PERCOLATION DIMENSION OF FRACTALS

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Abstract. "Percolation dimension" is introduced in this note. It characterizes certain fractals and its definition is based on the Hausdorff dimension. It is shown that percolation dimension and "boundary dimension" are in a sense independent from the Hausdorff dimension and, therefore, provide an additional tool for classification of fractals.

A unifying concept of fractal was introduced by Mandelbrot to provide a family of models for real world phenomena which cannot be understood in terms of "smooth mathematics" (see Mandelbrot (1982)). Mandelbrot used the Hausdorff dimension as a measure of departure of a fractal from a smooth model and as a convenient way to classify fractals. The purpose of this note is to introduce a new number ("percolation dimension") characterizing fractals. This, together with the Hausdorff dimension, may provide a more accurate classification scheme for fractals.

The impulse to write this note came from the author's research of Brownian paths (see e.g. Burdzy (1987 a,b)). Brownian trace seems to be an example of a fractal with nontrivial percolation dimension.

Percolation dimension of a fractal is, roughly speaking, the size of the shortest path (i.e. the Hausdorff dimension of a connected subset of the fractal) joining distinct points of the fractal.

The definitions and results will be stated for subsets of the plane only. Generalizing them to higher dimensions poses no problems.

Recall that a set $A \subset \mathbf{R}^2$ has Hausdorff dimension α, $\dim A = \alpha$, if α is the infimum of numbers β such that for each $\epsilon > 0$, A may be covered by balls with radii r_k, $k \geq 1$, and $\sum_{k\geq 1}(r_k)^\beta < \epsilon$.

The following definition is adapted from Mandelbrot (1982).

Definition 1. A set $A \subset \mathbf{R}^2$ will be called homogeneous if there exists α such that for each open set U either $U \cap A = \emptyset$ or $\dim(U \cap A) = \alpha$. This property will be denoted $!\text{-}\dim A = \alpha$.

Definition 2. It will be written $\text{b-}\dim A = \beta$ if $!\text{-}\dim(\partial F) = \beta$ for every connected component F of $\mathbf{R}^2 \setminus A$.

The number $\text{b-}\dim A$ may be called, somewhat imprecisely, the "boundary dimension" of A.

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For a connected set $A \subset \mathbf{R}^2$ and $x, y \in A$ let $\mathcal{F}_A(x,y)$ denote the family of all connected subsets of A which contain x and y.

Definition 3. Let

$$\text{p-}\underline{\dim}\, A = \inf_{\substack{x,y\in A\\ x\neq y}} \inf_{B\in\mathcal{F}_A(x,y)} \dim B,$$

$$\text{p-}\overline{\dim}\, A = \sup_{\substack{x,y\in A\\ x\neq y}} \inf_{B\in\mathcal{F}_A(x,y)} \dim B.$$

If $\text{p-}\underline{\dim}\, A = \text{p-}\overline{\dim}\, A$ then the common value will be denoted $\text{p-dim}\, A$ and called the percolation dimension of A.

Remarks. (i) If $A \subset \mathbf{R}^2$ is closed, contains at least 2 points and $!\text{-dim}\, A$, $\text{b-dim}\, A$ and $\text{p-dim}\, A$ exist then obviously

$$1 \le \text{p-dim}\, A \le \text{b-dim}\, A \le !\text{-dim}\, A \le 2.$$

The theorem below will show that there are no other relations between the three dimensions. Moreover, an effort will be made to construct a set A which is compact and locally connected, so that it appears quite "natural", at least from the topological point of view.

(ii) Let $A = X([0,1])$ be the set of all points visited by a 2-dimensional Brownian motion X between times 0 and 1. Intuition suggests that

$$1 < \text{p-dim}\, A < \text{b-dim}\, A < !\text{-dim}\, A = 2.$$

The last equality is well known. Mandelbrot (1982) conjectures that $\text{b-dim}\, A = 4/3$. The author plans to prove in a future article that $\text{b-dim}\, A \in [1.01, 1.6]$.

(iii) Fractals with simple geometric or topological structure have trivial percolation dimension. For example, the percolation dimension of a Koch curve is equal to its Hausdorff dimension; percolation dimension of the Sierpiński carpet is 1 (see Mandelbrot (1982) for definitions of these fractals).

(iv) The idea of using several numbers to characterize a fractal model appears in Mandelbrot (1982), see e.g. Sections 7 and 13.

