Formalization of Structural Properties of Triangulated Categories

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Abstract

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This dissertation explores several structural properties of triangulated categories. In particular, it develops theories of generation and dimension, decomposability, and support. Existing notions from the literature are surveyed, new notions are developed, and several interactions between these theories are described. Additionally, a majority of the results are formalized in the proof assistant Lean4.

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to Karl

Chapter 1

INTRODUCTION

A triangulated category is a homological-algebraic object which has become indispensable in the algebraist's toolkit. The first definition was given by Verdier in his thesis [Ver96] as a bookkeeping tool in order to concisely express complex statements such as Serre-Grothendieck duality, and there he considered the derived category of an abelian category, in particular, the category of quasicoherent sheaves on a scheme. Triangulated categories soon arose in fields other than algebraic geometry, such as the stable homotopy category in homotopy theory and the stable module category in representation theory.

Triangulated categories began to be understood as geometric objects in their own right, and understanding their geometry became central to the study of the objects they classified. Beilinson proved in [Bei79] the derived category of projective space admits a semi-orthogonal decomposition which mimics its cell-structure, and similar semi-orthogonal decompositions were later discovered for grassmanians, flag varieties, and blowups. Bondal and Orlov proved in [BO01] that a variety with (anti)-ample canonical is determined by its bounded derived category of coherent sheaves, and Balmer prove in [Bal05] proved that a scheme is determined by its tensor-triangulated category of bounded perfect complexes. The depth of these relationships is still subject to intense study. It is conjectured [Kaw02] that derived equivalence between two birationally equivalent varieties has a simple birational geometric characterization. It has also been conjectured [BGL21] varieties with nef and effective canonical divisors have indecomposable derived categories, further relating the study of derived categories to the minimal model program.

Meanwhile, powerful categorical techniques were developed by Neeman [Nee96] to prove representability of a very broad class of functors, yielding a cleaner proof of Grothendieck duality. These ideas, which required the categories in question to be large enough to admit arbitrary coproducts, were adapted to categories small enough to satisfy certain finiteness conditions by Bondal and van den Bergh [BvdB03]. Ideas from the latter inspired Rouquier [Rou08] to define a notion of dimension for triangulated categories. Orlov [Orl09] proved that the derived category of a smooth projective curve is one-dimensional, and conjectured that the dimension of the derived category of any smooth projective variety is equal to the variety's dimension.

This thesis grew out of the study of these three conjectures. I provide the necessary background and describe my original contributions to the subject.

In addition to this more traditional mathematical exposition, I have formalized a majority of the definitions and theorems presented here in the proof assistant Lean4. Formalization is the process of translating a mathematical proof into a formal language so that the correctness can be algorithmically checked by the computer. The standards for mathematical proof have undergone many transformations throughout the history of the discipline, and I believe we are at the beginning of another paradigm-shift. As the frontiers of mathematical research expand, the effort required to check all the details of an argument becomes untenable.

There are famous examples of peer-reviewed papers, published in the same journal, which contain contradictory statements. There are conjectures with claimed proofs which lack clear consensus about their correctness, resulting in unusual situations where academic politics, rather than mathematical rigor, determine whether a certain proposition is accepted as true. Entire theories are often built on highly-technical foundational results for which no detailed proof exists in writing. For an extreme example, consider the following quote from the introduction of [Con02]:

> *I initially tried to verify the commutativity of (1.1.3) by a direct calculation... This approach quickly gets stuck... When I asked Deligne about this difficulty, he agreed that the projective smooth case seemed puzzling if one tried to analyze it directly... A subsequent discussion about the general case (1.1.3) with some other experts was*

also inconclusive. Despite the fact that everyone believes that (1.1.3) commutes, no published proof seems to exist. Nevertheless, it is widely used.

Formalization provides a solution to this problem, and in a world where research journals require formal verifications, the refereeing process would only involve checking that the definitions and theorem statements have been correctly formalized, rather than meticulously checking every detail of a proof.

Of course, formalizing mathematical research is a daunting task. The technical background knowledge required to undertake a large-scale formalization project is formidable, and this difficulty is the primary barrier to transitioning to a world in which all new mathematical research is formalized. The formalization of definitions and theorems presented here not only lays the groundwork for a formalization of whatever proofs of the main conjectures eventually appear, but it could also allow for the tools of the future to assist mathematicians in their search for those proofs.

1.1 Conventions and Notations

We assume familiarity with triangulated categories. For an excellent general treatment of triangulated categories, we recommend [Nee01], [Wei94], or [GM13]. For a treatment of triangulated categories which arise specifically in algebraic geometry, we recommend [Huy06] or [Har66].

We reserve the notation $\mathcal{T}, \mathcal{S}$ for abstract triangulated categories, and in these cases we use symbols a, b, c to refer to objects. If X is a scheme, we write $D^b(X)$ to mean to the bounded derived category of the category of coherent sheaves on X, and we use symbols like $\mathcal{E}, \mathcal{F}, \mathcal{G}$ for objects of these categories. Similarly if R is a commutative ring, we write $D^b(R)$ to to mean the bounded derived category of the category of finitely generated R-modules, equivalently, $D^b(R)$ is shorthand for $D^b(\mathrm{Spec}(R))$. If $\mathcal{E}$ is a coherent sheaf on X, we will often abuse notation and consider $\mathcal{E}$ as an object of $D^b(X)$ via a complex with $\mathcal{E}$ in degree zero and zeroes elsewhere.

We almost always use the term "triangle" to mean "distinguished triangle", and if we mean to describe a not-necessarily-distinguished triangle, we will be explicit. We use brackets to represent the shift functor, so for example a triangle will be represented as $a \to b \to c \to a[1]$. We often identify subsets of the collection of objects of $\mathcal{T}$ with full subcategories, and we use set-theoretic notation to describe these subcategories. We usually use symbols $\mathcal{I}, \mathcal{J}, \mathcal{K}$ to denote full subcategories. We often identify an object $a \in \mathcal{T}$ with the full subcategory $\{a\} \subseteq \mathcal{T}$. When referring to a "functor" $F : \mathcal{T} \to \mathcal{S}$ between two triangulated categories, we will always mean "triangulated functor", i.e. an additive functor which commutes with the shift and takes triangles to triangles.

For a few concepts we decided to match the notation used in Lean4's mathematical library mathlib [mat20], rather than the notation more commonly used by mathematicians. In particular, we write $a \boxplus b$ rather than $a \oplus b$ for the biproduct of a and b in the category $\mathcal{T}$, and we write $f \gg g$ rather than $g \circ f$ for the composition of morphisms in a category. The exception is that we write $\mathcal{F} \oplus \mathcal{G}$ for the biproduct of complexes in $D^b(X)$.

We also assume some familiarity with notions from dependent type theory as they appear in Lean4. The two most important such notions are inductive types and typeclasses. To learn about type theory in general, we recommend [Uni13]. The language reference for Lean4 is [Dev]. For a less comprehensive, but more practical introduction to working with Lean4, we recommend [AdMK+24] or [Com24] The former is written from more of a computer-scientific point of view, while the latter is written by mathematicians for mathematicians.

Any theorems and definitions which have been formalized will be marked with " ↗ ". For the electronic version of this thesis, these symbols contain links to the corresponding formalization, currently hosted on GitHub as of May 13, 2026.

Chapter 2

GENERATION TIME AND DIMENSION

Following Rouquier [Rou08], we wish to define a notion of dimension for triangulated categories, and we hope that it will be a useful invariant. As triangulated categories are homological objects, we wish to draw inspiration from homological dimension. Recall that a Noetherian commutative ring R is regular if and only if it has finite homological dimension, and in this case homological dimension and Krull dimension coincide. In other words, me may detect Krull dimension in these cases by the existence of projective resolutions for any finitely generated module with a uniform bound. By considering successive truncation triangles in $D^b(R)$, we see that any finitely generated R-module M can be constructed from finite projective objects by using only d cones. Since a projective object is a summand of a free object, we see that any M can be constructed from sums and shifts of R, by taking d cones and a single summand.

The Rouquier dimension of a triangulated category $\mathcal{T}$ is then roughly the minimal number of cones required to construct any object of $\mathcal{T}$ starting from a single fixed generator. The above discussion hints at the fact that this theory is well-behaved for categories of the form $D^b(R)$ where R is a Noetherian ring. The story becomes much more complicated for categories of the form $D^b(X)$ where X is a proper scheme. In these cases, the category of coherent sheaves on X doesn't have enough projectives, so the above construction cannot be carried out, and it is unclear how to appropriately glue together generators over an affine cover of X.

In this chapter, we make these ideas precise and give the definition of Rouquier dimension. After developing the general theory by following the formalization, we specialize to geometric cases of interest.

2.1 Basic Definitions

Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{I}, \mathcal{J}, \mathcal{K} \subseteq \mathcal{T}$. We denote by $\overline{0}$ the strictly full triangulated subcategory of zero objects of $\mathcal{T}$.

Definition 2.1.1 (↗). Define $\mathrm{addc}(\mathcal{I})$ to be the smallest full subcategory satisfying the following properties:

1. If $a \in \mathcal{I}$, then $a \in \mathrm{addc}(\mathcal{I})$.

2. The zero object $0 \in \mathrm{addc}(\mathcal{I})$.

3. For any object a, $a \in \mathrm{addc}(\mathcal{I})$, if and only if $a[1] \in \mathrm{addc}(\mathcal{I})$.

4. If $a \in \mathrm{addc}(\mathcal{I})$ and $a \cong b$, then $b \in \mathrm{addc}(\mathcal{I})$.

5. If $a, b \in \mathrm{addc}(\mathcal{I})$, then $a \boxplus b \in \mathrm{addc}(\mathcal{I})$.

Definition 2.1.2 (↗). Define $\mathrm{smd}(\mathcal{T})$ to be the smallest full subcategory satisfying the following properties:

- If $a \in \mathcal{I}$, then $a \in \mathrm{smd}(\mathcal{I})$.

- If $a \boxplus b \in \mathcal{I}$, then $a \in \mathcal{I}$ and $b \in \mathcal{I}$.

Remark. Once of the principals of formalization is that definitions should be as general as possible. In [BvdB03], the operation "smd" is only defined on the class of strictly full subcategories, while my definition is valid for any collection of objects of $\mathcal{T}$.

In mathlib, the notation "$a \boxplus b$" denotes a fixed, noncanonical choice of a biproduct, and so $\mathrm{smd}(\mathcal{I})$ is only useful if $\mathcal{I}$ is closed under isomorphisms. As many of the formal properties still hold without this assumption, the formalization is cleaner with this definition. For almost all applications that require $\mathcal{I}$ to be closed under isomorphisms, this hypothesis can be managed by Lean4's typeclass inference system, making this definition easy to use.

Definition 2.1.3 (↗)**.** Define a full subcategory

$$\mathcal{I} \star \mathcal{J} := \{c \in \mathcal{T} \mid \text{there exists a triangle } a \to c \to b \to a[1] \text{ with } a \in \mathcal{I}, b \in \mathcal{J}\}$$

Definition 2.1.4 (↗)**.** Define a full subcategory $\mathcal{I} \diamond \mathcal{J} := \mathrm{smd}(\mathrm{addc}(\mathcal{I}) \star \mathrm{addc}(\mathcal{J}))$

Remark. In [BBDG18], the operation "$\diamond$" is only defined for strictly full subcategories closed under direct sums, and it is defined as $\mathcal{I} \diamond \mathcal{J} = \mathrm{smd}(\mathcal{I} \star \mathcal{J})$. We once again modify the definition to apply to arbitrary full subcategories. This leads to an equivalent definition of dimension which is more amenible to formalization.

Definition 2.1.5 (↗)**.** Let $\mathcal{I} \subseteq \mathcal{T}$. We inductively define

1. $\langle \mathcal{I} \rangle_0 := \mathrm{smd}(\mathrm{addc}(\{0\}))$.

2. $\langle \mathcal{I} \rangle_{k+1} := \langle \mathcal{I} \rangle_k \diamond \mathcal{I}$

3. $\langle \mathcal{I} \rangle := \bigcup_{k \geq 1} \langle \mathcal{I} \rangle_k$.

The full subcategory $\langle \mathcal{I} \rangle_k$ is sometimes called the *k-th thickening* of $\mathcal{I}$, and $\langle \mathcal{I} \rangle$ the *thick closure* of $\mathcal{I}$. If $\langle g \rangle = \mathcal{T}$, then we call g a *classical generator* of $\mathcal{T}$. If $\langle g \rangle_k = \mathcal{T}$ for some k, then we call g a *strong generator*.

Remark. In [BvdB03], the subcategories are only defined for $k \geq 1$. There the equality $\langle \mathcal{I} \rangle_1 = \mathrm{smd}(\mathrm{addc}(\mathcal{I}))$ is a definition, while here it is a theorem.

Definition 2.1.6 (↗)**.**

1. The *generation time* of $g \in \mathcal{T}$ is

$$U(g) := \inf\{k \in \mathbb{N} \mid \langle g \rangle_{k+1} = \mathcal{T}\}.$$

2. The *dimension spectrum* of $\mathcal{T}$ is

$$\mathrm{dimSpec}(\mathcal{T}) := \{U(g) \mid g \in \mathcal{T} \text{ is a strong generator}\}.$$

3. The *Rouquier dimension* of $\mathcal{T}$ is

$$\mathrm{Rdim}(\mathcal{T}) := \inf\{k \in \mathbb{N} \cup \infty \mid k \in \mathrm{dimSpec}(\mathcal{T})\}.$$

Remark. In mathlib, the infinimum of the empty set of natural numbers is defined to be 0. This design decision was made so that the "set infimum" function is defined for any set of naturals. On the other hand, the infimum of the empty set of extended natural numbers (i.e. $\mathbb{N} \cup \infty$) is ∞. In order to have cleaner theorem statements, we made $\mathrm{Rdim}(\mathcal{T})$ have type $\mathbb{N} \cup \infty$, and $\mathrm{Rdim}(\mathcal{T}) = \infty$ if $\mathcal{T}$ is not strongly generated.

Remark. The subcategories $\langle \mathcal{I} \rangle_k$ are not in general triangulated, as they are not closed under taking cones. In general, the cone of a map $a \to b$ with $a \in \langle \mathcal{I} \rangle_k$ and $b \in \langle \mathcal{I} \rangle_j$ is in $\langle \mathcal{I} \rangle_{k+j}$, and it is interesting to study when the cone is in $\langle \mathcal{I} \rangle_m$ for some $m < k + j$.

2.2 First Properties

The goal of this section is to prove that $\langle \mathcal{I} \rangle$ is a thick subcategory. Recall that a thick subcategory is a strictly full triangulated subcategory which is closed under taking summands. While the statement is intuitive enough that the details are not written down in the literature, its formalization was a non-trivial task. We recount many of the lemmas necessary to complete the formalization, in order to demonstrate the level of detailed required by formalization.

Proposition 2.2.1 (↗)**.** *The operation "$\star$" is associative :* $(\mathcal{I} \star \mathcal{J}) \star \mathcal{K} = \mathcal{I} \star (\mathcal{J} \star \mathcal{K})$.

Proof. This is nearly a restatement of the octohedral axiom. Let $c \in (\mathcal{I} \star \mathcal{J}) \star \mathcal{K}$. Choose a triangle $b_1 \to c \to b_2 \to b_1[1]$ with $b_1 \in (\mathcal{I} \star \mathcal{J})$ and $b_2 \in \mathcal{K}$. Choose a triangle $a_1 \to b_1 \to a_2 \to a_1[1]$ with $a_1 \in \mathcal{I}$ and $a_2 \in \mathcal{J}$. The octohedral axiom then gives the following diagram, where all straight lines belong to triangles.

$$\begin{tikzcd}
a_1 \arrow[d] \arrow[dr] & & & \\
b_1 \arrow[r] \arrow[dd] & c \arrow[rr] \arrow[dr] & & b_2 \\
& & d \arrow[ur] & \\
a_2 \arrow[urr] & & &
\end{tikzcd}$$

The triangle $a_2 \to d \to b_2 \to a_2[1]$ verifies $d \in (\mathcal{J} \star \mathcal{K})$, and the triangle $a_1 \to c \to d \to a_1[1]$ verifies $c \in \mathcal{I} \star (\mathcal{J} \star \mathcal{K})$.

Conversely, if $c \in \mathcal{I} \star (\mathcal{J} \star \mathcal{K})$, we choose a triangle $b_1 \to c \to b_2 \to b_1[1]$ with $b_2 \in (\mathcal{J} \star \mathcal{K})$ and $b_1 \in \mathcal{I}$, and then we choose a triangle $a_1 \to b_2 \to a_2 \to a_1[1]$ with $a_1 \in \mathcal{J}$ and $a_2 \in \mathcal{K}$. Applying the octohedral axiom to the composition $c \to b_2 \to a_2$ gives the below diagram.

$$\begin{tikzcd}
c \arrow[d] \arrow[dr] & & & \\
b_2 \arrow[r] \arrow[dd] & a_2 \arrow[rr] \arrow[dr] & & a_1[1] \\
& & d \arrow[ur] & \\
b_1[1] \arrow[urr] & & &
\end{tikzcd}$$

The shifted triangle $b_1 \to d[-1] \to a_1 \to b_1[1]$ verifies $d[-1] \in \mathcal{I} \star \mathcal{J}$, and the triangle $d[-1] \to c \to a_2 \to d$ verifies $c \in (\mathcal{I} \star \mathcal{J}) \star \mathcal{K}$. □

Remark. In mathlib, a (not necessarily distinguished) triangle is implemented as a structure containing three objects a, b, c, and three morphisms $f : a \to b$, $g : b \to c$, $h : c \to a[1]$. In particular, the triangle $d[-1] \to c \to a_2 \to d$ is really implemented as $d[-1] \to c \to a_2 \to d[-1][1]$, making us of the natural isomorphism $d[-1][1] \cong d$. Similarly, the shifted triangle $b_1 \to d[-1] \to a_1 \to b[1]$ is really implemented as $b_1[1][-1] \to d[-1] \to a_1[1][-1] \to b[1][-1][1]$, using the natural isomorphisms $b_1[1][-1] \cong b$, $a_1[1][-1]$ and $b[1][-1][1] \cong b[1]$.

