Stationary distribution for spinning reflecting diffusions

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A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2012

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Program Authorized to Offer Degree:

Mathematics

University of Washington

Abstract

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This dissertation studies two different types of interaction of diffusion processes with the boundary of a domain $D \subseteq \mathbb{R}^n$, which is assumed to be bounded, and of class $C^2(\mathbb{R}^n)$. The first process that is studied is obliquely reflected Brownian motion, and it is constructed as the unique Hunt process X properly associated with the following Dirichlet form:

$$\mathcal{E}(u,v) = \frac{1}{2}\int_D \nabla u \nabla(u\rho)dx + \frac{1}{2}\int_D \nabla u \cdot \vec{\tau}\ v\ \rho(x)\sigma(dx), \tag{1}$$

where $\vec{\tau} : \partial D \to \mathbb{R}^n$ is tangential to ∂D, and u, v belong to the Sobolev space $W^{1,2}(D)$. The reference measure $\rho(x)dx$ is assumed to be given by a harmonic function ρ whose gradient $\nabla\rho$ is uniformly bounded. It is shown that such process X admits a Skorohod decomposition

$$dX_t = dB_t + [\vec{n} + \vec{\tau}](X_t)dL_t. \tag{2}$$

Moreover, we show that the unique stationary distribution of X is the measure given by $\rho(x)dx$.

In the second part of the dissertation, we present a new reflection process X_t in a bounded domain D of class $C^2(\mathbb{R}^n)$ that behaves very much like oblique reflected Brownian motion, except that the directions of reflection depend on an external parameter S_t called spin. The spin is allowed to change only when the process X_t is on the boundary of D. The pair

(X, S) is called spinning Brownian motion and is found as the unique strong solution to the following stochastic differential equation:

$$\begin{cases} dX_t &= \sigma(X_t)dB_t + \vec{n}(X_t)dL_t + \vec{\tau}(X_t, S_t)dL_t \\ dS_t &= [\vec{g}(X_t) - S_t]\, dL_t \end{cases} \tag{3}$$

where L_t is the local time process of X_t, $\vec{n}$ is the interior unit normal to ∂D, and $\vec{\tau}$ is a vector field perpendicular to $\hat{n}$. The function $\sigma(\cdot)$ is a non-degenerate $(n \times n)$-matrix valued function, and $\vec{\tau}(\cdot)$ and $\vec{g}(\cdot)$ are Lipschitz bounded vector fields. We prove that a unique strong solution to (3) exists as the limit of a family of processes $(X^\varepsilon, S^\varepsilon)$ that satisfy an equation like (3), but in which the spin component dS has a noise εdW. With this added noise, the process $(X^\varepsilon, S^\varepsilon)$ is an obliquely reflected Brownian motion in an unbounded domain. It is also shown that spinning Brownian motion has a unique stationary distribution. The main tool of the proof is excursion theory, and an identification of the Local time of X_t as a component of an exist system for X_t.

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Many years ago, in the local school of a poor neighborhood in Santiago, a fifth grader came up to me and told me how happy he was to have finally met a mathematician. His compliment made me blush with embarrassment, but also make me feel sad. Honestly, if even now I could say I'm a mathematician, back then I wasn't even sure of what I wanted to do with my life career-wise. I figured, I was the first person attending college that the kid ever met.

I am thankful to that kid for helping me choose a path in life. For opening my eyes to the limited spectrum of educational opportunities that kids have in my country of Chile and to how often all of their expectations are never fulfilled, and replaced by a low key materialistic lifestyle instead. I decided, then, that I should work to fix that.

I want to thank my advisor, Krzystof Burdzy for never lowering his expectations of my work, and pushing me to expect great things of my research. He has been always kind, and I owe him for much of what I have learned in this past 5 years.

I will always be thankful of my parents Marcia and Marco, who brought me to this world, and were always inspiring me to stretch my thinking, experiment new things, and be different, above all. I learned about hard work, and dedication from you, but most importantly, about kindness and humbleness.

In my path of life I was lucky to find the best of all partners. Thank you Marita for all your love, for teaching me how to listen to my friends, and teaching me to concentrate my energy in one place. For always listening to me, and for freely choosing to make of our own paths a new single one for the future. I love you.

DEDICATION

to all of you who love me.

Chapter 1

INTRODUCTION

1.1 Motivation

Multidimensional reflection processes arose from queuing and storage theory, and were studied as early as in the sixties. Two main approaches have been developed in the literature, the submartingale problem and the Skorohod problem. Both approaches have their advantages with respect to one another, and they are rarely mixed in the study of a particular problem.

In this work, we present a new reflection process in a domain D that behaves very much like obliquely reflected Brownian motion (ORBM), except that the direction of reflection depends on an external parameter or spin, that changes only when the main process is on the boundary of D. Let $D \subseteq \mathbb{R}^n$ be a C^2 domain (an open connected set), and let B_t be a n-dimensional Brownian motion. A pair (X_t, S_t) with values in $\overline{D} \times \mathbb{R}^p$ is called **spinning Brownian motion (sBM)** if it solves the following stochastic differential equation

$$\begin{cases} dX_t &= \sigma(X_t)dB_t + \vec{n}(X_t)dL_t + \vec{\tau}(X_t, S_t)dL_t, \\ dS_t &= \left[\vec{g}(X_t) - S_t\right] dL_t \end{cases} \tag{4.1.1}$$

where L_t is a local time for X_t, and $\vec{n}$ is the interior unit normal to ∂D. The function $\sigma(\cdot)$ is a non-degenerated $n \times n$-matrix valued function, and $\vec{\tau}(\cdot)$ and $\vec{g}(\cdot)$ are well behaved vector valued functions on $\partial D \times \mathbb{R}^p$.

The process X_t behaves just like a Brownian diffusion inside D, and is reflected instantaneoously in the direction $\vec{\gamma} = \vec{n} + \vec{\tau}$ once it hits the boundary. The challenge is that said direction of reflection depends on an external parameter S_t which we call *spin*, which is updated every time the main process X_t hits the boundary of D.

This type of process arises naturally from a physical model that might be useful for applications: Consider a small ball that spins and moves around a planar strip following a Brownian path. On the boundary of the box, we put tiny wheels which rotate at different speeds, modifying the spin of the ball as well as pushing it in a certain (non-normal) direction. In this context, it is natural to think of the boundary wheel as an external forcing system that is not affected by the hit of the ball, namely, every wheel on the boundary rotates at a speed only dependent on its position.This 2-dimensional setting was our first inspiration to define spinning Brownian motion, and has been source of examples and questions presented in this dissertation.

1.2 Main results

The general setting of our work is presented next. The letter D represents a $C^2(\mathbb{R}^n)$ domain, that might be unbounded, and we will specify whenever boundedness is assumed. The matrix valued map $\sigma : \overline{D} \to M_{n\times n}$ is assumed to be strictly elliptic and bounded, that is, there are constants $\sigma_1, \sigma_2 > 0$ such that for any $x \in \overline{D}$, and $\xi \in \mathbb{R}^n$ we have $\sigma_1 |\xi|^2 \le \xi^T \sigma(x) \xi \le \sigma_2 |\xi|^2$. For points $x \in \partial D$, we denote by $\vec{n}(x)$ the interior normal at x. Also, $\vec{\tau} : \partial D \times \mathbb{R}^p \to \mathbb{R}^n$ is a Lipschitz vector field that satisfies the condition $\vec{n}(x) \cdot \vec{\tau}(x, s) = 0$ throughout the boundary ∂D. Sometimes $\vec{\tau}$ will not depend on s, which will be easy to deduce from the context. The last vector field we introduced is $\vec{g} : \partial D \to \mathbb{R}^p$, which will be assumed to be Lipschitz and bounded.

As a shorthand, we will use $\vec{\gamma}(x, s) = \vec{n}(x) + \vec{\tau}(x, s)$, and $\vec{\kappa}(x, s) = (\vec{\gamma}(x, s), \vec{g}(x) - s)^T$.

In the first part of this thesis, we prove strong existence and uniqueness for solutions to equation (4.1.1). Even though the domain is smooth enough to apply results from the classical theory of diffusion processes, the pair (X_t, S_t) is not generated by an elliptic differential operator, and thus the classical theory fails to produce the existence result we look for.

An alternative to the classical submartingale approach of Stroock and Varadhan [26] was introduced by Lions and Sznitman in Theorem 4.1 in [22], where existence of reflected diffusions driven by a general semimartingales was shown, but said result only holds for smooth, bounded domains. Although the result of Lions and Sznitman can be modified to work in more general domains, we take a mixed approach that combines elements from the submartingale and Skorokhod problems to obtain strong existence and uniqueness of spinning Brownian motion as a limit of obliquely reflected processes with elliptic generators. This is achieved by first showing strong existence and uniqueness of the following family of reflected diffusions

$$\begin{cases} dX_t^\varepsilon = \sigma(X_t^\varepsilon)dB_t + \vec{\gamma}(X_t^\varepsilon, S_t^\varepsilon)dL_t, \\ dS_t^\varepsilon = \varepsilon dW_t + \left[\vec{g}(X_t^\varepsilon) - S_t^\varepsilon\right]dL_t, \end{cases} \tag{4.2.1}$$

in Theorem 4.2.1. Since the pair (X_t, S_t) solving (4.2.1) can be seen to fit within the classical framework of the submartingale problem, the main part of this theorem is devoted to obtain the Itô decomposition and to prove pathwise uniqueness. We name this family of reflected diffusions the approximating processes. Notice that each member of this family is supported in the unbounded set $\overline{D} \times \mathbb{R}^p$.

Though our ultimate goal is to study the stationary distribution of spinning Brownian motion, we first prove uniqueness and existence of a stationary distribution for (4.2.1). Even though well known techniques such as the Lyapunov functions and estimates of hitting time distributions are broadly used to deduce existence of stationary distributions for processes in unbounded sets, a direct approach worked well in our setting. Namely, in Theorem 4.3.1 we show that, for fixed $\varepsilon > 0$, the family of occupation measures

$$\mu_{x,s}^\varepsilon(A) = \frac{1}{t}\int_0^t \mathbb{P}_{x,s}\left((X_u^\varepsilon, S_u^\varepsilon) \in A\right) du$$

is tight, and thus it admits at least a sub-sequential limit. This is achieved by estimating the second moment of both S_t and $\sup_{0\le u\le t}|S_u|$, through some analytical lemmas on the exponential decay of $\mathbb{E}e^{-L_t}$. Standard results from Markov and ergodic theories allow us to deduce uniqueness.

The next part of the thesis is devoted to show tightness of the family of approximating processes as $\varepsilon \to 0$, and to characterize its unique limit as a solution of equation (4.1.1). The main component of the proof is achieved in several Lemmas that show how to control the modulus of continuity of the local times associated to the family (see Lemma 4.4.1) by using repeatedly Itô's formula for some well chosen test functions. Some ideas in this part have been taken from [26], but they didn't have to deal with the supremum of increments as we need to do. Our precise results are the following.

Theorem 4.4.3 *Let $\mathbb{P}^{\varepsilon}_{x,s}$ denote the law of a solution $(X^{\varepsilon}, S^{\varepsilon})$ to (4.2.1), starting from (x, s). Then the family $\mathcal{P}_M = \left\{\mathbb{P}^{\varepsilon}_{x,s} : 0 < \varepsilon < 1,\ x \in \overline{D}, |s| \leq M\right\}$ is tight in the Skorohod space $\mathcal{D}\left([0,\infty), \mathbb{R}^{n+p}\right)$.*

Lemma 4.4.4 *Let $(Z^n_t = Z^{\varepsilon_n}_t)$ be any convergent subsequence of the family (indexed by ε) of strong solutions to (4.2.1), as $\varepsilon_n \to 0$, as in Theorem 4.4.3. Call the limit $Z = (X, S)$. Then, there exists a local time process L, such that for $Z = (X, S)$, we have that $(X, S; L)$ satisfy (4.1.1). Also, any solution to (4.1.1) is pathwise unique.*

Theorem 4.1.1 *Let $D \subseteq \mathbb{R}^n$ be a C^2 domain. Assume the coefficients $\sigma, \vec{\tau}, \vec{g}$ are of class C^2, uniformly Lipschitz, and bounded. Let B_t be a n-dimensional Brownian motion in $(\Omega, \mathcal{F}_t)$. Then, there exist a (strongly) unique, continuous, adapted processes (X, S) satisfying (4.1.1), with $X_t \in \overline{D}$.*

The last part of this thesis is devoted to uniqueness of the stationary distribution of spinning Brownian motion for (X, S), in a particular case of equation (4.1.1), and to some examples. Namely, we prove uniqueness when the equation for S_t is changed into

$$dS_t = \left[\vec{g}(X_t) - \alpha(X_t)S_t\right] dL_t,$$

and $\alpha : \partial D \to \mathbb{R}_+$ is a continuous function that is bounded below by a positive constant. The question of existence of a stationary measure is solved by the classical theory of Markov processes in compact sets, as it follows from Proposition 4.6.1, which identifies the spin process S_t as the unique process given by

$$S_t = Y_t^{-1} S_0 + Y_t^{-1} \int_0^t \frac{\vec{g}(X_u)}{\alpha(X_u)} dY_u, \tag{4.6.3}$$

where $Y_t = \exp \int_0^t \alpha(X_u) dL_u$. In particular, this implies that if S_0 is bounded, then S_t is uniformly bounded for $t > 0$.

To show uniqueness of the petitionary distribution, our inspiration comes from an analogous result for Brownian motion with inert drift, recently proved by Bass, Burdzy, Chen and Hairer in [2]. The main tool of the proof is excursion theory, and an identification of the local time of X_t as a component of an exit system of excursions from the boundary of D. This part of our work is presented in Section 4.5, which includes the following theorem.

Theorem 4.5.2 *Let $\mathbb{P}^D$ be the law of Brownian motion killed upon exiting D. Define*

$$\mathbf{H}_x = \lim_{\lambda \downarrow 0} \lambda^{-1} \mathbb{P}^D_{x+\lambda \hat{n}(x)} \tag{4.5.2}$$

and let L_t be the local time of (X, S). Then $\mathbf{H}_x$ is a sigma-finite measure, strongly Markovian, and $(L_t, \mathbf{H}_x)$ is an exit system from $\partial D \times \mathbb{R}^p$ for the process (X, S).

For a definition of exit system, see Section 4.5, or [24].

To prove uniqueness, our program splits into four steps. In the first one we use a support theorem and continuity results for the Skorohod map to show that for any given point $z \in D$, $T > 0$ and $\varepsilon > 0$, the probability of (X_T, S_T) to be in a ball of radius ε around the point $(z, 0)$ is positive, no matter what the initial position is. In the second step, we characterize the process of excursions from ∂D by means of an exit system and use excursion theory to show that a set of paths of X that has positive measure admits a decomposition into several excursions that are almost independent, and consequently we show how spinning Brownian motion up to the first hitting time of a ball $U \subseteq D$ can be obtained from sBM conditioned on never hitting U, and adding a suitable "last excursion" that hits U. This construction is then used in the third step to patch together a spinning Brownian motion from several independent spinning Brownian motions Y^j_t. In the final step, we show how to condition each of the Y^j's on hitting the boundary of D only at certain places and deduce from this that a component of the spin S_t has a density with respect to Lebesgue measure. The theorem then follows from standard results in ergodic theory.

Since S_t is bounded, it follows that the vector $\vec{\kappa}$ defined in the first paragraph of this section is bounded, and so the following theorem form the unpublished dissertation of Weiss [27] can be applied to our setting: Let $\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i}$ be a second order differential operator, where $a_{i,j}$ and b_i are bounded, Lipschitz functions. Assume that a bounded, Lipschitz vector field $\vec{\kappa}$ is given on the boundary of a $C^2(\mathbb{R}^d)$ domain G, such that $\vec{\kappa} \cdot \hat{n}(x) \geq \beta > 0$ for $x \in \partial G$. Let ϕ be a $C^2(\mathbb{R}^d)$ function defining the boundary of G.

Theorem 1.2.1 (Weiss). *Let $\overline{G}$ be compact in $\mathbb{R}^d$ and b_j and $\vec{\gamma}$ as before, suppose $(a_{i,j}(x))$ is bounded continuous positive semidefinite with $\nabla\phi(x)^T a(x)\nabla\phi(x) > 0$ for x in a neighborhood of ∂G (i.e. the diffusion has nonzero random component normal to the boundary). Suppose that μ is a probability measure on $\overline{G}$ with $\mu(\partial G) = 0$ and*

$$\int_G \mathcal{L}f(x)\mu(dx) \leq 0 \tag{2.6.3}$$

for all $f \in C_b^2(\overline{G})$ with $\nabla f \cdot \vec{\kappa}(x) \geq 0$ for $x \in \partial G$. Suppose that the submartingale problem for a, b and $\vec{\kappa}$ is uniquely solvable starting from any $x \in \overline{G}$. Then μ is an invariant measure of the diffusion.

Theorem 1.2.1 has been recently proved for non smooth domains, and general diffusions arising from a well-posed submartingale problem, by Kang and Ramanan [19].

Chapter 2

A SURVEY OF SOME USEFUL LITERATURE

In this chapter, we introduce the basic terminology we will use throughout this dissertation, and also provide some preliminary results that are used in our proofs.

2.1 Stochastic processes

A **stochastic process** is a collection of random variables $X = \{X_t : 0 \leq t < \infty\}$ on a measurable space $(\Omega, \mathcal{F})$ called the sample space, and taking values on a complete and separable metric space (E, d), called the state space. In this work we are interested in processes taking values in the the space $\mathbb{R}^n$, endowed with the Euclidean metric.

The index t admits a convenient interpretation a time. For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is called the **sample path** or trajectory of the process X associated with ω. The sample path, provides a mathematical model to study the evolution of some random experiment whose output can be observed continuously in time, for instance, the number of clients that get service at a restaurant, or the temperature in a fixed place observed and recorded over a period of time.

The stochastic process X is called measurable if the mapping $(t, \omega) \mapsto X_t(\omega)$ is measurable form the sigma algebra $\mathcal{B}([0, \infty)) \times \mathcal{F}$ to $\mathcal{B}(\mathbb{R}^n)$. Although measurability conditions are often technical, there is a nontechnical reason to include sigma algebras in the study of stochastic processes, and that is to keep track of information. Since the temporal feature of a stochastic process suggests evolution in time, it is important to define precisely what the concepts of past, present and future mean for stochastic processes, so that we can ask how much of the evolution an observer knows about the process' present, as compared to some other point in the past or the future. Our sample space $(\Omega, \mathcal{F})$ will be equipped with an increasing family of sub sigma algebras $\{\mathcal{F}_t : \ t \geq 0\}$ of $\mathcal{F}$, known as **filtration**: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s < t$, and all $t \geq 0$. For a stochastic process X, the simplest choice of filtration is that generated by the process itself:

$$\mathcal{F}_t^X = \sigma\left(X_s : 0 \leq s \leq t\right),$$

which is the smallest sigma algebra with respect to which X_s is measurable for every $s \in [0, t]$. Sets $A \in \mathcal{F}_t^X$ can be interpreted as events that an observer of X knows whether or not A has occurred by time t.

Given a filtration as above, for each $t \geq 0$ is it possible to define the sigma algebra of the immediate future $\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s$. We say that a filtration is right continuous if $\mathcal{F}_t = \mathcal{F}_{t^+}$. We will always assume that our filtrations are right continuous. Also, when a probability measure $\mathbb{P}$ is given on $(\Omega, \mathcal{F})$, we require that all the $\mathbb{P}$ null sets of $\mathcal{F}$ are contained in $\mathcal{F}$. When a given filtration does not satisfy this property for $\mathbb{P}$, we can add all the $\mathbb{P}$ null sets to each $\mathcal{F}_t$, and create another filtration that is called the **augmentation** of $\{\mathcal{F}_t\}$. In the future, we will always assume that our filtrations are augmented. Notice that this requirement is not the same as saying the $\mathcal{F}_0$ is complete, since some of the $\mathbb{P}$ null sets of $\mathcal{F}$ may not be in the completion of $\mathcal{F}_0$. We stress that all of our filtrations satisfy these two properties: they are right continuous and contain the $\mathbb{P}$ null sets. These two properties are often referred as the usual conditions is the literature.

The introduction of a filtration $\{\mathcal{F}_t\}$ allows us to strengthen the measurability requirements on a stochastic process. We say that the stochastic process X is **adapted** to the filtration $\{\mathcal{F}_t\}$ if, for each $t \geq 0$, X_t is an $\mathcal{F}_t$ measurable random variable.

When two processes X and Y are defined in a certain measure space $(\Omega, \mathcal{F}, \mathbb{P})$, we can ask how different X and Y are under the measure $\mathbb{P}$. We say that Y is a **modification** of X if, for every $t \geq 0$, we have $\mathbb{P}(X_t = Y_t) = 1$. In general, this property does not imply that the processes have the same trajectories. For example, if U is a random variable with uniform distribution in $[0, 1]$, then we can take $X_t \equiv 0$, and $Y_t = \mathbb{1}_{\{U\}}(t)$. Then Y is a modification of X, but their trajectories are not the same, since $\sup Y_t = 1$ and $\sup X_t = 0$. However, if we assume that X and Y are modifications of each other, and that both processes have a.s.

right continuous sample paths, then we can deduce that

$$\mathbb{P}(X_t = Y_t,\ \forall\, t \geq 0) = 1.$$

It also can be shown that right continuity of the paths of a process implies that a stochastic process X that is adapted to $\{\mathcal{F}_t\}$ has a modification Y such that the map $(t, \omega) \mapsto Y_t(\omega)$ is measurable from $\mathcal{B}([0, \infty)) \times \mathcal{F}_t$ to $\mathcal{B}(\mathbb{R}^n)$. This property is referred to as **progressively measurable**. Compare this property with our previous definition of measurable process. The proof of this fact can be found in [20, Proposition 1.1.13].

Our experience with different stochastic processes indicates that it is conceptually helpful to define a whole family of probabilities associated to a stochastic process X with values in E. Instead of our usual definition of X in a probability sample space, we will define a stochastic processes X, with values in E, in the space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \{\mathbb{P}_x\}_{x\in E})$, where $(\Omega, \mathcal{F})$ is a measurable space, $\{\mathcal{F}_t\}$ is a filtration to which the process is adapted, and $\mathbb{P}_x$ is a family of probability measures satisfying:

(i) for each $F \in \mathcal{F}$, the map $x \mapsto \mathbb{P}_x(F)$ is measurable,

(ii) $\mathbb{P}_x(X_0 = x) = 1$ for all $x \in E$.

Actually, these two properties do not need to hold for all $x \in E$ for a stochastic process to be a "nice" process. We need them to hold outside of a set of capacity zero [4], but we will avoid this technicality as much as possible in this dissertation. Whenever the introduction of capacities is needed, we will make it clear, and otherwise will assume that (i) and (ii) hold everywhere in E.

Stopping times

A mapping $T : \Omega \to [0, \infty]$, such that $\{T \leq t\} \in \mathcal{F}_t$, is called a **stopping time**. If T is a stopping time, we denote by $\mathcal{F}_T$ the collection of all sets $A \in \mathcal{F}$ such that $A \cap \{T \leq t\} \in \mathcal{F}_t$. Stopping times receive their name from a situation that can be exemplified in the following example: you are driving in Highway 101, and need to take the first exit after the toll station. We can model your position in time with a stochastic process, and whether the (random) time T of exiting the highway has occurred yet can be deduced from your current position. This means that $\{T \leq t\}$ is an event in the sigma algebra $\mathcal{F}_t^X$. On the other hand, if your instructions are to take the last exit before the toll station, you will have to get to the toll station and then back up to the last exit you passed. So the first time you drive by the exit you could have left the highway, but there was no way to know if this was the right exit: you needed some information about the future.

2.2 Convergence of processes

The Skorohod topology

Henceforth, we will consider trajectories of process in the space of functions $\omega : [0, \infty) \to \mathbb{R}^n$ that are right continuous with left limits, and it will be denoted by $\mathcal{D} = \mathcal{D}[0, \infty)$. The functions in $\mathcal{D}$ are often referred to as **càdlàg**, from the french *continue à droite, limite à gauche*. We endow this space with the topology given by the following metric:

$$d_S(f, g) = \inf_{\lambda \in \Lambda} \max \{\|\lambda - I\|, \|f - g \circ \lambda\|\},$$

where I is the identity map in $[0, \infty)$, and Λ is the collection of all bijections from $[0, \infty)$ to itself that are continuous and strictly increasing. This topology is known as the **Skorohod topology**.

The Borel sigma algebra $\mathcal{B}(\mathcal{D})$ associated to the Skorokhod topology, is the smallest sigma algebra containing the cylinder sets

$$\{\omega \in \mathcal{D} : \omega(t_1) \in A_1, \ldots, \omega(t_m) \in A_m\},$$

for $\mathcal{B}(\mathbb{R}^n)$ measurable sets $A_1, \ldots, A_m$. This property is essential when constructing probability measures, because it hints that it is enough to define a probability $\mathbb{P}$ in $\mathcal{D}$ by only defining the **finite dimensional distributions** $\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_m} \in A_m)$. Such construction can be made formal under some consistency conditions covered in the well known

Consistency theorem of Kolmogorov. We omit its detailed statement as we don't need in this dissertation, but the avid reader can find accessible expositions and proofs in [20, Section 2.2.2], or [3, Appendix II].

The Skorohod metric, does not make $\mathcal{D}$ into a complete space. Nonetheless, it can be shown (see [3]) that there is a metric d_S^* on $\mathcal{D}$ that generates the same topology as d_S, and such that $(\mathcal{D}, d_S^*)$ is a complete space. Since the Skorohod topology is also separable, we see that $\mathcal{D}$ is a Polish space for the Skorohod topology.

Since we will deal only with processes having continuous sample paths, it is worth noting that the space of continuous functions $C([0,\infty),\mathbb{R}^n)$ is a subspace of $\mathcal{D}$, and when the relative Skorohod topology is considered in $C([0,\infty),\mathbb{R}^n)$, we obtain the same topology as the one defined by the uniform norm, that is, $\|\omega\|_\infty = \sup_{t\geq 0}|\omega(t)|$. To simplify notation, we will often write $C[0,\infty)$ when the image space is $\mathbb{R}^n$ or can be easily deduced from the context.

Convergence of processes in $C([0,\infty))$

Only processes with continuous sample paths are studied in this dissertation. The extensive machinery of convergence in the Skorohod space $\mathcal{D}$ can be notably reduced in the continuous case, and so we will present results of convergence in $C([0,\infty),\mathbb{R}^n))$ rather than in the general case of the Skorohod topology in $\mathcal{D}$. Most of the definitions and results presented in this section are taken from the second chapter of *Convergence of probability measures* by Patrick Billingsley [3] and the second chapter of *Brownian motion and stochastic calculus* by Ioannis Karatzas and Steve Shreve [20]. We start with three definitions of fundamental importance in probability theory.

Let (E,d) be a metric space with Borel sigma algebra $\mathcal{B}(E)$. Let $\{\mathbb{P}_n\}$ be a sequence of probability measures on $(E,\mathcal{B}(E))$, and let $\mathbb{P}$ be another measure on this space. We say that $\{\mathbb{P}_n\}$ **converges weakly** to $\mathbb{P}$, if and only if

$$\lim_{n\to\infty}\int_E f(x)\mathbb{P}_n(dx) = \int_E f(x)\mathbb{P}(dx),$$

for every bounded, continuous real-valued function f on E. It follows that the weak limit $\mathbb{P}$

is a probability measure and that it is unique.

Let $\{(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)\}$ be a sequence of probability spaces, and on each of them consider a random variable X_n taking values in a metric space (E, d). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be another probability space, on which a random variable X taking values in (E, d) is given. We say that $\{X_n\}$ **converges to** X **in distribution**, and write $X_n \overset{d}{\to} X$, if the sequence of measures $\left\{\mathbb{P}_n X_n^{-1}\right\}$ converges weakly to the measure PX^{-1}, that is,

$$\lim_{n\to\infty} \mathbb{E}_n f(X_n) = \mathbb{E} f(X),$$

for every bounded, continuous real-valued function f on E, where $\mathbb{E}_n$ and $\mathbb{E}$ denote expectations with respect to $\mathbb{P}_n$ and $\mathbb{P}$ respectively.

Let Π be a family of probability measures on $(E, \mathcal{B}(E))$. We say that Π is **relatively compact** if every sequence of elements in Π contains a weakly convergent subsequence. We hay that Π is **tight** if for ever $\varepsilon > 0$, there exists a compact set $K \subseteq E$ such that $\mathbb{P}(K) \geq 1 - \varepsilon$, for every $\mathbb{P} \in \Pi$.

Theorem 2.2.1 (Prohorov)**.** *Let Π be a family of probability measures on a Polish space E. This family is relatively compact if and only if it is tight.*

Since we are interested in convergence of processes in the Polish space $C[0, \infty)$, we need a characterization of tightness that is easy to work with. To this end, the **modulus of continuity** on $[0, T]$ is defined as

$$\mathrm{mc}_T(\omega, \delta) = \max_{\substack{|s-t|<\delta \\ 0\leq s,t\leq T}} |\omega(t) - \omega(s)|\,,$$

for $0 < \delta \leq T$ and $\omega \in C[0, \infty)$. It is not hard to verify that the modulus of continuity defines a continuous function in $C[0, \infty)$ for each fixed $\delta > 0$, and that for fixed $\omega \in C[0, \infty)$ the modulus of continuity is increasing in δ, and $\lim_{\delta\downarrow 0} \mathrm{mc_T}(\omega, \delta) = 0$.

Theorem 2.2.2 (2.4.10 in [20], 2.8.2 in [3])**.** *The sequence $\{\mathbb{P}_n\}$ of probability measures on $C[0, \infty)$ is tight if and only if these two conditions hold for each $T > 0$:*

(i) For each positive η, there exists a λ such that

$$\mathbb{P}_n\left(\omega : |\omega(0)| > \lambda\right) \le \eta, \qquad n \ge 1. \tag{2.2.1}$$

(ii) For each positive ε and η, there exist $\delta \in (0,1)$, and an integer n_0 such that

$$\sup_{n \ge n_0} \mathbb{P}_n\left(\omega : \mathrm{mc_T}(\omega, \delta) \ge \varepsilon\right) \le \eta. \tag{2.2.2}$$

2.3 Markov processes

Let X be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x)$, and taking values in a Polish space E. Recall that the filtration $\{\mathcal{F}_t\}$ is assumed to be augmented and right continuous.

We say that X is a **Markov process** if for each $x \in E$

$$\mathbb{P}_x\left(X_{t+s} \in A|\mathcal{F}_t\right) = \mathbb{P}_x\left(X_{t+s} \in A|X_t\right), \tag{2.3.1}$$

for all $s, t \ge 0$ and $A \in \mathcal{B}(E)$. If $\mathcal{G}_t$ is a filtration with $\mathcal{F}_t \subseteq \mathcal{G}_t$, $t \ge 0$, then X is a Markov process with respect to $\{\mathcal{G}_t\}$ if (2.3.1) holds with $\mathcal{G}_t$ instead of $\mathcal{F}_t$. Equation (2.3.1) implies that $\mathbb{E}\left(f(X_{t+s})|\mathcal{F}_t\right) = \mathbb{E}\left(f(X_{t+s})|X_t\right)$ for every bounded, measurable function f on E.

A function $P(t, x, A)$ defined on $[o, \infty) \times E \times \mathcal{B}(E)$ is a time homogeneous **transition function** if (i) $P(t, x, \cdot)$ is a probablity measure on E for all $t \ge 0$ and $x \in E$; (ii) $P(0, x, \cdot =$ δ_x (the unit mass at $x \in E$); (iii) $P(\cdot, \cdot, A)$ is Borel measurable for each $A \in \mathcal{B}(E)$, and the Chapman-Kolmogorov equation

$$P(t+s, x, A) = \int P(s, y, A) P(t, x, dy) \tag{2.3.2}$$

is satisfied for $s, t \ge 0$, $x \in E$, and $A \in \mathcal{B}(E)$. A transition function $P(t, x, A)$ is a **transition function for a time-homogeneous Markov process** X if

$$P(X_{t+s} \in A|\mathcal{F}_t) = P(s, X_t, A)$$

for all $s, t \ge 0$ and $A \in \mathcal{B}(E)$.

Given a probability measure μ on $(E, \mathcal{B}(E))$, we define the probability measure $\mathbb{P}_\mu$ by

$$\mathbb{P}_\mu(F) = \int_E \mathbb{P}_x(F)\mu(dx),$$

where $B \in \mathcal{F}$. We denote by $\mathbb{E}_x$, and $\mathbb{E}_\mu$ the expectations corresponding respectively to $\mathbb{P}_x$ and $\mathbb{P}_\mu$. In this case, μ is called the **initial distribution** of X.

Theorem 2.3.1 (1.1 in [14])**.** *Let $P(t, x, A)$ be a time homogenous transition function and let ν be a probability measure on a polish space E, then there exists a Markov process X in E whose finite-dimensional distribution are uniquely determined by*

$$\begin{aligned} &\mathbb{P}\left(X_0 \in A_0, X_{t_1} \in A_1, \dots, X_{t_m} \in A_m\right) = \\ &\quad \int_{A_0} \cdots \int_{A_{m-1}} P(t_m - t_{m-1}, y_{m-1}, A_m) P(t_{m-1} - t_{m-2}, y_{m-2}, dy_{m-1}) \cdots P(t_1, y_0, dy_1)\nu(dy_0). \end{aligned}$$

Roughly speaking the Markov property (2.3.1), says that the future of the process X depends on the past only upon the present state of the process. In this sense, a Markov process is usually called a process "without memory" and this characteristic has been useful to model many phenomena with random evolution for which the "loss of memory" is observed in experiments. Often, though, a stronger Markovian property is needed in probability theory.

A progressively measurable process X defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x)$, with values on a Polish space $(E, \mathcal{B}(E))$ is said to be **strong Markov** if for $x \in E$, $t \geq 0$, $A \in \mathcal{B}(E)$, and any stopping time T of $\{\mathcal{F}_t\}$,

$$\mathbb{P}_x\left(X_{T+s} \in A | \mathcal{F}_T\right) = \mathbb{P}_x\left(X_{T+s} \in A | X_T\right), \tag{2.3.3}$$

$\mathbb{P}_x$ a.s. on $\{T < \infty\}$.

Given $\omega \in \Omega$ and $s > 0$, the map $t \mapsto X_t(\omega)$ is a measurable mapping from $(\Omega, \mathcal{F})$ into $(\mathcal{E}, \mathcal{B}(E))$. We can construct a family of **shift operators** $\theta_s : \Omega \to \Omega$, $s \geq 0$, such that each θ_s is measurable from $\mathcal{F}$ to $\mathcal{F}$ and

$$X_{s+t}(\omega) = X_t(\theta_s \omega) \qquad \forall\ \omega \in \Omega,\ \ s, t \geq 0.$$

An obvious example occurs when $\Omega = C[0, \infty)$ is the Skorohod space, and X is the coordinate mapping process $X_t(\omega) = \omega(t)$. We can define $\theta_s(\omega) = \omega(s + \cdot)$, i.e., $(\theta_s(\omega))(t) = \omega(t + s)$,

$t \geq 0$. The strong Markov property can be reformulated using the shift operators: for any bounded, measurable function f, and a $\{\mathcal{F}_t\}$ stopping time T,

$$\mathbb{E}_x\left[f(X_s \circ \theta_T)|\mathcal{F}_T\right] = \mathbb{E}_x\left[f(X_t)|X_T\right] = \mathbb{E}_{X_T}\left[f(X_t)\right]. \tag{2.3.4}$$

Semigroup associated to a Markov process

In this section, we assume that the state space E is a locally compact Polish space. The Markov property of a Markov process entails the following construction: for $t \geq 0$, and $x \in E$, and any bounded and measurable function f on E, define $P(t)f = \mathbb{E}_x f(X_t)$. By the Markov property, it is direct to see that

$$P(t+s)f(x) = P(t)P(s)f(x), \qquad \forall s,t \geq 0,\ x \in E.$$

Such collection of operators $\{P(t)\}$ is called the **semigroup** associated with X.

The semigroup is said to be a **contraction** when $\|P(t)f\|_\infty \leq \|f\|_\infty$ for every bounded, measurable function f, and it is called **strongly continuous** on a set of functions H is for every $f \in H$ we have $\lim_{t\to 0} \|P(t)f - f\|_\infty = 0$. The semigroup is said to be positive if $P(t)f \geq 0$ a.s. when $f \geq 0$ a.s., for all $t \geq 0$. Finally, the semigroup is said to be conservative if $P(t)\mathbb{1}_E = \mathbb{1}_E$ a.s for all $t \geq 0$.

We next introduce a class of functions. Recall that (E,d) is assumed to be a locally compact Polish space. The class of functions $C_0(E)$ is the collection of functions $f : E \to \mathbb{R}$ which are continuous and for which $d(x,0) \to \infty$ implies $f(x) \to 0$. The norm on $C_0(E)$ is given by $\|f\|_\infty = \sup_{x\in E} |f(x)|$.

The semigroup $\{P(t)\}$ linear, positive, conservative contraction operators is said to be a **Feller semigroup** if, for each $f \in C_0(E)$ and $x \in E$, we have $P(t)f \in C_0(E)$ and $\lim_{t\to 0} P(t)f(x) = f(x)$. The Feller property of the semigroup of a Markov process X reflects in the behavior of the trajectories of X. The fact that $P(t)$ maps C_0 into itself says that the dynamics are a smooth function of the initial state. This is similar as the continuous dependence on the initial state of the solution to a well-behaved differential equation. The second property, imposes some smoothness on the trajectories, without actually requiring that the paths are continuous, but just càdlàg. Some jumps, then, are allowed for Feller processes. The next section introduces the most important example of a process that is continuous, strong Markov and Feller.

2.4 Martingales and Brownian motion

An $\mathbb{R}^n$ valued stochastic process $M = \{M_t\}$ with $\mathbb{E}[|M_t|] < \infty$ for all $t \geq 0$, and adapted to a filtration $\{\mathcal{F}_t\}$ is a **martingale** with respect to $\{\mathcal{F}_t\}$ if

$$\mathbb{E}\left(M_{t+s}|\mathcal{F}_t\right) = M_s, \qquad t, s \geq 0. \tag{2.4.1}$$

The process X is called a **submartingale** (**supermartingale**) if the equal sign in (2.4.1) is replaced by $\geq$ ($\leq$). Notice that M is a martingale if and only if both M and $-M$ are submartingales, and that M is a supermartingale if and only if $-M$ is a submartingale. Hence, many results valid for submartingales yield analogous results for supermartingales and martingales. The following theorem is a common example of this property.

Theorem 2.4.1. *Let X be a non-negative right continuous submartingale. Then for $p > 1$ and $T > 0$,*

$$\mathbb{E}\left(\sup_{t\leq T} X_t{}^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left(X_T{}^p\right).$$

Given a martingale M, Jensen's inequality yields that $|M|$ is a real valued, non-negative submartingale, and thus the previous theorem applies to $X = |M|$, yielding Doob's inequality for martingales. For a martingale M, the process $M_t^* = \sup_{s\leq t} |M_s|$ is very important when proving with tightness of a family of martingales, and it will be a recurrent element in this dissertation. We next introduce the concept of quadratic variation, in order to exhibit a two sided inequality M^*, somewhat extending Theorem 2.4.1.

