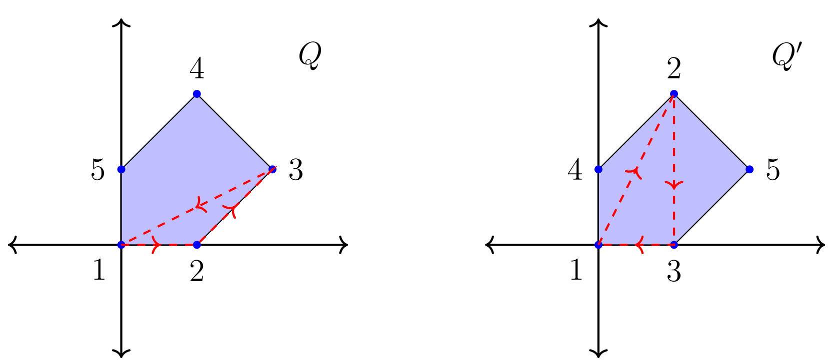
$$Gr_+(K) := \left\{ \begin{array}{ll} \Lambda \in Gr(d+1, v): & pl(\Lambda)_J = 0 \text{ if rays } J \text{ lie in a facet,} \\ & \Delta_J pl(\Lambda)_J > 0 \text{ if } J \in \overline{\mathcal{F}}(K) \end{array} \right\},$$

where $\Delta_J = \pm 1$ is a sign which depends on the orientation of simplex J.

Example 1.1.14. *Consider again the example of the cone* K *over the pentagon with facets* $\{1,2\}$*,* $\{2,3\}$*,* $\{3,4\}$*,* $\{4,5\}$*, and* $\{5,1\}$*. Since every set of* 3 *indices necessarily contains* 2 *consecutive indices, every* $J \subset \binom{[5]}{3}$ *is a facet extension. Thus*

$$Gr_+(K) := \left\{ \Lambda \in Gr(d+1, v): \ \ \Delta_J pl(\Lambda)_J > 0 \textit{ for all } J \right\}.$$

Notice that the choice of labeling for our polytope will affect the sign factors Δ_J*. To see this consider the following two pentagons,* Q, Q'*, whose homogenization cones have realizations* $\boldsymbol{R}, \boldsymbol{R}'$*, respectively. For* Q*, we see that simplices* 123 *and* 134 *are both positively oriented. However, in* Q'*,* 123 *is negatively oriented, while* 134 *is positively oriented.*



$$pl(\boldsymbol{R})_{123} = \det\begin{bmatrix}1&0&0\\1&1&0\\1&2&1\end{bmatrix} = 1 \qquad pl(\boldsymbol{R}')_{123} = \det\begin{bmatrix}1&0&0\\1&1&2\\1&1&0\end{bmatrix} = -2$$

$$pl(\boldsymbol{R})_{134} = \det\begin{bmatrix}1&0&0\\1&2&1\\1&1&2\end{bmatrix} = 3 \qquad pl(\boldsymbol{R}')_{134} = \det\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix} = 1.$$

Notice that the realization space $Gr_+(K)$ defines a semialgebraic subset of the Grassmannian, but can be relaxed to a subvariety, namely the Zariski closure of the following set.

Definition 1.1.15. *Let the* Grassmannian of K *be given by relaxing the inequalities in the above realization space to get*

$$Gr(K) := \left\{ \begin{array}{rr} \Lambda \in Gr(d+1,v): & pl(\Lambda)_J = 0 \textit{ if } J \in \mathcal{F}(K) \\ & pl(\Lambda)_J \neq 0 \textit{ if } J \in \overline{\mathcal{F}}(K) \end{array} \right\},$$

where we denote by $\mathcal{F}(K)$ the sets of $d+1$ extreme rays of K that are contained in individual facets of K.

Example 1.1.16. *Notice that the Grassmannian of the cone over the pentagon only requires all coordinates are nonzero; it contains no equations and so is full-dimensional in $Gr(3,5)$. Thus its Zariski closure is simply the whole Grassmannian $Gr(3,5)$.*

It is not hard to obtain the ideal of this subvariety from the ideal of the whole Grassmannian, the Plücker ideal $I_{d+1,v}$. From the definition of $Gr(K)$, we have

$$\begin{aligned} Gr(K) &= \Big(Gr(d+1,v) \cap \{\boldsymbol{p} \in \mathbb{P}^{\binom{[v]}{d+1}-1} : \boldsymbol{p}_J = 0 \quad \forall J \in \mathcal{F}(K)\}\Big) \setminus \{\boldsymbol{p} \in \mathbb{P}^{\binom{[v]}{d+1}-1} : \boldsymbol{p}_J = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall J \in \overline{\mathcal{F}}(K)\} \\ &= \Big(\mathcal{V}(I_{d+1,v}) \cap \mathcal{V}\big(\langle \boldsymbol{p}_J : J \in \mathcal{F}(K)\rangle\big)\Big) \setminus \mathcal{V}\big(\langle \boldsymbol{p}_J : J \in \overline{\mathcal{F}}(K)\rangle\big). \end{aligned}$$

Hence its ideal (see [12] for necessary correspondences) is given by

$$\mathcal{I}(Gr(K)) = \big(I_{d+1,v} + \langle \boldsymbol{p}_J : J \in \mathcal{F}(K)\rangle\big) : \left(\prod_{J \in \overline{\mathcal{F}}(K)} \boldsymbol{p}_J\right)^{\infty} \subset \mathbb{R}[\boldsymbol{p}]. \tag{1.2}$$

1.1.3 The slack matrix model

Our final model represents a polytope or cone by its slack matrix, and it will be further developed in Chapter 2. The following basic facts about slack matrices of polytopes and cones are from [19, 20]. If $Q = \text{conv}\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_v\}$ is a realization of a d-polytope with v vertices and f facets whose $\mathcal{H}$-representation is given by $Q = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{W}\boldsymbol{x} \leq \boldsymbol{w}\}$, then a slack matrix of Q is

$$S_Q = \begin{bmatrix} 1 & \boldsymbol{q}_1^\top \\ \vdots & \vdots \\ 1 & \boldsymbol{q}_v^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{w}^\top \\ -\boldsymbol{W}^\top \end{bmatrix} \in \mathbb{R}^{v \times f}.$$

Notice that the choice of $\mathcal{H}$-representation is not unique, and thus a realization does not have a unique slack matrix.

Similarly, if K is a realization of a $(d+1)$-dimensional cone with v extreme ray generators $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_v$, and f facets with $K = \{\boldsymbol{x} \in \mathbb{R}^{d+1} : \boldsymbol{x}^\top \boldsymbol{B} \geq \boldsymbol{0}\}$, then a slack matrix of K is

$$S_K = \begin{bmatrix} \boldsymbol{r}_1^\top \\ \vdots \\ \boldsymbol{r}_v^\top \end{bmatrix} \boldsymbol{B} \in \mathbb{R}^{v \times f}.$$

Notice that if $K = Q^h$, the homogenization cone of Q, that is, $K = \text{cone}\{(1, \boldsymbol{q}_1), \ldots, (1, \boldsymbol{q}_v)\} = \{(x_0, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^d : \boldsymbol{W}\boldsymbol{x} \leq x_0 \boldsymbol{w}\}$, then $S_K = S_Q$.

Example 1.1.17. *Recall our first realization of a pentagon from Example 1.1.1, which was given by $Q = \text{conv}\{(0,0), (1,0), (2,1), (1,2), (0,1)\}$. Its $\mathcal{H}$-representation is $\{\boldsymbol{x} \in \mathbb{R}^2 : y \geq 0, x - y \leq 1, x + y \leq 3, -x + y \leq 1, x \geq 0\}$, so that*

$$S_Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 & 1 \\ 1 & 2 & 2 & 0 & 0 \end{bmatrix}.$$

It is not hard to check that this is also a slack matrix of the cone over this pentagon.

Like polytopes, cones do not have unique slack matrices. In particular, recall that the vectors $\gamma_1 \boldsymbol{r}_1, \ldots, \gamma_v \boldsymbol{r}_v$ also generate the cone K for any $\gamma_1, \ldots, \gamma_v \in \mathbb{R}_{>0}$.

We know from [23] that we can identify equivalent realizations via their slack matrices as follows.

Lemma 1.1.18 (and [23, Corollary 1.5])**.** *Let Q and Q' be two realizations of an abstract polytope P. Then they are affinely equivalent if and only if they have the same set of slack matrices up to column scaling; that is, $S_{Q'} = S_Q D_f$, for some positive diagonal matrix D_f. They are projectively equivalent if and only if they have the same set of slack matrices up to row and column scaling; that is $S_{Q'} = D_v S_Q D_f$ for some positive diagonal matrices D_v, D_f.*

A similar result holds for cones and follows from the proof of [23, Corollary 1.5].

Lemma 1.1.19. *Let K_1 and K_2 be two realizations of an abstract polyhedral cone K. They are linearly equivalent if and only if they have the same set of slack matrices up to row and column scaling; that is $S_{K_2} = D_v S_{K_1} D_f$ for some positive diagonal matrices D_v, D_f.*

Example 1.1.20. *Consider what happens when we apply the linear transformation given by $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 2 & -3 \end{bmatrix}$ to the cone K over the pentagon of Example 1.1.17. We get*

$$S_{\boldsymbol{A}K} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 4 & 4 & 1 \\ 7 & 2 & -4 \\ 4 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 6 & 18 & 6 & 0 \\ 1 & -7 & -23 & -5 & 1 \\ -1 & 4 & 20 & 8 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 18 & 6 & 0 \\ 0 & 0 & 12 & 12 & 2 \\ 3 & 0 & 0 & 12 & 4 \\ 6 & 12 & 0 & 0 & 2 \\ 3 & 12 & 12 & 0 & 0 \end{bmatrix} = S_K \cdot \mathrm{diag}(3, 6, 6, 6, 2).$$

Since the affine hull of a realization of P is d-dimensional and the linear hull of a realization of K is $(d+1)$-dimensional, we have

$$\mathrm{rank}\left(\begin{bmatrix} 1 & \boldsymbol{q}_1^\top \\ \vdots & \vdots \\ 1 & \boldsymbol{q}_v^\top \end{bmatrix}\right) = \mathrm{rank}\left(\begin{bmatrix} \boldsymbol{r}_1^\top \\ \vdots \\ \boldsymbol{r}_v^\top \end{bmatrix}\right) = d+1,$$

which implies their slack matrices also have rank $d+1$. The zeros in each slack matrix also record the extreme point (or ray)-facet incidences of P (or K). Furthermore, we will always have $\mathbb{1}$ in the column space of the slack matrix of a polytope, whereas $\mathbb{1}$ is in the column space of S'_K if and only if K' is the homogenization cone of a realization of some polytope [19, Theorem 6]. Interestingly, it follows from [19, Theorems 22, 24] that the above properties are sufficient to characterize slack matrices of polytopes and cones.

Theorem 1.1.21. *(i) A nonnegative matrix $S \in \mathbb{R}^{v\times f}$ is the slack matrix of a cone with the same combinatorial types as the labeled $(d+1)$-dimensional cone K if and only if the following hold:*

(a) $\operatorname{support}(S) = \operatorname{support}(S_K)$,

(b) $\operatorname{rank}(S) = \operatorname{rank}(S_K) = d+1$, *and*

(ii) A nonnegative matrix $S \in \mathbb{R}^{v\times f}$ is the slack matrix of a polytope with the same combinatorial type as the labeled d-polytope P if and only if the following hold:

(a) $\operatorname{support}(S) = \operatorname{support}(S_P)$,

(b) $\operatorname{rank}(S) = \operatorname{rank}(S_P) = d+1$, *and*

(c) $\mathbb{1}$ *lies in the column span of* S.

To encode these conditions algebraically, we make the following definitions.

Definition 1.1.22. *The* symbolic slack matrix, $S_P(\boldsymbol{x})$, *of an abstract polytope P has entries*

$$(S_P(\boldsymbol{x}))_{i,j} = \begin{cases} x_{i,j} & \text{if vertex } i \text{ is not in facet } j \\ 0 & \text{else.} \end{cases}$$

We can think of it as being obtained from a slack matrix of a realization of P by replacing each nonzero entry with a distinct variable. Suppose there are t variables in $S_P(\boldsymbol{x})$. The slack ideal *of P is the saturation of the ideal generated by the $(d+2)$-minors of $S_P(\boldsymbol{x})$, namely*

$$I_P := \langle (d+2)\text{-}minors\ of\ S_P(\boldsymbol{x})\rangle : \left(\prod_{i=1}^{t} x_i\right)^{\infty} \subset \mathbb{C}[\boldsymbol{x}] := \mathbb{C}[x_1, \ldots, x_t]. \tag{1.3}$$

The slack variety *of P is the complex variety $\mathcal{V}(I_P) \subset \mathbb{C}^t$. If $\boldsymbol{s} \in \mathbb{C}^t$ is a zero of I_P, then we identify it with the matrix $S_P(\boldsymbol{s})$.*

The definitions for an abstract polyhedral cone K are the same. In fact, if $K = P^h$, then $S_P(\boldsymbol{x}) = S_K(\boldsymbol{x})$ so that their varieties are also the same, $\mathcal{V}(I_P) = \mathcal{V}(I_K)$.

Example 1.1.23. *Continuing to take P to be the pentagon, we have*

$$S_P(\boldsymbol{x}) = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & 0 \\ 0 & 0 & x_4 & x_5 & x_6 \\ x_7 & 0 & 0 & x_8 & x_9 \\ x_{10} & x_{11} & 0 & 0 & x_{12} \\ x_{13} & x_{14} & x_{15} & 0 & 0 \end{bmatrix}$$

which gives

$$\begin{aligned} I_P = \langle\ & x_4x_8x_{12}x_{14} + x_6x_8x_{11}x_{15} - x_5x_9x_{11}x_{15}, && x_2x_6x_{11}x_{13} + x_1x_4x_{12}x_{13} - x_2x_6x_{10}x_{14} + x_1x_6x_{10}x_{15} \\ & x_2x_8x_{12}x_{14} - x_3x_9x_{11}x_{15} - x_1x_8x_{12}x_{15} && x_3x_4x_{12}x_{14} - x_2x_5x_{12}x_{14} + x_3x_6x_{11}x_{15} + x_1x_5x_{12}x_{15} \\ & x_2x_8x_{12}x_{13} - x_3x_9x_{10}x_{15} + x_3x_7x_{12}x_{15} && x_2x_6x_8x_{13} + x_3x_4x_9x_{13} - x_2x_5x_9x_{13} + x_3x_6x_7x_{15} \\ & x_3x_4x_{12}x_{13} - x_2x_5x_{12}x_{13} + x_3x_6x_{10}x_{15} && x_4x_8x_{12}x_{13} + x_6x_8x_{10}x_{15} - x_5x_9x_{10}x_{15} + x_5x_7x_{12}x_{15} \\ & x_4x_8x_{11}x_{13} - x_4x_8x_{10}x_{14} + x_5x_7x_{11}x_{15} && x_2x_6x_8x_{11} + x_3x_4x_9x_{11} - x_2x_5x_9x_{11} + x_1x_4x_8x_{12} \\ & x_3x_6x_{11}x_{13} + x_1x_5x_{12}x_{13} - x_3x_6x_{10}x_{14} && x_2x_8x_{11}x_{13} - x_2x_8x_{10}x_{14} + x_1x_8x_{10}x_{15} + x_3x_7x_{11}x_{15} \\ & x_1x_4x_9x_{10} - x_2x_6x_7x_{11} - x_1x_4x_7x_{12} && x_1x_6x_8x_{10} - x_1x_5x_9x_{10} + x_3x_6x_7x_{11} + x_1x_5x_7x_{12} \\ & x_1x_4x_9x_{13} - x_2x_6x_7x_{14} + x_1x_6x_7x_{15} && x_3x_9x_{11}x_{13} + x_1x_8x_{12}x_{13} - x_3x_9x_{10}x_{14} + x_3x_7x_{12}x_{14} \\ & x_1x_6x_8x_{13} - x_1x_5x_9x_{13} + x_3x_6x_7x_{14} && x_1x_4x_8x_{13} + x_3x_4x_7x_{14} - x_2x_5x_7x_{14} + x_1x_5x_7x_{15} \\ & x_1x_4x_8x_{10} + x_3x_4x_7x_{11} - x_2x_5x_7x_{11} \\ & x_2x_6x_8x_{10} + x_3x_4x_9x_{10} - x_2x_5x_9x_{10} - x_3x_4x_7x_{12} + x_2x_5x_7x_{12} \\ & x_2x_6x_8x_{14} + x_3x_4x_9x_{14} - x_2x_5x_9x_{14} - x_1x_6x_8x_{15} + x_1x_5x_9x_{15} \\ & x_2x_6x_8x_{10} + x_3x_4x_9x_{10} - x_2x_5x_9x_{10} - x_3x_4x_7x_{12} + x_2x_5x_7x_{12} \\ & x_6x_8x_{11}x_{13} - x_5x_9x_{11}x_{13} - x_6x_8x_{10}x_{14} + x_5x_9x_{10}x_{14} - x_5x_7x_{12}x_{14} \\ & x_2x_9x_{11}x_{13} - x_2x_9x_{10}x_{14} + x_2x_7x_{12}x_{14} + x_1x_9x_{10}x_{15} - x_1x_7x_{12}x_{15} \\ & x_3x_4x_{11}x_{13} - x_2x_5x_{11}x_{13} - x_3x_4x_{10}x_{14} + x_2x_5x_{10}x_{14} - x_1x_5x_{10}x_{15}\rangle. \end{aligned}$$

Now Theorem 1.1.21 together with Lemmas 1.1.18 and 1.1.19 shows us how the slack variety leads to a realization space of a polytope or a cone.

Theorem 1.1.24 (See Corollary 2.3.4)**.** *(i) Given a polytope P, there is a bijection between the elements of $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ and the classes of projectively equivalent polytopes in the combinatorial class of P.*

(ii) Given a cone K, there is a bijection between elements of $\mathcal{V}_+(I_K)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ and the classes of linearly equivalent cones in the combinatorial class of K.

The space $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ is called the *slack realization space* of P. The space $\mathcal{V}_+(I_K)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ is called the *slack realization space* of K. Notice that when $K = P^h$, then these spaces coincide.

Example 1.1.25. *We construct slack realization space of the pentagon by first modding out row and column scalings (the action of $\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0}$) by setting a collection of variables in the slack matrix to 1. (See Lemma 3.5.2 in Chapter 3 for more details on how the entries can be chosen.)*

$$S_P(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 & x_2 & 1 & 0 \\ 0 & 0 & 1 & 1 & x_6 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & x_{11} & 0 & 0 & x_{12} \\ 1 & x_{14} & x_{15} & 0 & 0 \end{bmatrix}$$

Now taking the slack ideal of the above matrix, we find that the slack realization space is the positive part of the variety of

$$\begin{aligned} \langle x_{11}x_{15} + x_{11} - x_{14}, \quad & x_2x_{14} - x_{14} - x_{15} - 1, \quad & x_{12}x_{14} + x_{11} + x_{12} - x_{14}, \\ x_6x_{14} + x_6 - 1, \quad & x_2x_{12} + x_{12}x_{15} - x_{15}, \quad & x_6x_{15} + x_{12}x_{15} + x_{12} - x_{15}, \\ x_2x_{11} - x_{11} - 1, \quad & x_6x_{11} + x_6 + x_{12} - 1, \quad & x_2x_6 - x_{12}x_{15} - x_2 - x_{12} + x_{15} + 1\rangle. \end{aligned}$$

For the remainder of this chapter, we will focus on the realization space of a cone, since we have just seen that we can obtain the slack realization space of a polytope by considering its homogenization cone.

1.2 $\mathcal{R}(K)$ and $Gr_+(K)$

Next we will see how each of these realization spaces contains the same information about realizations of K in a very precise sense. (See Figure 1.2.)

Theorem 1.2.1. *There is a one-to-one correspondence between points in $Gr_+(K)$ and linear equivalence classes of sets of extreme rays that generate K.*

Proof. We have already seen that if $\boldsymbol{R} \in \mathcal{R}(K)$, then the Plücker vector of its column space satisfies the Plücker conditions defining $Gr_+(K)$. Furthermore, if $\boldsymbol{R}, \boldsymbol{Q} \in \mathcal{R}(K)$ are linearly equivalent with $\boldsymbol{Q} = \boldsymbol{R}\boldsymbol{A}$ for some $\boldsymbol{A} \in GL_{d+1}(\mathbb{R})$, then $pl(\boldsymbol{Q}) = \det(\boldsymbol{A}) \cdot pl(\boldsymbol{R})$; in other words, they are the same when considered as points in the Grassmannian via the map ρ.

Conversely, given a point $\Lambda \in Gr_+(K)$, there exists a matrix $\boldsymbol{X}_\Lambda \in \mathbb{R}^{v\times(d+1)}$ with that column space. We claim that the map

$$\begin{aligned} \mathbf{GrR} : Gr_+(K) &\to \mathcal{R}(K)/GL_{d+1}(\mathbb{R}) \\ \Lambda &\mapsto \boldsymbol{X}_\Lambda \end{aligned}$$

is inverse to ρ, giving the desired one-to-one correspondence. In fact, it is clear that this is the desired inverse, as long as we show that it is well-defined. For this, we need to show that the rows of $\boldsymbol{X}_\Lambda$ form a generating set for a cone in the combinatorial class of K. To see this, notice that each facet F of K is generated by some set of d rays $j_1, \dots, j_d$. Every other ray $j_0 \in [v]$ is either in that facet of K, or forms a facet extension with $j_1, \dots, j_d$. If $j_0 \in F$, then by definition of $Gr_+(K)$, rows $j_0, j_1, \dots, j_d$ of $\boldsymbol{X}_\Lambda$ are linearly dependent. If $j_0 \notin F$, then again by definition of $Gr_+(K)$, row j_0 of $\boldsymbol{X}_\Lambda$ is in the positive half-space of the hyperplane defined by the rows $j_1, \dots, j_d$. Thus rows of $\boldsymbol{X}_\Lambda$ generate a cone having the desired combinatorics. Finally, any other choice of matrix with $\rho(\boldsymbol{Y}) = \Lambda$ must be of the form $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{A}$ for some $\boldsymbol{A} \in GL_{d+1}(\mathbb{R})$ by (1.1), giving the correspondence up to linear equivalence, as desired. $\square$

Remark 1.2.2. Notice that even though rays $\{\boldsymbol{r}_1, \dots, \boldsymbol{r}_v\}$ and $\{\lambda_1\boldsymbol{r}_1, \dots, \lambda_v\boldsymbol{r}_v\}$ generate the *same* cone, they are not necessarily linearly equivalent for $\lambda_1, \dots, \lambda_v$ not all equal. This is reflected in their Plücker coordinates, as

$$pl(\operatorname{diag}(\lambda_1, \dots, \lambda_v)\boldsymbol{R})_{j_0 \cdots j_d} = pl(\boldsymbol{R})_{j_0 \cdots j_d} \cdot \prod_{i=0}^{d} \lambda_{j_i}.$$

Thus if there is some $i \neq k$ with $\lambda_i \neq \lambda_k$, we see that $pl(\boldsymbol{R})_{ij_1 \cdots j_d}$ and $pl(\boldsymbol{R})_{kj_1 \cdots j_d}$ for $i, k \notin \{j_1, \dots, j_d\}$ will be scaled differently, resulting in a different point in $Gr_+(K)$ under the map ρ.

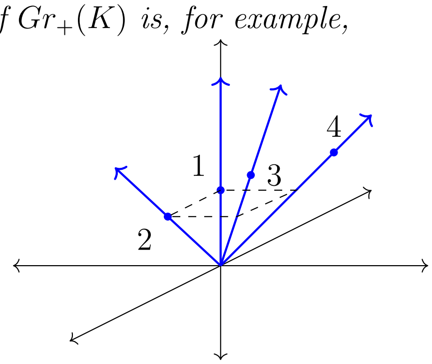
Example 1.2.3. *Let K be a cone over a square, with facets $\{12, 23, 34, 14\}$. Then since every set of 3 rays forms a facet extension, similarly to the pentagon we get*

$$Gr_+(K) = \{\Lambda \in Gr(3,4) : pl(\Lambda) > 0\} = \{[w : x : y : z] \in \mathbb{P}^3 : w, x, y, z > 0\},$$

where the last equality comes from the fact that there are no Plücker relations for $d + 1 = 3, v = 4$. From the proof of Theorem 1.2.1, we know that to obtain a realization of K from a point $[w : x : y : z] \in Gr_+(K)$ we want a matrix $\boldsymbol{X}$ such that $pl(\boldsymbol{X}) = [w : x : y : z]$.

To mod out by linear transformations, we will fix a choice of $d+1 = 3$ extreme rays in $\boldsymbol{X}_\Lambda$ for every Λ in advance. Notice that we can fix d arbitrary linearly independent vectors to be a spanning set for some facet, then the remaining vectors are determined by the Plücker coordinates we wish to impose. So we can choose for example $1 \mapsto (0, 0, 1)^\top$, and $2 \mapsto (1, 0, 1)^\top$. In addition, we can choose a final linear subspace, independent from the first d, for an extreme ray which forms a facet extension with those d indices. In this case, we choose ray 4 to be the span of vector $(0, 1, 1)^\top$ and the actual generator, $(0, x, x)^\top$ is determined by the scaling of our Plücker vector. Then a representative of the linear equivalence class of generating rays corresponding to each element $[w : x : y : z]$ of $Gr_+(K)$ is, for example,

$$\boldsymbol{X}_\Lambda = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ y/x & w & (w + y/x) - z/x \\ 0 & x & x \end{bmatrix}.$$



Remark 1.2.4. Notice that if P is such that $K = P^h$, Theorem 1.2.1 does not give us a one-to-one correspondence between points of $Gr_+(K)$ and equivalence classes of realizations of P. This is because a linear equivalence class of cone generators need not live in a single affine hyperplane and hence won't correspond to an actual realization of the polytope P.

Example 1.2.5. *Continuing Example 1.2.3, we see that the rows of $\boldsymbol{X}_\Lambda$ will give us a realization of a square if and only if they live in some hyperplane $\{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{\alpha}^\top \boldsymbol{x} = \gamma\}$ for some $\boldsymbol{\alpha} \in \mathbb{R}^3$ and $\gamma \in \mathbb{R}$. Solving the equation $\boldsymbol{X}_\Lambda \boldsymbol{\alpha} = \gamma$, tells us that this only occurs for $[w : x : y : z]$ satisfying $w + y - z = x$.*

It is easy to see in the above example that we may assume $\gamma = 1$, so that we only get a square when the all ones vector is in the column space of $\boldsymbol{X}_\Lambda$. In fact, this is true in general.

Corollary 1.2.6. *For $K = P^h$, there is a one-to-one correspondence between points of $Gr_+(K)$ which contain $\mathbb{1}$ and affine equivalence classes of realizations in $\mathcal{R}(P)$.*

Proof. A realization $\boldsymbol{R}$ of K, gives a realization of P if and only if the rows of $\boldsymbol{R}$ live in a single affine hyperplane of $\mathbb{R}^{d+1}$; that is, there exists $\boldsymbol{\alpha} \in \mathbb{R}^{d+1}$ so that $\boldsymbol{R}\boldsymbol{\alpha} = 1$. Since the subspace of $Gr_+(K)$ is the column space of the corresponding matrix in $\mathcal{R}(K)$, the result follows from Theorem 1.2.1 by restricting maps ρ and $\mathbf{GrR}$ to matrices with $\mathbb{1}$ in the column space and subspaces Λ containing $\mathbb{1}$, respectively. $\square$

It is not hard to see from the proof of Theorem 1.2.1 that as well as specializing to equivalence classes of actual polytope realizations, we can also generalize the map $\mathbf{GrR}$ to a map on all of $Gr(K)$, and then ρ generalizes to a map $\mathbf{RGr}$ on $v \times (d+1)$ matrices having "prescribed combinatorics" up to $GL_{d+1}(\mathbb{R})$-action.

Corollary 1.2.7. *There is a one-to-one correspondence between elements of $Gr(K)$ and full rank elements of $\mathbb{R}^{v\times(d+1)}/GL_{d+1}(\mathbb{R})$ whose rows satisfy the dependence/independence relations imposed by the combinatorics of K.*

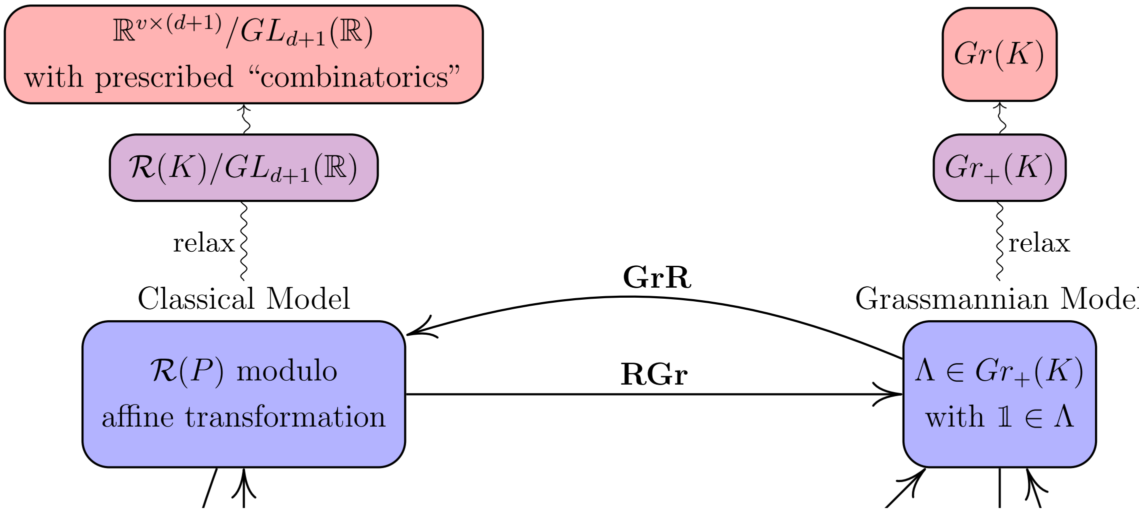


Figure 1.2: The relationships between realization space models: classical and Grassmannian.

1.3 Slack matrices from the Grassmannian

Recall from Section 1.1.3 that the slack matrix of a cone is the product of the matrix whose rows are its extreme rays with the matrix whose columns are its facet normals. Using Fact 1.1.12, this means given a realization $\boldsymbol{R} \in \mathbb{R}^{v\times f}$ and the combinatorics of K, we can calculate the entry of its slack matrix corresponding to vertex i and facet F by

$$(S_K)_{i,F} = \det\left[\boldsymbol{R}_{J_F}^\top | \boldsymbol{r}_i\right] \tag{1.4}$$

where J_F indexes a set of d rays which span facet F of K. From this formulation of the slack matrix we can see that the slack matrix of this realization of K can be obtained by filling a $v \times f$ matrix with Plücker coordinates of $\boldsymbol{R}$. (See also Chapter 5.)

Example 1.3.1. *Let K be a cone over a pentagon with extreme rays generated by the rows of matrix $\boldsymbol{X}$ of Example 1.1.11. The Plücker vector of $\boldsymbol{X}$ is*

$$\begin{array}{ccccccccccccc} {\scriptstyle i_0i_1i_2} & & {\scriptstyle 123} & {\scriptstyle 124} & {\scriptstyle 134} & {\scriptstyle 234} & {\scriptstyle 125} & {\scriptstyle 135} & {\scriptstyle 235} & {\scriptstyle 145} & {\scriptstyle 245} & {\scriptstyle 345} \\ pl(\boldsymbol{X}) & = & [1: & 2: & 3: & 2: & 1: & 2: & 2: & 1: & 2: & 2] \end{array}$$

and a slack matrix of K is given in Example 1.1.17. The facets of K are spanned by extreme rays $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,5\}$ and $\{5,1\}$, and we can calculate a slack matrix of K from $\boldsymbol{X}$ using (1.4) as follows.

$$S_K = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 12 & 23 & 34 & 45 & 51 \end{array} \\ \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 & 1 \\ 1 & 2 & 2 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} pl(\boldsymbol{X})_{121} & pl(\boldsymbol{X})_{231} & pl(\boldsymbol{X})_{341} & pl(\boldsymbol{X})_{451} & pl(\boldsymbol{X})_{511} \\ pl(\boldsymbol{X})_{122} & pl(\boldsymbol{X})_{232} & pl(\boldsymbol{X})_{342} & pl(\boldsymbol{X})_{452} & pl(\boldsymbol{X})_{512} \\ pl(\boldsymbol{X})_{123} & pl(\boldsymbol{X})_{233} & pl(\boldsymbol{X})_{343} & pl(\boldsymbol{X})_{453} & pl(\boldsymbol{X})_{513} \\ pl(\boldsymbol{X})_{124} & pl(\boldsymbol{X})_{234} & pl(\boldsymbol{X})_{344} & pl(\boldsymbol{X})_{454} & pl(\boldsymbol{X})_{514} \\ pl(\boldsymbol{X})_{125} & pl(\boldsymbol{X})_{235} & pl(\boldsymbol{X})_{345} & pl(\boldsymbol{X})_{455} & pl(\boldsymbol{X})_{515} \end{bmatrix}.$$

Unlike for the pentagon, most cones will have facets for which there are multiple choices for a spanning set J (e.g., cones over non-simplicial polytopes). Denote by $\mathcal{B}_K = \{J_F : F \in \text{facets}(P)\} \subset \binom{[v]}{d}$ one such set of choices; that is, J_F spans facet F of K. What (1.4) tells us is that given a choice of $\mathcal{B}_K$, we get a map $\mathbf{GrV}$ from $Gr_+(K)$ to the slack realization space of K given by

$$\mathbf{GrV}(\Lambda) := \Big[\Delta_{i,J}\, pl(\Lambda)_{i,J}\Big]_{i\in[v],J\in\mathcal{B}(K)},$$

where $\Delta_{i,J}$ is a sign that depends on the orientation of simplex J as well as the sign of the permutation that orders the elements of $\{i\} \cup J$.

Example 1.3.2. *Let K be a cone over a triangular prism with extreme rays given by the rows of the following matrix:*

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

Then, abbrieviating for example $\{1,2,3\}$ by 123, the facets of K are defined by extreme rays 123, 456, 1245, 1346, and 2356, and a possible choice of $\mathcal{B}_K$ is $\{123, 456, 124, 136, 236\}$. With this choice we find $\mathbf{GrV}(\rho(\boldsymbol{X}))$ gives

$$S_K = \begin{array}{c c} & \begin{array}{ccccc} \scriptstyle 123 & \scriptstyle 456 & \scriptstyle 124 & \scriptstyle 136 & \scriptstyle 236 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} pl(\boldsymbol{X})_{1123} & pl(\boldsymbol{X})_{1456} & pl(\boldsymbol{X})_{1124} & pl(\boldsymbol{X})_{1136} & pl(\boldsymbol{X})_{1236} \\ pl(\boldsymbol{X})_{1223} & pl(\boldsymbol{X})_{2456} & pl(\boldsymbol{X})_{1224} & pl(\boldsymbol{X})_{1236} & pl(\boldsymbol{X})_{2236} \\ pl(\boldsymbol{X})_{1233} & pl(\boldsymbol{X})_{3456} & pl(\boldsymbol{X})_{1234} & pl(\boldsymbol{X})_{1336} & pl(\boldsymbol{X})_{2336} \\ pl(\boldsymbol{X})_{1234} & pl(\boldsymbol{X})_{4456} & pl(\boldsymbol{X})_{1244} & -pl(\boldsymbol{X})_{1346} & pl(\boldsymbol{X})_{2346} \\ pl(\boldsymbol{X})_{1235} & pl(\boldsymbol{X})_{4556} & -pl(\boldsymbol{X})_{1245} & -pl(\boldsymbol{X})_{1356} & pl(\boldsymbol{X})_{2356} \\ pl(\boldsymbol{X})_{1236} & pl(\boldsymbol{X})_{4566} & -pl(\boldsymbol{X})_{1246} & pl(\boldsymbol{X})_{1366} & pl(\boldsymbol{X})_{2366} \end{bmatrix}.$$

Notice, we could just have easily chosen, say $\mathcal{B}_K = \{123, 456, 125, 134, 356\}$.

Relaxing this map from the realization space of K to $Gr(K)$, we get

$$\begin{aligned} \mathbf{GrV} : Gr(K) &\to \mathcal{V}(I_K)^* := \mathcal{V}(I_K) \cap (\mathbb{R}^*)^t \\ \Lambda &\mapsto \Big[\Delta_{i,J} pl(\Lambda)_{i,J}\Big]_{i\in[v], J\in\mathcal{B}(K)} \end{aligned} \tag{1.5}$$

Notice that this relaxation is well-defined, since $\mathbf{GrV}(\Lambda)$ has the correct zero pattern by definition of $Gr(K)$: any nonzero entry of a slack matrix is indexed by a facet extension and each zero entry is indexed by vertices contained in a facet of K. Furthermore, $\mathbf{GrV}(\Lambda)$ has rank $d+1$ since it is obtained by applying linear functionals to some rank $d+1$ matrix whose column span is Λ.

Since we are thinking about this map as filling a slack matrix with Plücker coordinates, we will use the additional notation $S_K(\Lambda_{\mathcal{B}})$ for denoting the image of Λ under $\mathbf{GrV}$ in order to emphasize that the choice of $\mathcal{B}$ determines which Plücker coordinates are used as slack entries. We will use the map $\mathbf{GrV}$ to prove the following theorem.

Theorem 1.3.3. *The nonzero real part of the slack variety of K, $\mathcal{V}(I_K)^*$, up to column scaling, is birationally equivalent to the Grassmannian of the cone K, $Gr(K)$. When $K = P^h$*

for polytope P, $Gr(K)$ is also equivalent to the nonzero real part of the slack variety of P, $\mathcal{V}(I_P)^*$, up to column scaling.

To prove this theorem, we define the following reverse map.

$$\begin{aligned} \mathbf{VGr} : \mathcal{V}(I_K)^* &\to Gr(K) \\ \boldsymbol{s} &\mapsto \rho(\boldsymbol{s}). \end{aligned} \tag{1.6}$$

That is, we map a matrix in the slack variety to its column space. This is a $(d+1)$-dimensional subspace of $\mathbb{R}^v$ by Theorem 2.3.2 and it will have the correct Plücker coordinates because the rows of $S_K(\boldsymbol{s})$ form a realization of K by the reasoning of [19, Theorem 14].