The main technical difficulty in formalizing this statement was juggling these natural isomorphisms when showing commutativity of certain diagrams.

Proposition 2.2.2 (↗). *If* $\mathcal{I} \subseteq \mathcal{J}$, *then* $\mathrm{addc}(\mathcal{I}) \subseteq \mathrm{addc}(\mathcal{J})$ *and* $\mathrm{smd}(\mathcal{I}) \subseteq \mathrm{smd}(\mathcal{J})$.

Proof. This is proved by structural induction on the predicates $\mathrm{addc}(\mathcal{I})$ and $\mathrm{smd}(\mathcal{I})$. □

Proposition 2.2.3 (↗). *If* $\mathcal{I} \subseteq \mathcal{J}$ *and* $\mathcal{I}' \subseteq \mathcal{J}'$, *then* $\mathcal{I} \star \mathcal{J} \subseteq \mathcal{I}' \star \mathcal{J}'$.

Proof. If $c \in \mathcal{I} \star \mathcal{J}$, then there is a triangle $a \to c \to b \to a[1]$ with $a \in \mathcal{I}$ and $b \in \mathcal{J}$. Then $a \in \mathcal{I}'$ and $b \in \mathcal{J}'$, so that $c \in \mathcal{I}' \star \mathcal{J}'$. □

Proposition 2.2.4 (↗). *If* $\mathcal{I} \subseteq \mathcal{J}$ *and* $\mathcal{I}' \subseteq \mathcal{J}'$, *then* $\mathcal{I} \diamond \mathcal{J} \subseteq \mathcal{I}' \diamond \mathcal{J}'$.

Proof. We have $\mathrm{addc}(\mathcal{I}) \subseteq \mathrm{addc}(\mathcal{I}')$ and $\mathrm{addc}(\mathcal{J}) \subseteq \mathrm{addc}(\mathcal{J}')$ by (2.2.2). We then have $\mathrm{addc}(\mathcal{I}) \star \mathrm{addc}(\mathcal{J}) \subseteq \mathrm{addc}(\mathcal{I}') \star \mathrm{addc}(\mathcal{J}')$ by (2.2.3). Finally, (2.2.2) implies

$$\mathcal{I} \diamond \mathcal{J} = \mathrm{smd}(\mathrm{addc}(\mathcal{I}) \star \mathrm{addc}(\mathcal{J})) \subseteq \mathrm{smd}(\mathrm{addc}(\mathcal{I}') \star \mathrm{addc}(\mathcal{J}')) = \mathcal{I}' \diamond \mathcal{J}'.$$

□

Proposition 2.2.5 (↗). $\mathcal{I} \subseteq \mathcal{I} \diamond \mathcal{J}$ *and* $\mathcal{J} \subseteq \mathcal{I} \diamond \mathcal{J}$

Proof. We only show the first claim, the second being similar. If $a \in \mathcal{I}$, then $a \in \mathrm{addc}(\mathcal{I})$. Since $0 \in \mathrm{addc}(\mathcal{J})$, the triangle $a \to a \to 0 \to a[1]$ verifies $a \in \mathrm{addc}(\mathcal{I}) \star \mathrm{addc}(\mathcal{J})$, and so $a \in \mathcal{I} \diamond \mathcal{J}$. □

Remark. If we had defined $\mathcal{I} \diamond \mathcal{J}$ as $\mathrm{smd}(\mathcal{I} \star \mathcal{J})$ as in [BBDG18], then (2.2.5) would not hold, as we would have $\mathcal{I} \diamond \emptyset = \mathrm{smd}(\mathcal{I} \star \emptyset) = \mathrm{smd}(\emptyset) = \emptyset$. With the original definition, the statement of (2.2.5) would only be true for $\mathcal{I}, \mathcal{J}$ nonempty, and it would be more cumbersome in the formalization to carry that hypothesis around.

Proposition 2.2.6 (↗). *If* $k \leq j$, *then* $\langle \mathcal{I} \rangle_k \subseteq \langle \mathcal{I} \rangle_j$

Proof. It suffices to show that $\langle \mathcal{I} \rangle_k \subseteq \langle \mathcal{I} \rangle_{k+1}$. Since $\langle \mathcal{I} \rangle_{k+1} = \langle \mathcal{I} \rangle_k \diamond \mathcal{I}$, this is (2.2.5). □

Proposition 2.2.7 (↗). *If $\mathcal{I} \subseteq \mathcal{J}$, then $\langle \mathcal{I} \rangle_k \leq \langle \mathcal{J} \rangle_k$.*

Proof. This follows by induction. The case $k = 0$ holds because $\langle \mathcal{I} \rangle_0 = \langle \mathcal{J} \rangle_0$. For the inductive step, if we assume $\langle \mathcal{I} \rangle_k \subseteq \langle \mathcal{J} \rangle_k$, then (2.2.4) implies

$$\langle \mathcal{I} \rangle_{k+1} = \langle \mathcal{I} \rangle_k \diamond \mathcal{I} \subseteq \langle \mathcal{J} \rangle_k \diamond \mathcal{J} = \langle \mathcal{J} \rangle_{k+1}.$$

□

Proposition 2.2.8 (↗). $\mathrm{smd}(\mathcal{I} \star \mathrm{smd}(\mathcal{J})) = \mathrm{smd}(\mathcal{I} \star \mathcal{J}) = \mathrm{smd}(\mathrm{smd}(\mathcal{I}) \star \mathcal{J})$

Proof. We prove the first equality, the second being similar. Monotonicity of '$\star$' and 'smd' imply that $\mathrm{smd}(\mathcal{I} \star \mathcal{J}) \subseteq \mathrm{smd}(\mathcal{I} \star \mathrm{smd}(\mathcal{J}))$, so we only show the reverse inclusion.

Let $c \in \mathrm{smd}(\mathcal{I} \star \mathrm{smd}(\mathcal{J}))$. Either $c \in \mathcal{I} \star \mathrm{smd}(\mathcal{J})$, or there is some c' with $c \boxplus c'$ or $c' \boxplus c$ in $\mathrm{smd}(\mathcal{I} \star \mathrm{smd}(\mathcal{J}))$. In the latter case, we can assume by induction that $c \boxplus c'$ or $c' \boxplus c$ is in $\mathrm{smd}(\mathcal{I} \star \mathcal{J})$, so that $c \in \mathrm{smd}(\mathcal{I} \star \mathcal{J})$.

In the former case, we obtain a triangle $a \to c \to b \to a[1]$ with $a \in \mathcal{I}$ and $b \in \mathrm{smd}(\mathcal{J})$. If $b \in \mathcal{J}$, then $c \in \mathcal{I} \star \mathcal{J}$, and so $c \in \mathrm{smd}(\mathcal{I} \star \mathcal{J})$. Otherwise, there is some b' with $b \boxplus b'$ or $b' \boxplus b \in \mathrm{smd}(\mathcal{J})$. Then the triangle $a \to c \boxplus b' \to b \boxplus b' \to a[1]$ implies that $c \boxplus b' \in \mathrm{smd}(\mathcal{I} \star \mathcal{J})$, or the triangle $a \to b' \boxplus c \to b' \boxplus b \to a[1]$ implies that $b' \boxplus c \in \mathrm{smd}(\mathcal{I} \star \mathcal{J})$. In either case, $c \in \mathrm{smd}(\mathcal{I} \star \mathcal{J})$. □

Proposition 2.2.9 (↗). *Suppose that $\mathcal{I}$ is closed under isomorphisms, closed under shifts, and closed under direct sums. Then* $\mathrm{addc}(\mathcal{I}) = \mathcal{I}$.

Proof. This is proved by structural induction on the predicate $\mathrm{addc}(\mathcal{I})$. □

Proposition 2.2.10 (↗). *Suppose that $\mathcal{I}$ is closed under isomorphisms. Then* $\mathrm{smd}(\mathcal{I})$ *is as well.*

Proof. Let $a \in \mathrm{smd}(\mathcal{I})$ and suppose $a \cong b$. If $a \in \mathcal{I}$, then since $\mathcal{I}$ is closed under isomorphisms, $b \in \mathcal{I}$, so that $b \in \mathrm{smd}(\mathcal{I})$. Otherwise there exists some a' with $a \boxplus a' \in \mathrm{smd}(\mathcal{I})$ or $a' \boxplus a \in \mathrm{smd}(\mathcal{I})$. In the first case, the isomorphism $a \boxplus a' \cong b \boxplus a'$ implies $b \boxplus a' \in \mathrm{smd}(\mathcal{I})$, so that $b \in \mathrm{smd}(\mathcal{I})$. The second case is similar to the first. □

Proposition 2.2.11 (↗). *If $\mathcal{I}$ is closed under shifts and closed under isomorphisms, then* $\operatorname{smd}(\mathcal{I})$ *is closed under shifts.*

Proof. Let $a \in \operatorname{smd}(\mathcal{I})$ and let $i \in \mathbb{Z}$. If $a \in \mathcal{I}$, then $a[i] \in \mathcal{I}$, so that $a[i] \in \operatorname{smd}(\mathcal{I})$. Otherwise there is some a' with $a \boxplus a' \in \operatorname{smd}(\mathcal{I})$ or $a' \boxplus a \in \operatorname{smd}(\mathcal{I})$. In the first case, we can assume by induction that $(a \boxplus a')[i] \in \operatorname{smd}(\mathcal{I})$. Since $(a \boxplus a')[i] \cong a[i] \boxplus a'[i]$, Lemma (2.2.10) implies that $a[i] \boxplus a'[i] \in \operatorname{smd}(\mathcal{I})$, so that $a[i] \in \operatorname{smd}(\mathcal{I})$. The second case is similar. □

Proposition 2.2.12 (↗). *If $\mathcal{I}$ is closed under direct sums and closed under isomorphisms, then* $\operatorname{smd}(\mathcal{I})$ *is closed under direct sums.*

Proof. Suppose $a, b \in \operatorname{smd}(\mathcal{I})$. If $a, b \in \mathcal{I}$, then $a \boxplus b \in \mathcal{I}$, so that $a \boxplus b \in \operatorname{smd}(\mathcal{I})$. If $a \notin \mathcal{I}$, there is some a' with $a \boxplus a' \in \operatorname{smd}(\mathcal{I})$ or $a' \boxplus a \in \operatorname{smd}(\mathcal{I})$. In the first case, we can assume by induction that $(a \boxplus a') \boxplus b \in \operatorname{smd}(\mathcal{I})$. Since $\operatorname{smd}(\mathcal{I})$ is closed under isomorphisms by (2.2.10), we see that the isomorphism $(a \boxplus a') \boxplus b \cong a' \boxplus (a \boxplus b)$ implies that $a \boxplus b \in \operatorname{smd}(\mathcal{I})$. The second case is similar, as are the two cases which arise if $b \notin \mathcal{I}$. □

Proposition 2.2.13 (↗). *If $\mathcal{I}$ and $\mathcal{J}$ are closed under shifts, then $\mathcal{I} \star \mathcal{J}$ is closed under shifts.*

Proof. Let $c \in \mathcal{I} \star \mathcal{J}$ and let $i \in \mathbb{Z}$. Choose a triangle $a \to c \to b \to a[1]$ with $a \in \mathcal{I}, b \in \mathcal{J}$. Then $a[i] \in \mathcal{I}$ and $b[i] \in \mathcal{J}$, so the the triangle $a[i] \to c[i] \to b[i] \to a[i+1]$ verifies $c[i] \in \mathcal{I} \star \mathcal{J}$. □

Proposition 2.2.14 (↗). *If $\mathcal{I}$ is closed under shifts, then*

$$\mathcal{I} \star \mathcal{J} = \{c \in \mathcal{T} \mid \textit{there exists a triangle } c \to b \to a \to c[1] \textit{ with } a \in \mathcal{I}, b \in \mathcal{J}\}.$$

If $\mathcal{J}$ is closed under shifts, then

$$\mathcal{I} \star \mathcal{J} = \{c \in \mathcal{T} \mid \textit{there exists a triangle } b \to a \to c \to b[1] \textit{ with } a \in \mathcal{I}, b \in \mathcal{J}\}.$$

Proof. The first statement follows by comparing the triangle $c \to b \to a \to c[1]$ to the triangle $a[-1] \to c \to b \to a$, since $a \in \mathcal{I}$ if and only if $a[-1] \in \mathcal{I}$. The second statement is similar. □

Proposition 2.2.15 (🗗). *If $\mathcal{I}$ and $\mathcal{J}$ are closed under biproducts, then $\mathcal{I}\star\mathcal{J}$ is closed under biproducts.*

Proof. Let $c, c' \in \mathcal{I}\star\mathcal{J}$. Choose triangles $a \to c \to b \to a[1], a' \to c' \to b' \to a'[1]$ with $a, a' \in \mathcal{I}$ and $b, b' \in \mathcal{J}$. Then the triangle $a \boxplus a' \to c \boxplus c' \to b \boxplus b' \to (a \boxplus a')[1]$ verifies $c \boxplus c' \in \mathcal{I}\star\mathcal{J}$. □

Proposition 2.2.16 (🗗). *If $\mathcal{I}$ is closed under isomorphisms, then $\mathcal{I}\star\overline{0} = \mathcal{I} = \overline{0}\star\mathcal{I}$.*

Proof. If $c \in \mathcal{I}\star\overline{0}$, then there is a triangle $a \to c \to b \to a[1]$ with $a \in \mathcal{I}$ and with b a zero object. It follows that the map $a \to c$ is an isomorphism, and so $c \in \mathcal{I}$. Conversely, if $c \in \mathcal{I}$, the triangle $c \to c \to 0 \to c[1]$ verifies $c \in \mathcal{I}\star\overline{0}$.

The second equality is similar. □

Proposition 2.2.17 (🗗). *The operation '$\diamond$' is associative: $(\mathcal{I}\diamond\mathcal{J})\diamond K = \mathcal{I}\diamond(\mathcal{J}\diamond\mathcal{K})$*

Proof. We have

$$\begin{aligned}
(\mathcal{I}\diamond\mathcal{J})\diamond\mathcal{K} &= \operatorname{smd}(\operatorname{addc}(\operatorname{smd}(\operatorname{addc}(\mathcal{I})\star\operatorname{addc}(\mathcal{J})))\star\operatorname{addc}(\mathcal{K})) \\
&= \operatorname{smd}(\operatorname{smd}(\operatorname{addc}(\mathcal{I})\star\operatorname{addc}(\mathcal{J}))\star\operatorname{addc}(\mathcal{K})) && \text{by (2.2.9)} \\
&= \operatorname{smd}(\operatorname{addc}(\mathcal{I})\star\operatorname{addc}(\mathcal{J})\star\operatorname{addc}(\mathcal{K})) && \text{by (2.2.8)}
\end{aligned}$$

A similar computation shows that $\mathcal{I}\diamond(\mathcal{J}\diamond\mathcal{K}) = \operatorname{smd}(\operatorname{addc}(\mathcal{I})\star\operatorname{addc}(\mathcal{J})\star\operatorname{addc}(\mathcal{K}))$. □

Proposition 2.2.18 (🗗). *For all $\mathcal{I}$ we have $\langle\mathcal{I}\rangle_0 = \overline{0}$.*

Proof. By definition $\langle\mathcal{I}\rangle_0 = \operatorname{smd}\operatorname{addc}\{0\}$, so it suffices to show that $\operatorname{addc}(\{0\}) = \overline{0}$ and that $\operatorname{smd}(\overline{0}) = \overline{0}$. This is proved by structural induction and the fact that sums, shifts, and summands of zero objects are zero objects. □

Proposition 2.2.19 (🗗). $\langle\mathcal{I}\rangle_n \diamond \langle\mathcal{I}\rangle_m = \langle\mathcal{I}\rangle_{n+m}$.