Let X be a stochastic process defined in $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\mathbb{R}^n$. The **quadratic variation** of X, denoted by $\langle X\rangle$, is the stochastic process defined by

$$\langle X\rangle_t = \lim_{|\Pi|\to 0} \sum_{j=1}^{m_\Pi - 1} \left|X_{t_{j+1}} - X_{t_j}\right|^2, \tag{2.4.2}$$

where $\Pi : 0 = t_0 < t_1 < \ldots < t_{m_\Pi} = t$ is a partition of $[0, t]$, and $|\Pi| = \sup |t_{j+1} - t_j|$. The limit, provided it exists, is taken in the sense of probability. More generally, the **quadratic covariation** or **cross quadratic variation** between stochastic processes X and Y is defined as

$$\langle X, Y \rangle_t = \frac{1}{2} \left(\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t \right).$$

Let X be a martingale defined in $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We say that X is continuous if $\mathbb{P}$-almost every path of X is continuous, and that X is square-integrable if $\mathbb{E}\left(|X_t|^2\right) < \infty$ for every $t \geq 0$. By Jensen's inequity, the process X^2 is a non-negative submartingale, and it can be shown (by using the Doob-Meyer decomposition) that the following unique decomposition holds:

$$X_t^2 = X_0^2 + M_t + A_t, \qquad 0 \leq t < \infty,$$

where M is a continuous martingale with respect to $\mathcal{F}_t$, and $A = \{A_t\}$ is a continuous, increasing process, adapted to $\mathcal{F}_t$. We normalize these processes so that $M_0 = A_0 = 0$.

A connection between all the objects defined in this section is given by the following theorem.

Theorem 2.4.2. *Let X be a square-integrable, continuous martingale with respect to $\{\mathcal{F}_t\}$. The quadratic variation $\langle X \rangle$ is a continuous, increasing process with $\langle X \rangle_0 = -0$ a.s., and $X^2 - \langle X \rangle$ is a continuous martingale with respect to $\{\mathcal{F}_t\}$.*

This theorem says that the continuous increasing process A appearing in the Doob-Meyer decomposition of X^2 is exactly the quadratic variation $\langle X \rangle$. By using the quadratic variation, we can obtain a double sided inequality, similar to that of Theorem 2.4.1.

Theorem 2.4.3 (Burkholder-Davis-Gundy inequality)**.** *Let M be a continuous, square integrable martingale with respect to $\{\mathcal{F}_t\}$. For every $p > 0$ there exists universal, positive constants c_p and C_p (depending only on p) such that the inequalities*

$$c_m \mathbb{E}\left(\langle M \rangle_T^p\right) \leq \mathbb{E}\left((M_T^*)^{2p}\right) \leq C_m \mathbb{E}\left[\langle M \rangle_T^p\right], \tag{2.4.3}$$

hold for any stopping time T with respect to $\{\mathcal{F}_t\}$.

Brownian motion

A process $B = \{B_t,\ t \geq 0\}$ with values in $\mathbb{R}^n$ is said to be an n-dimensional $\{\mathcal{F}_t\}$ Brownian motion if:

(i) $B_0 = 0$ a.s.

(ii) B is adapted to the filtration $\{\mathcal{F}_t\}$, and $\mathcal{F}_t$ is independent of $\sigma\left(W_u - W_t : u \geq t\right)$ for each $t \geq 0$.

(iii) $W_t - W_s$ is distributed as a multidimensional Gaussian random variable with mean 0 and covariance matrix $(t-s)I_n$, were I_n is the $(n \times n)$ identity matrix, for every $t > s \geq 0$.

(iv) W has sample paths in $C\left([0,\infty), \mathbb{R}^n\right)$.

The first problem one encounters with Brownian motion is its existence. One approach to this question is to write down what the finite dimensional distributions of this process must be, and then construct a probability measure and a process on an appropriate measurable space in such a way that we obtain the prescribed finite dimensional distributions. This

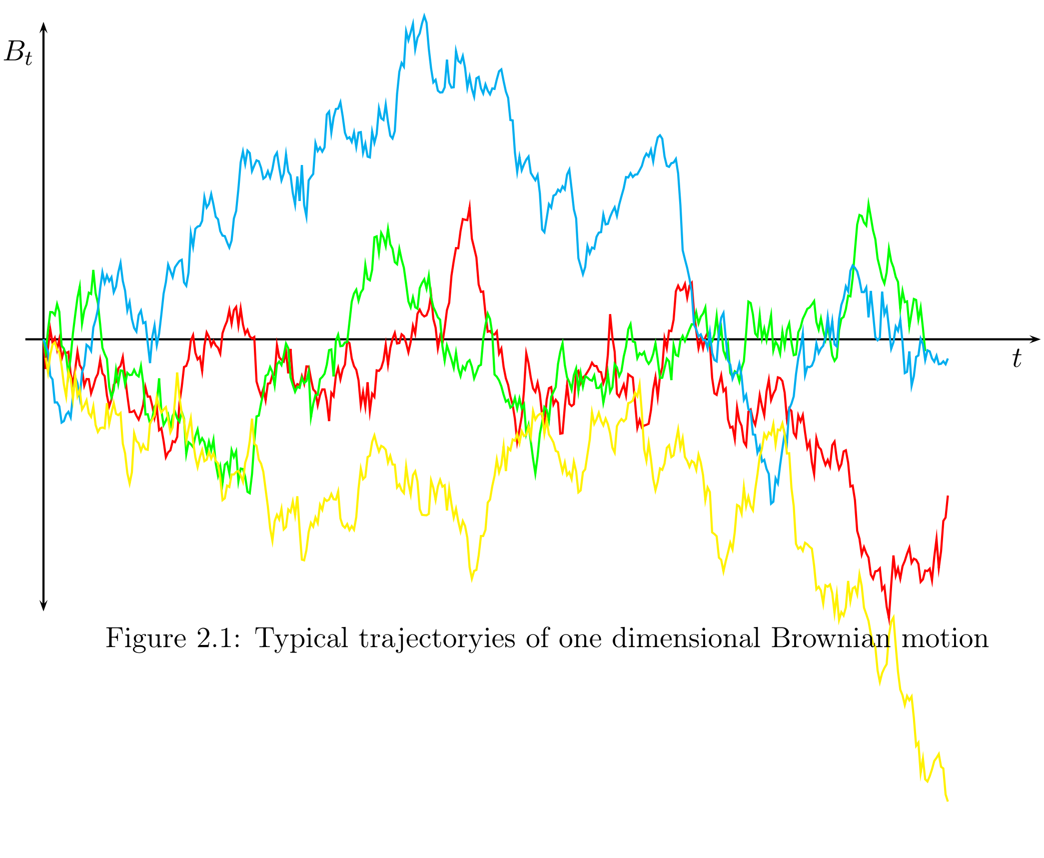

Figure 2.1: Typical trajectoryies of one dimensional Brownian motion

direct approach is often used to construct a Markov process as has been noted before, but its rather lengthy and technical. A more elegant approach for Brownian motion, which exploits the Gaussian features of this process, is based on Hilbert space theory, and is closer in spirit to Wiener's original construction.

Theorem 2.4.4. *An n-dimensional Brownian motion exists. Let B be an n-dimensional $\{\mathcal{F}_t\}$ Brownian motion. Then the following hold:*

(i) B is a strong Markov process with respect to $\{\mathcal{F}_t\}$, and corresponds to the semigroup induced by the transition function $P(t,x,A) = \frac{1}{\sqrt{2\pi t}^n} \int_A e^{-\frac{\|x-y\|^2}{2t}} dx$.

(ii) Writing $W = (W_1, \dots, W_n)$, each W_j is a continuous, square integrable martingale, and $\langle W_i, W_j \rangle_t = t\delta_{ij}$ for $i, j = 1, \dots, n$ and all $t \geq 0$.

(iii) Let $\xi_1, \xi_s, \dots$ be a sequence of iid, $\mathbb{R}^n$ valued random variables with mean 0 and covariance matrix I_n, the $(n \times n)$ identity matrix. Let $\{X_n\}$ be a family of processes in $D[0,\infty)$ defined as follows

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \xi_k, \qquad t \geq 0.$$

Then X_n converges weakly to B.

Given a n-dimensional Brownian motion $B_t = \left(B_t^1, \dots, B_t^n\right)$, adapted to $\mathcal{F}_t$, we can compute the cross quadratic variation between its component to obtain

$$\left\langle B^j, B^k \right\rangle_t = \delta_{jk} t, \qquad 1 \leq k, j \leq n;\ 0 \leq t < \infty.$$

It turns out that this property characterizes Brownian motion among continuous martingales as the next theorem states. It should be remarked that this theorem can be seen a particular property of the Brownian filtration, and continuity plays a fundamental role in the proof. The reader can see Theorem 3.3.16 in [20] for a proof.

Theorem 2.4.5. *Let* $X = \left\{X_t = \left(X_t^1, \ldots, X_t^n\right)\right\}$ *be a continuous, adapted process in* $\mathbb{R}^n$*, adapted to the filtration* $\{\mathcal{F}_t\}$*, which satisfies the usual conditions. Assume that for every component* $1 \leq k \leq n$*, the process*

$$M_t^k = X_t^k - X_0^k, \qquad 0 \leq t < \infty,$$

is a continuous local martingale with respect to $\{\mathcal{F}_t\}$*, and the cross quadratic variations are given by*

$$\left\langle M^k, M^j \right\rangle_t = \delta_{kj} t, \qquad 1 \leq k, j \leq n.$$

Then, $\{X_t\}$ *is an* n*-dimensional Brownian motion with respect to* $\{\mathcal{F}_t\}$*.*

To see how continuity is essential for this theorem, notice that the compensated Poisson process with intensity $\lambda = 1$ is a square-integrable martingale with $\langle M \rangle_t = t$, but it is not a continuous process.

Ito's formula

For a Brownian motion B, it can be shown that the trajectories $t \mapsto B_t$ have unbounded first variation a.s. This implies that the classical approach by Rieman-Stieltjes fails to define an integral of Brownian motion. In a famous work from, Itô realized how to define an integrals with respect to the trajectories of Brownian motion B_t, despite the fact that a.s. . The

Theorem 2.4.6 (Itô)**.** *Let* B_t *a Brownian motion adapted to* $\mathcal{F}_t$*, and let* $\{H_t\}$ *be a progressively measurable stochastic process with respect to* $\mathcal{F}_t$*. Then the following limit* $L^2(\mathbb{P}(d\omega) \otimes dt)$ *exists:*

$$\lim_{|\Pi_\delta| \to 0} \sum_{j=0}^{m} H_{t_j} \left(B_{t_{j+1}} - B_{t_j}\right),$$

where Π_δ *represents the partition* $t_0 = 0 < t_1 < t_2 < \cdots < t_m = t$ *of* $[0, t]$*, and is such that* $|\Pi_\delta| = \sup |t_{j+1} - t_j| < \delta$*. Moreover, this limit defines a continuous, square integrable martingale with respect to* $\{\mathcal{F}_t\}$*. We define* $\int_0^t H_u dB_u$ *as the limit above, and the following equation holds*

$$\mathbb{E}\left(\left[\int_0^t H_s dB_s\right]^2\right) = \mathbb{E}\left(\int_0^t H_u^2 du\right). \tag{2.4.4}$$

Also, for every $f \in C^2(\mathbb{R}^n)$,

$$f(B_t) = f(B_0) + \int_0^t \nabla f(B_u) dB_u + \frac{1}{2}\int_0^t \Delta f(B_u) du. \qquad (2.4.5)$$

This equation is often referred to as ***Itô's formula for Brownian motion****.*

A typical example of Itô's formula is obtained when setting $f(x) = \frac{1}{2}x^2$. For a one dimensional Brownian motion B,

$$\frac{1}{2}B_t^2 = \int_0^t B_u dB_u + \frac{1}{2}t.$$

Notice how this differs from the fundamental theorem of calculus, that would yield $\frac{1}{2}B_t^2 = \int_0^t B_u dB_u$, if applicable. The key difference between Itô's integral and the classical Riemann-Stieltjes integral is that the process $Y_t = \int_0^t B_u dB_u$ cannot be defined on a path by path basis because the Brownian paths have unbounded variation.

Itô's construction of the stochastic integral of Brownian motion can be extended to martingales. The reader is referred to Chapter 3 in [20] for a detailed account of stochastic integration. Other sources that can be consulted are Chapter 5 in [14] and the book "Stochastic Integration and Differential Equations" by Philip Protter. We will only list a couple of theorems that are necessary in our proofs.

A **continuous semimartingale** is a stochastic process X, adapted to $\{F_t\}$ which has the a.s. decomposition

$$X_t = X_0 + M_t + A_t, \qquad 0 \le t < \infty, \qquad (2.4.6)$$

where M is a continuous (local) martingale with respect to $\{\mathcal{F}_t\}$, and A is the difference of continuous, increasing, adapted processes A^+ and A^-, with $A^+ = A^- = 0$ a.s. The most general definition of semimartingale deals with local martingale, a concept that is not need in our presentation.

Since continuous increasing processes have bounded variation, the Rieman Stieltjes integral with respect to dA_u can be defined pathwise, and we only need to have a definition with respect to dM_u to extend Itô's integral to all semimartingales. This is taken care of by the following theorem.

Theorem 2.4.7. *Let M be a continuous martingale with respect to a filtration $\{\mathcal{F}_t\}$, and let X be a real valued, progressively measurable process adapted to $\{\mathcal{F}_t\}$, and satisfying*

$$\mathbb{E}\left(\left[\int_0^t X_u^2 d\langle M\rangle_u\right]\right) < \infty, \qquad \text{for all } 0 \leq t < \infty. \tag{2.4.7}$$

Then, there is a unique continuous, square-integrable martingale $(X \cdot M)$ with respect to $\{\mathcal{F}_t\}$ satisfying

$$\mathbb{E}\left((X \cdot M)_t^2\right) = \mathbb{E}\left(\int_0^t X_u^2 d\langle M\rangle_u\right), \qquad \text{for all } 0 \leq t < \infty.$$

For $\alpha \in \mathbb{R}$ and two processes X, Y satisfying (2.4.7), *we have*

$$((X + \alpha Y) \cdot M)_t = (X \cdot M)_t + \alpha(Y \cdot M)_t, \qquad \text{for all } 0 \leq t < \infty.$$

This linear property allows us to define $\int_0^t X_u dM_u = (X \cdot M)_t$. We have,

$$\left\langle \int_0^{\cdot} X_u dM_u \right\rangle_t = \int_0^t X_u^2 d\langle M\rangle_u. \tag{2.4.8}$$

We finish this section with the general version of Itô's formula for semimartingales.

Theorem 2.4.8 (Itô '44, Kunita & Watanabe '67)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 and let X be a continuous semimartingale adapted to $\{\mathcal{F}_t\}$ with decomposition* (2.4.6)*. Then, almost surely,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) dM_u + \int_0^t \nabla f'(X_u) dA_u + \frac{1}{2}\int_0^t f''(X_u) d\langle M\rangle_u, \tag{2.4.9}$$

for $0 \leq t < \infty$.

2.5 Diffusions with boundary conditions

Definition 2.5.1. *Let $D \subseteq \mathbb{R}^n$ be a $C^2(\mathbb{R}^n)$ domain with boundary ∂D and interior normal denoted by $\vec{n}(x)$ at a point x on its boundary. Given a Lipschitz, bounded map $\vec{\tau} : \partial D \to \mathbb{R}^n$ such that $\vec{\tau}(x)$ and $\vec{n}(x)$ are orthogonal; a Lipschitz, bounded, and strongly elliptic matrix $\sigma : \overline{D} \to \mathbb{R}^n \times \mathbb{R}^n$, and a $n-$dimensional Brownian motion B_t; an* ***Obliquely reflected***

Brownian motion *is a strong Markov process Z_t (with respect to the right continuous, complete filtration generated by B_t) with continuous sample paths, satisfying*

$$dZ_t = \sigma(Z_t)dB_t + [\vec{n}(Z_t) + \vec{\tau}(Z_t)]\, dL_t, \tag{2.5.1}$$

where L_t is an increasing, adapted process, that only increases when Z_t is on the boundary, that is, $dL_t = \mathbb{1}_{\partial D}(Z_t)dL_t$. To simplify notation, we will often use the shorthand $\vec{\gamma}(x) = \vec{n}(x) + \vec{\tau}(x)$.

The process just defined behaves as Brownian motion inside the domain D and reflects instantaneously at the boundary, in the direction given by $\vec{\gamma}$. The process L_t is known as **local time** and can be thought of as the minimal amount of push needed to keep the process within $\overline{D}$. Whenever Z_t hits the boundary of D, infinitely many tiny excursions occur and so local time is accumulated, even though Z spends zero (Lebesgue) time at the boundary. The name local time comes from the fact that its one dimension counterpart can be constructed as

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|x|<\varepsilon\}}(B_u)\, du,$$

when D is the positive part of the real line. In the multidimensional setting, the process of excursions from the boundary is a Poisson point process with a variable intensity that

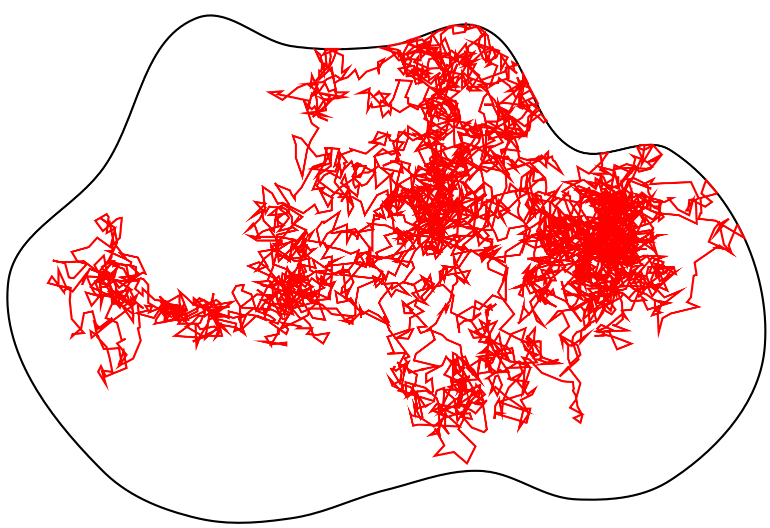

Figure 2.2: Typical path of obliquely reflected Brownian motion.

depends on the boundary point where the excursion starts. As a Poisson point process, the excursions have a clock, which can be roughly interpreted as the local time.

2.5.1 The submartingale problem

One very successful way of constructing diffusion processes with boundary conditions was developed by Stroock and Varadhan [26]. Their submartingale problem proved to be a successful extension of their ideas developed to treat the well-known martingale problem. The following survey on the submartingale problem is based on their original presentation.

Let D a non-empty, open subset of $\mathbb{R}^n$, such that:

(i) there exists $\phi \in C_b^2(\mathbb{R}^n;\mathbb{R})$ such that $D = \phi^{-1}(0,\infty)$, and $\partial D = \phi^{-1}(\{0\})$.

(ii) $\|\nabla\phi(x)\| \geq 1$ for all $x \in \partial D$.

The following functions will also be given:

(i) $a : [0,\infty) \times D \to M_n^+(\mathbb{R})$ which is bounded and continuous,

(ii) $b : [0,\infty) \times D \to \mathbb{R}^n$ which is bounded and continuous,

(iii) $\vec{\gamma} : [0,\infty) \times \partial D \to \mathbb{R}^n$ which is bounded, continuous, and satisfies $\langle \vec{\gamma}(t,x), \nabla\phi(x)\rangle \geq \beta > 0$ for $t \geq 0$ and $x \in \partial D$.

(iv) $\rho : [0,\infty) \times \partial D \to [0,\infty)$ which is bounded and continuous.

Define, for $u \geq 0$ and $x \in D$

$$\mathcal{L}_u = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(u,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(u,x)\frac{\partial}{\partial x_i}; \tag{2.5.2}$$

and, for $u \geq 0$ and $x \in \partial D$

$$B_u = \sum_{i=1}^{n} \gamma_i(u,x)\frac{\partial}{\partial x_i}.$$

We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ solves the submartingale problem on D for coefficients $a, b, \vec{\gamma}$ and ρ if $\mathbb{P}\left(X_t \in \overline{D}\right) = 1$, for $t \geq 0$, and

$$f(t, X_t) - \int_0^t \mathbb{1}_D(X_u)\left[\frac{\partial f}{\partial u} + L_u f\right](u, X_u)\, du$$

is a $\mathbb{P}$-submartingale for any $f \in C_0^{1,2}\left([0,\infty) \times \mathbb{R}^n\right)$ satisfying

$$\rho\frac{\partial f}{\partial t} + B_t f \geq 0 \qquad \text{on} \qquad [0,\infty) \times \partial D.$$

We say the the submartingale problem is well-posed if it has a unique solution.

Theorem 2.5.1 (Theorem 3.1 in [26])**.** *If the matrix $a(\cdot)$ is uniformly elliptic, and under the conditions stated above on the coefficients $a, b, \vec{\gamma}, \rho$, we have that for every $x \in \overline{D}$ there is a solution P, starting from x, to the submartingale problem.*

Theorem 2.5.2 (Theorem 5.4 in [26])**.** *Assume all the conditions on the coefficients a, b, γ, ρ hold, and also that γ is Lipschitz. If either $\rho \equiv 0$ or ρ is also a locally Lipschitz function which is strictly positive at each point of $[0,\infty) \times \partial D$, then the solution to the submartingale problem is unique and depends continuously on t and x.*

2.5.2 Oblique reflection in unbounded domains

Equation (2.5.1) fits very nicely the framework of the submartingale problem. In the case of our interest, the coefficients in the generator (2.5.2) do not depend on time, and we assume the drift term is zero for simplicity. We will drop the subindex u from $\mathcal{L}_u$ due to the independence of time. We have the following theorem, which is standard in the literature.

Theorem 2.5.3. *Assume that D is a domain of class $C^2(\mathbb{R}^n)$, not necessarily bounded. Obliquely reflected Brownian motion in D exists, as the unique strong solution to (2.5.1). The process is also square integrable when the initial distribution of Z_0 is square integrable.*

Proof. Fix $x \in \overline{D}$. By Theorems 2.5.1, 2.5.2, and 2.4 in [26], there exists a unique probability measure $\mathbb{P}_x$ on the canonical space of continuous functions that solves the submartingale problem for $\mathcal{L}$ with non-negative boundary derivative along $\vec{\gamma}$. Precisely, there exists a continuous, adapted process Z_t with values in $\overline{D}$; and a unique, continuous, non-decreasing, process L_t with values in $[0,\infty)$ such that $L_0 = 0$, $\mathbb{E}[e^{\lambda L_t}] < \infty$ for any $\lambda > 0$ and $t > 0$, and $dL_t = \mathbb{1}_{\partial D}(Z_t)dL_t$. Moreover, L_t is adapted to the right continuous, completed filtration determined by Z_t; and it satisfies that

(a) $h(Z_t) - \int_0^t \mathbb{1}_D(Z_s)\mathcal{L}h(Z_s)ds - \int_0^t \nabla h(Z_s)\cdot\vec{\gamma}(Z_s)dL_s$

(b) $\exp\left(\lambda\int_0^t \nabla h(Z_s)dZ_s - \frac{\lambda^2}{2}\int_0^t \mathbb{1}_D(Z_s)\left\langle \nabla h, [\sigma^T\sigma]\nabla h\right\rangle(Z_s)\; ds - \lambda\int_0^t \nabla h\cdot\vec{\gamma}(Z_s)dL_s\right)$

are $\mathbb{P}_x$-martingales for every $\lambda \in \mathbb{R}$ and every $h \in C_0^{2,1}(\mathbb{R}^n)$. Also, Theorem 2.5 in [26] gives us an Itô rule for integration with respect to dZ_t, namely

(c) $h(Z_t) - h(Z_0) = \int_0^t \nabla h(Z_s)dZ_s + \int_0^t \mathbb{1}_D(Z_s)\mathcal{L}h(Z_s)ds,$

for every $h \in C_0^{2,1}(\mathbb{R}^n)$.

Call G_t the martingale in (a), and set $J_t = \int_0^t \mathbb{1}_D\left\langle \nabla h, [\sigma^T\sigma]\nabla h\right\rangle(Z_s)ds$. Using (b) and (c) we obtain that $\exp\left[\lambda G_t - \frac{\lambda^2}{2}J_t\right]$ is a martingale for every $\lambda \in \mathbb{R}$. Looking at the coefficient of λ^2 in the power expansion of this exponential we readily see that $G_t^2 - J_t$ is a martingale. Since $J_0 = 0$, by the uniqueness of Doob's decomposition we get $\langle G, G\rangle_t = J_t$.

On the other hand, from the martingale representation theorem, we have that there is an n-dimensional, standard Brownian motion B_t in $(\Omega, \mathcal{F}_t, \mathbb{P})$ such that $G_t = \mathbb{E}(G_0) + \int_0^t g_s dB_s$ for some adapted process g_s. The calculation above shows that $G_t - \int_0^t \mathbb{1}_D(Z_s)[\nabla h\, \sigma](Z_s)dB_s$ is a square integrable martingale with zero quadratic variation, and so it is equal to the constant $G_0 = h(Z_0) = h(x)$ under $\mathbb{P}_x$. From this we get

$$h(Z_t) - h(Z_0) = \int_0^t \mathbb{1}_D(Z_s)\nabla h^T\sigma(Z_s)dB_s + \int_0^t \mathbb{1}_D(Z_s)\mathcal{L}h(Z_s)ds + \\ + \int_0^t \nabla h\cdot\vec{\gamma}(Z_s)dL_s, \qquad (2.5.3)$$

for any $h \in C_0^2(\mathbb{R}^n)$. This is enough to establish weak existence in (2.5.1). Uniqueness is established as a direct consequence of the uniqueness of the solution to the submartingale problem.

Using equation (2.5.3) we can show that Z_t is square integrable for each $t > 0$ under $\mathbb{P}_x$. To do this, we first need a sequence of C_0^2 functions $\varphi_k : \mathbb{R}^n \to [0, \infty)$ satisfying:

(i) $\varphi_k(x)$ defines an increasing sequence, with $\lim_{k\to\infty} \varphi_k(x) = |x|$ in the set $\{|x| \geq 1\}$.

(ii) $\sup_k \varphi_k(x) \leq 2$ for $|x| \leq 1$,

(iii) $|\nabla\varphi_k(x)| \leq 2$ for all $x \in \mathbb{R}^n$,

(iv) $\left|\partial_{x_i}\partial_{x_j}\varphi_k(x)\right| \leq C_n$ for some positive constant C_n, and all $x \in \mathbb{R}^n$.

To do so, it is enough to take care of the case $n = 1$, and then define a rotationally symmetric extension to $\mathbb{R}^n$. We stress that in (i) we only ask for monotone convergence away from zero. We have,

$$\begin{aligned}
\mathbb{E}\,|Z_t|^2 &\leq \mathbb{E}\left(1 + |Z_t|\,\mathbb{1}_{\{|x|\geq 1\}}(Z_t)\right)^2 \leq 2 + 2\mathbb{E}\,|Z_t|^2\,\mathbb{1}_{\{x\}\geq 1}(Z_t) \\
&= 2 + 2\lim_{k\to\infty} \mathbb{E}\left[\varphi_k(Z_t)^2\mathbb{1}_{\{|x|\geq 1\}}(Z_t)\right] \\
&\leq 2 + 2\ \limsup_{k\in\mathbb{N}}\ \mathbb{E}\left[\varphi_k(Z_t)^2\right].
\end{aligned}$$

Since each function φ_k has bounded second derivatives and compact support, equation (2.5.3) applies, and so

$$\begin{aligned}
\mathbb{E}\left[\varphi_k(Z_t)^2\right] \leq 2\mathbb{E}\left[\varphi_k(Z_0)^2\right] &+ 2\mathbb{E}\left[\left|\int_0^t \nabla\varphi_k(Z_u)^T\sigma(Z_u)dB_u\right|^2\right] + \\
&+ 2\mathbb{E}\left[\left|\int_0^t \nabla\varphi_k(Z_u)\cdot\vec{\gamma}(Z_u)dL_u\right|^2 + \frac{1}{4}\left|\int_0^t \mathcal{L}\varphi_k(Z_u)du\right|^2\right] \\
\leq 2\mathbb{E}\left[\varphi_k(Z_0)\right]^2 &+ 2\mathbb{E}\int_0^t |\nabla\varphi_k(Z_u)|^2\,\|\sigma(Z_u)\|^2\,du + \\
&+ 2\mathbb{E}\left[\left(\int_0^t |\nabla\varphi_k(Z_u)|\,|\vec{\gamma}(Z_u|\,dL_u\right)^2 + \frac{1}{4}\left(\int_0^t |\mathcal{L}\varphi_k(Z_u)|\,du\right)^2\right].
\end{aligned}$$

Condition (iii), (iv), and the fact that all the coefficients of σ and thus of $\mathcal{L}$ are bounded, imply that for some constant $C > 0$ we have

$$\mathbb{E}\left[\varphi_k(Z_t)^2\right] \leq 2\mathbb{E}\left[\varphi_k(Z_0)^2\right] + Ct + C\mathbb{E}\left[L_t^2\right] + Ct^2.$$

To get a uniform bound in k, we use condition (i) and (ii): $\varphi_k(Z_0) \leq 2+|Z_0|$. By monotone convergence, we conclude that $\mathbb{E}\,|Z_t|^2 \leq 8 + 4\mathbb{E}\,|Z_0|^2 + Ct(1+t) + C\mathbb{E}\left[L_t^2\right]$.

A standard argument shows that when we combine weak existence of solutions and pathwise uniqueness, we have strong existence and uniqueness for a stochastic differential equation. See, for example, [20, Section IX.1, thm 1.7]. Thus, we just need to show pathwise uniqueness of Z_t. Let $\tau_k = \inf\{t > 0 : |Z_t| > k\}$. Since Z_t is square integrable we know that that $\tau_n \to \infty$ a.s., and thus it is enough to show pathwise uniqueness up to time τ_n.

Define D_k as any bounded smooth domain in $\mathbb{R}^n$ such that $D_k \cap B(0,3k) = D \cap B(0,3k)$. Also, extend $\vec{\gamma}\mathbb{1}_{B(0,3k)}$ to a bounded, Lipschitz vector field $\vec{\gamma}_k$ that is defined in ∂D_k, and consider the following reflected Brownian motion in D_k:

$$dY_t = \sigma(Y_t)dB_t + \vec{\gamma}_k(Y_t)dL_t.$$

In [11] (see also its correction [12]), pathwise uniqueness of obliquely reflected Brownian motion has been showed, under less restrictive assumptions on the coefficients $\sigma, \vec{\gamma}$ than ours, but only for processes in bounded domains. Thus, Y_t satisfies path wise uniqueness, and since $Y_{t\wedge\tau_k} = Z_{t\wedge\tau_k}$ under $\mathbb{P}_x$ for $|x| < k$, we have that pathwise uniqueness for $Z_{t\wedge\tau_k}$ holds, as we wanted to prove.

$\blacksquare$

2.6 Stationary distributions

This section is mostly a collection of already known results on stationary distributions of difussions with boundary conditions. We start by some basic definitions and a very useful functional characterization due to Weiss [27] by following closely his presentation.

Definition 2.6.1. *Let $P(t)$ be the strongly continuous contraction semigroup associated to some Markov process with state space E. A probability measure μ on E is called a stationary distribution for such process if*

$$\mu(A) = \int_E P(t)\mathbb{1}_A(x)\mu(dx) \tag{2.6.1}$$

for all Borel sets A. When the property above holds for a general (non-finite) measure, we call μ an invariant measure. It is clear that a probability measure μ is stationary if and only if

$$\int_E f(x)\mu(dx) = \int_E P(t)f(x)\mu(dx), \tag{2.6.2}$$

for any $f \in C_b(E)$ and $t > 0$.

A stationary distribution can be often found as a limit of long time averages. Indeed, let $P(t)$ be the semigroup of a Markov process Z_t. For any probability measure μ_0 on the state space, define the **occupation time measures** by

$$\mu_t(A) = \frac{1}{t}\int_0^t \mathbb{P}_{\mu_0}\left(Z_s \in A\right)\, ds = \frac{1}{t}\int_0^t \int P(s)\mathbb{1}_A(x)\mu_0(dx)ds.$$

Whenever this family of measures has a convergent subsequence, for example, when the state space is compact, any of its limits will be a stationary distribution. Indeed, we have the following lemma,

Lemma 2.6.1. *Let P_t be a Feller semigroup associated to a process Z_t with state space E, and let μ be a limit point of the sequence of occupation time measures. Then μ is a stationary distribution for Z.*

Proof. Let $\mu^n = \mu_{t_n}$ be a sequence in the aforementioned family, converging weakly to a probaility measure μ, and let $f \in C_b(\mathbb{R}^n \times \mathbb{R}^p)$. For any $t > 0$

$$\begin{aligned}\int_E P(t)f(x)\mu(dx) &= \lim_n \int_E P(t)f(x)\mu^n(dx)\\ &= \lim_n \frac{1}{t_n}\int_0^{t_n} P(u)P(t)f(x)du\\ &= \lim_n \frac{1}{t_n}\int_0^{t_n} P(u+t)f(x)du\\ &= \lim_n \frac{1}{t_n}\int_t^{t_n+t} P(v)f(x)dv.\end{aligned}$$

Since $|P(t)f(x)| \leq \|f\|_\infty$, we have the following estimate

$$\frac{1}{t_n}\left|\int_0^t P(v)f(x)\mu(dx)\right| + \frac{1}{t_n}\left|\int_{t_n}^{t_n+t} P(v)f(x)\mu(dx)\right| \leq \frac{2\,\|f\|_\infty\, t}{t_n},$$

which converges to zero as $t_n \to \infty$. Therefore,

$$\int_E P(t)f(x)\mu(dx) = \lim_n \frac{1}{t_n}\int_0^{t_n} P(v)f(x)dv = \lim_n \int_E f(x)\mu_n(dx) = \int_E f(x)\mu(dx),$$

which shows that μ is stationary by (2.6.2) ■

Roughly speaking, this result states that a stationary distribution represents the average time that the diffusion spends in Borel sets. An interesting consequence of this result is that if there is only one stationary distribution, then it can be characterized as the limit of μ_t as t goes to infinity, for any starting measure μ_0. The result avoids the question of convergence of the occupation time measures, and one has to draw upon to techniques such as Lyapunov functions to obtain existence of limit points. For a detailed discussion of the matter we refer the reader to Kurtz and Ethier [14], chapter 4, section 9.

Consider the obliquely reflected Brownian motion Z_t solving (2.5.1). Assume μ is a stationary distribution. From (2.6.2), and the submartingale characterization of Z_t from 2.5.3, we have that

$$\mathbb{E}_x\left[f(Z_t) - \int_0^t \mathcal{L}f(Z_u)du\right] \geq \mathbb{E}_x\left[f(Z_0)\right],$$

for any $f \in C_b^2(\mathbb{R}^n)$. If Z_0 is distributed according to μ, then Z_u is distributed as μ for all $u \geq 0$. Thus, integrating the above with respect to $\mu(dx)$

$$\begin{aligned}\mathbb{E}_\mu\left[f(Z_t) - \int_0^t \mathcal{L}f(Z_u)du\right] &= \int_{\overline{D}} f(x)\mu(dx) - \int_0^t \int_D \mathcal{L}f(x)\mu(dx)dt \\ &\geq \int_{\overline{D}} f(x)\mu(dx) = \mathbb{E}_\mu\left[f(Z_0)\right].\end{aligned}$$

That is, for $f \in C^2(\overline{D})$ with $\nabla f \cdot \vec{\gamma}(x) \geq 0$ for $x \in \partial D$, if μ is invariant for the diffusion, then

$$\int_D \mathcal{L}f(x)\mu(dx) \leq 0.$$

The main result of [27] is the converse of the previous statement. It covers the case of a sticky boundary, in which the process spends positive Lebesgue time at the boundary. We give a specialized version that will be sufficient in our case: Let $\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^n a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} +$

$\sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}$ be a second order differential operator, where $a_{i,j}$ and b_i are bounded, Lipschitz functions. Assume that a bounded, Lipschitz vector field is given on the boundary of a $C^2(\mathbb{R}^d)$ domain G, such that $\vec{\kappa} \cdot \hat{n}(x) \geq \beta > 0$ for $x \in \partial G$. Let ϕ be a $C^2(\mathbb{R}^d)$ function defining the boundary of G.

Theorem 2.6.2 (Weiss)**.** *Let $\overline{G}$ be compact in $\mathbb{R}^d$ and b_j and $\vec{\gamma}$ as before, suppose $(a_{i,j}(x))$ is bounded continuous positive semidefinite with $\nabla\phi(x)^T a(x)\nabla\phi(x) > 0$ for x in a neighborhood of ∂G (i.e. the diffusion has nonzero random component normal to the boundary). Suppose that μ is a probability measure on $\overline{G}$ with $\mu(\partial G) = 0$ and*

$$\int_G \mathcal{L}f(x)\mu(dx) \leq 0 \tag{2.6.3}$$

for all $f \in C_b^2(\overline{G})$ with $\nabla f \cdot \vec{\kappa}(x) \geq 0$ for $x \in \partial G$. Suppose that the submartingale problem for a, b and $\vec{\kappa}$ is uniquely solvable starting from any $x \in \overline{G}$. Then μ is an invariant measure of the diffusion.

Theorem 2.6.2 has been successfully used by Harrison, Landau and Shepp [17] to give an explicit formula for the stationary distribution of obliquely reflected Brownian motion in planar domains, in two cases: (a) the domain is of class $C^2(\mathbb{C})$ and bounded, and the reflection coefficient $\vec{\gamma}$ has a global extension to a $C_b^2(\mathbb{R}^2)$ vector field; and (b) the domain is a convex polygon, and the reflection coefficient is constant in each face. Their technique to obtain an explicit representation is to assume that $\mu(dx) = \rho(x)dx$ and integrate (2.6.3) by parts to obtain a PDE with boundary conditions for ρ, and solve such equation.

To prove uniqueness of the stationary distribution, we can follow the scheme of Harrison and Williams in [18]. Their setting is different from ours in that they consider obliquely reflected Brownian motion in an orthant, which is a non-smooth domain, and the reflection vector γ is assumed to be constant along each face. Nonetheless, their proofs only are based in two facts: (a) the process behaves as Brownian motion inside of the domain, and (b) the process spend zero Lebesgue time on the boundary. Since these two facts are true in our case, it is possible to reproduce their proofs, and make them work in our setting. The following theorem, summarizes the properties we need.

Lemma 2.6.3. *For each $x \in \overline{D}$, and $t > 0$*

(a) $\mathbb{P}_x \left(Z_t \in \partial D \right) = 0$.

(b) Let m be the Lebesgue measure in $\mathbb{R}^n$. For any Borel set $A \subseteq \overline{D}$ we have

$$\mathbb{P}_x(Z_t \in A) = 0 \qquad \Longleftrightarrow \qquad m(A) = 0. \tag{2.6.4}$$

(c) Suppose μ is a stationary distribution for Z. Then μ and m are mutually absolutely continuous on D.

Proof. (a) is a result form Varadhan and Stroock. See Theorem XX in [26] for a proof.

To show (b), in view of part (a) and the fact that $m(\partial D) = 0$, it suffices to assume that $A \subseteq K$, where K is a compact subset of $D \setminus \partial D$. Let $\tau = \inf \left\{ u \geq 0 : Z_u \in \partial D \right\}$, $\sigma = \inf \left\{ u \geq 0 : Z_u \in K \right\}$, $\sigma_0 = 0$, and for each $k \geq 1$, let $\tau_k = \sigma_{k-1} + \tau \circ \theta_{\sigma_{k-1}}$ and $\sigma_k = \tau_k + \sigma \circ \theta_{\tau_k}$ where θ is the usual shift operator for Z. Then, for each $x \in \overline{D}$, $\tau_k \nearrow \infty$ $\mathbb{P}_x$ a.s. as $k \to \infty$ and

$$\mathbb{E}_x \left[\int_0^\infty \mathbb{1}_A(Z_t) dt \right] = \sum_{k=1}^\infty \mathbb{E}_x \left[\int_{\sigma_{k-1}}^{\tau_k} \mathbb{1}_A(Z_t) dt \right] = \sum_{k=1}^\infty \mathbb{E}_x \left[\mathbb{E}_{Z_{\sigma_{k-1}}} \left(\int_0^\tau \mathbb{1}_A(Z_t) dt \right) \right],$$

where the second equality holds by the strong Markov property. Since $Z_{\sigma_{k-1}} \in K$, and Z_t behaves as Brownian motion within D, we have

$$\mathbb{E}_x \left[\int_0^\infty \mathbb{1}_A(Z_t) dt \right] = \sum_{k=1}^\infty \mathbb{E}_x \left[\mathbb{E}_{Z_{\sigma_{k-1}}} \left(\int_0^\tau \mathbb{1}_A(B_t) dt \right) \right],$$

and so the left hand side is zero if and only if $m(A) = 0$, since the distribution of B_t is mutually absolutely continuous with the Lebesgue measure for all $t > 0$. By Fubini's theorem we deduce,

$$m(A) = 0 \iff \int_0^\infty \mathbb{P}_x(Z_t \in A) dt = 0 \iff \mathbb{P}_x(Z_t \in A) = 0,$$

for all $t > 0$, since the trajectories of Brownian motion are continuous. This shows part (b).