Remark 1.3.4. These are the maps implicitly defined in the proof of Theorem 5.3.8.

Proposition 1.3.5. *Two elements of $\mathcal{V}(I_K)^*$ are the same up to column scaling if and only if they have the same column space.*

Proof. Let $\boldsymbol{s}, \boldsymbol{t} \in \mathcal{V}(I_K)^*$. Clearly if $S_K(\boldsymbol{s}) = S_K(\boldsymbol{t}) \cdot D_f$ for some invertible diagonal matrix D_f, then

$$S_K(\boldsymbol{s}) \cdot \mathbb{R}^f = S_K(\boldsymbol{t}) \cdot D_f \cdot \mathbb{R}^f = S_K(\boldsymbol{t}) \cdot \mathbb{R}^f,$$

so their column spaces are the same.

Conversely, suppose they have the same column space. For each facet F_j, corresponding to column j of each slack matrix, there exists a flag (maximal chain) through F_j in the face lattice of K from which we obtain a $(d+1) \times (d+1)$ lower triangular submatrix of $S_K(\boldsymbol{s}), S_K(\boldsymbol{t})$ with nonzero diagonal, (see Lemma 2.3.1). The columns corresponding to each submatrix will span their respective column spaces, and since these spaces are the same there must be a change of basis matrix $\boldsymbol{A} \in GL_{d+1}(\mathbb{R})$ taking one to the other:

$$\underbrace{\begin{array}{cccc} & & & F_j \\ \lambda_1 & 0 & \cdots & 0 \\ * & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ * & & \cdots & \lambda_{d+1} \\ * & & \cdots & * \\ \vdots & & & \vdots \\ * & & \cdots & * \end{array}}_{S_K(\boldsymbol{s})} \cdot \boldsymbol{A} = \underbrace{\begin{array}{cccc} & & & F_j \\ \gamma_1 & 0 & \cdots & 0 \\ * & \gamma_2 & \cdots & 0 \\ \vdots & & \ddots & \\ * & & \cdots & \gamma_{d+1} \\ * & & \cdots & * \\ \vdots & & & \vdots \\ * & & \cdots & * \end{array}}_{S_K(\boldsymbol{t})}.$$

However, from the zero pattern, it is clear that the column corresponding to F_j in the $S_K(\boldsymbol{s})$ can only be a scaling of the same column in $S_K(\boldsymbol{t})$. Since this is true for each facet of K, the result follows. □

Since it is clear that $\mathbf{GrV}, \mathbf{VGr}$ are rational when we record elements of $Gr(K)$ by their Plücker coordinates, Theorem 1.3.3 becomes a corollary of the following result.

Lemma 1.3.6. *The maps* $\mathbf{GrV}, \mathbf{VGr}$ *defined in* (1.5),(1.6)*, respectively, are inverses; that is* $\mathbf{VGr} \circ \mathbf{GrV} = id_{Gr(K)}$ *and* $\mathbf{GrV} \circ \mathbf{VGr} = id_{\mathcal{V}(I_K)^*/\mathbb{R}^f}$*.*

Proof. First consider

$$\mathbf{VGr} \circ \mathbf{GrV}(\Lambda) = \rho(S_K(\Lambda_{\mathcal{B}})).$$

Let $\boldsymbol{X} \in \mathbb{R}^{v \times (d+1)}$ be a matrix whose columns form a basis for the subspace Λ, so that

$$S_K(\Lambda_{\mathcal{B}}) = \boldsymbol{X} \begin{bmatrix} | & & | \\ \boldsymbol{\alpha}_1 & \cdots & \boldsymbol{\alpha}_f \\ | & & | \end{bmatrix}$$

where $\boldsymbol{\alpha}_j$ is the normal to facet j calculated from $\boldsymbol{X}$ as in Fact 1.1.12. Since $\operatorname{rank}(S_K(\Lambda_{\mathcal{B}})) = \operatorname{rank}(\boldsymbol{X}) = d+1$, we must also have $\operatorname{rank}([\boldsymbol{\alpha}_1 \cdots \boldsymbol{\alpha}_f]) = d+1$, so that in particular, the column spaces of $S_K(\Lambda_{\mathcal{B}})$ and $\boldsymbol{X}$ are the same. Since the column space of $\boldsymbol{X}$ is Λ by definition, this gives

$$\mathbf{VGr} \circ \mathbf{GrV}(\Lambda) = \Lambda,$$

as desired.

Next consider

$$\mathbf{GrV} \circ \mathbf{VGr}(\boldsymbol{s}) = S_K\big(\rho(\boldsymbol{s})_{\mathcal{B}}\big).$$

By Proposition 1.3.5 it suffices to show that the column space of $S_K(\rho(\boldsymbol{s})_{\mathcal{B}})$ is the same as the column space of the slack matrix $S_K(\boldsymbol{s})$, but this follows from what we just showed, namely that $\rho(S_K(\Lambda_{\mathcal{B}})) = \Lambda$. □

Example 1.3.7. *Recall that the cone K over the pentagon with generators given by the rows of $\boldsymbol{X}$ had a slack matrix given by $\mathbf{GrV}(\rho(\boldsymbol{X}))$, as in Example 1.3.1.*

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad S_K = \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 & 1 \\ 1 & 2 & 2 & 0 & 0 \end{bmatrix}$$

Notice that the column space of S_K is spanned, for example, by the first three columns, and has Plücker vector $[2:4:6:4:2:4:4:2:4:4]$. *The column space of $\boldsymbol{X}$ is represented by Plücker vector* $[1:2:3:2:1:2:2:1:2:2]$, *which is the same as the previous vector in projective space, so that these column spaces are indeed the same.*

Remark 1.3.8. Notice that as in Theorem 1.2.1, there may be choices of points in $Gr(K)$ whose image under $\mathbf{GrV}$ is not an affine equivalence class of realizations of P, but is simply the orbit of a *generalized slack matrix* (see Section 3.2 of Chapter 2) of P under column scaling; that is, $S_K(\Lambda_{\mathcal{B}})$ might not have $\mathbb{1}$ in its column space. In fact, we see from the above proof that, as in Corollary 1.2.6, $\mathbb{1} \in \rho(S_K(\Lambda_{\mathcal{B}}))$ if and only if $\mathbb{1} \in \Lambda$. (See Figure 1.3.)

Example 1.3.9. *Recall the cone K over the square of Example 1.2.3. Letting $\Lambda = [w : x : y : z] \in Gr(K)$, we had*

$$\boldsymbol{X}_\Lambda = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ y/x & w & (w + y/x) - z/x \\ 0 & x & x \end{bmatrix}.$$

For a square, as with a pentagon, there is only once choice for facet bases $\mathcal{B}_K$, so that there is a single possible **GrV**, *which gives*

$$\mathbf{GrV}(\Lambda) = S_K(\Lambda_{\mathcal{B}}) = \begin{bmatrix} 0 & w & y & 0 \\ 0 & 0 & z & x \\ w & 0 & 0 & y \\ x & z & 0 & 0 \end{bmatrix}.$$

Even if we restrict to a point of $Gr_+(K)$, say $w = x = y = 1$, $z = 2$, it is not hard to check that neither $\boldsymbol{X}$, nor the resulting slack matrix, has $\mathbb{1}$ in its column space. In fact, we can see that the condition of Example 1.2.5 which guarantees the rows of $\boldsymbol{X}_\Lambda$ give the realization of a square, namely that $w + y - z = x$, also guarantee condition (ii)(c) of Theorem 1.1.21 on the resulting slack matrix.

Example 1.3.9 also illustrates another important use of the maps which give us the equivalence of Theorem 1.3.3, namely, they allow us to obtain useful parametrizations of the slack variety. By filling a slack matrix with Plücker coordinates, we are essentially imposing a collection of linear equalities on the slack

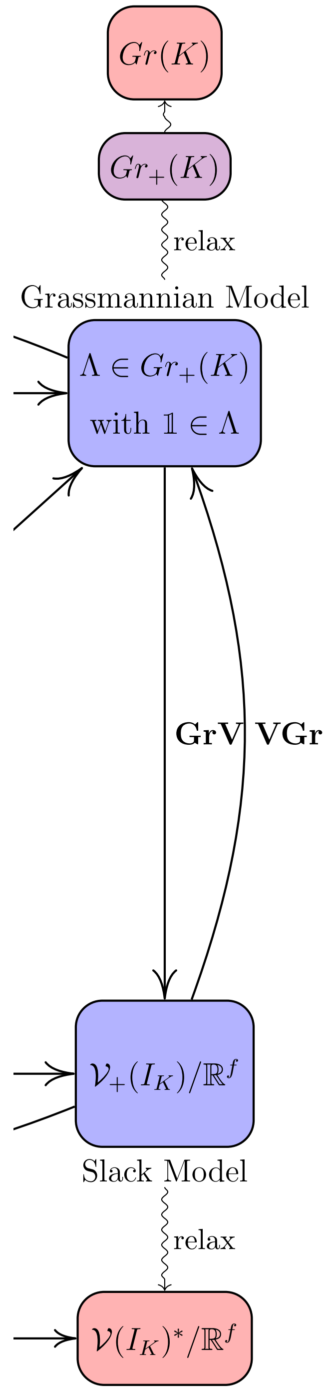


Figure 1.3: The relationships between realization space models: Grassmannian and slack.

variety determined by which entries are filled with the same Plücker coordinates. In some cases, we will see that restricting to such a parametrization allows us to greatly simplify the slack ideal.

If we fix a choice of $\mathcal{B}_K$ for the cone K, then we can consider the "symbolic image" of the map $\mathbf{GrV}$ as being a matrix in the Plücker variables $\boldsymbol{p}$. Namely, from (1.5), $\mathbf{GrV}$ evaluates the following symbolic matrix on the Plücker vector of Λ

$$\mathbf{GrV}(\boldsymbol{p}) := \Big[\Delta_{i,J} p_{i,J}\Big]_{i\in[v], J\in\mathcal{B}(K)}.$$

Recall that the Plücker coordinates of Λ, and hence the entries of $\mathbf{GrV}(\Lambda)$, satisfy the equations of the ideal of the Grassmannian of K given in (1.2), namely,

$$\mathcal{I}(Gr(K)) = \big(I_{d+1,v} + \langle \boldsymbol{p}_J : J \in \mathcal{F}(K)\rangle\big) : \left(\prod_{J\in\overline{\mathcal{F}}(K)} \boldsymbol{p}_J\right)^{\infty} \subset \mathbb{R}[\boldsymbol{p}].$$

Notice that the entries of $\mathbf{GrV}(\boldsymbol{p})$ need not use all the Plücker coordinates. In particular, entries are indexed by facets and facet extensions, so that we only require Plücker variables indexed by $\mathcal{F}(K), \overline{\mathcal{F}}(K) \subset \binom{[v]}{d+1}$, and it is often the case that

$$\mathcal{F}(K) \cup \overline{\mathcal{F}}(K) \neq \binom{[v]}{d+1}.$$

Furthermore, by definition of $Gr(K)$, only the variables $\boldsymbol{p}_J$ for $J \in \overline{\mathcal{F}}(K)$ will be nonzero. In light of this, we define the following ideal, which gives the conditions on only the nonzero Plücker coordinates that will be used in $\mathbf{GrV}(\Lambda)$.

Definition 1.3.10. *The* Grassmannian section ideal *of K, denoted $I_{d+1,v}(K)$ is given by eliminating the variables that are not necessary for* $\mathbf{GrV}$ *from $\mathcal{I}(Gr(K)$; that is,*

$$I_{d+1,v}(K) := \left(\big(I_{d+1,v} + \langle \boldsymbol{p}_J : J \in \mathcal{F}(K)\rangle\big) : \left(\prod_{J\in\overline{\mathcal{F}}(K)} \boldsymbol{p}_J\right)^{\infty}\right) \bigcap \mathbb{R}[\boldsymbol{p}_J : J \in \overline{\mathcal{F}}(K)].$$

Remark 1.3.11. Since $\mathbf{GrV}$ implicitly depends on the choice of $\mathcal{B}_K$, so does $I_{d+1,v}(K)$. Hence we could have several different section ideals for the same cone.

Example 1.3.12. *Recall that for a cone over a pentagon $\mathcal{F}(K) = \emptyset$, and all the Plücker variables are used to fill a slack matrix*

$$\mathbf{GrV}(\boldsymbol{p}) = \begin{bmatrix} 0 & p_{123} & p_{134} & p_{145} & 0 \\ 0 & 0 & p_{234} & p_{245} & p_{125} \\ p_{123} & 0 & 0 & p_{345} & p_{135} \\ p_{124} & p_{234} & 0 & 0 & p_{145} \\ p_{125} & p_{235} & p_{345} & 0 & 0 \end{bmatrix}.$$

So in this case, the Grassmannian section ideal is just the Plücker ideal.

Example 1.3.13. *Let K be a 4-dimensional cone over a triangular prism as in Example 1.3.2 with $\mathcal{B}_K = \{123, 456, 124, 136, 236\}$. Then*

$$\mathbf{GrV}(\boldsymbol{p}) = \begin{array}{c c} & \begin{array}{ccccc} 123 & 345 & 124 & 136 & 236 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & p_{1456} & 0 & 0 & p_{1236} \\ 0 & p_{2456} & 0 & p_{1236} & 0 \\ 0 & p_{3456} & p_{1234} & 0 & 0 \\ p_{1234} & 0 & 0 & -p_{1346} & p_{2346} \\ p_{1235} & 0 & -p_{1245} & -p_{1356} & p_{2356} \\ p_{1236} & 0 & -p_{1246} & 0 & 0 \end{bmatrix} \end{array},$$

and

$$\begin{aligned} I_{4,6}(K) \;=\; \langle\, & p_{2346}p_{1456} + p_{1246}p_{3456},\ p_{2346}p_{1356} + p_{1236}p_{3456},\ p_{1246}p_{1356} - p_{1236}p_{1456}, \\ & p_{1234}p_{1356}p_{2456} - p_{1235}p_{1246}p_{3456},\ p_{1235}p_{1246}p_{2346} + p_{1234}p_{1236}p_{2456}\,\rangle. \end{aligned}$$

The following lemma is an immediate consequence of Definitions 1.1.15 and 1.3.10.

Lemma 1.3.14. *The closure of the projection of the Grassmannian of K onto its nonzero coordinates is the variety of the Grassmannian section ideal of K; that is,*

$$\overline{\pi_{\overline{\mathcal{F}(K)}}(Gr(K))} = \mathcal{V}(I_{d+1,v}(K)).$$

Lemma 1.3.14 together with Theorem 1.3.3 gives us a geometric relationship between the slack variety and the Grassmannian section variety, namely that points in $\mathcal{V}(I_{d+1,v}(K))^*$ are in one-to-one correspondence with representatives of column scaling equivalence classes of $\mathcal{V}(I_P)^*$. However, we also find the following algebraic relationship that allows us to consider the Grassmannian section ideal as a potentially simplifying algebraic relaxation of the slack ideal.

Lemma 1.3.15. *When appropriate Plücker variables are substituted for slack variables in the slack ideal I_K, it is contained in the Grassmannian section ideal $I_{d+1,v}(K)$.*

Proof. Fix a choice of $\mathbf{GrV}$ for K. We have already seen that for each $\Lambda \in Gr(d+1, v)$, the matrix $\mathbf{GrV}(\Lambda) = S_K(\Lambda_{\mathcal{B}})$ has rank $d+1$. This means the polynomials given by the $(d+2)$-minors of the symbolic matrix $\mathbf{GrV}(\boldsymbol{p})$ vanish on every point of $\Lambda \in Gr(d+1, v)$, and hence, these minors must be in the Plücker ideal $I_{d+1,v}$. Setting appropriate Plücker variables to zero in the minors and in $I_{d+1,v}$ preserves this containment, and gives the desired result after saturation. ◻

Example 1.3.16. *Recall from Example 1.1.23 that the slack ideal of the pentagon had* 10 *three-term generators,* 9 *four-term generators and* 6 *five-term generators, all of degree* 4*. In contrast, the Grassmannian section ideal, which was simply the Plücker ideal $I_{3,5}$, is much simpler having only the following* 5 *trinomial generators of degree* 2*:*

$$p_{235}p_{145} - p_{135}p_{245} + p_{125}p_{345}$$
$$p_{234}p_{145} - p_{134}p_{245} + p_{124}p_{345}$$
$$p_{234}p_{135} - p_{134}p_{235} + p_{123}p_{345}$$
$$p_{234}p_{125} - p_{124}p_{235} + p_{123}p_{245}$$
$$p_{134}p_{125} - p_{124}p_{135} + p_{123}p_{145}.$$

Remark 1.3.17. It might seem that the effect of substituting Plücker variables into the slack matrix is simply to force certain entries to be equal. While certain equalities are forced by the choice of $\mathbf{GrV}$, the insistence that the entries come from a Plücker vector is in fact more restrictive than the slack matrix rank condition together with these inequalities. That is, the containment of Lemma 1.3.15 is strict in general.

Example 1.3.18. *Recall the section ideal of the cone over a triangular prism of Example 1.3.13,*

$$\begin{aligned} I_{4,6}(K) \;\; = \;\; & \langle\, p_{2346}p_{1456} + p_{1246}p_{3456},\; p_{2346}p_{1356} + p_{1236}p_{3456},\; p_{1246}p_{1356} - p_{1236}p_{1456}, \\ & p_{1234}p_{1356}p_{2456} - p_{1235}p_{1246}p_{3456},\; p_{1235}p_{1246}p_{2346} + p_{1234}p_{1236}p_{2456}\,\rangle. \end{aligned}$$

Simply setting the appropriate slack entries equal, we get

$$S_K(\boldsymbol{x}) = \begin{bmatrix} 0 & x_1 & 0 & 0 & x_2 \\ 0 & x_3 & 0 & x_2 & 0 \\ 0 & x_5 & x_6 & 0 & 0 \\ x_6 & 0 & 0 & 0 & x_8 \\ x_9 & 0 & 0 & x_{10} & 0 \\ x_2 & 0 & x_{12} & 0 & 0 \end{bmatrix}$$

and the resulting ideal is

$$\begin{aligned} \langle x_1x_8 - x_5x_{12}, x_3x_6x_{10} - x_5x_9x_{12}\rangle \;\; \mapsto \;\; & \langle p_{2346}p_{1456} + p_{1246}p_{3456}, p_{1234}p_{1356}p_{2456} - p_{1235}p_{1246}p_{3456}\rangle \\ \subsetneq \;\; & I_{4,6}(K) \end{aligned}$$

In fact, one can see that the matrix

$$S_K(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & x \\ 0 & 1 & 0 & x & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 & 0 \\ x & 0 & 1 & 0 & 0 \end{bmatrix}$$

has the appropriate slack variables set equal and that the rank of this matrix is 4 *for any choice of $x \neq 0$, hence it corresponds to a point in the slack variety $\mathcal{V}(I_K)^*$. However, now* *we check the following element of the Grassmannian section ideal:*

$$p_{2346}p_{1356} + p_{1236}p_{3456} = (-1)(2) + (-x)(-1) = -2 + x$$

which is only satisfied when $x = 2$. *That is, if we choose, say* $x = 1$, *the entries of* $S_K(x)$ *do not come from the Plücker vector of any point in* $Gr(K)$. *However, we know it is in the (column scaling) equivalence class of something that does, namely,*

$$\rho(S_K(1)) = \text{col sp}\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right) \overset{\mathbf{GrV}}{\longmapsto} \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} = S_K(1) \cdot \text{diag}(1, 2, 1, 1, 1).$$

This example illustrates the reason for including the word "section" in the Grassmannian section ideal. We have seen that the Grassmannian section ideal is in a sense a super ideal of the slack ideal (and a potentially simpler ideal) which cuts out a variety which corresponds to the "section" of the slack variety that contains an equivalence class representative whose entries come exactly from the Plücker coordinates of an element of $Gr(K)$. This is a direct consequence of Lemma 1.3.14 and Theorem 1.3.3.

Corollary 1.3.19. *The nonzero part of the Grassmannian section variety of* K, $\mathcal{V}(I_{d+1,v}(K))^*$, *is in one-to-one correspondence with equivalence class representatives of* $\mathcal{V}(I_K)^*/\mathbb{R}^f$ *of the form* $\mathbf{GrV}(\Lambda)$ *for* $\Lambda \in Gr(K)$.

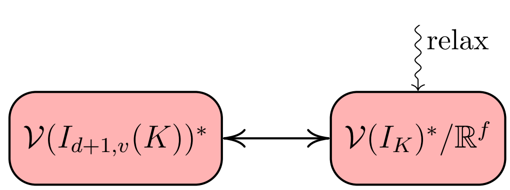


Figure 1.4: Relationships between realization space models: Grassmannian section and slack.

1.4 Gale Diagrams and Slack Varieties

Here we introduce one final realization space, the space of *Gale diagrams* of a polytope. Gale diagrams were developed by Perles in the 1960s (recorded by Grünbaum [28, Section 5.4]), and have long found use in the study of polytopes. They are particularly useful in the study of polytopes with relatively few vertices compared to their dimension. This is because they are in some sense a "dual" representation of the polytope. (The particular type of duality we speak of here is that of oriented matroids. For more details see [7], [45, Section 6].)

Given a realization $Q = \text{conv}\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_v\}$ of d-polytope P, let $\boldsymbol{B}$ be a matrix whose columns form a basis for the space of affine dependencies among the vertices of Q; that is,

$$\begin{bmatrix} 1 & \cdots & 1 \\ \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_v \end{bmatrix} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{B} \in \mathbb{R}^{v \times (v-d-1)},\ \text{rank}(\boldsymbol{B}) = v - d - 1.$$

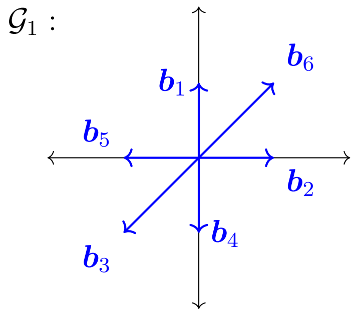
Let $\boldsymbol{b}_i^\top$ be the ith row of $\boldsymbol{B}$. The *Gale transform* of Q is the vector configuration $\mathcal{G} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_v\} \subset \mathbb{R}^{v-d-1}$. Recall that by definition of a slack matrix of Q, this implies

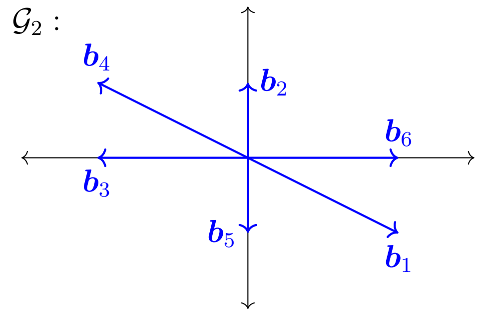
$$\begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_v \end{bmatrix} \cdot S_Q = 0.$$

Notice that for any $\boldsymbol{A} \in GL_{v-d-1}(\mathbb{R})$, the configuration $\{\boldsymbol{A}\boldsymbol{b}_1, \ldots, \boldsymbol{A}\boldsymbol{b}_v\}$ is also a Gale transform of Q, since $\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}^\top$ have the same column space.

Example 1.4.1. *Let Q be the realization of a (3-dimensional) triangular prism given by the rows of the matrix $\boldsymbol{X}$. Its Gale transform should be $6 - 3 - 1 = 2$ dimensional and it is not hard to check that the rows of $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are Gale transforms of Q, related by $\boldsymbol{A} \in GL_2(\mathbb{R})$.*

$$\boldsymbol{X} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \\ 0 & -1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -2 & 0 \\ -2 & 1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$





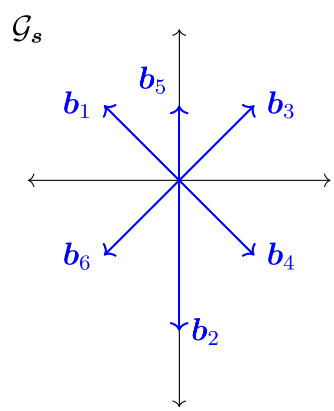
Definition 1.4.2. *By slight abuse of terminology, for each $\boldsymbol{s} \in \mathcal{V}(I_P)$ call a vector configuration $\mathcal{G}_{\boldsymbol{s}} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_v\}$ a* Gale transform *of $S_P(\boldsymbol{s})$ if $\begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_v \end{bmatrix} S_P(\boldsymbol{s}) = 0$ and the matrix $\boldsymbol{B}_{\boldsymbol{s}} := \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_v \end{bmatrix}^\top$ is full rank.*

Example 1.4.3. *Let P be the (abstract) triangular prism with facets* 123, 456, 1245, 1346, *and* 2356. *Consider the element of $\mathcal{V}(I_P)^*$ given by*

$$S_P(\boldsymbol{s}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is not hard to check that $\mathbb{1} \notin \rho(S_P(\boldsymbol{s}))$, so that by Theorem 1.1.21, $S_P(\boldsymbol{s})$ is not a slack matrix of an actual realization of P, but it has a Gale transform given by the rows of

$$\boldsymbol{B}_{\boldsymbol{s}} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \\ 1 & 1 \\ 1 & -1 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$



Remark 1.4.4. One can easily check from the Gale transform $\mathcal{G}_{\boldsymbol{s}} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_v\}$ whether it comes from an actual realization of P, since $\mathbb{1} \in \rho(S_P(\boldsymbol{s}))$ if and only if $\boldsymbol{b}_1 + \cdots + \boldsymbol{b}_v = 0$, by definition of $\mathcal{G}_{\boldsymbol{s}}$.

Denote by $\mathcal{G}(P)$ the set of all possible Gale transforms of a given abstract polytope P; that is,

$$\mathcal{G}(P) = \{\mathcal{G}_{\boldsymbol{s}} : \boldsymbol{s} \in \mathcal{V}(I_P)^*\}.$$

Of course, for $K = P^h$ we already have $\mathcal{V}(I_K)^* = \mathcal{V}(I_P)^*$, so that

$$\mathcal{G}(K) := \{\mathcal{G}_{\boldsymbol{s}} : \boldsymbol{s} \in \mathcal{V}(I_K)^*\} = \mathcal{G}(P). \tag{1.7}$$

From Proposition 1.3.5 and the definition of Gale transform, it is clear that there is a one-to-one correspondence between $GL_{v-d-1}(\mathbb{R})$ equivalence classes of $\mathcal{G}(K)$ and elements of $\mathcal{V}(I_K)^*$ up to column scaling. For this reason, and since for fixed $\boldsymbol{s}$, each $\mathcal{G}_{\boldsymbol{s}}$ just comes from a different choice of basis for the column space $\rho(\boldsymbol{B}_{\boldsymbol{s}})$, it makes sense to consider $\mathcal{G}(K)$ in the Grassmannian $Gr(v-d-1, v)$. Once again we consider how the combinatorics of K is encoded in the Plücker coordinates of each $\mathcal{G}_{\boldsymbol{s}} \in \mathcal{G}(K)$.

Lemma 1.4.5. *Let $\boldsymbol{s} \in \mathcal{V}(I_K)$. A set of rows $J \subsetneq [v]$ of $S_K(\boldsymbol{s})$ are dependent if and only if $[v] \backslash J$ indexes a set of rows of its Gale transform $\boldsymbol{B}_{\boldsymbol{s}}$ which live in a hyperplane through the origin.*

Remark 1.4.6. This lemma is weaker than the usual characterization of the combinatorics of a polytope from its Gale transform, namely that J indexes a face of P if and only if $J = [v]$ or if $\mathbf{0}$ is in the relative interior of Gale vectors $\{\boldsymbol{b}_j : j \in [v] \backslash J\}$. This lemma is also a consequence of the fact that matrices whose columns form bases for orthogonal vector spaces define dual matroids. However, we include an independent proof here.

Proof. Suppose that a set of vectors J in the Gale transform are coplanar; that is, there exists $\boldsymbol{c} \in \mathbb{R}^{v-d-1}$ such that

$$\boldsymbol{c}^\top \boldsymbol{b}_j = \begin{cases} 0 & \text{if } j \in J \\ \beta_j \neq 0 & \text{if } j \notin J. \end{cases} \tag{1.8}$$

Then

$$\begin{aligned} 0 &= \boldsymbol{c}^\top \boldsymbol{B}_{\boldsymbol{s}}^\top S_K(\boldsymbol{s}) && \text{by definiton of } \boldsymbol{B}_{\boldsymbol{s}} \\ &= \begin{bmatrix} \beta_1 & \cdots & \beta_n \end{bmatrix} S_K(\boldsymbol{s}) && \text{where } \beta_j = 0 \text{ for } j \in J \text{ by (1.8)} \\ &= \sum_{j \notin J} \beta_j \boldsymbol{y}_j && \text{where } \boldsymbol{y}_j \text{ is the } j\text{th row of } S_K(\boldsymbol{s}) \end{aligned}$$

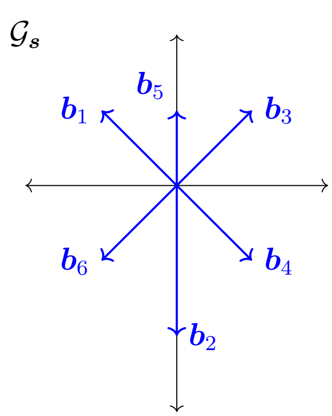
so that the complement of J indexes a dependent set of rows of $S_K(\boldsymbol{s})$.

Similarly, if there exists a linear dependence among the rows of $S_K(\boldsymbol{s})$, say

$$\sum_{j \in J} \boldsymbol{\alpha}_j \boldsymbol{y}_j = 0, \quad \boldsymbol{\alpha}_j \neq 0 \quad \forall j \in J,$$

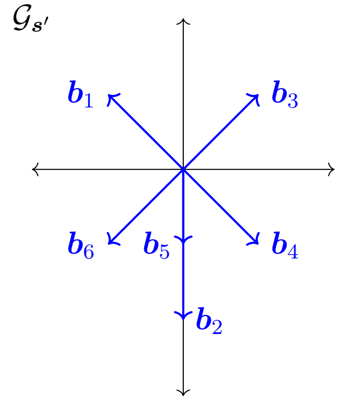
then the vector $\boldsymbol{a} \in \mathbb{R}^v$ with support indexed by J and corresponding entries $\boldsymbol{\alpha}_j$ is an element of the left kernel of $S_K(\boldsymbol{s})$. Thus, there exists $\boldsymbol{c} \in \mathbb{R}^{v-d-1}$ so that $\boldsymbol{c}^\top \boldsymbol{B}_{\boldsymbol{s}}^\top = \boldsymbol{a}^\top$, since $\boldsymbol{B}_{\boldsymbol{s}}^\top$ is a basis for the kernel. But now this means $\{\boldsymbol{b}_j\}_{j \notin J}$ are coplanar in the Gale transform, since $\boldsymbol{c}^T \boldsymbol{b}_j = \boldsymbol{a}_j = 0$ for $j \notin J$. □

Example 1.4.7. *Continuing Example 1.4.3, we note that trivially all sets of at least* 5 *rows of* S_P *are dependent, since* S_P *has rank* $d+1 = 4$*. This means each single vector of* $\mathcal{G}_{\boldsymbol{s}}$ *lives in a hyperplane through the origin, which is again trivially true. The only sets of non-trivially dependent rows of* S_P *are* $\{1,2,4,5\}$, $\{1,3,4,6\}$, *and* $\{2,3,5,6\}$, *which correspond to facets of* P*. Looking at the drawing of the Gale transform, we can see that the only pairs of vectors living in single hyperplanes (lines) through the origin are* $\{3,6\}$, $\{2,5\}$ *and* $\{1,4\}$, *as expected.*



Notice that for this example, we still have $\mathbf{0}$ *in the relative interior of* $\{\boldsymbol{b}_j : j \in [6]\backslash J\}$ *for faces* J *of* P*. However if we consider the slack matrix corresponding to* $\boldsymbol{s}' \in \mathcal{V}(I_P)^*$*, as given below, then its Gale transform does not have* $\mathbf{0} \in \text{rel int}\{\boldsymbol{b}_2, \boldsymbol{b}_5\}$ *even though* $\{1,3,4,6\}$ *represents a facet.*

$$S_P(\boldsymbol{s}') = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$



Using this lemma, we characterize the Gale transforms of combinatorial type K in the Grassmannian as follows.

Definition 1.4.8. *Define the* dual Grassmannian of cone K *to be*

$$\begin{aligned} Gr^*(K) := \{\Lambda \in Gr(v-d-1, v) : pl(\Lambda)_{[v]\setminus J} = 0 \quad \forall J \in \mathcal{F}(K), \\ pl(\Lambda)_{[v]\setminus J} \neq 0 \quad \forall J \in \overline{\mathcal{F}}(K)\}. \end{aligned}$$

Notice that $Gr^*(K)$ and $Gr(K)$ are isomorphic under the standard isomorphism of Grassmannians, $Gr(d+1, v) \cong Gr(v-d-1, v)$, which sends a subspace Λ to its orthogonal complement $\Lambda^\perp$. We will use this isomorphism in the proof of the following proposition which shows that $Gr^*(K)$ actually captures exactly the desired Gale transforms.

Proposition 1.4.9. *There is a one-to-one correspondence between elements of $Gr^*(K)$ and elements of $\mathcal{G}(K)$ modulo the action of $GL_{v-d-1}(\mathbb{R})$.*

Proof. Given an element $\mathcal{G}_{\boldsymbol{s}} \in \mathcal{G}(K)$, we map it to the column space of $\boldsymbol{B}_{\boldsymbol{s}}$. This space is an element of $Gr(v-d-1, v)$ since $\boldsymbol{B}_{\boldsymbol{s}}$ is full rank and satisfies the conditions of $Gr^*(K)$ by Lemma 1.4.5.

To see that each element of $Gr^*(K)$ represents the Gale transform of something in $\mathcal{V}(I_K)^*$, we use the well-known isomorphism of Grassmannians, namely,

$$\begin{aligned} Gr(d+1, v) &\cong Gr(v-d-1, v) \\ \Lambda &\leftrightarrow \Lambda^\perp. \end{aligned}$$

As a map on Plücker coordinates this translates to

$$\big(pl(\Lambda)_J\big)_{J\in\binom{[v]}{d+1}} \quad \leftrightarrow \quad \big(\operatorname{sgn}(J)\cdot pl(\Lambda)_{[v]\setminus J}\big)_{J\in\binom{[v]}{d+1}}, \tag{1.9}$$

where sgn(J) the sign of the permutation $(J, [v]-J)$. The result now follows from the fact that elements of $Gr(K)$ are in one-to-one correspondence with elements of $\mathcal{V}(I_K)^*$, up to column scaling, that have the same column space, by Theorem 1.3.3. □

Example 1.4.10. *Recall that for a cone K over a pentagon, all sets $J \in \binom{[v]}{d+1}$ are facet extensions and hence*

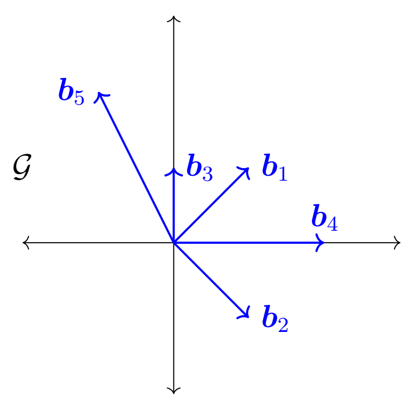
$$Gr(K) = \{\Lambda \in Gr(3,5) : pl(\Lambda)_J \neq 0 \quad \forall J \in \binom{[5]}{3}\}.$$

From this we also get

$$Gr^*(K) = \{\Lambda \in Gr(2,5) : pl(\Lambda)_J \neq 0 \quad \forall J \in \binom{[5]}{2}\}.$$

So for example, the matrix $\boldsymbol{B}$ below represents an element of $Gr^*(K)$ and hence a Gale transform of some pentagon.

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 2 & 0 \\ -1 & 2 \end{bmatrix}$$



To see to which element of the slack variety this Gale transform corresponds we recall that given $\Lambda \in Gr(K)$, we get a slack matrix via the map **GrV**, *which gives us a matrix of the form*

$$\mathbf{GrV}(\boldsymbol{p}) = \begin{bmatrix} 0 & p_{123} & p_{134} & p_{145} & 0 \\ 0 & 0 & p_{234} & p_{245} & p_{125} \\ p_{123} & 0 & 0 & p_{345} & p_{135} \\ p_{124} & p_{234} & 0 & 0 & p_{145} \\ p_{125} & p_{235} & p_{345} & 0 & 0 \end{bmatrix}.$$

Using (1.9)*, we easily obtain a slack matrix without constructing* Λ*.*

$$S_K = \begin{bmatrix} 0 & p_{45} & p_{25} & p_{23} & 0 \\ 0 & 0 & -p_{15} & -p_{13} & p_{34} \\ p_{45} & 0 & 0 & p_{12} & -p_{24} \\ -p_{35} & -p_{15} & 0 & 0 & p_{23} \\ p_{34} & p_{14} & p_{12} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & -3 & -1 & -2 \\ 4 & 0 & 0 & -2 & -2 \\ -1 & -3 & 0 & 0 & 1 \\ -2 & -2 & -2 & 0 & 0 \end{bmatrix}.$$

As a simple corollary, we a bijection from the Gale (dual Grassmannian) space to the slack variety (see Figure 1.5).

Corollary 1.4.11. *There is a one-to-one correspondence between elements of* $Gr^*(K)$ *and elements of* $\mathcal{V}(I_K)^*$ *up to column scaling.*

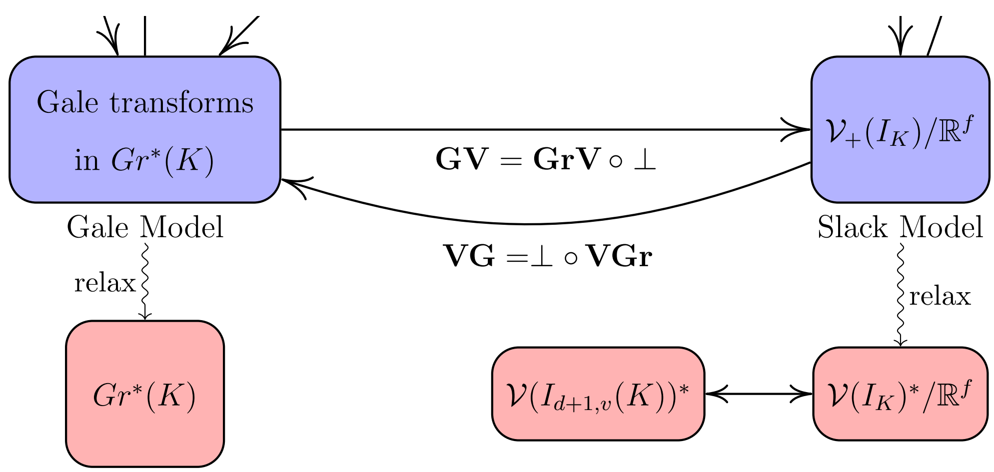


Figure 1.5: Relationships between realization space models: Gale and slack.

