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Higher-Dimensional Analogues of Two Theorems of Orponen on Exceptional Sets

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Abstract

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Let $A \subseteq \mathbb{R}^n$ be a Borel set of Hausdorff dimension $\dim_{\mathbb{H}} A = s$ and, for each m -plane V in the Grassmannian $\mathbf{Gr}(n, m)$, let $\pi_V: \mathbb{R}^n \rightarrow V$ be the orthogonal projection of $\mathbb{R}^n$ onto V . *Marstrand's projection theorem* states that $\dim_{\mathbb{H}} \pi_V(A) = \min\{s, m\}$ for almost every $V \in \mathbf{Gr}(n, m)$, and *Marstrand's slicing theorem* states that the set of $\mathbf{x} \in V$ such that $\dim_{\mathbb{H}}(\pi_V^{-1}(\mathbf{x}) \cap A) = s - m$ has positive Lebesgue measure for almost every $V \in \mathbf{Gr}(n, m)$ provided $s > m$. Informally, for all but a few subspaces V , the shadow of A on V has the largest possible dimension, and there are many sections of A orthogonal to V of essentially maximal dimension.

In this dissertation, we prove two results concerning the sizes of *exceptional sets* of projections and slices—the null sets of parameters V for which the conclusion of Marstrand's projection or slicing theorem fails, respectively. These are, in particular, higher-dimensional analogues of two theorems of Tuomas Orponen. The first result states that, if linear maps are *almost dimension conserving* for A , then the exceptional sets of orthogonal projections of A have small packing dimension. This applies, for example, to (weakly) Furstenberg homogeneous sets and to certain self-similar and graph-directed sets. The second con-

cerns slices by fibers of the *generalized projections* of Peres and Schlag. Roughly, the conclusion of Marstrand's slicing theorem holds for very general families of nonlinear projections, and the exceptional sets of slices have small Hausdorff dimension. In fact, not only is the set of exceptional slices of A small, but the union of the exceptional sets of all positive- $\mathcal{H}^s$ -measure Borel *subsets* of A is small.

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DEDICATION

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Chapter 1

INTRODUCTION

1.1 Overview

Broadly, the purpose of this dissertation is to quantify the exceptional parameter sets arising from projections and slices in geometric measure theory. First, we show how a certain form of structural regularity introduced by Furstenberg [23] entails that a set admits packing dimension bounds on its exceptional sets of orthogonal projections; second, we establish a “universal” slicing theorem for fibers of nonlinear projections in the spirit of Peres and Schlag [40]. In a sense, these exemplify the two ways of furthering mathematical knowledge—specialization and generalization of known results—while extending theorems of Tuomas Orponen [39, 38] to higher dimensions.

The aforementioned exceptional set estimate for orthogonal projections appeals to the “Fubini heuristic” $\mathcal{H}^s \times \mathcal{H}^t \approx \mathcal{H}^{s+t}$ for Hausdorff measures. These two measures are, in fact, far from equal in general, but by imposing an “(almost) dimension conservation” hypothesis of this flavor, one can indeed show that the orthogonal projections of a set have the generic, maximal dimension even more often than entailed by the classical projection theory that grew from the work of Marstrand [33]. Our main result in this vein is Theorem 2.3, which we prove in a discretized setting using only the geometry of the Grassmannian (Lemma 2.9) and some rudimentary pigeonholing (Lemma 2.11). Proposition 2.2 is introduced as a counterpoint to illustrate what could previously be said about the packing dimensions of exceptions using a result of Rams [42].

The slicing theorem concerns the classical Hausdorff dimension of the collection of exceptional parameters, but in the metric space framework devised by Peres and Schlag for generalized projections. Theorem 3.2 is our main contribution here: given a Borel set A and a parametrized family of mappings, it constrains the set of parameters such that the fibers of the corresponding maps have unusually small intersection with A . Furthermore, using a trick of Falconer and Mattila [17], we constrain the parameters for which those fibers have small intersection with even a single positive-measure subset of A . Theorem 3.9 omits this “strong” conclusion under a slightly weaker hypothesis on A , and Theorem 3.1 illustrates Theorem 3.2 through the special case of linear slices in $\mathbb{R}^n$.

The next few sections of this chapter contextualize these main results in the narrative at whose heart lie Marstrand’s projection theorem and Marstrand’s slicing theorem (Theorems 1.1 and 1.2, respectively). General background material pertaining to measures, dimensions, energies, and other fundamental concepts is collected in §1.5. Additional specialized background appears later in §2.2 and §3.2 where it is needed.

Chapter 2 marks the beginning of the research content with the primary goal of stating and proving Theorem 2.3. After the setup is established in §2.1–2.2, a short lemma on discretized projections is proved in §2.3. In §2.4 the proof of Theorem 2.3 is completed; it is, essentially, a corollary of Lemma 2.11.

Then in Chapter 3, we transition to the study of slices with an eye toward Theorem 3.2. After the introductory §3.1–3.2, the foundational lemmas in the proof of Theorem 3.2 appear in §3.3. This includes the central Lemma 3.7 in §3.3.2, which is essentially a measure version of Theorem 3.9. The proofs of Theorems 3.2 and 3.9 are by this point mostly complete, and the final strands are tied together in §2.4.

Finally, Appendix A proves the aforementioned Proposition 2.2 from [42] for purely illustrative purposes, as it is already a special case of Theorem 2.3, and Appendix B contains the lengthy and computationally-intensive proof of the technical Lemma 3.6 required for the main results on slices.

1.2 Prelude: Product sets

Let $\dim_{\text{H}}$ denote Hausdorff dimension and $\mathcal{H}^s$ the s -dimensional Hausdorff measure in a separable metric space. An early result of geometric measure theory due to Besicovitch and Moran [6] states that, if $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are measurable with $0 < \mathcal{H}^s(A), \mathcal{H}^t(B) < \infty$ for some s and t , then

$$\dim_{\text{H}} A + \dim_{\text{H}} B \leq \dim_{\text{H}}(A \times B). \quad (1.1)$$

Their proof was highly geometric in character, but the prevailing modern approach (see, for example, [34]) is capacity-theoretic. Using net measures, Marstrand [32] eliminated the measure and measurability hypotheses, and his method is still standard for attaining this full generality (but see also [31]).

Tricot [46], in the same paper in which he defined the packing dimension $\dim_{\text{P}}$, proved several more inequalities expressing the relationship between the dimension of a product set and the dimensions of its factors:

$$\dim_{\text{H}}(A \times B) \leq \dim_{\text{H}} A + \dim_{\text{P}} B \leq \dim_{\text{P}}(A \times B) \leq \dim_{\text{P}} A + \dim_{\text{P}} B. \quad (1.2)$$

Then Howroyd [26, 27], building on the work of Larman [30] and Wegmann [47], extended both (1.1) and (1.2) to subsets of arbitrary metric spaces, once more using net measures.

One can interpret these as results about the behavior of dimension under projections, namely, the projections of product sets onto the factors of the ambient product space. Specifically, we have

$$A = \pi_{\mathbb{R}^n}(A \times B) \quad \text{and} \quad B = \pi_{\mathbb{R}^m}(A \times B)$$

in the Euclidean case, where π_V denotes the orthogonal projection of $\mathbb{R}^{n+m} \cong \mathbb{R}^n \times \mathbb{R}^m$ onto a lower-dimensional subspace V , and

$$A = \pi_X(A \times B) \quad \text{and} \quad B = \pi_Y(A \times B)$$

in the general metric space setting, $A \times B \subseteq X \times Y$. In this language, (1.1) and (1.2) tell us how the Hausdorff and packing dimensions of a product set relate to those of its images under projection, and how the regularity of the images—as measured by the difference between their Hausdorff and packing dimensions—affects the strength of those relationships. In particular, if $\dim_{\mathbf{P}} B = \dim_{\mathbf{H}} B$, then $\dim_{\mathbf{H}} A + \dim_{\mathbf{H}} B = \dim_{\mathbf{H}}(A \times B)$, and if $\dim_{\mathbf{P}} A = \dim_{\mathbf{H}} A$ as well, then all five expressions in (1.1) and (1.2) are equal.

1.3 Projections and intersections in geometric measure theory

A natural question at this point is what can be said when we replace the product set $A \times B$ with an arbitrary measurable subset of the ambient space. In view of (1.1) and (1.2), one might hope to ascertain the size or regularity of a set from knowledge of its projections, or conversely. In the absence of two distinguished projections, one is compelled to consider the projections onto *all* subspaces of $\mathbb{R}^n$ of a fixed dimension m —or at least a rich subcollection thereof—when working with orthogonal projections of Euclidean spaces. Incontrovertibly the most important development in this direction is a theorem due to Marstrand [33], Kaufman [29], and numerous others who in subsequent decades built on the foundations they laid. Denote by $\mathbf{Gr}(n, m)$ the Grassmannian of m -dimensional linear subspaces of $\mathbb{R}^n$ equipped with its usual metric structure, and for each $V \in \mathbf{Gr}(n, m)$ and $A \subseteq \mathbb{R}^n$ let $A_V := \pi_V(A)$.

Theorem 1.1 (Marstrand’s projection theorem). *Let $A \subseteq \mathbb{R}^n$ be a Borel set and let*

$$E_s := \{V \in \mathbf{Gr}(n, m) : \dim_{\mathbf{H}} A_V < s\},$$

$0 < s \leq \min \{\dim_{\mathbf{H}} A, m\}$. Then

$$\dim_{\mathbf{H}} E_s \leq m(n - m) - (\max \{\dim_{\mathbf{H}} A, m\} - s),$$

provided the right-hand side is nonnegative. If $\dim_{\mathbf{H}} A > m$, then similarly

$$\dim_{\mathbf{H}} \{V \in \mathbf{Gr}(n, m) : \mathcal{H}^m(A_V) = 0\} \leq m(n - m) - (\dim_{\mathbf{H}} A - m).$$

This dimension bound on E_s is only sharp in the endpoint case $s = \dim_{\text{H}} A$. The sharp bound for $n = 2$, $m = 1$, and general $0 < s < \dim_{\text{H}} A$ was only proved quite recently by Ren and Wang [44] as a corollary to their resolution of the Furstenberg set conjecture. Cholak et al. [9] subsequently bootstrapped this sharp estimate to projections onto lines and hyperplanes in all dimensions. However, Theorem 1.1 remains the strongest known statement for general $1 < m < n - 1$, apart from a few special cases in which $\dim_{\text{H}} A$ and s take particular values.

Another perspective from which to compare the dimensions of A , B , and $A \times B$ is to view the former as sections or “slices” of the latter,

$$A \cong \pi_{\mathbb{R}^m}^{-1}(\mathbf{y}) \cap (A \times B) \quad \text{and} \quad B \cong \pi_{\mathbb{R}^n}^{-1}(\mathbf{x}) \cap (A \times B),$$

where $\mathbf{y} \in B$ and $\mathbf{x} \in A$. This alternate viewpoint turns out to be especially fruitful in that, although axis-parallel projections and slices of product sets are equivalent, for arbitrary sets and arbitrary directions the slices frequently carry more information than do the corresponding projections. In particular, this is the case for sets with dimension larger than that of the codomain of the projection maps, and another theorem of Marstrand complementary to Theorem 1.1 identifies the typical structure of slices by planes in a quantitative way.

Theorem 1.2 (Marstrand’s slicing theorem). *Let $A \subseteq \mathbb{R}^n$ be a Borel set with $0 < \mathcal{H}^s(A) < \infty$ and let*

$$E_s := \{V \in \mathbf{Gr}(n, m) : \mathcal{H}^m(\{\mathbf{x} \in V : \dim_{\text{H}}(\pi_V^{-1}(\mathbf{x}) \cap A) = s - m\}) = 0\},$$

$s > m$. Then

$$\dim_{\text{H}} E_s \leq m(n - m) + m - s.$$

Marstrand’s projection theorem already asserts that most projections of a set of dimension $s > m$ have positive m -dimensional Hausdorff measure, and his slicing theorem augments this with the observation that many of those fibers are not merely nonempty, but actually substantial in Hausdorff dimension.

Both of these theorems concern orthogonal projections in $\mathbb{R}^n$ parametrized by the Grassmannian $\mathbf{Gr}(n, m)$. In the case of general metric spaces, the framework constructed by Peres and Schlag [40] in their seminal paper on generalized projections has become standard, and in this setting special hypotheses on the families of projections guarantee that they exhibit many of the essential analytical features of orthogonal projections. We shall touch on this again shortly in the next section, but a fuller exposition is postponed to Chapter 3.

1.4 Two theorems of Orponen

As laid out in §1.1, the foci of this dissertation are the higher-dimensional analogues of a pair of theorems of Orponen (see [37, 38, 39]) on exceptional parameter estimates in the theme of Theorems 1.1 and 1.2. Before proceeding to the technicalities in the next section, we present the context here and state Orponen’s two results. As needed, the reader should refer to §1.5 for general background, and to §2.2 and §3.2 for the more specialized background material pertinent to §1.4.1 and §1.4.2 below, respectively.

1.4.1 Orthogonal projections of homogeneous sets

Neither Kaufman’s [29] potential-theoretic method for proving Marstrand’s projection theorem nor Falconer’s [13] subsequent Fourier-analytic improvement of this method admits a straightforward adaptation to statements involving other notions of dimension. Could this a deficiency of the tools, or perhaps of the way they are used? There are many answers to this question according to what one means by “adaptation,” but the issue with replacing $\dim_{\mathbb{H}} E_s$ with $\dim_{\mathbb{P}} E_s$ in Theorem 1.1 is that the resulting statement is simply false—and in the strongest possible way. For each $e \in \mathbb{S}^1$, let $\pi_e : \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the orthogonal projection $\pi_e(x) := e \cdot x$.

Theorem 1.3 (Orponen [39] Theorem 1.1). *There exists a compact set $K \subseteq \mathbb{R}^2$ with $\mathcal{H}^1(K) > 0$ such that*

$$\dim_{\mathbb{P}} \{e \in \mathbb{S}^1 : \dim_{\mathbb{H}} K_e = 0\} = 1.$$

The set K that Orponen constructs is fairly tame by comparison to many other pathological fractals that geometers have constructed in various contexts. A natural line of investigation is the nature of the geometric obstructions to the strengthening of Theorem 1.1 that this counterexample rules out. It turns out that, if at many points the set K looks highly uniform along some sequence of scales, then its exceptional set of projections is also fairly small at all scales. This is the content of the next theorem.

Theorem 1.4 (Orponen [39] Theorem 1.8). *Let $K \subset \mathbb{R}^2$ be self-similar or homogeneous. Then*

$$\dim_{\mathbb{P}} \{e \in \mathbb{S}^1 : \dim_{\mathbb{H}} K_e \leq s\} \leq s$$

for all $0 \leq s < \dim_{\mathbb{H}} K$.

Homogeneity, or *Furstenberg homogeneity* (see [23] and Definition 2.2), implies that slices and projections are truly dual in a quantitative way. For such homogeneous sets, counting the points in a (discretized) projection is equivalent to counting the points in the slices in the orthogonal direction. Because the mass of a homogeneous set is distributed fairly evenly across scales, lower bounds on the number of points in a slice follow from the pigeonhole principle.

Theorem 2.3 extends Orponen's projection theorem to projections onto m -planes in $\mathbb{R}^n$. While this and the accompanying Corollary 2.4 are stated in somewhat greater generality, the proof of Theorem 1.4 in [39] allows for these same technical improvements in the planar case.

1.4.2 Slices of metric spaces by fibers of generalized projections

A celebrated paper [40] of Peres and Schlag constructed a theory of *generalized projections* with far-reaching consequences for Bernoulli convolutions, distance sets, and dynamically-defined fractals. Let (Ω, d) be a compact metric space and let $(\Pi_\lambda : \Omega \rightarrow \mathbb{R})_{\lambda \in J}$ be a parametrized family of maps with uniformly bounded derivatives $\frac{d^k}{d\lambda^k} \Pi_\lambda(\omega)$, $\lambda \in J \subseteq \mathbb{R}$. Roughly, for $\beta \in [0, 1)$, Peres and Schlag call the family *transversal of order β* if $|\frac{d}{d\lambda} \Pi_\lambda(\omega) - \frac{d}{d\lambda} \Pi_\lambda(\zeta)| \gtrsim d(\omega, \zeta)^\beta$ whenever $\Pi_\lambda(\omega)$ and $\Pi_\lambda(\zeta)$ are close, and they call it *regular of order β* if $|\frac{d^k}{d\lambda^k} \Pi_\lambda(\omega) - \frac{d^k}{d\lambda^k} \Pi_\lambda(\zeta)| \lesssim d(\omega, \zeta)^{-\beta k}$ whenever $\Pi_\lambda(\omega)$ and $\Pi_\lambda(\zeta)$ are close. See Definition 3.1.

In this special 1-dimensional case, a somewhat simplified statement of their main projection theorem reads as follows.

Theorem 1.5 (Peres–Schlag [40] Theorem 2.8). *There exists a constant $a > 0$ such that the following holds. Let $J \subseteq \mathbb{R}$ be an open interval and let $(\Pi_\lambda : \Omega \rightarrow \mathbb{R})_{\lambda \in J}$ be a family of generalized projections that is both transversal and regular of order $\beta \in [0, 1)$. For every Borel set $A \subseteq \Omega$ of Hausdorff dimension $s > 0$,*

$$\dim_{\text{H}} \left\{ \lambda \in J : \dim_{\text{H}} \Pi_\lambda(A) < t \right\} \leq \begin{cases} t & \text{if } 0 < t \leq s - 3\beta \\ 1 + t - \frac{s}{1+a\beta} & \text{if } \max \left\{ \frac{s}{1+a\beta} - 1, 0 \right\} < t \leq s. \end{cases}$$

Peres and Schlag’s methods are very general: many of the key lemmas leading up to their proof of Theorem 1.5 concern energies and exceptional sets without specific reference to projections. This provides new machinery for proving theorems amenable to these tools, and in fact their entire strategy serves as a road map toward slicing results of a very similar nature.

Say that $(\Pi_\lambda : \Omega \rightarrow \mathbb{R})_{\lambda \in J}$ is *strongly regular of order β* if $|\frac{d^k}{d\lambda^k} \Pi_\lambda(\omega) - \frac{d^k}{d\lambda^k} \Pi_\lambda(\zeta)| \lesssim d(\omega, \zeta)^{-\beta k}$ holds for all $\omega, \zeta \in \Omega$.

Theorem 1.6 (Orponen [38] Theorem 2.2). *There exists a constant $a \geq 1$ such that the following holds. Let $J \subseteq \mathbb{R}$ be an open interval and let $(\Pi_\lambda: \Omega \rightarrow \mathbb{R})_{\lambda \in J}$ be a family of generalized projections that is both transversal and strongly regular of order $\beta \in [0, 1)$. For every $1 < s < 2$ and every Borel set $A \subseteq \Omega$ with $\mathcal{H}^s(A) > 0$, there exists a set $E \subseteq J$ with $\dim_{\mathbb{H}} E \leq 2 - s + a\beta^{1/3}$ such that, for every $\lambda \in J \setminus E$, the inequality*

$$\dim_{\mathbb{H}}(A \cap \Pi_\lambda^{-1}(x)) \geq s - 1 - a\beta^{1/3}$$

holds for a positive-Lebesgue-measure set of $x \in \mathbb{R}$. If $\beta > 0$, the same conclusion holds assuming only $\dim_{\mathbb{H}} A \geq s$.

The $\sim \beta^{1/3}$ error term, left unspecified in the theorem statement in [38], appears explicitly in Orponen's proof.

Falconer and Mattila [17] proved a “universal” or “strong” generalized projection theorem that takes Peres and Schlag's proof one step further. Here is the 1-dimensional statement corresponding to Theorem 1.5:

Theorem 1.7 (Falconer–Mattila [17] Theorem 2.3). *There exists a constant $a > 0$ such that the following holds. Let $J \subseteq \mathbb{R}$ be an open interval, let $(\Pi_\lambda: \Omega \rightarrow \mathbb{R})_{\lambda \in J}$ be a family of generalized projections that is both transversal and regular of order $\beta \in [0, 1)$, and let $A \subseteq \Omega$ be a Borel set of finite s -dimensional Hausdorff measure. If $0 < t \leq s \leq m$, then there exists a set $E_t \subset J$ with*

$$\dim_{\mathbb{H}} E_t \leq \begin{cases} t & \text{if } 0 < t \leq s - 3\beta \\ 1 + t - \frac{s}{1+a\beta} & \text{if } \max\left\{\frac{s}{1+a\beta} - 1, 0\right\} < t \leq s. \end{cases}$$

such that, for every Borel set $B \subseteq A$ with $\mathcal{H}^s(B) > 0$,

$$\dim_{\mathbb{H}} \Pi_\lambda(B) \geq t \quad \text{for all } \lambda \in J \setminus E_t.$$

If instead $t = s > m$, then there exists a set $E \subset J$ with

$$\dim_{\mathbb{H}} E \leq 2 - \frac{s}{1+a\beta}$$

such that, under the same hypotheses on B ,

$$\mathcal{L}^m(\Pi_\lambda(B)) > 0 \quad \text{for all } \lambda \in J \setminus E_s.$$

Given the similarities between the proofs of Theorems 1.5 and 1.6, it is natural to expect that the latter also admits a strong variant in the spirit of Theorem 1.7, and this is indeed the case. This version of Theorem 1.6, once more extended to higher dimensions, appears as Theorem 3.2.

1.5 Preliminaries from geometric measure theory and Fourier analysis

This section surveys the mathematical tools—all of which can be found in the textbooks [34, 35]—that underpin the subsequent chapters. It is organized thematically, for which reason some of the material is more pertinent to one of the following two chapters than to the other. The more specialized and logically-independent background is relegated to the chapter in which it finds use.

Before we begin, we recall a common convention that shall be used in the sequel. Frequently quantities arise that are (un)equal only up to an unimportant multiplicative constant. For this reason, given parameters $w_1, \dots, w_k$ and quantities $a, b \in \mathbb{R} \cup \{\pm\infty\}$, we write $a \lesssim_{w_1, \dots, w_k} b$ if there exists a positive constant $c_{w_1, \dots, w_k}$ (depending only on those parameters) such that $a \leq c_{w_1, \dots, w_k} b$. Similarly, $a \gtrsim_{w_1, \dots, w_k} b$ means that $b \lesssim_{w_1, \dots, w_k} a$ and $a \sim_{w_1, \dots, w_k} b$ means that both $a \lesssim_{w_1, \dots, w_k} b$ and $a \gtrsim_{w_1, \dots, w_k} b$. For readability, some parameters (e.g., the dimension of the space in which we are working) may be suppressed from the notation when doing so poses no risk of error or confusion. Although this is less standard, we will also on occasion write $a \simeq_{w_1, \dots, w_k} b$ when $a = c_{w_1, \dots, w_k} b$. This is stronger than $a \sim_{w_1, \dots, w_k} b$, which only says that $c_{w_1, \dots, w_k} a \leq b \leq d_{w_1, \dots, w_k} a$.

1.5.1 Measures and dimensions

Let (Ω, d) be a separable metric space. In Chapter 2, Ω will always be Euclidean space or a Grassmannian, but the more general theory poses no additional challenges and will be essential in Chapter 3.

By a *measure* on Ω we mean what in other branches of measure theory is typically called an *outer measure*. Of exclusive concern to us are the *Radon measures*, i.e., the locally finite Borel measures that are inner regular on open sets and outer regular on all sets. We denote by $\mathcal{M}(A)$ the positive cone of all compactly-supported measures on a set $A \subseteq \Omega$, excluding the zero measure.

Denote by $|A|$ the diameter of a set $A \subseteq \Omega$. For each $s \geq 0$, the Carathéodory construction yields a family of functions $\mathcal{H}_\delta^s: 2^\Omega \rightarrow [0, \infty]$, $\delta \in (0, \infty]$, defined by

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{i=1}^{\infty} |A_i|^s : A_i \subseteq \Omega, |A_i| \leq \delta, A \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

The function $\mathcal{H}_\infty^s$, called the *s-dimensional Hausdorff content* on Ω , will be of particular use in Chapter 2. The resulting Carathéodory measure

$$\mathcal{H}^s(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A)$$

is Borel regular and is called the *s-dimensional Hausdorff measure* on Ω . For each $A \subseteq \Omega$, there is a unique $s \in [0, \infty]$ with the following property: for all $r < s < t$, we have

$$0 = \mathcal{H}^t(A) \leq \mathcal{H}^s(A) \leq \mathcal{H}^r(A) = \infty,$$

or, equivalently,

$$s = \sup \{t \in [0, \infty) : \mathcal{H}^t(A) > 0\} = \inf \{r \in [0, \infty) : \mathcal{H}^r(A) = 0\}.$$

We write $\dim_{\mathcal{H}} A := s$ and call this number the *Hausdorff dimension* of A . Although this is defined in terms of Hausdorff measure, which “sees” sets only at arbitrarily small scales, it is straightforward to show that Hausdorff content also determines Hausdorff dimension.

Proposition 1.8. *Let $A \subseteq \mathbb{R}^n$. Then $\mathcal{H}^s(A) > 0$ if and only if $\mathcal{H}_\infty^s(A) > 0$.*

It is also possible to recover partial information about Hausdorff dimension using covers not by all sets of diameter *at most* δ , but rather by open balls with radius *equal to* δ . For each nonempty bounded set $A \subseteq \Omega$ and each $\delta > 0$, let

$$N(A, \delta) := \min \left\{ k \in \mathbb{Z}_+ : \exists \omega_i \in \Omega \text{ s.t. } A \subseteq \bigcup_{i=1}^k B(\omega_i, \delta) \right\}. \quad (1.3)$$

We define the *upper box dimension* of A , also called its *upper Minkowski dimension*, by

$$\begin{aligned}\overline{\dim}_B A &:= \sup \left\{ t \in [0, \infty) : \limsup_{\delta \downarrow 0} N(A, \delta) \delta^{-t} > 0 \right\} \\ &= \inf \left\{ r \in [0, \infty) : \limsup_{\delta \downarrow 0} N(A, \delta) \delta^{-r} = 0 \right\}.\end{aligned}\tag{1.4}$$

Comparing the admissible covers of A in the definitions of Hausdorff and upper box dimensions yields the inequality

$$\dim_H A \leq \overline{\dim}_B A.$$

In Chapter 2 will often deal with sets A for which equality holds.

We now turn to packing dimension, a sort of “dual” to Hausdorff dimension. It is perhaps most natural to define this in terms of *packing measure*, but, to streamline the exposition, we present an alternative characterization—the only one we will use. The *packing dimension* of a nonempty subset $A \subseteq X$ is given by

$$\dim_P A := \inf \left\{ \sup_{1 \leq i < \infty} \overline{\dim}_B A_i : A = \bigcup_{i=1}^{\infty} A_i, |A_i| < \infty \right\}.$$

An important feature of packing dimension that upper box dimension lacks is *countable stability*:

$$\dim_P \bigcup_{i=1}^{\infty} A_i = \sup_{1 \leq i < \infty} \dim_P A_i.$$

Indeed, the definition of packing dimension above is simply a way of “forcing” upper box dimension to become countably stable.

1.5.2 Energies and capacities of measures

A fundamental principle of geometric measure theory and fractal geometry is that the dimension of a set is characterized by the degree of “concentration” of the Radon measures that it supports. Recall that $\mathcal{M}(\Omega)$ denotes the space of all nonzero Radon measures compactly supported in Ω . A measure $\mu \in \mathcal{M}(\Omega)$ is called *s-Frostman* if

$$\mu(B(\omega, r)) \lesssim_{\mu} r^s \quad \text{for all } \omega \in \Omega \text{ and } r > 0.$$

The following result, whose importance is difficult to understate, will be used frequently without explicit mention. This version for general metric spaces is due to Howroyd [26].