Finally, part (c) follows from part (b), and the fact that a stationary distribution μ satisfies

$$\mu(A) = \int_{\overline{D}} \mathbb{P}_x(Z_t \in A)\mu(dx), \qquad \text{for all } t > 0.$$

■

If both μ_1 and μ_2 are stationary for Z, then

$$\mu_j(A) = \int_{\overline{D}} \mathbb{P}_x(Z_t \in A)\mu_j(dx) = \int_D \mathbb{P}_x(Z_t \in A)\frac{d\mu_j}{dm}(x)dx,$$

where A is a Borel set and $\frac{d\mu_j}{dm}$ are Radon-Nikodym derivatives. The fact that μ_j and m are mutually absolutely continuous implies that μ_1 and μ_2 are mutually absolutely continuous. One the other hand, it follows from the ergodic decomposition theorem (2.2.8 in [1]) that any two stationary distributions must have disjoint supports. This contradiction shows that the stationary distribution for Z solving (2.5.1) is unique.

2.7 Non-symmetric Dirichlet forms

This section introduces the fundamental concepts of the theory of non-symmetric Dirichlet forms. The results that are reviewed are compiled from [8], [21] and [23].

Consider a measure space (X, λ). A subspace H of $L^2(X, \lambda)$ is called a space of base (X, λ), if it is endowed with the structure of a Hilbert space that is dense in $L^2(E, \lambda)$, such that if $u \in H$, then $|u|$, $\min\{u, 1\} \in H$; and such that the inclusion from H to $L^2(E, \lambda)$ is continuous.

Definition 2.7.1. *Let $\mathcal{E}$ be a bilinear form over $H \times H$. We say that $\mathcal{E}$ is a Dirichlet form if the following conditions are satisfied:*
Coercivity. *There exists constants $C_1, C_2 > 0$ such that for every $u \in H$ we have $\mathcal{E}(u, u) \geq C_1 \|u\|_H - C_2 \|u\|_{L^2}$.*
Continuity. *There is a constant $C_3 > 0$ such that $|\mathcal{E}(u, v)| \leq C_3 \|u\|_H \|v\|_H$ for any $u, v \in H$.*

Contraction. *For each $u \in H$, set $Tu = \min\{u^+, 1\}$. The following inequalities hold*

$$\mathcal{E}(Tu, u - Tu) \geq 0, \tag{C1}$$

$$\mathcal{E}(u - Tu, Tu) \geq 0. \tag{C2}$$

Given a Dirichlet form $\mathcal{E}$ on H, we associate with it the family of Dirichlet forms $(\mathcal{E}_\alpha)_\alpha$ given by $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v)_{L^2(\lambda)}$. Sometimes we will write $\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$ for $u, v \in H$.

A space of base (E, λ) is called **regular** if $H \cap C_0(E)$ is dense in H and in $C_0(E)$. If H is a (regular) space of base (E, λ), and $\mathcal{E}$ is a Dirichlet form, we say that $(\mathcal{E}, H)$ is a (regular) Dirichlet space of $L^2(E, \lambda)$. The form is also called local if $\mathcal{E}(u,v) = 0$ whenever $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset$.

By means of the Riesz representation theorem, it is possible to find a family of resolvents $\{G_\alpha\}_{\alpha>0}$ such that $\mathcal{E}_\alpha(G_\alpha u, v) = \langle u, v\rangle_{L^2}$, and so, a semigroup $\{D_t\}_{t>0}$ corresponding to the resolvent. The following theorem, asserts that those objects correspond to those of a Markov process.

Theorem 2.7.1 (Carrillo Menendez, [8])**.** *Given a local, regular Dirichlet form $(\mathcal{E}, H)$ on $L^2(E, \lambda)$, with resolvent $\{G_\alpha\}_{\alpha>0}$, there exists a conservative, strong Markov process $M = (X_t, \mathbb{P}_x)$ on E with continuous sample paths, whose resolvent is a version of G_α.*

2.7.1 Additive Functionals

It is crucial to notice that in the definition a Dirichlet form, there is some underlying symmetry, namely, that $\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$ is a Dirichlet form if and only if $\mathcal{E}$ is. Using theorem 2.7.1, we obtain then a pair of processes $(X, \hat{X})$, which are dual of each other, in the sense that their semigroups (and thus their resolvents) are in duality in $L^2(E, \lambda)$. Let Ω, $\{\mathcal{F}_t\}_{t\in[0,\infty]}$, ξ, θ_t be the sample space, the minimum completed admissible filtration, the life time and the shift operator respectively associated with the Hunt process X. The following definition is taken from [15].

Definition 2.7.2. *An extended real valued function $A_t(\omega)$ of $t \geq 0$ and $\omega \in \Omega$ is called an additive functional (AF in abbreviation) if it is $\{\mathcal{F}_t\}$-adapted and the following properties hold almost surely:*

(i) $A_0 = 0$, *A_t is càdlàg and finite on* $[0, \xi)$, $A_t = A_\xi$ *for* $t \geq \xi$,

(ii) $A_{s+t} = A_s + A_t \circ \theta_s$, $s, t \geq 0$.

An AF A_t is called positive and continuous (PCAF in abbreviation) if $A_t \geq 0$ and $t \mapsto A_t$ is continuous almost surely.

Definition 2.7.3. *Let A be an additive functional of X. We define its energy by*

$$e(A) = \lim_{\alpha \to \infty} \frac{\alpha}{2} \mathbb{E}_\lambda \int_0^\infty e^{-\alpha t} A_t^2 dt,$$

whenever this limit exists and is finite.

We denote by A_c^+ the set of continuous, positive additive functionals of X. Also, denote by H_b^+ the space of positive, bounded functions in H. Other definitions that are relevant are given next: For G, an open subset of E, the **capacity** of G is defined to be

$$\mathrm{Cap}(G) = \inf \{\mathcal{E}_1(u, u) : u \in H \text{ and } f \geq 1 \text{ on } G\}$$

with the understanding that the infimum of the empty set is $+\infty$. For subsets $B \subseteq E$, let

$$\mathrm{Cap}(B) = \inf \{\mathrm{Cap}(G) : G \text{ is open and } G \supset B\}.$$

A Borel set N is called **exceptional** if $\mathrm{Cap}(N) = 0$. A statement depending on $x \in B$ is said to be true **quasi-everywhere** on B if there is an exceptional set N such that the statement is true for $x \in B \setminus N$.

Definition 2.7.4. *A positive Borel measure μ on E is said to be smooth if the following conditions are satisfied:*

(a) $\mu(N) = 0$ *for every exceptional set* N, *and*

(b) there is an increasing sequence $\{F_k\}$ of compact sets such that $\mu(F_k) < \infty$, $\mu\left(E \setminus \bigcup_k F_k\right) = 0$, and $\lim_{k\to\infty} \mathrm{Cap}(K \setminus F_k) = 0$ for every compact set K.

We denote by S the space of smooth measures of X.

Theorem 2.7.2. *There is a one to one correspondence between S and A_c^+. The correspondence between $\mu \in S$ and $A \in A_c^+$ is characterized by the following relation*

$$\lim_{\alpha\to\infty} \alpha^2 \mathbb{E}_{h\cdot\lambda} \int_0^\infty e^{-\alpha t}(fA)_t dt = \langle f\mu, h\rangle$$

for any $h \in H_b^+$ and positive, Borel f in X.

For an exact definition of quasi-continuous functions and some more potential theory we refer the reader to [8, 21, 23].

Theorem 2.7.3. *Given $u \in H$, let $\tilde{u}$ be a quasi-continuous version of u. Define $A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0)$ for $t > 0$. Then, there is a unique decomposition*

$$A^{[u]} = M^{[u]} + N^{[u]},$$

where $M^{[u]}$ is a local, square-integrable martingale with zero mean; and $N^{[u]}$ is a continuous additive functional locally of zero energy.

The last theorem we present relates the Dirichlet form $\mathcal{E}$ to a certain additive functional.

Theorem 2.7.4. *Let $u \in H$ and A be an additive functional locally of zero energy. If*

$$\lim_{\alpha\to\infty} \alpha^2 \mathbb{E}_{v\cdot\lambda} \int_0^\infty e^{-\alpha t} A_t dt = -\mathcal{E}(u,v) \qquad \forall\ v \in H_b$$

then $A_t = N_t^{[u]}$. Also, for $u, f \in H_b$ we have

$$\int_X \tilde{f}\ d\mu_u = 2\mathcal{E}(u, fu) - \mathcal{E}(u^2, f),$$

where μ_u is the smooth measure associated to the quadratic variation of the local martingale $M^{[u]}$.

Chapter 3

OBLIQUE REFLECTION AND DIRICHLET FORMS

3.1 Preliminaries

Definition 3.1.1. *Let D be an open, connected subset of $\mathbb{R}^n$ (a domain). A cover of the boundary ∂D is a countable collection of triplets (x_k, B_k, ϕ_k), where $x_k \in \partial D$, B_k is a ball centered at x_k, and $\phi_k : B_k \to \mathbb{R}^n$ is such that*

$$\{x \in B_k : \phi_k(x) > 0\} = B_k \cap D.$$

Given a class of functions $\mathcal{A}(\mathbb{R}^n)$, we say that D is of class $\mathcal{A}$ if the maps ϕ_k belong to such class. For example, D is of class $C^p(\mathbb{R}^n)$ if the maps ϕ_k have continuous derivatives up to order p.

Definition 3.1.2. *Let D be a Lipschitz domain in $\mathbb{R}^n$ and $\vec{\tau} : \partial D \to \mathbb{R}^n$ be a vector field tangential to ∂D, that is, such that $\vec{\tau}(x) \cdot \vec{n}(x) = 0$ for almost every $x \in \partial D$. A function ξ, defined on ∂D in the sense of distributions, and satisfying*

$$\int_{\partial D} \nabla u \cdot \vec{\tau}(x)\ \sigma(dx) = -\int_{\partial D} u(x)\xi(x)\ \sigma(dx) \tag{3.1.1}$$

*for every $u \in C_c^\infty(\mathbb{R}^n)$, is called the **boundary divergence of** $\vec{\tau}$, and we denote it by $\xi = \mathrm{bdiv}(\vec{\tau})$.*

Unfortunately, the boundary divergence does not always exist in Lipschitz domains. Nonetheless, it exists in $C^{1,1}$-domains, or actually whenever a nice cover of the boundary can be found, as described by the following lemma.

Lemma 3.1.1. *Let (x_k, B_k, ϕ_k) be a cover of the boundary of D such that $\|\nabla\phi_k\|^2$ is differentiable and bounded for every k. Let $\vec{\tau}$, $\vec{\kappa}$ be bounded, Lipschitz vector fields that are* *tangential to ∂D. Assume that the components of these vectors and their derivatives are square integrable with respect to the surface measure $\sigma(dx)$. Then, their boundary divergence exists and it has the following properties:*

i. The boundary divergence is uniquely defined in $L^2(\partial D, \sigma(dx))$,

ii. bdiv *is a linear operator,*

iii. for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have $\mathrm{bdiv}(\varphi\vec{\tau}) = \nabla\varphi \cdot \vec{\tau} + \varphi \mathrm{bdiv}\vec{\tau}$.

Proof. Uniqueness is straightforward from existence, so we only prove the latter. Let (x_k, B_k, ϕ_k) be a cover of the boundary as in the statement, and consider a locally finite, smooth partition of the unity $\{\xi_k\}$ subordinated to $\{B_k\}$. It is well known that the unit normal is proportional to $(-\nabla\phi_k, 1)$ in the coordinates of B_k. Thus, $\vec{n} \cdot \vec{\tau} = 0$ is equivalent to $\sum_{j=1}^{n-1} \partial_j \phi_k \tau^j = \tau^n$ in B_k. This computation yields $\nabla u \cdot \vec{\tau} = \sum_{j=1}^{n-1} (\partial_j u + \partial_n u \partial_j \phi_k)\tau^j$ within B_k.

Let (y, x^n) be local coordinates in B_k, and define an operator T_k by $T_k u(y, s) = u(y, s + \phi_k(x))$. The operator T_k is introduced in [10] to prove the trace theorem for Sobolev spaces. It is shown there that T_k is an isomorphism between the Sobolev spaces $W^{1,2}(D \cap B_k)$ and $W^{1,2}(\mathbb{H}^n)$. We take advantage of this fact, but use it in a rather different manner. The key property of T_k is that in local coordinates, $\partial_j T_k u(x, 0) = \partial_j u + \partial_n u \partial_j \phi_k$, for $j = 1, \ldots, n-1$, and $(x, \phi_k(x)) \in \partial D \cap B_k$. So, $\nabla u \cdot \vec{\tau} = \sum_{j=1}^{n-1} \partial_j (T_k u)\tau^j(x, 0)$. As ξ_k has compact support,

$$\begin{aligned}\int_{\partial D} \nabla u \cdot \vec{\tau}\ \sigma(dx) &= \sum_k \int_{\partial D \cap B_k} \xi_k \nabla u \cdot \vec{\tau}\ \sigma(dx) \\ &= \sum_k \sum_{j=1}^{n-1} \int_{\mathbb{H}^n} T_k \xi_k \partial_j (T_k u) T_k \tau^j(y, 0)\ \sqrt{1 + \|\nabla\phi_k\|^2}\ dy.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_{\partial D} \nabla u \cdot \vec{\tau}\ \sigma(dx) = \sum_k \sum_{j=1}^{n-1} \int_{\mathbb{H}^n} -T_k u \xi_k \partial_j \left(T_k \tau^j \sqrt{1 + \|\nabla\phi_k\|^2} \right)\ dy\ + \\ - \sum_k \sum_{j=1}^{n-1} \int_{\mathbb{H}^n} \partial_j (T_k \xi_k) T_k u \tau^j \sqrt{1 + \|\nabla\phi_k\|^2} dy.\end{aligned}$$

The second term on the right hand side equals zero. Indeed, a previous computation shows that $\nabla\xi_k \cdot \vec{\tau} = \sum_{j=1}^{n-1} \partial_j(T_k\xi_k)\tau^j(x,0)$. Also, since ξ_k is supported in B_k,

$$\sum_k \sum_{j=1}^{n-1} \int_{\mathbb{H}^n} \partial_j(T_k\xi_k)T_k u\tau^j \sqrt{1+\|\nabla\phi_k\|^2}dy = \sum_k \int_{\partial D\cap B_k} \nabla\xi_k \cdot \vec{\tau}\ u(x)\sigma(dx)$$
$$= \int_{\partial D} \nabla\left(\sum_k \xi_k\right) \cdot \vec{\tau}\ u(x)\sigma(dx),$$

because the partition of unity is locally finite. But $\sum_k \xi_k = 1$, which shows our claim.

We now can write a local definition for the boundary divergence. For $x \in \partial D$,

$$\mathrm{bdiv}(\vec{\tau})(x) = \sum_k \sum_{j=1}^{n-1} \mathbb{1}_{B_k}(x)\xi_k(x)\partial_j\left(T_k\tau^j\sqrt{1+\|\nabla\phi_k\|^2}\right)\sqrt{1+\|\nabla\phi_k\|^2}^{-1}.$$

With this definition, (3.1.1) is satisfied.

Still, we have to show some more work to properly define bdiv in $L^2(\partial D, \sigma(dx))$. Continuing from the previous calculations, we will show that $u \mapsto \int_{\partial D} \nabla u \cdot \vec{\tau}\ \sigma(dx)$ is a continuous functional on the Sobolev space $H^1(D)$. By Hölder inequality in $L^2(\partial D, \sigma(dx))$, and since the sum in k is locally finite, we have

$$\left|\int_{\partial D} \nabla u \cdot \vec{\tau}\ \sigma dx\right| \leq \sum_k \sum_{j=1}^{n-1} \int_{\mathbb{H}^n} |T_k u\xi_k| \left|\partial_j\left(T_k\tau^j\sqrt{1+\|\nabla\phi_k\|^2}\right)\right| dy$$
$$\leq \sum_{j=1}^{n-1}\sum_k \left(\int_{\mathbb{H}^n} \left|T_k u^2\xi_k\right| dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^n} T_k\xi_k \left|\partial_j\left(T_k\tau^j\sqrt{1+\|\nabla\phi_k\|^2}\right)\right|^2 dy\right)^{\frac{1}{2}}$$

By assumption, the quantity $1+\|\nabla\phi_k\|^2$ is differentiable, and both itself and its derivative are unit ormly bounded in x and k. Let C_D an upper bound of these quantities. The product rules then yields,

$$\left|\int_{\partial D} \nabla u \cdot \vec{\tau}\ \sigma(dx)\right| \leq C_D \sum_{j=1}^{n-1}\sum_k \left(\int_{\mathbb{H}^n} \left|T_k u^2\xi_k\right| dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^n} T_k\xi_k \left[\left|T_k\tau^j\right|^2 + \left|\partial_j\left(T_k\tau^j\right)\right|^2\right] dy\right)^{\frac{1}{2}}$$
$$= C_D \sum_{j=1}^{n-1}\sum_k A_k^{\frac{1}{2}}B_k^{\frac{1}{2}} \leq C_D \sum_{j=1}^{n-1}\left(\sum_k A_k\right)^{\frac{1}{2}}\left(\sum_k B_k\right)^{\frac{1}{2}},$$

where the last inequality is the Cauchy-Schwartz inequality for summations. Next we compute

$$\begin{aligned}
\sum_k A_k &= \sum_k \int_{\mathbb{H}^n} T_k u^2 \xi_k(y,0) dy \\
&\le \sum_k \int_{\mathbb{H}^n} T_k u^2 \xi_k(y,0) \sqrt{1+\|\nabla \phi_k\|^2} dy \\
&= \sum_k \int_{\partial D \cap B_k} u^2 \xi_k(x) \sigma(dx) = \|u\|^2_{L^2(\partial D, \sigma(dx))},
\end{aligned}$$

and

$$\begin{aligned}
\sum_k B_k &= \sum_k \int_{\mathbb{H}^n} T_k \xi_k \left[\left| T_k \tau^j \right|^2 + \left| \partial_j \left(T_k \tau^j \right) \right|^2 \right] dy \\
&\le \sum_k \int_{\mathbb{H}^n} T_k \xi_k \left[\|\vec{\tau}\|^2 + \|D\vec{\tau}\|^2 \right] \sqrt{1+\|\nabla \phi_k\|^2} dy \\
&= \int_{\partial D \cap B_k} \xi_k(x) \left[\|\vec{\tau}\|^2 + \|D\vec{\tau}\|^2 \right](x)\ \sigma(dx), \\
&= \int_{\partial D} \left[\|\vec{\tau}\|^2 + \|D\vec{\tau}\|^2 \right](x)\ \sigma(dx).
\end{aligned}$$

By assumption $\sum_k B_k < \infty$, and therefore, we obtain the estimate

$$\left| \int_{\partial D} \nabla u \cdot \vec{\tau}\ \sigma(dx) \right| n \le C_D \left(\sum_k B_k \right)^{\frac{1}{2}} \|u\|_{L^2(\partial D, \sigma(dx))}.$$

It follows that $u \mapsto \int_{\partial D} u(x) \mathrm{bdiv}(\vec{\tau})(x)\sigma(dx)$ is a continuous map, and by self duality of L^2, we conclude.

The second property follows easily from the definition applied to $\lambda\vec{\tau} + \vec{\kappa}$, and the last assertion in the statement follows directly from (3.1.1) applied to $u\varphi$, where u is used as the test function. ∎

3.2 Construction of ORBM

Assume there is a positive, integrable function $\rho : \overline{D} \to \mathbb{R}_+$ satisfying the following conditions: $\rho \in H^1(D)$, the quantity $\nabla \ln(\rho)$ is bounded, and

$$\begin{cases} \Delta \rho = 0 & \text{in } D \\ \nabla \rho \cdot \vec{n} = \mathrm{bdiv}(\rho \vec{\tau}) & \text{on } \partial D \end{cases} \tag{3.2.1}$$

Consider the measure $d\mu = \rho(x)dx$, and the spaces $L^2(\rho) = L^2(D, \mu)$ and $H = H^1(D)$, and the following bilinear functional defined for $(u, v) \in H \times H$

$$\mathcal{E}(u, v) = \frac{1}{2} \int_D \nabla u \nabla (v\rho)\, dx - \frac{1}{2} \int_{\partial D} \vec{\tau} \cdot \nabla u \ v\rho \ \sigma(dx). \tag{3.2.2}$$

Theorem 3.2.1. *Under (3.2.1) and the assumptions above, the bilinear form $(\mathcal{E}, H)$ is a regular Dirichlet form in $L^2(\rho)$, and its associated Hunt process is an obliquely reflected Brownian motion X. Also, the measure μ defined by $d\mu = \rho(x)dx$ is the unique stationary distribution of X.*

Proof. It is clear from the assumptions, and from Cauchy-Schwartz inequality that there is a constant $C > 0$ such that

$$\begin{aligned} \left| \int_D \nabla u \nabla (v\rho) dx \right| &\leq \left| \int_D \nabla u \nabla v \ \rho dx \right| + \left| \int_D \nabla u v(x) \nabla \rho dx \right| \\ &\leq \|\nabla u\|_{L^2(\rho)} \|\nabla v\|_{L^2(\rho)} + C \int_D |\nabla u| \ |v(x)| \, \rho(x) dx \\ &\leq \|\nabla u\|_{L^2(\rho)} \|\nabla v\|_{L^2(\rho)} + C \|\nabla u\|_{L^2(\rho)} \|v\|_{L^2(\rho)} \\ &\leq C \|\nabla u\|_{L^2(\rho)} \|v\|_H . \end{aligned}$$

To check continuity of the second integral in $\mathcal{E}(u, v)$, select a cover (B_k, ϕ_k) of ∂D and let $\{\eta_k\}$ be a partition of unity subordinated to $\{B_k\}$, such that $\eta_k = \lambda_k^3$ for some smooth function λ_k. We have

$$\begin{aligned} \int_{\partial D} \vec{\tau} \cdot \nabla u \ v\rho \ \sigma(dx) &= \sum_k \int_{\partial D \cap B_k} \eta_k \left(\vec{\tau} \cdot \nabla u \right) \ v\rho \ \sigma(dx) \\ &= \sum_k \int_{\partial D \cap B_k} \left(\lambda_k \rho \vec{\tau} \cdot \nabla (\lambda_k u) \right) \lambda_k v \ \sigma(dx) + \int_{\partial D} G(x) uv \ \rho \ \sigma(dx) \end{aligned}$$

for certain bounded function G. The second integral on the right hand side defines a continuous bilinear function in H by the trace theorem [10], so it is enough to study each of the integrals in the sum on the right hand side, which we denote $V_k(u, v)$.

To reduce the load of notation, define $\vec{\xi_k} = \lambda_k \rho \ \left[1 + \|\nabla \phi_k\|^2 \right]^{1/2} \vec{\tau}$. For $(x, s) \in \mathbb{R}^{n-1} \times \mathbb{R}_+ = \mathbb{H}^n$ and $u \in H$ with $\operatorname{supp}(u) \subseteq B_k$, define $T_\phi u(x, s) = u(x, s + \phi(x))$. Then we have

$$V_k(u, v) = \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{n-1} (T_{\phi_k} \xi_k)^j \frac{\partial T_\phi (u\lambda_k)}{\partial x^j} \ T_{\phi_k}(v\lambda_k) \ dx,$$

as η^k vanishes outside B_k. Notice that the sum in j runs from 0 to $n-1$, therefore ξ_k^j has bounded weak derivatives, given the assumptions on $\vec{\tau}, \rho, \phi_k$. Proceeding as in [21, Theorem 6.2], we obtain

$$\begin{aligned} V_k(u,v) &\leq C \left\| T_\phi(u\lambda_k) \right\|_{W^{1,2}(\mathbb{H}^n)} \left\| T_\phi(v\lambda_k) \right\|_{W^{1,2}(\mathbb{H}^n)} \\ &\leq C \left\| u\lambda_k \right\|_{W^{1,2}(B_k \cap D)} \left\| v\lambda_k \right\|_{W^{1,2}(B_k \cap D)}, \end{aligned}$$

which proves continuity of the second term after summing over k, recalling that the partition B_k is locally finite. The second inequality follows since T_{ϕ_k} defines a continuous map from $W^{s,2}(D \cap B_k) \to W^{s,2}(\mathbb{H}^n)$ for $0 \leq s \leq 1$ ([10, Lemma 1]).

Next we show coercivity. First, assume that $u \in H$ is bounded. Integrating by parts,

$$\begin{aligned} 2\mathcal{E}(u,u) &= \left\| \nabla u \right\|^2_{L^2(\rho)} + \int_D u\nabla u \ \nabla\rho \ dx - \int_{\partial D} \frac{1}{2}\vec{\tau} \cdot \nabla(u^2) \ \rho \ \sigma(dx) \\ &\geq \left\| \nabla u \right\|^2_{L^2(\rho)} - C \left\| u \right\|_{L^2(\rho)} \left\| \nabla u \right\|_{L^2(\rho)} - \frac{1}{2} \int_{\partial D} \tau \cdot \nabla(u^2)\rho \ \sigma(dx) \\ &\geq \kappa \left\| \nabla u \right\|^2_{L^2(\rho)} - C \left\| u \right\|^2_{L^2(\rho)} - C \int_{\partial D} u^2 \ \rho \ \sigma(dx), \end{aligned}$$

where κ and C are independent of u. To show that the last inequality holds, first notice that for any $\varepsilon > 0$ and for $2\kappa = \min_{\partial G} \rho$, we have

$$\left\| \nabla u \right\|^2_{L^2(\rho)} - C \left\| u \right\|_{L^2(\rho)} \left\| \nabla u \right\|_{L^2(\rho)} \geq 2\kappa \left\| \nabla u \right\|^2_{L^2(\rho)} - \varepsilon \left\| \nabla u \right\|^2_{L^2(\rho)} - C_\varepsilon \left\| u \right\|^2_{L^2(\rho)}.$$

By applying the same methods and notation as in the proof of continuity, we estimate the last term

$$\begin{aligned} \int_{\partial D} \vec{\tau} \cdot \nabla(u^2)\rho \ \sigma(dx) &= \int_{\partial D} G(x)u^2 \ \rho \ \sigma(dx) + \sum_k V_k(u^2, 1) \\ &= \int_{\partial D} Gu^2 \ \rho \ \sigma(dx) + \sum_k \int_{\mathbb{R}^{N-1}} (T_{\phi_k}\xi_k)^j \partial_j T_{\phi_k} u^2 \lambda_k^2 \ dx \\ &= \int_{\partial D} Gu^2 \ \rho \ \sigma(dx) - \sum_k \int_{\mathbb{R}^{N-1}} \partial_j (T_{\phi_k}\xi_k)^j \ T_{\phi_k} u^2 \lambda_k^2 \ dx \\ &\geq C \int_{\partial D} u^2 \ \rho \ \sigma(dx). \end{aligned}$$

By using the trace inequality with $\varepsilon > 0$, we conclude that the form (3.2.2) is coercive.

Next we prove the contraction properties. Recall that $Tu = (u \wedge 1)^+$ for $u \in H$. A straight forward calculation shows that $\mathcal{E}(Tu, u - Tu) = 0$. On the other hand

$$\begin{aligned}2\mathcal{E}(u - Tu, Tu) &= \int_D \nabla(u-1)^+ \nabla\rho \, dx - \int_{\partial D} \vec{\tau} \cdot \nabla(u-1)^+ \, \rho\sigma(dx) \\ &= -\int_D (u-1)^+ \Delta\rho \, dx - \int_{\partial D} (u-1)^+ \left(\nabla\rho \cdot \vec{n}(x) - \mathrm{bdiv}(\rho\vec{\tau})\right) \sigma(dx)\end{aligned}$$

which is equal to zero by our assumptions (3.2.1) on the density ρ. This proves that $(\mathcal{E}, H)$ is a Dirichlet form.

The local property follows easily form the definition. Regularity follows from the well-known fact that $C_0^\infty(\overline{D})$ is a dense subspace of $W^{1,2}(D)$. See, for example, [21].

By Theorem IV.1.5 in [8], there is a Hunt process X properly associated with $(\mathcal{E}, H)$, which has continuous trajectories, as the local condition is trivially satisfied (Theorem III.1.2 in [8]). We next use the stochastic calculus for non-symmetric Dirichlet forms developed in [21] to find a Skorokhod decomposition of the process X.

Fix $\tilde{u}$, a quasi-continuous version of $u \in C^2(\mathbb{R}^n)$ with respect to the capacity induced by $(\mathcal{E}, H)$. Let L_t be the positive, continuous additive functional associated with the measure $\frac{1}{2}\rho \cdot \sigma$. We then have that L_t is supported in the boundary ∂D. Further, by (1.2) in [21],

$$\mathbb{E}_{h\cdot\rho dx} \int_0^t \mathbb{1}_{\partial\Gamma}(X_s) dL_s = \int_0^t \langle \mathbb{1}_{\partial D}\sigma, \hat{p}_s h\rangle \, ds = \int_0^t \langle \sigma, \hat{p}_s h\rangle \, ds = \mathbb{E}_{h\cdot\rho dx} L_t.$$

Since h can be any non-negative Borel function on $\overline{D}$, we deduce that

$$\mathbb{P}_x\left(L_t = \int_0^t \mathbb{1}_{\partial D}(X_s) dL_s\right) = 1 \qquad \text{for all } x \in \overline{D} \setminus N, \tag{3.2.3}$$

where N is a set of $\mathcal{E}_1-$capacity zero. This takes care of the existence of a continuous local time.

By Theorem 2.7.2, we have,

$$\lim_{\alpha\to\infty} \alpha^2 \mathbb{E}_{v\rho dx} \int_0^\infty e^{-\alpha t} (f \cdot L)_t dt = \frac{1}{2} \int_{\partial D} v(x) f(x) \rho(x) \sigma(dx). \tag{3.2.4}$$

Next, consider the PCAF $Q_t = t$. Use $f \equiv 1$ in Theorem 2.7.2, and that $\alpha^2 \int_0^\infty e^{-\alpha t} t dt = 1$ to get

$$\int_D v(x)\rho(x)dx = \lim_{\alpha\to\infty} \alpha^2 \mathbb{E}_{v\rho dx} \int_0^\infty e^{-\alpha t} t dt = \int_{\overline{D}} v(x) d\mu,$$

for all positive, bounded $h \in H^1(D)$. This shows that $\mu(dx) = \rho(x)dx$ and so,

$$\lim_{\alpha\to\infty} \alpha^2 \mathbb{E}_{v\rho dx} \int_0^\infty e^{-\alpha t}(f \cdot Q)_t dt = \int_D v(x)f(x)\rho(x)dx. \tag{3.2.5}$$

By the one-to-one correspondence we conclude that the measure ρdx is associated to the PCAF dt. Fix $u \in H$ and let $\tilde{u}$ be a quasi-continuous version of u. Consider the (signed) measure

$$\nu(dx) = \frac{1}{2}\Delta u(x)dx + \frac{1}{2}\left[\vec{n}(x) + \vec{\tau}(x)\right] \cdot \nabla u(x)\sigma(dx).$$

By (3.2.4) and (3.2.5) we deduce that the additive functional A_t associated to ν is given by

$$dA_t = \frac{1}{2}\Delta u(X_t)dt + [\vec{n}(X_t) + \vec{\tau}(X_t)] \cdot \nabla u(X_t)dL_t.$$

Integrating by parts we have

$$\begin{aligned}
\lim_{\alpha\to\infty} \alpha^2 \mathbb{E}_{v\rho dx} \int_0^\infty e^{-\alpha t}A_t dt &= \frac{1}{2}\int_D \Delta\tilde{u}\ v\rho\ dx + \frac{1}{2}\int_{\partial D} [\vec{n} + \vec{\tau}] \cdot \nabla u\ v\rho\ \sigma(dx) \\
&= -\frac{1}{2}\int_D \nabla u \cdot \nabla(v\rho)\ dx + \frac{1}{2}\int_{\partial D} (\vec{\tau} \cdot \nabla u)v\rho\ \sigma(dx) \\
&= -\mathcal{E}(u, v).
\end{aligned}$$

Set $K_t = \tilde{u}(X_t) - \tilde{u}(X_0)$. By the Fukushima decomposition, Theorem 2.7.3, and the previous computation, we have that $K_t = A_t + M_t$, where M_t is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. The quadratic variation process of M_t is a PCAF, and thus it has an associated smooth measure μ. To use Theorem 2.7.4 for bounded $f \in H$, and given that X is a continuous process, we first compute

$$\begin{aligned}
\mathcal{E}(u^2, f) &= \frac{1}{2}\int_D 2u\nabla u \cdot \nabla(f\rho)dx - \frac{1}{2}\int_{\partial D} 2u(\tau \cdot \nabla u)f\rho\ \sigma(dx) \\
&= \int_D \nabla u \cdot [\nabla(uf\rho) - f\rho\nabla u]\, dx - \int_{\partial D} (\tau \cdot \nabla u)uf\rho\ \sigma(dx) \\
&= 2\mathcal{E}(u, uf) - \int_D |\nabla u|^2 f\rho dx.
\end{aligned}$$

Applying Theorem 2.7.4 we obtain

$$\int_{\overline{D}} \tilde{f}(x)\mu(dx) = \int_D |\nabla u|^2 fv\rho dx,$$

and thus by (3.2.5) we have

$$\langle M \rangle_t = \int_0^t |\nabla u(X_s)|^2 \, ds \;\Rightarrow\; M_t = \int_0^t \nabla u(X_s) dB_s, \tag{3.2.6}$$

for some n-dimensional Brownian motion B_s, adapted to $(\mathcal{F}_t)_{t\geq 0}$. This implication follows from a standard construction of a Brownian motion based on Levy's theorem. Writing $K_t = M_t + A_t$ in terms of A_t and the martingale M_t just found, allows us to write Itô's formula for $\tilde{u}(X_t)$:

$$u(X_t) - u(X_0) = \int_0^t \nabla u(X_s) dB_s + \int_0^t [\vec{n} + \vec{\tau}]\,(X_s) \cdot \nabla u(X_s) dL_s + \frac{1}{2}\int_0^t \Delta u(X_s) ds.$$

This shows that the distribution of X is that of an obliquely reflected reflected Brownian motion.

To show that $\mu(dx) = \rho(x)dx$ is invariant, we will use (3.2.5) with $v \equiv 1$. We obtain by Fubini's theorem,

$$\begin{aligned}
\int_D h(x)f(x)\rho(x)dx &= \lim_{\alpha\to\infty} \alpha^2 \mathbb{E}_{h(x)\rho(x)dx} \int_0^\infty e^{-\alpha t} \int_0^t f(X_u) du dt \\
&= \lim_{\alpha\to\infty} \alpha^2 \mathbb{E}_{h(x)\rho(x)dx} \int_0^\infty \int_u^\infty e^{-\alpha t} f(X_u) dt du \\
&= \lim_{\alpha\to\infty} \alpha \mathbb{E}_{h(x)\rho(x)dx} \int_0^\infty e^{-\alpha u} f(X_u) du \\
&= \lim_{\alpha\to\infty} \alpha \int_D \int_0^\infty e^{-\alpha u} \mathbb{E}_x f(X_u) du h(x)\rho(x)dx \\
&= \lim_{\alpha\to\infty} \alpha \left\langle G_\alpha f, h\mu \right\rangle,
\end{aligned}$$

which is condition (vii) in Theorem 5.1.3 in [16], which is equivalent to condition (iii) in that theorem:

$$\mathbb{E}_{h(x)\rho(x)dx}(f \cdot t)_t = \int_0^t \langle f\mu, P_u h\rangle \, du,$$

where P_u is the semigroup associated to X. Setting $h \equiv 1$ we obtain, by Fubini's theorem,

$$\begin{aligned}
\int_D \mathbb{E}_x \int_0^t f(X_u) du \; \rho(x)dx &= t \int_D f(x)\rho(x)dx \\
\int_0^t \int_D P_u f(x)\rho(x) dx du &= t \int_D f(x)\rho(x)dx.
\end{aligned}$$

Taking derivatives with respect to t yields that $\rho(x)dx$ is the stationary distribution of X. Uniqueness comes from a comment at the end of Section 2.6. ∎

Corollary 3.2.2. *Under the assumptions stated above, the stochastic differential equation in* $\overline{D}$

$$X_t = x + B_t + \int_0^t (\vec{n} + \vec{\tau})(X_s) dL_s \tag{3.2.7}$$

has a weak solution, that is, there exists a Brownian motion B_t *and a local-time process* L_t *such that* (3.2.7) *holds, and* $X_t \in \overline{D}$ *for* $t \geq 0$. *This solution exist for all* $x \in \overline{D}$, *except in a set of* $\mathcal{E}_1-$*capacity zero.*

Proof. By using quasi-continuous versions of the coordinate functions e_j in Itô's formula (see last line of previous theorem), and redefining X_t^j by $\tilde{e_j}(X_t)$ we obtain the desired solution. ■

This last result should be interpreted as a way to construct an obliquely reflected diffusion with a prescribed stationary distribution. Indeed, given a harmonic function ρ in D, one would have to find a boundary push $\vec{\tau}$ with the appropriate smoothness and satisfying the second equation in (3.2.1). Then, we obtain the ORBM with stationary density ρ.

Chapter 4

SPINNING BROWNIAN MOTION

4.1 Introduction

To set things precisely, consider a C^2 domain $D \subseteq \mathbb{R}^n$, that is, there is $\phi \in C_b^2(\mathbb{R}^n)$ such that $D = \phi^{-1}(0,\infty)$ and $\partial D = \phi^{-1}(0)$. Let Ω be the space of continuous functions on $[0,\infty)$ with values in $\mathbb{R}^{n+p}$. For $\omega \in \Omega$, let $X_t(\omega) = \omega(t)$ be the coordinate map. The space Ω is endowed with the topology of uniform convergence on compact time intervals, which for continuous processes coincides with the Skorohod topology. From now on, $\mathcal{F}_t$ will be the right continuous, completed σ-algebra generated by X_u, for $0 \leq u \leq t$.

We say that the pair (X_t, S_t) with values in $\overline{D} \times \mathbb{R}^p$ is a **Spinning Brownian motion (sBM)** if there exists an $n-$dimensional Brownian motion B_t such that

$$\begin{cases} dX_t &= \sigma(X_t)dB_t + \vec{\gamma}(X_t, S_t)dL_t, \\ dS_t &= \left[\vec{g}(X_t) - S_t\right] dL_t, \end{cases} \tag{4.1.1}$$

where $\vec{n}$ is the interior unit normal to ∂D, $\sigma : \overline{D} \to M_{n\times n}(\mathbb{R})$ is a matrix valued map, $\vec{\gamma} = \vec{n} + \vec{\tau}$ where $\vec{\tau} : \partial D \times \mathbb{R}^p \to \mathbb{R}^n$, and $\vec{g} : \partial D \to \mathbb{R}^p$ are Lipschitz, bounded, functions satisfying:

(i) $X_t \in \overline{D}$, for all $t \geq 0$ a.s.

