Birational Functors in the Derived Category

Alexander D. Voet

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Reading Committee:

Max Lieblich, Chair

Jarod Alper

Sándor Kovács

Program Authorized to Offer Degree:
Mathematics

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Abstract

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Chair of the Supervisory Committee:
Professor Max Lieblich
Mathematics

In this thesis, we study a class of derived equivalences that naturally induce birational maps. We give several equivalent criteria for a birational correspondence to exist, and prove the correspondence induces a K-equivalece, extending a result of Kawamata. With the introduction of a canonical natural transformation from the right to left adjoint, we are also able to extend this study to the fully faithful situation. This natural transformation gives us context to provide a new proof of Bridgland's equivalence criterion and of the indecomposability of the derived category in the trivial canonical bundle. Additionally, we construct an algebraic moduli space of birational integral transforms inside of Lieblich's moduli of complexes.

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DEDICATION

to Dinosaur Dad

Chapter 1

INTRO

This is thesis is a new perspective on a well studied problem:

Question 1.0.1. What information is about the geometry of a scheme X is contained in its bounded derived category of coherent sheaves $\mathrm{D}^b(X)$?

The starting point of any answer to this question is Bondal and Orlov's Theorem, which says a scheme can be entirely reconstructed from its derived category if the canonical bundle is ample.

Theorem 1.0.2. *[BO01] Let $\Phi : D^b(X) \to D^b(Y)$ be a derived equivalence, and assume K_X or $-K_X$ is ample. Then $X \cong Y$.*

The link between the derived category and the canonical bundle is through Serre duality, studied abstractly by the categorical notion of a Serre functor.

Definition 1.0.3. A *Serre functor* for a k-linear category $\mathcal{C}$ is a functor $S : \mathcal{C} \to \mathcal{C}$ satisfying

$$\mathrm{Hom}(A, SB) = (\mathrm{Hom}(B, A))^\vee$$

for every pair of objects $A, B \in \mathcal{C}$, natural in both arguments.

This is a the categorified notion of classical Serre duality. For the derived category $D^b(X)$, the Serre functor is given by

$$S_X(\mathcal{E}) = \mathcal{E} \otimes \omega_X[\mathrm{dim}X]$$

It is readily checked that a Serre functor is intrinsic. That is to say: it is unique up to natural isomorphism. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are two k-linear categories and $F : \mathcal{A} \to \mathcal{B}$ is an equivalence, there is an isomorphism ([Huy06, Lem 1.30]):

$$F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$$

This result is often stated as "an equivalence commutes with Serre functors", and has been applied with great success to the study of derived categories, particularly due to Orlov's celebrated result:

Theorem 1.0.4. *[Orl97, Thm 2.2]*
Let X and Y be smooth projetive varieties over a field k. Let $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be a fully faithful exact functor. Then there exists an object $\mathcal{P} \in \mathrm{D}^b(X \times Y)$, unique up to isomorphism, such that $\Phi \cong \Phi_{\mathcal{P}}$, where

$$\Phi_{\mathcal{P}}(\mathcal{E}) = \pi_{Y*}(\mathcal{P} \otimes \pi_X^* \mathcal{E})$$

Orlov's original result required the existence of an adjoint, but due to [BvdB03, Thm 1.1], it turns out that any exact functor between derived categories has both a left and right adjoint, and furthermore we have the formulas for adjoints to a transform of the form $\Phi_{\mathcal{P}}$.

$$\mathcal{P}_R = \mathcal{P}^\vee \otimes \pi_X^* \omega_X[\mathrm{dim} X] \qquad \mathcal{P}_L = \mathcal{P}^\vee \otimes \pi_Y^* \omega_Y[\mathrm{dim} Y]$$

While in general, the functor which assigns an object $\mathcal{P}$ to the integral transform $\Phi_{\mathcal{P}}$ is poorly behaved (see the survey [CS12a]), Orlov's result shows that it is essentially injective when restricted to kernels mapping to fully faithful functors. This assignment is neither full nor faithful in general as shown in [CS12b]. However, it does reflect isomorphisms:

Theorem 1.0.5. *[LO11, Lem. 3.4]*
Let $f : \mathcal{P} \to \mathcal{Q}$ be a morphism in $D(X \times Y)$. Then f is an isomorphism if and only if the induced morphism of functors $\Phi(f) : \Phi_{\mathcal{P}} \to \Phi_{\mathcal{Q}}$ is an isomorphism.

The literature relating the derived category and canonical bundle is generally concerned with the an equivalence $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$, in which case the right and left adjoints are isomorphic (and also give equivalences), and we have an isomorphism of kernels

$$\mathcal{P}_L = \mathcal{P}^\vee \otimes \pi_X^* \omega_X[\mathrm{dim} X] \cong \mathcal{P}^\vee \otimes \pi_Y^* \omega_Y[\mathrm{dim} Y] = \mathcal{P}_R$$

Bondal and Orlov's reconstruction result sparked great interest in the derived category. In this thesis, we ask the following question:

Question 1.0.6. When does a derived equivalence induce a birational map?

Our methods also allow us to study the following

Question 1.0.7. When does a birational map extend to a derived equivalence?

Orlov first studied the behavior of the derived category with respect to blowings-up in [Orl93]. Specifically, he described the derived category of the blow up via semiorthogonal decompositions. The components of his decomposition are derived category of the base, and the derived category of a projective bundle over the blow up locus, and the . His proof proceeded by showing $\mathrm{L}\pi^* : \mathrm{D}(X) \to \mathrm{D}(Y)$ is fully faithful and realizes $\mathrm{D}(X)$ as an admissible subcategory of the blow up.

Bondal and Orlov continued this study in [BO95], where they show for a special class of flips and flops that flips induce fully faithful functors and flops induce derived equivalences. This sparked a "categorical" minimal model program, whose goal is to minimize the derived category within a given birational class. The blow up calculation from the previous paper and the reconstruction result in this one provide a link to the classical MMP.

The story for flops has gathered considerable interest, though it is still an open question whether two arbitrary varieties connected by a flop are derived equivalent. A starting point for our studies is Kawamata's DK-hypothesis [Kaw02], or more recently [Kaw18]. Two varieties X and Y are K equivalent if there is a simultaneous resolution $X \leftarrow Z \to Y$ where the pullbacks of their canonical bundles coincide. Similarly we have the notion of a K inequality if the difference is equivalent to an effective divisor.

Conjecture 1.0.8. *[Kaw18] If $X \sim_K Y$, then $D(X) \cong D(Y)$. If $X \leq_K Y$, there is a fully faithful functor $D(X) \to D(Y)$.*

The motivating result in this direction is a main result from [Kaw02]. The first part of the conjecture is true if X and Y are of general type. While we have not been successful thus far, we strongly believe our methods will be able to show the second part also holds for varieties of general type.

In this thesis, our starting point is an integral transform that induces a birational map. We call such a functor a *birational integral transform*. An arbitrary transform can be difficult to work with due to the uniqueness issues discussed above, so we restrict ourselves to case of fully faithful functors and equivalences. Most of the work in the derived category has been concerned with an equivalence Φ and relies on the adjoints $\Phi_R \cong \Phi_L$ being isomorphic, and Φ commuting with Serre functors. Perhaps the most important contribution in this paper is the introduction of a tool to extend the story of integral transforms and the canonical bundle to the fully faithful situation. The starting point is a canonical natural transformation between adjoints, whose existence was known previously [Joh11] but has not been applied to Algebraic Geometry.

Proposition 1.0.9. *Let F be a fully faithful functor with right adjoint R and left adjoint L. There is a canonically defined natural transformation $\theta : R \to L$, equivalently defined in terms of the unit and counit maps of the opposing adjunctions.*

$$R \xrightarrow{R\alpha} RFL \xrightarrow{\eta^{-1}L} L \qquad R \xrightarrow{\beta^{-1}R} LFR \xrightarrow{L\epsilon} L$$

Furthermore, we have the following identities:

$$\eta^{-1} = \beta \circ \theta F \qquad \beta^{-1} = \theta F \circ \eta$$

We give a different proof of this result, and as the adjunction mappings in te derived category are induced by morphisms of kernels, we present

Theorem 1.0.10. *Let $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ be a fully faithful integral transform. There is a canonical morphism in the derived category*

$$\theta : \mathcal{P}^\vee \otimes \pi_X^* \omega_X \to \mathcal{P}^\vee \otimes \pi_Y^* \omega_Y$$

We believe that a thorough understanding of this morphism will allow us to generalize many results known previously only for equivalences. One milestone would be an answer to the question:

Question 1.0.11. Does this morphism induce an inequality of degrees? For a curve C contained in $\mathrm{Supp}(\mathcal{P})$, or more specifically an irreducible component of $\mathrm{Supp}(\mathcal{P})$ which dominates X, does θ induce an inequality $\deg(\pi_X^*\omega_X) \leq \deg(\pi^*\omega_Y)$?

For the rest of the paper, we concern ourselves with fully faithful integral transforms that extend birational maps. In particular, we make the following definition:

Definition 1.0.12. An integral transform $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$ is a *birational integral transform* if there is an open set $U \subseteq Y$ where $\mathcal{H}^0(\mathcal{P}|_U) \cong \mathcal{O}_\Gamma$ is the graph of an open immersion $U \to X$.

Of course we could relax this description up to standard autoequivalences and achieve the same results, but for clarity of concept we restrict ourselves to this setting. With this restriction, the we answer the question above in the affirmative, and we are able to prove

Theorem 1.0.13. *Assume there is a birational, fully faithful integral transform $\Phi_{\mathcal{P}} : D(X) \to D(Y)$. Then $\Phi_{\mathcal{P}}$ induces an inequality $X \leq_K Y$.*

This immediately implies that birationally derived equivalent curves and surfaces are isomorphic. We can make an even stronger conclusion in dimension 1 situation, as the K equivalence is induced by a component of the $\mathrm{Supp}(\mathcal{P})$.

Corollary 1.0.14. *If $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is a BD equivalence and $\dim(X) = 1$, then $\Phi_{\mathcal{P}} = \mathrm{L}f^*$ for an isomorphism $f : Y \to X$.*

In dimension 2 the situation changes, as the spherical twist about a (-2) curve in a surface S is a birational derived equivalence which extends the identity morphism but is a nonstandard autoequivalence of $\mathrm{D}(S)$ — see "examples and nonexamples".

In dimension 3, less is known in terms of the classification of autoequivalences, however due to [Kaw02, Thm. 4.6] and [Bri02] we can conclude

Theorem 1.0.15. *Let X and Y be projective varieties with terminal Gorenstein singularities. Then $X \sim_K Y$ if and only if $X \sim_{BD} Y$.*

Kawamata shows that a K equivalence can be decomposed as a sequence of flops, and Bridgland shows that such flops induce derived equivalences. Furthermore, the universal perverse point sheaf in Bridgeland's construction is the graph of an isomorphism away from the contracted locus and thus we find this equivalance is a BD equivalence.

It is also worth mentioning that Uehara in [Ueh12] has produced examples of birational and derived equivalent Calabi Yau threefolds that are not K equivalent, which shows that there is more to the story than contained in our definition.

Structure of the thesis. We break the thesis into three main chapters, followed by a discussion of related problems. Chapter 2 is preliminary material and can safely be skipped by the expert. As the construction of the morphism from right to left adjoint is purely categorical, we give a brief review of adjoint functors and some properties. We provide references and statements of relevant results in birational geometry, particularly concerning the K partial ordering in a birational equivalence class. The chapter concludes with a brief dicussion of the derived category as pertaining to this thesis.

In Chapter 3 we discuss integral transforms. We reserve the name Fourier Mukai transforms for derived equivalences, and review some basic properties of integral kernels. Here we give a full proof of the existence of a natural transformation from right to left adjoint for an arbitrary fully faithful functor, which provides the basis for our understanding of birationally fully faithful maps. We then move to a discussion of integral transforms in families, where we are able to prove some interesting results on base change, and conclude by constructing a moduli space of fully faithful kernels inside of Lieblich's moduli of complexes.

Chapter 4 is concerned with those integral transforms that extend birational maps. We discuss how the geometry of X and Y is related to the support of an integral kernel, then proceed to show that a birational derived equivalence induces a K equivalence. After going over a few examples, we realize an algebraic moduli space of birational integral transforms which are fully faithful or give an equivalence.

The studies presented in this thesis seemed to raise more questions than answers; we discuss some of these in chapter 5. This research direction started through an attempt to answer "is liftability a derived invariant?". We give a short account of the lifting problem as pertaining to this question, and give evidence indicating that liftability is invariant under BD equivalences.

Chapter 2

PRELIMINARIES

This purpose of this chapter is to set the stage for the remainder of this thesis. We gather and organize results from the literature while giving perspective toward the problems tackled in this thesis. Most of the results here should be familiar to the practicing algebraic geometer, but we provide references, proofs, and examples when possible for ease of reading.

2.1 Categorical Constructions

Philosophically, an object is equivalently defined by its internal structure and by its interactions with other objects in the category. Mathematically, we have the following statement:

Lemma 2.1.1 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category, and write $Psh_{\mathcal{C}}$ for the functor category $[\mathcal{C}^{op}, Set]$. The functor $h_{-} : \mathcal{C} \to Psh_{\mathcal{C}}$, defined by $h_X = \mathrm{Hom}_{\mathcal{C}}(-, X)$ is fully faithful.*

A common proof strategy is to appeal to the full faithfulness as in following corollary.

Corollary 2.1.2. *Two objects X and Y are isomorphic if and only if h_X and h_Y are isomorphic as functors.*

The Yoneda lemma is the starting point for moduli theory, and the relevant formulation along with a proof sketch appears in the following section. This version will be enough for the remainder of our abstract categorical discussion.

In classical algebraic geometry, we are frequently concerned with the notion of a "base field". What is the characteristic? Is it algebraically closed? How does this affect the geometry? In the school of Grothendieck, we are now more aware of the importance of working over an arbitrary base. These notions are taken care of by the abstract machinery of a slice category.

