A Hyperresolution-Free Characterization of the Deligne-Du Bois Complex

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Abstract

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Let Y be a reduced, finite type scheme over $\mathbb{C}$, X a closed subscheme of Y and $\pi : \widetilde{Y} \to Y$ a projective morphism which is an isomorphism outside of X with $E = (\pi^{-1}(X))_{\mathrm{red}}$. In this paper, we provide a construction of the Deligne-Du Bois complex of X in terms of the Deligne-Du Bois complexes of $Y, \widetilde{Y}$ and E. In the case that Y is smooth and π is a log resolution of X in Y, this will provide a hyperresolution-free construction of $\underline{\Omega}^{\bullet}_X$ and its graded pieces.

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And to my husband, Nathan. Words will never be able to express how grateful I am for you and what you have done for me. I love you.

Contents

1 Introduction

1.1 The de Rham Complex

By the first fundamental theorem of calculus, we know that for $f : \mathbb{R} \to \mathbb{R}$, a real valued continuous function,

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

for any real number a. We can rephrase this result to say that all closed differential forms are exact on $\mathbb{R}^1$. It is natural to wonder whether this is a general property of differential forms or a property of the topological space on which we are working. On $\mathbb{R}^2 \setminus \{0\}$, there is a classical example of a closed 1-form that is not exact (namely the derivative of the argument function), suggesting that this is actually a property of the topological space. In fact, this property is related to the ability of differential forms to capture the "niceness" of the topology of a manifold. We might then wonder what topological properties exactly are being captured. This gives rise to the theory of de Rham cohomology, allowing us to piece together local information of differential forms to global phenomenon.

For M a real manifold of dimension n, the (smooth) de Rham complex of M is a complex of vector spaces of differential forms where the p-th graded piece of the complex gives the vector space of smooth differential p-forms. The Poincaré Lemma gives sufficient conditions for closed differential forms to be exact and, in fact, shows that this complex gives a resolution of $\mathbb{R}$ so that

$$0 \to \mathbb{R} \to \mathscr{O}_M \to \Omega_M^1 \to \Omega_M^2 \to \cdots \to \Omega_M^{n-1} \to \Omega_M^n \to 0$$

is an exact sequence. By taking cohomology of the de Rham complex, we obtain the smooth de Rham cohomology of M. The de Rham cohomology groups are invariants of a topological space, stable under homotopy and topological equivalence. Through analysis and leveraging cohomology theory, we can recover the genus, dimension and many other topological properties of M.

While de Rham cohomology is classically defined for real manifolds, a similar construction works for smooth complex analytic spaces with some key differences to account for the added complexity[1]. Most notably, we will use holomorphic forms instead of smooth differential forms and hypercohomology instead of cohomology to compute de Rham cohomology. Allowing these changes, we obtain a cohomology theory with properties reminiscent of the smooth de Rham complex and its cohomology groups. This complex and its properties will be discussed in more detail in sections 2 and 3.

While the absence of singularities is not necessary for the definition of the de Rham complex, many of the properties most useful break down for singular topological spaces. The attempt to understand what happens in the singular case and generalize the properties of the smooth de Rham complex is what lead to the Deligne-Du Bois complex.

1.2 Du Bois Singularities

The study of singularities, whether it is in their presence or absence, is intrinsic to algebraic geometry and, in particular, the minimal model program (MMP) [KM98]. A major goal of the MMP is the classification of the most "simple" birational models of varieties. While singularities, almost by definition, are more locally complex than smooth points, complex local behavior does not ensure complex global behavior. In fact, these simple representatives

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[1] A joke.

are rarely smooth. Even when dealing with smooth objects, their deformations and degenerations are often singular. Understanding singularities and their behavior thus becomes integral to solving many problems.

Specializing to characteristic zero, many of the singularities naturally appearing in moduli problems and higher dimensional geometry are Du Bois singularities. Du Bois singularities sit as an encompassing class of singularities and account for many different desirable behaviors and structures exhibited by other singularity classes.

$$\begin{tikzcd}
\text{Terminal} \arrow[r, Rightarrow] & \text{Canonical} \arrow[r, Rightarrow] & \text{Log Terminal} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Rational} \arrow[dd, Rightarrow] \\
 & & \text{Log Canonical} \arrow[d, Rightarrow] & \\
 & & \text{Semi-Log Canonical} \arrow[r, Rightarrow] & \text{Du Bois}
\end{tikzcd}$$

Under relatively tame conditions, Du Bois singularities deform well under small perturbations and provide structure to spaces, see [KK20] for more detail. It may be of interest to note that while Du Bois singularities deform well, this is not true for every subclass. For example, log canonical singularities are not invariant under small perturbations without stricter hypotheses.

While the definition is quite technical, we can think of Du Bois singularities as those that force algebraic behavior to be ruled by topological data. We can see this under nice enough hypotheses (such as X a proper variety over $\mathbb{C}$), where X being Du Bois implies that the natural map

$$H^i_{\text{sing}}(X^{an}, \mathbb{C}) \to H^i(X, \mathscr{O}_X)$$

is surjective for all i. Heuristically, Du Bois singularities can then be thought of as the largest class of singularities where such a surjection holds, see [Kov12] for details. While this makes for a poor (and even faulty) definition, it gives good intuition for how such singularities behave.

Du Bois singularities provide a motivation to study the Deligne-Du Bois complex, as their classical definition relies upon the construction of the zeroth graded piece of the Deligne-Du Bois complex. In fact, this relationship accounts for much of the knowledge we have of how the graded pieces of the Deligne-Du Bois complex behave.

1.3 Main Result

Theorem 1.1. *Let X be a reduced separated scheme of finite type over a field of characteristic 0. Embed X in a smooth scheme Y and let $\pi : \widetilde{Y} \to Y$ be a log resolution of X in Y that is an isomorphism outside of X with $E = \pi^{-1}(X)_{red}$. Then $\underline{\Omega}^\bullet_X$ is quasi-isomorphic to the complex*

$$\mathcal{R}\pi_*\Omega^\bullet_{\widetilde{Y}}[1] \oplus \mathcal{R}\pi_*\Omega^\bullet_E \oplus \Omega^\bullet_Y$$

with differential $(d_{\mathcal{R}\pi_*\Omega^\bullet_{\widetilde{Y}}}[1], \beta[1] + (d_{\mathcal{R}\pi_*\Omega^\bullet_E}, \alpha[1] + d_{\Omega^\bullet_Y}))$. Similarly, for each p, $\underline{\Omega}^p_X$ is quasi-isomorphic to the complex

$$\mathcal{R}\pi_*\Omega^p_{\widetilde{Y}}[1] \oplus \mathcal{R}\pi_*\Omega^p_E \oplus \Omega^p_Y$$

with differential $(d_{\mathcal{R}\pi_*\Omega^p_{\widetilde{Y}}}[1], \beta_p[1] + (d_{\mathcal{R}\pi_*\Omega^p_E}, \alpha_p[1] + d_{\Omega^p_Y}))$.

This new characterization allows us to study the Deligne-Du Bois complex of a singular scheme by studying the de Rham complexes of smooth and mildly singular schemes. This is actually a corollary of a stronger result, Theorem 3.14, which gives new distinguished triangles that $\underline{\Omega}_X^\bullet$ and $\underline{\Omega}_X^p$ fit into. Direct computation of cohomology with this characterization still proves challenging, however the placement of $\underline{\Omega}_X^\bullet$ and $\underline{\Omega}_X^p$ into distinguished triangles allows us to "bound" the bad behavior and analyze cohomology through comparisons with objects with a richer history of study (namely the de Rham complex). From this, we can recover Schwede's simple characterization of Du Bois singularities, see Theorem 3.9, as well as see why this characterization fails for the higher graded pieces. While our understanding of $\underline{\Omega}_X^0$ has benefited from its close relationship to Du Bois singularities, $\underline{\Omega}_X^p$ has remained more mysterious. It is our hope that this new construction will allow for a more hands-on approach and applications, particularly in leveraging the degeneration of the Frölicher-type spectral sequence, see Theorem 3.7.

