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Time-domain analysis of multiple scattering effects on the radar cross section (RCS) of objects in a random medium

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A dissertation

submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

University of Washington

2020

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Program Authorized to Offer Degree:

Electrical Engineering

University of Washington

Abstract

Time-domain analysis of multiple scattering effects on the radar cross section (RCS) of objects in a random medium

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This dissertation presents a theory of the time-domain radar cross section (RCS) of large conducting objects in discrete random media. The time-domain formula is obtained by applying the inverse Fourier transform of the two-frequency mutual coherence function (MCF) which is derived from both the 2nd order Rytov approximation and the strong fluctuation theory. The general formulation contains the 4th order moment which includes the correlation between the forward and the backward waves. The 4th order moment can be reduced to the summation of the 2nd order moments by assuming the fields are circular complex Gaussian random variables. The stochastic Green's function is simplified using the parabolic equation (PE) approximation, and the sizes of the conducting objects are large in terms of wavelength; therefore, the Kirchhoff approximation is applicable for calculating the surface fields. This theory includes both the backscattering enhancement and the time-domain shower curtain effect that are not normally considered in the conventional theory.

Numerical examples of the time-domain RCS of a conducting square plate in a discrete random medium characterized by the Gaussian phase function are shown to highlight the random media effects on the time-domain waveforms including time delay and pulse broadening in terms of optical depth and random medium location. Numerical results show that both pulse arrival time and pulse broadening increase significantly when the random media is placed far away from the object. This degradation of the image quality, known as the shower curtain effect, can be explained by the characteristics of the incoherent component.

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ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor, Prof. Yasuo Kuga for his guidance, patience and support during my Ph.D. study. He introduced me an excellent and exciting Ph.D. research project on imaging through random medium. I would like to thank Prof. Akira Ishimaru, who is a giant in electromagnetics. He reviewed my derivations carefully and gave many invaluable suggestions. I wish to thank Prof. Leung Tsang, who is a master in electromagnetics. He taught world class electromagnetics courses and provided excellent training for the students who will later do research in electromagnetics. I also wish to thank Prof. Chris Rudell for offering me an opportunity to work on the full-duplex radio project in his lab.

I want to thank the department of electrical & computer engineering to provide the financial support during my Ph.D. study.

I want to say thank you to Prof. Jenq-Neng Hwang and Prof. James A. Ritcey for joining my committee.

I am grateful to my colleagues from Laboratory of Applications and Computations in Electromagnetics and Optics (LACEO), Laboratory of Future Analog System Technologies (FAST) and Electromagnetics and Remote Sensing Laboratory (ERSL).

DEDICATION

To my family

Chapter 1 Introduction

Extensive research has been conducted on time-domain imaging of objects located within a discrete random media [1-4]. Techniques to obtain time-domain imaging in clutter environment have also been proposed by using either polarization diversity [5] or early-time diffusion (ETD) signal [6-7]. However, it is observed that the previous studies of the effects of the random media on the radar cross section (RCS) were limited to the continuous wave (CW) case [8]. Since the radar systems working at millimeter-wave (mm-wave) and infrared typically have a large bandwidth, it is desirable to characterize the random media effects in time-domain. In this study, we will give the definition and derive the formulas for the time-domain radar cross section (RCS) and apply the theory to study the random media effects on the time-domain RCS. The problem statement is visualized in Figure 1.1. An imaging system which can be either a radar or a lidar is located at the origin. The target is on the z-axis separated by a distance of L from the imaging system. The target is assumed to be a large Dirichlet object with piecewise smooth surface. A discrete random medium is sitting in between the imaging system and the target. Its distances to the imaging system and to the target can be varied.

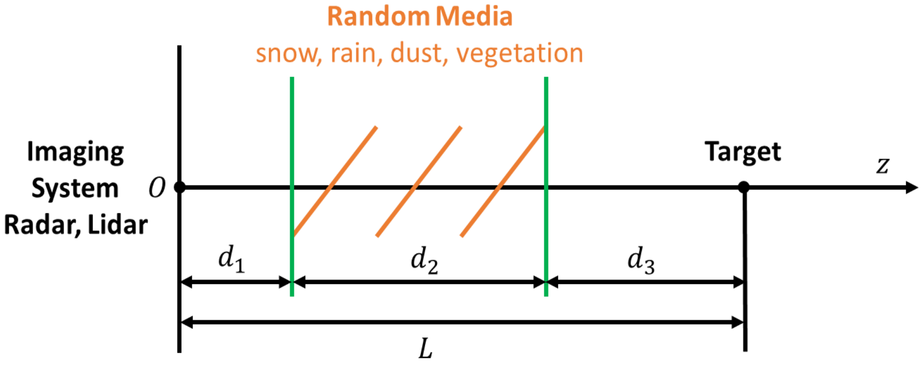

Figure 1.1 The geometry of the problem

The most commonly used technique to solve the problem of scattering from a Dirichlet object in presence of the random medium is the stochastic surface integral equation method. In this method, the scattered field is determined from the equivalent surface current on the object which is generated by the incident field. For a Dirichlet object in the random medium, the induced surface current depends not only on the geometry of the object but also on the characteristics of the random medium. In general cases, the resulted stochastic integral equations are too complicated to be solved analytically. Therefore, some simplifications must be made. In this study, we assume that the object size and the surface radius of the curvature are both much greater than the center wavelength of the imaging system. Under this assumption, the Kirchhoff approximation is applicable to obtain the induced surface current. It is noticed that this assumption should be valid for a variety of targets at mm-wave frequencies.

We start with the extended Huygens-Fresnel principle and the 2nd order Rytov approximation [1-4, 10] for the incident field and calculate the scattered field by using the reciprocity theory. Then the two-frequency RCS will be defined and computed. Finally, the time-domain RCS is found by taking the inverse Fourier transform of the two-frequency RCS. The formula of the time-domain RCS contains the 4th order moment which can be reduced to the summation of the 2nd order moments by assuming that both incident and scattered fields are circular complex Gaussian random variables [1-3, 8-9]. The reduced expression includes the correlation between the incident and scattered fields which leads to results in the backscattering enhancement effect and shower curtain effect. The backscattering enhancement effect is important to correctly predict the RCS of an object in the random medium while the shower curtain effect gives correct dependence of RCS on the location of the random medium. The time dependent factor $exp(-i\omega t)$ is assumed and suppressed in this study.

The thesis is organized in the following way. First of all, we will derive the two-frequency MCF using extended Huygens-Fresnel principle and the 2nd order Rytov approximation in Chapter 2. Secondly, we will discuss the correlation functions of the output signals of the imaging system in Chapter 3. Thirdly, we will extend the concept of the traditional frequency-domain RCS and give the formal definition of the time-domain RCS in Chapter 4. Fourthly, we will calculate the time-domain RCS using the strong fluctuation theory in Chapter 5. Fifthly, we will compute the time-domain RCS for different geometrical configurations. By comparing the simulation results for those cases, the effects of the random medium on the time-domain RCS will be analysed. Finally, we will discuss our future work.

Chapter 2 Two-frequency Mutual Coherence Function for Random Discrete Scatters

The two-frequency mutual coherence function (MCF) describes the correlation of fields at two points on a plane transverse to the direction of propagation of the waves at two frequencies. It has been shown that the two-frequency MCF is critical in studying the pulse propagation through random medium [1-4].

From the literature, two analytic techniques are most commonly used to evaluate the two-frequency MCF in a random medium. The first one is the parabolic equation method [4, 12-23]. The second one is the extended Huygens-Fresnel principle [1-4, 24-35]. In the parabolic equation method, the differential equation for the MCF is formed first. Then the Fourier transform method is applied to solve the differential equation. Combining with the proper boundary condition, the MCF can be finally determined. It has been shown that this method is applicable for both strong and weak cases of fluctuations. In the extended Huygens-Fresnel principle, the correlation between fields is given by the Huygens-Fresnel principle. Next the cumulent expansion is applied to calculate the statistical moments. Subsequently, the MCF is determined by combining the corresponding 1st order and 2nd order statistical moments. It is known that this method is only valid for the case of weak fluctuations. It is should be noticed that both methods are originally developed to study the line-of-sight propagation of optical waves through random medium. However, as pointed out in [39], the resulted derived for the optical case should also be valid at mm-wave.

In this chapter, we will derive the two-frequency MCF using the extended Huygens-Fresnel principle and the 2nd order Rytov approximation. We choose the extended Huygens-Fresnel principle because it is mathematically more convenient than the parabolic equation method for the problem under consideration. More specifically, the effects on the two-frequency MCF caused by varying the location of the random medium can be easily taken into account. This will be explained in details later.

To derive the two-frequency MCF for the random medium contains discrete scatters, we will use following steps. Frist, the formula of the two-frequency MCF for a continuous medium with random refractive index fluctuations is found by using the extended Huygens-Fresnel principle and the 2nd order Rytov approximation. Second, the relationship between the covariance function of the refractive index fluctuations and the scattering cross section of the discrete scatter is established. Finally, the two-frequency MCF is obtained for the random medium containing discrete scatters with Gaussian angular scattering characteristics.

2.1 Wave Equations in a Random Medium

We will derive the two-frequency mutual coherence function (MCF) by starting with the Maxwell's equations [3-4]

$$\nabla \times \bar{E}(\bar{r}) = \mathrm{i}\omega\mu_0\bar{H}(\bar{r}) \tag{2.1}$$

$$\nabla \times \bar{H}(\bar{r}) = -\mathrm{i}\omega\varepsilon_0\varepsilon_r(\bar{r})\bar{E}(\bar{r}) \tag{2.2}$$

where ε_0 is the free-space permittivity, μ_0 is the free-space permeability and $\varepsilon_r(\bar{r})$ is the dielectric constant which is a random function of position $\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Combining (2.1) and (2.2) gives

$$\nabla \times \nabla \times \bar{E}(\bar{r}) - \omega^2\varepsilon_0\varepsilon_r(\bar{r})\mu_0\bar{E}(\bar{r}) = 0 \tag{2.3}$$

Applying Gauss's law

$$\nabla \cdot \left(\varepsilon_0 \epsilon_r(\bar{r})\bar{E}(\bar{r})\right) = \varepsilon_0 \varepsilon_r(\bar{r})\nabla \cdot \bar{E}(\bar{r}) + \varepsilon_0 \bar{E}(\bar{r}) \cdot \nabla \varepsilon_r(\bar{r}) = 0 \qquad (2.4)$$

According to (2.4), the divergence of the electric field can be written as

$$\nabla \cdot \bar{E}(\bar{r}) = -\frac{\bar{E}(\bar{r}) \cdot \nabla \varepsilon_r(\bar{r})}{\varepsilon_r(\bar{r})} \qquad (2.5)$$

Applying the vector calculus identity

$$\nabla \times \nabla \times \bar{E}(\bar{r}) = -\nabla^2 \bar{E}(\bar{r}) + \nabla\left(\nabla \cdot \bar{E}(\bar{r})\right) \qquad (2.6)$$

Substituting (2.5)-(2.6) into (2.3), we obtain

$$\nabla^2 \bar{E}(\bar{r}) + \omega^2 \varepsilon_0 \varepsilon_r(\bar{r}) \mu_0 \bar{E}(\bar{r}) + \nabla\left(\frac{\nabla \varepsilon_r(\bar{r})}{\varepsilon_r(\bar{r})} \cdot \bar{E}(\bar{r})\right) = 0 \qquad (2.7)$$

The dielectric constant is related to the refractive index through

$$\varepsilon_r(\bar{r}) = n^2(\bar{r}) \qquad (2.8)$$

Combining (2.7) and (2.8) gives

$$\nabla^2 \bar{E}(\bar{r}) + k_0^2 \varepsilon_r(\bar{r}) \bar{E}(\bar{r}) + \nabla\left(\frac{\nabla n^2(\bar{r})}{n^2(\bar{r})} \cdot \bar{E}(\bar{r})\right) = 0$$

$$\nabla^2 \bar{E}(\bar{r}) + k_0^2 \varepsilon_r(\bar{r}) \bar{E}(\bar{r}) + 2\nabla\left(\frac{\nabla n(\bar{r})}{n(\bar{r})} \cdot \bar{E}(\bar{r})\right) = 0$$

$$\nabla^2 \bar{E}(\bar{r}) + k_0^2 \varepsilon_r(\bar{r}) \bar{E}(\bar{r}) + 2\nabla\left(\nabla \ln\left(n(\bar{r})\right) \cdot \bar{E}(\bar{r})\right) = 0 \qquad (2.9)$$

where k_0 is the free-space wavenumber.

The refractive index is a random function which contains the average and the fluctuation components

$$\mathrm{n}(\bar{r}) = \langle \mathrm{n}(\bar{r}) \rangle + n_1(\bar{r}) = 1 + n_1(\bar{r}) \tag{2.10}$$

where the angle brackets symbol denotes the ensemble average. $\langle \mathrm{n}(\bar{r}) \rangle$ is the mean refractive index which is assumed to be 1. $n_1(\bar{r})$ is the refractive index fluctuation which satisfies $\langle n_1(\bar{r}) \rangle = 0$.

If we assume the random medium is both statistically homogeneous and delta correlated in the direction of propagation and consider the direction of propagation is the positive-z axis, the covariance function of refractive index fluctuation may be written as

$$\langle n_1(\bar{r}_1) n_1(\bar{r}_2) \rangle = B_n(\bar{r}_1 - \bar{r}_2) = \delta(z_1 - z_2) A_n(\bar{\rho}_1 - \bar{\rho}_2) \tag{2.11}$$

where $A_n(\bar{\rho}_1 - \bar{\rho}_2)$ is the two-dimensional covariance function and $\bar{\rho} = x\hat{x} + y\hat{y}$ is the position vector in the plane transverse to the direction of propagation. Because of the statistically homogeneous property, the covariance function $A_n(\bar{\rho}_1 - \bar{\rho}_2)$ only depends on spatial difference $\bar{\rho}_1 - \bar{\rho}_2$.

When the correlation distance of the refractive index fluctuation is much larger than the wavelength, the last term in equation (2.9) may be neglected and the mathematical complexity of derivations will be greatly reduced [4, 10-13, 36-38]. However, by dropping the last term in (2.9), we will neglect the depolarization effect at the same time. We should emphasise that this will not introduce noticeable errors in the analysis. Considering the practical situations, the change of polarization state of an electromagnetic wave is caused primarily by the reflection from the object surface rather than the propagation of the wave through the random medium. Moreover, since the Kirchhoff approximation is applied to calculate the surface field, only co-polarized component of the scattered field can be obtained. In other words, our model for the surface reflection is not able to predict the cross-polarized component of the scattered field. Because the polarization information has already been excluded when calculating the scattered field from the object surface, we do not need to include depolarization term in the equation. Therefore, we verify the conclusion made before.

By neglecting the depolarization effect, the equation (2.9) becomes

$$\nabla^2 \bar{E}(\bar{r}) + k_0^2 \varepsilon_r(\bar{r}) \bar{E}(\bar{r}) = 0 \tag{2.12}$$

The equation (2.12) can be further reduced to a scalar wave equation. Let $U(\bar{r})$ be either x or y components of the electric field, that is, $U(\bar{r}) = E_\xi(\bar{r}), \xi = x, y$. The scalar stochastic Helmholtz equation may be written as

$$\nabla^2 U(\bar{r}) + k_0^2 n^2(\bar{r}) U(\bar{r}) = 0 \tag{2.13}$$

In weak fluctuation region, we can use the approximation

$$n^2(\bar{r}) = \left(1 + n_1(\bar{r})\right)^2 \approx 1 + 2n_1(\bar{r}) \tag{2.14}$$

where the refractive index fluctuation satisfies $|n_1(\bar{R})| \ll 1$.

Combining (2.13) and (2.14) gives

$$\nabla^2 U(\bar{r}) + k_0^2 \left(1 + 2n_1(\bar{r})\right) U(\bar{r}) = 0 \tag{2.15}$$

where k_0 is the free-space wavenumber and $n_1(\bar{r})$ is the refractive index fluctuation.

Under weak fluctuation condition, the scalar stochastic Helmholtz equation (2.15) can be solved by using the perturbation method. More specifically, we will write the solution in terms of series expansion. By substituting the lower-order terms into the equation (2.15), the succeeding higher-order terms can be calculated. By repeating the process, each term in the series can be determined. Depending on the perturbation series used in the iterative processes, we may have either Born solution or Rytov solution. We will discuss these two solutions and the relation between them in details later.

2.2 Covariance Function of Refractive Index Fluctuation

For a lossless random medium, the refractive index fluctuation $n_1(\bar{r})$ is a real function. And we can write [4, 10, 21]

$$B_n(\bar{r}_1 - \bar{r}_2) = \langle n_1(\bar{r}_1)n_1^*(\bar{r}_2)\rangle = \langle n_1(\bar{r}_1)n_1(\bar{r}_2)\rangle \qquad (2.16)$$

For the statistically homogeneous random fluctuations, the covariance function can be written as,

$$B_n(\bar{r}_1 - \bar{r}_2) = \iiint d\bar{K}\Phi_n(\bar{K})exp\big(i\bar{K}\cdot(\bar{r}_1 - \bar{r}_2)\big) \qquad (2.17)$$

where $\Phi_n(\bar{K})$ is the three-dimensional spatial power spectrum of the refractive index fluctuations. Because of the statistically homogeneity, the covariance function $B_n(\bar{r}_1 - \bar{r}_2)$ only depends on spatial difference $\bar{r}_1 - \bar{r}_2$.

It is convenient to separate the three-dimensional vectors into the component along with the direction of propagation and the component transverse to the direction of propagation, $\bar{r} = \bar{\rho} + z\hat{z}$ and $\bar{K} = \bar{\kappa} + K_z\hat{z}$.

We have

$$\bar{K}\cdot(\bar{r}_1 - \bar{r}_2) = \bar{\kappa}\cdot(\bar{\rho}_1 - \bar{\rho}_2) + K_z(z_1 - z_2) \qquad (2.18)$$

Substituting (2.18) into (2.17), we obtain

$$B_n(\bar{r}_1 - \bar{r}_2) = \iiint d\bar{K}\Phi_n(\bar{K})exp\big(i\bar{\kappa}\cdot(\bar{\rho}_1 - \bar{\rho}_2) + iK_z(z_1 - z_2)\big) \qquad (2.19)$$

Using the Fourier transform identity

$$2\pi\delta(z_1 - z_2) = \int_{-\infty}^{+\infty} exp\big(iK_z(z_1 - z_2)\big)dK_z \qquad (2.20)$$

Combining (2.19) and (2.20) gives

$$B_n(\bar{r}_1-\bar{r}_2)=\int exp\big(iK_z\cdot(z_1-z_2)\big)dk_z\iint\Phi_n(\bar{\kappa},0)exp\big(i\bar{\kappa}\cdot(\bar{\rho}_1-\bar{\rho}_2)\big)d\bar{\kappa}$$

$$=2\pi\delta(z_1-z_2)\iint\Phi_n(\bar{\kappa},0)exp\big(i\bar{\kappa}\cdot(\bar{\rho}_1-\bar{\rho}_2)\big)d\bar{\kappa}\tag{2.21}$$

Combining (2.11) and (2.21) gives

$$B_n(\bar{\rho}_1-\bar{\rho}_2,z_1-z_2)=2\pi\delta(z_1-z_2)\iint\Phi_n(\bar{\kappa},0)exp\big(i\bar{\kappa}\cdot(\bar{\rho}_1-\bar{\rho}_2)\big)d\bar{\kappa}$$

$$=\delta(z_1-z_2)A_n(\bar{\rho}_1-\bar{\rho}_2)\tag{2.22}$$

We have

$$A_n(\bar{\rho})=2\pi\iint\Phi_n(\bar{\kappa},0)exp(i\bar{\kappa}\cdot\bar{\rho})d\bar{\kappa}\tag{2.23}$$

where $\bar{\rho}=\bar{\rho}_1-\bar{\rho}_2$.

The covariance function of the refractive index fluctuation is

$$B_n(\bar{\rho},z)=\delta(z)A_n(\bar{\rho})=2\pi\delta(z)\iint\Phi_n(\bar{\kappa},0)exp(i\bar{\kappa}\cdot\bar{\rho})d\bar{\kappa}\tag{2.24}$$

where $z=z_1-z_2$ and $\bar{\rho}=\bar{\rho}_1-\bar{\rho}_2$.

The refractive index fluctuation can also be expressed in terms of the two-dimensional Riemann-Stieltjes integral

$$n_1(\bar{r})=n_1(\bar{s},z)=\iint exp(i\bar{\kappa}\cdot\bar{\rho})dv(\bar{\kappa},z)\tag{2.25}$$

where $\bar{\rho}=x\hat{x}+y\hat{y}$, $\bar{\kappa}=\kappa_x\hat{x}+\kappa_y\hat{y}$, and $dv(\bar{\kappa},z)$ is the random amplitude of the refractive index fluctuations.

The random amplitude of the refractive index fluctuations satisfies

$$\langle dv(\bar{\kappa},z)\rangle=0\tag{2.26a}$$

$$\langle dv(\bar{\kappa},z)dv^*(\bar{\kappa}',z')\rangle=F_n(\bar{\kappa},z-z')\delta(\bar{\kappa}-\bar{\kappa}')d\bar{\kappa}d\bar{\kappa}'\tag{2.26b}$$

where $F_n(\bar{\kappa}, z)$ is the two-dimensional spectral density.