Proof. We argue by induction on m. For the base case, we see that (2.2.9) applies to $\langle\mathcal{I}\rangle_n$, so that

$$\begin{aligned}
\langle\mathcal{I}\rangle_n \diamond \langle\mathcal{I}\rangle_0 &= \operatorname{smd}(\operatorname{addc}(\langle\mathcal{I}\rangle_n) \star \operatorname{addc}(\overline{0})) && \text{by (2.2.18)} \\
&= \operatorname{smd}(\langle\mathcal{I}\rangle_n \star \overline{0}) && \text{by (2.2.9)} \\
&= \operatorname{smd}(\langle\mathcal{I}\rangle_n) && \text{by (2.2.16)} \\
&= \langle\mathcal{I}\rangle_n
\end{aligned}$$

For the inductive step, if we assume that $\langle\mathcal{I}\rangle_n \diamond \langle\mathcal{I}\rangle_m = \langle\mathcal{I}\rangle_{n+m}$, then

$$\begin{aligned}
\langle\mathcal{I}\rangle_n \diamond \langle\mathcal{I}\rangle_{m+1} &= \langle\mathcal{I}\rangle_n \diamond (\langle\mathcal{I}\rangle_m \diamond \mathcal{I}) \\
&= (\langle\mathcal{I}\rangle_n \diamond \langle\mathcal{I}\rangle_m) \diamond \mathcal{I} && \text{by (2.2.17)} \\
&= \langle\mathcal{I}\rangle_{n+m} \diamond \mathcal{I} \\
&= \langle\mathcal{I}\rangle_{n+m+1}
\end{aligned}$$

□

Proposition 2.2.20 (↗). *The subcategory $\langle\mathcal{I}\rangle$ is thick.*

Proof. First we have $0 \in \overline{0} = \langle\mathcal{I}\rangle_0 \subseteq \langle\mathcal{I}\rangle$. If $a \in \langle\mathcal{I}\rangle$ and $i \in \mathbb{Z}$, then $a \in \langle\mathcal{I}\rangle_k$ for some k. Since $\langle\mathcal{I}\rangle_k$ is closed under shifts, $a[i] \in \langle\mathcal{I}\rangle_k \subseteq \langle\mathcal{I}\rangle$. If $a \in \langle\mathcal{I}\rangle$ and $a \cong b$, then $a \in \langle\mathcal{I}\rangle_k$ for some k. Since $\langle\mathcal{I}\rangle_k$ is closed under isomorphisms, $b \in \langle\mathcal{I}\rangle_k \subseteq \langle\mathcal{I}\rangle$.

If $a \to b \to c \to a[1]$ is a triangle with $a, b \in \langle\mathcal{I}\rangle$, then $a \in \langle\mathcal{I}\rangle_n$ and $b \in \langle\mathcal{I}\rangle_m$ for some n, m. By (2.2.14), we then have $c \in \langle\mathcal{I}\rangle_m \star \langle\mathcal{I}\rangle_n \subseteq \langle\mathcal{I}\rangle_m \diamond \langle\mathcal{I}\rangle_n = \langle\mathcal{I}\rangle_{m+n} \subseteq \langle\mathcal{I}\rangle$. Finally, if $a \boxplus b \in \langle\mathcal{I}\rangle$, then $a \boxplus b \in \langle\mathcal{I}\rangle_k$ for some k. Since $\langle\mathcal{I}\rangle_k$ is closed under summands, we conclude that $a \in \langle\mathcal{I}\rangle_k \subseteq \langle\mathcal{I}\rangle$. □

2.3 Ghost Lemma

Computing the dimension of a triangulated category is generally an intractable problem, and solutions are only known for a few special cases. The most basic problem we want to solve is, given $a, g \in \mathcal{T}$ and $k \in \mathbb{Z}$, how to determine if $a \in \langle g \rangle_k$. There is a satisfying theoretical answer to this question, using an idea that first appeared in [Kel65]. Here we present a complete formalization the so-called "Ghost Lemma", loosely following the exposition in [BFK12].

Definition 2.3.1 (↗). A morphism $f : a \to b$ is $\mathcal{I}$-*ghost* if for all $d \in \mathcal{I}$ and all $h : d \to a$, the composition $h \gg f$ vanishes. If $g \in \mathcal{T}$ is an object, we say "g-ghost" to mean $\langle g \rangle_1$-ghost.

Remark. In the formalization, we define a type of $\mathcal{I}$-ghosts with a fixed source a. This is defined as a structure containing an object $b \in \mathcal{T}$, a morphims $f : a \to b$, and a proof that f is $\mathcal{I}$-ghost.

Proposition 2.3.2 (↗). *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{T}$, and let $f : x \to y$ and $g : y \to z$ be $\mathcal{I}$-ghost and $\mathcal{J}$-ghost, respectively. Then $f \gg g$ is $(\mathcal{I} \star \mathcal{J})$-ghost.*

Proof. Let $c \in \mathcal{I} \star \mathcal{J}$ and let $\alpha : c \to x$. Choose a distinguished triangle $a \xrightarrow{j} c \xrightarrow{k} b \xrightarrow{l} a[1]$ with $a \in \mathcal{I}, b \in \mathcal{J}$, and consider the following diagram.

$$\begin{tikzcd} a \arrow[r, "j"] & c \arrow[r, "k"] \arrow[dl, "\alpha"'] & b \arrow[r, "l"] \arrow[dl, dashed, "\beta"'] & a[1] \\ x \arrow[r, "f"'] & y \arrow[r, "g"'] & z \end{tikzcd}$$

Since f is $\mathcal{I}$-ghost and $a \in \mathcal{I}$ we see that $j \gg \alpha \gg f = 0$. Since $\mathrm{Hom}_{\mathcal{T}}(-, y)$ is exact, i.e. takes triangles to long exact sequences, there is some $\beta : b \to y$ such that $k \gg \beta = \alpha \gg f$. Now $b \in \mathcal{J}$, and g is $\mathcal{J}$-ghost, so that $\beta \gg g = 0$. Thus, $\alpha \gg f \gg g = k \gg \beta \gg g = 0$. $\square$

Proposition 2.3.3 (↗). *If $\mathcal{I}$ is closed under shifts and $f : a \to b$ is $\mathcal{I}$-ghost, then f is* $\mathrm{addc}(\mathcal{I})$*-ghost.*

Proof. We will show a slightly more general statement, that for any $d \in \mathrm{addc}(\mathcal{I})$, $i \in \mathbb{Z}$, and any $h : d[i] \to a$, the composition $h \gg f$ vanishes. If $d = 0$, then $d[i] \cong 0$, and any morphism whose source is a zero object vanishes. If $d \in \mathcal{I}$, then $d[i] \in \mathcal{I}$, and $h \gg f$ vanishes because f is $\mathcal{I}$-ghost. If $d = d'[j]$ for some $d' \in \mathrm{addc}(\mathcal{I})$, then we may assume by induction that for all i

and all $h' : d'[i] \to a$, $h' \gg f = 0$. In particular, $h \gg f : d'[j][i] \cong d'[j+i] \to a \to b$ vanishes. If $d' \cong d$ with $d' \in \mathrm{addc}(\mathcal{I})$, then the composition $d \to a \to b$ factors as $d \to d' \to d \to a \to b$. By induction, $d' \to d \to a \to b$ vanishes, and so $d \to a \to b$ vanishes as well. If $d = d' \boxplus d''$ with $d', d'' \in \mathrm{addc}(\mathcal{I})$, then by induction, both compositions $d' \to d' \boxplus d'' \to a \to b$ and $d'' \to d' \boxplus d'' \to a \to b$ vanish. It follows that $d' \boxplus d'' \to a \to b$ vanishes. □

Remark. There is a subtlety in the formalization of this lemma. Structural induction on the statement "for all $d \in \mathrm{addc}(\mathcal{I})$, for all $h : d \to a, h \gg f = 0$", would not have worked. This is because the case where $d = d'[i]$ has for its inductive hypothesis "for all $h : d' \to a, h \gg f = 0$", which would not apply to $h : d'[i] \to a$.

Proposition 2.3.4 (↗). *If f is $\mathcal{I}$-ghost, then f is* $\mathrm{smd}(\mathcal{I})$*-ghost.*

Proof. Let $d \in \mathrm{smd}(\mathcal{I})$, and let $h : d \to a$. If $d \in \mathcal{I}$, then $h \gg f = 0$ because f is $\mathcal{I}$-ghost. If $d \boxplus d' \in \mathcal{I}$, then the inductive hypothesis is that any $h' : d \boxplus d' \to a$ satisfies $h' \gg f = 0$. We can then factor h as $d \to d \boxplus d' \to d \to a$, so that $h \gg f = 0$ as needed. The case where $d' \boxplus d \in \mathcal{I}$ is similar. □

Proposition 2.3.5 (↗). *If $\mathcal{I}, \mathcal{J}$ are closed under shifts, if $f : a \to b$ is $\mathcal{I}$-ghost, and if $g : b \to c$ is $\mathcal{J}$-ghost, then $f \gg g$ is $\mathcal{I} \diamond \mathcal{J}$-ghost.*

Proof. By (2.3.4), it suffices to show the $f \gg g$ is $\mathrm{addc}(\mathcal{I}) \star \mathrm{addc}(\mathcal{J})$-ghost. By (2.3.2), it suffices to show that f is $\mathrm{addc}(\mathcal{I})$-ghost and g is $\mathrm{addc}(\mathcal{J})$-ghost. Both follow from (2.3.3). □

Definition 2.3.6 (↗). An *$\mathcal{I}$-pseudo-adjoint* for $a \in \mathcal{T}$ is a morphism $f : x \to a$ with $x \in \mathcal{I}$ such that any $h : d \to a$ with $h \in \mathcal{I}$ factors through f.

Remark. In the formalization, we define a type of $\mathcal{I}$-pseudo-adjoints for a as a structure with four fields: the object x, the proof that $x \in \mathcal{I}$, the morphism h, and a proof of the factorization property for h.

Proposition 2.3.7 (↗). *If we complete a $\mathcal{I}$-pseudo-adjoint $f : x \to a$ to a triangle $x \to a \to b \to x[1]$, then the map $a \to b$ is an $\mathcal{I}$-ghost.*

Proof. Factor any $d \to a$ with $d \in \mathcal{I}$ through f, and the result follows since the composition of any two morphisms in a triangle vanishes.

$$\begin{tikzcd} & d \arrow[dl, dashed] \arrow[d] & & \\ x \arrow[r] & a \arrow[r] & b \arrow[r] & x[1] \end{tikzcd}$$

□

Remark. In the formalization, this is implemented as a function from the type of $\mathcal{I}$-pseudo-adjoints for a to the type of $\mathcal{I}$-ghosts with source a. This function is non-computable, as it requires the axiom of choice to pick a specific triangle completing the morphism $x \to a$.

Proposition 2.3.8 (↗)**.** *Suppose $\mathcal{I}$ is closed under shifts. If we are given a sequence of $\mathcal{I}$-ghosts $a_0 \to a_1 \to a_2 \to \ldots$, then for any n, the composition $a_0 \to a_n$ is $\langle \mathcal{I} \rangle_n$-ghost.*

Proof. We argue by induction on n. For the case $n = 0$, we need to show the identity $a_0 \to a_0$ is $\langle \mathcal{I} \rangle_0$-ghost. This holds because $\langle \mathcal{I} \rangle_0 = \overline{0}$ by (2.2.18), and because any map out of a zero-object is zero. For the inductive step, assume that $a_0 \to a_n$ is $\langle \mathcal{I} \rangle_n$-ghost. Since $a_n \to a_{n+1}$ is $\mathcal{I}$-ghost, (2.3.5) implies $a_0 \to a_{n+1}$ is ghost for $\langle \mathcal{I} \rangle_n \diamond \mathcal{I} = \langle \mathcal{I} \rangle_{n+1}$. □

Proposition 2.3.9 (↗)**.** *Suppose $\mathcal{I}$ is closed under shifts, and suppose we are given a sequence of $\mathcal{I}$-ghosts $a_0 \to a_1 \to a_2 \to \ldots$. If the sequence was constructed as in* (2.3.7)*, i.e. if each morphism is part of a triangle $c_i \to a_i \to a_{i+1} \to c_i[1]$ with $c_i \in \mathcal{I}$, then for any triangle completing the composition $a_0 \to a_n \to c \to a_0[1]$, we have $c \in \langle \mathcal{I} \rangle_n$.*

Proof. We argue by induction on n. For $n = 0$, any triangle $a_0 \to a_0 \to c \to a_0[1]$ completing the identity must satisfy $c \cong 0$, and so $c \in Z = \langle \mathcal{I} \rangle_0$. Now assuming the result holds for n, we apply the octahedral axiom to the composition $a_0 \to a_n \to a_{n+1}$ to obtain the following diagram.

$$\begin{tikzcd}
a_0 \arrow[d] \arrow[r] & a_{n+1} & & \\
a_n \arrow[r] \arrow[dd] & a_{n+1} \arrow[rr] \arrow[dr] & & c_n[1] \\
& & d \arrow[ur] & \\
c \arrow[urr] & & &
\end{tikzcd}$$

We have $c_n \in \mathcal{I}$, so that $c_n[1] \in \langle \mathcal{I} \rangle_1$, and the inductive hypothesis implies $c \in \langle \mathcal{I} \rangle_n$. Then $d \in \langle \mathcal{I} \rangle_n \diamond \langle \mathcal{I} \rangle_1 = \langle \mathcal{I} \rangle_{n+1}$. □

Proposition 2.3.10 (↗). *Suppose that for each $a \in \mathcal{T}$ we have an associated $\mathcal{I}$-pseudo-adjoint $c \to a$. For some $a_0 \in \mathcal{T}$, repeat the construction of* (2.3.7) *to obtain a sequence $a_0 \to a_1 \to a_2 \to \ldots$ of $\mathcal{I}$-ghosts. Then for all n, we have $a_0 \in \langle \mathcal{I} \rangle_n$ if and only if the composition $a_0 \to a_n$ vanishes.*

Proof. Suppose $a_0 \in \langle \mathcal{I} \rangle_n$. By (2.3.8), the composition $f_n : a_0 \to a_n$ is $\langle \mathcal{I} \rangle_n$-ghost, and so $f_n = \mathrm{id}_{a_0} \gg f_n = 0$. Conversely, if $a_0 \to a_n = 0$, then if we complete to a triangle $a_0 \to a_n \to c \to a_0[1]$, we see that $c \cong a_n \boxplus a_0[1]$. By (2.3.9), we have $c \in \langle \mathcal{I} \rangle_n$. Since $\langle \mathcal{I} \rangle_n$ is closed under summands and shifts, we then have $a_0 \in \langle \mathcal{I} \rangle_n$. □

The remaining piece of the story is showing that for a general class of categories, there exist pseudo-adjoints. Recall that $\mathcal{T}$ is *k-linear* for some field k if the Hom sets are k-vector spaces and pre- and post-composition by any morphism is k-linear. If $\mathcal{T}$ is k-linear, then we say $\mathcal{T}$ is *proper* if for any $a, b \in \mathcal{T}$, the vector space $\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(a, b[i])$ is finite-dimensional. In the literature, proper categories are sometimes instead called Hom-finite or Ext-finite.

Proposition 2.3.11. *Suppose $\mathcal{T}$ is k-linear and proper. Then for any $a, g \in \mathcal{T}$, there exists a $\langle g \rangle_1$-pseudo-adjoint for a.*

Proof. Let $x = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(g, a[i]) \otimes_k g[-i]$. By the properness condition, this is a finite direct sum of shifts of g, hence $x \in \langle g \rangle_1$. There is also a tautological evaluation map $\mathrm{ev} : x \to a$, and any map $h \to a$ with $h \in \langle g \rangle_1$ factors through ev by construction. □

2.4 Functors

We would like to be able to understand how thickenings of objects, generation times of generators, and dimension are affected by functors. The main result which I have formalized is that Rouquier dimension decreases under application of dense functors, which was a key ingredient in the computation [BFK12] of the Orlov spectrum of an elliptic curve. Throughout this section, let $\mathcal{T}, \mathcal{S}$ be two triangulated categories, and let $F : \mathcal{T} \to \mathcal{S}$ be a functor.

Definition 2.4.1 (↗). F is *dense* if for every $c \in \mathcal{S}$ there exists some $a \in \mathcal{T}$ with c a summand of of $F(a)$.

Remark. In the formalization, we define F to be dense if $\operatorname{smd}(\overline{F(\mathcal{T})}) = \mathcal{S}$, where $\overline{F(\mathcal{T})}$ is the essential image of F, i.e. the isomorphism closure of the image of F.

Proposition 2.4.2 (↗). *Suppose $F : \mathcal{T} \to \mathcal{S}$ is a dense functor, and suppose $\mathcal{I} \subseteq \mathcal{S}$ is closed under isomorphisms and summands. If $F(\mathcal{T}) \subseteq \mathcal{I}$, then $\mathcal{I} = \mathcal{S}$.*

Proof. It suffices to show that $\operatorname{smd}(\overline{F(\mathcal{T})}) \subseteq \mathcal{I}$. Let $c \in \operatorname{smd}(\overline{F(\mathcal{T})})$. If $c \in \overline{F(\mathcal{T})}$, then there is some $a \in \mathcal{T}$ with $c \cong F(a)$. Since $F(a) \in \mathcal{I}$ and $\mathcal{I}$ is closed under isomorphism, we see that $c \in \mathcal{I}$. If there is some c' with $c \boxplus c' \in \overline{F(\mathcal{T})}$, then we may assume by induction that $c \boxplus c' \in \mathcal{I}$. Since $\mathcal{I}$ is closed under summands, we conclude that $c \in \mathcal{I}$. The remaining case, where some c' satisfies $c \boxplus c' \in \overline{F(\mathcal{T})}$, is similar. □

Remark. Formalization blurs the lines between programming and doing mathematics, and under this identification the process of a programmer abstracting a repeated block of code into a subroutine corresponds with a the process of a mathematician abstracting a repeated logical argument into a lemma. This lemma, which has a somewhat odd statement from the mathematician's perspective, grew out of a natural need during the formalization process. It ends up being used in several places, and leads to a cleaner formalization of the main theorem of this section. The hypotheses that $\mathcal{I}$ be closed under isomorphisms and summands are very effectively handled by Lean4's typeclass inference system.