Remark 1.4.12. For polytopes, it is well-known that two realizations Q, Q' are projectively equivalent if and only if their Gale transforms $\mathcal{G}, \mathcal{G}'$ are related by a linear transformation and positive scaling of the vectors; that is, $\boldsymbol{b}'_i = \lambda_i \boldsymbol{A}\boldsymbol{b}_i$ for $\boldsymbol{A} \in GL_{v-d-1}(\mathbb{R})$, $\lambda_i \in \mathbb{R}_{>0}$. An analogous result holds for our generalized Gale transforms. Two elements of $\mathcal{V}(I_K)^*$, differ by row and column scaling if and only if their Gale transforms are related by a linear transformation and nonzero scaling of the vectors. This can be seen from the definition, since

$$\boldsymbol{B}_{\boldsymbol{s}}^{\top} S_K(\boldsymbol{s}) = 0 \Leftrightarrow (D_{v-d-1}\boldsymbol{A}\boldsymbol{B}_{\boldsymbol{s}}^{\top} D_v^{-1})(D_v S_K(\boldsymbol{s}) D_f) = 0$$

for invertible diagonal matrices $D_{v-d-1} \in \mathbb{R}^{(v-d-1)\times(v-d-1)}$, $D_v \in \mathbb{R}^{v\times v}$, $D_f \in \mathbb{R}^{f\times f}$.

Remark 1.4.13. It is also known that one can obtain the slack matrix of a realization Q by calculating the minimal positive circuits of its Gale transform $\mathcal{G} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_v\}$. That is, the columns of S_Q are vectors $\boldsymbol{y}$ such that $\boldsymbol{y}^{\top}\begin{bmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_v\end{bmatrix} = \boldsymbol{0}$, $\boldsymbol{y} \geq \boldsymbol{0}$, and $\boldsymbol{y}$ has minimal support. (For more details, see Section 6 of [45].)

Cone Realization Space	Restriction to Polytopes ($K = P^h$)	Algebraic Relaxation	Relevant Results Between Spaces
$\mathcal{R}(K)/GL_{d+1}(\mathbb{R})$	$\mathcal{R}(P)$ modulo affine transformations	$\mathbb{R}^{v\times(d+1)}/GL_{d+1}(\mathbb{R})$ with full rank and prescribed "combinatorics"	Theorem 1.2.1, Corollaries 1.2.6, 1.2.7
$Gr_+(K)$	$\Lambda \in Gr_+(K)$ with $\mathbb{1} \in \Lambda$	$Gr(K)$	Theorems 1.2.1, 1.3.3, Corollaries 1.2.6, 1.2.7
$\mathcal{V}_+(I_K)/\mathbb{R}^f$	$\boldsymbol{s} \in \mathcal{V}_+(I_P)/\mathbb{R}^f$ with $\mathbb{1} \in \rho(S_P(\boldsymbol{s}))$	$\mathcal{V}(I_K)^*/\mathbb{R}^f$	Theorem 1.3.3, Definition 1.4.2, (1.7), Corollaries 1.3.19, 1.4.11
		$\mathcal{V}I_{d+1,v}(K)$	Corollary 1.3.19
		$Gr^*(K)$	Proposition 1.4.9, (1.7), Corollary 1.4.11
$\mathcal{G} \in \mathcal{G}(K)/GL_{v-d-1}(\mathbb{R})$ s.t. every open half space H with $\boldsymbol{0} \in \mathrm{bd}(H)$ has $\lvert H \cap \mathcal{G}\rvert \geq 2$	$\mathcal{G} \in \mathcal{G}(K)/GL_{v-d-1}(\mathbb{R})$ s.t. every open half space H with $\boldsymbol{0} \in \mathrm{bd}(H)$ has $\lvert H \cap \mathcal{G}\rvert \geq 2$ and $\sum_{\boldsymbol{b}\in\mathcal{G}} \boldsymbol{b} = 0$	$\mathcal{G}(K)/GL_{v-d-1}(\mathbb{R})$	Proposition 1.4.9, Definition 1.4.2, (1.7)

Figure 1.6: Summary of results: relationships between realization spaces.

1.5 Organization

The remainder of this thesis is organized as follows.

The contents of Chapter 2 are from [20] written with Antonio Macchia, João Gouveia, and Rekha Thomas. This chapter gives more extensive background on slack matrices of polytopes and develops the slack realization space model in greater detail. It includes various classical realizability questions and how the slack ideal can be used as a new computational framework to answer these questions.

The contents of Chapter 3 are from [21] written with Antonio Macchia, João Gouveia, and Rekha Thomas. In this chapter we introduce a toric ideal T_P that can be associated to polytope P, namely the toric ideal of its vertex-facet non-incidence graph. We study the relationship of T_P to the slack ideal and show that among polytopes with toric slack ideals, we can identify the projectively unique ones as those whose slack ideal is equal to T_P. We also show that containment of the slack ideal in T_P is equivalent to a generalization of the 2-level property for polytopes. Finally, we look at a well-known example of a non-rational polytope due to Perles and show that it provides the first example of a non-prime slack ideal.

Chapter 4 considers the operations on polytopes introduced in [32] that preserve projective uniqueness. We classify all polytopes in dimension 5 that can be obtained from these operations. We also show that unlike what is known currently in dimensions 2, 3 and 4, there exists a projectively unique polytope that cannot be obtained from these constructions.

The contents of Chapter 5 are from [10] written with Madeline Brandt. This chapter generalizes slack matrices and slack ideals to the setting of matroids. We develop a realization space model for matroids analogous to that of Chapter 2 for polytopes. We relate this model to the classical Grassmannian realization space model and show how the slack ideal of a matroid can also be used to answer various questions about realizability. Furthermore, we generalize the notion of of the toric ideal of the non-incidence graph to the cycle ideal of a matroid and study its relationship to projective uniqueness.

Chapter 6 is based on work with Antonio Macchia. This chapter describes a `Macaulay2` package we have developed for creating and manipulating slack ideals.

Chapter 2

THE SLACK REALIZATION SPACE OF A POLYTOPE

2.1 Introduction

An important focus in the study of polytopes is the investigation of their realization spaces. Given a d-polytope $P \subset \mathbb{R}^d$, its face lattice determines its combinatorial type. A realization space of P is, roughly speaking, the set of all geometric realizations of the combinatorial type of P. This set, usually defined by fixing an affinely independent set of vertices in every realization of P, is a primary basic semialgebraic set, meaning that it is defined by a finite set of polynomial equations and strict inequalities.

Foundational questions about polytopes such as whether there is a polytope with rational vertices in the combinatorial class of P, whether a combinatorial type has any realization at all as a convex polytope, or whether faces of a polytope can be freely prescribed, are all questions about realization spaces. In general, many of these questions are hard to settle and there is no straightforward way to answer them by working directly with realization spaces. Each instance of such a question often requires a clever new strategy; indeed, the polytope literature contains many ingenious methods to find the desired answers.

In this paper, we introduce a model for the realization space of a polytope in a given combinatorial class modulo projective transformations. This space arises from the positive part of an algebraic variety called the *slack variety* of the polytope. An explicit model for the realization space of the projective equivalence classes of a polytope does not exist in the literature, although several authors have implicitly worked modulo projective transformations [3, 4, 38]. Using a related idea, we also construct a model for the realization space for a polytope that is rationally equivalent to the classical model for the realization space of the polytope. The ideal giving rise to the slack variety is called the *slack ideal* of the polytope and was introduced in [23]. The slack ideal in turn was inspired by the *slack matrix* of a polytope. This is a nonnegative real matrix with rows (and columns) indexed by the vertices (and facets) of the polytope and with (i, j)-entry equal to the slack of the ith vertex in the jth facet inequality. Each vertex/facet representation of a d-polytope P gives rise to a slack matrix S_P of rank $d+1$. Slack matrices have found remarkable use in the theory of extended formulations of polytopes (see for example, [44], [17], [39], [22], [30]). Their utility in creating a realization space model for polytopes was also observed in [14].

2.1.1 Our contribution

By passing to a symbolic version of the slack matrix S_P, wherein we replace every positive entry by a distinct variable in the vector of variables $\boldsymbol{x}$, one gets a symbolic matrix $S_P(\boldsymbol{x})$. The slack ideal I_P is the ideal obtained by saturating the ideal of $(d+2)$-minors of $S_P(\boldsymbol{x})$ with respect to all variables. The complex variety of I_P, $\mathcal{V}(I_P)$, is the slack variety of P. We prove that modulo a group action, the positive part of $\mathcal{V}(I_P)$ is a realization space for the projective equivalence classes of polytopes that are combinatorially equivalent to P. This is the slack realization space of P and it provides a new model for the realizations of a polytope modulo projective transformations. Working with a slightly modified ideal called the *affine slack ideal* of P, we also obtain a realization space for P that is rationally equivalent to the classical realization space of P. We call this the *affine slack realization space* of P. By the positive part of a complex variety we mean the intersection of the variety with the positive real orthant of the ambient space.

The slack realization space has several nice features. The inequalities in its description are simply nonnegativities of variables in place of the determinantal inequalities in the classical model. By forgetting these inequalities one can study the entire slack variety, which is a natural algebraic relaxation of the realization space. The slack realization space naturally mods out affine equivalence among polytopes and, unlike in the classical construction, does not depend on a choice of affine basis. The construction leads to a natural way to study polytopes up to projective equivalence. Further, it serves as a realization space for both the polytope it was constructed from as well as the polar of the polytope.

Additionally, the slack ideal provides a computational engine for establishing several types of results one can ask about the combinatorial class of a polytope. We exhibit three concrete applications of this machinery to determine non-rationality, non-prescribability of faces, and non-realizability of polytopes. We expect that further applications and questions on the important and difficult topic of realization spaces will be amenable to our algebraic geometry based approach.

2.1.2 Organization of the paper

In Section 2.2, we summarize the results on slack matrices needed in this paper. We also define the slack ideal and affine slack ideal of a polytope. In Section 2.3, we construct the slack and affine slack realization spaces of a polytope. We show that the affine slack realization space is rationally equivalent to the classical realization space of the polytope. In Section 2.4 we illustrate how the slack ideal provides a computational framework for many classical questions about polytopes such as convex realizability of combinatorial polytopes, rationality, and prescribability of faces.

2.1.3 Acknowledgements

We thank Arnau Padrol and Günter Ziegler for helpful pointers to the literature and valuable comments on the first draft of this paper. The `SageMath` and `Macaulay2` software systems were invaluable in the development of the results below. All computations described in this paper were done with one of these two systems [13], [27].

2.2 Background: Slack Matrices and Ideals of Polytopes

In this section we first present several known results about slack matrices of polytopes needed in this paper. Many of these results come from [19]. We then recall the slack ideal of a polytope from [23] which will be our main computational engine. While much of this section is background, we also present new objects and results that play an important role in later sections.

Suppose we are given a polytope $P \subset \mathbb{R}^d$ with v labelled vertices and f labelled facet inequalities. Assume that P is a d-polytope, meaning that $\dim(P) = d$. Recall that P has two usual representations: a $\mathcal{V}$-representation $P = \mathrm{conv}\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_v\}$ as the convex hull of vertices, and an $\mathcal{H}$-representation $P = \{\boldsymbol{x} \in \mathbb{R}^d : W\boldsymbol{x} \leq \boldsymbol{w}\}$ as the common intersection of the half spaces defined by the facet inequalities $W_j\boldsymbol{x} \leq w_j$, $j = 1, \ldots, f$, where W_j denotes the jth row of $W \in \mathbb{R}^{f\times d}$. Let $V \in \mathbb{R}^{v\times d}$ be the matrix with rows $\boldsymbol{p}_1{}^\top, \ldots, \boldsymbol{p}_v{}^\top$, and let $\mathbb{1}$ denote a vector (of appropriate size) with all entries equal to 1. Then the combined data of the two representations yields a *slack matrix* of P, defined as

$$S_P := \begin{bmatrix} \mathbb{1} & V \end{bmatrix} \begin{bmatrix} \boldsymbol{w}^\top \\ -W^\top \end{bmatrix} \in \mathbb{R}^{v\times f}. \tag{2.1}$$

The name comes from the fact that the (i, j)-entry of S_P is $w_j - W_j\boldsymbol{p}_i$ which is the *slack* of the ith vertex $\boldsymbol{p}_i$ of P with respect to the jth facet inequality $W_j\boldsymbol{x} \leq w_j$ of P. Since P is a d-polytope, $\mathrm{rank}(\begin{bmatrix} \mathbb{1} & V \end{bmatrix}) = d+1$, and hence, $\mathrm{rank}(S_P) = d+1$. Also, $\mathbb{1}$ is in the column span of S_P. While the $\mathcal{V}$-representation of P is unique, the $\mathcal{H}$-representation is not, as each facet inequality $W_j\boldsymbol{x} \leq w_j$ is equivalent to the scaled inequality $\lambda W_j\boldsymbol{x} \leq \lambda w_j$ for $\lambda > 0$, and hence P has infinitely many slack matrices obtained by positive scalings of the columns of S_P. Let D_t denote a diagonal matrix of size $t \times t$ with all positive diagonal entries. Then all slack matrices of P are of the form $S_P D_f$ for some D_f.

A polytope Q is *affinely equivalent* to P if there exists an invertible affine transformation ψ such that $Q = \psi(P)$. If Q is affinely equivalent to P, then S_P is a slack matrix of Q and thus P and Q have the same slack matrices (see Example 2.2.6). In fact, a slack matrix of P offers a representation of the affine equivalence class of P by the following result.

Lemma 2.2.1 ([19, Theorem 14])**.** *If S is any slack matrix of P, then the polytope $Q =$ $\mathrm{conv}(\mathrm{rows}(S))$, is affinely equivalent to P.*

By the above discussion, we may translate P so that $0 \in \text{int}(P)$ without changing its slack matrices. Subsequently, we may scale facet inequalities to set $\boldsymbol{w} = \mathbb{1}$. Then the affine equivalence class of P can be associated to the slack matrix

$$S_P^{\mathbb{1}} = [\mathbb{1}\ V] \begin{bmatrix} \mathbb{1} \\ -W^\top \end{bmatrix} \tag{2.2}$$

which has the special feature that the all-ones vector of the appropriate size is present in both its row space and column space. Again this matrix is not unique as it depends on the position of $0 \in \text{int}(P)$.

Recall that the polar of P is $P^\circ = \{\boldsymbol{y} \in (\mathbb{R}^d)^* : \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1 \ \forall\, \boldsymbol{x} \in P\}$. Under the assumption that $0 \in \text{int}(P)$ and that $\boldsymbol{w} = \mathbb{1}$, P° is again a polytope with 0 in its interior and representations [45, Theorem 2.11]:

$$P^\circ = \text{conv}\{W_1^\top, \ldots, W_f^\top\} = \{\boldsymbol{y} \in (\mathbb{R}^d)^* : V\boldsymbol{y} \leq \mathbb{1}\}.$$

This implies that $(S_P^{\mathbb{1}})^\top$ is a slack matrix of P° and all slack matrices of P° are of the form $(D_v S_P^{\mathbb{1}})^\top$.

We now pass from the fixed polytope P to its combinatorial class. Note that the zero-pattern in a slack matrix of P, or equivalently, the support of S_P, encodes the vertex-facet incidence structure of P, and hence the entire combinatorics (face lattice) of P [29]. A labelled polytope Q is *combinatorially equivalent* to P if P and Q have the same face lattice under the identification of vertex $\boldsymbol{p}_i$ in P with vertex $\boldsymbol{q}_i$ in Q and the identification of facet inequality f_j in P with facet inequality g_j in Q. The *combinatorial class* of P is the set of all labelled polytopes that are combinatorially equivalent to P. A *realization* of P is a polytope Q, embedded in some $\mathbb{R}^k$, that is combinatorially equivalent to P. By our labelling assumptions, all realizations of P have slack matrices with the same support as S_P. Further, since each realization Q of P is again a d-polytope, all its slack matrices have rank $d+1$ and contain $\mathbb{1}$ in their column span. Interestingly, the converse is also true and is a consequence of [19, Theorem 22].

Theorem 2.2.2. *A nonnegative matrix S is a slack matrix of some realization of the labelled d-polytope P if and only if all of the following hold:*

1. $\operatorname{supp}(S) = \operatorname{supp}(S_P)$
2. $\operatorname{rank}(S) = \operatorname{rank}(S_P) = d + 1$
3. $\mathbb{1}$ *lies in the column span of S.*

This theorem will play a central role in this paper. It allows us to identify the combinatorial class of P with the set of nonnegative matrices having the three listed properties.

A polytope Q is *projectively equivalent* to P if there exists a projective transformation ϕ such that $Q = \phi(P)$. Recall that a projective transformation is a map

$$\phi : \mathbb{R}^d \to \mathbb{R}^d, \;\; \boldsymbol{x} \mapsto \frac{B\boldsymbol{x} + \mathbf{b}}{\mathbf{c}^\top \boldsymbol{x} + \gamma}$$

for some $B \in \mathbb{R}^{d\times d}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, $\gamma \in \mathbb{R}$ such that

$$\det \begin{bmatrix} B & \mathbf{b} \\ \mathbf{c}^\top & \gamma \end{bmatrix} \neq 0. \tag{2.3}$$

The polytopes P and $Q = \phi(P)$ are combinatorially equivalent. Projective equivalence within a combinatorial class can be characterized in terms of slack matrices.

Lemma 2.2.3 ([23, Corollary 1.5])**.** *Two polytopes P and Q are projectively equivalent if and only if $D_v S_P D_f$ is a slack matrix of Q for some positive diagonal matrices D_v, D_f.*

Notice that Lemma 2.2.3 does not say that *every* positive scaling of rows and columns of S_P is a slack matrix of a polytope projectively equivalent to P, but rather that there is *some* scaling of rows and columns of S_P that produces a slack matrix of Q. In particular, condition (3) of Theorem 2.2.2 requires $\mathbb{1}$ to be in the column span of the scaled matrix. Not all row scalings will preserve $\mathbb{1}$ in the column span. Regardless, we will be interested in all row and column scalings of slack matrices.

Definition 2.2.4. *A* generalized slack matrix *of P is any matrix of the form $D_v S_Q D_f$, where Q is a polytope that is combinatorially equivalent to P and D_v, D_f are diagonal matrices with positive entries on the diagonal. Let $\mathfrak{S}_P$ denote the set of all generalized slack matrices of P.*

Theorem 2.2.5. *The set $\mathfrak{S}_P$ of generalized slack matrices of P consists precisely of the nonnegative matrices that satisfy conditions* (1) *and* (2) *of Theorem 2.2.2.*

Proof. By construction, every matrix in $\mathfrak{S}_P$ satisfies conditions (1) and (2) of Theorem 2.2.2. To see the converse, we need to argue that if S is a nonnegative matrix that satisfies conditions (1) and (2) of Theorem 2.2.2, then there exists some D_v, D_f such that $S = D_v S_Q D_f$ for some polytope Q that is combinatorially equivalent to P, or equivalently, that there is some row scaling of S that turns it into a slack matrix of a polytope combinatorially equivalent to P. By Theorem 2.2.2, this is equivalent to showing that $\mathbb{1}$ lies in the column span of $D_v^{-1}S$. Choose the diagonal matrix D_v^{-1} so that $D_v^{-1}S$ divides each row of S by the sum of the entries in that row. Note that this operation is well-defined, as a row of all zeros would correspond to a vertex which is part of every facet. Then the sum of the columns of $D_v^{-1}S$ is $\mathbb{1}$ making $D_v^{-1}S$ satisfy all three conditions of Theorem 2.2.2. Therefore, by the theorem, $D_v^{-1}S = S_Q$ for some polytope Q in the combinatorial class of P. $\square$

We illustrate the above results on a simple example.

Example 2.2.6. *Consider two realizations of a quadrilateral in $\mathbb{R}^2$,*

$$P_1 = \text{conv}\{(0,0),(1,0),(1,1),(0,1)\}, \text{ and}$$
$$P_2 = \text{conv}\{(1,-2),(1,2),(-1,2),(-1,-2)\},$$

where $P_2 = \psi(P_1)$ for the affine transformation $\psi(\boldsymbol{x}) = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. The most obvious *choice of facet representation for P_1 yields the slack matrix*

$$S_{P_1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

which, by calculating the effect of ψ on the facets of P_1, one finds is the same as the slack matrix for P_2,

$$S_{P_2} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}.$$

Since P_2 also contains the origin in its interior, we can scale each column of its $\mathcal{H}$-representation from above by 2 to obtain a slack matrix of the form $S^1_{P_2}$. Finally, consider the following nonnegative matrix

$$S = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$

Since S satisfies all three conditions of Theorem 2.2.2, it must be the slack matrix of some realization of a quadrilateral. In fact, it is easy to check that S is the slack matrix of the quadrilateral with vertices $\{(0,0),(1,0),(2,1),(0,1)\}$. Since all quadrilaterals are projectively equivalent, by Lemma 2.2.3 we must be able to obtain S_{P_1} by scaling the columns and rows of S and, in fact, multiplying its first column by 2 and its last two rows by $1/2$ we recover S_{P_1}.

We now recall the *symbolic slack matrix* and *slack ideal* of P which were defined in [23]. Given a d-polytope P, its *symbolic slack matrix* $S_P(\boldsymbol{x})$ is the sparse generic matrix obtained by replacing each nonzero entry of S_P by a distinct variable. Suppose there are t variables in $S_P(\boldsymbol{x})$. The *slack ideal* of P is the saturation of the ideal generated by the $(d+2)$-minors of $S_P(\boldsymbol{x})$, namely

$$I_P := \langle (d+2)\text{-minors of } S_P(\boldsymbol{x}) \rangle : \left(\prod_{i=1}^{t} x_i \right)^{\infty} \subset \mathbb{C}[\boldsymbol{x}] := \mathbb{C}[x_1, \ldots, x_t]. \tag{2.4}$$

The *slack variety* of P is the complex variety $\mathcal{V}(I_P) \subset \mathbb{C}^t$. The saturation of I_P by the product of all variables guarantees that there are no components in $\mathcal{V}(I_P)$ that live entirely in coordinate hyperplanes. If $\boldsymbol{s} \in \mathbb{C}^t$ is a zero of I_P, then we identify it with the matrix $S_P(\boldsymbol{s})$.

Lemma 2.2.7. *The set $\mathfrak{S}_P$ of generalized slack matrices is contained in the real part of the slack variety $\mathcal{V}(I_P)$.*

Proof. By Theorem 2.2.5, all matrices in $\mathfrak{S}_P$ have real entries, support equal to $\mathrm{supp}(S_P)$, and rank $d+1$. Therefore, $\mathfrak{S}_P$ is contained in the real part of $\mathcal{V}(I_P)$. □

To focus on "true slack matrices" of polytopes in the combinatorial class of P, meaning matrices that satisfy all conditions of Theorem 2.2.2, we define the *affine slack ideal*

$$\widetilde{I}_P = \langle (d+2)\text{-minors of } [S_P(\boldsymbol{x})\ \mathbb{1}] \rangle : \left(\prod_{i=1}^{t} x_i \right)^{\infty} \subset \mathbb{C}[\boldsymbol{x}], \tag{2.5}$$

where $[S_P(\boldsymbol{x})\ \mathbb{1}]$ is the symbolic slack matrix with a column of ones appended. By construction, $\mathcal{V}(\widetilde{I}_P)$ is a subvariety of $\mathcal{V}(I_P)$.

Definition 2.2.8. *Let $\widetilde{\mathfrak{S}}_P$ denote the set of true slack matrices of polytopes in the combinatorial class of P, or equivalently, the set of all nonnegative matrices that satisfy the three conditions of Theorem 2.2.2.*

Lemma 2.2.9. *The set $\widetilde{\mathfrak{S}}_P$ of true slack matrices is contained in the real part of $\mathcal{V}(\widetilde{I}_P)$.*

Proof. By definition, all elements $S \in \widetilde{\mathfrak{S}}_P$ have real entries and $\mathrm{supp}(S) = \mathrm{supp}(S_P)$. It remains to show that $\mathrm{rank}([S\ \mathbb{1}]) \leq d+1$. This follows immediately from the fact that S satisfies properties (2) and (3) of Theorem 2.2.2. □

Example 2.2.10. *For our quadrilateral P_1 from Example 2.2.6 and in fact any quadrilateral P labeled in the same way as P_1, we have*

$$S_P(\boldsymbol{x}) = \begin{bmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & x_3 & x_4 \\ x_5 & 0 & 0 & x_6 \\ x_7 & x_8 & 0 & 0 \end{bmatrix}.$$

Its slack ideal is

$$I_P = \langle 4\text{-}minors\ of\ S_P(\boldsymbol{x})\rangle : \left(\prod_{i=1}^{8} x_i\right)^{\infty} = \langle x_2x_4x_5x_8 - x_1x_3x_6x_7\rangle \subset \mathbb{C}[x_1, \ldots, x_8].$$

The affine slack ideal of P is

$$\begin{aligned}\widetilde{I}_P = \langle 4\text{-}minors\ of\ [S_P(\boldsymbol{x})\ \mathbb{1}]\rangle : \left(\prod_{i=1}^{8} x_i\right)^{\infty} = \langle & x_1x_3x_6 - x_2x_4x_8 + x_2x_6x_8 - x_3x_6x_8, \\ & x_2x_4x_5 - x_2x_4x_7 + x_2x_6x_7 - x_3x_6x_7, \\ & x_1x_4x_5 - x_1x_4x_7 + x_1x_6x_7 - x_4x_5x_8, \\ & x_1x_3x_5 - x_1x_3x_7 + x_2x_5x_8 - x_3x_5x_8\rangle.\end{aligned}$$

Notice, for example, that the generalized slack matrix which corresponds to $\boldsymbol{s} = (2, 2, 2, 1, 8, 2, 2, 1)$ is a zero of I_P but not of $\widetilde{I}_P$ and indeed $\mathbb{1}$ is not in the column span of $S_P(\boldsymbol{s})$. □

2.3 Realization spaces from Slack Varieties

Recall that a realization of a d-polytope $P \subset \mathbb{R}^d$ is a polytope Q that is combinatorially equivalent to P. A *realization space* of P is, essentially, the set of all polytopes Q which are realizations of P, or equivalently, the set of all "geometrically distinct" polytopes which are combinatorially equivalent to P. We say "essentially" since it is typical to mod out by affine equivalence within the combinatorial class.

The standard construction of a realization space of $P = \text{conv}\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_v\}$ is as follows (see [37]). Fix an *affine basis* of P, that is, $d + 1$ vertex labels $B = \{b_0, \ldots, b_d\}$ such that the vertices $\{\boldsymbol{p}_b\}_{b\in B}$ are necessarily affinely independent in every realization of P. Then the *realization space of P with respect to B* is

$$\mathcal{R}(P,B) = \{\text{realizations } Q = \text{conv}\{\boldsymbol{q}_1,\dots,\boldsymbol{q}_v\} \text{ of } P \text{ with } \boldsymbol{q}_i = \boldsymbol{p}_i \text{ for all } i \in B\}.$$

Fixing an affine basis ensures that just one Q from each affine equivalence class in the combinatorial class of P occurs in $\mathcal{R}(P,B)$.

Realization spaces of polytopes are *primary basic semialgebraic sets*, that is, they are defined by finitely many polynomial equations and strict inequalities. Recording each realization Q by its vertices, we can think of $\mathcal{R}(P,B)$ as lying in $\mathbb{R}^{d\cdot v}$. Two primary basic semialgebraic sets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^{m+n}$ are *rationally equivalent* if there exists a homeomorphism $f: X \to Y$ such that both f and f^{-1} are rational functions. The important result for us is that if B_1, B_2 are two affine bases of a polytope P, then $\mathcal{R}(P,B_1)$ and $\mathcal{R}(P,B_2)$ are rationally equivalent [37, Lemma 2.5.4]. Thus one can call $\mathcal{R}(P,B) \subset \mathbb{R}^{d\cdot v}$, *the* realization space of P.

The main goal of this section is to construct models of realization spaces for P from the slack variety $\mathcal{V}(I_P) \subset \mathbb{C}^t$ and affine slack variety $\mathcal{V}(\widetilde{I}_P)$ defined in Section 3.2. Recall that we identify an element $\boldsymbol{s}$ in either variety with the matrix $S_P(\boldsymbol{s})$. Then by Lemma 2.2.7, $\mathfrak{S}_P$, the set of all generalized slack matrices of all polytopes in the combinatorial class of P, is contained in $\mathcal{V}(I_P)$. Similarly, by Lemma 2.2.9, $\widetilde{\mathfrak{S}}_P$, the set of all true slack matrices of polytopes in the combinatorial class of P, is contained in $\mathcal{V}(\widetilde{I}_P)$. In fact, $\mathfrak{S}_P$ is contained in the positive part of $\mathcal{V}(I_P)$, defined as

$$\mathcal{V}_+(I_P) := \mathcal{V}(I_P) \cap \mathbb{R}^t_{>0} \tag{2.6}$$

and $\widetilde{\mathfrak{S}}_P$ is contained in the positive part of $\mathcal{V}(\widetilde{I}_P)$ defined as

$$\mathcal{V}_+(\widetilde{I}_P) := \mathcal{V}(\widetilde{I}_P) \cap \mathbb{R}^t_{>0}. \tag{2.7}$$

These positive spaces are going to lead to realization spaces of P. In order to get there, we first describe these sets more explicitly. We start with a well-known lemma, whose proof we include for later reference.

Lemma 2.3.1. *Let S be a matrix with the same support as S_P. Then* $\operatorname{rank}(S) \geq d+1$.

Proof. Consider a flag of P, i.e., a maximal chain of faces in the face lattice of P. Choose a sequence of facets $F_0, F_1, \ldots, F_d$ so that the flag is

$$\emptyset \;=\; F_0 \cap \cdots \cap F_d \;\subset\; F_1 \cap \cdots \cap F_d \;\subset\; \cdots \;\subset\; F_{d-1} \cap F_d \;\subset\; F_d \;\subset\; P.$$

Next choose a sequence of vertices so that $v_0 = F_1 \cap \cdots \cap F_d$ is the 0-face in the flag, making $v_0 \notin F_0$. Then choose $v_1 \in F_2 \cap \cdots \cap F_d$ but $v_1 \notin F_1$, $v_2 \in F_3 \cap \cdots \cap F_d$ but $v_2 \notin F_2$ and so on, until $v_{d-1} \in F_d$ but not in F_{d-1}. Finally, choose v_d so that $v_d \notin F_d$. Then the $(d+1) \times (d+1)$ submatrix of S_P indexed by the chosen vertices and facets is lower triangular with a nonzero diagonal, hence has rank $d+1$.

Now if S is a matrix with $\operatorname{supp}(S) = \operatorname{supp}(S_P)$, S will also have this lower triangular submatrix in it, thus $\operatorname{rank}(S) \geq d+1$. $\square$

We remark that the vertices chosen from the flag in the above proof form a suitable affine basis to fix in the construction of $\mathcal{R}(P, B)$.

Theorem 2.3.2. *The positive part of the slack variety, $\mathcal{V}_+(I_P)$, coincides with $\mathfrak{S}_P$, the set of generalized slack matrices of P. Similarly, $\mathcal{V}_+(\widetilde{I}_P)$ coincides with $\widetilde{\mathfrak{S}}_P$, the set of true slack matrices of P.*

Proof. We saw that $\mathfrak{S}_P \subseteq \mathcal{V}_+(I_P)$ and by Theorem 2.2.5, $\mathfrak{S}_P$ is precisely the set of nonnegative matrices with the same support as S_P and rank $d+1$. On the other hand, if $\boldsymbol{s} \in \mathcal{V}_+(I_P)$, then $S_P(\boldsymbol{s})$ is nonnegative and $\operatorname{supp}(S_P(\boldsymbol{s})) = \operatorname{supp}(S_P)$. Therefore, by Lemma 2.3.1, $\operatorname{rank}(S_P(\boldsymbol{s})) = d+1$. Thus, $\mathcal{V}_+(I_P) = \mathfrak{S}_P$.

We saw that $\widetilde{\mathfrak{S}}_P \subseteq \mathcal{V}_+(\widetilde{I}_P)$. Also recall that $\mathcal{V}_+(\widetilde{I}_P)$ is contained in $\mathcal{V}_+(I_P)$. Therefore, by the first statement of the theorem, if $\boldsymbol{s} \in \mathcal{V}_+(\widetilde{I}_P)$, then $S_P(\boldsymbol{s})$ is nonnegative, $\operatorname{supp}(S_P(\boldsymbol{s})) = \operatorname{supp}(S_P)$ and $\operatorname{rank}(S_P(\boldsymbol{s})) = d+1$. From the definition of $\widetilde{I}_P$, we have $\operatorname{rank}([S_P(\boldsymbol{s})\ \mathbb{1}]) \leq d+1$, so it follows that $\operatorname{rank}([S_P(\boldsymbol{s})\ \mathbb{1}]) = d+1$, or equivalently, $\mathbb{1}$ lies in the column span of $S_P(\boldsymbol{s})$. Therefore, the matrices in $\mathcal{V}_+(\widetilde{I}_P)$ satisfy all three conditions of Theorem 2.2.2, hence $\mathcal{V}_+(\widetilde{I}_P) = \widetilde{\mathfrak{S}}_P$. $\square$

Since positive row and column scalings of a generalized slack matrix of P give another generalized slack matrix of P, we immediately get that $\mathcal{V}_+(I_P)$ is closed under row and column scalings. Similarly, $\mathcal{V}_+(\widetilde{I}_P)$ is closed under column scalings.

Corollary 2.3.3.

1. *If $\boldsymbol{s} \in \mathcal{V}_+(I_P)$, then $D_v \boldsymbol{s} D_f \in \mathcal{V}_+(I_P)$, for all positive diagonal matrices D_v, D_f.*

2. *Similarly, if $\boldsymbol{s} \in \mathcal{V}_+(\widetilde{I}_P)$, then $\boldsymbol{s} D_f \in \mathcal{V}_+(\widetilde{I}_P)$, for all positive diagonal matrices D_f.*

Corollary 2.3.3 tells us that the groups $\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0}$ and $\mathbb{R}^f_{>0}$ act on $\mathcal{V}_+(I_P)$ and $\mathcal{V}_+(\widetilde{I}_P)$, respectively, via multiplication by positive diagonal matrices. Modding out these actions is the same as setting some choice of variables in the symbolic slack matrix to 1, which means that we may choose a representative of each equivalence class (affine or projective) with ones in some prescribed positions.

Corollary 2.3.4.

1. *Given a polytope P, there is a bijection between the elements of $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ and the classes of projectively equivalent polytopes of the same combinatorial type as P. In particular, each class contains a true slack matrix.*

2. *Given a polytope P, there is a bijection between the elements of $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$ and the classes of affinely equivalent polytopes of the same combinatorial type as P.*

The last statement in Corollary 2.3.4 (1) follows from the fact that every generalized slack matrix admits a row scaling that makes it satisfy all three conditions of Theorem 2.2.2, thereby making it a true slack matrix. An explicit example of such a scaling can be seen in the proof of Theorem 2.2.5.

By the above results we have that $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ and $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$ are parameter spaces for the projective (respectively, affine) equivalence classes of polytopes in the combinatorial class of P. Thus they can be thought of as realization spaces of P.

Definition 2.3.5. *Call* $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ *the* slack realization space *of the polytope* P*, and* $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$ *the* affine slack realization space *of the polytope* P*.*

We will see below that the affine slack realization space $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$ is rationally equivalent to the classical model of realization space $\mathcal{R}(P,B)$ of the polytope P. On the other hand, our main object, the slack realization space $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$, does not have an analog in the polytope literature. This is partly because in every realization of P, fixing a projective basis does not guarantee that the remaining vertices in the realization are not at infinity. The slack realization space is a natural model for the realization space of projective equivalence classes of polytopes. We note that in [25] the authors investigate the projective realization space of combinatorial hypersimplices and find an upper bound for its dimension. However they do not present an explicit model for it.

Theorem 2.3.6. *The affine slack realization space* $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$ *is rationally equivalent to the classical realization space* $\mathcal{R}(P,B)$ *of the polytope* P*.*

Proof. We will show that $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$ is rationally equivalent to $\mathcal{R}(P,B)$ for a particular choice of B. By [37, Lemma 2.5.4], this is sufficient to show rational equivalence for any choice of basis.

We have already shown that realizations of P modulo affine transformations are in bijective correspondence with the elements of both $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$ and $\mathcal{R}(P,B)$. So we just have to prove that this bijection induces a rational equivalence between these spaces, i.e., both the map and its inverse are rational.

We will start by showing the map sending a polytope in $\mathcal{R}(P,B)$ to its slack matrix is rational. Fix a flag in P, as in the proof of Lemma 2.3.1. Suppose the sequence of vertices and facets chosen from the flag in the proof are indexed by the sets I and J respectively. The vertices $\{\boldsymbol{p}_i\}_{i\in I}$ are affinely independent, so that $B = I$ is an affine basis of P. Moreover, by applying an affine transformation to P, we may assume that 0 is in the convex hull of $\{\boldsymbol{p}_i\}_{i\in I}$, hence is in the interior of every element of $\mathcal{R}(P,B)$. Consider the map

$$g : \mathcal{R}(P,B) \to \mathcal{V}_+(\widetilde{I}_P), \quad Q \mapsto S^1_Q.$$

The polytope Q is recorded in $\mathcal{R}(P,B)$ by its list of vertices, which in turn are the rows of the matrix V. Also, recall that $S_Q^1 = [\mathbb{1}\ V]\begin{bmatrix}\mathbb{1}\\ -W^\top\end{bmatrix}$. To prove that g is a rational map, we need to show that the matrix of facet normals W is a rational function of V. Since we know the combinatorial type of P, we know the set of vertices that lie on each facet. For facet j, let $V(j)$ be the submatrix of V whose rows are the vertices on this facet. Then the normal of facet j, or equivalently W_j, is obtained by solving the linear system $V(j)\cdot \boldsymbol{x} = \mathbb{1}$ which proves that W_j is a rational function of V. Then $\widetilde{g} = \pi \circ g$ is the desired rational map from $\mathcal{R}(P,B)$ to $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$, where π is the standard quotient map $\pi : \mathcal{V}_+(\widetilde{I}_P) \to \mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$. It sends the representative in $\mathcal{R}(P,B)$ of an affine equivalence class of polytopes in the combinatorial class of P to the representative of that class in $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^f_{>0}$.