Definition 2.1.3. Let $\mathcal{C}$ be an arbitrary category and take S to be an object of $\mathcal{C}$. The *slice category* $\mathcal{C}/S$ is the category whose objects are pairs (X, f), where X is an object of $\mathcal{C}$ and $f : X \to S$. A morphism $\phi : (X, f) \to (X', f')$ is then given by a commuting triangle in $\mathcal{C}$.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & S & \end{array}$$

The next relevant notion is that of an adjoint pair of functors. The two most known examples are probably the Free - Forgetful adjunction for various algebraic structures defined in terms of sets, and the Tensor-Hom adjunction that is taught in a first year algebra course. However, many important notions and correspondences between unexpected objects can often be phrased in terms of an adjunction. For this thesis, we rely on Grothendieck duality – a statement about the existence of a right adjoint to f_* in the derived category– to guarantee existence of right and left adjoints to arbitrary Fourier-Mukai transforms. We then show that certain types of fully faithful FM transforms induce K-inequalities in the sense of Kawamata by appealing to the canonical morphism from the right adjoint to the left described at the end of this section.

Notation. In what follows, F will always be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. We will use C for an object of $\mathcal{C}$ and D for an object of $\mathcal{D}$ If we only care about one adjunction, we will denote its right adjoint by G. If we are discussing multiple adjunctions, the right adjoint will be written R and the left adjoint will be written L. After the definition we will drop the subscript on the hom spaces.

Definition 2.1.4. We say that F is *left adjoint* to G if for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$, we have an isomorphism

$$\phi : \mathrm{Hom}_{\mathcal{D}}(FC, D) \cong \mathrm{Hom}_{\mathcal{C}}(C, GD)$$

which is natural in both C and in D. In this situation we also say G is *right adjoint* to F.

Example 2.1.5. Consider the functor $F : \text{Set} \to \text{Vect}_{\mathbb{R}}$ which takes any set to the $\mathbb{R}$-vector space with that set as basis. The right adjoint to F is the forgetful functor, which takes a vector space to the underlying set. The isomorphism in the the definition is a restatement of the fact that linear maps are uniquely determined by the image of any basis.

Definition 2.1.6. We say that F is *left adjoint* to G if there exist natural transformations

$$\eta : id \to GF \qquad \epsilon : FG \to id$$

which satisfy the triangle identities

$$\epsilon F \circ F\eta = 1_F \quad \text{and} \quad F\epsilon \circ \eta G = 1_G.$$

We call η the *unit*, and ϵ the *counit* (of the adjunction).

Example 2.1.7. Let A be a commutative ring and M be an A module. Then the functor $(-) \otimes M$ is left adjoint to $\text{Hom}(M, -)$. The symbol ϵ for the counit of the adjunction is suggestive of "evaluation". Notice that in this situation $FG(M') = \text{Hom}(M, M') \otimes M$ and $\epsilon_{M'}(f \otimes m) = f(m) \in id(M')$. The unit is the map $GF(M') = \text{Hom}(M, M' \otimes M)$. Then $\eta_{M'}(m')(m) = m' \otimes m$

Proposition 2.1.8. *The two definitions of adjoints presented are equivalent.*

Proof. Starting with the standard definition, define $\eta(C)$ as the image of the identity morphism $id \in \text{Hom}(FC, FC)$ under the adjunction. Define $\epsilon(D)$ similarly. To prove the triangle identities, follow the identity both directions around the commutative squares

$$\begin{array}{ccc} \text{Hom}(GFC, GFC) & \xrightarrow{\sim} & \text{Hom}(FGFC, FC) \\ \downarrow{\scriptstyle (-)\circ\eta_C} & & \downarrow{\scriptstyle (-)\circ F\eta_C} \\ \text{Hom}(C, GFC) & \xrightarrow{\sim} & \text{Hom}(FC, FC) \end{array} \qquad \begin{array}{ccc} \text{Hom}(FGD, FGD) & \xrightarrow{\sim} & \text{Hom}(GD, GFGD) \\ \downarrow{\scriptstyle \epsilon_D\circ(-)} & & \downarrow{\scriptstyle G\epsilon_D\circ(-)} \\ \text{Hom}(FGD, D) & \xrightarrow{\sim} & \text{Hom}(GD, GD) \end{array}$$

On the other hand, from the unit and counit we define an isomorphism $\phi : \text{Hom}(FC, D) \to \text{Hom}(C, GD)$ by the rule $\phi(g) = G(g) \circ \eta_C$. The inverse is given by $\phi^{-1}(f) = \epsilon_D \circ F(f)$. $\square$

In what follows, we will often be concerned with a fully faithful functor between derived categories of certain schemes. As discussed in the section on Fourier-Mukai transforms, Grothendieck duality guarantees that such a functor will always have a right and left adjoint. We appeal frequently to the following characterization, proof available as [Sta20, 07RB]

Proposition 2.1.9. *Let F be a functor with right adjoint R and/or left adjoint L. Then*

(i) F is fully faithful if and only if the unit $\eta : id \to RF$ is an isomorphism

(ii) F is fully faithful if and only if the counit $\beta : LF \to id$ is an isomorphism.

Remark 2.1.10. By a universality argument, in fact it is true that the unit is an isomorphism if there is *any* isomorphism $id \cong RF$. A similar statement is true for the counit.

We discuss less known properties of fully faithful functors in the following chapter, in the applied context of integral transforms.

2.2 Representable Functors on Schemes

We make frequent use of the Yoneda lemma in the following form.

Lemma 2.2.1 (Yoneda). *Let $\mathcal{C}$ be a locally small category, and write $Psh_{\mathcal{C}}$ for the functor category $[\mathcal{C}^{op}, \mathrm{Set}]$. For any $F \in Psh_{\mathcal{C}}$, and for any $X \in \mathcal{C}$, there is a natural isomorphism*

$$\mathrm{Hom}_{Psh_{\mathcal{C}}}(h_X, F) \cong F(X)$$

where $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X)$.

Proof. A morphism in $Psh_{\mathcal{C}}$ is a natural transformation of functors. The key to the proof is to show that a natural transformation $\phi : h_X \to F$ is uniquely determined by the image of $id \in h_X(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$. □

The most common statement and usage of Yoneda's lemma is for the special case where F above is representable, i.e. $F \cong h_Y$ for some $Y \in \mathcal{C}$.

Corollary 2.2.2 (Yoneda)**.** *Let* $\mathcal{C}$*,* $Psh_{\mathcal{C}}$*. The assignment* $X \mapsto h_X$ *induces a fully faithful embedding*

$$\mathcal{C} \hookrightarrow Psh_{\mathcal{C}}$$

Proof. By the lemma above, we have $\mathrm{Hom}_{Psh_{\mathcal{C}}}(h_X, h_Y) \cong h_Y(X) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The result follows. □

Example 2.2.3 (Projective Space)**.** The functor $(\mathrm{Sch}/k)^{op} \to \mathrm{Set}$ which takes

$$X/k \mapsto \{\mathcal{L} \in \mathrm{Pic}X \text{ and } (s_0, \ldots, s_N) \in \mathrm{H}^0(X, \mathcal{L}) \text{ which generate } \mathcal{L}\}$$

isn't quite represented by projective space. We obtain the functor $\mathbb{P}^n_k$ by taking the quotient of these sets by the equivalence relation $(\mathcal{L}, (s_0, \ldots, s_N)) \sim (\mathcal{M}, (t_0, \ldots, t_N))$ if there is an isomorphism $\phi : \mathcal{L} \to \mathcal{M}$ such that $\phi^*(t_i) = s_i$ for every i.

Example 2.2.4 (General Linear Group)**.** Consider the functor $\mathrm{Rings} = (\mathrm{Aff})^{op} \to \mathrm{Set}$ which takes $R \mapsto GL_n(R)$. An element of $GL_n(R)$ is an invertible $n \times n$ matrix with entries in R. To specify such a matrix, we can equivalently give a map

$$\phi^{\#} : \mathbb{Z}[(x_{ij})] \to R,$$

where the image of x_{ij} corresponds to the element in the ij position of the matrix. In other words: $\mathrm{Spec}(\mathbb{Z}[(x_{ij})])$ represents $\mathrm{Mat}_{n,n}(-)$. The condition that the matrix be invertible is precisely the condition of a polynomial expression (the determinant) in the $(\phi^{\#}(x_{ij}))$ be nonzero. Equivalently, $\phi^{\#}$ must be well defined over the localization $\mathbb{Z}[(x_{ij})]_{det}$. We conclude that matrices in $GL_n(R)$ are in bijection with morphisms

$$\phi : \mathrm{Spec}(R) \to \mathrm{Spec}((\mathbb{Z}[(x_{ij})])_{det})$$

and that GL_n is representable by an affine scheme. Now we can study $GL_n(X)$ for any scheme X by looking at morphisms $X \to GL_n$ in the category of schemes!

The example of the general linear group is instructive for two reasons, which we will proceed to discuss:

1. It is our first example of a group scheme.
2. It is our first example of an open subfunctor.

In this thesis, we will be studying a subgroup of autoequivalences of the derived category. Once we prove our representability result, it immediately follows that the moduli space of such equivalences is a group object.

Definition 2.2.5. A **group object** in a category $\mathcal{C}$ is an object $X \in \mathcal{C}$ whose corresponding functor $h_X \in [\mathcal{C}^{op}, \text{Set}]$ factors through the forgetful functor $F : \text{Group} \to \text{Set}$.

Example 2.2.6 (General Linear Group)**.** For any ring R, the set $GL_n(R)$ has the structure of a group. For any map $\phi^{\#} : \text{Spec}S \to \text{Spec}R$, we have the induced map $\varphi : GL_n(R) \to GL_n(S)$. To see that GL_n is a group object, it suffices to show this induced map is indeed a homomorphism of groups. Consider two matrices $A, B \in GL_n(R)$ Looking elementwise, we know $(AB)_{ij} = \sum_k a_{ik}b_{kj}$, so $(\varphi(AB))_{ij}) = \phi(\sum_k a_{ik}b_{kj})$. It quickly follows from the fact that ϕ is a ring homomorphism that $\varphi(AB) = \varphi(A)\varphi(B)$.

Example 2.2.7 (Automorphisms)**.** Consider the automorphism group of a flat and projective scheme X/S, denoted $\text{Aut}_S(X)$. For a morphism of schemes $X \to Y$, we can pull back an S-automorphism of Y to get an S-automorphism of X. In this manner we see $\text{Aut} : (Sch)^{op} \to \text{Set}$ factors through the forgetful functor $\text{Grp} \to \text{Set}$. This only shows that Aut is a group functor, the representability follows from the representability of the Hom scheme.

We show that the moduli space of birational derived equivalences representable by realizing it as an open subfunctor of a known representable functor.

Definition 2.2.8. Let $F, G \in [\mathcal{C}^{op}, \text{Set}]$. We say F is a **subfunctor** of G and write $F \subseteq G$ if the following conditions are satisfied:

1. $F(X) \subseteq G(X)$ for every $X \in \mathcal{C}$

2. For any morphism $f : X \to Y$, $F(f)$ is the restriction of $G(f)$ to to $F(X)$.

Example 2.2.9 (General Linear Group)**.** Every invertible matrix is, in particular, a matrix. So as functors, we find $GL_n \subseteq Mat_{n,n}$. Recall that we identified $GL_n = \mathrm{D}(det) \subseteq \mathrm{Spec}(\mathbb{Z}[(x_{ij})] = Mat_{n,n}$ as the non-vanishing locus of the determinant. So not only is $h_{GL} \subseteq h_{Mat}$, but the corresponding morphism $GL_n \hookrightarrow Mat_{n,n}$ is an open immersion. Considering the fact that an open subset of a scheme comes with a natural scheme structure, we are led to the following categorical notion of an open immersion.

Definition 2.2.10 (Stacks 01JF)**.** Take our category above $\mathcal{C}$ to be (Sch)/S for some base scheme S.

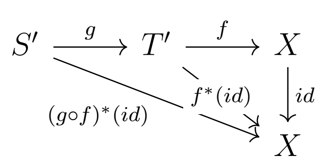
A subfunctor $F \subseteq G$ is an **open subfunctor** if for all pairs (T, ξ), where T is a scheme and $\xi \in G(T)$ there is an open subscheme $U_\xi \subseteq T$ with the property:

- A morphism $f : T' \to T$ factors through U_ξ if and only if $f^*\xi \in F(T')$.

Proposition 2.2.11. *An open subfunctor of a representable functor represents an open subscheme.*

If $F \subseteq h_X$ is open, then $F \cong h_U$ for some open subscheme $U \subseteq X$.

Proof. Consider the pair (X, id) and write $U \subseteq X$ for the subscheme U_{id} in the definition. We then claim that $F \cong h_U$. Let's be very careful here. Consider a scheme T'. Since $F \subseteq h_X$, there is a bijection between elements $\xi \in F(T')$ and a particular subset of $\mathrm{Hom}(T', X)$. This means we may write $\xi \in F(T')$ and $\xi : T' \to X$ with no ambiguity. Now $\xi^*(id) = \xi$ (see diagram below), which we know is in $F(T')$. By the definition above, this is equivalent to the claim that ξ factors through $U = U_{id}$.



Morphisms $T' \to X$ with image in U are in natural bijection with morphisms $T' \to U$, so we have shown $F(T')$ is in bijection with $\mathrm{Hom}(T', U)$ for every T'. The proof of naturality is contained in the diagram above, and we have an isomorphism of functors. □

A moduli space is a special type of representable functor. We make the following not-very-specific definition.

Definition 2.2.12. A moduli functor $\mathcal{M} : (\mathrm{Sch}/S)^{op} \to \mathrm{Set}$ is a functor where

$$\mathcal{M}(T) = \{\text{“Families” over } T\}$$

where we have an object for each point $t \in T$ that vary continuously with T.

In what follows, we construct the moduli functor of derived equivalences (and more generally, fully faithful functors) which extend birational maps. We do not show that it is representable by a scheme, but we do prove algebraicity by embedding it in Lieblich's “mother of all moduli” [Lie06]. This shows algebraicity as a stack, and since the kernel of a fully faithful functor is necessarily simple, we can consider these kernels up to isomorphism and achieve representability as an algebraic space. As we do not go beyond this result, we will not discuss algebraic spaces in this thesis.

2.3 Birational Geometry

The literature on birational geometry is vast and we make no attempt at a complete discussion. For basic properties of birational maps, we refer to [Sta20, Tags 01RR, 01RN], and for the K-equivalence relation we follow[Wan98] and the notes [Pop11].

Definition 2.3.1. Let X and Y be schemes with finitely many irreducible components. A *rational map* $\varphi : X \dashrightarrow Y$ is an equivalence class of morphisms $f : U \to Y$ on open sets $U \subseteq X$. We say $f \sim f'$ if there is an open $W \subseteq U \cap U'$ such that $f'|_W = f|_W$. We say φ is *defined at* x if there is a representative (U, f) with $x \in U$. The *domain of definition* is the set of points where φ is defined.