Proposition 2.4.3 (↗). *$F(\operatorname{addc}(\mathcal{I})) \subseteq \operatorname{addc}(F(\mathcal{I}))$.*

Proof. Let $c \in \operatorname{addc}\mathcal{I}$, and we must show $F(c) \in \operatorname{addc}(F(\mathcal{I}))$. If $c = 0$, then $F(0) \cong 0$ because F is additive. Since $0 \in \operatorname{addc}(F(\mathcal{I}))$, we see $F(0) \in \operatorname{addc}(\mathcal{I})$. If $c \in \mathcal{I}$, then $F(c) \in F(\mathcal{I})$, and so $F(c) \in \operatorname{addc} F(\mathcal{I})$. If $c = c'[i]$ for some $c' \in \operatorname{addc}(\mathcal{I})$, then by induction we may assume $F(c') \in \operatorname{addc}(F(\mathcal{I}))$. Then $F(c'[i]) \cong F(c')[i] \in \operatorname{addc}(F(\mathcal{I}))$. If $c \cong c'$ for some $c' \in \operatorname{addc}(\mathcal{I})$, then by induction we may assume $F(c') \in \operatorname{addc}(F(\mathcal{I}))$. Since $F(c) \cong F(c')$, we conclude that $F(c) \in \operatorname{addc} F(\mathcal{I})$. Finally, if $c = a \boxplus b$ for some $a, b \in \operatorname{addc}(\mathcal{I})$, then by induction we may assume $F(a), F(b) \in \operatorname{addc}(F(\mathcal{I}))$. Then $F(a) \boxplus F(b) \in \operatorname{addc}(F(\mathcal{I}))$. Since $F(a) \boxplus F(b) \cong F(a \boxplus b)$, we conclude $F(a \boxplus b) \in \operatorname{addc}(F(\mathcal{I}))$. $\square$

Proposition 2.4.4 (↗). *Suppose that $\mathcal{K} \subseteq \mathcal{S}$ is closed under isomorphisms and $F(\mathcal{I}) \subseteq \mathcal{K}$. Then $F(\operatorname{smd}(\mathcal{I})) \subseteq \operatorname{smd}(\mathcal{K})$.*

Proof. Let $c \in \operatorname{smd}(\mathcal{I})$. We must show that $F(c) \in \operatorname{smd}(\mathcal{K})$. If $c \in \mathcal{I}$, then we see that $F(c) \in F(\mathcal{I}) \subseteq \mathcal{K} \subseteq \operatorname{smd}(\mathcal{K})$. If there is some c' with $c \boxplus c' \in \operatorname{smd}(\mathcal{I})$, then by induction we may assume $F(c \boxplus c') \in \operatorname{smd}(\mathcal{K})$. Since F is additive, $F(c \boxplus c') \cong F(c) \boxplus F(c')$, and so (2.2.10) implies $F(c) \boxplus F(c') \in \operatorname{smd}(\mathcal{K})$, from which we see that $F(c) \in \operatorname{smd}(\mathcal{K})$. The final case, where some c' satisfies $c' \boxplus c \in \operatorname{smd}(\mathcal{I})$, is similar to the second case. $\square$

Remark. A lemma like $F(\operatorname{smd}(\mathcal{I})) \subseteq \operatorname{smd}(F(\mathcal{I}))$ would be preferable, but it is not true without additional hypotheses on $\mathcal{I}$. For example, if $a \boxplus b \in \mathcal{I}$, then $F(a) \in F(\operatorname{smd}(\mathcal{I}))$. Since there is no way to upgrade the isomorphism $F(a \boxplus b) \cong F(a) \boxplus F(b)$ to an equality, we cannot conclude that $F(a) \in \operatorname{smd}(F(\mathcal{I}))$.

Proposition 2.4.5 (↗). *$F(\mathcal{I} \star \mathcal{J}) \subseteq F(\mathcal{I}) \star F(\mathcal{J})$.*

Proof. Let $c \in \mathcal{I} \star \mathcal{J}$. We must show that $F(c) \in F(\mathcal{I}) \star F(\mathcal{J})$. Choose a triangle $a \to c \to b \to a[1]$ with $a \in \mathcal{I}$ and $b \in \mathcal{J}$, and then since F is triangulated, the triangle $F(a) \to F(c) \to F(b) \to F(a)[1]$ verifies $F(c) \in F(\mathcal{I}) \star F(\mathcal{J})$. $\square$

Proposition 2.4.6 (↗). *$F(\mathcal{I} \diamond \mathcal{J}) \subseteq F(\mathcal{I}) \diamond F(\mathcal{J})$*

Proof. By (2.4.4) it suffices to show that $F(\mathrm{addc}(\mathcal{I}) \star \mathrm{addc}(\mathcal{J})) \subseteq \mathrm{addc}(F(\mathcal{I})) \star \mathrm{addc}(F(\mathcal{J}))$. By transitivity and (2.4.5), it suffices to show that $F(\mathrm{addc}(\mathcal{I})) \star F(\mathrm{addc}(\mathcal{J})) \subseteq \mathrm{addc}(F(\mathcal{I})) \star \mathrm{addc}(F(\mathcal{J}))$. By (2.2.3), we need only show that $F(\mathrm{addc}(\mathcal{I})) \subseteq \mathrm{addc}(F(\mathcal{I}))$ and similarly for $\mathcal{J}$. This holds by (2.4.3). $\square$

Proposition 2.4.7 (). *For all n we have $F(\langle \mathcal{I} \rangle_n) \subseteq \langle F(\mathcal{I}) \rangle_n$.*

Proof. Argue by induction on n. For the case $n = 0$, we see that because F preserves zero objects, we have $F(\langle \mathcal{I} \rangle_0) = F(\overline{0}) \subseteq \overline{0} = \langle F(\mathcal{I}) \rangle_0$. For the inductive step, we have $F(\langle \mathcal{I} \rangle_{n+1}) = F(\langle \mathcal{I} \rangle_n \diamond \mathcal{I}) \subseteq F(\langle \mathcal{I} \rangle_n) \diamond F(\mathcal{I}) \subseteq \langle F(\mathcal{I}) \rangle_n \diamond F(\mathcal{I}) = \langle F(\mathcal{I}) \rangle_{n+1}$, where the first subset relation follows from 2.4.6, and the second follows from the inductive hypothesis and (2.2.4). $\square$

Proposition 2.4.8 (). *We have $F(\langle \mathcal{I} \rangle) \subseteq \langle F(\mathcal{I}) \rangle$.*

Proof. Let $c \in \langle \mathcal{I} \rangle$, and we must show that $F(c) \in \langle F(\mathcal{I}) \rangle$. We have $c \in \langle \mathcal{I} \rangle_k$ for some k, so that $F(c) \in \langle F(\mathcal{I}) \rangle_k$ by (2.4.7). Thus, $F(c) \in \langle F(\mathcal{I}) \rangle$. $\square$

Proposition 2.4.9 (). *If F is dense and $g \in \mathcal{T}$ is a classical generator, then $F(g)$ is a classical generator.*

Proof. To show that $\langle F(g) \rangle = \mathcal{S}$, it suffices by (2.4.2) to show that $F(\mathcal{T}) \subseteq \langle F(g) \rangle$. Since g is a generator, this amounts to showing $F(\langle g \rangle) \subseteq \langle F(g) \rangle$. This is exactly (2.4.8). $\square$

Proposition 2.4.10 (). *If F is dense and $g \in \mathcal{T}$ is a strong generator, then $F(g)$ is a strong generator.*

Proof. Choose an n for which $\langle g \rangle_n = \mathcal{T}$. We claim that $\langle F(g) \rangle_n = \mathcal{S}$. This follows from (2.4.2) and (2.4.7), since $F(\mathcal{T}) = F(\langle g \rangle_n) \subseteq \langle F(g) \rangle_n$. $\square$

Proposition 2.4.11 (). *If F is dense and g is a strong generator, then $U(F(g)) \leq U(g)$.*

Proof. A similar argument to (2.4.10) shows that $\langle F(g) \rangle_{U(g)+1} = \mathcal{S}$. $\square$

Proposition 2.4.12 (). *If F is dense, then* $\mathrm{Rdim}(\mathcal{S}) \leq \mathrm{Rdim}(\mathcal{T})$.

Proof. If $\mathrm{Rdim}(\mathcal{T}) = \infty$, then the statement is trivial. If $\mathrm{Rdim}(\mathcal{T}) = n \leq \infty$, there exists a strong generator g of $\mathcal{T}$ with $U(g) = n$. Then $U(F(g)) \leq n$, so that $\mathrm{Rdim}(\mathcal{S}) \leq n$. □

2.5 Varieties

In this section we apply the theory of dimension to the triangulated categories $D^b(X)$, where X is a smooth projective variety. While the algebraic geometry library in mathlib is not developed enough yet to formalize the results of this section, the gaps are rapidly being closed. Schemes [BHL+22] were recently formalized, as were derived categories [Rio25], and both have been merged into mathlib. The Auslander-Buchsbaum-Serre theorem characterizing regular local rings as those with finite homological dimension was formalized in [GH25]. As this theorem is the critical ingredient in Rouquier's [Rou08] lower bound of the dimension of categories $D^b(X)$, it may soon be possible to formalize more of these results.

Rather than formalization, we focus on an exposition of the main results and conjectures existing in the literature, and then we give original constructions meant to aid in the study of these conjectures. The biggest conjecture in this field is due to Orlov [Orl09]

Conjecture 2.5.1. *If X is a smooth projective variety, then* $\mathrm{Rdim}(D^b(X)) = \dim(X)$.

The first known case of this conjecture is implicitly due to Beilinson [Bei79], whose results immediately imply $\mathrm{Rdim}(D^b(\mathbb{P}^n)) = n$. This is due to a startlingly concrete construction. The identity functor on $D^b(\mathbb{P}^n)$ is a Fourier-Mukai functor with kernel $\mathcal{O}_\Delta$ the structure sheaf of the diagonal in $D^b(\mathbb{P}^n \times \mathbb{P}^n)$. The diagonal admits an explicit resolution in terms of box products, and computing the identify functor using this resolution reveals that $\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)$ is a generator with generation time n.

Theorem 2.5.2. *If X is a smooth projective variety, then* $\dim(X) \leq \mathrm{Rdim}(D^b(X)) \leq 2\dim(X)$.

This was first proved by Rouquier [Rou08] in the same paper where he defined Rouquier dimension. The lower bound follows from a homological dimension argument. If $\mathcal{G} \in D^b(X)$

is a complex, there is some smooth point $p \in X$ for which all cohomology sheaves of $\mathcal{G}$ are locally free at p. Now there is an essentially surjection restriction functor $D^b(X) \to D^b(\mathcal{O}_{X,p})$, so by (2.4.12) it suffices to show $\operatorname{Rdim} D^b(\mathcal{O}_{X,p}) \geq \dim X$. The upper bound comes from a similar resolution of the diagonal $\mathcal{O}_\Delta \in D^b(X \times X)$, and the $2d$ comes from the homological dimension of $\operatorname{Coh}(X \times X)$.

Theorem 2.5.3. *If C is a smooth projective curve, then* $\operatorname{Rdim}(D^b(C)) = 1$.

This result, proved by Orlov [Orl09], inspired him to make his famous conjecture. Orlov's construction is to take a line bundle $\mathcal{L}$ of degree at least $8g(C)$, and to show that $\mathcal{G} := \mathcal{L}^{-1} \boxplus \mathcal{O}_C \boxplus \mathcal{L} \boxplus \mathcal{L}^2$ is a strong generator with generation time 1. Since any object of $D^b(C)$ is isomorphic to the direct sum of its cohomology, it suffices to show any coherent sheaf is in $\langle \mathcal{G} \rangle_2$. Semi-stable locally free sheaves with sufficiently low degree admit a two-term left resolution by objects of $\langle \mathcal{G} \rangle_1$, and semi-stable locally free sheaves with sufficiently high degree admit a two-term right resolution by objects of $\langle \mathcal{G} \rangle_1$. The clever idea of Orlov was to use the Harder-Narisihman filtration (see, for example [HL10]) to split any coherent sheaf into a high-degree part and a low-degree part, resolve the two parts individually, and then stitch the resolutions together.

In the same paper, Orlov defines dimSpec and computes $\operatorname{dimSpec}(D^b(\mathbb{P}^1)) = \{1, 2\}$. He also asks if the dimension spectrum of $D^b(X)$ is always bounded for X a smooth projective variety.

Theorem 2.5.4. *If E is an elliptic curve, then* $\operatorname{dimSpec}(E) = \{1, 2, 3, 4\}$.

This theorem was proved in [BFK12]. The idea is that the auto-equivalence group of $D^b(E)$ is large enough that any generator $\mathcal{G}$ can be rewritten as direct sum of a torsion sheaf and a locally free sheaf, after which (2.4.12) shows that $U(\mathcal{G}) \leq U(\mathcal{O}_e \boxplus \mathcal{O}_C)$. The latter generation time is computed explicitly using (2.3.10).

Theorem 2.5.5. *Orlov's conjecture holds for normal toric varieties.*

This theorem was proved independently by [FH23] and [HHL24]. Both proofs use ideas from mirror symmetry, and the second proof uses a similar resolution-of-the-diagonal style argument. A new notion of dimension, namely the diagonal dimension, was introduced in [EL21]. Olander [Ola24] proved that the diagonal dimension of a curve of positive genus is two, effectively showing that Orlov's conjecture cannot be settled by a resolution-of-the-diagonal style argument.

Theorem 2.5.6. *If X is a Noetherian regular scheme of dimension d, and if $\mathcal{L}$ is an ample line bundle on X, then $\langle \{\mathcal{L}^i \mid i \geq 0\}\rangle_{d+1} = D^b(X)$.*

This was proved in [Ola23]. The idea is similar to that of (2.3.10), where to any object $\mathcal{E}$ one attaches a sequence $\mathcal{E} \to \mathcal{E}_1 \to \ldots \to \mathcal{E}_{d+1}$ with cones in $\langle \{\mathcal{L}^i \mid i \geq 0\}\rangle_1$ and such that the composition $\mathcal{E} \to \mathcal{E}_{d+1}$ vanishes. The technical backbone is a functorial filtration on the Hom sets, where the degree one piece is all maps which vanish on cohomology, the degree $d+1$ piece vanishes, and degree is additive over composition. The maps $\mathcal{E}_i \to \mathcal{E}_{i+1}$ are obtained from cones of maps $\bigoplus_{i\geq 0}(\mathcal{L}^{d_i})^{\oplus r_i}[i]$ which are constructed to vanish on cohomology. A corollary of this theorem is that $\mathrm{Rdim}(D^b(X)) = \dim(X)$ whenever X is Noetherian, regular, and quasi-affine, as in this case the structure sheaf $\mathcal{O}_X$ is ample.

This result strongly suggests the following attempt at resolving Orlov's conjecture. Let X be a smooth projective variety of dimension d, and let $\mathcal{L}$ be a line bundle $\mathcal{L}$ on X. For all $n \geq 0$, let $\mathcal{G}_n = \bigoplus_{i=0}^n \mathcal{L}^i$. The sequence $U(\mathcal{G}_n)$ is then bounded below by d and decreasing, so it has some limit $d_{\mathcal{L}} := \lim_{n\to\infty} U(\mathcal{G}_n)$. If $\mathcal{L}$ is ample, then we are very nearly in the situation of (2.5.6), which leads us to the following.

Conjecture 2.5.7. *For any ample line bundle $\mathcal{L}$, $d_{\mathcal{L}} = d$.*

While proving equality is out of reach, the following two propositions provide is heuristic evidence for equality, and they also give a feel for how to work with the generators $U(\mathcal{G}_n)$.

Proposition 2.5.8. *If $\mathcal{L}$ is basepoint-free, then for any $k \in \mathbb{Z}$ and $N \gg 0$, $\mathcal{L}^k \in \langle \mathcal{G}_N\rangle_{d+1}$.*

Proof. We see that $\mathcal{L}$ defines a map $\phi : X \to \mathbb{P}^n$ for some n. If $n > d$, we may project from a point to obtain a map $X \to \mathbb{P}^{n-1}$, and so we may assume $n \leq d$. From here we notice that

$$\mathcal{L}^k = \phi^*(\mathcal{O}_{\mathbb{P}^n}(k)) \in \phi^*(\langle \mathcal{O}_{\mathbb{P}^n} \oplus \ldots \mathcal{O}_{\mathbb{P}^n}(n)\rangle_{n+1}) \subseteq \langle \mathcal{O}_X \oplus \ldots \oplus \mathcal{L}^n\rangle_{n+1} \subseteq \langle \mathcal{G}_N\rangle_{k+1}$$

whenever $N \geq n$. ☐

Proposition 2.5.9. *If $\mathcal{L}$ is ample and $p \in X$ and $N \gg 0$, then $\mathcal{O}_p \in \langle \mathcal{G}_N\rangle_{d+1}$.*

Proof. The key is that for every $p \in X$ there exists some finite flat map $\phi : X \to \mathbb{P}^d$, defined by some power $\mathcal{L}^k$, which is not ramified at p. Then $\mathcal{O}_p$ is a summand of

$$\phi^*(\mathcal{O}_{\phi(p)}) \subseteq \phi^*(\langle \mathcal{O} \oplus \ldots \oplus \mathcal{O}(d)\rangle_{d+1}) \subseteq \langle \mathcal{O} \oplus \ldots \oplus \mathcal{L}^{kd}\rangle_{d+1} \subseteq \langle G_N\rangle_{d+1}$$

whenever $N \geq kd$.

