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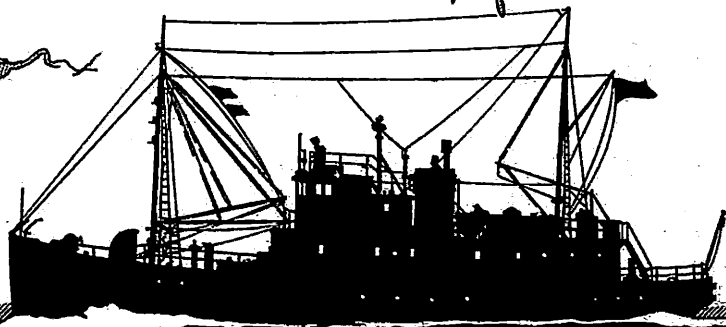
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Technical Report No. 82

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A Similarity Solution for Circulation in an Estuary¹

Maurice Rattray, Jr. and Donald V. Hansen

*Department of Oceanography
University of Washington
Seattle, Washington*

ABSTRACT

A set of partial differential equations and boundary conditions is written to describe flow and mixing in a simple class of estuaries. Two sets of similarity transformations are given which, with the appropriate boundary conditions, reduce the partial differential equations to a pair of simultaneous ordinary differential equations. One of these sets of transformations is used to obtain an approximate solution that is physically realistic.

Introduction. The relative importance of the various processes controlling the mean flow and mixing rates in two different types of estuaries has been determined by the investigations of Pritchard (1954, 1956) and of McAlister, *et al.* (1959). Their results provide a basis for the simplification of the differential equations to describe any particular estuarine circulation. Even in the simplest cases, the estuarine circulation must be described by a pair of simultaneous nonlinear partial differential equations. These equations have no known general solutions.

The close resemblance of the equations for estuarine circulation to those in thermal boundary-layer problems suggests that some of the techniques applied in that field may be equally useful here. In particular, a group of special solutions can be obtained when the form of the boundary conditions is appropriate. In such cases, ordinary differential equations are obtained from the partial differential equations by suitable transformations of variables. The solutions to these ordinary differential equations in the transformed variables are known as similarity solutions.

In this paper, similarity solutions are obtained to the equations which describe the circulation in an estuary under conditions where acceleration and

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vertical advective salt flux are negligible. These conditions are satisfied in a large variety of estuarine circulations. In particular, Pritchard has shown that they are applicable to circulation in coastal plain estuaries. In some situations, terms for these variables are also negligible in the equations which describe the circulation in fjord estuaries.

Formulation of the Problem. For mathematical convenience, the investigation is restricted to an estuary which has a rectangular cross-section of uniform width and variable depth. The estuary is considered to be straight and narrow so that lateral homogeneity will exist. The equations describing this flow are written in terms of a Cartesian coordinate system with the x axis horizontal, parallel to the longitudinal axis of the estuary, positive seaward, and with the z axis vertical, positive downward. Under the stated conditions, the equations which govern the estuarine circulation are

$$\rho u_0 \frac{\partial u_0}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z} \left(A \frac{\partial u}{\partial z} \right), \quad (1)$$

$$0 = -\frac{\partial p}{\partial z} + \rho g, \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (3)$$

$$u \frac{\partial S}{\partial x} = \frac{\partial}{\partial z} \left(K \frac{\partial S}{\partial z} \right), \quad (4)$$

where u and w are the two component velocities; u_0 the rms amplitude of tidal velocity; p the pressure; ρ the density of the water; g the acceleration due to gravity; S the salinity; and A and K the vertical eddy coefficients of viscosity and diffusivity, respectively.