(ii) There is $\nu > 0$ such that $\xi \cdot \sigma(y)\xi \geq \nu\, |\xi|^2$, for all $y \in \overline{D}$, $\xi \in \mathbb{R}^n$;

(iii) $\vec{\tau}(x,s) \cdot \vec{n}(x) = 0$ for all $s \in \mathbb{R}^d$, $x \in \partial D$.

Also, L_t is the local time of X_t: L_t is an increasing, continuous process starting at 0, such that $dL_t = \mathbb{1}_{\partial D}(X_t)dL_t$.

The following is one of the main theorems of Section 4.4.

Theorem 4.1.1. *Let $D \subseteq \mathbb{R}^n$ is a C^2 domain. Assume the coefficients $\sigma, \vec{\tau}, \vec{g}$ are of class C^2, uniformly Lipschitz, and bounded. Let B_t be a n-dimensional Brownian motion in $(\Omega, \mathcal{F}_t)$. Then, there exist a (strongly) unique, continuous, $\mathcal{F}_t$-adapted processes (X, S) satisfying* (4.1.1)*, with $X_t \in \overline{D}$.*

The process solving (4.1.1) belongs to a larger family of processes with randomized oblique reflection. One might attempt to study a similar system in which the function $\vec{\gamma}$ is random and depending just on X_t, instead of depending also on S_t. But the connection is not clear at this point to the author. One of the difficulties of dealing with such an equation is that the standard theory of stochastic differential equations covers mostly n-dimensional diffusions for which the driving Brownian motion is also n-dimensional. In our case, there is a gap of p dimensions between the process (X_t, S_t) and B_t, so existence does not follow directly from any of the classical approaches to solving SDE's. Nonetheless, our approach takes advantage of the classical theory: we add a small noise εdW_t to dS_t and take $\varepsilon \to 0$. We describe this construction in the next sections.

4.2 The approximating process ...

4.2.1 Existence results

The following construction will be of fundamental importance to show existences of spinning Brownian motion. It does not trivially follow from the submartingale problem as the reflection coefficient is unbounded, but a cut-off scheme turns out to be successful. For reasons that will be apparent in Section 4.4, we call the solution of (4.2.1) an **approximating process**.

Theorem 4.2.1. *Let D and $\vec{\tau}$ be as in Theorem 2.5.3, and let $\vec{g}$ be a bounded, Lipschitz vector field on ∂D. Consider n and p dimensional, independent Brownian motion processes*

B_t and W_t. For each $\varepsilon > 0$, the equation

$$\begin{cases} dX_t = \sigma(X_t)dB_t + \vec{\gamma}(X_t, S_t)dL_t, \\ dS_t = \varepsilon dW_t + [\vec{g}(X_t) - S_t]\, dL_t, \end{cases} \tag{4.2.1}$$

has a unique strong solution satisfying:

(i) $X_t \in \overline{D}$ *for all* $t \geq 0$,

(ii) L_t *is a boundary local time process that only increases when* $X_t \in \partial D$, *that is,* $dL_t = \mathbb{1}_{\partial D}(X_t)dL_t$.

(iii) The process has infinite lifetime.

Proof. Let $\psi^n \in C_b^\infty(\mathbb{R}^p, \mathbb{R}^p)$ be ion satisfying $\psi^n(s) = s$ for $|s| \leq n$, and consider the obliquely reflected Brownian motion $Z^n = (X^n, S^n)$ in $D \times \mathbb{R}^p$ solving

$$\begin{cases} dX_t = \sigma(X_t)dB_t + \vec{\gamma}(X_t, S_t)dL_t, \\ dS_t = \varepsilon dW_t + [\vec{g}(X_t) - \psi^n(S_t)]\, dL_t. \end{cases} \tag{4.2.2}$$

Define the sequence of stopping times $T_n = \{t > 0 : |S_t^n| > n\}$. It is clear that for $m > n$ we have that $Z^m_{t\wedge T_n} = Z^n_{t\wedge T_n}$ and $T_m > T_n$ a.s. Let $\xi = \sup T_n$. It is clear that a process Z can be defined for $t < \xi$ such that $Z_{t\wedge T_n} = Z^n_{t\wedge T_n}$, and so $Z = (X, S)$ solves (4.2.1) for $t < \xi$.

The corollary follows if $\xi = \infty$ a.s. Let ϕ be a defining function for the set D, that is $\phi \in C_b^2(\mathbb{R}^n)$ with $D = \phi^{-1}((0, \infty))$, $\partial D = \phi^{-1}(\{0\})$, and $|\nabla\phi(x)| \geq 1$ on ∂D. By Itô's formula

$$\begin{aligned} \phi(X_{t\wedge T_m}) - \phi(X_0) &= \int_0^{t\wedge T_m} \nabla\phi(X_u)dB_u + \int_0^{t\wedge T_m} \nabla\phi(X_u)\vec{\gamma}(X_u, S_u)dL_u + \frac{1}{2}\int_0^{t\wedge T_m} \Delta\phi(X_u)du \\ &\geq \int_0^{t\wedge T_m} \nabla\phi(X_u)dB_u + L_{t\wedge T_m} + \frac{1}{2}\int_0^{t\wedge T_m} \Delta\phi(X_u)du. \end{aligned}$$

Since D is bounded, the quantity $\phi(X_{t\wedge T_m}) - \phi(X_0)$ is totally bounded by $2\|\phi\|_\infty$. Since $\Delta\phi$ is also bounded, we arrive at

$$C_\phi(1 + t) - \int_0^{t\wedge T_m} \nabla\phi(X_u)dB_u \geq L_{t\wedge T_m},$$

where C_ϕ is a positive constant depending only on ϕ. By using the exponential martingale $\exp\left[-\int_0^t \alpha\nabla\phi(X_u)dB_u - \frac{\alpha^2}{2}\int_0^t |\nabla\phi(X_u)|^2\,du\right]$ we get that for any $\alpha > 0$

$$\begin{aligned}
\mathbb{E}\left[e^{\alpha L_{t\wedge T_m}}\right] &\le \exp\left[C_\phi\alpha(1+t)\right]\mathbb{E}\exp\left(-\alpha\int_0^{t\wedge T_m}\nabla\phi(X_u)dB_u\right)\\
&\le \exp\left[C_\phi\alpha(1+t)\right]\mathbb{E}\exp\left(\frac{\alpha^2}{2}\int_0^t|\nabla\phi(x_u)|^2\,du\right)\\
&\le \exp\left(C_\phi\alpha(1+t)+\alpha^2C'_\phi t\right),
\end{aligned}$$

and therefore we obtain the estimate,

$$\mathbb{E}\left[e^{\alpha L_{t\wedge T_m}}\right] \le C^\alpha e^{C\alpha(1+\alpha)t}, \tag{4.2.3}$$

where the constant C does not depend on the starting point of the process.

We next deduce a very useful representation of S_t. By Itô's formula, we have for $\varepsilon \ge 0$ and $t < \xi$

$$d\left[e^{L_t}S_t\right] = e^{L_t}S_tdL_t + \varepsilon e^{L_t}dW_t + e^{L_t}\left[\vec{g}(X_t) - S_t\right]dL_t = \varepsilon e^{L_t}dW_t + e^{L_t}\vec{g}(X_t)dL_t.$$

We obtain for $t < \xi$,

$$S_t = S_0e^{-L_t} + \varepsilon e^{-L_t}\int_0^t e^{L_u}dW_u + e^{-L_t}\int_0^t \vec{g}(X_u)e^{L_u}dL_u. \tag{4.2.4}$$

Under $\mathbb{P}_{x,s}$ the first term on the right hand side of last equation is bounded, and we can uniformly bound the third term as follows:

$$\left|e^{-L_t}\int_0^t \vec{g}(X_u)e^{L_u}dL_u\right| = \left|e^{-L_t}\int_0^t \vec{g}(X_u)de^{L_u}\right| \le \|\vec{g}\|_\infty e^{-L_t}\int_0^t de^{L_u} \le \|\vec{g}\|_\infty.$$

In the second inequality below, we use Itô's isometry to deduce that for large m and some constant $C > 0$,

$$\begin{aligned}
\mathbb{E}\left|S_{t\wedge T_m}\right|^2 &\le C + 2\mathbb{E}\left|e^{-L_u}\int_0^{t\wedge T_m} e^{L_u}dW_u\right|^2\\
&\le C + 2\mathbb{E}\left|\int_0^{t\wedge T_m} e^{L_u}dW_u\right|^2\\
&= C + 2\mathbb{E}\int_0^{t\wedge T_m} e^{2L_u}du\\
&\le\ C + 2\mathbb{E}\left[te^{2L_t}\right],
\end{aligned} \tag{4.2.5}$$

where the constant C is independent of both the starting point of (X, S) and t. On $\{\xi < t\}$ we have $|S_{t\wedge T_m}| \nearrow \infty$ as m grows to ∞ and t stays fixed. If $\mathbb{P}(\xi < t) > 0$ for some $t \geq 0$, we have by the monotone convergence theorem that $\lim_{m\to\infty} \mathbb{E}\left|S_{t\wedge T_m}\right|^2 = \infty$, which contradicts (4.2.5), and so $\mathbb{P}(\xi < t) = 0$. Since $\{\xi < \infty\} = \bigcup_n \{\xi < n\}$, we are done.

■

Corollary 4.2.2. *Let* $(X^\varepsilon, S^\varepsilon)$ *be a solution to* (4.2.1). *For any* $t \geq 0$,

$$S_t = e^{-L_t} S_0 + \varepsilon e^{-L_t} \int_0^t e^{L_u} dW_u + e^{-L_t} \int_0^t \vec{g}(X_u) e^{L_u} dL_u. \tag{4.2.6}$$

Proof. Since the life time $\xi = \infty$ almost surely, the claim follows from (4.2.4). ■

Fix $T > 0$. We would like to point out is a very raw bound for the supremum of $|S_t^\varepsilon|$, for times $0 \leq t \leq T$. It comes out from the well known Burkholder-Davis-Gundy martingale inequality.

$$\begin{aligned} |S_t|^2 &\leq 3\left|S_0\right|^2 e^{-2L_t} + 3e^{-2L_t} \left|\int_0^t e^{L_u} dW_u\right|^2 + 3e^{-2L_t} \left(\int_0^t \left|\vec{g}(X_u)\right| de^{L_u}\right)^2 \\ &\leq 3\left|S_0\right|^2 + 3\left|\int_0^t e^{L_u} dW_u\right|^2 + 3\left\|\vec{g}\right\|_\infty \left(1 - e^{-L_t}\right). \end{aligned}$$

If $\mathbb{E}\left|S_0\right|^2 < \infty$, by estimate (4.2.3) and the Burkholder-Davis-Gundy inequality, it follows that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |S_t|^2\right] < \infty. \tag{4.2.7}$$

Before moving on, we would also like to point ou that the process (4.2.2) converges weakly to the approximating process (4.2.1). To this end, we first need the following lemma, which is taken form the doctoral work of Echeverria.

Lemma 4.2.3 (Echeverria [13])**.** *Let* $\mathcal{D}\left([0,\infty,\overline{G}]\right)$ *denote the space of right continuous paths on* $\overline{G}$ *endowed with the Skorohod topology. Let* $\mathbb{P}$, $\mathbb{P}^n$, $\mathbb{P}^{n,k}$ *be probability measures on* $\mathcal{D}$, *and let* $\{\tau_k\}$ *be a sequence of stopping times on* $\mathcal{D}$, *such that, for all* $t > 0$,

(i) $\mathbb{P}\left(\tau_k \textit{ is lower semicontinuous}\right) = 1$

(ii) $\lim_{k\to\infty} \mathbb{P}(\tau_k \le t) = 0.$

Suppose that $\mathbb{P}^n = \mathbb{P}^{n,k}$ on $\mathcal{F}_{\tau_k}$, that for each fixed k, $\left\{\mathbb{P}^{n,k}\right\}_{n\ge 1}$ forms a weakly compact set in $\mathcal{D}$, and denote $\mathbb{Q}^k$ any limit of $\mathbb{P}^{n,k}$. Suppose that $\mathbb{Q}^k = \mathbb{P}$ on $\mathcal{F}_{\tau_k}$. Then $\mathbb{P}^n$ tend weakly to $\mathbb{P}$.

Proof. We have to show that

$$\lim_{n\to\infty} \int \psi(\omega)\mathbb{P}^n(d\omega) = \int \psi(\omega)\mathbb{P}(d\omega) \tag{4.2.8}$$

for all bounded continuous $\psi : \mathcal{D}\left([0,\infty,\overline{G}]\right) \to \mathbb{R}$. Actually it suffices to show (4.2.8) for all ψ that are in addition $\mathcal{F}_t$-measurable for any given t. So assume ψ is $\mathcal{F}_t$-measurable. Then

$$\begin{aligned}\int \psi(\omega)\mathbb{P}^n(d\omega) &= \int_{\{\tau_k\le t\}} \psi(\omega)\mathbb{P}^n(d\omega) + \int_{\{\tau_k> t\}} \psi(\omega)\mathbb{P}^n(d\omega)\\ &= \int_{\{\tau_k\le t\}} \psi(\omega)\mathbb{P}^n(d\omega) + \int_{\{\tau_k> t\}} \psi(\omega)\mathbb{P}^{n,k}(d\omega).\end{aligned}$$

So,

$$\left|\int \psi(\omega)\mathbb{P}^n(d\omega) - \int_{\{\tau_k> t\}} \psi(\omega)\mathbb{P}^{n,k}(d\omega)\right| \le \sup_{\omega\in\mathcal{D}} |\psi(\omega)|\, \mathbb{P}^n\left(\tau_k \le t\right),$$

and thus

$$\begin{aligned}\left|\int \psi(\omega)\mathbb{P}^n(d\omega) - \int \psi(\omega)\mathbb{P}(d\omega)\right| &\le \left|\int_{\{\tau_k> t\}} \psi(\omega)\mathbb{P}^{n,k}(d\omega) - \int \psi(\omega)\mathbb{P}(d\omega)\right| +\\ &\quad + \sup_{\omega\in\mathcal{D}} |\psi(\omega)|\, \mathbb{P}^n\left(\tau_k \le t\right)\\ &\le \left|\int_{\{\tau_k> t\}} \psi(\omega)\mathbb{P}^{n,k}(d\omega) - \int_{\{\tau_k> t\}} \psi(\omega)\mathbb{P}(d\omega)\right| +\\ &\quad + \sup_{\omega\in\mathcal{D}} |\psi(\omega)| \left[\mathbb{P}\left(\tau_k \le t\right) + \mathbb{P}^n\left(\tau_k \le t\right)\right].\end{aligned}$$

Next, we use that $\mathbb{P} = \mathbb{Q}^k$ on $\mathcal{F}_{t_k}$, that $\mathbb{P}^{n,k}\left(\tau_k \le t\right) = \mathbb{P}^n\left(\tau_k \le t\right)$ and $\mathbb{Q}^k\left(\tau_k \le t\right) = \mathbb{P}\left(\tau_k \le t\right)$ to deduce

$$\begin{aligned}\left|\int \psi(\omega)\mathbb{P}^n(d\omega) - \int \psi(\omega)\mathbb{P}(d\omega)\right| &\le \left|\int \phi(\omega)\mathbb{P}^{n,k}(d\omega) - \int \phi(\omega)\mathbb{Q}^k(d\omega)\right| +\\ &\quad + 2\sup_{\omega\in\mathcal{D}} |\psi(\omega)| \left[\mathbb{P}\left(\tau_k \le t\right) + \mathbb{P}^n\left(\tau_k \le t\right)\right].\end{aligned}$$

If τ_k are lower semicontinuous (which is automatic if the paths of the process are a.s. continuous), then $\{\tau_k \leq t\}$ is be closed. By Prohorov's theorem

$$\limsup_{n\to\infty} \mathbb{P}^n\left(\tau_k \leq t\right) = \limsup_{n\to\infty} \mathbb{P}^{n,k}\left(\tau_k \leq t\right) \leq \mathbb{Q}^k(\tau_k \leq t) = \mathbb{P}(\tau_k \leq t).$$

It follows that

$$\limsup_{n\to\infty} \left| \int \psi(\omega)\mathbb{P}^n(d\omega) - \int \psi(\omega)\mathbb{P}(d\omega) \right| \leq 4 \sup_{\omega\in\mathcal{D}} |\psi(\omega)|\, \mathbb{P}\left(\tau_k \leq t\right).$$

This shows that (4.2.8) is indeed true. ■

Corollary 4.2.4. *Let $\psi^n \in C_b^\infty(\mathbb{R}^p, \mathbb{R}^p)$ be a function satisfying $\psi^n(s) = s$ for $|s| \leq n$, and consider the obliquely refelcted Brownian motion $Z^n = (X^n, S^n)$ in $D \times \mathbb{R}^p$ solving*

$$\begin{cases} dX_t = \sigma(X_t)dB_t + \vec{\gamma}(X_t, S_t)dL_t \\ dS_t = \varepsilon dW_t + \left[\vec{g}(X_t) - \psi^n(S_t)\right] dL_t \end{cases}$$

As $n \to \infty$ the process above converges weakly to the approximating process (4.2.1)*.*

Proof. Let $\mathbb{P}$ be the law of the approximating process as constructed in Theorem 4.2.1. Let $\mathbb{P}^n$ be the law of the process $Z^n = (X^n, S^n)$ given in the statement of this corollary, and let $\mathbb{P}^{n,k}$ be the law of Z^n stopped at time $\tau_k = \inf\{t > 0 : |S_t^n| > k\}$.

It is clear form the first part of the proof of Theorem 4.2.1 that for $m > n > k$ we have that $Z_{t\wedge\tau_k} = Z^m_{t\wedge\tau_k} = Z^n_{t\wedge\tau_k}$ for all $t \geq 0$, and so $\mathbb{P}^{n,k} = \mathbb{P}^{m,k}$. It trivially follows that $\mathbb{P}^{n,k}$ converges as $n \to \infty$, and that its limits coincide with $\mathbb{P}$ on $\mathcal{F}_{\tau_k}$.

Since each of the processes Z_t^n and Z_t are continuous, the sets $\{\tau_k \leq t\}$ are closed for each k and t, which means that the stopping times τ_k are lower semicontinuous almost surely.

The last condition in Lemma 4.2.3 ($\lim_{k\to\infty} \mathbb{P}(\tau_k \leq t) = 0$) has been shown in the proof of Theorem 4.2.1. Our claim follows from said lemma. ■

4.2.2 Second moment estimates

The following lemma provides a finer estimate for the second moment of S_t, and will be used later to show existence of a stationary distribution for the approximating process.

Lemma 4.2.5. *Fix a compact set $M \subseteq \mathbb{R}^p$ containing a nieghborhood of zero. There is a constant $C > 0$, independent of $t > 0$, such that for $x \in \overline{D}$ and $s \in M$ the estimate $\mathbb{E}_{x,s} |S_t|^2 \leq C$, holds.*

Proof. Under $\mathbb{P}_{x,s}$ we see that both the first and third terms from the right hand side of equation (4.2.6) are uniformly bounded. It is enough to bound the expectation of Y_t^2, where εY_t is the middle term. To estimate it, we will discretize the Itô integral involved. Let $t_k = tk/N$, for $k = 0, \dots, N$, then

$$\begin{aligned}
Y_t^2 &= \lim_{N\to\infty} \left[\sum_{k=0}^{N-1} e^{-(L_t - L_{t_{k+1}})} \left(W_{t_{k+1}} - W_{t_k} \right) \right]^2 \\
&= \lim_{N\to\infty} \sum_{k=0}^{N-1} e^{-2(L_t - L_{t_{k+1}})} \left(W_{t_{k+1}} - W_{t_k} \right)^2 + \\
&\quad + 2 \sum_{k=0}^{N-1} \sum_{j=k+1}^{N-1} e^{-(L_t - L_{t_{k+1}}) - (L_t - L_{t_{j+1}})} \left(W_{t_{j+1}} - W_{t_j} \right) \left(W_{t_{k+1}} - W_{t_k} \right).
\end{aligned}$$

where the limits are taken in $L^1(\mathbb{P})$ sense.

Let $\mathcal{F}_s^t = \sigma\left((X_u, S_u) : u \in [s,t] \right)_0^+$ be the right continuous, completed augmentation of the sigma algebra of events happening between times s and t. Since $L_t - L_s = L_{t-s} \circ \theta_s$, where θ is the standard shift operator, we have that both $L_t - L_s$ and $W_t - W_s$ are $\mathcal{F}_s^t$-measurable. Also, since Brownian motion is time reversible and its increments are independent,

$$\mathbb{E}_{x,s} \left(W_{t_{k+1}} - W_{t_k} | \mathcal{F}_{t_{k+1}}^t \right) = 0.$$

By conditioning each term of the double sum above on $\mathcal{F}_{t_{k+1}}^t$, we readily see that its expectation equals zero. Therefore, by independence of Brownian increments again,

$$\begin{aligned}
\mathbb{E}_{x,s} \left(Y_t^2 \right) &= \lim_{N\to\infty} \sum_{k=0}^{N-1} \mathbb{E}_{x,s} \left[e^{-2(L_t - L_{t_{k+1}})} \mathbb{E}_{x,s} \left[\left(W_{t_{k+1}} - W_{t_k} \right)^2 | \mathcal{F}_{t_{k+1}}^t \right] \right] \\
&= \lim_{N\to\infty} \sum_{k=0}^{N-1} \mathbb{E}_{x,s} \left[e^{-2(L_t - L_{t_{k+1}})} \right] \left(t_{k+1} - t_k \right).
\end{aligned}$$

Conditioning this time on the sigma algebra $\mathcal{F}_{t_{k+1}}$, by the Markov property,

$$\begin{aligned}\mathbb{E}_{x,s}\left(Y_t^2\right) &= \lim_{N\to\infty}\sum_{k=0}^{N-1}\mathbb{E}_{x,s}\left[\mathbb{E}_{x,s}\left[e^{-2(L_t-L_{t_{k+1}})}\middle|\mathcal{F}_{t_{k+1}}\right]\right](t_{k+1}-t_k)\\ &= \lim_{N\to\infty}\sum_{k=0}^{N-1}\mathbb{E}_{x,s}\left[\mathbb{E}_{X_{t_{k+1}},S_{t_{k+1}}}\left[e^{-2L_{t-t_{k+1}}}\right]\right](t_{k+1}-t_k)\\ &\leq \limsup_{N\to\infty}\sum_{k=0}^{N-1}\sup_{x,s}\mathbb{E}_{x,s}\left[e^{-2L_{t-t_{k+1}}}\right](t_{k+1}-t_k)\,.\end{aligned}$$

Define $\varphi(u)=\sup_{x,s}\mathbb{E}_{x,s}\left[e^{-2L_u}\right]$. It follows from the calculation above and a simple change of variable that

$$\mathbb{E}_{x,s}\left(Y_t^2\right)\leq\int_0^t\varphi(u)du.$$

Our main claim would follow if the integral in the last line remains bounded as $t\to\infty$. Since $\varphi(u)\geq 0$, this fact follows from next lemma, which concludes our proof.

$\blacksquare$

Lemma 4.2.6. *Let* $\overline{\varphi}(u)=\sup\limits_{x\in\partial D,s}\mathbb{E}_{x,s}\left(e^{-2L_u}\right)$, *and* $\varphi(u)=\sup\limits_{x,s}\mathbb{E}_{x,s}\left(e^{-2L_u}\right)$. *Then,*

(i) For any $u>0$, *we have* $\overline{\varphi}(u)<1$.

(ii) For all $u>0$, $\varphi(u)<1$.

(iii) The integral $\int_0^\infty\varphi(u)du$ *is finite.*

Proof. Let $x\in\partial D$. By Fubini's theorem

$$\begin{aligned}\mathbb{E}_{x,s}\left[e^{-2L_u}\right] &= 2\int_0^\infty e^{-2h}\mathbb{P}_{x,s}\left[L_u<h\right]dh\\ &\leq \mathbb{P}_{x,s}\left(L_u\leq q\right)\left(1-e^{-2q}\right)+e^{-2q}\\ &= 1-\left(1-e^{-2q}\right)\mathbb{P}_{x,s}\left(L_u>q\right),\end{aligned}$$

where in the second to last step, we have split the integral into two: before and after $h=q$, used the fact that L_u is increasing, and that $\mathbb{P}_{x,s}\left(L_u<h\right)\leq 1$.

A result by Stroock and Varadhan [26], corollary 2.3, says that almost surely, for any $u > 0$ we have $L_u > 0$. Then, for each $x \in \partial D$ we can choose $q > 0$ such that the right hand side above is strictly less than one. Claim (i) would follow from the continuity in (x, s) of the solution to the submartingale problem (see [26], Theorem 5.4) in case our process lived in a compact set. Nonetheless, since $\vec{\gamma}$ is bounded, we have that

$$\{L_u \leq q\} \subseteq \left\{ \sup_{0 \leq v \leq u} |B_v| \leq \operatorname{diam} D + \|\vec{\gamma}\|_\infty q \right\}.$$

The set on the right hand side is independent of the starting point (x, s), and has probability less than one. Thus, claim (i) is true.

To show (ii) it is enough to consider $x \in D$. Let T_D be the first hitting time of ∂D by the process X. We know that $\mathbb{P}_{x,s}(T_D > 0) = \mathbb{P}_x(T_D > 0) = 1$, as the process X_t is independent of S_t up to time T_D. For $u > 0$, since the local time is increasing we have that $e^{-2L_{2u}} \mathbb{1}_{\{T_D < u\}} \leq e^{-2L_{u+T_D}} \mathbb{1}_{\{T_D < u\}}$. Then, by the strong Markov property

$$\begin{aligned}
\mathbb{E}_{x,s}\left[e^{-2L_{2u}}\right] &\leq \mathbb{P}_{x,s}\left(T_D \geq u\right) + \mathbb{E}_{x,s}\left[\mathbb{E}\left[e^{-2L_{u+T_D}} \mathbb{1}_{\{T_D<u\}} \middle| \mathcal{F}_{T_D}\right]\right] \\
&= \mathbb{P}_{x,s}\left(T_D \geq u\right) + \mathbb{E}_{x,s}\left[\mathbb{E}_{X_{T_D}, S_{T_D}}\left[e^{-2L_u}\right] \mathbb{1}_{\{T_D<u\}}\right] \\
&\leq \mathbb{P}_{x,s}\left(T_D \geq u\right) + \overline{\varphi}(u) \mathbb{P}_{x,s}\left(T_D < u\right) \\
&= 1 - \left(1 - \overline{\varphi}(u)\right) \mathbb{P}_{x,s}(T_D < u).
\end{aligned}$$

Notice that $\mathbb{P}_{x,s}(T_D < u)$ is independent of s, and depends continuously on x. Thus, since $\overline{D}$ is compact, we have that $p = \inf_{x \in D} \mathbb{P}_x(T_D < u) > 0$, because the infimum is achieved at some point away from the boundary. By assertion (i), $\varphi(2u) \leq 1 - (1 - \overline{\varphi}(u))p < 1$, proving (ii).

To see (iii), first notice that for $t, u \geq 0$ the Markov property easily implies that $\varphi(u+t) \leq \varphi(u)\varphi(t)$. Indeed,

$$\begin{aligned}
\mathbb{E}_{x,s}\left[e^{-2L_{t+u}}\right] &= \mathbb{E}_{x,s}\left[e^{-2[L_{t+u} - L_t] - 2L_t}\right] \\
&= \mathbb{E}_{x,s}\mathbb{E}_{x,s}\left[e^{-2L_u \circ \theta_t - 2L_t} | \mathcal{F}_t\right] \\
&= \mathbb{E}_{x,s}\left[\mathbb{E}_{X_t, S_t}\left[e^{-2L_u}\right] e^{-2L_t}\right] \\
&\leq \mathbb{E}_{x,s}\left[\varphi(u) e^{-2L_t}\right] \leq \varphi(u)\varphi(t).
\end{aligned}$$

It follows that $\ln \varphi(t)$ is subadditive and so $\lim_{t\to\infty} \varphi(t)^{1/t}$ exists and equals $\inf_{t>0} \varphi(t)^{1/t}$. By (ii), such limit is less than or equal to $\varphi(1) = 1 - 2\delta < 1$. It follows that for some $t_0 > 0$, and all $t > t_0$ we have $\varphi(t)^{1/t} \leq 1 - \delta$. Choose $\alpha > 0$ such that $1 - \delta = e^{-\alpha}$. For $t > t_0$ we have $\varphi(t) \leq e^{-\alpha t}$. Since $\varphi(u) \leq 1$, it follows that

$$\int_0^\infty \varphi(t)\ dt \leq t_0 - \frac{e^{-\alpha t}}{\alpha}\Big|_{t_0}^{\infty} = t_0 + \frac{e^{-\alpha t_0}}{\alpha} < \infty.$$

■

4.3 ... and its stationary distribution

In Section 4.4, we will show that the approximating process (4.2.1) converges weakly to spinning Brownian motion, which is a recurrent process, as we will also prove. In the approximating case, the space process X_t lives in a bounded domain and it's somehow straightforward to see that it is recurrent. On the other hand, S_t can be anywhere in $\mathbb{R}^p$ with positive probability and it is not trivial to see whether it is recurrent or transient. Nonetheless, the following theorem ensures there is a unique stationary distribution associated to this process.

Theorem 4.3.1. *There is a unique stationary distribution for the approximating process.*

Proof. To find a stationary distribution, we will consider the average occupation time measures:

$$\mu_t^{x,s}(A) = \frac{1}{t}\int_0^t \mathbb{P}_{x,s}\left((X_u, S_u) \in A\right) du = \frac{1}{t}\int_0^t P(u)\mathbb{1}_A(x,s)du, \tag{4.3.1}$$

where A is a Borel set of $\overline{D} \times \mathbb{R}^p$, and P is the strongly continuous semigroup associated to the diffusion (X, S). We assume that the starting point (x, s) belongs to a fixed, but arbitrary, compact set M.

We claim that the family $\left\{\mu_t^{x,s} : t > 0, x \in \overline{D}, s \in M\right\}$ is tight. Indeed, for the compact set $K = \overline{D} \times \{s \in \mathbb{R}^p : |s| \leq \beta\}$ we have by Chebyshev's inequality

$$\mu_t^{x,s}(K^c) = \frac{1}{t}\int_0^t \mathbb{P}_{x,s}\left(|S_u| > \beta\right) du \leq \sup_{u\in[0,t]} \frac{\mathbb{E}_{x,s}\left|S_u\right|^2}{\beta^2}.$$

By Lemma 4.2.5, we conclude that given $\varepsilon > 0$, there is $\beta > 0$ such that for all $(x,s) \in M$, and all $t > 1$ the bound $\mu_t^{x,s}(K) > 1 - \varepsilon$ holds. The claim now follows from Prohorov's theorem.

Recall that the solution to a submartingale problem is a Feller process. Then, Lemma 2.6.1 applies and it follows that any limit point of $\{\mu_t^{x,s}\}$ is a stationary distribution.

Finally, we show that such limit is unique. Let μ be any stationary distribution. For $\delta > 0$, define $A(\delta) = \left\{(x,s) \in D \times \mathbb{R}^p : \operatorname{dist}(x, \partial D) > \delta,\ \|s\| < \delta^{-1}\right\}$. It is clear that $A(\delta)$ increases to $D \times \mathbb{R}^p$ as $\delta \downarrow 0$, and so by continuity of the measure we have that $\mu(A(\delta)) \uparrow 1$. Notice that here we use the fact that

$$\mu(\partial D \times \mathbb{R}^p) = \int_{D \times \mathbb{R}^p} \mathbb{P}_{x,s}\left(X_t \in \partial D\right) \mu(dxds) = 0,$$

since the solution to the submartingale problem spends zero Lebesgue time at the boundary.

It follows that we can choose a $\delta > 0$ such that $\mu(A(\delta)) > \frac{1}{2}$, which we will fix from now on. Choose an open set U whose closure is a compact subset of D and an open set $V \subseteq \mathbb{R}^p$ that is contained in the unit ball. Let T^D be the exit time of the process X from D. By stationarity and Fubini's theorem, for any $t > 0$

$$\begin{aligned}
\mu(U \times V) &= \int_{D \times \mathbb{R}^p} \mathbb{P}_{x,s}\left(X_t \in U, S_t \in V\right) \mu(dxds) \\
&\geq \int_{A(\delta)} \mathbb{P}_{x,s}\left(X_t \in U, S_t \in V\right) \mu(dxds) \\
&\geq \int_{A(\delta)} \mathbb{P}_{x,s}\left(X_t \in U, S_t \in V, t < T^D\right) \mu(dxds) \\
&\geq \int_{A(\delta)} \mathbb{P}_{x,0}\left(X_t \in U, s + \varepsilon W_t \in V, t < T^D\right) \mu(dxds).
\end{aligned}$$

In the last step, we have used that $S_t = s + \varepsilon W_t$ for $t < T^D$. Let X^D be the unique process satisfying $dX_t^D = \sigma(X_t^D) dB_t$. It is clear that $X_t^D = X_t$ for $t < T^D$, and since W_t and B_t are independent,

$$\begin{aligned}
\mu(U \times V) &\geq \int_{A(\delta)} \mathbb{P}_x^D\left(X_t^D 2 \in U\right) \mathbb{P}_0(s + \varepsilon W_t \in V) \mu(dxds) \\
&= \int_{A(\delta)} \int_V \mathbb{P}_x^D\left(X_2^D \in U\right) p(2\sqrt{\varepsilon}; s, y) dy \mu(dxds)
\end{aligned}$$

where $\mathbb{P}^D$ is the law of X^D killed at the boundary of D, and $p(t; s, y)$ is the transition density for p-dimensional Brownian motion. For $(x, s) \in A(\delta)$ we have $\|s\| < \delta^{-1}$. For $y \in V$, we have that $\|y\| \leq 1$, and so we have $p(2\sqrt{\varepsilon}; 0, s) > c_1$ for some positive constant c_1 that depends only on ε and δ. Let m^p be the Lebesgue measure in $\mathbb{R}^p$, and p^D be the density of X^D killed upon hitting the boundary of D. So far we have proved the following lower bound:

$$
\begin{aligned}
\mu(U \times V) &\geq c_1 m^p(V) \int_{A(\delta)} \mathbb{P}_x^D \left(X_2^D \in U \right) \mu(dxds) \\
&= c_1 m^p(V) \int_{A(\delta)} \int_U p^D(2; x, z) dz \mu(dxds).
\end{aligned}
$$

Recall that the function $z \mapsto p^D(t; x, z)$ is the solution to the heat equation

$$
\frac{\partial}{\partial t} p(t; x, z) - \sum_{i,j=1}^{n} [\sigma^T(z)\sigma(z)]_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} p(t, x, z) = 0
$$

with initial condition $\lim_{t \downarrow 0} p(t; x, z) = \delta(x - z)$ and boundary condition $p(t; x, z) = 0$ for $z \in \partial D$. This equation satisfies a maximum principle, and its unique solution takes positive values in D for all $t > 0$. It follows that $m^{n+p}|_{U \times V}$ is absolutely continuous with respect to any stationary distribution μ.

Let μ and ν be any two stationary distributions. A well known consequence of the ergodic decomposition theorem (see [1] for results in ergodic theory) is that any two stationary distributions must be mutually singular, which means that there is a set $K \subseteq \mathbb{R}^{n+p}$ such that $\mu(K) = 0$ and $\nu(K^c) = 0$. From the previous paragraph, it follows that $m^{n+p}|_{U \times V}(K) = 0$ and $m^{n+p}|_{U \times V}(K^c) = 0$. This contradicts our assumption that U and V are open sets, and so there is at most one stationary distribution. ■

4.4 Convergence to SBM

In Section 4.2, we have proved strong existence and uniqueness of solutions to the equation

$$
\begin{cases}
dX_t = \sigma(X_t)dB_t + \vec{\gamma}(X_t, S_t)dL_t, \\
dS_t = \varepsilon dW_t + [\vec{g}(X_t) - S_t]\, dL_t.
\end{cases}
$$

Its solution $(X^\varepsilon, S^\varepsilon)$ has been named the approximating process.

To prove Theorem 4.1.1, we will show that as $\varepsilon \to 0$, the unique limit of the approximating processes exists and such limit is a solution to our original equation (4.1.1). In the foregoing, it will be useful to write $Z_t^\varepsilon = (X_t^\varepsilon, S_t^\varepsilon)$, and to notice that to solve (4.2.1) is equivalent to solve $dZ_t^\varepsilon = \sigma_\varepsilon(Z_t)d\tilde{B}_t + \vec{\kappa}(Z_t^\varepsilon)dL_t^z$ for $\varepsilon > 0$, with $Z_t^\varepsilon \in \overline{D \times \mathbb{R}^d}$, where

$$\sigma_\varepsilon(x,s) = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \varepsilon \mathbf{I}_p \end{pmatrix} \qquad\qquad \vec{\kappa}(x,s) = \begin{pmatrix} \vec{\gamma}(x,s) \\ \vec{g}(x) - s \end{pmatrix}.$$

This is clear since $\overline{D \times \mathbb{R}^d} = \overline{D} \times \mathbb{R}^d$, and the unit normal $\hat{n}(x,s)$ at $(x,s) \in \partial(D \times \mathbb{R}^d) = \partial D \times \mathbb{R}^d$ equals the unit normal $\hat{n}(x)$ at $x \in \partial D$, for any $s \in \mathbb{R}^p$. This last identification also allows us to readily see that the local time L_t of X_t^ε, equals that of Z_t^ε.

In this context, one would be tempted to say that spinning Brownian motion (4.1.1) is the process Z_t^0, but this identification would be very imprecise because the driving Brownian motion for SBM in (4.1.1) is n-dimensional, whereas $\tilde{B}_t$ above is $(n+p)$-dimensional. Without this difference in the dimension, we would not be able to prove strong uniqueness for spinning Brownian motion.

We proceed to prove several lemmas that will imply together tightness of the approximating processes. The following lemma shows the strong equicontinuity of the local times.

Lemma 4.4.1. *There exists a constant $C = C(D, \sigma, \vec{\tau}, \vec{g})$, independent of ε, such that for any* $0 < \delta < 1$

$$\mathbb{E}\left(\sup_{|t-s|\le\delta} |L_t^\varepsilon - L_s^\varepsilon|^2 \right) \le C\delta^{1/2}. \tag{4.4.1}$$

Proof. As usual, the letter C will represent a constant that may vary from line to line, only depending on $D, \sigma, \vec{f}, \vec{g}$. The letter C_T will denote a finite constant also dependent on T.