2 Preliminaries

2.1 Derived Categories

Let X be a scheme of finite type over a field k. For a complex of $\mathscr{O}_X$-modules on X,

$$A := \quad \cdots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \cdots$$

we denote by $h^i(A) = \ker d^i / \mathrm{im} d^{i-1}$ the ith cohomology sheaf. An $\mathscr{O}_X$-module $\mathscr{F}$ can be considered as a complex $\mathscr{F}^\bullet$ with $\mathscr{F}^0 = \mathscr{F}$ and $\mathscr{F}^i = 0$ for $i \neq 0$. A map $\alpha : A \to B$ of two such complexes is a *quasi-isomorphism* if it induces an isomorphism on cohomology $h^i(A) \simeq h^i(B)$ for all i. If $\alpha : (A, F_A) \to (B, F_B)$ is a map of filtered complexes, we say that α is a *filtered quasi-isomorphism* if all of the induced graded maps are quasi-isomorphisms.

We denote by $D^b_{\text{filt}}(X)$ the derived category of bounded filtered complexes of $\mathscr{O}_X$-modules and by $D^b_{\text{filt,coh}}(X)$ the subcategory of $D^b_{\text{filt}}(X)$ of complexes A such that for all i, the cohomology sheaves of $Gr^i_{\text{filt}}A$ are coherent. Given a bounded chain complex of $\mathscr{O}_X$-modules, we can consider it as an object of $D^b_{\text{filt}}(X)$ by equipping it with the *filtration bête.* For a complex A, this is the filtration $(A, F^{\bullet})$ defined by the graded pieces

$$F^p(A) := \quad 0 \to \cdots \to 0 \to A^p \to A^{p+1} \to A^{p+2} \to \cdots \to A^n \to 0$$

Unless otherwise specified, we will assume all complexes are objects in $D^b_{\text{filt}}(X)$ (equipped with the filtration bête when necessary) and all morphisms are filtered. In particular, all quasi-isomorphisms are filtered quasi-isomorphisms.

Given a left exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories, the right derived functor (when it exists) will be denoted by $\mathcal{R}F : D(\mathcal{A}) \to D(\mathcal{B})$. Furthermore, we will use the notation $R^iF = h^i \circ \mathcal{R}F$ and $\mathbb{H}^i = R^i\Gamma$ for the cohomology of a derived functor and the hypercohomology, respectively. For $\iota : \Sigma \hookrightarrow X$ a closed embedding of schemes , ι_* is exact and so $R^i\iota_* = 0$ for $i > 0$. For any $A \in \mathsf{Ob}(D^b_{\text{filt}}(\Sigma))$, we will make a small abuse of notation and drop the ι_* from ι_*A, allowing us to consider A as an object in $D^b_{\text{filt}}(X)$. We will use this frequently to compare complexes of subschemes with those on the ambient space.

As $D^b_{\text{filt}}(X)$ is a triangulated category, we have a class of distinguished triangles. Moreover, any morphism of complexes $\alpha : A \to B$, can be completed to a distinguished triangle

$$A \xrightarrow{\alpha} B \longrightarrow \text{Cone}(\alpha) \xrightarrow{+1}$$

and every distinguished triangle

$$A \longrightarrow B \longrightarrow C \xrightarrow{+1}$$

induces a long exact sequence in cohomology

$$\cdots \to h^i(A) \to h^i(B) \to h^i(C) \to h^{i+1}(A) \to \cdots$$

of $\mathscr{O}_X$ modules. Distinguished triangles will play an important role in the results of this paper. In particular, the following lemmas will be key to the results in section 3.4.

Lemma 2.1. *Given a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

in a triangulated category, we can extend to a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & A'[1] \arrow[d] \\
A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & A''[1] \arrow[d] \\
A[1] \arrow[r] & B[1] \arrow[r] & C[1] \arrow[r] & A[2]
\end{tikzcd}$$

where all of the rows and columns form distinguished triangles. Moreover, every square is commutative bar the bottom right square, which is anticommutative.

Proof. See [sta18, Tag 05R0]. $\square$

Remark. In the proof of Lemma 2.1, we are allowed a choice of C, C', A'' and B'' up to quasi-isomorphism. We then have that C'' is determined by the base morphism $C \to C'$ induced by the triangulated structure.

Lemma 2.2. *Let A, B, C, A', B', and C' be objects in any derived category such that the rows of the following commutative diagram form distinguished triangles:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & A[1] \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

Then there exists an object D fitting into an exact triangle

$$D \longrightarrow B' \oplus C \longrightarrow C' \xrightarrow{+1}$$

and a map $\delta : B \to D$. Furthermore, α is a quasi-isomorphism if and only if δ is a quasi-isomorphism.

Proof. See [KK10] Lemma 2.1. □

2.2 Hyperresolutions

Hyperresolutions encompass simplicial, cubic, and polyhedral resolutions and act as a "topology preserving" resolution of singularities. The original construction of the Deligne-Du Bois complex was given by Du Bois in [DB81] using simplicial resolutions. Subsequent constructions were given by [Car85] and [GNAPGP88] using cubical and polyhedral resolutions. Since the construction of the Deligne-Du Bois complex is independent of choice of hyperresolution and the details of hyperresolutions can be quite technical, we will give a brief overview of only cubic hyperresolutions following the exposition in [KS11] (which follows the construction in [GNAPGP88]).

Let $\mathsf{Sch}_{\mathsf{red}}$ denote the category of reduced schemes and let $\square_n^+$ denote the product of $n+1$ copies of the category $\{0 \to 1\}$ for $n \geq -1$. By convention, $\square_{-1}^+ = \{0\}$ and $\square_0^+ = \{0 \to 1\}$. Objects in $\square_n^+$ can be identified with sequences $(\alpha_0, \alpha_1, ..., \alpha_n)$ with each $\alpha_i \in \{0, 1\}$. The full subcategory consisting of all objects of $\square_n^+$ except the initial object corresponding to $(0, ..., 0)$ will be denoted by $\square_n$.

For any I-scheme $X_{\bullet} : I^{op} \to \mathsf{Sch}_{\mathsf{red}}$ and $i \in \mathsf{Ob}\,(I)$, the scheme corresponding to i will be denoted by X_i. Similarly, for any $\varphi \in \mathsf{Mor}(I)$ where $\varphi : j \to i$, the corresponding morphism will be denoted by $X_\varphi : X_j \to X_i$. For any morphism of $f : Y_{\bullet} \to X_{\bullet}$ of I-schemes, denote by f_i the induced morphism $f_i : Y_i \to X_i$ on schemes. A property $\mathscr{P}$ of a morphism of schemes can be extended to a property of morphisms of I-schemes by requiring that for each object i in I, the corresponding induced morphism on schemes has the property $\mathscr{P}$.

Definition 2.3. [GNAPGP88] Given a morphism $f : Y_{\bullet} \to X_{\bullet}$ of I-schemes, define the *discriminant of* f to be the smallest closed sub-I-scheme $Z_{\bullet}$ of $X_{\bullet}$ such that $f_i : Y_i \backslash f_i^{-1}(Z_i) \to X_i \backslash Z_i$ is an isomorphism for all i.

Definition 2.4. [GNAPGP88] Let $S_{\bullet}$ be an I-scheme, $f : X_{\bullet} \to S_{\bullet}$ a proper morphism of I-schemes and let $D_{\bullet}$ be the discriminant of f. We say that f is a *resolution of* $S_{\bullet}$ if $X_{\bullet}$ is a smooth I-scheme and $\dim f_i^{-1}(D_i) < \dim S_i$ for all $i \in \mathsf{Ob}\,(I)$.