It remains to show the existence of such a map. There is some very ample $\mathcal{L}^k$ which defines an embedding $X \to \mathbb{P}^N$. If $N > d$, then projection away from a point not in the tangent space of X at p gives a map $X \to \mathbb{P}^{N-1}$, unramified at p, still defined by $\mathcal{L}^k$. Iterate this process to obtain a map $X \to \mathbb{P}^d$, and the resulting map is flat by miracle flatness.

☐

We next turn our attention to a smooth projective curve C of genus $g > 1$. As the dimension spectra of $\mathbb{P}^1$ and and elliptic curves are known, this is the next interesting case. While $\mathrm{Rdim}(D^b(C)) = 1$ by (2.5.3), the behavior of other strong generators is worthy of study. We are particularly interested in whether $\mathrm{dimSpec}(D^b(C))$ is bounded. It turns out that the challenge in finding a generator with high generation time lies in avoiding generating any torsion sheaves, since soon as a torsion sheaf is generated, the rest of the category follows quickly. On a curve of high genus, there is a natural choice of generator which appears to avoid points, giving an example of a family of generators whose generation time is likely linear in the genus. My contribution is a more sophisticated construction which gives a sequence of generators over a fixed curve whose generation time conjecturally tends to infinity. This would show that the dimension spectrum of a curve of genus at least two is unbounded.

While the second example is more extreme than the first, it is important to study both, since the two examples share many formal properties which could be axiomatized and used to study generators with high generation time in other contexts.

Proposition 2.5.10. *There exists a constant λ, depending only on C, such that for any generator $\mathcal{G} \in D^b(C)$, if $\langle \mathcal{G} \rangle_k$ contains a torsion sheaf, then $U(\mathcal{G}) \leq \lambda k$.*

Proof. Choose a divisor D in C such that $\mathcal{O}_D$ is a summand of some torsion $T \in \langle \mathcal{G} \rangle_k$. Then $\mathcal{O}_D \in \langle \mathcal{G} \rangle_k$ as well. Since $\mathcal{G}$ is a generator, it cannot be torsion, so there is some locally free $\mathcal{F}$ with $\mathcal{F} \oplus \mathcal{O}_D \in \langle \mathcal{G} \rangle_k$.

The functor $- \otimes \mathcal{F} : D^b(C) \to D^b(C)$ is dense and sends $\mathcal{O}_C \oplus \mathcal{O}_D$ to $\mathcal{F} \oplus \mathcal{O}_D$. Then $U(\mathcal{F} \oplus \mathcal{O}_D) \leq U(\mathcal{O}_C \oplus \mathcal{O}_D)$ by (2.4.11). The short exact sequences

$$0 \to \mathcal{O}_C((n-1)D) \to \mathcal{O}_C(nD) \to \mathcal{O}_D \to 0$$

show that $\mathcal{E} := \mathcal{O}_C(-8gD) \oplus \mathcal{O}_C \oplus \mathcal{O}_C(8gD) \oplus \mathcal{O}_C(16gD) \in \langle \mathcal{O}_C \oplus \mathcal{O}_D \rangle_{16g+1}$. The proof of (2.5.3) implies that $D^b(C) = \langle \mathcal{E} \rangle_2 \subseteq \langle \mathcal{O}_C \oplus \mathcal{O}_D \rangle_{32g+2} \subseteq \langle \mathcal{G} \rangle_{(32g+2)k}$ so that $\lambda := 32g+2$ works. $\square$

A key player in the above lemma was the generator $\mathcal{O}_C \oplus \mathcal{O}_D$. We would like to better understand the generation time of such a generator. The simplest case to analyze is where $D = p$ is a single reduced Weierstrass point, e.g. a point where $h^0(C, \mathcal{O}_C(kp)) = 1$ for $k \leq g$. In this case, intuition would suggests that it is difficult to generate more sheaves. Objects are generated by taking cones of morphisms, so if there aren't many morphisms, as is the case here, there are not many ways to generate more objects.

To make some precise statements, we notice that there is a sequence of maps

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(p) \to \mathcal{O}_C(2p) \to \ldots \to \mathcal{O}_C(gp)$$

where the cone of the first map $0 \to \mathcal{O}_C$ is $\mathcal{O}_C$ and the cone of every other map is $\mathcal{O}_p$. We therefore see that $\mathcal{O}_C(gp) \in \langle \mathcal{O}_C \oplus \mathcal{O}_p \rangle_{g+1}$. Moreover, this filtration of $\mathcal{O}_C(gp)$ seems to be minimal in some sense.

Proposition 2.5.11. *The following properties hold for the above sequence.*

1. *Each composition $\mathcal{O}_C \to \mathcal{O}_C(kp)$ is injective and a generator for the one-dimensional k vector space* $\mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_C(kp))$.

2. *In each triangle $\mathcal{O}_C(kp) \to \mathcal{O}_C((k+1)p) \to \mathcal{O}_p \to \mathcal{O}_C(kp)[1]$, the latter two maps are generators for the one-dimensional k vector spaces* $\mathrm{Hom}(\mathcal{O}_C((k+1)p), \mathcal{O}_p)$ *and* $\mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_C(kp)[1])$.

3. *There are no non-zero maps* $\mathrm{Hom}(\mathcal{O}_C((k+1)p), \mathcal{O}_p[i])$ *for $i \neq 0$, and there are no non-zero maps* $\mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_C(kp)[i])$ *for $i \neq 1$.*

4. *In particular, each map $\mathcal{O}_C(kp) \to \mathcal{O}_C((k+1)p)$ is $\mathcal{O}_p$-ghost, and each $\mathcal{O}_p \to \mathcal{O}_C(kp)$ is $\mathcal{O}_C((k+1)p)$-ghost.*

Proof.

1. This is immediate from the hypothesis $h^0(C, \mathcal{O}_C(kp)) = 1$ for $k \leq g$.

2. First notice that neither of the two maps is zero, for otherwise the triangle would be a direct sum triangle. It therefore suffices to show both vector spaces are one-dimensional. Since $\mathcal{O}_p$ is fixed by tensoring by any line bundle, we have $\dim \mathrm{Hom}(\mathcal{O}_C(kp), \mathcal{O}_p) = h^0(\mathcal{O}_p) = 1$. Similarly, Serre duality implies

$$\dim \mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_C(kp)[1]) = \dim \mathrm{Hom}(\mathcal{O}_C(kp), \omega_C \otimes \mathcal{O}_p) = h^0(\mathcal{O}_p) = 1.$$

3. For all sheaves $\mathcal{F}, \mathcal{G}$ on C and all $i \notin \{0, 1\}$ we have $\mathrm{Hom}(\mathcal{F}, \mathcal{G}[i])$, so we need only show the case $i = 1$ for the first case and $i = 0$ for the second. We have $\dim \mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_C(kp)) = 0$ because there are no maps from a torsion sheaf to a torsion-free sheaf. Similarly,

$$\dim \mathrm{Hom}(\mathcal{O}_C((k+1)p, \mathcal{O}_p[1])) = \dim \mathrm{Hom}(\mathcal{O}_p, \omega_C \otimes \mathcal{O}_C((k+1)p)) = 0.$$

4. This is similar to the proof of (2.3.11). Since $\mathcal{O}_p \xrightarrow{\alpha} \mathcal{O}_C(kp)[1]$ generates the graded vector space $\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}(\mathcal{O}_p, \mathcal{O}_C(kp)[i])$, it is an $\mathcal{O}_p$-pseudo-adjoint for $\mathcal{O}_C(kp)$, so its cone $\mathcal{O}_C(kp) \to \mathcal{O}_C((k+1)p)$ is $\mathcal{O}_p$-ghost. In the same way we see that $\mathcal{O}_p \to \mathcal{O}_C(kp)$ is $\mathcal{O}_C((k+1)p)$-ghost.

□

Conjecture 2.5.12. $\mathcal{O}_C(gp) \notin \langle \mathcal{O}_C \oplus \mathcal{O}_p \rangle_g$

If this conjecture is true, then any curve of genus g satisfies $\max \operatorname{dimSpec}(C) \geq g$. This implies that there is no uniform bound on generation time of an arbitrary generator as a function of the Rouquier dimension. We next turn to a stronger statement: we seek to construct generators with arbitrarily high generation time. By (2.5.10), we need the generators to be "far away" from any torsion sheaves. We want a construction with similar nice properties to (2.5.11), with vanishing of several Hom groups. My original construction is as follows.

Define $\mathcal{F}_0 = \mathcal{O}_p$, and $\mathcal{F}_1 = \mathcal{O}_C(p)$. Now suppose by induction we have constructed a sheaf $\mathcal{F}_n$ with degree 1 and rank n for $n > 1$. Then $\dim \operatorname{Ext}^1(\mathcal{F}_n, \mathcal{O}_C) = h^1(C, \mathcal{F}_n^*)$ and by Riemann-Roch,

$$\chi(C, \mathcal{F}_n^*) = n(1-g) - 1 < 0$$

so that $h^1(C, \mathcal{F}_n^*) > 0$. We may then define $\mathcal{F}_{n+1}$ by choosing a non-split extension

$$0 \to \mathcal{O}_C \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to 0.$$

To begin a more careful analysis of this construction, we recall some classical definitions and properties of (semi)-stable vector bundles on curves.

Definition 2.5.13. The *slope* of a vector bundle $\mathcal{F}$ on a curve C is $\mu(\mathcal{F}) := \deg(\mathcal{F})/\operatorname{rk}(\mathcal{F})$. We say $\mathcal{F}$ is *stable* (resp, *semi-stable*) if for any non-zero proper subsheaf $\mathcal{F} \subseteq \mathcal{F}$ we have $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$ (resp, $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$).

Semi-stability is a useful notion for us here because it creates restrictions on morphisms. Stability is equivalent to all non-trivial quotient sheaves having greater slope. If $\mu(\mathcal{F}) > \mu(\mathcal{G})$ and $\mathcal{F}, \mathcal{G}$ are semi-stable, then $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$, because the image $\mathcal{H}$ of any non-zero morphism would have to satisfy $\mu(\mathcal{F}) \leq \mu(\mathcal{H}) \leq \mu(\mathcal{G})$. A similar argument shows that morphism between two stable sheaves of the same slope is either zero or an isomorphism.

Proposition 2.5.14. *A sheaf $\mathcal{F}$ of degree 1 is stable if and only if every proper subsheaf $\mathcal{G}$ has degree ≤ 0.*

Proof. Since $\mu(\mathcal{F}) = 1/\operatorname{rk}(\mathcal{F})$, we see that $\mu(\mathcal{G}) < \mu(\mathcal{F})$ is equivalent to $\deg(\mathcal{G}) < \operatorname{rk}(\mathcal{G})/\operatorname{rk}(\mathcal{F})$. The result now holds since $0 < \operatorname{rk}(\mathcal{G})/\operatorname{rk}(\mathcal{F}) \leq 1$. $\square$

Proposition 2.5.15. *Let $0 \to \mathcal{O}_C \to \mathcal{F} \to \overline{\mathcal{F}} \to 0$ be a non-split short exact sequence of sheaves, and suppose $\overline{\mathcal{F}}$ is stable of degree 1. Then $\mathcal{F}$ is stable.*

Proof. Let $\mathcal{G} \subseteq \mathcal{F}$ be a proper subsheaf. We must show $\deg(\mathcal{G}) \leq 0$. Let $\overline{\mathcal{G}}$ be the image of $\mathcal{G}$ in $\overline{\mathcal{F}}$, and let $\mathcal{G}'$ be the kernel. If $\overline{\mathcal{G}} = \overline{\mathcal{F}}$, then $\mathcal{G}'$ must be a proper subsheaf of $\mathcal{O}_C$, for otherwise we would have $\mathcal{G} = \mathcal{F}$. Then $\deg(\mathcal{G}') < 0$, but $\deg(\overline{\mathcal{G}}) = 1$, and so $\deg(\mathcal{G}) \leq 0$. Otherwise, $\overline{\mathcal{G}}$ is a proper subsheaf of $\overline{\mathcal{F}}$. Since $\overline{\mathcal{F}}$ is stable, we see that $\deg(\overline{\mathcal{G}}) \leq 0$. Since $\deg(\mathcal{G}') \leq 0$ as well, we see $\deg(\mathcal{G}) \leq 0$. $\square$

Proposition 2.5.16. *We can choose the $\mathcal{F}_n$ so that $h^0(\mathcal{F}_n) = 1$ for all n.*

Proof. We use induction, noticing that $\mathcal{F}_1 = \mathcal{O}_C(p)$ already satisfies the statement. So assume $h^0(\mathcal{F}_n) = 1$ for some $n \geq 1$, and let s be a non-zero section. Applying $\operatorname{Hom}(-, \mathcal{O}_C)$ to the short exact sequence $0 \to \mathcal{O}_C \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to 0$, we obtain the exact sequence

$$\operatorname{Ext}^1(\mathcal{F}_n, \mathcal{O}_C) \xrightarrow{s^*} \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \to \operatorname{Ext}^2(\mathcal{F}_{n-1}, \mathcal{O}_C)$$

Since $\operatorname{Ext}^2(\mathcal{F}_{n-1}, \mathcal{O}_C) = 0$, we see that $s^*\colon \operatorname{Ext}^1(\mathcal{F}_n, \mathcal{O}_C) \to \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{O}_C)$ is surjective, so we may choose $\mathcal{F}_{n+1}$ so that the class $[\mathcal{F}_{n+1}] \in \operatorname{Ext}^1(\mathcal{F}_n, \mathcal{O}_C)$ doesn't map to 0.

Now suppose $h^0(\mathcal{F}_{n+1}) > 1$. From the exact sequence

$$0 \to H^0(\mathcal{O}_C) \to H^0(\mathcal{F}_{n+1}) \to H^0(\mathcal{F}_n) \to H^1(\mathcal{O}_C)$$

and the fact that $h^0(\mathcal{O}_C) = h^0(\mathcal{F}_n) = 1$, we see that $h^0(\mathcal{F}_{n+1}) = 2$, and we can choose a basis s_1, s_2 of $H^0(\mathcal{F}_{n+1})$ for which the following diagram commutes with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r, "i_1"] \arrow[d, equal] & \mathcal{O}_C \oplus \mathcal{O}_C \arrow[r, "p_2"] \arrow[d, "\begin{pmatrix} s_1 & s_2 \end{pmatrix}"] & \mathcal{O}_C \arrow[r] \arrow[d, "s"] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r, "s_1"] & \mathcal{F}_{n+1} \arrow[r] & \mathcal{F}_n \arrow[r] & 0
\end{tikzcd}$$

But then the right commutative square is cartesian, and this diagram demonstrates that $s^*([\mathcal{F}_{n+1}]) = 0$, which is a contradiction. $\square$

If we define $\mathcal{G}_n = \mathcal{O}_C \oplus \mathcal{F}_n$, then each $\mathcal{G}_n$ is a generator of $D^b(C)$. We also have the following sequence

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \ldots \to \mathcal{F}_2 \to \mathcal{O}_C(p) \to \mathcal{O}_p$$

where the cone of $0 \to \mathcal{F}_n$ is $\mathcal{F}_n$ and the cone of each map $\mathcal{F}_{i+1} \to \mathcal{F}_i$ is $\mathcal{O}_C[1]$. This sequence satisfies many similar properties as the sequence from (2.5.11), and so we expect the following.

Conjecture 2.5.17. *$\mathcal{O}_p \notin \langle \mathcal{F}_n \oplus \mathcal{O}_C \rangle_n$.*

This conjecture, if true, would imply that dimSpec(C) is unbounded for whenever $g > 1$. This would provide the first example of an unbounded dimension spectrum, and the existence of generators with arbitrarily high generation time would be an extremely interesting property.

Chapter 3
SUPPORT

In order to treat a triangulated category as a geometric object in its own right, it is desirable to attach a topological space to it. For triangulated categories of the form $D^b(X)$ where X is a scheme, a natural choice of associated topological space would be the space X. The category $D^b(X)$ then interfaces with the space through the notion of support.

The support of a bounded complex of coherent $\mathcal{O}_X$-modules (that is, the union of the supports of the cohomology sheaves) is a closed subset of X, and support behaves well with respect to several categorical operations: taking cones, direct sums, and tensor products. The insight of Balmer [Bal05] was that these properties can be packaged into the objects of a category, the category of support data, and then X can be recovered as the final object in that category.

Many of the triangulated categories which arise in algebraic geometry do not have a tensor structure, and effort has been made in the past two decades to exploit similar ideas in the absence of a tensor product. In the first section, we produce a complete formalization of a result implicitly due to [GS23] and stated explicitly by [BO24]. In the second section we give an original new characterization of classical generators in terms of support data. We also show that support cannot detect generation time, by exhibiting an example of two triangulated categories with different dimensions but equivalent categories of support data.

3.1 The Lattice of Thick Subcategories

Definition 3.1.1 (↗). Let $\mathrm{Th}(\mathcal{T})$ be the collection of all thick subcategories of $\mathcal{T}$.

Definition 3.1.2 (↗). If $\mathcal{I} \subseteq \mathcal{T}$, let

$$D(\mathcal{I}) := \{P \in \mathrm{Th}(\mathcal{T}) \mid P \cap \mathcal{I} \neq \emptyset\}$$

Proposition 3.1.3. *The sets $D(\mathcal{I})$ form a topology on* $\mathrm{Th}(\mathcal{T})$.