Before the main result is stated, a few classical definitions will be reviewed. The complex plane $\mathbf{C}$ will be identified with $\mathbf{R}^2$, the imaginary unit will be denoted i. A set homeomorphic to $[0,1]$ will be called a Jordan arc. A set homeomorphic to a circle will be called a closed Jordan arc. For a Jordan arc Γ, $\operatorname{Int}\Gamma$ will denote Γ without its endpoints. An open, simply connected set whose boundary is a closed Jordan arc will be called a Jordan domain. An analytic, one-to-one function defined on a Jordan domain may be continuously extended to the boundary.

Theorem. *For all α, β, γ satisfying*

$$1 \le \alpha \le \beta \le \gamma \le 2$$

there exists a compact, connected and locally connected set $F \subset \mathbf{R}^2$ such that

$$\textit{p-dim}\, F = \alpha, \quad \textit{b-dim}\, F = \beta \quad \textit{and} \quad \textit{!-dim}\, F = \gamma.$$

Proof. The proof will consist of several steps.

Step 1. For each $\alpha \in [1,2]$ there exists a Jordan arc Γ with endpoints 0 and 1, such that $!\text{-}\dim \Gamma = \alpha$ and

$$\operatorname{Int}\Gamma \subset E^0 \overset{\text{df}}{=} \{z \in \mathbf{C} : \Re z \in (0,1), \Im z \in (-1,1)\}.$$

Proof. The construction is given in Mandelbrot (1982) Section 6. A few more details will be supplied here.

It is easy to see that for $k \geq 1$ one can choose positive integers N_k and n_k and closed line segments Δ_k^m, $m = 1, 2, \dots, N_k$, so that

(i) $|\alpha - \log N_k / \log n_k| < 2^{-k}$,

(ii) Δ_k^m has endpoints

$$p/n_k + iq/n_k \quad \text{and} \quad (p+1)/n_k + iq/n_k$$

or

$$p/n_k + iq/n_k \quad \text{and} \quad p/n_k + i(q+1)/n_k$$

for some integers p and q,

(iii) $T_k \overset{\text{df}}{=} \bigcup_{m=1}^{N_k} \Delta_k^m$ is a Jordan arc with endpoints 0 and 1,

(iv) $T_k \subset \{z \in \mathbf{C} : |\Im z| \leq \Re z/2,\ |\Im(1-z)| \leq \Re(1-z)/2\}$.

Let Γ_2 be a Jordan arc obtained from T_1 by replacing every line segment Δ_1^m with a rescaled copy of T_2 so that the endpoints of the rescaled (and rotated, if necessary) T_2 coincide with the endpoints of Δ_1^m. By induction, Γ_k is obtained from Γ_{k-1} by replacing every line segment in Γ_{k-1} (which has length $(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})^{-1}$) with a rescaled copy of T_k.

It is easy to check that Γ_k's converge to a Jordan arc Γ with endpoints 0 and 1, $\operatorname{Int}\Gamma \subset E^0$.

Observe that Γ_k (and Γ as well) may be covered by $N_1 \cdot N_2 \cdot \ldots \cdot N_k$ balls of radius $(n_1 \cdot n_2 \cdot \ldots \cdot n_k)^{-1}$ centered at the endpoints of line segments constituting Γ_k. Standard techniques show that

$$!\text{-}\dim \Gamma = \lim_{k \to \infty} \log(N_1 \cdot N_2 \cdot \ldots \cdot N_k) / \log(n_1 \cdot n_2 \cdot \ldots \cdot n_k) = \alpha.$$

Step 2. Suppose that $D \subset \mathbf{C}$ is a Jordan domain, $x, y \in \partial D$, $x \neq y$, and $\alpha \in [1,2]$. Then there exists a Jordan arc Γ with endpoints x and y, such that $\operatorname{Int}\Gamma \subset D$ and $!\text{-}\dim \Gamma = \alpha$.

Proof. Let $f : E^0 \to D$ be analytic, one-to-one and onto. Moreover, choose f so that it maps 0 and 1 onto x and y. Let Γ_1 be a Jordan arc with endpoints 0 and 1, $!\text{-}\dim \Gamma_1 = \alpha$, $\operatorname{Int}\Gamma_1 \subset E^0$. Define $\Gamma = f(\Gamma_1)$.