When we consider our moduli problem, we use the following relative notion.

Definition 2.3.2. If X and Y are schemes over a base scheme S, we call the map S-*rational* if there is a representative which is an S-morphism.

Definition 2.3.3. We say X and Y are (S-)*birational* if they are isomorphic in the category of schemes and (S-)rational maps.

Another useful characterization is the following, which that a rational map is completely determined by the image of the generic points of the components.

Lemma 2.3.4 (Stacks 0BX8)**.** *Let S be a scheme. Let X and Y be schemes over S. Assume X has finitely many irreducible components with generic points $x_1, \dots, x_n$, and let $s_i \in S$ be the image of x_i. If $Y \to S$ is finitely presented, we have a bijection*

$$\left\{ \begin{array}{c} S\text{-rational maps} \\ \text{from } X \text{ to } Y \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} (y_1, \varphi_1, \dots, y_n, \varphi_n) \text{ where } y_i \in Y \text{ lies over } s_i \text{ and} \\ \varphi_i : \mathcal{O}_{Y,y_i} \to \mathcal{O}_{X,x_i} \text{ is a local } \mathcal{O}_{S,s_i}\text{-algebra map} \end{array} \right\}.$$

Definition 2.3.5. A *resolution* of a rational map $\varphi : X \dashrightarrow Y$ is a scheme Z along with rational morphisms $p : Z \to X$ and $q : Z \to Y$ such that $q \circ p^{-1} \sim \varphi$.

A resolution can always be obtained by considering the graph of φ, which is the closure of the graph of $f : U \to Y$ in $X \times Y$ for $(f, U) \in [\varphi]$. Especially important is the situation when we can relate the canonical bundles of X and Y on this resolution.

Definition 2.3.6. We say two smooth projective varieties X and Y are K-equivalent, and write $X \sim_K Y$, if there is a smooth scheme Z with projective birational morphisms $p : Z \to X$, $q : Z \to Y$ with $p^*\omega_X \cong q^*\omega_Y$. We also have the notion of K-inequality, where we write $X \leq_K Y$ if with the same setup we find $H^0(Z, q^*\omega_Y \otimes (p^*\omega_X)^{-1}) \neq 0$, i.e. $q^*K_Y - p^*K_X$ is linearly equivalent to an effective divisor.

Wang discusses this notion in much greater generality in his paper [Wan98], but for clarity of concept we restrict ourselves to the smooth case.

Example 2.3.7. The most fundamental example of a K-inequality is that of a blowup Y of a smooth variety X along a smooth subvariety of codimension c. In this case $Z \cong Y$ and $K_Y - p^*K_X$ is equivalent to a multiple the exceptional divisor of the blow up by the standard formula $K_Y = p^*K_X + (c-1)E$.

This partial ordering fits into the minimal model program via the following theorem

Theorem 2.3.8. *[Wan98, Thm 1.4]*
Let $f : X \dashrightarrow Y$ be a birational map between two varieties with canonical singularities. Suppose the exceptional locus Z is proper and K_X is nef along Z. Then $X \leq_K Y$ and furthermore, if Y is terminal we have $\mathrm{codim}_Z X \geq 2$.

The codimension statement is true for any K-inequality with mild singularities; it is a consequence of the canonical bundle formula for the morphisms in the correspondence $X \leftarrow Z \rightarrow Y$. In low dimension, the K equivalence relation is very restrictive.

Proposition 2.3.9. *If X and Y are smooth projective surfaces and $X \sim_K Y$, then $X \cong Y$.*

This follows from the fact that a proper birational morphism of smooth surfaces can be factored into a sequence of one point blow ups. For a complete proof, see [Pop11, Prop 1.4]. The notes [Pop11, Chapters 4-6] also outline the following corollary to K-equivalence, due to Kontsevich

Corollary 2.3.10. *If $X \sim_K Y$, then X and Y have equal Hodge and Betti numbers.*

More implications of K-equivalence and inequality are discussed in [Wan98], including a comparison of $\mathbb{F}_q$ rational points

Corollary 2.3.11 (Wang Cor 3.3)**.** *If $X \leq_K Y$ then $|\bar{X}(\mathbb{F}_q)| \leq |\bar{Y}(\mathbb{F}_q)|$*

In this thesis, we start with an integral transform $\Phi_{\mathcal{P}} : \mathrm{D}^b(X) \rightarrow \mathrm{D}^b(Y)$ and discuss the situation when $\mathrm{Supp}(\mathcal{P})$ has an irreducible component which is the graph of a birational map φ. In this situation, we say $\Phi_{\mathcal{P}}$ is an extension of φ to the derived category. By modifying

Kawamata's argument in [Kaw02], we are able to show the induced birational correspondence is a K-equivalence. We also discuss the the situation of a fully faithful functor extending a birational map and conclude that the birational map must induce a K-inequality, owing to the canonical natural transformation from right to left adjoint discussed in the next chapter.

2.4 Derived Categories

We will avoid a full discussion of the theory of derived categories. In particular, we will not discuss the construction of the category or derived functors, instead referring to standard texts on the subject: [Huy06], [BBR09, Appendix A], [Sta20, Tag 05QI]. This section serves to gather and highlight several important results and compatibilities that appear in later proofs.

Notation. We write $\mathrm{D}^b(X) = \mathrm{D}^b(Coh(\mathcal{O}_X)$ for the bounded derived category of coherent sheaves of $\mathcal{O}_X$ modules. If our variety X is smooth, then any object is isomorphic to a bounded complex of locally free sheaves: a perfect complex. As we only consider derived functors, we drop the decorations and write f^* for $\mathrm{L}f^*$, $\otimes$ for $\otimes^{\mathbb{L}}$, etc., with the exception of Hom. We write $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ for the set of morphisms in the derived category $R\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ for the derived functor. For the internal hom, we only consider the derived version, but we still will write $R\mathcal{H}om(\mathcal{F}, \mathcal{G})$ in this situation. The cohomology sheaves of a complex $\mathcal{P}$ will be written $\mathcal{H}^i(\mathcal{P})$.

Consider morphisms $f : X \to Y$ and $g : Y \to Z$ between smooth and projective schemes. Then the following identities hold:

- (Projection Formula) – If f is proper, $f_*\mathcal{E} \otimes \mathcal{F} \cong f_*(\mathcal{E} \otimes f^*\mathcal{F})$
- (Adjunction of f_* and f^*) – If f is proper, $\mathrm{Hom}(f^*\mathcal{F}, \mathcal{E}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$
- (Local and Global Hom) – $R\Gamma \circ R\mathcal{H}om(\mathcal{F}, -) = R\mathrm{Hom}(\mathcal{F}, -)$

- (Internal Adjunction) – If f is proper, $f_*\mathcal{H}om(f^*\mathcal{F},\mathcal{E}) \cong \mathcal{H}om(\mathcal{F}, f_*\mathcal{E}))$
- (Pullback commutes with tensor product) – $f^*\mathcal{E}\otimes f^*\mathcal{F} \cong f^*(\mathcal{E}\otimes\mathcal{F})$
- (Functoriality of pullback) – $g^*\circ f^* \cong (g\circ f)^*$
- (Tensor Hom adjunction) – $R\mathcal{H}om(\mathcal{E}\otimes\mathcal{F},\mathcal{G}) \cong R\mathcal{H}om(\mathcal{E},\mathcal{H}om(\mathcal{F},\mathcal{G}))$
- (Tensor compatibility) – $R\mathcal{H}om(\mathcal{E},\mathcal{F}\otimes\mathcal{G}) \cong R\mathcal{H}om(\mathcal{E},\mathcal{F})\otimes\mathcal{G}$
- (Grothendieck Duality) – $f_*R\mathcal{H}om(\mathcal{F}, f^*\mathcal{E}\otimes\omega_f[dim(f)]) \cong R\mathcal{H}om(f_*\mathcal{F},\mathcal{E})$

When studying integral transforms in families, the base change isomorphism is indispensible. This map, and the proof of isomorphism, is given in full generality in [Sta20, Tag 08IB], and is a formal consequence of adjunction and functoriality of pullback.

Setup. Consider the fiber square below, with f proper.

$$\begin{array}{ccc} X\times_Z Y & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Z \end{array}$$

Theorem 2.4.1. *There is a canonical map*

$$g'^*f_*\mathcal{F} \xrightarrow{\sim} f'_*g^*\mathcal{F}$$

Which is an isomorphism if f or u is flat.

The map is constructed to be the adjoint of the map $f'^*\mathbf{L}g^*f_*\mathcal{F}\to g'^*\mathcal{F}$. Owing to the isomorphisms of functors $f'^*g^*\cong(g\circ f')\cong g'^*f^*$, we equivalently define a map $g'^*f^*f_*\mathcal{F}\to g'^*\mathcal{F}$. This map is simply g'^* applied to the map $f^*f_*\mathcal{F}\to\mathcal{F}$ coming from the counit of the adjunction.

In our study of integral transforms and adjoints, a particularly important construction is that of the derived dual object. We give a few important properties. Since we are working with the bounded derived category, we assume that the scheme X is smooth so the dual is bounded above.

Definition 2.4.2. The dual of an object in the derived category $\mathcal{P} \in \mathrm{D}^b(X)$ is the object $\mathcal{P}^\vee = R\mathcal{H}om(\mathcal{P}, \mathcal{O}_X)$

Remark 2.4.3. For a sheaf $\mathcal{F} \in \mathrm{D}^b(X)$, the dual in the derived category may not agree with the classical dual sheaf. Indeed, for $i : X \subseteq Y$ a smooth closed subvariety of codimension c, we can compute $(i_*\mathcal{O}_X)^\vee \cong i_*\omega_X \otimes \omega_Y^{-1}[-c]$ via a standard duality argument as in [Huy06, Cor 3.40].

Despite this, the derived dual has several pleasing properties that follow from the analogous properties of locally free sheaves or more formally from the identities above.

Proposition 2.4.4. *The functor $R\mathcal{H}om(\mathcal{F}, -)$ is isomorphic to the functor $\mathcal{F}^\vee \otimes (-)$.*

Proposition 2.4.5. *[BBR09, Prop A.88]*
The functor $(-) \otimes \mathcal{P}^\vee$ is both left and right adjoint to the functor $(-) \otimes \mathcal{P}$.

Proposition 2.4.6. *For any morphism $f : T \to X$ there is a natural isomorphism $(f^*\mathcal{P})^\vee \cong f^*(\mathcal{P}^\vee)$.*

Proof. We have $f^*R\mathcal{H}om(P, \mathcal{O}_X) = R\mathcal{H}om(f^*\mathcal{P}, f^*\mathcal{O}_X) = R\mathcal{H}om(f^*\mathcal{P}, \mathcal{O}_T)$. □

Proposition 2.4.7. *The double dual of an object is canonically isomorphic to the original object: $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$*

Another relevant construction is the support of an object $\mathcal{F} \in \mathrm{D}(X)$. We will see how the support of an integral kernel $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ and its adjoints relate the geometry of X and Y.

Definition 2.4.8. The *support* of an object $\mathcal{P} \in \mathrm{D}^b(X)$ is the union of the supports of the cohomology sheaves $\mathrm{Supp}(\mathcal{P}) = \bigcup_i \mathcal{H}^i(\mathcal{P})$

The structure sheaves of points form a spanning class in the derived category, which gives us a useful characterization of supports.

Proposition 2.4.9. *[BBR09, Prop A.91]*
A closed point $x \in X$ is in the support of $\mathcal{P} \in D^b(X)$ if and only if $\mathrm{Hom}(\mathcal{P}, \mathcal{O}_x[i]) \neq 0$ for some i.

As a consequence of the above and the reflexivity of objects in the bounded derived category of smooth schemes we have the following:

Proposition 2.4.10. *[Huy06, Lem 3.32]*
For any $\mathcal{F} \in D^b(X)$, we have

$$Supp(\mathcal{F}) = Supp(\mathcal{F}^\vee)$$

Since we are only considering bounded complexes with coherent cohomology, we note:

Proposition 2.4.11. *For a bounded complex $\mathcal{P}$, the set $\mathrm{Supp}(\mathcal{P})$ is closed in X.*

Related to our support discussion we have the following fiberwise criterion for checking if an object is zero when working over a base scheme.

Lemma 2.4.12. *[RMdS07, Lemma 2.3] Let $X \to S$ be a proper morphism and consider an object $\mathcal{E} \in D^b(X)$. If $j_s^*\mathcal{E} = 0$ for every closed point $s \in S$.*

Chapter 3

FOURIER–MUKAI TRANSFORMS

Here we discuss a fundamental class of functors between derived categories of schemes. Introduced by Mukai in [Muk81], integral transforms (Fourier-Mukai transforms) have proved an indispensible tool in the study of derived functors and derived categories. For the standard material we refer to [Huy06] or [BBR09]. We review a few key proofs and results from these sources, and additionally we are able to leverage the canonical natural transformation $\Theta : \Phi_R \to \Phi_L$ from right to left adjoint to provide some new proofs of known theorems. After gathering some important results, we proceed with a discussion on base change and describe the moduli space of fully faithful integral transforms. This space, denoted $FF(X, Y)$, is realized as an open substack of Lieblich's moduli of complexes constructed in [Lie06]. In the next chapter, we will introduce the birationality condition and construct the open substack of birational fully faithful transforms.

3.1 First Properties

Definition 3.1.1. Let X and Y be proper varieties over a field k. Write $X \times Y$ for the product over k and p, q for the projections from the product onto X and Y respectively. Let $\mathcal{P} \in \mathrm{D}(X \times Y)$ be an object. The *integral transform* with *kernel* $\mathcal{P}$ is the functor

$$\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y) \quad \text{with} \quad \Phi_{\mathcal{P}}(\mathcal{E}) = \mathrm{R}q_*(\mathcal{P} \otimes^{\mathrm{L}} \mathrm{L}\pi^*\mathcal{E})$$

We reserve the phrase *Fourier-Mukai Transform* for an integral transform which induces an equivalence of categories. To avoid cumbersome notation, we will drop the R and L decorations with the understanding that we only deal with derived functors in this context.

Example 3.1.2. For a proper morphism $f : X \to Y$, the kernels of f_* and f^* are both given by the structure sheaf of the graph $\mathcal{O}_{\Gamma_f} \in \mathrm{D}^b(X \times Y)$. Thus we begin to see the connection between the support of a kernel and the geometry of X and Y. We will generalize this example by considering kernels which are isomorphic to the structure sheaf of a graph on an open subcheme of $X \times Y$.