Theorem 1.9 (Frostman’s lemma). *If Ω is a σ -compact metric space with $\mathcal{H}^s(\Omega) > 0$, then Ω supports an s -Frostman measure.*

A related tool for describing the scaling of a measure $\mu \in \mathcal{M}(\Omega)$ is the s -energy

$$I_s(\mu) := \iint_{\Omega \times \Omega} \frac{d(\mu \times \mu)(\omega, \zeta)}{d(\omega, \zeta)^s}. \quad (1.5)$$

If μ is finite and s -Frostman, then it has finite t -energy for all $0 \leq t < s$. Conversely, if μ is finite and $I_s(\mu) < \infty$, then there exists a positive-measure subset $B \subseteq \Omega$ such that $\mu \llcorner B$, the restriction of μ to B , is s -Frostman.

A useful characterization and extension of the s -energy of a Borel measure μ on $\mathbb{R}^n$ is the identity

$$I_s(\mu) \simeq_{n,s} \int_{\mathbb{R}^n} |\mathbf{z}|^{s-n} |\hat{\mu}(\mathbf{z})|^2 d\mathbf{z}, \quad 0 < s \leq k, \quad (1.6)$$

where $\hat{\mu}: \mathbb{R}^n \rightarrow \mathbb{C}$ is the Fourier transform of the measure μ . In fact, the right-hand side of the equation can in fact converge for *any* value of $s \in \mathbb{R}$ (simply choose μ to be a Schwartz function), whereas the integral in (1.5) always diverges when $s > n$. The right-hand side of (1.6) is therefore denoted $\mathcal{I}_s(\mu)$ and given its own name, the s -Sobolev energy of μ . The *Sobolev dimension* of μ is then defined as the supremum over all s such that μ has finite s -Sobolev energy:

$$\dim_S \mu := \sup \{s \in \mathbb{R} : \mathcal{I}_s(\mu) < \infty\},$$

where $\sup \emptyset := 0$. Clearly $\mathcal{I}_s(\mu) < \infty$ for all $s < 0$, so $\dim_S \mu \geq 0$.

1.5.3 The Grassmannian

Let $\mathbf{Gr}(n, m)$ be the *Grassmannian*—the manifold of dimension $m(n-m)$ whose elements are the m -dimensional linear subspaces of $\mathbb{R}^n$. The Grassmannian carries the structure

of a metric space with metric $d(V, W) := \|\pi_V - \pi_W\|$, where $\|\cdot\|$ is the operator norm induced by the Euclidean norm on $\mathbb{R}^n$. For computational purposes, it is often convenient to identify V and W with $\mathbb{R}^m$; this is possible using coordinate charts on $\mathbf{Gr}(n, m)$ —something we do implicitly without further mention. The Grassmannian also enjoys an invariant measure induced by the action of the orthogonal group $O(n)$: if θ_n is the Haar measure on $O(n)$ and $V \in \mathbf{Gr}(n, m)$, then

$$\gamma_{n,m}(A) := \theta_n(\{T \in O(n) : T(V) \in A\}),$$

where the definition does not depend on the specific choice of V . This Radon measure is *uniformly distributed* in the sense that $\gamma_{n,m}(B(V, r)) = \gamma_{n,m}(B(W, r))$ for all $V, W \in \mathbf{Gr}(n, m)$ and all $r > 0$, and it is *$m(n-m)$ -Ahlfors regular* in the sense that $\gamma_{n,m}(B(V, r)) \sim r^{m(n-m)}$ for all $r \in (0, 1]$.

A key estimate used in proving projection theorems is the following.

Lemma 1.10. *If $\nu \in \mathcal{M}(\mathbf{Gr}(n, m))$ is s -Frostman, then*

$$\nu(\{V \in \mathbf{Gr}(n, m) : |\pi_V(\mathbf{x})| \leq \delta\}) \lesssim_n \left(\frac{\delta}{|\mathbf{x}|} \right)^{s-m(n-m-1)}$$

and

$$\nu(\{V \in \mathbf{Gr}(n, m) : \text{dist}(x, V) \leq \delta\}) \lesssim_n \left(\frac{\delta}{|\mathbf{x}|} \right)^{s-(m-1)(n-m)}$$

for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and all $0 < \delta < \infty$.

Its proof is an elementary geometric argument whose crux is an estimate the number of balls in a $\frac{\delta}{|\mathbf{x}|}$ -cover of the set $\{V \in \mathbf{Gr}(n, m) : \mathbf{x} \in V\}$.

1.6 Other directions in the study of projections and intersections

Estimating the size of exceptional sets is a lucrative topic in geometric measure theory, fractal geometry, and potential theory, but it is just one of many aspects of the study of incidences from the measure-theoretic vantage point. To follow up on the relationship

between projections and regularity touched on in §1.2 and to offer a glimpse of the surrounding landscape, we end this chapter with a brief discussion of some topics independent of the rest of this dissertation.

Perhaps the most important notion of regularity in geometric measure theory is *rectifiability*. Let $1 \leq m \leq n - 1$. A set $A \subset \mathbb{R}^n$ is called *countably m -rectifiable* if, up to an $\mathcal{H}^m$ -null set, it can be covered by a countable union of Lipschitz images of $\mathbb{R}^m$; it is called *purely m -unrectifiable* if every Lipschitz image of $\mathbb{R}^m$ intersects A in an $\mathcal{H}^m$ -null set.

Historically, the first major theorem concerning the projections of planar sets came in [5]—the final result in a groundbreaking trio of papers—which Besicovitch asserted to have stature comparable to the characterization of rectifiable and purely unrectifiable sets in terms of the a.e.-existence of weak tangents. Recall that $\pi_e: \mathbb{R}^2 \rightarrow \mathbb{R}$ denotes the orthogonal projection of $\mathbb{R}^2$ onto the unit vector $e \in \mathbb{S}^1$ and, when $A \subseteq \mathbb{R}^2$, $A_e := \pi_e(A)$.

Proposition 1.11 (Besicovitch [5] Theorem 4). *Let $A \subset \mathbb{R}^2$ be an $\mathcal{H}^1$ -measurable set with $0 < \mathcal{H}^1(A) < \infty$.*

- a.** *The set A is purely unrectifiable if and only if $\mathcal{L}^1(A_e) = 0$ for $\mathcal{H}^1$ -a.e. $e \in \mathbb{S}^1$.*
- b.** *The set A has a countably rectifiable subset of positive $\mathcal{H}^1$ -measure if and only if $\mathcal{L}^1(A_e) > 0$ for $\mathcal{H}^1$ -a.e. $e \in \mathbb{S}^1$.*

By an elementary geometric argument, a countably rectifiable 1-set projects onto a set of measure 0 in at most one direction, and this simple observation sharpens Proposition 1.11 in an extreme way. See, e.g., [14].

Corollary 1.12. *Let $A \subset \mathbb{R}^2$ be an $\mathcal{H}^1$ -measurable set with $0 < \mathcal{H}^1(A) < \infty$.*

- a.** *The set A is purely unrectifiable if and only if $\mathcal{L}^1(A_e) = 0$ for at least two distinct $e \in \mathbb{S}^1$.*
- b.** *If A has a countably rectifiable subset of positive $\mathcal{H}^1$ -measure, then $\mathcal{L}^1(A_e) = 0$ for at most one $e \in \mathbb{S}^1$.*

Federer [20] extended Proposition 1.11 to higher dimensions as follows.

Theorem 1.13 (Besicovitch–Federer projection theorem). *Let $A \subset \mathbb{R}^n$ be an $\mathcal{H}^m$ -measurable set with $0 < \mathcal{H}^m(A) < \infty$.*

- a.** *The set A is purely m -unrectifiable if and only if $\mathcal{L}^m(A_V) = 0$ for $\gamma_{n,m}$ -a.e. $V \in \mathbf{Gr}(n, m)$.*
- b.** *The set A has a countably rectifiable subset of positive $\mathcal{H}^m$ -measure if and only if $\mathcal{L}^m(A_V) > 0$ for $\gamma_{n,m}$ -a.e. $V \in \mathbf{Gr}(n, m)$.*

In fact, Federer proved something more general and abstract—see Theorem 3.2.27 of [21]—that pairs with Theorem 1.13 to give a straightforward analogue of Corollary 1.12.

Heuristically, the slices of a set should say at least as much about a set as do its projections: one can rebuild a set from its sections, but not from its shadows. Theorem 1.13 says that, as far as the Lipschitz regularity of an integer-dimensional set is concerned, projections say just as much as one could hope to learn from its sections by Lipschitz surfaces. Favard length (or even the “generalized Favard length” of Bongers and Taylor [7]) offers a more quantitative means of studying purely unrectifiable sets using only projections, whereas intersections by surfaces with higher regularity are a natural way to investigate the finer differential structure of rectifiable sets. Davey, Ghinassi, and Wilson [11] recently introduced the study of the “degree of irregularity” of an unrectifiable set through the *dimension* of its intersections with Lipschitz surfaces. They showed, for example, that Ahlfors–David regular sets contain pieces that are “close” to regular in this sense.

Theorem 1.14 (Davey–Ghinassi–Wilson [11] Proposition 2.8). *If $A \subset \mathbb{R}^2$ is an Ahlfors–David regular set with $0 < \mathcal{H}^1(A) < \infty$, then there exists a Lipschitz curve $\gamma \subset \mathbb{R}^2$ such that $\dim_{\mathbb{H}}(A \cap \gamma) = 1$.*

By definition, $\mathcal{H}^1(A \cap \gamma) = 0$ if A is purely 1-unrectifiable, but this result guarantees that A cannot somehow be “even more unrectifiable” if it is Ahlfors–David regular.

Chapter 2

EXCEPTIONAL PROJECTIONS OF SETS EXHIBITING
ALMOST DIMENSION CONSERVATION

This chapter concerns a class of highly structured sets that, owing to their uniform behavior across scales, admit much stronger bounds on their exceptional sets of projections than do typical Borel sets. Most of the text is adapted from [8], to which the reader is also referred.¹

2.1 Main result*2.1.1 Motivation: A theorem of Rams*

A dynamical result due to Rams [42] implies a result similar to Marstrand's projection theorem that bounds the *packing* dimension of the exceptional set, but only when the set under consideration is highly regular and $m = n - 1$. His theorem concerns parametrized families of conformal iterated function systems (IFS), but we state it here more narrowly for parametrized families of similarities. The terminology involved is heavy, so we refer the reader to §1 and Definition 4.4 of [42] and Appendix A of this dissertation for additional background. Henceforth, *IFS* without qualification refers to a *self-similar* IFS (see §2.2.2 and Appendix A).

Theorem 2.1 (Rams [42] Theorem 1.1). *Let $U \subset \mathbb{R}^m$ be a bounded open set, and for each $\lambda \in \overline{U}$, let $(f_i(\cdot; \lambda))_{i=1}^N$ be a family of IFS on $\mathbb{R}^m$ with limit set K_λ . Assume that each f_i is smooth in all m variables and m parameters, and denote by $\sigma(\lambda)$ the solution*

¹The author thanks the *Annales Fennici Mathematici* for their permission to reproduce this material.

to Hutchinson's equation

$$\sum_{i=1}^N a_i(\boldsymbol{\lambda})^{\sigma(\boldsymbol{\lambda})} = 1,$$

where $a_i(\boldsymbol{\lambda}) \in (0, 1)$ is the similarity ratio of $f_i(\cdot; \boldsymbol{\lambda})$. If $(f_i(\cdot; \boldsymbol{\lambda}))_{i=1}^N$ satisfies the transversality condition (A.1), then

$$\dim_P \{ \boldsymbol{\lambda} \in \overline{U} : \dim_H K_{\boldsymbol{\lambda}} \leq s \} \leq s \quad \text{for all } 0 \leq s < \min \left\{ m, \sup_{\boldsymbol{\lambda} \in \overline{U}} \sigma(\boldsymbol{\lambda}) \right\}.$$

Let $\rho_{\mathbf{e}} : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the orthogonal projection onto the hyperplane with unit normal vector $\mathbf{e} \in \mathbb{S}^{n-1}$. Suppose $K \subset \mathbb{R}^n$ is the limit set of an IFS $(g_i)_{i=1}^N$ that satisfies the strong separation condition (SSC), and for each $\mathbf{e} \in \mathbb{S}^{n-1}$, let $f_i(\cdot; \mathbf{e}) := \rho_{\mathbf{e}} \circ g_i \circ \rho_{\mathbf{e}}^{-1} : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$. Then each collection $(f_i(\cdot; \mathbf{e}))_{i=1}^N$ is an IFS in $\mathbb{R}^{n-1}$ with limit set $K_{\mathbf{e}} := \rho_{\mathbf{e}}(K)$. Therefore, $(f_i(\cdot; \mathbf{e}))_{i=1}^N$ is a transverse, smoothly parametrized $(n-1)$ -parameter family of IFS on $\mathbb{R}^{n-1}$, so the problem of determining the exceptional set of projections for K is equivalent to determining the exceptional set of the IFS $(f_i(\cdot; \mathbf{e}))_{i=1}^N$.

This setup allows for an application of Rams' theorem to obtain the following.

Proposition 2.2. *Let $K \subset \mathbb{R}^n$ be the limit set of a self-similar IFS containing no rotations or reflections and satisfying the SSC. Then*

$$\dim_P \{ \mathbf{e} \in \mathbb{S}^{n-1} : \dim_H K_{\mathbf{e}} \leq s \} \leq s \quad \text{for all } 0 \leq s < \dim_H K.$$

Our main result subsumes this as a special case; nevertheless, we include its proof in Appendix A to shed light on the relationship between his work and our own, and to give a sense of just how strong the transversality condition in Theorem 2.1 is.

2.1.2 Orponen's packing dimension bound on exceptional sets

For planar sets, Orponen's result in [39] allows us to forgo any separation conditions in the case that K is self-similar, or to instead assume that K is homogeneous (see Definition 2.2). Recall the statement from §1.4.1:

Theorem (Theorem 1.4, Restated). *Let $K \subset \mathbb{R}^2$ be self-similar or homogeneous. Then*

$$\dim_{\mathbf{P}} \{e \in \mathbb{S}^1 : \dim_{\mathbf{H}} K_e \leq s\} \leq s \quad \text{for all } 0 \leq s < \dim_{\mathbf{H}} K.$$

This result generalizes nicely to higher dimensions in the homogeneous case, but the self-similar case is far more delicate: there is a dichotomy between planar self-similar sets containing dense rotations in the orthogonal group and those that do not, whereas no such dichotomy exists in dimension $n \geq 3$. Peres and Shmerkin [41] showed that the exceptional set is *empty* for self-similar sets in $\mathbb{R}^2$ with dense rotations, leaving only the case of non-dense rotations—a case that readily reduces to the case of homogeneous sets.

The key to Theorem 1.4 is that linear maps are *dimension conserving* for homogeneous sets, and the equality of Hausdorff and upper box dimension of homogeneous sets allows one to utilize this principle in a discretized setting. Our main purposes are to generalize Theorem 1.4 to higher dimensions and to weaken the dimension conservation principle to “*almost* dimension conservation” (cf. Definitions 2.1 and 2.3). Recall that $A_V := \pi_V(A)$ denotes the image of a set $A \subseteq \mathbb{R}^n$ under the orthogonal projection $\pi_V : \mathbb{R}^n \rightarrow V \cong \mathbb{R}^m$.

Theorem 2.3. *Let $A \subset \mathbb{R}^n$ be a bounded set with $\dim_{\mathbf{H}} A = \overline{\dim}_{\mathbf{B}} A$ and let $0 \leq s < \dim_{\mathbf{H}} A$. If there exists a set $F_s \subseteq \mathbf{Gr}(n, m)$ with $\dim_{\mathbf{P}} F_s \leq m(n - m) - (m - s)$ such that π_V is almost dimension conserving for A for all $V \in \mathbf{Gr}(n, m) \setminus F_s$, then*

$$\dim_{\mathbf{P}} \{V \in \mathbf{Gr}(n, m) : \dim_{\mathbf{H}} A_V \leq s\} \leq m(n - m) - (m - s). \quad (2.1)$$

For examples of sets whose projections exhibit almost dimension conservation but (to the author’s knowledge) not necessarily “true” dimension conservation in the sense of Furstenberg, see [16] and [18].

As a corollary to Theorem 2.3, we obtain the following analogue of Theorem 1.4.

Corollary 2.4. *Let $K \subset \mathbb{R}^n$ be either a (Furstenberg) homogeneous set, or a graph-directed set with one of the following two properties: either (1) its transformation group is*

finite or (2) the action of the transformation group on $\mathbf{Gr}(n, m)$ has a dense orbit. Then

$$\dim_{\mathbf{P}} \{V \in \mathbf{Gr}(n, m) : \dim_{\mathbf{H}} K_V \leq s\} \leq m(n - m) - (m - s) \quad \text{for all } 0 \leq s < \dim_{\mathbf{H}} K.$$

Proof. In both cases, $\dim_{\mathbf{H}} K = \overline{\dim}_{\mathbf{B}} K$. If K is homogeneous, then every linear map (in particular, every orthogonal projection) is (classically) dimension conserving for K by Furstenberg's Theorem 2.6. Likewise, if K is a graph-directed set with finite transformation group, then every linear map is almost dimension conserving by Farkas' Theorem 2.8. Lastly, if K is graph-directed and has a dense orbit in the Grassmannian, then [18] Theorem 1.15 implies that the exceptional set of projections is empty. $\square$

If one replaces almost dimension conservation with *bona fide* dimension conservation in Theorem 2.3, one can still conclude Corollary 2.4 using [18] Theorem 1.3, which implies that graph-directed sets with finite transformation groups are well-approximated from within by homogeneous sets. However, the theorem is stated as is in the interests of generality and future examples of almost dimension conservation.

The hypothesis that the transformation group of K is finite in the graph-directed case is essential for the justification made in the proof above: Rapaport [43] constructed a self-similar set in the plane with infinite rotation group whose projections are not dimension conserving in a set of directions with full packing measure, although by [18] this set necessarily has projections of maximal dimension for other reasons, as noted in the proof of Corollary 2.4. The question of the “intermediate” cases in which some $V \in \mathbf{Gr}(n, m)$ have discrete orbits under the action of the closed transformation group of the IFS while others do not remains unanswered. Algom and Shmerkin [1] introduced a sort of transversality criterion on the orbit of V that guarantees $\dim_{\mathbf{H}} K_V$ is maximal, but it is unknown what happens when V has a non-discrete orbit that does not satisfy this criterion.²

²The author thanks Balázs Bárány for explaining the status of this problem. It seems the general belief among fractal geometers is that orthogonal projections should “generically” be dimension conserving for self-similar sets, but no useful characterization of those directions is known. See [4] and [24].

Our proof of Theorem 2.3 is similar to Orponen’s proof of Theorem 1.4. Philosophically, dimension conservation explains the dimension drop of a set under projection onto a subspace: the missing dimension did not simply “vanish,” but rather it is largely accounted for by the mass of the fibers of the projection. This information enables us to deduce what is happening upstairs in $\mathbb{R}^n$ from what we see downstairs in $\mathbb{R}^m$. While one might hope for a criterion implying (2.1) that does not rely on a dimension conservation principle, any such criterion would have to rule out the counterexample of Theorem 1.3. In that construction, it is the *exceptional set* that is highly nonuniform across scales, so the crux would be finding a hypothesis that precludes this without stringent requirements on the regularity of A itself.

Kaufman’s approach to the $s < m$ case of Theorem 1.1 relies on potential theory, and our proof of Theorem 2.3 can be understood as a discretization thereof. The $\dim_{\text{H}} A > m$ case of Theorem 1.1 suggests a the following extension of Theorem 2.3; however, as that case is not known to be provable without use of the Fourier transform, it is probable that any modification of our proof leading to such an extension will also need to emulate the role played by the Fourier transform. A plausible strategy is the discretized L^2 -energy method; see, for example, [19].

Conjecture. *Under the hypotheses of Theorem 2.3, if $\dim_{\text{H}} A > m$, then*

$$\dim_{\text{P}} \{V \in \mathbf{Gr}(n, m) : \dim_{\text{H}} A_V < s\} \leq m(n - m) - (\dim_{\text{H}} A - s)$$

for all $m < s < \dim_{\text{H}} A$.

2.1.3 Outline

In §2.2, we lay down some preliminaries on dimension conservation, homogeneous sets, and iterated function systems, and graph-directed sets. A lemma on the sizes of discrete sets of points in $\mathbf{Gr}(n, m)$ follows in §2.3, and this is used in §2.4 in the proof of Theorem 2.3. Finally, the proof of Proposition 2.2 applying Rams’ Theorem 2.1 to orthogonal projections of self-similar sets appears in Appendix A.

2.2 Additional background

2.2.1 Dimension conservation and homogeneous sets

As a prelude to the introduction of *almost dimension conservation*, we discuss the relevant terminology from Furstenberg’s paper [23].

Definition 2.1. A Lipschitz map $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be ***dimension conserving*** (or ***DC***) for a set $A \subseteq \mathbb{R}^n$ if there exists $\Delta \geq 0$ such that

$$\Delta + \dim_{\mathbb{H}} \{ \mathbf{y} \in f(A) : \dim_{\mathbb{H}}(A \cap f^{-1}(\mathbf{y})) \geq \Delta \} \geq \dim_{\mathbb{H}} A,$$

where $\dim_{\mathbb{H}} \emptyset := -\infty$.

Heuristically, f is DC for A if the dimension of its fibers over A is “complementary” to the dimension of $f(A)$: it is a sort of “rank-nullity condition.” The pathological Example 7.8 of [15] shows that even the projection of a product set onto the coordinate axes may radically fail to be DC for that set.

The *Hausdorff metric* on the class $\mathcal{K}$ of nonempty compacta in $\mathbb{R}^n$ is defined by

$$\rho(H, K) := \inf \{ \varepsilon \geq 0 : H \subseteq K_\varepsilon \text{ and } K \subseteq H_\varepsilon \},$$

$H, K \in \mathcal{K}$, where A_ε is the closed ε -neighborhood of A . With the Hausdorff metric, $\mathcal{K}$ is a complete metric space.

We now define the archetypal class of compacta in the study of dimension conservation. Scaling and translating a set does not affect the dimension of the projection of a set in any direction, so we assume without loss of generality that $K \subseteq [0, 1]^n$.

Definition 2.2. A closed set $K' \subset [0, 1]^n$ is called a ***miniset*** of K if there exists an expanding homothety $\varphi(\mathbf{x}) = r\mathbf{x} + \mathbf{b}$ ($|r| \geq 1$) such that $K' \subseteq \varphi(K)$. A closed set $K'' \subseteq [0, 1]^n$ is called a ***microset*** of K if there exists a sequence $(K'_i)_{i=1}^\infty$ of minisets of K converging to K'' in the Hausdorff metric: $\rho(K'_i, K'') \rightarrow 0$. Finally, K is said to

be (*Furstenberg*) *homogeneous* if all its microsets are minisets; that is, the class of minisets of K is a closed in $\mathcal{K}$.

Loosely, K is homogeneous if it looks the same at all scales: even if the minisets K'_i must be contained in larger and larger expansions of K as $i \rightarrow \infty$ (meaning they resemble smaller and smaller subsets of K), there still exists a scale on which the limiting set K'' coincides with a subset of K at that scale. See [2] and [22] for further explication of the geometry of minisets.

While the precise statement of Definition 2.1 will be important in the sequel, the technical definition of a homogeneous set is not strictly necessary, as the only two properties we require are the following.

Proposition 2.5. *If K is homogeneous, then $\dim_{\text{H}} K = \overline{\dim_{\text{B}}} K$.*

Theorem 2.6 (Furstenberg [23] Theorem 6.2). *If $K \subset \mathbb{R}^n$ is homogeneous and $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then f is DC for K .*

In particular, Theorem 2.6 implies that every projection map is DC for K . By [12], the same is true if K is merely a *weakly homogeneous set*—a closed subset of $[0, 1]^n$ whose minisets are all *finite unions* of microsets. (By [12] Proposition 5.9, this includes self-similar sets satisfying the weak separation condition.) Thus, “homogeneous” can be related to “weakly homogeneous” in the statement of Corollary 2.4.

There is also a straightforward weakening of the notion of dimension conservation due to Farkas [18].

Definition 2.3. We call a Lipschitz function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ *almost dimension conserving* for a set $A \subseteq \mathbb{R}^n$ if there exists $\Delta \geq 0$ such that, for every $\varepsilon > 0$,

$$\Delta + \dim_{\text{H}} \{ \mathbf{y} \in f(A) : \dim_{\text{H}}(A \cap f^{-1}(\mathbf{y})) \geq \Delta - \varepsilon \} \geq \dim_{\text{H}} A, \quad (2.2)$$

where $\dim_{\text{H}} \emptyset := -\infty$ as before.

See also [16] for the more restrictive concept of *weak* dimension conservation. In §2.4, A is fixed and we let $\Delta'(V)$ denote the set of all $\Delta \geq 0$ such that (2.2) holds with $f = \pi_V$.

2.2.2 Symbolic dynamics

A family $(f_i)_{i=1}^k$ of contractions of $\mathbb{R}^n$ is called an *iterated function system (IFS)*. It is a fact that there exists a unique nonempty compact set $K \subset \mathbb{R}^n$ that is invariant under the system, i.e., such that

$$\bigcup_{i=1}^k f_i(K) = K;$$

this is called the *limit set* or *attractor* of the IFS. In the event that the sets $f_i(K)$ are pairwise disjoint, we say that the IFS satisfies the *strong separation condition (SSC)*. If the f_i are all conformal maps, we call K *self-conformal*; if they are similarities (i.e., if $f_i(\mathbf{x}) = a_i \mathbf{x} + \mathbf{b}_i$, $a_i \in \mathbb{R}$, $\mathbf{b}_i \in \mathbb{R}^n$), we call K *self-similar*. The middle-thirds Cantor set in $\mathbb{R}$ and the four-corner Cantor set in $\mathbb{R}^2$ are perhaps the most popular examples of self-similar sets, and both are generated by IFS satisfying the SSC.

What follows is closely adapted from [42], and presenting the main result therein is our reason for laying down this notation. In particular, these definitions will help us to articulate what it means for a parametrized family of IFS to be “degenerate” for some values of the parameter.

Let $(f_i)_{i=1}^k$ be a conformal (or self-similar) IFS on $\mathbb{R}^n$ that is $C^{1,\alpha}$ for some $\alpha > 0$ and whose local contraction ratios $|\det Df_i|^{1/n}$ are uniformly bounded.

The set $\Sigma := \{1, \dots, k\}^{\mathbb{Z}_+}$ of all (infinite) sequences of integers 1 through k is called the *symbol space* of the IFS. Given $\omega = (\omega_1, \omega_2, \dots) \in \Sigma$, we denote $\omega^m := (\omega_1, \dots, \omega_m)$ and $f_{\omega^m} := f_{\omega_1} \circ \dots \circ f_{\omega_m}$. We moreover define

$$f_{\omega}(\mathbf{x}) := \lim_{m \rightarrow \infty} f_{\omega^m}(\mathbf{x});$$

the limit always exists by the Cantor intersection theorem, as the f_i are all contractions.