Combining (2.25) and (2.26) gives

$$B_n(\bar{r}_1 - \bar{r}_2) = \langle \iint exp(i\bar{\kappa} \cdot \bar{\rho}_1) dv(\bar{\kappa}, z_1) \iint exp(-i\bar{\kappa}' \cdot \bar{\rho}_2) dv^*(\bar{\kappa}', z_2) \rangle$$

$$= \iint \iint exp(i\bar{\kappa} \cdot \bar{\rho}_1 - i\bar{\kappa}' \cdot \bar{\rho}_2) \langle dv(\bar{\kappa}, z_1) dv^*(\bar{\kappa}', z_2) \rangle$$

$$= \iint F_n(\bar{\kappa}, z_1 - z_2) exp\big(i\bar{\kappa} \cdot (\bar{\rho}_1 - \bar{\rho}_2)\big) d\bar{\kappa} \tag{2.27}$$

The covariance function of the refractive index fluctuation is

$$B_n(\bar{\rho}, z) = \iint F_n(\bar{\kappa}, z) exp(i\bar{\kappa} \cdot \bar{\rho}) d\bar{\kappa} \tag{2.28}$$

The three-dimensional spectral density is the inverse Fourier transform of the three-dimensional covariance function

$$\Phi_n(\bar{K}) = \frac{1}{(2\pi)^3} \iiint d\bar{R} B_n(\bar{r}) exp(-i\bar{K} \cdot \bar{r}) \tag{2.29}$$

Similarly, the two-dimensional spectral density can be written as

$$F_n(\bar{\kappa}, z) = \frac{1}{(2\pi)^2} \iint d\bar{\rho} B_n(\bar{\rho}, z) exp(-i\bar{\kappa} \cdot \bar{\rho}) \tag{2.30}$$

Combining (2.29) and (2.30) gives

$$\Phi_n(\bar{K}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} exp(-iK_z z) F_n(\bar{\kappa}, z) dz \tag{2.31}$$

If the random fluctuation is isotropic, we can express the two-dimensional integral in the polar coordinate system

$$\iint d\bar{\kappa} = \int_0^{\infty} \kappa d\kappa \int_0^{2\pi} d\theta \tag{2.32}$$

Combining (2.23) and (2.24) gives

$$A_n(\bar{\rho}) = 2\pi \int_0^\infty \kappa d\kappa \int_0^{2\pi} d\theta\, \Phi_n(\bar{\kappa}, 0) exp\big(i\kappa\rho cos(\theta)\big)$$

$$= 2\pi \int_0^\infty \kappa d\kappa \Phi_n(\bar{\kappa}, 0) \int_0^{2\pi} d\theta\, exp\big(i\kappa\rho cos(\theta)\big) \qquad (2.33)$$

Employing the integral identity

$$\int_0^{2\pi} d\theta\, exp\big(\pm ixcos(\theta)\big) = 2\pi J_0(x) \qquad (2.34)$$

where $J_0(x)$ is Bessel function of the first kind of order zero.

Then the two-dimensional covariance function becomes

$$A_n(\bar{\rho}) = 4\pi^2 \int_0^\infty \kappa \Phi_n(\bar{\kappa}, 0) J_0(\kappa\rho) d\kappa \qquad (2.35)$$

Therefore, for the statistically homogeneous and isotropic random fluctuations, we have

$$\Phi_n(K) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} exp(-iK_z z) F_n(\kappa, |z|) dz \qquad (2.36)$$

where $K = |\bar{K}|$ and $\kappa = |\bar{\kappa}|$.

Since (2.36) should hold for arbitrary K_z, we can set $K_z = 0$ and obtain

$$2\pi \Phi_n(K) = \int_{-\infty}^{+\infty} F_n(\kappa, |z|) dz \qquad (2.37)$$

It should be noticed that (2.37) is applicable to statistically homogeneous and isotropic random fluctuations.

2.3 Born Approximation

In Born approximation, we write $U(\bar{r})$ in forms of the expansion [3-4, 10, 21]

$$U(\bar{r}) = U_0(\bar{r}) + U_1(\bar{r}) + U_2(\bar{r}) + \cdots \qquad (2.38)$$

where $U_0(\bar{r})$ is the field in absence of the random medium, $U_j(\bar{r}),\ j = 1,2,\cdots$ is jth order scattering field caused by the random fluctuations.

The convergence of the series expansion requires that

$$|U_0(\bar{r})| \ll |U_1(\bar{r})| \ll |U_2(\bar{r})| \ll \cdots \tag{2.39}$$

Substituting (2.38) into (2.15), we have

$$\nabla^2(U_0(\bar{r}) + U_1(\bar{r}) + U_2(\bar{r}) + \cdots)$$

$$+k^2\big(1 + 2n_1(\bar{R})\big)(U_0(\bar{r}) + U_1(\bar{r}) + U_2(\bar{r}) + \cdots) = 0 \tag{2.40}$$

Equating terms of equal order to zero and writing down equations up to 2nd order, we obtain

$$\nabla^2 U_0(\bar{r}) + k^2 U_0(\bar{r}) = 0 \tag{2.41a}$$

$$\nabla^2 U_1(\bar{r}) + k^2 U_1(\bar{r}) = -2k^2 n_1(\bar{r}) U_0(\bar{r}) \tag{2.41b}$$

$$\nabla^2 U_2(\bar{r}) + k^2 U_2(\bar{r}) = -2k^2 n_1(\bar{r}) U_1(\bar{r}) \tag{2.41c}$$

The inhomogeneous scalar wave equations (2.41a)-(2.41c) can be converted into the integral equations using the Green's function.

The 1st order scattered field is written as

$$U_1(\bar{r}) = \iiint G(\bar{r},\bar{r}')[2k^2 n_1(\bar{r}') U_0(\bar{r}')]d\bar{r}' \tag{2.42}$$

where $G(\bar{r},\bar{r}')$ is the free-space Green's function.

The 2nd order scattered field is written as

$$U_2(\bar{r}) = \iiint G(\bar{r},\bar{r}')[2k^2 n_1(\bar{r}')U_2(\bar{r}')]d\bar{r}' \tag{2.43}$$

where $G(\bar{r},\bar{r}')$ is the free-space Green's function.

The free-space Green's function $G(\bar{r},\bar{r}')$ is given by

$$G(\bar{r},\bar{r}') = \frac{exp(ik|\bar{r}-\bar{r}'|)}{4\pi|\bar{r}-\bar{r}'|} \tag{2.44a}$$

$$|\bar{r}-\bar{r}'| = \sqrt{|\bar{\rho}-\bar{\rho}'|^2 + (z-z')^2} \tag{2.44b}$$

where $\bar{\rho} = x\hat{x} + y\hat{y}$. Because the longitudinal distance between the source and the observation point $|z-z'|$ is much larger than the lateral distance $|\bar{\rho}-\bar{\rho}'|$, the term $|\bar{r}-\bar{r}'|$ may be expanded in a binomial series [4, 10-11, 21].

Therefore, we have

$$G(\bar{r},\bar{p}) \approx \frac{1}{4\pi(L-z)} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right) \tag{2.45a}$$

$$\bar{r} = \bar{\rho} + L\hat{z},\ \bar{p} = \bar{s} + z\hat{z} \tag{2.45b}$$

where $\bar{r}$ denotes the observation point, $\bar{p}$ denotes the source and L is the longitudinal distance between the observation point and the source.

Combining (2.42) and (2.45), we obtain the 1st order scattered field

$$U_1(\bar{\rho},L) = \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)\frac{n_1(\bar{s},z)U_0(\bar{s},z)}{L-z} \tag{2.46}$$

Taking the ensemble average of (2.46) gives

$$\langle U_1(\bar{\rho},L)\rangle = \langle\frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)\frac{n_1(\bar{s},z)U_0(\bar{s},z)}{L-z}\rangle$$

$$= \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)\frac{\langle n_1(\bar{s},z)\rangle U_0(\bar{s},z)}{L-z} = 0 \quad (2.47)$$

It is observed that the mean of the 1st order scattered field is zero.

Combine (2.43) and (2.45), we obtain the 2nd order scattered field

$$U_2(\bar{\rho},L) = \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)\frac{n_1(\bar{s},z)U_1(\bar{s},z)}{L-z} \quad (2.48)$$

Taking the ensemble average of (2.48) gives

$$\langle U_2(\bar{\rho},L)\rangle = \langle\frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)\frac{n_1(\bar{s},z)U_1(\bar{s},z)}{L-z}\rangle$$

$$= \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)\frac{\langle n_1(\bar{s},z)U_1(\bar{s},z)\rangle}{L-z} \neq 0 \quad (2.49)$$

It is observed that the mean of the 2nd order scattered field is not zero.

2.4 Rytov Approximation for Random Fluctuations

In Rytov approximation, we write $U(\bar{r})$ in the form [3-4, 10, 21]

$$U(\bar{r}) = U_0(\bar{r}) exp\big(\psi(\bar{r})\big) \quad (2.50)$$

where $\psi(\bar{r})$, the complex phase perturbation, takes an expansion of the form

$$\psi(\bar{r}) = \psi_1(\bar{r}) + \psi_2(\bar{r}) + \cdots \quad (2.51)$$

where $\psi_j(\bar{r}), j = 1, 2, \cdots$ is the jth order perturbation.

It is known that terms in expansion of the complex phase perturbation can be related to corresponding terms in Born approximation [10, 24, 31]. In order to do so, it is convenient to define the normalized Born approximation

$$\Phi_j(\bar{r}) = \frac{U_j(\bar{r})}{U_0(\bar{r})},\ j = 1, 2,\ \cdots \quad (2.52)$$

Equating the 1st order Rytov and 1st order Born approximations, we obtain

$$U_0(\bar{r})exp(\psi_1(\bar{r})) = U_0(\bar{r}) + U_1(\bar{r}) = U_0(\bar{r})(1 + \Phi_1(\bar{r})) \qquad (2.53)$$

Applying Taylor series expansion

$$U_0(\bar{r})exp(\psi_1(\bar{r})) \approx U_0(\bar{r})(1 + \psi_1(\bar{r})) \qquad (2.54)$$

Combining (2.53) into (2.54) gives

$$\psi_1(\bar{r}) \approx \Phi_1(\bar{r}) \qquad (2.55)$$

It shows that the 1st order Rytov approximation is equal to the 1st order normalized Born approximation.

Then the 1st order normalized Born approximation can be written as

$$\Phi_1(\bar{\rho}, L) = \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s}\frac{1}{L-z} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)$$

$$\times \frac{n_1(\bar{s},z)}{L-z}\frac{U_0(\bar{s},z)}{U_0(\bar{\rho},L)} \qquad (2.56)$$

Equating the 2nd order Rytov and the 2nd Born approximations gives

$$U_0(\bar{r})exp(\psi_1(\bar{r}) + \psi_2(\bar{r})) = U_0(\bar{r}) + U_1(\bar{r}) + U_2(\bar{r})$$

$$= U_0(\bar{r})(1 + \Phi_1(\bar{r}) + \Phi_2(\bar{r})) \qquad (2.57)$$

For mathematical convenience, a dummy variable ϵ is introduced to represent the order of each term in the perturbation series.

Applying Taylor series expansion

$$U_0(\bar{r})exp(\epsilon\psi_1(\bar{r}) + \epsilon^2\psi_2(\bar{r})) = U_0(\bar{r})(1 + \epsilon\Phi_1(\bar{R}) + \epsilon^2\Phi_2(\bar{r}))$$

$$1 + \epsilon\psi_1(\bar{r}) + \epsilon^2\psi_2(\bar{r}) + \frac{1}{2}\left(\epsilon\psi_1(\bar{r}) + \epsilon^2\psi_2(\bar{r})\right)^2 = 1 + \epsilon\Phi_1(\bar{r}) + \epsilon^2\Phi_2(\bar{r})$$

$$\psi_1(\bar{r}) + \psi_2(\bar{r}) + \frac{1}{2}\epsilon^2\psi_1^2(\bar{r}) \approx \epsilon\Phi_1(\bar{r}) + \epsilon^2\Phi_2(\bar{r}) \qquad (2.58)$$

Combining (2.55) and (2.58) gives

$$\psi_2(\bar{r}) \approx \Phi_2(\bar{r}) - \frac{1}{2}\Phi_1^2(\bar{r}) \qquad (2.59)$$

Then the 2nd order normalized Born approximation can be written as

$$\Phi_2(\bar{\rho}, L) = \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s}\, exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)$$

$$\times \frac{n_1(\bar{s},z)}{L-z}\Phi_1(\bar{s},z)\frac{U_0(\bar{s},z)}{U_0(\bar{\rho},L)} \qquad (2.60)$$

2.5 1st Order Spectral Representation of Scattered Field

The 1st order normalized Born approximation is [4, 10, 21]

$$\Phi_1(\bar{\rho}, L) = \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s}\frac{1}{L-z} exp\left(ik(L-z) + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)}\right)$$

$$\times \frac{n_1(\bar{s},z)}{L-z}\frac{U_0(\bar{s},z)}{U_0(\bar{\rho},L)} \qquad (2.61)$$

For mathematical convenience, we introduce the normalized variable $\gamma = \frac{z}{L}$.

We have

$$\frac{U_0(\bar{s},z)}{U_0(\bar{r},L)} = \frac{1}{\gamma} exp\left(-ik(L-z) + \frac{ik\bar{s}\cdot\bar{s}}{2z} - \frac{ik\bar{\rho}\cdot\bar{\rho}}{2L}\right) \qquad (2.62)$$

We have

$$\frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)} + \frac{iks^2}{2z} - \frac{ik\rho^2}{2L} = \frac{ik\left(\rho^2 - 2\bar{\rho}\cdot\bar{s} + s^2\right)}{2(L-z)} + \frac{iks^2}{2z} - \frac{ik\rho^2}{2L}$$

$$= \frac{ik\rho^2}{2(L-z)} - \frac{ik\rho^2}{2L} + \frac{iks^2}{2(L-z)} - \frac{ik\bar{\rho}\cdot\bar{s}}{L-z} + \frac{iks^2}{2z}$$

$$= \frac{ik\gamma\rho^2}{2(L-z)} - \frac{ik\bar{\rho}\cdot\bar{s}}{L-z} + \frac{iks^2}{2(L-z)\gamma} \tag{2.63}$$

Thus

$$exp\left(\frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)} + \frac{iks^2}{2z} - \frac{ik\rho^2}{2L}\right) = exp\left(\frac{ik\gamma\rho^2}{2(L-z)}\right) exp\left(-\frac{ik\bar{\rho}\cdot\bar{s}}{L-z} + \frac{iks^2}{2(L-z)\gamma}\right) \tag{2.64}$$

Combining (2.61), (2.62) and (2.64) gives

$$\Phi_1(\bar{\rho}, L) = \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s} exp\left(\frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)} + \frac{iks^2}{2z} - \frac{ikr^2}{2L}\right)\frac{n_1(\bar{s},z)}{(L-z)\gamma}$$

$$= \frac{k^2}{2\pi}\int_0^L dz \iint d\bar{s} \iint \frac{dv(\bar{\kappa},z)}{(L-z)\gamma} exp\left(i\bar{\kappa}\cdot\bar{s} + \frac{ik(\bar{\rho}-\bar{s})\cdot(\bar{\rho}-\bar{s})}{2(L-z)} + \frac{iks^2}{2z} - \frac{ik\rho^2}{2L}\right)$$

$$= \frac{k^2}{2\pi}\int_0^L dz \iint \frac{dv(\bar{\kappa},z)}{(L-z)\gamma} exp\left(\frac{ik\gamma\rho^2}{2(L-z)}\right) \iint d\bar{s} exp\left(i\bar{\kappa}\cdot\bar{s} + \frac{iks^2}{2(L-z)\gamma} - \frac{ik\bar{\rho}\cdot\bar{s}}{L-z}\right) \tag{2.65}$$

It will be helpful to introduce the integral identity

$$\int_0^\infty exp(iax^2)dx = exp\left(i\frac{\pi sgn(a)}{4}\right)\sqrt{\frac{\pi}{4|a|}} \tag{2.66}$$

where sgn is the sign function. Like Fresnel integral, this integral can be evaluated by the complex contour integration [40].

We have

$$\iint d\bar{s} exp\left(i\bar{\kappa}\cdot\bar{s} + \frac{iks^2}{2(L-z)\gamma} - \frac{ik\bar{\rho}\cdot\bar{s}}{L-z}\right) = exp\left(-\frac{ik\gamma}{2(L-z)}\left(\frac{L-z}{k}\bar{\kappa} - \bar{\rho}\right)\cdot\left(\frac{L-z}{k}\bar{\kappa} - \bar{\rho}\right)\right)$$

$$\times \iint d\bar{s} exp\left(\frac{ik}{2(L-z)\gamma}\left(\bar{s} + \gamma\left(\frac{L-z}{k}\bar{\kappa} - \bar{\rho}\right)\right)\cdot\left(\bar{s} + \gamma\left(\frac{L-z}{k}\bar{\kappa} - \bar{\rho}\right)\right)\right) \tag{2.67}$$

Making use of the formula (2.66), we can evaluate the integral

$$\iint d\bar{s} exp\left(\frac{ik\left(\bar{s}+\gamma\left(\frac{L-z}{k}\bar{\kappa}-\bar{\rho}\right)\right)\cdot\left(\bar{s}+\gamma\left(\frac{L-z}{k}\bar{\kappa}-\bar{\rho}\right)\right)}{2(L-z)\gamma}\right) = \frac{i2\pi(L-z)\gamma}{k} \tag{2.68}$$

We have

$$exp\left(-\frac{ik\gamma\left(\frac{L-z}{k}\bar{\kappa}-\bar{\rho}\right)\cdot\left(\frac{L-z}{k}\bar{\kappa}-\bar{\rho}\right)}{2(L-z)}\right) = exp\left(-\frac{ik\gamma}{2(L-z)}\left(\left(\frac{L-z}{k}\right)^2\kappa^2 - 2\frac{L-z}{k}\bar{\kappa}\cdot\bar{\rho}+\rho^2\right)\right)$$

$$= exp\left(-\frac{i\gamma\kappa^2}{2k}(L-z) + i\gamma\bar{\kappa}\cdot\bar{\rho} - \frac{i\gamma k\rho^2}{2(L-z)}\right) \tag{2.69}$$

Combining (2.65), (2.68) and (2.69) gives

$$\Phi_1(\bar{\rho}, L) = \frac{k^2}{2\pi}\int_0^L dz \iint \frac{dv(\bar{\kappa},z)}{(L-z)\gamma} exp\left(\frac{ik\gamma\rho^2}{2(L-z)}\right)$$

$$\times \iint d\bar{s} exp\left(i\bar{\kappa}\cdot\bar{s} + \frac{iks^2}{2(L-z)\gamma} - \frac{ik\bar{\rho}\cdot\bar{s}}{L-z}\right)$$

$$= ik\int_0^L dz \iint dv(\bar{\kappa},z)\, exp\left(-\frac{i\gamma\kappa^2}{2k}(L-z) + i\gamma\bar{\kappa}\cdot\bar{\rho}\right) \tag{2.70}$$

Therefore, the spectral representation of the normalized 1st order Born approximation is

$$\Phi_1(\bar{\rho}, L) = ik\int_0^L dz \iint dv(\bar{\kappa},z)\, exp\left(-\frac{i\gamma\kappa^2}{2k}(L-z) + i\gamma\bar{\kappa}\cdot\bar{\rho}\right) \tag{2.71}$$

2.6 2nd Order Spectral Representation of Scattered Field

For mathematical convenience, we introduce the normalized variable $\gamma' = \frac{z'}{z}$.