A general functor between derived categories can prove difficult to work with, as calculation often hinges on working with a particular resolution of a complex. The class of Fourier-Mukai transforms is much better behaved: by the uniqueness result in Orlov's theorem, along with heavy use of the projection formula and a dollop of Grothendieck duality, we obtain several workable formulas for computing with FM transforms.

Proposition 3.1.3. *The composition of integral transforms is an integral transform. The kernel of $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}}$ will be written $\mathcal{Q} \circ \mathcal{P}$ and is given by the formula*

$$\mathcal{Q} \circ \mathcal{P} = \pi_{13*}(\pi_{23}^* \mathcal{Q} \otimes \pi_{12}^* \mathcal{P})$$

Proof. [BBR09, Prop 1.3] □

As a straightforward generalization of [BBR09, Prop 1.13] to the relative setting, we have the following:

Proposition 3.1.4. *Let $\Phi_{\mathcal{P}}$ be an integral transform with kernel $\mathcal{P} \in D^b(X \times_S Y)$. The objects*

$$\mathcal{P}_R = \mathcal{P}^\vee \otimes \pi_X^* \omega_{X/S}[\dim X/S] \qquad \mathcal{P}_L = \mathcal{P}^\vee \otimes \pi_Y^* \omega_{Y/S}[\dim Y/S]$$

induce the right and left adjoints to $\Phi_{\mathcal{P}}$, which will be denoted Φ_R and Φ_L respectively.

Remark 3.1.5. A more general discussion on the existence and form of adjoints to exact functors is given [Riz17], including a discussion of the nonprojective and singular cases. Of particular interest is the discussion of when the kernel of $\Phi_{\mathcal{P}}$ is a perfect complex. We content ourselves to the bounded derived categories of smooth schemes, in which case the kernel of a fully faithful functor is necessarily perfect. This allows us to construct the moduli space of fully faithful functors inside of Lieblich's moduli of complexes.

It turns out that a large class of exact functors between derived categories are integral transforms. The paper [CS12a] gives an excellent survey of first properties and existence questions for kernels, but for this thesis we restrict ourselves to the simplest situation. The most relevant and well-known result on the existence and uniqueness of kernels is due to Orlov.

Theorem 3.1.6. *[Orl97, Thm 2.2]*
Let X and Y be smooth projetive varieties over a field k. Let $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ be a fully faithful exact functor [with a left (or right) adjoint]. Then there exists an object $\mathcal{P} \in \mathrm{D}^b(X \times Y)$, unique up to isomorphism, such that $\Phi \cong \Phi_{\mathcal{P}}$.

This theorem has been extended in numerous ways. It is now known that such a functor always has left and right adjoints and this assumption can be removed. This is obtained as a corollary to the main result of Bondal and van Der Bergh in [BvdB03]. See, for instance, [BBR09, Prop 1.20]. There have been other generalizations, including a version for DG-categories; it would be interesting to see how our techniques of studying birationality problems transfer to that setting.

Remark 3.1.7. There exist exact functors between derived categories which are not integral transforms. See [RVdBN19] and [Vol19].

3.2 Right to Left Adjoint

The canonical bundle is the cornerstone of birational geometry. This object, which also plays the role of the dualizing sheaf (or complex), is central to our understanding of integral transforms as seen in the adjoint kernel formula. In this section, we make the connection between fully faithful functors and canonical (K-) inequality through the following categorical lemma.

Lemma 3.2.1 ([Joh11])**.** *Let F be a fully faithful functor with right adjoint R and left adjoint L. There is a canonical natural transformation $R \to L$.*

This natural transformation is, in fact, induced by a morphism of kernels and we can conclude:

Theorem 3.2.2. *Let $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ be a fully faithful integral transform. There is a canonically defined morphism in the derived category*

$$\mathcal{P}^{\vee} \otimes \pi_X^* \omega_X[\mathrm{dim} X] \to \mathcal{P}^{\vee} \otimes \pi_Y^* \omega_Y[\mathrm{dim} Y]$$

Johnstone describes his lemma as a folklore result; indeed discussion of this natural transformation appears earlier in [Law07], but his is the only proof available in the literature and it appears the result has not yet been applied to algebraic geometry. We proceed by giving a new proof of Johnstone's lemma, which has the added benefit of describing the relation between the inverse of the unit $\eta : id \to RF$ and the inverse of the counit $\beta : LF \to id$ via the canonical natural transformation. In the following chapter, we will discuss the connection to birational geometry.

Proposition 3.2.3. *Let F be a fully faithful functor with right adjoint R and left adjoint L. There is a canonically defined natural transformation $\theta : R \to L$, equivalently defined in terms of the unit and counit maps of the opposing adjunctions.*

$$R \xrightarrow{R\alpha} RFL \xrightarrow{\eta^{-1}L} L \qquad R \xrightarrow{\beta^{-1}R} LFR \xrightarrow{L\epsilon} L$$

Furthermore, we have the following identities:

$$\eta^{-1} = \beta \circ \theta F \qquad \beta^{-1} = \theta F \circ \eta$$

Proof. First, we define $\theta = \eta^{-1}L \circ R\alpha$. Using this definition, we then prove the formula for β^{-1} conclude the second definition is equivalent.

Note that β and η are assumed invertible, so we proceed by giving a one sided inverse. This proof is best presented using the visual language of string diagrams ([Cat07]); we provide the classical notation below.

$$\begin{aligned}
\beta \circ (\eta^{-1}L \circ R\alpha)F \circ \eta &= \beta \circ \eta^{-1}LF \circ R\alpha F \circ \eta \\
&= \eta^{-1} \circ RF\beta \circ R\alpha F \circ \eta \\
&= \eta^{-1} \circ R(F\beta \circ \alpha F) \circ \eta \\
&\cong \eta^{-1} \circ R1_F \circ \eta \\
&= id
\end{aligned}$$

With the formula for β^{-1} in hand, we compute $\text{Ł}\epsilon \circ \beta^{-1}R$ and show it is naturally isomorphic to $\eta^{-1}L \circ R\alpha$. Again, the should be viewed using string diagrams.

$$\begin{aligned}
L\epsilon \circ \beta^{-1}R &= L\epsilon \circ (\theta F \circ \eta)R \\
&= L\epsilon \circ \theta FR \circ \eta R \\
&= \theta \circ R\epsilon \circ \eta R \\
&= (\eta^{-1}L \circ R\alpha) \circ R\epsilon \circ \eta R \\
&\cong \eta^{-1}L \circ R\alpha \circ 1_R \\
&= \eta^{-1}L \circ R\alpha
\end{aligned}$$

□

Now we apply this canonical natural transformation to the context of integral transforms. We appeal to the following:

Lemma 3.2.4. *Let X and Y be smooth proper schemes over a field k and $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ be an integral transform. The counit (and unit) are induced by a morphism of kernels.*

This was proven in great detail for separated schemes over an algebraically closed field of characteristic 0 in [AL12], though the assumptions can likely be relaxed further due to progress in foundations of grothendieck duality. The construction of the morphism is also discussed in [LO11, Sec 3].

Corollary 3.2.5. *With the conditions above, assume furthermore that $\Phi_{\mathcal{P}}$ is fully faithful. Then there is a canonically defined morphism in the derived category:*

$$\mathcal{P}^{\vee} \otimes \pi_X^* \omega_X[\text{dim}X] \to \mathcal{P}^{\vee} \otimes \pi_Y^* \omega_Y[\text{dim}Y]$$

Question 3.2.6. What additional assumptions are necessary to conclude F is fully faithful from the existence of a natural transformation $R \to L$? Is it sufficient that the induced map $id \to RF \to LF \to id$ is the identity natural transformation?

We conclude this section with a couple of example calculations. Particularly relevant to this discussion is the following.

Lemma 3.2.7. *Let $f : Y \to X$ be a morphism between smooth projective varieties. Then $\mathrm{R}f_* \mathcal{O}_Y = \mathcal{O}_X$ if and only if $\mathrm{L}f^* : \mathrm{D}(X) \to \mathrm{D}(Y)$ is fully faithful.*

Proof. Assume that $\mathrm{R}\pi_* \mathcal{O}_Y \cong \mathcal{O}_X$. Then for any object $\mathcal{E}$, we have a functorial isomorphism

$$\mathrm{R}\pi_*(\mathcal{O}_Y \otimes \mathrm{L}\pi^* \mathcal{E}) \cong \mathcal{O}_X \otimes \mathcal{E} \cong \mathcal{E}$$

by the projection formula. Thus we conclude $\mathrm{R}\pi_* \circ \mathrm{L}\pi^* \cong id$ and $\mathrm{L}\pi^*$ is fully faithful.
If $\mathrm{R}\pi_* \circ \mathrm{L}\pi^* \cong id$, then in particular $\mathrm{R}\pi_* \mathrm{L}\pi^*(\mathcal{O}_X) \cong \mathcal{O}_X$. Expanding the left side and applying the projection formula, we conclude

$$\mathrm{R}\pi_*(\mathcal{O}_Y \otimes \mathrm{L}\pi^* \mathcal{O}_X) \cong \mathrm{R}\pi_* \mathcal{O}_Y \cong \mathcal{O}_X$$

□

Example 3.2.8. (Projective Space)
Let k be a field and consider $f : \mathbb{P}_k^n \to k$. The functor $\Phi = \mathrm{L}f^* : \mathrm{D}(k) \to \mathrm{D}(\mathbb{P}^n)$ is fully faithful, with kernel given by the graph of f. The right adjoint to Φ has kernel $\mathcal{O}_{\mathbb{P}_k^n}$ and the left adjoint has kernel $\mathcal{O}_{\mathbb{P}_k^n}(-n-1)[n]$, so the canonical morphism is a nonzero map

$$\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-n-1)[n]$$

We can realize this as a nonzero class in $\mathrm{Ext}^n(\mathcal{O}_{\mathbb{P}^n}, \omega_{\mathbb{P}^n}) = H^n(\mathbb{P}^n, \omega_{\mathbb{P}^n}) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) = k$. It may be worth noting that for $n = 1$ this is the class of the euler exact sequence.

Example 3.2.9. (Blowing Up)

Let $f : Y \to \mathbb{P}^n$ be the blow up along a smooth subvariety $Z \subseteq \mathbb{P}^n$ of codimension 2, and write $E \subseteq Y$ for the exceptional divisor. Again, $\Phi = \mathrm{L}f^*$ is fully faithful. We have $\omega_Y = f^*\omega_{\mathbb{P}^n} \otimes \mathcal{O}(E)$ and in fact the morphism from right to left adjoint is given by a morphism of kernels

$$\mathcal{O}_{\Gamma_f}^\vee \otimes \pi_X^*\mathcal{O}(-3)[2] \to \mathcal{O}_{\Gamma_f}^\vee \otimes \pi_Y^* \left(f^*\mathcal{O}(-3) \otimes \mathcal{O}_Y(E)\right)[2]$$

By restricting to the graph of the blowup, we find this is induced by a section

$$\mathcal{O}_Y \to \mathcal{O}_Y(E)$$

This is also nonzero class in a 0 dimensional space of sections.

3.3 Full Faithfulness and Equivalence Criteria

Here, we gather a few criteria for detecting when an integral transform is fully faithful or an equivalence. We are able to give a new and simplified proof of Bridgeland's criterion for a fully faithful functor to be an equivalence by using the canonical natural transformation described in the previous section.

When we are describing our moduli functors, the most useful characterization of fully faithful functors and equivalences will be:

- A functor F with left adjoint L is fully faithful if and only if the counit $\epsilon : FL \to id$ is an isomorphism.
- A functor F with left adjoint L is an equivalence if and only if F and L are fully faithful.

We continue appeal to [AL12] and freely use the fact that for any integral transform $\Phi_{\mathcal{P}}$, the counit map is induced by a morphism of kernels $\mathcal{P}_L \circ \mathcal{P} \to \mathcal{O}_\Delta$.

For most of this thesis, the above characterizations will suffice. Specific to the derived category, we also have the following "orthogonality conditions", originally due to Bondal and Orlov [BO95].

Theorem 3.3.1. *Let X and Y be smooth projective varieties over a field k and $\mathcal{P} \in D(X \times_k Y)$. Then $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ is fully faithful if and only if*

i) $\mathrm{Hom}^i(\Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_y)) = 0 \quad \forall i$ *and* $x \neq y$*; and*

ii) $\mathrm{Hom}^0(\Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_x)) = k$

$$\mathrm{Hom}^i(\Phi_{\mathcal{P}}(\mathcal{O}_x), \Phi_{\mathcal{P}}(\mathcal{O}_x)) = 0 \quad \forall i \notin [0, \dim(X)].$$

Remark 3.3.2. Another proof is available in [BBR09, Sec 1.3.2], where the conditions above are formalized as the notion of "strong simplicity" of the kernel $\mathcal{P}$. The orthogonality conditions are rephrased in this context: $\Phi_{\mathcal{P}}$ is fully faithful if and only if $\mathcal{P}$ is strongly simple over X.

As a corollary, we have established the following result which allows us to view the moduli stack of fully faithful complexes inside the "mother of all moduli".

Corollary 3.3.3. *The kernel $\mathcal{P}$ of a fully faithful functor $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ is a simple and gluable complex in the sense of [Lie06, Def 2.1.8].*

The following result was first proven by Bridgeland in [Bri99]; we use the canonical morphism described in the previous section to give a new proof.