Lastly, we define a projection map $\Pi: \Sigma \rightarrow \mathbb{R}^n$ by

$$\Pi(\omega) := f_\omega(\mathbf{0}).$$

When working with families of IFS indexed by a parameter $\mathbf{t}$, the subscripted symbol $\Pi_{\mathbf{t}}$ will denote the corresponding projection.

We now introduce a generalization of self-similar sets that motivates our focus on *almost* dimension conservation. Let $(\mathcal{E}, \mathcal{V})$ be a directed graph with vertices $1, \dots, N$, possibly with loops. For all $i, j \in \mathcal{V}$, let $\mathcal{E}_{i,j}$ be the set of all edges from i to j and $\mathcal{E}_{i,j}^\ell$ the set of all paths of length ℓ from i to j . We assume that $(\mathcal{E}, \mathcal{V})$ is *transitive* (or *strongly connected*), i.e., that for all distinct $i, j \in \mathcal{V}$, at least one of the $\mathcal{E}_{i,j}^\ell$ is nonempty. Given a family $(g_e)_{e \in \mathcal{E}}$ of contracting similarities, there exists a unique tuple $(K_1, \dots, K_N)$ of compacta with the invariance property

$$K_i = \bigcup_{j=1}^N \bigcup_{e \in \mathcal{E}_{i,j}} g_e(K_j)$$

for each i . The system $(g_e)_{e \in \mathcal{E}}$ is called a *graph-directed IFS*, the tuple $(K_1, \dots, K_N)$ is called its *attractor*, and each K_i is called a *graph-directed set*. Finally, if we write $g_e(\mathbf{x}) = a_e T_e(\mathbf{x}) + \mathbf{b}_e$, where $a_e \in (0, 1)$ are the similarity ratios, $T_e \in O(n)$ are orthogonal transformations, and $\mathbf{b}_e \in \mathbb{R}^n$ are translation vectors, then the group $\mathcal{T}_i \leq O(n)$ generated by the set $\{T_{e_1} \circ \dots \circ T_{e_\ell} \in O(n) : (e_1, \dots, e_\ell) \in \mathcal{E}_{i,i}^\ell, \ell \in \mathbb{Z}_+\}$ is called the *transformation group* of K_i . Of particular importance are the case that $\mathcal{T}_i$ is finite and the case that some (hence, every) orbit in $\mathbf{Gr}(n, m)$ under the action of $\mathcal{T}_i$ is dense. What happens between these two extremes, while interesting, is not well-understood (but see [45] §7.2 and [1]).

Analogous to Proposition 2.5 and Theorem 2.6 are the following. They are the only properties of graph-directed sets important for our purposes.

Proposition 2.7. *If K is a graph-directed set, then $\dim_{\mathrm{H}} K = \overline{\dim}_{\mathrm{B}} K$.*

Theorem 2.8 (Farkas [18] Theorem 1.14). *Let $(K_1, \dots, K_\ell)$ be the attractor of a graph-directed IFS in $\mathbb{R}^n$. If $i \in \{1, \dots, \ell\}$ is any vertex such that the transformation group of K_i is finite and if $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then f is almost DC for K_i .*

Thus, under the hypotheses of Theorem 2.8, every projection map is almost DC for K_i .

2.3 Counting points on $\mathbf{Gr}(n, m)$

We will have occasion to ask, given two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and a δ -separated set $E \subset \mathbf{Gr}(n, m)$, for how many $V \in E$ we have $|\pi_V(\mathbf{x}) - \pi_V(\mathbf{y})| \leq c\delta$, where $c > 0$ is given. The following lemma—a discrete version of a key step in the proof of Marstrand’s projection theorem—addresses this question in greater generality.

Lemma 2.9. *Let $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $\delta_1, \delta_2 \in (0, 1]$ and let $E \subset \mathbf{Gr}(n, m)$ be δ_2 -separated. Then*

$$\text{card} \{V \in E : |\pi_V(\mathbf{x})| \leq \delta_1\} \lesssim_{n,m} \delta_1^m \delta_2^{-m(n-m)} |\mathbf{x}|^{-m}.$$

Proof. Since the invariant measure $\gamma_{n,m}$ and the $m(n-m)$ -dimensional Hausdorff measure $\mathcal{H}^{m(n-m)}$ are both uniformly distributed measures on $\mathbf{Gr}(n, m)$, they are equal up to a constant. Consequently, $r^{m(n-m)} \lesssim \gamma_{n,m}(B(V, r))$ for all $V \in \mathbf{Gr}(n, m)$ and $r \in (0, 1]$, and it follows from the separation hypothesis on E that

$$\delta_2^{m(n-m)} \text{card} \{V \in E : |\pi_V(\mathbf{x})| \leq \delta_1\} \lesssim \gamma_{n,m}(\{V \in \mathbf{Gr}(n, m) : |\pi_V(\mathbf{x})| \leq \delta_1\}).$$

Lemma 3.11 of [34] states that

$$\gamma_{n,m}(\{V \in \mathbf{Gr}(n, m) : |\pi_V(\mathbf{x})| \leq \delta_1\}) \lesssim_{n,m} \delta_1^m |\mathbf{x}|^{-m},$$

and combining this inequality with the previous gives

$$\delta_2^{m(n-m)} \text{card} \{V \in E : |\pi_V(\mathbf{x})| \leq \delta_1\} \lesssim \delta_1^m |\mathbf{x}|^{-m}.$$

Dividing through by $\delta_2^{m(n-m)}$ yields the desired inequality. $\square$

The question at the start of this section is answered by replacing $\mathbf{x}$ with $\mathbf{x} - \mathbf{y}$ and applying the linearity of π_V . Curiously, the proof of Theorem 2.3 (and, in particular, of Lemma 2.11) does not seem to benefit from the full generality of Lemma 2.9: discretizing in the

Grassmannian and in Euclidean space at the same scale (i.e., letting $\delta_1 \sim \delta_2$) does not seem to affect the strength of the conclusion.

2.4 Proof of Theorem 2.3

Theorem 2.3 will follow readily from the following two lemmas. The first is essentially true by definition, but it quantitatively formalizes the idea that, if K is homogeneous and the dimension of K_V is small, then the dimensions of the fibers of π_V over K must be large.

Lemma 2.10. *Let $A \subseteq \mathbb{R}^n$. If there exists a set $F \subset \mathbf{Gr}(n, m)$ with $t := \dim_{\mathbb{P}} F$ such that π_V is almost DC for A for all $V \in \mathbf{Gr}(n, m) \setminus F$, then*

$$\begin{aligned} & \dim_{\mathbb{P}} \{V \in \mathbf{Gr}(n, m) : \dim_{\mathbb{H}} A_V \leq s\} \\ & \leq \max\left(t, \dim_{\mathbb{P}} \{V \in \mathbf{Gr}(n, m) \setminus F : \Delta + s \geq \dim_{\mathbb{H}} A \ \forall \Delta \in \Delta'(V)\}\right). \end{aligned}$$

for all $s \geq 0$.

Proof. Let $V \in \mathbf{Gr}(n, m) \setminus F$ be such that $\dim_{\mathbb{H}} A_V \leq s$ and let $\Delta \in \Delta'(V)$. Then, by the definition of almost dimension conservation,

$$\Delta + \dim_{\mathbb{H}} \{\mathbf{y} \in A_V : \dim_{\mathbb{H}}(A \cap \pi_V^{-1}(\mathbf{y})) \geq \Delta - \varepsilon\} \geq \dim_{\mathbb{H}} A$$

for every $\varepsilon > 0$. It follows from the monotonicity of dimension that

$$\Delta + s \geq \Delta + \dim_{\mathbb{H}} A_V \geq \dim_{\mathbb{H}} A.$$

This proves the inclusion

$$\begin{aligned} & \{V \in \mathbf{Gr}(n, m) \setminus F : \dim_{\mathbb{H}} A_V \leq s\} \\ & \subseteq \{V \in \mathbf{Gr}(n, m) \setminus F : \Delta + s \geq \dim_{\mathbb{H}} A \ \forall \Delta \in \Delta'(V)\}, \end{aligned}$$

whence

$$\begin{aligned}
& \dim_{\mathbf{P}} \{V \in \mathbf{Gr}(n, m) : \dim_{\mathbf{H}} A_V \leq s\} \\
& \leq \max(\dim_{\mathbf{P}} F, \dim_{\mathbf{P}} \{V \in \mathbf{Gr}(n, m) \setminus F : \dim_{\mathbf{H}} A_V \leq s\}) \\
& \leq \max(t, \dim_{\mathbf{P}} \{V \in \mathbf{Gr}(n, m) \setminus F : \Delta + s \geq \dim_{\mathbf{H}} A \ \forall \Delta \in \Delta'(V)\}). \quad \square
\end{aligned}$$

Lemma 2.11. *Let $A \subset \mathbb{R}^n$ be a bounded set with $\gamma := \dim_{\mathbf{H}} A = \overline{\dim}_{\mathbf{B}} A$, and denote*

$$E_s := \{V \in \mathbf{Gr}(n, m) : \exists \Delta \in \Delta'(V) \text{ s.t. } \Delta + s \geq \gamma\}.$$

Then

$$\dim_{\mathbf{P}} E_s \leq m(n - m) - (m - s) \quad (2.3)$$

for all $0 \leq s < \gamma$.

As the proof of this lemma is much more involved, we first show how it (almost immediately) implies Theorem 2.3.

Proof of Theorem 2.3. Let $A \subset \mathbb{R}^n$ be a bounded set with $\gamma := \dim_{\mathbf{H}} A = \overline{\dim}_{\mathbf{B}} A$ whose orthogonal projections π_V are almost DC for $V \in \mathbf{Gr}(n, m)$ outside a set F of packing dimension t . Then A satisfies the hypotheses of Lemmas 2.10 and 2.11, whence

$$\begin{aligned}
& \dim_{\mathbf{P}} \{V \in \mathbf{Gr}(n, m) : \dim_{\mathbf{H}} A_V \leq s\} \\
& \leq \max\left(t, \dim_{\mathbf{P}} \{V \in \mathbf{Gr}(n, m) \setminus F : \Delta \geq \gamma - s \ \forall \Delta \in \Delta'(V)\}\right) \\
& \leq \max\left(t, \dim_{\mathbf{P}} \{V \in \mathbf{Gr}(n, m) : \exists \Delta \in \Delta'(V) \text{ s.t. } \Delta \geq \gamma - s\}\right) \\
& \leq \max(t, m(n - m) - (m - s)). \quad \square
\end{aligned}$$

We conclude by proving Lemma 2.11, wherein the dimension conservation hypothesis and the geometry of the Grassmannian come into play.

Proof of Lemma 2.11. The $s = 0$ case follows from the $s > 0$ case by letting $s \downarrow 0$, so we assume that $s > 0$.

STEP 1. We begin by reducing the lemma to the following claim: that

$$\overline{\dim}_B E_{s,\varepsilon} \leq m(n-m) - (m-s) + 3\varepsilon \quad (2.4)$$

for all sufficiently small $\varepsilon > 0$, where

$$E_{s,\varepsilon} := \left\{ V \in \mathbf{Gr}(n, m) : \exists \Delta \geq \gamma - s \text{ s.t.} \right. \\ \left. \mathcal{H}_\infty^{\gamma-\Delta-\varepsilon}(\{\mathbf{y} \in \mathbb{R}^m : \mathcal{H}_\infty^{\Delta-\varepsilon}(A \cap \pi_V^{-1}(\mathbf{y})) > \varepsilon\}) > \varepsilon \right\}.$$

To prove the validity of this reduction, suppose that $\Delta \geq \gamma - s$ for some $\Delta \in \Delta'(V)$. (In particular, π_V is almost DC for A .) Then, for every $\varepsilon > 0$,

$$\dim_H \{\mathbf{y} \in \mathbb{R}^m : \dim_H(A \cap \pi_V^{-1}(\mathbf{y})) \geq \Delta - \frac{\varepsilon}{2}\} > \gamma - \Delta - \varepsilon.$$

Consequently, if $0 < \varepsilon < \gamma - s \leq \Delta$, the set on the left-hand side of this inequality has infinite $(\gamma - \Delta - \varepsilon)$ -dimensional Hausdorff measure and, in particular, positive $(\gamma - \Delta - \varepsilon)$ -dimensional Hausdorff content:

$$\mathcal{H}_\infty^{\gamma-\Delta-\varepsilon}(\{\mathbf{y} \in \mathbb{R}^m : \dim_H(A \cap \pi_V^{-1}(\mathbf{y})) \geq \Delta - \frac{\varepsilon}{2}\}) > 0.$$

Similarly, $\dim_H(A \cap \pi_V^{-1}(\mathbf{y})) \geq \Delta - \frac{\varepsilon}{2}$ implies $\mathcal{H}_\infty^{\Delta-\varepsilon}(A \cap \pi_V^{-1}(\mathbf{y})) > 0$, so

$$\mathcal{H}_\infty^{\gamma-\Delta-\varepsilon}(\{\mathbf{y} \in \mathbb{R}^m : \mathcal{H}_\infty^{\Delta-\varepsilon}(A \cap \pi_V^{-1}(\mathbf{y})) > 0\}) > 0.$$

Writing

$$\{\mathbf{y} \in \mathbb{R}^m : \mathcal{H}_\infty^{\Delta-\varepsilon}(A \cap \pi_V^{-1}(\mathbf{y})) > 0\} = \bigcup_{k=1}^{\infty} \{\mathbf{y} \in \mathbb{R}^m : \mathcal{H}_\infty^{\Delta-\varepsilon}(A \cap \pi_V^{-1}(\mathbf{y})) > k^{-1}\},$$

we can by Proposition 1.8 find $k \in \mathbb{Z}_+$ such that

$$\mathcal{H}_\infty^{\gamma-\Delta-\varepsilon}(\{\mathbf{y} \in \mathbb{R}^m : \mathcal{H}_\infty^{\Delta-\varepsilon}(A \cap \pi_V^{-1}(\mathbf{y})) > k^{-1}\}) > 0.$$

The left-hand side is decreasing in ε and increasing in k , so we may adjust these parameters accordingly so that $k^{-1} = \varepsilon$ for simplicity. Indeed, for $\varepsilon > 0$ sufficiently small, we will have

$$\mathcal{H}_\infty^{\gamma-\Delta-\varepsilon}(\{\mathbf{y} \in \mathbb{R}^m : \mathcal{H}_\infty^{\Delta-\varepsilon}(A \cap \pi_V^{-1}(\mathbf{y})) > \varepsilon\}) > \varepsilon,$$

from which we conclude that $V \in E_{s,\varepsilon}$. In particular, we can write

$$E_s \subseteq \bigcup_{k=N}^{\infty} E_{s,k-1}$$

for any $N \in \mathbb{Z}_+$. It then follows from our definition of packing dimension that (2.4) implies (2.3), so we set out to prove (2.4).

STEP 2. Let $0 < \varepsilon < \gamma - s$. We discretize the problem and define a family of relations, indexed by $V \in \mathbf{Gr}(n, m)$, that relate distant points $\mathbf{x}, \mathbf{y} \in A$ whose projections $\pi_V(\mathbf{x}), \pi_V(\mathbf{y})$ are close.

Let $\gamma' > \gamma$, $d := (\gamma - s - \varepsilon)^{-1}$, and

$$\delta < \eta := \frac{\varepsilon^d}{n^{1/2} 2^{d+2} (2^{n-m} + 1)^d}.$$

(The significance of this requirement on δ will become apparent later.) By the definition of upper box dimension, there exists a finite subset $A' \subseteq A$ such that $\text{card } A' \lesssim \delta^{-\gamma'}$ and

$$A \subseteq \bigcup_{\mathbf{x} \in A'} B(\mathbf{x}, \delta).$$

For each $V \in \mathbf{Gr}(n, m)$, let $\mathcal{T}_V$ be the family of δ -fat $(n-m)$ -planes of the following form:

$$\pi_V^{-1} \left(\prod_{i=1}^m [j_i \delta, (j_i + 1) \delta) \right) \subset \mathbb{R}^n,$$

$j_1, \dots, j_m \in \mathbb{Z}$. (For succinctness, we will call these “fat planes” or “elements of $\mathcal{T}_V$.”) These are half-open neighborhoods of fibers of π_V over points of the lattice $(\mathbb{Z} + \frac{1}{2})^m \delta$, and their disjoint union is all of $\mathbb{R}^n$. We define relations $\sim_V$ on $\mathbb{R}^n$ by

$$\begin{aligned} \mathbf{x} \sim_V \mathbf{y} \quad &\Longleftrightarrow \quad |\mathbf{x} - \mathbf{y}| > 2\eta = \frac{\varepsilon^d}{2^{d+1} (2^{n-m} + 1)^d} \quad \text{and} \quad \exists T \in \mathcal{T}_V \quad \text{s.t.} \\ &B(\mathbf{x}, \delta) \cap T \neq \emptyset \quad \text{and} \quad B(\mathbf{y}, \delta) \cap T \neq \emptyset. \end{aligned} \tag{2.5}$$

This states that $\sim_V$ relates points of $\mathbb{R}^n$ that are not too close to each other, but that nevertheless belong to the same fat plane, adjacent fat planes, or fat planes with a common neighboring fat plane. In particular, although the points are fairly distant from each other, their projections onto V are quite close.

STEP 3. Let $E' \subseteq E_{s,\varepsilon}$ be any δ -separated subset and define the energy of E' by

$$\mathcal{E} := \sum_{V \in E'} \text{card} \{(\mathbf{x}, \mathbf{y}) \in (A')^2 : \mathbf{x} \smile_V \mathbf{y}\}. \quad (2.6)$$

We use this energy to bound $\text{card } E'$ and, in turn, $\overline{\dim}_{\mathbb{B}} E_{s,\varepsilon}$.

To obtain an upper bound, note that, given $\mathbf{x}, \mathbf{y} \in A'$, the number of m -planes $V \in E'$ such that $\mathbf{x} \smile_V \mathbf{y}$ is $\lesssim \delta^{-m(n-m)+m} |\mathbf{x} - \mathbf{y}|^{-m}$ by Lemma 2.9. Hence, for a fixed $\mathbf{x} \in A'$, the bound $\text{card } A' \lesssim_{n,\gamma} \delta^{-\gamma'}$ and the trivial estimate $\eta^{-m} \sim_{\varepsilon,n,m} 1$ give

$$\begin{aligned} \sum_{V \in E'} \text{card} \{\mathbf{y} \in A' : \mathbf{x} \smile_V \mathbf{y}\} &= \sum_{\mathbf{y} \in A'} \text{card} \{V \in E' : \mathbf{x} \smile_V \mathbf{y}\} \\ &\lesssim \delta^{-m(n-m)+m} \left(\max_{\substack{\mathbf{y} \in A', V \in E' : \\ \mathbf{x} \smile_V \mathbf{y}}} |\mathbf{x} - \mathbf{y}|^{-m} \right) \text{card } A' \\ &\leq \delta^{-m(n-m)+m} (2\eta)^{-m} \delta^{-\gamma'} \sim \delta^{-m(n-m)+m-\gamma'}, \end{aligned}$$

where the implicit constants depend on n, m, γ, s , and ε , but not γ' or δ . Summing over all $\mathbf{x} \in A'$ gives

$$\mathcal{E} \lesssim \delta^{-m(n-m)+m-\gamma'} \text{card } A' \lesssim \delta^{-m(n-m)+m-2\gamma'}.$$

To place a lower bound on $\mathcal{E}$, we estimate the individual terms in the sum (2.6). Let $V \in E' \subseteq E_{s,\varepsilon}$ and $\Delta \in \Delta'(V)$. Unwinding the definition of $E_{s,\varepsilon}$, we see that there exist $j \gtrsim \delta^{\Delta-\gamma+\varepsilon}$ fat planes $T_i \in \mathcal{T}_V$ and points $\mathbf{y}_i \in \pi_V(T_i)$, $i = 1, \dots, j$, with the following property: if $W_i := \pi_V^{-1}(\mathbf{y}_i)$ denotes the $(n-m)$ -plane contained in T_i that “passes through” $\mathbf{y}_i$, then

$$\mathcal{H}_{\infty}^{\Delta-\varepsilon}(A \cap W_i) > \varepsilon. \quad (2.7)$$

To count the number of relations $\mathbf{x} \smile_V \mathbf{y}$ that hold on A' , we checkerboard each fat plane $T_i \in \mathcal{T}_V$ with boxes or “checkerboard squares”

$$R = T_i \cap \pi_{V^\perp}^{-1} \left(\prod_{\ell=1}^{n-m} [4i_\ell \eta, 4(i_\ell + 1)\eta) \right),$$

$i_1, \dots, i_{n-m} \in \mathbb{Z}$ (see Figure 2.1). Recalling that we chose $\delta < \eta$, we see that

$$|R| = \left(\sum_{\ell=1}^{n-m} (4\eta)^2 + \sum_{\ell=n-m+1}^n \delta^2 \right)^{1/2} < \left(\sum_{\ell=1}^n (4\eta)^2 \right)^{1/2} = n^{1/2} \cdot 4\eta = \frac{\varepsilon^d}{2^d (2^{n-m} + 1)^d}$$

and, consequently, that

$$\mathcal{H}_{\infty}^{\Delta-\varepsilon}(R) \leq |R|^{\Delta-\varepsilon} < \frac{\varepsilon}{2(2^{n-m} + 1)}, \quad (2.8)$$

per our choice of d . It then follows from (2.7) and (2.8) that, for any choice of squares $R_1, \dots, R_{2^{n-m}+1}$,

$$\mathcal{H}_{\infty}^{\Delta-\varepsilon} \left((A \cap W_i) \setminus \bigcup_{\ell=1}^{2^{n-m}+1} R_{\ell} \right) > \frac{\varepsilon}{2},$$

so any cover of $(A \cap W_i) \setminus R$ by δ -balls contains $\gtrsim \delta^{\varepsilon-\Delta}$ balls.

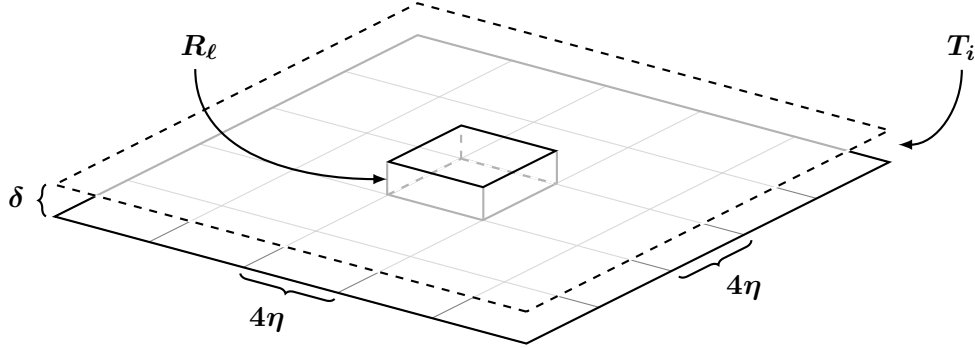


Figure 2.1: A “checkerboard square” R_{ℓ} in the fat plane T_i , $(n, m) = (3, 1)$.

Now, (2.7) and (2.8) also entail that there exist distinct $R_1, \dots, R_{2^{n-m}+1}$ whose intersections with $A \cap W_i$ each have positive $(\Delta - \varepsilon)$ -dimensional Hausdorff content. In particular,

$$\text{card} \{ \mathbf{x} \in A' : B(\mathbf{x}, \delta) \cap (A \cap W_i \cap R_p) \neq \emptyset \} \gtrsim \delta^{\varepsilon-\Delta} \quad (2.9)$$

for $p = 1, \dots, 2^{n-m} + 1$ because $\{B(\mathbf{x}, \delta) : \mathbf{x} \in A'\}$ is a cover of A . Necessarily, at least 2 of these squares R_p, R_q are mutually non-adjacent, so they are 4η -separated. Therefore, if $\mathbf{x}, \mathbf{y} \in A'$ are such that

$$B(\mathbf{x}, \delta) \cap (A \cap W_i \cap R_p) \neq \emptyset \quad \text{and} \quad B(\mathbf{y}, \delta) \cap (A \cap W_i \cap R_q) \neq \emptyset,$$

then

$$|\mathbf{x} - \mathbf{y}| > 4\eta - 2\delta > 2\eta,$$

so that $\mathbf{x} \sim_V \mathbf{y}$. In conjunction with (2.9), this yields the estimate

$$\begin{aligned} \text{card } \{(\mathbf{x}, \mathbf{y}) \in (A')^2 : B(\mathbf{x}, \delta) \cap (A \cap W_i \cap R_p) \neq \emptyset, \\ B(\mathbf{y}, \delta) \cap (A \cap W_i \cap R_q) \neq \emptyset, \mathbf{x} \sim_V \mathbf{y}\} \gtrsim \delta^{2(\varepsilon - \Delta)}. \end{aligned}$$

No δ -ball intersects more than 3^m fat planes in $\mathcal{T}_V$, so we may sum the previous over all $i \in \{1, \dots, j\}$ to get

$$\text{card } \{(\mathbf{x}, \mathbf{y}) \in (A')^2 : \mathbf{x} \sim_V \mathbf{y}\} \gtrsim j(\delta^{\varepsilon - \Delta})^2 \gtrsim \delta^{(\Delta - \gamma + \varepsilon) + 2(\varepsilon - \Delta)} \geq \delta^{3\varepsilon + s - 2\gamma},$$

where the final inequality follows from our original hypothesis that $\Delta \geq \gamma - s$. This is the desired lower bound on the individual summands in (2.6), and multiplying by $\text{card } E'$ yields the desired bound on $\mathcal{E}$ itself:

$$\mathcal{E} \gtrsim \delta^{3\varepsilon + s - 2\gamma} \text{card } E'.$$

STEP 4. In combination with our upper bound $\delta^{-m(n-m)+m-2\gamma'} \gtrsim \mathcal{E}$, this at last provides a concrete upper bound on $\text{card } E'$ in terms of δ , namely,

$$\text{card } E' \lesssim \delta^{-m(n-m)+m-2\gamma'-(3\varepsilon+s-2\gamma)} = \delta^{-m(n-m)+(m-s)-3\varepsilon-2(\gamma'-\gamma)}.$$

Since $\gamma' > \gamma$ was arbitrary and does not appear in the implicit constant, the estimate $\text{card } E' \lesssim \delta^{-m(n-m)+(m-s)-3\varepsilon}$ follows at once. This holds for every δ -separated subset $E' \subseteq E_{s,\varepsilon}$, so we conclude (2.4) and, in turn, (2.3). $\square$

Chapter 3

STRONG EXCEPTIONAL PARAMETERS FOR NONLINEAR SLICES

In this chapter, we redirect our attention from linear projections to nonlinear slices.

3.1 Main result

3.1.1 The statements

Our purpose is to build on a paper [38] of Orponen that sharpens Marstrand's slicing theorem and extends it to slices by fibers of generalized projections in the sense of Peres–Schlag [40]. The terminology of the Peres–Schlag paper, although standard, is also substantial, so we first state the main result in the classical setting of slices by affine subspaces of Euclidean space. Recall that, for $1 \leq m < n$, the symbol $\mathbf{Gr}(n, m)$ denotes the Grassmannian of m -dimensional subspaces of $\mathbb{R}^n$ and, for each $V \in \mathbf{Gr}(n, m)$, the symbol π_V denotes the orthogonal projection of $\mathbb{R}^n$ onto $V \cong \mathbb{R}^m$.