Resolutions of I-schemes are more subtle than individually resolving each X_i and, a priori, do not always exist. However, by restricting ourselves to characteristic zero and finite categories where all isomorphism and endomorphisms are the identity, resolutions of I-schemes exist.

Theorem 2.5 ([GNAPGP88])**.** *Let S be an I-scheme of finite type over a field k of characteristic zero and let I be a finite ordered category.Then there exists a resolution of S.*

Constructing a cubic hyperresolution is an iterative process with steps referred to 2-resolutions.

Definition 2.6. [GNAPGP88] Let S be an I-scheme and $Z_\bullet$ a $\square_1^+ \times I$-scheme. We say $Z_\bullet$ is a *2-resolution of S* if $Z_\bullet$ is defined by the Cartesian square of morphisms of I-schemes

$$\begin{tikzcd}
Z_{11} \arrow[r, hook] \arrow[d] & Z_{01} \arrow[d, "f"] \\
Z_{10} \arrow[r, hook] & Z_{00}
\end{tikzcd}$$

where

i) $Z_{00} = S$.

ii) Z_{01} is a smooth I-scheme.

iii) The horizontal arrows are closed immersions of I-schemes.

iv) f is a proper I-morphism.

v) Z_{10} contains the discriminant of f.

Definition 2.7. [GNAPGP88] Let r be a positive integer and for $1 \leq n \leq r$, let $X_\bullet^n$ be a $\square_n^+ \times I$-scheme. Suppose that for all such n, the $\square_{n-1}^+ \times I$-schemes $X_{00\bullet}^{n+1}$ and $X_{1\bullet}^n$ are equal.

Then, by induction on r, define a $\square_r^+ \times I$-scheme

$$Z_\bullet = \operatorname{red}(X_\bullet^1, X_\bullet^2, ..., X_\bullet^r)$$

called the *reduction* of $(X_\bullet^1, X_\bullet^2, ..., X_\bullet^r)$ in the following way: If $r = 1$, $Z_\bullet = X_\bullet^1$. If $r = 2$, $Z_\bullet = \operatorname{red}(X_\bullet^1, X_\bullet^2)$ is defined

$$Z_{\alpha\beta} = \begin{cases} X_{0\beta}^1 & \text{if } \alpha = (0,0) \\ X_{\alpha\beta}^2 & \text{if } \alpha \in \square_1 \end{cases}$$

for all $\beta \in \square_0^+$. If $r > 2$, define $Z_\bullet$ recursively as $\operatorname{red}(\operatorname{red}(X_\bullet^1, X_\bullet^2, ..., X_\bullet^{r-1}), X_\bullet^r)$.

Definition 2.8. [GNAPGP88] Let S be an I-scheme. A *cubic hyperresolution augmented over* S is a $\square_r^+ \times I$-scheme $Z_\bullet$ such that

$$Z_\bullet = \operatorname{red}(X_\bullet^1, X_\bullet^2, ..., X_\bullet^r)$$

where

i) $X_\bullet^1$ is a 2-resolution of S,

ii) for $1 \leq n < r$, $X_\bullet^{n+1}$ is a 2-resolution of X_1^n, and

iii) Z_α is smooth for all $\alpha \in \square_r$.

Finally, we note that cubic hyperresolutions for schemes exist under modest hypotheses. In particular, it should be noted that these hypotheses are satisfied under the "classical" hypotheses for the construction of the Deligne-Du Bois complex (,i.e., a reduced, finite type scheme over a field of characteristic zero).

Theorem 2.9 ([GNAPGP88]). *Let S be an I-scheme. Suppose that k is a field of characteristic zero and that I is a finite ordered category. Then there exists $Z_\bullet$, a cubic hyperresolution augmented over S such that*

$$\dim Z_\alpha \leq \dim S - |\alpha| + 1$$

for all $\alpha \in \square_r$.

For the purposes of this paper, the existence of hyperresolutions (and thus the existence of the Deligne-Du Bois complex) will be of more import than the details of the construction.

Example 2.10. Let X be a curve. In this case, we can construct a cubic hyperresolution of X that consists only of a single 2-resolution $X_\bullet$ of X

$$\begin{tikzcd}
X_{11} = E \arrow[r, hook] \arrow[d] & \widetilde{X} = X_{01} \arrow[d, "\pi"] \\
X_{10} = \Sigma \arrow[r, hook] & X = X_{00}
\end{tikzcd}$$

where Σ is the set of singular points of X, $\pi : \widetilde{X} \to X$ is the normalization of X and $E = (\pi^{-1}\Sigma)_{\mathrm{red}}$. Since each X_{01}, X_{10} and X_{11} is smooth, $X_\bullet$ is a cubic hyperresolution of X.

Remark. While the normalization of plane curves is not too difficult to write down, it becomes more challenging in higher dimensions. Blowups also provide good examples of hyperresolutions, so long as the reduced exceptional set and the singular set are both smooth. However, the typical hyperresolution is difficult to write down or compute and often requires ad hoc means to be determined efficiently.

2.3 Hodge Theory

The Deligne-Du Bois complex was born from Deligne's work [Del74] in Hodge theory, a major component of which was the development and study of mixed Hodge structures.

Theorem 2.11 (Deligne)**.** *The cohomology groups of algebraic varieties carry mixed Hodge structures.*

For the results of this paper, we can think of a *pure Hodge structure* as a bi-grading on cohomology groups and a *Hodge structure of weight* n as one that behaves like the cohomology of a Kähler manifold of dimension n. A *mixed Hodge structure* is then a filtration on cohomology groups such that the graded pieces have pure Hodge structure of weight k in each degree k. Hodge structures arise naturally when examining the de Rham cohomology groups of compact Kähler manifolds.

Theorem 2.12 (Hodge Decomposition)**.** *Let M be a compact Kähler manifold. Then there is a direct sum decomposition*

$$H^k_{dR}(M) = \bigoplus_{p+q=k} H^q(M, \Omega^p_M)$$

of the de Rham cohomology of M.

In Hodge theoretic terminology, the Hodge decomposition gives a Hodge structure of weight k to the de Rham cohomology group $H^k_{dR}(M)$. It should be noted that Hodge decomposition is typically phrased in terms of singular and Dolbeault cohomology groups. However, Grothendieck's singular-de Rham comparison theorem (see Theorem 3.3) and Dolbeault's theorem allow us the above rephrasing.

The Hodge decomposition is a consequence of the degeneration of the Frölicher spectral sequence (also known as the Hodge-de Rham spectral sequence).

Theorem 2.13 (Frölicher Spectral Sequence). *Let M be a complex manifold. There exists a spectral sequence*

$$H^q(M,\Omega^p_M) \Longrightarrow H^{p+q}_{sing}(M).$$

of sheaf cohomology groups abutting to singular cohomology. Moreover, if M is a compact Kähler manifold, this sequence degenerates at the E_1-page.

This spectral sequence is a particularly powerful tool in cohomological comparisons, allowing us to relate topological and anaylitic data.

3 The Deligne-Du Bois Complex

3.1 Cohomological Origins

While the de Rham complex was first defined for real manifolds, it has a natural generalization to complex analytic spaces.

Definition 3.1. Let M be a complex analytic space of dimension n. The *holomorphic de Rham complex of M*, denoted by $\Omega^{\bullet}_{M/\mathbb{C}}$, is the complex

$$0 \longrightarrow \Omega^0_{M/\mathbb{C}} \xrightarrow{d} \Omega^1_{M/\mathbb{C}} \xrightarrow{d} \Omega^2_{M/\mathbb{C}} \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1}_{M/\mathbb{C}} \xrightarrow{d} \Omega^n_{M/\mathbb{C}} \longrightarrow 0,$$

where $\Omega^p_{M/\mathbb{C}}$ is the sheaf of holomorphic p-forms on M and d is the usual exterior differentiation. The *holomorphic de Rham cohomology of M* is the hypercohomology $H^i_{dR}(M/\mathbb{C}) := \mathbb{H}^i(M,\Omega^{\bullet}_{M/\mathbb{C}})$ of the holomorphic de Rham complex.