It is elementary to check that the Hausdorff dimension is conformal invariant. Thus $!\text{-}\dim \Gamma = \alpha$ and Γ has all the desired properties.

Step 3. Let $M^0 = \overline{E^0}$, $B^0 = \partial E^0$, $C^0 = \{z \in M^0 : \Im z = 0\}$,

$$\begin{aligned} A^0 = & \bigcup_{m=-1,0,1} \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} \{z \in M^0 : |z - k2^{-n} - im| = 2^{-n}\} \\ & \cup \bigcup_{m=0,1} \bigcup_{n=1}^{\infty} \bigcup_{k=-\infty}^{\infty} \{z \in M^0 : |z - ik2^{-n} - m| = 2^{-n}\}, \\ D^0 = & A^0 \cup B^0 \cup C^0. \end{aligned}$$

It is evident from the definition that $A^0 = \bigcup_{k=1}^{\infty} A^0_k$ for some Jordan arcs A^0_k, which have only countably many intersection points.

There exists a homeomorphism $h : M^0 \to M^1$ such that if $h(A^0_k) = A^1_k$, $h(B^0) = B^1$, $h(C^0) = C^1$, $h(D^0) = D^1$, and $h(E^0) = E^1$ then !-dim $A^1_k = \alpha$, !-dim $B^1 = \beta$ and !-dim $C^1 = \gamma$.

Proof. First, use the method of Step 1 to find a closed Jordan arc B^1 with !-dim $B^1 = \beta$. Then use Step 2 to find a Jordan arc C^1 so that $B^1 \cup C^1$ is homeomorphic to $B^0 \cup C^0$ and !-dim $C^1 = \gamma$. Add sets $A^1_1, A^1_2, \dots$ one by one, each time choosing A^1_k so that $B^1 \cup C^1 \cup \bigcup_{n=1}^{k} A^1_n$ is homeomorphic to $B^0 \cup C^0 \cup \bigcup_{n=1}^{k} A^0_n$ and !-dim $A^1_k = \alpha$. It is easy to see that A^1_k's may be chosen so that $D^1 \overset{\text{df}}{=} B^1 \cup C^1 \cup \bigcup_{k=1}^{\infty} A^1_k$ is homeomorphic to D^0 and, moreover, there exists a homeomorphism h with the properties specified above.

Step 4. In this step, the set F will be constructed.

The set $M^1 \setminus D^1$ consists of a countable number of Jordan domains. Denote them G^1_k, $k \geq 1$.

Now it will be shown how to obtain inductively A^{n+1}, B^{n+1}, C^{n+1}, D^{n+1} and $\{G^{n+1}_k\}_{k\geq 1}$ given A^n, B^n, C^n, D^n and $\{G^n_k\}_{k\geq 1}$.

The set A^1 has dimension α, by Step 3. It will follow, by induction, that $\dim A^n = \alpha$ for every n. Cover the set A^n with balls P'_k with radii r'_k so that $\sum_k (r'_k)^{\alpha_n} < 2^{-n} 2^{-\alpha_n}$ where $\alpha_n = \alpha + 2^{-n}$. Let P_k be an open ball concentric with P'_k and radius $r_k = 2r'_k$, and let $U^n = \bigcup_k P_k$. Thus, A^n has an open superset U^n which may be covered by balls which have radii r_k and $\sum_k (r_k)^{\alpha_n} < 2^{-n}$.

Consider a set G^n_k and choose a sequence of distinct points $x_m \in \partial G^n_k$, $m \geq 1$, which is dense in ∂G^n_k. Note that $\partial G^n_k \subset A^n \subset U^n$. Therefore, there exists a Jordan domain $J \subset U \cap G^n_k$ such that $\overline{J} \cap \partial G^n_k = \{x_1\}$. Let K be a Jordan domain containing E^1 and such that $\partial K \cap D^1 = \{y\}$. Find an analytic, one-to-one function f which maps K onto J and y onto x_1. Let $f(A^1) = A^{n+1}_{k,1}, f(B^1) = B^{n+1}_{k,1}, f(C^1) = C^{n+1}_{k,1}, f(D^1) = D^{n+1}_{k,1}, f(M^1) = M^{n+1}_{k,1}$. Repeat the construction for each point x_m and make sure that $M^{n+1}_{k,m}$ and $M^{n+1}_{k,j}$ (which correspond to x_m and x_j) are disjoint for $j \neq m$. Then repeat the whole construction for each G^n_k, $k \geq 1$. As a result, one obtains analytic, one-to-one functions $f_{k,m} : K \to G^n_k$ and sets $A^{n+1}_{k,m} \overset{\text{df}}{=} f_{k,m}(A^1)$, $B^{n+1}_{k,m} \overset{\text{df}}{=} f_{k,m}(B^1)$, $C^{n+1}_{k,m} \overset{\text{df}}{=} f_{k,m}(C^1)$, $D^{n+1}_{k,m} \overset{\text{df}}{=} f_{k,m}(D^1)$, $M^{n+1}_{k,m} \overset{\text{df}}{=} f_{k,m}(M^1)$ with the following properties:

(i) $M^{n+1}_{k,m} \subset \overline{G^n_k} \cap U^n$,
(ii) $M^{n+1}_{k,m} \cap M^{n+1}_{j,p} = \emptyset$ if $(k,m) \neq (j,p)$,
(iii) $D^{n+1}_{k,m} \cap \partial G^n_k = \{y_{k,m}\}$ and $y_{k,m}$'s are dense in ∂G^n_k,
(iv) !-dim $A^{n+1}_{k,m} = \alpha$, !-dim $B^{n+1}_{k,m} = \beta$, and !-dim $C^{n+1}_{k,m} = \gamma$.

Let $A^{n+1} = \bigcup_{k,m} A^{n+1}_{k,m}$, $B^{n+1} = \bigcup_{k,m} B^{n+1}_{k,m}$, $C^{n+1} = \bigcup_{k,m} C^{n+1}_{k,m}$, $D^{n+1} = \bigcup_{k,m} D^{n+1}_{k,m}$, $M^{n+1} = \bigcup_{k,m} M^{n+1}_{k,m}$, $H^n_k = G^n_k \setminus \bigcup_m M^{n+1}_{k,m}$.

In the above construction, choose functions $f_{k,m}$ so that the diameter of $M^{n+1}_{k,m}$ is less than 2^{-n-m-k}. Then the set $M^1 \setminus D^{n+1}$ will consist of a countable family of Jordan domains. Some of them are H^n_k's. Label the remaining ones G^{n+1}_k, $k \geq 1$. This ends the inductive step of the definition.

Let F be the closure of $\bigcup\limits_{n\geq 1} D^n$. Since the diameter of $M^{n+1}_{k,m}$ is less than 2^{-n-m-k} for each n and m, the set F is contained in $M^{k+1} \cup \bigcup\limits_{n\leq k} D^n$, for each k.

Step 5. It will be proved that the set F constructed in the previous step satisfies the theorem.

(i) The set F is bounded and closed so it is compact. It is easy to see that it is connected (even arc-connected) and locally connected.

(ii) It will be proved that !-dim $F = \gamma$. Note that $F \subset U^n \cup \bigcup\limits_{k=1}^{\infty} (B^k \cup C^k)$ for every n. Fix some n. Since the set $\bigcup\limits_{k=1}^{\infty} (B^k \cup C^k)$ has dimension γ, it can be covered by balls with the radii ρ_m such that $\sum\limits_m (\rho_m)^{\gamma_n} < 2^{-n}$ where $\gamma_n = \gamma + 2^{-n}$. By definition, the set U^n may be covered by balls with radii r_k, $\sum\limits_m (r_m)^{\alpha_n} < 2^{-n}$, $\alpha_n = \alpha + 2^{-n} \leq \gamma + 2^{-n}$. It follows that F may be covered by balls with radii p_m, $\sum\limits_m (p_m)^{\gamma_n} < 2^{-n+1}$. Since n is arbitrary, $\dim F \leq \gamma$.

To see that !-dim $F = \gamma$, observe that the set $\bigcup\limits_{k=1}^{\infty} C^k \stackrel{\text{df}}{=} C^\infty$ is dense in F and !-dim $C^\infty = \gamma$.

(iii) The complement of F is the disjoint union of the complement of M^1 and H^n_k's for $k,\, n \geq 1$. By construction, !-dim $\partial M^1 = \beta$. Suppose that S is an open set which has a nonempty intersection with ∂H^n_k. Then it has a nonempty intersection with $\partial M^{n+1}_{k,m}$ for some m. Since !-dim $\partial M^{n+1}_{k,m} = \beta$, it follows that $\dim(S \cap \partial H^n_k) \geq \beta$.