Theorem 3.1. *Let $m < s < n$ and let $A \subseteq \mathbb{R}^n$ be a Borel set with $0 < \mathcal{H}^s(A) < \infty$. There exists a set $E \subset \mathbf{Gr}(n, m)$ with*

$$\dim_{\mathbf{H}} E \leq m(n - m) + m - s$$

such that, for every subspace $V \in \mathbf{Gr}(n, m) \setminus E$ and every Borel set $B \subseteq A$ with $\mathcal{H}^s(B) > 0$, the equation

$$\dim_{\mathbf{H}}(B \cap \pi_V^{-1}(\mathbf{x})) = s - m \tag{3.1}$$

holds for a positive- $\mathcal{H}^m$ -measure set of $\mathbf{x} \in V$.

Informally, for all $V \in \mathbf{Gr}(n, m)$ outside a small “bad” set E , the intersections of A with the orthogonal spaces $\mathbf{x} + V^\perp$ are large (relative to s) for many points $\mathbf{x} \in V$. Moreover, E is *universal* for A in the sense that this same statement holds for all subsets $B \subseteq A$ of positive $\mathcal{H}^s$ -measure: E contains the exceptional set of directions $V \in \mathbf{Gr}(n, m)$ corresponding to *every* such B . Following Falconer and Mattila [17], we accordingly refer to the estimate in Theorem 3.1 as “strong.”

The upper bound on the dimension of slices follows immediately from the well-known sublevel set inequality for the Hausdorff measure of Lipschitz maps (see, for example, [34] Theorem 7.7). In particular, since π_V is Lipschitz for *all* $V \in \mathbf{Gr}(n, m)$, the set $A \cap \pi_V^{-1}(\mathbf{x})$ (hence, every subset thereof) has finite $(s - m)$ -dimensional Hausdorff measure for $\mathcal{H}^m$ -a.e. $\mathbf{x} \in V$. *A fortiori* there are *no* $V \in \mathbf{Gr}(n, m)$ such that the upper bound in (3.1) fails for $\mathcal{H}^m$ -positively-many $\mathbf{x}$.

The lower bound $\dim_{\mathcal{H}}(B \cap \pi_V^{-1}(\mathbf{x})) \geq s - m$ in Theorem 3.1 is a special case of our main result, which is situated in the Peres–Schlag framework. In both the statement and the subsequent proofs, it may be efficacious to begin by considering only the case $\beta = 0$. See §3.2 for a review of the definitions, and see Theorem 3.9 for a simpler statement that forgoes the universality of the exceptional set.

Theorem 3.2. *Let $U \subseteq \mathbb{R}^N$ be an open set and let $(\Pi_{\lambda} : \Omega \rightarrow \mathbb{R}^m)_{\lambda \in U}$ be a family of generalized projections that is transversal and strongly regular of degree $\beta \in [0, 1)$. There exists a constant $a \geq 1$, depending only on N and m , such that the following holds. If $m < s < N + m$, then for every Borel set $A \subseteq \Omega$ with $0 < \mathcal{H}^s(A) < \infty$, there exists a set $E \subset U$ with*

$$\dim_{\mathcal{H}} E \leq N + m - s + a\beta^{1/3}$$

such that, for every $\lambda \in U \setminus E$ and every Borel set $B \subseteq A$ with $\mathcal{H}^s(B) > 0$, the inequality

$$\dim_{\mathcal{H}}(B \cap \Pi_{\lambda}^{-1}(\mathbf{x})) \geq (1 - \beta)(s - m) \tag{3.2}$$

holds for a positive-Lebesgue-measure set of $\mathbf{x} \in \mathbb{R}^m$.

Note that the dimension bound $N + m - s + \varepsilon(\beta)$ on E , where $\varepsilon(\beta) \rightarrow 0$ as $\beta \rightarrow 0$, is precisely the form that the corresponding projection result takes (see Theorem 3.3 below). This will be important in §3.4.

As a side note, since the ambient dimension $\dim_{\mathbb{H}} \Omega$ plays no role and the transversality and regularity of $(\Pi_{\lambda})_{\lambda \in U}$ are preserved under restriction of the domain Ω , it would suffice to take $A = \Omega$.

3.1.2 Overview and outline

The methods of this paper are already well-established in geometric measure theory. Järvenpää–Järvenpää–Niemelä [28] proved a nonlinear slicing theorem for mappings between manifolds using the same basic definitions as Peres and Schlag, but their methods were not Fourier-analytic and did not yield dimension bounds on the exceptional sets. Orponen [38] then proved Theorem 3.2 for $m = N = 1$ and $B = A$ (recall Theorem 1.6 from the introduction), and the proof of this generalization follows his closely. Mattila [35] also proved the $B = A$ case of Theorem 3.1 by importing some of Orponen’s machinery into higher dimensions (for which reason he attributes the result to Orponen), and the obstacles in extending their slicing result for orthogonal projections to generalized projections are more practical than theoretical. Once the details are worked out, the strong aspect of Theorem 3.2 follows from a simple adaptation of the argument in [17].

After laying out the preliminaries in §3.2, we state several technical lemmas in §3.3.1 that adapt the Peres–Schlag framework to slices. Most of the proof of Theorem 3.2 is contained in that of Lemma 3.7 in §3.3.2, and in turn the main hurdle in proving this lemma is verifying the hypotheses of Lemma 3.5. Since the proof of Lemma 3.7 is quite lengthy as is, Lemma 3.6 is set aside to facilitate the application of Lemma 3.5, and following Orponen’s example we relegate the cumbersome proof of Lemma 3.6 to Appendix B. With the main Lemma 3.7 in place, the proof of Theorem 3.2 is completed in two steps: first in §3.4.1 without the “universal” exceptional set, which is then introduced in §3.4.2

using the method of Falconer and Mattila, thereby completing the proof of the theorem. Finally, §3.5 presents an application of Theorem 3.2 to vertical slices in the Heisenberg groups $\mathbb{H}^n$.

3.2 Additional background

3.2.1 Pushforwards and slices of measures

If $\Psi : \Omega \rightarrow \mathbb{R}^d$ is a continuous map and $\mu \in \mathcal{M}(\Omega)$, then there is a Radon measure $\Psi_{\#}\mu \in \mathcal{M}(\mathbb{R}^d)$ —the *pushforward* of μ by Ψ —defined on Borel sets $X \subseteq \mathbb{R}^d$ by $\Psi_{\#}\mu(X) := \mu(\Psi^{-1}(X))$. The pushforward measure satisfies $\text{spt } \Psi_{\#}\mu = \Psi(\text{spt } \mu)$ and, for all Borel functions $g : \mathbb{R}^d \rightarrow [0, \infty]$,

$$\int_{\mathbb{R}^d} g(\mathbf{z}) d(\Psi_{\#}\mu)(\mathbf{z}) = \int_{\Omega} (g \circ \Psi)(\omega) d\mu(\omega).$$

From this formula one can derive the powerful identity

$$\int_{\mathbb{R}^d} g(\mathbf{z}) |\widehat{\Psi_{\#}\mu}(\mathbf{z})|^2 d\mathbf{z} = \iint_{\Omega \times \Omega} \widehat{g}(\Psi(\omega) - \Psi(\zeta)) d(\mu \times \mu)(\omega, \zeta) \quad (3.3)$$

using Plancherel’s theorem. Incidentally, (1.6) is a special case of this.

There is a straightforward procedure for defining the slices of a measure by the fibers of a continuous map by means of its pushforward under said map. However, we will later relocate our analysis from Ω to a Euclidean space in such a way that the slices of a measure $\mu \in \mathcal{M}(\Omega)$ correspond to the slices of its “surrogate” in $\mathbb{R}^d$ by affine subspaces. (Incidentally, this surrogate is itself a pushforward measure—the pushforward of μ by the map Ψ_{λ} defined in §3.2.3 below.) For that reason, we need only to understand the slices of measures on $\mathbb{R}^d$ by the fibers of orthogonal projections.

To that end, for each $V \in \mathbf{Gr}(m+n, m)$, let $\nu_V := \pi_{V\#}\nu \in \mathcal{M}(\mathbb{R}^m)$ be the pushforward of the measure $\nu \in \mathcal{M}(\mathbb{R}^{m+n})$ by π_V , where we identify V with $\mathbb{R}^m$ using local coordinates on $\mathbf{Gr}(m+n, m)$. For $\mathbf{x} \in \mathbb{R}^m$, we define the *slice of ν by $\pi_V^{-1}(\mathbf{x})$* , denoted $\nu_{V,\mathbf{x}}$, via its

action on functions $f \in C_c^+(\mathbb{R}^{m+n})$:

$$\int_{\mathbb{R}^{m+n}} f(\mathbf{z}) d\nu_{V,\mathbf{x}}(\mathbf{z}) := \lim_{\delta \rightarrow 0} \frac{1}{\mathcal{L}^m(B(\mathbf{x}, \delta))} \int_{\pi_V^{-1}(B(\mathbf{x}, \delta))} f(\mathbf{w}) d\nu(\mathbf{w}),$$

provided this limit exists. This is precisely the Radon–Nikodym derivative $D(\pi_{V\#}\nu_f, \mathbf{x})$ of the pushforward by π_V of the Radon measure $\nu_f(A) := \int_A f(\mathbf{z}) d\nu(\mathbf{z})$ on $\mathbb{R}^{m+n}$, so for a *fixed* $f \in C_c^+(\mathbb{R}^{m+n})$, the limit exists for $\mathcal{L}^m$ -a.e. $\mathbf{x} \in \mathbb{R}^m$. A standard argument from the separability of $C_c(\mathbb{R}^{m+n})$ then implies the existence of this limit for *all* $f \in C_c^+(\mathbb{R}^{m+n})$, so the sliced measure $\nu_{V,\mathbf{x}}$ exists as a positive linear functional on $C_c^+(\mathbb{R}^{m+n})$ for almost every $\mathbf{x}$. By the Riesz representation theorem, it is an honest Radon measure.

3.2.2 Bessel functions and spherical harmonics

Special functions make a brief appearance in the proof of the main lemma. All the facts utilized are “well-known” and they are unimportant in the broader scope of this chapter, so we only record only the very basics here. See [36] for a full exposition.

For each $\nu > -\frac{1}{2}$, let $J_\nu: [0, \infty) \rightarrow \mathbb{R}$ be the *Bessel function of the first kind of order ν* :

$$J_\nu(u) := \frac{u^\nu}{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 e^{iut} (1 - t^2)^{\nu-1/2} dt,$$

where Γ is the gamma function. It behaves roughly like $u^{-1/2} \cos(u - b)$ for some phase shift $b = b(\nu)$. The Bessel functions will only arise in the *Hankel transform of order ν* : for any $f \in L_{\text{loc}}^1([0, \infty))$,

$$\mathbf{H}_\nu f(s) := \int_0^\infty f(r) J_\nu(sr) r dr,$$

where the integral is understood in the limiting sense as an improper integral. The Hankel transform of a polynomial has an explicit expression that we shall state and use later.

A class of functions that “play well” with the Bessel functions are the spherical harmonics on $\mathbb{S}^{d-1}$ —the restrictions to $\mathbb{S}^{d-1}$ of the homogeneous harmonic polynomials on $\mathbb{R}^d$ —which will play the role of multidimensional Fourier series. More precisely, a function $Y_\ell: \mathbb{S}^{d-1} \rightarrow \mathbb{C}$ is called a *spherical harmonic of degree ℓ* if there exists a homogeneous

polynomial $P: \mathbb{R}^d \rightarrow \mathbb{C}$ of degree $\ell \in \mathbb{N}$ satisfying $\Delta P = 0$ such that $P|_{\mathbb{S}^{d-1}} = Y_\ell$, where Δ is the Laplacian on $\mathbb{R}^d$. We shall fix an orthonormal basis $\bigcup_{\ell=0}^\infty \{Y_{\ell,k}\}_{k=1}^{d(\ell)}$ for $L^2(\mathbb{S}^{d-1})$, where each $Y_{\ell,k}$ is a spherical harmonic of degree ℓ and $d(\ell)$ is the dimension of the space of spherical harmonics of degree ℓ .

3.2.3 Generalized projections

We now turn to the requisite terminology from the Peres–Schlag framework for generalize projections. Let $N, m \in \mathbb{Z}_+$, let $\emptyset \neq U \subseteq \mathbb{R}^N$ be an open set, let (Ω, d) be a compact metric space, and let $\Pi: \Omega \times U \rightarrow \mathbb{R}^m$ be a continuous map. Writing $\Pi_\lambda := \Pi(\cdot, \lambda)$, we consider Π as a parametrized family $(\Pi_\lambda: \Omega \rightarrow \mathbb{R}^m)_{\lambda \in U}$ of mappings from Ω to $\mathbb{R}^m$.

Definition 3.1. The family $(\Pi_\lambda)_{\lambda \in U}$ is called a **family of generalized projections** if $\lambda \mapsto \Pi_\lambda(\omega)$ is C^∞ for all $\omega \in \Omega$ and the following condition is satisfied:

- i. For every compact set $K \subset U$ and every $\alpha \in \mathbb{N}^N$, the inequality

$$|\partial_\lambda^\alpha \Pi_\lambda(\omega)| \lesssim_{K, |\alpha|} 1 \quad (3.4)$$

holds for all $\lambda \in K$ and $\omega \in \Omega$.

Supposing $(\Pi_\lambda)_{\lambda \in U}$ is a family of generalized projections, let

$$\Phi_\lambda(\omega, \zeta) := \frac{\Pi_\lambda(\omega) - \Pi_\lambda(\zeta)}{d(\omega, \zeta)}$$

for $\omega \neq \zeta$ and $\Phi_\lambda(\omega, \omega) := \mathbf{0}$, and let $\beta \in [0, 1)$.

- ii. The family is said to be **transversal of degree β** if, for every compact $K \subset U$, every $\lambda \in K$, and all $\omega \neq \zeta$,

$$\begin{aligned} |\Phi_\lambda(\omega, \zeta)| &\lesssim_{\beta, K} d(\omega, \zeta)^\beta \\ \implies \det \left[D_\lambda \Phi_\lambda(\omega, \zeta) (D_\lambda \Phi_\lambda(\omega, \zeta))^t \right] &\gtrsim_{\beta, K} d(\omega, \zeta)^{2\beta}. \end{aligned} \quad (3.5)$$

- iii. The family is said to be **regular of degree β** if, for every compact $K \subset U$, every $\lambda \in K$, all $\omega \neq \zeta$, and all $\alpha \in \mathbb{N}^N$,

$$\begin{aligned} |\Phi_\lambda(\omega, \zeta)| &\lesssim_{\beta, K} d(\omega, \zeta)^\beta \\ \implies |\partial_\lambda^\alpha \Phi_\lambda(\omega, \zeta)| &\lesssim_{K, \beta, |\alpha|} d(\omega, \zeta)^{-\beta|\alpha|}. \end{aligned} \quad (3.6)$$

- iv. The family is said to be **strongly regular of degree β** if, for every compact $K \subset U$, every $\lambda \in K$, all $\omega \neq \zeta$, and all $\alpha \in \mathbb{N}^N$,

$$|\partial_\lambda^\alpha \Phi_\lambda(\omega, \zeta)| \lesssim_{K, \beta, |\alpha|} d(\omega, \zeta)^{-\beta|\alpha|} \quad (3.7)$$

(regardless of the size of $|\Phi_\lambda(\omega, \zeta)|$).

The main results of [40] concern the Sobolev dimension of projections (i.e., pushforwards) of Radon measures. Having a large projection in the direction λ is certainly requisite for a measure on Ω to have many large slices by Π_λ , so we record (a portion of) one of the main Peres–Schlag theorems that bounds the exceptional sets of projections.

Theorem 3.3 (Peres–Schlag [40] Theorem 7.3). *There exists a constant $a_0 > 0$ depending only on N and m such that the following holds. Let $(\Pi_\lambda : \Omega \rightarrow \mathbb{R}^m)_{\lambda \in U}$ be a family of generalized projections that is both transversal and regular of some degree $\beta \in [0, 1)$. If $\mu \in \mathcal{M}(\Omega)$ has finite s -energy for some $s > m$, then*

$$\dim_{\text{H}} \{ \lambda \in U : \Pi_{\lambda\#} \mu \not\ll \mathcal{H}^r \} \leq N + r - \frac{s}{1 + a_0 \beta}$$

for all $r \in (0, m]$.

An important family of objects in connection with the generalized projections, originally introduced in [38], will be the mappings $\Psi_\lambda : \Omega \rightarrow \mathbb{R}^{m+mN}$ defined by

$$\Psi_\lambda(\omega) := (\Pi_\lambda(\omega), D_\lambda \Pi_\lambda(\omega)), \quad (3.8)$$

where the total derivative $D_\lambda \Pi_\lambda(\omega)$ is identified with an element of $\mathbb{R}^{mN}$. In [40], Π_λ is adequate to relocate the entire projection problem into Euclidean space: the transversality

and regularity conditions black-box the fact that preimages of the sets $\Pi_{\lambda}(A)$ belong to an abstract metric space. In the slicing problem, on the other hand, the objects of study are precisely those sets that collapse under the application of Π_{λ} , so finding another “natural” mapping of Ω to a Euclidean space is the first step in adapting the Peres–Schlag framework to the study of slices by means of Fourier analysis.

The maps Ψ_{λ} do the trick because the transversality condition (3.5) requires that, modulo a loss inflicted by a positive β , the first partial derivatives of Φ_{λ} pick up the information about pairs of points that Φ_{λ} itself loses. It follows from (3.5) that

$$|\Phi_{\lambda}(\omega, \zeta)| \lesssim_{\beta, K} d(\omega, \zeta)^{\beta} \implies \max_{1 \leq i \leq N} |\partial_{\lambda_i} \Phi_{\lambda}(\omega, \zeta)|_{\infty} \gtrsim_{\beta, K} d(\omega, \zeta)^{\beta},$$

so packaging the partial derivatives of the components of Π_{λ} into Ψ_{λ} ensures that it always distinguishes between pairs of points in the same fiber of Π_{λ} . Due to (3.7), Ψ_{λ} is sufficiently regular as not to stretch the fibers beyond recognition.

We can summarize the relevant properties of Ψ_{λ} as a single bi-Hölder condition

$$d(\omega, \zeta)^{\beta} \lesssim_{\beta, K} \frac{|\Psi_{\lambda}(\omega) - \Psi_{\lambda}(\zeta)|}{d(\omega, \zeta)} \lesssim_{\beta, K} d(\omega, \zeta)^{-\beta}, \quad (3.9)$$

valid for all λ belonging to a compact set $K \subset U$, where the lower and upper bounds owe themselves to the transversality and regularity of Π_{λ} , respectively. Note that the upper Hölder bound would not follow if we only required that only (3.6) hold, as in [40], rather than (3.7). However, an important step in the proof of Lemma 3.6 requires that $|\Psi_{\lambda}(\omega) - \Psi_{\lambda}(\zeta)|$ be small whenever $d(\omega, \zeta)$ is—not just the $|\Pi_{\lambda}(\omega) - \Pi_{\lambda}(\zeta)|$ component. In particular, if it is possible to relax this condition, doing so would require a more nuanced strategy for ensuring that the derivative bounds in the hypotheses of Lemma 3.5 are satisfied.

3.2.4 Notation and conventions

Throughout, $(\Pi_{\lambda})_{\lambda \in U}$ denotes a family of generalized projections $\Pi_{\lambda} : \Omega \rightarrow \mathbb{R}^m$, where (Ω, d) is a compact metric space and $\emptyset \neq U \subseteq \mathbb{R}^N$ is an open set. Typically $(\Pi_{\lambda})_{\lambda \in U}$

will be both transversal and strongly regular of the same order $\beta \geq 0$. Except in §3.5, the integers $N, m \in \mathbb{Z}_+$ and $n := mN$ will remain fixed throughout this chapter.

3.3 A few lemmas

The purpose of this section is to begin the proof of Theorem 3.2 by establishing Lemma 3.7. In broad strokes, the logic parallels that of [40] §7 on generalized projections in arbitrary dimension, but we closely follow the adaptations made by Orponen [38] for generalized slices. Lemma 3.4 is essentially the “Fubini inequality” for sliced measures and does not have an explicit analogue in [40], whereas Lemma 3.5 is taken from [40] Lemma 3.1 essentially unchanged. The last main ingredient, Lemma 3.6, requires proof but plays the role of [40] Lemma 7.10.

With $N, m \in \mathbb{Z}_+$ and $n := mN$ fixed as in §3.2.4, let $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$, $\mathbf{z}^{(1)} := \mathbf{x} \in \mathbb{R}^m$, and $\mathbf{z}^{(2)} := \mathbf{y} \in \mathbb{R}^n$. For $\nu \in \mathcal{M}(\mathbb{R}^{m+n})$ and $\mathbf{x} \in \mathbb{R}^m$, we denote by $\nu_{\mathbf{x}}^{(1)} \in \mathcal{M}(\mathbb{R}^{m+n})$ the slice of ν by the coordinate map $(\mathbf{x}', \mathbf{y}') \mapsto \mathbf{x}'$ over the point $\mathbf{x}$.

3.3.1 Technical lemmas

Lemma 3.7 hinges on three results. The first is taken from Mattila [35], where it is given a tedious but fairly straightforward proof. This will serve as our starting point in the proof of Lemma 3.7.

Lemma 3.4 (Mattila [35] Proposition 6.3). *For every $\nu \in \mathcal{M}(\mathbb{R}^{m+n})$ and every $m < \sigma < m + n$,*

$$\int_{\mathbb{R}^m} I_{\sigma-m}(\nu_{\mathbf{x}}^{(1)}) d\mathbf{x} \lesssim_{N,m,\sigma} \int_{\mathbb{R}^{m+n}} |\mathbf{z}^{(2)}|^{\sigma-m-n} |\widehat{\nu}(\mathbf{z})|^2 d\mathbf{z}.$$

The second lemma—a result on the divergence of power series—is paraphrased from [40], and it is one of the main pillars in their proofs of their generalized projection theorems. The primary difference between their statement and the one below is the parameter L ,

which they take to be ∞ ; however, as they remark at the beginning of §3.2, their proof implies this more general, lower-regularity statement.

Lemma 3.5 (Peres–Schlag [40] Lemma 3.1). *Let $U \subseteq \mathbb{R}^N$ be an open set, $A, R > 1$, and $L \in \mathbb{Z}_+ \cup \{\infty\}$, and suppose $(h_i)_{i=1}^\infty$ is a sequence in $C^L(U)$ satisfying*

$$\|\partial_\lambda^\alpha h_i\|_{L^\infty(K)} \lesssim_{K,L} A^{i|\alpha|} \quad \text{for all } |\alpha| \leq L \quad \text{and} \quad \|h_i\|_{L^1(K)} \lesssim_K R^{-i} \quad (3.10)$$

for all $i \in \mathbb{N}$ and every compact set $K \subset U$. If $1 \leq r < R$ and $0 < q < N$ are such that $A^q r^{q/L} \leq \frac{R}{r} \leq A^N r^{N/L}$, then

$$\dim_{\text{H}} \left\{ \lambda \in U : \sum_{i=1}^\infty r^i |h_i(\lambda)| = \infty \right\} \leq N - q.$$

Verifying the hypotheses of Lemma 3.5 for certain functions h_j to be defined later constitutes the greater part of the proof of Lemma 3.7 in §3.3.2 below. Both the derivative and the integral estimates in (3.10) will demand extensive computations, but the integral bound in particular requires a statement that we break off here as a lemma of its own. Let $\eta \in C_c^\infty((0, \infty))$ be a bump function for some bounded interval. In addition, let $\eta^{(1)}(\mathbf{x}) := \eta(|\mathbf{x}|)$ for all $\mathbf{x} \in \mathbb{R}^m$ and $\eta^{(2)}(\mathbf{y}) := \eta(|\mathbf{y}|)$ for all $\mathbf{y} \in \mathbb{R}^n$.

Lemma 3.6. *There exists a constant $a_1 \geq 1$ (depending only on N and m) such that, for all $\rho \in C_c^\infty(U)$, all $c \geq 0$, and all $\omega, \zeta \in \Omega$,*

$$\left| \int_U \rho(\lambda) \widehat{\eta^{(1)}}(2^j d(\omega, \zeta) \Phi_\lambda(\omega, \zeta)) \widehat{\eta^{(2)}}(2^{j-i} d(\omega, \zeta) D_\lambda \Phi_\lambda(\omega, \zeta)) d\lambda \right| \lesssim_{\rho,c} (1 + 2^j d(\omega, \zeta)^{1+a_1\beta})^{-c}. \quad (3.11)$$

This is the higher-dimensional analogue of Orponen [38] Lemma 4.2, which in turn is an adaptation of Peres–Schlag [40] Lemma 4.6. The higher-dimensional analogue of the latter also appears in [40] as Lemma 7.10, and the proof of Lemma 3.6 is a lengthy but straightforward amalgam of Orponen’s and Peres and Schlag’s proofs. For this reason, the argument appears in Appendix B.

3.3.2 The main lemma

Recall the family of maps $\Psi_{\lambda}: \Omega \rightarrow \mathbb{R}^m$ defined in (3.8). For each $\mu \in \mathcal{M}(\Omega)$,

$$\mu_{\lambda, \mathbf{x}} := (\Psi_{\lambda\#}\mu)_{\mathbf{x}}^{(1)}$$

is the slice of the pushforward measure $\Psi_{\lambda\#}\mu \in \mathcal{M}(\mathbb{R}^{m+n})$ by the fiber of the coordinate map $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$ over the point $\mathbf{x} \in \mathbb{R}^m$.

Lemma 3.7. *There exists a constant $a_2 \geq 1$, depending only on N and m , such that the following holds. Let $(\Pi_{\lambda})_{\lambda \in U}$ be a family of generalized projections that is both transversal and strongly regular of order β . If $\mu \in \mathcal{M}(\Omega)$ is finite and has finite $(s + a_2\beta^{1/3})$ -energy for some $m < s < N + m$, then*

$$\dim_{\text{H}} \left\{ \lambda \in U : \int_{\mathbb{R}^m} I_{s+\beta^{1/3}-m}(\mu_{\lambda, \mathbf{x}}) d\mathbf{x} = \infty \right\} \leq N + m - s. \quad (3.12)$$

Proof. With the notation and hypotheses of the lemma statement and $n := mN$ as before, let $t \geq s$ be such that $I_t(\mu) < \infty$ and let $\sigma := s + \beta^{1/3}$. By Lemma 3.4,

$$\int_{\mathbb{R}^m} I_{\sigma-m}(\mu_{\lambda, \mathbf{x}}) d\mathbf{x} \lesssim_{\beta, s} \int_{\mathbb{R}^{m+n}} |\mathbf{z}^{(2)}|^{\sigma-m-n} |\widehat{\Psi_{\lambda\#}\mu}(\mathbf{z})|^2 d\mathbf{z}$$

for all $\lambda \in U$. Using a countable exhaustion of U by compact sets, we may (and shall) restrict our attention from U to a compact subset $K \subset U$. For each $i \in \mathbb{N}$, let

$$\mathbf{C}_i = \left\{ \mathbf{z} \in \mathbb{R}^{m+n} : \frac{1}{2^{i+1}} < \frac{|\mathbf{z}^{(2)}|}{|\mathbf{z}|} \leq \frac{1}{2^i} \right\}$$

be the set of points in $\mathbb{R}^{m+n}$ whose position vectors make an angle between $\cos^{-1}(2^{-i-1})$ and $\cos^{-1}(2^{-i})$ with the coordinate plane $\{0\}^m \times \mathbb{R}^n$. These partition $\mathbb{R}^{m+n}$ up to a set of Lebesgue measure 0, so

$$\begin{aligned} \int_{\mathbb{R}^{m+n}} |\mathbf{z}^{(2)}|^{\sigma-m-n} |\widehat{\Psi_{\lambda\#}\mu}(\mathbf{z})|^2 d\mathbf{z} &= \sum_{i=0}^{\infty} \int_{\mathbf{C}_i} |\mathbf{z}^{(2)}|^{\sigma-m-n} |\widehat{\Psi_{\lambda\#}\mu}(\mathbf{z})|^2 d\mathbf{z} \\ &\sim_{\beta, s} \sum_{i=0}^{\infty} 2^{i(n+m-\sigma)} \int_{\mathbf{C}_i} |\mathbf{z}|^{\sigma-m-n} |\widehat{\Psi_{\lambda\#}\mu}(\mathbf{z})|^2 d\mathbf{z}; \end{aligned} \quad (3.13)$$

the second line uses the fact that $|\mathbf{z}^{(2)}| \sim 2^{-i}|\mathbf{z}|$ on $\mathbf{C}_i$. Our goal is to determine how large t must be if this series is to converge for all but a few $\lambda \in K$; later, t will take the form $s + a_2\beta^{1/3}$ for some $a_2 \geq 1$ to be determined.