The holomorphic de Rham complex has a variety of nice properties. In particular, $\Omega^{\bullet}_{M/\mathbb{C}}$ forms a resolution of $\mathbb{C}$ when M is a smooth complex analytic space (,i.e., a complex manifold). The holomorphic de Rham complex has been a major object of study in Hodge theory, leading to many interesting and powerful results.

As we know, complex analytic spaces and varieties are closely tied to one another and, given a complex variety X, there exists a complex analytic space X^{an}. Moreover, while the exterior differentiation d of the de Rham complex is not an $\mathscr{O}_M$-linear morphism, the graded pieces are $\mathscr{O}_M$-modules. With an eye towards GAGA [Ser56], it might then be unsurprising to learn that there is an algebraic notion of the de Rham complex and de Rham cohomology.

Definition 3.2. Let X be a complex scheme of dimension n. The *algebraic de Rham complex of* X, denoted by $\Omega^{\bullet}_{X/\operatorname{Spec}\mathbb{C}}$, is the complex

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega^1_{X/\operatorname{Spec}\mathbb{C}} \xrightarrow{d} \Omega^2_{X/\operatorname{Spec}\mathbb{C}} \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1}_{X/\operatorname{Spec}\mathbb{C}} \xrightarrow{d} \Omega^{n}_{X/\operatorname{Spec}\mathbb{C}} \longrightarrow 0,$$

where $\Omega^p_{X/\operatorname{Spec}\mathbb{C}} = \bigwedge^p \Omega_{X/\operatorname{Spec}\mathbb{C}}$ is given by the p-th exterior power of the sheaf of differentials. The *algebraic de Rham cohomology* is the hypercohomology $H^i_{dR}(X/\operatorname{Spec}\mathbb{C}) := \mathbb{H}^i(X, \Omega^{\bullet}_{X/\operatorname{Spec}\mathbb{C}})$ of the algebraic de Rham complex of X.

Theorem 3.3 (de Rham-Singular Comparison Theorem)**.** *Let $X/\mathbb{C}$ be a smooth variety. Then there are isomorphisms*

$$H^i_{sing}(X^{an}, \mathbb{C}) \simeq H^i_{dR}(X^{an}/\mathbb{C}) \simeq H^i_{dR}(X/Spec\,\mathbb{C})$$

for all $i \geq 0$.

Proof. See [Gro66]. $\square$

When X is smooth and the topology is clear, this allow us to justify a small abuse of notation and refer simply to the de Rham complex of X, denoted $\Omega^{\bullet}_X$, and the de Rham cohomology of X, denoted $H^{\bullet}_{dR}(X)$. We can now utilize the vast knowledge of the Hodge theoretic behavior of smooth complex manifolds to study the topological and analytic behavior of algebraic varieties. In particular, we have an algebraic version of the Frölicher spectral sequence.

Theorem 3.4 (Algebraic Frölicher Spectral Sequence)**.** *Let X be a complex variety. There exists a spectral sequence*

$$H^q(X, \Omega^p_X) \Longrightarrow H^{p+q}_{sing}(X^{an}, \mathbb{C}).$$

of sheaf cohomology groups abutting to singular cohomology groups. Moreover, if X is proper, this sequence degenerates at the E_1-page.

As in section 2.3, we can leverage the degeneration of this spectral sequence to obtain a decomposition

$$H^k_{\mathrm{sing}}(X^{an}, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega^p_X).$$

of singular cohomology in terms of sheaf cohomology. The left hand side of this decomposition is topological while the right hand side is purely algebraic. In this way, the analytic and topological behaviors of X are ruled by the algebraic.

Unfortunately, these relationships break down for singular schemes. Deligne's work in Hodge theory [Del74], particularly his contribution to mixed Hodge theory, was critical to determining how to generalize the cohomological behavior of the de Rham complex to singular schemes. Deligne and Illusie's seminal paper [DI87] gave a purely algebraic proof of the decomposition of the Frölicher spectral sequence for smooth proper varieties over fields of characteristic zero. In fact, Deligne and Illusie proved a stronger result over perfect fields of any characteristic from which the degeneration follows. Unfortunately, in generalizing Hodge theory to singular complex algebraic varieties, it became clear that the same properties could not be expected to hold for the algebraic de Rham complex. The Deligne-Du Bois complex is sometimes referred to as the generalized de Rham complex as it exhibits the same cohomological behavior as the de Rham complex and agrees (up to isomorphism) under smoothness hypotheses, see Theorem 3.7 for more details.

3.2 Classical Construction

Du Bois's original construction of the Deligne-Du Bois complex can be found in [DB81] but a more accessible introduction can be found in [Ste85]. While Du Bois's construction, and subsequently the definition for Du Bois singularities, was for a single scheme X, the evolution of the MMP has shown that singularities are often best defined and studied in pairs. This requires a notion of the Deligne-Du Bois complex of a pair, see [Kov11], which we overview below.

Notation 1. For the remainder of this section, unless otherwise specified, all schemes we consider will be reduced schemes of finite type over a field of characteristic zero. A reduced pair is a pair of schemes (X, Σ) where X is a reduced scheme of finite type over a field of characteristic 0 and Σ is a reduced, closed subscheme of X. Note that there is no requirement on equidimensionality.

Let (X, Σ) be a reduced pair with Σ a divisor in X and let

$$\Omega^p_{X,\Sigma} := \Omega^p_X(\log \Sigma)(-\Sigma)$$

be the $\mathscr{O}_X$-module where $\Omega^p_X(\log \Sigma)$ is the sheaf of log differential p-forms of X with simple poles along Σ. We can then form a complex $\Omega^{\bullet}_{X,\Sigma} \in D_{\text{filt}}(X)$ with the natural differential arising from exterior differentiation on the log differentials. This can be thought of as a de Rham complex for the pair (X, Σ), see [Del70] for more details.

Definition 3.5. Let (X, Σ) be a reduced pair. A *good hyperresolution of* Σ *in* X is a hyperresolution $X_{\bullet}$ of X

$$\begin{tikzcd}
\Sigma_{\bullet} := (\varepsilon_{\bullet}^{-1}(\Sigma))_{\text{red}} \arrow[r, hook] \arrow[d] & X_{\bullet} \arrow[d, "\varepsilon_{\bullet}"] \\
\Sigma \arrow[r, hook] & X
\end{tikzcd}$$

where, for all α, either $\Sigma_\alpha := \varepsilon_\alpha^{-1}(\Sigma)$ is an snc divisor or $\Sigma_\alpha = X_\alpha$.

Definition 3.6. Let (X, Σ) be a reduced pair of finite type over $\mathbb{C}$ and $\varepsilon_{\bullet}$ a good hyperresolution of Σ in X. Then the *Deligne-Du Bois complex of* (X, Σ) is defined as

$$\underline{\Omega}^{\bullet}_{X,\Sigma} := \mathcal{R}\varepsilon_{\bullet *}\Omega^{\bullet}_{X_{\bullet},\Sigma_{\bullet}}.$$

and is an object in $D_{\text{filt}}(X)$ with graded quotients

$$\underline{\Omega}^p_{X,\Sigma} := Gr^p_{\text{filt}}\underline{\Omega}^{\bullet}_{X,\Sigma}[p].$$

Taking $\Sigma = \varnothing$, we obtain the definition of the *Deligne-Du Bois complex of* X, $\underline{\Omega}^{\bullet}_X$.