The 2nd order normalized Born approximation is [10, 21]

$$\Phi_2(\bar{\rho}, L) = \frac{ik^3}{2\pi}\int_0^L dz \int_0^z dz' \iint dv(\bar{\kappa},z) \iint dv(\bar{\kappa}',z')$$

$$\times \frac{1}{\gamma(L-z)} exp\left(\frac{i\gamma k\rho^2}{2(L-z)} - \frac{i\gamma'\kappa'^2}{2k}(z-z')\right)$$

$$\times \iint d\bar{s} exp\left(i\left(\bar{\kappa} + \gamma'\bar{\kappa}' - \frac{k\bar{\rho}}{L-z}\right)\cdot\bar{s} + \frac{iks^2}{2\gamma(L-z)}\right) \quad (2.72)$$

We have

$$\iint d\bar{s} exp\left(i\left(\bar{\kappa} + \gamma'\bar{\kappa}' - \frac{k\bar{\rho}}{L-z}\right)\cdot\bar{s} + \frac{iks^2}{2\gamma(L-z)}\right)$$

$$= exp\left(-\frac{i\gamma(L-z)}{2k}(\bar{\kappa} + \gamma'\bar{\kappa}')\cdot(\bar{\kappa} + \gamma'\bar{\kappa}') - \frac{ik\gamma\rho^2}{2(L-z)} + i\gamma(\bar{\kappa} + \gamma'\bar{\kappa}')\cdot\bar{\rho}\right)$$

$$\times \iint d\bar{s} exp\left(\frac{ik\left(\bar{s}+2\gamma\left(\frac{L-z}{k}(\bar{\kappa}+\gamma'\bar{\kappa}')-\bar{\rho}\right)\right)\cdot\left(\bar{s}+2\gamma\left(\frac{L-z}{k}(\bar{\kappa}+\gamma'\bar{\kappa}')-\bar{\rho}\right)\right)}{2\gamma(L-z)}\right) \quad (2.73)$$

Making use of the integral identity (2.66), the integral can be evaluated

$$\iint d\bar{s} exp\left(\frac{ik\left(\bar{s}+2\gamma\left(\frac{L-z}{k}(\bar{\kappa}+\gamma'\bar{\kappa}')-\bar{\rho}\right)\right)\cdot\left(\bar{s}+2\gamma\left(\frac{L-z}{k}(\bar{\kappa}+\gamma'\bar{\kappa}')-\bar{\rho}\right)\right)}{2\gamma(L-z)}\right) = \frac{i2\pi(L-z)\gamma}{k} \quad (2.74)$$

We have

$$\iint d\bar{s} exp\left(i\left(\bar{\kappa} + \gamma'\bar{\kappa}' - \frac{k\bar{\rho}}{L-z}\right)\cdot\bar{s} + \frac{iks^2}{2\gamma(L-z)}\right) = \frac{i2\pi(L-z)\gamma}{k}$$

$$\times exp\left(-\frac{i\gamma(L-z)}{2k}(\bar{\kappa} + \gamma'\bar{\kappa}')\cdot(\bar{\kappa} + \gamma'\bar{\kappa}') - \frac{ik\gamma\rho^2}{2(L-z)} + i\gamma(\bar{\kappa} + \gamma'\bar{\kappa}')\cdot\bar{\rho}\right) \quad (2.75)$$

Combining (2.72), (2.73) and (2.75) gives

$$\Phi_2(\bar{\rho}, L) = -k^2 \int_0^L dz \int_0^z dz' \iint dv(\bar{\kappa}, z) \iint dv(\bar{\kappa}', z') \times exp\left(-\frac{i\gamma'\kappa'^2}{2k}(z-z')\right)$$

$$\times exp\left(-\frac{i\gamma(L-z)}{2k}(\bar{\kappa} + \gamma'\bar{\kappa}')\cdot(\bar{\kappa} + \gamma'\bar{\kappa}') + i\gamma(\bar{\kappa} + \gamma'\bar{\kappa}')\cdot\bar{\rho}\right) \quad (2.76)$$

Therefore, the spectral representation of the normalized 2nd order Born approximation is

$$\Phi_2(\bar{\rho}, L) = -k^2 \int_0^L dz \int_0^z dz' \iint dv(\bar{\kappa}, z) \iint dv(\bar{\kappa}', z') exp\left(-\frac{i\gamma'\kappa'^2}{2k}(z - z')\right)$$

$$\times\, exp\left(-\frac{i\gamma(L-z)}{2k}(\bar{\kappa} + \gamma'\overline{\kappa}') \cdot (\bar{\kappa} + \gamma'\overline{\kappa}') + i\gamma(\bar{\kappa} + \gamma'\overline{\kappa}') \cdot \bar{\rho}\right) \qquad (2.77)$$

2.7 Statistical Moments

Let us define the 2nd order statistical moments [10, 24]

$$E_1(\bar{\rho}, \bar{\rho}, k) = \langle \Phi_2(\bar{\rho}, L, k) \rangle = \langle \psi_2(\bar{\rho}, L, k) \rangle + \frac{1}{2}\langle \psi_1^2(\bar{\rho}, L, k) \rangle \qquad (2.78)$$

$$E_2(\bar{\rho}_1, k_1 \bar{\rho}_2, k_2) = \langle \Phi_1(\bar{\rho}_1, L, k_1) \Phi_1^*(\bar{\rho}_2, L, k_2) \rangle = \langle \psi_1(\bar{\rho}_1, L, k_1) \psi_1^*(\bar{\rho}_2, L, k_2) \rangle \quad (2.79)$$

Combining (2.71) and (2.78) gives

$$E_1(\bar{\rho}, \bar{\rho}, k) = \langle \Phi_2(\bar{\rho}, L, k) \rangle$$

$$= -k^2 \int_0^L dz \int_0^z dz' \iint \iint \langle dv(\bar{\kappa}, z) dv(\bar{\kappa}', z') \rangle\, exp\left(-\frac{i\gamma'\kappa'^2}{2k}(z - z')\right)$$

$$\times\, exp\left(-\frac{i\gamma(L-z)}{2k}(\bar{\kappa} + \gamma'\overline{\kappa}') \cdot (\bar{\kappa} + \gamma'\overline{\kappa}') + i\gamma(\bar{\kappa} + \gamma'\overline{\kappa}') \cdot \bar{\rho}\right) \qquad (2.80)$$

Combining (2.77) and (2.79) gives

$$E_2(\bar{\rho}_1, k_1 \bar{\rho}_2, k_2) = \langle \Phi_1(\bar{\rho}_1, L, k_1) \Phi_1^*(\bar{\rho}_2, L, k_2) \rangle$$

$$= k_1 k_2 \int_0^L dz \int_0^L dz' \iint \iint \langle dv(\bar{\kappa}, z) dv^*(\bar{\kappa}', z') \rangle\, exp(i\gamma(z)\bar{\kappa} \cdot \bar{\rho}_1 - i\gamma^*(z')\bar{\kappa}' \cdot \bar{\rho}_2)$$

$$\times\, exp\left(-\frac{i\gamma(z)\kappa^2}{2k_1}(L - z) + \frac{i\gamma^*(z')\kappa'^2}{2k_2}(L - z')\right) \qquad (2.81)$$

For statistically homogeneous and isotropic random fluctuations, we have

$$\langle dv(\bar{\kappa},z)dv^*(\bar{\kappa}',z')\rangle = F_n(\bar{\kappa},|z-z'|)\delta(\bar{\kappa}-\bar{\kappa}')d\bar{\kappa}d\bar{\kappa}' \tag{2.82}$$

where $F_n(\bar{\kappa},|z-z'|)$ is the two dimensional spatial power spectrum.

Combining (2.81) and (2.82) gives

$$E_2(\bar{\rho}_1,k_1\bar{\rho}_2,k_2) = \langle \Phi_1(\bar{\rho}_1,L,k_1)\Phi_1^*(\bar{\rho}_2,L,k_2)\rangle$$

$$= k_1k_2\int_0^L dz\int_0^L dz' \iint d\bar{\kappa}F_n(\bar{\kappa},|z-z'|)\, exp(i\gamma(z)\bar{\kappa}\cdot\bar{\rho}_1 - i\gamma^*(z')\bar{\kappa}\cdot\bar{\rho}_2)$$

$$\times\, exp\left(-\frac{i\gamma(z)\kappa^2}{2k_1}(L-z)+\frac{i\gamma^*(z')\kappa^2}{2k_2}(L-z')\right) \tag{2.83}$$

It is noticed that the expressions of the 2nd order statistical moments contain integrals of spectrum $F_n(\bar{\kappa},|z-z'|)$ which is a function of the difference coordinate $z-z'$ only. In order to simplify the integration with respect to z and z' further, it is convenient to introduce the average and the difference coordinates which are defined as,

$$\mu = z-z', \eta = \frac{1}{2}(z+z') \tag{2.84}$$

The region of integration can be written as [4, 10-11]

$$\int_0^L dz\int_0^L dz' = \int_0^L d\eta\int_{\xi_2(\eta)}^{\xi_1(\eta)} d\mu \tag{2.85}$$

Since the spectrum $F_n(\bar{\kappa},|\mu|)$ has an appreciable value only when $|\mu|$ is small comparing with the correlation distance of the refractive index fluctuation, then we may able to extend the limits of integration $\xi_1(\eta)$ and $\xi_2(\eta)$ to $-\infty$ and $+\infty$ without incurring significant error. Therefore, we have [4, 10-11]

$$\int_0^L dz\int_0^L dz' \approx \int_0^L d\eta\int_{-\infty}^{+\infty} d\mu \tag{2.86}$$

Combining (2.83) and (2.86) gives

$$E_2(\bar{\rho}_1, k_1\bar{\rho}_2, k_2) = k_1 k_2 \iint d\bar{\kappa} \int_0^L d\eta\, exp(i\gamma(\eta)\bar{\kappa}\cdot\bar{\rho}_1 - i\gamma^*(\eta)\bar{\kappa}\cdot\bar{\rho}_2)$$

$$\times\, exp\left(-\frac{i\gamma(\eta)\kappa^2}{2k_1}(L-\eta) + \frac{i\gamma^*(\eta)\kappa^2}{2k_2}(L-\eta)\right)\int_{-\infty}^{+\infty} d\mu\, F_n(\bar{\kappa}, |\mu|) \qquad (2.87)$$

Combining (2.36) and (2.87) gives

$$E_2(\bar{\rho}_1, k_1\bar{\rho}_2, k_2) = 2\pi k_1 k_2 \iint d\bar{\kappa}\, \Phi_n(\bar{\kappa}) \int_0^L d\eta\, exp(i\gamma(\eta)\bar{\kappa}\cdot\bar{\rho}_1 - i\gamma^*(\eta)\bar{\kappa}\cdot\bar{\rho}_2)$$

$$\times\, exp\left(-\frac{i\gamma(\eta)\kappa^2}{2k_1}(L-\eta) + \frac{i\gamma^*(\eta)\kappa^2}{2k_2}(L-\eta)\right) \qquad (2.88)$$

Combine equations (2.16) and (2.25) gives

$$\iint exp(i\bar{\kappa}\cdot\bar{s})dv(\bar{\kappa}, z) = \iint exp(-i\bar{\kappa}'\cdot\bar{s})dv^*(\bar{\kappa}', z)$$

$$\iint exp(i\bar{\kappa}\cdot\bar{s})dv(\bar{\kappa}, z) = \iint exp(i\bar{\kappa}'\cdot\bar{s})dv^*(-\bar{\kappa}', z)$$

$$dv(\bar{\kappa}, z) = dv^*(-\bar{\kappa}', z) \qquad (2.89)$$

Combining (2.82) and (2.89) gives

$$\langle dv(\bar{\kappa}, z)dv(\bar{\kappa}', z')\rangle = F_n(\bar{\kappa}, |z-z'|)\delta(\bar{\kappa}+\bar{\kappa}')d\bar{\kappa}d\bar{\kappa}' \qquad (2.90)$$

Substituting (2.90) into (2.80) gives

$$E_1(\bar{\rho}, \bar{\rho}, k) = -k^2 \int_0^L dz \int_0^z dz' \iint F_n(\bar{\kappa}, |z-z'|)d\bar{\kappa}\, exp\left(-\frac{i\gamma'\kappa^2}{2k}(z-z')\right)$$

$$\times\, exp\left(-\frac{i\gamma(L-z)}{2k}(\bar{\kappa}-\gamma'\bar{\kappa})\cdot(\bar{\kappa}-\gamma'\bar{\kappa}) + i\gamma(\bar{\kappa}-\gamma'\bar{\kappa})\cdot\bar{\rho}\right) \qquad (2.91)$$

For further simplification, we make the approximation that $\eta \approx z \approx z'$. Setting $\gamma' =$

$\frac{z'}{z} \approx 1$, we obtain the formula

$$E_1(\bar{\rho}, \bar{\rho}, k) = -k^2 \int_0^L dz \int_0^z dz' \iint F_n(\bar{\kappa}, |z - z'|) d\bar{\kappa} \tag{2.92}$$

Changing the variable of integration from z' to $\mu = z - z'$

$$E_1(\bar{\rho}, \bar{\rho}, k) = -k^2 \int_0^L dz \int_0^z d\mu \iint F_n(\bar{\kappa}, |\mu|) d\bar{\kappa}$$

$$\approx -\frac{k^2}{2} \int_0^L dz \iint d\bar{\kappa} \int_{-z}^z d\mu \, F_n(\bar{\kappa}, |\mu|) \tag{2.93}$$

It should be noticed that the 2nd order statistical moment $E_1(\bar{\rho}, \bar{\rho}, k)$ does not depend on spatial coordinate $\bar{\rho}$. Therefore, we may set $\bar{\rho} = \bar{0}$ in expression (2.93).

If the refractive index fluctuations are statistically homogeneous and isotropic in the plane transverse to the direction of propagation, the two-dimensional spatial power spectrum depends on $\kappa = |\bar{\kappa}|$. It is convenient to express the two-dimensional integral in the polar coordinate system.

Combining (2.36) and (2.93) gives

$$E_1(\bar{0}, \bar{0}, k) = -2\pi^2 k^2 \int_0^L dz \int_0^\infty \kappa d\kappa \Phi_n(\kappa) \tag{2.94}$$

Combining (2.36) and (2.88) gives

$$E_2(\bar{\rho}_1, k_1 \bar{\rho}_2, k_2) = 2\pi k_1 k_2 \int_0^L d\eta \int_0^\infty \kappa d\kappa \Phi_n(\kappa)$$

$$\times exp\left(-\frac{i\gamma(\eta)\kappa^2}{2k_1}(L - \eta) + \frac{i\gamma^*(\eta)\kappa^2}{2k_2}(L - \eta)\right)$$

$$\times \int_0^{2\pi} d\theta exp\big(i\bar{\kappa} \cdot (\gamma(\eta)\bar{\rho}_1 - \gamma^*(\eta)\bar{\rho}_2)\big) \tag{2.95}$$

Applying integral identity (2.32), we obtain

$$E_2(\bar{\rho}_1, k_1\bar{\rho}_2, k_2) = 4\pi^2 k_1 k_2 \int_0^L d\eta \int_0^\infty \kappa d\kappa\, \Phi_n(\kappa) J_0(\kappa|\gamma(\eta)\bar{\rho}_1 - \gamma^*(\eta)\bar{\rho}_2|)$$

$$\times\, exp\left(-\frac{i\gamma(\eta)\kappa^2}{2k_1}(L-\eta) + \frac{i\gamma^*(\eta)\kappa^2}{2k_2}(L-\eta)\right) \qquad (2.96)$$

where J_0 is the Bessel function of the first kind of order zero.

2.8 Rytov Approximation for Refractive Index Fluctuations

The 2nd order Rytov approximation is [10, 24, 31-32]

$$U(\bar{r}) = U_0(\bar{r}) exp\big(\psi(\bar{r})\big) = U_0(\bar{r}) exp\big(\psi_1(\bar{r}) + \psi_2(\bar{r})\big) \qquad (2.97)$$

where $U_0(\bar{r})$ is the field in absence of the refractive index fluctuations, $\psi_1(\bar{r})$ and $\psi_2(\bar{r})$ represent the 1st and 2nd order perturbations caused by the random inhomogeneities.

Taking the ensemble average of (2.97), we obtain the coherent field

$$\langle U(\bar{r})\rangle = U_0(\bar{r})\langle exp\big(\psi(\bar{r})\big)\rangle = U_0(\bar{r})\langle exp\big(\psi_1(\bar{r}) + \psi_2(\bar{r})\big)\rangle \qquad (2.98)$$

The ensemble average of the 2nd order complex phase perturbation is

$$\langle exp\big(\psi(\bar{r}_1) + \psi^*(\bar{r}_2)\big)\rangle = \langle exp\big(\psi_1(\bar{r}_1) + \psi_2(\bar{r}_1) + \psi_1^*(\bar{r}_2) + \psi_2^*(\bar{r}_2)\big)\rangle \qquad (2.99)$$

where $*$ denotes the complex conjugate.

As shown in formula (2.99), we need to calculate the ensemble average of the exponential function. This could be done by using the method of cumulants.

The average of the exponential function can be expanded as [10]

$$\langle exp(\psi)\rangle = exp\left(K_1 + \frac{1}{2!}K_2 + \frac{1}{3!}K_3 + \cdots\right) \qquad (2.100a)$$

$$K_1 = \langle \psi \rangle \tag{2.100b}$$

$$K_2 = \langle \psi^2 \rangle - \langle \psi \rangle^2 \tag{2.100c}$$

$$K_3 = \langle \psi^3 \rangle - 3\langle \psi \rangle \langle \psi^2 \rangle + 2\langle \psi \rangle^3 \tag{2.100d}$$

where $K_j,\ j = 1, 2,\ \cdots$ are cumulants.

Including 1st and 2nd order perturbations

$$\langle exp(\psi) \rangle \approx exp\left(K_1 + \frac{1}{2}K_2\right) = exp\left(\langle \psi \rangle + \frac{1}{2}(\langle \psi^2 \rangle - \langle \psi \rangle^2)\right) \tag{2.101}$$

For mathematical convenience, a dummy variable ϵ will be introduced to represent the order of each term in the perturbation series.

Combining (2.97) and (2.101) gives

$$\langle \psi \rangle + \frac{1}{2}(\langle (\psi)^2 \rangle - \langle \psi \rangle^2) = \langle \epsilon\psi_1 + \epsilon^2\psi_2 \rangle + \frac{1}{2}(\langle (\epsilon\psi_1 + \epsilon^2\psi_2)^2 \rangle - \langle \epsilon\psi_1 + \epsilon^2\psi_2 \rangle^2)$$

$$\approx \epsilon\langle \psi_1 \rangle + \epsilon^2\langle \psi_2 \rangle + \frac{1}{2}(\epsilon^2\langle \psi_1^2 \rangle - \epsilon^2\langle \psi_1 \rangle^2)$$

$$= \epsilon^2\langle \psi_2 \rangle + \frac{1}{2}\epsilon^2\langle \psi_1^2 \rangle \tag{2.102}$$

In expression (2.102), we keep 1st and 2nd order perturbations and neglect all higher order terms.

Combining (2.78) and (2.102) gives

$$\langle exp\big(\psi(\bar{\rho}, L, k)\big) \rangle = exp\big(E_1(\bar{\rho}, \bar{\rho}, k)\big) \tag{2.103}$$

Since $E_1(\bar{\rho}, \bar{\rho}, k)$ does not depend on spatial coordinate $\bar{\rho}$, we can write

$$\langle exp\big(\psi(\bar{\rho}, L, k)\big) \rangle = exp\big(E_1(\bar{0}, \bar{0}, k)\big) \tag{2.104}$$

Combining (2.99) and (2.101) gives

$$\langle\psi+\psi'^{*}\rangle+\frac{1}{2}(\langle(\psi+\psi'^{*})^{2}\rangle-\langle\psi+\psi'^{*}\rangle^{2})$$

$$\approx\epsilon\langle\psi_1\rangle+\epsilon^2\langle\psi_2\rangle+\epsilon\langle\psi_1'^{*}\rangle+\epsilon^2\langle\psi_2'^{*}\rangle+\frac{1}{2}\epsilon^2(\langle(\psi_1+\psi_1'^{*})^2\rangle-\langle\psi_1+\psi_1'^{*}\rangle^2)$$

$$=\epsilon^2\langle\psi_2\rangle+\epsilon^2\langle\psi_2'^{*}\rangle+\frac{1}{2}\epsilon^2(\langle\psi_1^2\rangle+2\langle\psi_1\psi_1'^{*}\rangle+\langle\psi_1'^{*2}\rangle)$$

$$=\epsilon^2\langle\psi_2\rangle+\frac{1}{2}\epsilon^2\langle\psi_1^2\rangle+\epsilon^2\langle\psi_2'^{*}\rangle+\frac{1}{2}\epsilon^2\langle\psi_1'^{*2}\rangle+\epsilon^2\langle\psi_1\psi_1'^{*}\rangle \qquad (2.105)$$

where $*$ denotes complex conjugate, and perturbations without prime symbol are functions of $\bar{\rho}$ whereas perturbations with prime symbol are functions of $\bar{\rho}'$.

Combining (2.99) and (2.105) gives

$$\langle exp\big(\psi(\bar{\rho}_1,L,k_1)+\psi^{*}(\bar{\rho}_2,L,k_2)\big)\rangle=exp(\langle\psi_1(\bar{\rho}_1,L,k_1)\psi_1^{*}(\bar{\rho}_2,L,k_2)\rangle)$$

$$\times\, exp\left(\langle\psi_2(\bar{\rho}_1,L,k_1)\rangle+\frac{1}{2}\langle\psi_1^2(\bar{\rho}_1,L,k_1)\rangle+\psi_2^{*}(\bar{\rho}_2,L,k_2)+\frac{1}{2}\langle\psi_1^{*2}(\bar{\rho}_2,L,k_2)\rangle\right)$$

$$=exp\big(E_1(\bar{\rho}_1,\bar{\rho}_1,k_1)+E_1(\bar{\rho}_2,\bar{\rho}_2,k_2)+E_2(\bar{\rho}_1,k_1,\bar{\rho}_2,k_2)\big) \qquad (2.106)$$

Since $E_1(\bar{\rho},\bar{\rho},k)$ does not depend on spatial coordinate $\bar{\rho}$, we can write

$$\langle exp\big(\psi(\bar{\rho}_1,L,k_1)+\psi^{*}(\bar{\rho}_2,L,k_2)\big)\rangle$$

$$=exp\big(E_1(\bar{0},\bar{0},k_1)+E_1(\bar{0},\bar{0},k_2)+E_2(\bar{\rho}_1,k_1,\bar{\rho}_2,k_2)\big) \qquad (2.107)$$

Combining (2.94), (2.98) and (2.104) gives

$$\langle U(\bar{\rho},L,k)\rangle=U_0(\bar{\rho},L,k)exp\big(E_1(\bar{0},\bar{0},k)\big)$$

$$=U_0(\bar{\rho},L,k)exp\left(-2\pi^2k^2\int_0^L dz\int_0^\infty \kappa d\kappa\Phi_n(\kappa)\right) \qquad (2.108)$$

The expression (2.108) shows that the coherent field attenuates exponentially as it propagates through the random medium. The amount of attenuation depends on the frequency, the thickness of the random medium and the three-dimensional spectral density of the refractive index fluctuations.