Theorem 3.3.4 (Bridgeland)**.** *Suppose $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ is fully faithful. Then it is an equivalence if and only if $\mathcal{P}_y \otimes \omega_X \cong \mathcal{P}_y$ for every $y \in Y$.*

Proof. If $\mathcal{P}$ is an equivalence, then we see $\mathcal{P}_y \otimes \omega_X \cong \mathcal{P}_y$ by restricting the isomorphism $\Phi_R \cong \Phi_L$. On the other hand, if $\Phi_{\mathcal{P}}$ is fully faithful we have the canonical morphism $\Phi_R \to \Phi_L$. This is induced by a morphism of kernels

$$\mathcal{P}^\vee \otimes \pi_X^* \omega_X \to \mathcal{P}^\vee \otimes \pi_Y^* \omega_Y$$

Let $\mathcal{K}$ denote the cone of this morphism. By assumption, $\mathcal{K}_y \cong 0$ for every $y \in Y$ and we conclude $\mathcal{K} \cong 0$. □

Bridgeland's proof proceeds by appealing to the following

Theorem 3.3.5 (Bridgeland Thm 3.3)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful exact functor between triangulated categories. Suppose $\mathcal{B}$ is indecomposable, and that not every object of $\mathcal{A}$ is isomorphic to 0. Then F is an equivalence of categories if and only if F has a left adjoint L and right adoint R such that for any object $b \in \mathcal{B}$,*

$$Rb \cong 0 \Rightarrow Lb \cong 0$$

From our perspective we have the following reformulation:

Corollary 3.3.6. *Let X and Y be smooth projective schemes over a field k, and assume Y is irreducible. Let F be a fully faithful exact functor $F : D^b(X) \to D^b(Y)$ and $\Theta : R \to L$ the canonical natural transformation from the right adjoint to the left adjoint. Then Θ is an isomorphism if and only if $Rb \cong 0 \Rightarrow Lb \cong 0$*

Question 3.3.7. For a fully faithful functor $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ which is not an equivalence, what is the structure of those objects $b \in \mathrm{D}^b(Y)$ with $Rb \cong 0$ but $Lb \ncong 0$?
If $F = Lf^*$ for $f : Bl_Z(X) \to X$ a blow up of a smooth variety along a smooth subvariety, one such object is $\mathcal{O}(E)_E$, where E denotes the exceptional divisor.

We can also recover a classical result on the non-existence of semiorthogonal decompositions in the derived category of a Calabi-Yau variety.

Proposition 3.3.8. *Let X and Y be smooth varieties such that $S_X = (-)[n]$ and $S_Y = (-)[n]$. Then any fully faithful functor $\Phi : D(X) \to D(Y)$ is an equivalence.*

Proof. As Φ is fully faithful, it is an integral transform. Furthermore, it has both right and left adjoints, and we have the canonical natural transformation $\Theta : \Phi_R \to \Phi_L$. This natural transformation is induced by a morphism of kernels, which is a nonzero map $\theta \in \mathrm{Hom}(\mathcal{P}^\vee, \mathcal{P}^\vee)$. By simplicity of $\mathcal{P}^\vee$, this map is an isomorphism, so we find $\Phi_R \cong \Phi_L$ and conclude. □

3.4 Base Change Properties

Setup. Assume that $X \to B$ and $Y \to B$ are proper. For any closed point $b \in B$ write X_b, Y_b for the fibers of the structure morphisms. Let $\mathcal{P} \in \mathrm{D}(X \times_B Y)$, and write

$$\Phi = \Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$$

for the associated transform. We wish to study the restriction of this transform to the fibers, which we will denote

$$\Phi_b : \mathrm{D}(X_b) \to \mathrm{D}(Y_b)$$

and is defined by the complex $\mathcal{P}_b = Lj^*\mathcal{P}$, where $j : X_b \times_b Y_b \hookrightarrow X \times_B Y$ is the natural inclusion.

When discussing arbitrary base change $B' \to B$, we write $\Phi' = \Phi_{\mathcal{P}'} : \mathrm{D}(X_{B'}) \to \mathrm{D}(Y_{B'})$, where $\mathcal{P}'$ is the derived pullback of $\mathcal{P}$ along the map $X_{B'} \times_{B'} Y_{B'} \to X \times_B Y$

Remark 3.4.1. Notice that when $\mathcal{P}$ has finite homological dimension, so does the pullback $\mathcal{P}'$. In this case, each of these functors restrict to the bounded derived categories.

We collect a few base change lemmas for integral transforms. The first two are required for the proof above and stated without proof in [BBR09]. The last is new, but not used in the rest of the thesis.

Notice that if $\Phi = \mathbf{R}g_*$, the first lemma is exactly the usual derived base change isomorphism.

Lemma 3.4.2 (Base Change I)**.** *Let $f : B' \to B$ and write X', Y' for the base changes of X and Y to B'. Assume that $f : B' \to B$ or $p : X \to B$ is flat. Then base change holds for Φ, in the sense that for any $\mathcal{E} \in D^b(X)$ there is a functorial isomorphism*

$$\mathbf{L}f_Y^*\Phi(\mathcal{E}) \cong \Phi'(\mathbf{L}f_X^*\mathcal{E})$$

in the derived category of Y'.

Proof. Write q and p for the projections from $X \times Y$ onto X and Y respectively, and use q' and p' for their base changes. All functors are derived, so we drop the R's and L's.

$$\begin{aligned} f_Y^*\Phi(\mathcal{E}) &= f_Y^* p_*(\mathcal{P} \otimes q^*\mathcal{E}) \\ &\cong p'_* f_{X\times Y}^*(\mathcal{P} \otimes q^*\mathcal{E}) \\ &\cong p'_* \left(\mathcal{P}' \otimes f_{X\times Y}^* q^*\mathcal{E}\right) \\ &\cong p'_* \left(\mathcal{P}' \otimes q'^*(f_X^*\mathcal{E})\right) \quad = \Phi'(f_X^*\mathcal{E}) \end{aligned}$$

The first step is an application of the base change isomorphism, which is where the flatness hypotheses come in; the second step is the compatibility of tensor product and derived pullback; and the third step relies on functoriality of pullback. □

Lemma 3.4.3 (Base Change II)**.** *With the same notation as the previous lemma, but now assuming $f : B' \to B$ or $p : X \to B$ is flat. Then pushforward commutes with Φ, in the sense that for any $\mathcal{E} \in D^b(X)$ there is a functorial isomorphism*

$$\mathbf{R}f_{Y*}\Phi'(\mathcal{G}) \cong \Phi\left(\mathbf{R}f_{X*}\mathcal{G}\right)$$

in the derived category of Y

Proof. The proof proceeds similarly to the previous one, using the projection formula in a critical moment to show compatibility of twisting by the kernel before or after pushing forward. Again, we will drop the derived decorations for our functors.

$$\begin{aligned} f_{X*}\Phi'(\mathcal{G}) &= f_{Y*}p'_*\left(\mathcal{P}' \otimes q'^*\mathcal{G}\right) \\ &\cong p_* f_{X\times Y,*}\left(\mathcal{P}' \otimes q'^*\mathcal{G}\right) \\ &\cong p_*\left(\mathcal{P} \otimes f_{X\times Y,*} q'^*\mathcal{G}\right) \\ &\cong p_*\left(\mathcal{P} \otimes q^* f_{Y*}\mathcal{G}\right) \qquad = \Phi(f_{Y*}\mathcal{G}) \end{aligned}$$

In the first step we use the isomorphism $R(f \circ g)_* \cong Rf_* \circ Rg_*$. □

Now see what we can say about an arbitrary pair of morphisms $f : X' \to X$ and $g : Y' \to Y$. This is not necessary for the remainder of this thesis. Given a functor $\Phi = \Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$, where $\mathcal{P} \in D^b(X \times Y)$, we define $\Phi' = \Phi_{\mathcal{P}'} : \mathrm{D}(X) \to \mathrm{D}(Y)$ via the kernel $\mathcal{P}' = (f \times g)^*\mathcal{P})$. It is helpful to stare at the following diagram when reviewing proofs.

$$\begin{array}{ccccc}
X' \times Y' & \xrightarrow{g''} & X' \times Y & \xrightarrow{q'} & X' \\
\big\downarrow{\scriptstyle f''} & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\
X \times Y' & \xrightarrow{g'} & X \times Y & \xrightarrow{q} & X \\
\big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} & & \\
Y' & \xrightarrow{\quad g \quad} & Y & &
\end{array}$$

Note that $\Phi'(\mathcal{G}) = p'_* f''_*(\mathcal{P}' \otimes g''^* q'^* \mathcal{G})$ and that $\mathcal{P}' \cong (f \times g)^*\mathcal{P} \cong f''^* g'^* \mathcal{P} \cong g''^* f'^* \mathcal{P}$

Lemma 3.4.4 (Base Change III)**.** *With the setup above. Given a complex $\mathcal{E} \in D(Z')$, there is a natural morphism*

$$\Phi(f_*\mathcal{E}) \to g_*\Phi'(\mathcal{E})$$

which is adjoint to an isomorphism in each of the following cases

a) The maps $f : X' \to X$ and $X \to S$ are flat,

b) The maps $g : Y' \to Y$ and $Y \to S$ are flat, or

c) The maps $f : X' \to X$ and $g : Y' \to Y$ are flat

Proof. We first prove a compatibility between twists and base change in the top left square of the diagram. Namely that there is a natural map

$$g'^*(\mathcal{P} \otimes f'_*\mathcal{E}) \to f''_*(\mathcal{P}' \otimes g''^*\mathcal{E})$$

which is an isomorphism if f' or g' is flat (more generally if they are tor independent). This follows from the usual base change and the projection formula as follows.

$$\begin{aligned} g'^*(\mathcal{P} \otimes f'_*\mathcal{E}) &\cong g'^*\mathcal{P} \otimes g'^* f'_*\mathcal{E} \\ &\to g'^*\mathcal{P} \otimes f''_* g''^*\mathcal{E} \\ &\cong f''_*(f''^* g'^*\mathcal{P} \otimes g''^*\mathcal{E}) \\ &\cong f''_*(\mathcal{P}' \otimes g''^*\mathcal{E}) \end{aligned}$$

Now for the full calculation of the map $\Phi(f_*\mathcal{E}) \to g_*\Phi'(\mathcal{E})$. We construct the adjoint map $g^*\Phi(f_*\mathcal{E}) \to \Phi'(\mathcal{E})$ which is an isomorphism in the cases mentioned.

$$\begin{aligned} g^*\Phi(f_*\mathcal{E}) &= g^* p_*(\mathcal{P} \otimes q^* f_*\mathcal{E}) \\ &\to p'_* g'^*(\mathcal{P} \otimes f'_* q'^*\mathcal{E}) \\ &\to p'_* f''_*(\mathcal{P}' \otimes g''^*\mathcal{E}) \quad = \Phi'(\mathcal{E}) \end{aligned}$$

The first step is two applications of the base change morphism and the second step is the compatibility calculation above. The conditions $(a), (b), (c)$ each guarantee that all applications of the base change morphism are isomorphisms. □

3.5 Moduli

We continue our discussion of integral transforms in families with an eye on the following moduli functor:

Definition 3.5.1. Fix a base scheme S and morphisms of schemes $X \to B$ and $Y_T = Y \to B$ over S. For a B-scheme T, write $X_T = X \times_B T$ and $Y \times_B T$. Assume that X or Y is flat over B, and define a functor

$$FF_B(X,Y) : (Sch/B)^{opp} \to Sets, \quad T \mapsto \{\mathcal{P} \in \mathrm{D}(X_T \times_T Y_T) \mid \Phi_{\mathcal{P}} \text{ is fully faithful}\}$$

We gather some results on full faithfulness in families of integral transforms. The first says that full faithfulness of a relative integral transform can be checked on closed points.

Then, we establish some conditions for the pullback of a fully faithful kernel to induce a fully faithful functor, and conclude by realizing the moduli functor of fully faithful transforms as an open subfunctor of Lieblich's moduli of complexes [Lie06].

Proposition 3.5.2. *[RMdS07, Prop 2.15] Assume that $X \to B$ and $Y \to B$ are proper and flat and that $\mathcal{P} \in D^b(X \times_B Y)$ has finite homological dimension. Then the relative integral functor $\Phi = \Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is fully faithful (resp. an equivalence) if and only if $\Phi_b : D^b(X_b) \to D^b(Y_b)$ is fully faithful (resp. an equivalence) for every closed point $b \in B$.*

Proof. The key step of this proof is to consider functors $\Phi_L, \Phi_R : \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$ which are left and right adjoint to Φ respectively: $\langle \Phi_L, \Phi, \Phi_R \rangle$ and observe that $\Phi_{\mathcal{P}}$ is fully faithful (an equivalence) if and only if the natural transformation $Id \to \Phi_R \circ \Phi$ (along with $\Phi \circ \Phi_L \to Id$) is an isomorphism. We will only prove the fully faithful part; the proof for an equivalence is nearly identical.

($\Rightarrow$) : If Φ is fully faithful, then for every closed point $b \in B$ and $\mathcal{G} \in \mathrm{D}^b(X_b)$ we have a natural isomorphism $(j_*\mathcal{G})_b \to j_*\mathcal{G}_b$ and by the first base change lemma, we have $j_* (H_b \circ \Phi_b)(\mathcal{G}) \cong (H \circ \Phi)(j_*\mathcal{G})$. Since j is a closed immersion, it reflects isomorphisms and we conclude the existence of a natural isomorphism $(H_s \circ \Phi_s)(\mathcal{G}) \cong \mathcal{G}$

($\Leftarrow$) : Let $\mathcal{F} \in \mathrm{D}^b(X)$ and consider C fitting into the exact triangle

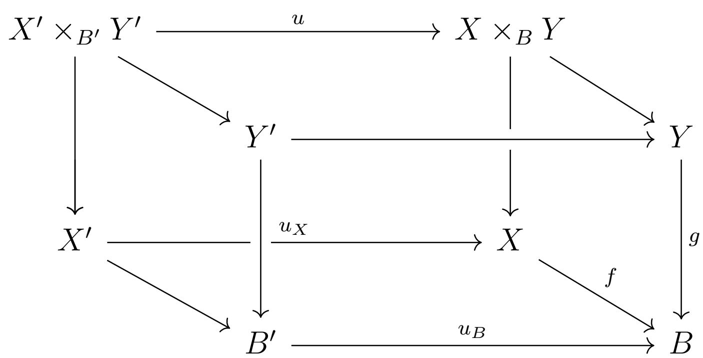
$$\mathcal{F} \to (H \circ \Phi)(\mathcal{F}) \to C \to \mathcal{F}[1]$$

For a closed point $b \in B$ we apply Lj^* and use the second base change lemma to obtain a triangle in $\mathrm{D}(X_b)$

$$\mathbf{L}j^*\mathcal{F} \to (H_b \circ \Phi_b)(\mathbf{L}j^*\mathcal{F}) \to \mathbf{L}j^*C \to \mathbf{L}j^*\mathcal{F}[1]$$

If Φ_b is fully faithful for every closed point $b \in B$, then the map $id \to (H_b \circ \Phi_b)$ is an isomorphism, so we find $\mathbf{L}j^*C = 0$ for every fiber $j : X_b \hookrightarrow X$. This is enough to guarantee $C \cong 0$. Indeed, if C is not 0 in the derived category, there is a maximum integer q_0 such that $\mathcal{H}^{q_0}(C) \neq 0$. If b is a point in the image of $\mathrm{Supp}\,(\mathcal{H}^{q_0}(C)) \to B$, then we have $j_b^*\mathcal{H}^{q_0}(C) \neq 0$, which contradicts the vanishing of $\mathcal{H}^{q_0}(\mathbf{L}j_b^*C)$ □

Setup. Consider a morphism $u_B : B' \to B$ inducing the diagram below, with all faces cartesian.