An approximate relation between density and salinity, suitable for estuaries, is

$$\rho = \rho_0 + kS. \quad (5)$$

It is convenient to introduce a salinity defect, σ , defined by

$$\sigma = \rho_0 g \left(1 - \frac{S}{S_b} \right), \quad (6)$$

where S_b is a base salinity, taken to be that of the oceanic water. A stream function, ψ , is introduced, such that the velocities are given by

$$\left. \begin{aligned} u &= -\frac{\partial \psi}{\partial z} \\ w &= \frac{\partial \psi}{\partial x} \end{aligned} \right\} \quad (7)$$

Considering A and K constant and the left side of (1) independent of z , the equations describing the circulation become

$$A \psi_{zzzz} - \varepsilon \sigma_x = 0, \quad (8)$$

$$K \sigma_{zz} + \psi_x \sigma_x = 0, \quad (9)$$

where ε denotes the differential density ratio, $k \delta \rho / \rho_0$, a small but fundamental parameter of the density current.

Consideration is restricted to those estuarine reaches in which the net transport is but a small fraction of the total circulation. For this condition, an adequate approximation can be obtained by setting the net transport equal to zero. Boundary conditions on the velocity are taken to be: at the free surface, flow parallel to the surface and shearing stress equal to wind stress; at the bottom, velocity equal to zero. The salt flux through the free surface and the bottom must be zero. These boundary conditions are expressed by the following equations.

At the free surface, $z = 0$:

$$\psi(x, 0) = \sigma_z(x, 0) = 0, \quad (10)$$

$$A \psi_{zz}(x, 0) = \tau_w(x), \quad (11)$$

where $\tau_w(x)$ is the wind stress on the surface.

At the bottom, $z = D(x)$:

$$\psi(x, D) = \psi_x(x, D) = \sigma_z(x, D) = 0. \quad (12)$$

The Similarity Transformations. The procedure outlined by Hansen and Herzig (1956) is followed in obtaining special solutions in the form

$$\psi(x, z) = M(x) F(\eta), \quad (13)$$

$$\sigma(x, z) = N(x) G(\eta), \quad (14)$$

where

$$\eta = z L(x). \quad (15)$$

The functions of x may be chosen either as power or exponential functions as follows:

$$L(x) \propto x^\alpha, \quad M(x) \propto x^{\alpha+1}, \quad N(x) \propto x^{5\alpha+2}; \quad (16)$$

or

$$L(x) \propto e^{\alpha x}, \quad M(x) \propto e^{\alpha x}, \quad N(x) \propto e^{5\alpha x}. \quad (17)$$

In the present case, it is convenient to define a characteristic length, l_0 , such that the dimensionless variable, η , varies from zero at the surface to unity at the bottom of the estuary, a provision that specifies the depth of the estuary as proportional to $x^{-\alpha}$. Then if the functions of x are written as

$$L(x) = l_0^{-1} \xi^\alpha, \quad (18)$$

$$M(x) = K \xi^{\alpha+1}, \quad (19)$$

$$N(x) = AK l_0^{-3} \xi^{5\alpha+2}, \quad (20)$$

where $\xi = x/l_0$ is the dimensionless distance along the estuary, the differential equations, (8) and (9), become

$$F'''' - \varepsilon [(5\alpha + 2)G + \alpha\eta G'] = 0, \quad (21)$$

$$G'' + F' [(5\alpha + 2)G + \alpha\eta G'] = 0, \quad (22)$$

in which (22) will be replaced by the more convenient form

$$\varepsilon G'' + F' F'''' = 0, \quad (23)$$

where primes denote differentiation with respect to the independent variable, η . The boundary conditions are expressed as follows:

$$F(1) = F(0) = F'(1) = G'(0) = G'(1) = 0, \quad (24)$$

$$\frac{F''(0)}{\varepsilon} = \frac{l_0^2}{\varepsilon AK} \frac{\tau_w(x)}{\xi^{3\alpha+1}} = Q_1, \text{ a constant.} \quad (25)$$

The surface stress condition (25) has been taken to make the wind-circulation the same order in ε as the density-circulation. Thus a system is described in which the wind has a modifying effect on the basic density-circulation. The wind stress must then have an x -dependence given by

$$\tau_w(x) = \tau_0 \xi^{3\alpha+1}. \quad (26)$$

To specify the solution completely, it is necessary to have some measure of the rate of change of properties along the estuary. Such a measure is given by the change in surface salinity over the length of the estuarine reach. Or, equivalently, the condition,

$$G(0) = \frac{I_0^3}{AK} \frac{\sigma(x, 0)}{\xi^{5\alpha+2}} = Q_2, \text{ a constant,} \quad (27)$$

specifies a particular x -distribution for the surface salinity. From (15), (18), and (27), it is apparent that α must be less than or equal to -0.4 in order to satisfy, with increasing values of x , the realistic estuarine conditions of increasing depth and approach of the salinity to the oceanic salinity.