For the reverse map, we have to send a slack matrix S_Q of a realization Q of P to the representative of its affine equivalence class in $\mathcal{R}(P,B)$. We saw in Lemma 2.2.1 that the rows of S_Q are the vertices of a realization Q' of P that is affinely equivalent to Q. So we just have to show that Q' can be rationally mapped to the representative of Q in $\mathcal{R}(P,B)$. To do that, denote by $\widehat{S}_Q$ the $(d+1)\times(d+1)$ lower triangular submatrix of S_Q from our flag, with rows indexed by I and columns indexed by J. Then $(\widehat{S}_Q)^{-1}$ consists of rational functions in the entries of $\widehat{S}_Q$. Let $\mathcal{B}$ be the $(d+1)\times d$ matrix whose rows are the vertices of P indexed by B. Recall that these vertices are common to all elements of $\mathcal{R}(P,B)$, and in particular, they form an affine basis for the representative of Q in $\mathcal{R}(P,B)$. Then the linear map

$$\psi_{S_Q} : \mathbb{R}^f \to \mathbb{R}^d, \quad \boldsymbol{x} \mapsto \boldsymbol{x}_J^\top \widehat{S}_Q^{-1}\mathcal{B},$$

where $\boldsymbol{x}_J$ is the restriction of $\boldsymbol{x} \in \mathbb{R}^f$ to the coordinates indexed by J, is defined rationally in terms of the entries of S_Q, and maps row i of S_Q to the affine basis vertex $\boldsymbol{p}_i$, for all $i \in I$. Now since ψ_{S_Q} is a linear map, $\psi_{S_Q}(Q')$ is affinely equivalent to Q' which is itself affinely equivalent to Q. Furthermore, ψ_{S_Q} sends an affine basis of Q' to the corresponding affine basis in Q, so in fact it must be a bijection between the two polytopes. Hence, $\psi_{S_Q}(\text{rows of } S_Q)$ equals the representative of Q in $\mathcal{R}(P,B)$, completing our proof. $\square$

The slack realization space is especially elegant in the context of polarity. Let P° be the polar polytope of P. It is not immediately obvious from the standard model of a realization space, how $\mathcal{R}(P, B_1)$ and $\mathcal{R}(P^\circ, B_2)$ are related. In [37], it is shown that the realization spaces of P and P° are *stably equivalent*, a coarser notion of equivalence than rational equivalence (see [37, Definition 2.5.1]); however, the proof of this fact in [37, Theorem 2.6.3] is non-trivial. Now consider the slack model. Recall we know that one slack matrix of P° is $(S_P^1)^\top$, so that $S_{P^\circ}(\boldsymbol{x}) = S_P(\boldsymbol{x})^\top$. In particular, this means that $I_{P^\circ} = I_P$, so that the slack varieties and realization spaces of P and P° are actually the same when considered as subsets of $\mathbb{R}^t$. We simply need to interpret $\boldsymbol{s} \in \mathcal{V}_+(I_P) = \mathcal{V}_+(I_{P^\circ})$ as a realization of P or P° by assigning its coordinates to $S_P(\boldsymbol{x})$ along rows or columns.

Example 2.3.7. *Let us return to the realization space of the unit square P_1 from Example 2.2.6. Suppose we fix the affine basis $B = \{1, 2, 4\}$, where we had $\boldsymbol{p}_1 = (0,0), \boldsymbol{p}_2 = (1,0)$ and $\boldsymbol{p}_4 = (0,1)$. Then the classical realization space $\mathcal{R}(P_1, B)$ consists of all quadrilaterals $Q = \mathrm{conv}\{\boldsymbol{p}_1, \boldsymbol{p}_2, (a,b), \boldsymbol{p}_4\}$, where $a, b \in \mathbb{R}$ must satisfy $a, b > 0$ and $a + b > 1$ in order for Q to be convex.*

In the slack realization spaces, modding out by row and column scalings is equivalent to fixing some variables in $S_P(\boldsymbol{x})$ to 1. So for example, we could start with the following scaled symbolic slack and affine slack matrices

$$S_P(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & x_8 & 0 & 0 \end{bmatrix}, \quad [S_P(\boldsymbol{x}) \; \mathbb{1}] = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & x_3 & x_4 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ x_7 & x_8 & 0 & 0 & 1 \end{bmatrix}.$$

Computing the 4-minors of these scaled symbolic slack matrices and saturating with all variables produces the scaled slack ideals

$$\begin{aligned} I_P^{\text{scaled}} &= \langle x_8 - 1 \rangle, \text{ and} \\ \widetilde{I}_P^{\text{scaled}} &= \langle x_3x_8 + x_4x_8 - x_3 - x_8,\, x_4x_7 + x_4x_8 - x_4 - x_7,\, x_3x_7 - x_4x_8 \rangle. \end{aligned}$$

Therefore the slack realization space, $\mathcal{V}_+(I_P)/(\mathbb{R}^4_{>0} \times \mathbb{R}^4_{>0})$, has the unique element $(1,1,1,1,1,1,1,1)$, *and indeed, all convex quadrilaterals are projectively equivalent to P_1.*

From the generators of $\widetilde{I}_P^{\text{scaled}}$ one sees that the affine slack realization space, $\mathcal{V}(\widetilde{I}_P)/\mathbb{R}^4_{>0}$, is two-dimensional and parametrized by x_3, x_4 with

$$x_7 = \frac{x_4}{x_3 + x_4 - 1} \quad \text{and} \quad x_8 = \frac{x_3}{x_3 + x_4 - 1}.$$

Since all the four variables have to take on positive values in a (scaled) slack matrix of a quadrilateral, we get that the realization space $\mathcal{V}_+(\widetilde{I}_P)/\mathbb{R}^4_{>0}$ is cut out by the inequalities $x_3 > 0,\ x_4 > 0,\ x_3 + x_4 > 1$. This description coincides exactly with that of $\mathcal{R}(P_1, B)$ that we saw earlier.

Example 2.3.8. *Consider the 5-polytope P with vertices $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_8$ given by*

$$e_1, e_2, e_3, e_4, -e_1 - 2e_2 - e_3, -2e_1 - e_2 - e_4, -2e_1 - 2e_2 + e_5, -2e_1 - 2e_2 - e_5$$

where $e_1, \ldots, e_5$ are the standard basis vectors in $\mathbb{R}^5$. It can be obtained by splitting the distinguished vertex v of the vertex sum of two squares, $(\square, v) \oplus (\square, v)$ in the notation of [32]. This polytope has 8 vertices and 12 facets and its symbolic slack matrix has the zero-pattern below

$$\begin{bmatrix}
0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & * & * \\
* & 0 & 0 & * & 0 & * & 0 & 0 & * & 0 & * & 0 \\
* & * & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & * & 0 & * & 0 & 0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & *
\end{bmatrix}.$$

By [32, Theorem 5.3] P is not projectively unique, meaning its slack realization space $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ will consist of more than a single point. Indeed, by fixing ones in the maximum number of positions, marked in bold face below, we find that $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ is a one-dimensional *space of projectively inequivalent realizations parametrized by slack matrices of the following form*

$$S_P(a) = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 1 & 0 & 0 & 1 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & a & 0 & a & 0 \\ \mathbf{1} & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & a & 0 & a \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}.$$

If we wish to look at a representative of each equivalence class which is a true slack matrix, then we can scale the above to guarantee that $\mathbb{1}$ is in the column space.

Remark 2.3.9. We have shown that $\mathcal{V}_+(I_P)$ is a natural model for the realization space of P, but it could be that I_P is not the biggest ideal that vanishes on its Zariski closure. In other words, we have not proved that $\mathcal{V}_+(I_P)$ is Zariski dense in the slack variety $\mathcal{V}(I_P)$. Determining the vanishing ideal of $\mathcal{V}_+(I_P)$ would allow one to transfer invariants from the variety of this ideal to the realization space. For instance, whether one can compute the dimension of a realization space is an important and largely open question, and having the correct ideal would provide an algebraic tool for answering this question.

2.4 Applications

In this section we illustrate the computational power of the slack ideal in answering three types of questions that one can ask about realizations of polytopes. We anticipate further applications.

2.4.1 Abstract polytope with no realizations

Checking if an abstract polytopal complex is the boundary of an actual polytope is the classical *Steinitz problem*, and an important ingredient in cataloging polytopes with few vertices. In [5], Altshuler and Steinberg enumerated all 4-polytopes and 3-spheres with 8 vertices. The first non-polytopal 3-sphere in [5, Table 2] has simplices and square pyramids as facets, and these facets have the following vertex sets

$$12345, 12346, 12578, 12678, 14568, 34578, 2357, 2367, 3467, 4678.$$

If there was a polytope P with these facets, its symbolic slack matrix would be

$$S_P(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & 0 & x_6 & x_7 & 0 & 0 & x_8 & x_9 \\ 0 & 0 & x_{10} & x_{11} & x_{12} & 0 & 0 & 0 & 0 & x_{13} \\ 0 & 0 & x_{14} & x_{15} & 0 & 0 & x_{16} & x_{17} & 0 & 0 \\ 0 & x_{18} & 0 & x_{19} & 0 & 0 & 0 & x_{20} & x_{21} & x_{22} \\ x_{23} & 0 & x_{24} & 0 & 0 & x_{25} & x_{26} & 0 & 0 & 0 \\ x_{27} & x_{28} & 0 & 0 & x_{29} & 0 & 0 & 0 & 0 & 0 \\ x_{30} & x_{31} & 0 & 0 & 0 & 0 & x_{32} & x_{33} & x_{34} & 0 \end{bmatrix}.$$

One can compute that the would-be slack ideal I_P in this case is trivial, meaning that there is no rank five matrix with the support of $S_P(\boldsymbol{x})$. In particular, there is no polytope with the given facial structure. In fact, there is not even a hyperplane-point arrangement in $\mathbb{R}^4$ or $\mathbb{C}^4$ with the given incidence structure.

In some other cases, one can obtain non-empty slack varieties that have no positive part. A simple example of that behaviour can be seen in the *tetrahemihexahedron*, a polyhedralization of the real projective plane with 6 vertices, and facets with vertex sets 235, 346, 145, 126, 2456, 1356, 1234. Its slack matrix is therefore

$$S_P(\boldsymbol{x}) = \begin{bmatrix} x_1 & x_2 & 0 & 0 & x_3 & 0 & 0 \\ 0 & x_4 & x_5 & 0 & 0 & x_6 & 0 \\ 0 & 0 & x_7 & x_8 & x_9 & 0 & 0 \\ x_{10} & 0 & 0 & x_{11} & 0 & x_{12} & 0 \\ 0 & x_{13} & 0 & x_{14} & 0 & 0 & x_{15} \\ x_{16} & 0 & x_{17} & 0 & 0 & 0 & x_{18} \end{bmatrix}.$$

Computing the slack ideal from the 5-minors of $S_P(\boldsymbol{x})$ we find that I_P is generated by the binomials

$$
\begin{array}{ccc}
x_8x_{15}x_{17} + x_7x_{14}x_{18} & x_4x_{15}x_{17} + x_5x_{13}x_{18} & x_{11}x_{15}x_{16} + x_{10}x_{14}x_{18} \\
x_2x_{15}x_{16} + x_1x_{13}x_{18} & x_5x_{12}x_{16} + x_6x_{10}x_{17} & x_7x_{11}x_{16} - x_8x_{10}x_{17} \\
x_3x_7x_{16} + x_1x_9x_{17} & x_2x_5x_{16} - x_1x_4x_{17} & x_6x_{11}x_{13} + x_4x_{12}x_{14} \\
x_1x_{11}x_{13} - x_2x_{10}x_{14} & x_5x_8x_{13} - x_4x_7x_{14} & x_3x_8x_{13} + x_2x_9x_{14} \\
x_6x_7x_{11} + x_5x_8x_{12} & x_3x_8x_{10} + x_1x_9x_{11} & x_2x_6x_{10} + x_1x_4x_{12}
\end{array}
$$

$$
\begin{array}{cc}
x_6x_{11}x_{15}x_{17} - x_5x_{12}x_{14}x_{18} & x_2x_9x_{15}x_{17} - x_3x_7x_{13}x_{18} \\
x_4x_{12}x_{15}x_{16} - x_6x_{10}x_{13}x_{18} & x_3x_8x_{15}x_{16} - x_1x_9x_{14}x_{18} \\
x_2x_7x_{14}x_{16} - x_1x_8x_{13}x_{17} & x_5x_{11}x_{13}x_{16} - x_4x_{10}x_{14}x_{17} \\
x_2x_6x_9x_{11} - x_3x_4x_8x_{12} & x_2x_5x_8x_{10} - x_1x_4x_7x_{11} \\
x_3x_6x_7x_{10} - x_1x_5x_9x_{12} & x_3x_4x_7 + x_2x_5x_9
\end{array}
$$

Since the slack ideal contains binomials whose coefficients are both positive, it has no positive zeros. In fact, by fixing some coordinates to one, it has a unique zero up to row and column scalings, where all entries are either 1 or -1.

2.4.2 Non-prescribable faces of polytopes

Another classical question about polytopes is whether a face can be freely prescribed in a realization of a polytope with given combinatorics.

We begin by observing that there is a natural relationship between the slack matrix/ideal of a polytope and those of each of its faces. For instance, if F is a facet of a d-polytope P, a symbolic slack matrix $S_F(\boldsymbol{x})$ of F is the submatrix of $S_P(\boldsymbol{x})$ indexed by the vertices of F and the facets of P that intersect F in its $(d-2)$-dimensional faces. Let $\boldsymbol{x}_F$ denote the vector of variables in that submatrix. All $(d+1)$-minors of $S_F(\boldsymbol{x})$ belong to the slack ideal I_P. To see this, consider a $(d+2)$-submatrix of $S_P(\boldsymbol{x})$ obtained by enlarging the given $(d+1)$-submatrix of $S_F(\boldsymbol{x})$ by a row indexed by a vertex $\boldsymbol{p} \notin F$ and the column indexed by F. The column of F in this bigger submatrix has all zero entries except in position $(\boldsymbol{p}, F)$. The minor of this $(d+2)$-submatrix in $S_P(\boldsymbol{x})$ after saturating out the variable in position $(\boldsymbol{p}, F)$, is the

$(d+1)$-minor of $S_F(\boldsymbol{x})$ that we started with. Therefore,

$$I_F \subseteq I_P \cap \mathbb{C}[\boldsymbol{x}_F].$$

By induction on the dimension, this containment is true for all faces F of P.

A face F of a polytope P is *prescribable* if, given any realization of F, we can complete it to a realization of P. In our language, a face F is prescribable in P if and only if

$$\mathcal{V}_+(I_F) = \mathcal{V}_+(I_P \cap \mathbb{C}[\boldsymbol{x}_F]).$$

Consider the four-dimensional prism over a square pyramid, for which it was shown in [6] that its only cube facet F is non-prescribable. This polytope P has 10 vertices and 7 facets and its symbolic slack matrix is

$$S_P(\boldsymbol{x}) = \begin{bmatrix} \mathbf{x_1} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{0} \\ \mathbf{x_4} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & \mathbf{x_5} & \mathbf{x_6} \\ \mathbf{x_7} & \mathbf{0} & 0 & \mathbf{x_8} & \mathbf{0} & \mathbf{0} & \mathbf{x_9} \\ \mathbf{x_{10}} & \mathbf{0} & 0 & \mathbf{x_{11}} & \mathbf{x_{12}} & \mathbf{0} & \mathbf{0} \\ x_{13} & 0 & x_{14} & 0 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{x_{15}} & 0 & \mathbf{0} & \mathbf{x_{16}} & \mathbf{x_{17}} & \mathbf{0} \\ \mathbf{0} & \mathbf{x_{18}} & 0 & \mathbf{0} & \mathbf{0} & \mathbf{x_{19}} & \mathbf{x_{20}} \\ \mathbf{0} & \mathbf{x_{21}} & 0 & \mathbf{x_{22}} & \mathbf{0} & \mathbf{0} & \mathbf{x_{23}} \\ \mathbf{0} & \mathbf{x_{24}} & 0 & \mathbf{x_{25}} & \mathbf{x_{26}} & \mathbf{0} & \mathbf{0} \\ 0 & x_{27} & x_{28} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In bold we mark $S_F(\boldsymbol{x})$ sitting inside $S_P(\boldsymbol{x})$. Computing I_P and intersecting with $\mathbb{C}[\boldsymbol{x}_F]$, we obtain an ideal of dimension 15. On the other hand, the slack ideal of a cube has dimension 16, suggesting an extra degree of freedom for the realizations of a cube, and the possibility that the cubical facet F cannot be arbitrarily prescribed in a realization of P. However, we need more of an argument to conclude this, since $I_F \neq I_P \cap \mathbb{C}[\boldsymbol{x}_F]$ does not immediately mean that $\mathcal{V}_+(I_F) \neq \mathcal{V}_+(I_P \cap \mathbb{C}[\boldsymbol{x}_F])$. We need to compute further to get Barnette's result.

We first note that one can scale the rows and columns of $S_F(\boldsymbol{x})$ to set 13 of its 24 variables to one, say $x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_{10}, x_{15}, x_{16}, x_{18}, x_{21}, x_{24}$. Guided by the resulting slack ideal we further set $x_{20} = 1, x_{11} = \frac{1}{2}, x_{17} = 2$ and $x_{25} = 1$. Now solving for the remaining variables from the equations of the slack ideal, we get the following true slack matrix of a cube:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 3/2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

However, making the above-mentioned substitutions for

$$x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_{10}, x_{11}, x_{15}, x_{16}, x_{17}, x_{18}, x_{20}, x_{21}, x_{24}, x_{25}$$

in $S_P(\boldsymbol{x})$ and eliminating x_{13}, x_{14}, x_{27} and x_{28} from the slack ideal results in the trivial ideal showing that the cube on its own admits further realizations than are possible as a face of P.

2.4.3 Non-rational polytopes

A combinatorial polytope is said to be *rational* if it has a realization in which all vertices have rational entries. This has a very simple interpretation in terms of slack varieties.

Lemma 2.4.1. *A polytope P is rational if and only if $\mathcal{V}_+(I_P)$ has a rational point.*

The proof is trivial since any rational realization gives rise to a rational slack matrix and any rational slack matrix is itself a rational realization of the polytope P. Recall that any point in $\mathcal{V}_+(I_P)$ can be row scaled to be a true slack matrix by dividing each row by the sum of its entries, so a rational point in $\mathcal{V}_+(I_P)$ will provide a true rational slack matrix of P.

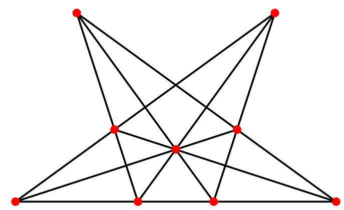


Figure 2.1: Non-rational line-point configuration

Unfortunately, the usual examples of non-rational polytopes tend to be too large for direct computations, so we illustrate our point on the non-rational point-line arrangement in the plane shown in Figure 3.2 from [28, Figure 5.5.1]. A true non-rational polytope can be obtained from this point-line arrangement by Lawrence lifting. We will show the non-rationality of this configuration by computing its slack ideal as if it were a 2-polytope. Its symbolic slack matrix is the 9×9 matrix

$$S(\boldsymbol{x}) = \begin{bmatrix} x_1 & 0 & x_2 & 0 & x_3 & x_4 & x_5 & x_6 & 0 \\ x_7 & x_8 & x_9 & 0 & x_{10} & 0 & 0 & x_{11} & x_{12} \\ x_{13} & x_{14} & 0 & x_{15} & x_{16} & x_{17} & x_{18} & 0 & 0 \\ x_{19} & x_{20} & 0 & x_{21} & 0 & 0 & x_{22} & x_{23} & x_{24} \\ x_{25} & 0 & x_{26} & x_{27} & 0 & x_{28} & 0 & 0 & x_{29} \\ 0 & 0 & x_{30} & x_{31} & x_{32} & 0 & x_{33} & x_{34} & x_{35} \\ 0 & x_{36} & 0 & x_{37} & x_{38} & x_{39} & 0 & x_{40} & x_{41} \\ 0 & x_{42} & x_{43} & 0 & x_{44} & x_{45} & x_{46} & 0 & x_{47} \\ 0 & x_{48} & x_{49} & x_{50} & 0 & x_{51} & x_{52} & x_{53} & 0 \end{bmatrix}.$$

One can simplify the computations by scaling rows and columns to fix $x_i = 1$ for $i = 1, 2, 8, 14, 20, 26, 30, 36, 42, 44, 47, 48, 50, 51, 52, 53,$ as this does not affect rationality. Then one sees that the polynomial $x_{46}^2 + x_{46} - 1$ is in the slack ideal, so $x_{46} = \frac{-1\pm\sqrt{5}}{2}$, and there are no rational realizations of this configuration.

Remark 2.4.2. We note that as illustrated by the above example, the slack matrix and slack ideal constructions are not limited to the setting of polytopes, but in fact, are applicable to the more general setting of any point/hyperplane configuration.

Chapter 3

PROJECTIVELY UNIQUE POLYTOPES AND TORIC SLACK IDEALS

3.1 Introduction

An important focus in the study of polytopes is the investigation of their realization spaces. Given a d-polytope $P \subset \mathbb{R}^d$, its face lattice determines its combinatorial type. The realization space of P is the set of all geometric realizations of polytopes in the combinatorial class of P. A new model for the realization space of a polytope modulo projective transformations, called the *slack realization space*, was introduced in Chapter 1. This model arises as the positive part of the real variety of I_P, the *slack ideal* of P, which is a saturated determinantal ideal of a symbolic matrix whose zero pattern encodes the combinatorics of P. The slack ideal and slack realization space will be extended to matroids in Chapter 5.

The overarching goal of this paper is to initiate a study of the algebraic and geometric properties of slack ideals as they provide the main computational engine in our model of realization spaces. As shown in Chapter 2, slack ideals can be used to answer many different questions about the realizability of polytopes. These ideals were introduced in [23] where they were used to study the notion of *psd-minimality* of polytopes, a property of interest in optimization. Thus, developing the properties and understanding the implications of slack ideals can directly impact both polytope and matroid theory. Even as a purely theoretical object, slack ideals present a new avenue for research in commutative algebra.

In this paper, we focus on the simplest possible slack ideals, namely, toric slack ideals. Since slack ideals do not contain monomials, the simplest ones are generated by binomials. Toric ideals are precisely those binomial ideals that are prime. Toric slack ideals already form a rich class with important connections to projective uniqueness. In general, slack ideals offer a new classification scheme for polytopes via the algebraic properties and invariants of the ideal, and the toric case offers a nice example of this. The vertex-facet (non)-incidence structure of a polytope P can be encoded in a bipartite graph whose toric ideal, T_P, plays a special role in this context. We call T_P the *toric ideal of the non-incidence graph of* P, and say that I_P is *graphic* if it coincides with T_P. In Theorem 3.4.4 we prove that I_P is graphic if and only if I_P is toric and P is projectively unique. On the other hand, there are infinitely many combinatorial types in high enough dimension that are projectively unique but do not have toric slack ideals, as well as non-projectively unique polytopes with toric slack ideals. We give several concrete examples.

The toric ideal T_P has other interesting geometric connections. We prove that I_P is contained in T_P if and only if P is *morally 2-level*, which is a polarity-invariant property of a polytope that generalizes the notion of 2-level polytopes [8], [16], [26], [40]. Theorem 3.3.10 characterizes morally 2-level polytopes in terms of the slack variety. As a consequence we get that a polytope with no rational realizations cannot have a toric slack ideal.

An important feature of a toric ideal is that the positive part of its real variety is Zariski dense in its complex variety. This implies that the toric ideal is the vanishing ideal of the positive part of its variety. In general, it is not easy to determine whether I_P is the vanishing ideal of the positive part $\mathcal{V}_+(I_P)$, of its variety $\mathcal{V}(I_P)$. We show that the slack ideal of a classical polytope due to Perles is reducible and that in this case, $\mathcal{V}_+(I_P)$ is not Zariski dense in $\mathcal{V}(I_P)$. This eight-dimensional polytope is projectively unique and does not have rational realizations. It provides the first concrete instance of a slack ideal that is not prime.

Organization of the paper. In Section 3.2 we summarize the needed background on slack ideals of polytopes. In Section 3.3 we introduce T_P, the toric ideal of the non-incidence graph of a polytope P, and show its relationship to pure difference binomial slack ideals and morally 2-level polytopes. We prove in Section 3.4 that slack ideals are graphic if and only if they are toric and the underlying polytope is projectively unique. In particular, we show that all d-polytopes with $d+2$ vertices or facets have graphic slack ideals, but this property holds beyond this class. In this section we also illustrate toric slack ideals that do not come from projectively unique polytopes and the existence of projectively unique polytopes that do not have toric slack ideals. We conclude in Section 3.5 with the Perles polytope [28, Section 5.5]. We show that the Perles polytope has a reducible slack ideal despite being projectively unique, providing the first concrete example of a non-prime slack ideal. In this case, $\mathcal{V}_+(I_P)$ is not Zariski dense in $\mathcal{V}(I_P)$.

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3.2 Background: Slack Matrices and Ideals of Polytopes

We now give a brief introduction to slack matrices and slack ideals of polytopes. For more details see [19], [23] and Chapter 2.

A d-dimensional polytope $P \subset \mathbb{R}^d$ with v labelled vertices and f labelled facet inequalities has two usual representations: a $\mathcal{V}$-representation $P = \text{conv}\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_v\}$ as the convex hull of vertices, and an $\mathcal{H}$-representation $P = \{\boldsymbol{x} \in \mathbb{R}^d : W\boldsymbol{x} \leq \boldsymbol{w}\}$ as the intersection of the half spaces defined by the facet inequalities $W_j\boldsymbol{x} \leq \boldsymbol{w}_j$, $j = 1, \dots, f$, where W_j denotes the jth row of $W \in \mathbb{R}^{f\times d}$. Let $V \in \mathbb{R}^{v\times d}$ be the matrix with rows $\boldsymbol{p}_1{}^\top, \dots, \boldsymbol{p}_v{}^\top$, and let $\mathbb{1} \in \mathbb{R}^v$ be the vector of all ones. The combined data of the two representations yields a *slack matrix* of P, defined as

$$S_P := \begin{bmatrix} \mathbb{1} & V \end{bmatrix} \begin{bmatrix} \boldsymbol{w}^\top \\ -W^\top \end{bmatrix} \in \mathbb{R}^{v\times f}. \tag{3.1}$$

Since scaling the facet inequalities by positive real numbers does not change the polytope, P in fact has infinitely many slack matrices of the form $S_P D_f$ where D_f denotes a $f \times f$ diagonal matrix with positive entries on the diagonal. Also, affinely equivalent polytopes have the same set of slack matrices.

Slack matrices were introduced in [44]. The (i,j)-entry of S_P is $\boldsymbol{w}_j - W_j\boldsymbol{p}_i$ which is the *slack* of the ith vertex $\boldsymbol{p}_i$ of P with respect to the jth facet inequality $W_j^\top \boldsymbol{x} \leq w_j$ of P. Since P is a d-polytope, $\text{rank}(\begin{bmatrix} \mathbb{1} & V \end{bmatrix}) = d+1$, and hence, $\text{rank}(S_P) = d+1$. Also, $\mathbb{1}$ is in the column span of S_P. Further, the zeros in S_P record the vertex-facet incidences of P, and hence the entire combinatorics (face lattice) of P [29]. Interestingly, it follows from [19, Theorem 22] that any matrix with the above properties is in fact the slack matrix of a polytope that is combinatorially equivalent to P.

Theorem 3.2.1. *A nonnegative matrix $S \in \mathbb{R}^{v\times f}$ is the slack matrix of a polytope in the combinatorial class of the labelled polytope P if and only if the following hold:*

1. $\text{support}(S) = \text{support}(S_P)$,

2. $\text{rank}(S) = \text{rank}(S_P) = d+1$, *and*

3. $\mathbb{1}$ *lies in the column span of S.*

This theorem gives rise to a new model for the realization space of P, as observed in [23] and [14]. We briefly explain the construction of the slack model for the realization space of P from [23], developed further in Chapter 2.

The *symbolic slack matrix*, $S_P(\boldsymbol{x})$, of P is obtained by replacing each nonzero entry of S_P by a distinct variable. Suppose there are t variables in $S_P(\boldsymbol{x})$. The *slack ideal* of P is the saturation of the ideal generated by the $(d+2)$-minors of $S_P(\boldsymbol{x})$, namely

$$I_P := \langle (d+2)\text{-minors of } S_P(\boldsymbol{x})\rangle : \left(\prod_{i=1}^{t} x_i\right)^{\infty} \subset \mathbb{C}[\boldsymbol{x}] := \mathbb{C}[x_1,\dots,x_t]. \tag{3.2}$$

Note that since I_P is saturated, it does not contain any monomials. The *slack variety* of P is the complex variety $\mathcal{V}(I_P) \subset \mathbb{C}^t$. If $\boldsymbol{s} \in \mathbb{C}^t$ is a zero of I_P, then we identify it with the matrix $S_P(\boldsymbol{s})$.

By [23, Corollary 1.5], two polytopes P and Q in the same combinatorial class are projectively equivalent if and only if $D_v S_P D_f$ is a slack matrix of Q for some positive diagonal matrices D_v, D_f. Using this fact and Theorem 3.2.1, we see that the positive part of $\mathcal{V}(I_P)$, namely $\mathcal{V}(I_P) \cap \mathbb{R}^t_{>0} =: \mathcal{V}_+(I_P)$, leads to a realization space for P, modulo projective transformations.

Theorem 3.2.2. *Given a polytope P, there is a bijection between the elements of $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ and the classes of projectively equivalent polytopes in the combinatorial class of P.*

The space $\mathcal{V}_+(I_P)/(\mathbb{R}^v_{>0} \times \mathbb{R}^f_{>0})$ is called the *slack realization space* of P.

3.3 The toric ideal of the non-incidence graph of a polytope

We begin by defining the toric ideal T_P of the non-incidence graph of a polytope P. In the next section we characterize when T_P equals I_P which relies on the projective uniqueness of P. In this section we examine the relationship between I_P and T_P and the implications of I_P being contained in T_P.

First we recall the definition of a toric ideal. Let $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ be a point configuration in $\mathbb{Z}^d$. Sometimes we will identify $\mathcal{A}$ with the $d \times n$ matrix whose columns are the vectors $\boldsymbol{a}_i$. Consider the $\mathbb{C}$-algebra homomorphism

$$\pi : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}], \text{ such that } x_j \mapsto \mathbf{t}^{\boldsymbol{a}_j}.$$

The kernel of π, denoted by $I_\mathcal{A}$, is called the *toric ideal* of $\mathcal{A}$. The ideal $I_\mathcal{A}$ is binomial and prime (see [41, Chapter 4]). More precisely, $I_\mathcal{A}$ is generated by homogeneous binomials:

$$I_\mathcal{A} = \langle \boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} \in \mathbb{C}[x_1, \ldots, x_n] : \boldsymbol{u} \in \ker_\mathbb{Z}(\mathcal{A}) \rangle, \tag{3.3}$$

where $\ker_\mathbb{Z}(\mathcal{A}) = \{\boldsymbol{u} \in \mathbb{Z}^n : \mathcal{A}\boldsymbol{u} = \boldsymbol{0}\}$, $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$, with $\boldsymbol{u}^+, \boldsymbol{u}^- \in \mathbb{Z}^n_{\geq 0}$ the positive and the negative parts of $\boldsymbol{u}$.

Let $I_\mathcal{A}$ be a toric ideal and $V_\mathcal{A} = \mathcal{V}_\mathbb{C}(I_\mathcal{A})$ be its complex affine toric variety which is the Zariski closure of the set of points $\{(\boldsymbol{t}^{\boldsymbol{a}_1}, \ldots, \boldsymbol{t}^{\boldsymbol{a}_n}) : \boldsymbol{t} \in (\mathbb{C}^*)^d\}$. Define

$$\phi_\mathcal{A} : (\mathbb{C}^*)^d \to \mathbb{C}^n, \qquad \boldsymbol{t} \mapsto (\boldsymbol{t}^{\boldsymbol{a}_1}, \ldots, \boldsymbol{t}^{\boldsymbol{a}_n}),$$

so that $V_{\mathcal{A}} = \overline{\phi_{\mathcal{A}}((\mathbb{C}^*)^d)}$. We are interested in the positive part of $V_{\mathcal{A}}$, namely, $V_{\mathcal{A}} \cap \mathbb{R}^n_{>0}$. Note that this set contains $\phi_{\mathcal{A}}(\mathbb{R}^d_{>0})$.

The following result follows from the Zariski density of the positive part of a toric variety in its complex variety. However, we write an independent proof.

Lemma 3.3.1. *Let $I_{\mathcal{A}}$ be a toric ideal in $\mathbb{C}[x_1, \ldots, x_n]$. If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n$ and $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}}$ vanishes on the set of points $\phi_{\mathcal{A}}(\mathbb{R}^d_{>0})$, then $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in I_{\mathcal{A}}$.*

Proof. Notice that $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}}$ evaluated at any point $(\boldsymbol{t}^{\boldsymbol{a}_1}, \ldots, \boldsymbol{t}^{\boldsymbol{a}_n}) \in \phi_{\mathcal{A}}((\mathbb{C}^*)^d)$ is just $\boldsymbol{t}^{\mathcal{A}\boldsymbol{u}} - \boldsymbol{t}^{\mathcal{A}\boldsymbol{v}}$. Then, since $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}}$ vanishes on $\phi_{\mathcal{A}}(\mathbb{R}^d_{>0})$, we have that $\boldsymbol{t}^{\mathcal{A}\boldsymbol{u}} = \boldsymbol{t}^{\mathcal{A}\boldsymbol{v}}$ for all $\boldsymbol{t} \in \mathbb{R}^d_{>0}$. Thus, if we fix $i \in \{1, \ldots, d\}$ and specialize to $t_j = 1$ for all $j \neq i$, we get $t_i^{(\mathcal{A}\boldsymbol{u})_i} = t_i^{(\mathcal{A}\boldsymbol{v})_i}$ for all $t_i \in \mathbb{R}_{>0}$, which means we must have $(\mathcal{A}\boldsymbol{u})_i = (\mathcal{A}\boldsymbol{v})_i$. Since this holds for all i, it follows that $\mathcal{A}\boldsymbol{u} = \mathcal{A}\boldsymbol{v}$, hence $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in I_{\mathcal{A}}$ by (3.3). □

Definition 3.3.2. *Let P be a d-polytope in $\mathbb{R}^d$.*

1. *Define the* non-incidence graph *of P, denoted as G_P, to be the undirected bipartite graph on the vertices and facets of P with an edge connecting vertex i to facet j if and only if i* **does not** *lie on j.*

2. *Let T_P be the toric ideal of $\mathcal{A}_P$, the vertex-edge incidence matrix of G_P. We call T_P the toric ideal of the non-incidence graph of P.*

Note that G_P records the support of a slack matrix of P, and so we can think of its edges as being labelled by the corresponding entry of $S_P(\boldsymbol{x})$. Toric ideals of bipartite graphs have been studied in the literature.

Lemma 3.3.3 (**[34, Lemma 1.1]**, **[43, Theorem 10.1.5]**)**.** *The ideal T_P is generated by all binomials of the form $\boldsymbol{x}^{C^+} - \boldsymbol{x}^{C^-}$, where C is an (even) chordless cycle in G_P, and $C^+, C^- \in \mathbb{Z}^{|E|}$ are the incidence vectors of the two sets of edges that partition C into alternate edges (that is, if we orient edges from vertices to facets in G_P, then C^+ consists of the forward* *edges in a traversal of C, and C^- the backward edges). Thus, for every even closed walk W in G_P, and indeed any union of such, $\boldsymbol{x}^{W^+} - \boldsymbol{x}^{W^-} \in T_P$.*

Example 3.3.4. *Consider the 4-polytope $P = \mathrm{conv}(0, 2e_1, 2e_2, 2e_3, e_1 + e_2 - e_3, e_4, e_3 + e_4)$ [23, Table 1. #3] where e_i is the standard unit vector in $\mathbb{R}^4$. This polytope is projectively unique with f-vector (7,17,17,7). It has symbolic slack matrix*

$$S_P(\boldsymbol{x}) = \begin{bmatrix} 0 & x_1 & 0 & 0 & 0 & x_2 & 0 \\ x_3 & 0 & 0 & 0 & 0 & x_4 & 0 \\ x_5 & 0 & x_6 & 0 & 0 & 0 & x_7 \\ 0 & x_8 & x_9 & 0 & 0 & 0 & x_{10} \\ 0 & 0 & 0 & 0 & x_{11} & 0 & x_{12} \\ 0 & 0 & 0 & x_{13} & x_{14} & x_{15} & 0 \\ 0 & 0 & x_{16} & x_{17} & 0 & 0 & 0 \end{bmatrix}.$$

Its non-incidence graph G_P is given in Figure 3.1. Notice that each edge of G_P can be naturally labelled with the corresponding x_i from $S_P(\boldsymbol{x})$. Under this labelling, the chordless cycle marked with dashed lines in Figure 3.1 corresponds to the binomial $x_2x_3x_6x_8 - x_1x_4x_5x_9 \in T_P$. One can check that the remaining generators of T_P, corresponding to chordless cycles of G_P, are

$$\begin{array}{ll} x_7x_9 - x_6x_{10}, & x_{10}x_{11}x_{13}x_{16} - x_9x_{12}x_{14}x_{17}, \\ x_7x_{11}x_{13}x_{16} - x_6x_{12}x_{14}x_{17}, & x_2x_8x_{13}x_{16} - x_1x_9x_{15}x_{17}, \\ x_4x_5x_{13}x_{16} - x_3x_6x_{15}x_{17}, & x_2x_8x_{12}x_{14} - x_1x_{10}x_{11}x_{15}, \\ x_4x_5x_{12}x_{14} - x_3x_7x_{11}x_{15}, & x_2x_3x_7x_8 - x_1x_4x_5x_{10}. \end{array}$$

The toric ideal T_P can coincide with I_P as we will see in the next section. For the remainder of this section we focus on the connections between I_P and T_P.