STEP 1. We bound the second series in (3.13) from above by a series of functions amenable to the application of Lemma 3.5. But first, the $i = 0$ term warrants a separate and elementary treatment: the Plancherel identity (3.3) and the transversality constraint (3.9) together yield

$$\begin{aligned} \int_{\mathbf{C}_0} |\mathbf{z}|^{\sigma-m-n} |\widehat{\Psi_{\lambda\#}\mu}(\mathbf{z})|^2 d\mathbf{z} &\leq \int_{\mathbb{R}^{m+n}} |\mathbf{z}|^{\sigma-m-n} |\widehat{\Psi_{\lambda\#}\mu}(\mathbf{z})|^2 d\mathbf{z} \\ &\sim_{N,m,\beta,s} \iint_{\Omega \times \Omega} \frac{d(\mu \times \mu)(\omega, \zeta)}{|\Psi_{\lambda}(\omega) - \Psi_{\lambda}(\zeta)|^{\sigma}} \\ &\lesssim_{\beta,K} \iint_{\Omega \times \Omega} \frac{d(\mu \times \mu)(\omega, \zeta)}{d(\omega, \zeta)^{(1+\beta)\sigma}} \\ &= I_{(1+\beta)\sigma}(\mu). \end{aligned}$$

As μ has finite t -energy, this last integral is finite for all $\lambda \in K$ provided that $\sigma \leq \frac{t}{1+\beta}$.

To bound the remaining terms in (3.13), we increase the regularity of the integrals using smooth bump functions $\phi_i: \mathbb{R}^{m+n} \rightarrow [0, 1]$ satisfying

$$\chi_{\mathbf{C}_i} \leq \phi_i \leq \chi_{\mathbf{C}_{i-1} \cup \mathbf{C}_i \cup \mathbf{C}_{i+1}}.$$

In particular, we set

$$\phi_i(\mathbf{z}) := \phi(2^{i+1}|\mathbf{z}^{(2)}|/|\mathbf{z}|), \quad \text{where } \phi \in C_c^\infty((\tfrac{1}{2}, \tfrac{5}{2}))$$

is a smooth bump function for the interval $[1, 2]$; thus,

$$\int_{\mathbf{C}_i} |\mathbf{z}|^{\sigma-m-n} |\widehat{\Psi_{\lambda\#}\mu}(\mathbf{z})|^2 d\mathbf{z} \leq \int_{\mathbb{R}^{m+n}} \phi_i(\mathbf{z}) |\mathbf{z}|^{\sigma-m-n} |\widehat{\Psi_{\lambda\#}\mu}(\mathbf{z})|^2 d\mathbf{z}$$

for all $i \geq 1$. In view of our aim of proving that the value in (3.13) is finite for “most” $\lambda \in K$, our work is complete if we can show that

$$\dim_{\mathbb{H}} \left\{ \lambda \in K : \sum_{i=1}^{\infty} 2^{i(2n+m-\sigma)} h_i(\lambda) = \infty \right\} \leq N + m - s$$

for all sufficiently small $\beta \geq 0$, where

$$h_i(\boldsymbol{\lambda}) := \frac{1}{2^{in}} \int_{\mathbb{R}^{m+n}} \phi_i(\mathbf{z}) |\mathbf{z}|^{\sigma-m-n} |\widehat{\Psi_{\boldsymbol{\lambda}^\#} \mu}(\mathbf{z})|^2 d\mathbf{z}. \quad (3.14)$$

We do so by means of Lemma 3.5, and the remainder of this proof consists primarily of showing that, for some $0 < p < n$, these h_i satisfy (3.10) with $A = 2$, $R = 2^{p+n}$, $q < s-m$, and some large $L \in \mathbb{Z}_+$ to be determined. As such, we must bound the L^1 -norms of the $h_i|_K$ and the L^∞ -norms of their derivatives.

STEP 2. We verify the L^1 -bounds in (3.10). Let $A_{i,j} := \{\mathbf{z} \in \text{spt } \phi_i : 2^j \leq |\mathbf{z}| < 2^{j+1}\}$ for $j \in \mathbb{Z}$, so that

$$h_i(\boldsymbol{\lambda}) \sim \frac{1}{2^{in}} \sum_{j \in \mathbb{Z}} 2^{j(\sigma-m-n)} \int_{A_{i,j}} |\widehat{\Psi_{\boldsymbol{\lambda}^\#} \mu}(\mathbf{z})|^2 d\mathbf{z} \quad (3.15)$$

by the definition of h_i and the boundedness of the Fourier transform of a finite Borel measure. For a sufficiently large constant $b \geq 2$ depending only on N and m , the set $A_{i,j}$ is contained in the product annulus

$$B_{i,j} := \left\{ \mathbf{z} \in \mathbb{R}^{m+n} : \frac{1}{b} 2^j < |\mathbf{z}^{(1)}| \leq b 2^j, \frac{1}{b} 2^{j-i} < |\mathbf{z}^{(2)}| \leq b 2^{j-i} \right\}.$$

If $\eta : (\frac{1}{b^2}, b^2) \rightarrow [0, 1]$ is a smooth bump function for the interval $[\frac{1}{b}, b]$ and if $\eta_j^{(k)}(\mathbf{z}) := \eta(2^{-j} |\mathbf{z}^{(k)}|)$ for $\mathbf{z} \in \mathbb{R}^{m+n}$, then $\mathbf{z} \mapsto \eta_j^{(1)}(\mathbf{z}) \eta_{j-i}^{(2)}(\mathbf{z})$ is a smooth bump function for $B_{i,j}$. It then follows from the Plancherel identity (3.3), the definition of $\Psi_{\boldsymbol{\lambda}}$, and the tensor product factorization of the Fourier transform that

$$\begin{aligned} \int_{A_{i,j}} |\widehat{\Psi_{\boldsymbol{\lambda}^\#} \mu}(\mathbf{z})|^2 d\mathbf{z} &\leq \int_{\mathbb{R}^{m+n}} \eta_j^{(1)}(\mathbf{z}) \eta_{j-i}^{(2)}(\mathbf{z}) |\widehat{\Psi_{\boldsymbol{\lambda}^\#} \mu}(\mathbf{z})|^2 d\mathbf{z} \\ &= \iint_{\Omega \times \Omega} (\eta_j^{(1)} \eta_{j-i}^{(2)})^\wedge (\Psi_{\boldsymbol{\lambda}}(\boldsymbol{\omega}) - \Psi_{\boldsymbol{\lambda}}(\boldsymbol{\zeta})) d(\mu \times \mu)(\boldsymbol{\omega}, \boldsymbol{\zeta}) \\ &= 2^{mj+n(j-i)} \iint_{\Omega \times \Omega} \widehat{\eta^{(1)}}(2^j d(\boldsymbol{\omega}, \boldsymbol{\zeta}) \Phi_{\boldsymbol{\lambda}}(\boldsymbol{\omega}, \boldsymbol{\zeta})) \\ &\quad \cdot \widehat{\eta^{(2)}}(2^{j-i} d(\boldsymbol{\omega}, \boldsymbol{\zeta}) D_{\boldsymbol{\lambda}} \Phi_{\boldsymbol{\lambda}}(\boldsymbol{\omega}, \boldsymbol{\zeta})) d(\mu \times \mu)(\boldsymbol{\omega}, \boldsymbol{\zeta}). \end{aligned} \quad (3.16)$$

In the third line, $\eta^{(1)}(\mathbf{x}) := \eta_0^{(1)}(\mathbf{x}, \mathbf{y})$ for any $\mathbf{y} \in \mathbb{R}^n$ and $\eta^{(2)}(\mathbf{y}) := \eta_0^{(2)}(\mathbf{x}, \mathbf{y})$ for any $\mathbf{x} \in \mathbb{R}^m$ (note that $\eta_0^{(1)}$ is independent of $\mathbf{y}$ and $\eta_0^{(2)}$ is independent of $\mathbf{x}$), and the Fourier

transforms thereof denote the m - and n -dimensional Fourier transforms, respectively, rather than the full $(m+n)$ -dimensional Fourier transform.

Fix a value $0 < p < n$, to be chosen sufficiently close to n much later in the proof. Lemma 3.6, (3.15), and (3.16) together imply that, whenever $\rho \in C_c^\infty(U)$ is a smooth bump function for K ,

$$\begin{aligned}
2^{i(p+n)} \int_K h_i(\boldsymbol{\lambda}) d\boldsymbol{\lambda} &\lesssim_\rho 2^{ip} \sum_{j \in \mathbb{Z}} 2^{j(\sigma-m-n)} \int_U \rho(\boldsymbol{\lambda}) \int_{A_{i,j}} |\widehat{\Psi_{\boldsymbol{\lambda}^\#} \mu}(\mathbf{z})|^2 d\mathbf{z} d\boldsymbol{\lambda} \\
&\lesssim_K 2^{ip} \sum_{j \in \mathbb{Z}} 2^{j(\sigma-m-n)} \cdot 2^{mj+n(j-i)} \\
&\quad \cdot \iint_{\Omega \times \Omega} \left[\int_U \rho(\boldsymbol{\lambda}) \widehat{\eta^{(1)}}(2^j d(\boldsymbol{\omega}, \boldsymbol{\zeta}) \Phi_{\boldsymbol{\lambda}}(\boldsymbol{\omega}, \boldsymbol{\zeta})) \right. \\
&\quad \left. \cdot \widehat{\eta^{(2)}}(2^{j-i} d(\boldsymbol{\omega}, \boldsymbol{\zeta}) D_{\boldsymbol{\lambda}} \Phi_{\boldsymbol{\lambda}}(\boldsymbol{\omega}, \boldsymbol{\zeta})) d\boldsymbol{\lambda} \right] d(\mu \times \mu)(\boldsymbol{\omega}, \boldsymbol{\zeta}) \\
&\lesssim 2^{i(p-n)} \sum_{j \in \mathbb{Z}} 2^{j\sigma} \iint_{\Omega \times \Omega} (1 + 2^j d(\boldsymbol{\omega}, \boldsymbol{\zeta})^{1+a_1\beta})^{-c} d(\mu \times \mu)(\boldsymbol{\omega}, \boldsymbol{\zeta}).
\end{aligned} \tag{3.17}$$

(The implicit constant depends only on K and not ρ once a standard procedure for constructing bump functions chosen.) The series in j converges provided $c > \sigma$, and in fact it is bounded above by $\sim_{\beta,s,c} I_{(1+a_1\beta)\sigma}(\mu)$. To see this, write

$$\begin{aligned}
\sum_{j \in \mathbb{Z}} \frac{2^{js}}{(1 + 2^j d(\boldsymbol{\omega}, \boldsymbol{\zeta})^{1+a_1\beta})^c} &\sim d(\boldsymbol{\omega}, \boldsymbol{\zeta})^{-(1+a_1\beta)\sigma} \left(\sum_{k=0}^{\infty} \frac{2^{ks}}{(1 + 2^k)^c} + \sum_{k=-\infty}^{-1} \frac{2^{ks}}{(1 + 2^k)^c} \right) \\
&\leq d(\boldsymbol{\omega}, \boldsymbol{\zeta})^{-(1+a_1\beta)\sigma} \left(\frac{1}{1 - 2^{\sigma-c}} + \frac{1}{1 - 2^{-\sigma}} \right),
\end{aligned}$$

so that by (3.17)

$$2^{i(p+n)} \int_K h_i(\boldsymbol{\lambda}) d\boldsymbol{\lambda} \lesssim_{\beta,K,s,c} 2^{i(p-n)} \iint_{\Omega \times \Omega} \frac{d(\mu \times \mu)(\boldsymbol{\omega}, \boldsymbol{\zeta})}{d(\boldsymbol{\omega}, \boldsymbol{\zeta})^{(1+a_1\beta)\sigma}} = 2^{i(p-n)} I_{(1+a_1\beta)\sigma}(\mu).$$

With (say) $c = N + m > \sigma$, the power series estimate

$$\sum_{i=1}^{\infty} 2^{i(p+n)} \|h_i\|_{L^1(K)} \lesssim_{N,m,\beta,K,s} \frac{1}{1 - 2^{p-n}} I_{(1+a_1\beta)\sigma}(\mu)$$

follows at once, so

$$\|h_i\|_{L^1(K)} \lesssim_{N,m,\beta,K,s,p} 2^{-i(p+n)}.$$

Note that this imposes the new constraint $\sigma \leq \frac{t}{1+a_1\beta}$, which is stronger than previously-established the constraint $\sigma \leq \frac{t}{1+\beta}$ since $a_1 \geq 1$.

STEP 3. We verify the L^∞ -bounds (3.10) on the partial derivatives $\partial_{\lambda}^{\alpha} h_i$. To do so, we rewrite the integral in (3.14) using Plancherel's theorem and repeatedly differentiate under the integral sign. Plancherel's theorem tells us that

$$h_i(\lambda) = \frac{1}{2^{in}} \iint_{\Omega \times \Omega} \widehat{g}_i(\Psi_{\lambda}(\omega) - \Psi_{\lambda}(\zeta)) d(\mu \times \mu)(\omega, \zeta), \quad (3.18)$$

where

$$g_i(\mathbf{z}) := \phi_i(\mathbf{z})|\mathbf{z}|^{\sigma-m-n} = \phi(2^{i+1}|\mathbf{z}^{(2)}|/|\mathbf{z}|)|\mathbf{z}|^{\sigma-m-n}.$$

Our goal, then, is to establish pointwise bounds on the partial derivatives of the function

$$\gamma_i(\lambda) := \widehat{g}_i(\Psi_{\lambda}(\omega) - \Psi_{\lambda}(\zeta)), \quad (3.19)$$

where $\omega, \zeta \in \Omega$ are fixed. To do this, we first consider $\widehat{g}_i$ itself. Write $\mathbf{z} = r\boldsymbol{\nu}$ in spherical coordinates, so that

$$\widehat{g}_i(\mathbf{w}) = \int_0^\infty r^{\sigma-1} \int_{\mathbb{S}^{m+n-1}} \phi_i(\boldsymbol{\nu}) e^{-2\pi i r \boldsymbol{\nu} \cdot \mathbf{w}} d\mathcal{H}^{m+n-1}(\boldsymbol{\nu}) dr. \quad (3.20)$$

Since $\phi_i|_{\mathbb{S}^{m+n-1}}$ is C^∞ (it is a smooth bump function for a “band” on the sphere of width $\sim 2^{-i}$), it has an absolutely convergent spherical harmonic expansion

$$\phi_i(\boldsymbol{\nu}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{d(\ell)} a_{i,\ell,k} Y_{\ell,k}(\boldsymbol{\nu})$$

with rapidly-decaying coefficients. In fact, since $\phi_i(\boldsymbol{\nu}) = \phi(2^{i+1}|\boldsymbol{\nu}^{(2)}|)$, there exist $a_{\ell,k}$ such that $|a_{i,\ell,k}| \sim 2^{-in}|a_{k,\ell}|$ for all $i \in \mathbb{Z}_+$ —a fact we shall use shortly.

Now, the identity

$$\begin{aligned} \int_{\mathbb{S}^{m+n-1}} Y_{\ell,k}(\boldsymbol{\nu}) e^{-2\pi i r \boldsymbol{\nu} \cdot \mathbf{w}} d\mathcal{H}^{m+n-1}(\boldsymbol{\nu}) \\ = (2\pi)^{(m+n-1)/2} (-i)^\ell \frac{J_{\ell+(m+n-2)/2}(2\pi r|\mathbf{w}|)}{(r|\mathbf{w}|)^{(m+n-2)/2}} Y_{\ell,k}\left(\frac{\mathbf{w}}{|\mathbf{w}|}\right) \end{aligned}$$

is standard, from which it follows that

$$\begin{aligned} \int_{\mathbb{S}^{m+n-1}} \phi_i(\boldsymbol{\nu}) e^{-2\pi i r \boldsymbol{\nu} \cdot \mathbf{w}} d\mathcal{H}^{m+n-1}(\boldsymbol{\nu}) \\ = (2\pi)^{(m+n-1)/2} \sum_{\ell=0}^{\infty} \sum_{k=1}^{d(\ell)} (-i)^\ell a_{i,\ell,k} \frac{J_{\ell+(m+n-2)/2}(2\pi r |\mathbf{w}|)}{(r |\mathbf{w}|)^{(m+n-2)/2}} Y_{\ell,k} \left(\frac{\mathbf{w}}{|\mathbf{w}|} \right). \end{aligned}$$

Plugging this back into (3.20) then yields

$$\begin{aligned} \widehat{g}_i(\mathbf{w}) &\simeq_{m,N} \sum_{\ell=0}^{\infty} \sum_{k=1}^{d(\ell)} (-i)^\ell a_{i,\ell,k} Y_{\ell,k} \left(\frac{\mathbf{w}}{|\mathbf{w}|} \right) \int_0^\infty r^{\sigma-1} \frac{J_{\ell+(m+n-2)/2}(2\pi r |\mathbf{w}|)}{(r |\mathbf{w}|)^{(m+n-2)/2}} dr \\ &= |\mathbf{w}|^{-(m+n-2)/2} \sum_{\ell=0}^{\infty} \sum_{k=1}^{d(\ell)} (-i)^\ell a_{i,\ell,k} Y_{\ell,k} \left(\frac{\mathbf{w}}{|\mathbf{w}|} \right) \int_0^\infty r^{\sigma-(m+n)/2} J_{\ell+(m+n-2)/2}(2\pi r |\mathbf{w}|) dr. \end{aligned}$$

The integral in the second line is precisely the Hankel transform of order $\ell + (m+n-2)/2$ of the monomial $r^{\sigma-(m+n-2)/2}$, and it can be expressed simply in terms of the gamma function as

$$\frac{2^{\sigma-(m+n)/2}}{(2\pi |\mathbf{w}|)^{\sigma-(m+n+2)/2}} \frac{\Gamma(\frac{\ell+\sigma+1}{2})}{\Gamma(\frac{\ell-\sigma+m+n}{2})}.$$

Thus, we have shown that

$$\widehat{g}_i(\mathbf{w}) \simeq_{m,N,s,\beta} |\mathbf{w}|^{-\sigma} \sum_{\ell=0}^{\infty} \sum_{k=1}^{d(\ell)} (-i)^\ell a_{i,\ell,k} Y_{\ell,k} \left(\frac{\mathbf{w}}{|\mathbf{w}|} \right) \frac{\Gamma(\frac{\ell+\sigma+1}{2})}{\Gamma(\frac{\ell-\sigma+m+n}{2})}.$$

Now consider $\partial^\alpha \widehat{g}_i(\mathbf{w})$ for $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in \mathbb{N}^m \times \mathbb{N}^n$. We first have the elementary estimate

$$\partial^\alpha |\mathbf{w}|^{-\sigma} \lesssim_{|\boldsymbol{\alpha}|} \sigma^{|\boldsymbol{\alpha}|} |\mathbf{w}|^{-\sigma-|\boldsymbol{\alpha}|} \sim_{\beta,s,|\boldsymbol{\alpha}|} |\mathbf{w}|^{-\sigma-|\boldsymbol{\alpha}|}.$$

Denoting by $P_{\ell,k}$ an extension of $Y_{\ell,k}$ to a harmonic polynomial of degree ℓ on $\mathbb{R}^{m+n}$, so that $Y_{\ell,k}(\mathbf{w}/|\mathbf{w}|) = |\mathbf{w}|^{-\ell} P_{\ell,k}(\mathbf{w})$, we can similarly estimate the derivatives of the $Y_{\ell,k}$ factor:

$$\begin{aligned} \left| \partial^\alpha Y_{\ell,k} \left(\frac{\mathbf{w}}{|\mathbf{w}|} \right) \right| &\leq \sum_{|\boldsymbol{\gamma}| \leq |\boldsymbol{\alpha}|} \binom{|\boldsymbol{\alpha}|}{|\boldsymbol{\gamma}|} |\partial^{\boldsymbol{\gamma}} |\mathbf{w}|^{-\ell}| |\partial^{\boldsymbol{\alpha}-\boldsymbol{\gamma}} P_{\ell,k}(\mathbf{w})| \\ &\lesssim_{|\boldsymbol{\alpha}|} |\mathbf{w}|^{-\ell-|\boldsymbol{\alpha}|} \cdot \ell^{|\boldsymbol{\alpha}|} |\mathbf{w}|^\ell \|Y_{\ell,k}\|_{L^\infty(\mathbb{S}^{m+n-1})} \\ &\lesssim \ell^{|\boldsymbol{\alpha}|+(m+n-2)/2} |\mathbf{w}|^{-|\boldsymbol{\alpha}|}. \end{aligned}$$

The bound $\|Y_{\ell,k}\|_{L^\infty(\mathbb{S}^{m+n-1})} \lesssim \ell^{(m+n-2)/2}$ arises from the Sturm–Liouville construction of the basis of spherical harmonics from the Laplacian on $\mathbb{S}^{m+n-1}$. Together, the above inequalities, the asymptotic formula

$$\frac{\Gamma(x+r)}{\Gamma(x)} \sim x^r$$

for all real $x, r \geq 0$, and the Leibniz rule give

$$\begin{aligned} & |\partial^\alpha \widehat{g}_i(\mathbf{w})| \\ & \lesssim_{m,N,\beta,s} |\mathbf{w}|^{-\sigma-|\alpha|} \sum_{\ell=0}^{\infty} \sum_{k=1}^{d(\ell)} |a_{i,\ell,k}| \ell^{|\alpha|+(m+n-2)/2} |\mathbf{w}|^{-\ell} \left(\frac{\ell - \sigma + m + n}{2} \right)^{\sigma-(m+n-1)/2} \\ & \lesssim_{m,N} |\mathbf{w}|^{-\sigma-|\alpha|} \sum_{\ell=0}^{\infty} \sum_{k=1}^{d(\ell)} |a_{i,\ell,k}| \ell^{|\alpha|+\sigma-1/2} |\mathbf{w}|^{-\ell} \\ & \lesssim |\mathbf{w}|^{-\sigma-|\alpha|} \sum_{\ell=0}^{\infty} \sum_{k=1}^{d(\ell)} 2^{-in} |a_{\ell,k}| \ell^{|\alpha|+\sigma-1/2} |\mathbf{w}|^{-\ell} \\ & \lesssim 2^{i(|\alpha|-n)} |\mathbf{w}|^{-\sigma-|\alpha|}, \end{aligned}$$

where in the final line we have used the fact that the coefficients $|a_{\ell,k}|$ are rapidly-decaying and, consequently, are $\lesssim_{\alpha} \frac{1}{d(\ell)} \ell^{-|\alpha|-\sigma-1/2} \sim_{m,N} \ell^{-|\alpha|-\sigma-m-n+1/2}$. The factor $|\mathbf{w}|^{-\ell}$ can be eliminated independently of $\mathbf{w}$ bounded away from $\mathbf{0}$. In particular, evaluating at $\mathbf{w} = \Psi_{\lambda}(\omega) - \Psi_{\lambda}(\zeta)$ gives, by (3.9),

$$\begin{aligned} |\partial^\alpha \widehat{g}_i(\Psi_{\lambda}(\omega) - \Psi_{\lambda}(\zeta))| & \lesssim_{\beta,s,|\alpha|} 2^{i(|\alpha|-n)} (1 + d(\omega, \zeta)^{\beta+1})^{-\sigma-|\alpha|} \\ & \leq 2^{i(|\alpha|-n)} d(\omega, \zeta)^{-(1+\beta)(\sigma+|\alpha|)}. \end{aligned}$$

Recall that our intention was to establish bounds on the partial derivatives of γ_i . It is clear that, for every multi-index $\alpha \in \mathbb{N}^N$, the corresponding partial derivative $\partial_{\lambda}^{\alpha} \gamma_i(\lambda)$ is a finite sum of terms the form

$$\partial_{\mathbf{w}}^{\alpha_0} \widehat{g}_i(\mathbf{w}) \partial_{\lambda}^{\alpha_1} [\Psi_{1,\lambda}(\omega) - \Psi_{1,\lambda}(\zeta)] \cdots \partial_{\lambda}^{\alpha_{m+n}} [\Psi_{m+n,\lambda}(\omega) - \Psi_{m+n,\lambda}(\zeta)] \Big|_{\mathbf{w}=\Psi_{\lambda}(\omega)-\Psi_{\lambda}(\zeta)},$$

where $\Psi_{j,\lambda}$ is the j^{th} component of Ψ_{λ} , each $\alpha_j \in \mathbb{N}^{m+n}$ has length $|\alpha_j| \leq |\alpha|$, and (contrary to the usual convention) the j^{th} term is simply 1 when $\alpha_j = \mathbf{0}$. (See [38] Proposition 4.3.)