An Approximate Solution. An approximate solution to the problem is obtained by expanding F and G in power series in the small parameter ε ,

$$F = \varepsilon \sum_{n=0}^{\infty} \varepsilon^n F_n, \quad (28)$$

$$G = \sum_{n=0}^{\infty} \varepsilon^n G_n, \quad (29)$$

substituting these expressions into (21) and (23), and equating the coefficients of equal powers of ε . The zero and first order approximations are given by

$$G_0 = Q_2,$$

$$F_0 = \sum_{i=1}^4 A_i \frac{\eta^i}{i!},$$

$$G_1 = -(5\alpha+2)Q_2 \sum_{i=1}^4 A_i \frac{\eta^{i+1}}{(i+1)!},$$

$$F_1 = (5\alpha+2)Q_2 \sum_{j=1}^9 B_j \frac{\eta^j}{j!},$$

where

$$A_1 = -\frac{1}{4} \left[Q_1 - \frac{(5\alpha+2)Q_2}{12} \right],$$

$$A_2 = Q_1,$$

$$A_3 = -\frac{3}{2} \left[Q_1 + \frac{(5\alpha+2)Q_2}{4} \right],$$

$$A_4 = (5\alpha+2)Q_2,$$

and

$$B_2 = B_4 = B_5 = 0,$$

$$B_1 = \sum_{j=6}^9 \left(\frac{j-3}{2} \right) \frac{B_j}{j!},$$

$$B_3 = -3 \sum_{j=6}^9 (j-1) \frac{B_j}{j!},$$

$$B_{i+5} = [(i+6)\alpha + 2] A_i.$$

Higher order approximations can readily be obtained. They include terms in higher powers of η and have the effect of adding the small scale features to the vertical distributions of salinity and velocity. In the present model, with constant eddy coefficients, it is questionable that any meaning could be attached to these smaller scale features.

The general nature of the solution is shown by Figs. 1 through 4, where the values, $\alpha = -0.45$, $\varepsilon = 0.025$, have been used. These figures utilize only the zero order velocity, and the zero and first order salinity defect. An illustration of the effects of the next higher terms is shown in Figs. 5 and 6.

Fig. 1 shows the vertical velocity profile under conditions of zero wind stress. These curves show a surface outflow and a deeper inflow of water with a level of no motion at 0.42 of the depth to the bottom. The magnitudes of the velocities are proportional to the density gradients in the estuary and are inversely proportional to the eddy coefficient of viscosity.

