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Nonparametric Identification and Structural Estimation of Auction Models

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Abstract

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This dissertation contributes to the structural auction literature in two different auction models, namely the pure common value model and the affiliated private value model. The goal of structural analysis of auction data is to recover the model primitives and to provide policy guidance for welfare analysis. In Chapter 1, we study identification in the first-price and the second-price sealed-bid auctions within the pure common value framework. In Chapter 2, we apply the identification results and estimation method in Chapter 1 to analyze the U.S. Outer Continental Shelf (OCS) wildcat auction data and provide policy guidance for welfare analysis. In Chapter 3, we develop identification and partial identification results for the first-price and the second-price sealed-bid auction models with affiliated private values and incomplete sets of bids.

Chapter 1: In this chapter, we establish novel identification results for both the first-price and the second-price sealed-bid auction models within the pure common value framework. We show that the policy parameters, including the expected total welfare, the seller's expected revenue, and the bidders' expected surplus under any reserve price are identified for a general nonparametric class of latent joint distributions when the ex-post common value is unobserved. Moreover, we establish that these policy parameters are nonparametric identified without normalization assumption when the ex-post common value is observed. We propose a semiparametric estimation method

and establish consistency of the estimator. Results from Monte Carlo experiments reveal good finite sample performance of the estimator.

Chapter 2: In this chapter, we employ the identification strategy and estimation method in Chapter 1 to analyze data from the U.S. Outer Continental Shelf (OCS) wildcat auctions in the pure common value framework. We study the welfare implication of different counterfactual reserve prices, focusing on the cases with two and three bidders. The empirical results suggest that if the U.S. government had set reserve prices optimally using the newly-developed econometric method in Chapter 1, its expected revenue can be increased by around 34% and 30% for these two cases, respectively. Lastly, we compare our results with those estimated under the affiliated private value framework, and find that the estimated welfare curves under the two different frameworks are very different.

Chapter 3: In this chapter, we address the identification issue in the first-price sealed-bid affiliated private value model when an incomplete set of bids is observed. In the simple case with symmetric bidders and non-binding reserve price, we establish identification or partial identification results in two scenarios of practical interest. First, when the two highest bids are observed, we achieve identification of the joint distribution function of private values by assuming the copula function of private values to be a nonparametric Archimedean copula with weak requirement. Second, when only the highest bid is observed, we establish partial identification for the quantile function of private value and several policy parameters by parameterizing the copula function. Further, we extend the identification/partial identification results to the cases with asymmetric bidders and/or binding reserve price. We also extend our identification/partial identification results to the second-price sealed-bid auction.

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DEDICATION

To My Family

Chapter 1

NONPARAMETRIC IDENTIFICATION AND STRUCTURAL ESTIMATION OF PURE COMMON VALUE AUCTION MODELS

1.1 Introduction

Auctions are ubiquitous in market economies. For example, the U.S. Department of the Treasury conducts weekly auctions to sell long-term securities in order to finance the government's borrowing needs; the U.S. Forest Service conducts auctions to sell timbers; the U.S. federal government conducts auctions to sell mineral rights on oil and gas on the Outer Continental Shelf (OCS) off the coasts of Texas and Louisiana, and so on.

There are two frameworks in the auction theory literature: the private value framework and the common value framework (see Krishna (2010) for an excellent review). In the former framework, a private-value bidder observes her private value and bids for the object for personal use. In the latter framework, a common-value bidder observes a private signal that is a proxy for the object's unknown common value, and bids for the object for reselling purposes.

Structural econometrics of auction data was pioneered by Paarsch (1992) and Guerre, Perrigne, and Vuong (2000) (see Paarsch and Hong (2006), Athey and Haile (2007), Hendricks and Porter (2007), and Hickman, Hubbard, and Sağlam (2012) for surveys of the literature). While econometric identification and estimation have been well developed for the private value framework,[1] they are much less developed for the

[1]See Guerre, Perrigne, and Vuong (2000, 2009), Li, Perrigne, and Vuong (2000, 2002, 2003), Haile and Tamer (2003), Li (2005), Campo et al. (2011), Krasnokutskaya (2011), Komarova (2011),

common value framework. The common value framework has broad applicability in many real world auctions. Examples include the eBay auctions analyzed in Bajari and Hortacsu (2003), the U.S. OCS wildcat auctions of oil-drilling rights in Hendricks, Pinkse, and Porter (2003), and among many others. However, nonparametric identification and structural estimation remain challenging in this framework. As noted in Hickman, Hubbard, and Sağlam (2012): "work on estimation in the common value paradigm has been sparse after Paarsch (1992). Identification within the common value paradigm is considerably more difficult than under private values."

One leading case of the common value framework is the pure common value model, in which all bidders share the same ex-post common value. This model is particularly relevant when all bidders face the same market selling price for the object or the same project cost at a later date. One of the most important examples is the U.S. OCS wildcat auctions of oil-drilling rights. However, previous structural analysis of this data set has been conducted in the private value framework (see Li, Perrigne, and Vuong (2000, 2003), Campo, Perrigne, and Vuong (2003)). As suggested by the results in Hendricks, Pinkse, and Porter (2003), the OCS wildcat auction data is more consistent with the pure common value model than with the private value model. The goal of this paper is to develop econometric machinery to analyze the OCS wildcat auction data set, and more generally, any data set that falls into the pure common value framework.

The nonparametric identification problem in the pure common value model is challenging for two reasons. First, the dimension of the model primitive is greater than the dimension of observed bids. In the structural auction literature, the model primitive refers to the latent joint distribution of private values in the private value framework, and it refers to the latent joint distribution of the common value and private sig-

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Hubbard, Li, and Paarsch (2012), Marmer and Shneyerov (2012), Aradilla-López, Gandhi, and Quint (2013), Armstrong (2013), Gentry and Li (2014), Ma, Marmer, and Shneyerov (2016).

nals in the pure common value framework.[2] In the latter framework, the latent full joint distribution is of one more dimension than the observed joint distribution of bids, thus recovering it is in general impossible. Second, the standard transformation approach in the private value framework encounters a problem in the pure common value framework. Nonparametric identification in the private value framework relies on transforming the original first-order condition for the equilibrium bidding function into an equivalent form that only involves the observed distribution of bids. In this way, pseudo private values can be obtained to estimate the joint distribution of private values. In the pure common value model, however, the transformed first-order condition still involves an unknown function, which is the expectation of the common value conditional on a bidder's own signal and the highest competing signal.[3] This unknown conditional expectation function prevents one from identifying the joint distribution of private signals, and identification of the latent full joint distribution remains even more challenging.

In this paper, instead of targeting at the latent full joint distribution, we focus on policy parameters such as the expected total welfare, the seller's expected revenue, and the bidders' expected surplus under any reserve price. In practice, these parameters are more important than the latent distribution since they readily provide guidance for welfare analysis. Although these policy parameters can be expressed as functionals of the latent full joint distribution, we show that information on the latent full joint distribution is sufficient but not necessary for one to nonparametrically identify the policy parameters. We analyze the exact dependence of these policy parameters on the observed distributions and the unknown conditional expectation function, and establish

[2]In the following, we refer this joint distribution as the latent full joint distribution to distinguish it from the joint distribution of private signals.

[3]Other common-value bidders' private information can reveal extra information on the common value. Wilson (1977) showed that a rational common-value bidder will take this informational update into account and shade her bid to avoid the winner's curse. As a result, a common value bidder forms an expectation of the common value conditional on her own signal and the highest competing signal when maximizing her expected profit.

nonparametric identification of this unknown function and consequently the policy parameters. Specifically, we show that for a general nonparametric class of latent full joint distributions, the unknown conditional expectation function only depends on the joint distribution function of private signals. By Sklar's Theorem, this joint distribution function of private signals is decomposed into the copula function and the marginal distribution function of private signals. We identify the copula function of private signals from that of observed bids, and identify the inverse of the marginal distribution function–the quantile function, by a Volterra integral equation of the second kind. For estimation, we propose a semiparametric method in which the copula function is parameterized and the marginal distribution function is left nonparametric. The finite dimensional parameter of the copula function is estimated by a pseudo maximum likelihood method, and the quantile function of private signals is estimated by either a geometric series estimator or an iterative sieve estimator.

This paper is related to a few existing works in the literature. Paarsch (1992) imposed parametric assumptions on the private signal distribution to obtain tractable equilibrium bidding function, and used either the maximum likelihood method or the method of moment to estimate the finite dimensional parameter. Li, Perrigne, and Vuong (2000) assumed the log of the unknown conditional expectation function to be of log-linear form and achieved identification up to location and scale parameters. Hendricks, Pinkse, and Porter (2003) focused on testing the rational and equilibrium bidding assumption. Février (2008) assumed a specific form of the private signal density function conditional on the common value and established nonparametric identification in a particular class. Tang (2011) established bounds on the revenue distribution under counterfactual auction format and reserve prices by assuming each bidder's value to be degenerate conditional on other private signals but with an unknown link function. In a general interdependent cost model, Somaini (2015) exploited exclusion restriction on the covariates (cost shifters) and achieved identification of both the joint distribution of private signals and the full information expected completion cost conditional on covariates.[4]

This paper is different from these existing works in the following ways. First, we do not assume any parametric form of the unknown conditional expectation function as in Li, Perrigne, and Vuong (2000), nor do we impose the identity bidding function assumption as in Février (2008) and Tang (2011) to circumvent the problem caused by the unknown conditional expectation function. Instead, we use the data to identify this function in a nonparametric class of latent full joint distributions and thus establish nonparametric identification of the policy parameters. Second, we do not need covariates and exclusion restrictions as in Somaini (2015). As a result, our approach can deal with any data set that falls into the pure common value framework, provided that our identification assumption is plausible for that data set. We discuss the differences between our approach and these existing approaches in detail in Section 1.2.2.

The rest of this chapter is organized as follows. In Section 1.2, we review the first-price sealed-bid pure common value auction model and analyze its identification challenges. We argue that nonparametric identification of the full joint distribution of the common value and private signals is sufficient but not necessary for identification of the policy parameters. In Section 1.3, we show nonparametric identification of the policy parameters in both the first-price and the second-price sealed-bid auction models. In Section 1.4, we propose semiparametric estimation method and establish consistency. Monte Carlo experiments are conducted in Section 1.5. Section 1.6 concludes. All proofs are relegated to Appendix A.

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[4] In addition to these papers, in a procurement setting, Hong and Shum (2002) imposed parametric assumption on the joint distribution of private signals and costs, where each bidders' cost consists of a private value component and a common value component. They focused on empirically evaluating the competition effects and winner's curse effects as the number of bidders increases. Haile, Hong, and Shum (2006) developed nonparametric tests to differentiate between the private value and the common value frameworks in the first-price sealed-bid auction.

1.2 The Model, Identification Challenge, Policy Parameters

1.2.1 The Model

One indivisible good is auctioned by a first-price sealed-bid auction. Let the common value of the good be X_o with distribution function $F_{X_o}(x)$ and Lebesgue density function $f_{X_o}(x)$ on $[\underline{x}_o, \overline{x}_o]$. Each of M risk-neutral common-value bidders seeks to maximize her expected profit. Let the vector of private signals be $\underline{X} = (X_1, \cdots, X_M)$, distributed according to $F_{\underline{X}}(\cdot)$ with Lebesgue density function $f_{\underline{X}}(\cdot)$ on $[0, \overline{x}]^M$. The full vector $(X_o, \underline{X})$ is assumed to be affiliated (see Milgrom and Weber (1982)), and is distributed according to $F_{X_o\underline{X}}(\cdot)$ with Lebesgue density function $f_{X_o\underline{X}}(\cdot)$ on $[\underline{x}_o, \overline{x}_o] \times [0, \overline{x}]^M$. The common-value bidders are symmetric in the sense that the joint distribution function $F_{X_o\underline{X}}(\cdot)$ is invariant to any permutation of its last M arguments. Let all cdfs and conditional cdfs be denoted by upper-case letters and all corresponding pdfs and conditional pdfs be denoted by lower-case letters. Specifically, for two latent random variables X, Y, we use $F_{XY}(x, y)$ to denote the joint cdf, use $F_{X|Y}(x|y), f_{X|Y}(x|y)$ to denote the conditional cdf and pdf, respectively, and use $F_X(x), f_X(x)$ to denote the marginal cdf and pdf, respectively. For two observed random variables, we use G, g to replace F, f, respectively in the above notations.

We focus on bidder 1 due to symmetry among the bidders. In the pure common value framework, bidder 1 does not observe the realization of X_o prior to the auction, but observes her private signal $X_1 = x$. We focus on symmetric, strictly increasing, and differentiable Bayesian Nash equilibrium bidding strategy. Given that other bidders follow the same equilibrium bidding strategy $\beta(\cdot)$, bidder 1 chooses a bid b to maximize her expected profit

$$\pi(b; x) = \mathbb{E}[(X_o - b)\mathbb{1}(\beta(Y_1) \leq b)|X_1 = x], \tag{1.1}$$

where $Y_1 = \max X_{-1}$, $X_{-1} = (X_2, \cdots, X_M)$, and $\mathbb{1}(\cdot)$ is the indicator function. The first-order condition and definition of Bayesian Nash equilibrium lead us to the equilibrium bidding function that satisfies the differential equation

$$\beta'(x) = \Big[\overline{H}(x) - \beta(x)\Big]\rho_{Y_1|X_1}(x), \tag{1.2}$$

subject to the boundary condition $\beta(0) = \overline{H}(0)$, where $\rho_{Y_1|X_1}(x) = f_{Y_1|X_1}(x|x)/F_{Y_1|X_1}(x|x)$ is the reverse hazard function of Y_1 conditional on X_1 evaluated at the diagonal and $\overline{H}(x) = \mathbb{E}[X_o|X_1 = x, Y_1 = x]$. Under the affiliation assumption of $(X_o, \underline{X})$, there exists a unique solution to (1.2) that is strictly increasing and differentiable (see Milgrom and Weber (1982)). The equilibrium bidding function can be solved as

$$\beta(x) = \overline{H}(x) - \int_0^x J(a|x)d\overline{H}(a),\ x \in [0, \overline{x}], \tag{1.3}$$

where $J(a|x) = \exp\big(-\int_a^x \rho_{Y_1|X_1}(s)ds\big)$. The function $\overline{H}(x)$ represents bidder 1's expectation of the common value conditional on her signal and on her equilibrium bid being pivotal. Hong, Haile, and Shum (2006) termed it the conditional expected valuation. In the analysis below, we show that it plays a key role for the nonidentification result in the pure common value auction model.

1.2.2 Identification Challenges and Existing Approaches

In practice, we typically observe a random sample of bids $\{B_{1\ell}, \cdots, B_{M\ell}\}_{\ell=1}^{L}$ from L repeated auctions with non-binding reserve price. Let $B_1 = \beta(X_1)$, $M_1 = \beta(Y_1)$, and thus M_1 is the maximum bid from bidder 1's competitors. Let $G_{M_1|B_1}(m_1|b_1)$ and $g_{M_1|B_1}(m_1|b_1)$ be the distribution function and density function of M_1 conditional on B_1, and $\rho_{M_1|B_1}(b) = g_{M_1|B_1}(b|b)/G_{M_1|B_1}(b|b)$ be the reverse hazard function of M_1 conditional on B_1 evaluated at the diagonal. Applying the standard GPV type transformation (see Guerre, Perrigne, and Vuong (2000), Li, Perrigne, and Vuong (2002)), it can be easily shown that $\rho_{M_1|B_1}(\beta(x)) = \rho_{Y_1|X_1}(x)/\beta'(x)$, and we can write (1.2) as

$$x = \overline{H}^{-1}\left(b + \frac{1}{\rho_{M_1|B_1}(b)}\right), \tag{1.4}$$

where $b = \beta(x)$ and $\overline{H}^{-1}(x)$ is the inverse function of $\overline{H}(x)$.

The standard GPV type transformation for nonparametric identification encounters two challenges in the pure common value auction model. First, even if the private signals could be estimated from the observed bids by (1.4), using an M dimensional pseudo signals to recover the $M+1$ dimensional latent full joint distribution is not possible. Second, the key component, the conditional expected valuation function, is unknown. This prevents us from obtaining pseudo signals from the observed bids and bids distributions as represented by $\rho_{M_1|B_1}(b)$ above.

Several attempts have been made to deal with the above two challenges. First, it is common to adopt the mineral rights model, which is a special case of the pure common value model. In this model, the private signals are assumed to be i.i.d. conditional on the common value X_o. In this case, the joint distribution of the common value and the private signals is reduced to the marginal density function $f_{X_o}(x)$ and conditional density function $f_{X_1|X_o}(x_1|x_o)$. Previous works under this framework include Paarsch (1992), Li, Perrigne, and Vuong (2000), and Février (2008). For identification in the mineral rights model, more assumptions are needed. Paarsch (1992) parameterized the marginal and conditional density functions to special families in order to yield a tractable equilibrium bidding function and used either the maximum likelihood method or the method of moment to estimate the finite dimensional parameter. Li, Perrigne, and Vuong (2000) assumed a multiplicative decomposition of the form $X_m = X_o\epsilon_m$ with $X_o \perp \epsilon_m$ for i.i.d. $\epsilon_1, \cdots, \epsilon_M$, where "$\perp$" denotes the statistical independence. Their model was defined by $f_{X_o}(\cdot)$ and the density function $f_\epsilon(\cdot)$ of the idiosyncratic term. Février (2008) assumed a very specific nonparametric structure of the conditional density function.

Second, the unknown conditional expected valuation did not pose a problem in Paarsch (1992) since the inverse transformation was not needed in his parametric approach. Li, Perrigne, and Vuong (2000) assumed this function to be of the form $\overline{H}(x) = a_1 x^{a_2}$ for some constants a_1 and a_2, which restricted the latent full joint distribution to an unknown class of functions. Février (2008) and Tang (2011) both adopted a normalization assumption that the equilibrium bidding function is the identity function to circumvent the problem, and this also restricted the full joint distribution to an unknown class of functions. For the identification results, Paarsch (1992) achieved identification under parameterization; Li, Perrigne, and Vuong (2000) identified the functions $f_{X_o}(\cdot)$ and $f_\epsilon(\cdot)$ up to the parameters a_1, a_2 using the Kotlarski decomposition; Février (2008) achieved nonparametric identification in a particular class; and Tang (2011) employed a partial identification approach and focused on bounding the seller's expected revenue under counterfactual auction format and reserve price.

In this paper, we take a step back and pose two questions: First, instead of making the identity bidding function normalization or assuming certain forms to deal with the unknown conditional expected valuation function, can we identify it from the data under some weak assumptions on the latent full joint distribution? Second, although the full joint distribution is sufficient for identifying any functional of the model primitive, for particular functionals of interest, is it necessary to identify the full joint distribution? We address the two questions in the following sections.

1.2.3 Expected Total Welfare, Seller's Expected Revenue, and Bidders' Expected Surplus under Counterfactual Reserve Price

Let $\overline{L}(x) = \mathbb{E}[X_o|X_1 = x, Y_1 \leq x]$. From Milgrom and Weber (1982), the equilibrium bidding function in a first-price sealed-bid pure common value auction under reserve price $r \in [\overline{L}(0), \overline{L}(\overline{x})]$ is

$$\beta_r(x) = rJ(x_r^*|x) + \int_{x_r^*}^{x} \overline{H}(a)dJ(a|x),\ x \in [x_r^*, \overline{x}],$$

where $x_r^* = \inf_{x \in [0,\overline{x}]}\{\overline{L}(x) \geq r\}$ with $\beta_r(x_r^*) = r$. x_r^* is interpreted as the lowest signal at which a bidder believes that the value of the object conditional on winning is worth at least the reserve price. In the rest of this paper, we name $\overline{H}(x)$ and $\overline{L}(x)$ the high and low conditional expected valuations, respectively. Policy makers are often interested in how the expected total welfare, the seller's expected revenue, and the bidders' expected surplus change with reserve price since these policy parameters depend on $\beta_r(x)$. As the expected total welfare is the sum of the seller's expected revenue and the bidders' expected surplus, we focus on the latter two quantities in the following analysis. The example below illustrates the welfare implication of reserve price.

Example 1.2.1 Let the private signals $\{X_m\}_{m=1}^M$ be i.i.d. uniformly distributed on $(0,1)$, $X_o = \sum_{m=1}^M X_m/M + \epsilon$, where $M = 3$, $\epsilon \perp \underline{X}$ with $\mathbb{E}[\epsilon] = 0$. Let the seller's own valuation v_o be 0.25. In a first-price sealed-bid auction under the pure common value framework, it can be shown that $\beta_r(x) = \frac{3r^3}{8x^2} + \frac{5x}{9}$ for $(x,r) \in [\frac{3r}{2}, 1] \times [0, \frac{2}{3}]$, and

$$\mathbb{E}[\pi_S(r)] = \begin{cases} -\frac{243}{64}r^4 + \frac{63}{32}r^3 + \frac{5}{12} & r \in [0, \frac{2}{3}] \\ \frac{1}{4} & r \in (\frac{2}{3}, 1] \end{cases}, \ \mathbb{E}[\pi_B(r)] = \begin{cases} \frac{81}{64}r^4 - \frac{9}{8}r^3 + \frac{1}{12} & r \in [0, \frac{2}{3}] \\ 0 & r \in (\frac{2}{3}, 1] \end{cases},$$

where $\mathbb{E}[\pi_S(r)]$ and $\mathbb{E}[\pi_B(r)]$ denote the seller's expected revenue and the bidders' expected surplus under reserve price r, respectively. $\mathbb{E}[\pi_S(r)]$ is maximized at $r = 0.39$ with value 0.446. The expected total welfare, defined as $\mathbb{E}[\pi_S(r)] + \mathbb{E}[\pi_B(r)]$, is maximized at $r = v_o = 0.25$ with value 0.503. If r increases from 0.25 to 0.39, there will be a 3.3% increase in the seller's expected revenue, accompanied by a 1.9% loss in the expected total welfare.

In addition, we emphasize the important implications of model specification on the policy parameters. To illustrate, we compare the two policy parameters in the pure common value framework and in the private value framework in the setup of Example 1.2.1. It can be shown that in the private value framework, $\beta_r(x) = \frac{2}{3}x + \frac{r^3}{3x^2}$ for $x \in [r, 1]$, and

$$\mathbb{E}[\pi_S(r)] = -\frac{3}{2}r^4 + \frac{5}{4}r^3 + \frac{1}{2}, \ \mathbb{E}[\pi_B(r)] = \frac{3}{4}r^4 - r^3 + \frac{1}{4}, r \in [0,1].$$

Different curves are plotted in Figure 1.1. If the true framework is the pure common value model but incorrectly specified as a private value one, the maximized seller's expected revenue will move from point A to C with a 27% loss.

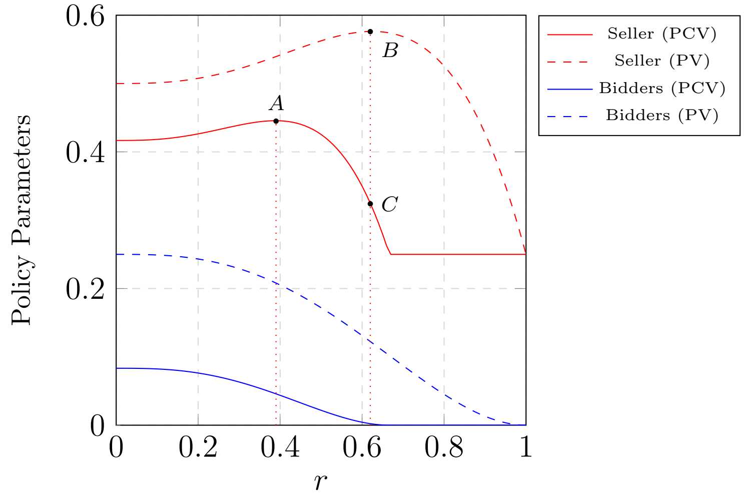

Figure 1.1: Reserve Price and Policy Parameters under Different Frameworks

Let v_o be the seller's own valuation of the object, in general, we can follow a similar idea as in Li, Perrigne, and Vuong (2003) to write the above two policy parameters in terms of observed quantities. We make the following assumption.

Assumption (CI): $\overline{H}(x)$ and $\overline{L}(x)$ are continuous and strictly increasing on $[0, \overline{x}]$.

Proposition 1.2.2 In a first-price sealed-bid pure common value auction, given any reserve price $r \in [\overline{L}(0), \overline{L}(\overline{x})]$, the seller's expected revenue and the bidders' expected surplus are

$$\mathbb{E}[\pi_S(r)] = v_o \mathbb{E}\left[\mathbb{1}(B^{(M)} < b_r^*)\right] + \mathbb{E}[\pi_P(r)], \tag{1.5}$$

$$\begin{aligned} \mathbb{E}[\pi_B(r)] &= \mathbb{E}\left[\mathbb{1}(M_1 \geq b_r^*)\mathbb{1}(B_1 \leq M_1)\overline{L}(\beta^{-1}(M_1))\right] \\ &+ \mathbb{E}\left[\mathbb{1}(B_1 \geq b_r^*)\mathbb{1}(M_1 \leq B_1)\overline{L}(\beta^{-1}(B_1))\right] - \mathbb{E}[\pi_P(r)], \end{aligned} \tag{1.6}$$

where $\mathbb{E}[\pi_P(r)] = \mathbb{E}\left[\left(B^{(M)} + (r - b_r^*)J^*(b_r^*|B^{(M)})\right)\mathbb{1}(B^{(M)} \geq b_r^*)\right]$ is the expected payment from the bidders when the object is sold. $B^{(M)} = B_1 \vee M_1$ is the maximum bid, $J^*(b_r^*|B^{(M)}) = \exp\left(-\int_{b_r^*}^{B^{(M)}} \rho_{M_1|B_1}(t)dt\right)$, $b_r^* = \beta(x_r^*)$, and $\beta^{-1}(b) = \overline{H}^{-1}(b + \frac{1}{\rho_{M_1|B_1}(b)})$.

Proof. See Appendix A. ■

Remark 1.2.3 Given a random sample of equilibrium bids, $\mathbb{E}[\pi_S(r)]$ and $\mathbb{E}[\pi_B(r)]$ will be nonparametrically identified if $\overline{H}(x), \overline{L}(x)$, and b_r^* are known. By construction, b_r^* solves

$$\overline{H}(x_r^*) = b_r^* + \frac{1}{\rho_{M_1|B_1}(b_r^*)}. \quad (1.7)$$

The definition of x_r^* implies that solving b_r^* requires information on the functions $\overline{H}(x)$ and $\overline{L}(x)$. Therefore, for both $\mathbb{E}[\pi_S(r)]$ and $\mathbb{E}[\pi_B(r)]$, the essential unknowns are the two conditional expected valuation functions $\overline{H}(x)$ and $\overline{L}(x)$. In the next section, we show that they are nonparametrically identified under a weak assumption on the joint distribution of the common value and the private signals. This implies that the seller's expected revenue and the bidders' expected surplus under any reserve price are nonparametrically identified.

1.3 Nonparametric Identification

1.3.1 Identification in the First-Price Sealed-Bid Pure Common Value Auction

First-price sealed-bid auctions are prevalent in the real world. Examples include the U.S. OCS wildcat auctions (Li, Perrigne, and Vuong (2000, 2003), Hendricks, Pinkse, and Porter (2003)), the U.S. highway procurement auctions (Li and Zheng, 2009), and the competitive sales of the U.S. municipal bonds (Tang, 2011). In this section, we show that in the first-price sealed-bid pure common value auction, both the seller's expected revenue and the bidders' expected surplus under any reserve price are nonparametrically identified under a weak assumption on the joint distribution of the common value and the private signals.

The basic idea of our identification approach is as follows. From the analysis in Section 1.2.3, the essential unknown quantities in evaluating the two policy parameters are the two conditional expected valuation functions, which are functionals of the latent full joint distribution. In general, the common value is unobserved thus posing difficulty for identifying the two conditional expected valuation functions. If we can reduce these two functions as functionals of the latent joint distribution $F_{\underline{X}}(\cdot)$ of private signals, then Sklar's theorem (see Nelsen (2006)) gives

$$F_{\underline{X}}(x_1, \cdots, x_M) = C_o(F_o(x_1), F_o(x_2), \cdots, F_o(x_M)), \tag{1.8}$$

where $C_o(\cdot)$ denotes the true copula function of private signals, and we use $F_o(\cdot)$ to denote the true marginal distribution function of any private signal to minimize notation. Given the continuity of $F_o(x)$ implied by the absolute continuity of $(X_o, \underline{X})$, the copula function is unique. Since the observed bids are continuous and strictly increasing transformations of the private signals, it is straightforward to show that the copula function of the private signals is the same as the copula function of the observed bids. As a result, the copula function of private signals is directly identified from the sample. The only unknown quantity is the marginal distribution function, but the first-order condition in either (1.2) or (1.4) can be used as a restriction to reduce the space of marginal distribution functions that the true one lies in. If the restricted space turns out to be a singleton, we achieve point identification of the marginal distribution function. To reduce the dimension such that the two conditional expected valuation functions are functionals of $F_{\underline{X}}(\cdot)$, it is most natural to assume the following.

Assumption (AS) $X_o = \frac{1}{M}\sum_{m=1}^{M} X_m + \epsilon$, where $\epsilon \perp \underline{X}$ and $\mathbb{E}[\epsilon] = 0$.[5]

The idea of Assumption (AS) is that all the bidders' partial information pulled together provides a good estimate of the common value. For example, bidders in the OCS wildcat oil-drilling auction might conduct their own seismic surveys of an oil tract, bidders in an automobile auction might bring their own mechanics to learn about conditions of used cars. In such examples, bidders' signals are equally informative. Each bidder forms an imprecise estimate of the common value, which is likely to be the average of these partial information up to some stochastic error term. Previous

[5]Alternatively, we can assume an independent multiplicative error, that is, $X_o = (\frac{1}{M}\sum_{m=1}^{M} X_m)\epsilon$, where $\epsilon \perp \underline{X}$ and $\mathbb{E}[\epsilon] = 1$. Moreover, from our derivation in Appendix A, the simple mean function in Assumption (AS) can be extended to other known function of $\underline{X}$, but the resulting integral equation that restricts the quantile function of private signals could be nonlinear.

papers that have used the average formulation include Klemperer (1998), Goeree and Offerman (2002, 2003), where a stronger form that $X_o = \frac{1}{M}\sum_{m=1}^{M} X_m$ was assumed.

Remark 1.3.1 Assumption (AS) implies that the conditional mean of the common value is the simple average of private signals, that is, $\mathbb{E}[X_o|\underline{X}] = \frac{1}{M}\sum_{m=1}^{M} X_m$. This implication is in a similar spirit of the normalization assumption made in Hendricks, Pinkse, and Porter (2003) when they examine if the bidders bid less aggressively on tracts with more bidders.

Lemma 1.3.2 Under Assumption (AS),

$$\overline{H}(x) \equiv \overline{H}(x; F_o, C_o) = x - \frac{M-2}{M} \frac{\int_0^x C_{o,12}(F_o(x), F_o(x), F_o(t), F_o(x), \cdots, F_o(x))dt}{C_{o,12}(F_o(x), \cdots, F_o(x))},$$
$$\overline{L}(x) \equiv \overline{L}(x; F_o, C_o) = x - \frac{M-1}{M} \frac{\int_0^x C_{o,1}(F_o(x), F_o(t), F_o(x), \cdots, F_o(x))dt}{C_{o,1}(F_o(x), \cdots, F_o(x))},$$

where $F_o(x)$ is the distribution function of X_1, $C_o(\underline{u})$, $\underline{u} = (u_1, \cdots, u_M)$, is the true copula function of the private signals, $C_{o,1}(\underline{u}) = \partial C_o(\underline{u})/\partial u_1$, and $C_{o,12}(\underline{u}) = \partial^2 C_o(\underline{u})/\partial u_1 \partial u_2$.

Proof. See Appendix A. ■

Now $\overline{H}(x)$ and $\overline{L}(x)$ are known up to the marginal distribution function $F_o(x)$. If we are willing to make an assumption that the marginal distribution function is known, and in particular, is the uniform distribution on the unit interval as in Somaini (2015), then the two conditional expected valuation functions are identified. However, we will show in the following that under Assumption (AS), the first-order condition in (1.4) actually imposes a restriction on the marginal distribution function and we cannot arbitrarily assume a known marginal distribution.

Due to the fact that $B_1 = \beta(X_1)$, we have $\beta(x) = Q_{B_1}(F_o(x))$, where $Q_{B_1}(\cdot)$ is the quantile function of B_1. Combining this with the transformed first-order condition in (1.4), we obtain $R_1(x; F_o, C_o) = 0$, where

$$R_1(x; F, C_o) = \overline{H}(x; F, C_o) - Q_{B_1}(F(x)) - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(F(x)))}. \tag{1.9}$$

The relation $R_1(x; F_o, C_o) = 0$ imposes a restriction on the distribution function $F_o(x)$.[6] We show in the following theorem that $F_o(x)$ or equivalently, the quantile function $Q_o(\tau) = F_o^{-1}(\tau), \tau \in [0,1]$ of private signals, is nonparametrically identified by this restriction under Assumption (CU-1) below. Therefore, both $\overline{H}(x)$ and $\overline{L}(x)$ are nonparametrically identified and as a result, $\mathbb{E}[\pi_S(r)]$ and $\mathbb{E}[\pi_B(r)]$ are nonparametrically identified.

Let

$$\phi_{1o}(\tau) = \frac{M}{2}\left(Q_{B_1}(\tau) + \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))}\right), \; k_{1o}(\tau, s) = -\frac{(M-2)}{2} z_{1o}(\tau, s),$$

where $z_{1o}(\tau, s) = \frac{C_{o,123}(\tau,\tau,s,\tau,\cdots,\tau)}{C_{o,12}(\tau,\cdots,\tau)}$ and $C_{o,123}(\underline{u}) = \partial^3 C_o(\underline{u})/\partial u_1 \partial u_2 \partial u_3$. We further make the following assumption.

Assumption (CU-1) $\phi_{1o}(\tau)$ is continuous on $[0,1]$, $k_{1o}(\tau, s)$ is continuous on $0 \le s \le \tau \le 1$.

Theorem 1.3.3 In the first-price sealed-bid pure common value auction model, under Assumptions (AS) and (CU-1), the true quantile function $Q_o(\tau)$ of private signal is nonparametrically identified as the unique solution to the following Volterra integral equation of the second kind,

$$Q(\tau) - \int_0^\tau k_{1o}(\tau, s) Q(s) ds = \phi_{1o}(\tau). \tag{1.10}$$

Proof. see Appendix A.[7] ■

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[6]This is a more general view on the identification of the marginal distribution function (or equivalently the quantile function). In the affiliated private value framework (which nests the independent private value case), the private value can be expressed as a closed form function of observed distributions. As a result, the private value quantile function can be expressed as a closed form function of the observed quantile function of bids (see Marmer and Shneyerov (2012), Fan, Li, and Pesendorfer (2015), and Fan, He, and Li (2015)). In particular, we have $x = b + \frac{1}{\rho_{M_1|B_1}(b)}$ in the affiliated private value framework, where $b = \beta(x)$. Upon the substitution $\beta(x) = Q_{B_1}(F_o(x))$ and change of variable, we get $Q_o(\tau) = Q_{B_1}(\tau) + \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))}$.