Proof. Since 0 is in any thick subcategory, we have $D(\mathcal{T}) = \mathrm{Th}(T)$. If $\mathcal{I}, \mathcal{J} \subseteq \mathcal{T}$, we claim that $D(\mathcal{I}) \cap D(\mathcal{J}) = D(\{a \boxplus b \mid a \in \mathcal{I}, b \in \mathcal{J}\})$. This holds because for any thick subcategory P, $a, b \in P$ if and only if $a \boxplus b \in P$. If $\{\mathcal{I}_\alpha\}$ is a collection of subsets of $\mathcal{T}$, then we claim that $\bigcup(D(\mathcal{I}_\alpha)) = D(\bigcup \mathcal{I}_\alpha)$. Indeed, if $P \in \bigcup(D(\mathcal{I}_\alpha))$, then there is some α with $P \in D(\mathcal{I}_\alpha)$, i.e. there is some α and a with $a \in P \cap \mathcal{I}_\alpha$. Then $a \in P \cap \bigcup \mathcal{I}_\alpha$, so that $P \in D(\bigcup \mathcal{I}_\alpha)$. The other inclusion is similar. $\square$

Definition 3.1.4. A *support datum* for $\mathcal{T}$ is a pair (X, σ), where X is a topological space and σ associates to any $a \in \mathcal{T}$ a closed set $\sigma(a) \subseteq X$, subject to the following conditions.

1. $\sigma(0) = \emptyset$

2. For all $a, b \in \mathcal{T}$, $\sigma(a \boxplus b) = \sigma(a) \cup \sigma(b)$.

3. For all $a \in \mathcal{T}$, $\sigma(a[1]) = \sigma(a)$

4. For any triangle $a \to b \to c \to a[1]$, $\sigma(c) \subseteq \sigma(a) \cup \sigma(b)$.

Remark. If $\mathcal{I} \subseteq \mathcal{T}$, we will sometimes abuse notation and write $\sigma(\mathcal{I})$ to mean $\bigcup_{a \in \mathcal{I}} \sigma(a)$.

Proposition 3.1.5. *If we define $\sigma(a) = \{P \in \mathrm{Th}(\mathcal{T}) \mid a \notin P\}$, then $(\mathrm{Th}(\mathcal{T}), \sigma)$ is a support datum for $\mathcal{T}$.*

Proof. Since 0 is in any thick subcategory, we see that $\sigma(0) = \emptyset$. Since $a, b \in P$ if and only if $a \boxplus b \in P$, we see that $\sigma(a) \cup \sigma(b) = \sigma(a \boxplus b) \in P$. Since thick subcategories are closed under shifts, we see that $a \in P$ if and only if $a[1] \in P$, so that $\sigma(a) = \sigma(a[1])$. Finally, if $a \to b \to c \to a[1]$ is a triangle with $a, b \in P$, then $c \in P$, which shows $\sigma(c) \subseteq \sigma(a) \cup \sigma(b)$. $\square$

Proposition 3.1.6. *If $a \to b \to c \to a[1]$ is a triangle, then $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$ and also $\sigma(b) \subseteq \sigma(a) \cup \sigma(c)$.*

Proof. Considering the rotated triangles $b \to c \to a[1] \to b[1]$ and $c \to a[1] \to b[1] \to c[1]$ gives $\sigma(a[1]) \subseteq \sigma(b) \cup \sigma(c)$ and $\sigma(b[1]) \subseteq \sigma(a[1]) \cup \sigma(c)$. The result now follows since $\sigma(a[1]) = \sigma(a)$ and $\sigma(b[1]) = \sigma(b)$. $\square$

Proposition 3.1.7 (↗). *If $a \cong b$, then $\sigma(a) = \sigma(b)$.*

Proof. The triangle $a \to b \to 0 \to a[1]$ is distinguished, as it is isomorphic to the triangle $a \to a \to 0 \to a[1]$. Then (3.1.6) implies that $\sigma(a) \subseteq \sigma(b) \cup \sigma(0)$ and $\sigma(b) \subseteq \sigma(a) \cup \sigma(0)$. Since $\sigma(0) = \emptyset$, this implies $\sigma(a) \subseteq \sigma(b)$ and $\sigma(b) \subseteq \sigma(a)$. $\square$

Proposition 3.1.8 (↗). *For any $i \in \mathbb{Z}$ we have $\sigma(a[i]) = \sigma(a)$.*

Proof. The base case $i = 0$, we apply (3.1.7) to the natural isomorphism $a[0] \cong a$. For the inductive step, we must show that $\sigma(a[i]) = \sigma(a[i+1])$ and $\sigma(a[i]) = \sigma(a[i-1])$. Both equalities hold in light of (3.1.7) due to the natural isomorphisms $a[i][1] \cong a[i+1]$ and $a[i-1][1] \cong a[i]$, as then $\sigma(a[i]) = \sigma(a[i][1]) = \sigma(a[i+1])$ and $\sigma(a[i-1]) = \sigma(a[i-1][1]) = \sigma(a[i])$. $\square$

Remark. The definition of a shift functor in mathlib is rather complicated: it is a monoidal functor from discrete category of an additive monoid (in our case, $\mathbb{Z}$) to the category of endofunctors (in our case, on $\mathcal{T}$). The monoidal structure on the discrete category is given by the monoid operation, while the monoidal structure on the category of endofunctors is given by composition. The takeaway for us is that we do not have equality $a[0] = a$ or $a[i][j] = a[i+j]$, but rather natural isomorphisms satisfying certain compatibilities.

Definition 3.1.9 (↗). A morphism of support data $(X, \sigma) \to (Y, \tau)$ is a continuous function $f : X \to Y$ with the property that for any $a \in \mathcal{T}$ we have $\sigma(a) = f^{-1}(\tau(a))$.

With this definition, support data for $\mathcal{T}$ becomes a category.

Proposition 3.1.10 (↗). *$(\mathrm{Th}(\mathcal{T}), \sigma)$ is final in the category of support data for $\mathcal{T}$.*

Proof. Let (X, τ) be a support datum for $\mathcal{T}$, and we must show there is a unique morphism to $(\mathrm{Th}(\mathcal{T}), \sigma)$. For existence, if $x \in X$, define $f(x) := \{a \in \mathcal{T} \mid x \notin \tau(a)\}$. We must verify that $f(x)$ is a thick subcategory. As $x \notin \tau(0) = \emptyset$, we have $0 \in f(x)$. Since $\tau(a) = \tau(a[i])$ for all i by (3.1.8), we see that $f(x)$ is closed under shifts. Since $\tau(a) = \tau(b)$ whenever $a \cong b$ by (3.1.7), we see that $f(x)$ is closed under isomorphisms. If $a \to b \to c \to a[1]$ is a triangle and $c \notin f(x)$, then $x \in \tau(c) = \tau(a) \cup \tau(b)$, so either $a \notin f(x)$ or $b \notin f(x)$, and so $f(x)$ is closed under taking cones. If $a \boxplus b \in f(x)$, then $x \notin \tau(a \boxplus b) = \tau(a) \cup \tau(b)$, so $x \notin \tau(a)$ and $x \notin \tau(b)$, and so $f(x)$ is closed under taking summands.

To see that $f(x)$ is continuous, let $Z \subseteq \mathrm{Th}(X)$ be a closed set. Then we have $Z^c = D(\mathcal{I})$ for some $\mathcal{I} \subseteq \mathcal{T}$. We claim that $f^{-1}(Z) = \bigcap_{a \in \mathcal{I}} \tau(a)$. If $f(x) = \{a \in \mathcal{T} \mid x \notin \tau(a)\} \in Z$, then by definition of $D(\mathcal{I})$ we have $f(x) \cap \mathcal{I} = \emptyset$, so that any $a \in \mathcal{I}$ satisfies $x \in \tau(a)$, so that $x \in \bigcap_{a \in \mathcal{I}} \tau(a)$. All of the implications are reversible, and so the claim is shown.

To show that f is unique, suppose that $g : X \to \mathrm{Th}(X)$ is another support datum. If $a \in g(x)$, then $g(x) \notin \sigma(a)$, and so $x \notin g^{-1}(\sigma(a)) = \tau(a)$. As these implications are reversible, we see that $g(x) = \{a \in \mathcal{T} \mid x \notin \tau(a)\} = f(x)$. $\square$

3.2 A Characterization of Generators

The main result of this section is an original theorem giving a characterization of classical generators of a triangulated category $\mathcal{T}$ in terms of the category of support data for $\mathcal{T}$. Let (X, σ) be a support datum, and let $\mathcal{I}, \mathcal{J} \subseteq X$.

Proposition 3.2.1 (↗)**.** *If $\mathcal{I} \subseteq \mathcal{J}$, then $\sigma(\mathcal{I}) \subseteq \sigma(\mathcal{J})$.*

Proof. If $P \in \sigma(\mathcal{I})$, then $P \in \sigma(a)$ for some $a \in \mathcal{I}$. Then $a \in \mathcal{J}$, and so $P \in \sigma(\mathcal{J})$. $\square$

Proposition 3.2.2 (↗)**.** *If $\overline{0}$ is the collection of all zero objects in $\mathcal{T}$, then $\sigma(\overline{0}) = \emptyset$.*

Proof. We have $\sigma(0) = \emptyset$ by definition and σ is invariant under isomorphisms (3.1.7). $\square$

Proposition 3.2.3 (↗)**.** *We have $\sigma(\mathrm{addc}(\mathcal{I})) = \sigma(\mathcal{I})$.*

Proof. Since $\mathcal{I} \subseteq \operatorname{addc}(\mathcal{I})$, (3.2.1) implies $\sigma(\mathcal{I}) \subseteq \sigma(\operatorname{addc}(\mathcal{I}))$, and so we need only show the reverse inclusion. Suppose $x \in \sigma(\operatorname{addc}(\mathcal{I}))$. Then $x \in \sigma(a)$ for some $a \in \operatorname{addc}(\mathcal{I})$. We cannot have $a = 0$, because $\sigma(a) = \emptyset$. If $a \in \mathcal{I}$, then $x \in \sigma(\mathcal{I})$ as needed. If $a = a'[i]$ for some $i \in \mathbb{Z}$ and $a' \in \operatorname{addc}(\mathcal{I})$, then by induction it suffices to show that $x \in \sigma(a')$, which holds because $\sigma(a) = \sigma(a')$ by (3.1.8). If $a \cong b$ for some $b \in \operatorname{addc}(\mathcal{I})$, then by induction it suffices to show that $x \in \sigma(b)$, which holds because $\sigma(a) = \sigma(b)$ by (3.1.7). If $a = a' \boxplus a''$ for some $a', a'' \in \operatorname{addc}(\mathcal{I})$, then by induction it suffices to show that either $x \in \sigma(a')$ or that $x \in \sigma(a''')$, which holds because $\sigma(a) = \sigma(a') \cup \sigma(a'')$. $\square$

Proposition 3.2.4 (↗). *We have* $\sigma(\operatorname{smd}(\mathcal{I})) = \sigma(\mathcal{I})$.

Proof. Since $\mathcal{I} \subseteq \operatorname{smd}(\mathcal{I})$, (3.2.1) implies $\sigma(\mathcal{I}) \subseteq \sigma(\operatorname{smd}(\mathcal{I}))$, so we only need to show the reverse inclusion. Suppose $x \in \sigma(\operatorname{smd}(\mathcal{I}))$. Then $x \in \sigma(a)$ for some $a \in \operatorname{smd}(\mathcal{I})$. If $a \in \mathcal{I}$, then $x \in \sigma(\mathcal{I})$ as needed. Otherwise there is some $a' \in \mathcal{T}$ with $a \boxplus a' \in \operatorname{smd}(\mathcal{I})$ or $a' \boxplus a \in \operatorname{smd}(\mathcal{I})$. In the first case, it suffices by induction to prove that $x \in \sigma(a \boxplus a')$. This holds because $\sigma(a \boxplus a') = \sigma(a) \boxplus \sigma(a')$ and $x \in \sigma(a)$. The second case follows similarly. $\square$

Proposition 3.2.5 (↗). *We have* $\sigma(\mathcal{I} \star \mathcal{J}) \subseteq \sigma(\mathcal{I}) \cup \sigma(\mathcal{J})$.

Proof. Suppose $x \in \sigma(\mathcal{I} \star \mathcal{J})$. Then there is some $c \in \mathcal{I} \star \mathcal{J}$ with $x \in \sigma(c)$. Choose a triangle $a \to c \to b \to a[1]$ with $a \in \mathcal{I}, b \in \mathcal{J}$. By (3.1.6), $\sigma(c) \subseteq \sigma(a) \cup \sigma(b)$. Then $x \in \sigma(a) \subseteq \sigma(\mathcal{I})$ or $x \in \sigma(b) \subseteq \sigma(\mathcal{J})$. In either case, $x \in \sigma(\mathcal{I}) \cup \sigma(\mathcal{J})$, as needed. $\square$

Remark. In general, we don't have equality. For example, if $\sigma(\mathcal{I}) \neq \emptyset$, then $\sigma(\mathcal{I} \star \emptyset) = \sigma(\emptyset) = \emptyset$, while $\sigma(\mathcal{I}) \cup \sigma(\emptyset) \neq \emptyset$.

Proposition 3.2.6 (↗). *We have* $\sigma(\mathcal{I} \diamond \mathcal{J}) = \sigma(\mathcal{I}) \cup \sigma(\mathcal{J})$.

Proof. For one inclusion we have

$$\begin{aligned}
\sigma(\mathcal{I} \diamond \mathcal{J}) &= \sigma(\mathrm{smd}(\mathrm{addc}(\mathcal{I}) \star \mathrm{addc}(\mathcal{J}))) \\
&= \sigma(\mathrm{addc}(\mathcal{I}) \star \mathrm{addc}(\mathcal{J})) && \text{by (3.2.4)} \\
&\subseteq \sigma(\mathrm{addc}(\mathcal{I})) \cup \sigma(\mathrm{addc}(\mathcal{J})) && \text{by (3.2.5)} \\
&= \sigma(\mathcal{I}) \cup \sigma(\mathcal{J}) && \text{by (3.2.3)}
\end{aligned}$$

For the reverse inclusion, it suffices to show that $\sigma(\mathcal{I}), \sigma(\mathcal{J}) \subseteq \sigma(\mathcal{I} \diamond \mathcal{J})$. By (3.2.1), it suffices to show $\mathcal{I}, \mathcal{J} \subseteq \mathcal{I} \diamond \mathcal{J}$, and this is exactly (2.2.5). □

Proposition 3.2.7 (**↗**)**.** *For all $n > 0$, we have $\sigma(\langle \mathcal{I} \rangle_n) = \sigma(\mathcal{I})$.*

Proof. We argue by induction on n. For the base case $n = 1$, we have $\langle \mathcal{I} \rangle_1 = \mathrm{smd}(\mathrm{addc}(\mathcal{I}))$, so that $\sigma(\langle \mathcal{I} \rangle_1) = \sigma(\mathcal{I})$ by (3.2.3) and (3.2.4). For the inductive step, we have

$$\sigma(\langle \mathcal{I} \rangle_{n+1}) = \sigma(\langle \mathcal{I} \rangle_n \diamond \mathcal{I}) = \sigma(\langle \mathcal{I} \rangle_n) \cup \sigma(\mathcal{I}) = \sigma(\mathcal{I}) \cup \sigma(\mathcal{I}) = \sigma(\mathcal{I}).$$

□

Proposition 3.2.8 (**↗**)**.** *We have $\sigma(\langle \mathcal{I} \rangle) = \sigma(\mathcal{I})$.*

Proof. We have $\sigma(\langle \mathcal{I} \rangle) = \bigcup_{n \geq 0} \sigma(\langle \mathcal{I} \rangle_n) = \sigma(\langle \mathcal{I} \rangle_0) \cup \bigcup_{n>0} \sigma(\langle \mathcal{I} \rangle_n) = \emptyset \cup \bigcup \sigma(\mathcal{I}) = \sigma(\mathcal{I})$. □

Proposition 3.2.9 (**↗**)**.** *An object $g \in \mathcal{T}$ is a classical generator if and only if for any support datum (X, σ), $\sigma(g) = \sigma(\mathcal{T})$.*

Proof. If g is a generator, then by (3.2.8) we have $\sigma(\mathcal{T}) = \sigma(\langle g \rangle) = \sigma(g)$. Conversely, if g is not a generator, then by (2.2.20) $\langle g \rangle$ is a proper thick subcategory. Choosing some $a \in \mathcal{T} \setminus \langle g \rangle$, we see that $\langle g \rangle \in \sigma(a) \subseteq \sigma(\mathcal{T})$, while $\langle g \rangle \notin \sigma(g)$. □

One might wonder if support data can detect finer information, such as generation time of a certain generator. As the next proposition shows, we cannot hope for this in general.

Proposition 3.2.10. *There exist two triangulated categories $\mathcal{T}$ and $\mathcal{S}$ with equivalent categories of support data, but with $\mathrm{Rdim}(\mathcal{S}) \neq \mathrm{Rdim}(\mathcal{T})$.*

Proof. Let $\mathcal{T} = D^b(k)$, and let $\mathcal{S}$ be the full triangulated subcategory of $D^b(k[x])$ consisting of complexes whose cohomology modules are all annihilated by some power of x.