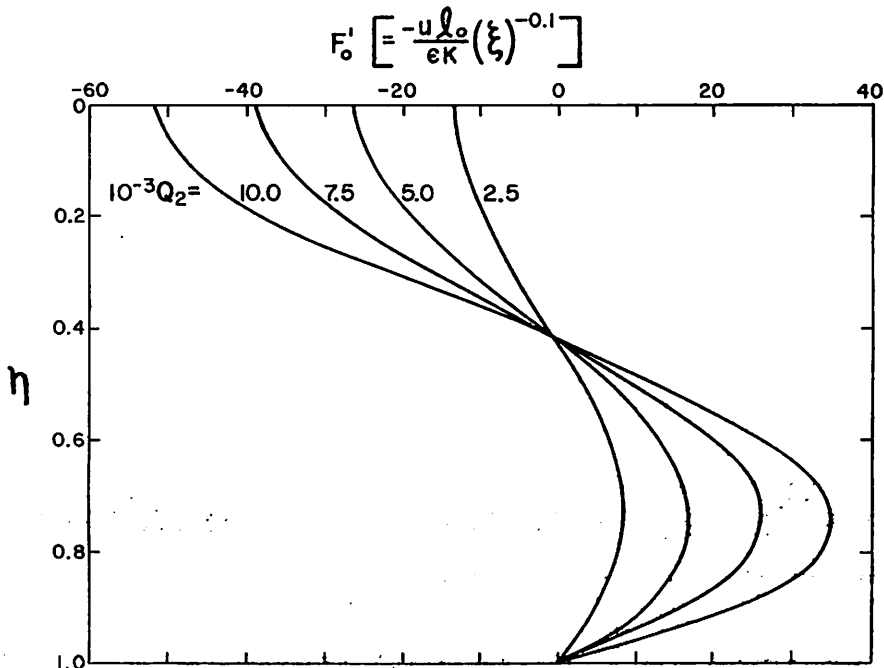


Figure 1. Variation of velocity profile with horizontal density gradient for zero wind stress.

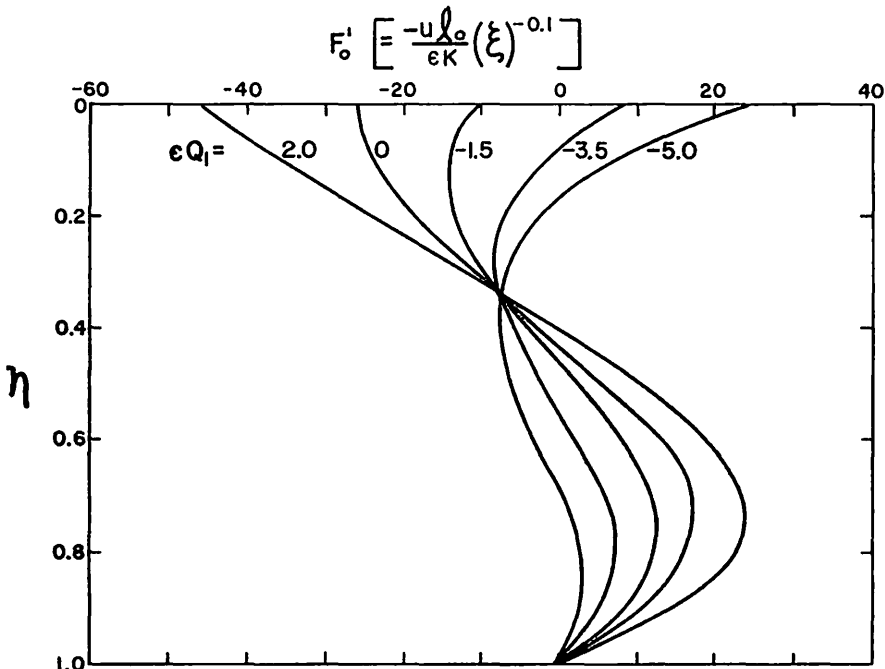


Figure 2. Variation of velocity profile with wind stress for $Q_1 = 5000$.

Fig. 2 shows the effect of wind stress when other parameters are kept constant. A positive wind stress increases the surface and bottom velocities and decreases the depth of no motion. A negative wind stress, acting in opposition to the density flow, will decrease the surface and bottom velocities and increase the depth of no motion. With a sufficiently large wind stress opposing the surface flow, a reversal of the surface flow occurs and a three-layer current regime results. In this case, there is an up-estuary flow at the surface and near the bottom, and a down-estuary flow at intermediate depths.