The two-frequency MCF is defined as the correlation between fields at two frequencies at two points on the plane transverse to the direction of propagation [1-4, 8, 10, 12-28, 32-34, 42]

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = \langle U(\bar{\rho}_1, L, k_1) U^*(\bar{\rho}_2, L, k_2) \rangle \tag{2.109}$$

Combining (2.104), (2.108) and (2.109) gives

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = U_0(\bar{\rho}_1, L, k_1) U_0^*(\bar{\rho}_2, L, k_2) \langle exp\big(\psi(\bar{\rho}_1, L, k_1) + \psi^*(\bar{\rho}_2, L, k_2)\big) \rangle$$

$$= U_0(\bar{\rho}_1, L, k_1) U_0^*(\bar{\rho}_2, L, k_2)$$

$$\times\, exp\big(E_1(\bar{0}, \bar{0}, k_1) + E_1(\bar{0}, \bar{0}, k_2) + E_2(\bar{\rho}_1, k_1, \bar{\rho}_2, k_2)\big) \tag{2.110}$$

In absence of the random fluctuations, the two-frequency MCF is reduced to the free-space two-frequency MCF which is given by [1-4, 8, 10, 15-18, 24-25, 42]

$$\Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = U_0(\bar{\rho}_1, L, k_1) U_0^*(\bar{\rho}_2, L, k_2) \tag{2.111}$$

Combining (2.110) and (2.111) gives

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = \Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2)$$

$$\times\, exp\big(E_1(\bar{0}, \bar{0}, k_1) + E_1(\bar{0}, \bar{0}, k_2) + E_2(\bar{\rho}_1, k_1, \bar{\rho}_2, k_2)\big) \tag{2.112}$$

Combining (2.45) and (2.110) gives

$$\Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = U_0(\bar{\rho}_1, L, k_1) U_0^*(\bar{\rho}_2, L, k_2)$$

$$= \left(\frac{1}{4\pi L} exp\left(ik_1L + \frac{ik_1\rho_1^2}{2L}\right)\right)\left(\frac{1}{4\pi L} exp\left(ik_2L + \frac{ik_2\rho_2^2}{2L}\right)\right)^*$$

$$= \frac{1}{(4\pi L)^2} exp\left(i(k_1 - k_2)L + \frac{i}{2L}(k_1\rho_1^2 - k_2\rho_2^2)\right) \quad (2.113)$$

It is convenient to define the function [1-3, 10, 24-25]

$$H(\bar{\rho}_1, k_1, \bar{\rho}_2, k_2) = -E_1(\bar{0}, \bar{0}, k_1) - E_1(\bar{0}, \bar{0}, k_2) - E_2(\bar{\rho}_1, k_1, \bar{\rho}_2, k_2)$$

$$= 2\pi^2(k_1^2 + k_2^2)\int_0^L dz \int_0^\infty \kappa d\kappa \Phi_n(\kappa) - 4\pi^2 k_1 k_2 \int_0^L dz \int_0^\infty \kappa d\kappa\, \Phi_n(\kappa)$$

$$\times J_0(\kappa|\gamma(z)\bar{\rho}_1 - \gamma^*(z)\bar{\rho}_2|) exp\left(-\frac{i\gamma(z)\kappa^2}{2k_1}(L-z) + \frac{i\gamma^*(z)\kappa^2}{2k_2}(L-z)\right)$$

$$= 4\pi^2 \int_0^L dz \int_0^\infty \kappa d\kappa \Phi_n(\kappa)\left(\frac{k_1^2 + k_2^2}{2}\right) - 4\pi^2 \int_0^L dz \int_0^\infty \kappa d\kappa\, \Phi_n(\kappa) k_1 k_2$$

$$\times exp\left(-\frac{iz\kappa^2(L-z)}{2L}\left(\frac{k_1 - k_2}{k_1 k_2}\right)\right) J_0\left(\kappa\left|\frac{z}{L}(\bar{\rho}_1 - \bar{\rho}_2)\right|\right) \quad (2.114)$$

It is noticed that the effects of the random fluctuations are contained in the H function.

Combining (2.112) and (2.114) gives

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = \Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) exp\big(-H(\bar{\rho}_1, k_1, \bar{\rho}_2, k_2)\big) \quad (2.115)$$

For most of the practical imaging systems working at carrier frequency f_0 with bandwidth Ω, the narrowband condition $\Omega \ll f_0$ is generally satisfied [1-4, 15-18, 24-25, 42].

It is convenient to define the average and the difference coordinates

$$k_c = \frac{1}{2}(k_1 + k_2),\ k_d = k_1 - k_2 \quad (2.116a)$$

$$\bar{\rho}_c = \frac{1}{2}(\bar{\rho}_1 + \bar{\rho}_2), \;\; \bar{\rho}_d = \bar{\rho}_1 - \bar{\rho}_2 \qquad (2.116b)$$

Combining (2.113) and (2.116) gives

$$\Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = U_0(\bar{\rho}_1, L, k_1)U_0^*(\bar{\rho}_2, L, k_2)$$

$$= \frac{1}{(4\pi L)^2} exp\left(\frac{i}{2L}(k_1\rho_1^2 - k_2\rho_2^2) + i(k_1z_1 - k_2z_2)\right) \qquad (2.117)$$

We have

$$k_1\rho_1^2 - k_2\rho_2^2 = k_c\left(\left(\bar{\rho}_c + \frac{\bar{\rho}_d}{2}\right)\cdot\left(\bar{\rho}_c + \frac{\bar{\rho}_d}{2}\right) - \left(\bar{\rho}_c - \frac{\bar{\rho}_d}{2}\right)\cdot\left(\bar{\rho}_c - \frac{\bar{\rho}_d}{2}\right)\right)$$

$$+ \frac{k_d}{2}\left(\left(\bar{\rho}_c + \frac{\bar{\rho}_d}{2}\right)\cdot\left(\bar{\rho}_c + \frac{\bar{\rho}_d}{2}\right) - \left(\bar{\rho}_c - \frac{\bar{\rho}_d}{2}\right)\cdot\left(\bar{\rho}_c - \frac{\bar{\rho}_d}{2}\right)\right)$$

$$= 2k_c\bar{\rho}_c\cdot\bar{\rho}_d + k_d\left(\bar{\rho}_c\cdot\bar{\rho}_c + \frac{1}{4}\bar{\rho}_d\cdot\bar{\rho}_d\right) \qquad (2.118)$$

We have

$$k_1z_1 - k_2z_2 = \left(k_c + \frac{k_d}{2}\right)z_1 - \left(k_c - \frac{k_d}{2}\right)z_2$$

$$= k_c(z_1 - z_2) + \frac{1}{2}k_d(z_1 + z_2) \qquad (2.119)$$

Combining (2.117), (2.118) and (2.119) gives

$$\Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = \frac{1}{(4\pi L)^2} exp\left(\frac{i}{2L}(k_1\rho_1^2 - k_2\rho_2^2) + i(k_1z_1 - k_2z_2)\right)$$

$$= \frac{1}{(4\pi L)^2} exp\left(ik_c\frac{\bar{\rho}_c\cdot\bar{\rho}_d}{L} + i\frac{k_d}{2L}\left(\bar{\rho}_c\cdot\bar{\rho}_c + \frac{1}{4}\bar{\rho}_d\cdot\bar{\rho}_d\right) + ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2)\right)$$

$$\approx \frac{1}{(4\pi L)^2} exp\left(ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2)\right) \qquad (2.120)$$

Under narrowband assumption, we have

$$k_1^2 + k_2^2 = \left(k_c + \frac{k_d}{2}\right)^2 + \left(k_c - \frac{k_d}{2}\right)^2 \approx 2k_c^2 \approx 2k^2 \qquad (2.121)$$

$$k_1 k_2 = k_c^2 - \frac{k_d^2}{4} \approx k_c^2 \approx k^2 \qquad (2.122)$$

where k is the wave number at the carrier frequency.

By using approximations (2.121) and (2.122), it can be shown that the H function given by (2.114) will depend only on the difference wavenumber k_d. This will greatly reduce the mathematical complexity in the subsequent derivations.

Combining (2.114), (2.121) and (2.122) gives

$$H(\bar{\rho}_d, k_d) \approx 4\pi^2 k^2 \int_0^L dz \int_0^\infty \kappa d\kappa \Phi_n(\kappa) - 4\pi^2 k^2 \int_0^L dz \int_0^\infty \kappa d\kappa$$

$$\times \Phi_n(\kappa) exp\left(-\frac{iz\kappa^2(L-z)}{2L}\left(\frac{k_d}{k^2}\right)\right) J_0\left(\kappa \left|\frac{z}{L}\bar{\rho}_d\right|\right) \qquad (2.123)$$

where $\bar{\rho}_d = \bar{\rho}_1 - \bar{\rho}_2$.

Combining (2.115), (2.120) and (2.123), we obtain the two-frequency MCF

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, k_c, k_d) = \Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, k_c, k_d) exp\big(-H(\bar{\rho}_d, k_d)\big)$$

$$= \frac{1}{(4\pi L)^2} exp\left(ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2) - H(\bar{\rho}_d, k_d)\right) \qquad (2.124)$$

The two-frequency MCF can be written as the summation of the coherent and the incoherent components [1-4, 15-18, 42].

The coherent two-frequency MCF is defined as

$$\Gamma_{2c}(\bar{\rho}_1, \bar{\rho}_2, L, k_c, k_d) = \frac{1}{(4\pi L)^2} exp\left(ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2) - \tau_0\right) \quad (2.125)$$

where τ_0 is the optical depth.

And the incoherent two-frequency MCF is defined as

$$\Gamma_{2i}(\bar{\rho}_1, \bar{\rho}_2, L, k_c, k_d) = \frac{1}{(4\pi L)^2} exp\left(ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2)\right)$$

$$\times \left(exp\big(-H(\bar{\rho}_d, k_d)\big) - exp(-\tau_0)\right) \quad (2.126)$$

where τ_0 is the optical depth.

According to formulas (2.125) and (2.126), the random medium has different effects on different components of the two-frequency MCF. More specifically, for the coherent component, the random medium causes attenuation. And for the incoherent component, the random medium causes both attenuation and dispersion.

2.9 Scattering Cross Section and Three Dimensional Power Spectrum

In previous sections, we discuss the two-frequency MCF of the continuous random medium with refractive index fluctuations. In order to derive the two-frequency MCF for the discrete scatters, it is necessary to relate three-dimensional spatial power spectrum of the refractive index fluctuations to the scattering cross-section of the discrete scatters [1-4, 11]. Here we assume that all the scatters have the same scattering characteristics.

In the far field of the scatter, the scattered field behaves as a spherical wave and is given by [3-4, 11]

$$\bar{E}_s(\bar{r}) = \bar{f}\big(\hat{0}, \hat{\imath}\big)\frac{exp(ikR)}{R}, R > \frac{D^2}{\lambda} \quad (2.127)$$

where $\bar{f}\big(\hat{0}, \hat{\imath}\big)$ is the scattering amplitude, $\hat{\imath}$ is the direction of the incident wave, $\hat{0}$ is the direction from the reference point to the observation point, R is the distance between the reference point and the observation point in direction $\hat{0}$, D is the characteristic dimension of the scatter, λ is the wavelength and k is the wave number. The scattering amplitude $\bar{f}(\hat{0},\hat{\imath})$ contains the amplitude, phase and polarization information of the scattered wave in the far field in the direction $\hat{0}$ when the scatter is illuminated by a plane wave propagating in direction $\hat{\imath}$ with unit amplitude.

The differential scattering cross section is defined as [3-4]

$$\sigma_d(\hat{0},\hat{\imath}) = \lim_{R\to\infty}\left(\frac{R^2|\bar{S}_s|}{|\bar{S}_i|}\right) = \left|\bar{f}(\hat{0},\hat{\imath})\right|^2 = \frac{\sigma_t}{4\pi}p(\hat{0},\hat{\imath}) \qquad (2.128)$$

where $\bar{S}_i$ and $\bar{S}_s$ are incident and scattering power flux density vectors

$$\bar{S}_i = \frac{1}{2}\bar{E}_i \times \bar{H}_i^* = \frac{|\bar{E}_i|^2}{2\eta_0}\hat{\imath},\ \bar{S}_s = \frac{1}{2}\bar{E}_s \times \bar{H}_s^* = \frac{|\bar{E}_s|^2}{2\eta_0}\hat{0} \qquad (2.129)$$

where $\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ is the characteristic impedance of the background medium.

The bistatic scattering cross section is [3-4]

$$\sigma_{bi} = 4\pi\sigma_d(\hat{0},\hat{\imath}) \qquad (2.130)$$

The scattering cross section can be found by integrating the observed scattered power at all angles surrounding the scatter

$$\sigma_s = \int_0^{4\pi}\sigma_d(\hat{0},\hat{\imath})d\omega \qquad (2.131)$$

where $d\omega$ is the differential solid angle.

Starting from (2.1) and (2.2), we have

$$\nabla \times \bar{E}(\bar{r}) = \mathrm{i}\omega\mu_0\bar{H}(\bar{r}) \qquad (2.132)$$

$$\nabla \times \bar{H}(\bar{r}) = -\mathrm{i}\omega\varepsilon_0 \bar{E}(\bar{r}) + \bar{J}_{eq}(\bar{r}) \tag{2.133}$$

where $\bar{J}_{eq}(\bar{r})$ the equivalent current source, is given by

$$\bar{J}_{eq}(\bar{r}) = \begin{Bmatrix} -\mathrm{i}\omega\varepsilon_0(\varepsilon_r(\bar{r}) - 1)\bar{E}(\bar{r}) & \text{inside scatter} \\ 0 & \text{outside scatter} \end{Bmatrix} \tag{2.134}$$

The solution to (2.132) and (2.133) can be written as

$$\bar{E}(\bar{r}) = \bar{E}_i(\bar{r}) + \bar{E}_s(\bar{r}) \tag{2.135}$$

$$\bar{H}(\bar{r}) = \bar{H}_i(\bar{r}) + \bar{H}_s(\bar{r}) \tag{2.136}$$

where $\bar{E}_i(\bar{r})$ and $\bar{H}_i(\bar{r})$ denote the incident field in absence of the scatter, and $\bar{E}_s(\bar{r})$ and $\bar{H}_s(\bar{r})$ denote the scattered field radiated from the scatter.

The scattered fields can be written as [3-4]

$$\bar{E}_s(\bar{r}) = \nabla \times \nabla \times \bar{\Pi}_s(\bar{r}) \tag{2.137}$$

$$\bar{H}_s(\bar{r}) = -\mathrm{i}\omega\varepsilon_0 \nabla \times \bar{\Pi}_s(\bar{r}) \tag{2.138}$$

where $\bar{\Pi}_s(\bar{r})$ is the Hertz vector given by

$$\bar{\Pi}_s(\bar{r}) = -\frac{1}{\mathrm{i}\omega\varepsilon_0} \iiint G(\bar{r},\bar{r}')\bar{J}_{eq}(\bar{r}')d\bar{r}'$$

$$= \iiint (\varepsilon_r(\bar{r}') - 1)\bar{E}(\bar{r}')G(\bar{r},\bar{r}')d\bar{r}' \tag{2.139}$$

where $G(\bar{r},\bar{r}')$, the Green's function in background medium, is given by

$$G(\bar{r},\bar{r}') = \frac{exp(ik|\bar{r}-\bar{r}'|)}{4\pi|\bar{r}-\bar{r}'|} \tag{2.140a}$$

$$|\bar{r}-\bar{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \tag{2.140b}$$

In the far field, we can apply the binomial expansion

$$|\bar{r}-\bar{r}'| \approx \sqrt{R^2-2\bar{r}\cdot\bar{r}'} \approx R-\hat{0}\cdot\bar{r}' \tag{2.141}$$

Combining (2.139), (2.140) and (2.141) gives

$$\bar{\Pi}_s(\bar{r}) \approx \frac{exp(ikR)}{4\pi R}\iiint(\varepsilon_r(\bar{r}')-1)\bar{E}(\bar{r}')exp\left(-ik\hat{0}\cdot\bar{r}'\right)d\bar{r}' \tag{2.142}$$

The permittivity is a random function which can be written as [3-4]

$$\varepsilon(\bar{r}) = \langle\varepsilon(\bar{r})\rangle\varepsilon_r(\bar{r}) = \langle\varepsilon(\bar{r})\rangle\left(1+\varepsilon_1(\bar{r})\right) \tag{2.143}$$

where $\langle\varepsilon(\bar{r})\rangle = \varepsilon_0$ is free-space permittivity and $\varepsilon_1(\bar{r})$ is the permittivity fluctuation with $\langle\varepsilon_1(\bar{r})\rangle = 0$.

Combining (2.142) and (2.143) gives

$$\bar{\Pi}_s(\bar{r}) \approx \frac{exp(ikR)}{4\pi R}\iiint \varepsilon_1(\bar{r}')\bar{E}(\bar{r}')exp\left(-ik\hat{0}\cdot\bar{r}'\right)d\bar{r}' \tag{2.144}$$

In the far field, the gradient of the Green's function is [3-4]

$$\nabla\left(\frac{exp(ikR)}{R}\right) = \hat{x}\frac{\partial}{\partial x}\left(\frac{exp(ikR)}{R}\right)+\hat{y}\frac{\partial}{\partial y}\left(\frac{exp(ikR)}{R}\right)+\hat{z}\frac{\partial}{\partial z}\left(\frac{exp(ikR)}{R}\right)$$

$$= \hat{x}\frac{exp(ikR)}{R}\left(\frac{ikR-1}{R}\right)\frac{x}{R}+\hat{y}\frac{exp(ikR)}{R}\left(\frac{ikR-1}{R}\right)\frac{y}{R}+\hat{z}\frac{exp(ikR)}{R}\left(\frac{ikR-1}{R}\right)\frac{z}{R}$$

$$\approx ik\frac{exp(ikR)}{R}\left(\hat{x}\frac{x}{R}+\hat{y}\frac{y}{R}+\hat{z}\frac{z}{R}\right) = ik\frac{exp(ikR)}{R}\hat{0} \tag{2.145}$$

Applying the vector calculus identity

$$\nabla\times(\psi\bar{A}) = \psi\nabla\times\bar{A}+\nabla\psi\times\bar{A} \tag{2.146}$$

where ψ is a scalar function and $\bar{A}$ is a constant vector.

Combining (2.145) and (2.146) gives

$$\nabla \times \left(\frac{exp(ikR)}{R}\bar{A}\right) = \left(\frac{exp(ikR)}{R}\right)\nabla \times \bar{A} + \nabla\left(\frac{exp(ikR)}{R}\right) \times \bar{A}$$

$$= \nabla\left(\frac{exp(ikR)}{R}\right) \times \bar{A} \approx ik\frac{exp(ikR)}{R}\hat{0} \times \bar{A} \qquad (2.147)$$

Combining (2.137), (2.144) and (2.147) gives

$$\bar{E}_s(\bar{r}) = \nabla \times \nabla \times \left(\frac{exp(ikR)}{4\pi R}\iiint \varepsilon_1(\bar{r}')\bar{E}(\bar{r}')exp\left(-ik\hat{0}\cdot\bar{r}'\right)d\bar{r}'\right)$$

$$= \iiint \nabla \times \nabla \times \left[\frac{exp(ikR)}{4\pi R}\bar{E}(\bar{r}')\right]\varepsilon_1(\bar{r}')exp\left(-ik\hat{0}\cdot\bar{r}'\right)d\bar{r}'$$

$$= \iiint \nabla \times \left[ik\frac{exp(ikR)}{4\pi R}\hat{0} \times \bar{E}(\bar{r}')\right]\varepsilon_1(\bar{r}')exp\left(-ik\hat{0}\cdot\bar{r}'\right)d\bar{r}'$$

$$= \frac{k^2 exp(ikR)}{4\pi R}\iiint\left[-\hat{0} \times \hat{0} \times \bar{E}(\bar{r}')\right]\varepsilon_1(\bar{r}')exp\left(-ik\hat{0}\cdot\bar{r}'\right)d\bar{r}' \qquad (2.148)$$

Combining (2.127) and (2.148) gives

$$\bar{f}\left(\hat{0},\hat{\imath}\right) = \frac{k^2}{4\pi}\iiint\left[-\hat{0} \times \hat{0} \times \bar{E}(\bar{r}')\right]\varepsilon_1(\bar{r}')exp\left(-ik\hat{0}\cdot\bar{r}'\right)d\bar{r}' \qquad (2.149)$$

where $\bar{E}(\bar{r}')$ is the total electric field at $\bar{r}'$ when the incident field has unit amplitude. The scattering cross section can also be defined using the scattering amplitude [3-4]

$$\sigma_s = \int_0^{4\pi}\left|f\left(\hat{0},\hat{\imath}\right)\right|^2 d\omega = \frac{\sigma_t}{4\pi}\int_0^{4\pi} p\left(\hat{0},\hat{\imath}\right)d\omega \qquad (2.150)$$

where σ_t is called the extinction cross section and $p\left(\hat{0},\hat{\imath}\right)$ is the phase function.

The albedo of a single particle is ratio of the scattering cross section to the extinction cross section [3-4]

$$W_0 = \frac{\sigma_s}{\sigma_t} = \frac{1}{\sigma_t}\int_0^{4\pi} \left|f\left(\hat{0},\hat{\imath}\right)\right|^2 d\omega = \frac{1}{4\pi}\int_0^{4\pi} p\left(\hat{0},\hat{\imath}\right) d\omega \qquad (2.151)$$

If the refractive index fluctuation is small, the Born approximation may be applied and the total electric field inside the scatter is approximately equal to the incident electric field. Therefore, we have

$$\bar{E}(\bar{r}') = \hat{e}_i exp(ik\hat{\imath}\cdot\bar{r}') \qquad (2.152)$$

where $\hat{e}_i$ is a unit vector in the direction of polarization.

The Born approximation is valid when [3-4]

$$2\pi\varepsilon_1\frac{D}{\lambda} \ll 1 \qquad (2.153)$$

where D is the characteristic dimension of the scatter and λ is the wavelength at carrier frequency.