Note first that $\mathrm{Rdim}(\mathcal{T}) = 0$, since the complex k concentrated in degree zero is generator with $\langle k \rangle_1 = \mathrm{Rdim}(\mathcal{T})$. We claim that $\mathrm{Rdim}(\mathcal{S}) = \infty$. If $\mathcal{E}$ is a complex in $\mathcal{S}$, then there is some k with x^k annihilating all the cohomology of $\mathcal{E}$. We claim via induction that any object in $\langle \mathcal{E} \rangle_n$ has cohomology annihilated by x^{nk}. This is true for $n = 0$. For the inductive step, suppose that $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to \mathcal{F}'[1]$ is a triangle where all cohomology sheaves of $\mathcal{F}'$ (resp, $\mathcal{F}''$) are annihilated by x^{nk} (resp, x^k). Now consider the exact coholomogy sequence $\mathcal{H}^i(\mathcal{F}') \xrightarrow{f} \mathcal{H}^i(\mathcal{F}) \xrightarrow{g} \mathcal{H}^i(\mathcal{F}'')$. If $m \in \mathcal{H}^i(\mathcal{F})$, then $g(x^k m) = x^k g(m) = 0$, so $x^k m = f(m')$ for some $m' \in \mathcal{H}^i(\mathcal{F}')$. Then $x^{(n+1)k} = x^{nk} x^k = x^{nk} f(m') = f(x^{nk} m) = f(0) = 0$. Since there are objects of $\mathcal{S}$ which are annihilated by arbitrarily high powers of x (take for example, $k[x]/(x^m)$ for m large), we see that $\langle \mathcal{E} \rangle_n \neq \mathcal{S}$ for all n, so that $\mathrm{Rdim}(\mathcal{S}) = \infty$ as needed.

We next describe the categories of support data for the two categories. Given a support datum (X, σ) for $\mathcal{T}$, any object $\mathcal{E}$ of $\mathcal{T}$ is either 0, and therefore $\sigma(\mathcal{E}) = \emptyset$, or $\mathcal{E}$ is a direct sum of shifts of k, and therefore $\sigma(\mathcal{E}) = \sigma(k)$. It follows that the category of support data for $\mathcal{T}$ is equivalent to the category of pairs (X, Z_X) of a topological space X and a closed $Z_X \subseteq X$ and continuous maps $f : X \to Y$ satisfying $Z_X = f^{-1}(Z_Y)$.

The support data for $\mathcal{S}$ admit a similar description, although the argument is a more involved. We first claim that the module k (where x acts as zero) is a classical generator. To see this, first note that because $k[x]$ has homological dimension 1, Corollary 3.15 of [Huy06] implies that any complex in $\mathcal{S}$ is isomorphic to the direct sum of its cohomology modules. Thus, it suffices to show that any finitely generated $k[x]$-module annihilated by a power of x is in $\langle k \rangle$. Since any such module is a sum of modules of the form $k[x]/(x^n)$, we need only show that these are in $\langle k \rangle$. Inducting on n, the case $n = 1$ is the statement $k \in \langle k \rangle$. For the inductive step, assuming that $k[x]/(x^n) \in \langle k \rangle$, the exact sequence $0 \to k \to k[x]/(x^{n+1}) \to k[x]/(x^n) \to 0$ induces a triangle which shows $k[x]/(x^{n+1}) \in \langle k \rangle$. We next show that any non-zero $\mathcal{E} \in \mathcal{S}$ satisfies $\sigma(\mathcal{E}) = \sigma(k)$. Since $\sigma(k)$ is a strong generator, $\sigma(\mathcal{E}) \subseteq \sigma(k)$, so we need only show the reverse inclusion. Once again, we may assume $\mathcal{E} = k[x]/(x^n)$. From here, the exact sequence

$$0 \to k \xrightarrow{x^{n-1}} k[x]/(x^n) \xrightarrow{x} k[x]/(x^n) \to k \to 0$$

gives rise to a triangle $k[x]/(x^n) \to k[x]/(x^n) \to k \oplus k[1] \to (k[x](x^n))[1]$, which implies $\sigma(k \oplus k[1]) \subseteq \sigma(k[x]/(x^n))$. As $\sigma(k \oplus k[1]) = \sigma(k)$, we are finished. $\square$

Chapter 4

SEMIORTHOGONAL DECOMPOSITIONS AND ADMISSIBLE SUBCATEGORIES

A semi-orthogonal decomposition is a natural way to break up a triangulated category into smaller pieces. Beilinson [Bei79] proved that $D^b(\mathbb{P}^n)$ has a semi-orthogonal decomposition which mimics the cell structure of complex projective space. Kapranov [Kap88] found similar semi-orthogonal decompositions for grassmannians and flag varieties which also reflect the cell structure of their complex-geometric counterparts. Bondal and Kapranov [BK89] defined admissible subcategories and Serre functors Orlov [Orl92] discovered a semi-orthogonal decomposition for the derived category of a blow-up which reflects the blowup structure. Belmans, Fu, and Raedschelders [BFR22] proved the existence of semi-orthogonal decompositions arising from flips in the minimal model program. These results indicate that semi-orthogonal decompositions play an important role in birational geometry. Semi-orthogonal decompositions play major roles in other fields as well. For example, Tevelev and Torres [TT24] proved the existence of a semi-orthogonal decomposition for the derived category of certain moduli spaces of sheaves on a smooth curve.

In the first section of this chapter, we define semi-orthogonal decompositions and admissible subcategories. We present a partial formalization of a certain equivalence between these two notions. In the second section, we present an original structural theorem about supports of exceptional objects, which connects the theories of semi-orthogonal decompositions and support data. In the final section, we present an original definition of admissible objects, which creates a bridge from the theory of admissible subcategories to the theory of generation.

4.1 Definitions

Let $\mathcal{T}$ be a triangulated category.

Definition 4.1.1 (↗). A *semi-orthogonal decomposition* is a pair of thick subcategories $\mathcal{A}, \mathcal{B}$ such that

1. For any $b \in \mathcal{B}$ and $a \in \mathcal{A}$ we have $\mathrm{Hom}_{\mathcal{T}}(b, a) = 0$.

2. For any $c \in \mathcal{T}$ there exists a distinguished $b \to c \to a \to b[1]$ with $b \in \mathcal{B}$ and $a \in \mathcal{A}$.

We write $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ to express that $\mathcal{A}, \mathcal{B}$ form a semi-orthogonal decomposition.

Remark. We note that while there is a similar notion of orthogonal decomposition, it is too restrictive to be interesting. For example, if X is a Noetherian scheme, then the category $D^b(X)$ admits a non-trivial orthogonal decomposition if and only if X is not connected, see Proposition 3.10 of [Huy06]. It is a surprise that semi-orthogonal decompositions arise in the wild, and that they have such lucid geometric significance.

Definition 4.1.2 (↗). A thick subcategory P is *left-admissible* (resp., *right-admissible*) if the inclusion functor $P \to \mathcal{T}$ admits a left (resp., right) adjoint.

Proposition 4.1.3 (↗). *Let $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semi-orthogonal decomposition, and let $a \in \mathcal{A}$. Then for any morphism $f : c \to a$, there exists a unique morphism $g : c_{\mathcal{A}} \to a$ with $\epsilon \gg g = f$.*

Proof. Consider the following diagram.

$$\begin{tikzcd} c_{\mathcal{B}} \arrow[r, "\eta"] & c \arrow[r, "\epsilon"] \arrow[d, "f"'] & c_{\mathcal{A}} \arrow[r, "\delta"] \arrow[dl, dashed, "g"] & c_{\mathcal{B}}[1] \\ & a & & \end{tikzcd}$$

Since $c_{\mathcal{B}} \in \mathcal{B}$ and $a \in \mathcal{A}$, we have $\eta \gg f = 0$. Since $\mathrm{Hom}_{\mathcal{T}}(-, a)$ is exact, we then obtain some $g : c_{\mathcal{A}} \to a$ with $\epsilon \gg g = f$, which shows existence. If g' with $\epsilon \gg g' = f$ for some g', then $\epsilon \gg (g' - g) = 0$, so exactness of $\mathrm{Hom}_{\mathcal{T}}(-, a)$ gives some $h : c_{\mathcal{B}}[1] \to a$ with $\delta \gg h = (g' - g)$. Since $c_{\mathcal{B}}[1] \in \mathcal{B}$ and $a \in \mathcal{A}$, we have $h = 0$, so that $(g' - g) = 0$. Then $g = g'$, and uniqueness is shown. $\square$

Proposition 4.1.4 (↗). *Let $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semi-orthogonal decomposition, and let $b \in \mathcal{A}$. Then for any morphism $f : b \to c$, there exists a unique $g : b \to c_{\mathcal{B}}$ which factors f.*

Proof. This is a similar diagram chase as the proof of (4.1.3), pondering instead the below diagram.

$$\begin{tikzcd} & b \arrow[dl, dashed, "g"'] \arrow[d, "f"] & & \\ c_{\mathcal{B}} \arrow[r, "\eta"'] & c \arrow[r, "\epsilon"'] & c_{\mathcal{A}} \arrow[r, "\delta"'] & c_{\mathcal{B}}[1] \end{tikzcd}$$

□

Remark. What we have really done here is the diagram chasing for a lemma which says that in an abelian category, an exact sequence $0 \to a \to b \to 0$ is an isomorphism. While the formalization ought to use this lemma, the API for homological algebra in mathlib is currently cumbersome, and so doing the diagram chase manually in this instance is more compact than interfacing with the more general mathlib lemmas.

Proposition 4.1.5 (↗). *If $c \in \mathcal{A}$, then the map $\epsilon : c \to c_{\mathcal{A}}$ is an isomorphism.*

Proof. Apply the existence part of (4.1.3) to the identity map on c to get some $g : c_{\mathcal{A}} \to c$ with $\epsilon \gg g = \mathrm{id}_c$. To show that $g \gg \epsilon = \mathrm{id}_{c_{\mathcal{A}}}$, we apply the uniqueness part of (4.1.3) to the map ϵ. Equality then holds, since $\epsilon \gg g \gg \epsilon = \epsilon = \epsilon \gg \mathrm{id}_{c_{\mathcal{A}}}$. □

Proposition 4.1.6 (↗). *If $c \in \mathcal{B}$, then the map $\eta : c_B \to c$ is an isomorphism.*

Proof. This proof is similar to (4.1.5), applying (4.1.4) to the identity on c to obtain the inverse. □

Definition 4.1.7. Given a morphism $\phi : c \to c'$, let $\phi_{\mathcal{A}} : c_{\mathcal{A}} \to c'_{\mathcal{A}}$ be the map obtained by applying (4.1.3) to the composition $\phi \gg \epsilon'$, and let $\phi_{\mathcal{B}}$ be the map obtained by apply (4.1.4) to the composition $\eta \gg \phi$, as shown in the below diagram.

$$\begin{tikzcd} c_{\mathcal{B}} \arrow[r, "\eta"] \arrow[d, dashed, "\phi_{\mathcal{B}}"'] & c \arrow[r, "\epsilon"] \arrow[d, "\phi"] & c_{\mathcal{A}} \arrow[r] \arrow[d, dashed, "\phi_{\mathcal{A}}"] & c_{\mathcal{B}}[1] \\ c'_{\mathcal{B}} \arrow[r, "\eta'"] & c' \arrow[r, "\epsilon'"] & c'_{\mathcal{A}} \arrow[r] & c'_{\mathcal{B}}[1] \end{tikzcd}$$

Proposition 4.1.8 (🔗). *The associations $\phi \to \phi_{\mathcal{A}}$ and $\phi \to \phi_{\mathcal{B}}$ are functorial.*

Proof. To see that $(\mathrm{id}_c)_{\mathcal{A}} = \mathrm{id}_{c_{\mathcal{A}}}$, it suffices by the uniqueness part of (4.1.3) to notice that $\epsilon \gg \mathrm{id}_{c_{\mathcal{A}}} = \epsilon \gg \mathrm{id}_c$. Similarly, if $\phi : c \to c'$ and $\phi' : c' \to c''$ are given, then $(\phi \gg \phi')_{\mathcal{A}} = \phi_{\mathcal{A}} \gg \phi'_{\mathcal{A}}$ follows from the observation

$$\epsilon'' \gg \phi_{\mathcal{A}} \gg \phi'_{\mathcal{A}} = \phi \gg \epsilon' \gg \phi'_{\mathcal{A}} = \phi \gg \epsilon \gg \phi'.$$

Thus, $\phi \to \phi_{\mathcal{A}}$ is functorial. The functoriality of $\phi \to \phi_{\mathcal{B}}$ is shown similarly. □

Proposition 4.1.9 (🔗). *If $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$, then $\mathcal{A}$ is left admissible and $\mathcal{B}$ is right admissible.*

Proof. We prove the first claim by providing a unit and counit natural transformation, and the proof of the second claim is similar. The unit is a natural transformation $c \to c_{\mathcal{A}}$, natural for $c \in \mathcal{T}$, which we define as the map ϵ. The counit is a natural transformation $a_{\mathcal{A}} \to a$ natural for $a \in \mathcal{A}$. By (4.1.5), we see that $\epsilon : a \to \mathcal{A}$ is an isomorphism, so we may define the counit as ϵ^{-1}. □

We next turn to a converse of the above theorem.

Definition 4.1.10. For any $\mathcal{I} \subseteq \mathcal{T}$ define

$$\mathcal{I}^{\perp} = \{b \in \mathcal{T} \mid \text{for all } a \in \mathcal{I}, \mathrm{Hom}_{\mathcal{T}}(a, b) = 0\}$$

$$^{\perp}\mathcal{I} = \{b \in \mathcal{T} \mid \text{for all } a \in \mathcal{I}, \mathrm{Hom}_{\mathcal{T}}(b, a) = 0\}$$

We call $\mathcal{I}^{\perp}$ (resp., $^{\perp}\mathcal{I}$) the *right* (resp., *left*) *semi-orthogonal complement* of $\mathcal{I}$.

Proposition 4.1.11 (🔗). *If $\mathcal{I}$ is closed under shift, then $\mathcal{I}^{\perp}$ and $^{\perp}\mathcal{I}$ are thick subcategories.*

Proof. We show that $\mathcal{I}^{\perp}$ is thick, the case $^{\perp}\mathcal{I}$ is similar. We have $0 \in \mathcal{I}^{\perp}$ since $\mathrm{Hom}_{\mathcal{T}}(a, 0) = 0$ for all a. If $b \in \mathcal{I}^{\perp}$ and $i \in \mathbb{Z}$, then for any $a \in \mathcal{I}$ we have $\mathrm{Hom}_{\mathcal{T}}(a, b[i]) \cong \mathrm{Hom}_{\mathcal{T}}(a[-i], b) = 0$, since $a[-i] \in \mathcal{I}$ as well. If $b \cong b' \in \mathcal{I}^{\perp}$, then for any $a \in \mathcal{I}$ we have $\mathrm{Hom}_{\mathcal{T}}(a, b) \cong \mathrm{Hom}_{\mathcal{T}}(a, b') = 0$, so $b \in \mathcal{I}^{\perp}$. If $b \boxplus b' \in \mathcal{I}^{\perp}$ and $a \in \mathcal{I}$, then we can rewrite any $a \to b$ as $a \to b \to b \boxplus b' \to b$, and since $a \to b \to b \boxplus b' = 0$, we see that $a \to b = 0$. Then $b \in \mathcal{I}^{\perp}$ as well. □

Proposition 4.1.12. *If $\mathcal{A}$ is left admissible, then we have a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{A}, {}^\perp \mathcal{A}\rangle$. If $\mathcal{B}$ is right admissible, then we have $\mathcal{T} = \langle \mathcal{B}^\perp, \mathcal{B}\rangle$.*

Proof. We only show the second claim, as the first is similar. By definition of ${}^\perp\mathcal{B}$, if $b \in \mathcal{B}$ and $a \in \mathcal{B}^\perp$, then $\mathrm{Hom}_{\mathcal{T}}(b, a) = 0$. Write $(-)_{\mathcal{B}}$ for the right adjoint to the inclusion, so that we have a natural isomorphism $\mathrm{Hom}_{\mathcal{T}}(b, c_{\mathcal{B}}) \cong \mathrm{Hom}_{\mathcal{T}}(b, c)$ for any $c \in \mathcal{T}$ and $b \in \mathcal{B}$. Now let $c \in \mathcal{T}$, and complete the counit $c_{\mathcal{B}} \to c$ of the adjunction to a triangle $c_{\mathcal{B}} \to c \to d$. We claim that $d \in \mathcal{B}^\perp$. If $b \in \mathcal{B}$, then applying $\mathrm{Hom}_{\mathcal{T}}(b, -)$ gives the exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(b, c_{\mathcal{B}}) \to \mathrm{Hom}_{\mathcal{T}}(b, c) \to \mathrm{Hom}_{\mathcal{T}}(b, d) \to \mathrm{Hom}_{\mathcal{T}}(b, c_{\mathcal{B}}[1]) \to \mathrm{Hom}_{\mathcal{T}}(b, c[1]).$$

Since the leftmost and rightmost arrows are isomorphisms by the adjunction, we see that $\mathrm{Hom}_{\mathcal{T}}(b, d) = 0$. $\square$

Remark. This proof contains a subtlety which makes the formalization very challenging. Specifically, the fact that $\mathrm{Hom}_{\mathcal{T}}(b, c_{\mathcal{B}}[1]) \to \mathrm{Hom}_{\mathcal{T}}(b, c[1])$ is an isomorphism depends on showing that $c_{\mathcal{B}}[1]$ is naturally isomorphic to $(c[1])_{\mathcal{B}}$. The general statement to formalize would be that if $\mathcal{T}, \mathcal{S}$ are categories with shifts, and if $F : \mathcal{T} \to \mathcal{S}$ and $G : \mathcal{S} \to \mathcal{T}$ is an adjoint pair, then F commutes with shifts if and only if G commutes with shifts. This is future work.