Therefore, to prove the desired estimates on the partial derivatives of γ_i , we need only to estimate the contributions of each of the $k + 1$ factors. Having treated the first factor already, we turn our attention to $\partial_{\lambda}^{\alpha_j} [\Psi_{j,\lambda}(\omega) - \Psi_{j,\lambda}(\zeta)]$. These follow readily from the regularity hypotheses on Π_{λ} : if $m + 1 \leq j \leq n$, then for some $1 \leq k \leq N$

$$\begin{aligned} |\partial_{\lambda}^{\alpha_j} [\Psi_{j,\lambda}(\omega) - \Psi_{j,\lambda}(\zeta)]| &= |\partial_{\lambda}^{\alpha_j} [d(\omega, \zeta) \partial_{\lambda_k} \Phi_{\lambda}(\omega, \zeta)]| \\ &= d(\omega, \zeta) |\partial_{\lambda}^{\alpha_j} \partial_{\lambda_k} \Phi_{\lambda}(\omega, \zeta)| \\ &\lesssim_{\beta, s, |\alpha|} d(\omega, \zeta) \cdot d(\omega, \zeta)^{-\beta(|\alpha_j|+1)} \\ &\lesssim d(\omega, \zeta)^{1-\beta(|\alpha|+1)}, \end{aligned}$$

and similarly when $1 \leq j \leq m$ except that $d(\omega, \zeta)^{1-\beta(|\alpha|+1)}$ is replaced by the “smaller” (up to a multiplicative constant depending on $|\Omega|$) factor $d(\omega, \zeta)^{1-\beta|\alpha|}$. There are $k = |\alpha_0|$ such factors, which together with the factor $\partial_{\mathbf{z}}^{\alpha_0} \widehat{g}_i(\mathbf{z})$ yields

$$\begin{aligned} |\partial_{\lambda}^{\alpha} \gamma_i(\lambda)| &\lesssim_{\beta, s, |\alpha|} 2^{i(|\alpha|-n)} d(\omega, \zeta)^{-(1+\beta)(\sigma+|\alpha_0|)} \cdot d(\omega, \zeta)^{|\alpha_0|(1-\beta(|\alpha|+1))} \\ &\lesssim_{|\alpha|} 2^{i(|\alpha|-n)} d(\omega, \zeta)^{-\sigma-\beta(|\alpha|^2+2|\alpha|+\sigma)}. \end{aligned}$$

On the other hand, by the basic uniform estimates on $\partial_{\lambda}^{\alpha} \Pi_{\lambda}$ in (3.4),

$$|\partial_{\lambda}^{\alpha_j} [\Psi_{j,\lambda}(\omega) - \Psi_{j,\lambda}(\zeta)]| \leq |\partial_{\lambda}^{\alpha} \partial_{\lambda_k} \Pi_{\lambda}(\omega)| + |\partial_{\lambda}^{\alpha} \partial_{\lambda_k} \Pi_{\lambda}(\zeta)| \lesssim_{K, |\alpha|, \beta} 1.$$

This gives rise to the bound

$$|\partial_{\lambda}^{\alpha} \gamma_i(\lambda)| \lesssim_{\beta, K, |\alpha|} 2^{i(|\alpha|-n)} d(\omega, \zeta)^{-(1+\beta)(\sigma+|\alpha|)},$$

which improves upon the previous when $\beta > 0$ and $|\alpha| + 1 > \beta^{-1}$. Thus, with $\tau = \tau_{\beta, s, |\alpha|} := \min \{ \sigma + \beta(|\alpha|^2 + 2|\alpha| + \sigma), (1 + \beta)(\sigma + |\alpha|) \}$,

$$|\partial_{\lambda}^{\alpha} \gamma_i(\lambda)| \lesssim_{\beta, K, s, |\alpha|} 2^{i(|\alpha|-n)} d(\omega, \zeta)^{-\tau}.$$

Recalling the definition of γ_i in (3.19), we may now differentiate (3.18) under the integral sign and use this inequality to conclude that

$$|\partial_{\lambda}^{\alpha} h_i(\lambda)| \lesssim \frac{2^{i(|\alpha|-n)}}{2^{in}} \iint_{\Omega \times \Omega} \frac{d(\mu \times \mu)(\omega, \zeta)}{d(\omega, \zeta)^{\tau}} = 2^{i(|\alpha|-2n)} I_{\tau}(\mu) \leq 2^{i|\alpha|} I_{\tau}(\mu)$$

for all $\lambda \in K$. The right-hand side is finite when $\tau \leq t$, which fact we shall use momentarily in deciding for which $L \in \mathbb{Z}_+$ Lemma 3.5 applies.

STEP 4. Having established the bounds in (3.10) on the functions h_i defined in (3.14), we at last apply Lemma 3.5 to conclude the proof. The case $\beta = 0$ does not quite follow from the case $\beta > 0$, and since the latter is slightly more involved, we begin with the former. Let

$$0 < q < s - m, \quad n - (s - m - q) < p < n, \quad A := 2, \quad R := 2^{p+n}, \quad \text{and} \quad r := 2^{2n+m-\sigma},$$

and let $L \in \mathbb{Z}_+$ be sufficiently large that

$$q + \left(\frac{q}{L} + 1\right)(2n + m - \sigma) \leq p.$$

We must verify that $A^q r^{q/L} \leq \frac{R}{r} \leq A^N r^{N/L}$. Since all three expressions are powers of 2, this is equivalent to

$$q + \left(\frac{q}{L} + 1\right)(2n + m - \sigma) \leq p + n \leq N + \left(\frac{N}{L} + 1\right)(2n + m - \sigma),$$

which is indeed consistent with the choices of parameters made above owing to the fact that $\sigma = s$:

$$\begin{aligned} q + \left(\frac{q}{L} + 1\right)(2n + m - \sigma) &= (n - (s - m - q)) + n + \frac{q}{L}(2n + m - s) \\ &\leq p + n = N + (p + n - N) \leq N + (2n - (s - m)) \\ &\leq N + \left(\frac{q}{L} + 1\right)(2n + m - \sigma). \end{aligned}$$

The constraints on t accumulated over the course of the proof are

$$\sigma \leq \frac{t}{1 + a_1 \beta} \quad \text{and} \quad \min \{ \sigma + \beta(L^2 + 2L + t), (1 + \beta)(\sigma + L) \} \leq t, \quad (3.21)$$

both of which hold trivially with $t = s$ when $\beta = 0$. Thus, our parameter choices satisfy the hypotheses of Lemma 3.5, so by the results of Step 1, (3.12) holds.

Now suppose instead that $0 < \beta < 1$ and let

$$q := s - m, \quad p := n - \frac{1}{2}\beta^{1/3}, \quad A := 2, \quad R := 2^{p+n}, \quad \text{and} \quad r := 2^{2n+m-\sigma}.$$

If $L \in [\frac{4nN}{\beta^{1/3}}, \frac{8nN}{\beta^{1/3}}] \cap \mathbb{Z}$, then the hypotheses of Lemma 3.5 are satisfied just as in the $\beta = 0$ case:

$$\begin{aligned}
q + \left(\frac{q}{L} + 1\right)(2n + m - \sigma) &< \left(n - \frac{1}{2}\beta^{1/3}\right) + n - \frac{1}{2}\beta^{1/3} + \frac{s-m}{L}(2n + m - s - \beta^{1/3}) \\
&< \left(n - \frac{1}{2}\beta^{1/3}\right) + n - \frac{1}{2}\beta^{1/3} + \frac{2nN}{L} \\
&\leq p + n \\
&= N + \left(2n - N - \frac{1}{2}\beta^{1/3}\right) \\
&\leq N + \left(\frac{N}{L} + 1\right)(2n + m - \sigma).
\end{aligned}$$

In addition, the first inequality in (3.21) is satisfied by any $t \geq s + b_0\beta^{1/3}$ with $b_0 \geq a_1N + a_1 + 1$, and the second inequality in (3.21) is satisfied by any $t \geq s + b_1\beta^{1/3}$ with $b_1 \geq 64n^2N^2 + 16nN + N + 2$. Thus, (3.12) holds in the $\beta > 0$ case with $a_2 := \max\{b_0, b_1\}$. $\square$

Replacing s with $s - a_2\beta^{1/3}$ —and increasing the constant by 1 for the sake of symmetry in the two error terms—gives the following equivalent statement.

Corollary 3.8. *There exists a constant $a_3 \geq 1$, depending only on N and m , such that the following holds. Let $(\Pi_{\lambda})_{\lambda \in U}$ be a family of generalized projections that is both transversal and strongly regular of order β . If $\mu \in \mathcal{M}(\Omega)$ is finite and has finite s -energy for some $m < s < N + m$, then*

$$\dim_{\mathbb{H}} \left\{ \lambda \in U : \int_{\mathbb{R}^m} I_{s-m-a_3\beta^{1/3}}(\mu_{\lambda, \mathbf{x}}) d\mathbf{x} = \infty \right\} \leq N + m - s + a_3\beta^{1/3}.$$

Although this statement is perhaps easier to read, a direct application of Lemma 3.7 is more convenient for the purposes of proving the main results of this chapter.

3.4 Proof of Theorem 3.2

3.4.1 The “weak” version

First we prove a version of Theorem 3.2 in which the set $E \subset U$ contains the exceptional parameters for a fixed $A \subseteq \Omega$ but not necessarily for all the positive- $\mathcal{H}^s$ -measure subsets of A . As a result, though, the $0 < \mathcal{H}^s(A) < \infty$ hypothesis can be relaxed to $\dim_{\mathbb{H}} A \geq s$ when $\beta > 0$.

Theorem 3.9. *There exists a constant $a \geq 1$, depending only on N and m , such that the following holds. Let $(\Pi_{\lambda}: \Omega \rightarrow \mathbb{R}^m)_{\lambda \in U}$ be a family of generalized projections that is both transversal and strongly regular of degree $\beta \in [0, 1)$. If $m < s < N + m$, then for every Borel set $A \subseteq \Omega$ with $\mathcal{H}^s(A) > 0$, there exists a set $F \subset U$ with*

$$\dim_{\mathbb{H}} F \leq N + m - s + a\beta^{1/3} \quad (3.22)$$

such that, for every $\lambda \in U \setminus F$, the inequality

$$\dim_{\mathbb{H}}(A \cap \Pi_{\lambda}^{-1}(\mathbf{x})) \geq (1 - \beta)(s - m)$$

holds for a positive-Lebesgue-measure set of $\mathbf{x} \in \mathbb{R}^m$. Moreover, if $\beta > 0$, the hypothesis $\mathcal{H}^s(A) > 0$ can be relaxed to $\dim_{\mathbb{H}} A \geq s$.

Proof. We begin with the case $\beta = 0$. With $A \subseteq \Omega$ as above, there exists an s -Frostman measure $\mu \in \mathcal{M}(A)$, so $I_{\sigma}(\mu) < \infty$ for all $\sigma < s$. Thus,

$$\dim_{\mathbb{H}} \{ \lambda \in U : \Pi_{\lambda\#} \mu \not\ll \mathcal{L}^m \} \leq N + m - \sup_{\sigma < s} \sigma = N + m - s$$

by Theorem 3.3, which together with Lemma 3.7 implies that

$$\underbrace{\dim_{\mathbb{H}} \left\{ \lambda \in U : \int_{\mathbb{R}^m} I_{\sigma-m}(\mu_{\lambda, \mathbf{x}}) d\mathbf{x} = \infty \ \forall s > \sigma > s - \varepsilon \text{ or } \Pi_{\lambda\#} \mu \not\ll \mathcal{L}^m \right\}}_{F_{\varepsilon}} \leq N + m - s + \varepsilon.$$

Now, by the Lebesgue differentiation theorem and the definition of a slice measure,

$$\mathcal{L}^m\left(\underbrace{\{\mathbf{x} \in \mathbb{R}^m : \mu_{\lambda, \mathbf{x}} \text{ is defined and } \mu_{\lambda, \mathbf{x}} \not\equiv 0\}}_{N_\lambda}\right) > 0 \quad (3.23)$$

for all $\lambda \in U$ with $\Pi_{\lambda\#}\mu \not\ll \mathcal{L}^m$. Whenever $\lambda \in U \setminus F_\varepsilon$ and $\mathbf{x} \in N_\lambda$, the inequality $I_{\sigma-m}(\mu_{\lambda, \mathbf{x}}) < \infty$ holds for all $s - \varepsilon < \sigma < s$; thus, for all $\lambda \in U \setminus F_\varepsilon$, it follows from the Hölder lower bound (3.9)—which is a *Lipschitz* bound when $\beta = 0$ —that

$$\begin{aligned} \dim_{\mathbb{H}}(A \cap \Pi_\lambda^{-1}(\mathbf{x})) &= \dim_{\mathbb{H}}(\Psi_\lambda(A) \cap \{\mathbf{z} \in \mathbb{R}^{m+n} : \mathbf{z}^{(1)} = \mathbf{x}\}) \\ &\geq \dim_{\mathbb{H}}(\text{spt } \mu_{\lambda, \mathbf{x}}) \geq \sup_{\sigma < s} \sigma - m = s - m \end{aligned}$$

for $\mathcal{L}^m$ -positively-many $\mathbf{x} \in \mathbb{R}^m$. Hence,

$$\mathcal{L}^m\left(\{\mathbf{x} \in \mathbb{R}^m : \dim_{\mathbb{H}}(A \cap \Pi_\lambda^{-1}(\mathbf{x})) \geq s - m\}\right) \geq \mathcal{L}^m(N_\lambda) > 0,$$

so λ does *not* belong to the exceptional set

$$F := \left\{ \lambda \in U : \mathcal{L}^m\left(\{\mathbf{x} \in \mathbb{R}^m : \dim_{\mathbb{H}}(A \cap \Pi_\lambda^{-1}(\mathbf{x})) \geq s - m\}\right) = 0 \right\};$$

that is, $F \subseteq F_\varepsilon$, whence $\dim_{\mathbb{H}} F \leq N + m - s + \varepsilon$ for all $\varepsilon > 0$.

Now let

$$a := \max\{(N + m)a_0, a_2\},$$

suppose $\beta > 0$, and suppose $\dim_{\mathbb{H}} A \geq s$ (but not necessarily $\mathcal{H}^s(A) > 0$) for some $m < s < N + m - a\beta^{1/3}$. (If $s \geq N + m - a\beta^{1/3}$, then (3.22) is trivial.) As in the $\beta = 0$ case, for any $\mu \in \mathcal{M}(A)$ with finite $(s - \beta^{1/3})$ -energy, the inequality

$$\underbrace{\dim_{\mathbb{H}} \left\{ \lambda \in U : \int_{\mathbb{R}^m} I_{s-m}(\mu_{\lambda, \mathbf{x}}) d\mathbf{x} = \infty \text{ or } \Pi_{\lambda\#}\mu \not\ll \mathcal{L}^m \right\}}_{\widetilde{F}} \leq N + m - s + a\beta^{1/3}$$

holds by Theorem 3.3, Lemma 3.7, and our choice of a . For all $\lambda \in U \setminus \widetilde{F}$, (3.23) remains valid as in the $\beta = 0$ case, and for these λ and all $\mathbf{x} \in N_\lambda$, the upper Hölder condition (3.9) on Ψ_λ entails that

$$\dim_{\mathbb{H}}(A \cap \Pi_\lambda^{-1}(\mathbf{x})) \geq (1 - \beta) \dim_{\mathbb{H}} \{\mathbf{y} \in \mathbb{R}^n : (\mathbf{x}, \mathbf{y}) \in \Psi_\lambda(A)\}$$

$$\begin{aligned}
&\geq (1 - \beta) \dim_{\mathbb{H}}(\text{spt } \mu_{\lambda, \mathbf{x}}) \\
&\geq (1 - \beta)(s - m).
\end{aligned}$$

This again implies that λ does not belong to the exceptional set

$$F := \left\{ \lambda \in U : \mathcal{L}^m \left(\left\{ \mathbf{x} \in \mathbb{R}^m : \dim_{\mathbb{H}}(A \cap \Pi_{\lambda}^{-1}(\mathbf{x})) \geq (1 - \beta)(s - m) \right\} \right) = 0 \right\},$$

so $F \subseteq \tilde{F}$ and $\dim_{\mathbb{H}} F \leq N + m - s + a\beta^{1/3}$. $\square$

3.4.2 The full version

At last we complete the proof of Theorem 3.2. The trick is identical to that of Falconer and Mattila [17]: rerun the proof of the “basic” version—in our case, Theorem 3.9—with a countable family of s -Frostman measures μ_j such that $\sum_{j=0}^{\infty} \mu_j$ is the restriction of $\mathcal{H}^s$ to A .

Proof of Theorem 3.2. Since $\mathcal{H}^s(A) < \infty$, the upper s -density

$$\Theta^{*s}(A, \omega) := \limsup_{r \rightarrow 0} \frac{\mathcal{H}^s(A \cap B(\omega, r))}{(2r)^s}$$

is less than or equal to 1 at $\mathcal{H}^s$ -a.e. $\omega \in A$. For each $r > 0$, the function $\omega \mapsto \mathcal{H}^s(A \cap B(\omega, r))$ is upper semicontinuous (and, hence, $\omega \mapsto \Theta^{*s}(A, \omega)$ is Borel measurable), so the set of all $\omega \in A$ with upper density at most 1 can be written as a countable union of (countable intersections of) Borel sets:

$$A' := \{\omega \in A : \Theta^{*s}(A, \omega) \leq 1\} = \bigcup_{j=1}^{\infty} \bigcap_{r \in \mathbb{Q}_+} \{\omega \in A : \mathcal{H}^s(A \cap B(\omega, r)) \leq jr^s\}.$$

Thus, with $A_1 := \{\omega \in A : \mathcal{H}^s(A \cap B(\omega, r)) \leq r^s \ \forall r \in \mathbb{Q}_+\}$,

$$A_j := \left\{ x \in A \setminus \bigcup_{i=1}^{j-1} A_i : \mathcal{H}^s(A \cap B(\omega, r)) \leq jr^s \ \forall r \in \mathbb{Q}_+ \right\} \quad \text{for all } j \geq 2,$$

and $\mu_j := \mathcal{H}^s \llcorner A_j$, we have

$$A' = \bigsqcup_{j=1}^{\infty} A_j \quad \text{and, consequently,} \quad \mathcal{H}^s \llcorner A' = \mathcal{H}^s \llcorner A = \sum_{j=0}^{\infty} \mu_j.$$

Moreover, each μ_j that is not identically 0 is s -Frostman (with coefficient j), so every μ_j has finite σ -energy for all $0 \leq \sigma < s$. As in the proof of Theorem 3.9,

$$\underbrace{\dim_{\mathbb{H}} \left\{ \lambda \in U : \int_{\mathbb{R}^m} I_{(1-\beta)(\sigma-m)}(\mu_{j,\lambda,\mathbf{x}}) d\mathbf{x} = \infty \text{ or } \Pi_{\lambda\#} \mu_j \not\ll \mathcal{H}^m \right\}}_{E_{\sigma,j}} \leq N + m - \sigma + a\beta^{1/3},$$

where $\mu_{j,\lambda,\mathbf{x}} := (\mu_j)_{\lambda,\mathbf{x}}$.

Now let $B \subseteq A$ be a Borel set with $\mathcal{H}^s(B) > 0$. Then there exists some $j \in \mathbb{Z}_+$ such that $\mu_j(B) > 0$. Continuing to follow the proof of Theorem 3.9 with $\mu_j \llcorner B$ in place of μ , one sees that, for all $\lambda \in U \setminus E_{\sigma,j}$,

$$\dim_{\mathbb{H}}(B \cap \Pi_{\lambda}^{-1}(\mathbf{x})) \geq \sigma - m - a\beta^{1/3}$$

for a positive- $\mathcal{L}^m$ -measure set of $\mathbf{x} \in \mathbb{R}^m$, namely, for all $\mathbf{x} \in \mathbb{R}^m$ such that $\mu_{j,\lambda,\mathbf{x}} \llcorner B \neq 0$ is defined and $I_{\sigma-m-a\beta^{1/3}}(\mu_{j,\lambda,\mathbf{x}})$ is finite. It follows that, for all Borel sets $B \subseteq A$ with $\mathcal{H}^s(B) > 0$,

$$\left\{ \lambda \in U : \mathcal{L}^m \left(\left\{ \mathbf{x} \in \mathbb{R}^m : \dim_{\mathbb{H}}(B \cap \Pi_{\lambda}^{-1}(\mathbf{x})) \geq \sigma - m - a\beta^{1/3} \right\} \right) > 0 \right\} \subseteq \bigcup_{j=1}^{\infty} E_{\sigma,j}.$$

In the usual way, taking the union over $\sigma = s - \frac{1}{k}$, $s - \frac{1}{k+1}$, $\dots$ and the intersection over all $k \in \mathbb{Z}_+$ gives

$$\bigcup_{\substack{B \subseteq A \text{ Borel} \\ \mathcal{H}^s(B) > 0}} \left\{ \lambda \in U : \mathcal{L}^m \left(\left\{ \mathbf{x} \in \mathbb{R}^m : \dim_{\mathbb{H}}(B \cap \Pi_{\lambda}^{-1}(\mathbf{x})) \geq s - m - a\beta^{1/3} \right\} \right) > 0 \right\} \subseteq \bigcup_{j=1}^{\infty} E_{s,j},$$

where $\dim_{\mathbb{H}} \bigcup_{j=1}^{\infty} E_{s,j} \leq N + m - s$ by countable stability. This concludes the proof. $\square$

3.5 An application to the Heisenberg group

Equip $\mathbb{R}^{2n}$ with the standard symplectic form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ and let $G_h(n, m) \subseteq \mathbf{Gr}(2n, m)$ be the *isotropic Grassmannian*—the collection of m -dimensional subspaces $V \subseteq \mathbb{R}^{2n}$ such that $\omega(V, V) = \{0\}$ (the *isotropic subspaces* of $\mathbb{R}^{2n}$). It is well-known that

$G_h(n, m)$ is a Riemannian submanifold of $\mathbf{Gr}(2n, m)$ of dimension $2nm - \frac{m(3m-1)}{2}$. By Hovila [25] Theorem 1.1, the family $(\pi_V)_{V \in G_h(n, m)}$ of orthogonal projections is transversal and strongly regular of degree 0.¹ (Hovila only claims that (3.7) holds through $|\alpha| = 2$, but it is clear that it also holds for all α . In fact, strong regularity is inherited directly from $(\pi_V)_{V \in \mathbf{Gr}(2n, m)}$, so the only novelty is that only the transversality condition (3.5) also holds.)

The n^{th} Heisenberg group $\mathbb{H}^n$ is a Lie group that, under a pointwise identification $\mathbb{H}^n \cong \mathbb{R}^{2n} \times \mathbb{R}$, carries the group operation

$$(\mathbf{z}, \tau) * (\mathbf{w}, \sigma) := (\mathbf{z} + \mathbf{w}, \tau + \sigma + 2\omega(\mathbf{z}, \mathbf{w})).$$

With the *Heisenberg metric*

$$d_H((\mathbf{z}, \tau), (\mathbf{w}, \sigma)) := \|(\mathbf{z}, \tau)^{-1} * (\mathbf{w}, \sigma)\|_H, \quad \|(\mathbf{z}, \tau)\|_H := (\|\mathbf{z}\|^4 + \tau^2)^{1/4},$$

$\mathbb{H}^n$ has Hausdorff dimension $2n + 2$. Roughly, this is because the Heisenberg metric d_H restricted to the $(2n + 1)^{\text{st}}$ coordinate axis is the square root of the Euclidean metric on $\mathbb{R}$, so the contributions to dimension of a set owing to its extent in this vertical direction are doubled.

Call a subgroup $\mathbb{V} := V \times \{0\} \subset \mathbb{H}^n$ with $V \in G_h(n, m)$ a *horizontal subgroup* and define the *horizontal projection* $\pi_{\mathbb{V}} : \mathbb{H}^n \rightarrow \mathbb{V}$ by $\pi_{\mathbb{V}}(\mathbf{z}, \tau) := (\pi_V(\mathbf{z}), 0)$. Since $d_H|_{\mathbb{V}}$ is precisely the Euclidean metric on $\mathbb{V}$, such a subgroup has Hausdorff dimension m , whereas the fibers of $\pi_{\mathbb{V}}$ have dimension $2n - m + 2$ rather than $2n - m$. The isotropic Grassmannian parametrizes the family of horizontal projections in $\mathbb{H}^n$ via $V \mapsto \pi_{\mathbb{V}}$, so Hovila's transversality result allows for an immediate application of Theorem 3.2 to horizontal projections in the Heisenberg group:

¹The author thanks Tuomas Orponen for drawing his attention to this result.

Corollary 3.10. *For every Borel set $A \subseteq \mathbb{R}^{2n}$ with $0 < \mathcal{H}^s(A) < \infty$ for some $s > m$,*

$$\begin{aligned} \dim_{\mathbb{H}} \bigcup_{\substack{B \subseteq A \text{ Borel} \\ \mathcal{H}^s(B) > 0}} \left\{ V \in G_h(n, m) : \mathcal{H}^m(\{\mathbf{x} \in V : \dim_{\mathbb{H}}(\pi_V^{-1}(\mathbf{x}) \cap B) = s - m\}) = 0 \right\} \\ \leq 2nm - \frac{m(3m+1)}{2} - s. \end{aligned}$$

Consequently, if $A' \subseteq \mathbb{H}^n$ is Borel with $0 < \mathcal{H}^s(A') < \infty$ for some $s > m + 2$, then

$$\begin{aligned} \dim_{\mathbb{H}} \bigcup_{\substack{B' \subseteq A' \text{ Borel} \\ \mathcal{H}^s(B') > 0}} \left\{ V \in G_h(n, m) : \mathcal{H}^m(\{\mathbf{p} \in V : \dim_{\mathbb{H}}(\pi_V^{-1}(\mathbf{p}) \cap B') = s - m\}) = 0 \right\} \\ \leq 2nm - \frac{3m(m-1)}{2} - s + 2. \end{aligned}$$

Theorem 3.2 states that the set

$$\{\mathbf{x} \in V : \dim_{\mathbb{H}}(\pi_V^{-1}(\mathbf{x}) \cap B) \geq s - m\}$$

has positive $\mathcal{H}^m$ -measure for all but a few $V \in G_h(n, m)$. However, the set

$$\{\mathbf{x} \in V : \dim_{\mathbb{H}}(\pi_V^{-1}(\mathbf{x}) \cap B) > s - m\}$$

never has positive $\mathcal{H}^m$ -measure (see the discussion following Theorem 3.1), for which reason we are guaranteed that, generically, $\dim_{\mathbb{H}}(\pi_V^{-1}(\mathbf{x}) \cap B)$ is *exactly* $s - m$ for a positive-measure set of $\mathbf{x} \in V$.

Since $\pi_{\mathbb{V}}^{-1}(\mathbf{p}) = \mathbb{V}^{\perp} * \mathbf{p}$, Corollary 3.10 is a statement about the Hausdorff dimension of slices by right cosets of the *vertical subgroups* of the Heisenberg group. This improves on [3] Theorem 1.5 concerning slices in the Heisenberg group.

Appendix A

PROOF OF PROPOSITION 2.2

This section details the proof of Proposition 2.2 from Theorem 2.1. The differences in notation compared to [42] are fairly minor, to which the reader is referred for complete statements of the relevant definitions. In summary:

- $\rho_{\mathbf{e}}: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is the orthogonal projection onto $(\text{span } \mathbf{e})^\perp \cong \mathbb{R}^{n-1}$.
- Given $N \in \mathbb{Z}_+$, $\Sigma := \{1, \dots, N\}^{\mathbb{N}}$ denotes our symbol space, and a typical element of Σ is written as $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots)$. We also denote $\boldsymbol{\omega}^m := (\omega_1, \dots, \omega_m)$.
- Given an IFS $(g_i)_{i=1}^N$ on $\mathbb{R}^n$, $\Pi: \Sigma \rightarrow \mathbb{R}^{n-1}$ indicates the projection map

$$\Pi(\boldsymbol{\omega}) := \lim_{m \rightarrow \infty} g_{\boldsymbol{\omega}^m}(\mathbf{0}),$$

where $g_{\boldsymbol{\omega}^m} := g_{\omega_1} \circ \dots \circ g_{\omega_m}$. This limit always exists by the Cantor intersection theorem. When working with a parametrized family $(f_i(\cdot; \mathbf{t}))_{i=1}^N$, the projections $\Pi_{\mathbf{t}}$ are distinguished with a subscript $\mathbf{t}$.