[7]A special case occurs when $M = 2$. In this case, identification of the quantile function reduces exactly to that in the affiliated private value framework. This is because $\overline{H}(x) = x$ when $M = 2$

Remark 1.3.4 In the nonparametric instrumental regression problem, a Fredholm integral equation of the first kind is typically involved. In that case, the inverse problem is ill-posed and a regularized estimator is needed, see Darolles, Fan, Florens, and Renault (2011). In contrast, the above Volterra integral equation of the second kind is well-posed in the sense that the inverse operator $(I - K_{1o})^{-1}$ of $I - K_{1o}$ exists and is a continuous, where $K_{1o}Q(\tau) = \int_0^\tau k_{1o}(\tau, s)Q(s)ds$ (see more discussion in Section 1.4.2). Given that $Q_o(\cdot)$ is nonparametrically identified, the joint distribution function $F_{\underline{X}}(\cdot)$ is nonparametrically identified by (1.8). As a result, the two conditional expected valuation functions are nonparametrically identified by Lemma 1.3.2 and the seller's expected revenue and the bidders' expected surplus under any reserve price are nonparametrically identified by Proposition 1.2.2. In addition, our identification approach can be extended to the case when only the highest two bids are observed as in Fan, He, and Li (2015). By assuming the copula function to be in a nonparametric Archimedean class with weak requirement, the results in Theorem 3.2 of Fan, He, and Li (2015) can be used to identify the copula function, and our Theorem 1.3.3 above implies identification of the joint distribution of private signals and thus the two policy parameters.

Example 1.3.5 Consider the same setup as in Example 1.2.1. The equilibrium bidding strategy in a first-price sealed-bid auction is $\beta(x) = 5x/9$ for $x \in [0, 1]$. In this case, $\rho_{M_1|B_1}(b) = 2/b$ for $b \in [0, 5/9]$, $Q_{B_1}(\tau) = 5\tau/9$ for $\tau \in [0, 1]$, and $\overline{H}(x; F, C_\perp) = x - \int_0^x F(t)dt/[3F(x)]$, where $C_\perp$ denotes the independence copula. The Volterra integral equation becomes

$$Q(\tau) + \frac{1}{2\tau}\int_0^\tau Q(s)ds = \frac{5\tau}{4},$$

in the pure common value model under Assumption (AS), which is the same as that in the private value model. This implies that the two models will generate the same equilibrium bidding function under no reserve price and thus the same observed distributions. However, this does not imply that they will generate the same equilibrium bidding functions under any reserve price and as a consequence, the seller's expected revenue and the bidders' expected surplus under any reserve price will be different. In fact, using the setup of Example 1.2.1, a simple calculation would reveal that the two models generate very different policy parameters even when $M = 2$.

with the unique solution $Q_o(\tau) = \tau, \tau \in [0, 1]$.

1.3.2 Identification in Second-Price Sealed-Bid Pure Common Value Auction

We consider the second-price sealed-bid auction in this section. In the pure common value framework, it is well known that the equilibrium bidding function is $\beta(x) = \overline{H}(x)$, $x \in [0, \overline{x}]$. Under Assumption (AS), we can make use of the relation $\beta(x) = Q_{B_1}(F_o(x))$ and write

$$R_2(x; F, C_o) = \overline{H}(x; F, C_o) - Q_{B_1}(F(x)). \tag{1.11}$$

Then the function $F_o(\cdot)$ satisfies the restriction $R_2(x; F_o, C_o) = 0$. We show in the following theorem that in the second-price sealed-bid auction, the quantile function $Q_o(\cdot)$ of private signal is also nonparametrically identified by an integral equation similar to that in Theorem 1.3.3. Let

$$\phi_{2o}(\tau) = \frac{MQ_{B_1}(\tau)}{2},\ k_{2o}(\tau, s) = k_{1o}(\tau, s),$$

where $k_{1o}(\tau, s)$ is defined in Theorem 1.3.3. We make the following assumption which is similar to Assumption (CU-1).

Assumption (CU-2) $\phi_{2o}(\tau)$ is continuous on $[0, 1]$, $k_{2o}(\tau, s)$ is continuous on $0 \leq s \leq \tau \leq 1$.

Theorem 1.3.6 In a second-price sealed-bid pure common value auction, under Assumptions (AS) and (CU-2), the true quantile function $Q_o(\tau)$ of private signal is nonparametrically identified as the solution to the following Volterra integral equation of the second kind,

$$Q(\tau) - \int_0^\tau k_{2o}(\tau, s)Q(s)ds = \phi_{2o}(\tau), \tag{1.12}$$

Proof. Follows a similar argument as in the proof of Theorem 1.3.3. ■

Example 1.3.7 We continue Example 1.3.5 but in a second-price sealed-bid auction. The equilibrium bidding strategy is $\beta(x) = \frac{M+2}{2M}x$ for $x \in [0, 1]$. In this case, $\rho_{M_1|B_1}(b) =$

$(M-1)/b$ for $b \in [0, \frac{M^2+M-2}{2M^2}]$, $Q_{B_1}(\tau) = \frac{M+2}{2M}\tau$ for $\tau \in [0,1]$, and $\overline{H}(x; F, C_\perp) = x - (M-2)\int_0^x F(t)dt/[MF(x)]$. The Volterra integral equation becomes

$$Q(\tau) + \frac{M-2}{2\tau}\int_0^\tau Q(s)ds = \frac{(M+2)\tau}{4}.$$

It is easy to verify that the unique solution is $Q_o(\tau) = \tau, \tau \in [0,1]$.

In a second-price sealed-bid auction, as shown in Milgrom and Weber (1982), the equilibrium bidding strategy under reserve price $r \in [\overline{L}(0), \overline{L}(\overline{x})]$ is $\beta_r(x) = \overline{H}(x), x \in [x_r^*, \overline{x}]$ and $\beta_r(x) < r, x \in [0, x_r^*)$. We show in the following proposition that identification of the two conditional expected valuations are sufficient for identifying the seller's expected revenue and the bidders' expected surplus under any reserve price.

Proposition 1.3.8 In a second-price sealed-bid pure common value auction, given any reserve price $r \in [\overline{L}(0), \overline{L}(\overline{x})]$, the seller's expected revenue and the bidders' expected surplus are

$$\mathbb{E}[\pi_S(r)] = v_o\mathbb{E}[\mathbb{1}(B^{(M)} < \overline{H}(x_r^*))] + \mathbb{E}[\pi_P(r)], \tag{1.13}$$

$$\begin{aligned} \mathbb{E}[\pi_B(r)] &= \mathbb{E}\left[\mathbb{1}(M_1 \geq \overline{H}(x_r^*))\mathbb{1}(B_1 \leq M_1)\overline{L}(\overline{H}^{-1}(M_1))\right] \\ &+ \mathbb{E}\left[\mathbb{1}(B_1 \geq \overline{H}(x_r^*))\mathbb{1}(M_1 \leq B_1)\overline{L}(\overline{H}^{-1}(B_1))\right] - \mathbb{E}[\pi_P(r)], \end{aligned} \tag{1.14}$$

where $\mathbb{E}[\pi_P(r)] = r\mathbb{E}[\mathbb{1}(B^{(M-1)} < \overline{H}(x_r^*) \leq B^{(M)})] + \mathbb{E}[B^{(M-1)}\mathbb{1}(B^{(M-1)} \geq \overline{H}(x_r^*))]$ is the expected payment from the bidders when the object is sold, $B^{(M)}$ and $B^{(M-1)}$ are the highest and second-highest bids, respectively.

Proof. See Appendix A. ■

1.3.3 Extension to the Case with Observed Ex-Post Common Value

In certain cases, the researcher might observe the ex-post common value or can construct some engineer's ex-post estimate of the common value, see Hendricks, Pinkse, and Porter (2003) for an example. Here we abstract away from the potential measurement error issue and assume that the researcher observes the exact ex-post common value. The natural question rises: can this information aid identification in the pure common value auction models without the normalization in Assumption (AS)? The answer is yes and we discuss this situation in details in the following.

First, consider the first-price sealed-bid auction, we know from Proposition 1.2.2 that the unknown quantities are $\overline{H}(x)$, $\overline{L}(x)$, and b_r^*, where b_r^* is subject to $\overline{H}(x_r^*) = b_r^* + [\rho_{M_1|B_1}(b_r^*)]^{-1}$. For simplicity, we assume that $\overline{L}(x)$ is continuous, then $x_r^* = \overline{L}^{-1}(r)$, and b_r^* is subject to

$$\overline{H}(\overline{L}^{-1}(r)) = b_r^* + \frac{1}{\rho_{M_1|B_1}(b_r^*)}. \tag{1.15}$$

We want to use the information on X_o to learn about $\overline{H}(x)$ and $\overline{L}(x)$. To this end, write $\overline{H}(x) = \overline{H}^*(\beta(x))$, $\overline{L}(x) = \overline{L}^*(\beta(x))$, where

$$\overline{H}^*(b) \equiv \mathbb{E}[X_o|B_1 = M_1 = b] = \mathbb{E}[X_o|X_1 = Y_1 = \beta^{-1}(b)] = \overline{H}(\beta^{-1}(b)) = b + \frac{1}{\rho_{M_1|B_1}(b)},$$

$$\overline{L}^*(b) \equiv \mathbb{E}[X_o|B_1 = b, M_1 \leq b] = \mathbb{E}[X_o|X_1 = \beta^{-1}(b), Y_1 \leq \beta^{-1}(b)] = \overline{L}(\beta^{-1}(b)).$$

Both $\overline{H}^*(b)$ and $\overline{L}^*(b)$ are nonparametrically identified from the sample. It can be easily shown that $\overline{H}^{-1}(\overline{H}^*(b)) = \overline{L}^{-1}(\overline{L}^*(b))$, which in turn yields

$$\overline{H}(\overline{L}^{-1}(x)) = \overline{H}^*(\overline{L}^{*-1}(x)),\ \overline{L}(\overline{H}^{-1}(x)) = \overline{L}^*(\overline{H}^{*-1}(x)). \tag{1.16}$$

Then (1.15) becomes

$$\overline{H}^*(\overline{L}^{*-1}(r)) = b_r^* + \frac{1}{\rho_{M_1|B_1}(b_r^*)} = \overline{H}^*(b_r^*),$$

which implies that $b_r^* = \overline{L}^{*-1}(r)$. From the expressions in (1.5) and (1.6), since b_r^* and $\overline{L}(\beta^{-1}(\cdot))$ are identified, then the seller's expected revenue and the bidders' expected surplus are identified and we summarize it in the following theorem.

Theorem 1.3.9 When the ex-post common value is also observed in a first-price sealed-bid pure common value auction, given any reserve price $r \in [\overline{L}(0), \overline{L}(\overline{x})]$, the seller's expected revenue and the bidders' expected surplus are nonparametrically identified as

$$\mathbb{E}[\pi_S(r)] = v_o\mathbb{E}\left[\mathbb{1}(B^{(M)} < b_r^*)\right] + \mathbb{E}[\pi_P(r)], \tag{1.17}$$

$$\begin{aligned} \mathbb{E}[\pi_B(r)] &= \mathbb{E}\left[\mathbb{1}(M_1 \geq b_r^*)\mathbb{1}(B_1 \leq M_1)\overline{L}^*(M_1)\right] \\ &+ \mathbb{E}\left[\mathbb{1}(B_1 \geq b_r^*)\mathbb{1}(M_1 \leq B_1)\overline{L}^*(B_1)\right] - \mathbb{E}[\pi_P(r)], \end{aligned} \tag{1.18}$$

where $\mathbb{E}[\pi_P(r)] = \mathbb{E}\left[\left(B^{(M)} + (r - b_r^*)J^*(b_r^*|B^{(M)})\right)\mathbb{1}(B^{(M)} \geq b_r^*)\right]$ is the expected payment from the bidders when the object is sold. $B^{(M)} = B_1 \vee M_1$ is the maximum bid, $J^*(b_r^*|B^{(M)}) = \exp\left(-\int_{b_r^*}^{B^{(M)}} \rho_{M_1|B_1}(t)dt\right)$, and $b_r^* = \overline{L}^{*-1}(r)$.

Next, consider the second-price sealed-bid auction, from (1.13) and (1.14), to identify the seller's expected revenue and the bidders' expected surplus, it is sufficient to identify $\overline{H}(x_r^*)$ and $\overline{L}(\overline{H}^{-1}(\cdot))$. In the second-price sealed-bid auction, since $\beta(x) = \overline{H}(x)$, then $\overline{H}^*(b) = \overline{H}(\beta^{-1}(b)) = b$, which implies that

$$\overline{H}(x_r^*) = \overline{H}(\overline{L}^{-1}(r)) = \overline{H}^*(\overline{L}^{*-1}(r)) = \overline{L}^{*-1}(r),\ \overline{L}(\overline{H}^{-1}(x)) = \overline{L}^*(\overline{H}^{*-1}(x)) = \overline{L}^*(x).$$

Therefore, the seller's expected revenue and the bidders' expected surplus are identified and we summarize it in the following theorem.

Theorem 1.3.10 When the ex-post common value is also observed in a second-price sealed-bid pure common value auction, given any reserve price $r \in [\overline{L}(0), \overline{L}(\overline{x})]$, the seller's expected revenue and the bidders' expected surplus are nonparametrically identified as

$$\mathbb{E}[\pi_S(r)] = v_o\mathbb{E}[\mathbb{1}(B^{(M)} < \overline{L}^{*-1}(r))] + \mathbb{E}[\pi_P(r)], \tag{1.19}$$

$$\begin{aligned} \mathbb{E}[\pi_B(r)] &= \mathbb{E}\left[\mathbb{1}(M_1 \geq \overline{L}^{*-1}(r))\mathbb{1}(B_1 \leq M_1)\overline{L}^*(M_1)\right] \\ &+ \mathbb{E}\left[\mathbb{1}(B_1 \geq \overline{L}^{*-1}(r))\mathbb{1}(M_1 \leq B_1)\overline{L}^*(B_1)\right] - \mathbb{E}[\pi_P(r)], \end{aligned} \tag{1.20}$$

where $\mathbb{E}[\pi_P(r)] = r\mathbb{E}\left[\mathbb{1}(B^{(M-1)} < \overline{L}^{*-1}(r) \leq B^{(M)})\right] + \mathbb{E}\left[B^{(M-1)}\mathbb{1}(B^{(M-1)} \geq \overline{L}^{*-1}(r))\right]$ is the expected payment from the bidders when the object is sold, $B^{(M)}$ and $B^{(M-1)}$ are the highest and second-highest bids, respectively.

1.4 Semiparametric Estimation

In this section, we consider estimation of the model primitives, that is, the copula function and the quantile function of private signals. We focus on the case when the ex-post common value is unobserved. To estimate the quantile function of private signals, we first need to estimate the kernel functions $k_{jo}(\tau, s)$ and the functions $\phi_{jo}(\tau)$ for $j = 1, 2$. Recall that $z_{1o}(\tau, s)$ and $z_{2o}(\tau, s)$ appear in the kernel functions and different partial derivatives of the copula function are involved in the definitions of $z_{1o}(\tau, s)$ and $z_{2o}(\tau, s)$. In principle, we could estimate the copula function and its partial derivatives nonparametrically following an idea similar to the local polynomial estimation for a regression function. In practice, however, the number of common-value bidders can be greater than three and auction data typically observed is not very large for a given number of bidders. To avoid the curse of dimensionality, we consider a semiparametric approach. Specifically, we parameterize the copula function and leave the quantile function nonparametric.

1.4.1 Estimation of the Copula Function

First, we consider estimating the copula function and its partial derivatives. We make the following assumption.

Assumption (PC) The true copula function $C_o(\underline{u}) = C_o(u_1, \cdots, u_M)$ lies in a parametric family indexed by $\theta \in \Theta$, with the true parameter θ_o.

Under Assumption (PC), estimating the copula function reduces to estimating the parameter θ_o. Following Oakes (1994) and Genest, Ghoudi, and Rivest (1995), we can estimate θ_o by

$$\widehat{\theta}_L = \arg\max_{\theta \in \Theta} \mathcal{L}(\theta),$$

where

$$\mathcal{L}(\theta) = \sum_{\ell=1}^{L} \log c\left(\widehat{G}_{B_1}(B_{1\ell}), \cdots, \widehat{G}_{B_1}(B_{M\ell}); \theta\right),$$

$c(\underline{u};\theta)$ is the copula density function with parameter θ, $\widehat{G}_{B_1}(b) = \frac{1}{1+L}\sum_{\ell=1}^{L} \mathbb{1}(B_{1\ell} \leq b)$. Notice that $\widehat{G}_{B_1}(b)$ is $L/(1+L)$ times the usual empirical distribution function. This rescaling avoids difficulties caused by the potential unboundedness of $\log c(u_1, \cdots, u_M; \theta)$ when some of the u_ms approach one. Genest, Ghoudi, and Rivest (1995) established the root-n consistency and asymptotic normality of $\widehat{\theta}_L$ (see also Chen, Fan, and Tsyrennikov (2006) for a semiparametric efficient sieve estimator of θ_o). Let $C(\underline{u};\theta) = C(u_1, \cdots, u_M; \theta)$, then we can estimate $C(\underline{u};\theta_o)$ by $C(\underline{u};\widehat{\theta}_L)$, estimate $z_{jo}(\tau, s) \equiv z_j(\tau, s; \theta_o)$ by $z_j(\tau, s; \widehat{\theta}_L)$, where $z_j(\tau, s; \theta) = \frac{C_{123}(\tau,\tau,s,\tau,\cdots,\tau;\theta)}{C_{12}(\tau,\cdots,\tau;\theta)}$, $j = 1, 2$.

One practical question is the choice of the copula family. In pure common value auction models, private signals are assumed to be positively correlated and the level of dependence can range from independent to perfectly positively correlated. In choosing the family of copula functions in practice, a good candidate is the Archimedean family, which is flexible in the sense that it can allow any level of positive dependence. We thus further make the following assumption.

Assumption (AC) The true copula function of the private signals is an Archimedean copula function with strictly decreasing, twice continuously differentiable generator function $\varphi_{\theta_o}(\cdot)$, with its inverse function $\varphi_{\theta_o}^{-1}(\cdot)$ completely monotone on $[0, \infty)$.[8]

Under Assumption (AC), the true copula function is of the form

$$C_o(\underline{u}) = \varphi_{\theta_o}^{-1}\Big[\sum_{m=1}^{M} \varphi_{\theta_o}(u_m)\Big], \tag{1.21}$$

where $\varphi_{\theta_o} : [0, 1] \to [0, \infty)$ with $\varphi_{\theta_o}(1) = 0$. Let $\varphi_{\theta i}(\underline{u}) = \varphi_\theta^{(i)}[\varphi_\theta^{-1}(\sum_{m=1}^{M} \varphi_\theta(u_m))], i = 1, 2, 3$, where $\varphi_\theta^{(i)}(t)$ denotes the i-th partial derivative of $\varphi_\theta(t)$ with respect to t.

[8]Complete monotonicity of $\varphi_{\theta_o}^{-1}(\cdot)$ is a sufficient condition to guarantee that the expression in (1.21) actually generates an M-dimensional Archimedean copula function for any $M \geq 3$. This assumption suffices for the purpose of this paper. For a necessary and sufficient condition on this issue, see McNeil and Neslehova (2009).

Straightforward calculation gives

$$C_1(\underline{u};\theta) = \frac{\varphi_\theta^{(1)}(u_1)}{\varphi_{\theta 1}(\underline{u})}, C_{12}(\underline{u};\theta) = -\frac{\varphi_\theta^{(1)}(u_1)\varphi_\theta^{(1)}(u_2)\varphi_{\theta 2}(\underline{u})}{[\varphi_{\theta 1}(\underline{u})]^3},$$
$$C_{123}(\underline{u};\theta) = \frac{\varphi_\theta^{(1)}(u_1)\varphi_\theta^{(1)}(u_2)\varphi_\theta^{(1)}(u_3)}{[\varphi_{\theta 1}(\underline{u})]^4}\left[-\varphi_{\theta 3}(\underline{u}) + \frac{3[\varphi_{\theta 2}(\underline{u})]^2}{\varphi_{\theta 1}(\underline{u})}\right],$$

and $z_1(\tau, s;\theta)$, $z_2(\tau, s;\theta)$ can be calculated accordingly.

Remark 1.4.1 While we assume parametric Archimedean copula family in this paper for convenience, our estimation method does not rely on this assumption, and it can readily accommodate other choices of parametric copula family in practice.

In addition to the copula parameter θ under Assumption (PC), certain dependence measures are of their own interest as well in practice. For example, in the U.S. OCS wildcat auction, one might be interested in the dependence level among the private signals which could reflect the preciseness of technology in conducting the seismic survey. In automobile auctions, the level of dependence among the private signals could reflect how widely the market information is diffused among the used car dealers. Common measures of dependence level such as Kendall's τ_k and Spearman's ρ are closely related to the copula parameter θ. Consider Kendall's τ_k, one version of the multivariate Kendall's τ_k is simply the average of pairwise Kendall's τ_k.[9] That is,

$$\tau_k(X_1, \cdots, X_M) = \frac{1}{\binom{n}{2}} \sum_{1\le i<j\le M} \tau_k(X_i, X_j).$$

In the symmetric bidders' case, the multivariate Kendall's τ_k is simply $\tau_k(X_i, X_j)$ for any pair (X_i, X_j). Under Assumption (AC), it is known from Corollary 5.1.4 in Nelsen (2006) that

$$\tau_k(\theta) = 1 + 4\int_0^1 \frac{\varphi_\theta(t)}{\varphi_\theta^{(1)}(t)}dt.$$

[9] For an alternative Kendall's τ_k formula in the multivariate case, see Genest, Nešlehová, and Ghorbal (2011).

The copula generator functions and the relation between Kendall's τ_k and θ for three popular parametric families, namely, Clayton, Frank, and Gumbel families, are summarized in Table 1.1 and Figure 1.2. Given estimate $\widehat{\theta}_L$ of θ_o, we can estimate Kendall's τ_k among the private signals.

	$\varphi_\theta(t)$	Kendall's τ_k
Clayton	$\frac{t^{-\theta}-1}{\theta}$	$\frac{\theta}{\theta+2}$
	$\theta \in [0, \infty)$	–
Frank	$-\log \frac{e^{-\theta t}-1}{e^{-\theta}-1}$	$\frac{\theta-4}{\theta} + \frac{4}{\theta^2}\int_0^\theta \frac{t}{e^t-1}dt$
	$\theta \in [0, \infty)$	–
Gumbel	$(-\log t)^\theta$	$\frac{\theta-1}{\theta}$
	$\theta \in [1, \infty)$	–

Table 1.1: Generator and Kendall's τ_k

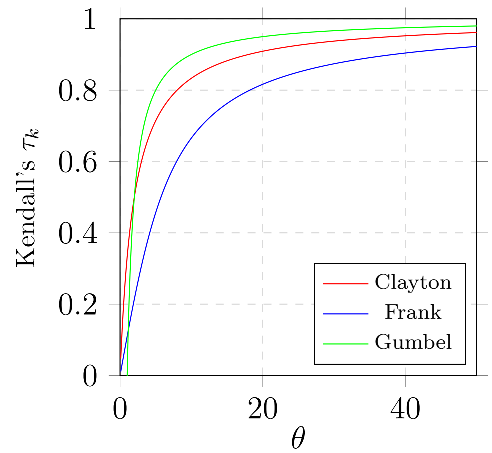


Figure 1.2: $\tau_k(\theta)$

The ranges of $\tau_k(\theta)$ are $[0, 1]$ for the above three families. Hence the Archimedean copula families are rich enough to accommodate any level of positive dependence for empirical works. In sum, Assumptions (PC) and (AC) reduce the dimension of the space of copula functions without much loss of generality. In a different context, Fan and Liu (2013) demonstrated that the estimation results are robust against mis-specification of the copula family (see also Zheng and Klein (1995), Huang and Zhang (2008), Chen (2010), and Hubbard, Li, and Paarsch (2012)). We will also show this robustness in both the simulation in the following and the empirical application in Chapter 2.

1.4.2 Estimation of the Signal Quantile Function

In this section, we consider estimating the quantile function $Q_o(\tau)$. The first-price and second-price pure common value auction models in Sections 1.3.1 and 1.3.2 induce the same linear inverse problem. That is, $Q_o(\tau)$ is subject to the restriction

$$(I - K_{jo})Q_o(\tau) = \phi_{jo}(\tau), j = 1, 2, \tag{1.22}$$

where I is the identity operator and $K_{jo}Q(\tau) = \int_0^\tau k_{jo}(\tau, s)Q(s)ds$, $j = 1, 2$, is a linear operator. For an excellent review on the linear inverse problem in structural econometrics, see Florens (2003) and Carrasco, Florens, and Renault (2007). Under Assumptions (CU-1) or (CU-2), the Volterra integral operator K_{jo} does not have nonzero spectral values. This implies that 1 is not an eigenvalue of K_{jo} (see Kress (1999)). Therefore, $I - K_{jo}$ is invertible and it admits a linear continuous inverse $(I - K_{jo})^{-1}$. Unlike the linear inverse problem encountered in nonparametric instrumental regression problem such as that in Darolles, Fan, Florens, and Renault (2011), our linear inverse problem is well-posed and regularization is not needed.

Denote the estimator of K_{jo} as $\widehat{K}_j$, where

$$\widehat{K}_j Q(\tau) = \int_0^\tau \widehat{k}_j(\tau, s)Q(s)ds,$$

and $\widehat{k}_j(\tau, s)$ is the plug-in estimator of $k_{jo}(\tau, s)$ given the estimator $\widehat{\theta}_L$ of θ_o. Let the estimator of $\phi_{jo}(\tau)$ be $\widehat{\phi}_j(\tau)$, then the estimator of $Q_o(\tau)$ is defined as

$$\widehat{Q}(\tau) = (I - \widehat{K}_j)^{-1}\widehat{\phi}_j(\tau). \tag{1.23}$$

Intuitively, if $\widehat{K}_j$ is close to K_{jo} w.p.a.1, then $(I - \widehat{K}_j)^{-1}$ is close to $(I - K_j)^{-1}$ w.p.a.1. Further, the eigenvalues of $\widehat{K}_j$ should be close to the eigenvalues of K_{jo}. Therefore, 1 is not an eigenvalue of $\widehat{K}_j$ w.p.a.1. and $(I - \widehat{K}_j)^{-1}$ is continuous w.p.a.1. If $\widehat{\phi}_j$ is also close to ϕ_{jo}, then $(I - \widehat{K}_j)^{-1}\widehat{\phi}_j(\tau)$ should be close to $Q_o(\tau)$.

Consider first the estimation of $Q_o(\tau)$ in the first-price sealed-bid pure common value auction model. Recall that

$$K_{1o}Q(\tau) = -\frac{(M-2)}{2}\int_0^\tau z_{1o}(\tau, s)Q(s)ds,\ \phi_{1o}(\tau) = \frac{M}{2}\left(Q_{B_1}(\tau) + \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))}\right).$$

We use the standard empirical quantile function to estimate $Q_{B_1}(\tau)$, let $\widehat{Q}_{B_1}(\tau) = B_{1\lceil L\tau\rceil}$, where $\lceil a \rceil$ is the smallest integer greater than or equal to a. For the estimation of $\rho_{M_1|B_1}(b)$, we follow Li, Perrigne, and Vuong (2002) and Haile, Hong, and Shum (2006) to use

$$\widehat{\rho}_{M_1|B_1}(b) = \frac{\widehat{g}_{M_1 B_1}(b)}{\widehat{G}_{M_1\times B_1}(b)},$$

where

$$\widehat{G}_{M_1\times B_1}(b) = \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{M}\sum_{m=1}^{M}\mathbb{1}(M_{m\ell}\leq b)k_{G,h_G}(B_{m\ell}-b),$$
$$\widehat{g}_{M_1B_1}(b) = \frac{1}{L}\sum_{\ell=1}^{L}\frac{1}{M}\sum_{m=1}^{M}k_{g,h_g}(M_{m\ell}-b)k_{g,h_g}(B_{m\ell}-b),$$

and $k_{G,h_G}(x) = k_G(\frac{x}{h_G})/h_G, k_{g,h_g}(x) = k_g(\frac{x}{h_g})/h_g$. Here, $k_G(\cdot)$ and $k_g(\cdot)$ are two kernel density functions and $h_G, h_g \to 0$ are two bandwidth sequences. Let

$$\widehat{K}_1Q(\tau) = -\frac{M-2}{2}\int_0^{\tau} z_1(\tau,s;\widehat{\theta}_L)Q(s)ds,\ \widehat{\phi}_1(\tau) = \frac{M}{2}\Big(\widehat{Q}_{B_1}(\tau) + \frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(\tau))}\Big),$$

then $Q_o(\tau)$ is estimated by

$$\widehat{Q}(\tau) = (I-\widehat{K}_1)^{-1}\widehat{\phi}_1(\tau).$$

We make the following regularity assumptions.

Assumption (KS) $k_G(\cdot), k_g(\cdot)$ are symmetric density functions with bounded support and have continuous and bounded first derivatives.

Assumption (BS) $h_G = c_G(\log L/L)^{\frac{1}{2d+2M-3}}$, $h_g = c_g(\log L/L)^{\frac{1}{2d+2M-2}}$ for some constants c_G, c_g, and d is the differentiability order of the private signals' joint distribution.

Assumption (UB)

(i) $\rho_{M_1|B_1}(b)$ is uniformly bounded away from zero on $[\underline{b},\overline{b}]$;

(ii) $g_{M_1B_1}(b)$ is uniformly bounded away from zero on $[\underline{b},\overline{b}]$;

(iii) $G_{M_1\times B_1}(b) = \partial G_{M_1B_1}(m_1,b_1)/\partial b_1|_{m_1=b_1=b}$ is uniformly bounded from above on $[\underline{b},\overline{b}]$.

Assumption (RA-1) Given $\epsilon>0$,

(i) $w_{1\epsilon}(\theta) = \sup\limits_{t\in[\epsilon,1-\epsilon]}\int_0^1|z_1(t,s;\theta)-z_{1o}(t,s)|ds$ is continuous at θ_o;

(ii) $w_{2\epsilon}(\theta) = \sup\limits_{t\in[\epsilon,1-\epsilon]}\int_0^1|z_1(t,s;\theta)-z_{1o}(t,s)|Q_o(s)ds$ is continuous at θ_o;

(iii) $||(I-K_{1o})^{-1}||_\epsilon = \sup_{||\varphi||_\epsilon\neq 0}||(I-K_{1o})^{-1}\varphi||_\epsilon/||\varphi||_\epsilon<\infty$, where $||\varphi||_\epsilon = \sup_{t\in[\epsilon,1-\epsilon]}|\varphi(t)|$.

Assumption (RA-2) $w_{10}(\theta), w_{20}(\theta)$ are continuous at θ_o for $w_{1\epsilon}(\theta), w_{2\epsilon}(\theta)$ in Assumption (RA-1) evaluated at $\epsilon = 0$.

Theorem 1.4.2 In a first-price sealed-bid pure common value auction, under Assumptions (PC), (KS), (BS), (UB), and (RA-1),

$$\sup_{\tau\in[\epsilon,1-\epsilon]} \left|\widehat{Q}(\tau) - Q_o(\tau)\right| = o_P(1).$$

Proof. See Appendix A. ■

Next, we consider estimating $Q_o(\tau)$ in the second-price sealed-bid pure common value auction model. Recall that

$$K_{2o}Q(\tau) = -\frac{M-2}{2}\int_0^\tau z_{1o}(\tau, s)Q(s)ds, \ \ \phi_{2o}(\tau) = \frac{MQ_{B_1}(\tau)}{2}.$$

We estimate $Q_o(\tau)$ by

$$\widehat{Q}(\tau) = (I - \widehat{K}_2)^{-1}\widehat{\phi}_2(\tau),$$

where

$$\widehat{K}_2 Q(\tau) = -\frac{M-2}{2}\int_0^\tau \widehat{z}_1(\tau, s; \widehat{\theta}_L)Q(s)ds, \ \ \widehat{\phi}_2(\tau) = \frac{M\widehat{Q}_{B_1}(\tau)}{2}.$$

Theorem 1.4.3 In a second-price sealed-bid pure common value auction, under Assumptions (CU-2), (PC), and (RA-2),

$$\sup_{\tau\in[0,1]} \left|\widehat{Q}(\tau) - Q_o(\tau)\right| = o_P(1).$$

Proof. See Appendix A. ■

1.5 Simulation

In this section, we conduct a Monte Carlo simulation to evaluate the finite sample performance of our estimators. We focus on the first-price sealed-bid pure common value auction in both the simulation and the empirical application. In the simulation designs, we set the number of bidders to be 3 and let the true copula function be the independent copula. For the marginal distribution function of private signals, we follow Marmer and Shneyerov (2012) to let $F_o(x) = x^\alpha, x \in [0,1]$ for $\alpha = 0.5, 1, 2$. Table 1.5 summarizes the true quantile functions and equilibrium bidding functions in each design. Three different sample sizes, $L = 100, 200,$ and 500, are used. The number of repetitions is set to be 1000.

α	$Q_o(\tau)$	$\beta(x)$
0.5	τ^2	$\frac{7}{72}(8x - 4\sqrt{x} - e^{-4\sqrt{x}} + 1)$
1	τ	$5x/9$
2	$\tau^{\frac{1}{2}}$	$\frac{16}{9} e^{\frac{2}{x}} \int_{2/x}^{\infty} \frac{e^{-t}}{t} dt$

Table 1.2: Equilibrium Bidding Functions under Different Marginal Distributions

In estimating the copula parameter, we employ three popular copula families, namely the Clayton, Frank, and Gumbel families. Each of them nests the independent copula as a special case. In estimating the observed conditional reserve hazard function, we follow Li, Perrigne, and Vuong (2002) to use the triweight kernel for both $k_G(\cdot)$ and $k_g(\cdot)$,

$$k_G(s) = k_g(s) = \frac{35}{32}(1 - s^2)^3 \mathbb{1}(|s| \leq 1).$$

Bandwidth choice of h_G, h_g follows from Assumption (BS) with $d = 3$. The constants in bandwidth are set to be $c_G = c_g = 2.978 \times 1.06\widehat{\sigma}_B$ due to our choice of triweight kernel, where $\widehat{\sigma}_B$ is the empirical standard deviation of the bids.