Let $\phi \in C_b^2(\mathbb{R}^n)$ be a defining function for the boundary of D, and define $\Phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ by $\Phi(x,s) = \phi(x)$. Then $\Phi \in C_b^2(\mathbb{R}^{n+p})$ and it defines the boundary of $D \times \mathbb{R}^p$. By using

Itô's formula on both ϕ and ϕ^2

$$\phi(X^\varepsilon_t) - \phi(X^\varepsilon_s) = \int_s^t \nabla\phi^T\sigma(X^\varepsilon_t)dB_u + \int_s^t \mathcal{L}_u\phi(X^\varepsilon_u)du + \int_s^t \nabla\phi\cdot\hat{n}(X^\varepsilon_u)dL^\varepsilon_u, \tag{4.4.2}$$

$$\phi(X^\varepsilon_t)^2 - \phi(X^\varepsilon_s)^2 = 2\int_s^t \phi\nabla\phi^T\sigma(X^\varepsilon_t)dB_u + \int_s^t \mathcal{L}_u\phi^2(X^\varepsilon_u)du. \tag{4.4.3}$$

Therefore, by the Burkholder-Davis-Gundy inequality [25, Chapter 4, thm 4.1],

$$\begin{aligned}\mathbb{E}\left(\sup_{|t-s|\le\delta}\left|\phi(X^\varepsilon_u)^2 - \phi(X^\varepsilon_s)^2\right|^2\right) &\le 4\mathbb{E}\left(\sup_{|t-s|\le\delta}\left|\int_s^t \phi\nabla\phi^T\sigma(X^\varepsilon_t)dB_u\right|^2\right) + C\delta^2\\ &\le C\mathbb{E}\left(\sup_{|t-s|\le\delta}\int_s^t \left|\phi\nabla^T\phi\sigma(X^\varepsilon_u)\right|^2 du\right) + C\delta^2,\end{aligned}$$

which is less that $C\delta$, as $\phi\in C^2_b(\mathbb{R}^n)$ and σ is bounded. Recall that $\phi(x)>0$ in D, and that $\nabla\phi\cdot\hat{n}(x)\ge 1$ for all $x\in\partial D$. Thus, if $t\ge s$, we have from (4.4.2) that

$$-2\phi(X^\varepsilon_s)\left(\phi(X^\varepsilon_t)-\phi(X^\varepsilon_s)\right) \le C\left|\int_s^t \nabla\phi^T\sigma(X^\varepsilon_t)dB_u\right| + C(t-s).$$

Notice that $\left(\phi(X^\varepsilon_t)-\phi(X^\varepsilon_s)\right)^2 = \phi(X^\varepsilon_t)^2 - \phi(X^\varepsilon_s)^2 - 2\phi(X^\varepsilon_s)\left(\phi(X^\varepsilon_t)-\phi(X^\varepsilon_s)\right)$. Using this decomposition, the previous estimates, Hölder inequality, and Burkholder-Davis-Gundy inequality once more, we arrive at

$$\begin{aligned}\mathbb{E}\left(\sup_{|t-s|\le\delta}|\phi(X^\varepsilon_t)-\phi(X^\varepsilon_s)|^2\right) &\le C\delta^{1/2} + C\mathbb{E}\left(\sup_{|t-s|\le\delta}\left|\int_s^t \nabla\phi^T\sigma(X^\varepsilon_t)dB_u\right|^2\right)^{1/2}\\ &\le C\delta^{1/2}.\end{aligned}$$

This estimate helps to show that the local times L^ε are (uniformly) equicontinuous. Once more, as $\nabla\Phi\cdot\vec{\kappa}\ge 1$ on the boundary of the domain, by using (4.4.2) we obtain

$$\begin{aligned}\frac{1}{2}|L^\varepsilon_t - L^\varepsilon_s|^2 &\le \frac{1}{2}\left|\int_s^t \nabla\phi\cdot\hat{n}(X^\varepsilon_u)dL^\varepsilon_u\right|^2\\ &\le (\phi(X^\varepsilon_t)-\phi(X^\varepsilon_s))^2 + \left|\int_s^t \nabla\phi^T\sigma(X^\varepsilon_u)dB_u\right|^2 + \left|\int_s^t \mathbb{1}_D\mathcal{L}_u\phi(X^\varepsilon_u)du\right|^2.\end{aligned}$$

Since $t\mapsto L^\varepsilon_t$ is increasing, we readily get from last estimate and Itô's isometry that

$$\mathbb{E}\left(\sup_{|t-s|\le\delta}|L^\varepsilon_t - L^\varepsilon_s|^2\right) \le C\delta^{1/2}.$$

Notice that the right hand side does not depend on ε, or on the starting point of the process. ■

The estimate found in the previous lemma will be used to prove tightness of the family $(X^\varepsilon, S^\varepsilon)$. Equicontinuity of X^ε follows by a very straightforward calculation by using once more the Burkholder-Davis-Gundy inequality:

$$\begin{aligned}\frac{1}{2}|X_t^\varepsilon - X_s^\varepsilon|^2 &\leq \left|\int_s^t \mathbb{1}_D(X_u^\varepsilon) dX_u^\varepsilon\right|^2 + \left|\int_s^t \vec{\gamma}(X_u^\varepsilon) dL_u^\varepsilon\right|^2 \\ &= \left|\int_s^t \mathbb{1}_D(X_u^\varepsilon)\sigma(Z_u^\varepsilon) dB_u\right|^2 + \left|\int_s^t \vec{\gamma}(Z_u^\varepsilon) dL_u^\varepsilon\right|^2.\end{aligned}$$

Since both σ and $\vec{\gamma}$ are uniformly bounded, by Itô's isometry we deduce that

$$\mathbb{E}\left[\sup_{|t-s|<\delta} |X_t^\varepsilon - X_s^\varepsilon|^2\right] \leq C\mathbb{E}\left[\sup_{|t-s|<\delta} |B_t - B_s|^2 + |L_t^\varepsilon - L_s^\varepsilon|^2\right] \leq C\delta^{1/2}, \tag{4.4.4}$$

for $0 < \delta < 1$.

Unfortunately, we didn't obtain the same type of L^2 estimate for S^ε. The main issue is that we haven't found a successful way to express S^ε in terms of (sub)martingales that would allow us to use the Burkholder-Davis-Gundy inequality, in order to estimate the supremum of $|S^\varepsilon|^2_{0<t<T}$ in terms of $\mathbb{E}\left(|S_T^\varepsilon|^2\right)$. Instead, we will obtain an estimate in L^1 that is still sufficient to prove tightness of the approximating process, as we'll see in the following two lemmas.

Lemma 4.4.2. *Fix $T > 0$, and $\beta \in (0, 1/4)$. There exist constants $C = C(D, \sigma, \vec{\tau}, \vec{g})$ and $\nu > 0$, independent of ε, such that for any* $0 < \delta < 1$,

$$\mathbb{E}_{x,s}\left(\sup_{\substack{|t-r|\leq\delta \\ 0\leq t,r\leq T}} |S_t^\varepsilon - S_r^\varepsilon|\right) \leq C_T(1 + |s|^2)^{1/2}\delta^\nu. \tag{4.4.5}$$

Proof. By the product rule we see that $d[e^{L_t}S_t^\varepsilon] = \varepsilon e^{L_t^\varepsilon} dW_t + \vec{g}(X_t^\varepsilon) de^{L_t^\varepsilon}$. For $0 \leq r < t \leq T$, we have that

$$S_t^\varepsilon - S_r^\varepsilon = \left(e^{-(L_t^\varepsilon - L_r^\varepsilon)} - 1\right) S_r^\varepsilon + \varepsilon e^{-L_t^\varepsilon} \int_r^t e^{L_u^\varepsilon} dW_u + e^{-L_t^\varepsilon} \int_r^t \vec{g}(X_u^\varepsilon) e^{L_u^\varepsilon} dL_u^\varepsilon.$$

We'll handle the three terms above on the right hand side independently.

First term: Set $E^\varepsilon_{r,t} = \left|\left(e^{-(L^\varepsilon_t - L^\varepsilon_r)} - 1\right) S^\varepsilon_r\right|$. For $x \geq 0$, the function e^{-x} is decreasing, and its derivative is bounded above by 1, thus, for $x > y \geq 0$ we have $0 \leq 1 - e^{-(x-y)} \leq x - y$. We estimate the expectation of E^ε_δ as follows

$$\begin{aligned}
\mathbb{E}_{x,s}\left[\sup_{|t-r|<\delta} E^\varepsilon_{r,t}\right] &\leq \mathbb{E}_{x,s}\left(\sup_{|t-r|<\delta} |L^\varepsilon_t - L^\varepsilon_r| \cdot \sup_{r\in[0,T]} |S^\varepsilon_r|\right) \\
&\leq \mathbb{E}_{x,s}\left[\sup_{|t-r|<\delta} |L^\varepsilon_t - L^\varepsilon_r|^2\right]^{1/2} \mathbb{E}_{x,s}\left[\sup_{r\in[0,T]} |S^\varepsilon_r|^2\right]^{1/2} \\
&\leq C\delta^{1/4}\left(|s|^2 + (1+\varepsilon^2)C_T\right)^{1/2},
\end{aligned}$$

where the bound on the increment of local time comes from Lemma 4.4.1 and the bound on S^ε comes from equation (4.2.7).

Second term: Let's call this term $M^\varepsilon_{r,t}$, ignoring the ε that multiplies it. We compute,

$$M^\varepsilon_{r,t} = e^{-(L^\varepsilon_t - L^\varepsilon_r)} \int_r^t e^{L^\varepsilon_u - L^\varepsilon_r} dW_u = e^{-L^\varepsilon_{t-r}} \int_0^{t-r} e^{L^\varepsilon_u} dW_u \circ \theta_r,$$

where θ_r is the usual shift operator. By the Markov property:

$$\begin{aligned}
\mathbb{E}_{x,s}\left[\sup_{|t-r|<\delta} \left|M^\varepsilon_{r,t}\right|\right] &= \mathbb{E}_{x,s}\left[\mathbb{E}_{X_r,S_r} \sup_{b\in(0,\delta)} e^{-L^\varepsilon_b}\left|\int_0^b e^{L^\varepsilon_u} dW_u\right|\right] \\
&\leq \mathbb{E}_{x,s}\left[\mathbb{E}_{X_r,S_r} \sup_{b\in(0,\delta)} \left|\int_0^b e^{L^\varepsilon_u} dW_u\right|^2\right]^{1/2} \\
&\leq C\mathbb{E}_{x,s}\left[\mathbb{E}_{X_r,S_r} \int_0^\delta e^{2L^\varepsilon_u} du\right]^{1/2} \\
&\leq C\delta^{1/2} \sup_{x,s} \mathbb{E}_{x,s}\left[e^{2L^\varepsilon_1}\right]^{1/2}.
\end{aligned}$$

The third line follows from Burkholder-Davis inequality, and thus, from (4.2.3) we get the estimate

$$\mathbb{E}_{x,s}\left[\sup_{|t-r|<\delta} \left|M^\varepsilon_{r,t}\right|\right] \leq C\delta^{1/2}, \qquad \delta < 1.$$

Third term: This is the simplest one to handle, by using the fact that the local time is increasing and $\vec{g}$ is bounded

$$\left|e^{-L^\varepsilon_t} \int_r^t \vec{g}(X^\varepsilon_u) e^{L^\varepsilon_u} dL^\varepsilon_u\right| \leq C\left|L^\varepsilon_t - L^\varepsilon_r\right|.$$

The main claim of this lemma then follows by Lemma 4.4.1, Hölder inequality, and the previous estimates for $E^{\varepsilon}_{r,t}$ and $M^{\varepsilon}_{r,t}$. ■

Theorem 4.4.3. *Let $\mathbb{P}^{\varepsilon}_{x,s}$ denote the law $(X^{\varepsilon}, S^{\varepsilon})$ starting from (x,s). Then, the family $\mathcal{P}_M = \left\{\mathbb{P}^{\varepsilon}_{x,s} : 0 < \varepsilon < 1,\ x \in \overline{D}, |s| \leq M\right\}$ is tight in the Skorohod space $D\left([0,\infty), \mathbb{R}^{n+p}\right)$.*

Proof. Condition (i) in Theorem 2.2.2 is automatically satisfied by the family $\mathcal{P}_M$. To show the second condition, notice that for $\lambda > 0$ and $\delta \in (0,1)$

$$\begin{aligned} \mathbb{P}_{x,s}\left(\sup_{|t-r|<\delta} |Z^{\varepsilon}_t - Z^{\varepsilon}_r| > \lambda\right) &\leq \frac{1}{\lambda}\mathbb{E}_{x,s}\left[\sup_{|t-r|<\delta} |Z^{\varepsilon}_t - Z^{\varepsilon}_r|\right] \\ &\leq \frac{1}{\lambda}\mathbb{E}_{x,s}\left[\sup_{|t-r|<\delta} |X^{\varepsilon}_t - X^{\varepsilon}_r| + |S^{\varepsilon}_t - S^{\varepsilon}_r|\right]. \end{aligned}$$

By the remark after Lemma 4.4.1 and Lemma 4.4.2, we have that the right hand side above is bounded above by $C_{T,M}\lambda^{-1}\delta^{1/4}$, where $C_{T,M}$ only depends on the coefficients of (4.2.1), and the constants T and M, of course. Therefore, for any $\lambda, \eta > 0$, by choosing $\delta < \left(C^{-1}_{T,M}\lambda\eta\right)^4 \wedge 1$, we have that

$$\sup_{\varepsilon>0} \mathbb{P}_{x,s}\left(\sup_{\substack{|t-r|<\delta \\ 0\leq r,t\leq T}} |Z^{\varepsilon}_t - Z^{\varepsilon}_r| > \lambda\right) \leq \eta,$$

which is the condition we were looking for. ■

The next lemma will show that any limit of the tight family (Z^{ε}) is a (weak) solution of our original equation (4.1.1). Also, we show that pathwise uniqueness for (4.1.1) holds, which combined with weak existence gives strong existence and uniqueness, proving theorem 4.1.1.

Lemma 4.4.4. *Let $(Z^n_t = Z^{\varepsilon_n}_t)$ be any convergent subsequence of the family (indexed by ε) of strong solutions to* (4.2.1)*, as $\varepsilon_n \to 0$, as in Theorem 4.4.3. Call the limit $Z = (X,S)$. Then, there exists a local time process L, such that for $Z = (X,S)$, we have that $(X,S;L)$ satisfy* (4.1.1)*. Also, any solution to* (4.1.1) *is pathwise unique.*

Proof. We first prove that any limit satisfies (4.1.1). By Itô's formula for Z_t^ε, we have that for any $h \in C_0^2(\mathbb{R}^{n+p})$

$$h(Z_t^\varepsilon) - h(Z_0^\varepsilon) = \int_0^t \mathbb{1}_{D\times\mathbb{R}^p} \nabla h(Z_u^\varepsilon)\sigma_\varepsilon(Z_u^\varepsilon) d\tilde{B}_u + \int_0^t \mathbb{1}_{D\times\mathbb{R}^p} \mathcal{L}_u h(Z_u^\varepsilon) du + \\ + \int_0^t \nabla h \cdot \vec{\kappa}(Z_u^\varepsilon) dL_u^\varepsilon. \tag{4.4.6}$$

Our idea is to pass to the limit as $\varepsilon_n \to 0$ in this equation. First, we take care of the term involving local time.

We can actually get stronger convergence of the family of local times L_t^ε if we use (4.4.1) together with Arzela-Ascoli's theorem in the Banach space $L^2\left(C([0,T])\right)$. It follows that there is a Cauchy subsequence $(L_t^{\varepsilon_n})$, and $L \in L^2\left(C([0,T])\right)$, such that

$$\mathbb{E}\left(\sup_{s\in[0,T]} |L_s^{\varepsilon_n} - L_s|^2\right) \to 0.$$

This sequence has a further subsequence that converges pointwise to L. Thus the map $t \mapsto L_t$ is increasing and continuous, and also satisfies an estimate analogous to (4.4.1).

Let $f \in C_0(\mathbb{R}^{n+p})$ be an arbitrary function. We have

$$\begin{aligned} &\mathbb{E}\left(\left|\int_0^t f(Z_u^n) dL_u^n - \int_0^t f(Z_u) dL_u\right|^2\right) \\ &\qquad \le 2\mathbb{E}\left(\left|\int_0^t f(Z_u^n)\left[dL_u^n - dL_u\right]\right|^2\right) + 2\mathbb{E}\left(\left|\int_0^t \left[f(Z_u^n) - f(Z_u)\right] dL_u\right|^2\right) \\ &\qquad \le 2t\,\|f\|_\infty^2\, \mathbb{E}\left(\sup_{u\in[0,t]} |L_u^n - L_u|^2\right) + 2\mathbb{E}\left(\left|\int_0^t \left[f(Z_u^n) - f(Z_u)\right] dL_u\right|^2\right). \end{aligned}$$

We know that the first term above goes to zero as $n \to \infty$. The second one also does, as $f(Z_u^n) \to f(Z_u)$ pointwise, and L_u is continuous with bounded variation (see [25, Chapter IV, thm 2.12]). We use this convergence to prove two different things: first, if a sequence of smooth functions $\varphi_k \nearrow \mathbb{1}_D$, then, as $\mathbb{1}_D(X_u)$ is locally bounded

$$0 = \int_0^t \varphi_k(X_u^n) dL_u^n \to \int_0^t \varphi_k(X_u) dL_u,$$

so the latter term is a.s. zero. By taking $k \to \infty$, monotone convergence ensures that $dL_t = \mathbb{1}_{\partial D}(X_t) dL_t$, and so L_t is a local time process for X_t (and Z_t).

Second, by setting $f(z) = \nabla h(z) \cdot \vec{\kappa}(z)$ we have that

$$\int_0^t \nabla h \cdot \vec{\kappa}(Z_u^\varepsilon) dL_u^\varepsilon \to \int_0^t \nabla h \cdot \vec{\kappa}(Z_u) dL_u,$$

almost surely. Assume that $Z_0^n = (x, s)$ a.s. Then $Z_0 = (x, s)$, and for any $h \in C_0^2(\mathbb{R}^{n+p})$ we have that, almost surely,

$$\begin{aligned} h(Z_t^n) - h(Z_0^n) &\to h(Z_t) - h(Z_0), \\ \nabla h(Z_u^n)^T \sigma_{\varepsilon_n}(Z_u^n) &\to \nabla_x h(Z_u) \sigma(X_u), \\ \sum_{i,j=1}^{n+d} a_{\varepsilon_n}^{i,j}(Z_u) \partial_i \partial_j h(Z_u) &\to \sum_{i,j=1}^{n} (\sigma^T \sigma)^{i,j}(X_u) \partial_i \partial_j h(Z_u). \end{aligned}$$

It follows from (4.4.6), and the dominated convergence theorem that the following equation holds

$$\begin{aligned} h(Z_t) - h(Z_0) = \lim_{n\to\infty} \int_0^t \mathbb{1}_{D\times\mathbb{R}^p} \nabla h(Z_u^n) \sigma_{\varepsilon_n}(X_u^n) d\tilde{B}_u + \int_0^t \mathbb{1}_{D\times\mathbb{R}^p} \mathcal{L}_u^0 h(Z_u) du \, + \\ + \int_0^t \nabla h \cdot \vec{\kappa}(Z_u) dL_u. \end{aligned}$$

By Ito's isometry, we have

$$\mathbb{E}\left(\int_0^t \mathbb{1}_{D\times\mathbb{R}^p} \left[\nabla h \sigma_{\varepsilon_n}(Z_u^n) - \nabla_x h \sigma(Z_u) \right] d\tilde{B}_u \right)^2 \le \int_0^t \mathbb{E} \left| \nabla h \sigma_{\varepsilon_n}(Z_u^n) - \nabla_x h \sigma(Z_u) \right|^2 du.$$

Since $\nabla h \sigma_{\varepsilon_n}(Z_u^n) - \nabla_x h \sigma(Z_u) \to 0$ almost surely, and $h \in C_0^2$, we have that by the dominated convergence theorem in $\mathbb{P} \otimes [0, t]$ the right hand side in last inequality goes to zero. Therefore,

$$\begin{aligned} h(Z_t) - h(Z_0) = \int_0^t \mathbb{1}_{D\times\mathbb{R}^p} \nabla_x h(Z_u) \sigma(X_u) dB_u + \int_0^t \mathbb{1}_{D\times\mathbb{R}^p} \mathcal{L}_u^0 h(Z_u) du + \\ + \int_0^t \nabla h \cdot \vec{\kappa}(Z_u) dL_u, \end{aligned} \tag{4.4.7}$$

where $\mathcal{L}_t^0 h = \frac{1}{2} \sum_{i,j=1}^n (\sigma^T \sigma)^{i,j} \partial_i \partial_j h$. This proves weak existence for (4.1.1).

Proof of pathwise uniqueness

To prove pathwise uniqueness, we will use some ideas from [22]. According to Lemma 4.1 in [22], there exists a symmetric-matrix valued function $\Lambda(z)$, with $z \in \mathbb{R}^{n+p}$, such that

$\Lambda(\cdot)$ is uniformly elliptic and bounded, and $\Lambda(z)^T\vec{\gamma}(z) = \hat{n}(z)$ for $z \in \partial D \times \mathbb{R}^p$. Moreover, $\Lambda_{ij}(\cdot) \in C_b^2(\mathbb{R}^{n+p})$, and for some $C_0 > 0$

$$C_0 |z-y|^2 + \vec{\gamma}(z)^T \Lambda(z)(z-y) \geq 0 \qquad z \in \partial D \times \mathbb{R}^p, y \in \overline{D} \times \mathbb{R}^p. \tag{4.4.8}$$

Notice that, for some A to be determined, as $\vec{\kappa}$ and Λ are Lipschitz, and bounded, we get for $z \in \partial D \times \mathbb{R}^p, y \in \overline{D} \times \mathbb{R}^p$,

$$\begin{aligned} A|z-y|^2 + \vec{\kappa}(z)^T \Lambda(y)(z-y) &= A|z-y|^2 + \vec{\kappa}(z)^T[\Lambda(y) - \Lambda(z)](z-y) \\ &\quad + \vec{\kappa}(z)^T \Lambda(z)(z-y) \\ &\geq A|z-y|^2 - [C_{\vec{\kappa},\Lambda} + C_0]|z-y|^2, \end{aligned}$$

thus we get for some $C_0' > 0$, that $C_0' |z-y|^2 + \vec{\kappa}(y)^T\Lambda(y)(z-y) \geq 0$. Adding this last inequality with (4.4.8) together, and putting $A(z,y) = \Lambda(z) + \Lambda(y)$, we finally obtain

$$C_0 |z-y|^2 + \vec{\kappa}(z) A(z,y)(z-y) \geq 0 \qquad z \in \partial D \times \mathbb{R}^p, y \in \overline{D} \times \mathbb{R}^p. \tag{4.4.9}$$

Let ϕ be a $C_b^2(\mathbb{R}^n)$ function defining ∂D. Suppose that $(X, S; L)$ and $(X', S'; L')$ solve (4.1.1), and set

$$\begin{aligned} \eta(z,z') &\stackrel{\text{def}}{=} \exp\left[\lambda(\phi(x) + \phi(x'))\right], \\ \xi(z,z') &\stackrel{\text{def}}{=} \eta(z,z')(z-z')^T\left[\Lambda(z) + \Lambda(z')\right](z-z'). \end{aligned}$$

Let λ be a negative number whose value will be specified later. By Itô's formula,

$$\begin{aligned} \xi(Z_t, Z_t') = &\int_0^t \nabla_z \xi(Z_s, Z_s') dZ_s + \int_0^t \nabla_{z'} \xi(Z_s, Z_s') dZ_s' + \\ &+ \frac{1}{2}\int_0^t \sum_{i,j=1}^n \frac{\partial^2 \xi}{\partial z^i \partial z^j}(Z_s, Z_s') d\left\langle Z^i, Z'^j \right\rangle_s + \\ &+ \frac{1}{2}\int_0^t \sum_{i,j=1}^n \frac{\partial^2 \xi}{\partial z^i \partial z'^j}(Z_s, Z_s') d\left\langle Z^i, Z'^j \right\rangle_s + \\ &+ \frac{1}{2}\int_0^t \sum_{i,j=1}^n \frac{\partial^2 \xi}{\partial z'^i \partial z^j}(Z_s, Z_s') d\left\langle Z'^i, Z^j \right\rangle_s + \\ &+ \frac{1}{2}\int_0^t \sum_{i,j=1}^n \frac{\partial^2 \xi}{\partial z'^i \partial z'^j}(Z_s, Z_s') d\left\langle Z'^i, Z'^j \right\rangle_s. \end{aligned}$$

The sum in the quadratic-variation component is only up to n because the remaining components of Z and Z' are continuous, with bounded variation. For $1 \le i, j \le n$ we see that $d\left\langle Z^i, Z'^j \right\rangle_s = [\sigma(X_s)\sigma(X'_s)^T]^{i,j} ds$, and analogous formulae hold for the other terms.

Next we compute the second derivatives. It will be useful to denote $A(z,z') = \Lambda(z) + \Lambda(z')$, which is a symmetric matrix. In the following calculations $z = (x,s)$, and $A_{i\cdot}$ represents the i-th row of the matrix A. Similar notation will be used for the matrix Λ.

$$\frac{\partial \xi}{\partial z^i} = \lambda \frac{\partial \phi}{\partial z^i} \xi(z,z') + \eta(z,z') \left(2A(z,z')_{ij}(z_j - z'_j) + (z-z')^T \frac{\partial \Lambda}{\partial z^i} \right)(z-z'). \qquad (4.4.10)$$

Thus,

$$\begin{aligned}
\frac{\partial^2 \xi}{\partial z^j \partial z^i} = & \lambda \frac{\partial^2 \phi}{\partial z^j \partial z^i} \xi + \lambda \frac{\partial \phi}{\partial z^i} \frac{\partial \xi}{\partial z^j} + 2\eta(z,z') A(z,z')_{ij} + \\
& + \eta(z,z') \left[2\frac{\partial \Lambda_{i\cdot}}{\partial z^j} + 2\frac{\partial \Lambda_{j\cdot}}{\partial z^i} + (z-z')^T \frac{\partial^2 \Lambda}{\partial z^j \partial z^i} \right](z-z') + \\
& + \lambda \frac{\partial \phi}{\partial z^j} \eta(z,z') \left(2A(z,z')_{i\cdot} + (z-z')^T \frac{\partial \Lambda}{\partial z^i} \right)(z-z') \\
\frac{\partial^2 \xi}{\partial z'^j \partial z^i} = & \lambda \frac{\partial \phi}{\partial z^i} \frac{\partial \xi}{\partial z'^j} - 2\eta(z,z') A(z,z')_{ij} + 2\eta(z,z') \left[\frac{\partial \Lambda_{i\cdot}}{\partial z'^j} - \frac{\partial \Lambda_{j\cdot}}{\partial z^i} \right](z-z') + \\
& + \lambda \frac{\partial \phi}{\partial z'^j} \eta(z,z') \left(2A(z,z')_{i\cdot} + (z-z')^T \frac{\partial \Lambda}{\partial z^i} \right)(z-z').
\end{aligned}$$

Interchanging z and z' we obtain $\frac{\partial^2 \xi}{\partial z^j \partial z'^i}$ and $\frac{\partial^2 \xi}{\partial z'^j \partial z'^i}$.

We aim to get quadratic bounds of the form $|z-z'|^2$ for these terms, so we will substitute any term that is bounded in this sense by the standard $O(|z-z'|^2)$.

$$\begin{aligned}
\frac{\partial^2 \xi}{\partial z^j \partial z^i} = & O(\left|z-z'\right|^2) + 2\eta(z,z') A(z,z')_{ij} + \\
& + 2\eta(z,z') \left(\lambda \frac{\partial \phi}{\partial z^i} A(z,z')_{j\cdot} + \lambda \frac{\partial \phi}{\partial z^j} A(z,z')_{i\cdot} + \frac{\partial \Lambda_{i\cdot}}{\partial z^j} + \frac{\partial \Lambda_{j\cdot}}{\partial z^i} \right)(z-z'), \\
\frac{\partial^2 \xi}{\partial z'^j \partial z^i} = & O(\left|z-z'\right|^2) - 2\eta(z,z') A(z,z')_{ij} + \\
& + 2\eta(z,z') \left(-\lambda \frac{\partial \phi}{\partial z^i} A(z,z')_{j\cdot} + \lambda \frac{\partial \phi}{\partial z'^j} A(z,z')_{i\cdot} + \frac{\partial \Lambda_{i\cdot}}{\partial z'^j} - \frac{\partial \Lambda_{j\cdot}}{\partial z^i} \right)(z-z').
\end{aligned}$$

Define, for $z, z' \in \mathbb{R}^{n+p}$

$$\begin{aligned}
\Gamma_1^{ij}(z,z') &= [\sigma(x)\sigma(x)^T]_{ij} \frac{\partial^2 \xi}{\partial z^j \partial z^i} + [\sigma(x)\sigma(x')^T]_{ij} \frac{\partial^2 \xi}{\partial z'^j \partial z^i}, \\
\Gamma_2^{ij}(z,z') &= [\sigma(x')\sigma(x')^T]_{ij} \frac{\partial^2 \xi}{\partial z'^j \partial z'^i} + [\sigma(x')\sigma(x)^T]_{ij} \frac{\partial^2 \xi}{\partial z^j \partial z'^i}.
\end{aligned}$$

Notice how Γ_2 is obtained from Γ_1 by reversing the roles of z and z', including the variables in the derivatives.

From the computations we have done, we group similar terms together and used the fact that $\sigma(\cdot)$ is Lipschitz and bounded to obtain the second equality below:

$$\begin{aligned}
\Gamma_1^{ij}(z,z') &= O(\left|z-z'\right|^2) + 2\left(A(z,z')_{ij} + \left(\lambda\frac{\partial\phi}{\partial z^i}A(z,z')_{j\cdot} + \frac{\partial\Lambda_{j\cdot}}{\partial z^i}\right)(z-z')\right)\cdot \\
&\qquad \cdot\eta(z,z')\left[\sigma(x)\left(\sigma(x)-\sigma(x')\right)^T\right]_{ij} + \\
&\qquad + 2\eta(z,z')\left(\lambda\frac{\partial\phi}{\partial z^j}A(z,z')_{i\cdot} + \frac{\partial\Lambda_{i\cdot}}{\partial z^j}\right)(z-z')\left[\sigma(x)\left(\sigma(x)+\sigma(x')\right)^T\right]_{ij} \\
&= O(\left|z-z'\right|^2) + 2A(z,z')_{ij}\eta(z,z')\left[\sigma(x)\left(\sigma(x)-\sigma(x')\right)^T\right]_{ij} + \\
&\qquad + 2\eta(z,z')\left(\lambda\frac{\partial\phi}{\partial z^j}A(z,z')_{i\cdot} + \frac{\partial\Lambda_{i\cdot}}{\partial z^j}\right)(z-z')\left[\sigma(x)\left(\sigma(x)+\sigma(x')\right)^T\right]_{ij}
\end{aligned}$$

In a similar way, we can write an estimate for Γ_2. Adding the two together, and using that $A(z,z') = A(z',z)$, we arrive at

$$\begin{aligned}
\Gamma_1^{ij} + \Gamma_2^{ij}(z,z') &= O(\left|z-z'\right|^2) + 2\eta(z,z')A(z,z')_{ij}\left[(\sigma(x)-\sigma(x'))(\sigma(x)-\sigma(x'))^T\right]_{ij} + \\
&\quad + 2\eta(z,z')\left(\lambda\frac{\partial\phi}{\partial z^j}A(z,z')_{i\cdot} + \frac{\partial\Lambda_{i\cdot}}{\partial z^j}\right)(z-z')\left[\sigma(x)\left(\sigma(x)+\sigma(x')\right)^T\right]_{ij} + \\
&\quad - 2\eta(z,z')\left(\lambda\frac{\partial\phi}{\partial z'^j}A(z,z')_{i\cdot} + \frac{\partial\Lambda_{i\cdot}}{\partial z'^j}\right)(z-z')\left[\sigma(x')\left(\sigma(x)+\sigma(x')\right)^T\right]_{ij}.
\end{aligned}$$

As $\sigma(\cdot)$ is Lipschitz, the second term is $O(|z-z'|^2)$. As both $\phi(\cdot)$ and $A(\cdot)$ have bounded second derivatives, we have that $y \mapsto \lambda\frac{\partial\phi}{\partial z^j}(y)A(z,z')_{i\cdot} + \frac{\partial\Lambda_{i\cdot}}{\partial z^j}$ is Lipschitz, so the third and four line together can be are $O(|z-z'|^2)$, as well. Summing up, there is a constant $C > 0$, independent of t, such that

$$\xi(Z_t,Z'_t) \le \int_0^t \nabla_z\xi(Z_s,Z'_s)dZ_s + \int_0^t \nabla_{z'}\xi(Z_s,Z'_s)dZ'_s + C\int_0^t \left|Z_s-Z'_s\right|^2 ds$$

Since any solution of (4.1.1) is square integrable, then we have from (4.4.10) that $M_t = \int_0^t \nabla_z\xi(Z_s,Z'_s)dB_s$ is a martingale. Using (4.4.10), the fact that $\nabla\phi\cdot\vec{\gamma} \ge 1$ on ∂D, and

(4.4.9) we get for $\lambda < 0$,

$$\begin{aligned}
\xi(Z_t, Z'_t) \le{}& M_t + C\int_0^t \left|Z_s - Z'_s\right|^2 ds + \lambda\int_0^t \xi(Z_s, Z'_s)d(L_s + L'_s) + \\
&- 2\int_0^t \eta(Z_s, Z'_s)(Z_s - Z'_s)^T A(Z_s, Z'_s)\vec{\gamma}(Z_s)dL_s + \\
&- 2\int_0^t \eta(Z_s, Z'_s)(Z'_s - Z_s)^T A(Z_s, Z'_s)\vec{\gamma}(Z'_s)dL'_s \\
\le{}& M_t + C\int_0^t \left|Z_s - Z'_s\right|^2 ds + \lambda\int_0^t \xi(Z_s, Z'_s)d(L_s + L'_s) + \\
&+ 2C_0\int_0^t \eta(Z_s, Z'_s)\left|Z_s - Z'_s\right|^2 d(L_s + L'_s).
\end{aligned}$$

Recall that Λ is elliptic, say of constant $\nu > 0$. As $\lambda < 0$ and $\phi \ge 0$ in $\overline{D}$, we have that $\eta \le 1$ and therefore,

$$\xi(Z_t, Z'_t) \le M_t + C\int_0^t \left|Z_s - Z'_s\right|^2 ds + (C_0 + 2\lambda\nu)\int_0^t \eta(Z_s, Z'_s)\left|Z_s - Z'_s\right|^2 d(L_s + L'_s).$$

At this point we pick $\lambda = -(2\nu)^{-1}C_0$. Using ellipticity once more, we arrive at

$$\mathbb{E}\left|Z_t - Z'_t\right|^2 \le C\int_0^t \mathbb{E}\left|Z_s - Z'_s\right|^2 ds,$$

and we conclude by Gronwall's inequality that, almost surely, $Z_t = Z'_t$ for all $t \ge 0$. ∎

4.5 Excursions from the boundary

We introduce the notion of Exit System, first developed by Maisonneuve in [24]. Let Z be a standard Markov process taking values in a domain $E \subseteq \mathbb{R}^n$ with boundary ∂E. We attach to E a "cemetery" point Δ outside of $\overline{E}$, and we denote by $\mathcal{C}$ the set of functions $f : [0, \infty) \to \mathbb{R}^n \cup \{\Delta\}$ that are continuous in some interval $[0, \zeta)$ taking values in $\mathbb{R}^n$, and are equal to Δ in $[\zeta, \infty)$.

An exit system for the process Z from ∂E is a pair $(L^*_t, \mathbf{H}_x)$, where L^*_t is a positive additive functional of Z, and $\{\mathbf{H}_x\}_{x \in \partial E}$ is a family of sigma-finite measures on $\mathcal{C}$ such that the canonical process is strong Markov on (t_0, ∞) under $\mathbf{H}_x$. These measures are called **excursion laws**.

Excursions of Z from ∂E will be denoted e or e_s, i.e, if $s < u$ and $Z_s, Z_u \in \partial E$, and $Z_t \notin \partial E$ for $t \in (s,u)$, then $e_s = \{e_s(t) = X_{t+s},\ t \in [0, u-s)\}$ and the lifetime of such excursion is given by $\zeta(e_s) = u - s$. By convention, $e_s(t) = \Delta$ for $t \geq \zeta$.

Let $\sigma_t = \inf\{s \geq 0 : L_s^* \geq t\}$ and let I be the set of left endpoints of all connected components of $(0,\infty) \setminus \{t \geq 0 : X_t \in \partial D\}$. The following is a special case of the exit system formula.

Theorem 4.5.1 (Theorem 1 in [24])**.** *There exists a positive, continuous additive functional L^* of (X,S) such that, for every $x \in \overline{D}$, any positive, bounded, predictable process V, and any universally measurable function $f : \mathcal{C} \to [0,\infty)$ that vanishes on excursions e_t identically equal to Δ,*

$$\mathbb{E}_x\left[\sum_{t\in I} V_t f(e_t)\right] = \mathbb{E}_x\left[\int_0^\infty V_{\sigma_s}\mathbf{H}_{Z(\sigma_s)}(f)ds\right] = \mathbb{E}_x\left[\int_0^\infty V_t \mathbf{H}_{Z_t}(f)dL_t^*\right]. \tag{4.5.1}$$

Standard notation is used for $\mathbf{H}_x(f) = \int_{\mathcal{C}} f d\mathbf{H}_x$.

For spinning Brownian motion in a domain D, note that excursions of (X,S) from $\partial D \times \mathbb{R}^p$ correspond to excursions of X from ∂D as S doesn't change within any excursion that goes inside D. In view of (4.5.1), it is enough to consider excursion laws $\{\mathbf{H}_x\}_{x\in\partial D}$ for an exit system for X.

Theorem 4.5.2. *Let $\mathbb{P}^D$ be the law of Brownian motion killed upon exiting D. Define*

$$\mathbf{H}_x \stackrel{\text{def}}{=} \lim_{\lambda\downarrow 0} \lambda^{-1}\mathbb{P}^D_{x+\lambda\vec{n}(x)} \tag{4.5.2}$$

and let L_t be the local time of (X,S), satisfying equation (4.1.1)*. Then $\mathbf{H}_x$ is a sigma-finite measure, strongly Markovian with respect to the filtration of the driving Brownian motion B_t, and $(L_t, \mathbf{H}_x)$ is an exit system from $\partial D \times \mathbb{R}^p$ for the process (X,S).*

Remark. When the underlying process is reflected Brownian motion in a half space, excursions from the boundary hyperplane are well understood. It is known that the local time L_t appearing in the Skorohod decomposition equals the positive additive functional L_t^* in the exit system of these excursions, except for a multiplicative constant. See [6], or Theorem 7.2 in [5] for details. The proof of Theorem 4.5.2 follows from this fact when D is a half plane.

Indeed, if $D = \mathbb{H}^n$, the standard n-dimensional half-space, fix an additive functional L_t^* such that $(dL_t^*, \mathbf{H}_x)$ is an exit system from ∂D for X. We will prove that is possible to replace L_t^* by the local time L_t that appears in equation (4.1.1). For $K \subseteq \partial D$, and $\varepsilon, T > 0$, let $A_K^{\varepsilon,T}$ be the sets of excursions starting from K before time T that eventually reach level ε above the boundary, that is $A_K^{\varepsilon,T} \overset{\text{def}}{=} \{e_t : t < T,\ e_t(0) \in K,\ \sup_{s>0} e_t^n(s) \geq \varepsilon\}$. Here e^n represents the n-th component of e.