The vertical salinity profiles under the above conditions are shown in Figs. 3 and 4. For zero wind stress, the salinity profiles are all similar, the only distinctions being in the magnitude of the salinity differences. However, the wind stress has a considerable effect on the shape of the vertical salinity profile. Wind stress in the direction of surface outflow increases the surface to bottom salinity difference. An opposing wind stress acts to decrease the surface to bottom salinity difference; in fact, these curves indicate that a near-surface instability would result for wind stresses above a certain critical value. This result cannot be expected in nature because the assumption of constant eddy coefficients would certainly break down when such conditions were approached. However, this result does suggest a mechanism for the wind

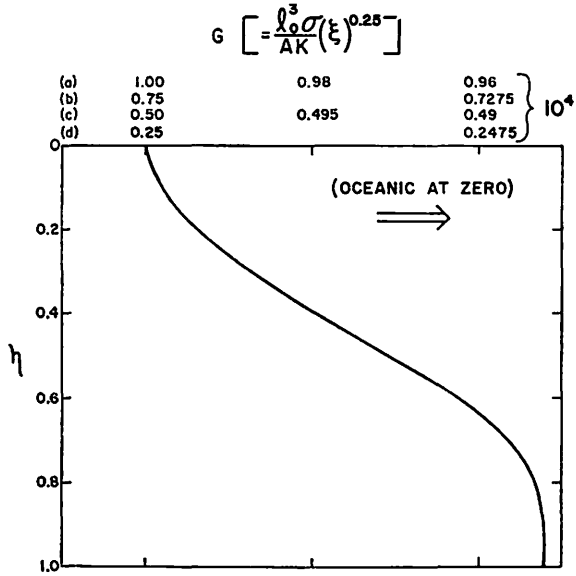


Figure 3. Vertical profiles of salinity defect for zero wind stress with four values of Q_2 : (a) 1.00×10^4 (b) 0.75×10^4 (c) 0.50×10^4 (d) 0.25×10^4 .

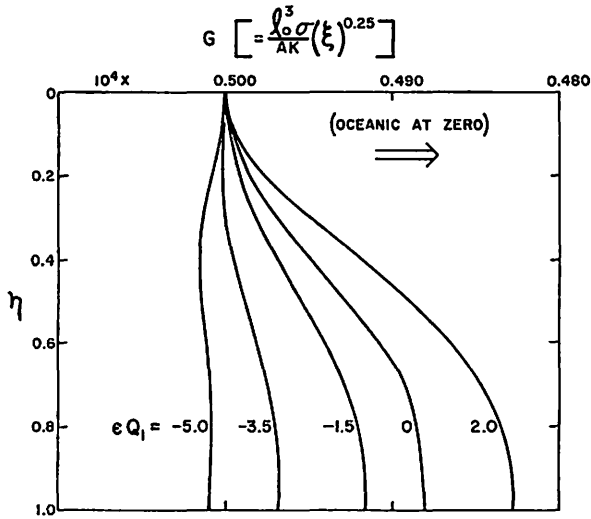


Figure 4. Variation of salinity defect profile with wind stress for $Q_2 = 5000$.

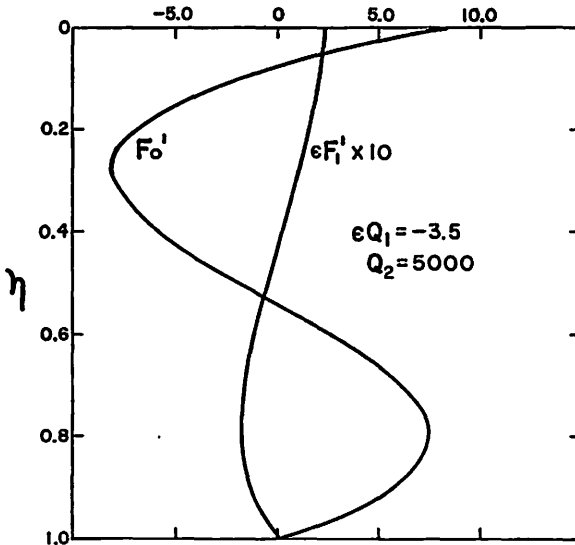


Figure 5. Comparative magnitudes, zero and first order velocity.

mixing of surface layers in an estuary; that is, differential advection in the surface layers will tend to develop a density instability in these layers, giving a much increased vertical coefficient of diffusivity. Photographs from a model study by Mortimer (1952; fig. 14) indicate an experimental realization of this mechanism.