For estimation of the quantile function, using the relation $Q_o(\tau) = (I - K_{1o})^{-1}\phi_{1o}(\tau)$ and replacing the unknown true quantities with their estimators, we get

$$\widehat{Q}(\tau) = (I - \widehat{K}_1)^{-1}\widehat{\phi}_1(\tau) = \sum_{j=0}^{\infty} \widehat{K}_1^j \widehat{\phi}_1(\tau). \tag{1.24}$$

We name this geometric series estimator (GSE).[10] In the implementation, let $\widehat{Q}^{(J)}(\tau) =$

[10]In the implementation of both the geometric series estimator and the iterative sieve estimator

$\sum_{j=0}^{J} \widehat{K}_1^j \widehat{\phi}_1(\tau)$, and we use the following convergence criterion: stop the iteration if

$$\frac{\sum_{p=1}^{P}\left[\widehat{Q}^{(J+1)}(x_p) - \widehat{Q}^{(J)}(x_p)\right]^2}{\sum_{p=1}^{P}\left[\widehat{Q}^{(J)}(x_p)\right]^2 + 0.0001} < 0.001,$$

where $x_p, p = 1, \cdots, P$ are evaluation points. Previous works using this criterion include Nielsen and Sperlich (2005), Henderson et al. (2008), Mammen et al. (2009), and Su and Lu (2013). Following Su and Lu (2013), we choose 100 evaluation points evenly distributed between the 0.2 and 0.8 quantiles of the private signals. The evaluation points are fixed across repetitions. In our simulation designs, the geometric series estimator typically hits the convergence criterion when $J = 3$.

Furthermore, under Assumption (CU-1), $Q_o(\tau)$ is the unique minimizer of

$$\mathcal{M}(Q) = \int_0^1 [(I - K_{1o})Q(\tau) - \phi_{1o}(\tau)]^2 d\tau.$$

This suggests another estimator of $Q_o(\tau)$, namely, $\widetilde{Q}(\tau) = \arg\min_Q \widehat{\mathcal{M}}(Q)$, where

$$\widehat{\mathcal{M}}(Q) = \int_0^1 \left[(I - \widehat{K}_1)Q(\tau) - \widehat{\phi}_1(\tau)\right]^2 d\tau. \tag{1.25}$$

In principle, a weighting function in the definition of $\widehat{\mathcal{M}}(Q)$ can be used for different τ to improve efficiency, we choose the unweighted criterion as a baseline and leave the proper choice of weighting function to future research. We estimate $Q_o(\tau)$ by sieve method (see Chen (2007) for an excellent review on the sieve estimation).

Note that the quantile function is defined on $[0, 1]$, and we use the Bernstein polynomial sieve basis. A Bernstein polynomial sieve of order H_L is defined as

$$B_{H_L}(t) = \sum_{j=0}^{H_L} \alpha_j \binom{H_L}{j} t^j (1-t)^{H_L - j}, t \in [0, 1],$$

with Bernstein cofficients $\alpha_j, j = 1, \cdots, H_L$. In the implementation, we follow Gentry, Li, and Lu (2015) to approximate the integral in (1.25) by specifying a discrete grid

below, computing multiple integral is needed. We approximate the integral by dividing the interval $[0, 1]$ into 100 subintervals and compute the discretized sum.

$g_\tau \subset [0,1]$ and use the discretized criterion function

$$\widehat{\mathcal{M}}^d(Q) = \sum_{\tau \in g_\tau} \left[(I - \widehat{K}_1)Q(\tau) - \widehat{\phi}_1(\tau) \right]^2.$$

Due to the bias of $\widehat{\phi}_1(\tau)$ on the boundary, we use a grid with 100 evenly spaced points between the 0.05 and 0.95 quantiles of the private signals. In the simulation, we experimented with different orders and found that the estimator when $H_L = 2$ performs the best.

Let $\widetilde{Q}^{(0)} = \arg\min_Q \widehat{\mathcal{M}}(Q)$. Note that $Q_o(\tau) = K_{1o}Q_o(\tau) + \phi_{1o}(\tau)$, under conditions such that the geometric series estimator is convergent, the sequence of approximations

$$Q^{(J)}(\tau) = K_{1o}Q^{(J-1)}(\tau) + \phi_{1o}(\tau), J = 1, 2, \cdots,$$

is close to $Q_o(\tau)$ from any starting point $Q^{(0)}(\tau)$. In addition, if $\widehat{K}_1$ and $\widehat{\phi}_1(\tau)$ are sufficiently close to K_{1o} and $\phi_{1o}(\tau)$, respectively, then

$$\widetilde{Q}^{(J)}(\tau) = \widehat{K}_1 \widetilde{Q}^{(J-1)}(\tau) + \widehat{\phi}_1(\tau), J = 1, 2, \cdots \tag{1.26}$$

is close to $\widehat{Q}(\tau)$. We name this iterative sieve estimator (ISE). The same convergence criterion as in the geometric series estimator is used and the iterative sieve estimator typically hits the convergence criterion when $J = 2$ in our simulation designs.

In Tables 1.3, 1.4, 1.5, we report the estimated RMSEs of both the geometric series estimator and the iterative sieve estimator at different quantiles of the private signals. We focus on discussing the simulation results in Table 1.4. In this case, $\alpha = 1$ and the true quantile function is the identity function $Q_o(\tau) = \tau, \tau \in [0,1]$. First, for the same sample size, the estimated RMSEs are very close between the geometric series estimator and the iterative sieve estimator. Consider for example the median RMSE when $L = 100$, the two estimated RMSEs are 0.0479 and 0.0486 for the Clayton copula, 0.0477 and 0.0484 for the Frank copula, and 0.0477 and 0.0483 for the Gumbel copula. Second, the robustness against copula specification is confirmed in our simulation. Consider the median RMSE when $L = 200$. For the geometric series

L	100		200		500	
Clayton	GSE	ISE	GSE	ISE	GSE	ISE
25% quantile	0.0295	0.0278	0.0235	0.0209	0.0179	0.0146
50% quantile	0.0613	0.0597	0.0487	0.0463	0.0379	0.0349
75% quantile	0.1344	0.1330	0.1230	0.1211	0.1147	0.1125
Frank						
25% quantile	0.0295	0.0277	0.0235	0.0209	0.0178	0.0145
50% quantile	0.0617	0.0601	0.0490	0.0466	0.0381	0.0351
75% quantile	0.1336	0.1322	0.1224	0.1205	0.1143	0.1121
Gumbel						
25% quantile	0.0295	0.0278	0.0234	0.0209	0.0178	0.0145
50% quantile	0.0618	0.0602	0.0490	0.0467	0.0381	0.0351
75% quantile	0.1343	0.1330	0.1234	0.1215	0.1151	0.1128

Table 1.3: RMSE of GSE and ISE, $\alpha = 0.5$

estimator, the estimated RMSEs are $0.0329, 0.0328, 0.0328$ for the Clayton, Frank and Gumbel copula families, respectively. For the iterative sieve estimator, the estimated RMSEs are $0.0334, 0.0333, 0.0333$ for the three copula families, respectively. Third, as the sample size increases, the estimation precision increases. For example, when the Clayton copula is used, the estimated median RMSEs of the geometric series estimator are $0.0479, 0.0329$, and 0.0227 when $L = 100, 200$, and 500, while the estimated median RMSEs of the iterative sieve estimator are $0.0486, 0.0334$, and 0.0233 when $L = 100, 200$, and 500. The cases when $\alpha = 0.5$ and 2 follow similar patterns and discussions.

1.6 Conclusion

In this chapter, we study the identification problem in the pure common value auction models. We analyze the main challenges in the nonparametric identification

L	100		200		500	
Clayton	GSE	ISE	GSE	ISE	GSE	ISE
25% quantile	0.0457	0.0453	0.0319	0.0320	0.0219	0.0223
50% quantile	0.0479	0.0486	0.0329	0.0334	0.0227	0.0233
75% quantile	0.0917	0.0931	0.0778	0.0788	0.0596	0.0607
Frank						
25% quantile	0.0456	0.0453	0.0319	0.0320	0.0219	0.0223
50% quantile	0.0477	0.0484	0.0328	0.0333	0.0227	0.0232
75% quantile	0.0909	0.0924	0.0773	0.0784	0.0592	0.0603
Gumbel						
25% quantile	0.0456	0.0453	0.0318	0.0319	0.0219	0.0223
50% quantile	0.0477	0.0483	0.0328	0.0333	0.0227	0.0232
75% quantile	0.0916	0.0931	0.0782	0.0793	0.0601	0.0613

Table 1.4: RMSE of GSE and ISE, $\alpha = 1$

of the full joint distribution of the common value and private signals. We argue that identifying the full joint distribution is sufficient but not necessary for certain policy parameters. In particular, we show that in order to identify the expected total welfare, the seller's expected revenue, and the bidders' expected surplus under any reserve price, information on the two conditional expected valuation functions are sufficient.

First, in both the first-price and second-price sealed-bid auction models, we establish nonparametric identification of the two conditional expected valuation functions under a weak assumption on the joint distribution of the common value and the private signals. Identifying these two functions is essentially due to direct identification of the private signals' copula function from the observed bids' copula function and due to identification of the signal's quantile function by a Volterra integral equation of a second kind. As a result, all the three policy parameters are nonparametrically identified in both models. Second, we propose a semiparametric estimation method

L	100		200		500	
Clayton	GSE	ISE	GSE	ISE	GSE	ISE
25% quantile	0.0610	0.0618	0.0486	0.0511	0.0431	0.0452
50% quantile	0.0754	0.0703	0.0690	0.0644	0.0696	0.0641
75% quantile	0.0445	0.0467	0.0318	0.0336	0.0230	0.0194
Frank						
25% quantile	0.0617	0.0617	0.0492	0.0512	0.0436	0.0453
50% quantile	0.0749	0.0691	0.0687	0.0636	0.0695	0.0636
75% quantile	0.0442	0.0462	0.0318	0.0334	0.0233	0.0194
Gumbel						
25% quantile	0.0617	0.0617	0.0492	0.0512	0.0436	0.0454
50% quantile	0.0748	0.0691	0.0686	0.0636	0.0694	0.0636
75% quantile	0.0438	0.0462	0.0314	0.0334	0.0225	0.0190

Table 1.5: RMSE of GSE and ISE, $\alpha = 2$

and establish consistency of the estimator for the signal's quantile function. Monte Carlo experiments are conducted to show the estimator's satisfactory finite sample performances.

Several future research directions can be considered. First, although we assume that the common value is the simple average of the private signals up to some independent stochastic error, it is expected that the simple average can be generalized to other forms. In such situations, the quantile function of private signals is expected to be subject to a similar but possibly nonlinear Volterra integral equation. Given identification of the signal's quantile function, our approach in identifying the seller's expected revenue and the bidders' expected surplus applies. Second, although we focus on the expected value of the seller's revenue and the bidders' surplus in this paper, it would be of interest to study the distribution function of these two quantities. In fact, from the derivation in Appendix A, the seller's revenue under any reserve price, as a random variable, is a known function of the observed quantities. Similar discussion applies to the bidders' surplus when the ex-post common value is observed. These are beyond the scope of the current paper and are left to future research.

Chapter 2

STRUCTURAL ANALYSIS OF U.S. OCS WILDCAT AUCTIONS IN A PURE COMMON VALUE FRAMEWORK

2.1 Introduction

The United States federal government has been selling gas and oil exploration rights on the Outer Continental Shelf (OCS) by auctions since 1954. As documented in Weaver et al. (1973), these auctions have consisted of a significant fraction of U.S. domestic hydrocarbon production. For example, by 1970, 16.7 percent of the U.S. domestic oil and lease condensate production and 15 percent of marketed gas production came from offshore wells. Bids paid by oil firms in these auctions became an important source of the federal government's revenue.

In this chapter, we focus on the OCS wildcat auction on offshore lands off the coasts of Texas and Louisiana. There are three types of oil and gas lease sales: wildcat sales, developmental sales, and drainage sales. A wildcat sale refers to tracts located in previously unexplored areas with unknown geological and seismic characteristics. A developmental sale refers to tracts previously sold but re-offered due to either the government's rejection of the winning bid or the winner's relinquishment. A drainage sale is the sale of tracts in which oil or gas deposits have been found. Among the three types, the wildcat auction fits into the symmetric pure common value framework (Hendricks, Pinkse, and Porter (2003)) for two reasons. First, the future selling price of gas or oil from one tract is the same for different bidders. Second, the exact volume of deposit is unknown to each bidder and no bidder has more information than others, thus bidders can be approximately viewed as symmetric. Therefore, we focus on the wildcat auctions to fit into the framework in Chapter 1.

In the offshore drilling auctions, the presence of winner's curse has been a concern. The winner' curse refers to the situation in which winning an auction would actually incur a loss for the winner. This is due to the fact that one bidder's value of the object being auctioned depends on other bidders' private information. On the one hand, winning the auction represents an update on other bidders' private information. This negative informational update makes the winner overpay for the object whose value is not expected to be high in the future. On the other hand, if the bidders take this informational update into account and bid according to the equilibrium bidding function, there won't be winner's curse effect. The issue that whether the bidders actually follow the equilibrium bidding function was addressed in Hendricks, Pinkse, and Porter (2003). They focused on the OCS wildcat auctions and developed testing procedure for the equilibrium bidding hypothesis. Their main conclusion is that the bidders are aware of the winner's curse and take it into account when forming their bidding strategies.

Given that the bidders follow a symmetric equilibrium bidding strategy in the pure common value framework, we are interested in the design of the auctions in order to achieve certain policy goal. In particular, we want to know the counterfactual welfare implication in terms of the government's expected revenue, the bidders' expected surplus, and the expected total welfare if the government had set the reserve price at different levels. This analysis would enable us to quantify the loss of welfare in terms of different policy parameters under the actual auction design, represented by the actual reserve price, and to quantify the size of room for improvement. Further, the OCS wildcat auction data has been previously analyzed in Li, Perrigne, and Vuong (2000, 2003) in the private value framework. The correct framework for this data is subject to hot debate in the literature, as noted in Hendricks, Pinkse, and Porter (2003): "Bidding behavior appears to be largely consistent with a symmetric pure common value environment,... In contrast, some features of bidding behavior appear to be inconsistent with a private values environment. To repeat, the bidders' valuations in OCS auctions probably have both private and common components, but the common components appear to be important." We analyze the data in the symmetric pure common value framework, and will also conduct similar analysis in the affiliated private value framework for comparison purposes.

2.2 Data Description

One tract is sold in each wildcat auction. A tract is defined as either a block or half a block and a block is usually either 5000 or 5760 acres of land. Potential participants in wildcat sales are allowed to carry out a seismic investigation before the sale date, but they are not allowed to drill any exploratory wells. Each firm evaluates the tracts by analyzing its seismic survey. This provides noisy but roughly equally informative signals about the amount of oil and gas on a lease. In a given sale, all of the announced tracts are sold simultaneously by first-price sealed-bid auctions. The Department of the Interior announces the values of all submitted bids and identities of the firms. The winner of the lease has to pay a nominal rental fee until the production begins, normally $3 per acre per year for wildcat tracts (Porter, 1995). A fixed portion of the revenue from oil production is claimed by the government as royalty payment.

In the wildcat auction, there may be a reservation price of $15 or $25 per acre (Porter, 1995). The reservation price is the same across all tracts within a sale but may be different across sales. The reserve price has long been perceived as too low (see McAfee and Vincent (1992)), thus we follow Li, Perrigne, and Vuong (2000) to view the bids as from auctions with non-binding reserve price. Given our nonparametric identification of the seller's expected revenue under any reserve price, we are interested in whether or not the actual reserve price is indeed too low, and if it is, what is the optimal reserve price that could have generated the maximum expected revenue for the government.

The data is obtained from the website of the Center for the Study of Auctions, Procurements and Competition Policy at Penn State. In the following sections, we focus on the subsets of the auctions held between 1954 and 1970 with two or three bidders, respectively.

2.3 Structural Analysis and Empirical Findings

2.3.1 Two Bidders' Case

In this section, we focus on a subset of the auctions with two bidders. Table 2.1 and Figure 2.1 provide some descriptive statistics of the sub data set.[1] In particular, the bids are concentrated to the left in Figure 2.1 and the histogram implies that the density function of bids decreases as we move towards the right. Given that bids are strictly increasing transformation of private signals, we expect the density function of private signals to have similar pattern.

In the OCS wildcat auction, each bidder has partial information on the unknown exact volume of deposit and their partial information are correlated. First, when firms jointly hire a geophysical company to shoot the seismic survey for a tract, although different firms may have different algorithms and analysts to interpret the survey data, it is expected that their outputs are correlated. Second, even when each firm conducts its own seismic survey, given similar available technologies, the estimates of oil volume of the same area or tract from different firms should be correlated since each one is an estimate of the true volume. It is therefore of interest to know whether their private signals are correlated and the degree of correlation. To this end, we estimate the copula parameter θ as well as Kendall's τ_k and summarize results in Table 2.2. The

[1] The original data set includes 262 auctions with two bidders, where the maximum bid is 52.21 million dollars per tract in 1982$ and the minimum bid is merely 0.000724 million dollars. To avoid the possible contamination of these outliers, we trim the data so that auctions with bids smaller than a certain lower threshold level or larger than a certain upper threshold level are dropped. This trimming makes the auctions in the data set more homogenized. However, this results in a trade-off between more homogenized auctions and smaller sample sizes. After some preliminary analysis, we set the lower threshold to be 0.2 million dollars, and the upper threshold level to be 40 million dollars. This leaves us with 231 auctions.

estimated Kendall's τ_ks are around 0.11 for the Clayton and the Gumbel families, while it is a little larger for the Frank family. The 95% bootstrapped confidence intervals are reported in the parentheses and they reveal that the Kendall's τ_k is statistically significantly different from zero, suggesting a positive dependence among the private signals.

$M = 2$	
#Tracts	231
Mean	1.938
Median	0.800
Min Bid	0.200
Max Bid	22.984
Std	3.227
Million in 1982$ per Tract	

Table 2.1: Summary Statistics

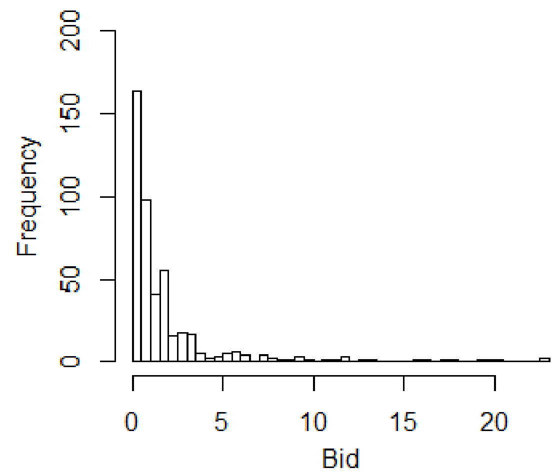
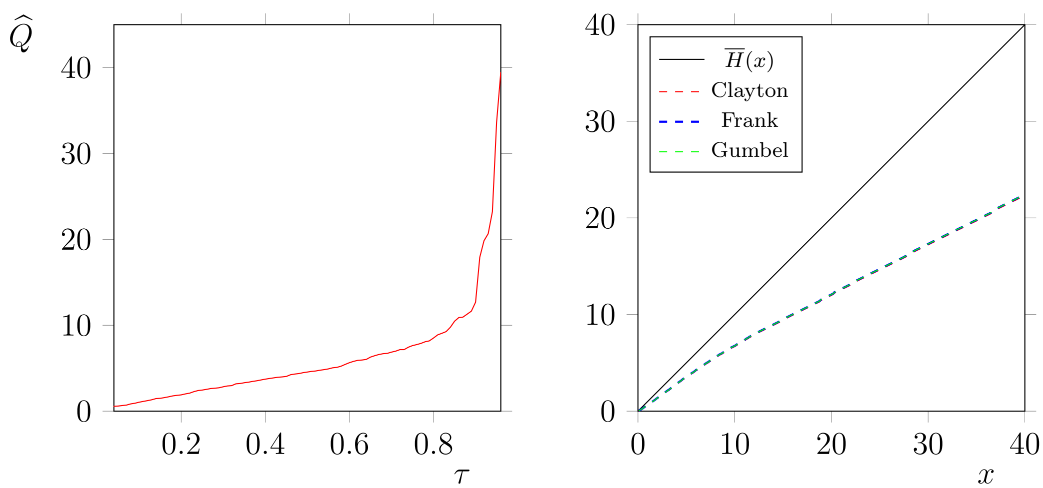


Figure 2.1: Histogram of Bids

	Clayton	Frank	Gumbel
θ	0.264	1.518	1.123
	$(0.072, 0.456)$	$(0.785, 2.251)$	$(1.032, 1.214)$
τ_k	0.117	0.165	0.109
	$(0.044, 0.190)$	$(0.089, 0.241)$	$(0.036, 0.182)$

Table 2.2: Estimated θ and Kendall's τ_k

Next, we estimate the quantile function of private signals and the two conditional expected valuation functions. In the two bidders' case, first, it can be easily shown that the estimated quantile function in the pure common value framework is the same as that in the affiliated private value framework due to the fact that the kernel function in the integral equation is the zero function. Second, the high conditional expected valuation function is the identity function in this case due to a similar reason. However, the low conditional expected valuation function still depends on the estimated quantile function of private signals and the choice of copula function. The estimated quantile function of private signals is shown in Panel (a) of Figure 2.2. From the estimated quantile function, the upper support of private signals is around 40 million dollars. The first quartile, median and third quartile are around 2.45, 4.55 and 7.63 million dollars with interquartile range to be around 5.18 million dollars. This implies that the private signals have a large probability of taking small values. The slope of the estimated quantile function is strictly increasing almost everywhere when τ increases, implying that the density function of private signals decreases as we move towards the right. This is in line with the histogram of bids in Figure 2.1 since bids are strictly increasing transformation of private signals.



(a) Quantile Function

(b) Conditional Expected Valuations

Figure 2.2: Estimates of Signal Quantile and Low Conditional Expected Valuation Function

The estimated curves of the conditional expected valuation functions are shown in Panel (b) of Figure 2.2. The solid line represents the high conditional expected valuation function (which is the identity function and not estimated), while the dashed lines represent the estimates of the low conditional expected valuation function. The estimated curves almost overlay each other when different copula families are used, which suggests the little loss in parameterizing the copula function. Based on the closeness of the estimated low conditional expected valuation functions, we expect the robustness of estimated policy parameters against choice of the copula family and present the results as follows.

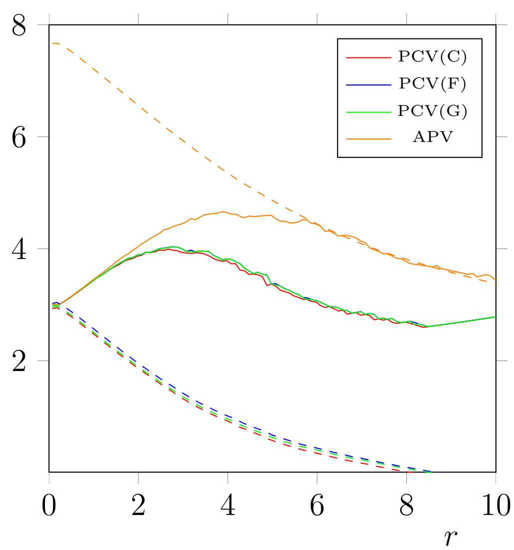


Figure 2.3: Estimated Seller's Expected Revenue and Bidders' Expected Surplus

	Optimal Reserve	Max Revenue
Clayton	2.71	4.00
	–	(33.2%)
Frank	2.77	4.03
	–	(34.2%)
Gumbel	2.77	4.66
	–	(54.8%)
APV	3.87	3.81
	–	(26.8%)
Actual	0.075	3.01

Table 2.3: Optimal Reserve Price and Maximized Revenue (Million $ per Tract)

First, in Figure 2.3, "C, F, G" represent the Clayton, Frank, and Gumbel copulae, respectively. The solid lines represent the seller's expected revenue and the dashed lines represent the bidders' expected surplus. The estimated curves of the seller's expected revenue peak at a point above the seller's own valuation v_o, which is set to be the actual reserve price by convention in the literature. This is consistent with the theoretical prediction that a reserve price higher than v_o can be used as a tool to screen out low-type bidders and pressure high-type bidders into bidding more aggressively and thus generates a higher expected revenue. When the reserve price is greater than the optimal reserve price, the seller's expected revenue decreases since too few bidders will actually participate in the auction. The estimated curves of the bidders' expected surplus decrease from the beginning. This is also consistent with the theory that it should be maximized when there is no reserve price. The estimated curves are very close to each other when different copula families are used.

Second, we present the results of optimal reserve price and maximized expected revenue in Table 2.3. The estimated optimal reserve prices are around 2.7 million dollars per tract, which are quite close when different copula families are used. The actual reserve price is either 0.075 or 0.125 million dollars per tract (or equivalently either 15 or 25$ per acre), which has been long perceived to be too low. Using our estimates, if the optimal reserve price were used, the government's expected revenue would be around 4.03 million dollars per tract, which amounts to an increase of 34.2% upon the actual revenue.

Lastly, for comparison purposes, we also estimate the two policy parameters under the affiliated private values framework. The estimated curves are also shown in Figure 2.3, and they are quite different from those under the pure common value framework. The estimated optimal reserve price is 3.87, which is much higher than those under the pure common value framework. If the private value framework is used to guide the choice of optimal reserve price, the government's expected revenue will be around 3.81, which merely amounts to an 26.8% increase upon the actual revenue. This leads to an loss of 50.8 million dollars compared to the maximized expected revenue that our optimal reserve price can generate. This comparison re-emphasizes the important implications of model specification on the policy parameters. In practice, if it is uncertain which framework is more appropriate, policy makers are suggested to estimate these policy parameters under both frameworks and use the results as complements.

2.3.2 *Three Bidders' Case*

In this section, we focus on analyzing the subset of data with three bidders. The analysis will be more complicated since it involves estimating the quantile function of private signals by the Volterra integral equation. Table 2.4 and Figure 2.4 provide some descriptive statistics of the sub data set.[2]

$M = 3$	
#Tracts	121
Mean	2.593
Median	1.082
Min Bid	0.223
Max Bid	30.472
Std	4.199
Million in 1982$ per Tract	

Table 2.4: Summary Statistics

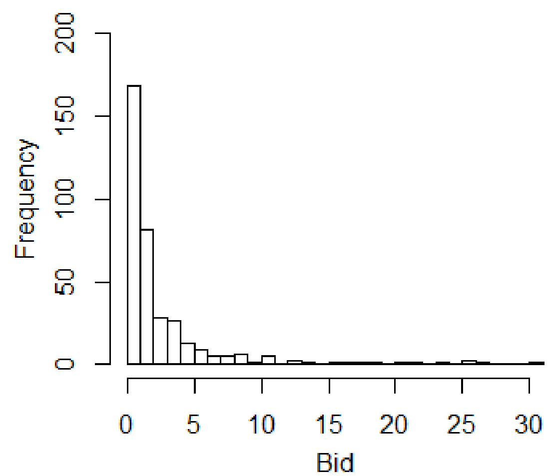


Figure 2.4: Histogram of Bids

We estimate the copula parameter θ as well as Kendall's τ_k and summarize results in Table 2.5. The estimated Kendall's τ_ks are around 0.14 for the Frank and the Gumbel families, while it is a little smaller for the Clayton family. The 95% bootstrapped confidence intervals are reported in the parentheses and they reveal that the Kendall's τ_k is statistically significantly different from zero, suggesting a positive dependence among the private signals.

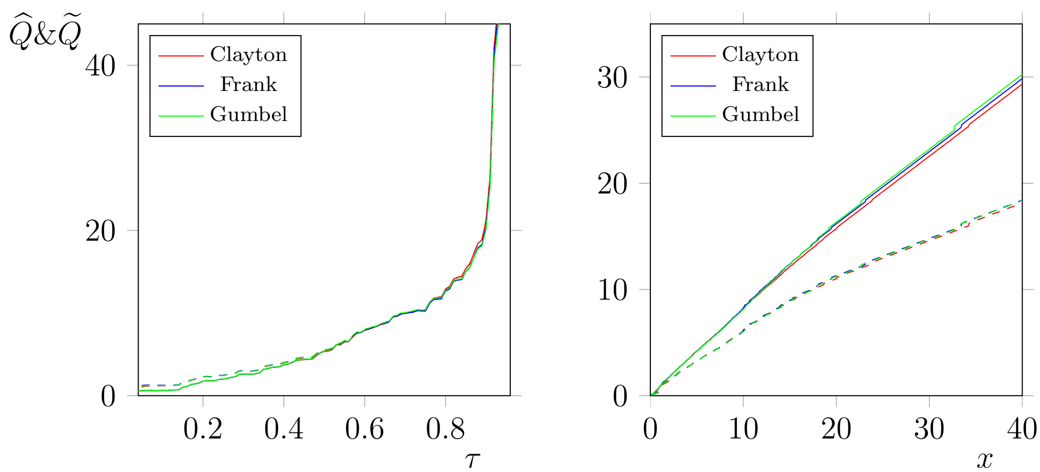
Next, we estimate the quantile function of private signals. As in the simulation in Chapter 1, we employ the two methods (GSE and ISE) and the three Archimedean copula families (Clayton, Frank, and Gumbel). We are interested in whether the

[2]The original data set includes 147 auctions with three bidders, where the maximum bid is 79.95 million dollars per tract in 1982$ and minimum bid is merely 0.000543 million dollars. We do similar trimming as in the two bidders' case, and this leaves us with 121 auctions.

	Clayton	Frank	Gumbel
θ	0.255	1.213	1.155
	$(0.034, 0.476)$	$(0.336, 2.090)$	$(1.045, 1.265)$
τ_k	0.105	0.141	0.143
	$(0.027, 0.199)$	$(0.040, 0.226)$	$(0.049, 0.219)$

Table 2.5: Estimated θ and Kendall's τ_k with 95% Bootstrap Confidence Interval

estimation results are robust against the choice of copula family. This is important since fully nonparametric estimation of high dimensional copula and its partial derivatives would be difficult in practice due to the curse of dimensionality.



(a) Quantile Function (b) Conditional Expected Valuations

Figure 2.5: Estimates of Signal Quantile and Conditional Expected Valuation Functions

The estimated curves are shown in Panel (a) of Figure 2.5. The geometric series estimates are solid lines and the iterative sieve estimates are dashed lines. The estimated curves almost coincide with each other. The robustness against estimation methods is expected since in theory, the two estimators should converge to the same function in the limit. The robustness against choice of copula family suggests that little loss is incurred when we parameterize the copula function in practice. From the estimated quantile functions, the upper support of private signals is a little above 40 million dollars. The first quartile, median and third quartile are around 2.45, 5.50 and 10.39 million dollars with interquartile range to be around 7.94 million dollars. This implies that the private signals have a large probability of taking small values. The slope of the estimated quantile functions are strictly increasing almost everywhere when τ increases, implying that the density function of private signals decreases as we move towards the right. This is in line with the histogram of bids in Figure 2.4 since bids are strictly increasing transformation of private signals.

Given the estimated copula function and quantile function, we are ready to estimate the two conditional expected valuation functions. The estimated curves are shown in Panel (b) of Figure 2.5. Solid lines represent the estimates of the high conditional expected valuation function and the dashed lines represent the estimates of the low conditional expected valuation function.[3] Based on the closeness of the two estimated conditional expected valuation functions, we expect the robustness of estimated policy parameters against choice of the copula family and present the results in Figure 2.6 and Table 2.6.

First, the estimated curves of policy parameters are shown in Figure 2.6 using the iterative sieve estimator (estimated curves from the geometric series estimator are very similar thus omitted). In Figure 2.6, "C, F, G" represent the Clayton, Frank, and Gumbel copulae, respectively. The solid lines represent the seller's expected revenue and the dashed lines represent the bidders' expected surplus. Similar to the case with two bidders, the estimated curves of the seller's expected revenue peak at a point above

[3]Given the robustness of estimated quantile function against estimation methods, we show the estimated conditional expected valuation functions with the quantile function estimated from the iterative sieve method in Panel (b) of Figure 2.5, the estimates of conditional expected valuation functions with the quantile function estimated from the geometric series method are very similar and thus omitted to save space.

the seller's own valuation v_o, which is consistent with the theoretical prediction that a reserve price higher than v_o can be used as a tool to screen out low-type bidders and pressure high-type bidders into bidding more aggressively and thus generate a higher expected revenue. When the reserve price is greater than the optimal reserve price, the seller's expected revenue decreases since too few bidders will actually participate in the auction. The estimated curves of the bidders' expected surplus decrease from the beginning. This is also consistent with the theory that it should be maximized when there is no reserve price. The estimated curves are very close to each other when different copula families are used.

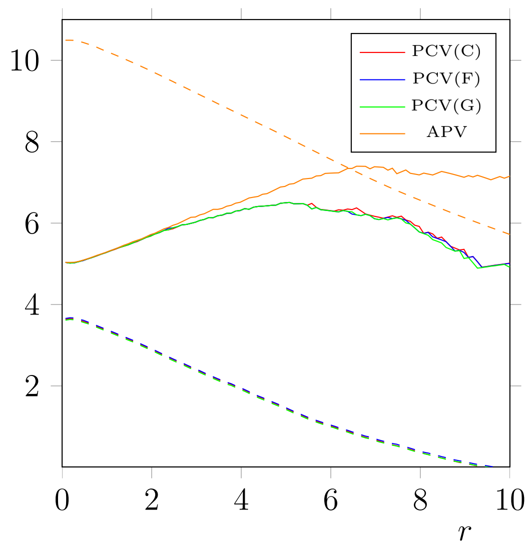


Figure 2.6: Estimated Seller's Expected Revenue and Bidders' Expected Surplus

	Optimal Reserve		Max Revenue	
	GSE	ISE	GSE	ISE
Clayton	5.01	5.14	6.47	6.54
	–	–	(28.6%)	(30.0%)
Frank	4.95	5.09	6.44	6.51
	–	–	(28.0%)	(29.4%)
Gumbel	4.98	5.11	6.46	6.52
	–	–	(28.4%)	(29.6%)
APV	6.58		7.39	
	–		(46.9%)	
Actual	0.075		5.03	

Table 2.6: Optimal Reserve Price and Maximized Revenue (Million $ per Tract)

Second, we present the results of optimal reserve price and maximized expected revenue in Table 2.6. The estimated optimal reserve prices are around 5.00 million dollars per tract, which are quite close when different estimators and/or different copula families are used. The actual reserve price is either 0.075 or 0.125 million dollars per tract (or equivalently either 15 or 25$ per acre), which has been long perceived to be too low. Using our estimates, if the optimal reserve price were used, the government's expected revenue would be around 6.5 million dollars per tract, which amounts to an increase around 30% upon the actual revenue.[4]

Lastly, for comparison purposes, we also estimate the two policy parameters under the affiliated private values framework. The estimated curves are also shown in Figure 2.6, which are quite different from those under the pure common value framework. The estimated optimal reserve price is 6.58, which is much higher than those under the pure common value framework. If the private value framework is used to guide the choice of optimal reserve price, the government's expected revenue will be around 6.14, which amounts to an increase of 22% upon the actual revenue. This leads to an loss of 48.4 million dollars compared to the maximized expected revenue that our optimal reserve price can generate. This comparison re-emphasizes the important implications of model specification on the policy parameters. In practice, if it is uncertain which framework is more appropriate, policy makers are suggested to estimate these policy parameters under both frameworks and use the results as complements.