We will apply the exit formula (4.5.1) to the function $f = \mathbb{1}_{A_K^{\varepsilon,T}}$. To this end, we note that $\mathbb{1}_{A_K^{\varepsilon,T}}(e_t) = \mathbb{1}_K(X_t)\mathbb{1}_{A_{\partial D}^{\varepsilon,T}}(e_t)$. Next, we consider a reflected Brownian motion process Y_t in D, driven by the same Brownian motion B_t that drives the process X_t. Recall that our filtration is defined as the right continuous completion of $\mathcal{F}_t = \sigma(B_s,\ s \leq t)$. In particular, $\tilde{V}_t = V_t \mathbb{1}_K(X_t)$ is predictable. Therefore,

$$\mathbb{E}_{x_0}\left[\int_0^T V_s \mathbf{H}_{X_s}(A_K^{\varepsilon,T}) dL_s^*\right] = \mathbb{E}_{x_0}\left[\sum_{s<T} V_s \mathbb{1}_{A_K^{\varepsilon,T}}(e_t)\right] = \mathbb{E}_{x_0}\left[\sum_{s<T} \tilde{V}_s \mathbb{1}_{A_{\partial D}^{\varepsilon,T}}(e_t)\right].$$

Because of the geometry of $D = \mathbb{H}^n$, the stochastic differential equation for X decouples into components, and the n-th component X^n can be seen to be a one dimensional reflected Brownian notion at zero. The fact that $X_t^n = Y_t^n$ leads to two important points: first, it implies that the local times for X_t and Y_t are the same, which can be seen from uniqueness of the solution to the stochastic Skorokhod equation. In view of [5], Theorem 7.2, $(L_t, \mathbf{H}_y)$ is an exit system for excursions of Y. Second, excursions of X_t and Y_t start at the same time, as $X_t \in \partial D$ if and only if $X_t^n = 0 = Y_t^n$, which happens if and only if $Y_t \in \partial D$. Thus $\mathbb{1}_{A_{\partial D}^{\varepsilon,T}}(e_t) = \mathbb{1}_{A_{\partial D}^{\varepsilon,T}}(e_t^Y)$, where e^Y represents excursions of Y_t, as $A_{\partial D}^{\varepsilon,T}$ depends only on the n-th component of the process. This fact also shows that $\mathbf{H}_{Y_s}(A_{\partial D}^{\varepsilon,T}) = \mathbf{H}_{X_s}(A_{\partial D}^{\varepsilon,T})$. We put together all these facts to compute,

$$\begin{aligned}
\mathbb{E}_{x_0}\left[\int_0^T V_s \mathbf{H}_{X_s}(A_K^{\varepsilon,T}) dL_s^*\right] &= \mathbb{E}_{x_0}\left[\sum_{s<T} \tilde{V}_s \mathbb{1}_{A_{\partial D}^{\varepsilon,T}}(e_t^Y)\right] \\
&= \mathbb{E}_{x_0}\left[\int_0^T \tilde{V}_s \mathbf{H}_{Y_s}(A_{\partial D}^{\varepsilon,T}) dL_s\right] \\
&= \mathbb{E}_{x_0}\left[\int_0^T V_s \mathbb{1}_D(X_s) \mathbf{H}_{X_s}(A_{\partial D}^{\varepsilon,T}) dL_s\right] \\
&= \mathbb{E}_{x_0}\left[\int_0^T V_s \mathbf{H}_{X_s}(A_K^{\varepsilon,T}) dL_s\right].
\end{aligned}$$

Let F be a Borel measurable set in $\mathcal{C}$, and set $T_\varepsilon = \inf\{t > 0 : \mathrm{dist}(X_t, \partial D) > \varepsilon\}$. Since, under $\mathbf{H}_x$, we have that $B \circ T_\varepsilon = X \circ T_\varepsilon = Y \circ T_\varepsilon$, we can repeat the previous argument for the set $A_K^{\varepsilon,T} \cap (F \circ T_\varepsilon)$, where F is any universally measurable set in $\mathcal{C}$,

$$\mathbb{E}_{x_0}\left[\int_0^T V_s \mathbf{H}_{X_s}(A_K^{\varepsilon,T} \cap (F \circ T_\varepsilon)) dL_s^*\right] = \mathbb{E}_{x_0}\left[\int_0^T V_s \mathbf{H}_{X_s}(A_K^{\varepsilon,T} \cap (F \circ T_\varepsilon)) dL_s\right]. \quad (4.5.3)$$

By continuity of the paths, we know that $T_\varepsilon > 0$ and $T_\varepsilon \to 0$ as $\varepsilon \to 0$, $\mathbf{H}_x$-a.s. Thus, (4.5.3) guarantees that

$$\mathbb{E}_{x_0}\left[\sum_{s<T} V_s \mathbb{1}_{F_0}(e_s)\right] = \mathbb{E}_{x_0}\left[\int_0^T V_s \mathbf{H}_{X_s}(F_0) dL_s^*\right] = \mathbb{E}_{x_0}\left[\int_0^T V_s \mathbf{H}_{X_s}(F_0) dL_s\right], \quad (4.5.4)$$

holds for any F_0 that belongs to the collection $\mathcal{A} = \bigcup_{\varepsilon>0} \mathcal{F}_{T_\varepsilon}$. By the strong Markov property, the sigma algebra $\mathcal{F}_\infty$ of universally measurable sets in $\mathcal{C}$ and $\mathcal{A}$ differ only by sets of $\mathbb{P}$ measure zero, which shows that (4.5.4) is valid not only for $F_0 \in \mathcal{A}$, but also for any $F \in \mathcal{F}_\infty$. A standard extension using simple functions to approximate measurable functions shows that

$$\mathbb{E}_{x_0}\left[\sum_{s<T} V_s f(e_s)\right] = \mathbb{E}_{x_0}\left[\int_0^T V_s \mathbf{H}_{X_s}(f) dL_s\right], \quad (4.5.5)$$

for any universally measurable function $f : \mathcal{C} \to [0, \infty)$ which vanishes on excursions e_t identically equal to Δ. This shows that $(dL_t, \mathbf{H}_x)$ is an exit system.

■

Exit system for general smooth sets

For general smooth domains, the proof is much more complicated as the local time does not depend exclusively on one fixed component of the driving Brownian motion. The events of excursions reaching distance ε away from the boundary and only hitting the boundary within a fixed set are not independent anymore. Nonetheless, smooth sets are “locally flat” and thus, by conditioning X_t to hit the boundary only at a very small set, we can reproduce the result.

Proof. The fact that $\mathbf{H}_x$ is sigma-finite for all $x \in \partial D$, and strongly Markovian is proved in Theorem 7.2 in [5]. The proof actually applies as these are properties of the measures and do not have anything to do with the local time.

Let $(dL_t^*, \mathbf{H}_x)$ be an exit system for (X, S), where L^* is the additive functional from Theorem 4.5.1. As before, we will prove that it is possible to replace L_t^* by the local time L_t from equation (4.1.1). Let $K \subseteq \partial D$ be open in the relative topology, and set $C = \|\vec{\gamma} - \vec{n}\|_\infty$. Let $\varepsilon > 0$ be a very small number. For $T > 0$, denote by A^ε the set of excursions from ∂D that start before time T and reach a level ε from ∂D:

$$A^\varepsilon \stackrel{\text{def}}{=} \left\{ e_t : t < T,\ \sup_{s<\zeta}\ \operatorname{dist}\left(e_t(s), \partial D\right) > \varepsilon \right\}.$$

Let $x \in \partial D$ be fixed. Pick $\delta > 0$ so small such that the following is possible: Choose a small r and coordinates $(x^i)_{i=1}^n$ that differ from the canonical coordinates in $\mathbb{R}^n$ by a translation and a rotation, such that $x = 0$ in these coordinates, and

- $D \cap B_r(x) = \{(y, s) : s > \phi(y)\} \cap B_r(x)$, that is, D is locally is the graph above a function ϕ,
- $\partial D \cap B_r(x) = \big\{(y, \phi^i(y))\big\} \cap B_r(x)$,
- $\nabla\phi(x) = 0$.

We call the above a normal set of coordinates centered at x.

Next we decompose K into sets closed set $K_1, \ldots, K_{m_\delta}$ such that the surface measure of the symmetric difference between K_j and K_i is zero $(i \neq j)$, and such that in each K_j we have sets of normal coordinates centered at $x_1, \ldots, x_{m_\delta}$, and respective boundary defining functions $\phi^1, \ldots, \phi^{m_\delta}$ as above, and

$$\vec{n}(z) \cdot \vec{n}(x_j) > 1 - \delta^2 \qquad \forall\ z \in K_j. \tag{4.5.6}$$

To simplify notation, for $z = (y, \phi^j(y)) \in \partial D$, we will write $\phi^j(z)$ instead of $\phi^j(y)$. This is obviously an abuse of notation that can be justified by identifying ϕ^j with $\phi^j \circ \imath_{n-1}$ on the boundary of D, where $\imath_{n-1}$ is the projection into the first $n-1$ coordinates. By our choice of $\nabla\phi^j(x_j) = 0$, we have that $\vec{n}(x_j) = \mathbf{e}_n$. With this convention, condition (4.5.6) reads

$$1 - \delta^2 < \vec{n}(z) \cdot \mathbf{e}_n = \frac{\left(-\nabla\phi^j(z), 1\right)}{\sqrt{1 + \|\nabla\phi^j(z)\|^2}} \cdot \mathbf{e}_n = \sqrt{1 + \|\nabla\phi^j(z)\|^2}^{-1}.$$

We deduce that for small δ, the estimate $\left\|\nabla\phi^j(z)\right\| \leq 2\delta$ holds, and by the mean value theorem

$$\left|\phi^j(z)\right| = \left|\phi^j(z) - \phi^j(x_j)\right| \leq 2\delta \|x_j - z\| \leq 2\delta r_\delta.$$

Let T_{x_j} be the tangent plane to ∂D at x_j. The computation above says that the piece of the boundary $\partial D \cap B_{r_\delta}(x_j)$ is contained in the cylinder $[T_{x_j} \cap B_{r_\delta}(x_j)] \times [-2\delta r_\delta, 2\delta r_\delta]$. Thus, $K_j \subseteq C_j \cap \partial D$. In our proof, it will be useful to set $\lambda = \lambda(\delta) \stackrel{\text{def}}{=} 2r_\delta\sqrt{\delta}$ and define the cylinder $C_j = [T_{x_j} \cap B_{r_\delta}(x_j)] \times [-\lambda, \lambda]$. The "top" of the cylinder C_j will be denoted by C_j^T.

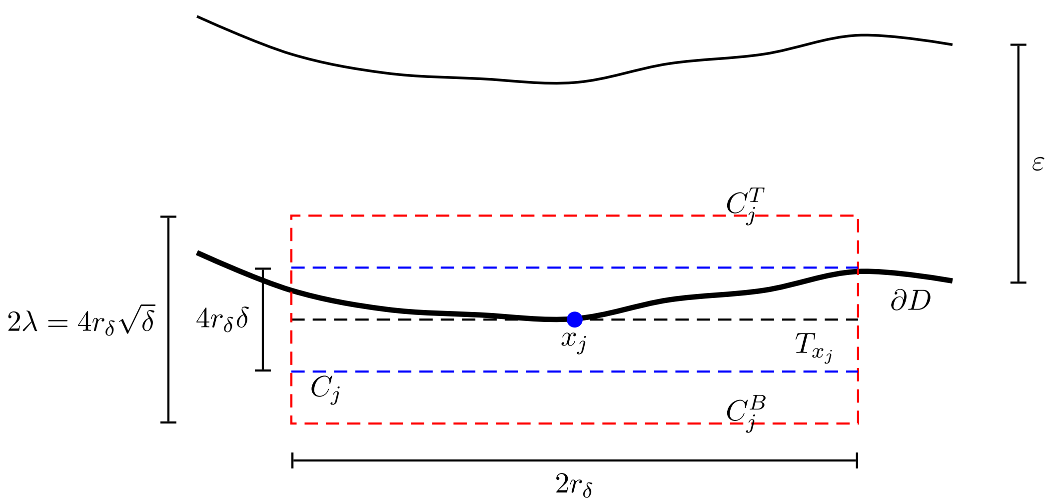

Figure 4.1: Cylinder C_j, and other related objects.

By the exit system formula with $V_u = 1$, and the construction above, we have that

$$\begin{aligned} \mathbb{E}_{x,s} \int_0^T \mathbb{1}_K(X_u) \mathbf{H}_{X_u}(A^\varepsilon) dL_u^* &= \mathbb{E}_{x,s} \sum_{u<T} \mathbb{1}_K(X_u) \mathbb{1}_{A^\varepsilon}(e_u) \\ &= \sum_{j=1}^{m_\delta} \mathbb{E}_{x,s} \sum_{u<T} \mathbb{1}_{K_j}(X_u) \mathbb{1}_{A^\varepsilon}(e_u). \end{aligned} \tag{4.5.7}$$

for any Borel set $R \subseteq \mathbb{R}^p$, and any $T > 0$.

Let $\sigma^j = \inf\{t \geq 0 : X_t \in K_j\}$ and $\tau^j = \inf\{t \geq 0 : X_t \notin C_j\}$. Set $\sigma_0^j = \sigma_0^j$, and for each integer $k \geq 1$, let $\tau_k^j = \tau^j \circ \theta(\sigma_k^j) + \sigma_k^j$ and $\sigma_{k+1}^j = \sigma_0^j \circ \theta(\tau_k^j) + \tau_k^j$, where θ is the usual shift operator. It is a well known fact that all these objects are stopping times. Since K_j is closed, the process $\mathbb{1}_{K_j}(X_u)$ is predictable as X is continuous. Then, $\tilde{V}_u = V_u \mathbb{1}_{K_j}(X_u)$ is non-negative, predictable and bounded, and so, by the exit formula for $(dL_t^*, \mathbf{H}_x)$, and a simple change of variable,

$$
\begin{aligned}
\mathbb{E}_{x,s} \sum_{u<T} \mathbb{1}_{K_j}(X_u)\mathbb{1}_{A^\varepsilon}(e_u) = \mathbb{E}_{x,s} \int_0^T \mathbb{1}_{K_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^* \\
&= \sum_{k=0}^\infty \mathbb{E}_{x,s}\mathbb{1}_{\{\sigma_k^j<T\}} \int_{\sigma_k^j}^{\tau_{k+1}^j} \mathbb{1}_{K_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^* \\
&= \sum_{k=0}^\infty \mathbb{E}_{x,s}\mathbb{1}_{\{\sigma_k^j<T\}} \left(\int_0^{\tau^j} \mathbb{1}_{K_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^* \circ \theta(\sigma_k^j)\right) \\
&= \sum_{k=0}^\infty \mathbb{E}_{x,s}\mathbb{1}_{\{\sigma_k^j<T\}} \mathbb{E}_{X_{\sigma_k^j}, S_{\sigma_k^j}} \left(\int_0^{\tau^j} \mathbb{1}_{K_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^*\right),
\end{aligned}
$$

where the last equality holds by the strong Markov property applied at time σ_k^j.

Assume that for all $z = (x, s)$ with $x \in C_j \cap \partial D$ the following equation holds,

$$
\begin{aligned}
\mathbb{E}_z \int_0^{\tau^j} \mathbb{1}_{K_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^* = (1 + O(\delta))\mathbb{E}_z \int_0^{\tau^j} \mathbb{1}_{K_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u \\
+ O(\delta)\mathbb{E}_z \left(L_{\tau^j}\right),
\end{aligned}
\tag{4.5.8}
$$

where $O(\delta)$ is standard notation for a bounded function that converges to zero as $\delta \to 0$. Then, we can trace back all of our computations up to (4.5.7) to obtain the following property: for all measurable sets $K \subseteq \partial D$ and $v < t$,

$$
\mathbb{E}_{x,s} \int_v^t \mathbb{1}_K(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^* = O(\delta)\mathbb{E}_{x,s}L_t + (1 + O(\delta))\mathbb{E}_{x,s} \int_v^t \mathbb{1}_K(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u.
$$

Since K, $\varepsilon > 0$, and $v < t$ are independent of δ, we can take the limit as $\delta \to 0$ in last equation to obtain

$$
\mathbb{E}_{x,s} \int_v^t \mathbb{1}_K(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^* = \mathbb{E}_{x,s} \int_v^t \mathbb{1}_K(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u.
$$

A standard argument involving the monotone class theorem, shows that this last equation is not only valid for $\mathbb{1}_K$, but also for all bounded, measurable functions $f : \partial D \to \mathbb{R}$. Since for each $\varepsilon > 0$, the function $x \mapsto \mathbf{H}_x(A^\varepsilon)$ is bounded away from zero (see [5]), we can take $f(x) = e^{-\alpha t}\mathbf{H}_x(A^\varepsilon)^{-1}$ to get

$$\mathbb{E}_{x,s}\int_v^t e^{-\alpha t} dL_u^* = \mathbb{E}_{x,s}\int_v^t e^{-\alpha t} dL_u, \tag{4.5.9}$$

for any positive α, and arbitrary $v < t$. We can extend (4.5.9) in the following way: for $N \in \mathbb{N}$ and $T > 0$, define simple functions by

$$f_{N,T}(t) = \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{N} \mathbb{1}_{\left(\frac{kT}{N},\frac{(k+1)T}{N}\right]}(t)\ e^{-\alpha\frac{(k+1)T}{N}}.$$

It follows from (4.5.9) that $\mathbb{E}_{x,s}\int_0^\infty f_{N,T}(u)dL_u^* = \mathbb{E}_{x,s}\int_0^\infty f_{N,T}(u)dL_u$. It is clear that $f_{N,T}(u)$ increases to $e^{-\alpha u}\mathbb{1}_{[0,T]}(u)$ for all $T > 0$, and so, by the monotone convergence theorem, we obtain

$$\mathbb{E}_{x,s}\int_0^\infty e^{-\alpha u} dL_u^* = \mathbb{E}_{x,s}\int_0^\infty e^{-\alpha u} dL_u,$$

that is, L^* and L have the same α-potential functions. Since both L^* and L are continuous, it follows by [4], Chapter 4, Theorem 2.13, that $L = L^*$ a.s. This shows that $(dL_t, \mathbf{H}_x)$ is an exit system, an the theorem is proved.

It remains to show that equation (4.5.8) holds. It might seem that the exit formula (4.5.1), and equation (4.5.8) just differ on the upper limit of the integral. In the former, the upper limit is a fixed time T whereas in the latter, the upper limit is the random time τ^j. The advantage of having τ^j as an upper limit is that *at most one excursion* that reaches level ε away from ∂D starts before time τ^j. We will drop the super index j from τ^j for notational simplicity.

Call I_z to the left hand side of (4.5.8). The process $\mathbb{1}_{\{\tau<u\}}$, and thus $\mathbb{1}_{\{u\le\tau\}}$, are predictable. Since at time τ the process X cannot be on the boundary, we see by the exit

formula (4.5.1) applied with $V_u = \mathbb{1}_{\{u\le\tau\}}\mathbb{1}_{K_j}(X_u)$

$$\begin{aligned}
I_z &= \mathbb{E}_z \int_0^\infty \mathbb{1}_{\{u<\tau\}}\mathbb{1}_{K_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^* = \mathbb{E}_z \int_0^\infty \mathbb{1}_{\{u\le\tau\}}\mathbb{1}_{K_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u^* \\
&= \mathbb{E}_z \sum_{u<\infty} \mathbb{1}_{\{u\le\tau\}}\mathbb{1}_{K_j}(X_u)\mathbb{1}_{A^\varepsilon}(e_u) = \mathbb{E}_z \sum_{u<\infty} \mathbb{1}_{\{u<\tau\}}\mathbb{1}_{K_j}(X_u)\mathbb{1}_{A^\varepsilon}(e_u) \\
&= \mathbb{E}_z \sum_{u<\tau} \mathbb{1}_{K_j}(X_u)\mathbb{1}_{A^\varepsilon}(e_u),
\end{aligned}$$

which roughly says that the exit formula is also valid if we change $T > 0$ for the stopping time τ.

Since only one excursion in A^ε can happen before time τ, we see that I_z is equal to the probability that, after escaping C_j, an excursion reaches distance ε away from ∂D. Intuitively, a reflected Brownian motion Y starting at z accumulate roughly the same local time as X before exiting C_j. Since the cylinder C_j is very thin, both Y and X are likely to exit the cylinder through C_j^T and be away from each other no more than the amount of local time accumulated up to time τ. Since both X and Y are the same Brownian motion inside of D, the probability that, after exiting C_j, an excursion of X reaches distance ε from ∂D is roughly the same as if such probability is computed with respect to Y. This idea yields that I_z can be estimated by using Y instead of X. For reflected Brownian motion, it is known that L^* can be chosen to be the local time from its Skorohod decomposition, that is, Y satisfies (4.5.8). So, X should satisfy (4.5.8) as well. We will formalize this idea next.

Define $\Lambda = \{\exists\ u < \tau,\ X_u \in K_j,\ e_u \in A^\varepsilon\}$. Then,

$$I_z = \mathbb{E}_z \sum_{u<\tau} \mathbb{1}_{K_j}(X_u)\mathbb{1}_{A^\varepsilon}(e_u) = \mathbb{P}_z(\Lambda) = \mathbb{P}_z(\Lambda, X_\tau \in C_j^T)\left(1 + \frac{\mathbb{P}_z(\Lambda, X_\tau \notin C_j^T)}{\mathbb{P}_z(\Lambda, X_\tau \in C_j^T)}\right).$$

Call D^ε to the set of points in D at distance at least ε from ∂D. Let T_{D^ε} be the hitting time of D^ε and T_{C_j} the hitting time of C_j. By the strong Markov property,

$$\frac{\mathbb{P}_z(\Lambda, X_\tau \notin C_j^T)}{\mathbb{P}_z(\Lambda, X_\tau \in C_j^T)} = \frac{\mathbb{P}_z\left(\mathbb{P}_{X_\tau}(T_{D^\varepsilon} < T_{\partial D})\mathbb{1}_{\partial C_j \setminus C_J^T}(X_\tau)\right)}{\mathbb{P}_z\left(\mathbb{P}_{X_\tau}(T_{D^\varepsilon} < T_{\partial D})\mathbb{1}_{C_j^T}(X_\tau)\right)}.$$

The process $\tilde{B}_t = B\left(T_{D^\varepsilon} - t\right)$ is also a Brownian motion by the strong Markov property and independence of increments. Therefore,

$$\frac{\mathbb{P}_z(\Lambda, X_\tau \notin C_j^T)}{\mathbb{P}_z(\Lambda, X_\tau \in C_j^T)} \le \frac{\sup_{x\in D^\varepsilon} \mathbb{P}_x(\tilde{B}_{T_{C_j}} \in \partial C_j \setminus C_j^T)}{\inf_{x\in D^\varepsilon} \mathbb{P}_x(\tilde{B}_{T_{C_j}} \in C_j^T)} \le C_\varepsilon \frac{\omega^y(\partial C_j \setminus C_j^T)}{\omega^y(C_j^T)},$$

where ω^y represents the harmonic measure of $D \setminus C_j$, for an arbitrary point $y \in D^\varepsilon$, and $C_\varepsilon > 0$ only depends on ε. The surface area of the side of the cylinder C_j is about $\sqrt{\delta}$ times less the surface area of the top of C_j, so we conclude that

$$I_z = (1 + O(\delta))\mathbb{P}_z \left(\mathbb{P}_{X_\tau}(T_{D^\varepsilon} < T_{\partial D})\mathbb{1}_{C_j^T}(X_\tau)\right). \tag{4.5.10}$$

Next we introduce a reflected Brownian motion Y starting from z, driven by the same Brownian motion B that drives X. We proceed to do some estimate to compare X and Y.

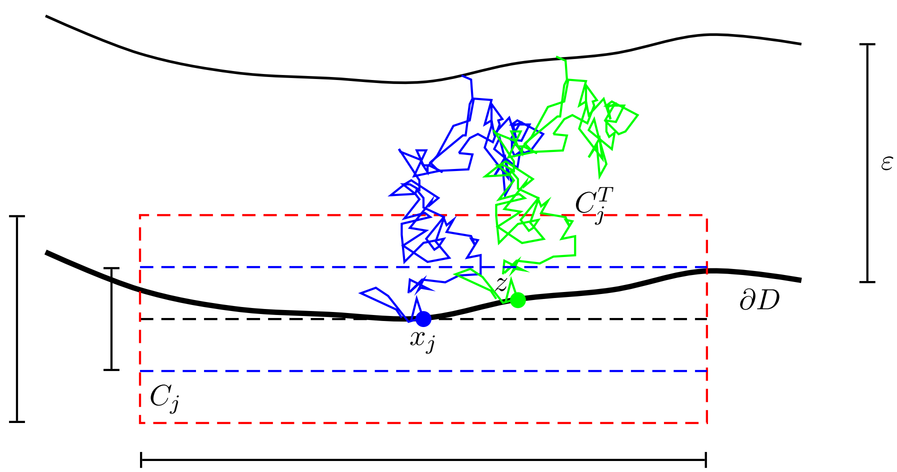

Figure 4.2: Comparing excursions of X and Y.

We claim that $\mathbb{P}_z(L_\tau > 2r_\delta\sqrt{\delta}) = O(\delta)$. Otherwise, there is a sequence $\delta_n \downarrow 0$ and $p > 0$ such that $\mathbb{P}_z(L_\tau > 2r_{\delta_n}\sqrt{\delta_n}) > p$. But any function $\omega \in C[0,1)$ with $L_\tau(\omega) > 2r_\delta\sqrt{\delta}$ satisfies that $B_t^n(\omega) < 0$ for $t < T_\delta(\omega)$. This only occurs with probability $O(\delta)$, which contradicts the existence of the sequence δ_n. Since

$$\|X_t - Y_t\|^2 - |X_t^n - Y_t^n|^2 \le \delta^2(L_\tau + L_\tau^Y)^2 + \|\vec{\gamma}\|_\infty^2 L_\tau^2,$$

the process Y leaves the cylinder C_j through the side only when X_t gets at distance $(\delta^2 + r_\delta\sqrt{\delta})O(1)$ from the side of C_j. This event has probability of order $\sqrt{\delta}$ by a harmonic measure argument similar to that that lead to (4.5.10). Therefore, the event $Y_u \in C_j$ for $u \le \tau$ has probability $1 - O(\delta)$. Actually, the event holds true if we request that $|z - x_j| < r_\delta - 2r_\delta\sqrt{\delta}$, as long as $L_\tau \le 2r_\delta\sqrt{\delta}$. Thus, since for $y \in C_j \cap \partial D$ we have $1 \ge \vec{n}(y)\mathbf{e}_n \ge 1 - \delta^2$,

$$X_t^n - Y_t^n = \int_0^t \vec{n}(X_u)\mathbf{e}_n dL_u - \int_0^t \vec{n}(Y_u)\mathbf{e}_n dL_u^Y = (1 + O(\delta))[L_t - L_t^Y],$$

for all $t \le \tau$, on the set $\left\{L_\tau \le 2r_\delta\sqrt{\delta}\right\}$.

It is clear that $\operatorname{dist}(X_\tau, \partial D)$ is comparable to $r_\delta\sqrt{\delta}$, and that $\operatorname{dist}(Y_\tau, \partial D)$ is comparable to Y_τ^n, since Y_τ^n does not leave the cylinder C_j on the event $\left\{L_\tau \le 2r_\delta\sqrt{\delta}\right\}$. We will argue that $|X_\tau^n - Y_\tau^n| \le 4r_\delta\delta$. Indeed, assume that $X_t^n \ge 2r_\delta\delta$. Since no piece of the boundary is above level $2r_\delta\delta$, no local time L is accumulated and so $X^n - Y^n$ does no increase. Similarly $Y^n - X^n$ does not increase when $Y^n \ge 2r_\delta\delta$. But if both X^n and Y^n are less than $2r_\delta\delta$, then it is clear than $|X^n - Y^n| \le 4r_\delta\delta$, since the piece of boundary $C_j \cap \partial D$ is between levels $-2r_\delta\delta$ and $2r_\delta\delta$. Thus,

$$\left|\frac{Y_\tau^n}{X_\tau^n} - 1\right| = \left|\frac{Y_\tau^n - X_\tau^n}{X_\tau^n}\right| \le 2\sqrt{\delta},$$

and so, for some positive constants a_δ, b_δ, depending on δ,

$$a_\delta < \frac{\operatorname{dist}(Y_\tau, \partial D)}{\operatorname{dist}(X_\tau, \partial D)} < b_\delta.$$

Thus, by the boundary Harnack principle, there are positive constants c_δ, C_δ such that

$$c_\delta < \frac{\mathbb{P}_{X_\tau}(T_{D^\varepsilon} < T_{\partial D})}{\mathbb{P}_{Y_\tau}(T_{D^\varepsilon} < T_{\partial D})} < C_\delta.$$

Moreover, since $\|X_\tau - Y_\tau\| \to 0$ as $\delta \to 0$, Lemma 1 in [7] shows that both c_δ and C_δ can be chosen arbitrarily close to 1 as $\delta \to 0$. Using this fact in (4.5.10) we obtain

$$\begin{aligned} I_z = (1 + O(\delta))\mathbb{P}_z\left(\mathbb{P}_{Y_\tau}(T_{D^\varepsilon < T_{\partial D}})\mathbb{1}_{C_j}(Y_\tau)\right) \\ + (1 + O(\delta))\mathbb{P}_z\left(\mathbb{P}_{X_\tau}(T_{D^\varepsilon} < T_{\partial D})\mathbb{1}_{\left\{L_\tau > 2r_\delta\sqrt{\delta}\right\}}\right). \end{aligned} \tag{4.5.11}$$

Since $\operatorname{dist}(X_\tau, \partial D)$ is about $2r_\delta\sqrt{\delta}$, standard estimates for Brownian motion show that $\mathbb{P}_{X_\tau}(T_{D^\varepsilon} < T_{\partial D})$ is comparable to $\varepsilon^{-1}r_\delta\sqrt{\delta}$, so

$$\mathbb{P}_z\left(\mathbb{P}_{X_\tau}(T_{D^\varepsilon} < T_{\partial D})\mathbb{1}_{\left\{L_\tau > 2r_\delta\sqrt{\delta}\right\}}\right) \le C\mathbb{E}_z\left(L_\tau\mathbb{1}_{\left\{L_\tau > 2r_\delta\sqrt{\delta}\right\}}\right) = O(\delta)\mathbb{E}_z(L_\tau).$$

By using this last equation in (4.5.11), we are allowed to write the following equation, by the same arguments that led to (4.5.10),

$$\begin{aligned} I_z &= (1+O(\delta))\mathbb{P}_z\left(\mathbb{P}_{Y_\tau}(T_{D^\varepsilon<T_{\partial D}})\mathbb{1}_{C_j}(Y_\tau)\right) + O(\delta)\mathbb{E}_x(L_\tau) \\ &= (1+O(\delta))\mathbb{E}_z\left(\sum_{u<\tau}\mathbb{1}_{C_j}(Y_u)\mathbb{1}_{A^\varepsilon}(e_u^Y)\right) + O(\delta)\mathbb{E}_x(L_\tau), \end{aligned}$$

where e^Y denotes excursions of the reflected Brownian motion Y. Theorem 7.2 in [5] says that $(dL_t^Y, \mathbf{H}_y)$ is an exit system for Y, therefore

$$I_z = (1+O(\delta))\mathbb{E}_z\left(\int_0^\tau \mathbb{1}_{C_j}(Y_u)\mathbf{H}_{Y_u}(A^\varepsilon)dL_u^Y\right) + O(\delta)\mathbb{E}_z(L_\tau). \tag{4.5.12}$$

We will use the Harnack boundary principle for the harmonic function $x \mapsto \mathbb{P}_x^D(A^\varepsilon)$, which vanishes on $C_j \cap \partial D$. Since

$$\frac{\mathbf{H}_y(A^\varepsilon)}{\mathbf{H}_{x_k}(A^\varepsilon)} = \lim_{\lambda\downarrow 0}\frac{\mathbb{P}^D_{y+\lambda\mathbf{e}_n}(A^\varepsilon)}{\mathbb{P}^D_{x_k+\lambda\mathbf{e}_n}(A^\varepsilon)}.$$

By the Harnack boundary principle, and Lemma 1 in [7], we obtain that the right hand side above is of order $1+O(\delta)$. It follows that

$$\mathbb{E}_z\int_0^\tau \mathbb{1}_{C_j}(Y_u)\mathbf{H}_{Y_u}(A^\varepsilon)dL_u^Y = (1+O(\delta))\,\mathbf{H}_{x_j}(A^\varepsilon)\mathbb{E}_z(L_\tau^Y). \tag{4.5.13}$$

The same argument allows us to write

$$\mathbb{E}_z\int_0^\tau \mathbb{1}_{C_j}(X_u)\mathbf{H}_{X_u}(A^\varepsilon)dL_u = (1+O(\delta))\,\mathbf{H}_{x_j}(A^\varepsilon)\mathbb{E}_z(L_\tau). \tag{4.5.14}$$

By previous estimates and the optional sampling theorem applied to B_τ^n,

$$\begin{aligned} \frac{\mathbb{E}_z(L_\tau)}{\mathbb{E}_z(L_\tau^Y)} &= (1+O(\delta))\frac{\mathbb{E}_z\left(X_\tau^n - B_\tau^n + \delta^2 L_\tau\right)}{\mathbb{E}_z(Y_\tau^n - B_\tau^n)} \\ &= (1+O(\delta))\frac{\mathbb{E}_z\left(X_\tau^n + \delta^2 L_\tau\right)}{\mathbb{E}_z(Y_\tau^n)} \\ &= (1+O(\delta))\left(1 + \frac{\mathbb{E}_z(X_\tau^n - Y_\tau^n + \delta^2 L_\tau)}{\mathbb{E}_z(Y_\tau^n - X_\tau^n + X_\tau^n)}\right). \end{aligned}$$

But,

$$\left|\frac{\mathbb{E}_z(X_\tau^n - Y_\tau^n + \delta^2 L_\tau)}{\mathbb{E}_z(Y_\tau^n - X_\tau^n + X_\tau^n)}\right| \le \frac{4r_\delta\delta + \delta^2\mathbb{E}_z(L_\tau)}{-4r_\delta\delta + 2r_\delta\sqrt{\delta}} = O(\delta),$$

which shows that $\mathbb{E}_z(L_\tau) = (1+O(\delta))\mathbb{E}_z(L_\tau^Y)$. Using this fact, equations (4.5.12), (4.5.13), and (4.5.14), we obtain (4.5.8), as desired, and the theorem is proved.

■

4.6 The stationary distribution

Our main goal of the section is to prove existence and uniqueness of the stationary distribution in a slightly modified version of the spinnig Brownian motion (4.1.1). Consider vector fields $\vec{\tau} : \partial D \times \mathbb{R}^p \to \mathbb{R}^n$ and $\vec{g} : \partial D \to \mathbb{R}^p$ that are Lipschitz continuous and bounded. Let $\alpha : \partial D \to \mathbb{R}$ be a uniformly positive, continuous function. Assume the following stochastic differential equation has a strong Markov solution

$$\begin{cases} dX_t &= dB_t + \hat{n}(X_t)dL_t + \vec{\tau}(X_t, S_t)dL_t, \\ dS_t &= \left(\vec{g}(X_t) - \alpha(X_t)\vec{S}_t\right) dL_t. \end{cases} \tag{4.6.1}$$

Notice that if $\alpha \equiv 1$ we recover spinning Brownian motion. One of the advantages of this equation is that it allows us to solve explicitly for S_t in terms of the process X and its local time L. Indeed, set $Y_t = \exp\left[\int_0^t \alpha(X_s)dL_s\right]$, and apply Itô's formula to the product Y_tS_t. Since both S_t and Y_t have bounded variation a.s., we get that

$$S_t = Y_t^{-1}S_0 + Y_t^{-1}\int_0^t Y_s\ \vec{g}(X_s)\ dL_s. \tag{4.6.2}$$

Proposition 4.6.1. *The spin process S_t in* (4.6.1) *is the unique process given by*

$$S_t = Y_t^{-1}S_0 + Y_t^{-1}\int_0^t \frac{\vec{g}(X_u)}{\alpha(X_u)}dY_u, \tag{4.6.3}$$

where $Y_t = \exp\int_0^t \alpha(X_u)dL_u$.

Proof. Let S_t' be the process defined by (4.6.3). We will show that $dS_t' = [\vec{g}(X_t) - \alpha(X_t)S_t]\, dL_t$ and conclude by path wise uniqueness that $S_t' = S_t$.

As Y_t is positive for every $t > 0$, and almost surely finite, the definition of S_t' yields $d(S_t'Y_t) = \vec{g}(X_t)/\alpha(X_t)dY_t$, since both Y_t and S_t' have paths with finite variation. By the product rule, and $dY_t = \alpha(X_t)Y_t dL_t$, we get

$$Y_t dS'_t = \frac{\vec{g}(X_t)}{\alpha(X_t)} dY_t - S'_t dY_t = \left[\vec{g}(X_t) - \alpha(X_t)S'_t\right] Y_t dL_t,$$

so S'_t satisfies the second equation in (4.6.1), as we claimed. ■

One of the issues with the diffusion (X, S) is the lack of a driving Brownian motion for the coordinates related to the spin. At an intuitive level, this means that the spin S_t could be confined to very small regions of the space, regions having Hausdorff dimension less than p, and consequently the support of the stationary distribution of the process could be singular with respect to Lebesgue measure. To make sure this is not the case, we need to impose some condition on the infinitesimal change of S_t, more precisely, on the function $\vec{g}$:

A1 There are $p+1$ points $x_1, \ldots, x_{p+1}$ on the boundary of D such that for every $y \in \mathbb{R}^p$, there exist non negative coefficients λ_j such that $y = \sum_{j=1}^{p+1} \lambda_j \vec{g}(x_j)$.

From now on, we assume that A1 holds, and we fix the points $x_1, \ldots, x_{p+1}$ that realize it. Notice that if $y = \sum_{j=1}^{p+1} \vec{g}(x_j)$, then A1 implies that $-y$ has an expansion with non-negative coefficients, and so we have that for every $\varepsilon > 0$ there are coefficients $\eta_j > 0$ such that $0 = \sum_{j=1}^{p+1} \eta_j \vec{g}(x_j)$, and $\sum_{j=1}^{p+1} \eta_j < \varepsilon$.

Lemma 4.6.2. *The set* $U_\varepsilon = \left\{ \sum_{j=1}^{p+1} \eta_j \vec{g}(x_j) : \eta_j \geq 0,\ \sum_{j=1}^{p+1} \eta_j < \varepsilon \right\}$ *is an open neighborhood of zero for every* $\varepsilon > 0$.

Proof. Since $U_\varepsilon = \varepsilon U_1$ it suffices to show that U_1 is open. The previous discussion shows that $0 \in U_1$. By **A1**, $\mathbb{R}^p = \cup_{n \in \mathbb{N}} U_n$, and so one of the sets U_n contains an open set by Baire's category theorem. As $U_n = nU_1$, we deduce that U_1 contains an open set, which we call V.

Let $w \in V$. Since we can write $w = \sum \eta_j \vec{g}(x_j)$ with $\sum \eta_j < 1$, it follows that $0 \in U_1 - w \subseteq U_2 = 2U_1$, and we deduce that U_1 contains an open set around zero, and so does every U_ε. We need to show that a similar property holds at every point $z \in U_1$. Let $z = \sum \lambda_j \vec{g}(x_j) \in U_1$ with $\sum \lambda_j = 1 - \delta$. Then $z + U_\delta$ is a neighborhood of z contained in U_1, which shows that U_1 is open, as we wanted to show. ■

For $y \in \partial D$, set $\vec{g}_\alpha(y) = \vec{g}(y)/\alpha(y)$. The convex combinations of $\vec{g}_\alpha$ will play a fundamental role in the characterization of the support of the stationary distribution of spinning Brownian motion. Such set is commonly referred as the *convex hull of* $\{\vec{g}_\alpha\}$:

$$H_{\vec{g},\alpha} = \left\{ \sum_{j=1}^{\infty} \lambda_j \vec{g}(y_j) : y_j \in \partial D,\ \lambda_j \geq 0,\ \sum_{j=1}^{\infty} \lambda_j = 1 \right\}.$$

The spin process S_t lives eventually in the closure of this set. To see this at an intuitive level, for initial condition s_0 we have

$$S_t = s_0 Y_t^{-1} + Y_t^{-1} \int_0^t \vec{g}_\alpha(X_u) dY_u.$$

The first term on the right hand side goes to zero and the process eventually becomes very close to the one starting at zero, which corresponds to the integral term above. The latter clearly belongs to the closure of $H_{\vec{g},\alpha}$, as it is a positive combination of elements in $\{\vec{g}_\alpha\}$ whose coefficients add up to $1 - Y_t^{-1}$. But as the discussion following A1 shows, the point zero can be written as a positive combination with coefficients adding up to Y_t^{-1}, and so S_t eventually belongs to the closure of $H_{\vec{g},\alpha}$.

Notice that the set $U_{1/2}$ from Lemma 4.6.2 is contained in $H_{\vec{g},\alpha}$. The following proposition says that the support of the stationary distribution of spinning Brownian motion lies within the closure of $D \times H_{\vec{g},\alpha}$.