The formula of vector triple product is

$$\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C} \qquad (2.153)$$

Combining (2.149), (2.152) and (2.153) gives

$$\bar{f}\left(\hat{0},\hat{\imath}\right) = \frac{k^2}{4\pi}\iiint\left[-\hat{0}\ \times\hat{0}\ \times\hat{e}_i\right] exp\left(ik\left(\hat{\imath}-\hat{0}\right)\cdot\bar{r}'\right)d\bar{r}'$$

$$= \frac{k^2}{4\pi}\iiint\left[\hat{e}_i - \hat{0}\left(\hat{0}\cdot\hat{e}_i\right)\right] exp\left(ik\left(\hat{\imath}-\hat{0}\right)\cdot\bar{r}'\right)d\bar{r}' \qquad (2.154)$$

It is convenient to define the directional cosine

$$cos(\chi) = \hat{0}\cdot\hat{e}_i \qquad (2.155)$$

The vector dot product is

$$\left(\hat{e}_i - \hat{0}\left(\hat{0}\cdot\hat{e}_i\right)\right)\cdot\left(\hat{e}_i - \hat{0}\left(\hat{0}\cdot\hat{e}_i\right)\right) = \left(\hat{e}_i - \hat{0}cos(\chi)\right)\cdot\left(\hat{e}_i - \hat{0}cos(\chi)\right)$$

$$= 1 - cos^2(\chi) = sin^2(\chi) \qquad (2.156)$$

It is convenient to define

$$\hat{e}_s sin(\chi) = \hat{e}_i - \hat{0}\left(\hat{0}\cdot\hat{e}_i\right) \qquad (2.157)$$

$$\bar{k}_s = k\left(\hat{\imath} - \hat{0}\right) \qquad (2.158)$$

It can be shown that

$$k_s^2 = k^2\left(\hat{\imath} - \hat{0}\right)\cdot\left(\hat{\imath} - \hat{0}\right) = 2k^2\left(1 - cos(\theta)\right) = 4k^2 sin^2\left(\frac{\theta}{2}\right) \qquad (2.159)$$

where θ is the angle between $\hat{\imath}$ and $\hat{0}$.

Combining (2.154), (2.158) and (2.157) gives

$$\bar{f}\left(\hat{0},\hat{\imath}\right) = \hat{e}_s sin(\chi)\frac{k^2}{4\pi}\iiint exp\left(i\bar{k}_s\cdot\bar{r}'\right)d\bar{r}' \qquad (2.160)$$

The differential cross section per unit volume of the random medium is given by [4, 11]

$$\sigma_d'\left(\hat{0},\hat{\imath}\right) = \frac{1}{\delta V}\langle\bar{f}\left(\hat{0},\hat{\imath}\right)\cdot\bar{f}^*\left(\hat{0},\hat{\imath}\right)\rangle \qquad (2.161)$$

where δV is the elementary volume of the random medium. It is required that δV much be small enough so that the incident wave can be considered as a plane wave, meanwhile, δV should be much greater than the correlation distance of the random medium.

Combining (2.10) and (2.143) gives

$$\varepsilon_r(\bar{r}) = n^2(\bar{r})$$

$$1+\varepsilon_1(\bar{r})\approx 1+2n_1(\bar{r})$$

$$\varepsilon_1(\bar{r})=2n_1(\bar{r}) \tag{2.162}$$

Combining (2.161) and (2.162) gives

$$\sigma_d'(\hat{0},\hat{\imath})=\frac{k^4 sin^2(\chi)}{(4\pi)^2\delta V}\iiint\iiint\langle\varepsilon_1(\bar{r}_1)\varepsilon_1^*(\bar{r}_2)\rangle exp\left(i\bar{k}_s\cdot(\bar{r}_1-\bar{r}_2)\right)d\bar{r}_1 d\bar{r}_2$$

$$=\frac{4k^4 sin^2(\chi)}{(4\pi)^2\delta V}\iiint\iiint\langle n_1(\bar{r}_1)n_1^*(\bar{r}_2)\rangle exp\left(i\bar{k}_s\cdot(\bar{r}_1-\bar{r}_2)\right)d\bar{r}_1 d\bar{r}_2$$

$$=\frac{4k^4 sin^2(\chi)}{(4\pi)^2\delta V}\iiint\iiint B_n(\bar{r}_1,\bar{r}_2)exp\left(i\bar{k}_s\cdot(\bar{r}_1-\bar{r}_2)\right)d\bar{r}_1 d\bar{r}_2 \tag{2.164}$$

where the region of integration is the volume of the random medium δV. If the random medium is statistically homogenous and isotropic, then the covariance function $B_n(\bar{r}_1-\bar{r}_2)$ is a function of difference $|\bar{r}_1-\bar{r}_2|$. It is helpful to make the change of variables

$$\bar{r}_c=\frac{1}{2}(\bar{r}_1+\bar{r}_2),\bar{r}_d=\bar{r}_1-\bar{r}_2 \tag{2.165}$$

Combining (2.164) and (2.165) gives

$$\iiint\iiint\langle n_1(\bar{r}_1)n_1^*(\bar{r}_2)\rangle exp\left(i\bar{k}_s\cdot(\bar{r}_1-\bar{r}_2)\right)d\bar{r}_1 d\bar{r}_2$$

$$=\iiint\iiint B_n(\bar{r}_d)exp\left(i\bar{k}_s\cdot\bar{r}_d\right)d\bar{r}_d d\bar{r}_c \tag{2.166}$$

Since the covariance function has negligible value when $\bar{r}_d$ is greater than the correlation distance of the random medium, the region of integration of $\bar{r}_d$ can be extended to infinite without incurring error.

Combining (2.29), (2.164) and (2.166) gives

$$\sigma_d'(\hat{0},\hat{\imath}) = \frac{4k^4 sin^2(\chi)}{(4\pi)^2\delta V} \iiint \iiint B_n(\bar{r}_d) exp(i\bar{k}_s \cdot \bar{r}_d) d\bar{r}_d d\bar{r}_c$$

$$= \frac{4k^4 sin^2(\chi)}{(4\pi)^2\delta V} \iiint d\bar{r}_c \iiint B_n(\bar{r}_d) exp(i\bar{k}_s \cdot \bar{r}_d) d\bar{r}_d$$

$$= \frac{2\pi k^4 sin^2(\chi)}{\delta V} \Phi_n(\bar{k}_s) \iiint d\bar{r}_c = 2\pi k^4 sin^2(\chi) \Phi_n(\bar{k}_s) \qquad (2.167)$$

It is noticed that the angular dependence $sin^2(\chi)$ is due to the radiation pattern of a dipole. This is to be expected because the scattered electric field is generated by the equivalent current source $\bar{J}_{eq}(\bar{r})$ can be considered as an electric dipole which the electric polarization $\varepsilon_0(\varepsilon_r(\bar{r}) - 1)\bar{E}(\bar{r})$.

For a scalar wave, the angular dependence $sin^2(\chi)$ is unit [4]. Therefore, we obtain

$$\sigma_d'(\hat{0},\hat{\imath}) = 2\pi k^4 \Phi_n(\bar{k}_s) \qquad (2.168)$$

For statistically homogeneous and isotropic random fluctuations, the three-dimensional spatial power spectrum $\Phi_n(k_s)$ is a function of $k_s = |\bar{k}_s|$

$$\sigma_d'(\hat{0},\hat{\imath}) = 2\pi k^4 \Phi_n(k_s) \qquad (2.169)$$

where $k_s = 2k \sin\left(\frac{\theta}{2}\right)$.

The attenuation constant α_r is defined as by integrating the differential scattering cross section over 4π solid angle.

$$\alpha_r = \int_0^{4\pi} \sigma_d'(\hat{0},\hat{\imath}) d\omega \qquad (2.170)$$

The differential solid angle can be written as

$$d\omega = \sin(\theta)\, d\theta d\phi = \frac{2k \sin\left(\frac{\theta}{2}\right) k \cos\left(\frac{\theta}{2}\right) d\theta d\phi}{k^2} = \frac{k_s dk_s d\phi}{k^2} \qquad (2.171)$$

Combining (2.169), (2.170) and (2.171) gives

$$\alpha_r = 2\pi k^2 \int_0^{2k} \int_0^{2\pi} \Phi_n(k_s)\, k_s dk_s d\phi = 4\pi^2 k^2 \int_0^{2k} \Phi_n(k_s)\, k_s dk_s \quad (2.172)$$

By equating the differential cross section per unit volume of the reflection index fluctuations and that of a tenuous distribution of random discrete scatters, we obtain

$$\sigma_d'(\hat{0}, \hat{\imath}) = \rho_n \sigma_d(\hat{0}, \hat{\imath})$$

$$2\pi k^4 \Phi_n(k_s) = \frac{\rho_n \sigma_t}{4\pi} p(\hat{0}, \hat{\imath}) \quad (2.173)$$

where ρ_n is the number of scatters per unit volume.

It is noticed that the independent scattering assumption is made and the fractional volume of the discrete scatters in the random medium should be less than 0.1%.

It is convenient to define variable

$$s = 2sin\left(\frac{\theta}{2}\right) \quad (2.174)$$

where θ is the angle between $\hat{\imath}$ and $\hat{0}$.

The normalized phase function is

$$\beta(s) = \frac{\sigma_t}{\sigma_s} p(s) \quad (2.175)$$

which satisfies the normalization condition

$$\frac{1}{2}\int_0^2 \beta(s) s ds = 1 \quad (2.176)$$

Combining (2.150) and (2.152) gives

$$2\pi k^4\Phi_n(\kappa) = \frac{\rho_n\sigma_t}{4\pi}p(s) = \frac{\rho_n\sigma_s}{4\pi}\beta(s) = \frac{b}{4\pi}\beta(s) \qquad (2.177)$$

where ρ_n is the number density and b is the scattering coefficient.

2.10 Rytov Approximation for Discrete Scatters

By replacing the three-dimensional spatial power spectrum with the normalized phase function, we can get the two-frequency MCF for the random discrete scatters and the coherent field [1-2, 8]. First, we will calculate the H function for the discrete random medium.

To simplify the notations, it is convenient to define function

$$g = exp\left(-\frac{iz\kappa^2(L-z)}{2L}\frac{k_d}{k^2}\right) \qquad (2.178)$$

We relate the integration variable κ in the spectral representation to the wavenumber k_s in the scattering amplitude [1-2, 8].

$$\kappa = 2k\sin\left(\frac{\theta}{2}\right) = ks \qquad (2.179)$$

where θ is the scattering angle.

Combining (2.178) and (2.179) gives

$$g = exp\left(-\frac{iz(L-z)}{2L}k_d s^2\right) \qquad (2.180)$$

Combining (2.123), (2.179) and (2.180) gives

$$H(\bar{\rho}_d, k_d) = 4\pi^2\int_0^L dz\int_0^\infty \kappa d\kappa\Phi_n(\kappa)\left(k^2 - k^2 g J_0\left(\kappa\left|\frac{z\bar{\rho}_d}{L}\right|\right)\right)$$

$$= 2\pi\int_0^L dz\int_0^2 sds\left(2\pi k^4\Phi_n(ks)\right)\left(1 - gJ_0\left(ks\left|\frac{z\bar{\rho}_d}{L}\right|\right)\right)$$

$$= \int_0^L dz \frac{b}{2} \int_0^2 sds\beta(s) \left(1 - gJ_0 \left(ks \left|\frac{z\bar{\rho}_d}{L}\right|\right)\right) \tag{2.181}$$

For mathematical convenience, we assume the scattering characteristics of the discrete scatters can be described by a Gaussian function [1-4, 8, 15-18, 42]

$$\beta(s) = 4\alpha_p exp\left(-\alpha_p s^2\right) \tag{2.182}$$

where $\alpha_p = \frac{4ln(2)}{\theta_p^2}$ and θ_p is the half-power beamwidth of the scattering pattern.

The Gaussian function satisfies the normalization condition [1-2, 8]

$$\frac{1}{2}\int_0^2 \beta(s)sds \approx \frac{1}{2}\int_0^\infty \beta(s)sds = \alpha_p \int_0^\infty exp\left(-\alpha_p s^2\right)ds^2 = 1 \tag{2.183}$$

Even a closed form expression could be obtained by using the Gaussian phase function. For most practical cases, the more realistic Henyey-Greenstein phase function should be used to give better representations of the scatter characteristics of the discrete scatters. The Henyey-Greenstein phase function is given by [1-4, 8]

$$\beta(s) = \frac{1+g_a}{(1-g_a)^2} \frac{1}{[1+(s/s_0)^2]^{3/2}} \tag{2.184}$$

where $s_0^2 = (1 - g_a)^2/g_a$ and g_a is the anisotropy factor.

In the derivation, we assume the characteristic dimension of the scatters D is much larger than the wavelength λ at the carrier frequency of the imaging system. Therefore, the scattering power is concentrated in a small cone region in the forward direction. If the Gaussian phase function gives approximately the same amount of scattering in the forward direction as that predicted by the Henyey-Greenstein phase function, more accurate results could be obtained. For this reason, we determine the parameter α_p by equating the half power beamwidth of the Gaussian phase function to the half-power beamwidth of the Henyey-Greenstein phase function for a given anisotropy factor g_a [1-2].

$$\alpha_p = \frac{ln(2)g_a}{(1-g_a)^2(2^{2/3}-1)} \qquad (2.185)$$

Combining (2.181) and (2.182) gives

$$\frac{1}{2}\int_0^2 sds\beta(s)\left(1 - gJ_0\left(ks\left|\frac{z\bar{\rho}_d}{L}\right|\right)\right)$$

$$\approx 2\alpha_p \int_0^\infty \left(1 - gJ_0\left(ks\left|\frac{z\bar{\rho}_d}{L}\right|\right)\right) exp\left(-\alpha_p s^2\right)sds \qquad (2.186)$$

In formula (2.186), the limit of integration is extended from 2 to $+\infty$. Since the integrand decreases exponentially with respect to s, little error will be introduced.

It is convenient to define function

$$A(k_d, z) = \frac{iz(L-z)}{2L}k_d \qquad (2.187)$$

Combining (2.180) and (2.187) gives

$$g = exp\left(\frac{iz(L-z)}{2L}k_d s^2\right) = exp(A(k_d)s^2) \qquad (2.188)$$

It is convenient to define function

$$P = \left|\frac{\bar{\rho}_d z}{L}\right| \qquad (2.189)$$

Combining (2.186), (2.188) and (2.189) gives

$$2\alpha_p \int_0^\infty \left(1 - exp(A(k_d, z)s^2)J_0(ksP)\right) exp\left(-\alpha_p s^2\right)sds$$

$$= \alpha_p \int_0^\infty exp\left(-\alpha_p s^2\right)ds^2 - \alpha_p \int_0^\infty exp\left(-\left(\alpha_p - A(k_d, z)\right)s^2\right)J_0(ksP)ds^2$$

$$= 1 - \alpha_p \int_0^\infty exp\left(-\left(\alpha_p - A(k_d, z)\right) s^2\right) J_0(ksP) ds^2 \qquad (2.190)$$

Applying the integral identity

$$\int_0^\infty J_v(at) exp(-p^2t^2) t^{v+1} dt = \frac{a^v}{(2p^2)^{v+1}} exp\left(-\frac{a^2}{4p^2}\right) \qquad (2.191)$$

Combining (2.186), (2.190) and (2.191) gives

$$\frac{1}{2}\int_0^2 sds\beta(s)\left(1 - gJ_0(ksP)\right) = 1 - \frac{1}{1-A(k_d,z)/\alpha_p} exp\left(-\frac{1}{4}\frac{k^2P^2}{\alpha_p - A(k_d,z)}\right) \quad (2.192)$$

Combining (2.181) and (2.192) gives

$$H(\bar{\rho}_d, k_d) = \int_0^L dzb\left(1 - \frac{1}{1-A(k_d,z)/\alpha_p} exp\left(-\frac{1}{4}\frac{k^2P^2}{\alpha_p - A(k_d,z)}\right)\right) \qquad (2.193)$$

It is convenient to define function

$$B(k_d, z) = \frac{1}{1-A(k_d,z)/\alpha_p} \qquad (2.194)$$

Combining (2.193) and (2.194) gives

$$H(\bar{r}_d, k_d) = \int_0^L dzb\left(1 - B(k_d, z) exp\left(-\frac{1}{4}\frac{k^2P^2}{\alpha_p - A(k_d,z)}\right)\right) \qquad (2.195)$$

The H function in formula (2.195) includes the effects of scattering. If the random medium is also lossy, an additional term corresponding to the dielectric loss can be added to the H function [1-2, 4, 8]

$$H(\bar{\rho}_d, k_d) = \int_0^L adz + \int_0^L dzb\left(1 - B(k_d, z) exp\left(-\frac{1}{4}\frac{k^2P^2}{\alpha_p - A(k_d,z)}\right)\right) \quad (2.196)$$

where a is the absorption coefficient. It is noticed that the limits of integration is from 0 to L in the equation. This implies that the random medium occupies the entire space between the transceiver and the object. If the random medium is uniform from d_1 to d_1+d_2, we can make the change of the limits of the integration. More specifically, the H function is written as

$$H(\bar{\rho}_d,k_d)=\int_{d_1}^{d_1+d_2} adz+\int_{d_1}^{d_1+d_2} dzb\left(1-B(k_d,z)exp\left(-\frac{1}{4}\frac{k^2P^2}{\alpha_p-A(k_d,z)}\right)\right) \quad (2.197)$$

The expression (2.197) is derived based on the assumption that the transceiver has isotropic 180° antenna pattern. If the antennas of the imaging system have relatively broad beamwidth, then this assumption should be valid. However, for narrow beam antennas, an additional antenna pattern factor must be included in the integral to incorporate the finite beamwidth. The most commonly used the antenna pattern factor is Gaussian function, because of its mathematical convenience and its ability to represent the main lobe of a high gain antenna [4, 41],

Combining (2.124) and (2.197) gives

$$\Gamma_2(\bar{\rho}_d,L,k_c,k_d)=\Gamma_2^0(\bar{\rho}_1,\bar{\rho}_2,L,k_c,k_d)exp\big(-H(\bar{\rho}_d,k_d)\big)$$

$$=\frac{1}{(4\pi L)^2}exp\left(ik_c(z_1-z_2)+i\frac{k_d}{2}(z_1+z_2)-H(\bar{\rho}_d,k_d)\right) \quad (2.198)$$

where $\bar{\rho}_d=\bar{\rho}_1-\bar{\rho}_2$.

The coherent part Γ_{2c} and the incoherent part Γ_{2i} of the two-frequency MCF can be written as

$$\Gamma_{2c}(\bar{\rho}_d,L,k_c,k_d)=\frac{1}{(4\pi L)^2}exp\left(ik_c(z_1-z_2)+i\frac{k_d}{2}(z_1+z_2)-\tau_0\right) \quad (2.199)$$

$$\Gamma_{2i}(\bar{\rho}_d,L,k_1,k_2)=\frac{1}{(4\pi L)^2}exp\left(ik_c(z_1-z_2)+i\frac{k_d}{2}(z_1+z_2)\right)$$

$$\times \left(exp\left(-H(\bar{\rho}_d, k_d, k)\right) - exp(-\tau_0) \right) \tag{2.200}$$

where τ_0, the optical depth, is given by

$$\tau_0 = \int_{d_1}^{d_1+d_2} (a+b)dz \tag{2.201}$$

where a is the absorption coefficient and b is the scattering coefficient.

In an analogous way, the coherent field can be evaluated.

Combining (2.108), (2.177) and (2.178) gives

$$\langle U(\bar{\rho}, L, k) \rangle = U_0(\bar{\rho}, L, k) exp\left(-\pi \int_0^L dz \int_0^2 sds\left(2\pi k^4 \Phi_n(ks)\right)\right)$$

$$= U_0(\bar{\rho}, L, k) exp\left(-\frac{1}{4}\int_0^L bdz \int_0^2 sds\beta(s)\right)$$

$$= U_0(\bar{\rho}, L, k) exp\left(-\frac{1}{2}\int_0^L bdz\right) \tag{2.202}$$

For a layer of lossy random medium from d_1 to $d_1 + d_2$, we obtain

$$\langle U(\bar{\rho}, L, k) \rangle = U_0(\bar{\rho}, L, k) exp\left(-\frac{1}{2}\int_{d_1}^{d_1+d_2} (a+b)dz\right) \tag{2.203}$$

where a is the absorption coefficient.

Combining (2.201) and (2.203) gives

$$\langle U(\bar{\rho}, L, k) \rangle = U_0(\bar{\rho}, L, k) exp\left(-\frac{\tau_0}{2}\right) \tag{2.204}$$

It can be observed that the coherent field attenuates exponentially as it propagates through the random medium.

Chapter 3 Correlation Function of Imaging System Output

For practical time-domain imaging systems, the modulated waveform is commonly used to probe the target characteristics. The images of the target can be formed based on statistical information extracted from the complex envelope measurements. Therefore, it is important to determine the effects of the random medium on the statistics of the complex envelopes. In this chapter, we will the discuss relationship between the correlation function of the output complex envelope and the two-frequency MCF. It will be shown that the for a given input signal, the correlation function of the output complex envelope can be calculated from the two-frequency MCF by taking the inverse Fourier transform.

This chapter is organized in following. Frist, the correlation function of the output complex envelop is derived and its relationship to the two-frequency MCF is demonstrated. Second, a simplified correlation function of the output complex envelop is obtained by assuming the transmitting signal is modulated Gaussian.

3.1 Correlation Function of Output Complex Envelope

For a time-domain imaging system transmitting the modulated signals, we have [4, 25]

$$p_i(t) = f_i(t)exp(-i\omega_0 t) \tag{3.1}$$

where $f_i(t)$ is the complex envelope of the transmitting signal and $f_0 = \frac{\omega_0}{2\pi}$ is the carrier frequency.

The receiving signal is

$$p_o(t) = f_o(t)exp(-i\omega_0 t) \tag{3.2}$$

where $f_o(t)$ is the complex envelope of the receiving signal and $f_0 = \frac{\omega_0}{2\pi}$ is the carrier frequency.