2.4 Conclusion

In this chapter, we apply the identification results and structural estimation method developed in Chapter 1 to analyze the U.S. Outer Continental Offshore wildcat auction data in the pure common value framework. Given that the bidders are assumed to bid according to the equilibrium bidding strategy, we primarily focus on the welfare implication under different counterfactual reserve prices, and in particular the seller's

[4]Moreover, it is easy to see that the expected total welfare, defined as the sum of the seller's expected revenue and the bidders' expected surplus, is maximized when the reserve price is set to be the seller's own valuation v_o. Consequently, any reserve price above v_o would incur a loss in the expected total welfare. If the reserve price is set to maximize the seller's expected revenue, then the bidders would pay a higher price on average. In this case, although the government's revenue increases, the higher price paid by the oil company would translate into a higher gasoline price that the consumers need to pay. Therefore, from a society's point of view, using the optimal reserve price might not be a good choice. In practice, the government could choose a reserve price above v_o but below the optimal reserve price to balance the tradeoff between its own revenue and the expected total welfare. This tradeoff can be analyzed using the estimated curves for the seller's expected revenue and the bidders' expected surplus.

expected revenue, the bidders' expected surplus, and the expected total welfare.

We are the first to conduct structural analysis of this data set in the pure common value framework. Our results suggest that the private signals are positively correlated, with the estimated Kendall's τ_k to be between 0.109 and 0.165 for the two bidders' case and between 0.105 and 0.143 for the three bidders' case. We estimate the seller's expected revenue and the bidders' expected surplus to perform counterfactual analysis, and compare our results with those obtained in the affiliated private values framework. The estimated welfare curves using our method are very close to each other under different choice of parametric copula families and different estimation methods of the quantile function of private signals. While it has long been perceived in the literature that the actual reserve price is too low to generate enough government revenue, we show that there is much room for improvement. Specifically, our results suggest that the government's expected revenue can be increased by around 34% and 30% for the auctions with two and three bidders considered in our sample, respectively.

The estimated welfare curves under the affiliated private value framework are quite different from those under the pure common value framework. This comparison re-emphasizes the important implications of model specification on the policy parameters. In practice, if it is uncertain which framework is more appropriate, policy makers are suggested to estimate these policy parameters under both frameworks and use the results as complements.

Chapter 3

A SENSITIVITY ANALYSIS IN AUCTION MODELS WITH AFFILIATED PRIVATE VALUES AND INCOMPLETE SETS OF BIDS

3.1 Introduction

Most of the previous works on structural analysis of auction data assume that all bids are available, see Guerre, Perrigne and Vuong (2000, 2009), Li, Perrigne and Vuong (2000, 2002), and Marmer and Shneyerov (2012). The scenario in which all bids are observed could most likely occur in a first-price sealed-bid auction. For the other three common auction formats, namely, the second-price sealed-bid auction, the ascending auction, and the descending auction, usually only the transaction price is observed.[1] Moreover, even in some first-price sealed-bid auctions, the econometrician may only have access to an incomplete set of bids such as the winning bids or the two highest bids. When only incomplete information on bids are available, identification and estimation procedures developed in existing works do not apply, and it is of interest to know what can be learned given an incomplete set of bids.

There have been only a couple of papers addressing the non-identification issues in auction models with dependent private values and incomplete sets of bids.[2] In a symmetric ascending auction framework with correlated private values, Aradilla-Lopez,

[1]On the one hand, in a descending auction which is strategically equivalent to a first-price sealed-bid auction, the transaction price is the winning bid. On the other hand, in a second-price sealed-bid auction or an (strategically equivalent) ascending auction, the transaction price is the second highest private value. Due to these strategic equivalences, we focus on the first-price (second-price) sealed-bid auctions in the rest of this paper.

[2]For the first-price sealed-bid auction with independent private values (IPV) and unobserved auction heterogeneity, Armstrong (2013) provides bounds on several quantities using data on a single bidder or on the winning bid.

Gandhi and Quint (2013) assume only the availability of transaction price and use exogenous participation to develop bounds on the seller's expected revenue and the bidders' expected surplus. In an asymmetric second-price sealed-bid auction framework, Komarova (2013a) establishes bounds on the joint and marginal distribution functions of private values under scenarios with different levels of incompleteness of the bidding data. The partial identification results in these two papers are established only in the ascending (or equivalently the second-price sealed-bid) auction framework, where the Bayesian Nash equilibrium bidding strategy is trivially the identity function. The identification issues for the first-price sealed-bid auction with dependent private values and incomplete sets of bids have not been explored.

In this paper, we address the identification issues in a first-price sealed-bid auction with affiliated private values (APV) and incomplete sets of bids. The APV model of a first-price sealed-bid auction is nonparametrically identified when all bids are observed, as established in Li, Perrigne, and Vuong (2002) for the symmetric case, and Campo, Perrigne, and Vuong (2003) for the asymmetric case. However, as shown in Athey and Haile (2002), the APV model is not nonparametrically identified with incomplete sets of bids observed. In this paper, we further address this identification issue and obtain the following novel results. First, in the simple case with symmetric bidders and non-binding reserve price, when the two highest bids are observed, we achieve identification of the copula function of private values in a large and flexible nonparametric Archimedean class. Therefore, combining it with identification of the marginal distribution function of the private value, we attain identification of the joint distribution function of private values and thus all policy-relevant quantities.[3] In contrast, for the same scenario, Athey and Haile (2002) only establish identification of the joint distribution function of the two highest private values and show that the joint distribution function of all private values is in general not identified. Second, in the same simple

[3]The copula approach has recently also been used in a couple of papers in modeling affiliation in auctions. See, e.g., Hubbard, Li, and Paarsch (2012), Li and Zhang (2013).

case but with only the transaction price observed, we parameterize the Archimedean copula function and explore information on the level of dependence to establish bounds on the quantile function of each bidder's private value, the total welfare of a first-price sealed-bid auction, its efficiency gain over the random assignment mechanism, the seller's expected revenue, the bidders' expected surplus under any counterfactual reserve price, and the optimal reserve price. Third, we show that some of our results can be readily extended to the case with asymmetric bidders and/or binding reserve price, although the derivations and results are more involved notationally. Finally, we extend our idea to study identification/partial identification in a second-price sealed-bid auction to complement the works by Aradilla-Lopez, Gandhi, and Quint (2013), and Komarova (2013a).

Our work contributes to the literature on partial identification of auction models, which has recently drawn attention. Haile and Tamer (2003) relax assumptions on the bidding behavior in oral English auctions to obtain bounds on model fundamentals and seller's counterfactual expected revenue as well as the optimal reserve price. Tang (2011) uses complete data to provide bounds on the distribution function of seller's counterfactual revenue in common-value auctions. Armstrong (2013), Komarova (2013a) and Aradilla-Lopez, Gandhi, and Quint (2013) establish bounds on various quantities of interest in different auction models with different informational assumptions and with incomplete sets of bids. Gentry and Li (2014) study the IPV model with selective entry and show that the model is point-identified with continuous entry variation and partially identified with discrete entry variation.

Our work also contributes to the growing literature on partial identification pioneered by Charles Manski. For a summary of early contributions, see Manski (2003). Recent developments include Manski and Tamer (2002), Tamer (2003), Molinari (2008), Fan and Park (2009), Santos (2012), Kline and Santos (2013), Fan and Liu (2013), Henry, Kitamura, and Salanie (2013), among many others. Our work is in spirit more closely related to Fan and Liu (2013), Kline and Santos (2013). In a linear quantile regression model with dependent censoring, Fan and Liu (2013) provide the identified set and inference procedure for the quantile regression coefficient where an Archimedean copula is used to capture the dependent censoring mechanism. They obtain identified set for the quantile regression coefficient by letting the copula function vary in some parametric Archimedean family. In a missing data problem, by restricting the degree of departure from the missing-at-random assumption, Kline and Santos (2013) establish the identified sets for the conditional quantile function and the coefficient of the best linear approximation to the true conditional quantile function. Similar in spirit to these two papers, in the case when only the transaction price is observed in auction models, we obtain the identified sets for several quantities of interest by varying the copula function in some known parametric Archimedean family.

Finally, our work contributes to the classical competing risks literature, to which the scenario that only the two highest bids are observed is closely related.[4] In the classical competing risks model, a machine breaks down when one of its crucial components fails. The observed data pertains to the machine's lifetime and the component that caused the failure. For identification results in the classical competing risks literature, see Tsiatis (1975), Peterson (1976), Crowder (1991), Bedford and Meilijson (1997), Zheng and Klein (1995), Braekers and Veraverbeke (2005), among others. Braekers and Veraverbeke (2005) show that if the copula function of the competing risks is an Archimedean copula with a known generator function, then the marginal distribution of each potential risk can be identified from data on the failure time and the cause of failure. In this paper, we make use of Lemma 1 in Braekers and Veraverbeke (2005) to achieve identification in auction models with asymmetric bidders and/or binding reserve price when the highest two bids and the winner's identity are observed. It is straightforward to apply our result to the competing risks model in which the machine breaks down when two of its crucial components fail.

[4]For other related works and discussions of the relation between auction models and competing risks models, see Komarova (2013b), Lee and Lewbel (2013).

The rest of the paper is organized as follows. The model is presented in Section 3.2. Identification/partial identification results in a first-price sealed-bid auction with symmetric bidders and non-binding reserve price are discussed in Section 3.3. In Section 3.4, we extend our results to the cases with asymmetric bidders and/or binding reserve price. We also use our idea to explore the partial identification in a second-price sealed-bid auction with symmetric bidders and non-binding reserve price. Section 3.5 concludes.

3.2 The Model

A single and indivisible object is auctioned to n risk-neutral bidders by the first-price sealed-bid auction, in which all bidders simultaneously submit bids and the bidder with highest bid wins the auction and pays her bid. The bidders' private values $V = (V_1, \cdots, V_n)$ for the object is distributed according to cumulative distribution function (cdf) $\overline{F}(\cdot)$ on the support $[\underline{v}, \overline{v}]^n$, with probability density function (pdf) $\overline{f}(\cdot)$ with respect to the Lebesgue measure. V is assumed to be affiliated as defined in Milgrom and Weber (1982). Loosely speaking, affiliation in V means that large values for some of its components make the other components more likely to be large rather than small. We allow the bidders to be asymmetric in the sense that $\overline{F}(\cdot)$ may not be exchangeable in its arguments. Denote the marginal cdf (pdf) of V_i by $F_i(\cdot)$ ($f_i(\cdot)$), $i = 1, \cdots, n$. By Sklar's Theorem (see Nelsen (2006)), there exists a unique copula function $C_o(u_1, \cdots, u_n)$ on $[0, 1]^n$ such that $\overline{F}(v_1, \cdots, v_n) = C_o[F_1(v_1), \cdots, F_n(v_n)]$ for any $(v_1, \cdots, v_n) \in [\underline{v}, \overline{v}]^n$.

To simplify the exposition, we focus on the case with non-binding reserve price, and extend the analysis to binding reserve price in Section 3.4.1. We focus on strictly increasing and differentiable Bayesian Nash equilibrium bidding strategies. Before bidding, each bidder i knows the distribution function $\overline{F}(\cdot)$ and her realized private value v_i but does not know other bidders' private values. Given that all other bidders $j, j \neq i$ bid according to their Bayesian Nash equilibrium bidding strategies $s_j(\cdot)$, for bidder i with private value v_i, her expected profit by bidding b_i is

$$\begin{aligned}\pi(b_i|v_i) &= (v_i - b_i)\mathbb{P}\left[s_j(V_j) \le b_i, \forall j \ne i | V_i = v_i\right] \\ &= (v_i - b_i)F_{-i|i}\big(s_j^{-1}(b_i), \forall j \ne i | v_i\big) \\ &= (v_i - b_i)C_{o,i}(\Delta(b_i, v_i)).\end{aligned} \tag{3.1}$$

In the above expression, $s_j^{-1}(\cdot)$ is the inverse function of $s_j(\cdot)$, $F_{-i|i}(v_{-i}|v_i)$ is the cdf of $V_{-i} = (V_1, \cdots, V_{i-1}, V_{i+1}, \cdots, V_n)$ conditional on $V_i = v_i$, $C_{o,i}(\cdot) = \partial C_o(\cdot)/\partial u_i$, and $\Delta(b_i, v_i) = [F_1(s_1^{-1}(b_i)), \cdots, F_{i-1}(s_{i-1}^{-1}(b_i)), F_i(v_i), F_{i+1}(s_{i+1}^{-1}(b_i)), \cdots, F_n(s_n^{-1}(b_i))]$. The third equality in (3.1) follows from the fact that $F_{-i|i}(v_{-i}|v_i) = C_{o,i}[F_1(v_1), \cdots, F_n(v_n)]$, and its proof can be found in Lemma 1 of Hubbard, Li, and Paarsch (2012).

Taking derivative of $\pi(b_i|v_i)$ with respect to b_i, the first-order condition is

$$(v_i - b_i)\sum_{j \ne i} C_{o,ij}(\Delta(b_i, v_i))\frac{f_j(s_j^{-1}(b_i))}{s_j'(s_j^{-1}(b_i))} = C_{o,i}(\Delta(b_i, v_i)), \tag{3.2}$$

where $C_{o,ij}(\cdot) = \partial^2 C_o(\cdot)/\partial u_i \partial u_j$. When the reserve price is not binding, the equilibrium bidding functions are subject to the boundary condition $s_i(\underline{v}) = \underline{v}, i = 1, \cdots, n$. Let $\underline{b} = s_i(\underline{v})$, and $\bar{b} = s_i(\bar{v}) = s_j(\bar{v})$ for any $i \ne j$, then $F_i(s_i^{-1}(b)) = G_i(b)$ and $f_i(s_i^{-1}(b))/s_i'(s_i^{-1}(b)) = g_i(b)$, $b \in [\underline{b}, \bar{b}]$, $i = 1, \cdots, n$, where $G_i(\cdot)$ is the cdf of $B_i = s_i(V_i)$ with corresponding pdf $g_i(\cdot)$.

In a spirit similar to that in Guerre, Perrigne, and Vuong (2000), Li, Perrigne, and Vuong (2002), and Campo, Perrigne, and Vuong (2003), we can rewrite (3.2) as

$$v_i = s_i^{-1}(b_i) = b_i + \frac{C_{o,i}(\underline{G}(b_i))}{\sum_{j \ne i} C_{o,ij}(\underline{G}(b_i))g_j(b_i)}, \tag{3.3}$$

where $\underline{G}(b_i) = [G_1(b_i), \cdots, G_n(b_i)]$. Further, for $p \in [0, 1]$, let $Q_i(p), q_i(p)$ denote the p-th quantile functions of V_i and B_i, respectively, then the fact that $V_i = s_i^{-1}(B_i)$ implies $Q_i(p) = s_i^{-1}(q_i(p))$, that is,

$$Q_i(p) = q_i(p) + \frac{C_{o,i}(\underline{G}(q_i(p)))}{\sum_{j \ne i} C_{o,ij}(\underline{G}(q_i(p)))g_j(q_i(p))}. \tag{3.4}$$

Equation (3.4) is important and will be the starting point for our analysis in the next section.

3.3 Identification/Partial Identification with Symmetric Bidders and Non-Binding Reserve Price

By assuming that all bids in each auction are observed in the sample, several previous works, including Li, Perrigne, and Vuong (2002) and Hubbard, Li, and Paarsch (2012), study the identification and estimation of the first-price sealed-bid auction model in the symmetric APV framework. Li, Perrigne, and Vuong (2002) use a two-step nonparametric approach to estimate the joint density function of private values. Hubbard, Li, and Paarsch (2012) assume a parametric family for the copula function, and essentially use a three-step semiparametric approach to estimate the joint density function of private values. In the real-world auctions, nevertheless, observations on all bids may not always be available. One leading example is the prevalent descending auction, in which only the highest bid (transaction price) is observed in each auction. When only incomplete information on bids are available, identification and estimation procedures developed in existing works do not apply, and it is of interest to know what can be learned given an incomplete set of bids.

In this section, we focus on the simple case with symmetric bidders and non-binding reserve price, and will point out the results that can be readily generalized to the case with asymmetric bidders and/or binding reserve price. Under the symmetry, $\overline{F}(\cdot)$ is exchangeable among its arguments, and all bidders have the same private value distribution function, denoted as $F_o(\cdot)$, with corresponding quantile function $Q_o(\cdot)$. The quantile relation in (3.4) reduces to

$$Q_o(p) = q(p) + \frac{1}{n-1}\frac{C_{o,1}(\underline{p})}{C_{o,12}(\underline{p})}q'(p), p \in [0,1], \tag{3.5}$$

where $\underline{p} = (p, \cdots, p)$, $q(p)$ and $q'(p) = 1/g(q(p))$ are quantile function and quantile density function of B_i, respectively. $g(\cdot) = G'(\cdot)$, and $G(\cdot)$ is the distribution function of B_i.

Suppose the incomplete set of bids at least contains the highest bid in each auction, then we seek to express (3.5) in terms of observed quantities pertaining to the highest bids. Let us first introduce some notations. For $k = 1, \cdots, n$, let $B^{(k)}$ denote the k-th order statistic of $B_1, \cdots, B_n$, with $B^{(n)}$ being the highest. Let $G^{(k)}(\cdot)$, $g^{(k)}(\cdot)$ and $q^{(k)}(\cdot)$ be the distribution function, density function and quantile function of $B^{(k)}$, respectively. It is easy to see that $G^{(n)}(b) = C_o(G(b), \cdots, G(b))$ and $G^{(n)}(q(p)) = C_o(\underline{p})$. Consequently, we have $q(p) = q^{(n)}(C_o(\underline{p}))$ and $q'(p) = nq^{(n)\prime}(C_o(\underline{p}))C_{o,1}(\underline{p})$, where $q^{(n)\prime}(\cdot)$ is the derivative of $q^{(n)}(\cdot)$. Therefore, (3.5) can be written as

$$Q_o(p) = q^{(n)}\left(C_o(\underline{p})\right) + \frac{n}{n-1}\frac{C_{o,1}^2(\underline{p})}{C_{o,12}(\underline{p})}q^{(n)\prime}\left(C_o(\underline{p})\right), p \in [0,1]. \tag{3.6}$$

This relation suggests that $Q_o(p)$ is point identified as long as the sample information identifies $q^{(n)}(\cdot), q^{(n)\prime}(\cdot)$ and the copula function $C_o(\cdot)$, which in turn implies identification of the joint distribution function of private values. This is consistent with the identification results in both Li, Perrigne, and Vuong (2002) and Hubbard, Li, and Paarsch (2012), since it is clear that all of the three quantities are identified when all bids are observed in each auction.

Remark 3.3.1 One special case is the independent private value (IPV) paradigm, for which $C_o(u_1, \cdots, u_n) = \prod_{i=1}^n u_i, (u_1, \cdots, u_n) \in [0,1]^n$. In this case, (3.6) simplifies to

$$Q_o(p) = q^{(n)}(p^n) + \frac{n}{n-1}p^n q^{(n)\prime}(p^n), p \in [0,1].$$

Thus observing only the highest bids is sufficient for identification of the private value distribution in the IPV framework, which is in line with the results in Guerre, Perrigne, and Vuong (1995) and Athey and Haile (2002).

Given the highest bids, $q^{(n)}(\cdot)$ and $q^{(n)\prime}(\cdot)$ are identified. Now the natural question arises: what information are necessary and/or sufficient for achieving identification of the copula function? From Remark 3.3.1, identification is achieved in the IPV framework. The IPV case is at one extreme of the spectrum of APV models, and the other extreme of APV models is the perfectly positively correlated private values case.[5]

[5] It can be easily shown that in this case, the Bayesian Nash equilibrium bidding function is the identity function. All bids are the same, then the highest bids also identify the joint distribution function of private values.

Now our aim is to find a class of copula functions rich enough to contain the two extreme cases and the sample information of an incomplete set of bids allow for identification of the copula function within this class. Further, if such a rich class exists, what extra sample information besides the highest bids is needed for identification of the copula function? In the following subsection, we give a novel identification result to provide one set of answers to these questions. First, observe that in the IPV case, the copula function is in the Archimedean family with strict generator function $\varphi_o(t) = -\ln t, t \in (0, 1]$. In the case with perfectly positively correlated private values, the copula function is the Fréchet-Hoeffding upper bound $C_o(u_1, \cdots, u_n) = \min\{u_1, \cdots, u_n\}$, which is a limiting case of many Archimedean families. This observation motivates Assumption (AS) below. Second, we show that under weak requirement on the Archimedean copula generator function, the extra information of the second highest bids enables us to achieve identification of the copula function.

Assumption (AS) The true copula function of private values is an Archimedean copula function with strict, twice continuously differentiable generator function $\varphi_o(\cdot)$, with its inverse function $\varphi_o^{-1}(\cdot)$ completely monotone on $[0, \infty)$.[6]

Under Assumption (AS), the true copula function is of the form

$$C_o\left(u_1, \cdots, u_n\right) = \varphi_o^{-1}\Big[\sum_{i=1}^{n} \varphi_o\left(u_i\right)\Big], \tag{3.7}$$

where $\varphi_o : [0, 1] \to [0, \infty)$ is a continuous, strictly decreasing function with $\varphi_o(1) = 0$.

3.3.1 Two Highest Bids Observed

Auction participants often refer to the difference between the top two bids as

[6]Complete monotonicity of $\varphi_o^{-1}(\cdot)$ is a sufficient condition to guarantee that the expression in (3.7) actually generates an n-dimensional Archimedean copula function for any $n \geq 2$. This assumption suffices for the purpose of this paper. For a necessary and sufficient condition on this issue, see McNeil and Neslehova (2009). Furthermore, it can be easily shown that the private values are affiliated (or $\overline{f}(\cdot)$ is of multivariate total positivity of order two (MTP$_2$)) if the density function of $C_o(\cdot)$ is MTP$_2$, which is in turn implied by the complete monotonicity of $\varphi_o^{-1}(\cdot)$ (see Müller and Scarsini (2005)).

"money left on the table", which could reveal information relevant to bidding strategies. As argued in Athey and Haile (2002), the auctioneer has incentive to record the top two bids. For example, in procurement auctions, the auctioneer may keep records of the top two bids in case that the winner defaults. In the following theorem, we consider the additional information of the second highest bid $B^{(n-1)}$ and show that the two highest bids suffice to identify the copula function under Assumption (AS). Therefore, combining this result with (3.6), the joint distribution function of private values and hence all policy parameters are identified.

Theorem 3.3.2 In a first-price sealed-bid auction with symmetric bidders and non-binding reserve price, if the highest two bids $B^{(n)}, B^{(n-1)}$ are observed, then under Assumption (AS), the copula generator function is identified as

$$\varphi_o(t) = \int_1^t \alpha \exp\Big(- \int_0^s z(u)du \Big) ds, t \in (0,1],$$

where $\alpha < 0$, and

$$\begin{aligned} z(u) &= \frac{n}{n-1}\phi\left(q^{(n)}(u)\right)\left(q^{(n)\prime}(u)\right)^2, u \in [0,1], \\ \phi(b) &= \left[\frac{\partial^2 \mathbb{P}(B^{(n-1)} \le m_1, B^{(n)} \le b_1)}{\partial m_1 \partial b_1}\right]_{|m_1=b_1=b}, b \in [\underline{b}, \overline{b}]. \end{aligned}$$

Proof. See Appendix B. ■

Remark 3.3.3 The message of the above result is essentially that for the class of distributions generated by copula function under Assumption (AS), the joint distribution function is informationally equivalent to the joint distribution function of the highest two order statistics. When the lowest two order statistics are observed, it is straightforward to identify the copula generator function if the survival copula satisfies Assumption (AS). This implies that our results can be readily applied to identification of the competing risks model in which the machine breaks down when two of its crucial components fail.

Our identification result is stronger than that in Athey and Haile (2002). They identify each bidder's inverse equilibrium bidding function and thus the joint distribution function of $V^{(n)}, V^{(n-1)}$ in the asymmetric case (when the winner's identity is known). In general, however, Athey and Haile (2002) show that with the highest two bids, the joint distribution function of private values is not identified even in the case of symmetric bidders. In sharp contrast, we show in Section 3.4.1 that, by borrowing result from Braekers and Veraverbeke (2005), our identification result generalizes to the case with both asymmetric bidders (when the winner's identity is known) and binding reserve price.

3.3.2 Only the Highest Bids Observed

Partial Identification of the Quantile Function of Private Value

We have shown that under Assumption (AS), the joint distribution function of private values is identified with the highest two bids. This result depends on the availability of $B^{(n)}, B^{(n-1)}$, and it obviously does not apply to the descending auction, in which the auction ends once a bidder stops the dropping price. The natural question is what can be learned when only the highest bids are available, and we explore this situation in the following. Recall that (3.6) is

$$Q_o(p) = q^{(n)}\left(C_o(\underline{p})\right) + \frac{n}{n-1}\frac{C_{o,1}^2(\underline{p})}{C_{o,12}(\underline{p})} q^{(n)\prime}\left(C_o(\underline{p})\right), p \in [0,1].$$

In the absence of information other than the highest bids, $Q_o(\cdot)$ is not identified in general since the true copula function is unknown. This situation is new. Usually, when we use restriction to identify the parameter of interest, the restriction is completely known, and we study the uniqueness of solution. If the solution is not unique, we obtain an identified set. In our situation, however, the restriction itself is only known up to some component. To be specific, the copula function is unknown. Now our idea is to let the copula function vary in a pre-specified and flexible family, so that we can partially identify the parameters of interest. Let $C_o(\cdot) \in \mathcal{C}$, a known family of copula functions, and for any $C(\cdot) \in \mathcal{C}$, let

$$Q^F(p;C) = q^{(n)}\left(C(\underline{p})\right) + \frac{n}{n-1}\frac{C_1^2(\underline{p})}{C_{12}(\underline{p})} q^{(n)\prime}\left(C(\underline{p})\right), p \in [0,1]. \tag{3.8}$$

The superscript "F" indicates the first-price sealed-bid auction. It follows from (3.6) that the identified set for $Q_o(\cdot)$ is

$$\mathcal{Q}_{\mathcal{C}}^F = \left\{Q \in \mathcal{Q} : Q(\cdot) = Q^F(\cdot;C) \text{ for some } C(\cdot) \in \mathcal{C}\right\},$$

where $\mathcal{Q}$ is the class of quantile functions.

One practical question is how to choose the family of copula functions. As argued in previous sections, there is a large spectrum of APV auction models between the IPV case and the case with perfectly positively correlated private values, each with a level of dependence among the private values. Therefore, the level of dependence is crucial in the APV framework, and the choice of $\mathcal{C}$ could be based upon either the researcher's prior knowledge on the dependence level of bidders' private values or the set of dependence structures that the researcher chooses to check against the IPV paradigm. Common measures of dependence level include Kendall's τ and Spearman's ρ, and it is natural to let one dependence measure vary in certain range to get a pre-specified family of copula functions. For example, Zheng and Klein (1995) analyze data from a clinical trial of patients with non-Hodgkin's lymphoma. Based on discussions with a physician at the Ohio State Medical College, they believe that the Kendall's τ is between 0.25 and 0.5. They use this information to get bounds on the survival function for the medical treatment. In choosing the family of copula functions, a good candidate is again the Archimedean family. This family is flexible in the sense that it can allow any level of positive dependence ranging from independent to perfectly positively correlated. We make the following assumptions.

Assumption (AP) The true generator function $\varphi_o(t), t \in (0,1]$ belongs to a known parametric family Φ_Θ of generator functions with a single parameter $\theta \in \Theta$, and the true generator function is denoted as $\varphi_{\theta_o}(t)$.

Assumption (IN) The true Archimedean copula generator $\varphi_{\theta_o}(t)$ and the marginal distribution function $F_o(v)$ of private value are invariant to the number of bidders.

The class of copula generator functions is given by $\Phi_\Theta = \{\varphi : \varphi(\cdot) = \varphi_\theta(\cdot) \text{ for some } \theta \in \Theta\}$ under Assumption (AP). Upon specifying the parametric class of copula functions, the level of dependence among the private values can be conveniently represented by the parameter θ. Let us illustrate this using Kendall's τ. One version of the multivariate Kendall's τ is simply the average of pairwise Kendall's τ,[7] that is,

$$\tau(V_1, \cdots, V_n) = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \tau(V_i, V_j).$$

In the symmetric bidders' case, the multivariate Kendall's τ is simply $\tau(V_i, V_j)$ for any pair (V_i, V_j). Under Assumption (AP), it is known from Corollary 5.1.4 in Nelsen (2006) that

$$\tau(\theta) = 1 + 4 \int_0^1 \frac{\varphi_\theta(t)}{\varphi'_\theta(t)} dt.$$

For example, for the popular Archimedean copula families such as the Clayton, Frank, and Gumbel families, we have

$$\tau_C(\theta) = \frac{\theta}{\theta+2}, \theta > 0;\ \tau_F(\theta) = 1 - \frac{4}{\theta}\left(1 - \frac{1}{\theta}\int_0^\theta \frac{t}{e^t - 1}\right) dt, \theta > 0;\ \tau_G(\theta) = \frac{\theta - 1}{\theta}, \theta \ge 1.$$

First, all of the three functions are monotonic. Given this, specifying the range of Kendall's τ is equivalent to specifying the range of the copula parameter θ. Therefore, in the rest of this paper, we work with the copula parameter θ instead and assume that the true parameter θ_o lies in some known interval $[\underline{\theta}, \overline{\theta}]$. Second, the ranges of these functions are $[0, 1]$, hence the Archimedean copula families are flexible in that it can achieve any level of positive dependence, and thus they are rich enough for empirical works. In sum, Assumption (AP) reduces dimension of the space of copula generator functions and enables us to represent the level of dependence in a convenient way without much loss of generality. In a different context, Fan and Liu (2013) demonstrate

[7]For an alternative Kendall's τ formula in the multivariate case, see Genest et al. (2011).

numerically that the correct specification of the range of a dependence measure such as Kendall's τ is more important than the correct specification of the copula function form, see also Zheng and Klein (1995), Huang and Zhang (2008), Chen (2010), Hubbard, Li, and Paarsch (2012).

Assumption (IN) is an exogenous participation assumption, which will be used to tighten the bounds on the quantile function of the private value.

Now we are in a position to bound the quantile function of private value. Recall that $\varphi_o^*(p) = \varphi_o^{-1}(n\varphi_o(p))$ for $p \in [0,1]$. Calculating $C_{o,1}(\cdot)$ and $C_{o,12}(\cdot)$ under Assumption (AP), (3.8) is written as

$$Q^F(p;\theta) = q^{(n)}(\varphi_\theta^*(p)) - \frac{n\varphi_\theta'[\varphi_\theta^*(p)]}{(n-1)\varphi_\theta''[\varphi_\theta^*(p)]} q^{(n)\prime}(\varphi_\theta^*(p)), (p,\theta) \in [0,1] \times \Theta.$$

we have $Q_o(p) = Q^F(p;\theta_o)$, and the identified set for $Q_o(\cdot)$ is $\mathcal{Q}_\Theta^F = \{Q \in \mathcal{Q} : Q(\cdot) = Q^F(\cdot;\theta) \text{ for some } \theta \in \Theta\}$. The theorem below characterizes an outer set of $\mathcal{Q}_\Theta^F$ when $\Theta = [\underline{\theta}, \overline{\theta}]$.

Theorem 3.3.4 In a first-price sealed-bid auction with only the winning bid observed, let the true value $\theta_o \in [\underline{\theta}, \overline{\theta}]$. Under Assumptions (AS) and (AP), for any $p \in [0,1]$, $Q_o(p)$ satisfies that

$$\underline{Q}_n(p) \le Q_o(p) \le \overline{Q}_n(p),$$

where

$$\underline{Q}_n(p) = \inf_{\theta \in [\underline{\theta}, \overline{\theta}]} Q^F(p;\theta) \text{ and } \overline{Q}_n(p) = \sup_{\theta \in [\underline{\theta}, \overline{\theta}]} Q^F(p;\theta).$$

In addition, if Assumption (IN) holds, then the bounds on $Q_o(p)$ can be further tightened as

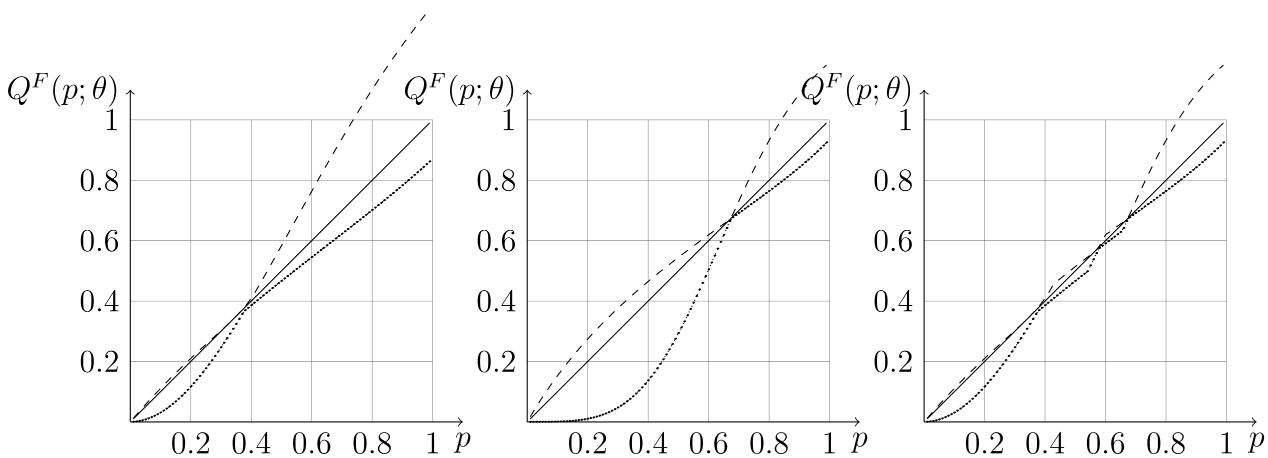
$$\sup_{n \in \mathcal{N}} \underline{Q}_n(p) \le Q_o(p) \le \inf_{n \in \mathcal{N}} \overline{Q}_n(p)$$

where $\mathcal{N}$ is the set of all observed numbers of bidders in all auctions.

Proof. The result is immediate from the expression of $Q^F(p;\theta)$ and the tightened bounds is trivial under Assumption (IN). ■

We illustrate our bounds in the following example.

Example 3.3.5 Let each private value be uniformly distributed on $[0,1]$, and the true copula be in the Clayton family with $\theta_o = 1$ ($\tau_o = 1/3$). For the Clayton family, we have $\theta = 2\tau/(1-\tau)$. Suppose the researcher's information is that $\tau_o \in (0, 1/2]$ (equivalently $[\underline{\theta}, \overline{\theta}] = (0, 2]$). The bounds for the true quantile function are depicted in Figure 3.1. Panels (a) and (b) show the bounds when $n = 2$ and $n = 4$, respectively, and panel (c) shows the tightened bounds when the number of bidders varies exogenously in $\{2, 3, 4\}$.