An ideal is said to be a *pure difference binomial ideal* if it is generated by binomials of the form $\boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}}$. It follows from (3.3) that toric ideals are pure difference binomial ideals. We now prove that if I_P is toric, or more generally, a pure difference binomial ideal, then I_P is always contained in T_P.

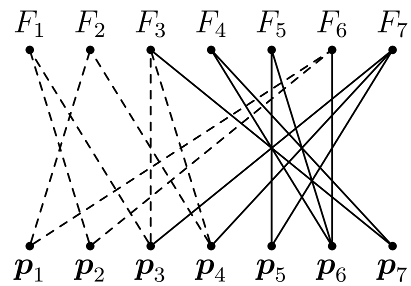


Figure 3.1: Non-incidence graph G_P

Lemma 3.3.5. *If a binomial $\boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}}$ belongs to I_P, then it also belongs to T_P.*

Proof. Let $p = \boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{\boldsymbol{b}}$. Each component a_i of $\boldsymbol{a}$ and b_j of $\boldsymbol{b}$ appears as the exponent of a variable in the symbolic slack matrix $S_P(\boldsymbol{x})$ and is hence indexed by an edge of G_P. Recall that all matrices obtained by scaling rows and columns of S_P by positive scalars also lie in the real variety of I_P, and hence must vanish on p. This implies that the sum of the components of $\boldsymbol{a}$ appearing as exponents of variables in a row (column) of $S_P(\boldsymbol{x})$ equals the sum of the components of $\boldsymbol{b}$ appearing as exponents of variables in the same row (column).

Now think of the edges of G_P in the support of $\boldsymbol{a}$ as oriented from vertices of P to facets of P and edges in the support of $\boldsymbol{b}$ as oriented in the opposite way. Then the previous statement is equivalent to saying that p is supported on an oriented subgraph of G_P (possibly with repeated edges) with the property that the in-degree and out-degree of every node in the subgraph are equal. Therefore, this subgraph is the vertex-disjoint union of closed walks in G_P, which by Lemma 3.3.3 implies that p is in T_P. $\square$

Corollary 3.3.6. *If I_P is a pure difference binomial ideal, then $I_P \subseteq T_P$.*

This containment can be strict as we see in the following example.

Example 3.3.7. *Consider the 5-polytope P with vertices $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_8$ given by*

$$e_1, e_2, e_3, e_4, -e_1 - 2e_2 - e_3, -2e_1 - e_2 - e_4, -2e_1 - 2e_2 + e_5, -2e_1 - 2e_2 - e_5$$

where $e_1, \ldots, e_5$ are the standard basis vectors in $\mathbb{R}^5$. It can be obtained by splitting the distinguished vertex v of the vertex sum of two squares, $(\square, v) \oplus (\square, v)$ in the notation of [32]. This polytope has 8 vertices and 12 facets and its symbolic slack matrix has the zero-pattern below

$$\begin{bmatrix} 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 & * & * \\ * & 0 & 0 & * & 0 & * & 0 & 0 & * & 0 & * & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & * & 0 & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{bmatrix}.$$

One can check using `Macaulay2`*[27] that I_P is toric and $I_P \subsetneq T_P$. In fact,* $\dim \mathbb{C}[\boldsymbol{x}]/I_P = 20$, *while* $\dim \mathbb{C}[\boldsymbol{x}]/T_P = 19$.

At first glance it might seem that if I_P is contained in T_P then I_P is a pure difference binomial ideal, but this is not true in general.

Example 3.3.8. *For the* 3*-cube, $I_P \subsetneq T_P$. The toric ideal T_P is minimally generated by* 80 *binomials, each corresponding to a chordless cycle in G_P, while I_P is minimally generated by* 222 *polynomials many of which are not binomials.*

In fact, one can attach a geometric meaning to polytopes for which $I_P \subseteq T_P$. A polytope P is said to be 2-*level* if it has a slack matrix in which every positive entry is one, i.e., $S_P(\mathbb{1})$ is a slack matrix of P. This class of polytopes have received a great deal of attention in the literature [8], [16], [26],[40] and are also known as *compressed polytopes.*

Definition 3.3.9. *We call a polytope P morally* 2*-level if $S_P(\mathbb{1})$ lies in the slack variety of P.*

Note that if P is morally 2-level, it might not be that $S_P(\mathbb{1})$ is a slack matrix of P, but merely that $\mathbb{1} \in \mathcal{V}_+(I_P)$. Hence, morally 2-level polytopes contain 2-level polytopes. For example, all regular d-cubes are 2-level and hence any polytope that is combinatorially a d-cube is morally 2-level but not necessarily 2-level. Being morally 2-level does not require that there is a polytope in the combinatorial class of P that is a 2-level polytope. For example, a bisimplex in $\mathbb{R}^3$ is morally 2-level, but no polytope in its combinatorial class is 2-level. This is since $S_P(\mathbb{1})$ can lie in the slack variety of P even though it may not have the all-ones vector in its column space. A very attractive feature of the set of morally 2-level polytopes is that it is closed under polarity unlike the set of 2-level polytopes, but preserves many of the properties of 2-level polytopes such as psd-minimality [24], [23].

Theorem 3.3.10. *A polytope P is morally 2-level if and only if $I_P \subseteq T_P$.*

Proof. Notice that the ideal $J_P = \langle (d+2)\text{-minors of } S_P(\boldsymbol{x})\rangle$ is contained in the slack ideal I_P. Suppose that $S_P(\mathbb{1}) \in \mathcal{V}(I_P)$. Then any $(d+2)$-minor p of $S_P(\boldsymbol{x})$ must have the same number of monomials with coefficient $+1$ as those with coefficient -1 since p must vanish on $S_P(\mathbb{1})$, which sets each monomial to one. This implies that we can write p as a sum of pure difference binomials. Since p is a minor, each of these pure difference binomials corresponds to a pair of permutations that induce two perfect matchings on the same set of vertices. The union of these two matchings is a subgraph of G_P, which we can view as a directed graph by orienting the two matchings in opposite directions. Then each vertex will have equal in-degree and out-degree, which shows that these edges form a union of closed walks in G_P, and thus the corresponding binomial is in T_P by Lemma 3.3.3. Therefore $p \in T_P$, so that $J_P \subseteq T_P$. Since toric ideals are saturated with respect to all variables, the result follows.

Conversely, suppose $I_P \subseteq T_P$. Since T_P is generated by pure difference binomials, which vanish when evaluated at $S_P(\mathbb{1})$, we have $S_P(\mathbb{1}) \in \mathcal{V}(T_P)$. But $I_P \subseteq T_P$ implies that $\mathcal{V}(I_P) \supseteq \mathcal{V}(T_P) \ni S_P(\mathbb{1})$, which is the desired result. ∎

We have talked about pure difference binomial slack ideals as a superset of toric slack ideals. A slack ideal is binomial if it is generated by binomials of the form $\boldsymbol{x}^{\boldsymbol{a}} - \gamma \boldsymbol{x}^{\boldsymbol{b}}$, where γ is a non-zero scalar. Therefore, one might extend the study of toric slack ideals to the following hierarchy of binomial slack ideals:

$$\text{toric} \subseteq \text{pure difference binomial} \subseteq \text{binomial}.$$

So far, we have not encountered a pure difference binomial slack ideal that is not toric, nor a binomial slack ideal which is not pure difference, but it might be possible that all containments are strict. It follows from Corollaries 2.2 and 2.5 in [15] that, if the slack ideal I_P is binomial, then it is a radical lattice ideal. This implies that the slack variety is a union of scaled toric varieties.

3.4 Projective uniqueness and toric slack ideals

Recall that a polytope P is said to be *projectively unique* if any polytope Q that is combinatorially equivalent to P is also projectively equivalent to P, i.e., there is a projective transformation that sends Q to P. This corresponds to saying that the slack realization space of P is a single positive point.

Every d-polytope with $d+2$ vertices or facets is projectively unique [28, Exercise 4.8.30 (i)]. In particular, all products of simplices are projectively unique. We first prove that the slack ideal of a d-polytope with $d+2$ vertices or facets coincides with T_P, and is thus toric.

Proposition 3.4.1. *Let P be a polytope in $\mathbb{R}^d$ with $d+2$ vertices or facets. Then its slack ideal I_P equals the toric ideal T_P.*

Proof. Up to polarity we may consider P to be a polytope with $d+2$ vertices. In this case P is combinatorially equivalent to a repeated pyramid over a free sum of two simplices, $\operatorname{pyr}_r(\Delta_k \oplus \Delta_\ell)$, with $k, \ell \geq 1$, $r \geq 0$ and $r + k + \ell = d$ [28, Section 6.1]. Since taking pyramids preserves the slack ideal, it is enough to study the slack ideals of free sums of simplices (respectively, product of simplices). By [23, Lemma 5.7], if $P = \Delta_k \oplus \Delta_\ell$, then $S_P(\boldsymbol{x})$ has the zero pattern of the vertex-edge incidence matrix of the complete bipartite graph $K_{k+1,\ell+1}$.

From [23, Proposition 5.9], it follows that I_P is generated by the binomials

$$\det(M_C) = \begin{vmatrix} x_1 & 0 & 0 & \cdots & 0 & x_2 \\ x_3 & x_4 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{2c-2} & 0 \\ 0 & 0 & 0 & \cdots & x_{2c-1} & x_{2c} \end{vmatrix},$$

where M_C is a $c \times c$ symbolic matrix whose support is the vertex-edge incidence matrix of the simple cycle C (of size c) in $K_{k+1,\ell+1}$.

On the other hand, T_P is generated by the binomials $\boldsymbol{x}^{D^+} - \boldsymbol{x}^{D^-}$ corresponding to chordless cycles D of the non-incidence graph G_P by Lemma 3.3.3. Thus, it suffices to show that there exists a bijection between simple cycles C in $K_{k+1,l+1}$ and chordless cycles D in G_P such that $\det(M_C) = \boldsymbol{x}^{D^+} - \boldsymbol{x}^{D^-}$.

Let $v_1, \ldots, v_{k+\ell+2}$ be the vertices of P and $F_1, \ldots, F_{(k+1)(\ell+1)}$ be its facets. Since $S_P(\boldsymbol{x})$ has the support of the vertex-edge incidence matrix of $K_{k+1,\ell+1}$, we can consider $K_{k+1,\ell+1}$ to be a bipartite graph on the vertices $v_1, \ldots, v_{k+\ell+2}$ where each edge $\{v_{i_1}, v_{i_2}\}$ corresponds exactly to the facet F_j of P containing neither v_{i_1} nor v_{i_2}. Notice that the non-incidence graph G_P can be obtained by subdividing each edge $\{v_{i_1}, v_{i_2}\}$ of $K_{k+1,\ell+1}$ into two edges $\{v_{i_1}, F_j\}$ and $\{F_j, v_{i_2}\}$.

Now, let C be a simple cycle of size c in $K_{k+1,\ell+1}$ with vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_c}$ and assume that $F_{j_1}, F_{j_2}, \ldots, F_{j_c}$ are the facets corresponding to the edges of C. Then in G_P there is a cycle D of size $2c$ on vertices $v_{i_1}, F_{j_1}, v_{i_2}, F_{j_2}, \ldots, F_{j_{c-1}}, v_{i_c}, F_{j_c}$. In fact, one can see that the subgraph induced by these vertices is exactly a chordless cycle in G_P. This is because from the support of S_P we know each facet in P corresponds to a vertex of degree 2 in G_P; furthermore, every edge in G_P must be between a vertex and a facet, but since every facet already has degree 2 in the cycle D, this subgraph must consist only of this cycle. Hence from a simple cycle C in $K_{k+1,\ell+1}$, we get a chordless cycle D in G_P, as desired. The reverse correspondence is analogous. $\square$

The class of polytopes for which $I_P = T_P$ is larger than those with $d+2$ vertices or facets.

Example 3.4.2. *For the polytope given in Example 3.3.4, which was* 4*-dimensional but with* 7 *vertices and* 7 *facets, one can check that* I_P *is the toric ideal* T_P*.*

In $\mathbb{R}^2$ the only projectively unique polytopes are triangles and squares. In $\mathbb{R}^3$ there are four combinatorial classes of projectively unique polytopes — tetrahedra, square pyramids, triangular prisms and bisimplices. The number of projectively unique 4-polytopes is currently unknown. There are 11 known combinatorial classes, attributed to Shephard by McMullen [32], and listed in full in [2]. Beyond the 4-polytopes with $4+2=6$ vertices or facets, this list has three additional combinatorial classes. One of them is the polytope seen in Example 3.4.2. It was shown in [23] that all of the 11 known projectively unique polytopes in $\mathbb{R}^4$ have toric slack ideals. This discussion suggests that there might be a connection between projective uniqueness of a polytope and its slack ideal being toric. In this section we establish the precise result. The toric ideal T_P of the non-incidence graph G_P will again play an important role.

Definition 3.4.3. *We say that the slack ideal* I_P *of a polytope* P *is graphic if it is equal to the toric ideal* T_P*.*

Theorem 3.4.4. *The slack ideal* I_P *of a polytope* P *is graphic if and only if* P *is projectively unique and* I_P *is toric.*

Proof. Suppose that I_P is graphic. Then, I_P is toric, so we only need to show that P is projectively unique. Pick a maximal spanning forest F of the bipartite graph G_P. By Lemma 3.5.2 we may scale the rows and columns of S_P so that it has ones in the entries indexed by F. Take an edge of G_P outside of F and consider the binomial corresponding to the unique cycle this edge forms together with F. Since $I_P = T_P$, this binomial is in I_P, therefore it must vanish on the above scaled slack matrix of P. This implies that the entry in the slack matrix indexed by the chosen edge must also be 1. Repeating this argument we see that the entire slack matrix has 1 in every non-zero entry which implies that there is only one possible slack matrix for P up to scalings, hence only one polytope in the combinatorial class of P up to projective equivalence.

Conversely, suppose that P is projectively unique and I_P is toric, say $I_P = I_{\mathcal{A}}$ for some point configuration $\mathcal{A}$. Let $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}}$ be a generator of T_P. Notice this generator vanishes when each $x_i = 1$, and by Lemma 3.3.3, $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} = \boldsymbol{x}^{C^+} - \boldsymbol{x}^{C^-}$ for some chordless cycle C of G_P. Now, since I_P is toric, by Corollary 3.3.6 we have that $I_P \subseteq T_P$, and then by Theorem 3.3.10, $S_P(\mathbb{1}) \in \mathcal{V}(I_P)$. Since P is projectively unique, every element of $\mathcal{V}_+(I_P)$ is obtained by positive row and column scalings of $S_P(\mathbb{1})$. Therefore, $\phi_{\mathcal{A}}(\mathbb{R}^d_{>0}) \subseteq \mathcal{V}_+(I_P)$ consists of row and column scalings of $S_P(\mathbb{1})$. Since a binomial of the form $\boldsymbol{x}^{C^+} - \boldsymbol{x}^{C^-}$, where C is a chordless cycle, contains in each of its monomials exactly one variable from each row and column of $S_P(\boldsymbol{x})$ on which it is supported, it must also vanish on all row and column scalings of $S_P(\mathbb{1})$. It follows that the generator $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}}$ vanishes on $\phi_{\mathcal{A}}(\mathbb{R}^d_{>0})$. By Lemma 3.3.1, this means that $\boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \in I_P$, thus all generators of T_P are contained in I_P, which completes the proof. $\square$

Theorem 3.4.4 naturally leads to the question whether P can have a toric slack ideal even if it is not projectively unique and whether all projectively unique polytopes have toric slack ideals. In the rest of this section, we discuss these two questions.

All d-polytopes with toric slack ideals for $d \leq 4$ were found in [23]. These polytopes all happen to be projectively unique, and hence have graphic slack ideals. Therefore the first possible non-graphic toric slack ideal has to come from a polytope of dimension at least five. Indeed, we saw that the polytope in Example 3.3.7 has a toric slack ideal but is not graphic. Hence, this polytope is not projectively unique by Theorem 3.4.4, recovering a result implied by a theorem of McMullen [32, Theorem 5.3].

In the next section we will see a concrete 8-polytope that is projectively unique but does not have a toric slack ideal. However, this is not an isolated instance as there are infinitely many such examples in high enough dimension.

Proposition 3.4.5. *For $d \geq 69$ there exist infinitely many projectively unique d-polytopes* *that do not have a toric (even pure difference binomial) slack ideal.*

Proof. In [2], Adiprasito and Ziegler have shown that for $d \geq 69$ there are infinitely many projectively unique d-polytopes. On the other hand, it follows from results in [23] concerning semidefinite lifts of polytopes that in any dimension, there can only be finitely many combinatorial classes of polytopes whose slack ideal is a pure difference binomial ideal. □

3.5 The Perles polytope has a reducible slack ideal

We now consider a classical example of a projectively unique polytope with no rational realization due to Perles [28, p.94]. This is an 8-polytope with 12 vertices and 34 facets with the additional feature that it has a non-projectively unique face. It is minimal in the sense that every d-polytope with at most $d+3$ vertices is rationally realizable. We will show that the Perles polytope does not have a toric slack ideal and that in fact, its slack ideal is not prime, providing the first such example.

The non-existence of rational realizations of a polytope immediately implies that its slack ideal is not toric. This is a corollary of Theorem 3.3.10.

Corollary 3.5.1. *Let P be a polytope in $\mathbb{R}^d$ with no rational realization. Then I_P cannot be a pure difference binomial ideal and, in particular, cannot be toric.*

Proof. If P has no rational realization, then $S_P(\mathbb{1})$ does not lie in the slack variety of P, since a rational point in $\mathcal{V}_+(I_P)$ yields a rational realization of P by [20, Lemma 4.1]. Therefore, by Theorem 3.3.10, I_P is not contained in T_P. Now applying Corollary 3.3.6, we can conclude that I_P is not a pure difference binomial ideal and, in particular, is not toric. □

The Perles polytope P is constructed in [28, p.95] from its affine Gale diagram shown in Figure 3.2. This planar configuration stands in for the vector configuration in $\mathbb{R}^3$ (Gale diagram) consisting of 12 vectors — the eight vectors A, B, C, D, E, F, G, H indicated with black dots that have $x_3 = 1$ and the four vectors $-F, -G, -H, -I$ indicated with open circles that have $x_3 = -1$. This means that P has 12 vertices and is of dimension $12 - 3 - 1 = 8$.

The facets of P are in bijection with the 34 minimal positive circuits of the Gale diagram. Computing these, we get the support of the slack matrix S_P shown below.

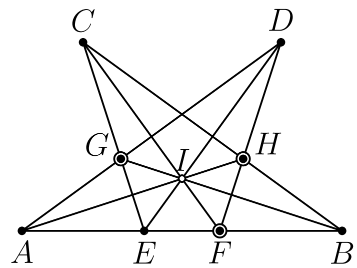


Figure 3.2: The Perles Gale diagram.

$$
\begin{bmatrix}
0&0&0&*&*&*&0&0&0&0&0&0&0&*&*&*&*&*&*&*&*&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&*&0&0&*&*&*&0&0&0&0&*&*&*&*&0&0&0&0&*&*&*&*&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&*&0&0&*&*&0&0&*&*&0&0&*&*&0&0&*&0&0&0&*&*&0&0&0&0&0&0&0\\
0&0&0&0&*&0&0&0&0&0&0&*&*&0&0&*&*&0&0&*&0&0&*&*&0&0&0&*&*&*&0&0&0&0\\
0&0&0&0&0&0&0&*&0&*&0&*&0&0&0&0&0&0&0&0&0&*&*&0&0&*&0&*&*&0&*&*&*&0\\
&0&0&0&0&0&0&0&0&0&&0&*&0&0&0&0&*&0&*&0&0&0&0&0&0&*&0&0&*&0&0&0&*\\
0&*&0&0&0&0&0&0&*&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&0&0&0&0&*&*&0&0&*\\
0&0&*&0&0&*&0&0&0&0&0&0&0&0&0&0&0&0&*&0&*&0&0&0&0&*&0&0&0&0&*&*&*&*\\
&0&0&&0&0&0&*&0&0&0&0&0&0&0&0&0&0&0&0&*&0&0&0&*&0&0&*&0&0&0&*&*&0\\
0&*&0&0&*&0&0&0&0&*&0&0&0&*&0&0&0&*&*&0&*&0&0&0&0&0&*&*&*&0&0&*&0&0\\
0&0&*&0&0&0&*&0&0&0&0&0&*&0&0&*&0&0&0&0&0&0&*&*&*&0&*&0&*&0&0&0&0&0\\
0&0&0&0&0&*&0&0&*&0&*&*&0&0&*&0&*&0&0&*&0&*&0&0&0&*&0&0&0&*&*&0&*&*
\end{bmatrix}
$$

It is straightforward to obtain $S_P(\boldsymbol{x})$ from the above matrix, but a direct calculation of the slack ideal of this example is challenging. Therefore, we resort to a scaling technique that makes slack ideal computations easier. The idea is to work with a subvariety of the slack variety that contains a representative for every orbit under row and column scalings. We do this by fixing as many entries as possible in $S_P(\boldsymbol{x})$ to one. Having less variables, the slack ideal becomes easier to compute. The non-incidence graph G_P from Section 3 provides a systematic way to scale a maximal number of entries in $S_P(\boldsymbol{x})$ to one.

Lemma 3.5.2. *Given a polytope P, we may scale the rows and columns of its slack matrix so that it has ones in the entries indexed by the edges in a maximal spanning forest F of the graph G_P.*

Proof. For every tree T in the forest, pick a vertex to be its root, and orient the edges away from it. Now for each tree, pick the edges leaving the root and set to one the corresponding entry of S_P by scaling the row or column corresponding to the destination vertex of the edge. Continue the process with the edges leaving the vertices just used and so on, until the trees are exhausted. Notice that once we fix an entry, the only way for us to change it again is by scaling either its row or column, which would mean in the graph that we would revisit one of the nodes of its corresponding edge. But this would imply the existence of a cycle in F, so by the time this process ends we have precisely the intended variables set to one. □

Even after the above scaling trick, the symbolic slack matrix of the Perles polytope has 75 variables which is challenging to work with. Therefore, we will work with a subideal of I_P.

Consider the following submatrix of $S_P(\boldsymbol{x})$ coming from its first 13 columns.

$$\begin{bmatrix}
0 & 0 & 0 & x_1 & x_2 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_4 & 0 & 0 & x_5 & x_6 & x_7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_8 & 0 & 0 & x_9 & x_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & x_{12} & x_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{14} & 0 & x_{15} & 0 & x_{16} & 0 \\
x_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{18} & 0 & x_{19} \\
0 & x_{20} & 0 & 0 & 0 & 0 & 0 & 0 & x_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & x_{22} & 0 & 0 & x_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_{24} & 0 & 0 & x_{25} & 0 & 0 & 0 & x_{26} & 0 & 0 & 0 & 0 & 0 \\
0 & x_{27} & 0 & 0 & x_{28} & 0 & 0 & 0 & 0 & x_{29} & 0 & 0 & 0 \\
0 & 0 & x_{30} & 0 & 0 & 0 & x_{31} & 0 & 0 & 0 & 0 & 0 & x_{32} \\
0 & 0 & 0 & 0 & 0 & x_{33} & 0 & 0 & x_{34} & 0 & x_{35} & x_{36} & 0
\end{bmatrix}.$$

The ideal of 10×10 minors of this submatrix, saturated by all its variables is clearly a subideal of I_P. Using the scaling lemma we first set $x_i = 1$ for $i = 1, 4, 5, 6, 7, 8, 9, 10, 13, 15, 16, 17, 18, 21, 22, 26, 27, 28, 29, 30, 31, 32, 33, 35$. The resulting scaled slack subideal is:

$$\langle \boldsymbol{x_{36}^2 + x_{36} - 1}, x_{34} - x_{36} - 1, x_{25} - x_{36}, x_{24} - x_{36}, x_{23} - 1, x_{20} - x_{36},$$
$$x_{19} - x_{36}, x_{14} - x_{36} - 1, x_{12} - x_{36}, x_{11} - 1, x_3 - 1, x_2 - x_{36} - 1\rangle.$$

This means that after scaling, the first 13 columns of every matrix $S_P(\mathbf{s})$ obtained from $\mathbf{s} \in \mathcal{V}(I_P)$ with full support must have the form

$$\begin{bmatrix}
0 & 0 & 0 & 1 & \alpha+1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha+1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha \\
0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & 0 & 0 & \alpha & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \alpha+1 & 0 & 1 & \alpha & 0
\end{bmatrix} \tag{3.4}$$

where $\alpha = \frac{-1 \pm \sqrt{5}}{2}$ is a root of $x^2 + x - 1$. One can check that there is a unique way to extend the above 12×13 matrix to a 12×34 matrix with rank nine and the support of the Perles slack matrix, provided we scale one variable to one in each of the new columns, as allowed by Lemma 3.5.2. The resulting parametrized matrix is shown in Figure 3.3. Up to scaling, the two matrices corresponding to the two values of α are therefore the only elements in the slack variety.

$$
\begin{array}{r}
A\\ B\\ C\\ D\\ E\\ F\\ G\\ H\\ -F\\ -G\\ -H\\ -I
\end{array}
\left[\begin{array}{cccccccccccccccccccccccccccccccccc}
0&0&0&1&\alpha+1&1&0&0&0&0&0&0&0&1&1&\alpha&\alpha+1&\alpha&1&1&\alpha+1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&1&1&0&0&0&0&1-\alpha&1&\alpha&\alpha&0&0&0&0&\alpha&1-\alpha&1&\alpha+2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&1&0&0&1&1&0&0&1&\alpha&0&0&1&0&0&0&1-\alpha&\alpha+1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&\alpha&1&0&0&1&1&0&0&1&0&0&1&1&0&0&0&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&\alpha+1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&\alpha&0&0&1&0&\alpha+2&1&0&1&\alpha+2&\alpha+1&0\\
1&0&0&0&0&0&0&0&0&0&1&0&\alpha&0&0&0&0&1-\alpha&0&\alpha&0&0&0&0&0&0&1&0&0&\alpha+1&0&0&0&1\\
0&\alpha&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1-\alpha&1&0&0&0&0&1&1-\alpha&0&0&1\\
0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1-\alpha&0&1&0&0&0&0&\alpha&0&0&0&0&1&1&1&1\\
\alpha&0&0&\alpha&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1-\alpha&0&0&0&1&0&0&\alpha&0&0&0&1&1-\alpha&0\\
0&1&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&1&0&1&0&0&0&0&0&1&1&1-\alpha&0&0&1&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&\alpha+1&\alpha+1&0&\alpha&0&\alpha&0&0&0&0&0\\
0&0&0&0&0&1&0&0&\alpha+1&0&1&\alpha&0&0&1&0&1&0&0&1&0&1&0&0&0&1&0&0&0&\alpha+2&\alpha+1&0&1&\alpha+2
\end{array}\right]
$$

Figure 3.3: Slack matrix of the Perles polytope

Theorem 3.5.3. *The slack ideal of the Perles polytope is not prime.*

Proof. Let us consider the polynomial

$$\begin{aligned} f(\boldsymbol{x}) &= (x_{10}x_{15}x_{36})^2 + (x_{10}x_{15}x_{36})(x_9x_{16}x_{35}) - (x_9x_{16}x_{35})^2 \\ &= (x_{10}x_{15}x_{36} - \alpha_1 x_9x_{16}x_{35})(x_{10}x_{15}x_{36} - \alpha_2 x_9x_{16}x_{35}), \end{aligned}$$

where $\alpha_1 = \frac{-1+\sqrt{5}}{2}, \alpha_2 = \frac{-1-\sqrt{5}}{2}$ are the roots of $x^2 + x - 1$. We see that the linear factor of $f(\boldsymbol{x})$ containing α_1 will not vanish on the submatrix (3.4) when we set $\alpha = \alpha_2$, and vice versa. Therefore neither of the linear factors will vanish on the slack variety. On the other hand, one can check that evaluating $f(\boldsymbol{x})$ on the matrix in Figure 3.3 reduces it to $\alpha^2 + \alpha - 1$ which is zero. Since $f(\boldsymbol{x})$ is homogeneous with respect to each row and column of the matrix, it will also vanish on the whole slack variety. Therefore, the vanishing ideal of the slack variety is not prime which implies that I_P is not prime. □

3.6 Conclusion

We have shown that the slack ideal of a polytope P may not be prime. However, the following question remains.

Problem 3.6.1. Is I_P a radical ideal? If not, what are the simplest counterexamples?

We have seen in Section 3.5 that $\mathcal{V}_+(I_P)$ need not be Zariski dense in the slack variety $\mathcal{V}(I_P)$, and hence I_P is not always the vanishing ideal of $\mathcal{V}_+(I_P)$. In the case that I_P is toric, we know that I_P is indeed the vanishing ideal of $\mathcal{V}_+(I_P)$, providing a perfect correspondence between algebra and geometry. Many further questions remain. In particular, what sort of polytope combinatorics lead to simple algebraic structure in slack ideals?

Problem 3.6.2. What conditions on P make its slack ideal toric, or pure difference binomial, or binomial?

In fact, so far, we have not been able to find any slack ideal that is in one of these classes but not in the others.

Problem 3.6.3. Is any of the inclusions

$$\text{toric} \subseteq \text{pure difference binomial} \subseteq \text{binomial}$$

of classes of slack ideals strict?

We also characterized the toric slack ideals that come from projectively unique polytopes as being T_P, the toric ideal of G_P, the graph of vertex-facet non-incidences of P. Such slack ideals were called graphic. The fact that testing and certifying projective uniqueness is easy for toric slack ideals is very interesting, as that is in general a hard problem. This raises the question of finding other classes of polytopes for which one can certify projective uniqueness easily.

Problem 3.6.4. Is there another class of polytopes, beyond those with graphic slack ideals, for which one can characterize projective uniqueness?

Apart from these concrete questions, many others could be formulated. There are, in particular, two general directions of study that can potentially be very fruitful: strengthening the correspondence between algebraic invariants and combinatorial properties, and revisiting the literature on realization spaces in our new language to see if further insights can be gained or open questions can be answered.

Chapter 4

MCMULLEN-TYPE PROJECTIVELY UNIQUE 5-POLYTOPES

4.1 Introduction

In this chapter, we classify all McMullen-type projectively unique polytopes in dimension 5. It is known that not all projectively unique polytopes are McMullen-type, but current examples are somewhat unwieldy (e.g., dimension at least 8 with at least 12 vertices and at least 34 facets). We give a simple example of a non-McMullen type projectively unique polytope in dimension 5. It has 9 vertices, 10 facets and can be realized with rational coordinates.

Recall that we call a polytope P *projectively unique* if any two realizations of P are related by a projective transformation. The only projectively unique 2-polytopes are the triangle and square; the only projectively unique 3-polytopes are those having at most 9 edges [28, Exercise 4.8.30]; the only known projectively unique 4-polytopes are 11 combinatorial types listed in [2] and attributed to Shephard by McMullen [32, Section 7].

Two main questions regarding projectively unique polytopes are as follows.

Question 4.1.1 (Perles & Shephard [36]). Is it true that, for each fixed $d \geq 2$, the number of distinct combinatorial types of projectively unique d-polytopes is finite?

Question 4.1.2 (McMullen & Shephard [32]). Is Shephard's list of 11 projectively unique combinatorial types in dimension 4 complete?

The first question was recently settled by Adiprasito & Ziegler [2] who showed that for each $d \geq 69$, there exists an infinite family of distinct projectively unique d-polytopes. The second question remains unsettled.

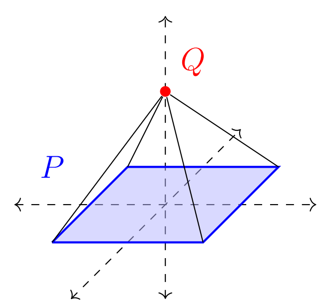
Further complicating matters is the limited number of constructions known to produce projectively unique polytopes. This chapter focuses on one such set of constructions described by McMullen in [32].

4.2 McMullen constructions

First we review four basic constructions from [32] and the conditions under which they produce projectively unique polytopes. Throughout this chapter we denote by Δ_k the k-dimensional simplex.

(M1) *Join*: If P and Q are two polytopes such that $\dim(P \cup Q) = \dim P + \dim Q + 1$ (that is, subspaces $\operatorname{aff}(P)$ and $\operatorname{aff}(Q)$ do not intersect nor are parallel; they are skew), the polytope $P * Q := \operatorname{conv}(P \cup Q)$ is called the join of P and Q and it has dimension $\dim P + \dim Q + 1$. Its faces are the joins of the faces of P with those of Q.

Example 4.2.1. *If $P \subset \mathbb{R}^k$ is any polytope and Q is a single vertex in a new dimension, then $P * Q \subset \mathbb{R}^{k+1}$ is just a pyramid over P.*



Theorem 4.2.2 ([32, Theorem 3.1])**.** *The join of P and Q is projectively unique if and only if both P and Q are.*

The next three constructions are all instances of McMullen's subdirect sum. Given polytopes P and Q with faces F and G, respectively, such that $\operatorname{aff}(P)$ and $\operatorname{aff}(Q)$ intersect in a single point which is in the relative interior of both F and G, the subdirect sum is

$$(P, F) \oplus (Q, G) := \operatorname{conv}(P \cup Q).$$

It has dimension $\dim P + \dim Q$. Its faces are joins of faces of P, Q which don't contain F, G, respectively, and subdirect sums of faces which do contain them.

(M2) *Direct sum of simplices*: A direct sum of simplices is a subdirect sum of the form $(\Delta_k, \Delta_k) \oplus (\Delta_\ell, \Delta_\ell)$. Its dimension is $k + \ell$, and its proper faces are all of the joins of the proper faces of Δ_k and Δ_ℓ.

Remark 4.2.3. If we assume that the origin is in the relative interior of both $P = \Delta_k \subset \mathbb{R}^k$ and $Q = \Delta_\ell \subset \mathbb{R}^\ell$, then their direct sum is

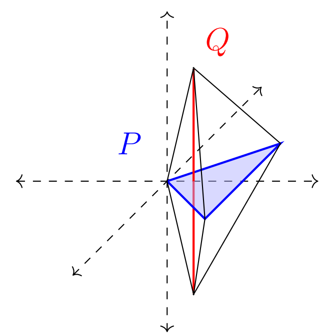
$$P \oplus Q := \operatorname{conv}\big((P \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times Q)\big) \subset \mathbb{R}^{k+\ell},$$

and the proper faces are

$$F * G = \operatorname{conv}\big((F \times \{\mathbf{0}\}) \cup (\{\mathbf{0}\} \times G)\big),$$

where $F \subset P$ and $G \subset Q$ are proper faces, and $\dim(F * G) = \dim(F) + \dim(G) + 1$.

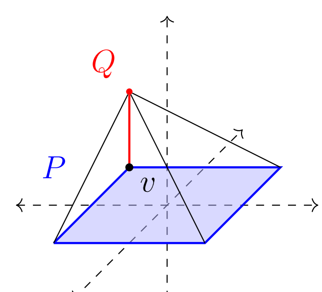
Example 4.2.4. *Let $P = \Delta_2$ and $Q = \Delta_1$ be as pictured, intersecting at a point along the y-axis. Then their direct sum is simply a bisimplex.*



Theorem 4.2.5 ([32, Theorem 5.1])**.** *The direct sum of two simplices is projectively unique.*

(M3) *Vertex sum*: Let P and Q be two polytopes that intersect in a single vertex v. The vertex sum of P and Q is $P \oplus_v Q := (P, v) \oplus (Q, v) = \operatorname{conv}(P \cup Q)$, and v is called the *distinguished vertex*.

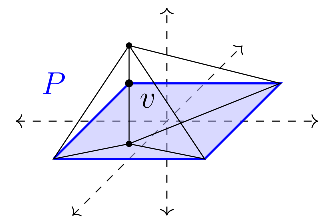
Example 4.2.6. *The vertex sum of a polytope P and a line segment $Q = \Delta_1$, is simply a pyramid over P.*



Theorem 4.2.7 ([32, Theorem 5.2])**.** *A vertex sum of two projectively unique polytopes is projectively unique.*

(M4) *Vertex split*: Let P be a polytope and $Q = \Delta_1$ a line segment. Then the vertex split of P at v is $P_v := (P, v) \oplus (Q, Q) = \mathrm{conv}(P \cup Q)$, where v is a vertex of P. It has dimension $\dim P + 1$. (More generally, this construction is called "adding a simplex at v".)

Example 4.2.8. *Let P be a square, as pictured. Then the vertex split of P at v is a bisimplex for all choices of v.*



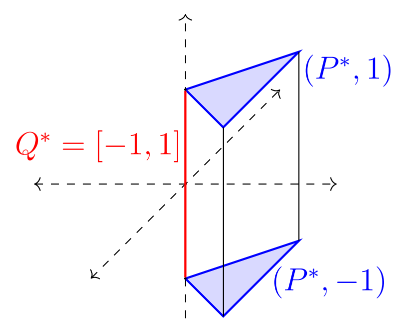
Theorem 4.2.9 ([32, Theorem 5.3])**.** *Let P be projectively unique and v a vertex of P. Then P_v is projectively unique except when P is a vertex sum with distinguished vertex v.*

The operations (M2), (M3), and (M4) also have dual operations (specific instances of subdirect product operation, which is the dual of subdirect sum), but since the dual of a projectively unique polytope is projectively unique [32, Theorem 3.3], we restrict to the above operations.

Example 4.2.10. *Consider the direct sum of Example 4.2.4. Its dual polytope is the the direct product of the duals of P and Q:*

$$(P \oplus Q)^* = P^* \times Q^*.$$

Since simplices are self-dual, we get a direct product of simplices, such as the one pictured at the right.



Definition 4.2.11. *Call a projectively unique polytope* McMullen-type *if it (or its dual) can be constructed via the above constructions (M1) – (M4) starting from simplices.*

All known projectively unique polytopes in dimension at most 4 are McMullen-type. However, McMullen notes that his constructions will never produce non-rational polytopes starting from rational input. In particular, the well-studied *Perles polytope* (see Section 3.5 of Chapter 3) which is a non-rational, projectively unique 8-polytope is not McMullen-type. McMullen also posits that a rational projectively unique $(d_1 + \cdots + d_k)$-polytope with $d_1 \cdots d_k + d_1 + \cdots + d_k + 1$ vertices is not obtainable from his constructions for $d_i = 3$ for all i and $k \geq 3$. (See [32, Section 7] for details.)

Recall from Chapter 3, that another way to identify certain projectively unique polytopes (whether or not they are produced by the above constructions) is via their slack ideals. Namely, a polytope P is projectively unique and its slack ideal I_P is toric if and only if I_P is graphic (equals the toric ideal of its vertex-facet non-incidence graph). All known projectively unique polytopes in dimension at most 4 also have graphic slack ideals.