The spectrum of the transmitting signal can be obtained by taking the Fourier transform

$$P_i(\omega) = \int_{-\infty}^{+\infty} p_i(t)\, exp(i\omega t) dt \tag{3.3}$$

Combining (3.1) and (3.3) gives

$$P_i(\omega) = \int_{-\infty}^{+\infty} f_i(t)\, exp(i(\omega - \omega_0)t) dt = F_i(\omega - \omega_0) \tag{3.4}$$

The spectrums of the transmitting and the receiving signals are related by the transfer function

$$P_0(\omega) = H(\omega) P_i(\omega) \tag{3.5}$$

Combining (3.2), (3.4) and (3.5) gives

$$f_o(t) exp(-i\omega_0 t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_i(\omega - \omega_0) H(\omega) exp(-i\omega t) d\omega$$

$$f_o(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_i(\omega - \omega_0) H(\omega) exp(-i(\omega - \omega_0)t) d\omega$$

$$f_o(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_i(\omega) H(\omega + \omega_0) exp(-i\omega t) d\omega \tag{3.6}$$

The complex envelope of the receiving signal is

$$f_o(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_i(\omega) H(\omega + \omega_0) exp(-i\omega t) d\omega \tag{3.7}$$

In the random medium, the transfer function is related to the complex amplitude of the field

$$H(\omega) = U(\bar{\rho}, L, \omega) \tag{3.8}$$

Combining (3.7) and (3.8) gives

$$f_o(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_i(\omega)U(\bar{r}, L, \omega + \omega_0)exp(-i\omega t)d\omega \qquad (3.9)$$

The correlation function of the receiving complex envelope is

$$B_f(\bar{\rho}_1, \bar{\rho}_2, L, t_1, t_2) = \langle f_o(\bar{\rho}_1, L, t_1) f_o^*(\bar{\rho}_2, L, t_2)\rangle$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2 \langle U(\bar{\rho}_1, L, \omega_1 + \omega_0)U^*(\bar{\rho}_2, L, \omega_2 + \omega_0)\rangle$$

$$\times F_i(\omega_1)F_i^*(\omega_2)exp(-i\omega_1 t_1 + i\omega_2 t_2) \qquad (3.10)$$

It is noticed that two-frequency MCF is defined as

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, \omega_1, \omega_2) = \langle U(\bar{\rho}_1, L, \omega_1)U^*(\bar{\rho}_2, L, \omega_2)\rangle \qquad (3.11)$$

Combining (3.10) and (3.11) gives

$$B_f(\bar{\rho}_1, \bar{\rho}_2, L, t_1, t_2) = \langle f_o(\bar{\rho}_1, L, t_1) f_o^*(\bar{\rho}_2, L, t_2)\rangle$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2\, \Gamma_2(\bar{r}_1, \bar{r}_2, L, \omega_1 + \omega_0, \omega_2 + \omega_0)$$

$$\times F_i(\omega_1)F_i^*(\omega_2)exp(-i\omega_1 t_1 + i\omega_2 t_2) \qquad (3.12)$$

3.2 Correlation Function of a Modulated Gaussian Waveform

If the transmitting signal is a modulated Gaussian pulse, then we have [1-2]

$$p_i(t) = f_i(t)exp(-i\omega_0 t) = A_0 exp\left(-\frac{t^2}{T_0^2} - i\omega_0 t\right) \qquad (3.13)$$

where A_0 is the amplitude and $f_0 = \frac{\omega_0}{2\pi}$ is the carrier frequency.

The spectrum of transmitting complex envelope is

$$F_i(\omega) = \int_{-\infty}^{+\infty} f_i(t) exp(i\omega t) dt = A_0 \int_{-\infty}^{+\infty} exp\left(-\frac{t^2}{T_0^2} + i\omega t\right) dt \qquad (3.14)$$

Applying the integral identity

$$\int_{-\infty}^{+\infty} exp(-at^2 + bt) dt = \sqrt{\frac{\pi}{a}} exp\left(\frac{b^2}{4a}\right),\ Re(a) > 0 \qquad (3.15)$$

Combining (3.14) and (3.15) gives

$$F_i(\omega) = A_0 \int_{-\infty}^{+\infty} exp\left(-\frac{t^2}{T_0^2} + i\omega t\right) dt = A_0 \sqrt{\pi} T_0 exp\left(-\frac{T_0^2 \omega^2}{4}\right) \qquad (3.16)$$

It is convenient to define variable

$$\Delta f = \frac{2}{T_0} = \frac{\Delta\omega}{2\pi} \qquad (3.17)$$

The spectrum of transmitting complex envelope is

$$F_i(\omega) = A_0 \frac{2\sqrt{\pi}}{\Delta f} exp\left(-\frac{\omega^2}{(\Delta f)^2}\right) \qquad (3.18)$$

The correlation of the transmitting complex envelope spectrum is

$$F_i(\omega_1) F_i^*(\omega_2) = A_0^2 \frac{4\pi}{(\Delta f)^2} exp\left(-\frac{\omega_1^2}{(\Delta f)^2} - \frac{\omega_2^2}{(\Delta f)^2}\right) \qquad (3.19)$$

It is convenient to make change of variables

$$t_c = \frac{1}{2}(t_1 + t_2), t_d = t_1 - t_2 \qquad (3.20)$$

$$\omega_c = \frac{1}{2}(\omega_1 + \omega_2), \omega_d = \omega_1 - \omega_2 \qquad (3.21)$$

We have

$$\omega_1 t_1 - \omega_2 t_2 = \left(\omega_c + \frac{\omega_d}{2}\right)\left(t_c + \frac{t_d}{2}\right) - \left(\omega_c - \frac{\omega_d}{2}\right)\left(t_c - \frac{t_d}{2}\right)$$

$$= \omega_d t_c + \omega_c t_d \tag{3.22}$$

Combining (3.19) and (3.21) gives

$$F_i(\omega_1)F_i^*(\omega_2) = \frac{4\pi A_0^2}{(\Delta f)^2} exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} - \frac{\omega_d^2}{2(\Delta f)^2}\right) \tag{3.23}$$

It is noticed that the formula (3.12) assumes infinite bandwidth. However, the practical systems can only have finite bandwidth. Therefore, we need to modify the limits of integration

$$\omega_1 \in \left[-\frac{\Delta\omega}{2},\ +\frac{\Delta\omega}{2}\right] \tag{3.24a}$$

$$\omega_2 \in \left[-\frac{\Delta\omega}{2},\ +\frac{\Delta\omega}{2}\right] \tag{3.24b}$$

where $\Delta\omega$ is the bandwidth.

Making change of variables, the limits of integration become

$$\omega_c \in \left[-\frac{\Delta\omega}{2},\ +\frac{\Delta\omega}{2}\right] \tag{3.25a}$$

$$\omega_2 \in [-\Delta\omega,\ \Delta\omega] \tag{3.25b}$$

The boundaries of the region of integration are

$$\omega_c = +\frac{\omega_d}{2} + \frac{\Delta\omega}{2} \tag{3.26a}$$

$$\omega_c = -\frac{\omega_d}{2} - \frac{\Delta\omega}{2} \tag{3.26b}$$

$$\omega_c = -\frac{\omega_d}{2} + \frac{\Delta\omega}{2} \tag{3.26c}$$

$$\omega_c = +\frac{\omega_d}{2} - \frac{\Delta\omega}{2} \tag{3.26d}$$

The limits of integration are

$$\int_{-\frac{\Delta\omega}{2}}^{+\frac{\Delta\omega}{2}} d\omega_1 \int_{-\frac{\Delta\omega}{2}}^{+\frac{\Delta\omega}{2}} d\omega_2 = \int_{-\Delta\omega}^{0} d\omega_d \int_{-\frac{\omega_d}{2}-\frac{\Delta\omega}{2}}^{+\frac{\omega_d}{2}+\frac{\Delta\omega}{2}} d\omega_c$$

$$+\int_{0}^{\Delta\omega} d\omega_d \int_{+\frac{\omega_d}{2}-\frac{\Delta\omega}{2}}^{-\frac{\omega_d}{2}+\frac{\Delta\omega}{2}} d\omega_c \tag{3.27}$$

Combining (3.12), (3.22) and (2.27) gives

$$B_f(\bar{\rho}_1, \bar{\rho}_2, L, t_c, t_d) = \frac{A_0^2}{\pi(\Delta f)^2} \iint d\omega_c d\omega_d \Gamma_2(\bar{r}_1, \bar{r}_2, L, \omega_c + \omega_0, \omega_d)$$

$$\times\, exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} - \frac{\omega_d^2}{2(\Delta f)^2}\right) exp(-i\omega_d t_c - i\omega_c t_d) \tag{3.28}$$

By setting $t_d = 0$, the measured intensity can be written as

$$B_f(\bar{\rho}_1, \bar{\rho}_2, L, t_c, t_d = 0) = \frac{A_0^2}{\pi(\Delta f)^2} \iint d\omega_c d\omega_d \Gamma_2(\bar{r}_1, \bar{r}_2, L, \omega_c + \omega_0, \omega_d)$$

$$\times\, exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} - \frac{\omega_d^2}{2(\Delta f)^2}\right) exp(-i\omega_d t_c) \tag{3.29}$$

Combining (3.27) and (3.29) gives

$$B_f(\bar{\rho}_1, \bar{\rho}_2, L, t_c, t_d = 0) = \int_{-\Delta\omega}^{0} d\omega_d \int_{-\frac{\Delta\omega+\omega_d}{2}}^{\frac{\Delta\omega+\omega_d}{2}} d\omega_c\, \Gamma_2(\bar{r}_1, \bar{r}_2, L, \omega_c + \omega_0, \omega_d)$$

$$\times\, exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} - \frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c\right) + \int_{0}^{\Delta\omega} d\omega_d \int_{-\frac{\Delta\omega-\omega_d}{2}}^{\frac{\Delta\omega-\omega_d}{2}} d\omega_c$$

$$\times\, \Gamma_2(\bar{r}_1, \bar{r}_2, L, \omega_c + \omega_0, \omega_d) exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} - \frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c\right)$$

$$= \int_{-\Delta\omega}^{0} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c\right) \int_{-\frac{\Delta\omega+\omega_d}{2}}^{\frac{\Delta\omega+\omega_d}{2}} d\omega_c \, exp\left(-\frac{2\omega_c^2}{(\Delta f)^2}\right)$$

$$\times \Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c + \omega_0, \omega_d) + \int_{0}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c\right) \int_{-\frac{\Delta\omega-\omega_d}{2}}^{\frac{\Delta\omega-\omega_d}{2}} d\omega_c$$

$$\times exp\left(-\frac{2\omega_c^2}{(\Delta f)^2}\right) \Gamma_2(\bar{r}_1, \bar{r}_2, L, \omega_c + \omega_0, \omega_d) \qquad (3.30)$$

It can be seen that the integrand decreases exponentially with respect to variable ω_c. Thus, we may extend the limits of integration to infinite

$$\int_{-\frac{\Delta\omega+\omega_d}{2}}^{\frac{\Delta\omega+\omega_d}{2}} d\omega_c \approx \int_{-\infty}^{+\infty} d\omega_c \qquad (3.31a)$$

$$\int_{-\frac{\Delta\omega-\omega_d}{2}}^{\frac{\Delta\omega-\omega_d}{2}} d\omega_c \approx \int_{-\infty}^{+\infty} d\omega_c \qquad (3.31b)$$

Combining (3.30) and (3.31) gives

$$B_f(\bar{\rho}_1, \bar{\rho}_2, L, t_c, t_d = 0) = \int_{-\Delta\omega}^{0} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c\right) \int_{-\infty}^{+\infty} d\omega_c \, exp\left(-\frac{2\omega_c^2}{(\Delta f)^2}\right)$$

$$\times \Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c + \omega_0, \omega_d) + \int_{0}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c\right) \int_{-\infty}^{+\infty} d\omega_c$$

$$exp\left(-\frac{2\omega_c^2}{(\Delta f)^2}\right) \Gamma_2(\bar{r}_1, \bar{r}_2, L, \omega_c + \omega_0, \omega_d)$$

$$= \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c\right) \int_{-\infty}^{+\infty} d\omega_c \, exp\left(-\frac{2\omega_c^2}{(\Delta f)^2}\right)$$

$$\times \Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c + \omega_0, \omega_d) \qquad (3.32)$$

Chapter 4 Radar Cross Section in Frequency and Time Domain

4.1 Kirchhoff Approximation and Scattered Power

Consider a large Dirichlet object located in the random medium. The scattered field at position $\overline{R}$ is given by [3-4, 8]

$$\psi_s(\bar{r}) = -\int G(\bar{r}, \bar{r}')\frac{\partial\psi(\bar{r}')}{\partial n'}dS' \tag{4.1}$$

where $G(\bar{r}, \bar{r}')$ is the stochastic Green's function. The random surface field $\frac{\partial\psi(\bar{r}')}{\partial n'}$ is related to the incident field through the random transition operator

$$\frac{\partial\psi(\bar{r}')}{\partial n'} = \int T(\bar{r}', \bar{r}'')\frac{\partial\psi_i(\bar{r}'')}{\partial n''}dS'' \tag{4.2}$$

where $\psi_i(\bar{r}'')$ is the incident field at position $\bar{r}''$ on object surface.

If the object size and the surface radius of the curvature are both much greater than the wavelength at the carrier frequency, then the Kirchhoff approximation can be applied to find the surface field,

$$T(\bar{r}', \bar{r}'') = 2\delta(\bar{r}' - \bar{r}'') \tag{4.3a}$$

$$\frac{\partial\psi(\bar{r}')}{\partial n'} = 2\frac{\partial\psi_i(\bar{r}')}{\partial n'} \tag{4.3b}$$

Combining (4.2) and (4.3) gives

$$\frac{\partial\psi(\bar{r}')}{\partial n'} = 2\int \delta(\bar{r}' - \bar{r}'')\frac{\partial\psi_i(\bar{r}'')}{\partial n''}dS'' = 2\frac{\partial\psi_i(\bar{r}')}{\partial n'} \tag{4.4}$$

The scattered field at position $\bar{r}$ is given by

$$\psi_s(\bar{r}) = -2\int G(\bar{r}, \bar{r}')\frac{\partial\psi_i(\bar{r}')}{\partial n'}dS' \tag{4.5}$$

The scattered power at position $\bar{R}$ is found to be

$$\langle|\psi_s(\omega)|^2\rangle = \langle\psi_s(\bar{r})\psi_s^*(\bar{r})\rangle = 4\iint \left\langle G_1 \frac{\partial\psi_{i1}}{\partial n_1} G_2^* \frac{\partial\psi_{i2}^*}{\partial n_2}\right\rangle dS_1 dS_2 \qquad (4.6)$$

where $G_1 = G_1(\bar{r},\bar{r}_1)$, $\psi_{i1} = \psi_i(\bar{r}_1)$, $G_2 = G_2(\bar{r},\bar{r}_2)$ and $\psi_{i2} = \psi_i(\bar{r}_2)$.

The expression for the scattered power contains the 4th order moment. By applying the circular complex Gaussian assumption, the 4th order moment can be expressed in terms of the 2nd order moments [8-9]

$$\left\langle G_1 \frac{\partial\psi_{i1}}{\partial n_1} G_2^* \frac{\partial\psi_{i2}^*}{\partial n_2}\right\rangle = \langle G_1 G_2^*\rangle \left\langle \frac{\partial\psi_{i1}}{\partial n_1} \frac{\partial\psi_{i2}^*}{\partial n_2}\right\rangle + \left\langle G_1 \frac{\partial\psi_{i2}^*}{\partial n_2}\right\rangle \left\langle \frac{\partial\psi_{i1}}{\partial n_1} G_2^*\right\rangle$$

$$-\langle G_1\rangle \left\langle \frac{\partial\psi_{i1}}{\partial n_1}\right\rangle \langle G_2^*\rangle \left\langle \frac{\partial\psi_{i2}^*}{\partial n_2}\right\rangle \qquad (4.7)$$

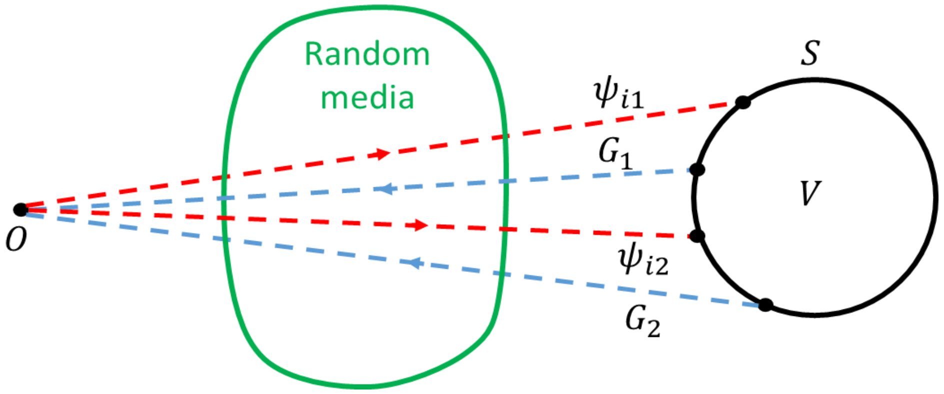

Figure 4.1 The circular complex Gaussian assumption includes the correlation between the incident field ψ_i and the scattered field G.

The second and third terms of equation (4.7) represent the correlations between the incident field ψ_i and the scattered fields G which give the backscattering enhancement. If the correlation between the incident field and the scattered field is neglected, then the 4th order moment becomes [8]

$$\left\langle G_1 \frac{\partial\psi_{i1}}{\partial n_1} G_2^* \frac{\partial\psi_{i2}^*}{\partial n_2}\right\rangle = \langle G_1 G_2^*\rangle \left\langle \frac{\partial\psi_{i1}}{\partial n_1} \frac{\partial\psi_{i2}^*}{\partial n_2}\right\rangle \qquad (4.8)$$

The incident field ψ_i on the object is equal to the stochastic Green's function and can be written as,

$$\psi_{i1} = \psi_i(\bar{r}_1) = G(\bar{r}_1\,, \bar{r}) = G(\bar{r}, \bar{r}_1) \tag{4.9a}$$

$$\psi_{i2} = \psi_i(\bar{r}_2) = G(\bar{r}_2\,, \bar{r}) = G(\bar{r}, \bar{r}_2) \tag{4.9b}$$

where the reciprocity is applied $G(\bar{r}, \bar{r}') = G(\bar{r}', \bar{r})$ [26, 47-48].

Combining (4.7) and (4.9) gives

$$\left\langle G_1 \frac{\partial \psi_{i1}}{\partial n_1} G_2^* \frac{\partial \psi_{i2}^*}{\partial n_2} \right\rangle = \langle G_1 G_2^* \rangle \left\langle \frac{\partial G_1}{\partial n_1} \frac{\partial G_2^*}{\partial n_2} \right\rangle + \left\langle G_1 \frac{\partial G_2^*}{\partial n_2} \right\rangle \left\langle \frac{\partial G_1}{\partial n_1} G_2^* \right\rangle$$

$$-\langle G_1 \rangle \left\langle \frac{\partial G_1}{\partial n_1} \right\rangle \langle G_2^* \rangle \left\langle \frac{\partial G_2^*}{\partial n_2} \right\rangle \tag{4.10}$$

Under the parabolic equation approximation, the Green's function is written as,

$$G(\bar{r}, \bar{r}') \approx \frac{1}{4\pi(z-z')} exp\left(ik(z-z') + \frac{ik}{2} \frac{(x-x')^2 + (y-y')^2}{z-z'} + \psi \right) \tag{4.11a}$$

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z},\ \bar{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z} \tag{4.11b}$$

where ψ is the complex phase perturbation which represents the effects of the random medium.

The normal derivative of the Green's function becomes

$$\frac{\partial}{\partial n} = \hat{n} \cdot \nabla = \hat{n} \cdot \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \tag{4.12}$$

Combining (4.11) and (4.12) gives

$$\frac{\partial G(\bar{r}, \bar{r}')}{\partial x} = \frac{ik(x-x')}{4\pi(z-z')^2} exp\left(ik(z-z') + \frac{ik}{2} \frac{(x-x')^2 + (y-y')^2}{z-z'} + \psi \right) \tag{4.13a}$$

$$\frac{\partial G(\bar{r},\bar{r}')}{\partial y} = \frac{ik(y-y')}{4\pi(z-z')^2} exp\left(ik(z-z') + \frac{ik}{2}\frac{(x-x')^2+(y-y')^2}{z-z'} + \psi\right) \quad (4.13b)$$

$$\frac{\partial G(\bar{r},\bar{r}')}{\partial z} \approx \frac{ik}{4\pi(z-z')} exp\left(ik(z-z') + \frac{ik}{2}\frac{(x-x')^2+(y-y')^2}{z-z'} + \psi\right) \quad (4.13c)$$

Here we assume the partial derivatives of the complex phase perturbation ψ have negligible contributions.