Lastly, we restate Definition 4.4 of [42]. A smoothly parametrized family $(f_i(\cdot; \mathbf{t}))_{i=1}^N$ as above is said to be *transverse* (or to satisfy the *transversality condition*) if there exists a constant $L > 0$ with the following property: for each parameter value $\mathbf{u}$ and each pair $\boldsymbol{\omega}, \boldsymbol{\zeta} \in \Sigma$ with $\omega_1 \neq \zeta_1$,

$$|\Pi_{\mathbf{u}}(\boldsymbol{\omega}) - \Pi_{\mathbf{u}}(\boldsymbol{\zeta})| < L \implies \left| \det D_{\mathbf{t}}(\Pi_{\mathbf{t}}(\boldsymbol{\omega}) - \Pi_{\mathbf{t}}(\boldsymbol{\zeta})) \Big|_{\mathbf{t}=\mathbf{u}} \right| > L. \quad (\text{A.1})$$

Here, $D_{\mathbf{t}}$ denotes the total derivative with respect to $\mathbf{t}$. Loosely, this definition states that, whenever the points $\Pi_{\mathbf{t}}(\boldsymbol{\omega}), \Pi_{\mathbf{t}}(\boldsymbol{\zeta})$ are close for some $\mathbf{t} = \mathbf{u}$, they do not remain close for long as the parameter $\mathbf{t}$ changes. The condition $\omega_1 \neq \zeta_1$ suggests that $\Pi_{\mathbf{t}}(\boldsymbol{\omega})$ and $\Pi_{\mathbf{t}}(\boldsymbol{\zeta})$ “should” land in different regions of the attractor $K_{\mathbf{t}}$ for any value of $\mathbf{t}$, although these regions may overlap for values of $\mathbf{t}$ at which the IFS is degenerate.

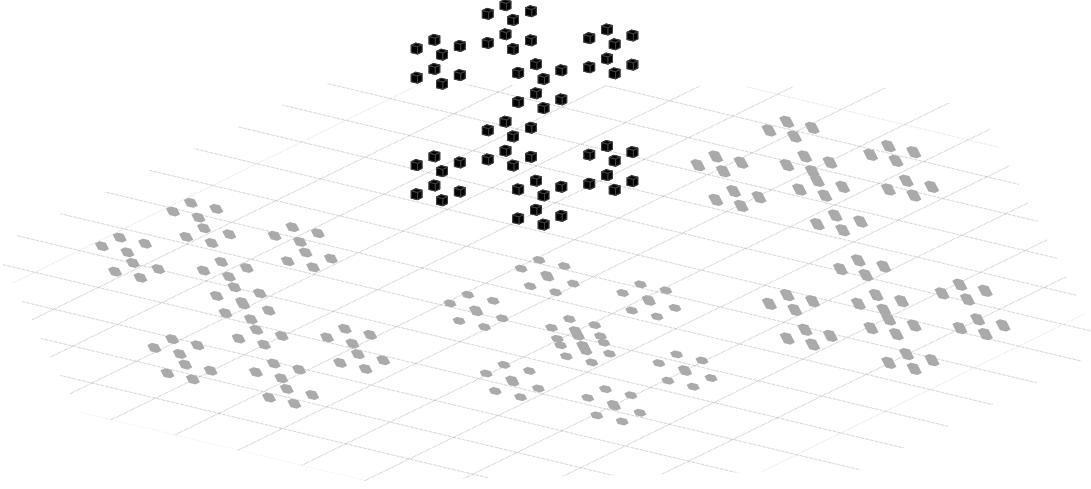


Figure A.1: Projecting the attractor of an IFS (in black) onto the hyperplane $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ from various angles gives rise to a parametrized family of self-similar sets (in gray) in an ambient space of one lower dimension. For the purposes of verifying transversality, one must exploit the SSC, for which reason one parametrizes the corresponding IFS.

Proof of Proposition 2.2. It suffices to work in local coordinates, so we let $V \subset \mathbb{S}^{n-1}$ be an open set whose closure $\overline{V}$ is diffeomorphic to a bounded subset of $\mathbb{R}^{n-1}$. These local coordinates also afford us a consistent identification of the tangent hyperplanes to $\mathbb{S}^{n-1}$ with $\mathbb{R}^{n-1}$. Dispensing with these technicalities, we simply refer to our parameter space as $\mathbb{S}^{n-1}$ and use the formula $\rho_{\mathbf{e}}(\mathbf{x}) = \mathbf{x} - (\mathbf{e} \cdot \mathbf{x})\mathbf{e}$ for the orthogonal projections.

STEP 1. Let $(g_i)_{i=1}^N$ be an IFS on $\mathbb{R}^n$ with limit set K and satisfying the SSC. We seek to produce a smooth family of IFS on $\mathbb{R}^{n-1}$ to which we can apply Theorem 1.1 of [42].

To this end, we define $(f_i(\cdot; \mathbf{e}))_{i=1}^N$ by

$$f_i(\boldsymbol{\xi}; \mathbf{e}) := (\rho_{\mathbf{e}} \circ g_i)(\rho_{\mathbf{e}}^{-1}(\boldsymbol{\xi}))$$

for each $\mathbf{e} \in \mathbb{S}^{n-1}$, where $\rho_{\mathbf{e}}^{-1}(\boldsymbol{\xi})$ is any preimage of the point $\boldsymbol{\xi} \in \mathbb{R}^{n-1}$. This definition is unambiguous because g_i takes the form $g_i(\mathbf{x}) = a\mathbf{x} + \mathbf{b}$ for some $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^n$,

whence

$$(\rho_{\mathbf{e}} \circ g_i)(\rho_{\mathbf{e}}^{-1}(\boldsymbol{\xi})) = \rho_{\mathbf{e}}(a\rho_{\mathbf{e}}^{-1}(\boldsymbol{\xi}) + \mathbf{b}) = a\rho_{\mathbf{e}}(\rho_{\mathbf{e}}^{-1}(\boldsymbol{\xi})) + \rho_{\mathbf{e}}(\mathbf{b}) = a\boldsymbol{\xi} + \rho_{\mathbf{e}}(\mathbf{b})$$

for any choice of $\rho_{\mathbf{e}}^{-1}(\boldsymbol{\xi})$. This also shows that $(f_i(\cdot; \mathbf{e}))_{i=1}^N$ is smooth in both $\mathbf{x}$ and $\mathbf{e}$, as $\rho_{\mathbf{e}}(\mathbf{b}) = \mathbf{b} - (\mathbf{e} \cdot \mathbf{b})\mathbf{e}$.

STEP 2. We show that $(f_i(\cdot; \mathbf{e}))_{i=1}^N$ is a transverse family. Let $\boldsymbol{\omega}, \boldsymbol{\zeta} \in \Sigma$, where $\omega_1 \neq \zeta_1$, and let $f_{\boldsymbol{\omega}^m}(\boldsymbol{\xi}; \mathbf{e})$ denote the composite map

$$f_{\omega_1}(\cdot; \mathbf{e}) \circ \cdots \circ f_{\omega_m}(\cdot; \mathbf{e})$$

evaluated at $\boldsymbol{\xi}$. Then

$$\begin{aligned} f_{\boldsymbol{\omega}^m}(\boldsymbol{\xi}; \mathbf{e}) &= ((\rho_{\mathbf{e}} \circ g_{\omega_1} \circ \rho_{\mathbf{e}}^{-1}) \circ \cdots \circ (\rho_{\mathbf{e}} \circ g_{\omega_m} \circ \rho_{\mathbf{e}}^{-1}))(\boldsymbol{\xi}) \\ &= (\rho_{\mathbf{e}} \circ (g_{\omega_1} \circ \cdots \circ g_{\omega_m}) \circ \rho_{\mathbf{e}}^{-1})(\boldsymbol{\xi}) \\ &= (\rho_{\mathbf{e}} \circ g_{\boldsymbol{\omega}^m} \circ \rho_{\mathbf{e}}^{-1})(\boldsymbol{\xi}) \end{aligned}$$

for any section $\rho_{\mathbf{e}}^{-1}$ of $\rho_{\mathbf{e}}$. Therefore, by the continuity of $\rho_{\mathbf{e}}$,

$$f_{\boldsymbol{\omega}}(\boldsymbol{\xi}; \mathbf{e}) := \lim_{m \rightarrow \infty} f_{\boldsymbol{\omega}^m}(\boldsymbol{\xi}; \mathbf{e}) = \rho_{\mathbf{e}}\left(\lim_{m \rightarrow \infty} g_{\boldsymbol{\omega}^m}(\rho_{\mathbf{e}}^{-1}(\boldsymbol{\xi}))\right) = (\rho_{\mathbf{e}} \circ g_{\boldsymbol{\omega}})(\rho_{\mathbf{e}}^{-1}(\boldsymbol{\xi})).$$

In particular, we can take $\rho_{\mathbf{e}}^{-1}(\mathbf{0}) = \mathbf{0}$, so

$$\Pi_{\mathbf{e}}(\boldsymbol{\omega}) = f_{\boldsymbol{\omega}}(\mathbf{0}; \mathbf{e}) = (\rho_{\mathbf{e}} \circ g_{\boldsymbol{\omega}})(\mathbf{0}) = \rho_{\mathbf{e}}(\Pi(\boldsymbol{\omega}));$$

likewise for $\boldsymbol{\zeta}$.

Denote $\mathbf{z} = (z_1, \dots, z_n) = \Pi(\boldsymbol{\omega}) - \Pi(\boldsymbol{\zeta})$, and suppose

$$|\rho_{\mathbf{u}}(\mathbf{z})| = |\rho_{\mathbf{u}}(\Pi(\boldsymbol{\omega})) - \rho_{\mathbf{u}}(\Pi(\boldsymbol{\zeta}))| = |\Pi_{\mathbf{u}}(\boldsymbol{\omega}) - \Pi_{\mathbf{u}}(\boldsymbol{\zeta})| < \frac{c}{\sqrt{2}} \quad (\text{A.2})$$

for some $\mathbf{u} \in \mathbb{S}^{n-1}$, where $c \in (0, 1]$ is a constant such that $\text{dist}(g_i(K), g_j(K)) > c$ for all $i \neq j$. Such an c exists because $(g_i)_{i=1}^N$ satisfies the SSC. Since $\Pi(\boldsymbol{\omega}) \in g_{\omega_1}(K)$, $\Pi(\boldsymbol{\zeta}) \in g_{\zeta_1}(K)$, and $\omega_1 \neq \zeta_1$, it follows that $|\mathbf{z}| > c$ —a fact we shall use shortly.

To show transversality, we must compute

$$\det D_{\mathbf{e}}(\rho_{\mathbf{e}}(\mathbf{z}))|_{\mathbf{e}=\mathbf{u}}.$$

The determinant is invariant under a linear change of coordinates, so we can rotate our coordinate system so that $\mathbf{u} = \mathbf{e}_n = (0, \dots, 0, 1)$. Consider $h: \mathbf{e} \mapsto \rho_{\mathbf{e}}(\mathbf{z})$ as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e., by extending $\rho_{\mathbf{e}}(\mathbf{z})$ to take parameter values in $\mathbb{R}^n$. Considered as an $n \times n$ matrix, the j^{th} column of the derivative $D_{\mathbf{e}}h(\mathbf{e})|_{\mathbf{e}=\mathbf{e}_n}$ is given by the directional derivative

$$\begin{aligned} \frac{d}{dr} \rho_{\mathbf{e}_n + r\mathbf{e}_j}(\mathbf{z})|_{r=0} &= \frac{d}{dr} (\mathbf{z} - (\mathbf{z} \cdot (\mathbf{e}_n + r\mathbf{e}_j))(\mathbf{e}_n + r\mathbf{e}_j))|_{r=0} \\ &= -z_j(\mathbf{e}_n + r\mathbf{e}_j) - (z_n + rz_j)\mathbf{e}_j|_{r=0} = -z_j\mathbf{e}_n - z_n\mathbf{e}_j, \end{aligned}$$

yielding

$$D_{\mathbf{e}}h(\mathbf{e})|_{\mathbf{e}=\mathbf{e}_n} = \begin{pmatrix} -z_n & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -z_n & 0 \\ -z_1 & -z_2 & \cdots & -2z_n \end{pmatrix}.$$

Since $D_{\mathbf{e}}h(\mathbf{e})|_{\mathbf{e}=\mathbf{e}_n}$ restricts to an automorphism of the tangent plane $(\text{span } \mathbf{e}_n)^\perp \cong T_{\mathbf{e}_n}\mathbb{S}^{n-1}$, and since the standard coordinate frame $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is adapted to $\mathbb{S}^{n-1}$ at the north pole $\mathbf{e}_n$, the matrix of this restricted linear map is obtained simply by omitting the n^{th} row and n^{th} column of the matrix. That is, $D_{\mathbf{e}}(\rho_{\mathbf{e}}(\mathbf{z}))|_{\mathbf{e}=\mathbf{e}_n} = -z_n I_{n-1}$ and, consequently,

$$\det D_{\mathbf{e}}(\rho_{\mathbf{e}}(\mathbf{z}))|_{\mathbf{e}=\mathbf{e}_n} = \det(-z_n I_{n-1}) = (-z_n)^{n-1}.$$

Now, since $\mathbf{z} = \rho_{\mathbf{e}_n}(\mathbf{z}) + z_n \mathbf{e}_n$, $|\rho_{\mathbf{e}_n}(\mathbf{z})|^2 < 2^{-1}c^2$, and $|\mathbf{z}|^2 > c^2$, it must be that $|z_n|^2 = |z_n \mathbf{e}_n|^2 > 2^{-1}c^2$ and, in turn,

$$\left\| D_{\mathbf{e}}(\rho_{\mathbf{e}}(\mathbf{z}))|_{\mathbf{e}=\mathbf{e}_n} \right\| = |z_n|^{n-1} > \frac{c^{n-1}}{2^{(n-1)/2}} \geq \frac{c}{\sqrt{2}}.$$

In view of Equation (A.2), we conclude that, whenever $\omega_1 \neq \zeta_1$,

$$|\Pi_{\mathbf{u}}(\omega) - \Pi_{\mathbf{u}}(\zeta)| < \frac{c}{\sqrt{2}}$$

$$\implies \left| \det D_{\mathbf{e}}(\Pi_{\mathbf{e}}(\boldsymbol{\omega}) - \Pi_{\mathbf{e}}(\boldsymbol{\zeta}))|_{\mathbf{e}=\mathbf{u}} \right| = \left\| D_{\mathbf{e}}(\rho_{\mathbf{e}}(\mathbf{z}))|_{\mathbf{e}=\mathbf{u}} \right\| > \frac{c}{\sqrt{2}},$$

so $(f_i(\cdot; \mathbf{e}))_{i=1}^N$ satisfies the transversality condition.

STEP 3. We apply Theorem 2.1 to get

$$\dim_{\mathbb{P}} \{ \mathbf{e} \in \mathbb{S}^{n-1} : \dim_{\mathbb{H}} K_{\mathbf{e}} \leq s \} \leq s \quad \forall 0 \leq s < \min \left\{ n-1, \sup_{\mathbf{e} \in \mathbb{S}^{n-1}} \sigma(\mathbf{e}) \right\}.$$

If $\dim_{\mathbb{H}} K \geq n-1$, then $\sup_{\mathbf{e} \in \mathbb{S}^{n-1}} \sigma(\mathbf{e}) = n-1$ and the desired conclusion holds for all $0 \leq s < \dim_{\mathbb{H}} K$, the values of s in the interval $(n-1, \dim_{\mathbb{H}} K)$ giving a trivial bound. If instead $\dim_{\mathbb{H}} K < n-1$, then $\sup_{\mathbf{e} \in \mathbb{S}^{n-1}} \sigma(\mathbf{e}) = \dim_{\mathbb{H}} K$, and again the bound on the exceptional set holds for all $0 \leq s < \dim_{\mathbb{H}} K$. $\square$

Appendix B

PROOF OF LEMMA 3.6

Before beginning the proof of Lemma 3.6, let us take a moment to understand why the steps that follow are necessary. The analogue of Lemma 3.6 in Peres–Schlag [40] is Lemma 7.10, in which the dependence on $D_{\lambda}\Phi_{\lambda}(\omega, \zeta)$ is not present. This alone could be rectified by assuming higher-order regularity on $(\Pi_{\lambda})_{\lambda \in U}$, and for us virtually no modification would be necessary since we assume that the family is C^{∞} in λ . The factor of 2^{j-i} is the more serious issue, since the dependence on i is not present in [40] and the right-hand side of (3.11) also does not depend on i . Reworking the computation of Lemma 7.10 to verify that the resulting estimate does not depend on i is all that needs to be done, and although one finds that the only necessary changes are essentially notational, we follow Orponen’s example ([38] Lemma 4.2) and write out the details here in the interest of completeness.

First we require a somewhat cumbersome statement about the distribution of critical points of $\Phi_{\lambda}(\omega, \zeta)$ for fixed, distinct $\omega, \zeta \in \Omega$. It takes the form of an implicit function theorem that, leveraging transversality and regularity, establishes quantitative bounds on the size and number of balls in U required for the implicit functions and their inverses to be bi-Hölder on each ball with the same Hölder constant across balls.

Peres and Schlag phrase the result in terms of what they call L, δ -regularity; the terminology suppresses the dependence on β , so we refer to this as L, δ, β -regularity for clarity. If $(\Pi_{\lambda})_{\lambda \in U}$ is ∞, δ, β -regular for some $\delta \in [0, 1)$, then it is ∞, δ, β -regular for *all* $\delta \in [0, 1)$. Rather than transcribe [40] Lemma 7.7 verbatim, we avoid L, δ, β -regularity and state the conclusion with $\delta = 1$. Peres and Schlag do not define $L, 1, \beta$ -regularity (doing so would not improve the statements of their main results), but their proof of Lemma 7.7 works

with no modifications when $\delta = 1$ (see also the proof of [38] Lemma A.1). We therefore state their result with $\delta = 1$ as Lemma B.1 below and refer the reader to [40] for the proof.

As in previous sections, we assume throughout this appendix that $(\Pi_\lambda)_{\lambda \in U}$ is transversal and regular of order $\beta \in [0, 1)$. All implicit constants are assumed to depend on this family. Without loss of generality, each compact set $K \subset U$ referred to below is assumed to be the closure of its interior. The symbol $\delta = \delta_{\beta, K} > 0$ denotes the implicit constant in (3.5) of Definition 3.1, i.e., a number such that

$$|\Phi_\lambda(\omega, \zeta)| \leq \delta d(\omega, \zeta)^\beta \implies \det \left[D_\lambda \Phi_\lambda(\omega, \zeta) (D_\lambda \Phi_\lambda(\omega, \zeta))^\mathbf{t} \right] \geq \delta^2 d(\omega, \zeta)^{2\beta}.$$

In all that follows, $\omega, \zeta \in \Omega$ are fixed and $r := d(\omega, \zeta)$.

Finally, we ask the reader's forgiveness for our choices of notation in this appendix. Symbols will frequently be recycled without mention, but we hope that the intended meanings are always clear from context.

Lemma B.1 (Peres–Schlag [40] Lemma 7.7). *Let $K \subseteq U$ be a compact set. There exist constants $C_0, C_1, C_2 > 0$ depending only on K such that, for all distinct $\omega, \zeta \in \Omega$, there exist $\lambda_1, \dots, \lambda_M \in U$ with $M \leq C_2 r^{-(m+2)N}$ such that*

$$\begin{aligned} \{\lambda \in K : |\Phi_\lambda(\omega, \zeta)| \leq C_0 r^{(m+2)\beta}\} &\subseteq \bigcup_{k=1}^M B(\lambda_k, C_1 r^{(m+2)\beta}) \quad \text{and} \\ \bigcup_{k=1}^M \underbrace{B(\lambda_k, 2C_1 r^{(m+2)\beta})}_{B_k} &\subseteq \left\{ \lambda \in U : |\Phi_\lambda(\omega, \zeta)| \leq \delta r^\beta \right\}. \end{aligned}$$

Moreover, for each $k \in \{1, \dots, M\}$ there exist coordinate directions $1 \leq i_1 \leq \dots \leq i_{N-m} \leq N$ such that, for all $\kappa \in \mathbb{R}^{N-m}$, the mapping F_κ defined by restricting $\lambda \mapsto \Phi_\lambda(\omega, \zeta)$ to the set $\{\lambda \in B_k : \lambda_{i_1} = \kappa_1, \dots, \lambda_{i_{N-m}} = \kappa_{N-m}\}$ is a diffeomorphism onto its image that satisfies

$$\left| \det \left[(DF_\kappa)^{-1} \right] \right| \leq C_2 r^{-\beta} \quad \text{and} \quad \left\| (DF_\kappa)^{-1} \right\| \leq C_2 r^{-m\beta}. \quad (\text{B.1})$$

Denote $\Phi_\lambda := \Phi_\lambda(\omega, \zeta)$ and recall what we intend to show:

Lemma (Lemma 3.6, Restated). *There exists a constant $a_1 \geq 1$ (depending only on N and m) such that, for all $\rho \in C_c^\infty(U)$, all $c \geq 0$, and all $\omega, \zeta \in \Omega$,*

$$\left| \int_U \rho(\lambda) \widehat{\eta^{(1)}}(2^j r \Phi_\lambda(\omega, \zeta)) \widehat{\eta^{(2)}}(2^{j-i} r D_\lambda \Phi_\lambda(\omega, \zeta)) d\lambda \right| \lesssim_{\rho, c} (1 + 2^j r^{1+a_1\beta})^{-c}.$$

As before, η is a smooth bump function compactly supported in $(0, \infty)$. The support of ρ will be denoted K , and we assume that $\rho \not\equiv 0$.

Proof of Lemma 3.6. Let

$$\phi(\mathbf{x}) := \widehat{\eta^{(1)}}(\mathbf{x}), \quad \psi(\mathbf{y}) := \widehat{\eta^{(2)}}(\mathbf{y}), \quad \text{and} \quad R(\lambda) := \rho(\lambda) \psi(2^{j-i} r D_\lambda \Phi_\lambda).$$

Suppressing the dependence of implicit constants on ρ , we write the desired inequality as

$$\left| \int_K R(\lambda) \phi(2^j r \Phi_\lambda) d\lambda \right| \lesssim_c (1 + 2^j r^{1+a_1\beta})^{-c}.$$

If $2^j r^{1+a'_1\beta} \leq 1$, where $a'_1 \geq 1$ is fixed but yet-to-be-specified, then this is immediate: both ϕ and ψ are bounded, so

$$\begin{aligned} \left| \int_K R(\lambda) \phi(2^j r \Phi_\lambda) d\lambda \right| &\leq \|\rho\|_{L^1(U)} \|\psi\|_{L^\infty(\mathbb{R}^n)} \|\phi\|_{L^\infty(\mathbb{R}^m)} \\ &\lesssim 1 \sim_c 2^{-c} \leq (1 + 2^j r^{1+a_1\beta})^{-c}. \end{aligned}$$

Assume, therefore, that $2^j r^{1+a'_1\beta} > 1$ and let $f \in C^\infty(\mathbb{R})$ be a bump function for $[-1, 1]$ supported in $[-2, 2]$. Then

$$\begin{aligned} &\int_K R(\lambda) \phi(2^j r \Phi_\lambda) d\lambda \\ &= \int_K R(\lambda) \phi(2^j r \Phi_\lambda) f\left(\frac{1}{\delta r^\beta} \Phi_\lambda\right) d\lambda \end{aligned} \tag{B.2}$$

$$+ \int_K R(\lambda) \phi(2^j r \Phi_\lambda) \left[1 - f\left(\frac{1}{\delta r^\beta} \Phi_\lambda\right)\right] d\lambda, \tag{B.3}$$

so we can bound (B.2) and (B.3) individually. The second integral is straightforward to handle: $|\Phi_\lambda| \geq \delta r^\beta$ on the support of the integrand by the definition of f , and since ϕ is a Schwartz function,

$$|\phi(2^j r \Phi_\lambda)| \lesssim_c (1 + \delta 2^j r \Phi_\lambda)^{-c} \leq (1 + \delta 2^j r^{1+\beta})^{-c} \lesssim_{|\Omega|, \beta, c} (1 + 2^j r^{1+a_1\beta})^{-c}.$$

Thus,

$$\begin{aligned} \int_K R(\boldsymbol{\lambda}) \phi(2^j r \Phi_{\boldsymbol{\lambda}}) [1 - f(\frac{1}{\delta r^\beta} \Phi_{\boldsymbol{\lambda}})] d\boldsymbol{\lambda} &\lesssim \int_K R(\boldsymbol{\lambda}) (1 + 2^j r^{1+a_1\beta})^{-c} d\boldsymbol{\lambda} \\ &\leq \|\rho\|_{L^1(U)} \|\psi\|_{L^\infty(\mathbb{R}^n)} (1 + 2^j r^{1+a_1\beta})^{-c} \sim_c (1 + 2^j r^{1+a_1\beta})^{-c}. \end{aligned} \quad (\text{B.4})$$

Our goal, then, is to bound (B.2) by this same expression. Recalling Lemma B.1, let $(\chi_k)_{k=1}^M$ be a smooth partition of unity for $\{\boldsymbol{\lambda} \in K : |\Phi_{\boldsymbol{\lambda}}| \leq C_0 r^{(m+2)\beta}\}$ subordinate to $(B_k)_{k=1}^M$. By the mean value theorem, we may choose these χ_k so that

$$\max_{1 \leq k \leq M} \|\partial^\alpha \chi_k\|_{L^\infty(U)} \lesssim_{|\alpha|} r^{-|\alpha|(m+2)\beta}, \quad (\text{B.5})$$

as each B_k has radius $\sim r^{(m+2)\beta}$. Since $f(\frac{1}{\delta r^\beta} \Phi_{\boldsymbol{\lambda}}) = 1$ for all $\boldsymbol{\lambda} \in \text{supp } \chi_k$ and all $1 \leq k \leq M$,

$$\int_K R(\boldsymbol{\lambda}) \phi(2^j r \Phi_{\boldsymbol{\lambda}}) f(\frac{1}{\delta r^\beta} \Phi_{\boldsymbol{\lambda}}) d\boldsymbol{\lambda} = \sum_{k=1}^M \int_K R(\boldsymbol{\lambda}) \phi(2^j r \Phi_{\boldsymbol{\lambda}}) \chi_k(\boldsymbol{\lambda}) d\boldsymbol{\lambda} \quad (\text{B.6})$$

$$+ \int_K R(\boldsymbol{\lambda}) \phi(2^j r \Phi_{\boldsymbol{\lambda}}) \left[1 - \sum_{k=1}^M \chi_k(\boldsymbol{\lambda}) \right] f(\frac{1}{\delta r^\beta} \Phi_{\boldsymbol{\lambda}}) d\boldsymbol{\lambda}. \quad (\text{B.7})$$

The second line admits a simple estimate in the same manner as (B.3): the inequality $|\Phi_{\boldsymbol{\lambda}}| \geq C_0 r^{(m+2)\beta}$ holds on the support of the integrand in (B.7) by Lemma B.1, so by the same argument as that used to derive (B.4) (with $C_0 r^{(m+2)\beta}$ in place of δr^β),

$$\int_K R(\boldsymbol{\lambda}) \phi(2^j r \Phi_{\boldsymbol{\lambda}}) \left[1 - \sum_{k=1}^M \chi_k(\boldsymbol{\lambda}) \right] f(\frac{1}{\delta r^\beta} \Phi_{\boldsymbol{\lambda}}) d\boldsymbol{\lambda} \sim_c (1 + 2^j r^{1+a_0\beta})^{-c}.$$