Proposition 4.6.3. *Let S_t be the spin process in* (4.6.1). *Then, for arbitrary initial conditions (x, s), there is $\tau < \infty$ such that $S_t \in H_{\vec{g},\alpha}$ for every $t > \tau$ almost surely.*

Proof. By Corollary 4.6.6 below the hitting time of $U_{1/2}$ is finite almost surely since $U_{1/2}$ is a neighborhood of zero. Also, $U_{1/2} \subseteq H_{\vec{g},\alpha}$ so we conclude that the hitting time τ of $H_{\vec{g},\alpha}$ is finite almost surely.

For $t > \tau$ we have that $S_t = Y_t^{-1} Y_\tau S_\tau + Y_t^{-1} \int_\tau^t \vec{g}_\alpha(X_u) dY_u$. By continuity, $S_\tau \in H_{\vec{g},\alpha}$, and so for some $\left\{x_j^\tau\right\}$ and $\left\{\lambda_j^\tau\right\}$ with $\sum_j \lambda_j^\tau = 1$ we have by the definition of stochastic integral

$$S_t = Y_t^{-1} Y_\tau \sum_{j=1}^{\infty} \lambda_j^\tau \vec{g}_\alpha(x_j^\tau) + Y_t^{-1} \lim_{|\Delta| \to 0} \sum_{j=1}^{N} \vec{g}_\alpha(X_{t_j}) \left(Y_{t_{j+1}} - Y_{t_j}\right),$$

where $\Delta = \{t_1, \dots, t_{N+1}\}$ is a partition of $[\tau, t]$. The sum of the coefficients above equals

$$Y_t^{-1} Y_\tau + Y_t^{-1} (Y_t - Y_\tau) = 1,$$

and as stochastic integral with respect to bounded variation processes can be computed pointwise, we deduce that $S_t \in \overline{H_{\vec{g},\alpha}}$ for all $t > \tau$. ■

We prepare to prove the counterpart of the previous proposition. The proof consists of four steps. In the first one (Proposition 4.6.5), we use a support theorem and continuity results for the Skorohod map to show that for any given point $z \in D$, $T > 0$ and $\varepsilon > 0$, the probability of (X_T, S_T) to be in a ball of radius ε around the final point $(z, 0)$ is positive, no matter what the initial position is. In the second step, we use the results of Section 4.5 and excursion theory to show how the path of X_t can be decomposed into several excursions, and how spinning Brownian motion up to the first hitting time of a ball $U \subseteq D$ can be obtained from sBM conditioned on never hitting U, and adding a suitable "last excursion" that hits U. This construction is then used in the third step to patch together a spinning Brownian motion from several independent spinning Brownian motions Y_t^j. In the final step, we show how to condition each of the Y^j's on hitting the boundary of D only at certain places and deduce from this that a component of the spin S_t has a density with respect to Lebesgue measure. This procedure is detailed in Theorem 4.6.7.

Lemma 4.6.4. *Let D, $\vec{\tau}$ and $\vec{g}$ be as above. Let $r, T > 0$, and $z \in D$. Assume that A1 holds. Then, for any $(x_0, s_0) \in \overline{D} \times \mathbb{R}^p$ there is $\omega \in C([0,T]; \mathbb{R}^n)$ with bounded variation such that there is a unique $(x, s) \in C([0,T]; \overline{D} \times \mathbb{R}^p)$ satisfying $(x(0), s(0)) = (x_0, s_0)$, $(x(T), s(T)) = (z, 0)$, and for $t \in [0,T]$.*

$$x(t) = x_0 + \omega(t) + \int_0^t \vec{\gamma}(x(u), k(u)) dl(u),$$
$$k(t) = k_0 + \int_0^t \vec{g}(x(u)) - \alpha(x(u)) k(u) \; dl(u),$$

Here, $l(\cdot)$ is a continuous and increasing function, satisfying $l(t) = \int_0^t \mathbb{1}_{\partial D}(x(u)) dl(u)$, that is, it only increases when $x(t) \in \partial D$.

Proof. In virtue of Theorem 4.1 in [22], we would expect uniqueness to hold in the bounded variation case. The issue to apply such theorem directly is that the reflection vector $\vec{\gamma}$

depends on the value of $k(\cdot)$, but it is easy to adapt the proof of uniqueness in such theorem to our setting, just like we did in the last step of Lemma 4.4.4.

Next we construct a function $\overline{\omega} \in C([0,T];\mathbb{R}^n)$ with bounded variation, and a solution (x,s) of the system above. Consider the uniform partition $0 < a_1 < b_1 < a_2 < \cdots < b_{p+1} < T$ of $[0,T]$.

To construct $\overline{\omega}$ and the associated solution, set $\overline{\omega}(0) = 0$ and for $t \in (0, a_1]$, let $\overline{\omega}(t)$ be defined as any fixed continuous function with bounded variation, such that $x_0 + \overline{\omega}(t) \in D$, and $x_0 + \overline{\omega}(a_1) = x_1$. It is clear that any solution (x,s) has to satisfy $x(t) = x_0 + \overline{\omega}(t)$, $k(t) = k_0$, and $l(t) = 0$ up to time a_1. Next we want to keep $x(t)$ at x_1 from a_1 to b_1. In view of (4.6.2), for $t \in [a_1, b_1]$ we set $y_1(t) = \exp \int_{a_1}^t \alpha(x(u)\ dl(u) = e^{\alpha(x_1)[l(t)-l(a_1)]}$ and so

$$
\begin{aligned}
k(t) &= y_1(t)^{-1}k_0 + \vec{g}(x_1)y_1(t)^{-1}\int_{a_1}^t e^{\alpha(x_1)l(u)}dl(u) \\
&= y_1(t)^{-1}k_0 + \frac{\vec{g}(x_1)}{\alpha(x_1)}y_1(t)^{-1}\left[y_1(t) - 1\right].
\end{aligned}
$$

By setting $l(t) = 0 + \eta_1(t - a_1)$ for $t \in [a_1, b_1]$, where η_1 is to be determined, we obtain that both l and k are continuous. All this implies that we need to define

$$
\overline{\omega}(t) = x_1 - x_0 - \eta_1 \int_{a_1}^t \vec{\gamma}\left(x_1, k(t)\right) ds.
$$

Uniqueness in $[a_1, b_1]$ follows directly form the fact that the equation above defines a continuous function with bounded variation. Thus, the functions (x,s) defined above correspond to the unique solutions to the Skorokhod problem for $\overline{\omega}$ in $[a_1, b_1]$.

Next, we want to keep $k(t)$ constant in $[b_1, a_2]$, while we move $x(t)$ from x_1 to x_2. To this end, pick a curve $\zeta_1 : [b_1, a_2] \to \overline{D}$ such that $\zeta_1(b_1) = x_1$, $\zeta_1(a_2) = x_2$ and $\zeta_1(t) \in D$ for other values of t. Set $l(t) = l(b_1)$ for $t \in [b_1, a_2]$, and $\overline{\omega}(t) = \zeta(t) - x_1 + \overline{\omega}(b_1)$. It is clear that the only solution with bounded variation in this interval is $x(t) = \zeta_1(t)$ and $k(t) = k(b_1)$.

We iterate this process by keeping $x(t)$ at x_j from $[a_j, b_j]$, and by defining $l(t) = l(b_{j-1})+$

$\eta_j(t-a_j)$, $y_j(t) = \exp[\alpha(x_j)\eta_j(t-a_j)]$ in that interval so that $k(t)$ must satisfy

$$k(t) = [y_1(b_1)\cdots y_{j-1}(b_{j-1})y_j(t)]^{-1}k_0 + \qquad (4.6.4)$$
$$+\sum_{m=1}^{j}\frac{\vec{g}(x_m)}{\alpha(x_m)}[y_m(b_m)\cdots y_{j-1}(b_{j-1})y_j(t)]^{-1}[y_m(b_m)-1]$$

for $t\in[a_j,b_j]$. The calculation leading to such equation, though tedious, is straight-forward to carry out by splitting the integral in $[0,t]$ into integral in the sets $[a_j,b_j]$ and $[b_j,b_{j+1}]$, and using our definition of $l(u)$ in each of those intervals. In the interval $[a_j,b_j]$, define $\overline{\omega}(t)$ by

$$\overline{\omega}(t) = \overline{\omega}(a_j) - \eta_j\int_{a_j}^{t}\vec{\gamma}(x_j,k(u))ds.$$

Once again the unique solution in this interval for this $\overline{\omega}$ is $(x_j,k(t))$.

From b_j to a_{j+1} we find a curve ζ_j going from x_j to x_{j+1} through D, and set $\overline{\omega}(t) = \zeta_j(t)-x_j+\overline{\omega}(b_j)$ and $l(t)=(b_j)$. The unique solution is then $(\zeta_j(t),k(b_j))$. This procedure can be also done so that $x(T)=z$.

It remains to show that we can choose the values of $\eta_1,\ldots,\eta_{p+1}\geq 0$ such that $k(T)=0$. At time T we find that

$$k(T) = k_0\prod_{m=1}^{p+1}y_m(b_m)^{-1} + \sum_{m=1}^{p+1}\frac{\vec{g}(x_m)}{\alpha(x_m)}[y_m(b_m)-1]\prod_{i=m}^{p+1}y_i(b_i)^{-1},$$

and to obtain $k(T)=0$ we need

$$-k_0 = \sum_{m=1}^{p+1}\frac{\vec{g}(x_m)}{\alpha(x_m)}[y_m(b_m)-1]\prod_{i=1}^{m-1}y_i(b_i).$$

By hypothesis there are non-negative $\lambda_1,\ldots,\lambda_{p+1}$ such that $-k_0 = \sum_{m=1}^{p+1}\vec{g}(x_m)\lambda_m$, so we need to choose the numbers η_m so that $y_m(b_m)$ satisfies $\lambda_m\alpha(x_m) = [y_m(b_m)-1]\prod_{i=1}^{m-1}y_i(b_i)$. This is easily achieved by an inductive procedure. ■

Proposition 4.6.5. *Let D, $\vec{\tau}$, $\vec{g}$; $r,T>0$, and $z\in D$ as in Lemma 4.6.4. Then, for every $(x_0,s_0)\in D\times\mathbb{R}^p$ there exists $p>0$ such that*

$$\mathbb{P}_{x_0,k_0}\left((X_T,S_T)\in B(z,r)\times B(0,r)\right)\geq q,$$

where q depends on T and r, but is independent of (x_0,s_0).

Proof. Let $\mathbb{P}$ be the law of standard Brownian motion in $\mathbb{R}^n$. By pathwise uniqueness, we know that for a.e. $\omega \in \operatorname{supp}(\mathbb{P})$ there is unique $(x,s) \in C([0,T]; \overline{D} \times \mathbb{R}^p)$, such that for $t \in [0,T]$

$$x(t) = x_0 + \omega(t) + \int_0^t \vec{\gamma}(x(u), s(u)) dl(u),$$
$$s(t) = s_0 + \int_0^t \vec{g}(x(u)) - \alpha(x(u))k(u)\ dl(u),$$

where, $l(\cdot)$ is a continuous and increasing, satisfying $l(t) = \int_0^t \mathbb{1}_{\partial D}(x(u))dl(u)$, that is, it only increases when $x(t) \in \partial D$. It is standard to call this function $l(\cdot)$ the local time.

Let Ω be the set of continuous $\omega \in D([0,T]; \mathbb{R}^n)$ such that this uniqueness hold. We emphasize that $\mathbb{P}(\Omega) = 1$ and that the $\overline{\omega}$ constructed in Lemma 4.6.4 belongs to Ω. Define Γ in Ω by the assignment $\omega \mapsto (x,s)$ as above. We claim that Γ is continuous at $\overline{\omega}$, where continuity is taken in the sense of uniform convergence in compact sets. Indeed, let $\omega_j \in \Omega$ be a sequence converging uniformly in $[0,T]$ to $\overline{\omega}$. Then, by setting $z(t) = (x,s)(t)$ and $\vec{\zeta}(z) = (\vec{\gamma}(x,s), \vec{g}(x))$, we have that the $\mathbb{R}^{n+p}-$valued functions $\eta_j = (\omega_j, 0)$ converge uniformly to $\overline{\eta} = (\overline{\omega}, 0)$. By Theorem 3.1 in [9], we have that the unique solutions (x_j, k_j) to the Skorokhod problem with reflecting vector $\vec{\zeta}$ in $\overline{D} \times \mathbb{R}^p$, and corresponding driving function η_j is relatively compact, and any limit is a solution of the corresponding problem with driving function $\overline{\omega}$. By uniqueness ($\overline{\omega} \in \Omega$), we deduce that $(x_j, k_j) \to (\overline{x}, \overline{k})$ in $D([0,T], \mathbb{R}^{n+p})$. But as all the involved functions are continuous, we actually deduce that the latter convergence is uniform in $[0,T]$.

In particular, there is $\delta > 0$ such that if $\omega \in \Omega \cap B_C(\overline{\omega}, \delta)$, then the associated solution to the Skorohod problem $(x,s) \in B_{C[0,T]}((\overline{x}, \overline{k}), r)$. Thus we have

$$\begin{aligned}\mathbb{P}_{x_0,k_0}\left((X_T, S_T) \in B(z,r) \times B(y,r)\right) &\geq \mathbb{P}_{x_0,k_0}\left((x,s) \in B_C((\overline{x}, \overline{k}), r)\right) \\ &\geq \mathbb{P}_{x_0}\left(\omega \in B_C(\overline{\omega}, \delta)\right),\end{aligned}$$

which is greater than some positive constant q, independent of x_0, by the support theorem of Brownian motion. ■

Corollary 4.6.6. *Let $r > 0$ and $\tau = \inf\{t > 0 : S_t \in B(0,r)\}$. Then τ is finite almost surely.*

Proof. Let $N[a,b]$ be the event "S_t is not in $B(0,r)$ for any $t \in [a,b]$". Then $\mathbb{P}_{x,s}(\tau < \infty) = 1 - \lim_{b\to\infty} \mathbb{P}_{x,s}(N[0,b])$, where the limit is obviously decreasing.

By Proposition 4.6.5 we have that $\mathbb{P}_{x,s}(N[0,T]) \leq 1-q$, where $q \in (0,1)$ depends on T but not on x, s. By the Markov property we have

$$\begin{aligned}
\mathbb{P}_{x,s}(N[0,(n+1)T]) &= \mathbb{P}_{x,s}(N[0,nT] \cap N[nT,(n+1)T]) \\
&= \mathbb{P}_{x,s}(\mathbb{P}_{x,s}(N[0,nT] \cap N[nT,(n+1)T] | \mathcal{F}_{nT})) \\
&= \mathbb{P}_{x,s}(N[0,nT]\ \mathbb{P}_{X_{nT},S_{nT}}(N[0,T])) \\
&\leq \mathbb{P}_{x,s}(N[0,nT])\,(1-q).
\end{aligned}$$

It follows by induction that $\mathbb{P}_{x,s}(N[0,nT]) \leq (1-q)^n$, and $\mathbb{P}_{x,s}(\tau < \infty) = 1$. ∎

4.6.1 Uniqueness of the stationary measure

We next proceed to prove some interesting results about the stationary distribution of the spinning Brownian motion. Our method is very much an adaptation of the proof of Theorem 6.1 in [2]. Such argument involves a decomposition of (the law of) X_t in several reflecting processes that are somewhat independent of each other. To the reader familiar with excursion theory, "independence" is achieved by using suitable exit systems (see [24, 5]). This decomposition allows us to control both the local time and the trajectory of the process before hitting a fixed open set U, and deduce that no stationary measure can be null in U.

Theorem 4.6.7. *Let (X,S) be spinning Brownian motion solving* (4.1.1) *with $S_0 = 0$. Then, for every $T > 0$, there is an open set $U \subseteq H_{\vec{g},\alpha}$ containing zero, and $c > 0$ such that for every open $B \subseteq U$ it holds that $\mathbb{P}_{x,s}(S_T \in B) > c\, m^p(B)$.*

Proof. In Theorem 4.5.2 we have obtained an exit system for $Z_t \doteq (X_t, S_t)$ representing excursions from $\partial D \times \mathbb{R}^p$. As S_t does not change within an excursion of Z_t away from $\partial D \times \mathbb{R}^p$, we can think of the excursion law $\mathbf{H}_{x,s}$ as a measure representing paths of X only. The unique (up to a multiplicative constant) excursion law $\mathbf{H}_{x,s}$ associated to the local time

L_t is given by

$$\mathbf{H}_{x,s}(A) = c_1 \lim_{\eta \downarrow 0} \frac{1}{\eta} \mathbb{P}^D_{x+\eta \hat{n}(x)}(A), \tag{4.6.5}$$

for some constant $c_1 > 0$, independent of x. Notice that $\mathbf{H}_{x,s}$ is independent of s, so we are allowed to drop the subindex s from it.

From now on we closely follow part of the proof of Theorem 6.1 in [2]. To simplify the notation, call $Z = (X, S)$, and we use the standard nomenclature T_U for first hitting time of a set U, and $\sigma_t = \inf \{s \geq 0 : L_s \geq t\}$ for the right inverse of local time.

We proceed to describe an exit system for a different, though related, process X'. Let $z_0 \in D$ and $r > 0$ be arbitrary but fixed, so that $\overline{B}_r(x_0) \subseteq D$ and set $U = \overline{B}_r(z_0)$. Let X' be the process X conditioned by the event $\left\{T^X_U > \sigma_1\right\}$. It follows from Proposition 4.6.5 and the strong Markov property that for any starting point in D, the probability of $\left\{T^X_U > \sigma_1\right\}$ is greater than zero. It is easy to see that (X'_t, L_t) is a time homogeneous Markov process for $\mathbb{P}_{z_0,s_0}$ in $(\mathcal{F}_t)_{t \geq 0}$. For notational consistency, we will write (X'_t, L'_t) instead of (X'_t, L_t).

We will now describe an exit system $(L'_t, \mathbf{H}'_{x,l})$ for (X'_t, L'_t) from the closed set $\overline{D} \times \mathbb{R}^p \times [0, \infty)$. We will construct this exit system on the basis of $(L_t, \mathbf{H}_x)$ because of the way that X' has been defined in relation to X. It is clear that L' does not change within any excursion interval of X' away from ∂D, so we will assume that $\mathbf{H}'_{x,l}$ is a measure on paths representing X' only. For $l \geq 1$ we let $\mathbf{H}'_{x,l} = \mathbf{H}_x$. Let $\widehat{\mathbb{P}}^D_y$ denote the distribution of Brownian motion starting from $y \in D \setminus \overline{B}_r(z_0)$, conditioned to hit ∂D before hitting $\overline{B}_r(z_0)$, and killed upon exiting D. For $l < 1$, we have

$$\mathbf{H}'_{x,l}(A) = c_1 \lim_{\eta \downarrow 0} \frac{1}{\eta} \widehat{\mathbb{P}}^D_{x+\eta \hat{n}(x)}(A). \tag{4.6.6}$$

Let $A_\star \subseteq C$ be the event that the path of X' hits U. It follows from the definition of $\mathbf{H}_x$ in Theorem 4.5.2 and (4.6.6) that for $l < 1$,

$$\mathbf{H}'_{x,l}(A) = \mathbf{H}_x(A \setminus A_\star). \tag{4.6.7}$$

One can deduce easily from standard estimates for Brownian motion that for some constants

$c_3, c_4 > 0$ and all $x \in \partial D$,

$$c_3 < \mathbf{H}_x(A_\star) < c_4. \tag{4.6.8}$$

Let $\sigma'_t = \inf\{s \geq 0 : L'_s \geq t\}$. The exit system formula (4.5.1) and (4.6.7) imply that we can construct X using X' as a building block, in the following way. Suppose that X' is given. We enlarge the probability space, if necessary, and construct a Poisson point process $\mathcal{E}$ with state space $[0,\infty) \times C$ whose intensity measure conditional on the whole trajectory $\{X'_t,\ t \geq 0\}$ is given by

$$\mu\left([a,b] \times F\right) = \int_{1\wedge a}^{1\wedge b} \mathbf{H}_{X'_{\sigma'_t}}(F \cap A_\star)dt. \tag{4.6.9}$$

Since $\mu([0,\infty] \times C) < c_4$, the Poisson process $\mathcal{E}$ may be empty; that is, if the Poisson process is viewed as a random measure, then the support of that measure may be empty. Consider the case when it is not empty and let S_1 be the minimum of the first coordinates of points in $\mathcal{E}$. Note that there can be only one point $(S_1, \varepsilon_{S_1}) \in \mathcal{E}$ with first coordinate S_1, because of (4.6.8). By convention, let $S_1 = \infty$ if $\mathcal{E} = \emptyset$. Recall that $T_U^X = \inf\{t > 0 : X_t \in U\}$ and let

$$T_U^{X'} = \inf\left\{t > 0 : X'_t \in U\right\},$$
$$T_\star = \sigma'_{S_1} + \inf\{t > 0 : e_{S_1}(t) \in U\}.$$

It follows from the exit system formula that the distribution of the process

$$\widehat{X}_t = \begin{cases} X'_t & \text{if } 0 \leq t \leq T_U^{X'} \wedge \sigma'_{S_1}, \\ e_{S_1}(t - \sigma'_{S_1}) & \text{if } \mathcal{E} \neq \emptyset \text{ and } \sigma'_{S_1} < t \leq T_\star, \end{cases}$$

is the same as the distribution of $\left\{X_t, 0 \leq t \leq T_U^X\right\}$.

We will now construct spinning Brownian motion in D from several trajectories, including a family of independent paths.

Let $U_j = \overline{B}_r(z_j)$ for $j = 1, \ldots, p+1$, where $z_j \in D$ are chosen so that $U_j \cap U_k = \emptyset$ for $j \neq k$, and $\bigcup_{1 \leq j \leq p+1} U_j \subseteq D$.

Recall how the process X was constructed from a process X'. Fix some $x_1 \in U_1$ and let X^1 be a process starting from $X_0^1 = x_1$, with the same transition probabilities as X', relative to U_2. We then construct Y^1 based on X^1, by adding an excursion that hits U_2, in the same way as X was constructed from X'. We thus obtain a process $\{Y^1_t, 0 \le t \le T_1\}$, where $T_1 = \inf\{t > 0 : Y^1_t \in U_2\}$, whose distribution is that of spinning Brownian motion in D starting with the uniform distribution on U_1, observed until the first hit of U_2.

We will next construct a family of independent spinning Brownian motions $\{Y^j\}_{1\le j\le p+1}$. For a fixed $j = 2, \dots, p+1$, we let X^j be a process with the same transition probabilities as X', relative to U_{j+1}, and initial distribution uniform in U_j. We then construct Y^j based on X^j, by adding an excursion that hits U_{j+1}, in the same way as X was constructed from X'. We thus obtain a process $\left\{Y^j_t, 0 \le t \le T_j\right\}$, where $T_j = \inf\left\{t > 0 : Y^j_t \in U_{j+1}\right\}$, whose distribution is that of spinning Brownian motion in D, observed until the first hit of U_{j+1}.

Note that for some $c_5 > 0$ and all $x, y \in U_{j+1},\ j = 1, \dots, p+1$,

$$\mathbb{P}_x\left(X_1 \in dy \text{ and } X_t \notin \partial D \text{ for } t \in [0,1]\right) \ge c_5 dy.$$

We can assume that all X^j's and Y^j's are defined on the same probability space. The last formula and standard coupling techniques show that on an enlarged probability space, there exist spinning Brownian motions W^j, $j = 1, \dots, p+1$, with the following properties. For $1 \le j \le p$, $W^j_0 = Y^j_{T_j}$, and for some $c_6 > 0$

$$\mathbb{P}\left(W^j_1 = Y^{j+1}_0 \text{ and } W^j_t \notin \partial D \text{ for } t \in [0,1] \,\middle|\, \{Y^k\}_{k=1}^{j},\ \{W^k\}_{k=1}^{j-1}\right) \ge c_6. \tag{4.6.10}$$

The process W^j does not depend otherwise on $\{Y^k\}_{k=1}^{p+1}$ and $\{W^k\}_{k\ne j}$. We define W^{d+1} as a spinning Brownian motion in D with $W^{p+1}_0 = Y^{p+1}_T$ but otherwise independent of $\{Y^k\}_{k=1}^{p+1}$ and $\{W^k\}_{k=1}^{p}$.

Let

$$F_j = \left\{W^j_1 = Y^{j+1}_0 \text{ and } W^j_t \notin \partial D \text{ for } t \in [0,1]\right\}.$$

We define a process X^* as follows. We let $X^*_t = Y^1_t$ for $0 \le t \le T_1$. If F^c_1 holds, then we let $X^*_t = W^1_{t-T_1}$ for $t \ge T_1$. If F_1 holds, then we let $X^*_t = W^1_{t-T_1-1}$ for $t \in [T_1, T_1+1]$ and $X^*_t = Y^2_{t-T_1-1}$ for $t \in [T_1+1, T_1+1+T_2]$. We proceed by induction. Suppose that X^*_t has been defined so far only for $t \in [0, T_1+1+T_2+1+\cdots+T_k]$, for some $k < n$. If F^c_k holds then we let

$$X_t^* = W^k_{t-T_1-1-T_2-1-\cdot-T_k}$$

for $t \geq T_1 + 1 + T_2 + 1 + \cdots + T_k$. If F_k holds, then we let

$$X_t^* = W^k_{t-T_1-1-T_2-1-\cdot-T_k}$$

for $t \in [T_1 + 1 + T_2 + 1 + \cdots + T_k, T_1 + 1 + T_2 + 1 + \cdots + T_k + 1]$, and

$$X_t^* = Y^{k+1}_{t-T_1-1-T_2-1-\cdots-T_k-1}$$

for $t \in [T_1 + 1 + T_2 + 1 + \cdots + T_k + 1, T_1 + 1 + T_2 + 1 + \cdots + T_k + 1 + T_{k+1}]$. We let

$$X_t^* = W^{p+1}_{t-T_1-1-T_2-1-\cdots-T_{p+1}}$$

for $t \geq T_1 + 1 + T_2 + 1 + \cdots + T_{p+1}$.

By construction, X^* is a spinning Brownian motion in D starting from x_1. Note that in view of (4.6.10), conditional on $\{X_t^j, t \geq 0\}$, $j = 1, \ldots, p+1$, there is at least probability ${c_6}^n$ that X^* is a time-shifted path of X_t^j on an appropriate interval, for all $j = 1, \ldots, p+1$.

In the last step of this proof, we show that with a positive probability, the process S can have "almost" independent and "almost" perpendicular increments over disjoint time intervals. Moreover, the distributions of the increments have densities in an appropriate sense.

Recall our assumption A1: there are $p+1$ points $x_1, \ldots, x_{p+1} \in \partial D$ such that the $\vec{g}(x_j)$'s positively span the whole $\mathbb{R}^p$. We claim that d of those points are linearly independent. Indeed, the rank of the $p \times (p+1)$ matrix $[\vec{g}(x_1)| \cdots |\vec{g}(x_{p+1})]$ is p and so one of its columns, say $\vec{g}(X_{p+1})$ without loss of generality, is a linear combination of the others. Then, the rank of $[\vec{g}(x_1)| \cdots |\vec{g}(x_p)]$ is still d, which means that its columns are linearly independent.

For $1 \leq j \leq d$, let $C_j = \{z \in \mathbb{R}^n : \angle(\vec{g}(x_j), z) \leq \delta_0\}$, for some $\delta_0 > 0$ so small that for any $z_j \in C_j$, $j = 1, \ldots, d$, the vectors $\{z_j\}$ are still linearly independent. Let $\delta_1 > 0$ be so small that for every $j = 1, \ldots, d$, and any $x \in \partial D \cap B_{\delta_1}(x_j)$, we have $\vec{g}(x) \in C_j$.

Let L^j the local time of X^j on ∂D and $\sigma_t^j = \inf\left\{s \geq 0 : L_s^j \geq t\right\}$. It is easy to see that there exists $p_2 > 0$ such that with probability greater than p_2 , for every $j = 1, \ldots, p+1$ we have $X_t^j \notin \partial D \setminus B_{\delta_1}(y_j)$, for $t \in [0, \sigma_t^j]$. Let

$$R_j = \sup\left\{t < T_j : Y_t^j \in \partial D\right\} \quad \text{and} \quad Q_j = L_{R_j}^j.$$

Let $F_\star$ be the event that for every $j = 1, \ldots, p$, we have $X_t^j \notin \partial D \setminus B_{\delta_1}(y_j)$ for $t \in [0, \sigma_1^j]$ and $R_j < \sigma_1^j$. Then (4.6.8) shows that $\mathbb{P}_{z_0}(F_\star) \geq p_2(1 - e^{-c_3})^p$.

Let $Y_t^j = \int_0^t \alpha(X_s^j) dL_s$, and consider the following collection of random variables

$$S^j(t_j, \ldots, t_p) = \left(Y_{\sigma_{t_j}^j}^j \cdots Y_{\sigma_{t_p}^p}^p\right)^{-1} \int_0^{t_j} \vec{g}\left(X_{\sigma_u^j}^j\right) Y_{\sigma_u^j}^j du.$$

Notice that if $F_\star$ holds, then $S^j(t_j, \ldots, t_p) \in C_j$ for all $t_j, \ldots, t_p \in (0, 1]$ and $j = 1, \ldots, p$. Define for any $0 \leq a_k < b_k \leq Q_k$ for $k = 1, \ldots, p$

$$\Lambda\left([a_1, b_1], \ldots, [a_p, b_p]\right) = \left\{\sum_{j=1}^p S^j(t_j, \ldots, t_d) : t_k \in [a_k, b_k]\right\}.$$

It is not difficult to estimate the d-dimensional volume of $\Lambda([a_1, b_1], \ldots, [a_p, b_p])$ by using the definition of C_j's. First, by continuity, it follows that under $F_\star$ there is a positive constants q such that $(1 - q\delta_0) \exp[\alpha(x_j) t_j] \leq Y_{\sigma_{t_j}^j}^j \leq (1 + q\delta_0) \exp[\alpha(x_j) t_j]$ for all $1 \leq j \leq p$. By definition of the set C_j, it follows that for some positive constant β,

$$(1 - \beta\delta_0) \frac{\vec{g}(x_j)}{\alpha(x_j)} \left[e^{\alpha(x_j) t_j} - 1\right] \leq \int_0^{t_j} \vec{g}(X_{\sigma_u^j}^j) Y_{\sigma_u^j}^j du \leq (1 + \beta\delta_0) \frac{\vec{g}(x_j)}{\alpha(x_j)} \left[e^{\alpha(x_j) t_j} - 1\right].$$

Define a function $\vec{v}(t_1, \ldots, t_d)$ by

$$\vec{v} = \sum_{j=1}^d e^{-\sum_{k=j}^d \alpha(x_k) t_k} \frac{\vec{g}(x_j)}{\alpha(x_j)} \left[e^{\alpha(x_j) t_j} - 1\right].$$

It follows from the inequalities in this paragraph that for some constant $\eta > 0$, independent of δ_0, we have

$$(1 - \eta\delta_0) \vec{v}(t_1, \ldots, t_d) \leq \sum_{j=1}^p S^j(t_j, \ldots, t_d) \leq (1 + \eta\delta_0) \vec{v}(t_1, \ldots, t_d).$$

If δ_0 is small enough, we have that there exist a constant c_3 independent of a_k, b_k such that

$$c_3^{-1} \leq \frac{m^d\left(\Lambda([a_1, b_1], \ldots, [a_p, b_p])\right)}{m^d\left\{\vec{v}(t_1, \ldots, t_p) : t_k \in [a_k, b_k]\right\}} \leq c_3$$

To compute the volume of the set in the denominator, we will calculate the Jacobian of $\vec{v}(\cdot)$ in Lemma 4.6.8. We obtain that $\det D\vec{v} = C\exp\left(-\sum_{k=1}^{p} k\alpha(x_k)t_k\right)$, which, as $t_k \in [0,1]$, readily yields that the d-dimensional volume of the random set $\Lambda([a_1, b_1], \ldots, [a_p, b_p])$ is bounded above by $c_4(b_1 - a_1)\cdots(b_p - a_p)$ and below by $c_5(b_1 - a_1)\cdots(b_p - a_p)$.

Let us consider the processes X^* defined above, conditioned on the sigma field

$$\mathcal{G} = \sigma\left(\left\{S^j(t_j, \ldots, t_d),\ t_k \in [0,1]\right\}_{j=1}^{d}\right).$$

By (4.6.8) and (4.6.9), conditional on $\mathcal{G}$, the random variables Q_j's have distributions whose densities in $[0,1]$ are bounded below. In view of our remarks on the volume of Λ, it follows that conditional on $\mathcal{G}$, the vector

$$S(Q_1, \ldots, Q_d) = S^1(Q_1, \ldots, Q_d) + S^2(Q_2, \ldots, Q_d) + \cdots + S^d(Q_d)$$

has a density with respect to d-dimensional Lebesgue measure that is bounded below by $c_5 > 0$ on the open set $\Lambda_1 = \Lambda((0,1), \ldots, (0,1))$. Even though $0 \notin \Lambda_1$, we have that $0 \in \partial\Lambda_1$, and it still holds that for small $r > 0$ the random variable $S(Q_1, \ldots, Q_d)$ has a density bounded below in $B(0,r)$. We now remove the conditioning and fix $r_0 > 0$ to conclude that $S(Q_1, \ldots, Q_d)$ has a component with a density with respect to d-dimensional Lebesgue measure that is bounded below on $U = B(0, r_0)$.

Define $Y_t^* = \exp\int_0^t \alpha(X_s^*)dL_s^*$ and $S_t^* = Y_t^{*-1}\int_0^t \vec{g}(X_s^*)Y_s^* dL_s^*$ and $T_\star = \sum_{j=1}^{d} T_j$, where L^* is the boundary local time for spinning Brownian motion X^*. Using conditioning on $F_\star$, we see that the distribution of $S_{T_\star}^*$ has a component with density greater than c_9 on U.

The previous argument can be modified to show that for any fixed $t_0 > 0$, the random variable $S_{t_0/2}^*$ has a component with a strictly positive density with respect to d-dimensional Lebesgue measure on a non-empty set U, which proves the theorem. All that is need to do is, for small $\varepsilon > 0$, find times $t_j > 0$ such that $T_j \in (t_j - \varepsilon, t_j + \varepsilon)$, with uniformly (in j) positive probability p_ε, and then further condition the stitched process Y^* to satisfy

$T_j \in (t_j - \varepsilon, t_j + \varepsilon)$. Set $t_* = \sum_{j=1}^d t_j$. This way, $T_* = \sum_{j=1}^d T_j \in (t_* - \varepsilon d, t_* + \varepsilon d)$ and since $Y^*_{T_*} \in U_d$, and U_d is away from the boundary, we can condition on Y^* to not to hit ∂D in $[t_* - \varepsilon d, t^* + \varepsilon_d]$ and thus have $S^*_{t_*} = S^*_{T_*}$. We then choose $t_0 = 2t_*$. ■

Lemma 4.6.8. *Let $\vec{v}(\cdot)$ be the function defined in the last step of the proof of Theorem 4.6.7. There exists a constant $C \neq 0$, depending only on the vectors $\vec{g}(x_k)$ such that*

$$\det D\vec{v}(t_1, \ldots, t_d) = C \exp \sum_{k=1}^{d} -k\alpha(x_k)t_k$$

for all $t_1, \ldots, t_d \in (0, 1]$

Proof. We calculate the derivatives with respect to t_i, for $i = 1, \ldots, d$:

$$\begin{aligned}
\frac{\partial \vec{v}}{\partial t_i} &= \sum_{j=1}^{d} e^{-\sum\limits_{k=j}^{d} \alpha(x_k)t_k} \frac{\vec{g}(x_j)}{\alpha(x_j)} \left(\alpha(x_j) e^{\alpha(x_j)t_j} \delta_{ij} - \alpha(x_i) \mathbb{1}_{[j,d]}(i) \left[e^{\alpha(x_j)t_j} - 1 \right] \right) \\
&= e^{-\sum\limits_{k=i+1}^{d} \alpha(x_k)t_k} \vec{g}(x_i) - \alpha(x_i) \sum_{j=1}^{i} e^{-\sum\limits_{k=j}^{d} \alpha(x_k)t_k} \frac{\vec{g}(x_j)}{\alpha(x_j)} \left[e^{\alpha(x_j)t_j} - 1 \right] \\
&= e^{-\sum\limits_{k=i}^{d} \alpha(x_k)t_k} \vec{g}(x_i) - \alpha(x_i) \sum_{j=1}^{i-1} e^{-\sum\limits_{k=j}^{d} \alpha(x_k)t_k} \frac{\vec{g}(x_j)}{\alpha(x_j)} \left[e^{\alpha(x_j)t_j} - 1 \right].
\end{aligned}$$

Let $\lambda_{j,i}$ the coefficient of $\vec{g}(x_j)$ in the expansion of $\frac{\partial \vec{v}}{\partial t_i}$ above. Because the vectors $\vec{g}(x_k)$ are linearly independent, these numbers are well defined. We specially remark that $\lambda_{j,i} = 0$ for $j > i$. Let $T_{\vec{g}}$ the $d \times d$ matrix whose $j-$th column is $\vec{g}(x_j)$ and let Λ be the matrix whose (j, i) component is $\lambda_{j,i}$. The calculation above then simply says that $D\vec{v} = T_{\vec{g}}\Lambda$. Therefore, as Λ is triangular

$$\det D\vec{v}(t_1, \ldots, t_d) = \det T_{\vec{g}} \cdot \prod_{i=1}^{d} \lambda_{i,i} = \det T_{\vec{g}} \cdot \exp \left(- \sum_{i=1}^{d} \sum_{k=i}^{d} \alpha(x_k)t_k \right).$$

The double sum on the right hand side equals $\sum_{k=1}^d k\alpha(x_k)t_k$, by Fubini's theorem, and the proof is complete. ■

Corollary 4.6.9. *Spinning Brownian motion has a unique stationary distribution, supported in the closure of $D \times H_{\vec{g},\alpha}$.*

Proof. Fix $T > 0$, and U given by the previous theorem. Since S does not change when X is inside the domain D and X behaves like Brownian motion within excursions, we conclude from the Markov property that (X_{T+1}, S_{T+1}) has a component with a density with respect to $(n+d)$-dimensional Lebesgue measure on a non-empty open subset of $E \times U$. By Proposition 4.6.3, we can assume that $U \subseteq H_{\vec{g},\alpha}$.

We can now combine this with the result of Proposition 4.6.5 using the Markov property to see that for some non-empty set $\tilde{U}$ and any starting point $(X_0, S_0) = (z_0, k_0)$, the process (X_{t_0}, S_{t_0}) has a positive density with respect to $(n+d)$-dimensional Lebesgue measure on $\tilde{U}$ under $\mathbb{P}_{z_0,k_0}$. This property is generally referred to as Harris irreducibility of the process (X, S).