Combining (4.12) and (4.13) gives

$$\frac{\partial G(\bar{r},\bar{r}')}{\partial n} = \hat{n} \cdot \left(\hat{x}\frac{ik(x-x')}{4\pi(z-z')} + \hat{y}\frac{ik(y-y')}{4\pi(z-z')} + ik\hat{z}\right) G(\bar{r},\bar{r}')$$

$$\frac{\partial G(\bar{r},\bar{r}')}{\partial n} \approx ik(\hat{n}\cdot\hat{z})G(\bar{r},\bar{r}') \quad (4.14)$$

Taking the complex conjugate of (4.14), we have

$$\frac{\partial G^*(\bar{r},\bar{r}')}{\partial n} \approx -ik(\hat{n}\cdot\hat{z})G^*(\bar{r},\bar{r}') \quad (4.15)$$

Combining (4.10), (4.14) and (4.15) gives

$$\left\langle G_1 \frac{\partial \psi_{i1}}{\partial n_1} G_2^* \frac{\partial \psi_{i2}^*}{\partial n_2}\right\rangle = k^2(\hat{n}_1\cdot\hat{z})(\hat{n}_2\cdot\hat{z})[2\langle G_1G_2^*\rangle\langle G_1G_2^*\rangle - \langle G_1\rangle^2\langle G_2^*\rangle^2] \quad (4.17)$$

We may resolve the Green’s function into coherent and incoherent components

$$G_1 = \langle G_1\rangle + G_{f1} = G(\bar{r},\bar{r}_1) = \langle G(\bar{r},\bar{r}_1)\rangle + G_f(\bar{r},\bar{r}_1) \quad (4.18a)$$

$$G_2 = \langle G_2\rangle + G_{f2} = G(\bar{r},\bar{r}_2) = \langle G(\bar{r},\bar{r}_2)\rangle + G_f(\bar{r},\bar{r}_2) \quad (4.18b)$$

Combining (4.17) and (4.18) gives

$$\left\langle G_1 \frac{\partial \psi_{i1}}{\partial n_1} G_2^* \frac{\partial \psi_{i2}^*}{\partial n_2}\right\rangle = k^2(\hat{n}_1\cdot\hat{z})(\hat{n}_2\cdot\hat{z})$$

$$\times \left[\langle G_1\rangle^2\langle G_2^*\rangle^2 + 4\langle G_1\rangle\langle G_2^*\rangle\langle G_{f1}G_{f2}^*\rangle + 2\langle G_{f1}G_{f2}^*\rangle^2\right] \tag{4.19}$$

Combining (4.6) and (4.19) gives

$$\langle|\psi_s(\omega)|^2\rangle = 4k^2 \iint \left(\langle G_1\rangle^2\langle G_2^*\rangle^2 + 4\langle G_1\rangle\langle G_2^*\rangle\left\langle G_{f1}G_{f2}^*\right\rangle + 2\left\langle G_{f1}G_{f2}^*\right\rangle^2\right)$$

$$\times (\hat{z}\cdot\hat{n}_1)(\hat{z}\cdot\hat{n}_2)dS_1dS_2 \tag{4.20}$$

If the correlation between the incident and the scattered fields is neglected as in (4.8), we obtain

$$\langle|\psi_s(\omega)|^2\rangle = 4k^2 \iint \left(\langle G_1\rangle^2\langle G_2^*\rangle^2 + 2\langle G_1\rangle\langle G_2^*\rangle\left\langle G_{f1}G_{f2}^*\right\rangle + \left\langle G_{f1}G_{f2}^*\right\rangle^2\right)$$

$$\times (\hat{z}\cdot\hat{n}_1)(\hat{z}\cdot\hat{n}_2)dS_1dS_2 \tag{4.21}$$

Comparing (4.20) and (4.21), it can be seen that (4.20) includes the terms corresponding to the backscattering enhancement effect while (4.21) does not.

The coherent component of Green's function is [8]

$$\langle G(\bar{\rho}, L, \omega\,)\rangle = \frac{1}{4\pi L} exp\left(\frac{ik\rho^2}{2z} + ikz - \frac{\tau_0}{2}\right) \tag{4.22}$$

where L is the distance between the transceiver and the object and τ_0 is the optical depth.

It is observed that the equation (4.21) gives the scattered power at a single frequency. In order to get the time-domain response, the cross-correlation between scattered fields at different frequencies must be evaluated. Therefore, we may write

$$\langle\psi_s(\omega_1)\psi_s^*(\omega_2)\rangle = 4k_1k_2 \iint dS_1dS_2(\hat{z}\cdot\hat{n}_1)(\hat{z}\cdot\hat{n}_2)\langle G_1(\omega_1)\rangle^2\langle G_2^*(\omega_2)\rangle^2$$

$$+16k_1k_2 \iint dS_1dS_2(\hat{z}\cdot\hat{n}_1)(\hat{z}\cdot\hat{n}_2)\langle G_1(\omega_1)\rangle\langle G_2^*(\omega_2)\rangle\left\langle G_{f1}(\omega_1)G_{f2}^*(\omega_2)\right\rangle$$

$$+8k_1k_2\iint dS_1dS_2(\hat{z}\cdot\hat{n}_1)(\hat{z}\cdot\hat{n}_2)\left\langle G_{f1}(\omega_1)G_{f2}^*(\omega_2)\right\rangle^2 \tag{4.23}$$

The correlation between the Green's functions at two frequencies is

$$\langle G_1(\omega_1)G_2^*(\omega_2)\rangle = \langle G_1(\omega_1)\rangle\langle G_2^*(\omega_2)\rangle + \langle G_{f1}(\omega_1)G_{f2}^*(\omega_2)\rangle \tag{4.24}$$

The product of the coherent Green's function is

$$\langle G_1(\omega_1)\rangle\langle G_2^*(\omega_2)\rangle = \frac{1}{(4\pi L)^2}exp\left(\frac{i}{2L}(k_1\rho_1^2 - k_2\rho_2^2)\right)$$

$$\times\, exp(i(k_1z_1 - k_2z_2) - \tau_0) \tag{4.25}$$

It is convenient to make change of variables

$$k_c = \frac{1}{2}(k_1 + k_2),\ k_d = k_1 - k_2 \tag{4.26a}$$

$$\bar{\rho}_c = \frac{1}{2}(\bar{\rho}_1 + \bar{\rho}_2),\ \bar{\rho}_d = \bar{\rho}_1 - \bar{\rho}_2 \tag{4.26b}$$

We have

$$k_1\rho_1^2 - k_2\rho_2^2 = 2k_c\bar{\rho}_c\cdot\bar{\rho}_d + k_d\left(\bar{\rho}_c\cdot\bar{\rho}_c + \frac{1}{4}\bar{\rho}_d\cdot\bar{\rho}_d\right) \tag{4.27}$$

We have

$$k_1z_1 - k_2z_2 = k_c(z_1 - z_2) + \frac{1}{2}k_d(z_1 + z_2) \tag{4.28}$$

Combining (4.25), (4.27) and (4.28) gives

$$\langle G_1(\omega_1)\rangle\langle G_2^*(\omega_2)\rangle = \frac{1}{(4\pi L)^2}exp\left(\frac{ik_c}{L}\bar{\rho}_c\cdot\bar{\rho}_d + \frac{ik_d}{2L}\left(\bar{\rho}_c\cdot\bar{\rho}_c + \frac{1}{4}\bar{\rho}_d\cdot\bar{\rho}_d\right)\right)$$

$$\times\, exp\left(ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2) - \tau_0\right)$$

$$\approx \frac{1}{(4\pi L)^2} exp\left(ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2) - \tau_0\right) \qquad (4.29)$$

Combining (2.199) and (4.29) gives

$$\langle G_1(\omega_1)\rangle\langle G_2^*(\omega_2)\rangle = \Gamma_{2c}(\bar{\rho}_d, L, \omega_c, \omega_d) \qquad (4.30)$$

The product of the average part of the Green's functions and its complex conjugate at two frequencies is the same as the coherent part of the two-frequency MCF.

Combining (2.200), (4.24) and (4.30) gives

$$\langle G_{f1}(\omega_1) G_{f2}^*(\omega_2)\rangle = \Gamma_{2i}(\bar{\rho}_d, L, \omega_c, \omega_d) \qquad (4.31)$$

The correlation between the fluctuation parts of the Green's functions at two frequencies is the same as the incoherent part of the two-frequency MCF.

Combining (4.23) and (4.31) gives

$$\langle \psi_s(\omega_1)\psi_s^*(\omega_2)\rangle = 4k_1k_2 \iint dS_1 dS_2 (\hat{z}\cdot\hat{n}_1)(\hat{z}\cdot\hat{n}_2)\langle G_1(\omega_1)\rangle^2\langle G_2^*(\omega_2)\rangle^2$$

$$+16k_1k_2 \iint dS_1 dS_2 (\hat{z}\cdot\hat{n}_1)(\hat{z}\cdot\hat{n}_2)\langle G_1(\omega_1)\rangle\langle G_2^*(\omega_2)\rangle\Gamma_{2i}(\omega_c, \omega_d)$$

$$+8k_1k_2 \iint dS_1 dS_2 (\hat{z}\cdot\hat{n}_1)(\hat{z}\cdot\hat{n}_2)\Gamma_{2i}^2(\omega_c, \omega_d) \qquad (4.32)$$

where $\omega_c = \frac{c}{2}(k_1 + k_2)$ and $\omega_d = c(k_1 - k_2)$.

This equation relates the correlation between the scattered fields at two frequencies to the two-frequency MCF.

4.2 Two-frequency Mutual Coherence Function

The two-frequency MCF is defined as

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, \omega_1, \omega_2) = \langle G(\bar{\rho}_1,\ L, \omega_1) G^*(\bar{\rho}_2, L, \omega_2)\rangle$$

$$= G_0(\bar{\rho}_1,\ L, \omega_1) G_0^*(\bar{\rho}_2, L, \omega_2) exp\big(-H(\omega_d, \rho_d\ )\big) \tag{4.33a}$$

$$G_0(\bar{\rho}, L, \omega\ ) = \frac{1}{4\pi L} exp\left(\frac{ik\rho^2}{2z} + ikz\right) \tag{4.33b}$$

where $\bar{\rho} = x\hat{x} + y\hat{y}$ denotes spatial point on the plane transverse to the z axis and L is the distance between the imaging system and the object. The effects of the random medium are included in function H. Since the random medium is assumed to be statistically homogeneous and isotropic, the function H depends only on the frequency separation $\omega_d = \omega_1 - \omega_2$ and the spatial distance $\rho_d = |\bar{\rho}_1 - \bar{\rho}_2|$. The function G_0 is the free-space Green's function under the parabolic equation approximation. A further simplification is achieved by noting that the distance between the imaging system and the object is much larger than the dimensions of the object. Therefore, the term $\frac{ik\rho^2}{2z}$ is negligible small and can be dropped [1-2, 8].

Substituting the (4.33b) into (4.33a) and using the center-of-mass and difference angular frequencies, we obtain

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c, \omega_d) = \frac{1}{(4\pi L)^2} exp\left(i\frac{\omega_c}{c}(z_1 - z_2) + i\frac{\omega_d}{2c}(z_1 + z_2) - H(\omega_d, \rho_d\ )\right) \tag{4.34}$$

where $\omega_c = \frac{\omega_1 + \omega_2}{2}$, $\omega_d = \omega_1 - \omega_2$, $\rho_d = |\bar{\rho}_1 - \bar{\rho}_2|$and c is the free-space light speed. The two-frequency MCF can resolved into the coherent part Γ_{2c} and the incoherent part Γ_{2i} [3-4, 8, 11]

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c, \omega_d) = \Gamma_{2c}(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c, \omega_d) + \Gamma_{2i}(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c, \omega_d) \tag{4.35}$$

$$\Gamma_{2c}(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c, \omega_d) = \frac{1}{(4\pi L)^2} exp\left(i\frac{\omega_c}{c}(z_1 - z_2) + i\frac{\omega_d}{2c}(z_1 + z_2)\right)$$

$$\times\ exp(-\tau_0) \tag{4.36}$$

$$\Gamma_{2i}(\bar{\rho}_1, \bar{\rho}_2, L, \omega_c, \omega_d) = \frac{1}{(4\pi L)^2} exp\left(i\frac{\omega_c}{c}(z_1 - z_2) + i\frac{\omega_d}{2c}(z_1 + z_2)\right)$$

$$\times \left(exp(-\tau_0) - H(\omega_d, \rho_d\,)\right) \tag{4.37}$$

The random medium is composed of the discrete scatters with the Gaussian phase function which is given by [1-4, 8, 15-18, 42]

$$p(s) = 4\alpha_p exp\left(-\alpha_p s^2\right) \tag{4.38}$$

where $s = 2\sin\left(\frac{\theta}{2}\right)$ and θ is the scattering angle. The Gaussian phase function satisfies the normalization condition $\frac{1}{2}\int_0^\infty p(s)sds = 1$.

For a layer of random medium contains discrete scatters with the Gaussian phase function, the H function can be written as

$$H(\bar{\rho}_d, k_d) = \int_{d_1}^{d_1+d_2} adz$$

$$+\int_{d_1}^{d_1+d_2} dzb\left[1 - B(\omega_d, z)exp\left(-\frac{P^2(\rho_d,z)}{4\left(\alpha_p - A(\omega_d,z\,)\right)}\left(\frac{\omega_0^2}{c^2} - \frac{\omega_d^2}{4c^2}\right)\right)\right] \tag{4.39a}$$

$$P(\rho_d, z) = \frac{\rho_d z}{L} \tag{4.39b}$$

$$A(\omega_d, z\,) = \frac{iz(L-z)}{2L}\frac{\omega_d}{c} \tag{4.39c}$$

$$B(\omega_d, z) = \left(1 - \frac{A(\omega_d,z\,)}{\alpha_p}\right)^{-1} \tag{4.39d}$$

4.3 Radar Cross Section in Frequency and Time Domain

The single-frequency monostatic radar cross section (RCS) is defined as [3-4, 8]

$$RCS(\omega) = \frac{4\pi L^2\langle\psi_s(\omega)\psi_s^*(\omega)\rangle}{|\psi_0|^2} \tag{4.40}$$

where ψ_0 is the free-space incident field at the object, ψ_s is the scattered field at the observation point and L is the distance between the observation point and the object. In this formula the correlation between the scattered fields is evaluated at the same frequency.

We can extend this definition and calculate the correction between the scattered fields at two frequencies. The two-frequency monostatic radar cross section (RCS) is defined as

$$RCS(\omega_1, \omega_2) = \frac{4\pi L^2 \langle \psi_s(\omega_1) \psi_s^*(\omega_2) \rangle}{|\psi_0|^2} \tag{4.41}$$

Combining (4.32) and (4.41) gives

$$RCS(\omega_1, \omega_2) = \frac{4\pi L^2 \langle \psi_s(\omega_1) \psi_s^*(\omega_2) \rangle}{|\psi_0|^2}$$

$$= I_1(\omega_1, \omega_2) + I_2(\omega_1, \omega_2) + I_3(\omega_1, \omega_2) \tag{4.42}$$

The first term $I_1(\omega_1, \omega_2)$ is called the coherent term, the second term $I_2(\omega_1, \omega_2)$ is called the mixed term and the third term $I_3(\omega_1, \omega_2)$ is called the incoherent term.

We use the notations

$$\int dS_1 (\hat{n}_1 \cdot \hat{z}) = \int d\hat{\rho}_1 \, , \int dS_2 (\hat{n}_2 \cdot \hat{z}) = \int d\hat{\rho}_2 \tag{4.43}$$

The incident intensity is

$$|\psi_0|^2 = \psi_0 \psi_0^* = \left(\frac{exp(ikL)}{4\pi L} \right) \left(\frac{exp(-ikL)}{4\pi L} \right) = \frac{1}{16\pi^2 L^2} \tag{4.44}$$

The coherent term $I_1(\omega_1, \omega_2)$ is

$$I_1(\omega_1, \omega_2) = \frac{4\pi L^2}{|\psi_0|^2} [4k_1 k_2 \iint d\hat{\rho}_1 d\hat{\rho}_2 \, (\langle G_1 \rangle^2 \langle G_2^* \rangle^2)]$$

$$= 256\pi^3 L^4 k_1 k_2 \iint d\bar{\rho}_1\, d\bar{\rho}_2 \left(\frac{exp\left(\frac{ik_1\rho_1^2}{2L} + ik_z z_1 - \frac{\tau_0}{2}\right)}{4\pi L} \right)^2 \left(\frac{exp\left(-\frac{ik_2\rho_2^2}{2L} - ik_z z_2 - \frac{\tau_0}{2}\right)}{4\pi L} \right)^2$$

$$\approx \frac{4\pi}{\lambda_1\lambda_2} exp(-2\tau_0) \iint d\hat{\rho}_1 d\hat{\rho}_2 exp\big(i2(k_1 z_1 - k_2 z_2)\big) \tag{2.45}$$

The mixed term $I_2(\omega_1, \omega_2)$ is

$$I_2(\omega_1, \omega_2) = \frac{4\pi L^2}{|\psi_0|^2}\left[4k_1k_2 \iint d\hat{\rho}_1 d\hat{\rho}_2 \left(4\langle G_1\rangle\langle G_2^*\rangle \Gamma_{2i}(\omega_1, \omega_2)\right)\right]$$

$$\approx 1024\pi^3 L^4 k_1 k_2 \iint d\hat{\rho}_1 d\hat{\rho}_2 \left(\frac{1}{16\pi^2 L^2} exp(i(k_1 z_1 - k_2 z_2) - \tau_0)\right)$$

$$\times \left(\frac{1}{16\pi^2 L^2}\Big(exp\big(-H(\bar{\rho}_d, \omega_1, \omega_2)\big) - exp(-\tau_0)\Big) exp(ik_1 z_1 - ik_2 z_2)\right)$$

$$= \frac{16\pi}{\lambda_1\lambda_2} exp(-\tau_0) \iint d\hat{\rho}_1 d\hat{\rho}_2 exp\big(i2(k_1 z_1 - k_2 z_2)\big)$$

$$\times \Big(exp\big(-H(\bar{\rho}_d, \omega_1, \omega_2)\big) - exp(-\tau_0)\Big) \tag{2.46}$$

The incoherent term $I_3(\omega_1, \omega_2)$ is

$$I_3(\omega_1, \omega_2) = \frac{4\pi L^2}{|\psi_0|^2}\left[4k_1k_2 \iint d\hat{\rho}_1 d\hat{\rho}_2 \left(2\Gamma_{2i}^2(\omega_1, \omega_2)\right)\right]$$

$$= 512\pi^3 L^4 k_1 k_2 \iint d\hat{\rho}_1 d\hat{\rho}_2 \left(\frac{1}{(16\pi^2 L^2)^2}\Big(exp\big(-H(\bar{r}_d, \omega_1, \omega_2)\big) - exp(-\tau_0)\Big)^2\right)$$

$$\times exp\big(i2(k_1 z_1 - k_2 z_2)\big)$$

$$= \frac{8\pi}{\lambda_1\lambda_2} \iint d\hat{\rho}_1 d\hat{\rho}_2 \Big(exp\big(-H(\bar{r}_d, \omega_1, \omega_2)\big) - exp(-\tau_0)\Big)^2$$

$$\times exp\big(i2(k_1 z_1 - k_2 z_2)\big) \tag{4.47}$$

It is convenient to make change of variables

$$k_c = \frac{1}{2}(k_1 + k_2),\ k_d = k_1 - k_2 \tag{4.48}$$

We have

$$k_1 z_1 - k_2 z_2 = k_c(z_1 - z_2) + \frac{1}{2} k_d (z_1 + z_2) \tag{4.49}$$

Under narrowband assumption, we have

$$\frac{\pi}{\lambda_1 \lambda_2} = \frac{\pi f_1 f_2}{c^2} \approx \frac{\pi f_c^2}{c^2} = \frac{\omega_c^2}{4\pi c^2} \tag{4.50}$$

where c is the free-space light speed.

The coherent term $I_1(\omega_c, \omega_d)$ is

$$I_1(\omega_c, \omega_d) = \frac{\omega_c^2}{\pi c^2} exp(-2\tau_0) \iint d\hat{\rho}_1 d\hat{\rho}_2$$

$$\times\, exp\left(i2\frac{\omega_c}{c}(z_1 - z_2) + i\frac{\omega_d}{c}(z_1 + z_2)\right) \tag{4.51}$$

The mixed term $I_2(\omega_c, \omega_d)$ is

$$I_2(\omega_c, \omega_d) = \frac{4\omega_c^2}{\pi c^2} exp(-\tau_0) \iint d\hat{\rho}_1 d\hat{\rho}_2 \left(exp\big(-H(\bar{r}_d, \omega_d)\big) - exp(-\tau_0)\right)$$

$$\times\, exp\left(i2\frac{\omega_c}{c}(z_1 - z_2) + i\frac{\omega_d}{c}(z_1 + z_2)\right) \tag{4.52}$$

The incoherent term $I_3(\omega_c, \omega_d)$ is

$$I_3(\omega_c, \omega_d) = \frac{2\omega_c^2}{\pi c^2} \iint d\hat{\rho}_1 d\hat{\rho}_2 \left(exp\big(-H(\bar{r}_d, \omega_d)\big) - exp(-\tau_0)\right)^2$$

The time-domain monostatic RCS is found by taking the inverse Fourier transform of the two-frequency monostatic RCS

$$RCS(t_1,t_2)=\frac{1}{(2\pi)^2}\iint d\omega_1 d\omega_2\, F(\omega_1)F^*(\omega_2)RCS(\omega_1,\omega_2)$$

$$\times\, exp(-i\omega_1 t_1+i\omega_2 t_2) \qquad (4.53)$$

where $F(\omega)$ is the spectrum of the transmitted waveform.

We assume the transmitted waveform is the modulated Gaussian pulse

$$f(t)=exp\left(-\frac{t^2}{T_0^2}-i\omega_0 t\right) \qquad (4.54)$$

$$F(\omega)=\sqrt{\pi}T_0 exp\left(-\frac{(\omega-\omega_0)^2}{(\Delta f)^2}\right) \qquad (4.55)$$

where $\Delta f=\frac{2}{T_0}$ is the bandwidth.