(a) $n = 2$ (b) $n = 4$ (c) $n = 2, 3, 4$

— true quantile - - - upper bound ······ lower bound

Figure 3.1: Bounds on the Quantile Function of Private Value

Partial Identification of Policy Parameters

While the model primitives are of their own theoretical interest, they serve as bases to answer certain policy questions related to welfare analysis. First, from the policy makers' point of view, they are interested in assessing the total welfare of a first-price sealed-bid auction under any reserve price, and in comparing its efficiency with other allocation mechanisms, such as the random assignment mechanism. Second, given that a first-price sealed-bid auction is chosen, the seller is interested in knowing how her expected revenue varies if she sets different reserve prices, and this is related to the so-called optimal reserve price. Third, the policy makers are interested in knowing how the bidders' expected surplus varies with the reserve price. Fourth, from the perspective of a potential participant of the auction, she is interested in knowing whether it is beneficial in the long run to participate in an auction similar to those observed in the sample. Answers to these questions are of practical importance. For example, in the APV framework, Li, Perrigne, and Vuong (2003) estimate the optimal reserve price for the U.S. Outer Continental Shelf (OCS) wildcat auctions. They find that the federal government's expected revenue would be around 50% larger than the actual revenue if the optimal reserve price were used.

Example 3.3.6 To illustrate the importance of the reserve price r on the policy parameters, we use the same setup as in Example 3.3.5 with $n = 2$. For the seller's own valuation, we set $v_o = 0$. The relations between the policy parameters and reserve price r are depicted in Figure 3.2.

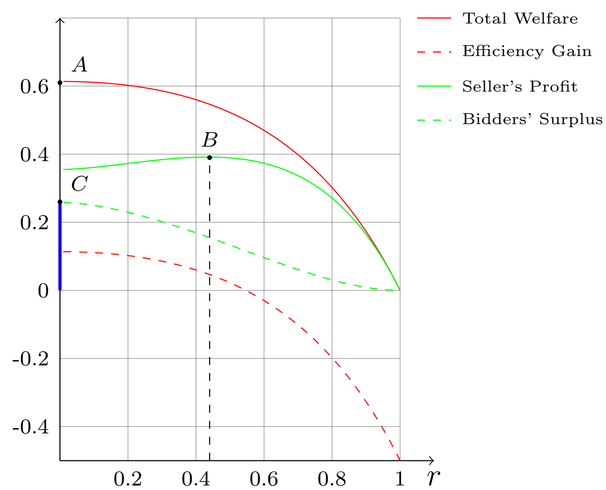


Figure 3.2: Welfare Implication of Reserve Price

In Figure 3.2, the total welfare is maximized at $r = v_o = 0$, represented by point A; the seller's expected revenue is maximized at $r^* = 0.44$, represented by point B;

and the bidders' expected surplus is maximized at $r = \underline{v} = 0$, represented by point C. As a special case of the bidders' expected surplus, the length of the blue segment measures the size of the bidders' expected surplus from a first-price sealed-bid auction without reserve price, which is proportional to the expected benefit of a potential bidder to participate in an auction similar to those observed in the sample. When $r = v_o$, the total welfare of a first-price sealed-bid auction, its efficiency gain over the random assignment mechanism, the seller's expected revenue, and the bidders' expected surplus are 0.614, 0.114, 0.355, and 0.259, respectively. When $r = r^*$, the four parameters are 0.547, 0.047, 0.392, and 0.155, respectively. As r changes from v_o to r^*, while the seller's expected revenue increases by 10.4%, the expected total welfare actually decreases by 10.9%, and the bidders' expected surplus decreases by 40.2%.

Given the practical importance of these policy parameters, we address their identification/partial identification issue in the following. On the one hand, Theorem 3.3.4 implies that if the true parameter θ_o is known, then the joint distribution of the private values is identified, so are these policy parameters expressible as functionals of the joint distribution function. On the other hand, in the case that θ_o is only known to lie in the interval $[\underline{\theta}, \overline{\theta}]$, these policy parameters can only be partially identified instead of being pinned down to a singleton.

With a reserve price $r \in [v_o, \overline{v}]$, the expected total welfare of a first-price sealed-bid auction, its efficiency gain over the random assignment mechanism, the seller's expected revenue, and the bidders' expected surplus are written as[8]

$$\omega_T(r) = \omega_S(r) + \omega_B(r) = v_o F^{(n)}(r) + \int_r^{\overline{v}} v dF^{(n)}(v),\ \omega_E(r) = \omega_T(r) - \mathbb{E}[V_i],$$

[8]By taking derivative of $\omega_T(r)$ or $\omega_E(r)$ with respect to r, it is easy to see that the expected total welfare and its efficiency gain over the random assignment mechanism are both maximized at $r = v_o$. Then identification or partial identification of the two functions are unnecessary if choosing r to maximize them is the only aim. However, knowing the whole functions of $\omega_T(r)$ or $\omega_E(r)$ is still of interest. For example, the policy makers might be interested in how fast $\omega_T(r)$ decreases as r increases starting from v_o. If $\omega_T(r)$ does not decrease very fast, then the policy makers have an option to choose $r \in (v_o, r^*)$ to balance the gain of the seller and the bidders without much loss in the expected total welfare.

$$\omega_S(r) = v_o\mathbb{P}(V^{(n)} \le r) + \mathbb{E}[s_r(V^{(n)})|V^{(n)} \ge r]\mathbb{P}(V^{(n)} \ge r) = v_oF^{(n)}(r) + \int_r^{\overline{v}} s_r(v)dF^{(n)}(v),$$

$$\omega_B(r) = \mathbb{E}[V^{(n)} - s_r(V^{(n)})|V^{(n)} \ge r]\mathbb{P}(V^{(n)} \ge r) = \int_r^{\overline{v}} (v - s_r(v))dF^{(n)}(v),$$

where $V^{(n)}$ is the highest private value with distribution function $F^{(n)}(\cdot)$, $s_r(v), v \in [r, \overline{v}]$ is the equilibrium bidding function under reserve price r. From Milgrom and Weber (1982), we have

$$s_r(v) = v - \int_r^v L(s|v)ds, v \in [r, \overline{v}],$$

where $L(s|v) = \exp[-\int_s^v f_{Y_1|V_1}(u|u)/F_{Y_1|V_1}(u|u)du]$, $Y_1 = \max_{j\neq 1} V_j$, $F_{Y_1|V_1}(y_1|v_1)$ is the distribution function of Y_1 conditional on $V_1 = v_1$, with the conditional density function $f_{Y_1|V_1}(y_1|v_1)$. We make the following assumption.

Assumption (PO) For any $t \in (0, 1]$, $\varphi'_\theta(t)/\varphi''_\theta(t)$ is strictly increasing in θ and $\varphi'_\theta(t)/\varphi''_\theta(t) \to 0$ as $\theta \to \infty$, where $\varphi'_\theta(t), \varphi''_\theta(t)$ denote the first and second partial derivatives of $\varphi_\theta(t)$ with respect to t, respectively.

In Assumption (PO), $\varphi'_\theta(t)/\varphi''_\theta(t)$ being monotonically increasing in θ is equivalent to $\varphi'_{\theta_1}(t)/\varphi'_{\theta_2}(t)$ being monotonically increasing in t on $(0, 1]$ for $\theta_1 < \theta_2$. By a straightforward generalization of Corollary 4.4.6 in Nelsen (2006) to the multivariate case, we have $C_{\theta_1}(u_1, \cdots, u_n) \le C_{\theta_2}(u_1, \cdots, u_n)$ for any $(u_1, \cdots, u_n) \in [0, 1]^n$, denoted as $C_{\theta_1} \prec C_{\theta_2}$. Therefore, the parametric family Φ_Θ is positively ordered under Assumption (PO). Many parametric families of Archimedean copula, including the Clayton, Frank and Gumbel families, satisfy Assumption (PO).[9] The following theorem establishes that the four policy parameters above are partially identified.

Theorem 3.3.7 In a first-price sealed-bid auction, if the true parameter θ_o lies in a known interval $[\underline{\theta}, \overline{\theta}]$, then under Assumptions (AS), (AP), and (PO), the expected total

[9] In fact, any Archimedean copula family in Table 4.1 of Nelsen (2006) with the Fréchet-Hoeffding upper bound as limit satisfies Assumption (PO). Moreover, besides the popular Clayton, Frank and Gumbel families, copula families #2, #6, #13, #15, #17, #20, #21 in Table 4.1 of Nelsen (2006) both satisfy Assumption (PO) and allow the level of dependence to range from independent to perfectly positively correlated.

welfare $\omega_T(r)$, the efficiency gain $\omega_E(r)$, the seller's expected revenue $\omega_S(r)$ and the bidders' expected surplus $\omega_B(r)$ are partially identified. Specifically, for any $r \in [\underline{v}, \overline{v}]$,

$$\omega_K(r) \in \left[\inf_{\theta\in\Theta_r} \omega_K(\theta; r), \sup_{\theta\in\Theta_r} \omega_K(\theta; r)\right], K = T, E, S, B,$$

where

$$\omega_T(\theta; r) = v_o\lambda(\theta) + \int_{\lambda(\theta)}^1 q^{(n)}(t)dt - \frac{n}{n-1}\int_{\lambda(\theta)}^1 \frac{\varphi_\theta'(t)}{\varphi_\theta''(t)} dq^{(n)}(t),$$

$$\omega_E(\theta; r) = \omega_T(\theta; r) - \int_0^1 \left(q^{(n)}(t) - \frac{n\varphi_\theta'(t)}{(n-1)\varphi_\theta''(t)} q^{(n)\prime}(t)\right) d\varphi_\theta^{*-1}(t),$$

$$\omega_S(\theta; r) = v_o\lambda(\theta) + \int_{\lambda(\theta)}^1 q^{(n)}(t)dt - \frac{n}{n-1}\frac{\varphi_\theta'(\lambda(\theta))}{\varphi_\theta''(\lambda(\theta))} q^{(n)\prime}(\lambda(\theta)) \int_{\lambda(\theta)}^1 \left[\frac{\varphi_\theta'(t)}{\varphi_\theta'(\lambda(\theta))}\right]^{\frac{n-1}{n}} dt,$$

$$\omega_B(\theta; r) = -\frac{n}{n-1}\int_{\lambda(\theta)}^1 \frac{\varphi_\theta'(t)}{\varphi_\theta''(t)} dq^{(n)}(t) + \frac{n}{n-1}\frac{\varphi_\theta'(\lambda(\theta))}{\varphi_\theta''(\lambda(\theta))} q^{(n)\prime}(\lambda(\theta)) \int_{\lambda(\theta)}^1 \left[\frac{\varphi_\theta'(t)}{\varphi_\theta'(\lambda(\theta))}\right]^{\frac{n-1}{n}} dt,$$

$\varphi_\theta^{*-1}(t) = \varphi_\theta^{-1}(\frac{1}{n}\varphi_\theta(t))$, $\lambda(\theta)$ is the implicit function determined by the constraint $R_1(\lambda(\theta), \theta) = r$, $r \in [\underline{v}, \overline{v}]$, where

$$R_1(t, \theta) = q^{(n)}(t) - \frac{n}{n-1}\frac{\varphi_\theta'(t)}{\varphi_\theta''(t)} q^{(n)\prime}(t).$$

Given that $\theta_o \in [\underline{\theta}, \overline{\theta}]$, under Assumption (PO), existence of such an implicit function depends on the values of θ and r in the following way: for $r \in [\underline{v}, \overline{b}]$, $\lambda(\theta)$ exists for any θ; for $r \in (\overline{b}, \overline{v}]$, $\lambda(\theta)$ exists for $\theta \in [\underline{\theta}, \overline{\theta}(r)]$, where $\overline{\theta}(r)$ is the value of θ such that $\lim_{t\uparrow 1} R_1(t, \theta) = r$. Let $\Theta_r = [\underline{\theta}, \overline{\theta}]$ if $r \in [\underline{v}, \overline{b}]$ and $\Theta_r = [\underline{\theta}, \overline{\theta} \wedge \overline{\theta}(r)]$ if $r \in (\overline{b}, \overline{v}]$. The existence of such an implicit function is illustrated in Figure 3.3.

As a special case of the bidders' expected surplus, when $r = \underline{v}$, $\omega_B(\underline{v})/n = \mathbb{E}[V^{(n)} - B^{(n)}]/n$ measures the expected profit for a potential participant to enter an auction similar to those in the sample, and it is partially identified as

$$\frac{\omega_B(\underline{v})}{n} \in \left[-\frac{1}{n-1}\int_0^1 \frac{\varphi_{\overline{\theta}}'(t)}{\varphi_{\overline{\theta}}''(t)} dq^{(n)}(t), -\frac{1}{n-1}\int_0^1 \frac{\varphi_{\underline{\theta}}'(t)}{\varphi_{\underline{\theta}}''(t)} dq^{(n)}(t)\right].$$

Proof. Refer to the proof of Theorem 3.4.4 in Appendix B where we show the results for the general case with binding reserve price. ∎

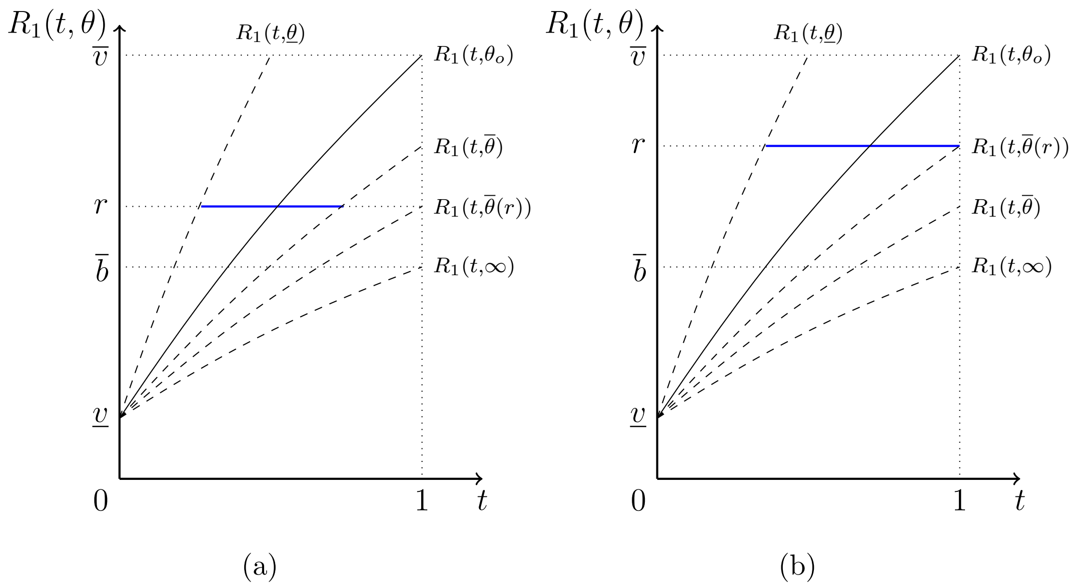


Figure 3.3: $R_1(t,\theta)$ and Existence of $\lambda(\theta)$

Further, we establish in the following theorem that the optimal reserve price is partially identified.

Theorem 3.3.8 Under the same setup as that in Theorem 3.3.7, the optimal reserve price, denoted as r^*, is partially identified. Specifically,

$$r^* \in \left[\inf_{[\underline{\theta},\overline{\theta}]} \{r : r = \delta(r;\theta)\}, \sup_{[\underline{\theta},\overline{\theta}]} \{r : r = \delta(r;\theta)\} \right],$$

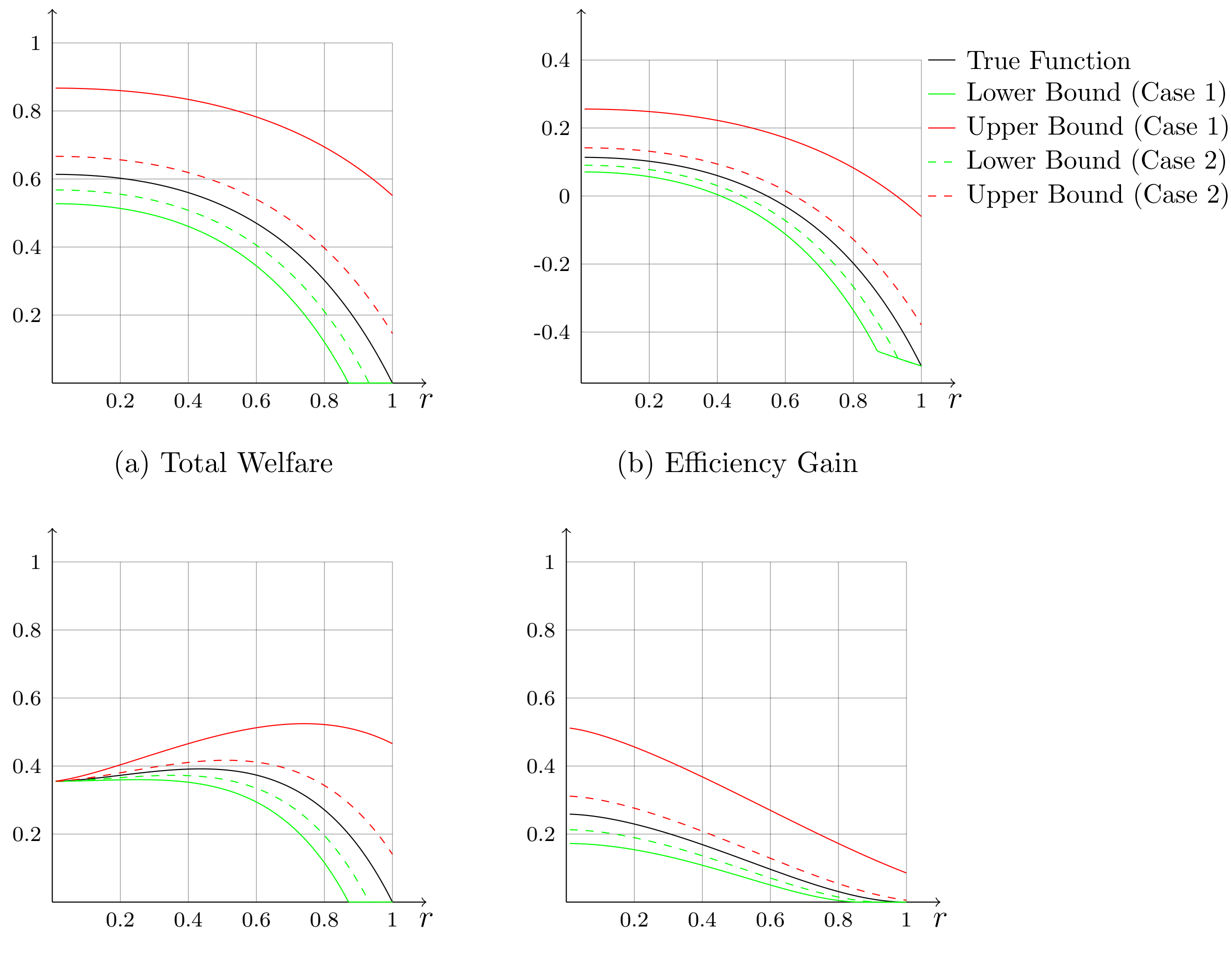
where

$$\delta(r;\theta) = v_o + \frac{1}{\lambda'(\theta)} \int_{\lambda(\theta)}^{1} \left[\frac{\varphi'_\theta(t)}{\varphi'_\theta(\lambda(\theta))} \right]^{\frac{n-1}{n}} dt,$$

$$\frac{1}{\lambda'(\theta)} = -\frac{q^{(n)\prime}(\lambda(\theta))}{n-1} + \frac{n}{n-1} \frac{\varphi'_\theta(\lambda(\theta))}{\varphi''_\theta(\lambda(\theta))} \left[\frac{\varphi'''_\theta(\lambda(\theta)) q^{(n)\prime}(\lambda(\theta))}{\varphi''_\theta(\lambda(\theta))} - q^{(n)\prime\prime}(\lambda(\theta)) \right],$$

and $\lambda(\theta)$ is subject to $R_1(\lambda(\theta),\theta) = r$.

Proof. Refer to the proof of Theorem 3.4.6 in Appendix B where we show the result for the general case with binding reserve price. ■

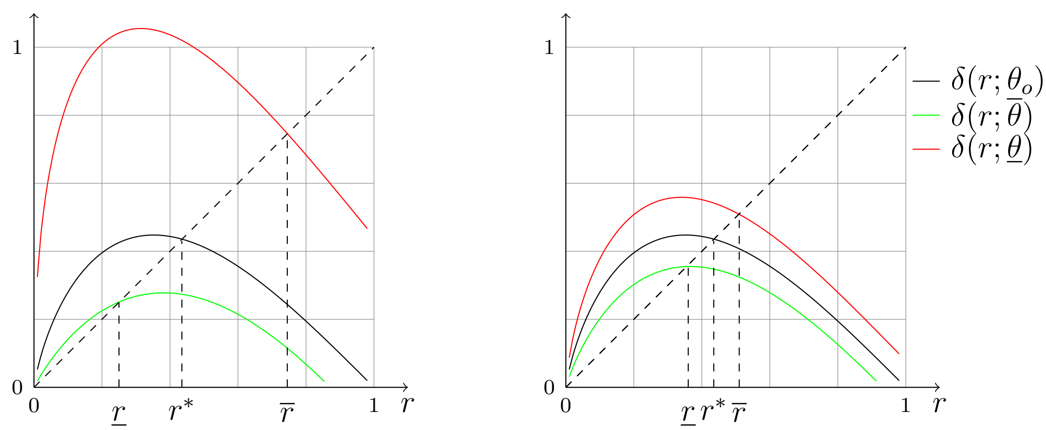


(a) Total Welfare (b) Efficiency Gain

(c) Seller's Expected Revenue (d) Bidders' Expected Surplus

Figure 3.4: Partial Identification of the Policy Parameters

Example 3.3.9 We continue Example 3.3.5 to illustrate our bounds in Theorem 3.3.7, and the bounds are graphed in Figure 3.4. We set $v_o = 0$, $\tau_o = 1/3$ (or $\theta_o = 1$), $n = 2$. In each subfigure of Figure 3.4, we graph the bounds for two cases, where Case 1 corresponds to the information that $\tau_o \in (0, 1/2]$ and thus $[\underline{\theta}, \overline{\theta}] = (0, 2]$, and Case 2 corresponds to the information that $\tau_o \in [1/4, 5/12]$ and thus $[\underline{\theta}, \overline{\theta}] = [2/3, 10/7]$. Therefore, one advantage of our approach is its flexibility. Whenever more precise information on the dependence level is known, the bounds are tighter, and the two bounds collapse when θ_o is known. Further, we illustrate our bounds on the optimal reserve price in Figure 3.5 under the same setup. The optimal reserve price in this example is $r^* = 0.44$. In Case 1, the bounds interval is $[0.29, 0.75]$, while in Case 2 when we have more precise information on $[\underline{\theta}, \overline{\theta}]$, the bounds interval shrinks to $[0.38, 0.51]$.[10]



(a) Case 1

(b) Case 2

Figure 3.5: Partial Identification of the Optimal Reserve Price

3.4 Extensions

3.4.1 Extension to Asymmetric Bidders and/or Binding Reserve Prices in the First-Price Sealed-Bid Auction

In the above discussion, we focused on the simple case with symmetric bidders and non-binding reserve price. However, this is usually not the case in real-world auctions. On the one hand, bidders are usually asymmetric ex-ante in that the joint distribution

[10] From Figure 3.5, while $\underline{r}$ and $\overline{r}$ are obtained by $\delta(r;\overline{\theta})$ and $\delta(r;\underline{\theta})$ in our example when the copula function is in the Clayton family, the bounds on the optimal reserve price are in general not obtained at the endpoints of $[\underline{\theta}, \overline{\theta}]$.

function $\overline{F}(\cdot)$ is not exchangeable in its arguments. On the other hand, in order to take advantage of her monopoly power to increase expected revenue from the auction, the seller often sets a reserve price that is high enough to be binding, i.e., $r_o \in (\underline{v}, \overline{v})$. In this case, only bidders with private values higher than the reserve price submit bids. The observed bids are truncated from below at the reserve price. To avoid confusion, we use r and r_o to denote counterfactual and observed reserve prices, respectively. The sample is from repeated auctions with reserve price r_o, and our interest lies in what would have happened if the reserve price were r. The natural question is which results in the previous section can be extended to the case with asymmetric bidders and/or binding reserve price. We address this in the following.

When the bidders are asymmetric and the reserve price r_o is binding, let $B_{ir_o} = \underline{v}$ if $V_i \in [\underline{v}, r_o)$ and $B_{ir_o} = s_{ir_o}(V_i)$ if $V_i \in [r_o, \overline{v}]$, let $B_{r_o}^{(n)} = \underline{v}$ if $V^{(n)} \in [\underline{v}, r_o)$ and $B_{r_o}^{(n)} = \max_i B_{ir_o}$ if $V^{(n)} \in [r_o, \overline{v}]$, where the subscript "$r_o$" denotes quantities from auctions with binding reserve price r_o, and $s_{ir_o}(\cdot)$ is bidder i's equilibrium bidding function under reserve price r_o, with the boundary condition $s_{ir_o}(r_o) = r_o$. Given that the copula function of the private values is $C_o(\cdot)$, it is easy to show that with binding reserve price, the joint distribution function of $(B_{1r_o}, \cdots, B_{nr_o})$ can still be expressed as $\mathbb{P}(B_{1r_o} \le b_1, \cdots, B_{nr_o} \le b_n) = C_o(G_{1r_o}(b_1), \cdots, G_{nr_o}(b_n))$ for $b_i \in [r_o, \overline{b}_{r_o}], i = 1, \cdots, n$, where $G_{ir_o}(b) = F_i(r_o)$ if $b \in [\underline{v}, r_o)$ and $G_{ir_o}(b) = F_i(s_{ir_o}^{-1}(b))$ if $b \in [r_o, \overline{b}_{r_o}]$, is the distribution function of B_{ir_o}, and $\overline{b}_{r_o} = s_{ir_o}(\overline{v}) = s_{jr_o}(\overline{v})$ for any $i \neq j$. Then the cdf of $B_{r_o}^{(n)}$, denoted as $G_{r_o}^{(n)}(b)$, is $G_{r_o}^{(n)}(b) = F^{(n)}(r_o)$ if $b \in [\underline{v}, r_o)$ and $G_{r_o}^{(n)}(b) = F^{(n)}(s_{r_o}^{-1}(b)) = C_o(F_1(s_{1r_o}^{-1}(b)), \cdots, F_n(s_{nr_o}^{-1}(b)))$ if $b \in [r_o, \overline{b}_{r_o}]$.

Given that we observe the highest bid $B_{r_o}^{(n)}$ and the winner's identity D provided that $B_{r_o}^{(n)} > r_o$, the two conditional distribution functions $G^{*(n)}(b) = \mathbb{P}(B_{r_o}^{(n)} \le b | B_{r_o}^{(n)} > r_o)$ and $G_{D,j}^*(b) = \mathbb{P}(B_{r_o}^{(n)} \le b, D = j | B_{r_o}^{(n)} > r_o)$ are identified directly from the sample as

$$G^{*(n)}(b) = \frac{G_{r_o}^{(n)}(b) - \sigma_1}{1 - \sigma_1}, G_{D,j}^*(b) = \frac{G_{D,j}(b; r_o) - G_{D,j}(r_o; r_o)}{1 - \sigma_1}, b \in [r_o, \overline{b}_{r_o}],$$

where $G_{D,j}(b;r_o) = \mathbb{P}(B_{r_o}^{(n)} \leq b, D = j)$ for $b \in [r_o, \overline{b}_{r_o}]$, and $\sigma_1 = G_{r_o}^{(n)}(r_o)$ is identified from data directly by the proportion of auctions without submission of any bids above r_o. Consequently, we also observe $q^{*(n)}(t) = G^{*(n)-1}(t)$, and $g^*_{D,j}(b) = \partial G^*_{D,j}(b)/\partial b = g_{D,j}(b;r_o)/(1-\sigma_1)$.

Further, if $B_{r_o}^{(n-1)}$ is observed in addition to $B_{r_o}^{(n)}$ and D, then $\mathbb{P}(B_{r_o}^{(n-1)} \leq m_i, B_{r_o}^{(n)} \leq b_i, D = i | B_{r_o}^{(n-1)} > r_o), r_o \leq m_i \leq b_i \leq \overline{b}_{r_o}$ and $\sigma_2 = G_{r_o}^{(n-1)}(r_o) = \mathbb{P}(B_{r_o}^{(n-1)} \leq r_o)$ are also identified from the sample. The following theorem establishes that when the highest two bids and the winner's identity are observed, the copula generator function $\varphi_o(t)$ is identified on the interval $[\sigma_1, 1]$ in the case with both asymmetric bidders and binding reserve price.

Theorem 3.4.1 When the highest two bids $B_{r_o}^{(n)}, B_{r_o}^{(n-1)}$ and the winner's identity D are observed in a first-price sealed-bid auction with asymmetric bidders and binding reserve price r_o, under Assumption (AS), the true copula generator function is identified as

$$\varphi_o(t) = \int_1^t \alpha \exp\left(-\int_0^s z_{ir_o}(u)du\right) ds,\ t \in [\sigma_1, 1], \forall\ i = 1, \cdots, n,$$

where $\alpha < 0$. In the above expression,

$$z_{ir_o}(u) = \frac{1-\sigma_2}{(1-\sigma_1)^2} \frac{\phi_{ir_o}\left(q^{*(n)}\left(\frac{u-\sigma_1}{1-\sigma_1}\right)\right)}{g^*_{D,i}\left(q^{*(n)}\left(\frac{u-\sigma_1}{1-\sigma_1}\right)\right)\sum_{j\neq i} g^*_{D,j}\left(q^{*(n)}\left(\frac{u-\sigma_1}{1-\sigma_1}\right)\right)}, u \in [\sigma_1, 1],$$

$$\phi_{ir_o}(b) = \left[\frac{\partial^2 \mathbb{P}(B_{r_o}^{(n-1)} \leq m_i, B_{r_o}^{(n)} \leq b_i, D = i | B_{r_o}^{(n-1)} > r_o)}{\partial m_i \partial b_i}\right]|_{m_i=b_i=b}, b \in [r_o, \overline{b}_{r_o}],$$

where σ_2 is identified from data directly by the proportion of auctions without submission of any bids or with only one bid above r_o.

Proof. See Appendix B. ■

Therefore, we have shown that the identification result of $\varphi_o(t)$ in Theorem 3.3.2 can be generalized to the case with both asymmetric bidders and binding reserve price. The copula generator function $\varphi_o(t)$ is identified on the interval $[\sigma_1, 1]$. Intuitively this is because we only have information on the joint distribution of $B_{r_o}^{(n)}, B_{r_o}^{(n-1)}$ on the support $\{(b_i, m_i) : m_i \in [r_o, \bar{b}_{r_o}], b_i \geq m_i\}$, so that the dependence structure below σ_1 is not revealed by the sample. As a special case, if the reserve price is not binding, we identify the copula generator function on $[0, 1]$. Further, it is obvious that $\varphi_o(t)$ is overidentified in the asymmetric bidders case. Therefore, Theorem 3.4.1 can serve as a base to test for the validity of Assumption (AS).

In order to discuss the implications of Theorem 3.4.1, we state the following lemma which adapts Lemma 1 in Braekers and Veraverbeke (2005) for the problem of competing risks with two risks.

Lemma 3.4.2 Suppose $\varphi_o(\cdot)$ is strict and $\varphi_o'(\cdot)$ exists on $(0, 1]$, then with asymmetric bidders and binding reserve price, for $i = 1, \cdots, n$,

$$G_{ir_o}(b) = G_{ir_o}(b; \varphi_o), b \in \left[r_o, \bar{b}_{r_o}\right],$$

where

$$G_{ir_o}(b; \varphi) = \varphi^{-1}\left[-(1-\sigma_1)\int_b^{\bar{b}_{r_o}} g_{D,i}^*(s)\,\varphi'\left((1-\sigma_1)G^{*(n)}(s)+\sigma_1\right)ds\right], b \in \left[r_o, \bar{b}_{r_o}\right].$$

Proof. See Appendix B. ■

Remark 3.4.3 In the case with asymmetric bidders with binding reserve price, the true copula generator function $\varphi_o(t)$ is identified on the interval $[\sigma_1, 1]$, thus by Lemma 3.4.2, $G_{ir_o}(b)$ and $g_{ir_o}(b) = G_{ir_o}'(b)$ are identified on $[r_o, \bar{b}_{r_o}]$. Or equivalently, $q_{ir_o}(p) = G_{ir_o}^{-1}(p)$ is identified on $[G_{ir_o}(r_o), 1]$. Under Assumption (AS), we have for $p \in [G_{ir_o}(r_o), 1]$,

$$Q_i(p) = q_{ir_o}(p) - \frac{\left(\varphi_o'\left[\varphi_o^{-1}\left(\sum_{j=1}^n \varphi_o(G_{jr_o}(q_{ir_o}(p)))\right)\right]\right)^2}{\sum_{j\neq i}\varphi_o'(G_{jr_o}(q_{ir_o}(p)))\varphi_o''\left[\varphi_o^{-1}\left(\sum_{j=1}^n \varphi_o(G_{jr_o}(q_{ir_o}(p)))\right)\right] g_{jr_o}(q_{ir_o}(p))}.$$

Then $Q_i(p)$ is identified on $[G_{ir_o}(r_o), 1]$ in the case with asymmetric bidders and binding reserve price. As a special case, when the bidders are asymmetric and the reserve price is not binding, the copula generator function as well as the quantile functions of private value bidders are all identified on $[0,1]$. Therefore, we identify the joint distribution function of private values and hence all the policy parameters.

In the following, we assume that the second highest bid $B_{r_o}^{(n-1)}$ is not observed and focus on the policy parameters, and we will show that the previous partial identification results on the expected total welfare, the seller's expected revenue as well as the bidders' expected surplus, can be extended to the case with symmetric bidders and binding reserve price. However, they do not extend to the asymmetric bidders' case even if the reserve price is not binding. This is because in the asymmetric bidders' case, each bidder has a different bidding function, and the highest bid is not necessarily submitted by the bidder with the highest value. From the proof of Theorem 3.3.7, it is also clear that the efficiency gain of a first-price sealed-bid auction with reserve price r over the random assignment mechanism cannot be extended to either the case with asymmetric bidders or the case with binding reserve price. Therefore, D is irrelevant in the following and only $B_{r_o}^{(n)}$ is observed in the sample.