Proposition 3.5.3. *Adjoint kernels commute with base change with the following additional assumptions:*

1. *If X is flat over B, then $u^*(\mathcal{P}_R) \cong (u^*\mathcal{P})_R$.*

2. *If Y is flat over B, then $u^*(\mathcal{P}_L) \cong (u^*\mathcal{P})_L$.*

3. *If $u_B : B' \to B$ is flat, then $u^*(\mathcal{P}_L) \cong (u^*\mathcal{P})_L$ and $u^*(\mathcal{P}_R) \cong (u^*\mathcal{P})_R$.*

Proof. We show the relevant calculation for the right adjoint. The calculation for the left is completely analagous, and the flatness conditions can be weakened, as they are only present to ensure an isomorphism $u_X^* f^! \cong f'^! u_T^*$.

$$\begin{aligned}
u^*(\mathcal{P}_R) &= u^*(\mathcal{P}^\vee \otimes \pi_{X_T}^* f^! \mathcal{O}_T) \\
&= (u^*\mathcal{P})^\vee \otimes (u^*\pi_{X_T}^*) f^! \mathcal{O}_T \\
&= (u^*\mathcal{P})^\vee \otimes \pi_{X_{T'}}^* (u_X^* f^! \mathcal{O}_T) \\
&= (u^*\mathcal{P})^\vee \otimes \pi_{X_{T'}}^* (f^! u_T^* \mathcal{O}_T) \\
&= (u^*\mathcal{P})^\vee \otimes \pi_{X_{T'}}^* (f^! \mathcal{O}_{T'}) \\
&= (u^*\mathcal{P})_R
\end{aligned}$$

□

Proposition 3.5.4. *Assume X and Y are flat over B, and let $T' \to T$ be an arbitrary morphism. Let $\Phi : D(X_T) \to D(Y_T)$ and $\Psi : D(Y_T) \to D(X_T)$ be integral transforms and Φ', Ψ' be their base change. Then $(\Psi \circ \Phi)' \cong \Psi' \circ \Phi'$*

Proof. This is a straightforward application of the base change isomorphism and functoriality of pullback. If Φ and Ψ have kernels $\mathcal{P}$ and $\mathcal{Q}$, the kernel of their composition is given by $\pi_{13*}(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{Q})$. □

Corollary 3.5.5. *With the same setup and assumptions as the previous proposition, assume that Φ is fully faithful. Then Φ' is fully faithful.*

Proof. Write $\mathcal{P}$ for the kernel of Φ and $\mathcal{P}_L$ for the kernel of the left adjoint. Since Φ is fully faithful, the counit $\Phi_L \circ \Phi \to id$ is an isomorphism. So we find $\mathcal{P}_L \circ \mathcal{P} \cong \mathcal{O}_\Delta$. By the previous two propositions, we can pull back this isomorphism to obtain an isomorphism $\mathcal{P}'_L \circ \mathcal{P}' \cong \mathcal{O}_{\Delta'}$, which induces an isomorphism of functors $\Phi'_L \circ \Phi' \cong id$, and we conclude. □

Setup. In what follows, we assume that X and Y are flat, proper, and finitely presented schemes over a base scheme B. We write $\mathcal{M}_{X \times_B Y}$ for Lieblich's moduli of simple, relatively perfect, universally gluable complexes defined in [Lie06]

Proposition 3.5.6. *The functor $FF_B(X, Y)$ is an open subfunctor of $\mathcal{M}_{X \times_B Y}$.*

Proof. First, note that the kernel of a fully faithful functor is strongly simple by [BBR09, Prop 1.27]. A strongly simple complex in our situation is necessarily universally glueable by [Lie06, Prop 2.1.9]. Now, let $\mathcal{P} \in \mathcal{M}_{X \times_B Y}(T)$. Write Φ for the integral transform induced by $\mathcal{P}$ and Φ_L for the left adjoint to Φ. The functor Φ is fully faithful if and only if the counit map $\epsilon : \Phi_L \circ \Phi \to id$ is an isomorphism. The counit map is induced by a morphism of kernels in $\mathrm{D}(X_T \times_T X_T)$, which we complete to an exact triangle:

$$\mathcal{O}_\Delta \to \mathcal{P}_L \circ \mathcal{P} \to \mathcal{G} \to \mathcal{O}_\Delta[1]$$

The set $\mathrm{Supp}(\mathcal{G})$ is closed in $X_T \times_T X_T$, so by properness of X_T, we see the image of $\mathrm{Supp}(\mathcal{G})$ in T is closed. Take $U \subseteq T$ to be its complement. Now let $f : T' \to T$ be an arbitrary morphism and write $\Phi' = \Phi_{f^*\mathcal{P}}$. To conclude, we show the map f factors through $U \hookrightarrow T$ if and only if Φ' is fully faithful. Full faithfulness of $\Phi_{\mathcal{P}_U}$ can be checked on closed points of U as in [RMdS07, Thm 2.4], by checking that the cone of the counit vanishes on each fiber. Thus, if f factors through $U \hookrightarrow T$ we see $f^{-1}\mathrm{Supp}(\mathcal{G}) = 0$ and so Φ' is fully faithful. On the other hand, $\Phi_{\mathcal{P}_U}$ is necessarily fully faithful, so Φ' is fully faithful for any f factoring through $U \hookrightarrow T$. □

Proposition 3.5.7. *The functor $FM_B(X, Y)$ is an open subfunctor of $FF_B(X, Y)$*

Proof. A fully faithful integral functor $\Phi : \mathrm{D}(X) \to \mathrm{D}(Y)$ with kernel $\mathcal{P}$ is an equivalence exactly when the natural map $\Phi_R \to \Phi_L$ is an isomorphism. This map is induced by a morphism of kernels, so we see the openness of FM $\subseteq$ FF is equivalent to openness of the isomorphism locus of this morphism. □

Chapter 4

BIRATIONAL INTEGRAL TRANSFORMS

In this chapter, we provide a new perspective on the connection between derived categories and birational geometry. The material in the first section should be familiar to experts; most of the material follows from discussion in [Huy06, $Ch.6$], though we give several important birationality criteria at the end of the section. In the remainder of the chapter, we study the class of exact functors which naturally extend a birational map.

Definition 4.0.1. Let X and Y be geometrically integral, Gorenstein, projective schemes over a field k. We say an exact functor $\Phi : \mathrm{D}(X) \to \mathrm{D}(Y)$ with kernel $\mathcal{P}$ is *birationally fully faithful* (a *birational derived equivalence*) if it is fully faithful (resp. a derived equivalence) and there is a line bundle a dense open $U \subseteq X$ such that $\mathcal{P}|_{U\times Y}$ is the graph of an open immersion.

Remark 4.0.2. We use the abbreviations BFF and BD, and write $X \leq_{BD} Y$ (resp. $X \sim_{BD} Y$) if there is a birationally fully faithful functor (resp. equivalence) from $\mathrm{D}^b(X)$ to $\mathrm{D}^b(Y)$.

Remark 4.0.3. As discussed in the previous section, the birationality assumption is equivalent to the existence of closed k-points $x \in X(k)$ and $y \in Y(k)$, and an isomorphism $\Phi_k(\kappa(x)) \cong \kappa(y)$.

One of the motivating questions for this thesis was "When does a birational map extend to a derived equivalence"? An important result in this study is the following theorem of Kawamata.

Theorem 4.0.4 ([Kaw02, Thm 1.4])**.** *Let X and Y be smooth projective varieties. Assume $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$. Then the following hold:*

1. $\dim(X) = \dim(Y)$

2. *If K_X is nef, then K_Y is also nef, and an equality of numerical Kodaira dimension holds*

3. *If $\kappa(X) = \dim(X)$, then X and Y are K-equivalent.*

We are able to generalize the third part of this theorem to birationally fully faithful functors and prove the following:

Theorem 4.0.5. *Let X and Y be irreducible, smooth projective varieties. Assume $X \leq_{BD} Y$. Then $X \leq_K Y$*

One immediate consequence of K-equivalence is that BD-equivalent varieties are isomorphic in codimension one. This alone severely restricts BD equivalence in low dimensions. After proving the theorem above, we spend the rest of this chapter classifying BD equivalences in low dimension, constructing a moduli space of BFF/BD kernels, and discussing further consequences of a birational relation between derived categories.

4.1 Geometry and Supports

Given an integral transform $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$, one lens through which we can study the geometry of X and Y is through the support of the kernel $\mathrm{Supp}(\mathcal{P})$. We collect a few results from [Huy06, Sec. 6.2], most of which are extracted from proofs in [Kaw02]. We will discuss Kawamata's results in the next chapter, as his paper was one of the starting points for this thesis. Huybrechts works only with an equivalence, but many of his proofs carry over verbatim to the fully faithful case.

Lemma 4.1.1. *If $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ is fully faithful, then the map $Supp(\mathcal{P}) \to X$ induced by projection is surjective.*

Proof. Assume there is a closed point $x \in X$ such that $\{x\} \times Y \cap \mathrm{Supp}(\mathcal{P}) = \emptyset$. Then $\mathcal{P}_x = 0$, but by full faithfulness we have $0 \neq \mathrm{Hom}(k(x), k(x)) \cong \mathrm{Hom}(\Phi(k(x), \Phi(k(x))) \cong \mathrm{Hom}(\mathcal{P}_x, \mathcal{P}_x) = 0$, a contradiction. $\square$

With a similar argument, we can also conclude

Lemma 4.1.2. *If $\Phi_{\mathcal{P}}$ is fully faithful, the fibers of Supp$(\mathcal{P}) \to X$ are connected.*

Remark 4.1.3. Since the supports of $\mathcal{P}$ and $\mathcal{P}_L$ are the same, in the case of an equivalence we can make the same statements about the map $\mathrm{Supp}(\mathcal{P}) \to Y$.

The complete picture remains an active area of research, but we may impose some additional conditions to ensure birationality. The following two lemmas ([Huy06, Cor 6.12 & 6.14]) are particularly relevant.

Lemma 4.1.4 (Cor 6.12)**.** *Let $\Phi_{\mathcal{P}}$ be fully faithful, and $Z \subseteq$ Supp$(\mathcal{P})$ be an irreducible component that surjects onto X. If $\dim(Z) = \dim(X)$, then $p : Z \to X$ is a birational morphism. Moreover, if such a component exists, then no other component of* $\mathrm{Supp}(\mathcal{P})$ *dominates X.*

Lemma 4.1.5 (Cor 6.14)**.** *Suppose there exists a closed point $x_0 \in X$ such that $\Phi_{\mathcal{P}}(k(x_0)) \cong k(y_0)$ for a certain closed point $y_0 \in Y$. Then one finds an open neighborhood $x_0 \in U \subseteq X$ and a morphism $f : U \to Y$ with $f(x_0) = y_0$ and such that $\Phi_{\mathcal{P}}(k(x)) \cong k(f(x))$ for all closed points $x \in U$.*

For an equivalence, we can make a symmetric argument about the component Z in and appeal to the equality $\mathrm{Supp}(\mathcal{P}_L) = \mathrm{Supp}(\mathcal{P})$ to conclude $Z \to Y$ is also surjective. Using a different argument, we can extend the same result to the fully faithful case, with the important assumption that X and Y are of the same dimension.

Proposition 4.1.6. *Let $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ be a fully faithful functor with X and Y smooth projective varieties of the same dimension. Assume there is an irreducible component $Z \subseteq$ Supp$(\mathcal{P})$ that surjects onto X with $\dim(Z) = \dim(X)$. Then the projections $Z \to X$ and $Z \to Y$ are birational morphisms.*

Proof. Following Huybrechts, first show that $Z \to X$ is birational and that $\mathrm{Supp}(\Phi_{\mathcal{P}}(\mathcal{O}_x))$ is connected and zero dimensional for any point x. After a suitable shift, we may assume there is a point x_0 with $\Phi_{\mathcal{P}}(k(x_0)) \cong k(y_0)$, by [Huy06, Lemma 4.5]. Then applying [Huy06, Cor 6.14], we find an open neighborhood U of x_0 and a function $f : U \to Y$ with $\Phi_{\mathcal{P}}(k(x)) \cong k(f(x))$ for $x \in U$. By the full faithfulness of $\Phi_{\mathcal{P}}$, we find f is universally injective and étale, whence an open immersion. In particular, the projection $q : Z \to Y$ contains this nonempty open set and is a dominant, birational map. □

This question was also addressed by Calabrese in [Cal17], with an extension to the mildly singular case:

Theorem 4.1.7. *Let X and Y be two geometrically integral, Gorenstein, projective varieties over a field k and let $\Phi : D(X) \to D(Y)$ be a fully faithful functor between their derived categories. Suppose there are closed k-points $x \in X(k)$ and $y \in Y(k)$, and an isomorphism $\Phi_k(\kappa(x)) \cong \kappa(y)$. Then X and Y are birational.*

This is a derived version of a classical fact about varieties: If there is an isomorphism of k-algebras $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$, then X and Y are birational, specifically in a neighborhood of x and y. See [Sta20, Tag 01RN], specifically [Sta20, Tag 0552] for related results over an arbitrary base.

proof idea. To construct the common open set, take the fiber product of X and Y over the moduli space of sheaves on Y. Specifically we have the diagram:

$$\begin{array}{ccccccc} U & \longrightarrow & & \longrightarrow & & \longrightarrow & Y \\ \downarrow & & & & & & \downarrow \\ X & \longrightarrow & \mathcal{S}_X & \longrightarrow & \mathcal{M}_X & \longrightarrow & \mathcal{M}_Y \end{array}$$

Where $\mathcal{S}_X$ is the rigidification of the stack of simple coherent sheaves on X and $\mathcal{M}_X$ is the rigidification of Lieblich's stack of relatively perfect, universally gluable complexes in $\mathrm{D}(X)$. The full faithfulness assumption guarantees all arrows in the diagram are open immersions, and the condition on skyscrapers is enough to show U is nonempty. □

In the following section we restrict our study to integral transforms satisfying the above conditions, called birational integral transforms.