As consequence of the definition, we have

$$\underline{\Omega}^p_{X,\Sigma} \simeq_{\mathrm{qis}} \mathcal{R}\varepsilon_{\bullet *}\Omega^p_{X_\bullet,\Sigma_\bullet}.$$

3.3 Properties

While the construction of the Deligne-Du Bois complex can be made to work for any field of characteristic zero, see [GNAPGP88] for a more detailed discussion, the properties it exhibits are best described over $\mathbb{C}$, and so we will specialize to $\mathbb{C}$-schemes. A selection of relevant and particularly nice properties are listed in the following theorem. See [Kol13, Theorem 6.5] for more details.

Theorem 3.7 ([DB81], [Kov11]). *Let (X,Σ) be a reduced pair of finite type over $\mathbb{C}$. Then $\underline{\Omega}^{\bullet}_{X,\Sigma} \in \mathit{Ob}\, D_{\mathit{filt}}(X)$ satisfies the following properties*

(1) Let $j : X\setminus\Sigma \hookrightarrow X$ denote the natural embedding. Then

$$\underline{\Omega}^{\bullet}_{X,\Sigma} \simeq_{qis} j_!\mathbb{C}_{X\setminus\Sigma}$$

(2) If $\varepsilon_\bullet : X_\bullet \to X$ is any embedded hyperresolution of $\Sigma \subset X$, then

$$\underline{\Omega}^{\bullet}_{X,\Sigma} \simeq_{qis} \mathcal{R}\varepsilon_{\bullet *}\Omega^{\bullet}_{X_\bullet,\Sigma_\bullet}.$$

In particular, $h^i(\underline{\Omega}^p_{X,\Sigma}) = 0$ for $i < 0$.

(3) If X is smooth and Σ is a normal crossing divisor, then

$$\underline{\Omega}^{\bullet}_{X,\Sigma} \simeq_{qis} \Omega^{\bullet}_{X,\Sigma},$$

and hence,

$$\underline{\Omega}^{p}_{X,\Sigma} \simeq_{qis} \Omega^{p}_{X,\Sigma}.$$

(4) $\underline{\Omega}^{\bullet}_{(-),(-)}$ is functorial, that is, if $\varphi : (Y,\Gamma) \to (X,\Sigma)$ is a morphism of reduced pairs of finite type, then there exists a natural map φ^* of filtered complexes

$$\varphi^* : \underline{\Omega}^{\bullet}_{X,\Sigma} \to \mathcal{R}\varphi_* \underline{\Omega}^{\bullet}_{Y,\Gamma}.$$

Furthermore, $\underline{\Omega}^{\bullet}_{X,\Sigma} \in \mathsf{Ob}\,(D^b_{\mathit{filt,coh}}(X))$, in particular, for any p, the cohomology sheaves $h^i(\Omega^p_{X,\Sigma})$ are coherent, and if φ is proper, then φ^* is a morphism in $D^b_{\mathit{filt,coh}}(X)$.

(5) Let $U \subset X$ be an open subscheme of X. Then

$$\underline{\Omega}^{\bullet}_{X,\Sigma}|_U \simeq_{qis} \underline{\Omega}^{\bullet}_{U,U\cap\Sigma}.$$

In particular,

$$\underline{\Omega}^{\bullet}_{X,\Sigma}|_{X\setminus\Sigma} \simeq_{qis} \underline{\Omega}^{\bullet}_{X\setminus\Sigma}.$$

(6) There exists a distinguished triangle,

$$\underline{\Omega}^{\bullet}_{X,\Sigma} \longrightarrow \underline{\Omega}^{\bullet}_{X} \longrightarrow \underline{\Omega}^{\bullet}_{\Sigma} \xrightarrow{+1}$$

and for each p there exists a distinguished triangle,

$$\underline{\Omega}^p_{X,\Sigma} \longrightarrow \underline{\Omega}^p_X \longrightarrow \underline{\Omega}^p_\Sigma \xrightarrow{+1} .$$

(7) *If X is proper, then there exists a spectral sequence degenerating at E_1 and abutting to the singular cohomology with compact support of $X\backslash\Sigma$:*

$$E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}^p_{X,\Sigma}) \implies H_c^{p+q}(X\backslash\Sigma, \mathbb{C}).$$

(8) *Let $\pi : \widetilde{X} \to X$ be a projective morphism and $\Sigma \subset X$ a reduced closed subscheme such that π is an isomorphism outside of Σ. Set $E = \pi^{-1}(\Sigma)_{red}$. Then there exists a distinguished triangle,*

$$\underline{\Omega}^\bullet_X \xrightarrow{\alpha} \underline{\Omega}^\bullet_\Sigma \oplus \mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{X}} \xrightarrow{\beta} \mathcal{R}\pi_*\underline{\Omega}^\bullet_E \xrightarrow{+1} ,$$

where α is the sum and β is the difference of the natural maps from (4). Similarly for each p there exists a distinguished triangle,

$$\underline{\Omega}^p_X \xrightarrow{\alpha} \underline{\Omega}^p_\Sigma \oplus \mathcal{R}\pi_*\underline{\Omega}^p_{\widetilde{X}} \xrightarrow{\beta} \mathcal{R}\pi_*\underline{\Omega}^p_E \xrightarrow{+1} .$$

As was originally desired, the properties given in Theorem 3.7 that the Deligne-Du Bois complex admits replicate the properties of the De Rham complex of a smooth variety. Numerous results, including vanishing of cohomology, have been proved for the Deligne-Du Bois complex, see [Kol13] for an overview and see [Ste83], [Kol14, Chapter 12], [Kov99], and [Kov00] for some applications. In particular, (7) tells us that a Frölicher-type spectral sequence holds for singular varieties when using the Deligne-Du Bois complex. As in the smooth case, this spectral sequence allows us to connect topological, analytic and algebraic data of a scheme. We could then try to determine in what situations the degeneration allows us to actually compute topological data, such as the singular cohomology of X^{an}, algebraically and vice versa. One such circumstance, and perhaps the most well understood situation, is when X has Du Bois singularities.

Definition 3.8. A scheme X has *Du Bois singularities* if the natural map $\mathscr{O}_X \to \underline{\Omega}^0_X$ is a quasi-isomorphism.

Remark. As consequence of the definition and the degeneration of the Frölicher-type spectral sequence, we recover our surjection "characterizing" Du Bois singularities from section 1.2. Namely, that if a proper complex variety X has Du Bois singularities, the natural map

$$H^i(X^{an}, \mathbb{C}) \to H^i(X, \mathscr{O}_X)$$

is surjective for all i.

The issue remains that $\underline{\Omega}^0_X$ can be difficult to compute outside of when the de Rham complex and the Deligne-Du Bois complex happen to agree. By constraining ourselves to when X is embedded in a smooth scheme, Schwede gives the following simple characterization.

Theorem 3.9. *[Sch07, Cor 4.6] Let X be a reduced separated scheme of finite type over a field of characteristic zero. Suppose that $X \subset Y$ where Y is smooth and suppose that $\pi : \widetilde{Y} \to Y$ is a log resolution of X in Y that is an isomorphism outside of X. If E is the reduced preimage of X in $\widetilde{Y}$, then $\underline{\Omega}^0_X \simeq_{qis} \mathcal{R}\pi_*\mathscr{O}_E$. In particular, X has DB singularities if and only if the natural map $\mathscr{O}_X \to \mathcal{R}\pi_*\mathscr{O}_E$ is a quasi-isomorphism.*

Definition 3.10. A scheme X has *higher p-Du Bois singularities* if the natural map $\Omega^k_X \to \underline{\Omega}^k_X$ is a quasi-isomorphism for all $0 \leq k \leq p$.