It therefore remains to bound (B.6). Fix $1 \leq k \leq M$ and recall from Lemma B.1 that, for all $\boldsymbol{\lambda}'' \in \mathbb{R}^{N-m}$, the map $\boldsymbol{\lambda}' \mapsto F_{\boldsymbol{\lambda}''}(\boldsymbol{\lambda}') := \Phi_{(\boldsymbol{\lambda}', \boldsymbol{\lambda}'')}$ is a diffeomorphism of $\{\boldsymbol{\lambda}' \in \mathbb{R}^m : (\boldsymbol{\lambda}', \boldsymbol{\lambda}'') \in B_k\}$ onto its image B'_k , possibly after a permutation of the coordinate axes. Henceforth denoting $\boldsymbol{\lambda} =: (\boldsymbol{\lambda}', \boldsymbol{\lambda}'')$, we assume that our smooth function ρ has the form $\rho(\boldsymbol{\lambda}) = \rho_1(\boldsymbol{\lambda}') \rho_2(\boldsymbol{\lambda}'')$; since functions of this form constitute a basis for both $L^1(K)$ and $C_c(K)$, this poses no loss of generality. With $1 \leq k \leq M$ fixed, let

$$g_{\boldsymbol{\lambda}''}(\mathbf{u}) := \rho_1(F_{\boldsymbol{\lambda}''}^{-1}(\mathbf{u})) \chi_k(F_{\boldsymbol{\lambda}''}^{-1}(\mathbf{u})) h_{\boldsymbol{\lambda}''}(\mathbf{u}) |\det D_{\mathbf{u}} F_{\boldsymbol{\lambda}''}^{-1}(\mathbf{u})|, \quad (\text{B.8})$$

for all $\mathbf{u} \in B'_k$, where

$$\begin{aligned} h_{\lambda''}(\mathbf{u}) &:= \psi \left(2^{j-i} r \left[\left(D_{\mathbf{u}} F_{\lambda''}^{-1}(\mathbf{u}) \right)^{-1} D_{\lambda''} \Phi_{(\lambda', \lambda'')} \Big|_{\lambda' = F_{\lambda''}^{-1}(\mathbf{u})} \right] \right) \\ &= \psi \left(2^{j-i} r \left[D_{F_{\lambda''}^{-1}(\mathbf{u})} F_{\lambda''}^{-1}(\mathbf{u}) D_{\lambda''} \mathbf{u} \right] \right). \end{aligned}$$

We use the change of variables $\lambda' = F_{\lambda''}^{-1}(\mathbf{u})$ (equivalently, $\mathbf{u} = \Phi_{\lambda'}$) to write

$$\begin{aligned} &\int_K R(\lambda) \phi(2^j r \Phi_{\lambda}) \chi_k(\lambda) d\lambda \\ &= \int_K \rho_1(\lambda') \chi_k(\lambda) \phi(2^j r \Phi_{\lambda}) \psi(2^{j-i} r D_{\lambda} \Phi_{\lambda}) \rho_2(\lambda'') d\lambda \\ &= \int_{\mathbb{R}^{N-m}} \rho_2(\lambda'') \int_{\mathbb{R}^m} g_{\lambda''}(\mathbf{u}) \phi(2^j r \mathbf{u}) d\mathbf{u} d\lambda''. \end{aligned} \tag{B.9}$$

To estimate this integral, we will bound the inner integral uniformly in λ'' via a Taylor expansion and use the fact that ρ_2 is Schwartz to handle the outer integral. For this reason, we require two general differentiation formulas. The first concerns the derivatives of inverse functions:

Lemma B.2 (Peres–Schlag [40] Lemma 7.9). *Let $V \subseteq \mathbb{R}^m$ be an open set and let $F: V \rightarrow \mathbb{R}^m$ be a C^L -diffeomorphism onto its image. For all $\alpha \in \mathbb{N}^m$ with $|\alpha| = L$,*

$$\partial^{\alpha}(F^{-1}) = \sum_{k=0}^{L-1} \sum_{j=1}^{L-1} \sum_{\sigma, \mathbf{q}} (DF \circ F^{-1})^{-L-j} \mathbf{v}_{\alpha, j}(\sigma, \mathbf{q}) \prod_{i=1}^j (\partial^{\sigma_i} F_{q_i}) \circ F^{-1},$$

where:

- the third sum runs over all $\mathbf{q} \in \{1, \dots, m\}^j$ and over all $\sigma \in \{2, \dots, L\}^{m \times j}$ (i.e., all j -tuples consisting of multi-indices $\sigma_1, \dots, \sigma_j \in \{2, \dots, L\}^m$) such that $|\sigma| := \sum_{i=1}^j |\sigma_i| \leq 2(L-1)$;
- $(DF \circ F^{-1})^{-L-j}$ is the inverse of DF , evaluated at $F^{-1}(\cdot)$, composed with (i.e., multiplied by) itself $L+j$ times;
- $\mathbf{v}_{\alpha, j}(\sigma, \mathbf{q}) \in \mathbb{R}^m$ are vectors being multiplied by $(DF \circ F^{-1})^{-L-j}$; and
- σ_i is the i^{th} row of σ and F_{q_i} is the q_i^{th} component function of F .

From the regularity condition (3.7) and the norm bound in (B.1), we conclude that

$$\begin{aligned}
|\partial_{\mathbf{u}}^{\alpha} F_{\lambda''}^{-1}| &= \left| \sum_{k=0}^{L-1} \sum_{j=1}^{L-1} \sum_{\sigma, \mathbf{q}} (D_{\lambda} F_{\lambda''} \circ F_{\lambda''}^{-1})^{-L-j} \mathbf{v}_{\alpha, j}(\sigma, \mathbf{q}) \prod_{i=1}^j (\partial^{\sigma_i} (F_{\lambda''})_{q_i}) \circ F_{\lambda''}^{-1} \right| \\
&\leq \sum_{k=0}^{L-1} \sum_{j=1}^{L-1} \sum_{\sigma, \mathbf{q}} \|D_{\lambda} F_{\lambda''} \circ F_{\lambda''}^{-1}\|^{-L-j} |\mathbf{v}_{\alpha, j}(\sigma, \mathbf{q})| \prod_{i=1}^j |(\partial^{\sigma_i} (F_{\lambda''})_{q_i}) \circ F_{\lambda''}^{-1}| \\
&\lesssim_{\beta, L} \sum_{k=0}^{L-1} \sum_{j=1}^{L-1} \sum_{\sigma, \mathbf{q}} r^{-\beta(L+j)} |\mathbf{v}_{\alpha, j}(\sigma, \mathbf{q})| \prod_{i=1}^j r^{-\beta|\sigma_i|} \\
&\lesssim_L \sum_{k=0}^{L-1} \sum_{j=1}^{L-1} \sum_{\sigma, \mathbf{q}} r^{-\beta(L+j+|\sigma|)} \lesssim_{|\Omega|, L} r^{-\beta(L+(L-1)+2(L-1))} \\
&= r^{-\beta(4L-3)} \lesssim_{|\Omega|, \beta} r^{-4\beta L}.
\end{aligned} \tag{B.10}$$

This will allow us to bound the derivatives of $g_{\lambda''}$, all of whose factors are functions of $F_{\lambda''}^{-1}(\mathbf{u})$ alone, but we will also need a multivariable chain rule to carry out this computation. We introduce this second differentiation formula now, starting with some notation.

Let $\prec$ denote the lexicographic order on the set of N -tuples over $\mathbb{N}$. For $\alpha \in \mathbb{N}^N$ and $\gamma \in \mathbb{N}^M$ (where M plays the role of $m+n$), define the following collection of pairs consisting of an L -tuple of N -tuples and an L -tuple of M -tuples, $L := |\alpha|$:

$$\begin{aligned}
P(\alpha, \gamma) := & \left\{ (\sigma, \tau) \in (\mathbb{N}^N)^L \times (\mathbb{N}^M)^L : \sum_{i=1}^L \tau_i = \gamma, \sum_{i=1}^L |\tau_i| \sigma_i = \alpha, \text{ and} \right. \\
& \exists 1 \leq p \leq L \text{ s.t. } \sigma_1 = \dots = \sigma_{L-p} = \mathbf{0}, \tau_1 = \dots = \tau_{L-p} = \mathbf{0}, \\
& \left. |\tau_{L-p+1}|, \dots, |\tau_L| > 0, \text{ and } \mathbf{0} \prec \sigma_{L-p+1} \prec \dots \prec \sigma_L \right\}.
\end{aligned}$$

Here $\sigma_i \in \mathbb{N}^N$ is the i^{th} component of σ for $i = 1, \dots, L$ (and similarly for $\tau_i \in \mathbb{N}^M$). For each $(\sigma, \tau) \in P(\alpha, \gamma)$ we denote

$$b_{\sigma, \tau} := \alpha! \prod_{i=1}^L \frac{1}{\tau_i! (\sigma_i!)^{|\tau_i|}},$$

where $\alpha! := \alpha_1 \cdots \alpha_N$. The only important points are that there $\lesssim_L 1$ coefficients $b_{\sigma, \tau}$, all of which are positive and $\lesssim_L 1$, and that the pairs $(\sigma, \tau) \in P(\alpha, \gamma)$ satisfy certain arithmetic conditions.

Lemma B.3 (Multivariate Faà di Bruno formula [10]). *Let $N, M, L \in \mathbb{Z}_+$. If $V \subseteq \mathbb{R}^N$ is an open set, $f \in C^L(V; \mathbb{R})$, $G \in C^L(\mathbb{R}^N; \mathbb{R}^M)$, and $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq L$, then*

$$\partial_{\lambda}^{\alpha}(f \circ G)(\lambda) = \sum_{1 \leq |\gamma| \leq |\alpha|} \partial_{\mathbf{z}}^{\gamma} f(\mathbf{z})|_{\mathbf{z}=G(\lambda)} \sum_{\sigma, \tau} b_{\sigma, \tau} \prod_{i=1}^{|\alpha|} [\partial_{\lambda}^{\tau_i} G(\lambda)]^{\sigma_i}$$

for all $\lambda \in V$, where the second sum runs over all $(\sigma, \tau) \in P(\alpha, \gamma)$.

Here, $[\mathbf{z}]^{\rho} := z_1^{\rho_1} \cdots z_M^{\rho_M}$. We continue our estimation of $g_{\lambda''}$ and its derivatives by treating each factor in (B.8) individually. First take $f = \rho_1$ (so m plays the role of N) and $G = F_{\lambda''}^{-1}$ (so $M = m$ as well). There are $\sim_{|\alpha|} 1$ terms in the outermost sum in the Faà di Bruno formula and $|\partial_{\lambda'}^{\gamma} \rho_1(\lambda')| \lesssim_{K, \rho, |\gamma|} 1$ for all γ and λ' , so

$$\begin{aligned} & |\partial_{\mathbf{u}}^{\alpha}(\rho_1 \circ F_{\lambda''}^{-1})(\mathbf{u})| \\ & \leq \sum_{1 \leq |\gamma| \leq |\alpha|} |\partial_{\lambda'}^{\gamma} \rho_1(\lambda')|_{\lambda' = F_{\lambda''}^{-1}(\mathbf{u})} \sum_{\sigma, \tau} b_{\sigma, \tau} \prod_{i=1}^{|\alpha|} |\partial_{\mathbf{u}}^{\tau_i} F_{\lambda''}^{-1}(\mathbf{u})|^{\sigma_i} \\ & \lesssim_{|\alpha|} \sum_{\sigma, \tau} b_{\sigma, \tau} \prod_{i=1}^{|\alpha|} |\partial_{\mathbf{u}}^{\tau_i} F_{\lambda''}^{-1}(\mathbf{u})|^{\sigma_i} \\ & \lesssim_{|\alpha|} \sum_{\sigma, \tau} b_{\sigma, \tau} \prod_{i=1}^{|\alpha|} r^{-4\beta|\tau_i| \sigma_i} \\ & = \sum_{\sigma, \tau} b_{\sigma, \tau} r^{-4\beta|\alpha|} \lesssim_{|\alpha|} r^{-4\beta|\alpha|}. \end{aligned}$$

This same computation and the derivative bounds in (B.5) likewise imply

$$|\partial_{\mathbf{u}}^{\alpha}(\chi_k \circ F_{\lambda''}^{-1})(\mathbf{u})| \lesssim r^{-(m+6)\beta|\alpha|}.$$

Before treating $h_{\lambda''}$ and its derivatives, we skip ahead to the Jacobian determinant factor. Since $F_{\lambda''}$ is a diffeomorphism, assume without loss of generality that $J_{\lambda''}(\mathbf{u}) := \det D_{\mathbf{u}} F_{\lambda''}^{-1}(\mathbf{u})$ for all $\mathbf{u} \in B'_k$. Then $J_{\lambda''}$ is a sum of m^2 terms of m factors $\partial_{u_j} F_{\lambda'', \ell}^{-1}(\mathbf{u})$, the ℓ^{th} component of $\partial_{u_j} F_{\lambda''}^{-1}(\mathbf{u})$. Consequently, $\partial_{\mathbf{u}}^{\alpha} J_{\lambda''}(\mathbf{u})$ is a sum of m -fold products of factors $\partial_{\mathbf{u}}^{\gamma} F_{\lambda'', \ell}^{-1}(\mathbf{u})$ with $1 \leq |\gamma| \leq |\alpha|$, and the number of terms in the sum is a function of m and $|\alpha|$ alone. Since

$$|\partial_{\mathbf{u}}^{\gamma} F_{\lambda''}^{-1}(\mathbf{u})| \lesssim_{\beta, |\gamma|} r^{-\beta(4|\gamma|-3)} \sim_{|\Omega|, \beta, |\alpha|} r^{-4\beta|\alpha|},$$

the crude bound

$$|\partial_{\mathbf{u}}^{\alpha} J_{\lambda''}(\mathbf{u})| \lesssim_{m,|\alpha|} (r^{-4\beta|\alpha|})^m = r^{-4m\beta|\alpha|}$$

follows.

Now we double back to the factor $h_{\lambda''}$ that we skipped. To control this, we again apply Theorem B.3, now with $f(\lambda') = \psi(2^{j-i}rD_{(\lambda',\lambda'')}\Phi_{(\lambda',\lambda'')})$ and $G(\mathbf{u}) = F_{\lambda''}^{-1}$:

$$\begin{aligned} |\partial_{\mathbf{u}}^{\alpha} h_{\lambda''}(\mathbf{u})| &\leq \sum_{1 \leq |\gamma| \leq |\alpha|} |\partial_{\lambda'}^{\gamma} \psi(2^{j-i}rD_{(\lambda',\lambda'')}\Phi_{(\lambda',\lambda'')})| \Big|_{\lambda' = F_{\lambda''}^{-1}(\mathbf{u})} \\ &\quad \cdot \sum_{\sigma, \tau} b_{\sigma, \gamma} \prod_{k=1}^{|\alpha|} \left| [\partial_{\mathbf{u}}^{\tau_k} F_{\lambda''}^{-1}(\mathbf{u})]^{\sigma_k} \right|. \end{aligned} \quad (\text{B.11})$$

The expression in the second line is amenable to (B.10):

$$\sum_{\sigma, \tau} b_{\sigma, \gamma} \prod_{k=1}^{|\alpha|} \left| [\partial_{\mathbf{u}}^{\tau_k} F_{\lambda''}^{-1}(\mathbf{u})]^{\sigma_k} \right| \lesssim_{|\alpha|} \prod_{k=1}^{|\alpha|} r^{-4|\tau_k|\sigma_k} \lesssim_{|\Omega|} r^{-4\beta|\alpha|}.$$

To bound the expression in the first line, recall that ψ is a rapidly-decreasing function of mN variables, so $|\partial_{\mathbf{t}}^{\gamma} \psi(\mathbf{t})| \lesssim_{p,|\alpha|} |\mathbf{t}|^{-p}$ for $p \in \mathbb{N}$ and all $1 \leq |\gamma| \leq |\alpha|$. In particular, appealing to Theorem B.3 once more and applying this rapid decay bound with $p = |\alpha|$ yields

$$\begin{aligned} &|\partial_{\lambda'}^{\alpha} \psi(2^{j-i}rD_{(\lambda',\lambda'')}\Phi_{(\lambda',\lambda'')})| \\ &\leq \sum_{1 \leq |\gamma| \leq |\alpha|} |\partial_{\mathbf{t}}^{\gamma} \psi(\mathbf{t})| \Big|_{\mathbf{t} = 2^{j-i}rD_{\lambda}\Phi_{\lambda}} \sum_{\sigma, \tau} b_{\sigma, \gamma} \prod_{k=1}^{|\alpha|} \left| [2^{j-i}r\partial_{\lambda'}^{\tau_k} D_{\lambda'}\Phi_{(\lambda',\lambda'')}]^{\sigma_k} \right| \\ &\lesssim_{\beta, |\alpha|} \sum_{1 \leq |\gamma| \leq |\alpha|} \left\| 2^{j-i}rD_{\lambda}\Phi_{\lambda} \right\|^{-|\alpha|} \sum_{\sigma, \tau} b_{\sigma, \gamma} \prod_{k=1}^{|\alpha|} (2^{j-i}r)^{\sigma_k} r^{-\beta(|\tau_k|+1)\sigma_k} \\ &\lesssim_{|\Omega|} \sum_{1 \leq |\gamma| \leq |\alpha|} (2^{j-i}r)^{-|\alpha|} \sum_{\sigma, \tau} b_{\sigma, \gamma} (2^{j-i}r)^{|\alpha|} \cdot r^{-2\beta|\alpha|} \\ &\lesssim_{|\alpha|} r^{-2\beta|\alpha|}. \end{aligned}$$

Thus, by (B.11)

$$|\partial_{\mathbf{u}}^{\alpha} h_{\lambda''}(\mathbf{u})| \lesssim \sum_{1 \leq |\gamma| \leq |\alpha|} r^{-2\beta|\alpha|} \cdot r^{-4\beta|\alpha|} \sim_{|\alpha|} r^{-6\beta|\alpha|}.$$

This in hand, we can at last bound the derivatives of $g_{\lambda''}$. The number of terms that arise by application of the product rule to (B.8) is a constant depending only on $|\alpha|$, so we may combine the bounds on the derivatives of each of the four factors to conclude that

$$|\partial_{\mathbf{u}}^{\alpha} g_{\lambda''}(\mathbf{u})| \lesssim_{|\alpha|} r^{-4\beta|\alpha|} \cdot r^{-(m+6)\beta|\alpha|} \cdot r^{-6\beta|\alpha|} \cdot r^{-4m\beta|\alpha|} = r^{-(5m+16)\beta|\alpha|}. \quad (\text{B.12})$$

This is valid for all $|\alpha| \geq 1$. In the $\alpha = \mathbf{0}$ case, it is enough to note that $\rho_1 \circ F_{\alpha''}^{-1}$, $\chi_k \circ F_{\lambda''}^{-1}$, and $h_{\lambda''}$ are bounded independently of λ'' , so $|g_{\lambda''}(\mathbf{u})| \lesssim 1 \cdot 1 \cdot 1 \cdot r^{-\beta} = r^{-\beta}$ for all $\mathbf{u} \in B'_k$ by (B.1).

The final order of business is to write out the Taylor series approximation of $g_{\lambda''}$ and use it to bound (B.9). (Recall our purpose is to use this to control (B.6), which is a sum of terms of this form, and thereby prove the lemma.) The factor $\rho_2(\lambda'')$ will pose no issue, so we set it aside for now, and the other factor $\phi(2^j r F_{\lambda''}^{-1}(\mathbf{u}))$ will take a Schwartz class bound. With $M := 2[c] - m$,

$$\begin{aligned} & \int_{\mathbb{R}^m} g_{\lambda''}(\mathbf{u}) \phi(2^j r \mathbf{u}) d\mathbf{u} \\ &= \int_{\{|\mathbf{u}| \leq (2^j r)^{-1/2}\}} \phi(2^j r \mathbf{u}) \left(\sum_{0 \leq |\alpha| \leq M} \frac{\partial_{\mathbf{u}}^{\alpha} g_{\lambda''}(\mathbf{0})}{\alpha!} \mathbf{u}^{\alpha} \right. \\ & \quad \left. + \sum_{|\alpha|=M+1} \int_0^1 [\partial_{\mathbf{u}}^{\alpha} g_{\lambda''}(t\mathbf{u}) - \partial_{\mathbf{u}}^{\alpha} g_{\lambda''}(\mathbf{0})] (1-t)^M dt \frac{\mathbf{u}^{\alpha}}{\alpha!} \right) d\mathbf{u} \\ & \quad + \int_{\{|\mathbf{u}| > (2^j r)^{-1/2}\}} \phi(2^j r \mathbf{u}) g_{\lambda''}(\mathbf{u}) d\mathbf{u} \\ &=: A_1 + A_2. \end{aligned}$$

To complete the proof, we must estimate A_1 and A_2 uniformly in λ'' . The latter is simple owing the rapid decay of ϕ and the basic inequality $|g_{\lambda''}(\mathbf{u})| \lesssim r^{-\beta}$:

$$\begin{aligned} |A_2| &\lesssim r^{-\beta} \int_{\{|\mathbf{u}| > (2^j r)^{-1/2}\}} |\phi(2^j r \mathbf{u})| d\mathbf{u} \\ &\lesssim_c r^{-\beta} \int_{\{|\mathbf{u}| > (2^j r)^{-1/2}\}} (2^j r |\mathbf{u}|)^{-2c-2m} d\mathbf{u} \\ &\sim r^{-\beta} \cdot (2^j r)^{-2c-2m} \cdot (2^j r)^{c+2m} = r^{-\beta} (2^j r)^{-c}. \end{aligned}$$

To estimate A_1 , observe first that

$$\begin{aligned}
\int_{\mathbb{R}^m} \mathbf{u}^\alpha \phi(2^j r \mathbf{u}) d\mathbf{u} &= \mathcal{F}_{\mathbf{u}}[\mathbf{u}^\alpha \phi(2^j r \mathbf{u})](\mathbf{0}) \\
&= (-2\pi i)^{|\alpha|} \partial_{\xi}^\alpha \mathcal{F}_{\mathbf{u}}[\phi(2^j r \mathbf{u})](\xi)|_{\xi=\mathbf{0}} \\
&= (-2\pi i)^{|\alpha|} (2^j r)^{-m} \partial_{\xi}^\alpha \widehat{\phi}(2^{-j} r^{-1} \xi)|_{\xi=\mathbf{0}} \\
&= (-2\pi i)^{|\alpha|} (2^j r)^{-m} \partial^\alpha \eta^{(1)}(\mathbf{0}),
\end{aligned}$$

and this vanishes for all $\alpha \in \mathbb{N}^m$ because η is identically 0 in a neighborhood of the origin.

Therefore,

$$\begin{aligned}
&\int_{\{|\mathbf{u}| \leq (2^j r)^{-1/2}\}} \phi(2^j r \mathbf{u}) \sum_{0 \leq |\alpha| \leq M} \frac{\partial_{\mathbf{u}}^\alpha g_{\lambda''}(\mathbf{0})}{\alpha!} \mathbf{u}^\alpha d\mathbf{u} \\
&= - \int_{\{|\mathbf{u}| > (2^j r)^{-1/2}\}} \phi(2^j r \mathbf{u}) \sum_{0 \leq |\alpha| \leq M} \frac{\partial_{\mathbf{u}}^\alpha g_{\lambda''}(\mathbf{0})}{\alpha!} \mathbf{u}^\alpha d\mathbf{u}.
\end{aligned}$$

We already know that the $\alpha = \mathbf{0}$ term is $\lesssim_c r^{-\beta}$ in absolute value. To bound the remaining terms, recall that we assumed $2^j r^{1+a_1'\beta} > 1$. At last setting $a_1' := 5m + 16$, we use (B.12) and the rapid decay bound $|\phi(2^j r \mathbf{u})| \lesssim_M (2^j r |\mathbf{u}|)^{-2c-m-|\alpha|}$ to conclude that

$$\begin{aligned}
&\left| \int_{\{|\mathbf{u}| > (2^j r)^{-1/2}\}} \phi(2^j r \mathbf{u}) \sum_{1 \leq |\alpha| \leq M} \frac{\partial_{\mathbf{u}}^\alpha g_{\lambda''}(\mathbf{0})}{\alpha!} \mathbf{u}^\alpha d\mathbf{u} \right| \\
&\lesssim_M \sum_{1 \leq |\alpha| \leq M} \int_{\{|\mathbf{u}| > (2^j r)^{-1/2}\}} (2^j r |\mathbf{u}|)^{-2c-m-|\alpha|} \cdot r^{-(5m+16)\beta|\alpha|} \cdot |\mathbf{u}|^{|\alpha|} d\mathbf{u} \\
&= \sum_{1 \leq |\alpha| \leq M} (2^j r)^{-2c-m-|\alpha|} \cdot r^{-(5m+16)\beta|\alpha|} \int_{\{|\mathbf{u}| > (2^j r)^{-1/2}\}} |\mathbf{u}|^{-2c-m} d\mathbf{u} \\
&\sim_{m,c} \sum_{1 \leq |\alpha| \leq M} (2^j r)^{-2c-m-|\alpha|} \cdot r^{-(5m+16)\beta|\alpha|} \cdot (2^j r)^c \\
&= \sum_{1 \leq |\alpha| \leq M} (2^j r)^{-c-m-|\alpha|} \cdot r^{-(5m+16)\beta|\alpha|} \\
&\leq \sum_{1 \leq |\alpha| \leq M} r^{-m(5m+16)\beta} (2^j r)^{-c} \\
&\sim_c r^{-m(5m+16)\beta} (2^j r)^{-c}.
\end{aligned}$$

It remains only to bound the Taylor error term. By Taylor's theorem and the boundedness of ϕ ,

$$\begin{aligned}
& \left| \int_{\{|\mathbf{u}| \leq (2^j r)^{-1/2}\}} \phi(2^j r \mathbf{u}) \sum_{|\alpha|=M+1} \int_0^1 [\partial_{\mathbf{u}}^\alpha g_{\lambda''}(t\mathbf{u}) - \partial_{\mathbf{u}}^\alpha g_{\lambda''}(\mathbf{0})] (1-t)^M dt \frac{\mathbf{u}^\alpha}{\alpha!} d\mathbf{u} \right| \\
& \leq \int_{\{|\mathbf{u}| \leq (2^j r)^{-1/2}\}} O\left(\sup_{|\alpha|=M+1} \|\partial_{\mathbf{u}}^\alpha g_{\lambda''}\|_{L^\infty(B'_k)} |\mathbf{u}|^{M+1} \right) d\mathbf{u} \\
& \lesssim r^{-(5m+16)\beta(M+1)} \int_{\{|\mathbf{u}| \leq (2^j r)^{-1/2}\}} |\mathbf{u}|^{M+1} d\mathbf{u} \\
& \sim_c r^{-2c(5m+16)\beta} \cdot (2^j r)^{-(2\lceil c \rceil + 1)/2} \\
& \lesssim_c r^{-2c(5m+16)\beta} \cdot (2^j r)^{-c}.
\end{aligned}$$

This shows that

$$\begin{aligned}
|A_1| + |A_2| & \lesssim r^{-m(5m+16)\beta} (2^j r)^{-c} + r^{-2c(5m+16)\beta} \cdot (2^j r)^{-c} + r^{-\beta} (2^j r)^{-c} \\
& \lesssim_{|\Omega|, \beta, c} r^{-a_1 c \beta} \cdot (2^j r)^{-c} \sim (1 + 2^j r^{1+a_1 \beta})^{-c},
\end{aligned}$$

where $a_1 := \max\{m, 2\}(5m + 16)$. All implicit constants are independent of λ'' and $\rho_2 \in L^1(\mathbb{R}^{N-m})$, so

$$\int_{\mathbb{R}^{N-m}} \rho_2(\lambda'') \int_{\mathbb{R}^m} g_{\lambda''}(\mathbf{u}) \phi(2^j r \mathbf{u}) d\mathbf{u} d\lambda'' \lesssim (1 + 2^j r^{1+a_1 \beta})^{-c}$$

as we sought to show. □

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