4.3 McMullen-type projectively unique polytopes in dimension 5

We wish to enumerate all projectively unique polytopes in dimension 5 which can be constructed using the basic McMullen constructions above starting from simplices.

We start with the list of known projectively unique polytopes given in Figure 4.1. We note that any polytope of dimension at most 4 obtained from constructions (M1) – (M4) already exists in this list. Thus to enumerate all 5-dimensional McMullen-type polytopes, we need only consider the McMullen operations starting with polytopes in the list of Figure 4.1 which result in an exact dimension of 5 as follows.

(M1). Joins ($\dim P + \dim Q + 1 = 5$):

- $\dim P = 4$, $\dim Q = 0$ (i.e., pyramids over projectively unique 4-polytopes).
- $\dim P = 3$, $\dim Q = 1$. Since the only 1-dimensional projectively unique polytope is a line segment, this is will give bipyramids over projectively unique 3-polytopes.
- $\dim P = \dim Q = 2$.

Polytope	Dimension	Construction	Dual	f-vector
Δ_2	2		self	(3,3)
$\square$	2	$\Delta_1 \oplus \Delta_1$	self	(4,4)
Δ_3	3		self	(4,6,4)
Square pyramid	3	$\square * \Delta_0$	self	(5,8,5)
Bisimplex	3	$\Delta_2 \oplus \Delta_1$	Triangular prism	(5,9,6)
Triangular prism	3	$\Delta_2 \times \Delta_1$	Bisimplex	(6,9,5)
$C1$	4	Δ_4	self	(5,10,10,5)
$C2$	4	$\square * \Delta_1$	self	(6,13,13,6)
$C3$	4	$(\Delta_2 \times \Delta_1)_0$	self	(7,17,17,7)
$C4$	4	$\Delta_3 \times \Delta_1$	$C5$	(8,16,14,6)
$C5$	4	$\Delta_3 \oplus \Delta_1$	$C4$	(6,14,16,8)
$C6$	4	$\Delta_2 \times \Delta_2$	$C7$	(9,18,15,6)
$C7$	4	$\Delta_2 \oplus \Delta_2$	$C6$	(6,15,18,9)
$C8$	4	$(\Delta_2 \times \Delta_1) * \Delta_0$	$C9$	(7,15,14,6)
$C9$	4	$(\Delta_2 \oplus \Delta_1) * \Delta_0$	$C8$	(6,14,15,7)
$C10$	4	subdirect product	$C11$	(8,18,17,7)
$C11$	4	$(\square, v) \oplus (\square, v)$	$C10$	(7,17,18,8)

Figure 4.1: Projectively unique d-polytopes for $d \leq 4$.

(M2). Direct sum of simplices ($\dim P + \dim Q = 5$):

- $\Delta_4 \oplus \Delta_1$
- $\Delta_3 \oplus \Delta_2$

(M3). Vertex sum ($\dim P + \dim Q = 5$):

- $\dim P = 4$, $\dim Q = 1$. This is a pyramid over P, which is also constructed as a join.
- $\dim P = 3$, $\dim Q = 2$.

(M4). Vertex split ($\dim P = 4$): we split each (combinatorially) distinct vertex of P, except distinguished vertex v of the vertex sum $P = C_{11}$ since the result of this operation is not projectively unique by Theorem 4.2.9.

To properly classify the results of vertex splitting, we need the following result which we will prove by looking at how vertex splitting affects a slack matrix of P.

Theorem 4.3.1. *The vertex split of a sum of two simplices is again a sum of simplices; in particular, for $P = \Delta_k \oplus \Delta_\ell$, where $\Delta_k = \mathrm{conv}\{v_1, \ldots, v_{k+1}\}$ and $\Delta_\ell = \mathrm{conv}\{u_1, \ldots, u_{\ell+1}\}$, we have for each $1 \leq i \leq k, 1 \leq j \leq \ell$,*

$$P_{v_i} = \Delta_{k+1} \oplus \Delta_\ell \quad \text{and} \quad P_{u_j} = \Delta_k \oplus \Delta_{\ell+1}.$$

Now consider a symbolic slack matrix $S_P(\boldsymbol{x})$ of polytope P. We can obtain the symbolic slack matrix $S_{P_v}(\boldsymbol{x})$, of its vertex split at some vertex v, as follows.

Lemma 4.3.2. *Let P be a polytope with v vertices and f facets. Assume (by relabeling if necessary) that vertex 1 is in facets $k+1, \ldots, f$ for some k, so that*

$$S_P(\boldsymbol{x}) = \begin{bmatrix} x_1 & \cdots & x_k & 0 & \cdots & 0 \\ \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_k & \boldsymbol{c}_{k+1} & \cdots & \boldsymbol{c}_f \end{bmatrix} \subset \mathbb{R}[\boldsymbol{x}],$$

for some column vectors $\boldsymbol{c}_i \in (\mathbb{R}[\boldsymbol{x}])^{v-1}$. *Then the polytope* P_1 *obtained from* P *by splitting vertex* 1 *has vertices* $\{0, 1, \ldots, v\}$ *and* $f + k$ *facets which give the following symbolic slack matrix*

$$S_{P_1}(\boldsymbol{x}) = \begin{bmatrix} x_1' & \cdots & x_k' & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_1 & \cdots & x_k & 0 & \cdots & 0 \\ \boldsymbol{c}_1' & \cdots & \boldsymbol{c}_k' & \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_k & \boldsymbol{c}_{k+1} & \cdots & \boldsymbol{c}_f \end{bmatrix} \subset \mathbb{R}[\boldsymbol{x}, \boldsymbol{x}'],$$

where $\boldsymbol{c}_i'$ *is a copy of column vector* $\boldsymbol{c}_i$ *with each variable* x *replaced by* x'.

Proof. Let $P = \operatorname{conv}\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_v\}$. Recall that P_1 is the subdirect sum of P and a line segment $[q_0, q_1]$ with $\boldsymbol{p}_1$ in its relative interior. So

$$P_1 = \operatorname{conv}(P \cup [q_0, q_1]) = \operatorname{conv}(\{\boldsymbol{p}_2, \ldots, \boldsymbol{p}_v\} \cup \{q_0, q_1\}),$$

and from [32], we know that faces of P_1 have one of the following the forms

(a) $F * G$, where F is a face of P not containing $\boldsymbol{p}_1$ and $G \in \{\{q_0\}, \{q_1\}\}$,

(b) F_1, the vertex split of a face F of P which contains $\boldsymbol{p}_1$.

Furthermore, to obtain a facet, we must take F to be a facet of P. Thus from (a), for each of the first k facets F of P, we get that $F * \{q_0\} = \operatorname{conv}(\{q_0\} \cup F)$ and $F * \{q_1\} = \operatorname{conv}(\{q_1\} \cup F)$ are facets of P_1. Furthermore, we know that $q_1 \notin F * \{q_0\}$ and $q_0 \notin F * \{q_1\}$. Replacing the row corresponding to $\boldsymbol{p}_1$ (which is now in the relative interior of P_1) by rows for vertices q_0, q_1, we get that the first $2k$ columns of the slack matrix have the claimed form. From (b), for each of the last $f - k$ facets F of P, we get that $F_1 = \operatorname{conv}(F \cup \{q_0, q_1\})$ is a facet of P_1, which gives us the remaining columns of the slack matrix. □

Theorem 4.3.1 is now a corollary of Lemma 4.3.2.

Proof of Theorem 4.3.1. From [23, Lemma 5.7], we know that $S_P(\boldsymbol{x})$ has the zero pattern of the vertex-edge incidence matrix of the complete bipartite graph $K_{k,\ell}$, namely,

$$S_P(\boldsymbol{x}) = \begin{array}{cc} \left[\begin{array}{ccccccc} x_{1,1} & \cdots & x_{1,\ell} & \cdots & 0 & \cdots & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & \cdots & 0 & \cdots & x_{k,1} & \cdots & x_{k,\ell} \\ y_{1,1} & \cdots & 0 & \cdots & y_{k,1} & \cdots & 0 \\ & \ddots & & \vdots & & \ddots & \\ 0 & \cdots & y_{1,\ell} & \cdots & 0 & \cdots & y_{k,\ell} \end{array}\right] & \begin{array}{l} \left.\vphantom{\begin{array}{c} x \\ \vdots \\ x \end{array}}\right\} \Delta_k \\ \left.\vphantom{\begin{array}{c} y \\ \ddots \\ y \end{array}}\right\} \Delta_\ell . \end{array} \end{array}$$

We could also see this from facial structure noted in (M2). Then without lost of generality, applying Lemma 4.3.2 to vertex 1 of Δ_k, we get slack matrix

$$S_{P_{v_1}}(\boldsymbol{x}) = \left[\begin{array}{ccccccccccc} x_{1,1} & \cdots & x_{1,\ell} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x'_{1,1} & \cdots & x'_{1,\ell} & \cdots & 0 & \cdots & 0 \\ & \vdots & & \vdots & & \ddots & & \vdots & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & x_{k,1} & \cdots & x_{k,\ell} \\ y_{1,1} & \cdots & 0 & y'_{1,1} & \cdots & 0 & \cdots & y_{k,1} & \cdots & 0 \\ & \ddots & & \ddots & & \vdots & & \ddots & & \\ 0 & \cdots & y_{1,\ell} & 0 & \cdots & y'_{1,\ell} & \cdots & 0 & \cdots & y_{k,\ell} \end{array}\right],$$

which clearly gives us the slack matrix of $\Delta_{k+1} \oplus \Delta_\ell$, as claimed. Similarly, applying Lemma 4.3.2 to vertex 1 of Δ_ℓ, we get the slack matrix of $\Delta_k \oplus \Delta_{\ell+1}$. □

Using `Macaulay2` and `SageMath` we generate all the polytopes that result from the above constructions, removing duplicates of the same polytope that can arise from different constructions. We also list each of their duals, even those arising from the dual operations of (M1)–(M4), since they too will be projectively unique.

Theorem 4.3.3. *There are* 32 *McMullen-type projectively unique polytopes in dimension* 5*. They are all listed in Figure 4.2.*

We note that we can also independently verify the projective uniqueness of all of the above polytopes by checking that each of their slack ideals is graphic. In particular, this means their slack ideals are also all toric.

	Construction	dual	f-vector
M_1	Δ_5	self	(6,15,20,15,6)
M_2	$\Delta_4 \oplus \Delta_1$	$M_{19} := \Delta_4 \times \Delta_1$	(7,20,30,25,10)
M_3	$\Delta_3 \oplus \Delta_2$	$M_{20} := \Delta_3 \times \Delta_2$	(7,21,34,30,12)
M_4	$C2 * \Delta_0$	self	(7,19,26,19,7)
M_5	$C8 * \Delta_0$	$M_{21} := C9 * \Delta_0$	(8,22,29,20,7)
M_6	$\Box * \Box$	self	(8,24,34,24,8)
M_7	$C3 * \Delta_0$	self	(8,24,34,24,8)
M_8	$C4 * \Delta_0$	$M_{22} := C5 * \Delta_0$	(9,24,30,20,7)
M_9	$C6 * \Delta_0$	$M_{23} := C7 * \Delta_0$	(10,27,33,21,7)
M_{10}	$C10 * \Delta_0$	$M_{24} := C11 * \Delta_0$	(9,26,35,24,8)
M_{11}	$(C3)_0$	M_{25}	(8,25,38,28,9)
M_{12}	$(C3)_2$	M_{26}	(8,26,41,31,10)
M_{13}	$(C3)_4$	M_{27}	(8,24,36,27,9)
M_{14}	$(C3)_5$	M_{28}	(8,26,41,31,10)
M_{15}	$(C10)_0$	M_{29}	(9,29,43,31,10)
M_{16}	$(C10)_1$	M_{30}	(9,28,40,28,9)
M_{17}	$(C11)_1$	M_{31}	(8,25,38,29,10)
M_{18}	$(C11)_5$	M_{32}	(8,25,39,31,11)

Figure 4.2: McMullen-type 5-polytopes.

4.4 Non-McMullen-type projectively unique polytopes

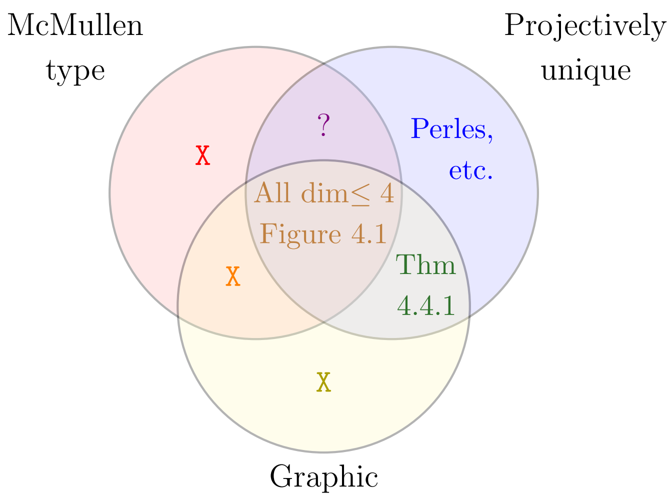
Finally, from Theorem 3.4.4, we know that any polytope with a graphic slack ideal is projectively unique. Using this characterization and a list of known 2-level polytopes, provided to us by Marco Macchia and available at `http://homepages.ulb.ac.be/~mmacchia/data.html`, we identify a projectively unique 5-polytope which is not in the list of Figure 4.2.

Theorem 4.4.1. *The* 5*-dimensional polytope, which is affinely equivalent to the rows of the following slack matrix, is projectively unique and not McMullen-type.*

$$\begin{bmatrix} 0&0&1&0&0&0&1&1&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 0&0&0&1&1&0&1&0&0&0\\ 0&1&0&1&0&0&0&0&0&1\\ 0&1&1&1&0&0&0&0&1&0\\ 1&0&0&0&1&1&0&0&0&0\\ 0&0&1&0&0&1&0&1&1&0\\ 0&0&0&0&0&1&0&1&0&1\\ 0&0&0&1&1&1&0&0&1&0 \end{bmatrix}.$$

Proof. It is not hard to check that the slack ideal of the above polytope is graphic so that it is projectively unique. It also has f-vector $(9, 30, 46, 33, 10)$, so that it (or its dual) is clearly not one of the McMullen-type polytopes of Figure 4.2. □

This is the first, though possibly not only, example of a non-McMullen type projectively unique polytope in dimension 5. A systematic classification of graphic slack ideals in dimension 5 has not been done. We also know that not all projectively unique polytopes need to have toric slack ideals, so it is possible that there exists a projectively unique 5-polytope which is not McMullen-type and whose slack ideal is not graphic. Furthermore, it is not known whether any polytope of McMullen-type must have a graphic slack ideal. We summarize with the following diagram, in which `X` means that no such polytope can exist and ? indicates such a polytope could exist but we do not have an example.



Chapter 5

THE SLACK REALIZATION SPACE OF A MATROID

5.1 Introduction

Realization spaces of matroids are well studied objects [7, 9, 33] which encode not only whether or not the matroid is realizable, but also carry additional information about the structure of the matroid. A realization (or representation) of a rank $d+1$ matroid M is a set of vectors in $\mathbb{k}^{d+1}$ which captures its independence structure. Roughly speaking, a realization space is the set of all such choices of vectors. Fundamental questions in the study of realization spaces of matroids include discovering whether or not a given matroid is realizable, determining over which field it is realizable, finding the structure of the set of realizations, and characterizing when realizations exist. A celebrated theorem of Mnëv [33] states that every semialgebraic set defined over the integers is stably equivalent to the realization space of some oriented matroid. That is, realization spaces of matroids can become arbitrarily complicated. In light of this, we aim to connect the combinatorics of the matroid to properties of its realization space.

Our realization space model will be based off the *slack matrix* of a matroid. This is a generalization of the slack matrix of a polytope [44], which has been used extensively in the study of extended formulations of polytopes; see for example [22, 39, 44]. In [23] the slack matrix of a polytope was used to construct a realization space for the polytope via its *slack ideal*. This realization space model and its properties were explored in detail in Chapter 2. We extend the results in Chapters 2 and 3 to matroids both defining the slack realization space of a matroid and examining its properties.

In Section 5.2 we introduce the main objects of study, as well as preliminary results and notation. In Section 5.3 we discuss two models for the realization space of a matroid.

One of our main theorems, Theorem 5.3.8, shows how the two realization space models can be described via a single overarching variety. In Section 5.4 we show how the slack realization model can be used to determine whether a matroid has a realization over a certain field. We also reframe the tools of final polynomials [9] in terms of slack ideals, and show how they can be used to improve computational efficiency of realizability checking. In Section 5.5 we introduce a toric ideal associated to a matroid and study its relationship to the projective uniqueness of the matroid. The computations in this paper are done in `Macaulay2` [27] with the help of the `Matroids` package [11]; the code we used can be found at `http://sites.math.washington.edu/∼awiebe`.

5.2 The slack matrix of a matroid

Much of this section is analogous to Section 2.2 of Chapter 2 to which we refer the reader for further details and excluded proofs. We assume the reader has familiarity with the basic definitions from matroid theory, see [35] or [18]. Throughout the paper, we assume all matroids are simple (having no loops or parallel elements).

Let $\Bbbk$ be a field. Let $M = (E, \mathcal{B})$ be a matroid of rank $d+1$ with ground set $E = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$, where each $\boldsymbol{v}_i \in \Bbbk^{d+1}$ and $\mathcal{B}$ is its set of bases. If V is the matrix with columns $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$, then the independent sets of M are the linearly independent subsets of columns, and we write $M = M[V]$.

Let $\mathcal{H}(M)$ denote the set of hyperplanes of M, which are the closed subsets (flats) of rank d. In $M[V]$, each hyperplane $H \in \mathcal{H}(M)$ corresponds to a linear subspace of $\Bbbk^{d+1}$, so is determined by some linear equation; that is, $H = \{\boldsymbol{x} \in E : \alpha_H^\top \boldsymbol{x} = 0\}$. For $\mathcal{H}(M) = \{H_1, \dots, H_h\}$, let W be the matrix whose columns are the hyperplane defining normals $\alpha_1, \dots, \alpha_h$, or some multiple, $\lambda_j \alpha_j$ for $\lambda_j \in \Bbbk^*$, thereof.

Definition 5.2.1. *The* slack matrix *of the matroid $M = M[V]$ over $\Bbbk$ is the $n \times h$ matrix $S_{M[V]} = V^\top W$.*

We wish to parametrize the set of realizations of a matroid by its slack matrices. So, we must determine the characteristics which define the set of all possible slack matrices of a given matroid. We begin by considering the rank of a slack matrix.

Lemma 5.2.2. *If S is a matrix having the same support as the slack matrix of some rank $d+1$ matroid $M = M[V]$, then* $\operatorname{rank}(S) \geq d+1$. *(See Lemma 2.3.1.)*

Corollary 5.2.3. *If $M = M[V]$ is a rank $d+1$ matroid then* $\operatorname{rank}(S_M) = d+1$.

Proof. The factored form of $S_M \in \Bbbk^{n\times(d+1)} \times \Bbbk^{(d+1)\times h}$ implies that $\operatorname{rank}(S_M) \leq d+1$. The result then follows from Lemma 5.2.2. □

Now, let $M = (E, \mathcal{B})$ be an abstract matroid of rank $d+1$. Unless otherwise stated, we take $E = [n] = \{1, \ldots, n\}$. A *realization* of M over $\Bbbk$ is a collection of vectors $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} \subset \Bbbk^\ell$ such that the independent subsets of V are indexed by the independent sets of the matroid, so $M = M[V]$. A matroid with a realization is called *realizable* (also *representable, linear* or *coordinatizable*).

Lemma 5.2.4. *The rows of a slack matrix S_M form a realization of the matroid M.*

Proof. It suffices to show that if we label the rows of S_M with $[n]$, the subsets indexing linearly independent rows of S_M are the independent sets of M. Since $S_M = V^T W$, if a subset J of E is dependent, then there exists a vector $\beta \in \Bbbk^n$ with support indexed by J such that $V\beta = 0$. But now, $\beta^\top S_M = (V\beta)^\top W = 0$, so J also indexes a dependent subset of the rows of S_M.

Conversely, suppose J indexes a dependent subset of the rows of S_M. Then for some $\beta \in \Bbbk^n$ with support indexed by J, we have $0 = \beta^\top S_M = (V\beta)^\top W$. Since W has full rank by Corollary 5.2.3, it must be the case that $V\beta = 0$, so that J also indexes a dependent set of M. □

From now on we assume that realizations come with a fixed labelling of ground set elements and hyperplanes, so that two slack matrices of the same matroid cannot differ by permutations of rows and columns. This also allows us to identify hyperplanes of a realization by vectors or the indices of those vectors. Now, we characterize the set of matrices which correspond to slack matrices of a matroid M.

Theorem 5.2.5. *Let M be a rank $d+1$ matroid with n elements and hyperplanes $\mathcal{H}(M) = \{H_1, \ldots, H_h\}$. A matrix $S \in \Bbbk^{n\times h}$ is the slack matrix of some realization of M if and only if both of the following hold:*

(i) $\mathit{supp}(S) = \mathit{supp}(S_M)$ *(ii)* $\operatorname{rank}(S) = d+1$.

Proof. Suppose S is the slack matrix of some realization of M. Then (i) holds trivially, and (ii) holds by Corollary 5.2.3.

Conversely, suppose S is a matrix satisfying (i) and (ii). By (ii), S has some rank factorization $S = AB$, where $A \in \Bbbk^{n\times(d+1)}$ and $B \in \Bbbk^{(d+1)\times h}$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \Bbbk^{d+1}$ be the rows of A and $\mathbf{b}_1, \ldots, \mathbf{b}_h \in \Bbbk^{d+1}$ be the columns of B. Then we claim that the rows of A give a realization of M; that is $M = M[A^\top]$. To see this, we show that the hyperplanes of M are also hyperplanes of $M[A^\top]$, and that $M[A^\top]$ can not contain more hyperplanes.

By (i), for each hyperplane H_j of M, there is a column of S with zeros in positions indexed by elements of H_j. Since $S = AB$, we have $\mathbf{b}_j^\top \mathbf{a}_i = 0$ if and only if $i \in H_j$. Thus

$$\{\boldsymbol{x} \in \Bbbk^{d+1} : \mathbf{b}_j^\top \boldsymbol{x} = 0\} \cap \{\text{rows}(A)\} = \{\mathbf{a}_i\}_{i\in H_j}$$

so that H_j is also a hyperplane of the matroid $M[A^\top]$.

Now suppose $M[A^\top]$ has an extra hyperplane $H \notin \mathcal{H}(M)$. Let $\{i_1, \ldots, i_d\} \subset H$ be any d distinct independent elements of H. Since $i_1, \ldots, i_d$ are also elements of matroid M, the flat $\overline{\{i_1, \ldots, i_d\}}$ is a hyperplane H' of M, and thus also a hyperplane of $M[A^\top]$, but $H' \neq H$. However, this means that $\{i_1, \ldots, i_d\}$ are contained in two distinct hyperplanes of $M[A^\top]$, which is not possible, so we arrive at a contradiction. $\square$

We now recall two equivalence relations on the set of realizations of a matroid M, and illustrate how these equivalences are reflected in slack matrices. For $A \in GL(\Bbbk^{d+1})$, it is easy to check that V and AV define the same matroid. We call these realizations *linearly* *equivalent*. If $P \in \Bbbk^{n\times n}$ is a permutation matrix which sends $i \mapsto \sigma(i)$, then V and VP define the same matroid up to relabelling the ground set $E = [n]$ with $\sigma(1), \ldots, \sigma(n)$. Thus if $A \in GL(\Bbbk^{d+1})$ and B is a permutation matrix with any element of $\Bbbk^*$ in the 1's positions, then V, AVB define the same matroid. We call the realizations V, AVB *projectively equivalent*. Call a matroid M *projectively unique* (over $\Bbbk$) if all realizations are projectively equivalent.

Lemma 5.2.6. *Two realizations of a matroid M are projectively equivalent if and only if their slack matrices are the same up to row and column scaling.*

Proof. Suppose we have projectively equivalent representations U, V of M. Then $U = AVB$, where $A \in GL(\Bbbk^{d+1})$ and without loss of generality B is an invertible $n \times n$ diagonal matrix (since we have assumed a fixed labelling of our matroid).

If $H = \{\boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_k}\}$ be a hyperplane of $M[V]$, then $H' = \{\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_k}\}$ is a hyperplane of $M[U]$. Furthermore, if $\alpha_H \in \Bbbk^{d+1}$ is normal to H, then since $\boldsymbol{u}_i = A\boldsymbol{v}_i \cdot b_i$, $A^{-\top}\alpha_H$ is normal to H', so that a slack matrix of $M[U]$ is

$$S_{M[U]} = U^\top \left[A^{-\top}\alpha_H\right]_{H\in\mathcal{H}} = B^\top V^\top A^\top \left[A^{-\top}\alpha_H\right]_{H\in\mathcal{H}} = B^\top V^\top W = B^\top S_{M[V]}.$$

Since we can always scale columns of a slack matrix, this completes the proof.

Conversely, suppose we have realizations U and V of the matroid M such that $S_{M[U]} = D_n S_{M[V]} D_h$ for invertible diagonal matrices $D_n \in \Bbbk^{n\times n}, D_h \in \Bbbk^{h\times h}$. By definition, $S_{M[U]} = U^\top W$ and $S_{M[V]} = V^\top W'$. Multiplying both sides of the above equation on the right by $W^\top(WW^\top)^{-1}$, we find

$$U^\top = D_n V^\top W' D_h W^\top (WW^\top)^{-1}.$$

We see that $W' D_h W^\top (WW^\top)^{-1}$ is a $(d+1) \times (d+1)$ invertible matrix, which makes U and V projectively equivalent, as desired. ∎

By taking B, D_n each to be the $n \times n$ identity matrix in the above proof, we recover the following lemma.

Lemma 5.2.7. *Two realizations of a matroid M are linearly equivalent if and only if their slack matrices are the same up to column scaling.*

We now define an analog of the slack matrix which can be constructed for any abstract matroid, even ones which are not realizable, as follows.

Definition 5.2.8. *Define the* symbolic slack matrix *of matroid M to be the matrix $S_M(\boldsymbol{x})$ with rows indexed by elements $i \in E$, columns indexed by hyperplanes $H_j \in \mathcal{H}(M)$ and (i,j)-entry*

$$S_M(\boldsymbol{x})_{ij} = \begin{cases} x_{ij} & \text{if } i \notin H_j \\ 0 & \text{if } i \in H_j. \end{cases}$$

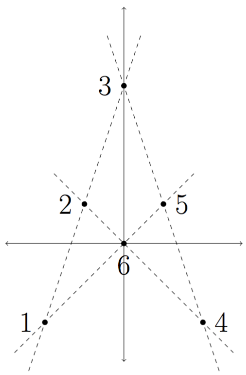
The slack ideal *of M is the saturation of the ideal generated by the $(d+2)$-minors of $S_M(\boldsymbol{x})$, namely*

$$I_M := \Big\langle (d+2) - \textit{minors of } S_M(\boldsymbol{x}) \Big\rangle : \left(\prod_{i=1}^{n} \prod_{j : i \notin H_j} x_{ij} \right)^{\infty} \qquad \subset \Bbbk[\boldsymbol{x}].$$

Suppose there are t variables in $S_M(\boldsymbol{x})$. The slack variety *is the variety $\mathcal{V}(I_M) \subset \Bbbk^t$. The saturation of I_M by the product of all the variables guarantees that there are no components of $\mathcal{V}(I_M)$ that live entirely in coordinate hyperplanes. If $\mathbf{s} \in \Bbbk^t$ is a zero of I_M, then we identify it with the matrix $S_M(\mathbf{s})$.*

Example 5.2.9. *Consider the rank 3 matroid $M_4 = M[V]$ for V whose columns are $\boldsymbol{v}_1 = (-2,-2,1)^\top$, $\boldsymbol{v}_2 = (-1,1,1)^\top$, $\boldsymbol{v}_3 = (0,4,1)^\top$, $\boldsymbol{v}_4 = (2,-2,1)^\top$, $\boldsymbol{v}_5 = (1,1,1)^\top$, $\boldsymbol{v}_6 = (0,0,1)^\top$. Projecting onto the plane $z = 1$, this can be visualized as the points of intersection of four lines in the plane, as in Figure 5.1. A slack matrix for this realization is then*

$$S_{M_4} = \begin{bmatrix} -2 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{c} \begin{array}{ccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ \scriptstyle 123 & \scriptstyle 246 & \scriptstyle 345 & \scriptstyle 156 & \scriptstyle 25 & \scriptstyle 14 & \scriptstyle 36 \end{array} \\ \begin{bmatrix} -3 & 3 & 6 & -3 & 0 & 0 & 4 \\ 1 & 3 & 2 & 3 & 2 & 4 & 0 \\ -4 & 0 & -8 & 0 & -2 & 8 & 0 \end{bmatrix} \end{array} = \begin{array}{c@{}c} & \begin{array}{ccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ \scriptstyle 123 & \scriptstyle 246 & \scriptstyle 345 & \scriptstyle 156 & \scriptstyle 25 & \scriptstyle 14 & \scriptstyle 36 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & -12 & -24 & 0 & -6 & 0 & -8 \\ 0 & 0 & -12 & 6 & 0 & 12 & -4 \\ 0 & 12 & 0 & 12 & 6 & 24 & 0 \\ -12 & 0 & 0 & -12 & -6 & 0 & 8 \\ -6 & 6 & 0 & 0 & 0 & 12 & 4 \\ -4 & 0 & -8 & 0 & -2 & 8 & 0 \end{bmatrix} \end{array},$$



$$\begin{array}{c c}
 & \begin{array}{ccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ 123 & 246 & 345 & 156 & 25 & 14 & 36 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & x_{12} & x_{13} & 0 & x_{15} & 0 & x_{17} \\ 0 & 0 & x_{23} & x_{24} & 0 & x_{26} & x_{27} \\ 0 & x_{32} & 0 & x_{34} & x_{35} & x_{36} & 0 \\ x_{41} & 0 & 0 & x_{44} & x_{45} & 0 & x_{47} \\ x_{51} & x_{52} & 0 & 0 & 0 & x_{56} & x_{57} \\ x_{61} & 0 & x_{63} & 0 & x_{65} & x_{66} & 0 \end{bmatrix}
\end{array}$$

Figure 5.1: The point-line configuration of Example 5.2.9, and its symbolic slack matrix.

where using $\{\boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_d}\} \subset H$ independent, we calculate each α_H as

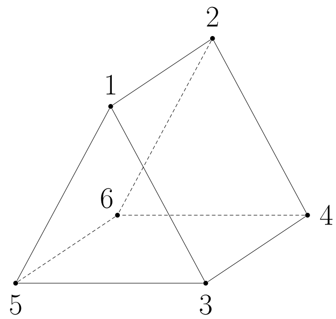
$$det \begin{bmatrix} \widehat{e_1} & | & \cdots & | \\ \vdots & \boldsymbol{v}_{j_1} & \cdots & \boldsymbol{v}_{j_d} \\ \widehat{e_{d+1}} & | & \cdots & | \end{bmatrix}. \tag{5.1}$$

The symbolic slack matrix of M_4 is in Figure 5.1. We take the ideal of 4-minors of this matrix, and saturate with respect to the product of all of the variables to get the slack ideal I_{M_4}. This has codimension 12, degree 293 and is generated by the 72 binomial generators in Table 5.1. In Section 5.5 we will see these correspond to the 72 cycles in the bipartite non-incidence graph of this configuration (Figure 5.6).

Remark 5.2.10. In Chapter 2, given a set of n vertices $V \subset \mathbb{k}^d$ defining a d-polytope $P = \text{conv}(V)$, they include only the facet defining hyperplanes in the slack matrix. We can also form a matroid associated to this polytope by considering all the hyperplanes; that is, we define the matroid $M = M[V']$ where $V' \subset \mathbb{k}^{(d+1)\times n}$ is the matrix obtained from V by appending a 1 to each vector. Then the symbolic slack matrix of P defined in Chapter 2 is the restriction of the symbolic slack matrix of matroid M to the subset of columns corresponding

deg 2	$x_{36}x_{65}+x_{35}x_{66},\ x_{26}x_{63}-x_{23}x_{66},\ x_{15}x_{63}-x_{13}x_{65},\ x_{56}x_{61}-x_{51}x_{66},\ x_{45}x_{61}-x_{41}x_{65},$ $x_{27}x_{56}+x_{26}x_{57},\ x_{36}x_{52}-x_{32}x_{56},\ x_{17}x_{52}-x_{12}x_{57},\ x_{47}x_{51}-x_{41}x_{57},\ x_{17}x_{45}+x_{15}x_{47},$ $x_{35}x_{44}-x_{34}x_{45},\ x_{27}x_{44}-x_{24}x_{47},\ x_{26}x_{34}-x_{24}x_{36},\ x_{15}x_{32}-x_{12}x_{35},\ x_{17}x_{23}-x_{13}x_{27}$
deg 3	$x_{47}x_{56}x_{65}-x_{45}x_{57}x_{66},\ x_{17}x_{56}x_{65}+x_{15}x_{57}x_{66},\ x_{12}x_{56}x_{65}+x_{15}x_{52}x_{66},\ x_{26}x_{47}x_{65}+x_{27}x_{45}x_{66},$ $x_{26}x_{44}x_{65}+x_{24}x_{45}x_{66},\ x_{17}x_{26}x_{65}-x_{15}x_{27}x_{66},\ x_{17}x_{56}x_{63}+x_{13}x_{57}x_{66},\ x_{12}x_{56}x_{63}+x_{13}x_{52}x_{66},$ $x_{27}x_{45}x_{63}+x_{23}x_{47}x_{65},\ x_{24}x_{45}x_{63}+x_{23}x_{44}x_{65},\ x_{12}x_{36}x_{63}+x_{13}x_{32}x_{66},\ x_{24}x_{35}x_{63}+x_{23}x_{34}x_{65},$ $x_{23}x_{57}x_{61}+x_{27}x_{51}x_{63},\ x_{15}x_{57}x_{61}+x_{17}x_{51}x_{65},\ x_{13}x_{57}x_{61}+x_{17}x_{51}x_{63},\ x_{35}x_{52}x_{61}+x_{32}x_{51}x_{65},$ $x_{15}x_{52}x_{61}+x_{12}x_{51}x_{65},\ x_{13}x_{52}x_{61}+x_{12}x_{51}x_{63},\ x_{26}x_{47}x_{61}+x_{27}x_{41}x_{66},\ x_{23}x_{47}x_{61}+x_{27}x_{41}x_{63},$ $x_{13}x_{47}x_{61}+x_{17}x_{41}x_{63},\ x_{36}x_{44}x_{61}+x_{34}x_{41}x_{66},\ x_{26}x_{44}x_{61}+x_{24}x_{41}x_{66},\ x_{23}x_{44}x_{61}+x_{24}x_{41}x_{63},$ $x_{35}x_{47}x_{56}+x_{36}x_{45}x_{57},\ x_{34}x_{47}x_{56}+x_{36}x_{44}x_{57},\ x_{17}x_{35}x_{56}-x_{15}x_{36}x_{57},\ x_{35}x_{47}x_{52}+x_{32}x_{45}x_{57},$ $x_{34}x_{47}x_{52}+x_{32}x_{44}x_{57},\ x_{27}x_{34}x_{52}+x_{24}x_{32}x_{57},\ x_{13}x_{26}x_{52}+x_{12}x_{23}x_{56},\ x_{36}x_{45}x_{51}+x_{35}x_{41}x_{56},$ $x_{32}x_{45}x_{51}+x_{35}x_{41}x_{52},\ x_{12}x_{45}x_{51}+x_{15}x_{41}x_{52},\ x_{36}x_{44}x_{51}+x_{34}x_{41}x_{56},\ x_{32}x_{44}x_{51}+x_{34}x_{41}x_{52},$ $x_{26}x_{44}x_{51}+x_{24}x_{41}x_{56},\ x_{27}x_{36}x_{45}-x_{26}x_{35}x_{47},\ x_{17}x_{32}x_{44}+x_{12}x_{34}x_{47},\ x_{15}x_{23}x_{44}+x_{13}x_{24}x_{45},$ $x_{17}x_{26}x_{35}+x_{15}x_{27}x_{36},\ x_{13}x_{26}x_{35}+x_{15}x_{23}x_{36},\ x_{15}x_{27}x_{34}+x_{17}x_{24}x_{35},\ x_{15}x_{23}x_{34}+x_{13}x_{24}x_{35},$ $x_{17}x_{26}x_{32}+x_{12}x_{27}x_{36},\ x_{13}x_{26}x_{32}+x_{12}x_{23}x_{36},\ x_{17}x_{24}x_{32}+x_{12}x_{27}x_{34},\ x_{13}x_{24}x_{32}+x_{12}x_{23}x_{34}$
deg 4	$x_{27}x_{35}x_{52}x_{63}-x_{23}x_{32}x_{57}x_{65},\ x_{17}x_{36}x_{44}x_{63}-x_{13}x_{34}x_{47}x_{66},\ x_{24}x_{35}x_{57}x_{61}-x_{27}x_{34}x_{51}x_{65},$ $x_{23}x_{34}x_{52}x_{61}-x_{24}x_{32}x_{51}x_{63},\ x_{12}x_{36}x_{47}x_{61}-x_{17}x_{32}x_{41}x_{66},\ x_{13}x_{32}x_{44}x_{61}-x_{12}x_{34}x_{41}x_{63},$ $x_{15}x_{26}x_{44}x_{52}-x_{12}x_{24}x_{45}x_{56},\ x_{13}x_{26}x_{45}x_{51}-x_{15}x_{23}x_{41}x_{56},\ x_{12}x_{23}x_{44}x_{51}-x_{13}x_{24}x_{41}x_{52}$

Table 5.1: The 72 generators of I_{M_4}.



$$S_P(\boldsymbol{x}) = \begin{array}{c|ccccc} & H_1 & H_2 & H_3 & H_4 & H_5 \\ & 1234 & 1256 & 3456 & 135 & 246 \\ \hline 1 & 0 & 0 & x_{13} & 0 & x_{15} \\ 2 & 0 & 0 & x_{23} & x_{24} & 0 \\ 3 & 0 & x_{32} & 0 & 0 & x_{35} \\ 4 & 0 & x_{42} & 0 & x_{44} & 0 \\ 5 & x_{51} & 0 & 0 & 0 & x_{55} \\ 6 & x_{61} & 0 & 0 & x_{64} & 0 \end{array}$$

Figure 5.2: The triangular prism and its slack matrix as a polytope.

to facet-defining hyperplanes. Thus the slack ideal of the polytope is always contained in the slack ideal of the matroid, $I_P \subseteq I_M$. We illustrate with the following example.