Theorem 3.4.4 In a first-price sealed-bid auction with symmetric bidders and binding reserve price $r_o \in (\underline{v}, \overline{v})$, suppose that the highest bid $B_{r_o}^{(n)}$ is observed given that it is larger than r_o. If the true parameter θ_o lies in a known interval $[\underline{\theta}, \overline{\theta}]$, then under Assumptions (AS), (AP) and (PO), the total welfare $\omega_T(r)$, the seller's expected revenue $\omega_S(r)$, and the bidders' expected surplus $\omega_B(r)$, are partially identified for $r \in [r_o, \overline{v}]$. Specifically,

$$\omega_K(r) \in \left[\inf_{\theta \in \Theta_{rr_o}} \omega_K(\theta; r), \sup_{\theta \in \Theta_{rr_o}} \omega_K(\theta; r) \right], K = T, S, B,$$

where

$$\omega_T(\theta;r)=\omega_S(\theta;r)+\omega_B(\theta;r),$$

$$\omega_S(\theta;r)=v_o\left[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1\right]+(1-\sigma_1)\int_{\lambda_{r_o}(\theta)}^1 q^{*(n)}(t)dt$$
$$-\frac{n}{n-1}\frac{\varphi'_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]}{\varphi''_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]}q^{*(n)\prime}(\lambda_{r_o}(\theta))\int_{\lambda_{r_o}(\theta)}^1\left[\frac{\varphi'_\theta[(1-\sigma_1)t+\sigma_1]}{\varphi'_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]}\right]^{\frac{n-1}{n}}dt,$$

$$\omega_B(\theta;r)=-\frac{n}{n-1}\int_{\lambda_{r_o}(\theta)}^1\frac{\varphi'_\theta[(1-\sigma_1)t+\sigma_1]}{\varphi''_\theta[(1-\sigma_1)t+\sigma_1]}dq^{*(n)}(t)$$
$$+\frac{n}{n-1}\frac{\varphi'_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]}{\varphi''_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]}q^{*(n)\prime}(\lambda_{r_o}(\theta))\int_{\lambda_{r_o}(\theta)}^1\left[\frac{\varphi'_\theta[(1-\sigma_1)t+\sigma_1]}{\varphi'_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]}\right]^{\frac{n-1}{n}}dt,$$

where $\lambda_{r_o}(\theta)$ is subject to the restriction $R_2(\lambda_{r_o}(\theta),\theta)=r$, $r\in[r_o,\overline{v}]$, where

$$R_2(t,\theta)=q^{*(n)}(t)-\frac{n}{(n-1)(1-\sigma_1)}\frac{\varphi'_\theta[(1-\sigma_1)t+\sigma_1]}{\varphi''_\theta[(1-\sigma_1)t+\sigma_1]}q^{*(n)\prime}(t).$$

Given that $\theta_o\in[\underline{\theta},\overline{\theta}]$, under Assumption (PO), existence of such an implicit function depends on the value of r in the following way: for $r\in[r_o,\overline{b}_{r_o}]$, $\lambda_{r_o}(\theta)$ exists for any θ; for $r\in(\overline{b}_{r_o},\overline{v}]$, $\lambda_{r_o}(\theta)$ exists for $\theta\in[\underline{\theta},\overline{\theta}_{r_o}(r)]$, where $\overline{\theta}_{r_o}(r)$ is the value of θ such that $\lim_{t\uparrow 1}R_2(t,\theta)=r$. Let $\Theta_{rr_o}=[\underline{\theta},\overline{\theta}]$ if $r\in[r_o,\overline{b}_{r_o}]$ and $\Theta_{rr_o}=[\underline{\theta},\overline{\theta}\wedge\overline{\theta}_{r_o}(r)]$. Thus in the case with with symmetric bidders and binding reserve price, the partial identification result in Section 3.3.2 can be extended, but only on the interval $r\in[r_o,\overline{v}]$.

As a special case of the bidders' expected surplus, when $r=r_o$, $\omega_B(r_o)/n$ measures the expected profit for a potential participant to enter an auction with binding reserve price r_o as observed in the sample, and it is partially identified as

$$\frac{\omega_B(r_o)}{n}\in\left[-\frac{1}{n-1}\int_0^1\frac{\varphi'_{\overline{\theta}}[(1-\sigma_1)t+\sigma_1]}{\varphi''_{\overline{\theta}}[(1-\sigma_1)t+\sigma_1]}dq^{*(n)}(t),-\frac{1}{n-1}\int_0^1\frac{\varphi'_{\underline{\theta}}[(1-\sigma_1)t+\sigma_1]}{\varphi''_{\underline{\theta}}[(1-\sigma_1)t+\sigma_1]}dq^{*(n)}(t).\right]$$

Proof. See Appendix B. ■

Remark 3.4.5 It is clear that when the reserve price is not binding, that is, $r_o=\underline{v}$, we have $\sigma_1=0$, $q^{*(n)}(t)=q^{(n)}(t)$, straightforward simplification gives the results of the expected total welfare, the seller's expected revenue, and the bidders' expected surplus in Theorem 3.3.7.

Theorem 3.4.6 Under the same setup as that in Theorem 3.4.4, the optimal reserve price r^* is partially identified provided that $r_o \leq r^*$. Specifically,

$$r^* \in \left[\inf_{[\underline{\theta},\overline{\theta}]} \{r : r = \delta_{r_o}(r;\theta)\}, \sup_{[\underline{\theta},\overline{\theta}]} \{r : r = \delta_{r_o}(r;\theta)\} \right],$$

where

$$\delta_{r_o}(r;\theta) = v_o + \frac{1}{\lambda'_{r_o}(\theta)} \int_{\lambda_{r_o}(\theta)}^{1} \left[\frac{\varphi'_\theta[(1-\sigma_1)t+\sigma_1]}{\varphi'_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]} \right]^{\frac{n-1}{n}} dt,$$

$$\frac{1}{\lambda'_{r_o}(\theta)} = -\frac{q^{*(n)\prime}(\lambda_{r_o}(\theta))}{n-1} + \frac{n}{n-1} \frac{\varphi'_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]}{\varphi''_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]} \left(\frac{\varphi'''_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1] q^{*(n)\prime}(\lambda_{r_o}(\theta))}{\varphi''_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]} - \frac{q^{*(n)\prime\prime}(\lambda_{r_o}(\theta))}{1-\sigma_1} \right),$$

and $\lambda_{r_o}(\theta)$ is subject to $R_2(\lambda_{r_o}(\theta), \theta) = r$.

Proof. See Appendix B. ■

3.4.2 Extension to Partial Identification in Second-Price Sealed-Bid Auctions with Symmetric Bidders

The second-price sealed-bid auction is also used in the real world, in which all bidders submit bids simultaneously, the bidder with highest bid wins the object and pays the second highest bid. In equilibrium, one bidder bids her own private value (see, e.g., Milgrom and Weber (1982)). The second-price sealed-bid auction has a long history in the markets of paper collectibles, from Civil War soldiers' letters to postage stamps, see Reiley (2000). For example, the auction house Sam Houston Philatelics uses the second-price sealed-bid auction to sell stamps. It is common that only the transaction price is available in a second-price sealed-bid auction data set. Another related auction format is the ascending auction. Following the button model in Milgrom and Weber (1982), the ascending auction becomes strategically equivalent to a second-price sealed-bid auction and the Bayesian Nash equilibrium bidding strategy is again to bid her own private value.[11] Examples of ascending auctions in which only the transaction price is observed include the wheat auctions in India studied by Banerji and Meenakshi (2004), the United States Forest Service (USFS) timber auctions studied by Aradilla-Lopez, Gandhi and Quint (2013), and the fish auctions analyzed by Brendstrup and Paarsch (2006). Given the practical importance of these two strategically equivalent auction formats, in the following, we use similar idea as in the previous sections to study what can be learned when only the transaction price is observed. we focus on the second-price sealed-bid auction and consider the simple case with symmetric bidders and non-binding reserve price.

Partial Identification of the Quantile Function of Private Value

In a second-price sealed-bid auction, the transaction price is the second highest private value, denoted as $V^{(n-1)}$. The distribution function $F^{(n-1)}(v)$ of $V^{(n-1)}$ is directly identified from the data. Using a copula representation, we can write

$$\begin{aligned} F^{(n-1)}(v) &= \mathbb{P}(V^{(n-1)} \leq v, V^{(n)} > v) + \mathbb{P}(V^{(n)} \leq v) \\ &= n\mathbb{P}(V_1 > v, V_2 \leq v, \cdots, V_n \leq v) + \mathbb{P}(V^{(n)} \leq v) \\ &= nC_o(1, F_o(v), \cdots, F_o(v)) - (n-1)C_o(F_o(v), \cdots, F_o(v)). \end{aligned} \tag{3.9}$$

Let $F_o(v) = p$, then $v = Q_o(p)$, and we get

$$Q_o(p) = Q^{(n-1)}\left(nC_o(1, p, \cdots, p) - (n-1)C_o(p, \cdots, p)\right), p \in [0, 1], \tag{3.10}$$

where $Q^{(n-1)}(p)$ is the quantile function of $V^{(n-1)}$. From the above expression, $Q_o(p)$ is identified if the copula function is known. One special case is the IPV paradigm, in which we have point identification as $Q_o(p) = Q^{(n-1)}(np^{n-1} - (n-1)p^n)$. Further, if

[11] As discussed in Haile and Tamer (2003), the button model rules out the possibilities of jumping bids and/or late bidding that could occur in oral English auctions in particular in some of the internet auctions. Haile and Tamer (2003) relax assumptions in the button model and provide bounds on the model primitives.

the highest bid is also recorded in a second-price sealed-bid auction, then the identification of the Archimedean copula generator function in Theorem 3.3.2 carries over in a straightforward manner. In the general case that only $V^{(n-1)}$ is observed, the identified set for $Q_o(\cdot)$ when $C_o(\cdot)$ belongs to a known class $\mathcal{C}$ is

$$\mathcal{Q}_{\mathcal{C}}^S = \{Q \in \mathcal{Q} : Q(\cdot) = Q^S(\cdot; C) \text{ for some } C(\cdot) \in \mathcal{C}\},$$

where $Q^S(p;C) = Q^{(n-1)}\left(nC(1,p,\cdots,p) - (n-1)C(p,\cdots,p)\right)$, $\mathcal{Q}$ is the class of quantile functions, and the superscript "S" denotes second-price sealed-bid auction. Moreover, under Assumption (AP), let us write

$$Q^S(p;\theta) = Q^{(n-1)}\left(n\varphi_\theta^{-1}[(n-1)\varphi_\theta(p)] - (n-1)\varphi_\theta^{-1}[n\varphi_\theta(p)]\right),$$

with $Q_o(p) = Q^S(p;\theta_o)$. Then the identified set for $Q_o(\cdot)$ is

$$\mathcal{Q}_\Theta^S = \{Q \in \mathcal{Q} : Q(\cdot) = Q^S(\cdot;\theta) \text{ for some } \theta \in \Theta\}.$$

Thus if the copula function is not known, we could partially identify $Q_o(p)$ in a pointwise sense, and the following theorem summarizes the result.

Theorem 3.4.7 In a second-price sealed-bid auction, assume that the true parameter θ_o lies in some known interval $[\underline{\theta}, \overline{\theta}]$. Then under Assumptions (AS) and (AP), for any $p \in [0,1]$, the quantile function of private value is partially identified. Specifically,

$$\underline{Q}_n^S(p) \leq Q_o(p) \leq \overline{Q}_n^S(p),$$

where

$$\underline{Q}_n^S(p) = \inf_{\theta \in [\underline{\theta},\overline{\theta}]} Q^S(p;\theta), \overline{Q}_n^S(p) = \sup_{\theta \in [\underline{\theta},\overline{\theta}]} Q^S(p;\theta).$$

In addition, if Assumption (IN) holds, then the bounds on $Q_o(p)$ can be further tightened as

$$\sup_{n \in \mathcal{N}} \underline{Q}_n^S(p) \leq Q_o(p) \leq \inf_{n \in \mathcal{N}} \overline{Q}_n^S(p),$$

where $\mathcal{N}$ is the set of all observed numbers of bidders in all auctions.

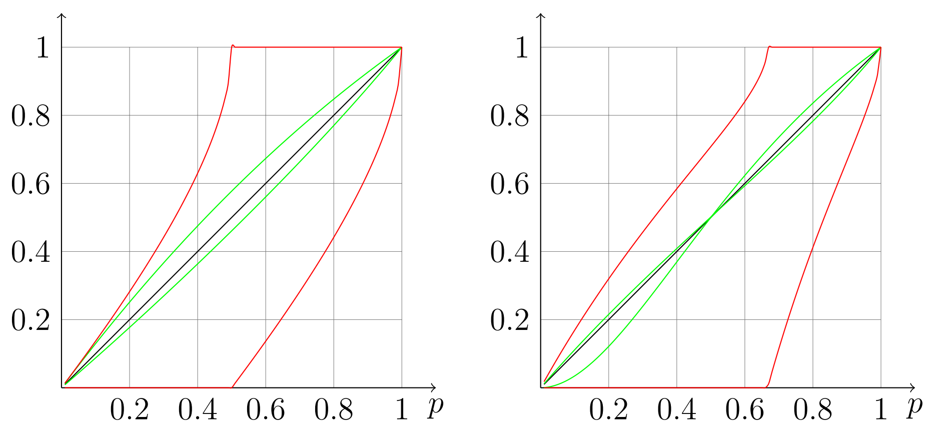
Notice that the bounds are generally not obtained at the endpoints of $[\underline{\theta},\overline{\theta}]$ even under Assumption (PO). This is because both $\varphi_\theta^{-1}[(n-1)\varphi_\theta(p)]$ and $\varphi_\theta^{-1}[n\varphi_\theta(p)]$ are increasing functions of θ under Assumption (PO), and the overall effect of θ on $Q^S(p;\theta)$ is not clear in general. One closely related work is Komarova (2013a), which allows for asymmetric bidders and arbitrary type of dependence among the bidders' private values. Her framework nests the general APV case, which in turn nests our model under Assumptions (AS) and (AP).[12] She provides bounds on the joint distribution function of private values for arbitrary subset of bidders. If we adapt the approach in Komarova (2013a) to our symmetric case for a single bidder, then it can be shown that the bounds on $Q_o(p)$ is

$$Q^{(n-1)}(np-n+1) \leq Q_o(p) \leq Q^{(n-1)}\left(\frac{np}{n-1}\right),$$

where $Q^{(n-1)}(np-n+1) = \underline{v}$ if $p \in [0,(n-1)/n]$ and $Q^{(n-1)}(np/(n-1)) = \overline{v}$ if $p \in [(n-1)/n, 1]$. Without additional information, bounds from this approach might be too wide since either the lower bound or the upper bound is at one of the endpoints of $[\underline{v},\overline{v}]$. While she mentions that it is difficult to use affiliation to construct bounds, here we provide an explicit way to explore the information on affiliation and use it to bound the quantile function of private value in the APV framework. It is natural that our bounds $[\underline{Q}_n^S(p), \overline{Q}_n^S(p)]$ is tighter than hers since we restrict the model to a smaller class and also restrict the level of dependence. The comparison between our bounds and the bounds $[Q^{(n-1)}(np-n+1), Q^{(n-1)}(np/(n-1))]$ is illustrated in Figure 3.6 using the same setup as in Example 3.3.5. It is seen that our bounds indeed are much narrower than the bounds in Komarova (2013a). It is worthwhile to mention that these are two different approaches. While Komarova (2013a) relies on certain constraints that observed bids impose on the private values to establish pointwise bounds on the distribution function, we express the quantile function explicitly in terms of quantities

[12] Assumption (AS) implies affiliation of private values, and Assumption (AP) further restricts it to a smaller class. Thus our model is a strict subset of the whole class of affiliated private values distributions.

identified from the sample and unknown component. we pin down the source of non-identification (which is the copula function) and vary it in a class to obtain partial identification. On the one hand, our approach is advantageous in the case that the level of dependence is partially known. On the other hand, her bounds are more robust and are suited for any type and any level of dependence. The two approaches complement with each other in bounding the marginal distribution function or quantile function of private values. Lastly, it is not clear how to translate her bounds on the joint distribution function of private values into bounds on the policy parameters. Our approach will be used to derive bounds explicitly on the policy parameters in the next section.



(a) $n = 2$ (b) $n = 3$

— True Quantile — Our Bounds — Bounds in Komarova (2013a)

Figure 3.6: Comparison between Bounds on the Quantile Function of Private Value in Second-Price Sealed-Bid Auction

Partial Identification of Policy Parameters

In a second-price sealed-bid auction setting, Aradilla-Lopez, Gandhi, and Quint (2013) provide partial identification results on the seller's expected revenue and the bidders' expected surplus by exploiting exogenous variation in the number of bidders. In the following, we show that there is another source to tighten their bounds on the two policy parameters. In addition, we also provide partial identification results for the other two policy parameters, namely, the total welfare of a second-price sealed-bid auction and its efficiency gain over the random assignment mechanism.

Let the seller's private value for the object be $v_o \in [\underline{v}, \overline{v}]$, with a reserve price $r \in [v_o, \overline{v}]$, the total welfare of a second-price sealed-bid auction, its efficiency gain over the random assignment mechanism, the seller's expected revenue and the bidders' expected surplus can be written respectively as

$$
\begin{aligned}
\varpi_T(r) &= \varpi_S(r) + \varpi_B(r) = \int_r^{\overline{v}} v dF^{(n)}(v) - v_o[1 - F^{(n)}(r)], \\
\varpi_E(r) &= \varpi_T(r) - \mathbb{E}[V_i] = \int_r^{\overline{v}} v dF^{(n)}(v) - v_o[1 - F^{(n)}(r)] - \int_{\underline{v}}^{\overline{v}} v dF_o(v), \\
\varpi_S(r) &= rF^{(n-1)}(r) + \int_r^{\overline{v}} v dF^{(n-1)}(v) - v_o - F^{(n)}(r)(r - v_o), \\
\varpi_B(r) &= \int_{\underline{v}}^{\overline{v}} \max\{r, v\} dF^{(n)}(v) - \int_{\underline{v}}^{\overline{v}} \max\{r, v\} dF^{(n-1)}(v).
\end{aligned}
$$

Aradilla-Lopez, Gandhi, and Quint (2013) focus on $\varpi_S(r)$ and $\varpi_B(r)$. In both $\varpi_S(r)$ and $\varpi_B(r)$, the only unknown quantity is $F^{(n)}(\cdot)$. They first show that for the general interdependent values case, $F^{(n)}(v) \in [(\psi_n(F^{(n-1)}(v)))^n, F^{(n-1)}(v)]$ for any $v \in [\underline{v}, \overline{v}]$, where the function $\psi_n : [0, 1] \to [0, 1]$ is defined implicitly by

$$t = \int_0^{\psi_n(t)} n(n-1)s^{n-2}(1-s)ds.$$

They argue that using these bounds on $F^{(n)}(v)$ might yield bounds on $\varpi_S(r)$ and $\varpi_B(r)$ that are too wide to be informative. Consequently, they exploit exogenous variation in the number of bidders to tighten the bounds. In the following, we show that there is another source to tighten their bounds. Notice that their original lower bound rely on the function $\psi_n(\cdot)$, which is essentially a map from the distribution function of the second-order statistic of n independent draws to the latent marginal distribution function. The lower bound on $F^{(n)}(v)$ corresponds to the IPV case, while upper bound on $F^{(n)}(v)$ corresponds to the case with perfectly positively correlated private values. These are the two extremes, and there is a large spectrum of cases in-between. It is natural to explore the cases between the IPV case and the case with perfectly positively correlated private values, which is particularly relevant when information on the range of the dependence level is available. In our framework, the level of dependence is represented by the copula parameter θ under Assumption (AP), and the bounds on $\varpi_S(r)$ and $\varpi_B(r)$ can be tightened using the information that $\theta_o \in [\underline{\theta}, \overline{\theta}]$. Following our approach, the four policy parameters above can be partially identified and the result is summarized in the following theorem.

Theorem 3.4.8 In a second-price sealed-bid, suppose the true copula parameter θ_o lies in a known interval $[\underline{\theta}, \overline{\theta}]$, then under Assumptions (AS) and (AP), the total welfare, its efficiency gain over the random assignment mechanism, the seller's expected revenue, and the bidders' expected surplus are partially identified for any reserve price $r \in [\underline{v}, \overline{v}]$. Specifically,

$$\varpi_K(r) \in \left[\inf_{\theta \in [\underline{\theta}, \overline{\theta}]} \varpi_K(\theta; r), \sup_{\theta \in [\underline{\theta}, \overline{\theta}]} \varpi_K(\theta; r) \right], K = T, E, S, B,$$

where

$$\begin{aligned}
\varpi_T(\theta; r) &= \int_{Q^{(n)-1}(r;\theta)}^{1} Q^{(n)}(t;\theta)dt - v_o \left[1 - Q^{(n)-1}(r;\theta)\right], \\
\varpi_E(\theta; r) &= \int_{Q^{(n)-1}(r;\theta)}^{1} Q^{(n)}(t;\theta)dt - v_o \left[1 - Q^{(n)-1}(r;\theta)\right] \\
&\quad - \int_0^1 Q^{(n-1)} \Big(n\varphi_\theta^{-1}[(n-1)\varphi_\theta(t)] - (n-1)\varphi_\theta^{-1}[n\varphi_\theta(t)] \Big) dt,
\end{aligned}$$

$$\begin{aligned}
\varpi_S(\theta; r) &= rQ^{(n-1)-1}(r) + \int_{Q^{(n-1)-1}(r)}^{1} Q^{(n-1)}(t)dt - v_o - Q^{(n)-1}(r;\theta)(r - v_o), \\
\varpi_B(\theta; r) &= \int_0^1 \max\{r, Q^{(n)}(t;\theta)\}dt - \int_0^1 \max\{r, Q^{(n-1)}(t)\}dt,
\end{aligned}$$

where $Q^{(n)-1}(t;\theta)$ is the inverse of $Q^{(n)}(t;\theta)$, $Q^{(n-1)-1}(t)$ is the inverse of $Q^{(n-1)}(t)$, and

$$Q^{(n)}(t;\theta) = Q^{(n-1)}\left[n\varphi_\theta^{-1}\left(\frac{n-1}{n}\varphi_\theta(t)\right) - (n-1)t\right].$$

Proof. See Appendix B. ■

3.5 Conclusion

In this paper, we address the interesting question that what can be learned when only an incomplete set of bids are available from the first-price sealed-bid auctions with affiliated private values. In the case with symmetric bidders and non-binding reserve price, we achieve novel identification result that for copula in the strict Archimedean family with generator function twice continuously differentiable and its inverse completely monotone, the highest two bids are sufficient for identifying the joint distribution function of private values. When only the highest bids are observed, the joint distribution function of private values is in general unidentified. The lack of identification in this scenario arises from the lack of information on the true copula function. We parameterize the copula function and introduce the level of dependence among the private values as a source for partial identification. Specifically, we vary this dependence level in a pre-specified set to partially identify the quantile function of private value as well as policy parameters, including the total welfare of an auction under any reserve price, its efficiency gain over the random assignment mechanism, the seller's expected revenue, the bidders' expected surplus, and the optimal reserve price. Further, we show that some of our results readily generalize to the cases with asymmetric bidders and/or binding reserve price. Finally, we extend our idea to establish partial identification in the second-price sealed-bid auction.

In our identification/partial identification approach, it is clear that the information contained in the copula function is crucial. On the one hand, while we restrict ourselves to the parametric Archimedean family when only the highest bids are observed, our partial identification approach applies to any parametric family of copula functions.

On the other hand, while we achieve identification of the joint distribution function of private values under Assumption (AS) when the highest two bids are observed, this has limitation since any Archimedean copula function is by definition symmetric among its arguments. It is of great interest to know that in more general family of copula functions that allows for asymmetry among the arguments, what information is necessary and/or sufficient for identification of the model primitives. This is left for future research.

Appendix A

PROOFS OF CHAPTER 1

Proof. (Proposition 1.2.2) Let the seller's own valuation be v_o, denote the equilibrium bidding function under r as $\beta_r(x), x \in [x_r^*, \overline{x}]$ and the equilibrium bidding function without reserve price as $\beta(x), x \in [0, \overline{x}]$. The seller's revenue under reserve price r is

$$\begin{aligned}
\pi_S(r) &= v_o \mathbb{1}\left(\beta_r(X^{(M)}) < r\right) + \beta_r\left(X^{(M)}\right) \mathbb{1}\left(\beta_r\left(X^{(M)}\right) \geq r\right) \\
&= v_o \mathbb{1}\left(X^{(M)} < x_r^*\right) + \beta_r\left(X^{(M)}\right) \mathbb{1}\left(X^{(M)} \geq x_r^*\right) \\
&\overset{(1)}{=} v_o \mathbb{1}\left(\beta(X^{(M)}) < \beta(x_r^*)\right) + \left(\beta(X^{(M)}) + (r - \beta(x_r^*)) J(x_r^* | X^{(M)})\right) \mathbb{1}\left(\beta(X^{(M)}) \geq \beta(x_r^*)\right) \\
&\overset{(2)}{=} v_o \mathbb{1}\left(B^{(M)} < b_r^*\right) + \left(B^{(M)} + (r - b_r^*) J^*(b_r^* | B^{(M)})\right) \mathbb{1}\left(B^{(M)} \geq b_r^*\right),
\end{aligned}$$

where $b_r^* = \beta(x_r^*)$, (1) follows from

$$\begin{aligned}
\beta_r(x) - \beta(x) &= \left[r - \overline{H}(x_r^*)\right] J(x_r^* | x) + \int_0^{x_r^*} J(a|x) d\overline{H}(a) \\
&= \left[r - \overline{H}(x_r^*)\right] J(x_r^* | x) + J(x_r^* | x) \int_0^{x_r^*} J(a|x_r^*) d\overline{H}(a) \\
&= \left[r - \overline{H}(x_r^*)\right] J(x_r^* | x) + J(x_r^* | x) \left[\overline{H}(x_r^*) - \beta(x_r^*)\right] \\
&= \left[r - \beta(x_r^*)\right] J(x_r^* | x), x \in [x_r^*, \overline{x}],
\end{aligned}$$

and (2) follows from

$$\begin{aligned}
J(x_r^* | x) &= \exp\left(-\int_{x_r^*}^{x} \rho_{Y_1|X_1}(s) ds\right) \\
&= \exp\left(-\int_{\beta(x_r^*)}^{\beta(x)} \rho_{M_1|B_1}(\beta(s)) \beta'(s) ds\right) \\
&= \exp\left(-\int_{\beta(x_r^*)}^{\beta(x)} \rho_{M_1|B_1}(t) dt\right) \equiv J^*(b_r^* | \beta(x)).
\end{aligned}$$

Taking expectation yields that the seller's expected revenue under reserve price r is

$$\mathbb{E}[\pi_S(r)] = v_o \mathbb{E}[\mathbb{1}(B^{(M)} < b_r^*)] + \mathbb{E}[\pi_P(r)],$$

where $\mathbb{E}[\pi_P(r)] = \mathbb{E}\left[\left(B^{(M)} + (r - b_r^*)J^*(b_r^*|B^{(M)})\right)\mathbb{1}\left(B^{(M)} \geq b_r^*\right)\right]$.

Next, the bidders' surplus under reserve price r is

$$\begin{aligned}\pi_B(r) &= \left(X_o - \beta_r(X^{(M)})\right)\mathbb{1}\left(\beta_r(X^{(M)}) \geq r\right)\\ &= X_o \mathbb{1}\left(\beta_r(X^{(M)}) \geq r\right) - \beta_r(X^{(M)})\mathbb{1}\left(\beta_r(X^{(M)}) \geq r\right).\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{E}[\pi_B(r)]\\ &= \mathbb{E}\left[X_o \mathbb{1}\left(\beta_r(X^{(M)}) \geq r\right)\right] - \mathbb{E}[\pi_P(r)]\\ &= \mathbb{E}[X_o|X^{(M)} \geq x_r^*]\mathbb{E}[\mathbb{1}(X^{(M)} \geq x_r^*)] - \mathbb{E}[\pi_P(r)]\\ &= \left(\int_{x_r^*}^{\overline{x}} \mathbb{E}[X_o|X^{(M)} = x]dF_{X^{(M)}}(x|X^{(M)} \geq x_r^*)\right)\mathbb{E}[\mathbb{1}(X^{(M)} \geq x_r^*)] - \mathbb{E}[\pi_P(r)]\\ &= \int_{x_r^*}^{\overline{x}} \mathbb{E}[X_o|X^{(M)} = x]dF_{X^{(M)}}(x) - \mathbb{E}[\pi_P(r)]\\ &\overset{(1)}{=} \int_{x_r^*}^{\overline{x}}\int_0^x \overline{L}(x) f_{X_1Y_1}(u,x)dudx + \int_{x_r^*}^{\overline{x}}\int_0^x \overline{L}(x) f_{X_1Y_1}(x,u)dudx - \mathbb{E}[\pi_P(r)]\\ &\overset{(2)}{=} \int_{b_r^*}^{\overline{b}}\int_{\underline{b}}^{m_1} \overline{L}(\beta^{-1}(m_1))g_{B_1M_1}(b_1,m_1)db_1dm_1 + \int_{b_r^*}^{\overline{b}}\int_{\underline{b}}^{b_1} \overline{L}(\beta^{-1}(b_1))g_{B_1M_1}(b_1,m_1)dm_1db_1 - \mathbb{E}[\pi_P(r)]\\ &= \mathbb{E}\left[\mathbb{1}(M_1 \geq b_r^*)\mathbb{1}(B_1 \leq M_1)\overline{L}(\beta^{-1}(M_1))\right] + \mathbb{E}\left[\mathbb{1}(B_1 \geq b_r^*)\mathbb{1}(M_1 \leq B_1)\overline{L}(\beta^{-1}(B_1))\right] - \mathbb{E}[\pi_P(r)],\end{aligned}$$

where $\beta^{-1}(b) = \overline{H}^{-1}\left(b + \frac{1}{\rho_{M_1|B_1}(b)}\right)$. (1) follows from

$$\begin{aligned}\mathbb{E}[X_o|X^{(M)} = x] &= \frac{\mathbb{E}[X_o|X_1 \leq x, Y_1 = x]\int_0^x f_{X_1Y_1}(u,x)du}{f_{X^{(M)}}(x)} + \frac{\mathbb{E}[X_o|X_1 = x, Y_1 \leq x]\int_0^x f_{X_1Y_1}(x,u)du}{f_{X^{(M)}}(x)}\\ &= \frac{\overline{L}(x)\int_0^x f_{X_1Y_1}(u,x)du}{f_{X^{(M)}}(x)} + \frac{\overline{L}(x)\int_0^x f_{X_1Y_1}(x,u)du}{f_{X^{(M)}}(x)},\end{aligned}$$

and the last line in the derivation of $\mathbb{E}[X_o|X^{(M)}=x]$ is due to

$$\begin{aligned}\mathbb{E}[X_o|X_1\le x, Y_1=x] &= \sum_{m=2}^{M}\frac{1}{M-1}\mathbb{E}[X_o|X_1\le x, X_2\le x,\cdots,X_m=x,X_{m+1}\le x,\cdots,X_M\le x]\\ &= \sum_{m=2}^{M}\frac{1}{M-1}\mathbb{E}[X_o|X_m=x,Y_m\le x]\\ &= \mathbb{E}[X_o|X_1=x,Y_1\le x]=\overline{L}(x)\end{aligned}$$

under the symmetric bidders assumption. (2) follows from changes of variables using the facts that $F_{X^{(M)}}(x)=G_{B^{(M)}}(\beta(x))$, $F_{X_1Y_1}(x_1,y_1)=G_{B_1M_1}(\beta(x_1),\beta(y_1))$ and thus $f_{X_1Y_1}(x_1,y_1)=g_{B_1M_1}(\beta(x_1),\beta(y_1))\beta'(x_1)\beta'(y_1)$.[1]

■

Proof. (Lemma 1.3.2) Under Assumption (AS), we can write

$$\begin{aligned}\overline{H}(x) &= \mathbb{E}[X_o|X_1=x,Y_1=x]\\ &= \mathbb{E}\Big[\frac{1}{M}\sum_{m=1}^{M}X_m|X_1=x,Y_1=x\Big]\\ &= \frac{1}{M}\left(x+\mathbb{E}\Big[\sum_{m=2}^{M}X_m|X_1=x,Y_1=x\Big]\right)\\ &= \frac{1}{M}\big(x+(M-1)\mathbb{E}\big[X_3|X_1=x,Y_1=x\big]\big)\\ &= \frac{1}{M}\left[x+(M-1)\left(\mathbb{E}\big[X_3|X_1=x,Y_1=x,Y_1\neq X_3\big]\frac{M-2}{M-1}+\frac{\mathbb{E}[X_3|X_1=x,Y_1=x,Y_1=X_3]}{M-1}\right)\right]\\ &= \frac{2x}{M}+\frac{(M-2)}{M}\mathbb{E}\big[X_3|X_1=x,Y_1=x,Y_1\neq X_3\big]\\ &= \frac{2x}{M}+\frac{(M-2)}{M}\mathbb{E}\big[X_3|X_1=x,X_2=x,X_3<x,X_4\le x,\cdot\cdot\cdot,X_M\le x\big]\\ &\overset{(1)}{=} \frac{2x}{M}+\frac{M-2}{M}\int_0^x td\left(\frac{C_{o,12}(F_o(x),F_o(x),F_o(t),F_o(x),\cdot\cdot\cdot,F_o(x))}{C_{o,12}(F_o(x),\cdot\cdot\cdot,F_o(x))}\right)\\ &\overset{(2)}{=} x-\frac{M-2}{M}\frac{\int_0^x C_{o,12}(F_o(x),F_o(x),F_o(t),F_o(x),\cdot\cdot\cdot,F_o(x))dt}{C_{o,12}(F_o(x),\cdot\cdot\cdot,F_o(x))},\end{aligned}$$

[1] Notice that $\mathbb{E}[X_o|X^{(M)}\ge x_r^*]\mathbb{E}[\mathbb{1}(X^{(M)}\ge x_r^*)]=\mathbb{E}[X_o|B^{(M)}\ge b_r^*]\mathbb{E}[\mathbb{1}(B^{(M)}\ge b_r^*)]$. When the ex-post common value is known, $\mathbb{E}[X_o|B^{(M)}\ge b_r^*]$ is immediately identified if $\overline{H}(x)$ and $\overline{L}(x)$ are known. One example is the "scaled sale" timber auction in U.S. and Canada, where the quantity of each species of timber extracted from a tract is recorded by an independent agent at the time of harvest, see Athey and Levin (2001).