Figs. 5 and 6 show that in this case higher order terms in ϵ are insignificant, and the zero and first order terms are sufficient to obtain a satisfactory solution.

Comparison with Field Observations. The most comprehensive data available for an estuarine system which obeys the simplified equations used in the preceding analysis are those obtained on the James River by Pritchard (1952, 1954, 1956). The ends of a 22-km section in this estuary are denoted by ξ_1 and ξ_2 . The following conditions are obtained from the published data (based on $S_b = 30 \text{ ‰}$):

$$\tau_w = 0, \quad (30)$$

$$\sigma(\xi_1, 0) = \frac{AKQ_2}{l_0^3} \xi_1^{5\alpha+2} = 720 \text{ gm cm}^{-2} \text{ sec}^{-2}, \quad (31)$$

$$\sigma(\xi_2, 0) = 430 \text{ gm cm}^{-2} \text{ sec}^{-2}, \quad (32)$$

$$l_0(\xi_2 - \xi_1) = 2.2 \times 10^6 \text{ cm}. \quad (33)$$

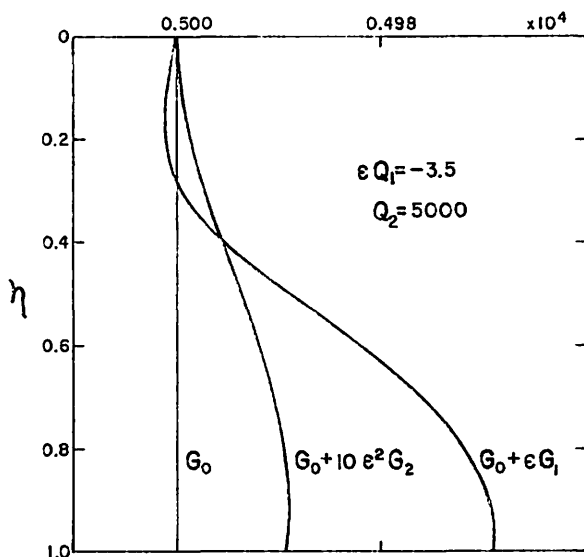


Figure 6. Comparative magnitudes, zero, first, and second order salinity defect.

The bottom profile of an estuary would normally indicate the value for α . In the case of the James River estuary, data on the bottom profile are not reported and, in any case, the cross-section shape of the estuary deviates sufficiently from the theoretical form that a determination of α from any mean profile would be rather doubtful. Thus, instead of determining α from the bottom profile, it is determined by matching the surface to bottom salinity difference and approximating the depth at one end of the section. Thus,

$$\frac{\varepsilon G_1(1)}{G_0(1)} = -0.15, \quad (34)$$

$$l_0 = 700 \xi_1 \alpha \text{ cm.} \quad (35)$$

With $K = 2 \text{ cm}^2/\text{sec}^{-1}$ and $A = 4 \text{ gm cm}^{-1} \text{ sec}^{-1}$ from Pritchard's data, and with $\varepsilon = 0.025$, the relation,

$$1.67 - (5\alpha + 2)^{-1} + 0.785 (5\alpha + 2)^{-1} = 1, \quad (36)$$

is obtained from (30)–(35). The only finite root of this equation is $\alpha = -0.53$, from which the other parameters take the values,

$$\xi_1 = 1.9 \times 10^7, \quad \xi_2 = 4.3 \times 10^7, \quad l_0 = 0.097 \text{ cm},$$