4.2 Some Indecomposability Results

A triangulated category is called *indecomposable* if it does not admit any non-trivial semi-orthogonal decompositions. One would hope for some mild finiteness conditions, satisfied by all the triangulated categories we care about, which imply the existence of a decomposition into finitely many indecomposables. A surprising fact as that such decompositions need not have equivalent components, see for example [BGvBS14] or [Kuz13]. Nevertheless, it is important to determine whether a given triangulated category is indecomposable. Almost all of the known examples follow from the same simple idea, which loosely speaking is to use Serre duality to pass from vanishing of maps from $\mathcal{B}$ to $\mathcal{A}$ to vanishing of maps from $\mathcal{A}$ to $\mathcal{B}$. Before stating the example theorem, recall that an *exceptional object* in a k-linear triangulated category is an object $\mathcal{E}$ such that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E}) \cong k$ and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E}[i]) = 0$ for $i \neq 0$. It is a fact that for any exceptional object, we have $\langle \mathcal{E} \rangle = \langle \mathcal{E} \rangle_1$ is an admissible subcategory, and exceptional objects were historically studied before admissible subcategories.

Proposition 4.2.1. *Let X be a smooth projective variety. Then any exceptional object $\mathcal{E} \in D^b(X)$ is supported in the base-locus of the canonical divisor ω_X.*

Proof. Suppose $s \in H^0(X, \omega_X)$ is a non-zero section. Then the map $E \xrightarrow{s} E \otimes \omega_X$ is Serre-dual to a map in $\mathrm{Hom}_{D^b(X)}(E, E[\dim X]) = 0$. Therefore s induces the zero map $E \to E \otimes \omega_X$. Since the restriction of s to $U = X \setminus V(s)$ is an isomorphism, we see that the identity on $E|_U$ vanishes, so that $E|_U \cong 0$. Therefore E is supported on $V(s)$. □

This result was first proved by Kawatani and Okawa [KO18]. They prove a more general statement, that if $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$, then one of $\mathcal{A}$ or $\mathcal{B}$ is supported on the base locus of the canonical. This result was generalized further by Lin [Lin26], who showed that one of $\mathcal{A}$ or $\mathcal{B}$ is supported on the base locus of any line bundle algebraically equivalent to the canonical. The same argument was generalized further by Pirozhkov [Pir25], roughly showing that admissible subcategories with proper support are controlled by sections of the restriction of the canonical to the support.

My contribution is the following generalization to categories which need not be of the form $D^b(X)$. Support should certainly be replaced with an arbitrary support datum, and the main idea is then how to understand the base-locus of the canonical divisor. To find the right analog of a global section $s : \mathcal{O}_X \to \omega_X$, we recall the notion of a Serre functor. A *Serre functor* [BK89] is a triangulated auto-equivalence $S : \mathcal{T} \to \mathcal{T}$ which admits a duality isomorphism $\mathrm{Hom}_{\mathcal{T}}(a, b) \cong \mathrm{Hom}_{\mathcal{T}}(b, S(a))^*$ natural in both a and b. Classical Serre duality can be reinterpreted as the statement that the functor $- \otimes \omega_X[\dim X]$ is a Serre functor for $D^b(X)$. The section $s : \mathcal{O}_X \to \omega_X$ can then be understood as a natural transformation between the identity and a shift of the Serre functor. The vanishing $V(s)$ of the section s can be understood as the support of $\mathcal{O}_{V(s)}$, which fits into a triangle induced by the short exact sequence $0 \to \mathcal{O}_X \to \omega_X \to \mathcal{O}_{V(s)} \to 0$. The generalization will then be in terms of supports of cones of components of natural transformations. In the following, we will say that a natural transformation α between two triangulated functors $F, G : \mathcal{T} \to \mathcal{S}$ commutes with the shift if for any $a \in \mathcal{T}$, we have an equality $\alpha_{a[1]} = \alpha_a[1]$ between the shift of the component and the component at the shift.

Proposition 4.2.2. *Let $F, G : \mathcal{T} \to \mathcal{S}$ be two functors, and let $\alpha : F \to G$ be a natural transformation commuting with the shift. Then*

$$\mathcal{I} := \{a \in \mathcal{T} \mid \alpha_a : F(a) \to G(a) \text{ is an isomorphism}\}$$

is a thick subcategory.

Proof. The shift functor is an equivalence, and so $\alpha_{A[1]} = \alpha_A[1]$ is an isomorphism if α_A is, showing that $\mathcal{I}$ is closed under shifts. Now let $a \to b \to c \to a[1]$ be a triangle and suppose that $a, b \in \mathcal{I}$. Then we get a morphism of triangles

$$\begin{tikzcd}
F(a) \arrow[r] \arrow[d, "\alpha_a"] & F(b) \arrow[r] \arrow[d, "\alpha_b"] & F(c) \arrow[r] \arrow[d, "\alpha_c"] & F(a)[1] \arrow[d, "\alpha_a[1]"] \\
G(a) \arrow[r] & G(b) \arrow[r] & G(c) \arrow[r] & G(a)[1]
\end{tikzcd} .$$

Since α_a, α_b are isomorphisms, the five lemma for triangulated categories implies α_c is an isomorphism as well. Thus, $\mathcal{I}$ is closed under cones.

It remains to show that $\mathcal{I}$ is closed under summands. Suppose $a \boxplus b \in \mathcal{I}$. We claim that the morphism $\alpha_{a \boxplus b} : F(a) \boxplus F(b) \to G(a) \boxplus G(b)$ is the direct sum of the morphisms α_a and α_b. To see this, consider for example the below diagram expressing naturality of α with respect to the structural morphisms $a \to a \boxplus b$ and $a \boxplus b \to a$:

$$\begin{tikzcd}
F(a) \arrow[r] \arrow[d, "\alpha_a"] & F(a) \boxplus F(a) \arrow[r] \arrow[d, "\alpha_{a \boxplus b}"] & F(a) \arrow[d, "\alpha_a"] \\
G(a) \arrow[r] & G(a) \boxplus G(b) \arrow[r] & G(a)
\end{tikzcd} .$$

This shows that the top-left entry of the matrix for $\alpha_{A \boxplus B}$ really is α_A. The other three entries may be found similarly.

From here we see that $\text{cone}(\alpha_{A \boxplus B}) = \text{cone}(\alpha_A) \boxplus \text{cone}(\alpha_B)$. In case $A \boxplus B \in \mathcal{I}$, all three cones therefore vanish, so that $A, B \in \mathcal{I}$ as well. $\square$

Corollary 4.2.3. *Suppose $\alpha : F \to G$ is a natural transformation commuting with the shift, and suppose the component α_g at some classical generator g is an isomorphism. Then α is a natural isomorphism.*

Proof. The collection of $a \in \mathcal{T}$ for which α_a is an isomorphism is a thick subcategory containing g. Since g is a classical generator, the only such subcategory is $\mathcal{T}$. $\square$

Corollary 4.2.4. *Let $\alpha : F \to G$ be a natural transformation, and let g be a classical generator of $\mathcal{T}$. If we complete the component at g to a distinguished triangle $F(g) \to G(g) \to c \to F(g)[1]$, then the completion of any component to a triangle $F(a) \to G(a) \to d \to F(a)[1]$ satisfies $d \in \langle c \rangle$.*

Proof. If π is the quotient functor $\mathcal{S} \to \mathcal{S}/\langle c \rangle$, then we obtain a natural transformation $\alpha \gg \pi : F \gg \pi \to G \gg \pi$ which is an isomorphism at the image of the object g, which is a classical generator by (2.4.9). Then $\alpha \gg \pi$ is a natural isomorphism by (4.2.3), so in particular, $(F \gg \pi)(a) \to (G \gg \pi)(a)$ is an isomorphism. Then $\pi(d) \cong 0$, so that $d \in \ker \pi = \langle c \rangle$. $\square$

Proposition 4.2.5. *Suppose $\mathcal{T}$ admits a Serre functor S, and let (X, σ) be a Serre-invariant support datum for X, i.e. such that $\sigma(S(a)) = \sigma(a)$ for all $a \in \mathcal{T}$). Fix a generator g for $\mathcal{T}$ and a natural transformation $\epsilon : \text{id}_{\mathcal{T}} \to S[i]$ where $i \neq 0$. Let c be a cone for the component of ϵ at g. Then any exceptional object $\mathcal{E}$ is supported on $\sigma(c)$.*

Proof. Complete the component of ϵ at $\mathcal{E}$ to a triangle $\mathcal{E} \xrightarrow{\epsilon_{\mathcal{E}}} S(\mathcal{E})[i] \to d \to \mathcal{E}[1]$. Then $\epsilon_{\mathcal{E}} \in \text{Hom}_{\mathcal{T}}(\mathcal{E}, S(\mathcal{E})[i]) \cong \text{Hom}_{\mathcal{T}}(\mathcal{E}[i], \mathcal{E})^* = 0$, so that $d \cong \mathcal{E}[1] \oplus S(\mathcal{E})[i]$. Then $\sigma(\mathcal{E}) = \sigma(\mathcal{E}[1]) \subseteq \sigma(d)$. Since $d \in \langle c \rangle$, we have $d \subseteq \sigma(c)$. $\square$

4.3 Admissible Objects

Our next task is to characterize admissible subcategories of $\mathcal{T}$ as those which admit strong generators. This characterization is valid under mild hypotheses that are satisfied, for example, by any triangulated categories of the form $D^b(X)$ where X is smooth and projective.

Definition 4.3.1. An object $a \in \mathcal{T}$ is *admissible* if $\langle a \rangle_k = \langle a \rangle_{k+1}$ for some k.

Proposition 4.3.2. *Suppose $\mathcal{T}$ is k-linear, proper, and admits a strong generator. Then a is admissible if and only if $\langle a \rangle$ is an admissible subcategory.*

Proof. If a is admissible, then $\langle a \rangle = \langle a \rangle_k$ for some k, and therefore is strongly generated. Then Theorem 1.3 from [BvdB03] implies that all functors on $\langle a \rangle$ satisfying certain finiteness conditions are representable. For any $c \in \mathcal{T}$, the functor $\mathrm{Hom}_{\mathcal{T}}(-, c)$ is then representable by some $c_a \in \langle a \rangle$, and the assignment $c \to c_a$ forms an adjoint to the inclusion.

Conversely, if $\langle a \rangle$ is admissible, then the projection functor $\mathcal{T} \to \langle a \rangle$ takes a strong generator of $\mathcal{T}$ to a strong generator of $\langle a \rangle$ by (2.4.10). □

This theorem connects the theory of admissible subcategories to the theories of dimension and generation time. It reduces the study of admissible subcategories to study of particular objects, and the focus is then to determine which objects are admissible and to determine which operations preserve admissibility. Strong generators are of course admissible, but admissible objects are much harder to deal with. Consider for example the following result.

Proposition 4.3.3. *Let $f : X \to Y$ be a morphism of smooth projective schemes such that $f^*(\mathcal{L})$ is ample for any ample $\mathcal{L}$. Then Lf^* takes strong generators to strong generators.*

Proof. Let $\mathcal{G}_X, \mathcal{G}_Y$ be strong generators of $D^b(X), D^b(Y)$. If $\mathcal{L}$ is ample in Y, then $f^*(\mathcal{L})$ is ample in X, so that $\{f^*(\mathcal{L}^k) \mid k \geq 0\}$ generates $D^b(X)$ by (2.5.6). In particular, $\mathcal{G}_X \in \langle \{f^*(\mathcal{L}^i) \mid i \geq 0\} \rangle_n$ for some n. If $\langle \mathcal{G}_Y \rangle_m = D^b(Y)$ and $\langle \mathcal{G}_X \rangle_k = D^b(X)$, then we have

$$D^b(X) = \langle \mathcal{G}_X \rangle_k \subseteq \langle \{f^*(\mathcal{L}^i) \mid i \geq 0\} \rangle_{nk} \subseteq \langle Lf^*(\mathcal{G}_Y) \rangle_{nmk}$$

□

The hypotheses of the theorem are satisfied in particular by finite morphisms and closed immersions. Neither of these classes however preserve admissible objects. If for example C is a curve of positive genus, then there exists a closed immersion $f : C \to \mathbb{P}^3$ as well as a finite $g : C \to \mathbb{P}^1$, although the structure sheaf is admissible for $\mathbb{P}^3$ and $\mathbb{P}^1$ but not for C.

While the interaction between admissible objects and functors is unclear, I have proved that large classes of objects are not admissible. Below we say that a closed subscheme $Z \subseteq X$ *moves* in X if the corresponding point $[Z]$ of the Hilbert scheme of X is not isolated.

Proposition 4.3.4. *Let $Z \subseteq X$ be an irreducible subvariety and suppose that Z moves in X. Then the structure sheaf $\mathcal{O}_Z$ is not admissible in $D^b(X)$.*

Proof. Since Z moves in X, there is a curve C in the Hilbert scheme of X passing through $[Z]$. This corresponds to a family $\mathcal{Z} \subseteq C \times X$ of subvarieties in X, flat over C. For some $p \in C$ the fiber $\mathcal{Z}_p \subseteq \{p\} \times X \cong X$ is isomorphic to Z, and also the image of $\mathcal{Z}$ in X properly contains Z. Let Z_n be the fiber over p_n, the n-th infinitesimial thickening of p in C. Since $Z_n \to p_n$ is flat, the filtration of $\mathcal{O}_{p_n}$ with subquotients $\mathcal{O}_p$ induces a filtration of $\mathcal{O}_{Z_n}$ with subquotients $\mathcal{O}_Z$, so that $\mathcal{O}_{Z_n} \in \langle \mathcal{O}_Z \rangle_n$. On the other hand, we claim that $\langle \mathcal{O}_Z \rangle_k \neq \langle \mathcal{O}_Z \rangle$ for any k. First notice that any complex in $\langle \mathcal{O}_Z \rangle_k$ has cohomology sheaves which are annihilated by the k-th power of the ideal sheaf $\mathcal{I}_Z$ defining Z. Since the image of $\mathcal{Z}$ in X properly contains Z, for any k there exists some Z_n with $\mathcal{O}_{Z_n}$ not annihilated by the $\mathcal{I}_Z^k$. Thus, $\mathcal{O}_{Z_n} \in \langle \mathcal{O}_Z \rangle_n \setminus \langle \mathcal{O}_Z \rangle_k$, which shows that $\mathcal{O}_Z$ is not admissible. $\square$

My result here is morally similar to a rigidity result from [KO18], which says that admissible categories are stable under tensor product by any algebraically trivial line bundle, and also stable under pullback by any automorphism in the connected component of the automorphism group scheme containing the identity. I expect a similar argument to work for any object with movable support.

Conjecture 4.3.5. *Let $\mathcal{F} \in D^b(X)$ and suppose that* $\operatorname{supp} \mathcal{F}$ *moves in X. Then $\mathcal{F}$ is not admissible.*

While (4.3.4) provides several examples of inadmissible objects, they are not exhaustive.

Proposition 4.3.6. *Let C be a curve of positive genus. Then $\mathcal{O}_C$ is not admissible in $D^b(C)$.*

Proof. Since $D^b(C)$ is indecomposable by [Oka11], it suffices to show that $\langle \mathcal{O}_C \rangle \neq D^b(C)$. This is true because any object of $\langle \mathcal{O}_C \rangle$ has cohomology sheaves which are semistable of slope zero.

□

It would be interesting to apply the previous results about movable support to this case, where the inadmissible object has full support. The idea would be to apply an appropriate Fourier-Mukai transform which takes $\mathcal{F}$ to an object with movable support. This is possible in some cases, for example:

Proposition 4.3.7. *There exists a smooth projective variety X and a Fourier-Mukai transform $\Phi : D^b(C) \to D^b(X)$ such that $\Phi(\mathcal{O}_C)$ has zero-dimensional support.*

Proof. If P is the desired kernel, then the kernel of the right adjoint P_R satisfies

$$\mathrm{Hom}^*_{D^b(C)}(\mathcal{O}_C, \Phi_{P_R}\mathcal{O}_{X,p}) = \mathrm{Hom}^*_{D^b(X)}(\Phi_P\mathcal{O}_C, \mathcal{O}_{X,p}) = 0$$

for all but finitely many $p \in X$. The condition $\mathrm{Hom}^*_{D^b(C)}(\mathcal{O}_C, \Phi_R\mathcal{O}_{X,p}) = 0$ is equivalent to $H^i(P_R|_{C\times\{x\}}) = 0$ for $i = 0, 1$.

Now consider $\mathrm{Pic}^{g-1}(C)$ and the locus $W_{g-1}(C)$ of effective degree $g-1$ divisors. By Riemann-Roch, $\chi(\mathcal{L}) = 0$ if $\deg(L) = g-1$, and so any $\mathcal{L} \in \mathrm{Pic}^{g-1}(C) \setminus W_{g-1}$ satisfies $H^0(C, L) = H^1(C, L) = 0$. We can then let X be any curve in Pic^{g-1} which is not contained in W_{g-1}, and let P_R be the restriction to X of the universal bundle. □

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