where (1) follows from the fact that for $t \leq x$,

$$\begin{aligned}
&\mathbb{P}(X_3 \leq t | X_1 = x, X_2 = x, X_3 < x, X_4 \leq x, \cdots, X_M \leq x) \\
=& \frac{\mathbb{P}(X_3 \leq t, X_4 \leq x, \cdots, X_M \leq x | X_1 = X_2 = x)}{\mathbb{P}(X_3 < x, X_4 \leq x, \cdots, X_M \leq x | X_1 = X_2 = x)} \\
=& \frac{C_{o,12}(F_o(x), F_o(x), F_o(t), F_o(x), \cdots, F_o(x))}{C_{o,12}(F_o(x), \cdots, F_o(x))},
\end{aligned}$$

and (2) is by a change of variable. Similarly, under Assumption (AS),

$$\begin{aligned}
\overline{L}(x) &= \mathbb{E}\Big[\frac{1}{M}\sum_{m=1}^{M} X_m | X_1 = x, Y_1 \leq x\Big] \\
&= \frac{1}{M}\left(x + \mathbb{E}\Big[\sum_{m=2}^{M} X_m | X_1 = x, Y_1 \leq x\Big]\right) \\
&= \frac{x}{M} + \frac{M-1}{M}\mathbb{E}\big[X_2 | X_1 = x, Y_1 \leq x\big] \\
&\overset{(1)}{=} \frac{x}{M} + \frac{M-1}{M}\int_0^x t d\left(\frac{C_{o,1}(F_o(x), F_o(t), F_o(x), \cdots, F_o(x))}{C_{o,1}(F_o(x), \cdots, F_o(x))}\right) \\
&\overset{(2)}{=} x - \frac{M-1}{M}\frac{\int_0^x C_{o,1}(F_o(x), F_o(t), F_o(x), \cdots, F_o(x))dt}{C_{o,1}(F_o(x), \cdots, F_o(x))},
\end{aligned}$$

where (1) follows from the fact that for $t \leq x$,

$$\begin{aligned}
\mathbb{P}(X_2 \leq t | X_1 = x, X_2 \leq x, \cdots, X_M \leq x) &= \frac{\mathbb{P}(X_2 \leq t, X_3 \leq x, \cdots, X_M \leq x | X_1 = x)}{\mathbb{P}(X_2 \leq x, \cdots, X_M \leq x | X_1 = x)} \\
&= \frac{C_{o,1}(F_o(x), F_o(t), F_o(x), \cdots, F_o(x))}{C_{o,1}(F_o(x), \cdots, F_o(x))},
\end{aligned}$$

and (2) is by a change of variable. ■

Proof. (Theorem 1.3.3) Under Assumption (AS), the restriction on $F_o(x)$ is $R_1(x; F_o, C_o) = 0$, where

$$\begin{aligned}
R_1(x; F, C_o) &= \overline{H}(x; F, C_o) - Q_{B_1}(F(x)) - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(F(x)))} \\
&= x - \frac{M-2}{M}\frac{\int_0^x C_{o,12}(F(x), F(x), F(t), F(x), \cdots, F(x))dt}{C_{o,12}(F(x), \cdots, F(x))} \\
&\quad - Q_{B_1}(F(x)) - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(F(x)))}.
\end{aligned}$$

By change of variables $F_o(x)=\tau$, $F_o(t)=s$ and thus $x=Q_o(\tau)$, $t=Q_o(s)$, we get the following equivalent restriction

$$
\begin{aligned}
& Q_o(\tau)-\frac{M-2}{M}\frac{\int_0^\tau C_{o,12}\left(\tau,\tau,s,\tau,\cdots,\tau\right)Q_o'(s)ds}{C_{o,12}\left(\tau,\cdots,\tau\right)}-Q_{B_1}(\tau)-\frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))}\\
=\ & Q_o(\tau)-\frac{M-2}{M}\left(Q_o(\tau)-\frac{\int_0^\tau C_{o,123}\left(\tau,\tau,s,\tau,\cdots,\tau\right)Q_o(s)ds}{C_{o,12}\left(\tau,\cdots,\tau\right)}\right)-Q_{B_1}(\tau)-\frac{1}{\rho_{M_1|B_1}(Q_{B_1}(\tau))}=0.
\end{aligned}
$$

Upon rearranging the restriction, we obtain

$$
Q_o(\tau)-\int_0^\tau k_{1o}(\tau,s)Q_o(s)ds=\phi_{1o}(\tau),
$$

where

$$
\phi_{1o}(\tau)=\frac{MQ_{B_1}(\tau)}{2}+\frac{M}{2\rho_{M_1|B_1}(Q_{B_1}(\tau))},\ \text{and}\ k_{1o}(\tau,s)=-\frac{(M-2)C_{o,123}\left(\tau,\tau,s,\tau,\cdots,\tau\right)}{2C_{o,12}\left(\tau,\cdots,\tau\right)}.
$$

The restriction above is a linear Volterra integral equation of the second kind. By definition, the true quantile function $Q_o(\tau)$ is solution to the integral equation. Further, under Assumption (CU-1), Theorem 2.1.1. in Burton (2005) or Theorem 3.12 in Kress (1999) can be readily applied and the above Volterra integral equation has a unique solution, thus the true quantile function $Q_o(\tau)$ of bidders' private signals is nonparametrically identified by the above Volterra integral equation. ■

Proof. (Proposition 1.3.8) Let the seller's own valuation be v_o, in a second-price sealed-bid pure common value auction, the equilibrium bidding function under r is $\beta_r(x)=\overline{H}(x)$, for $x\in[x_r^*,\overline{x}]$, and $\beta_r(x)<r$, for $x\in[0,x_r^*)$. The seller's revenue under reserve price r is

$$
\pi_S(r)=v_o\mathbb{1}(X^{(M)}<x_r^*)+r\mathbb{1}(X^{(M)}\geq x_r^*,X^{(M-1)}<x_r^*)+\overline{H}(X^{(M-1)})\mathbb{1}(X^{(M-1)}\geq x_r^*).
$$

Then

$$
\begin{aligned}
\mathbb{E}[\pi_S(r)] &= v_o\mathbb{E}[\mathbb{1}(X^{(M)}<x_r^*)]+r\mathbb{E}[\mathbb{1}(X^{(M)}\geq x_r^*,X^{(M-1)}<x_r^*)]+\mathbb{E}[\overline{H}(X^{(M-1)})\mathbb{1}(X^{(M-1)}\geq x_r^*)]\\
&= v_o\mathbb{E}[\mathbb{1}(B^{(M)}<\overline{H}(x_r^*))]+\mathbb{E}[\pi_P(r)],
\end{aligned}
$$

where $\mathbb{E}[\pi_P(r)] = r\mathbb{E}[\mathbb{1}(B^{(M)} \geq \overline{H}(x_r^*), B^{(M-1)} < \overline{H}(x_r^*))] + \mathbb{E}[B^{(M-1)}\mathbb{1}(B^{(M-1)} \geq \overline{H}(x_r^*))]$. On the other hand, for the bidders' surplus, we have

$$\pi_B(r) = (X_o - r)\mathbb{1}(X^{(M)} \geq x_r^*, X^{(M-1)} < x_r^*) + (X_o - \overline{H}(X^{(M-1)}))\mathbb{1}(X^{(M-1)} \geq x_r^*).$$

Then

$$\begin{aligned}
\mathbb{E}[\pi_B(r)] &= \mathbb{E}[(X_o - r)\mathbb{1}(X^{(M)} \geq x_r^*, X^{(M-1)} < x_r^*)] + \mathbb{E}[(X_o - \overline{H}(X^{(M-1)}))\mathbb{1}(X^{(M-1)} \geq x_r^*)] \\
&= \mathbb{E}[X_o \mathbb{1}(X^{(M)} \geq x_r^*)] - \mathbb{E}[\pi_P(r)] \\
&= \mathbb{E}\left[\mathbb{1}(M_1 \geq \overline{H}(x_r^*))\mathbb{1}(B_1 \leq M_1)\overline{L}(\overline{H}^{-1}(M_1))\right] \\
&\quad + \mathbb{E}\left[\mathbb{1}(B_1 \geq \overline{H}(x_r^*))\mathbb{1}(M_1 \leq B_1)\overline{L}(\overline{H}^{-1}(B_1))\right] - \mathbb{E}[\pi_P(r)],
\end{aligned}$$

where the last step follows similarly as that in the proof of Proposition 1.2.2, where we need to replace $\beta(\cdot)$ by $\overline{H}(\cdot)$ and to replace $b_r^* = \beta(x_r^*)$ by $\overline{H}(x_r^*)$. ■

Proof. (Theorem 1.4.2) Write

$$\begin{aligned}
\widehat{Q}(\tau) - Q_o(\tau) &= (I - \widehat{K}_1)^{-1}\widehat{\phi}_1(\tau) - (I - K_{1o})^{-1}\phi_{1o}(\tau) \\
&\overset{(1)}{=} (I - \widehat{K}_1)^{-1}(\widehat{\phi}_1 - \phi_{1o})(\tau) + \left[(I - \widehat{K}_1)^{-1} - (I - K_{1o})^{-1}\right]\phi_{1o}(\tau) \\
&\overset{(2)}{=} (I - \widehat{K}_1)^{-1}\left[(\widehat{\phi}_1 - \phi_{1o}) + (\widehat{K}_1 - K_{1o})(I - K_{1o})^{-1}\phi_{1o}\right](\tau) \\
&\overset{(3)}{=} (I - \widehat{K}_1)^{-1}\left[(\widehat{\phi}_1 - \phi_{1o}) + (\widehat{K}_1 - K_{1o})Q_o\right](\tau),
\end{aligned}$$

where (1) follows from the linearity of the operator $(I - \widehat{K}_1)^{-1}$, (2) uses the fact that $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$, (3) follows from the fact that $Q_o(\tau) = (I - K_{1o})^{-1}\phi_{1o}(\tau)$. The claim follows after we show the following two steps.

Step 1: To show $||(I - \widehat{K}_1)^{-1}||_\epsilon - ||(I - K_{1o})^{-1}||_\epsilon = o_P(1)$ for given $\epsilon > 0$, where $||(I - K_1)^{-1}||_\epsilon = \sup_{||\phi||_\epsilon \neq 0} \frac{||(I-K_1)^{-1}\phi||_\epsilon}{||\phi||_\epsilon}$. we first show $||\widehat{K}_1 - K_{1o}||_\epsilon = o_P(1)$, where $||\widehat{K}_1 - K_{1o}||_\epsilon = \sup_{||\varphi||_\epsilon \neq 0} \frac{||(\widehat{K}_1 - K_{1o})\varphi||_\epsilon}{||\varphi||_\epsilon}$ for φ taking values in the space of quantile functions. Here, $||\varphi||_\epsilon = \sup_{t\in[\epsilon,1-\epsilon]} |\varphi(t)|$ and $||(\widehat{K}_1 - K_{1o})\varphi||_\epsilon = \sup_{t\in[\epsilon,1-\epsilon]} |(\widehat{K}_1 -$

$K_{1o})\varphi(t)|$. we have

$$\begin{aligned}||(\widehat{K}_1 - K_{1o})\varphi||_\epsilon &\equiv \sup_{t\in[\epsilon,1-\epsilon]}\left|\int_0^t [\widehat{k}_1(t,s) - k_{1o}(t,s)]\varphi(s)ds\right| \\ &\leq \sup_{t\in[\epsilon,1-\epsilon]}\int_0^t |\widehat{k}_1(t,s) - k_{1o}(t,s)||\varphi(s)|ds \\ &\leq \sup_{t\in[\epsilon,1-\epsilon]}\int_0^t |\widehat{k}_1(t,s) - k_{1o}(t,s)| \sup_{s\in[\epsilon,1-\epsilon]}|\varphi(s)|ds \\ &\leq ||\varphi||_\epsilon \sup_{t\in[\epsilon,1-\epsilon]}\int_0^1 |\widehat{k}_1(t,s) - k_{1o}(t,s)|ds,\end{aligned}$$

where the third line follows from the fact that $\varphi(s)$ is increasing. Then it follows that

$$||\widehat{K}_1 - K_{1o}||_\epsilon \leq \frac{M-2}{2}\sup_{t\in[\epsilon,1-\epsilon]}\int_0^1 \left|z_1(t,s;\widehat{\theta}_L) - z_{1o}(t,s)\right| ds = \frac{M-2}{2} w_{1\epsilon}(\widehat{\theta}_L) = o_P(1)$$

under Assumption (RA-1) (i) and the fact that $\widehat{\theta}_L - \theta_o = o_P(1)$. Then

$$\begin{aligned}&\left| ||(I-\widehat{K}_1)^{-1}||_\epsilon - ||(I-K_{1o})^{-1}||_\epsilon \right| \\ &\overset{(1)}{\leq} ||(I-\widehat{K}_1)^{-1} - (I-K_{1o})^{-1}||_\epsilon \\ &\overset{(2)}{=} ||(I-\widehat{K}_1)^{-1}(\widehat{K}_1 - K_{1o})(I-K_{1o})^{-1}||_\epsilon \\ &\overset{(3)}{\leq} ||(I-\widehat{K}_1)^{-1}||_\epsilon ||\widehat{K}_1 - K_{1o}||_\epsilon ||(I-K_{1o})^{-1}||_\epsilon \\ &\leq \left| ||(I-\widehat{K}_1)^{-1}||_\epsilon - ||(I-K_{1o})^{-1}||_\epsilon \right| ||\widehat{K}_1 - K_{1o}||_\epsilon ||(I-K_{1o})^{-1}||_\epsilon \\ &\quad + ||\widehat{K}_1 - K_{1o}||_\epsilon ||(I-K_{1o})^{-1}||_\epsilon^2,\end{aligned}$$

where (1) follows from triangle inequality, (2) uses the fact that $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$, (3) follows from $||A\varphi|| \leq ||A||||\varphi||$. Let $Z_{1L} = \left| ||(I-\widehat{K}_1)^{-1}||_\epsilon - ||(I-K_{1o})^{-1}||_\epsilon \right|$, $Z_{2L} = ||\widehat{K}_1 - K_{1o}||_\epsilon$, $a_1 = ||(I-K_{1o})^{-1}||_\epsilon$. Upon rearrangement of the above inequality, we get $Z_{1L}(1 - a_1 Z_{2L}) \leq a_1^2 Z_{2L}$. Given any

$\delta > 0$,

$$
\begin{aligned}
\mathbb{P}(Z_{1L} > \delta) &= \mathbb{P}(Z_{1L} > \delta, 1 - a_1 Z_{2L} > 0) + \mathbb{P}(Z_{1L} > \delta, 1 - a_1 Z_{2L} \leq 0) \\
&\leq \mathbb{P}\left(\frac{a_1^2 Z_{2L}}{1 - a_1 Z_{2L}} \geq Z_{1L} > \delta, 1 - a_1 Z_{2L} > 0\right) + \mathbb{P}\left(Z_{2L} \geq \frac{1}{a_1}\right) \\
&\leq \mathbb{P}\left(\frac{a_1^2 Z_{2L}}{1 - a_1 Z_{2L}} > \delta, 1 - a_1 Z_{2L} > 0\right) + \mathbb{P}\left(Z_{2L} \geq \frac{1}{a_1}\right) \\
&\leq \mathbb{P}\left(Z_{2L} > \frac{\delta}{a_1^2 + a_1\delta}\right) + \mathbb{P}\left(Z_{2L} \geq \frac{1}{a_1}\right) \to 0, \text{ as } L \to \infty.
\end{aligned}
$$

Therefore, $||(I - \widehat{K}_1)^{-1}||_\epsilon - ||(I - K_{1o})^{-1}||_\epsilon = o_P(1)$.

Step 2: To show $||\widehat{\phi}_1 - \phi_{1o} + (\widehat{K}_1 - K_{1o})Q_o||_\epsilon = o_P(1)$. we have

$$
\begin{aligned}
&||\widehat{\phi}_1 - \phi_{1o} + (\widehat{K}_1 - K_{1o})Q_o||_\epsilon \\
&\leq ||\widehat{\phi}_1 - \phi_{1o}||_\epsilon + ||(\widehat{K}_1 - K_{1o})Q_o||_\epsilon \\
&\leq \frac{M}{2} \sup_{t\in[\epsilon,1-\epsilon]} |\widehat{Q}_{B_1}(t) - Q_{B_1}(t)| + \frac{M}{2} \sup_{t\in[\epsilon,1-\epsilon]} \left|\frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))}\right| \\
&\quad + \frac{M-2}{2} \sup_{t\in[\epsilon,1-\epsilon]} \int_0^1 |z_1(t, s; \widehat{\theta}_L) - z_{1o}(t, s)| Q_o(s) ds \\
&= o_P(1) + \frac{M}{2} \sup_{t\in[\epsilon,1-\epsilon]} \left|\frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))}\right|,
\end{aligned}
$$

where the last equality follows from Assumption (RA-1)(ii) and the fact that $\widehat{\theta}_L - \theta_o = o_P(1)$. For the second term, let $[\underline{b}_\epsilon, \overline{b}_\epsilon] \subset [\underline{b}, \overline{b}]$ be a compact strict subset, where $\underline{b}_\epsilon$ is such that $\mathbb{P}(\widehat{Q}_{B_1}(\epsilon) \geq \underline{b}_\epsilon) \to 1$ and $\overline{b}_\epsilon$ is such that $\mathbb{P}(\widehat{Q}_{B_1}(1-\epsilon) \leq \overline{b}_\epsilon) \to 1$. we have

$$
\begin{aligned}
&\sup_{t\in[\epsilon,1-\epsilon]} \left|\frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))}\right| \\
&\leq \sup_{t\in[\epsilon,1-\epsilon]} \left|\frac{1}{\widehat{\rho}_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(\widehat{Q}_{B_1}(t))}\right| + \sup_{t\in[\epsilon,1-\epsilon]} \left|\frac{1}{\rho_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))}\right| \\
&\equiv Z_{3L} + Z_{4L}.
\end{aligned}
$$

For Z_{3L}, let $Z_{3L}^* = \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{1}{\widehat{\rho}_{M_1|B_1}(b)} - \frac{1}{\rho_{M_1|B_1}(b)}\right|$, then for any $\delta > 0$,

$$\begin{aligned}\mathbb{P}(Z_{3L} \geq \delta) &\leq \mathbb{P}(Z_{3L} \geq \delta, Z_{3L}^* \geq Z_{3L}) + \mathbb{P}(Z_{3L} \geq \delta, Z_{3L}^* < Z_{3L})\\ &\leq \mathbb{P}(Z_{3L}^* \geq \delta, Z_{3L}^* \geq Z_{3L}) + \mathbb{P}(Z_{3L}^* < Z_{3L})\\ &\leq \mathbb{P}(Z_{3L}^* \geq \delta) + \mathbb{P}(Z_{3L}^* < Z_{3L}) \to 0,\end{aligned}$$

where $\mathbb{P}(Z_{3L}^* < Z_{3L}) \to 0$ follows from

$$\begin{aligned}\mathbb{P}(Z_{3L}^* \geq Z_{3L}) &\geq \mathbb{P}\left(\widehat{Q}_{B_1}(t) \in [\underline{b}_\epsilon, \bar{b}_\epsilon] \text{ for all } t \in [\epsilon, 1-\epsilon]\right)\\ &= \mathbb{P}\left(\inf_{t\in[\epsilon,1-\epsilon]} \widehat{Q}_{B_1}(t) \geq \underline{b}_\epsilon, \sup_{t\in[\epsilon,1-\epsilon]} \widehat{Q}_{B_1}(t) \leq \bar{b}_\epsilon\right)\\ &= \mathbb{P}\left(\widehat{Q}_{B_1}(\epsilon) \geq \underline{b}_\epsilon, \widehat{Q}_{B_1}(1-\epsilon) \leq \bar{b}_\epsilon\right)\\ &\geq \mathbb{P}\left(\widehat{Q}_{B_1}(\epsilon) \geq \underline{b}_\epsilon\right) - \mathbb{P}\left(\widehat{Q}_{B_1}(1-\epsilon) > \bar{b}_\epsilon\right) \to 1.\end{aligned}$$

we also need to show $\mathbb{P}(Z_{3L}^* \geq \delta) \to 0$. we have $\frac{1}{\widehat{\rho}_{M_1|B_1}(b)} = \frac{\widehat{G}_{M_1\times B_1}(b)}{\widehat{g}_{M_1B_1}(b)}$, $\frac{1}{\rho_{M_1|B_1}(b)} = \frac{G_{M_1\times B_1}(b)}{g_{M_1B_1}(b)}$, where $G_{M_1\times B_1}(b) = \partial G_{M_1B_1}(m_1, b_1)/\partial b_1|_{m_1=b_1=b}$. Li, Perrigne, and Vuong (2002) show that $\sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}|\widehat{G}_{M_1\times B_1}(b) - G_{M_1\times B_1}(b)| = o_P(1)$, and $\sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}|\widehat{g}_{M_1B_1}(b) - g_{M_1B_1}(b)| = o_P(1)$. Then

$$\begin{aligned}&\sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{1}{\widehat{\rho}_{M_1|B_1}(b)} - \frac{1}{\rho_{M_1|B_1}(b)}\right|\\ &= \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{\widehat{G}_{M_1\times B_1}(b)}{\widehat{g}_{M_1B_1}(b)} - \frac{\widehat{G}_{M_1\times B_1}(b)}{g_{M_1B_1}(b)}\right| + \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{\widehat{G}_{M_1\times B_1}(b)}{g_{M_1B_1}(b)} - \frac{G_{M_1\times B_1}(b)}{g_{M_1B_1}(b)}\right|\\ &\leq \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}|\widehat{G}_{M_1\times B_1}(b)| \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{1}{\overline{g}^2_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}|\widehat{g}_{M_1B_1}(b) - g_{M_1B_1}(b)|\\ &\quad + \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{1}{g_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\widehat{G}_{M_1\times B_1}(b) - G_{M_1\times B_1}(b)\right|\\ &\overset{(1)}{\leq} \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}|\widehat{G}_{M_1\times B_1}(b) - G_{M_1\times B_1}(b)| \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{1}{\overline{g}^2_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}|\widehat{g}_{M_1B_1}(b) - g_{M_1B_1}(b)|\\ &\quad + \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}|G_{M_1\times B_1}(b)| \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{1}{\overline{g}^2_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}|\widehat{g}_{M_1B_1}(b) - g_{M_1B_1}(b)|\\ &\quad + \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\frac{1}{g_{M_1B_1}(b)}\right| \sup_{b\in[\underline{b}_\epsilon,\bar{b}_\epsilon]}\left|\widehat{G}_{M_1\times B_1}(b) - G_{M_1\times B_1}(b)\right| \overset{(2)}{=} o_P(1),\end{aligned}$$

where $\overline{g}_{M_1B_1}(b)$ is between $\widehat{g}_{M_1B_1}(b)$ and $g_{M_1B_1}(b)$, (1) follows from triangle inequality, (2) follows from Assumptions (UB) (ii) and (iii). In sum, we have $Z_{3L} = o_P(1)$. On the other hand,

$$Z_{4L} = \sup_{t\in[\epsilon,1-\epsilon]} \left| \frac{1}{\rho_{M_1|B_1}(\widehat{Q}_{B_1}(t))} - \frac{1}{\rho_{M_1|B_1}(Q_{B_1}(t))} \right| = o_P(1)$$

due to the uniform continuity of $\frac{1}{\rho_{M_1|B_1}(b)}$ on $[\underline{b}, \overline{b}]$ under Assumption (UB) (i) and the fact that $\sup_{t\in[\epsilon,1-\epsilon]} \left|\widehat{Q}_{B_1}(t) - Q_{B_1}(t)\right| = o_P(1)$.

Now putting Step 1 and Step 2 together, we have

$$\begin{aligned}
\left|\left|\widehat{Q} - Q_o\right|\right|_\epsilon &\leq \left|\left|(I-\widehat{K}_1)^{-1}\right|\right|_\epsilon \left|\left|(\widehat{\phi}_1 - \phi_{1o}) + (\widehat{K}_1 - K_{1o})Q_o\right|\right|_\epsilon \\
&\leq \left(||(I-\widehat{K}_1)^{-1}||_\epsilon - \left|\left|(I-K_{1o})^{-1}\right|\right|_\epsilon\right) \left|\left|(\widehat{\phi}_1 - \phi_{1o}) + (\widehat{K}_1 - K_{1o})Q_o\right|\right|_\epsilon \\
&+ \left|\left|(I-K_{1o})^{-1}\right|\right|_\epsilon \left|\left|(\widehat{\phi}_1 - \phi_{1o}) + (\widehat{K}_1 - K_{1o})Q_o\right|\right|_\epsilon \\
&= o_P(1)o_P(1) + o(1)o_P(1) + O(1)o_P(1) = o_P(1),
\end{aligned}$$

where the last step follows from Assumption (RA-1)(iii). ∎

Proof. (Theorem 1.4.3) Similar to the proof of Theorem 1.4.2, write

$$\widehat{Q}(\tau) - Q_o(\tau) = (I - \widehat{K}_2)^{-1}\left[(\widehat{\phi}_2 - \phi_{2o}) + (\widehat{K}_2 - K_{2o})Q_o\right](\tau),$$

where

$$\begin{aligned}
\widehat{\phi}_2(\tau) - \phi_{2o}(\tau) &= \frac{M}{2}\left[\widehat{Q}_{B_1}(t) - Q_{B_1}(t)\right], \\
(\widehat{K}_2 - K_{2o})Q_o(\tau) &= -\frac{M-2}{2}\int_0^\tau \left[z_1(\tau, s; \widehat{\theta}_L) - z_{1o}(\tau, s)\right] Q_o(s)ds.
\end{aligned}$$

The claim follows from the following two steps.

Step 1: To show that $\left|\left|\widehat{\phi}_2 - \phi_{2o} + (\widehat{K}_2 - K_{2o})Q_o\right|\right|_\infty = o_P(1)$. we have

$$\begin{aligned}
&\left|\left|\widehat{\phi}_2 - \phi_{2o} + (\widehat{K}_2 - K_{2o})Q_o\right|\right|_\infty \\
&\leq \left|\left|\widehat{\phi}_2 - \phi_{2o}\right|\right|_\infty + \left|\left|(\widehat{K}_2 - K_{2o})Q_o\right|\right|_\infty \\
&\leq \frac{M}{2}\sup_{t\in[0,1]}\left|\widehat{Q}_{B_1}(t) - Q_{B_1}(t)\right| + \frac{M-2}{2}\sup_{t\in[0,1]}\int_0^1 \left|z_1(t, s; \widehat{\theta}_L) - z_{1o}(t, s)\right| Q_o(s)ds \\
&= o_P(1),
\end{aligned}$$

where the last equality follows from Assumption (RA-1)(ii) and the continuity of $Q_{B_1}(\tau)$ on $[0,1]$.

Step 2: To show that $||(I-\widehat{K}_2)^{-1}||_{op} - ||(I-K_{2o})^{-1}||_{op} = o_P(1)$. Following similar argument as in the proof of Theorem 1.4.2, we can show that under Assumption (RA-2)

$$\left|\left|\widehat{K}_2 - K_{2o}\right|\right|_{op} \leq \frac{M-2}{2} \sup_{\tau\in[0,1]} \int_0^{\tau} \left|z_1(\tau,s;\widehat{\theta}_L) - z_1(\tau,s)\right| ds = o_P(1).$$

Also, we can show that $Z_{5L}(1-a_2Z_{6L}) \leq a_2^2 Z_{6L}$, where $Z_{5L} = \Big|||(I-\widehat{K}_2)^{-1}||_{op} - ||(I-K_{2o})^{-1}||_{op}\Big|$, $Z_{6L} = ||\widehat{K}_2 - K_{2o}||_{op}$, $a_2 = ||(I-K_{2o})^{-1}||_{op}$. $a_2 < \infty$ since under Assumption (CU-2), $I - K_{2o}$ is invertible and its inverse $(I-K_{2o})^{-1}$ is continuous thus bounded. Given any $\delta > 0$, similar argument as in the proof of Theorem 1.4.2 leads to $\mathbb{P}(Z_{5L} > \delta) \to 0$. Now putting Step 1 and Step 2 together, we have

$$\begin{aligned}
\left|\left|\widehat{Q} - Q_o\right|\right|_{\infty} &\leq \left|\left|(I-\widehat{K}_2)^{-1}\right|\right|_{op} \left|\left|(\widehat{\phi}_2 - \phi_{2o}) + (\widehat{K}_2 - K_{2o})Q_o\right|\right|_{\infty} \\
&\leq \Big(||(I-\widehat{K}_2)^{-1}||_{op} - ||(I-K_{2o})^{-1}||_{op}\Big) \left|\left|(\widehat{\phi}_2 - \phi_{2o}) + (\widehat{K}_2 - K_{2o})Q_o\right|\right|_{\infty} \\
&+ ||(I-K_{2o})^{-1}||_{op} \left|\left|(\widehat{\phi}_2 - \phi_{2o}) + (\widehat{K}_2 - K_{2o})Q_o\right|\right|_{\infty} \\
&= o_P(1)o_P(1) + o_P(1) = o_P(1).
\end{aligned}$$

■

Appendix B

PROOFS OF CHAPTER 3

Proof. (Theorem 3.3.2): We focus on bidder 1 due to the symmetry, let $\underline{b} \leq m_1 \leq b_1 \leq \bar{b}$, $M_1 = \max_{j \neq 1} B_j$, then

$$\begin{aligned}
\phi(b) &= \left[\frac{\partial^2 \mathbb{P}(B^{(n-1)} \leq m_1, B^{(n)} \leq b_1)}{\partial m_1 \partial b_1}\right]_{|m_1=b_1=b} \\
&= \left[\frac{\partial^2 n\mathbb{P}(M_1 \leq m_1, B_1 \leq b_1)}{\partial m_1 \partial b_1}\right]_{|m_1=b_1=b} \\
&= \left[\frac{\partial^2 n\varphi_o^{-1}\left[(n-1)\varphi_o(G(m_1)) + \varphi_o(G(b_1))\right]}{\partial m_1 \partial b_1}\right]_{|m_1=b_1=b} \\
&= -\frac{n(n-1)[\varphi_o'(G(b))]^2 g^2(b) \varphi_o''(\varphi_o^*(G(b)))}{[\varphi_o'(\varphi_o^*(G(b)))]^3} \\
&= -\frac{(n-1)\varphi_o''(G^{(n)}(b))[g^{(n)}(b)]^2}{n\varphi_o'(G^{(n)}(b))} \\
&= -\frac{(n-1)\varphi_o''(G^{(n)}(b))}{n\varphi_o'(G^{(n)}(b))[q^{(n)\prime}(G^{(n)}(b))]^2},
\end{aligned}$$

where $\varphi_o^*(t) = \varphi_o^{-1}(n\varphi_o(t))$, and the second equality above follows from

$$\begin{aligned}
\mathbb{P}(B^{(n-1)} \leq m_1, B^{(n)} \leq b_1) &= n\mathbb{P}(M_1 \leq \min(m_1, B_1), B_1 \leq b_1) \\
&= n\mathbb{P}(M_1 \leq m_1, B_1 \leq b_1) - n\mathbb{P}(B_1 < M_1 \leq m_1).
\end{aligned}$$

The third equality in the derivation of $\phi(b)$ follows from Assumption (AS). Now a change of variable $G^{(n)}(b) = t$ gives

$$\frac{\varphi_o'(t)}{\varphi_o''(t)} = -\frac{1}{z(t)}, \text{ where } z(t) = \frac{n}{n-1}\phi\left(q^{(n)}(t)\right)\left(q^{(n)\prime}(t)\right)^2.$$

The expression for $\varphi_o(t)$ follows from solving the above differential equation subject to the boundary condition that $\varphi_o(1) = 0$. Here α can take any negative value and the restriction $\alpha < 0$ follows from the fact that $\varphi_o'(t) < 0$. In sum, $\varphi_o(t)$ is identified up to a constant α, hence the copula function is identified under Assumption (AS). ■

Proof. (Theorem 3.4.1): Let $M_{ir_o} = \max_{j\neq i} B_{jr_o}$, when the reserve price is binding, for $b \in [r_o, \bar{b}_{r_o}]$, we have

$$\begin{aligned}
\phi_{ir_o}(b) &= \left[\frac{\partial^2 \mathbb{P}(B_{r_o}^{(n-1)} \le m_i, B_{r_o}^{(n)} \le b_i, D = i | B_{r_o}^{(n-1)} > r_o)}{\partial m_i \partial b_i}\right]|_{m_i=b_i=b} \\
&= \frac{1}{1-\sigma_2}\left[\frac{\partial^2 \mathbb{P}(M_{ir_o} \le m_i, B_{ir_o} \le b_i)}{\partial m_i \partial b_i}\right]|_{m_i=b_i=b} \\
&= \frac{1}{1-\sigma_2}\left[\frac{\partial^2 \varphi_o^{-1}[\sum_{j\neq i}\varphi_o(G_{jr_o}(m_i)) + \varphi_o(G_{ir_o}(b_i))]}{\partial m_i \partial b_i}\right]|_{m_i=b_i=b} \\
&= -\frac{1}{1-\sigma_2}\frac{\varphi_o'(G_{ir_o}(b))g_{ir_o}(b)\left[\sum_{j\neq i}\varphi_o'(G_{jr_o}(b))g_{jr_o}(b)\right]\varphi_o''[\varphi_o^{-1}(\sum_{i=1}^n \varphi_o(G_{ir_o}(b)))]}{\{\varphi_o'[\varphi_o^{-1}(\sum_{i=1}^n \varphi_o(G_{ir_o}(b)))]\}^3} \\
&= -\frac{(1-\sigma_1)^2}{1-\sigma_2}\frac{\left[g_{D,i}^*(b)\sum_{j\neq i} g_{D,j}^*(b)\right]\varphi_o''\left[(1-\sigma_1)G^{*(n)}(b)+\sigma_1\right]}{\varphi_o'\left[(1-\sigma_1)G^{*(n)}(b)+\sigma_1\right]},
\end{aligned}$$

where the second equality follows from that for $r_o \le m_i \le b_i \le \bar{b}_{r_o}$, we have

$$\begin{aligned}
\mathbb{P}(r_o < B_{r_o}^{(n-1)} \le m_i, B_{r_o}^{(n)} \le b_i, D = i) &= \mathbb{P}(r_o < M_{ir_o} \le m_i, M_{ir_o} < B_{ir_o} \le b_i) \\
&= \mathbb{P}(M_{ir_o} \le m_i, B_{ir_o} \le b_i) - \mathbb{P}(B_{ir_o} \le M_{ir_o} \le m_i) \\
&\quad - \Big[\mathbb{P}(B_{ir_o} \le b_i, M_{ir_o} \le r_o) - \mathbb{P}(B_{ir_o} \le M_{ir_o} \le r_o)\Big].
\end{aligned}$$

The third equality in the derivation of $\phi_{ir_o}(b)$ follows from Assumption (AS), and the last equality in the derivation $\phi_{ir_o}(b)$ follows from the last equality in the derivation of $g_{D,i}^*(b)$ as well as the final expression of $g_{D,i}^*(b)$ in the proof of Lemma 3.4.2 below.