Example 5.2.11. *We consider the triangular prism P labelled as in Figure 5.2. As a 3-polytope, its facets are given by the hyperplanes* $1234, 1256, 3456, 135, 246$ *and the symbolic slack matrix is in Figure 5.2. Its slack ideal I_P is generated by 3 binomials.*

Considering P as a rank 4 matroid which has the 3 facets 1234, 1256, 3456 *of P as its non-bases, we obtain following symbolic slack matrix.*

$$S_M(\boldsymbol{x}) = \begin{array}{c|ccccc|cccccc} & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 & H_{10} & H_{11} \\ & 1234 & 1256 & 3456 & 135 & 246 & 136 & 146 & 145 & 245 & 235 & 236 \\ \hline 1 & 0 & 0 & x_{13} & 0 & x_{15} & 0 & 0 & 0 & x_{19} & x_{1,10} & x_{1,11} \\ 2 & 0 & 0 & x_{23} & x_{24} & 0 & x_{26} & x_{27} & x_{28} & 0 & 0 & 0 \\ 3 & 0 & x_{32} & 0 & 0 & x_{35} & 0 & x_{37} & x_{38} & x_{39} & 0 & 0 \\ 4 & 0 & x_{42} & 0 & x_{44} & 0 & x_{46} & 0 & 0 & 0 & x_{4,10} & x_{4,11} \\ 5 & x_{51} & 0 & 0 & 0 & x_{55} & x_{56} & x_{57} & 0 & 0 & 0 & x_{5,11} \\ 6 & x_{61} & 0 & 0 & x_{64} & 0 & 0 & 0 & x_{68} & x_{69} & x_{6,10} & 0 \end{array}$$

Not only is $I_P \subseteq I_M$ but in this case I_P is the elimination ideal given by eliminating the *variables in the columns indexed by the additional hyperplanes $H_6, \dots, H_{11}$.*

5.3 Realization space models

A realization space for a rank $d+1$ matroid M with n elements is, roughly speaking, a space whose points are in correspondence with (equivalence classes of) collections of vectors $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \subset \Bbbk^{d+1}$ whose matroid $M[V]$ is M. In this section we show that the slack variety defined in the §2 provides such a realization space, and we relate it to another realization space called the Grassmannian of the matroid.

Theorem 5.2.5 characterizes the slack matrices of realizations of a matroid. The next theorem shows that the slack variety captures exactly these matrices.

Theorem 5.3.1. *Let M be a rank $d+1$ matroid. Then V is a realization of M if and only if $S_{M[V]} = S_M(\boldsymbol{s})$ where $\boldsymbol{s} \in \mathcal{V}(I_M) \cap (\Bbbk^*)^t$.*

Proof. Let V be a realization of M. Then $S_{M[V]} = S_M(\boldsymbol{s})$ for some $\boldsymbol{s} \in (\Bbbk^*)^t$ by Theorem 5.2.5 (i). Furthermore, $\operatorname{rank}(S_{M[V]}) = d+1$ by Corollary 5.2.3, so that its $(d+2)$-minors vanish and thus $\boldsymbol{s} \in \mathcal{V}(I_M)$, as desired.

Let $V \in \Bbbk^{(d+1)\times n}$ be such that $S_{M[V]} = S_M(\boldsymbol{s})$ for some $\boldsymbol{s} \in \mathcal{V}(I_M) \cap (\Bbbk^*)^t$. Then $\operatorname{supp}(S_{M[V]}) = \operatorname{supp}(S_M)$ and $\operatorname{rank}(S_{M[V]}) \leq d+1$. But now by Lemma 5.2.2, $\operatorname{rank}(S_{M[V]}) \geq d+1$, making V a realization of M by Theorem 5.2.5. □

Since we know that the set of realizations of a matroid is closed under row and column scalings, Theorem 5.3.1 implies the following corollary. We denote the torus of row and column scalings, $(\Bbbk^*)^n \times (\Bbbk^*)^h$, by $T_{n,h}$.

Corollary 5.3.2. *The slack variety is closed under the action of the group $T_{n,h}$, where $(\Bbbk^*)^n$ acts by row scaling (left multiplication by diagonal matrices) and $(\Bbbk^*)^h$ acts by column scaling (right multiplication by diagonal matrices).*

Theorem 5.3.1 and Corollary 5.3.2 tell us that the slack variety is a realization space for matroid M and the slack variety modulo the action of $T_{n,h}$ is a realization space for the projective equivalence classes of realizations of M.

Definition 5.3.3. *The* slack realization space *of a rank $d+1$ matroid M on n elements with h hyperplanes is the image of the slack variety inside a product of projective spaces $\mathcal{V}(I_M) \cap (\Bbbk^*)^t \hookrightarrow (\mathbb{P}^{n-1})^h$, where $\boldsymbol{s}$ is sent to the columns of $S_M(\boldsymbol{s})$.*

Proposition 5.3.4. *Let M be a rank $d+1$ matroid on n elements with h hyperplanes. Then the points of its slack realization space are in one-to-one correspondence with linear equivalence classes of realizations of M.*

Proof. Under this embedding, two slack matrices which differ by column scaling are the same point in $(\mathbb{P}^{n-1})^h$. So, the result follows by Lemma 5.2.7. $\square$

Next we describe a known model for the realization space of a matroid arising from a subvariety of a Grassmannian. The *Grassmannian* $Gr(d+1,n)$ is a variety whose points correspond to $(d+1)$-dimensional linear subspaces of a fixed n-dimensional vector space Λ. It embeds into $\mathbb{P}^{\binom{n}{d+1}-1}$ as follows. Any $(d+1)$-dimensional linear subspace of Λ can be described as the row space of a $(d+1) \times n$ matrix of rank $d+1$. However, two matrices A and B have the same row space when then there is a matrix $G \in GL(d+1,\Lambda)$ such that $A = GB$. Thus, to ensure that we have a one-to-one correspondence between subspaces and points in the Grassmannian, we record a $(d+1)\times n$ matrix by its vector of $(d+1)$-minors. We call this the *Plücker vector*, and it has coordinates indexed by subsets σ of $[n]$ of size $d+1$. The *Plücker ideal* $P_{d+1,n} \subseteq \Bbbk[\boldsymbol{p}] = \Bbbk[p_\sigma \mid \sigma \subset [n],\ |\sigma| = d+1]$ is the set of all polynomials which vanish on every vector of $(d+1)$-minors coming from some $(d+1)\times n$ matrix. It is generated by the homogeneous quadratic Plücker relations and cuts out the Grassmannian as a variety inside $\mathbb{P}^{\binom{n}{d+1}-1}$.

If we have a rank $d+1$ matroid $M = (E, \mathcal{B})$ with realization $V \in \Bbbk^{(d+1)\times n}$, the Plücker coordinate p_σ of V is zero if and only if σ is a dependent set of M. Thus, realizations of M correspond to the subvariety of $Gr(d+1,n)$ defined by setting the Plücker coordinates of non-bases to 0. This variety is also cut out by an ideal, namely, the *Grassmannian ideal* $P_M \subset \Bbbk[\boldsymbol{p}_{\mathcal{B}}]$ of M,

$$P_M := (P_{d+1,n} + \langle p_\sigma \ :\ \sigma \notin \mathcal{B}\rangle) \cap \Bbbk[\boldsymbol{p}_{\mathcal{B}}],$$

where $\Bbbk[\boldsymbol{p}_{\mathcal{B}}] := \Bbbk[p_\sigma \mid \sigma \text{ is a basis of } M]$ is the ring with one variable for each basis $\sigma \in \mathcal{B}$. This is the ideal obtained from $P_{d+1,n} \subset \Bbbk[\boldsymbol{p}]$ by setting the variables indexed by non-bases of M to 0.

Definition 5.3.5. *The* Grassmannian of M*, denoted* $Gr(M)$*, is* $\mathcal{V}(P_M) \cap (\Bbbk^*)^{|\mathcal{B}|}$*. The points in* $Gr(M)$ *correspond to* $GL(\Bbbk^{d+1})$ *equivalence classes of* $(d+1) \times n$ *matrices which realize the matroid* M*.*

5.3.1 Universal realization ideal

Given a matroid M, we now define an ideal whose variety contains pairs $(\boldsymbol{s}, \boldsymbol{q})$, where $\boldsymbol{q}$ is a Plücker vector and $\boldsymbol{s}$ the nonzero entries of a slack matrix, and both come from the same realization of M.

If V is a realization of a rank $d+1$ matroid $M = (E, \mathcal{B})$, then a slack matrix $S_{M[V]}$ can be filled with the Plücker coordinates of V, which can be seen from (5.1). Given a hyperplane $H_j \in \mathcal{H}(M)$, we record all possible substitutions of Plücker variables for slack variables using a matrix M_{H_j} whose rows are indexed by $i \in E \backslash H_j$, and whose columns are indexed by subsets $J = \{j_1, \ldots, j_d\} \subset E$ with $\overline{J} = H$; that is,

$$M_{H_j} = \begin{bmatrix} x_{i_1 j} & \operatorname{sgn}(i_1, J_1) \cdot p_{i_1 \cup J_1} & \cdots & \operatorname{sgn}(i_1, J_k) \cdot p_{i_1 \cup J_k} \\ \vdots & \vdots & \cdots & \vdots \\ x_{i_m j} & \operatorname{sgn}(i_m, J_1) \cdot p_{i_m \cup J_1} & \cdots & \operatorname{sgn}(i_m, J_k) \cdot p_{i_m \cup J_k} \end{bmatrix},$$

where $\operatorname{sgn}(i, J)$ is the sign of the permutation putting $(i, j_1, \ldots, j_d)$ in increasing order.

Example 5.3.6. *Recall the matroid* M_4 *of Example 5.2.9 pictured in Figure 5.1. Consider the hyperplane* $H_2 = 246$*. It corresponds to slack variables* $x_{i,2}$ *for* $i = 1, 3, 5$ *and its independent subsets are* 24*,* 26*, and* 46*. So the matrix* M_{246} *has the form*

$$M_{246} = \begin{bmatrix} x_{12} & p_{124} & p_{126} & p_{146} \\ x_{32} & -p_{234} & -p_{236} & p_{346} \\ x_{52} & p_{245} & -p_{256} & -p_{456} \end{bmatrix}.$$

Definition 5.3.7. *Let $M = (E, \mathcal{B})$ be a matroid, P_M be the Grassmannian ideal of M and $I_2(M_{H_j})$ be the ideal of 2-minors of the matrix M_{H_j}. The* universal realization ideal *of M is*

$$U_M := P_M + \sum_{H_j \in \mathcal{H}(M)} I_2(M_{H_j}) \subseteq \Bbbk[\mathbf{x}, \mathbf{p}].$$

Intuitively, insisting that the matrices M_{H_j} have rank 1 corresponds to ensuring the columns of the slack matrix are simply scaled versions of the appropriate Plücker coordinates. We now state the main result of this section.

Theorem 5.3.8. *Let $M = ([n], \mathcal{B})$ be a rank $d+1$ matroid with universal realization ideal $U_M \subseteq \Bbbk[\boldsymbol{x}, \boldsymbol{p}]$. Then $\mathcal{V}(U_M) \in \Bbbk^t \times \Bbbk^{|\mathcal{B}|}$ with*

(i) the projection of $\mathcal{V}(U_M)$ onto the Plücker coordinates, $\pi_{\boldsymbol{p}} : \mathcal{V}(U_M) \to \Bbbk^{|\mathcal{B}|}$, is the Grassmannian of the matroid,

$$\overline{\pi_{\boldsymbol{p}}(\mathcal{V}(U_M))} = \overline{Gr(M)};$$

(ii) the projection of $\mathcal{V}(U_M) \cap \left((\Bbbk^*)^t \times (\Bbbk^*)^{|\mathcal{B}|}\right)$ onto the slack coordinates, $\pi_{\boldsymbol{x}} : \mathcal{V}(U_M) \to \Bbbk^t$, is the set of slack matrices of realizations of M,

$$\pi_{\boldsymbol{x}}\left(\mathcal{V}(U_M) \cap \left((\Bbbk^*)^t \times (\Bbbk^*)^{|\mathcal{B}|}\right)\right) = \mathcal{V}(I_M) \cap (\Bbbk^*)^t.$$

The proof of this theorem requires several preliminary lemmas. We first have the following result on Gröbner bases. (For notation and further details see [12].)

Lemma 5.3.9. *Fix an elimination order on $\Bbbk[\boldsymbol{x}, \boldsymbol{p}]$. Given two $\boldsymbol{x}$-homogeneous polynomials f, g and an $\boldsymbol{x}$-homogeneous set $\mathcal{G} \subset \Bbbk[\boldsymbol{x}, \boldsymbol{p}]$, if $h := \overline{S(f,g)}^{\mathcal{G}}$ with $h \neq 0$, then* $\deg_{\boldsymbol{x}}(h) \geq \max\{\deg_{\boldsymbol{x}}(f), \deg_{\boldsymbol{x}}(g)\}$.

Lemma 5.3.10. *The Grassmannian ideal of a matroid can be obtained by eliminating the slack variables from its universal realization ideal. That is, $P_M = U_M \cap \Bbbk[\mathbf{p}]$.*

Proof. We obtain one containment by the definition of U_M, since

$$P_M = P_M \cap \Bbbk[\mathbf{p}] \subseteq \left(P_M + \sum_{H \in \mathcal{H}(M)} I_2(M_H) \right) \cap \Bbbk[\mathbf{p}] = U_M \cap \Bbbk[\mathbf{p}].$$

It remains to show the reverse containment. Fix an elimination order on $\Bbbk[\boldsymbol{x}, \boldsymbol{p}]$ eliminating the $\boldsymbol{x}$ variables. Let $\mathcal{G}$ be a Gröbner basis for U_M with respect to this ordering. Then, it suffices to show that

$$\mathcal{G} \cap \Bbbk[\boldsymbol{p}] \subset P_M. \tag{5.2}$$

If we start with a generating set satisfying (5.2), then by by Lemma 5.3.9, any terms which are added to $\mathcal{G}$ after applying each step of Buchberger's algorithm with $\boldsymbol{x}$-degree 0 must be the reduction of an S-pair of elements which also have $\boldsymbol{x}$-degree 0, and are therefore also contained in P_M.

It remains to show that an initial generating set of U_M satisfies (5.2). Taking the generating set of the definition, it is enough to show that any minor in $I_2(M_H)$ not containing a slack variable is already in P_M. It is not hard to check that any such minor already arises in P_M as some 3-term Plücker relation having a term $p_\sigma p_\tau$ for some $\sigma \notin \mathcal{B}$ which gets set to zero. ☐

Proof of Theorem 5.3.8.

(i) This follows from the definition of $Gr(M)$ and Lemma 5.3.10.

(ii, $\subset$) Let $(\boldsymbol{s}, \boldsymbol{q}) \in \mathcal{V}(U_M) \cap \left((\Bbbk^*)^t \times (\Bbbk^*)^{|\mathcal{B}|}\right)$. From $\boldsymbol{q}$ we can obtain a $(d+1) \times n$ matrix V with Plücker vector whose nonzero coordinates come from $\boldsymbol{q}$. We claim that V is a realization of M, so that $S_{M[V]} \in \mathcal{V}(I_M) \cap (\Bbbk^*)^t$ by Theorem 5.3.1. Furthermore, we claim that $S_{M[V]}$ and $S_M(\boldsymbol{s})$ are the same up to column scaling, so that $\boldsymbol{s} \in \mathcal{V}(I_M) \cap (\Bbbk^*)^t$ by Corollary 5.3.2.

A subset of $d+1$ columns of V is independent if and only if the the corresponding Plücker coordinate is nonzero. Since $\boldsymbol{q} \in (\Bbbk^*)^{|\mathcal{B}|}$, the nonzero Plücker coordinates correspond to bases $B \in \mathcal{B}$ of M. Hence V is a realization of M.

Fix a hyperplane H and an independent subset $J = \{j_1, \dots, j_d\} \subset H$. Then the first column of M_H is $(s_{iH})_{i \notin H}^{\top}$, column J is $(\operatorname{sgn}(i, J) \cdot q_{i \cup J})_{i \notin H}^{\top}$, and by definition of U_M, there exists $\lambda_H \in \Bbbk^*$ such that $s_{iH} = \lambda_H \operatorname{sgn}(i, J) \cdot q_{i \cup J}$ for all i. Since $\boldsymbol{q}$ is the Plücker vector of V, this gives

$$s_{iH} = \lambda_H \operatorname{sgn}(i, J) \cdot q_{i \cup J} = \lambda_H \det(\boldsymbol{v}_i, \boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_d}), \quad \forall i$$

so that using subset J to calculate α_H, and hence the entries of $S_{M[V]}$, as in Example 5.2.9, we see that $S_{M[V]} D_h = S_M(\boldsymbol{s})$, for D_h the diagonal matrix with entries λ_H.

(ii, $\supset$) Let $\boldsymbol{s} \in \mathcal{V}(I_M) \cap (\Bbbk^*)^t$. By Theorem 5.3.1, $S_M(\boldsymbol{s})$ is the slack matrix of some realization V of M. Let $\boldsymbol{q}$ be the vector of Plücker coordinates of V. We claim $(\boldsymbol{s}, \boldsymbol{q}) \in \mathcal{V}(U_M) \cap \big((\Bbbk^*)^t \times (\Bbbk^*)^{|\mathcal{B}|}\big)$. To see this it suffices to show that the columns of M_H, with entries defined using $(\boldsymbol{s}, \boldsymbol{q})$, are scalar multiples of each other for each $H \in \mathcal{H}(M)$; that is, the $I_2(M_H)$ ideals are satisfied, since the Plücker coordinates necessarily satisfy the Plücker ideal equations, and hence the equations of P_M.

The (i, H) slack entry s_{iH} is of the form $\det(\boldsymbol{v}_i, \boldsymbol{w}_1, \dots, \boldsymbol{w}_d)$ for some choice of $\boldsymbol{w}_1, \dots, \boldsymbol{w}_d$ which span the hyperplane H. Each subsequent column of M_H has entries $\det(\boldsymbol{v}_i, \boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_d})$, obtained from $\boldsymbol{q}$ as above, for $j_1 < \cdots < j_d$ spanning H. Since each $\boldsymbol{v}_{j_k}$ lies on hyperplane the H, there is a sequence of elementary column operations that takes the matrix with columns $\boldsymbol{v}_i, \boldsymbol{w}_1, \dots, \boldsymbol{w}_d$ to the one with columns $\boldsymbol{v}_i, \boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_d}$ for each i. These column operations change the determinant by some scale factor $\lambda \in \Bbbk^*$ for all i, so that each column of M_H is a scalar multiple of the first column of slack entries as required.

□

Corollary 5.3.11. *Let $M = (E, \mathcal{B})$ be a rank $d + 1$ matroid. Then*

$$\sqrt{I_M} = \sqrt{U_M : \Bigg(\prod_{i \in E, H \in \mathcal{H}} x_{iH} \prod_{\sigma \in \mathcal{B}} p_\sigma \Bigg)^{\infty}} \bigcap \Bbbk[\boldsymbol{x}].$$

By universality [33], we do not expect that I_M is radical for every matroid. Thus, Corollary 5.3.11 may be the strongest relationship between I_M and U_M.

Example 5.3.12. *We now continue Example 5.2.9, for which we can verify Theorem 5.3.8 at the level of ideals. Let P_{M_4} be the Grassmannian ideal of this matroid, which is generated by the Plücker relations with variables $p_{123}, p_{246}, p_{345}, p_{156}$ set to 0. Let I be the ideal which guarantees each of the 7 matrices M_{123}, M_{246}, M_{345} M_{156}, M_{25}, M_{14}, M_{36} have rank 1; that is, $I = \sum_{H \in \mathcal{H}(M_4)} I_2(M_H)$. Then, the universal realization ideal is $U_M = I + P_{M_4}$. In this case, we can compute that $P_M = U_M \cap \Bbbk[\boldsymbol{p}]$ and $I_M = U_M : (\prod x_{iH} \prod p_\sigma)^\infty \cap \Bbbk[\boldsymbol{x}]$.*

5.4 Non-realizability

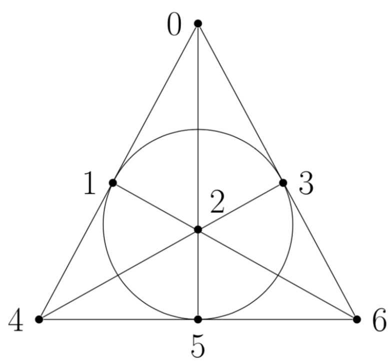
In this section we illustrate how the slack ideal can be used to determine matroid realizability over a given field. This is a well-studied problem [7, 9, 33] for which a complete characterization is only known in a very limited number of cases. Observe that Theorem 5.3.1 gives us the following criterion for realizability.

Corollary 5.4.1. *A matroid M is realizable over $\Bbbk$ if and only if $\mathcal{V}(I_M) \cap (\Bbbk^*)^t \neq \emptyset$.*

We now recast this into a test for realizability in terms of the slack ideal.

Proposition 5.4.2. *Let M be an abstract matroid and $\Bbbk$ be a field. If the slack ideal $I_M = \langle 1 \rangle$ over $\Bbbk$, then M is not realizable over $\Bbbk$. If $\Bbbk$ is algebraically closed and M is not realizable over $\Bbbk$, then $I_M = \langle 1 \rangle$.*

Proof. If M is realizable over $\Bbbk$, then there exists a slack matrix S_M which is an element of the variety $\mathcal{V}(I_M)$ by Theorem 5.3.1. Then we cannot have $I_M = \langle 1 \rangle$. On the other hand, if $I_M \neq \langle 1 \rangle$, then $\mathcal{V}(I_M)$ is not empty by the Nullstellensatz, and since I_M is saturated with respect to the product of the variables, $\mathcal{V}(I_M)$ is not contained entirely in the coordinate hyperplanes. Therefore, by Theorem 5.3.1 there is a slack matrix S_M, and the rows of S_M give a realization of M by Lemma 5.2.4. $\square$



$$\begin{array}{c} \\ \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 \\ 126 & 014 & 456 & 025 & 036 & 234 & 135 \end{array} \\
\begin{bmatrix}
x_{01} & 0 & x_{03} & 0 & 0 & x_{06} & x_{07} \\
0 & 0 & x_{13} & x_{14} & x_{15} & x_{16} & 0 \\
0 & x_{22} & x_{23} & 0 & x_{25} & 0 & x_{27} \\
x_{31} & x_{32} & x_{33} & x_{34} & 0 & 0 & 0 \\
x_{41} & 0 & 0 & x_{44} & x_{45} & 0 & x_{47} \\
x_{51} & x_{52} & 0 & 0 & x_{55} & x_{56} & 0 \\
0 & x_{62} & 0 & x_{64} & 0 & x_{66} & x_{67}
\end{bmatrix}
\end{array}$$

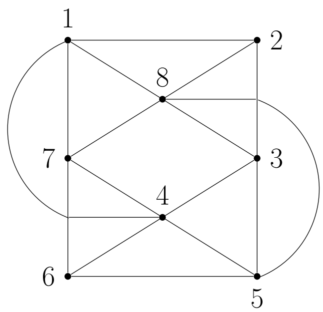
Figure 5.3: The Fano plane with its non-bases drawn as lines and a circle, and its symbolic slack matrix.

Example 5.4.3. *Consider the Fano plane M_F. It is a rank 3 matroid on 7 elements $E = \{0,1,2,3,4,5,6\}$, depicted in Figure 5.3 with its symbolic slack matrix. It is known that M_F is only realizable in characteristic 2. Using* `Macaulay2` *[27], we find $I_{M_F} = \langle 1 \rangle$ over $\mathbb{Q}$. So, the Fano plane is not realizable over $\mathbb{Q}$ by Proposition 5.4.2.*

Over $\mathbb{F}_2$, the slack ideal is generated by 126 *binomials of degrees 2,3, and 4. So, setting all variables to 1 in the slack matrix gives the realization of M_F in $\mathbb{F}_2^7$.*

Example 5.4.4. *Consider the* complex matroid *M_8 from [9, p. 33]. It is a rank 3 matroid on 8 elements with non-bases* 124, 235, 346, 457, 568, 167, 278, *and* 138. *It is depicted with its symbolic slack matrix in Figure 5.4.*

To simplify the computation, we use Corollary 5.3.2 and note that we can select a representative of each projective equivalence class by fixing certain variables in the slack matrix to be 1 (see §5.5.1 for more details). Fixing the variables x_{14}, x_{27}, x_{28}, $x_{3,12}$, x_{41}, x_{46}, x_{47}, $x_{4,10}$, x_{57}, x_{67}, x_{72}, x_{73}, x_{74}, x_{75}, x_{77}, x_{79}, $x_{7,11}$, $x_{7,12}$, x_{84} to 1 and computing the slack



$$\begin{array}{cc} & \begin{array}{cccccccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 & H_{10} & H_{11} & H_{12} \\ 167 & 235 & 568 & 346 & 124 & 278 & 138 & 457 & 48 & 37 & 26 & 15 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & 0 & x_{16} & 0 & x_{18} & x_{19} & x_{1,10} & x_{1,11} & 0 \\ x_{21} & 0 & x_{23} & x_{24} & 0 & 0 & x_{27} & x_{28} & x_{29} & x_{2,10} & 0 & x_{2,12} \\ x_{31} & 0 & x_{33} & 0 & x_{35} & x_{36} & 0 & x_{38} & x_{39} & 0 & x_{3,11} & x_{3,12} \\ x_{41} & x_{42} & x_{44} & 0 & 0 & x_{46} & x_{47} & 0 & 0 & x_{4,10} & x_{4,11} & x_{4,12} \\ x_{51} & 0 & 0 & x_{54} & x_{55} & x_{56} & x_{57} & 0 & x_{59} & x_{5,10} & x_{5,11} & 0 \\ 0 & x_{62} & 0 & 0 & x_{65} & x_{66} & x_{67} & x_{68} & x_{69} & x_{6,10} & 0 & x_{6,12} \\ 0 & x_{72} & x_{73} & x_{74} & x_{75} & 0 & x_{77} & 0 & x_{79} & 0 & x_{7,11} & x_{7,12} \\ x_{81} & x_{82} & 0 & x_{84} & x_{85} & 0 & 0 & x_{88} & 0 & x_{8,10} & x_{8,11} & x_{8,12} \end{bmatrix}. \end{array}$$

Figure 5.4: The complex matroid M_8 and its symbolic slack matrix $S_{M_8}(\boldsymbol{x})$.

ideal I_{M_8} in `Macaulay2` *[27], we find that it is not the unit ideal. However, it contains the polynomial $x_{8,12}^2 + x_{8,12} + 1$. Since this polynomial has only the complex roots $\frac{-1 \pm i\sqrt{3}}{2}$, we get by Corollary 5.4.1 that M_8 is not realizable over $\mathbb{R}$, but it is realizable over $\mathbb{C}$ by Proposition 5.4.2.*

5.4.1 Final Polynomials

The method of final polynomials introduced in [9, §4.2] certifies when a matroid has no realization. We define an analogous polynomial for the slack ideal, and show how it can be used to improve computational efficiency of checking non-realizability.

Definition 5.4.5. *Let M be any matroid of rank $d+1$. Let $\mathcal{S}$ be the multiplicatively closed set generated by taking finite products of the variables $\boldsymbol{x}$ in the symbolic slack matrix. A polynomial $f \in \Bbbk[\boldsymbol{x}]$ is a* slack final polynomial *if*

$$f \in I_{d+2}(S_M(\boldsymbol{x})) \cap (\mathcal{S} + I_M)$$

where $I_{d+2}(S_M(\boldsymbol{x}))$ is the ideal of $(d+2)$-minors of the symbolic slack matrix of M.

We now have the following result, which demonstrates that the existence of slack final polynomials gives a certificate for non-realizability.

Proposition 5.4.6. *Let M be a matroid of rank $d+1$. The following are equivalent.*

(i) $1 \in I_M \subseteq \Bbbk[\boldsymbol{x}]$,

(ii) There is a monomial $m \in \Bbbk[\boldsymbol{x}]$ *such that* $m \in I_{d+2}(S_M(\boldsymbol{x}))$,

(iii) A slack final polynomial $f \in I_{d+2}(S_M(\boldsymbol{x})) \cap (\mathcal{S} + I_M)$ *exists for* M.

Remark 5.4.7. Over an algebraically closed field, these conditions are equivalent to the matroid being non-realizable by Proposition 5.4.2. When $\Bbbk$ is not algebraically closed, these conditions imply non-realizability, but if a matroid is not realizable there may not be a slack final polynomial.

Example 5.4.8. *Recall that the complex matroid M_8 has a complex realization, as $1 \notin I_{M_8} \subseteq \mathbb{Q}[\boldsymbol{x}]$. However, by the above proposition, this means that even though M_8 is not realizable over $\mathbb{Q}$, it does not have a slack final polynomial.*

Proof.

(i) $\Rightarrow$ (iii) Suppose $1 \in I_M$. Since I_M is the saturation of $I_{d+2}(S_M(\boldsymbol{x}))$, this implies that there exists a monomial $m \in \Bbbk[\boldsymbol{x}]$ such that $m \cdot 1 \in I_{d+2}(S_M(\boldsymbol{x}))$. Then, we observe the m is already a slack final polynomial for M.

(ii) $\Rightarrow$ (i) If there is a monomial $m \in \Bbbk[\boldsymbol{x}]$ such that $m \in I_{d+2}(S_M(\boldsymbol{x}))$, then after saturation we find $1 \in I_M$.

(iii) $\Rightarrow$ (ii) Suppose f is a slack final polynomial for M. Since $f \in (\mathcal{S} + I_M)$, there exists a monomial m and a $g \in I_M$ with $f = m + g$. Since $g \in I_M$, there exists a monomial n such that $ng \in I_{d+2}(S_M(\boldsymbol{x}))$, so $nm = nf - ng \in I_{d+2}(S_M(\boldsymbol{x}))$ is a monomial in $I_{d+2}(S_M(\boldsymbol{x}))$.

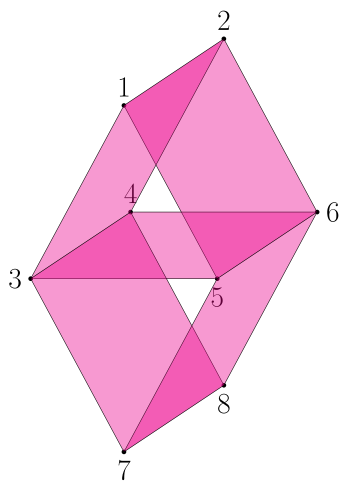
□

Remark 5.4.9. In practice saturation of the ideal $I_{d+2}(S_M(\boldsymbol{x}))$ can be quite slow, which often makes testing realizability via checking $1 \in I_M$ infeasible. Thus the real power of Proposition 5.4.6 is that one often finds relatively small monomials which are already contained in $I_{d+2}(S_M(\boldsymbol{x}))$. So, if one simply wants to certify non-realizability, a faster method is to compute $I_{d+2}(S_M(\boldsymbol{x}))$ and check, for example, if $\prod \boldsymbol{x} \in I_{d+2}(S_M(\boldsymbol{x}))$. In the following example we exhibit how this method can be useful for certifying non-realizability.

Example 5.4.10. *Consider the Fano matroid M_F of Example 5.4.3. If we compute $I_4(S_{M_F}(\boldsymbol{x}))$, then we can verify that the product of all of the variables is contained in this ideal. In fact, even the monomial $x_{07}x_{16}x_{25}x_{33}x_{41}x_{52}x_{64}$ is contained in $I_4(S_{M_F}(\boldsymbol{x}))$. Verifying this containment (using the laptop of one of the authors) in* `Macaulay2` *took 0.000067 seconds, while testing $1 \in I_M$ took 3.40765 seconds, indicating a speed up by a factor of 50,000.*

Example 5.4.11. *Consider the Vámos matroid pictured in Figure 5.5. It is a rank 4 matroid M_v on 8 elements whose non-bases are given by the sets* 1234*,* 1256*,* 3456*,* 3478*, and* 5678*. It is one of the smallest matroids known to be non-realizable over every field. However, the Vámos matroid has 41 hyperplanes, so that its slack matrix is an* 8×41 *matrix containing 200 distinct variables. Even computing the full set of minors of this matrix is computationally impractical.*

We note though, that it always suffices to show that Proposition 5.4.6 (ii) holds for some subideal of the ideal of $(d+2)$*-minors. In particular, we can look at the minors of a submatrix of $S_{M_v}(\boldsymbol{x})$. Consider the submatrix of the Vámos symbolic slack matrix in Figure 5.5. One can easily check with* `Macaulay2` *that the monomial given by the product of all the variables in this submatrix is already in the minor ideal of this submatrix (over $\mathbb{Q}$ and various finite fields), making M_v non-realizable over these fields by Propositions 5.4.6 and 5.4.2.*



$$
\begin{array}{c}
\\
\\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 \\ 3456 & 5678 & 1234 & 3478 & 1256 & 467 & 267 & 127 \end{array} \\
\begin{bmatrix}
x_{11} & x_{12} & 0 & x_{14} & 0 & x_{16} & x_{17} & 0 \\
x_{21} & x_{22} & 0 & x_{24} & 0 & x_{26} & 0 & 0 \\
0 & x_{32} & 0 & 0 & x_{35} & x_{36} & x_{37} & x_{38} \\
0 & x_{42} & 0 & 0 & x_{45} & 0 & x_{47} & x_{48} \\
0 & 0 & x_{53} & x_{54} & 0 & x_{56} & x_{57} & x_{58} \\
0 & 0 & x_{63} & x_{64} & 0 & 0 & 0 & x_{68} \\
x_{71} & 0 & x_{73} & 0 & x_{75} & 0 & 0 & 0 \\
x_{81} & 0 & x_{83} & 0 & x_{85} & x_{86} & x_{87} & x_{88}
\end{bmatrix}
\end{array}
$$

Figure 5.5: The Vámos matroid M_v with non-bases pictured as planes, and a submatrix its slack matrix.

5.5 *Projective uniqueness of matroids*

The simplest slack realization spaces are those belonging to projectively unique matroids. In this case, we know that there is a single realization up to projective transformations; in other words, $\mathcal{V}(I_M)$ is the toric variety which is the closure of the orbit of some realization under the action of $T_{n,h}$. This implies $\sqrt{I_M} = \mathcal{I}(\mathcal{V}(I_M))$ is a toric ideal; however, universality suggests that I_M need not be radical. A natural question which arises is whether projectively unique matroids correspond exactly to matroids with toric slack ideals. To study this question we introduce an intermediate toric ideal associated to a matroid.

Definition 5.5.1. *Define the* non-incidence graph of matroid M *as the bipartite graph* G_M *with one node for each element of the ground set of* M*, one node for each hyperplane, and an edge between element* i *and hyperplane* H_j *if and only if* $i \notin H_j$*. Notice that* G_M *records the support of the slack matrix* S_M*, and so we can think of its edges as being labelled by* *the corresponding entry of $S_M(\boldsymbol{x})$. (See Figure 5.6 for an example of the graph G_{M_4} for the matroid M_4 of Example 5.2.9, and Figure 5.7 for the non-incidence graph of the non-Fano matroid.)*

Let $\mathcal{A}_G$ be the set of vectors forming the columns of the vertex-edge incidence matrix of the graph G, and let T_G be the toric ideal of the vector configuration $\mathcal{A}_G$. The toric ideal of a bipartite graph is a well-studied object [34, 42]. If a matroid M has a slack matrix S_M which is a 0-1 matrix, then the toric ideal T_{G_M} associated to the graph G_M is the ideal of the orbit of S_M under the action of the torus $T_{n,h}$. So, T_{G_M} describes one projective equivalence class of slack matrices of M. We now define an analogous toric ideal for any projective equivalence class.

Definition 5.5.2. *Let M be an abstract matroid with realization V. Let $\boldsymbol{s} \in (\Bbbk^*)^t$ be such that $S_{M[V]} = S_M(\boldsymbol{s})$, where t is the number of variables in $S_M(\boldsymbol{x})$, the symbolic slack matrix of M. We define the* cycle ideal C_V *of $M[V]$ to be the ideal*

$$C_V = \left\langle \boldsymbol{x}^{c+} - \alpha_c \boldsymbol{x}^{c-} : c \text{ is a cycle in } G_M \text{ and } \alpha_c = \frac{\boldsymbol{s}^{c+}}{\boldsymbol{s}^{c-}} \right\rangle \subseteq \Bbbk[\boldsymbol{x}] \tag{5.3}$$

where $c+$ and $c-$ are alternating edges from the cycle c.

Theorem 5.5.3. *Let M be a matroid of rank $d+1$ on n elements with h hyperplanes. Let V be a realization of M with slack matrix $S_{M[V]} = [s_{i,j}]_{i=1,j=1}^{n,h}$. Then the ideal C_V is the (scaled) toric ideal which is the kernel of the $\Bbbk$-algebra homomorphism $\phi : \Bbbk[\boldsymbol{x}] \to \Bbbk[\boldsymbol{r}, \boldsymbol{t}, \boldsymbol{r}^{-1}, \boldsymbol{t}^{-1}]$, which sends $x_{ij} \mapsto s_{i,j} r_i t_j$.*

We note that the cycle ideal of a realization provides a way to distinguish projective equivalence classes of realizations of M, as well as detect projective uniqueness.

Lemma 5.5.4. *Let M be a matroid with realizations U and V. Then U, V are projectively equivalent if and only if $C_V = C_U$.*

Proof. First suppose that U and V are projectively equivalent. Let $S_{M[V]}$ have entries $s_{i,j}$ for elements i of the ground set of M and hyperplanes H_j of M. By Lemma 5.2.6 we know that $S_{M[U]}$ is obtained from $S_{M[V]}$ by scaling the rows by $r_1, \dots, r_n \in \Bbbk^*$ and scaling the columns by $t_1, \dots, t_h \in \Bbbk^*$. Then the entries of $S_{M[U]}$ are $r_i t_j s_{i,j}$. Since $S_{M[U]}$ and $S_{M[V]}$ have the same support, they have the same cycles. One may then show via elementary calculation that the coefficients α_c are the same when calculated from $S_{M[U]}$ and from $S_{M[V]}$ for every cycle $c \in G_M$.