From Proposition 4.6.3, we know that any stationary distribution of (X, S) has to be supported in the closure of $D \times H_{\vec{g},\alpha}$, a bounded set, therefore we deduce that (X, S) has at least one stationary distribution from the standard theory of Markov processes (see Lemma 2.6.1). Let μ be one of them. For the open set $\tilde{U}$ in the preceding paragraph

$$\mu(\tilde{U}) = \int_E \mathbb{P}_{x,k}\left[(X_{t_0}, S_{t_0}) \in \tilde{U}\right] \; \mu(dxdy) \geq c\, m^d(\tilde{U}) > 0,$$

which means that any stationary distribution contains $\tilde{U}$ in its support. This contradicts Birkhoff's ergodic theorem in case that more than one stationary distribution exist, so there is only one stationary distribution. ■

Corollary 4.6.10. *Let $A \times B \subseteq D \times H_{\vec{g},\alpha}$ be open. Then, the unique stationary distribution of spinning Brownian motion has a component with density that's bounded below on $A \times B$, and so its support is exactly the closure of $D \times H_{\vec{g},\alpha}$.*

Proof. Let $T > 0$ and U be the open neighborhood of zero from Theorem 4.6.7. Fix $t > T$, $z \in H_{\vec{g},\alpha}$ and a small $r > 0$. We have that for any $(x, s) \in D \times H_{\vec{g},\alpha}$,

$$\begin{aligned}
\mathbb{P}_{x,s}\left(S_t \in B(z,r)\right) &= \mathbb{P}_{x,s}\left(S_t - z \in \; B(0,r)\right) \\
&= \mathbb{P}_{x,s}\left(Y_t^{-1} Y_T S_T + Y_t^{-1} \int_T^t \vec{g}_\alpha(X_u) dY_u - z \in B(0,r)\right).
\end{aligned}$$

Notice that $Y_t^{-1}\int_T^t \vec{g}_\alpha(X_u)dY_u$ is an element of $H_{\vec{g},\alpha}$. Moreover, by the support theorem of Brownian motion, and computations similar to those in Lemma 4.6.4 and the arguments in Proposition 4.6.5 the event $F = \left\{\left|Y_t^{-1}\int_T^t \vec{g}_\alpha(X_u)dY_u - z\right| < \frac{r}{2}\right\}$ has positive probability, uniformly in (x, s). Conditioning on F we obtain

$$\begin{aligned}\mathbb{P}_{x,s}\left(S_t \in B(z,r)\right) &\geq c\,\mathbb{P}_{x,s}\left(Y_t^{-1}Y_T S_T \in B(0,r/2)|F\right)\\ &= c\,\mathbb{P}_{x,s}\left(S_T \in B(0, Y_t Y_T^{-1} r/2)|F\right) \geq c\,\mathbb{P}_{x,s}\left(S_T \in B(0,r/2)|F\right),\end{aligned}$$

where we use that $Y_t Y_T^{-1} = \exp(L_t - L_T) \geq 1$. The event F lies in the sigma algebra $\sigma\left((X_u, S_u) : T \leq u \leq t\right)$ of the future of S_T, of which S_T is independent by the Markov property. Hence,

$$\mathbb{P}_{x,s}\left(S_t \in B(z,r)\right) \geq c\,\mathbb{P}_{x,s}\left(S_T \in B(0,r/2)\right) \geq c\, r^d,$$

by our choice of S_T from Theorem 4.6.7.

We next use the previous estimate to compute the joint distribution of (X_t, S_t). Let $A \subseteq D$ and $B \subseteq H_{\vec{g},\alpha}$ be open, and pick a small $\varepsilon > 0$

$$\begin{aligned}\mathbb{P}_{x,s}\left(X_{t+1} \in A,\ S_{t+1} \in B\right) &\geq \mathbb{P}_{x,s}\left(X_{t+1} \in A,\ S_{t+1} \in B,\ L_{t+1} < \varepsilon + L_t\right)\\ &= \mathbb{P}_{x,s}\left(X_{t+1} \in A,\ S_t \in B^\varepsilon,\ L_{t+1} < \varepsilon + L_t\right)\\ &= \mathbb{P}_{x,s}\left(S_t \in B^\varepsilon;\ \mathbb{P}_{x,s}\left(X_{t+1} \in A,\ L_{t+1} < \varepsilon + L_t|\mathcal{F}_t\right)\right)\\ &= \mathbb{P}_{x,s}\left(S_t \in B^\varepsilon;\ \mathbb{P}_{X_t,S_t}\left(X_1 \in A,\ L_1 < \varepsilon\right)\right)\\ &\geq \mathbb{P}_{x,s}\left(S_t \in B;\ \mathbb{P}^D_{X_t}\left(X_1 \in A\right),\ d(X_t, \partial D) > \varepsilon\right).\end{aligned}$$

It is well known that there is $c_\varepsilon > 0$ such that for every $x \in D$ that is at least ε away from the boundary we have $\mathbb{P}^D_x(X_1 \in A) \geq c_\varepsilon m^n(A)$. So,

$$\mathbb{P}_{x,s}\left(X_{t+1} \in A,\ S_{t+1} \in B\right) \geq c_\varepsilon m^n(A)\mathbb{P}_{x,s}\left(S_t \in B^\varepsilon,\ d(X_t, \partial D) > \varepsilon\right),$$

as the probability of X_t being away from the boundary is positive, and there we can bound $P^D_x(X_1 \in A)$ below by $c\ m^n(A)$, uniformly in x. Finally, if μ is the stationary distribution of (X, S)

$$\mu(A \times B) = \int \mathbb{P}_{x,s}\left(X_{t+1} \in A,\ S_{t+1} \in B\right) d\mu \geq c\ m^n(A) m^d(B),$$

which concludes our proof. ∎

Chapter 5

EXAMPLES AND APPLICATIONS

5.1 Obliquely reflected Brownian motion

5.1.1 Uniform distribution in smooth domains

For a standard, n-dimensional Brownian motion, it is well known that the Lebesgue measure is stationary. This fact can be easily obtained from the invariance under rotations and translations of both Brownian motion and Lebesgue measure. In dimensions $n \geq 3$, Brownian motion is not Harris recurrent and thus some more stationary measures might arise. However, every such measure needs to be invariant under translations and rotations. The Lebesgue measure is the unique measure with these properties, and uniqueness holds.

Theorem 5.1.1. *Let X_t be an obliquely reflected Brownian motion on a $C^2(\mathbb{R}^2)$, bounded domain, with tangential boundary push $\vec{\tau}$ satisfying* $\operatorname{bdiv}(\vec{\tau}) = 0$. *Then, the uniform distribution in D is stationary for X.*

Proof. The proof is a straight-forward consequence of the definition of boundary divergence and Green's theorem: If $f \in C_0^2(\mathbb{R}^n)$ satisfies $\nabla f(x) \cdot [\vec{n} + \vec{\tau}](x) \geq 0$ on the boundary, then, by Green's theorem,

$$\begin{aligned}
\int_D \Delta f(x)dx &= -\int_{\partial D} \nabla f(x) \cdot \vec{n}(x)\sigma(dx) \\
&\leq -\int_{\partial D} \nabla f(x) \cdot \vec{\tau}(x)\sigma(dx) \\
&= \int_{\partial D} f(x)\mathrm{bdiv}(\tau)(x)\sigma(dx) = 0.
\end{aligned}$$

By Theorem **??**, we conclude that the Lebesgue measure is stationary for X. ■

Of course, choosing $\vec{\tau} \equiv 0$ gives us a uniform stationary distribution, as in this case the process X corresponds to reflected Brownian motion. Alternatively, going back to the proof of existence of the boundary divergence we can choose, locally, the components of $\vec{\tau}$ as follows:

$$\tau^j = \frac{1}{\sqrt{1 + \|\nabla \phi_k\|^2}},$$

and use a partition of unity to patch together this definition through ∂D.

5.1.2 Can we prescribe the stationary distribution?

Given a harmonic function $\rho : D \to \mathbb{R}$ satisfying the conditions of Theorem 3.2.1, can we find a vector $\vec{\tau}$ such that $dX_t = dB_t + [\vec{n} + \vec{\tau}]\,(X_t)dL_t$ has $\rho(x)$ as its stationary density?

The answer seems to be positive as long as the following procedure can be justified. For example this happens in $C^2(\mathbb{R}^n)$ domains: Integration by parts as in the proof of the contraction properties (Theorem 3.2.1 shows that, the density ρ and the tangential push $\vec{\tau}$ need to satisfy:

$$\nabla \rho \cdot \vec{n}(x) = \mathrm{bdiv}(\rho \vec{\tau})(x), \qquad\qquad x \in \partial D.$$

The problem then reduces to the following question: Given a continuous function $\varphi(x)$ on the boundary of D, is there a vector field $\vec{\kappa}$ such that $\varphi(x) = \mathrm{bdiv}(\vec{\kappa})(x)$? We do not know the most general conditions in which we can answer yes to the question, and we haven't found a substantially clarifying counter example either. Sometimes, we can answer positively to this question, as in the following example.

Example. Let D be the unit circle in $\mathbb{R}^2$. We claim that any strictly positive harmonic function in $\overline{D}$, with bounded derivatives is the stationary distribution of some obliquely reflected Brownian motion.

Indeed, if $\varphi(x, y)$ is any positive harmonic function in $\overline{D}$ with bounded derivatives, by working in polar coordinates and abusing the notation by writing $\varphi = \varphi(r, \theta)$, we look for a

2π-periodic function $\xi(\theta)$ such that

$$\int_0^{2\pi} -\partial_r\varphi(1,\theta)\ f(\theta)d\theta = -\int_0^{2\pi} \xi(\theta)\partial_\theta f(\theta)d\theta, \tag{5.1.1}$$

where f is any 2π-periodic function with continuous second derivatives. We can then set $\vec{\tau}(\theta) = \xi(\theta)\hat{\theta}$ as the tangential boundary push. Since the right hand side of (5.1.1) can be integrated by parts to get $\int_0^{2\pi} \partial_\theta\xi(\theta)\ f(\theta)$ it is sufficient to set

$$\xi(\theta) = \int_0^{2\pi} -\partial_r\varphi(1,\theta)d\theta.$$

This definition is consistent, as the resulting ξ is 2π periodic, since for $\theta_0 \in [0, 2\pi)$,

$$\begin{aligned}
\xi(\theta_0 + 2\pi) - \xi(\theta_0) &= \int_{\theta_0}^{2\pi+\theta_0} \partial_r\varphi(1,\theta)d\theta \\
&= \int_{\theta_0}^{2\pi} \partial_r\varphi(1,\theta)d\theta + \int_{2\pi}^{2\pi+\theta_0} \partial_r\varphi(1,\theta)d\theta \\
&= \int_{\theta_0}^{2\pi} \partial_r\varphi(1,\theta)d\theta + \int_0^{\theta_0} \partial_r\varphi(1,\theta+2\pi)d\theta \\
&= \int_{\theta_0}^{2\pi} \partial_r\varphi(1,\theta)d\theta + \int_0^{\theta_0} \partial_r\varphi(1,\theta)d\theta \\
&= \int_0^{2\pi} \partial_r(1,\theta)d\theta
\end{aligned}$$

By Green's theorem,

$$\xi(\theta_0 + 2\pi) - \xi(\theta_0) = \int_0^{2\pi} \partial_r(1,\theta)d\theta = -\int_D \Delta\varphi(r,\theta)rdrd\theta = 0.$$

■

5.2 Spinning Brownian motion

5.2.1 Getting to understand spin better

Consider the spinning Brownian motion in the strip $[-1, 1] \times \mathbb{R}$ with periodic boundary conditions of period 2π. This turns the strip into a compact domain and our construction from the previous section can be used to define SBM, and to prove that there exists only one stationary distribution. In this example we compute explicitly such stationary distribution.

Consider the function $g(x,y) = \alpha \mathbb{1}_{\{1\}}(y) - \beta \mathbb{1}_{\{-1\}}(y)$ for positive constants α, β, and $\tau(x,y;s) = \lambda \hat{x} \mathbb{1}_{\{1\}}(y)$, and the associated spinning BM solving the equation

$$\begin{cases} dX_t^y &= dB_t^y + \hat{n}(X_t) dL_t, \\ dX_t^x &= dB_t^x + \tau(X_t, S_t) dL_t, \\ dS_t &= \left(g(X_t) - S_t\right) dL_t. \end{cases}$$

Note that the normal depends only on the y-coordinate, and so X_t^y has the distribution of reflected Brownian motion in $[-1,1]$. In particular, L_t depends exclusively on B_t^y. Also, if we identify the points x and $x+2\pi$, the domain becomes a compact space and the existence of a unique stationary distribution follows from our theorem. It is clear from the equations that the law of (X,S) starting from $(x,y;s)$ is the same as the law of $(x+X^0,S)$, where (X^0,S) starts from $(0,y;s)$. A standard argument then shows that the stationary distribution is invariant under translations in the x-coordinate.

Looking for a candidate

It is well know that the reflecting Brownian motion process has a unique stationary distribution, that is absolutely continuous with respect to Lebesgue when the domain is smooth enough. See section 2.6 for details. Thus, the approximating process (4.2.1) has a stationary density $\rho_\varepsilon(y,s)$. Starting from the characterization of invariant measures introduced by A. Weiss [27], a straight forward calculation allows us to characterize ρ_ε as the unique positive solution of the following differential equation:

$$\Delta_{x,y}\rho_\varepsilon + \varepsilon \partial_{ss}\rho_\varepsilon = 0 \qquad \text{in } D \times \mathbb{R}^d \tag{5.2.1}$$

$$\nabla \rho_\varepsilon \cdot \hat{n}(y) = \operatorname{div}^T \left(\rho_\varepsilon \cdot (\tau(y,s), g(y) - s)\right) \qquad \text{on } \partial D \times \mathbb{R}^d. \tag{5.2.2}$$

As $\varepsilon \to 0$, we know that the approximating process converges to Spinning Brownian motion, which also entails convergence of the respective stationary probability distributions. A good guess would be that if the stationary distribution of spinning Brownian motion has a density $\rho(y;s)$, it satisfies the equation

$$\Delta_{x,y}\rho = 0 \qquad \text{in } D \times (-\beta,\alpha) \tag{5.2.3}$$

$$\nabla\rho \cdot \hat{n}(y) = \operatorname{div}^T\left(\rho \cdot (\tau(y,s), g(y) - s)\right) \qquad \text{on } \partial D \times (-\beta,\alpha). \tag{5.2.4}$$

For the given functions τ and g, the former equation has a unique positive solution up to a multiplicative constant. This solution is the integrable function given by $\rho(x,y;s) = a(s)y + b(s)$, with

$$\begin{aligned} a(s) &= \frac{2}{\alpha+\beta}\frac{s-\frac{1}{2}(\alpha-\beta)}{\sqrt{(\alpha-s)(\beta+s)}}, \\ b(s) &= \frac{1}{\sqrt{(\alpha-s)(\beta+s)}}. \end{aligned} \tag{5.2.5}$$

It is straight forward to check that $\rho(x,y;s) > 0$ in $D \times (-\beta,\alpha)$, and that it is integrable in that set. Also, it is worth noticing that the integral of $a(s)$ on the interval $(-\beta,\alpha)$ equals zero. This is intuitively expected, as then the corresponding integral of $\rho(x,y;s)$ is a constant, which agrees with the fact that the stationary distribution of the reflecting Brownian motion X_t is the Lebesgue measure, as its boundary push is constant.

To prove that the measure $\mu(dxdyds) = \mathbb{1}_{(-\beta,\alpha)}(s)\rho(x,y;s)dxdyds$ is stationary for the process, we use Theorem 2.6.2. Even though the operator associated to spinning Brownian motion is not strictly elliptic, we have shown that (4.1.1) has a unique strong solution, in particular, for each $x \in \overline{D}, s \in \mathbb{R}^p$ there is a unique measure $\mathbb{P}_{x,s}$ solving the submartingale problem for $\mathcal{L}, \vec{\kappa}$.

It remains to show that for $f \in C_c^2(\overline{D} \times \mathbb{R}^p)$ with $\nabla f \cdot \vec{\kappa}(x,s) \geq 0$ for all $x \in \partial D$ it holds that

$$\int_{D\times\mathbb{R}^p} \Delta_x f(x,y,s)\mu(dxdyds) \leq 0. \tag{5.2.6}$$

We proceed by direct computation: Since f has bounded derivatives and ρ is integrable

$$\begin{aligned}\int_{D\times\mathbb{R}^p}\Delta_x f(x,y,s)\mu(dxdyds) &= \int_{-\beta}^{\alpha}\int_{-1}^{1}\int_{0}^{2\pi}\Delta_x f(x,y,s)\left[a(s)y+b(s)\right]dxdyds\\ &= \int_0^{2\pi}\int_{-\alpha}^{\beta}\int_{-1}^{1}\partial_{yy}f(x,y,s)\left[a(s)y+b(s)\right]dydsdx\\ &= \int_0^{2\pi}\int_{-\alpha}^{\beta}\partial_y f(x,y,s)\left[a(s)y+b(s)\right]\Big|_{-1}^{1}dsdx-\int_0^{2\pi}\int_{-\alpha}^{\beta}\int_{-1}^{1}\partial_y f(x,y,s)a(s)dydsdx\\ &= \int_0^{2\pi}\int_{-\alpha}^{\beta}\partial_y f(x,y,s)\left[a(s)y+b(s)\right]\Big|_{-1}^{1}dsdx-\int_0^{2\pi}\int_{-\alpha}^{\beta}f(x,y,s)a(s)\Big|_{-1}^{1}dsdx.\end{aligned}$$

To move on, we need to write down explicitly the boundary condition $\nabla f\cdot\vec{\kappa}(x,y,s)\geq 0$ for $y\in(-1,1)$:

$$\begin{aligned}-\partial_y f+\lambda\partial_x f+(\alpha-s)\partial_s f(x,1,s)&\geq 0 \qquad\qquad y=1,\\ \partial_y f+\lambda\partial_x f+(-\beta-s)\partial_s f(x,-1,s)&\geq 0 \qquad\qquad y=-1.\end{aligned}$$

We see that

$$\partial_y f(x,y,s)\Big|_{-1}^{1}\leq\lambda\partial_x f+\left[g(y)-s\right]\partial_s f(x,y,s)\Big|_{-1}^{1}.$$

Since ρ does not depend on x, it is straightforward to obtain $\int_0^{2\pi}\lambda\partial_x f(x,y,s)\rho(y,s)dx=0$. Therefore

$$\int_{D\times\mathbb{R}^p}\Delta_x f(x,y,s)\mu(dxdyds)\leq\int_0^{2\pi}\int_{-\alpha}^{\beta}\left[g(y)-s\right]\rho(y,s)\partial_s f-a(s)f(x,y,s)\Big|_{-1}^{1}dsdx.$$

We'd like to do integration by parts in the s variable. By direct evaluation, we see that $\left[g(y)-s\right]\rho(y,s)=\frac{2y}{\alpha+\beta}\sqrt{(\alpha-s)(\beta+s)}$ at $y=-1,1$. This term vanishes both at $s=\alpha$ and $s=-\beta$. Its derivative with respect to s equals

$$\frac{y}{\alpha+\beta}\frac{-2s+\alpha-\beta}{\sqrt{(\alpha-s)(\beta+s)}}=-ya(s).$$

Doing integration by parts on the integral in the s variable, and using the previous computations, we see that the right hand above vanishes, and we obtain the desired inequality.

The following graphs show the density $\rho(y,s)$ from different perspectives. We have set $\alpha=\beta=1$ to simplify the plotting, but the general shape of the graphs is maintained.

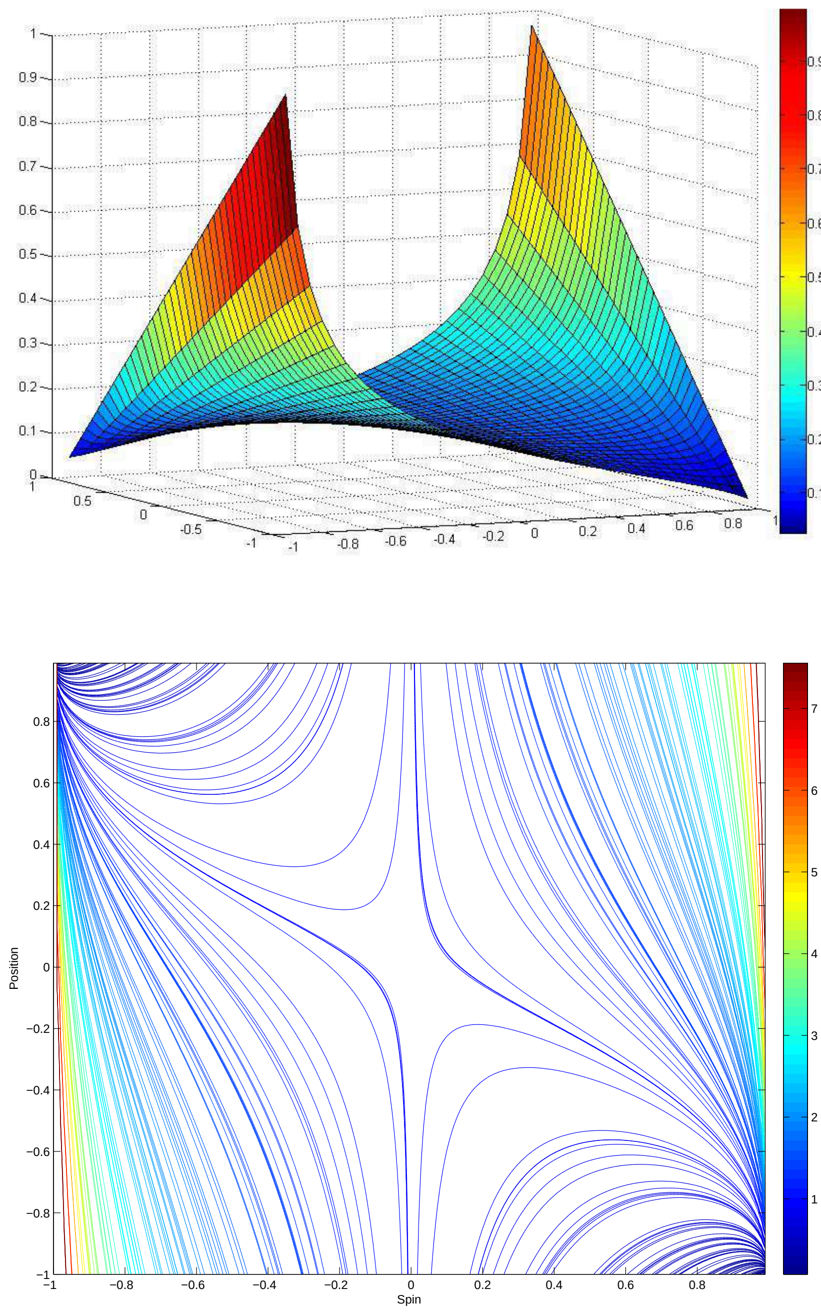

Figure 5.1: Graph and contour plot of the density $\rho(x, s)$ in (5.2.5).

It should be remarked that in the previous graph, the stationary density goes to infinity both at $(x, 1, \alpha)$ and $(x, -1, -\beta)$. Heuristically, when the process (X, S) is near $(x, 1, \alpha)$, the change in spin is little since $\vec{g}(X_t) - S_t \approx \vec{g}(x, 1) - \alpha = 0$. Thus, the spin stays around α for a "long" period of time, and thus the occupation measure has a lot of weight around $(x, 1, \alpha)$. A similar situation occurs at $(x, -1, -\beta)$. This observation does not seem to generalize trivially to higher dimensional case, but it inspires the examples we show in the next section.

5.2.2 Higher dimensional spin

The previous example involves one dimensional spin. We proceed to explore several examples of two dimensional spin, and their marginal stationary distributions to illustrate the impact of different vector fields $\vec{g}(x)$. Our setting is the following: consider the strip domain $\tilde{D} = \mathbb{R} \times [-1, 1]$. We will identify any point (x, y) in this domain with all x-translations by 2π, that is, $(x, y) = (x + 2\pi, y)$. Let D be the domain obtained from the strip $\tilde{D}$ after this identification of points.

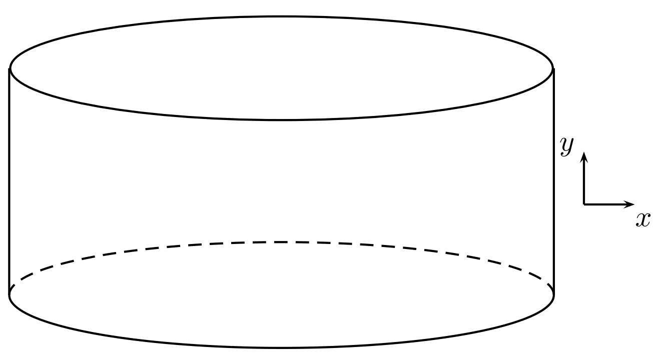

Figure 5.2: Wristband domain.

Let $\vec{\gamma}(x, y; s) = \vec{n}(y) + \tau(s)\vec{x}$, where $\vec{x}$ is the unit vector in the direction x, and $\vec{n}(y)$ is the normal at $y = \pm 1$. Despite this is not strictly speaking a domain in $\mathbb{R}^{n+p}$, our proof of existence and uniqueness of the stationary distribution apply with very little modifications.

Thus,

$$\begin{aligned} dX_t^x &= dB_t^x + \tau(S_t)dL_t, \\ dX_t^y &= dB_t^y + \vec{n}(X_t^y)dL_t, \\ dS_t &= \left[\mathbb{1}_{\{1\}}(X_t^y)\vec{g}(X_t^x) + \mathbb{1}_{\{-1\}}(X_t^y)\vec{g}(X_t^x) - S_t\right] dL_t, \end{aligned} \tag{5.2.7}$$

where B_t^x is a onc e dimensional Brownian motion modulo 2π.

Since X_t is on the boundary of D exactly when the component X_t^y takes the values 1 or -1, we have that X_t^y is a one dimensional reflected Brownian motion in the interval $[-1,1]$. This process can be obtained independently of S_t, and thus, the local time L_t can be constructed independently of the spin S_t:

$$dX_t^y = dB_t^y + dL_t^{-1} - dL_t^1, \tag{5.2.8}$$

where L_t^1 is the local time of the process at $y = 1$, and L_t^{-1} is the local time at $y = -1$.

We do not have explicit examples of the stationary distribution μ in this case. Instead, we have simulated spinning Brownian motion for different functions $\vec{g}$, and computed the average occupation time for the spin component S_t in order to estimate its marginal distribution. Precisely, Corollary (4.6.9) and Lemma 2.6.1 together say that the marginal μ_S of the stationary distribution of (X, S) is

$$\mu_S(U) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}\left(S_u \in U\right) du.$$

Even though, we have not proved an invariance principle for spinning Brownian motion, our approach to estimate μ_S is to discretize time to sample (B_t^x, B_t^y) at times discrete times $t_0 = 0$, $t_k = k\delta$, where $\delta > 0$ is fixed and small, and determine the increment $B_{t_{k+1}} - B_{t_k}$ as a two dimensional Normal distributed random variable with mean zero and variance δI_2. From equation (5.2.8) we can obtain the local times at -1 and at 1, and proceed to compute X_t^x and S_t.

Measure concentrated near a point. A key aspect of the behavior of the stationary distribution can be deduce from the differential equation $dS_t = [\vec{g}(X_t) - S_t]\, dL_t$, just as in the one dimensional case. Say that $\vec{g}(x) = \vec{g}_0$ is an extremal point of $H_{\vec{g}}$ and that the surface measure of the set $\Lambda_0 = \{x : \vec{g}(x) = \vec{g}_0\}$ is positive. Intuitively, this implies that X_t spends a lot of local time on Λ_0 and so S_t is frequently pushes towards the value $\vec{g}_0$. Further, if S_t takes a value close to $\vec{g}_0$, then the change dS_t is very small, since then $\vec{g}\,(X_t) - S_t$ is small. The process then is more likely to stay close to such points than to drift away, and it is natural to assume that neighborhoods of such points will have a large occupation measure.

Based on this heuristic, we simulated the spinning Brownian motion (5.2.7) for $\tau(s) = 1 - |s|^2$, and vector $\vec{g}$ given by $\vec{g}(x,1) = \frac{1}{2}(1,0)$ and $\vec{g}(x,-1) = \frac{1}{2}(\cos x, \sin x)$ and obtained the following graphs for the marginal stationary distribution of the spin S, where it can be seen that the point $(1/2, 1/2)$ gets most of the weight of the measure.

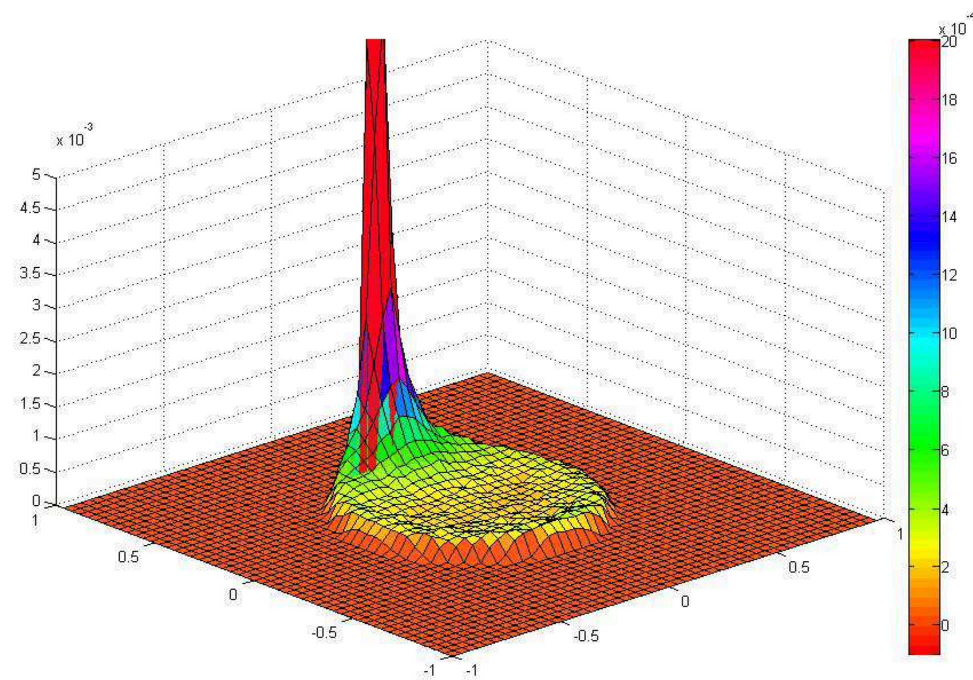

Figure 5.3: Graph of the spin marginal of the occupation measure.

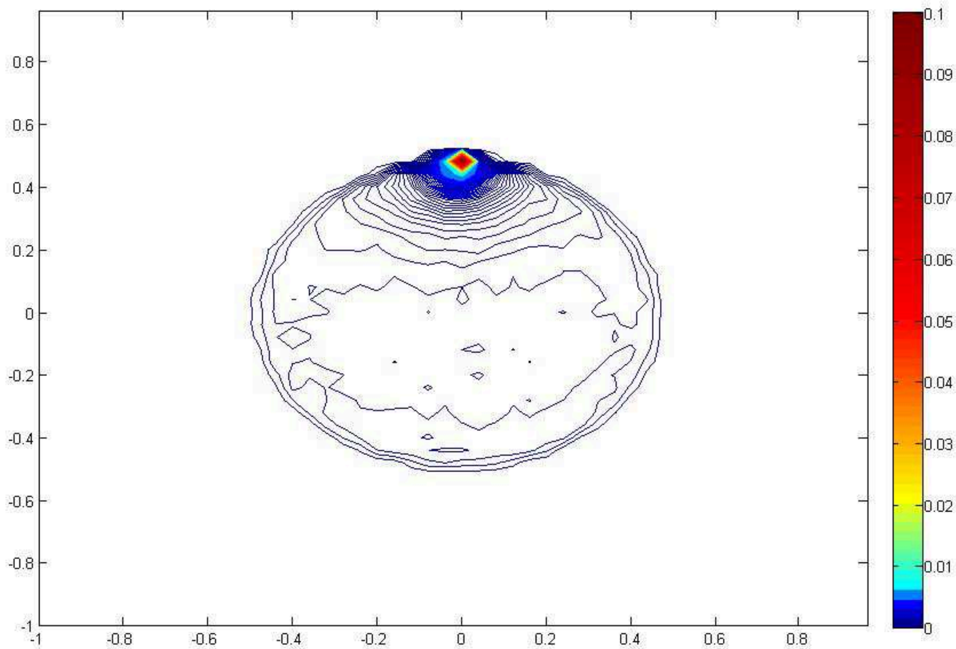

Figure 5.4: Contour plot of the spin marginal of the occupation measure.

Measure concentrated near the axes. Our next example, represent a different situation in which most of the measure seems to be accumulated in a particular set. This time, we want to force the spin to spend most of the time near the axes (diagonals) of a rhombus. We achieve this by only letting the function $\vec{g}$ be a pure vertical change at the top of the wristband, and only a horizontal change at the bottom of the wristband:

$$\vec{g}(x,y) = \begin{cases} (0, \sin x) & y = 1 \\ (\cos x, 0) & y = -1 \end{cases}.$$

Since most of the excursions of X_t from the boundary of the wristband both start and end on the same end of the wristband, the spin is rapidly pushed towards the axes. The expected graph of the spin marginal of the occupation measures, should show high concentration of density around the axes. This is exactly what was found through simulations.

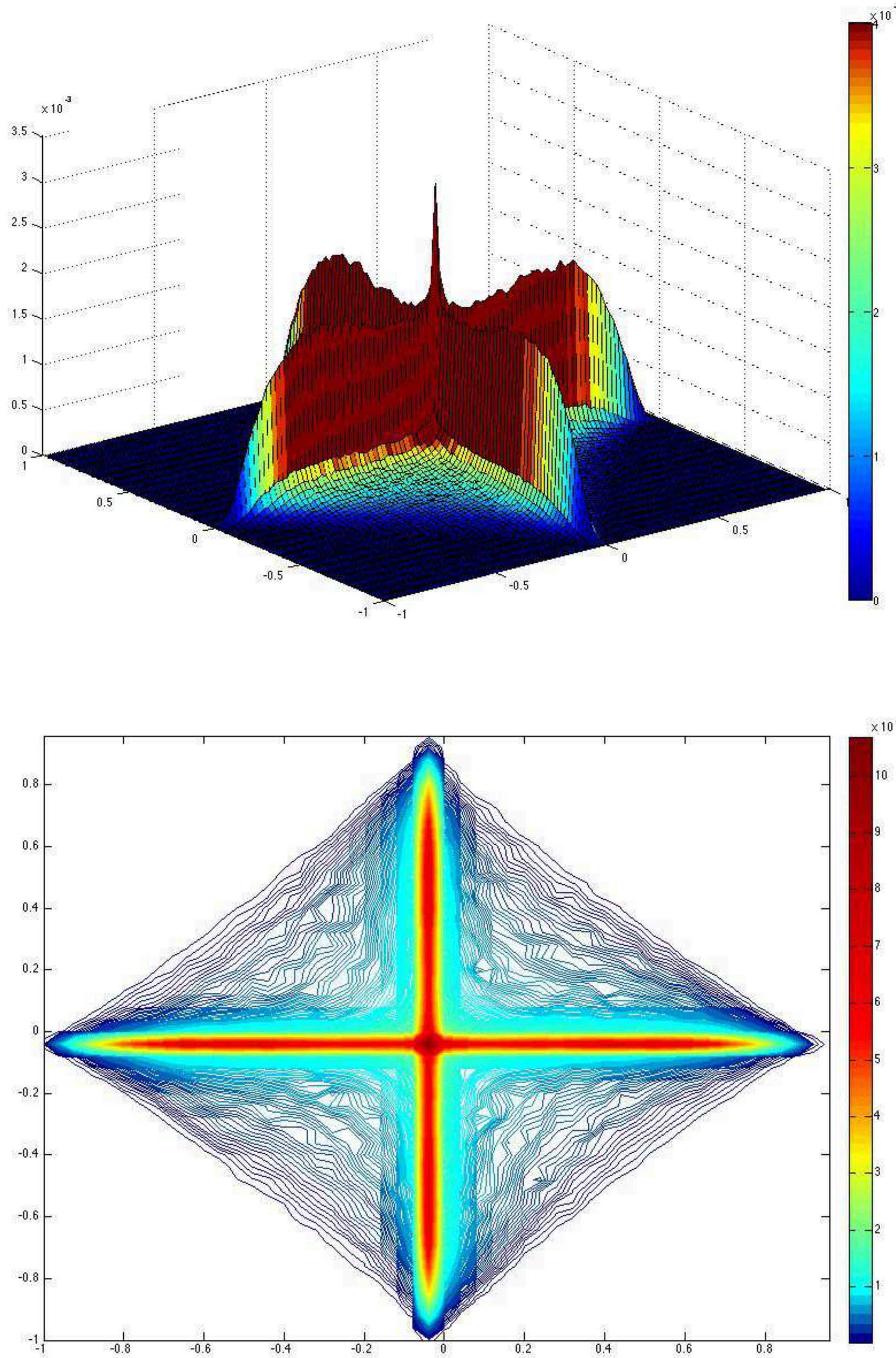

Figure 5.5: Graph and contour plot of the spin marginal of the occupation measure.

BIBLIOGRAPHY

[1] Jon Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[2] Richard F. Bass, Krzysztof Burdzy, Zhen-Qing Chen, and Martin Hairer. Stationary distributions for diffusions with inert drift. *Probab. Theory Related Fields*, 146(1-2):1–47, 2010.

[3] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[4] R. M. Blumenthal and R. K. Getoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York, 1968.

[5] K. Burdzy. *Multidimensional Brownian excursions and potential theory*, volume 164 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1987.

[6] Krzysztof Burdzy. Brownian excursions from hyperplanes and smooth surfaces. *Trans. Amer. Math. Soc.*, 295(1):35–57, 1986.

[7] Krzysztof Burdzy and Davar Khoshnevisan. Brownian motion in a Brownian crack. *Ann. Appl. Probab.*, 8(3):708–748, 1998.

[8] Santiago Carrillo-Menendez. Processus de Markov associé à une forme de Dirichlet non symétrique. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 33(2):139–154, 1975/76.

[9] C. Costantini. The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations. *Probab. Theory Related Fields*, 91(1):43–70, 1992.

[10] Zhonghai Ding. A proof of the trace theorem of Sobolev spaces on Lipschitz domains. *Proc. Amer. Math. Soc.*, 124(2):591–600, 1996.

[11] Paul Dupuis and Hitoshi Ishii. SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.*, 21(1):554–580, 1993.

[12] Paul Dupuis and Hitoshi Ishii. Correction: "SDEs with oblique reflection on nonsmooth domains" [Ann. Probab. **21** (1993), no. 1, 554–580]. *Ann. Probab.*, 36(5):1992–1997, 2008.

[13] Pedro Evilio Echeverria. *A test for invariant measures of Markov processes*. ProQuest LLC, Ann Arbor, MI, 1979. Thesis (Ph.D.)–New York University.

[14] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.

[15] Masatoshi Fukushima. On a decomposition of additive functionals in the strict sense for a symmetric Markov process. In *Dirichlet forms and stochastic processes (Beijing, 1993)*, pages 155–169. de Gruyter, Berlin, 1995.

[16] Masatoshi Fukushima, Yōichi Ōshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[17] J. M. Harrison, H. J. Landau, and L. A. Shepp. The stationary distribution of reflected Brownian motion in a planar region. *Ann. Probab.*, 13(3):744–757, 1985.

[18] J. M. Harrison and R. J. Williams. Brownian models of open queueing networks with homogeneous customer populations. *Stochastics*, 22(2):77–115, 1987.

[19] Weining Kang and Kavita Ramanan. Characterization of stationary distributions of reflected diffusions. arXiv:1204.4969v1, April 2012.

[20] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[21] Jai Heui Kim. Stochastic calculus related to nonsymmetric Dirichlet forms. *Osaka J. Math.*, 24(2):331–371, 1987.

[22] P.-L. Lions and A.-S. Sznitman. Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.*, 37(4):511–537, 1984.

[23] Zhi Ming Ma and Michael Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms*. Universitext. Springer-Verlag, Berlin, 1992.

[24] Bernard Maisonneuve. Exit systems. *Ann. Probability*, 3(3):399–411, 1975.

[25] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1994.

[26] Daniel W. Stroock and S. R. S. Varadhan. Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.*, 24:147–225, 1971.

[27] Alan Arthur Weiss. *Invariant measures of diffusion processes on domains with boundaries*. ProQuest LLC, Ann Arbor, MI, 1981. Thesis (Ph.D.)–New York University.

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