Remark. The notion of higher p-Du Bois singularities was originally defined for subvarieties of smooth varieties, see [JKSY22], but can be generalized to any scheme admitting a Deligne-Du Bois complex.

Our understanding of $\underline{\Omega}^0_X$ has benefited from its close relationship to Du Bois singularities. However, the nature of the technical construction makes it unclear if the higher graded pieces $\underline{\Omega}^p_X$ play a similar role or can be used in similar characterizations. Current techniques for analyzing $\underline{\Omega}^p_X$ rely upon Hodge theoretic techniques, often requiring strict hypotheses and a deep understanding of the minimal exponent. Most recent advances have been made for reduced hypersurfaces in smooth, irreducible, n-dimensional complex algebraic varieties, see [MOPW21] and [MP22]. In particular, it has been shown that in this setting, p-log canonical singularities are equivalent to higher p-Du Bois.

3.4 A Hyperresolution-Free Characterization

Lemma 3.11. *Let (Y, X) be a reduced pair of finite type over a field of characteristic 0. Let $\pi : \widetilde{Y} \to Y$ be a projective morphism such that π is an isomorphism outside of X and let $E = \pi^{-1}(X)_{red}$. Then there is a quasi-isomorphism of complexes $\underline{\Omega}^{\bullet}_{Y,X} \simeq_{qis} \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{\widetilde{Y},E}$. Moreover, for each p, there is a quasi-isomorphism of complexes, $\underline{\Omega}^p_{Y,X} \simeq_{qis} \mathcal{R}\pi_*\underline{\Omega}^p_{\widetilde{Y},E}$.*

Proof. We will prove that $\underline{\Omega}^{\bullet}_{Y,X} \simeq_{\text{qis}} \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{\widetilde{Y},E}$. The case of the p-th graded pieces will follow from the quasi-isomorphism being filtered. From Theorem 3.7, we have the distinguished triangle

$$\underline{\Omega}^{\bullet}_Y \longrightarrow \underline{\Omega}^{\bullet}_X \oplus \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{\widetilde{Y}} \longrightarrow \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_E \xrightarrow{+1}$$

and the commutative diagram of distinguished triangles

$$\begin{tikzcd}
\underline{\Omega}^{\bullet}_{Y,X} \arrow[r] \arrow[d] & \underline{\Omega}^{\bullet}_{Y} \arrow[r] \arrow[d] & \underline{\Omega}^{\bullet}_{X} \arrow[r, "+1"] \arrow[d] & {} \\
\mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{\widetilde{Y},E} \arrow[r] & \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{\widetilde{Y}} \arrow[r] & \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{E} \arrow[r, "+1"] & {}
\end{tikzcd}$$

Applying Lemma 2.2 gives the desired isomorphism. □

Notation 2. For any reduced pair (Y, X), a constructor of X in Y will be a triple, $(\pi : \widetilde{Y} \to Y, \widetilde{Y}, E)$, with $\widetilde{Y}$ and $E = \pi^{-1}(X)_{\text{red}}$ each admitting a Deligne-Du Bois complex and π a projective morphism that is an isomorphism outside of X (e.g. a log resolution of X in a smooth Y). Let (Y, X) be a reduced pair of finite type schemes over $\mathbb{C}$. Given a constructor $(\pi, \widetilde{Y}, E)$ of X in Y, denote by $\Xi^{\bullet}_{X\subset Y}$ the cone of the natural map $\mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{\widetilde{Y}} \xrightarrow{\beta} M(\alpha)$ where $M(\alpha)$ is the cone of the natural map $\mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{E}[-1] \xrightarrow{\alpha} \underline{\Omega}^{\bullet}_{Y}$. These maps come from leveraging the quasi-isomorphism in Lemma 3.11. Similarly, let $\Xi^{\bullet}_{p,X\subset Y}$ denote the cone of the natural map $\mathcal{R}\pi_*\underline{\Omega}^{p}_{\widetilde{Y}} \xrightarrow{\beta_p} M(\alpha_p)$ where $M(\alpha_p)$ is the cone of the natural map $\mathcal{R}\pi_*\underline{\Omega}^{p}_{E}[-1] \xrightarrow{\alpha_p} \underline{\Omega}^{p}_{Y}$. Unless otherwise specified, all maps between complexes will be the ones induced by the properties in Theorem 3.7.

Our goal is to construct the Deligne-Du Bois complex of a scheme using constructors. We will prove first that our construction is independent of choice of resolution.

Proposition 3.12. *Let (π_1, Y_1, E_1) and (π_2, Y_2, E_2) be constructors of X in Y. If $\Xi^{\bullet}_{\pi_1}$ denotes the construction of $\Xi^{\bullet}_{X\subset Y}$ using (π_i, Y_i, E_i), then $\Xi^{\bullet}_{\pi_1} \simeq_{qis} \Xi^{\bullet}_{\pi_2}$. Hence, $\Xi^{\bullet}_{X\subset Y} \in \mathit{Ob}(D_{\mathit{filt}}(Y))$ is independent of choice of constructor, up to quasi-isomorphism. Similarly, for each p, we have that $\Xi^{\bullet}_{p,X\subset Y} \in \mathit{Ob}(D_{\mathit{filt}}(Y))$ is independent of choice of constructor, up to quasi-isomorphism.*

Proof. We will prove that $\underline{\Omega}^{\bullet}_{Y,X}$ is independent of choice of constructor, up to quasi-isomorphism. The case of the p-th graded pieces will follow from the quasi-isomorphism being filtered. We note first that a constructor

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & \widetilde{Y} \arrow[d, "\pi"] \\
\Sigma \arrow[r, hook] & Y
\end{tikzcd}$$

gives a fibre diagram in the category of reduced schemes. Moreover, as Y is reduced, the blowup of Y along X is reduced and so the blowup gives a constructor $(\pi, \widetilde{Y}, E)$ of X in Y. Thus, there is a factorization of constructors

$$\begin{tikzcd}
Y_1 \arrow[r, "f_1"] \arrow[dr, "\pi_1"'] & \widetilde{Y} \arrow[d, "\pi"] & Y_2 \arrow[l, "f_2"'] \arrow[dl, "\pi_2"] \\
 & Y &
\end{tikzcd}$$

and so it is enough to show $\Xi^\bullet_{\pi_1} \simeq_{\text{qis}} \Xi^\bullet_{\pi}$. Lemma 3.11 implies $\underline{\Omega}^\bullet_{Y_1,E_1} \simeq_{\text{qis}} \mathcal{R}f_{1*}\underline{\Omega}^\bullet_{\widetilde{Y},E}$ and so, composing with $\mathcal{R}\pi_{1*}$, we have $\mathcal{R}\pi_{1*}\underline{\Omega}^\bullet_{Y_1,E_1} \simeq_{\text{qis}} \mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{Y},E}$. Thus, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{R}\pi_{1*}\underline{\Omega}^\bullet_{Y_1,E_1} \arrow[r] \arrow[d] & \mathcal{R}\pi_{1*}\underline{\Omega}^\bullet_{Y_1} \arrow[d] \\
\mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{Y},E} \arrow[r] & \mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{Y}}
\end{tikzcd}$$

where all of the morphisms are those induced by the properties in Theorem 3.7. Using Lemma 2.1, we can complete this to the (mostly) commutative diagram