$$Q_1 = 0, \quad Q_2 = 4550.$$

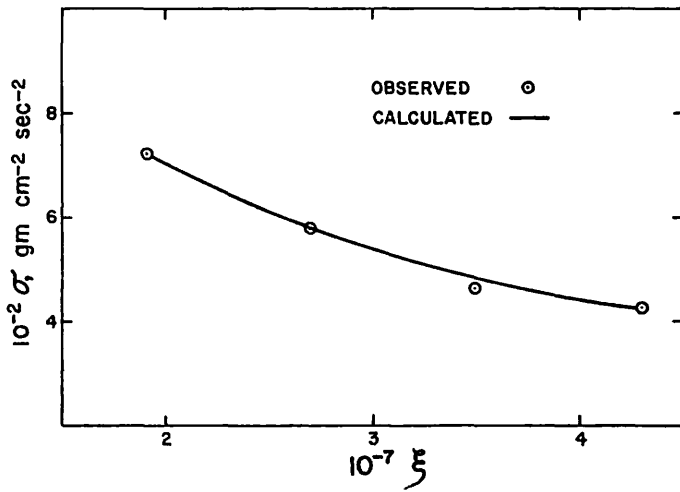


Figure 7. Longitudinal distribution of surface salinity defect in the James River estuary.

The representation of the salinity distribution given by these parameters is indicated by Figs. 7 and 8; Fig. 9 shows the horizontal velocity calculated from the theoretical formulas. For comparison, each figure also includes the observed values. In general, agreement is good, the main discrepancies appearing to be near the surface and the bottom in the velocity profile. These discrepancies can be easily explained by the crude approximations made in the

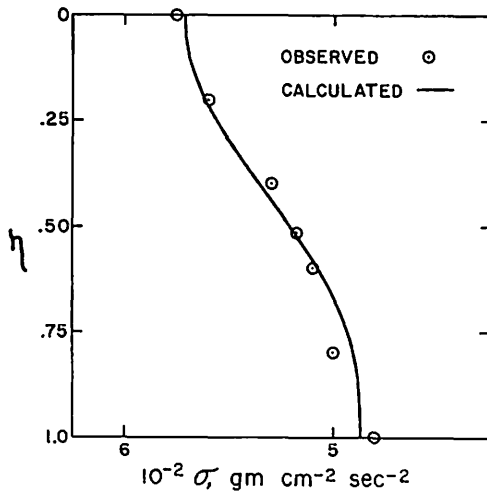


Figure 8. Vertical profile of salinity defect at station J-17, James River estuary.

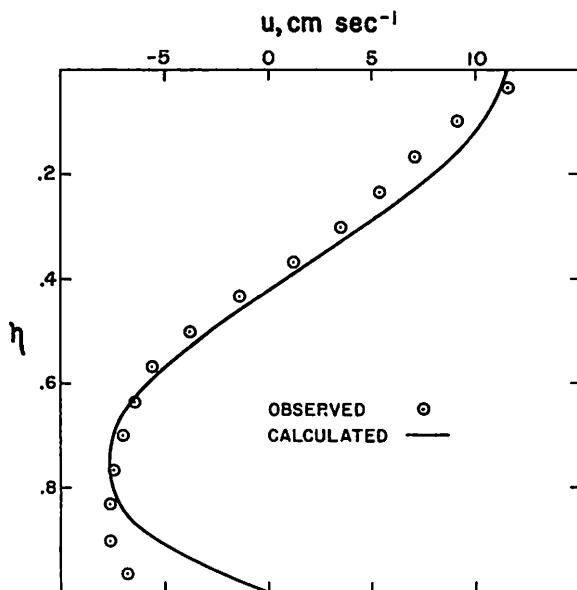


Figure 9. Mean velocity profile for station J-17, James River estuary.

theoretical model, where the shape of the channel is rectangular and the eddy coefficients are constant. There must also be some effect due to the net transport in the James River estuary. It is evident, however, that the mathematical model gives a good representation of the actual conditions in the estuary.

Conclusions. A method, which permits solution of the equations of estuarine circulation for a large variety of conditions which can occur in nature, has been applied to obtain the circulation in the simplest kinds of estuarine systems. For those systems in which the field accelerations and the vertical advective and horizontal diffusive fluxes are unimportant, the effects of horizontal density gradient and wind stress on the velocity and salinity distributions within the estuary have been determined. Calculations, using this method, have been shown to represent adequately the observed features in an estuary whose circulation is described by the same equations but whose geometry is rather different from that assumed in the theory. The results indicate that the distributions of density and velocity are not strongly affected by the geometry and that good results can be expected for estuarine circulations with a variety of cross-sectional shapes.

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