Conversely, suppose $C_V = C_U$. Then $\mathcal{V}(C_V) = \mathcal{V}(C_U)$, and in particular, $S_{M[V]} = S_M(\boldsymbol{s})$ and $S_{M[U]} = S_M(\boldsymbol{u})$ for $\boldsymbol{s}, \boldsymbol{u} \in \mathcal{V}(C_V) \cap (\Bbbk^*)^t$. We now argue that $S_{M[U]}$ can be row and column scaled to be equal to $S_{M[V]}$, which proves the results by Lemma 5.2.6. Fix a spanning forest T of G_M. By Lemma 5.5.11 we may scale the entries in $S_M(\boldsymbol{u})$ corresponding to edges in T to be equal to the corresponding entries of $S_{M[V]}$. Any remaining entry a of $S_M(\boldsymbol{u})$ will correspond to an edge e in G_M such that $T \cup \{e\}$ contains a unique cycle c, where $e \in c$. The equation $\boldsymbol{x}^{c+} - \alpha_c \boldsymbol{x}^{c-} \in C_V$ corresponding to this cycle must be satisfied by the scaling of $S_M(\boldsymbol{u})$, since $C_V = C_U$. Since all variables in the cycle except the one labelled by edge e have been fixed, we find that there is only one possible value for a. Furthermore, this value equals the corresponding entry in $S_{M[V]}$, since $S_{M[V]}$ satisfies the equations of C_V by definition. □

Corollary 5.5.5. *Let M be a matroid with realization V. Then, $\mathcal{V}(C_V) \cap (\Bbbk^*)^t$ consists of exactly the slack matrices of realizations projectively equivalent to V.*

Lemma 5.5.6. *Let $\Bbbk$ be algebraically closed and M be a matroid with realization V. Then the slack ideal I_M is contained in the cycle ideal C_V.*

Proof. By Corollary 5.5.5 we know that $\mathcal{V}(C_V) \subset \mathcal{V}(I_M)$. Then, $\mathcal{I}(\mathcal{V}(C_V)) \supset \mathcal{I}(\mathcal{V}(I_M))$. Since C_V is radical, and since $I_M \subset \sqrt{I_M}$, this gives $C_V \supset I_M$. □

Proposition 5.5.7. *Let M be a matroid with projectively unique realization V. Then $\mathcal{V}(I_M) = \mathcal{V}(C_V)$.*

Proof. By Corollary 5.5.5 and Theorem 5.3.1, we get $\mathcal{V}(I_M) \cap (\Bbbk^*)^t = \mathcal{V}(C_V) \cap (\Bbbk^*)^t$. Then since both varieties are irreducible, the result follows. □

In fact, we can have $I_M = C_V$ for a realization V of M. In this case, call I_M *cyclic*.

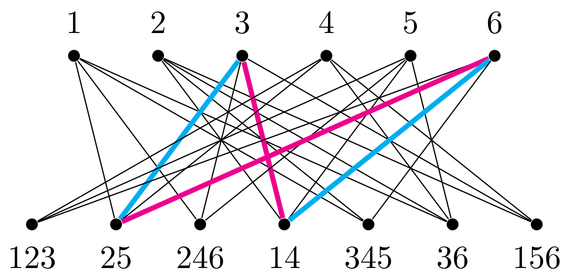


Figure 5.6: The graph G_{M_4} for matroid M_4 with the highlighted cycle corresponding to binomial $x_{36}x_{65} + x_{35}x_{66}$ of Table 5.1.

Theorem 5.5.8. *If the slack ideal of a matroid is cyclic then M is projectively unique and I_M is radical. The converse also holds when $\Bbbk$ is algebraically closed.*

Proof. Suppose that I_M is cyclic. Then M is projectively unique by Corollary 5.5.5 and I_M is radical, since it is prime by Theorem 5.5.3. Conversely, suppose that M is projectively unique and I_M is radical. By Proposition 5.5.7, $\mathcal{I}(\mathcal{V}(I_M)) = \mathcal{I}(\mathcal{V}(C_V))$, so that $I_M = C_V$, as both ideals are radical. $\square$

Example 5.5.9. *Recall the matroid M_4 from Example 5.2.9, whose non-incidence graph G_{M_4} is displayed in Figure 5.6. The matroid M_4 has a cyclic slack ideal. Hence I_{M_4} is a radical slack ideal, and M_4 is projectively unique.*

Example 5.5.10. *Recall the Fano plane discussed in Example 5.4.3. Over $\mathbb{F}_2$, the 126 binomial generators of I_{M_F} found in Example 5.4.3 correspond to each of the cycles in the graph G_{M_F}. Over $\mathbb{F}_2$ this is the projectively unique representation of M_F, and this ideal is equal to the cycle ideal of the representation.*

5.5.1 Scaled Slack Matrices

From Corollary 5.3.2 we know that quotienting by the action of $T_{n,h}$ on $\mathcal{V}(I_M) \cap (\Bbbk^*)^t$ gives us a realization space for projective equivalence classes of representations of M. We now give an explicit way of computing the variety of these equivalence classes. As in Lemma 3.5.2, we scale rows and columns of a slack matrix via the following lemma, to fix one representative of each projective equivalence class.

Lemma 5.5.11. *Given a realization of a matroid M, we may scale the rows and columns of its slack matrix S_M so that it has ones in the entries indexed by the edges in a maximal spanning forest F of the graph G_M; the resulting realization of M is projectively equivalent to the original realization of M.*

Definition 5.5.12. *Given a matroid M we can take a symbolic slack matrix and set variables corresponding to edges in a maximal spanning forest F to 1 as in Lemma 5.5.11 to obtain a* scaled symbolic slack matrix*. Then, the* scaled slack ideal *is obtained by taking the $(d+2)$-minors of this matrix and saturating with respect to the product of all the variables.*

Using the scaled symbolic slack matrix not only allows us to study the projective realization space of M, but also proves to be a useful tool for computations because this matrix will have considerably fewer variables.

Example 5.5.13. *Let M_{NF} be the non-Fano matroid. It is a rank 3 matroid on 7 elements depicted in Figure 5.7 with its symbolic slack matrix. It differs from the Fano plane by the inclusion of* 135 *as a basis.*

We now show that the non-Fano matroid is projectively unique, and write down a realization from the slack matrix. Let F be the spanning tree of $G_{M_{NF}}$ depicted in Figure 5.7. We set the corresponding variables x_{41}, x_{51}, x_{22}, x_{32}, x_{52}, x_{13}, x_{64}, x_{55}, x_{06}, x_{16}, x_{56}, x_{66}, x_{67}, x_{08}, x_{69} to 1 in the symbolic slack matrix in Figure 5.7. Taking the ideal of 4*-minors and saturating, we find that the ideal consists of equations of the form $x_{ij} - \alpha_{ij}$, for $\alpha_{ij} \in \mathbb{Q}$, so the configuration is projectively unique over $\mathbb{Q}$. The slack matrix corresponding to the single point in $\mathcal{V}(I_M) \cap (\mathbb{k}^*)^{39}/T_{7,9}$ is*

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

Example 5.5.14. *Consider the* Perles configuration $M_\star$ *of Figure 5.8. It is a matroid on 9 elements with hyperplanes given by* 0678, 347, 156, 128, 045, 358, 013, 246, 257, 48, 17, 36, 14, 23, 02*. Its symbolic slack matrix is shown in Figure 5.8 and has the matrix $S(\boldsymbol{x})$ studied in Section 3.5 of Chapter 3 as a submatrix.*

Since the ideal of $S(\boldsymbol{x})$ will be contained in the ideal of the whole matrix $S_{M_\star}(\boldsymbol{x})$, it follows from computation in Chapter 3 that $M_\star$ is not realizable over $\mathbb{Q}$. However, it is realizable over $\mathbb{R}$, and computing its scaled slack ideal we find that the slack variety consists of the following matrices, where α is a root of the polynomial $\alpha^2 - 3\alpha + 1$:

$$\begin{bmatrix} 0 & 1 & \alpha-3 & 3-\alpha & 0 & 3-\alpha & 0 & -1 & 1 & 3-\alpha & 1 & \alpha-2 & 2-\alpha & 2-\alpha & 0 \\ 2\alpha-5 & 1 & 0 & 0 & 3-\alpha & 3-\alpha & 0 & 2-\alpha & 3-\alpha & \alpha-2 & 0 & \alpha-2 & 0 & 2-\alpha & 5-2\alpha \\ 1 & \alpha & 1 & 0 & \alpha & \alpha-1 & 1 & 0 & 0 & \alpha & 1-\alpha & 1 & \alpha & 0 & 0 \\ 1 & 0 & \alpha-1 & 1-\alpha & \alpha-1 & 0 & 0 & \alpha-1 & -\alpha & 1 & 1-2\alpha & 0 & \alpha & 0 & -1 \\ 2-\alpha & 0 & 2-\alpha & \alpha-1 & 0 & \alpha-2 & 1 & 0 & \alpha-1 & 0 & \alpha & 2-\alpha & 0 & \alpha-1 & \alpha-1 \\ 2-\alpha & 1-\alpha & 0 & 1 & 0 & 0 & 1 & \alpha-1 & 0 & 2-\alpha & 1 & 1-\alpha & 1 & \alpha & \alpha-1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 3-\alpha & \alpha-2 & 1 & \alpha-2 & 1 & 1 & 0 & \alpha-2 & 0 & 2-\alpha & \alpha-1 & \alpha-1 & 1 \\ 0 & \alpha+1 & 1 & 0 & 1 & 0 & 1 & \alpha & 1-\alpha & 0 & 1-\alpha & 1-\alpha & \alpha & \alpha & 1 \end{bmatrix}.$$

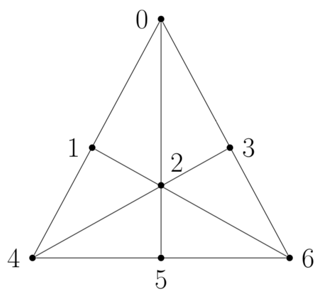
Over any field where 5 has a square root, the variety has degree 2 and dimension 0, so it consists of two points obtained by setting $\alpha = \frac{3\pm\sqrt{5}}{2}$.

5.5.2 *Acknowledgements*

We would like to thank Bernd Sturmfels and Rekha Thomas for their guidance and helpful discussion. We also thank Dan Corey for pointing us to examples of matroids with interesting realization spaces.

The software package `Macaulay2` [27] was invaluable in calculating all of the examples from this paper. In addition, the package "Matroids" written for `Macaulay2` by Justin Chen was indispensable.

We also acknowledge the Mathematical Sciences Research Institute, the Max-Planck-Institut für Mathematik in den Naturwissenschaften, and the University of Washington for facilitating our collaboration on this paper.



$$
\begin{array}{c|ccccccccc|}
 & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 \\
 & 126 & 014 & 456 & 025 & 036 & 234 & 35 & 13 & 15 \\
\hline
0 & x_{01} & 0 & x_{03} & 0 & 0 & x_{06} & x_{07} & x_{08} & x_{09} \\
1 & 0 & 0 & x_{13} & x_{14} & x_{15} & x_{16} & 0 & 0 & 0 \\
2 & 0 & x_{22} & x_{23} & 0 & x_{25} & 0 & x_{27} & x_{28} & x_{29} \\
3 & x_{31} & x_{32} & x_{33} & x_{34} & 0 & 0 & 0 & 0 & x_{39} \\
4 & x_{41} & 0 & 0 & x_{44} & x_{45} & 0 & x_{47} & x_{48} & x_{49} \\
5 & x_{51} & x_{52} & 0 & 0 & x_{55} & x_{56} & 0 & x_{58} & 0 \\
6 & 0 & x_{62} & 0 & x_{64} & 0 & x_{66} & x_{67} & x_{68} & x_{69}
\end{array}
$$

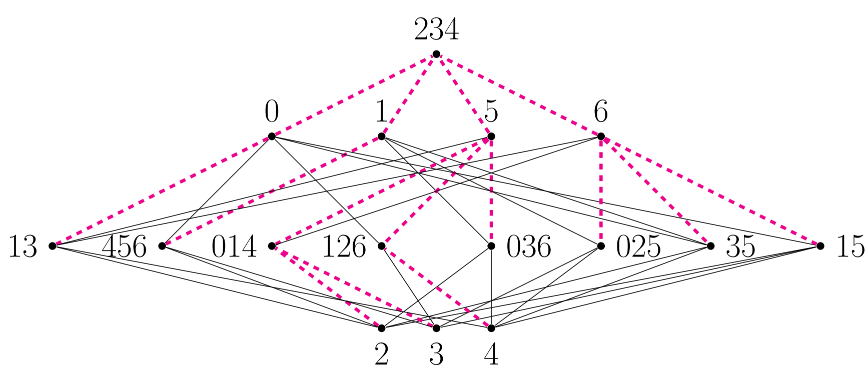
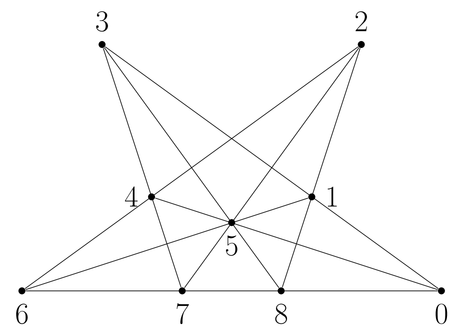


Figure 5.7: The non-Fano matroid, with its non-bases depicted as lines, together with its symbolic slack matrix and the spanning tree F selected of Example 5.5.13.



$$\begin{array}{cc}
 & \begin{array}{ccccccccccccccc} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 & H_{10} & H_{11} & H_{12} & H_{13} & H_{14} & H_{15} \\ 0678 & 347 & 156 & 128 & 045 & 358 & 013 & 246 & 257 & 48 & 17 & 36 & 14 & 23 & 02 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
0 & x_{02} & x_{03} & x_{04} & 0 & x_{06} & 0 & x_{08} & x_{09} & x_{0,10} & x_{0,11} & x_{0,12} & x_{0,13} & x_{0,14} & 0 \\
x_{11} & x_{12} & 0 & 0 & x_{15} & x_{16} & 0 & x_{18} & x_{19} & x_{1,10} & 0 & x_{1,12} & 0 & x_{1,14} & x_{1,15} \\
x_{21} & x_{22} & x_{23} & 0 & x_{25} & x_{26} & x_{27} & 0 & 0 & x_{2,10} & x_{2,11} & x_{2,12} & x_{2,13} & 0 & 0 \\
x_{31} & 0 & x_{33} & x_{34} & x_{35} & 0 & 0 & x_{38} & x_{39} & x_{3,10} & x_{3,11} & 0 & x_{3,13} & 0 & x_{3,15} \\
x_{41} & 0 & x_{43} & x_{44} & 0 & x_{46} & x_{47} & 0 & x_{49} & 0 & x_{4,11} & x_{4,12} & 0 & x_{4,14} & x_{4,15} \\
x_{51} & x_{52} & 0 & x_{54} & 0 & 0 & x_{57} & x_{58} & 0 & x_{5,10} & x_{5,11} & x_{5,12} & x_{5,13} & x_{5,14} & x_{5,15} \\
0 & x_{62} & 0 & x_{64} & x_{65} & x_{66} & x_{67} & 0 & x_{69} & x_{6,10} & x_{6,11} & 0 & x_{6,13} & x_{6,14} & x_{6,15} \\
0 & 0 & x_{73} & x_{74} & x_{75} & x_{76} & x_{77} & x_{78} & 0 & x_{7,10} & 0 & x_{7,12} & x_{7,13} & x_{7,14} & x_{7,15} \\
0 & x_{82} & x_{83} & 0 & x_{85} & 0 & x_{87} & x_{88} & x_{89} & 0 & x_{8,11} & x_{8,12} & x_{8,13} & x_{8,14} & x_{8,15}
\end{bmatrix}
\end{array}$$

Figure 5.8: The Perles configuration matroid $M_\star$ with non-bases shown as lines, and its symbolic slack matrix.

Chapter 6
COMPUTATIONAL ASPECTS

6.1 Introduction

In this chapter, we describe the `Macaulay2`[27] package `SlackIdeals.m2`, that is available at `https://sites.math.washington.edu/~awiebe/research.html`. It provides methods to define and manipulate slack matrices of polytopes, matroids, polyhedra, and cones; obtain a slack matrix directly from the Gale transform of a polytope; compute the symbolic slack matrix and the slack ideal from a slack matrix; compute the graphic ideal of a polytope and the cycle ideal and the universal ideal of a matroid. Moreover, it allows the user to speed up computation of the slack ideal by suitably setting to 1 as many entries of the slack matrix as possible. One can compute the slack ideal of this dehomogenized slack matrix and then homogenize the resulting ideal. The new ideal coincides with the original slack ideal if the latter is radical. Each of these functions will be illustrated with examples.

6.2 Slack matrices

Given the coordinates of the vertices of a polytope or the vectors of the ground set of a matroid, one can compute an associated slack matrix. When coordinates are given for the vectors of a matroid, they are always assumed to be an affine configuration which gets homogenized to form the matroid; in particular, this means that if $V = \mathrm{vert}(P)$, then the associated matroid is the matroid of the polytope P [7, Section 1.2].

```
i1 : needsPackage "SlackIdeals";
i2 : V = {{0,0},{0,1},{1,1},{1,0}};
-- Compute the slack matrix of P=conv(V)
i3 : slackMatrix(V)
o3 = | 0 1 0 1 |
```

```
     | 1 0 0 1 |
     | 0 1 1 0 |
     | 1 0 1 0 |
-- Compute the slack matrix of matroid of V
i4 : slackMatrix(V, Object=>"matroid")
o4 = | -1 -1 0 -1 0  0  |
     | -1 0  1 0  1  0  |
     | 0  1  1 0  0  -1 |
     | 0  0  0 -1 -1 -1 |
```

The `slackMatrix` command also takes a pre-computed matroid, polyhedron or cone object as input.

Another way to compute the slack matrix of a polytope is from its Gale transform. Let G be a matrix with real entries whose columns are the vectors of a Gale transform of a polytope P. A slack matrix of P is computed by finding the minimal positive circuits of G, see [28, Section 5.4].

```
-- Calculate slack matrix of triangular prism from its Gale transform,
   the column vectors of G
i5 : G = matrix(RR,{{0,1,-1,0,-1,1},{1,0,-1,-1,0,1}})
o5 = | 0 1 -1 0  -1 1 |
     | 1 0 -1 -1 0  1 |
i6 : slackFromGale(G)
-- warning: experimental computation over inexact field begun,
            results not reliable
o6 = | 1 0 0 1 0 |
     | 0 1 0 1 0 |
     | 0 0 1 1 0 |
     | 1 0 0 0 1 |
     | 0 1 0 0 1 |
     | 0 0 1 0 1 |
```

To approximate the real entries in the vectors of the Gale transform, the matrix G is transformed into the form `matrix(RR,...)`. Optionally one can choose a positive integer `Tolerance=>tol` to set entries less than 10^{-tol} to zero (the default value is `Tolerance=>14`).

Alternatively, the command `slackFromGalePlucker` applies the maps of Chapter 1 to fill a slack matrix with Plücker coordinates of the Gale transform.

```
-- Calculate a slack matrix from Plucker coordinates of Gale vectors G
   using spanning sets for each facet given in B
i7 : B = {{1,2,4},{0,2,3},{0,1,4},{3,4,5},{0,1,2}}
o7 = {{1, 2, 4}, {0, 2, 3}, {0, 1, 4}, {3, 4, 5}, {0, 1, 2}}
o7 : List
i8 : slackFromGalePlucker(G,B)
o8 = | 1 0  0 1 0  |
     | 0 -1 0 1 0  |
     | 0 0  1 1 0  |
     | 1 0  0 0 -1 |
     | 0 -1 0 0 -1 |
     | 0 0  1 0 -1 |
```

Remark 6.2.1. The `slackFromGalePlucker`, as well as `slackFromPlucker` commands don't check the orientation of the facet bases given in the argument B (see Example 1.1.14), so that slack matrices may differ by signs of each column.

The slack matrices of a few specific polytopes and matroids of theoretical importance are built-in. These are the slack matrices of Perles non-rational polytope (see Chapter 3), one of Barnette's spheres with a non-prescribable facet (see Chapter 2), a polytope with toric but not graphic slack ideal (see Chapter 3), a projectively unique polytope that is not of McMullen-type (see Chapter 4, the Fano and nonFano matroids, a matroid realizable over the complex numbers but not the reals , and the matroid of the Perles configuration (see Chapter 5).

```
i9 : specificSlackMatrix("barnette")
(dimension, 4)
o9 = | 1 0 0 0 1 1 0 |
     | 1 0 0 0 0 1 1 |
     | 1 0 0 1 0 0 1 |
     | 1 0 0 1 1 0 0 |
     | 1 0 1 0 0 0 0 |
```

```
| 0 1 0 0 1 1 0 |
| 0 1 0 0 0 1 1 |
| 0 1 0 1 0 0 1 |
| 0 1 0 1 1 0 0 |
| 0 1 1 0 0 0 0 |
```

6.3 Slack ideals

Given a (symbolic) slack matrix of a d-polytope, $(d+1)$-dimensional cone, or rank $d+1$ matroid, we can compute the associated slack ideal, specifying d as an input:

```
-- Compute slack ideal of d-polytope P=conv(V)
i10 : d = 2;
i11 : V = {{0,0},{0,1},{1,1},{1,0}};
i11 : slackIdeal(slackMatrix(V),d)
o11 = ideal(x x x x  - x x x x )
             1 4 6 7    2 3 5 8
```

We get the same result if we compute `slackIdeal(V,2)`, giving the only the list of vertices of a d-polytope or ground set vectors of a matroid instead of a slack matrix. As optional arguments, one can choose the object to be set as `"polytope"`, `"cone"`, or `"matroid"` (default is `Object=>"polytope"`), the field over which the polynomial ring is defined (default is `Field=>QQ`) and the name of the variables (which is useful for substituting the ideal into a pre-defined ring and where the default is `Vars=>{}` which creates new variables $x_1, \dots, x_t$). One may also use the optional argument `Saturate` where value `"all"` means that the saturation is computed with respect the product of all variables, and `"each"` means that it is computed with respect to each variable one at a time; various factors can affect which saturation method will be more computationally efficient.

Since slack ideals do not contain monomials, the simplest slack ideals are toric. In Chapter 3 we studied the relationship between projectively unique polytopes and toric slack ideals. For this, we introduced the vertex-facet non-incidence graph G_P of P and the corresponding toric ideal T_P of that graph. The command `graphFromSlack` takes a slack matrix of P

and returns the associated node-edge incidence matrix of G_P. The command `graphicIdeal` returns T_P given the vertices, or a slack matrix of P.

```
i12 : V = {{0,0},{1,0},{1,1},{0,1}};
i13 : graphFromSlack(symbolicSlack(V))
Graph computed from symbolic adjacency matrix: | 0   y_1 0   y_2 |
                                               | y_3 0   0   y_4 |
                                               | 0   y_5 y_6 0   |
                                               | y_7 0   y_8 0   |
o13 = | 1 1 0 0 0 0 0 0 |
      | 0 0 1 1 0 0 0 0 |
      | 0 0 0 0 1 1 0 0 |
      | 0 0 0 0 0 0 1 1 |
      | 0 0 1 0 0 0 1 0 |
      | 1 0 0 0 1 0 0 0 |
      | 0 0 0 0 0 1 0 1 |
      | 0 1 0 1 0 0 0 0 |
i14 : graphicIdeal(symbolicSlack(V))
Graph computed from symbolic adjacency matrix: | 0   y_1 0   y_2 |
                                               | y_3 0   0   y_4 |
                                               | 0   y_5 y_6 0   |
                                               | y_7 0   y_8 0   |
o14 = ideal(x x x x  - x x x x )
             1 4 6 7    2 3 5 8
```

Remark 6.3.1. The edges of the non-incidence graph (and hence the variables in T_P corresponding to them) are always assumed to be labeled in order by rows of the slack matrix.

More generally, for matroids, we define the *cycle ideal* (see Chapter 5.5) of the matroid, which is analogous to the graphic ideal, but allows for coefficients other than 1 in the binomial generators. Of course, if we start with a 0/1 slack matrix, the graphic and cycle ideals will coincide. The command `cycleIdeal` computes the cycle ideal given a slack matrix or realization of a matroid.

```
-- Compute cycle ideal of slack matrix S
o15 : S = matrix{{1,1,0},{0,1,2},{1,0,3}};
i16 : cycleIdeal(S)
```

```
Graph computed from symbolic adjacency matrix: | y_1 y_2 0   |
                                               | 0   y_3 y_4 |
                                               | y_5 0   y_6 |
o16 = ideal(3y y y  - 2y y y )
              2 4 5    1 3 6
```

As we have seen in Section 1.3 of Chapter 1, a slack matrix can be filled with Plücker coordinates of a matrix formed from the vertex coordinates (or extreme ray generators or ground set vectors). We also saw that the Grassmannian section ideal cuts out exactly a set of representatives of the slack variety that are constructed in this way. The command `grassmannSectionIdeal` computes this section ideal given a set of vertices of a polytope and the indices of vertices that span each facet. A list of these "facet bases" can be obtained from the vertices or a slack matrix using the `getFacetBases` command.

```
-- Get spanning sets for facets of P=conv(V) and compute corresponding
   section ideal
i17 : V = {{0,0},{1,0},{2,1},{1,2},{0,1}};
i18 : (VV,B) = getFacetBases V
Vertices have been reordered to
{{0, 0}, {1, 0}, {0, 1}, {2, 1}, {1, 2}}
o18 = ({{0, 0}, {1, 0}, {0, 1}, {2, 1}, {1, 2}},
       {{0, 2}, {0, 1}, {1, 3}, {2, 4}, {3, 4}})
o18 : Sequence
i19 : grassmannSectionIdeal(VV,B)
Order of vertices is
{{0, 0}, {1, 0}, {0, 1}, {2, 1}, {1, 2}}
o19 = ideal (p     p     - p     p     + p     p    , p     p     -
              1,2,4 0,3,4   0,2,4 1,3,4   0,1,4 2,3,4   1,2,3 0,3,4
      ---------------------------------------------------------------------
      p     p     + p     p    , p     p     - p     p     +
       0,2,3 1,3,4   0,1,3 2,3,4   1,2,3 0,2,4   0,2,3 1,2,4
      ---------------------------------------------------------------------
      p     p    , p     p     - p     p     + p     p    ,
       0,1,2 2,3,4   1,2,3 0,1,4   0,1,3 1,2,4   0,1,2 1,3,4
      ---------------------------------------------------------------------
      p     p     - p     p     + p     p    )
       0,2,3 0,1,4   0,1,3 0,2,4   0,1,2 0,3,4
```

6.4 On the dehomogenization of the slack ideal

Let P be a polytope and S_P its slack matrix. In Definition 3.3.2 we define the non-incidence graph G_P as the bipartite graph whose vertices are the vertices and facets of P, and whose edges are the vertex-facet pairs of P such that the vertex is not on the facet. This graphic structure provides a systematic way to scale a maximal number of entries in S_P to 1, as spelled out in Lemma 3.5.2. In particular, we may scale the rows and columns of $S_P(\boldsymbol{x})$ so that it has ones in the entries indexed by the edges in a maximal spanning forest of the graph G_P. This can be done using `setOnesForest`, which outputs the scaled symbolic slack matrix and the spanning forest used to scale it.

```
-- compute scaled slack matrix of P=conv(V) and forest used for scaling
i20 : V = {{0,0,0},{1,0,0},{0,1,0},{0,0,1},{1,0,1},{1,1,0}};
i21 : (T,F)=setOnesForest(symbolicSlack(V));
i22 : T
o22 = | 0 1 0   0    1 |
      | 1 0 0   0    1 |
      | 0 1 1   0    0 |
      | 1 0 x_8 0    0 |
      | 0 1 0   1    0 |
      | 1 0 0   x_12 0 |
i23 : F
o23 = Graph{edges => {{y , y }, {y , y }, {y , y }, {y , y }, {y , y },
                        2   7     4   7     6   7     1   8     3   8
                      {y , y }, {y , y }, {y , y  }, {y , y  }, {y , y  }}}
                        5   8     3   9     5   10    1   11    2   11
            ring => QQ[y , y , y , y , y , y , y , y , y , y  , y  ]
                        1   2   3   4   5   6   7   8   9   10   11
            vertices => {y , y , y , y , y , y , y , y , y , y  , y  }
                          1   2   3   4   5   6   7   8   9   10   11
o23 : Graph
```

This leads to a dehomogenized version of the slack ideal. Given S_P and a maximal spanning forest F of G_P, define a dehomogenized slack ideal I_P^F as follows. Let $S_P(\boldsymbol{x}^F)$ be the symbolic slack matrix of P with all the variables corresponding to edges in F set to 1.

Then I_P^F is slack ideal of this scaled slack matrix; that is,

$$I_P^F := \langle (d+2) - \text{minors of } S_P(\boldsymbol{x}^F)\rangle : \left(\prod \boldsymbol{x}^F\right)^\infty.$$

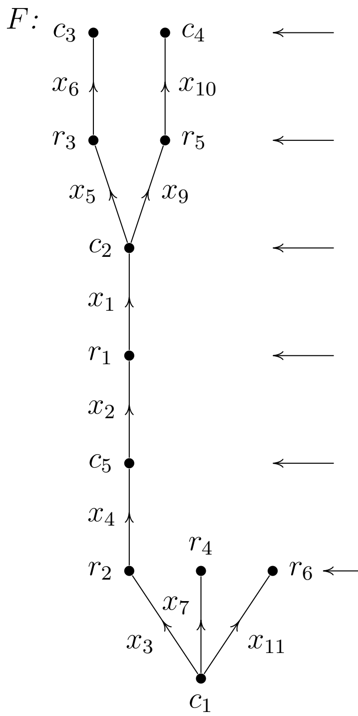
After calculating this scaled version of the slack ideal, it is natural to ask what is its relation to the original slack ideal. In particular, we might wish to know if we can recover the full slack ideal from I_P^F. From Lemma 3.5.2we know that any slack matrix in $\mathcal{V}(I_P)$ (or in fact any point in the slack variety with all coordinates that correspond to F being nonzero) can be scaled to a matrix in $\mathcal{V}(I_P^F)$, conversely it is clear that any point in $\mathcal{V}(I_P^F)$ can be thought of as a point in $\mathcal{V}(I_P)$. Thus in terms of the varieties we have $\mathcal{V}(I_P)^*/(\mathbb{R}^v \times \mathbb{R}^f) \cong \mathcal{V}(I_P^F)^*$, where $\mathcal{V}(I)^*$ denotes the part of the variety where all coordinates are nonzero.

To see the algebraic implications of this, let us introduce the following rehomogenization process. Recall in the proof of [20, Lemma 5.2], we dehomogenize by following the edges of forest F starting from some chosen root(s) and moving toward the leaves. The destination vertex of each edge tells us which row or column to scale, and the edge label is the variable by which we scale. Now given a polynomial in I_P^F, using the same forest and orientation we proceed in the reverse order: starting at the leaves, for each edge of the forest, we reintroduce the variable corresponding to it in order to rehomogenize the polynomial with respect to the row or column corresponding to the destination vertex of that edge.

Example 6.4.1. *Consider the slack matrix $S_P(\boldsymbol{x}^F)$ of the triangular prism P scaled according to forest F, pictured below. Then $I_P^F = \langle x_8 - 1, x_{12} - 1\rangle$. So we can rehomogenize, for example, the element $x_8 - x_{12}$ with respect to forest F as follows.*

$$S_P(\boldsymbol{x}^F) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & x_8 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & x_{12} & 0 \end{bmatrix}$$

First, consider the leaf corresponding to column 3. Its edge is labeled with x_6, so we reintroduce that variable to the monomial x_{12} since its degree in column 3 is currently 0*, while the* *degree of* x_8 *in that column is* 1*. We continue this process until all the edges of* F *have been used.*



$x_{10}x_8 - x_{12}x_6$, *both terms now have degree* 1 *in columns 3 and 4*

$x_5x_{10}x_8 - x_{12}x_6x_9$, *both terms now have degree* 1 *in rows 3 and 5*

$x_5x_{10}x_8 - x_{12}x_6x_9$, *both terms already have degree* 1 *in column 2*

$x_5x_{10}x_8 - x_{12}x_6x_9$, *both terms already have degree* 0 *in row 1*

$x_5x_{10}x_8 - x_{12}x_6x_9$, *both terms already have degree* 0 *in column 5*

$x_{11}x_5x_{10}x_8 - x_{12}x_6x_9x_7$, *both terms already have degree* 0 *in row 2*
both terms now have degree 0 *in rows 4 and 6*

Call the resulting ideal $H(I_P^F)$. By the tree structure, the rehomogenization process does indeed end with a polynomial that is homogeneous, as once we make it homogeneous for a row or column we never add variables in that row or column again. We now consider the effect of this rehomogenization on minors.

Lemma 6.4.2. *Let* p *be a minor of* $S_P(\boldsymbol{x})$ *and* p^F *its dehomogenization by* F*. Then its rehomogenization* $H(p^F)$ *equals* p *divided by the product of all variables in* F *that divide* p*.*

Proof. Note that all monomials in a minor have degree precisely one on every relevant row and column. In fact they can be interpreted as perfect matchings on the subgraph of G_P corresponding to the $(d+2)\times(d+2)$ submatrix being considered. Let $\boldsymbol{x}^{\boldsymbol{a}}$ and $\boldsymbol{x}^{\boldsymbol{b}}$ be two distinct monomials in the minor, then their dehomogenizations are also distinct. To see this, note that if we interpret $\boldsymbol{a}$ and $\boldsymbol{b}$ as matchings, a common dehomogenization would be a common submatching $\boldsymbol{c}$ of both, with all the remaining edges being in F. But $\boldsymbol{a}\setminus\boldsymbol{c}$ and $\boldsymbol{b}\setminus\boldsymbol{c}$

would then be distinct matchings on the same set of variables, hence their union contains a cycle, so they would not be both contained in the forest F.

Now note that when rehomogenizing a minor, we start with all degrees being zero or one for every row and column, and since we visit each node (corresponding to each of the rows/columns) exactly once by the tree structure, the degree of every row and column is at most one after homogenizing. In the first step of rehomogenizing, we start with a leaf of F, which means the variable $\boldsymbol{x}_i$ labeling its edge is the only variable in the row or column corresponding to that leaf which was set to 1. Thus if any monomial of the minor has degree zero on that row or column, it must be because $\boldsymbol{x}_i$ occurred in that monomial in the original minor. Hence rehomogenizing will just add that variable to the monomials where it originally was present, with the exception of the case where it was present on all monomials, in which case there will be no need to add it, as the dehomogenized polynomial would be homogeneous (of degree 0) for that particular row/column.

All degrees remain 0 or 1 after this process, and now the node incident to the leaf we just rehomogenized corresponds to a row/column with exactly one variable that is still dehomogenized. Thus we can repeat the argument on the entire forest to find that each monomial rehomogenizes to itself divided by the variables that were originally present in all monomials of the minor. □

Remark 6.4.3. It is important to note that $H(I_P^F)$ is the ideal of *all elements* of I_P^F rehomogenized. In general, this is different from the ideal generated by the rehomogenized generators of I_P^F.

For example, let V be the set of vertices of the triangular prism and let us compute the rehomogenized ideal $H(I_P^F)$.

```
i24 : V = {{0,0,0},{1,0,0},{0,1,0},{0,0,1},{1,0,1},{1,1,0}};
i25 : rehomIdeal(symbolicSlack(V),3)
o25 = ideal (x x x  x   - x x x x  , x x x  x   - x x x x  , x x x x  -
              5 8 10 11    6 7 9 12   1 4 10 11    2 3 9 12   1 4 6 7
      -----------------------------------------------------------------
```

```
x x x x )
 2 3 5 8
```

Example 6.4.4. *Recall that the generators of I_P^F for the triangular prism were $x_8 - 1$ and $x_{12} - 1$, which rehomogenize to $x_2x_3x_5x_8 - x_1x_4x_6x_7$ and $x_2x_3x_9x_{12} - x_1x_4x_{10}x_{11}$, respectively. However,*

$$\langle x_2x_3x_5x_8 - x_1x_4x_6x_7, x_2x_3x_9x_{12} - x_1x_4x_{10}x_{11}\rangle \neq H(I_P^F).$$

The relation between this rehomogenized ideal and the original slack ideal is given in the following lemma. The proof relies on the key fact about the rehomogenized ideal, which is that its variety is still the same as the slack variety that we started with.

Lemma 6.4.5. *Given a spanning forest F for the non-incidence graph of polytope P, the rehomogenization of its scaled slack ideal is an intermediate ideal between the slack ideal and its radical: $I_P \subseteq H(I_P^F) \subseteq \sqrt{I_P}$.*

Proof. To prove the inclusion $I_P \subseteq H(I_P^F)$ note that $p \in I_P$ happens if and only if $\boldsymbol{x}^{\boldsymbol{a}}p \in J$ for some exponent vector $\boldsymbol{a}$, where J is the ideal generated by all $(d+2)$-minors of the symbolic slack matrix of P. Dehomogenizing we get $\boldsymbol{x}^{\boldsymbol{b}}p^F \in J^F$. Which means p^F is in the saturation of J^F by the product of all variables, which is precisely the definition of I_P^F. That $p \in H(I_P^F)$ now follows from Lemma 6.4.2.

To prove that $H(I_P^F) \subseteq \sqrt{I_P}$ it is enough to show that any polynomial in $H(I_P^F)$ vanishes in the slack variety. By construction, any such polynomial must vanish on the points of the slack variety where the variables corresponding to the forest F are nonzero, $\mathcal{V}(I_P)\backslash\mathcal{V}(\boldsymbol{x}^F)$. But then they vanish on the Zariski closure of that set. Considering the following containments,

$$\mathcal{V}(I_P)\backslash\mathcal{V}(\boldsymbol{x}) \subset \mathcal{V}(I_P)\backslash\mathcal{V}(\boldsymbol{x}^F) \subset \mathcal{V}(I_P),$$

we get that this closure is exactly the slack variety since $\overline{\mathcal{V}(I_P)\backslash\mathcal{V}(\boldsymbol{x})} = \mathcal{V}(I_P : \langle\boldsymbol{x}\rangle^\infty) = \mathcal{V}(I_P)$. ∎

One would like to say that $I_P = H(I_P^F)$, and so far we have no counterexample for this equality, since it always holds if I_P is radical, and we also have no examples of non-radical slack ideals.

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