4.2 BD and K equivalence

First, we develop the relationship between Birational Derived equivalence and K equivalence, referencing [Kaw02]. We generalize one of the main results of Kawamata's paper:

Theorem 4.2.1. *[Kaw02, Thm 2.3(2)] Let X and Y be smooth and projective varieties of general type. Assume that $X \sim_D Y$. Then X and Y are K-equivalent.*

In the course of Kawamata's proof, he shows that some component of $\mathrm{Supp}(\mathcal{P})$ is the graph of a birational map, which gives us the following.

Corollary 4.2.2. *For general type varieties, every D equivalence is a BD equivalence.*

We believe that this result could be extended to the fully faithful case if X and Y are the same dimension through a closer study of the morphism $\mathcal{P}^\vee \otimes \pi_X^*\omega_X \to \mathcal{P}^\vee \otimes \pi_Y^*\omega_Y$. This is discussed more in the final chapter.

Without the general type assumption, we are able to conclude that any birationally fully faithful functor induces a K-inequality as a generalization to Kawamata's result.

Theorem 4.2.3. *Let X and Y be irreducible, smooth and projective varieties. Assume that $X \leq_{BD} Y$. Then $X \leq_K Y$, in the sense that for some common resolution $X \xleftarrow{p} \widetilde{Z} \xrightarrow{q} Y$, the difference $q^*K_Y - p^*K_X$ is linearly equivalent to an effective divisor.*

This result completely classifies BD equivalences among derived equivalences for surfaces and varieties of general type, and is a simple modification of Kodaira's original argument. Where he appeals to an an isomorphism between right and left adjoints, we appeal to the canonical morphism from right adjoint to left.

Proof. Let $\mathcal{P} \in \mathrm{D}(X \times Y)$ be the kernel of Φ. Since Φ is fully faithful, there is a canonical natural transformation $\Phi_R \to \Phi_L$. This transformation is induced by a morphism of kernels

$$\mathcal{P}^\vee \otimes \pi_X^*\omega_X[\dim X] \to \mathcal{P}^\vee \otimes \pi_Y^*\omega_Y[\dim Y].$$

By assumption, X and Y are of the same dimension. Additionally, there is an open $U \subseteq X$ where $\mathcal{P}|_{U\times Y} \cong \mathcal{O}_{\Gamma_f}$ for some open immersion $f : U \hookrightarrow Y$. Considering the cohomology sheaves, we obtain a morphism

$$\mathcal{H}^0(\mathcal{P}^\vee) \otimes \pi_X^*\omega_X \to \mathcal{H}^0(\mathcal{P}^\vee) \otimes \pi_Y^*\omega_Y$$

Now take $\tilde{Z}$ to be the normalization of $Z = \mathrm{Supp}(\mathcal{H}^0(\mathcal{P}^\vee))$. By assumption $\mathcal{P}$ is a locally free of rank 1 on the intersection $Z \cap (U \times Y)$. After restricting to the intersection, the free locus of $\mathcal{P}$ must have codimension ≥ 2. On this locus, we obtain a morphism $\nu^*p^*\omega_X \to \nu^*q^*\omega_Y$, which extends to all of $\tilde{Z}$ by normality. □

Corollary 4.2.4. *Let X and Y be smooth and projective varieties. Assume that $X \sim_{BD} Y$. Then X and Y are K-equivalent.*

4.3 Examples and Non-examples

Here we present a few results from the literature that give examples of BD equivalences. Working carefully through the blow up example was part of our motivation for this study, and provides the link between BFF transforms and the minimal model program. In the other examples, we summarize the results and provide reference to the paper for explicit details.

Example 4.3.1. The standard autoequivalences of $\mathrm{D}^b(X)$, generated by automorphisms of X, twists by line bundles, and the shift functor are all BD equivalences.

Example 4.3.2. (Blowing up) The simplest example captured by this definition is that of a blow-up, originally studied in [Orl93]. For $f : Y \to X$ a blowup of a smooth variety X at a smooth center, the functor $\mathrm{L}f^* : \mathrm{D}(X) \to \mathrm{D}(Y)$ is a fully faithful integral transform with kernel $\mathcal{O}_{\Gamma_f}$. Away from the blown up locus, f is an isomorphism, so we find $\mathrm{L}f^*$ is a BFF transform.

The first non-trivial (induced by a true morhpism) example of a BFF transform and BD equivalence comes from [BO95].

Example 4.3.3. (Standard flips and flops) Let X be a smooth algebraic variety with a smooth smooth subvariety $Y \cong \mathbb{P}^k$ with normal bundle $N_{X/Y} \cong \mathcal{O}_Y(-1)^{\ell+1}$ for $\ell \leq k$. The standard flip of X, written X^+, is obtained by blowing up X along Y and contracting the exceptional divisor in a different direction. Write $\tilde{X}$ for the blowup with contraction maps $\pi : \tilde{X} \to X$ and $\pi^+ : \tilde{X} \to X^+$. Bondal and Orlov show that the composition $\pi_* \circ \pi^{+*} : \mathrm{D}^b(X^+) \to \mathrm{D}^b(X)$ is fully faithful. Both π and π^+ are isomorphisms away from the exceptional locus, so we find $\pi_* \circ \pi^{+*}(k(x)) = k(x_+)$ for some points $x \in X \setminus Y$ and $x_+ \in X^+ \setminus Y^+$ and conclude $\pi_* \circ \pi^{+*}$ is a BFF transform.

In [Nam03], Namikawa generalizes the equivalence studied by Bondal and Orlov to Mukai flops.

Example 4.3.4. (Mukai Flop) Let X be a smooth variety of dimension $2n$ with a subvariety $\mathbb{P}^n \cong Y \subseteq X$. We construct the Mukai flop of X along Y again by blowing up X along Y and contracting in the other direction. Again write $\tilde{X}$ for the blowup with contraction maps $\pi : \tilde{X} \to X$ and $\pi^+ : \tilde{X} \to X^+$. Surprisingly, the functor $\pi_* \circ \pi^{+*} : \mathrm{D}^b(X^+) \to \mathrm{D}^b(X)$ is *not* fully faithful. It is still true, however, that $X \sim_{BD} X^+$. There are birational morphisms $X \to \bar{X}$ and : $X^+ \to \bar{X}$ which contract Y and Y^+ respectively. Let $\hat{X} = X \times_{\bar{X}} X^+$ be the fiber product, with projections q, q^+. Namikawa shows in Theorem 3.1 that $q_{+*} \circ q^* : \mathrm{D}^b(X^+) \to \mathrm{D}^b(X)$ is an equivalence of categories. Again, q and q^+ are generically isomorphisms and we find this functor is a BD equivalence.

Remark 4.3.5. In the preceding example, $\hat{X}$ is a normal crossing variety with irreducible components $\hat{X} = \tilde{X} \bigcup (\mathbb{P}^n \times \mathbb{P}^n)$. We consider $q_{+*} \circ q^*$ as an integral transform with kernel is $\mathcal{P} = j_* \mathcal{O}_{\hat{X}}$. We then see that the irreducible component of $\mathrm{Supp}(\mathcal{P})$ which dominates X is $\tilde{X}$, and indeed $\tilde{X}$ and the projections realize the K equivalence $X \sim_K X^+$. The significance of this example is that the "obvious" choice of kernel for relating K equivalent varieties does not necessarily induce a derived equivalence.

Example 4.3.6. (Stratified Mukai Flops) After proving the result for Mukai flops, Namikawa tried to same methods to study stratified Mukai flops in [Nam04]. A stratified Mukai flop is a certain relation between cotangent bundles of grassmanians, where $X = T^*(\mathbb{G}(k,N))$ and $X^+ = T^*(\mathbb{G}(N-k,N))$. In this case, he showed that the product $X \times_{\bar{X}} X^+$ induces an equivalence of Grothendeick groups, but does not induce a derived equivalence. This was studied further in [Kaw06], where Kawamata demonstrates an equivalence when $N = 4, k = 2$; and in [CKL13] with a complete proof of equivalence, using a categorical $\mathfrak{sl}_2$ action. An explicit description of the kernel is avialable in [Cau12], where it is shown that the transformation is induced by a line bundle on a common resolution $X \leftarrow Z \rightarrow X^+$, and so we find the equivalence is a BD-equivalence.

Example 4.3.7. (Spherical Twist) An object $\mathcal{E} \in \mathrm{D}^b(X)$ is spherical if $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ and $\mathrm{Hom}(\mathcal{E},\mathcal{E}) = k$ if $i = 0, \dim(X)$ and $= 0$ otherwise. It is shown that the cone of the canonical morphism $\pi_2^*\mathcal{E}^\vee \otimes \pi_1^*\mathcal{E} \rightarrow \mathcal{O}_\Delta$ is the kernel of an equivalence, which we denote $T_\mathcal{E}$, and furthermore that $T_\mathcal{E}(\mathcal{F})$ fits into an exact triangle as the cone of $\oplus_i(\mathrm{Hom}(\mathcal{E},\mathcal{F}[i]) \otimes \mathcal{E}[-i]) \rightarrow \mathcal{F}$. If we write $\mathcal{P}$ for the kernel of $T_\mathcal{E}$, we can compute $\mathrm{Supp}(\mathcal{P})$ via the supports of $T_\mathcal{E}(\mathcal{O}_x)$, viewed as sheaves on $\{x\} \times Y$. For $x \notin \mathrm{Supp}(\mathcal{E})$, $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_x[i]) = 0$ for all i, and so $T_\mathcal{E}(\mathcal{O}_x) = \mathcal{O}_x$. Otherwise we find $\mathrm{Supp}(T_\mathcal{E}(\mathcal{O}_x)) = \mathrm{Supp}(\mathcal{E})$. Thus we see that a spherical twist is a BD equivalence exactly when the support of $\mathcal{E}$ is not all of X, for instance, when $\mathcal{E} = \mathcal{O}_C$ for a (-2) curve on a smooth projective surface as in [Huy06, Example 8.10iii].

We now turn to a few nonexamples. We give find examples of a derived equivalence that is not birational, a birational map that cannot be part of a BD equivalence, and a derived equivalence between birational varieties that is not a BD equivalence.

(non-)Example 4.3.8. (Dual Abelian variety) Indeed, the original derived equivalence of [Muk81] is not a birational derived equivalence, as the Poincaré bundle maps point sheaves to line bundles.

(non-)Example 4.3.9. A line bundle on a Calabi Yau variety – $\omega_X \cong \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X)$ – is a spherical object. A spherical twist about such an object is not a BD equivalence, as the kernel is supported on all of $X \times X$.

(non-)Example 4.3.10. (Cremona transorm) The Cremona transform $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational map whose maximal domain of definition is the complement of 3 points. Assume that we have a fully faithful functor $\Phi_{\mathcal{P}} : \mathrm{D^b}(\mathbb{P}^2) \to \mathrm{D^b}(\mathbb{P}^2)$ which extends the Cremona transform. This would mean the existence of an irreducible component $Z \subseteq \mathrm{Supp}(\mathcal{P})$ and a birational correspondence $\mathbb{P}^2 \leftarrow Z \to \mathbb{P}^2$ resolving the cremona transform. Such a correspondence realizes a K-inequality and hence is an isomorphism in codimension 1, which is impossible. We could likely also argue more directly as follows: Let P_1, P_2, P_3 be the locus of indeterminacy of φ and write L_1, L_2, L_3 for the lines connecting those points. For two points $x, y \in L_1$ with $x \neq y \neq P_i$ we have $\mathrm{Hom}(k(x), k(y)) = 0$ but $\mathrm{Hom}(k(\varphi(x)), k(\varphi(y)) = k$. This seems to contradict the full faithfulness of an extension of φ to the derived category, but it is not a priori obvious that an extension of a rational map covers the whole domain of definition.

(non-)Example 4.3.11. (Birational and Derived Equivalent but not K-equivalent)
In [Ueh04], Uehara provides a collection of mutually birational and derived equivalent rational elliptic surfaces which are not K-equivalent and hence not BD equivalent.

4.4 Moduli

Birationality is a well behaved notion in families of integral transforms. This should be contrasted with the lack of a satisfactory algebraic structure on the set of birational automorphisms, as discussed in the survey paper [Bla15]. The main result of this section is the following:

Theorem 4.4.1. *Let X and Y be flat, proper, and finitely presented schemes over a base scheme B. The stack $BFF_B(X, Y)$ of birational fully faithful transforms from $\mathrm{D}(X) \to \mathrm{D}(Y)$ is an algebraic stack locally of finite presentation over B.*

We realize this space as an open substack of Lieblich's moduli of complexes in $\mathrm{D}(X \times_B Y)$, from [Lie06]. As the kernel of a fully faithful functor is necessarily simple, we find $BFF_B(X, Y)$ has the structure of a $\mathbb{G}_m$ gerbe over its coarse space, as in Corollary 4.3.3

of Lieblich's paper. We have already proved algebraicity of related stacks $FF_B(X,Y)$ and $FM_B(X,Y)$ parameterizing fully faithful functors and derived equivalences, and algeraicity of the stack $BD_B(X,Y)$ of birational derived equivalences follows as an immediate corollary to the theorem above.

After defining the stacks in question, we proceed with a short discussion on the conditions and then realize the expected poset of open subfunctors inside Lieblich's space.

Definition 4.4.2. Fix a base scheme S and morphisms of schemes $X \to B$ and $Y_T = Y \to B$ over S. For a B-scheme T, write $X_T = X \times_B T$ and $Y \times_B T$. Assume that X or Y is flat over B, and define a functor

$$BFF_B(X,Y) : (Sch/B)^{opp} \to Sets, \quad T \mapsto \{\mathcal{P} \in \mathrm{D}(X_T \times_T Y_T) \text{ satisfying (FF) and (B) below}\}$$

(FF) The transform $\Phi_{\mathcal{P}} : \mathrm{D}(X_T) \to \mathrm{D}(Y_T)$ is fully faithful. (Corollary 3.5.5)

(B) For every closed point $t \in T$, there are closed points of the fibers $x \in X_t$ and $y \in Y_t$ and an isomorphism $\Phi_{\mathcal{P}_t}(\kappa(x)) \cong \kappa(y)$.

Remark 4.4.3. As discussed in [Cal17], our condition (B) implies that X_t and Y_t are birational. This idea is also discussed in [Huy06, Cor 6.14] with a more explicit construction of the common open set.