Now let $G^{*(n)}(b)(1-\sigma_1) + \sigma_1 = t \in [\sigma_1, 1]$, then $b = q^{*(n)}(\frac{t-\sigma_1}{1-\sigma_1})$, and we get

$$\frac{\varphi_o'(t)}{\varphi_o''(t)} = -\frac{1}{z_{ir_o}(t)},$$

where for $t \in [\sigma_1, 1]$,

$$z_{ir_o}(t) = \frac{1-\sigma_2}{(1-\sigma_1)^2}\phi_{ir_o}\left(q^{*(n)}\Big(\frac{t-\sigma_1}{1-\sigma_1}\Big)\right)\left[g_{D,i}^*\left(q^{*(n)}\Big(\frac{t-\sigma_1}{1-\sigma_1}\Big)\right)\sum_{j\neq i} g_{D,j}^*\left(q^{*(n)}\Big(\frac{t-\sigma_1}{1-\sigma_1}\Big)\right)\right]^{-1}.$$

Then solving the above ordinary differential equation subject to the boundary condition $\varphi_o(1) = 0$ gives the solution of $\varphi_o(t)$ on $[\sigma_1, 1]$. ■

Proof. (Lemma 3.4.2): When the reserve price r_o is binding, let $G^*(b_1, \cdots, b_n) = \mathbb{P}(B_{jr_o} \leq b_j, \forall j | B_{r_o}^{(n)} > r_o), b_j \in [r_o, \bar{b}_{r_o}]$. Let $G_{D,i}^*(b) = \mathbb{P}(B_{r_o}^{(n)} \leq b, D = i | B_{r_o}^{(n)} > r_o)$, $g_{D,i}^*(b) = \partial G_{D,i}^*(b) / \partial b, b \in [r_o, \bar{b}_{r_o}]$ for $i = 1, \cdots, n$. Then

$$\begin{aligned}
g_{D,i}^*(b) &= \frac{\partial G^*(b_1, \cdots, b_n)}{\partial b_i} |_{b_1 = \cdots = b_n = b} \\
&= \frac{1}{1-\sigma_1} \frac{\mathbb{P}(B_{jr_o} \leq b_j, \forall\ j)}{\partial b_i} |_{b_1 = \cdots = b_n = b} \\
&= \frac{1}{1-\sigma_1} \frac{\partial \varphi_o^{-1} [\varphi_o(G_{1r_o}(b_1)) + \cdots + \varphi_o(G_{nr_o}(b_n))]}{\partial b_i} |_{b_1 = \cdots = b_n = b} \\
&= \frac{1}{1-\sigma_1} \frac{\varphi_o' [G_{ir_o}(b_i)] g_{ir_o}(b_i)}{\varphi_o' [\varphi_o^{-1} (\varphi_o(G_{1r_o}(b_1)) + \cdots + \varphi_o(G_{nr_o}(b_n)))]} |_{b_1 = \cdots = b_n = b} \\
&= \frac{1}{1-\sigma_1} \frac{\varphi_o' [G_{ir_o}(b)] g_{ir_o}(b)}{\varphi_o' [(1-\sigma_1) G^{*(n)}(b) + \sigma_1]}, b \in [r_o, \bar{b}_{r_o}],
\end{aligned}$$

where $G^{*(n)}(b) = \mathbb{P}(B_{r_o}^{(n)} \leq b | B_{r_o}^{(n)} > r_o)$. The first equality in the derivation of $g_{D,i}^*(b)$ is in the same spirit as the proof of Theorem 1 in Tsiatis (1975) for competing risks models: since

$$\begin{aligned}
g_{D,i}^*(b) = \lim_{\epsilon \to 0} \frac{G_{D,i}^*(b+\epsilon) - G_{D,i}^*(b)}{\epsilon} &= \lim_{\epsilon \to 0} \frac{P\left(b < B_{r_o}^{(n)} \leq b + \epsilon, D = i | B_{r_o}^{(n)} > r_o\right)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{P\left(b < B_{ir_o} \leq b + \epsilon, B_{jr_o} < B_{ir_o}, \forall\ j \neq i | B_{r_o}^{(n)} > r_o\right)}{\epsilon},
\end{aligned}$$

let $\epsilon^* > \epsilon > 0$, then

$$\begin{aligned}
P\left(b < B_{ir_o} \leq b + \epsilon, B_{jr_o} \leq b, \forall j \neq i | B_{r_o}^{(n)} > r_o\right) &\leq P\left(b < B_{ir_o} \leq b + \epsilon, B_{jr_o} < B_{ir_o}, \forall j \neq i | B_{r_o}^{(n)} > r_o\right) \\
&\leq P\left(b < B_{ir_o} \leq b + \epsilon, B_{jr_o} \leq b + \epsilon^*, \forall j \neq i | B_{r_o}^{(n)} > r_o\right),
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\partial G^*(b_1,\cdots,b_n)}{\partial b_i}|_{b_1=\cdots=b_n=b} &= \lim_{\epsilon\to 0}\frac{G^*\left(b,\cdots,b+\epsilon,\cdots,b\right)-G^*\left(b,\cdots,b,\cdots,b\right)}{\epsilon}\\
&\leq g^*_{D,i}\left(b\right)\\
&\leq \lim_{\epsilon^*\to 0}\lim_{\epsilon\to 0}\frac{G^*\left(b+\epsilon^*,\cdots,b+\epsilon,\cdots,b+\epsilon^*\right)-G^*\left(b+\epsilon^*,\cdots,b,\cdots,b+\epsilon^*\right)}{\epsilon}\\
&= \frac{\partial G^*(b_1,\cdots,b_n)}{\partial b_i}|_{b_1=\cdots=b_n=b}.
\end{aligned}$$

The second equality in the derivation of $g^*_{D,i}(b)$ follows from that for $b_j \in [r_o, \bar{b}_{r_o}], \forall\ j$, $\mathbb{P}(B_{jr_o} \leq b_j, \forall\ j, B^{(n)}_{r_o} > r_o) = \mathbb{P}(B_{jr_o} \leq b_j, \forall\ j) - \mathbb{P}(B_{jr_o} \leq b_j, \forall\ j, B^{(n)}_{r_o} \leq r_o) = \mathbb{P}(B_{jr_o} \leq b_j, \forall\ j) - \mathbb{P}(B^{(n)}_{r_o} \leq r_o)$.

Now for $b \in [r_o, \bar{b}_{r_o}]$,

$$\begin{aligned}
(1-\sigma_1)\int_b^{\bar{b}_{r_o}} g^*_{D,i}\left(s\right)\varphi_o'\left[(1-\sigma_1)G^{*(n)}\left(s\right)+\sigma_1\right]ds &= \int_b^{\bar{b}_{r_o}}\varphi_o'\left[G_{ir_o}\left(s\right)\right]dG_{ir_o}\left(s\right)\\
&= \varphi_o\left[G_{ir_o}(\bar{b}_{r_o})\right]-\varphi_o\left[G_{ir_o}\left(b\right)\right]\\
&= -\varphi_o\left[G_{ir_o}\left(b\right)\right],
\end{aligned}$$

because $\varphi_o\left[G_{ir_o}(\bar{b}_{r_o})\right] = \varphi_o\left(1\right) = 0$. Then we get

$$G_{ir_o}\left(b\right) = \varphi_o^{-1}\left[-(1-\sigma_1)\int_b^{\bar{b}_{r_o}} g^*_{D,i}\left(s\right)\varphi_o'\left[(1-\sigma_1)G^{*(n)}\left(s\right)+\sigma_1\right]ds\right], b\in\left[r_o,\bar{b}_{r_o}\right]$$

and thus Lemma 3.4.2 is proved.

Now we argue that $Q_i(p), p \in [G_{ir_o}(r_o), 1]$ is identified by combining Theorem 3.4.1 and Lemma 3.4.2. First, Theorem 3.4.1 implies that $\varphi_o'(\cdot)$ is identified on $[\sigma_1, 1]$, in the expression of $G_{ir_o}(b)$, we have $(1-\sigma_1)G^{*(n)}(s)+\sigma_1 \in [\sigma_1, 1]$. Second, Theorem 3.4.1 implies that $\varphi_o^{-1}(\cdot)$ is identified on $[0, \varphi_o(\sigma_1)]$, then we need to show that for $b \in [r_o, \bar{b}_{r_o}]$,

$$0 \leq -(1-\sigma_1)\int_b^{\bar{b}_{r_o}} g^*_{D,i}\left(s\right)\varphi_o'\left[(1-\sigma_1)G^{*(n)}\left(s\right)+\sigma_1\right]ds \leq \varphi_o(\sigma_1).$$

The first inequality is trivial. The second inequality follows from

$$\begin{aligned}
&-(1-\sigma_1)\int_b^{\bar{b}_{r_o}} g^*_{D,i}\left(s\right)\varphi_o'\left[(1-\sigma_1)G^{*(n)}\left(s\right)+\sigma_1\right]ds\\
&\leq -(1-\sigma_1)\int_{r_o}^{\bar{b}_{r_o}} g^*_{D,i}\left(s\right)\varphi_o'\left[(1-\sigma_1)G^{*(n)}\left(s\right)+\sigma_1\right]ds = \varphi_o[G_{ir_o}(r_o)] \leq \varphi_o(\sigma_1),
\end{aligned}$$

where the first inequality is due to the fact that $-(1-\sigma_1)\int_b^{\bar{b}_{r_o}} g^*_{D,i}(s)\,\varphi_o'\left[(1-\sigma_1)G^{*(n)}(s)+\sigma_1\right]ds$, as a function of b, is decreasing on $[r_o,\bar{b}_{r_o}]$, the last inequality is due to the fact that $\varphi_o(\cdot)$ is decreasing and $G_{ir_o}(r_o)\geq\sigma_1$. Now the identification of $G_{ir_o}(b)$ on $[r_o,\bar{b}_{r_o}]$ implies the identification of $q_{ir_o}(p)$ on $[G_{ir_o}(r_o),1]$, which in turn implies identification of $Q_i(p)$ on $p\in[G_{ir_o}(r_o),1]$. ■

Proof. (Theorem 3.4.4) Let r be the counterfactual reserve price and r_o be the observed reserve price from the auction. We have $G^{(n)}_{r_o}(b)=F^{(n)}(r_o)$ if $b\in[\underline{v},r_o)$ and $G^{(n)}_{r_o}(b)=F^{(n)}(s^{-1}_{r_o}(b))$ if $b\in[r_o,\bar{b}_{r_o}]$. Since the sample does not provide any information on $F^{(n)}(v)$ for $v\in[\underline{v},r_o)$, we can only identify the quantities of interest for $r\in[r_o,\overline{v}]$. we seek to write the quantities of interest in terms of observed distributions. For $r\in[r_o,\overline{v}]$,

$$
\begin{aligned}
&\omega_S(r)\\
&=v_oF^{(n)}(r)+\int_r^{\overline{v}} s_r(v)dF^{(n)}(v)\\
&=v_o[(1-\sigma_1)\lambda_{r_o}+\sigma_1]+(1-\sigma_1)\int_r^{\overline{v}}\left[s_{r_o}(v)+L(r|v)(r-s_{r_o}(r))\right]dG^{*(n)}(s_{r_o}(v))\\
&=v_o[(1-\sigma_1)\lambda_{r_o}+\sigma_1]+(1-\sigma_1)\int_r^{\overline{v}} s_{r_o}(v)dG^{*(n)}(s_{r_o}(v))\\
&\quad-\frac{n}{n-1}\int_r^{\overline{v}}\left[\frac{\varphi'_{\theta_o}[(1-\sigma_1)G^{*(n)}(s_{r_o}(v))+\sigma_1]}{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}\right]^{\frac{n-1}{n}}\left[\frac{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}q^{*(n)\prime}(\lambda_{r_o})\right]dG^{*(n)}(s_{r_o}(v))\\
&=v_o[(1-\sigma_1)\lambda_{r_o}+\sigma_1]+(1-\sigma_1)\int_{\lambda_{r_o}}^1 q^{*(n)}(t)dt\\
&\quad-\frac{n}{n-1}\frac{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}q^{*(n)\prime}(\lambda_{r_o})\int_{\lambda_{r_o}}^1\left[\frac{\varphi'_{\theta_o}[(1-\sigma_1)t+\sigma_1]}{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}\right]^{\frac{n-1}{n}}dt,
\end{aligned}
$$

where the second equality follows from (1): for $v\in[r_o,\overline{v}]$, $F^{(n)}(v)=G^{(n)}_{r_o}(s_{r_o}(v))=(1-\sigma_1)G^{*(n)}(s_{r_o}(v))+\sigma_1$, and we defines $\lambda_{r_o}=G^{*(n)}(s_{r_o}(r))$; (2): for $r\in[r_o,\overline{v}]$, $s_r(v)=s_{r_o}(v)+L(r|v)\int_{r_o}^r L(\alpha|r)d\alpha=s_{r_o}(v)+L(r|v)[r-s_{r_o}(r)]$. The third equality in the derivation of $\omega_S(r)$ follows from (1): for $r\in[r_o,\overline{v}]$,

$$
r-s_{r_o}(r)=-\frac{n}{n-1}\frac{\varphi'_{\theta_o}(G^{(n)}_{r_o}(s_{r_o}(r)))}{\varphi''_{\theta_o}(G^{(n)}_{r_o}(s_{r_o}(r)))g^{(n)}_{r_o}(s_{r_o}(r))}=-\frac{n}{(n-1)(1-\sigma_1)}\frac{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}q^{*(n)\prime}(\lambda_{r_o}),
$$

where the second equality is due to the fact that $[g_{r_o}^{(n)}(t)]^{-1} = q_{r_o}^{(n)\prime}(G_{r_o}^{(n)}(t)) = q^{*(n)\prime}[(G_{r_o}^{(n)}(t)-\sigma_1)/(1-\sigma_1)]/(1-\sigma_1)$. (2): under Assumptions (AS) and (AP), for $\alpha, v, r \in [r_o, \overline{v}]$,

$$\begin{aligned}
L(\alpha|v) &= \exp\left[-\int_{\alpha}^{v} \frac{f_{Y_1|V_1}(u|u)}{F_{Y_1|V_1}(u|u)} du\right] \\
&= \exp\left[-\int_{s_{r_o}(\alpha)}^{s_{r_o}(v)} \frac{g_{M_{1r_o}|B_{1r_o}}(t|t)}{G_{M_{1r_o}|B_{1r_o}}(t|t)} dt\right] \\
&= \exp\left[-(n-1)\int_{s_{r_o}(\alpha)}^{s_{r_o}(v)} \frac{C_{o,12}(G_{r_o}(t),\cdots,G_{r_o}(t))g_{r_o}(t)}{C_{o,1}(G_{r_o}(t),\cdots,G_{r_o}(t))} dt\right] \\
&= \exp\left[(n-1)\int_{s_{r_o}(\alpha)}^{s_{r_o}(v)} \frac{\varphi_{\theta_o}'(G_{r_o}(t))\varphi_{\theta_o}''[\varphi_{\theta_o}^{-1}(n\varphi_{\theta_o}(G_{r_o}(t)))]g_{r_o}(t)}{[\varphi_{\theta_o}'(\varphi_{\theta_o}^{-1}(n\varphi_{\theta_o}(G_{r_o}(t))))]^2} dt\right] \\
&= \exp\left[\frac{n-1}{n}\int_{s_{r_o}(\alpha))}^{s_{r_o}(v)} \frac{\varphi_{\theta_o}''(G_{r_o}^{(n)}(t))g_{r_o}^{(n)}(t)}{\varphi_{\theta_o}'(G_{r_o}^{(n)}(t))} dt\right] \\
&= \left[\frac{\varphi_{\theta_o}'(G_{r_o}^{(n)}(s_{r_o}(v)))}{\varphi_{\theta_o}'(G_{r_o}^{(n)}(s_{r_o}(\alpha)))}\right]^{\frac{n-1}{n}} = \left[\frac{\varphi_{\theta_o}'[(1-\sigma_1)G^{*(n)}(s_{r_o}(v))+\sigma_1]}{\varphi_{\theta_o}'[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}\right]^{\frac{n-1}{n}},
\end{aligned}$$

and

$$L(r|v) = \left[\frac{\varphi_{\theta_o}'[(1-\sigma_1)G^{*(n)}(s_{r_o}(v))+\sigma_1]}{\varphi_{\theta_o}'[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}\right]^{\frac{n-1}{n}}.$$

In the derivation of $L(\alpha|v)$, we used the facts that (1): for $y_1, v_1 \in [r_o, \overline{v}]$, $F_{Y_1|V_1}(y_1|v_1) = G_{M_{1r_o}|B_{1r_o}}(s_{r_o}(y_1)|s_{r_o}(v_1))$, and $f_{Y_1|V_1}(y_1|v_1) = g_{M_{1r_o}|B_{1r_o}}(s_{r_o}(y_1)|s_{r_o}(v_1))s_{r_o}'(y_1)$, where $M_{1r_o} = \max_{j\neq 1} B_{jr_o}$; (2): for $m_1, b_1 \in [r_o, \overline{b}_{r_o}]$, $G_{M_{1r_o}|B_{1r_o}}(m_1|b_1) = C_{o,1}(G_{r_o}(b_1), G_{r_o}(m_1),\cdots,G_{r_o}(m_1))$, and

$$g_{M_{1r_o}|B_{1r_o}}(m_1|b_1) = (n-1)C_{o,12}(G_{r_o}(b_1), G_{r_o}(m_1),\cdots,G_{r_o}(m_1))g_{r_o}(m_1),$$

where $g_{r_o}(m_1) = G_{r_o}'(m_1)$, and $G_{r_o}(\cdot)$ is the distribution function of any B_{ir_o} due to the symmetry. The fourth equality in the derivation of $\omega_S(r)$ follows from a change of variable $G^{*(n)}(s_{r_o}(v)) = t$.

There are two unknowns in $\omega_S(r)$, namely, θ_o and λ_{r_o}, which are subject to the restriction that

$$q^{*(n)}(\lambda_{r_o}) - \frac{n}{(n-1)(1-\sigma_1)} \frac{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]} q^{*(n)\prime}(\lambda_{r_o}) - r = 0, r \in [r_o, \overline{v}].$$

The restriction follows from the expression of $r - s_{r_o}(v)$ above. When θ_o is unknown, we use $\lambda_{r_o}(\theta)$ to replace λ_{r_o}, with $\lambda_{r_o}(\theta)$ to be the solution to the restriction $R_2(\lambda_{r_o}(\theta), \theta) = r$, $r \in [r_o, \overline{v}]$, where

$$R_2(t, \theta) = q^{*(n)}(t) - \frac{n}{(n-1)(1-\sigma_1)} \frac{\varphi'_{\theta}[(1-\sigma_1)t+\sigma_1]}{\varphi''_{\theta}[(1-\sigma_1)t+\sigma_1]} q^{*(n)\prime}(t).$$

Given that $\theta_o \in [\underline{\theta}, \overline{\theta}]$, under Assumption (PO), existence of such an implicit function $\lambda_{r_o}(\theta)$ depends on the values of θ and r in the following way: for $r \in [r_o, \overline{b}_{r_o}]$, $\lambda_{r_o}(\theta)$ exists for any θ; for $r \in (\overline{b}_{r_o}, \overline{v}]$, $\lambda_{r_o}(\theta)$ exists for $\theta \in [\underline{\theta}, \overline{\theta}_{r_o}(r)]$, where $\overline{\theta}_{r_o}(r)$ is the value of θ such that $\lim_{t \uparrow 1} R_2(t, \theta) = r$.[1]

For $\omega_B(r)$, after similar calculation, we have that for $r \in [r_o, \overline{b}_{r_o}]$,

$$\begin{aligned}
\omega_B(r) &= \int_r^{\overline{v}} [v - s_r(v)] dF^{(n)}(v) \\
&= (1-\sigma_1) \int_r^{\overline{v}} [v - s_{r_o}(v) - L(r|v)(r - s_{r_o}(r))] dG^{*(n)}(s_{r_o}(v)) \\
&= -\frac{n}{n-1} \int_r^{\overline{v}} \left[\frac{\varphi'_{\theta_o}[(1-\sigma_1)G^{*(n)}(s_{r_o}(v))+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)G^{*(n)}(s_{r_o}(v))+\sigma_1]} q^{*(n)\prime}(G^{*(n)}(s_{r_o}(v))) \right] dG^{*(n)}(s_{r_o}(v)) \\
&\quad - (1-\sigma_1) \int_r^{\overline{v}} [L(r|v)(r - s_{r_o}(r))] dG^{*(n)}(s_{r_o}(v)) \\
&= -\frac{n}{n-1} \int_{\lambda_{r_o}}^1 \frac{\varphi'_{\theta_o}[(1-\sigma_1)t+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)t+\sigma_1]} dq^{*(n)}(t) \\
&\quad + \frac{n}{n-1} \frac{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]} q^{*(n)\prime}(\lambda_{r_o}) \int_{\lambda_{r_o}}^1 \left[\frac{\varphi'_{\theta_o}[(1-\sigma_1)t+\sigma_1]}{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]} \right]^{\frac{n-1}{n}} dt,
\end{aligned}$$

where the third equality follows from

$$v - s_{r_o}(v) = -\frac{n}{(n-1)(1-\sigma_1)} \frac{\varphi'_{\theta_o}[(1-\sigma_1)G^{*(n)}(s_{r_o}(v))+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)G^{*(n)}(s_{r_o}(v))+\sigma_1]} q^{*(n)\prime}(G^{*(n)}(s_{r_o}(v))), v \in [r_o, \overline{v}].$$

---

[1] For Clayton copula, we have $\overline{\theta}_{r_o}(r) = \frac{nq^{*(n)\prime}(1)}{(r-\overline{b}_{r_o})(n-1)(1-\sigma_1)} - 1$.

For the special case that $r = r_o$, which measures the expected profit for a bidder to enter an auction with binding reserve price r_o as observed in the sample, we have $G^{*(n)}(s_{r_o}(r_o)) = G^{*(n)}(r_o) = 0$, and

$$\begin{aligned}\frac{\omega_B(r_o)}{n} &= -\frac{1}{n-1}\int_0^1 \frac{\varphi'_{\theta_o}[(1-\sigma_1)t+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)t+\sigma_1]}dq^{*(n)}(t) \\ &\quad + \frac{1}{n-1}\frac{\varphi'_{\theta_o}(\sigma_1)}{\varphi''_{\theta_o}(\sigma_1)}q^{*(n)\prime}(0)\int_0^1 \left[\frac{\varphi'_{\theta_o}[(1-\sigma_1)t+\sigma_1]}{\varphi'_{\theta_o}(\sigma_1)}\right]^{\frac{n-1}{n}} dt \\ &= -\frac{1}{n-1}\int_0^1 \frac{\varphi'_{\theta_o}[(1-\sigma_1)t+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)t+\sigma_1]}dt.\end{aligned}$$

The second equality above follows from the restriction that

$$0 = r_o - r_o = -\frac{n}{(n-1)(1-\sigma_1)}\frac{\varphi'_{\theta_o}(\sigma_1)}{\varphi''_{\theta_o}(\sigma_1)}q^{*(n)\prime}(0),$$

and the fact that

$$\int_0^1 \left[\frac{\varphi'_{\theta_o}(t(1-\sigma_1)+\sigma_1)}{\varphi'_{\theta_o}(\sigma_1)}\right]^{\frac{n-1}{n}} dt \leq \int_0^1 1dt = 1 < \infty,$$

where the first inequality is due to the fact that $\varphi'_{\theta_o}(t)$ is increasing and negative.

Further, the total welfare is simply

$$\begin{aligned}\omega_T(r) &= \omega_S(r) + \omega_B(r) \\ &= v_o[(1-\sigma_1)\lambda_{r_o}+\sigma_1] + (1-\sigma_1)\int_{\lambda_{r_o}}^1 q^{*(n)}(t)dt - \frac{n}{n-1}\int_{\lambda_{r_o}}^1 \frac{\varphi'_{\theta_o}[(1-\sigma_1)t+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)t+\sigma_1]}dq^{*(n)}(t).\end{aligned}$$

When the reserve price is not binding, i.e., $r_o = \underline{v}$, we have $\sigma_1 = 0$, $q^{*(n)}(t) = q^{(n)}(t)$. Straightforward simplification gives the results of the total welfare, the seller's expected revenue, and the bidders' expected surplus in Theorem 3.3.7, and the restriction $R_2(t,\theta)$ reduces to

$$R_1(t,\theta) = q^{(n)}(t) - \frac{n}{n-1}\frac{\varphi'_\theta(t)}{\varphi''_\theta(t)}q^{(n)\prime}(t).$$

Finally, for the efficiency gain of a first-price sealed-bid auction over the random assignment mechanism, we have that under Assumptions (AS) and (AP),

$$\mathbb{E}[V_i] = \int_{\underline{v}}^{\overline{v}} vdF(v) = \int_{\underline{v}}^{r_o} vd\varphi^{*-1}_{\theta_o}(F^{(n)}(v)) + \int_{r_o}^{\overline{v}} vd\varphi^{*-1}_{\theta_o}(F^{(n)}(v)).$$

Since we only observe the highest bid, if the data is from auctions with binding reserve price r_o, then it does not provide any information for $F^{(n)}(v)$ on $[\underline{v}, r_o]$. In order to proceed with our partial identification approach, the data must be from auctions without binding reserve price, that is, $r_o = \underline{v}$. In this case, we have

$$\begin{aligned}\mathbb{E}[V_i] &= \int_{\underline{v}}^{\overline{v}} v d\varphi_{\theta_o}^{*-1}(F^{(n)}(v)) \\ &= \int_{\underline{v}}^{\bar{b}} \left[b - \frac{n\varphi'_{\theta_o}(G^{(n)}(b))}{(n-1)\varphi''_{\theta_o}(G^{(n)}(b))} q^{(n)\prime}(G^{(n)}(b)) \right] d\varphi_{\theta_o}^{*-1}(G^{(n)}(b)) \\ &= \int_0^1 q^{(n)}(t) d\varphi_{\theta_o}^{*-1}(t) - \int_0^1 \frac{n\varphi'_{\theta_o}(t)}{(n-1)\varphi''_{\theta_o}(t)} q^{(n)\prime}(t) d\varphi_{\theta_o}^{*-1}(t),\end{aligned}$$

hence the result in Theorem 3.3.7 follows. ■

Proof. (Theorem 3.4.6) The seller's expected revenue under reserve price r is

$$\omega_S(r) = v_o F^{(n)}(r) + \int_r^{\overline{v}} s_r(v) dF^{(n)}(v).$$

Now for $r \in [r_o, \overline{v}]$, we can write the first-order condition of the seller's revenue maximization problem in terms of the observed quantities as

$$\begin{aligned}&\frac{d\omega_S(r)}{dr} \\ &= (v_o - r) f^{(n)}(r) + \int_r^{\overline{v}} \frac{\partial s_r(v)}{\partial r} f^{(n)}(v) dv \\ &= (1-\sigma_1)(v_o - r) g^{*(n)}(s_{r_o}(r)) s'_{r_o}(r) + (1-\sigma_1) \int_r^{\overline{v}} \left[\frac{\varphi'_{\theta_o}[(1-\sigma_1)G^{*(n)}(s_{r_o}(v)) + \sigma_1]}{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o} + \sigma_1]} \right]^{\frac{n-1}{n}} dG^{*(n)}(s_{r_o}(v)) \\ &= (1-\sigma_1)(v_o - r) \frac{d\lambda_{r_o}}{dr} + (1-\sigma_1) \int_{\lambda_{r_o}}^1 \left[\frac{\varphi'_{\theta_o}((1-\sigma_1)t + \sigma_1)}{\varphi'_{\theta_o}((1-\sigma_1)\lambda_{r_o} + \sigma_1)} \right]^{\frac{n-1}{n}} dt = 0.\end{aligned}$$

The second equality follows from the fact that $F^{(n)}(v) = (1-\sigma_1)G^{*(n)}(s_{r_o}(v)) + \sigma_1$, and thus $f^{(n)}(v) = (1-\sigma_1) g^{*(n)}(s_{r_o}(v)) s'_{r_o}(v)$ for $v \in [r_o, \overline{v}]$ together with the fact that

$$\frac{\partial s_r(v)}{\partial r} = L(r|v) = \left[\frac{\varphi'_{\theta_o}((1-\sigma_1)G^{*(n)}(s_{r_o}(v)) + \sigma_1)}{\varphi'_{\theta_o}((1-\sigma_1)\lambda_{r_o} + \sigma_1)} \right]^{\frac{n-1}{n}},$$

for $\underline{v} \leq r_o \leq r \leq v \leq \overline{v}$ as shown in the proof of Theorem 3.4.4.

In the above restriction for the optimal reserve price r^*, θ_o, λ_{r_o} and $d\lambda_{r_o}/dr$ are unknown. For $d\lambda_{r_o}/dr$, since $\lambda_{r_o} = G^{*(n)}(s_{r_o}(r))$, then

$$\begin{aligned} &\frac{d\lambda_{r_o}}{dr} \\ &= g^{*(n)}(s_{r_o}(r))s'_{r_o}(r) \\ &= \frac{1}{q^{*(n)\prime}(\lambda_{r_o})s_{r_o}^{-1\prime}(s_{r_o}(r))} \\ &= \left[-\frac{q^{*(n)\prime}(\lambda_{r_o})}{n-1} + \frac{n}{n-1}\frac{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}\left(\frac{\varphi'''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]q^{*(n)\prime}(\lambda_{r_o})}{\varphi''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]} - \frac{q^{*(n)\prime\prime}(\lambda_{r_o})}{1-\sigma_1}\right)\right]^{-1}, \end{aligned}$$

where the third equality follows from that for $b \in [r_o, \overline{b}_{r_o}]$,

$$\begin{aligned} s_{r_o}^{-1\prime}(b) = -\frac{1}{n-1} &+ \left(\frac{n}{n-1}\frac{\varphi'_{\theta_o}[(1-\sigma_1)G^{*(n)}(b)+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)G^{*(n)}(b)+\sigma_1]}\right) \\ &\times \left(\frac{\varphi'''_{\theta_o}[(1-\sigma_1)G^{*(n)}(b)+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)G^{*(n)}(b)+\sigma_1]} - \frac{1}{1-\sigma_1}\frac{q^{*(n)\prime\prime}(G^{*(n)}(b))}{q^{*(n)\prime}(G^{*(n)}(b))}\right), \end{aligned}$$

which in turn follows from taking derivative of $s_{r_o}^{-1}(b)$ with respect to b, where

$$s_{r_o}^{-1}(b) = b - \frac{n}{(n-1)(1-\sigma_1)}\frac{\varphi'_{\theta_o}[(1-\sigma_1)G^{*(n)}(b)+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)G^{*(n)}(b)+\sigma_1]}q^{*(n)\prime}[G^{*(n)}(b)], b \in [r_o, \overline{b}_{r_o}].$$

$d\lambda_{r_o}/dr$ essentially depends on θ_o and λ_{r_o}. For λ_{r_o}, we can obtain it as the solution to

$$q^{*(n)}(\lambda_{r_o}) - \frac{n}{(n-1)(1-\sigma_1)}\frac{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}{\varphi''_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}q^{*(n)\prime}(\lambda_{r_o}) - r = 0.$$

Now rearrange the first-order condition and write

$$r = v_o + \frac{1}{d\lambda_{r_o}/dr}\int_{\lambda_{r_o}}^{1}\left[\frac{\varphi'_{\theta_o}[(1-\sigma_1)t+\sigma_1]}{\varphi'_{\theta_o}[(1-\sigma_1)\lambda_{r_o}+\sigma_1]}\right]^{\frac{n-1}{n}} dt, r \in [r_o, \overline{v}].$$

If θ_o were known, λ_{r_o} as a function of $r \in [r_o, \overline{v}]$ is known. Consequently, the optimal reserve price is known as the zero of the above first-order condition, as long as $r_o \leq r^*$. When θ_o is unknown, we use $\lambda_{r_o}(\theta)$ to replace λ_{r_o}, where $\lambda_{r_o}(\theta)$ is subject to $R_2(\lambda_{r_o}(\theta), \theta) = r$.

Next, let

$$\delta_{r_o}(r;\theta) = v_o + \frac{1}{\lambda'_{r_o}(\theta)} \int_{\lambda_{r_o}(\theta)}^{1} \left[\frac{\varphi'_\theta[(1-\sigma_1)t+\sigma_1]}{\varphi'_\theta[(1-\sigma_1)\lambda_{r_o}(\theta)+\sigma_1]} \right]^{\frac{n-1}{n}} dt,$$

where $\lambda'_{r_o}(\theta)$ is obtained by replacing θ_o, λ_{r_o} with $\theta, \lambda_{r_o}(\theta)$ in the expression of $d\lambda_{r_o}/dr$, respectively. Then the bounds follows.

Finally, as a special case, when the observed reserve price is not binding, i.e. $r_o = \underline{v}$, $\delta_{r_o}(r;\theta)$ reduces to $\delta(r;\theta)$, where

$$\delta(r;\theta) = v_o + \frac{1}{\lambda'(\theta)} \int_{\lambda(\theta)}^{1} \left[\frac{\varphi'_\theta(t)}{\varphi'_\theta(\lambda(\theta))} \right]^{\frac{n-1}{n}} dt.$$

In the above expression,

$$\frac{1}{\lambda'(\theta)} = -\frac{q^{(n)\prime}(\lambda(\theta))}{n-1} + \frac{n}{n-1} \frac{\varphi'_\theta(\lambda(\theta))}{\varphi''_\theta(\lambda(\theta))} \left[\frac{\varphi'''_\theta(\lambda(\theta)) q^{(n)\prime}(\lambda(\theta))}{\varphi''_\theta(\lambda(\theta))} - q^{(n)\prime\prime}(\lambda(\theta)) \right],$$

where $\lambda(\theta)$ is subject to $R_1(\lambda(\theta),\theta) = r$. Then the bounds in Theorem 3.3.8 follows. ■

Proof. (Theorem 3.4.8) In a second-price sealed-bid auction, the quantile function $Q^{(n-1)}(p)$ of $V^{(n-1)}$ is directly identified from the data. Under Assumption (AS), (3.10) can be written as

$$Q_o(p) = Q^{(n-1)} \left(n\varphi_o^{-1}[(n-1)\varphi_o(p)] - (n-1)\varphi_o^{-1}[n\varphi_o(p)] \right).$$

Further, since $F_o(p) = \varphi_o^{-1}[\frac{1}{n}\varphi_o(F^{(n)}(p))]$, then

$$Q_o(p) = Q^{(n)} \left(\varphi_o^{-1}(n\varphi_o(p)) \right) = Q^{(n-1)} \left(n\varphi_o^{-1}[(n-1)\varphi_o(p)] - (n-1)\varphi_o^{-1}[n\varphi_o(p)] \right).$$

Let $\varphi_o^{-1}(n\varphi_o(p)) = t$, then $p = \varphi_o^{-1}(\frac{1}{n}\varphi_o(t))$, and

$$Q^{(n)}(t) = Q^{(n-1)} \left[n\varphi_o^{-1} \left(\frac{n-1}{n}\varphi_o(t) \right) - (n-1)t \right].$$

By change of variables, we can write

$$
\begin{aligned}
\varpi_T(r) &= \int_{Q^{(n)-1}(r)}^{1} Q^{(n)}(t)dt - v_o\left[1 - Q^{(n)-1}(r)\right], \\
\varpi_E(r) &= \int_{Q^{(n)-1}(r)}^{1} Q^{(n)}(t)dt - v_o\left[1 - Q^{(n)-1}(r)\right] - \int_0^1 Q_o(t)dt \\
&= \int_{Q^{(n)-1}(r)}^{1} Q^{(n)}(t)dt - v_o\left[1 - Q^{(n)-1}(r)\right] \\
&- \int_0^1 Q^{(n-1)}\Big(n\varphi_o^{-1}[(n-1)\varphi_o(t)] - (n-1)\varphi_o^{-1}[n\varphi_o(t)]\Big)dt, \\
\varpi_S(r) &= rQ^{(n-1)-1}(r) + \int_{Q^{(n-1)-1}(r)}^{1} Q^{(n-1)}(t)dt - v_o - Q^{(n)-1}(r)(r - v_o), \\
\varpi_B(r) &= \int_0^1 \max\{r, Q^{(n)}(t)\}dt - \int_0^1 \max\{r, Q^{(n-1)}(t)\}dt,
\end{aligned}
$$

where $Q^{(n)-1}(t)$ is the inverse of $Q^{(n)}(t)$ and $Q^{(n-1)-1}(t)$ is the inverse of $Q^{(n-1)}(t)$. Under Assumption (AP), when θ_o is unknown, we replace it with θ, then the bounds follows. ■

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