Note that $\partial H^n_k \subset \partial G^n_k \cup \bigcup\limits_{m\geq 1} \partial M^{n+1}_{k,m}$. Since $\dim \partial G^n_k = \alpha$ and $\dim \partial M^{n+1}_{k,m} = \beta$, it follows that $\dim \partial H^n_k \leq \beta$. This and the previous inequality prove that b-dim $F = \beta$.

(iv) Finally, it will be shown that p-dim $F = \alpha$.

The set $A^\infty \stackrel{\text{df}}{=} \bigcup\limits_{k\geq 1} A^k$ is connected (even arc-connected) and dense in F. Thus, for any $x, y \in F$, the set $\{x, y\} \cup A^\infty$ is connected and has dimension α; therefore, p-$\overline{\dim}\, F \leq \alpha$.

To prove that p-$\underline{\dim}\, F \geq \alpha$, consider first $x, y \in D^1$, $x \neq y$. Let Z be a connected set such that $Z \subset F$ and $x, y \in Z$. It is easy to see that $Z^1 \stackrel{\text{df}}{=} Z \cap D^1$ is also connected and $x, y \in Z^1$. It will be shown that Z^1 contains a Jordan arc which is a subset of B^1, C^1 or A^1_k.

Suppose otherwise. Find a closed Jordan arc Γ which partitions the plane into two open sets, with points x and y in the differential parts. The arc Γ may be chosen so that it does not intersect $B^1 \cap Z^1$ or $C^1 \cap Z^1$, because these sets do not contain arcs. Moreover, Γ need not pass through intersection points of A^1_k's, i.e. points in $A^1_k \cap A^1_j$, $k,\, j \geq 1$, since there are only countably many such points. Finally, modify Γ to obtain Γ', a closed Jordan arc passing through the same sequence of G^1_k's and not intersecting $A^1 \cap Z^1$ (again, this is possible since $A^1 \cap Z^1$ does not contain arcs). The arc Γ' can still separate x and y and not intersect Z^1. Therefore Z^1 is not connected, which is a contradiction. This proves that Z^1 must contain an arc of A^1, B^1 or C^1 and $\dim Z \geq \dim Z^1 \geq \alpha$.

A similar argument shows that if $x, y \in D^n_{k,m}$, $x \neq y$, then for every connected set Z such that $x, y \in Z \subset F$ one has $\dim Z \geq \alpha$.

Now consider $x, y \in F$, $x \neq y$, and suppose that there is no $D^n_{k,m}$ such that $x, y \in D^n_{k,m}$. Let n be the largest integer such that $x, y \in M^n_{k,m}$ for some k, m ($M^1_{k,m} \overset{\text{df}}{=} M^1$). Note that such an integer must exist, otherwise x and y would belong to $M^n_{k,m}$ for arbitrary large n and would have to be equal.

If $x, y \in M^n_{k,m}$ but there is no $M^{n+1}_{k,m}$ which contains x and y, three things may happen. The first one is that $x, y \in D^n_{k,m}$, which is ruled out. The other two are

(i) $x \in D^n_{k,m}$, $y \in M^{n+1}_{j,p}$, $y \notin D^n_{k,m}$, and $x \notin M^{n+1}_{j,p}$

or

(ii) $x, y \notin D^n_{k,m}$, so $x \in M^{n+1}_{j,p}$, $y \in M^{n+1}_{r,s}$, $(j,p) \neq (r,s)$.

The roles of x and y may be interchanged in (i).

Suppose that $Z \subset F$ is a connected set containing x and y. In case (i), Z must contain z, where $\{z\} = M^{n+1}_{j,p} \cap D^n_{k,m}$. Note that $z \neq x$ and $x, z \in D^n_{k,m} \cap Z$. An earlier part of the proof shows that $\dim Z \geq \alpha$. In case (ii) let $\{z_1\} = M^{n+1}_{j,p} \cap D^n_{k,m}$, $\{z_2\} = M^{n+1}_{r,s} \cap D^n_{k,m}$. Again, $z_1 \neq z_2$, $z_1, z_2 \in D^n_{k,m} \cap Z$, so $\dim Z \geq \alpha$. This finishes the proof that for every $x, y \in F$ and every connected set Z containing x and y, $\dim Z \geq \alpha$. It follows that $\text{p-dim}\, Z = \alpha$. $\square$

References

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