Given condition (FF), the the birationality condition in is equivalent to the following global property.

(B′) There is an open set $U \subseteq X$ mapping surjectively onto T and such that $\mathcal{P}|_{U\times Y} \cong \mathcal{O}_{\Gamma_f}$ for an open immersion $f : U \to Y$.

Proposition 4.4.4. *For any fully faithful integral transform Φ as above, conditions (B) and (B') are equivalent.*

Proof. (cf. [Huy06, Cor. 6.14])
(B′) ⇒ (B): follows from surjectivity of $U \to T$, as open immersions are stable under base change.
(B) ⇒ (B′): Consider a closed point $t \in T$. By assumption, there is a closed point $x \in X_t$ where the morphism $\mathrm{Supp}(\mathcal{P}_t) \to X_t$ is zero dimensional. Since x_0 is closed, this fiber is the same as the fiber of $\mathrm{Supp}(\mathcal{P}) \to X$ over x_0 and we can find an open neighborhood U of x_0 in X where the fibers are zero dimensional. This means $\Phi_{\mathcal{P}}(k(x))$ is concentrated in points for all $x \in U$, so by full faithfulness we can apply [Huy06, Lemma 4.5] and semicontinuity to conclude $\Phi_{\mathcal{P}}(k(x)) \cong k(y)$ for some y. The union of such open sets clearly surjects onto T, and we see $\mathcal{P}|_{U \times Y}$ is the graph of an open immersion. □

In what follows, we assume that X and Y are flat, proper, and finitely presented schemes over a base scheme B. We write $\mathcal{M}_{X \times_B Y}$ for Lieblich's moduli of simple, relatively perfect, universally gluable complexes.

Proposition 4.4.5. *The functor $BFF_B(X, Y)$ is an open subfunctor of $FF_B(X, Y)$*

Proof. Consider a complex $\mathcal{P}$ inducing a fully faithful functor $\Phi : \mathrm{D}(X_T) \to \mathrm{D}(Y_T)$. We must show there is an open subscheme $V \subseteq T$ such that a morhpism $f : T' \to T$ factors through V if and only if $f^*\mathcal{P}$ satisfies condition (B). Let U be the open locus of points in X where the dimension of the fibers of $\mathrm{Supp}(\mathcal{P}) \to X$ is zero-dimensional. If $g : X \to T$ is the structure morphism, take

$$V = T \setminus g(X \setminus U)$$

If f factors through $V \hookrightarrow T$, then $f^*(\mathcal{P})$ must satisfy (B′), as it must after restricting to V and (B′) is stable under base change. On the other hand, if $f^*(\mathcal{P}) \in BFF_B(X_{T'}, Y_{T'})$, then there is an open set in $X_{T'}$ where the fibers of $\mathrm{Supp}(\mathcal{P}_{T'}) \to X_{T'}$ are zero dimensional. Any closed point $t \in T$ in the image of $f : T' \to T$ must lie in V, since the fiber dimension is preserved under base change. □

Remark 4.4.6. For a moduli of sheaves powered proof, we could use the methods of [Cal17] to construct a common open set $U = X_T \times_{\mathcal{M}_{Y_T}} Y_T$ and again take $V = T \setminus g(X \setminus U)$

Corollary 4.4.7. *The functor $BD_B(X, Y)$ is an open subfunctor of $FM_B(X, Y)$*

A similar proof applies, or alternatively we can realize $BD_B(X, Y) = FM_B(X, Y) \times_{FF} BFF_B(X, Y)$

Chapter 5

OPEN PROBLEMS AND FUTURE DIRECTIONS

The derived category, while beautiful, continues to be a vexing and difficult muse. This study has raised more questions than it has answered, and we gather some of those questions here for continued study.

5.1 Birational Functors

We start with questions about the basic object described in this thesis: the birational integral transform. Recall Kawamata's theorem on general type varieties.

Theorem 5.1.1 (Kawamata)**.** *Any derived equivalence between general type varieties is a birational derived equivalence.*

We would like to say the same property holds for a fully faithful functor

Question 5.1.2. Let X and Y be smooth projective varieties and let $\Phi : \mathrm{D}(X) \to \mathrm{D}(Y)$. Assume furthermore that X is of general type and $\dim(X) = \dim(Y)$. Is Φ necessarily birationally fully faithful?

Note that it would suffice to prove (c.f. [Kaw02, Proof of Thm 2.3]): for a complete reduced curve C contained in $\mathrm{Supp}(\mathcal{P})$, there is an inequality of degrees

$$\deg(p^*\omega_X|_C) \leq \deg(q^*\omega_Y|_C)$$

Kawamata shows an equality of degrees, and uses this to show that for some component $Z \subseteq \mathrm{Supp}(\mathcal{P})$, the projection $q : Z \to Y$ is generically quasi-finite. For this, he appeals to the isomorphism between left and right adjoint kernels

$$\mathcal{P}^\vee \otimes \pi_X^*\omega_X[\mathrm{dim}X] \cong \mathcal{P}^\vee \otimes \pi_Y^*\omega_Y[\mathrm{dim}Y]$$

and shows equality by considering the determinant of a cohomology sheaf.

We have introduced the canonical map from right to left adjoint; it seems likely that a degree inequality can be realized by the map

$$\mathcal{P}^\vee \otimes \pi_X^*\omega_X[\dim X] \to \mathcal{P}^\vee \otimes \pi_Y^*\omega_Y[\dim Y]$$

The determinant argument does not generalize directly without a thorough understanding of the cone of this map, which might be computed by the methods of [AL12]. Assuming X and Y are the same dimension, we also note the simplicity of $\mathcal{P}$, and the induced map

$$R\mathcal{H}om(\mathcal{P},\mathcal{P}) \to \pi_Y^*\omega_Y \otimes (\pi_X^*\omega_X)^{-1}$$

Remark 5.1.3. The assumption that X and Y are of the same dimension is critical, as of course for any vector bundle $\mathcal{E}$ on a general type variety the pullback along $\mathbb{P}_X(\mathcal{E}) \to X$ is fully faithful.

We also might wish for a generalization of Bondal and Orlov's reconstruction theorem

Theorem 5.1.4. *Let $\Phi : D(X) \to D(Y)$ be a derived equivalence, and assume K_X or $-K_X$ is ample. Then $X \cong Y$.*

As shown in [Huy06, Cor 6.20] or [BBR09, Thm 2.51], one proof of this fact proceeds through Kawamata's K-equivalence argument. With the same setup above, if C is a curve contracted by $Z \to Y$, then $q^*\omega_Y|_C \cong \mathcal{O}_C$. But assuming K-inequality, we then have an inclusion of the ample bundle $p^*\omega_X|_C \subseteq \mathcal{O}_C$, which is ridiculous. We conclude that $Z \cong Y$ and that there is a proper birational morphism $Y \to X$.

Conjecture 5.1.5. *Let $\Phi : D(X) \to D(Y)$ be fully faithful. Assume that X and Y are of the same dimension and K_X or $-K_X$ is ample. Then there is a proper birational morphism $Y \to X$.*

Of course the end goal of this study would be an answer to Kawamata's most recent version of the DK hypothesis:

Conjecture 5.1.6 ([Kaw18])**.** *K -equivalent varieties are derived equivalent. A K-inequality induces a fully faithful functor between the derived categories.*

We can approach this problem from the other direction and ask

Question 5.1.7. For a birational fully faithful functor $\Phi_{\mathcal{P}} : \mathrm{D}(X) \to \mathrm{D}(Y)$, what is the structure of the exceptional locus? What structure is present in the components of $\mathrm{Supp}(\mathcal{P})$ that do not dominate X?

For instance: is there a small contraction that can be made on X? And how does the notion of a biratonal fully faithful functor relate to the study of perverse sheaves as in [Bri02]?

5.2 Moduli

Many similar questions can be asked of the stacks that were constructed. We refer to [Bla15] for a discussion on possible algebraic structures of the birational automorphism group of a variety, and we can ask

Question 5.2.1. What are the fibers of the natural map $BD(X, X) \to BirAut(X)$?

The example of a spherical twist about a (-2) curve shows that there can be multiple BD equivalences which extend the identity automorphism. Can this map have fibers of positive dimension?

This example may be related to the question of separatedness.

Question 5.2.2. Is $BD(X, Y)$ separated?

We could also define a moduli functor of varieties up to BD-equivalence, as in [Kol17].

$$\mathcal{V}ar_{BD}(S) = \{ \text{ smooth, proper families } X \to S \text{ modulo BD equivalences}\}$$

Since every derived equivalence between general type varieties is a birational derived equivalence, we may expect this functor to have similar properties to the Kollár's functor of general type varieties.

Question 5.2.3. Is $\mathcal{V}ar_{BD}$ separated?

See [Kol17, Prop 1.26].

5.3 Lifting Problems

This study began with a desire to understand the problem of lifting to characteristic 0.

Definition 5.3.1. Let X_0 be a proper, smooth scheme of finite type over an algebraically closed field k of characteristic p. A *lifting of* X_0 *to characteristic 0* is the data of a DVR A of characteristic 0 with residue field k, along with a proper, smooth scheme $X/\mathrm{Spec}(A)$ and an isomorphism $X_0 \cong \mathrm{Spec}(k) \times_{\mathrm{Spec}(A)} X$. If such a scheme exists, we say X_0 *is liftable.*

One of the original questions posed by my advisor was

Question 5.3.2. Is liftability a derived invariant?

This was refined to the question of whether or not liftability is a *birational* derived invariant, which still proved fairly intractable.

One positive result in this direction study says that we can "blow down" a lift.

Theorem 5.3.3. *[CvS09, Thm 3.1] Let $\pi : Y \to X$ be a morphisms of schemes over k and let A be an Artinian ring with residue field k. Assume that $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ and $R^1\pi_*\mathcal{O}_Y = 0$. Then for every lifting $\mathcal{Y} \to$ Spec(A) of Y there exists preferred and compatible lifting of $\mathcal{X}$.*

This leads us to the related question:

Question 5.3.4. If $X \leq_{BD} Y$ and there exists a formal lift of Y, does there exist a formal lift of X?

If we find that $\Phi_{\mathcal{P}}(\mathcal{O}_X) \cong \mathcal{O}_Y$, then $\Phi_R(\mathcal{O}_Y) \cong \mathcal{O}_X$ and we can proceed by analogy with Cynk and van Straten's result. We know that indeterminacy locus of the map $X \dashrightarrow Y$ has codimension ≥ 2 in X, so we can lift the structure sheaf $\mathcal{O}_X$ over an open set $U \subseteq X$ to algebra $A(U) = H^0(U \setminus U \cap E, \mathcal{O}_{\mathcal{Y}})$, which we must show provides a flat lift of $\mathcal{O}_X$.

The reverse question of extending a lift along a BFF transform was answered negatively in [Lie13]. Specifically, Leidtke and Satriano are able to show:

Theorem 5.3.5. *For every algebraically closed field k of positive characteristic and integer $d \geq 3$, there exists a smooth d dimensional ruled variety X/k that lifts projectively to $W(k)$, and a blow up $Y \to X$ along a smooth curve such that Y does not lift to $W_2(k)$.*

On the positive side, they show that formal liftability is preserved under Atiyah flops.

Theorem 5.3.6. *Let X be a 3 dimensional variety with one singular point that is an ordinary double point and Y_1, Y_2 the two small resolutions. Then Y_1 lifts formally if and only if Y_2 lifts formally.*

Another approach to lifting a birational derived equivalence $\mathrm{D}(X_0) \cong \mathrm{D}(Y_0)$ is to realize Y_0 as a moduli space of perverse sheaves on X_0, then to lift the moduli problem. This led to our construction of the moduli of BD equivalences, which is not quite the same idea, so perhaps there is more to this story. Perhaps we can reduce to the case of flops by the results of [Kaw07].

Theorem 5.3.7. *Suppose X and Y are projective, Q-factorial, terminal varieties. Assume that K_X and K_Y are nef, and that X and Y are K-equivalent. Then the K-equivalence decomposes as a sequence of flops.*

In the context of this thesis, we have the natural follow up

Question 5.3.8. If $X \sim_{BD} Y$, are X and Y connected by a sequence of flops?

Perhaps Kawamata's argument could be modified with a more thorough understanding of the birational geometry involved.

The last approach toward the lifting problem is through study the Hochschild cohomology. As discussed in [Căl03], [Căl05], the Hochschild cohomology is a derived invariant. The connection between Hochschild cohomology and deformation theory is given by the HKR isomorphism, which Căldăraru describes as a specific quasi isomorphism

$$L\Delta^*\mathcal{O}_\Delta \xrightarrow{\sim} \bigoplus_i \Omega^i_X[i]$$

He then proceeds to show:

Theorem 5.3.9. *[Căl03, Cor 8.3] Let X and Y be smooth projective varieties and $\Phi : D(X) \to D(Y)$ be an equivalence. Then Φ induces an isomorphism $\phi : \mathrm{HH}^*(X) \to \mathrm{HH}^*(Y)$*

In a more down to earth statement, the HKR isomorphism says that the n^{th} Hochschild cohomology has a direct sum decomposition:

$$HH^n(X) \cong \bigoplus_{p+q=n} H^p(X, \wedge^q T_X)$$

The relationship between deformations and Fourier Mukai transforms was studied by Toda in [Tod05]. He describes the direct summands of $HH^2(X)$ as parameterizing scheme deformations, noncommutative deformations, and twisted deformations of X and proves that a Fourier Mukai transform is compatible with these deformations. Interestingly, the induced transformation on Hochschild cohomology does not necessarily preserve direct summands. So, a scheme deformation on one side may become a twisted deformation on the other. Most relevant to our discussion he shows

Theorem 5.3.10. *[Tod05, Thm 7.1] Let X and Y be smooth projective varieties, which are connected by a flop $X \to W \leftarrow Y$. If $\Phi_{\mathcal{P}} : D(X) \to D(Y)$ is an equivalence, and the object $\mathcal{P}$ is supported on $X \times_W Y$, then the induced map on Hochschild Cohomology takes the direct summand $H^1(X, T_X) \subseteq HH^2(X)$ to $H^1(Y, T_Y) \subseteq HH^2(Y)$ under the HKR isomorphism.*

This shows that if both X and Y lift, the lifts are in one to one correspondence. He furthermore shows that such lifts are derived equivalent.

Question 5.3.11. Does a birational derived equivalence preserve the summands of HH^n under the HKR isomorphism?

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