$$\begin{tikzcd}
\mathcal{R}\pi_{1*}\underline{\Omega}^\bullet_{Y_1,E_1} \arrow[r] \arrow[d] & \mathcal{R}\pi_{1*}\underline{\Omega}^\bullet_{Y_1} \arrow[r] \arrow[d] & \mathcal{R}\pi_{1*}\underline{\Omega}^\bullet_{E_1} \arrow[r, "+1"] \arrow[d, "\gamma"] & {} \\
\mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{Y},E} \arrow[r] \arrow[d] & \mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{Y}} \arrow[r] \arrow[d] & \mathcal{R}\pi_*\underline{\Omega}_E \arrow[r, "+1"] \arrow[d] & {} \\
0 \arrow[r] \arrow[d, "+1"] & A^\bullet \arrow[r] \arrow[d, "+1"] & B^\bullet \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where γ can also be chosen to be the natural morphism arising from the properties of the Deligne-Du Bois complex. Note that our bottom distinguished triangle gives $A^\bullet \simeq_{\text{qis}} B^\bullet$. We now have the commutative diagram

$$\begin{tikzcd}
\mathcal{R}\pi_{1*}\underline{\Omega}^{\bullet}_{E_1}[-1] \arrow[r, "\alpha_1"] \arrow[d, "{\gamma[-1]}"] & \underline{\Omega}^{\bullet}_{Y} \arrow[d, "id"] \\
\mathcal{R}\pi_{*}\underline{\Omega}^{\bullet}_{E}[-1] \arrow[r, "\alpha"] & \underline{\Omega}^{\bullet}_{Y}
\end{tikzcd}$$

and so Lemma 2.1 gives the (mostly) commutative diagram

$$\begin{tikzcd}
\mathcal{R}\pi_{1*}\underline{\Omega}^{\bullet}_{E_1}[-1] \arrow[r, "\alpha_1"] \arrow[d, "{\gamma[-1]}"] & \underline{\Omega}^{\bullet}_{Y} \arrow[r] \arrow[d, "id"] & M(\alpha_1) \arrow[r, "+1"] \arrow[d, "\delta"] & {} \\
\mathcal{R}\pi_{*}\underline{\Omega}^{\bullet}_{E}[-1] \arrow[r, "\alpha"] \arrow[d] & \underline{\Omega}^{\bullet}_{Y} \arrow[r] \arrow[d] & M(\alpha) \arrow[r, "+1"] \arrow[d] & {} \\
B^{\bullet}[-1] \arrow[r] \arrow[d, "+1"] & 0 \arrow[r] \arrow[d, "+1"] & C^{\bullet} \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where δ (while not necessarily unique) is given by TR3 to make our top morphism of distinguished triangles commutative. Note that our bottom distinguished triangle gives $C^{\bullet} \simeq_{\text{qis}} B^{\bullet}$. Finally, we have the commutative square

$$\begin{tikzcd}
\mathcal{R}\pi_{1*}\underline{\Omega}^{\bullet}_{Y_1} \arrow[r, "\beta_1"] \arrow[d] & M(\alpha_1) \arrow[d, "\delta"] \\
\mathcal{R}\pi_{*}\underline{\Omega}^{\bullet}_{\widetilde{Y}} \arrow[r, "\beta"] & M(\alpha)
\end{tikzcd}$$

and so Lemma 2.1 gives the diagram

$$\begin{tikzcd}
\mathcal{R}\pi_{1*}\underline{\Omega}^{\bullet}_{Y_1} \arrow[r, "\beta_1"] \arrow[d] & M(\alpha_1) \arrow[r] \arrow[d, "\delta"] & \Xi^{\bullet}_{\pi_1} \arrow[r, "+1"] \arrow[d] & {} \\
\mathcal{R}\pi_{*}\underline{\Omega}^{\bullet}_{\widetilde{Y}} \arrow[r, "\beta"] \arrow[d] & M(\alpha) \arrow[r] \arrow[d] & \Xi^{\bullet}_{\pi} \arrow[r, "+1"] \arrow[d] & {} \\
A^{\bullet} \arrow[r] \arrow[d, "+1"] & C^{\bullet} \arrow[r] \arrow[d, "+1"] & D^{\bullet} \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd}.$$

Since $A^{\bullet} \simeq_{\text{qis}} B^{\bullet}$ and $B^{\bullet} \simeq_{\text{qis}} C^{\bullet}$, we can see that $D^{\bullet} \simeq_{\text{qis}} 0$ and so we must have $\Xi^{\bullet}_{\pi_1} \simeq_{\text{qis}} \Xi^{\bullet}_{\pi}$. □

Remark. We might naturally wonder if the choice of embedding $X \subset Y$ matters. Suppose that we have two reduced pairs (Y, X) and (Z, X). As X is a closed subscheme of Z and Y, there exists a pushout W such that the following diagram is Cartesian

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, hook] & Z \arrow[d, hook] \\
Y \arrow[r, hook] & W
\end{tikzcd}$$

with (W, X) a reduced pair, see [Fer03] for details.

Proposition 3.13. *The constructions of* $\Xi^{\bullet}_{X\subset Y}$ *and* $\Xi^{\bullet}_{p,X\subset Y}$ *are independent of choice of embedding* $X \subset Y$*.*

Proof. We will prove the result for $\Xi^{\bullet}_{X\subset Y}$. The case of the p-th graded pieces will follow from the quasi-isomorphism being filtered. With the above discussion for pushouts in mind, it is enough to show

$$\Xi^{\bullet}_{X\subset W} \simeq_{\text{qis}} \iota_*\Xi^{\bullet}_{X\subset Y}.$$

where $X \hookrightarrow Y$ and $\iota : Y \hookrightarrow W$ are closed embeddings of reduced schemes of finite type over $\mathbb{C}$. Let $(\pi, \widetilde{W}, E)$ be a constructor for X in W and let $\widetilde{Y} = (\pi^{-1}(Y))_{\text{red}}$. Note first that $(\pi|_{\widetilde{Y}}, \widetilde{Y}, E)$ is a constructor for X in Y and $\iota_*\mathcal{R}(\pi|_{\widetilde{Y}})_*\underline{\Omega}^{\bullet}_E[-1] \simeq_{\text{qis}} \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_E[-1]$, where $\iota : Y \hookrightarrow W$. Thus, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{R}\pi_*\underline{\Omega}^{\bullet}_E[-1] \arrow[r, "\alpha_W"] \arrow[d] & \underline{\Omega}^{\bullet}_W \arrow[d] \\
\underline{\Omega}^{\bullet}_Y \arrow[r, "id"] & \underline{\Omega}^{\bullet}_Y.
\end{tikzcd}$$

Applying Lemma 2.1, we have the diagram of distinguished triangles

$$\begin{tikzcd}
\mathcal{R}\pi_*\underline{\Omega}^\bullet_E[-1] \arrow[r, "\alpha_W"] \arrow[d, "\iota_*\alpha_Y"] & \underline{\Omega}^\bullet_W \arrow[r] \arrow[d] & M_{X\subset W} \arrow[r, "+1"] \arrow[d] & {} \\
\underline{\Omega}^\bullet_Y \arrow[r, "id"] \arrow[d] & \underline{\Omega}^\bullet_Y \arrow[r] \arrow[d] & 0 \arrow[r, "+1"] \arrow[d] & {} \\
\iota_*M_{X\subset Y} \arrow[r] \arrow[d, "+1"] & \mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{W},\widetilde{Y}}[1] \arrow[r] \arrow[d, "+1"] & A^\bullet \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where $M_{X\subset W}$ and $M_{X\subset Y}$ denote the cones over the relevant morphisms α_Y and α_W in the constructions of $\Xi^\bullet_{X\subset W}$ and $\Xi^\bullet_{X\subset Y}$. Thus, $A^\bullet \simeq_{\text{qis}} M_{X\subset Y}[1]$ and so, by Lemma 2.1, we can complete the commutative diagram

$$\begin{tikzcd}
\mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{W},\widetilde{Y}} \arrow[r] \arrow[d, "id"] & \mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{W}} \arrow[d, "\beta_W"] \\
\mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{W},\widetilde{Y}} \arrow[r] & M_{X\subset W}
\end{tikzcd}$$

to the diagram of distinguished triangles

$$\begin{tikzcd}
\mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{W},\widetilde{Y}} \arrow[r] \arrow[d, "id"] & \mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{W}} \arrow[r] \arrow[d, "\beta_W"] & \mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{Y}} \arrow[r, "+1"] \arrow[d, "\beta_Y"] & {} \\
\mathcal{R}\pi_*\underline{\Omega}^\bullet_{\widetilde{W},\widetilde{Y}} \arrow[r] \arrow[d] & M_{X\subset W} \arrow[r] \arrow[d] & \iota_*M_{X\subset Y} \arrow[r, "+1"] \arrow[d] & {} \\
0 \arrow[r] \arrow[d, "+1"] & \Xi^\bullet_{X\subset W} \arrow[r] \arrow[d, "+1"] & \iota_*\Xi^\bullet_{X\subset Y} \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

Thus, $\Xi^\bullet_{X\subset W} \simeq_{\text{qis}} \iota_*\Xi^\bullet_{X\subset Y}$. □

Theorem 3.14. *Let $(\pi, \widetilde{Y}, E)$ be a constructor of X in Y. Then $\underline{\Omega}^\bullet_X \simeq_{qis} \Xi^\bullet_{X\subset Y}$ and,* *moreover, for each p we have $\underline{\Omega}^p_X \simeq_{qis} \Xi^{\bullet}_{p,X\subset Y}$.*

Proof. We will prove the case that $\underline{\Omega}^{\bullet}_X \simeq_{\text{qis}} \Xi^{\bullet}_{X\subset Y}$. The case of the p-th graded pieces will follow from the quasi-isomorphism being filtered. Lemma 3.11 implies $\underline{\Omega}^{\bullet}_{Y,X} \simeq_{\text{qis}} \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{\widetilde{Y},E}$ and so we have the commutative diagram

$$\begin{tikzcd}
\mathcal{R}\pi_*\underline{\Omega}^{\bullet}_E[-1] \arrow[r] \arrow[d, "id"] & \underline{\Omega}^{\bullet}_{Y,X} \arrow[d] \\
\mathcal{R}\pi_*\underline{\Omega}^{\bullet}_E[-1] \arrow[r, "\alpha"] & \underline{\Omega}^{\bullet}_Y
\end{tikzcd} .$$

Applying Lemma 2.1 and Theorem 3.7, we have the diagram of distinguished triangles

$$\begin{tikzcd}
\mathcal{R}\pi_*\underline{\Omega}^{\bullet}_E[-1] \arrow[r] \arrow[d] & \underline{\Omega}^{\bullet}_{Y,X} \arrow[r] \arrow[d] & \mathcal{R}\pi_*\underline{\Omega}^{\bullet}_{\widetilde{Y}} \arrow[r, "+1"] \arrow[d, "\beta"] & {} \\
\mathcal{R}\pi_*\underline{\Omega}^{\bullet}_E[-1] \arrow[r, "\alpha"] \arrow[d] & \underline{\Omega}^{\bullet}_Y \arrow[r] \arrow[d] & M(\alpha) \arrow[r, "+1"] \arrow[d] & {} \\
0 \arrow[r] \arrow[d, "+1"] & \underline{\Omega}^{\bullet}_X \arrow[r] \arrow[d, "+1"] & \Xi^{\bullet}_{X\subset Y} \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

Thus, we see that $\underline{\Omega}^{\bullet}_X \simeq_{\text{qis}} \Xi^{\bullet}_{X\subset Y}$, as desired. ☐

Corollary 3.15. *If Y is smooth and $(\pi, \widetilde{Y}, E)$ is a log resolution of X in Y, then $\underline{\Omega}^{\bullet}_X$ is quasi-isomorphic to the complex*

$$\mathcal{R}\pi_*\Omega^{\bullet}_{\widetilde{Y}}[1] \oplus \mathcal{R}\pi_*\Omega^{\bullet}_E \oplus \Omega^{\bullet}_Y$$

with differential $(d_{\mathcal{R}\pi_*\Omega^{\bullet}_{\widetilde{Y}}}[1], \beta[1] + (d_{\mathcal{R}\pi_*\Omega^{\bullet}_E}, \alpha[1] + d_{\Omega^{\bullet}_Y}))$. Similarly, for each p, $\underline{\Omega}^p_X$ is quasi-isomorphic to the complex

$$\mathcal{R}\pi_*\Omega^p_{\widetilde{Y}}[1] \oplus \mathcal{R}\pi_*\Omega^p_E \oplus \Omega^p_Y$$

with differential $(d_{\mathcal{R}\pi_*\Omega^p_{\widetilde{Y}}}[1], \beta_p[1] + (d_{\mathcal{R}\pi_*\Omega^p_E}, \alpha_p[1] + d_{\Omega^p_Y}))$.

Corollary 3.16. *If Y is smooth and $(\pi, \widetilde{Y}, E)$ is a constructor of X in Y so that $\widetilde{Y}$ is smooth and E is Du Bois then X has Du Bois singularities if and only if*

$$\mathscr{O}_X \xrightarrow{qis} \mathcal{R}\pi_*\mathscr{O}_E.$$

Proof. Since E, Y and $\widetilde{Y}$ are Du Bois, we have the morphism of triangles:

$$\begin{tikzcd}
\mathcal{R}\pi_*\mathscr{O}_E[-1] \arrow[r, "\alpha_0"] \arrow[d] & \mathscr{O}_Y \arrow[r] \arrow[d] & M(\alpha_0) \arrow[r, "+1"] \arrow[d, "id"] & {} \\
\Xi^{\bullet}_{0,X\subset Y}[-1] \arrow[r] & \mathcal{R}\pi_*\mathscr{O}_{\widetilde{Y}} \arrow[r, "\beta_0"] & M(\alpha_0) \arrow[r, "+1"] & {}
\end{tikzcd}$$

Moreover, Y smooth (and hence rational) implies that $\mathscr{O}_Y \simeq_{\text{qis}} \mathcal{R}\pi_*\mathscr{O}_{\widetilde{Y}}$ and so $\mathcal{R}\pi_*\mathscr{O}_E \simeq_{\text{qis}} \Xi^{\bullet}_{0,X\subset Y}$. Thus X is Du Bois if and only if $\mathscr{O}_X \simeq_{\text{qis}} \mathcal{R}\pi_*\mathscr{O}_E$. This recovers Schwede's simple characterization of Du Bois singularities, see Theorem 3.9. □

Remark. A naive attempt to generalize this characterization to expressing $\underline{\Omega}^p_X$ in terms of $\mathcal{R}\pi_*\Omega^p_E$ highlights the difficulty in understanding the p-th graded pieces of the Deligne-Du Bois complex and why such strong assumptions might be needed when working with higher p-Du Bois singularities. Namely, with the same set up as above, if $\mathcal{R}\pi_*\Omega^p_E \simeq_{\text{qis}} \underline{\Omega}^p_X$ were to hold for some p, we would have the morphism of distinguished triangles

$$\begin{tikzcd}
\mathcal{R}\pi_*\Omega^p_E[-1] \arrow[r, "\alpha_p"] \arrow[d, "\simeq_{\text{qis}}"] & \Omega^p_Y \arrow[r] \arrow[d] & M(\alpha_p) \arrow[r, "+1"] \arrow[d, "id"] & {} \\
\Xi^{\bullet}_{p,X\subset Y}[-1] \arrow[r] & \mathcal{R}\pi_*\Omega^p_{\widetilde{Y}} \arrow[r, "\beta_p"] & M(\alpha_p) \arrow[r, "+1"] & {}
\end{tikzcd}.$$

This would force $\Omega^p_Y \simeq_{\text{qis}} \mathcal{R}\pi_*\Omega^p_{\widetilde{Y}}$, which would require stronger assumptions in general.

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