It is convenient to make change of variables

$$t_c=\frac{1}{2}(t_1+t_2), t_d=t_1-t_2 \qquad (4.56)$$

$$\omega_c=\frac{1}{2}(\omega_1+\omega_2), \omega_d=\omega_1-\omega_2 \qquad (4.57)$$

Combining (4.53), (4.56) and (4.57) gives

$$RCS(t_c,t_d)=\frac{1}{(2\pi)^2}\iint d\omega_c d\omega_d\, RCS(\omega_d,\omega_c)F\left(\omega_c+\frac{\omega_d}{2}\right)F^*\left(\omega_c-\frac{\omega_d}{2}\right)$$

$$\times\, exp(-i\omega_d t_c+i\omega_c t_d) \qquad (4.58)$$

The RCS is a slow varying function of variable ω_c in many practical problems. Therefore, we can integrate the RCS expression with respect to variable ω_c. Since the time-domain RCS depends on the correlation between the scattered fields at the same time instance, the difference time should be zero.

Since time-domain RCS is related to the complex envelope of the receiving signal, it will be convenient to make change of the variable

$$\omega_c \leftarrow \omega_c - \omega_0 \tag{4.59}$$

where $f_0 = \frac{\omega_0}{2\pi}$ is the carrier frequency. By doing the frequency shift, the integration with respect to frequency will be carried out around zero frequency.

$$RCS(t_c, t_d) = \frac{1}{(2\pi)^2} \iint d\omega_c d\omega_d \, exp(-i\omega_d t_c + i(\omega_c + \omega_0)t_d)$$

$$\times RCS(\omega_d, \omega_c + \omega_0) F\left(\omega_c + \omega_0 + \frac{\omega_d}{2}\right) F^*\left(\omega_c + \omega_0 - \frac{\omega_d}{2}\right) \tag{4.60}$$

When $t_d = 0$, the time-domain monostatic radar cross section (RCS) is

$$RCS(t_c) = I_1(t_c) + I_2(t_c) + I_3(t_c) \tag{4.61}$$

The coherent term $I_1(t_c)$ is

$$I_1(t_c) = \frac{A_0^2 exp(-2\tau_0)}{\pi^2 (\Delta f)^2 c^2} \iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1 + z_2)\right)$$

$$\times \int_{-\infty}^{+\infty} d\omega_c (\omega_c + \omega_0)^2 exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{\omega_c + \omega_0}{c}(z_1 - z_2) + i\frac{\omega_d}{c}(z_1 + z_2)\right)$$

$$= \frac{A_0^2 \omega_0^2 exp(-2\tau_0)}{\pi^2 (\Delta f)^2 c^2} \iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1 + z_2)\right)$$

$$\times \int_{-\infty}^{+\infty} d\omega_c \left(1 + \frac{\omega_c}{\omega_0}\right)^2 exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{\omega_c + \omega_0}{c}(z_1 - z_2) + i\frac{\omega_d}{c}(z_1 + z_2)\right)$$

$$\approx \frac{A_0^2\omega_0^2 exp(-2\tau_0)}{\pi^2(\Delta f)^2c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \int_{-\infty}^{+\infty} d\omega_c exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{\omega_c+\omega_0}{c}(z_1-z_2)\right) \qquad (4.62)$$

The mixed term $I_2(t_c)$ is

$$I_2(t_c) = \frac{4A_0^2 exp(-\tau_0)}{\pi^2(\Delta f)^2c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \left(exp\left(-H(\bar{r}_d,\omega_d)\right) - exp(-\tau_0)\right)$$

$$\times \int_{-\infty}^{+\infty} d\omega_c(\omega_c+\omega_0)^2 exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{(z_1-z_2)}{c}(\omega_0+\omega_c)\right)$$

$$= \frac{4A_0^2\omega_0^2 exp(-\tau_0)}{\pi^2(\Delta f)^2c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \left(exp\left(-H(\bar{r}_d,\omega_d)\right) - exp(-\tau_0)\right)$$

$$\times \int_{-\infty}^{+\infty} d\omega_c \left(1+\frac{\omega_c}{\omega_0}\right)^2 exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{(z_1-z_2)}{c}(\omega_0+\omega_c)\right)$$

$$\approx \frac{4A_0^2\omega_0^2 exp(-\tau_0)}{\pi^2(\Delta f)^2c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \left(exp\left(-H(\bar{r}_d,\omega_d)\right) - exp(-\tau_0)\right)$$

$$\times \int_{-\infty}^{+\infty} d\omega_c exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{(z_1-z_2)}{c}(\omega_0+\omega_c)\right) \qquad (4.63)$$

The incoherent term $I_3(t_c)$ is

$$I_3(t_c) = \frac{2A_0^2}{\pi^2(\Delta f)^2c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \left(exp\big(-H(\bar{r}_d,\omega_d)\big) - exp(-\tau_0)\right)^2$$

$$\times \int_{-\infty}^{+\infty} d\omega_c(\omega_0+\omega_c)^2 exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{(z_1-z_2)}{c}(\omega_0+\omega_c)\right)$$

$$= \frac{2A_0^2\omega_0^2}{\pi^2(\Delta f)^2c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \left(exp\big(-H(\bar{r}_d,\omega_d)\big) - exp(-\tau_0)\right)^2$$

$$\times \int_{-\infty}^{+\infty} d\omega_c \left(1+\frac{\omega_c}{\omega_0}\right)^2 exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{(z_1-z_2)}{c}(\omega_0+\omega_c)\right)$$

$$\approx \frac{2A_0^2\omega_0^2}{\pi^2(\Delta f)^2c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \left(exp\big(-H(\bar{r}_d,\omega_d)\big) - exp(-\tau_0)\right)^2$$

$$\times \int_{-\infty}^{+\infty} d\omega_c exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{(z_1-z_2)}{c}(\omega_0+\omega_c)\right) \qquad (4.64)$$

It is convenient to evaluate the integral

$$\int_{-\infty}^{+\infty} d\omega_c exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{\omega_c+\omega_0}{c}(z_1-z_2)\right)$$

$$= exp\left(i2\frac{\omega_0}{c}(z_1-z_2)\right)\int_{-\infty}^{+\infty} d\omega_c exp\left(-\frac{2\omega_c^2}{(\Delta f)^2} + i2\frac{\omega_c}{c}(z_1-z_2)\right) \qquad (4.65)$$

Applying the integral identity

$$\int_{-\infty}^{+\infty} exp(-at^2+bt)dt = \sqrt{\frac{\pi}{a}}exp\left(\frac{b^2}{4a}\right),\ Re(a)>0 \tag{4.66}$$

We obtain

$$\int_{-\infty}^{+\infty} d\omega_c exp\left(-\frac{2\omega_c^2}{(\Delta f)^2}+i2\frac{\omega_c+\omega_0}{c}(z_1-z_2)\right)$$

$$=\sqrt{\frac{\pi}{2}}\Delta f exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1-z_2)^2+i2\frac{\omega_0}{c}(z_1-z_2)\right) \tag{4.67}$$

Combining (4.62) and (4.67) gives

$$I_1(t_c)=\frac{A_0^2\omega_0^2 exp(-2\tau_0)}{\sqrt{2\pi}\pi\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2\int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2}-i\omega_d t_c+i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1-z_2)^2+i2\frac{\omega_0}{c}(z_1-z_2)\right) \tag{4.68}$$

Combining (4.63) and (4.67) gives

$$I_2(t_c)=\frac{4A_0^2\omega_0^2 exp(-\tau_0)}{\sqrt{2\pi}\pi\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2\int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2}-i\omega_d t_c+i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times\left(exp\big(-H(\bar{r}_d,\omega_d)\big)-exp(-\tau_0)\right)$$

$$\times exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1-z_2)^2+i2\frac{\omega_0}{c}(z_1-z_2)\right) \tag{4.69}$$

Combining (4.64) and (4.67) gives

$$I_3(t_c)=\frac{2A_0^2\omega_0^2}{\sqrt{2\pi}\pi\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2\int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2}-i\omega_d t_c+i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times\left(exp\big(-H(\bar{r}_d,\omega_d)\big)-exp(-\tau_0)\right)^2$$

$$\times exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1 - z_2)^2 + i2\frac{\omega_0}{c}(z_1 - z_2)\right) \tag{4.70}$$

It is observed that the each term in the time-domain RCS expressions contains two double integrals over the surface of the object and one integral over frequency. Therefore, the time-domain RCS is computationally intensive. In order to speed up the calculation, the integral over frequency will be computed using the fast Fourier transform (FFT).

Define the inverse Fourier transforms

$$I_1(t_c) = \frac{1}{2\pi}\int_{-\Delta\omega}^{\Delta\omega} d\omega_d I_1(\omega_d) exp(-i\omega_d t_c) \tag{4.71}$$

$$I_2(t_c) = \frac{1}{2\pi}\int_{-\Delta\omega}^{\Delta\omega} d\omega_d I_2(\omega_d) exp(-i\omega_d t_c) \tag{4.72}$$

$$I_3(t_c) = \frac{1}{2\pi}\int_{-\Delta\omega}^{\Delta\omega} d\omega_d I_3(\omega_d) exp(-i\omega_d t_c) \tag{4.73}$$

The coherent term $I_1(\omega_d)$ is

$$I_1(\omega_d) = \frac{2A_0^2\omega_0^2 exp(-2\tau_0)}{\sqrt{2\pi}\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2\, exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} + i\frac{\omega_d}{c}(z_1 + z_2)\right)$$

$$\times exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1 - z_2)^2 + i2\frac{\omega_0}{c}(z_1 - z_2)\right) \tag{4.74}$$

The mixed term $I_2(\omega_d)$ is

$$I_2(\omega_d) = \frac{8A_0^2\omega_0^2 exp(-\tau_0)}{\sqrt{2\pi}\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2\, exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} + i\frac{\omega_d}{c}(z_1 + z_2)\right)$$

$$\times \left(exp\left(-H(\bar{r}_d, \omega_d)\right) - exp(-\tau_0)\right)$$

$$\times\, exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1-z_2)^2+i2\frac{\omega_0}{c}(z_1-z_2)\right) \tag{4.75}$$

The incoherent term $I_3(\omega_d)$ is

$$I_3(\omega_d)=\frac{4A_0^2\omega_0^2}{\sqrt{2\pi}\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 exp\left(-\frac{\omega_d^2}{2(\Delta f)^2}+i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times\left(exp\left(-H(\bar{r}_d,\omega_d)\right)-exp(-\tau_0)\right)^2$$

$$\times\, exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1-z_2)^2+i2\frac{\omega_0}{c}(z_1-z_2)\right) \tag{4.76}$$

It is worth mentioning here that our theory offers two distinct advantages compared to the other numerical methods, such as the parabolic equation (PE) method and the Monte Carlo (MC) method. First of all, our theory provides a more clearly physical picture. As shown in (4.61) and (4.71)-(4.76), the total time-domain RCS contains three terms and each term is corresponding to a unique scattering mechanism. By comparing the contributions from different terms, we could have a better physical understanding of the effects of the random medium on wave propagation. In contrast, the numerical methods in general only provide the total response and it may not be possible to separate the contributions from different mechanisms. Secondly, our theory is computational more efficient. In our theory, the statistical moments are calculated analytically and it does not need to run multiple realizations to get the time-domain RCS. On the other hand, averaging over many realizations are required for the numerical methods in order to achieve statistical convergence.

4.4 Conventional Radar Cross Section in Time Domain

If the correlation between the incident field and the scattered field is neglected, then the time-domain monostatic radar cross section (RCS) becomes

$$RCS(t_c) = I_1(t_c) + I_2(t_c) + I_3(t_c) \tag{4.77}$$

The coherent term $I_1(t_c)$ is

$$I_1(t_c) = \frac{A_0^2\omega_0^2 exp(-2\tau_0)}{\sqrt{2\pi}\pi\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times\, exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1-z_2)^2 + i2\frac{\omega_0}{c}(z_1-z_2)\right) \tag{4.78}$$

The mixed term $I_2(t_c)$ is

$$I_2(t_c) = \frac{2A_0^2\omega_0^2 exp(-\tau_0)}{\sqrt{2\pi}\pi\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \left(exp\big(-H(\bar{r}_d,\omega_d)\big) - exp(-\tau_0)\right)$$

$$\times\, exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1-z_2)^2 + i2\frac{\omega_0}{c}(z_1-z_2)\right) \tag{4.79}$$

The incoherent term is

$$I_3(t_c) = \frac{A_0^2\omega_0^2}{\sqrt{2\pi}\pi\Delta f c^2}\iint d\hat{\rho}_1 d\hat{\rho}_2 \int_{-\Delta\omega}^{\Delta\omega} d\omega_d exp\left(-\frac{\omega_d^2}{2(\Delta f)^2} - i\omega_d t_c + i\frac{\omega_d}{c}(z_1+z_2)\right)$$

$$\times \left(exp\big(-H(\bar{r}_d,\omega_d)\big) - exp(-\tau_0)\right)^2$$

$$\times\, exp\left(-\frac{(\Delta f)^2}{2c^2}(z_1-z_2)^2 + i2\frac{\omega_0}{c}(z_1-z_2)\right) \tag{4.80}$$

By comparing equations (4.68) - (4.70) with (4.77) - (4.79), it can be observed that a factor of 2 enhancement for both mixed term and incoherent term is missing in the conventional time-domain RCS expression due to the assumption that no correlations exist between the incident field and the scattered field.

4.5 Numerical Integration Using Fast Fourier Transform

As indicated in equations (4.71) - (4.73), the time-domain RCS can be calculated by taking the inverse Fourier transforms of the corresponding frequency-domain terms. It is convenient to apply the Fast Fourier Transform (FFT) algorithm to speed up the computation. Since most of the FFT routines are implemented based on equations using the engineering convention $exp(j\omega t)$ rather than the physics convention $exp(-i\omega t)$, our will change the conventions from physics to engineering in the following discussions. The switching between two conventions can be done by substituting $i \rightarrow -j$ in the expressions.

The inverse Fourier transform is defined as

$$f(t) = \frac{1}{2\pi}\int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} F(\omega)exp(j\omega t)d\omega = \int_{-\frac{B_W}{2}}^{\frac{B_W}{2}} F(f)exp(j2\pi ft)df \qquad (4.81)$$

Where $B_W = \frac{\Omega}{2\pi}$ is the bandwidth.

The integral can be approximated by a Riemann sum

$$f(n\Delta t) \approx \Delta f \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} F(k\Delta f)exp(j2\pi kn\Delta f\Delta t)$$

$$= \Delta f \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} F(k\Delta f)exp\left(j\frac{2\pi}{N}kn\right) \qquad (4.82)$$

The discrete inverse Fourier transform is defined as

$$x[n] = \mathcal{F}^{-1}\{X[k]\} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]exp\left(j\frac{2\pi}{N}kn\right) \qquad (4.83)$$

Combining (4.82) and (4.83) gives

$$f(t) \approx N\Delta f\left(\frac{1}{N}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} F(k\Delta f)exp\left(j\frac{2\pi}{N}kn\right)\right) = B_W\mathcal{F}^{-1}\{F[f]\} \qquad (4.84)$$

where the N periodic prosperity of $X[k]$ is applied.

Chapter 5 Time-Domain Radar Cross Section Based on Strong Fluctuation Theory

The time-domain RCS can also be derived using the strong fluctuation theory when the optical depth of the random media is large and the scattering effect is dominant over the absorption effect. In this chapter, we derive the time-domain RCS by following the procedure given in [42]. To minimize the confusion, we will use the same notations used in [42]. The relationship with our previous equations is described when necessary.

5.1 Differential Equation for the Two-frequency Mutual Coherence Function

We consider the discrete random medium composed of scatters whose characteristics dimensions are much larger than the carrier wavelength of the imaging system. Under this assumption, most of the scattering power will concentrate in the forward region of the scatters and the parabolic equation approximation is applied.

If the z axis is chosen as the direction of propagation, the parabolic differential equation for the two frequency MCF can be written as

$$\left(\frac{\partial}{\partial z}-\frac{i}{2}\left(\frac{\nabla_1^2}{k_1}-\frac{\nabla_2^2}{k_2}\right)-i(k_1-k_2)+P(\bar{\rho}_1-\bar{\rho}_2)\right)\Gamma_2(\bar{\rho}_1,\bar{\rho}_2,z,\omega_1,\omega_2)=0 \quad (5.1)$$

where $k_1=\frac{\omega_1}{c}$ and $k_2=\frac{\omega_2}{c}$ are wavenumbers. c is the light speed in the background medium. ∇_1^2 and ∇_2^2 the Laplacians in the transverse plane with respect to $\bar{\rho}_1$ and $\bar{\rho}_2$ where $\bar{\rho}_1=x_1\hat{x}+y_1\hat{y}$ and $\bar{\rho}_2=x_2\hat{x}+y_2\hat{y}$.

The P function contains the random medium effects and is given by

$$P(r_d)=a+b\frac{\int_0^\infty \beta_s(s)(1-J_0(ks\rho_d))sds}{\int_0^\infty \beta_s(s)sds} \quad (5.2)$$

where $\rho_d = |\bar{\rho}_1 - \bar{\rho}_2|$, a is the absorption coefficient, b is the scattering coefficient and β_s is the volume scattering function.

The scattering coefficient b and the volume scattering function β_s are related

$$b = 2\pi \int_0^\infty \beta_s(s) s ds \tag{5.3}$$

where $s = 2sin\left(\frac{\theta}{2}\right)$ and θ is the scattering angle.

The parabolic approximation of the spherical wave due to a point source located at the origin is

$$U(\bar{\rho}, z, \omega) = \frac{1}{z} exp\left(i\frac{k\rho^2}{2z} + ikz\right) = U_0(\bar{\rho}, z, \omega) exp(ikz) \tag{5.4}$$

It is convenient to define function

$$\Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, z, \omega_1, \omega_2) = \langle U(\bar{\rho}_1, z, \omega_1) U^*(\bar{\rho}_2, z, \omega_2)\rangle$$

$$= \frac{1}{z^2} exp\left(i(k_1 - k_2)z + i\frac{k_1\rho_1^2}{2z} - i\frac{k_2\rho_2^2}{2z}\right) \tag{5.5}$$

The function Γ_2^0 is the two-frequency MCF for a spherical wave in background medium and satisfies the parabolic differential equation

$$\left(\frac{\partial}{\partial z} - \frac{i}{2}\left(\frac{\nabla_1^2}{k_1} - \frac{\nabla_2^2}{k_2}\right) - i(k_1 - k_2) + P(\rho_d)\right)\Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, z, \omega_1, \omega_2) = 0 \tag{5.6}$$

We let the two-frequency MCF take the form

$$\Gamma_2(\bar{\rho}_1, \bar{\rho}_2, z, \omega_1, \omega_2) = \Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, z, \omega_1, \omega_2)\Gamma_s(z, \bar{\rho}_d, \omega_d) \tag{5.7}$$

where $\bar{\rho}_d = \bar{\rho}_1 - \bar{\rho}_2$ and $\omega_d = \omega_1 - \omega_2$

For the free-space where no scatters are in the presence, we have $\Gamma_2 = \Gamma_2^0$ and $\Gamma_s = 1$.

Therefore, the boundary condition for Γ_s at the interface between the free-space and random medium should be $\Gamma_s = 1$ at $z = 0$.

Applying the vector calculus identity

$$\nabla^2(\phi\psi) = \psi\nabla^2\phi + 2\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi \tag{5.8}$$

Combining equations (5.7) and (5.8) gives

$$\nabla_1^2\Gamma_2 = \nabla_1^2(\Gamma_2^0\Gamma_s) = \Gamma_s\nabla_1^2\Gamma_2^0 + 2\nabla_1\Gamma_2^0\cdot\nabla_1\Gamma_s + \Gamma_2^0\nabla_1^2\Gamma_s \tag{5.9a}$$

$$\nabla_2^2\Gamma_2 = \nabla_2^2(\Gamma_2^0\Gamma_s) = \Gamma_s\nabla_2^2\Gamma_2^0 + 2\nabla_2\Gamma_2^0\cdot\nabla_2\Gamma_s + \Gamma_2^0\nabla_2^2\Gamma_s \tag{5.9b}$$

Combining equations (5.1) and (5.9) gives

$$\left(\frac{\nabla_1^2}{k_1} - \frac{\nabla_2^2}{k_2}\right)\Gamma_2^0\Gamma_s = \Gamma_s\left(\frac{1}{k_1}\nabla_1^2\Gamma_2^0 - \frac{1}{k_2}\nabla_2^2\Gamma_2^0\right) + 2\left(\frac{1}{k_1}\nabla_1\Gamma_2^0\cdot\nabla_1\Gamma_s - \frac{1}{k_2}\nabla_2\Gamma_2^0\cdot\nabla_2\Gamma_s\right)$$

$$+\Gamma_2^0\left(\frac{1}{k_1}\nabla_1^2\Gamma_s - \frac{1}{k_2}\nabla_2^2\Gamma_s\right) \tag{5.10}$$

Combining equations (5.1), (5.7) and (5.10) gives

$$\left(\frac{\partial}{\partial z} - \frac{i}{2}\left(\frac{\nabla_1^2}{k_1} - \frac{\nabla_2^2}{k_2}\right) - i(k_1 - k_2) + P(\rho_d)\right)\Gamma_2^0\Gamma_s = 0$$

$$\Gamma_2^0\frac{\partial\Gamma_s}{\partial z} - i\left(\frac{1}{k_1}\nabla_1\Gamma_2^0\cdot\nabla_1\Gamma_s - \frac{1}{k_2}\nabla_2\Gamma_2^0\cdot\nabla_2\Gamma_s\right) - \frac{i}{2}\left(\frac{1}{k_1}\nabla_1^2\Gamma_s - \frac{1}{k_2}\nabla_2^2\Gamma_s\right)$$

$$+P(\bar{\rho}_1 - \bar{\rho}_2)\Gamma_2^0\Gamma_s = 0 \tag{5.11}$$

Combining equations (5.5) and (5.7) gives

$$\nabla_1\Gamma_2^0\cdot\nabla_1\Gamma_s = \left(\hat{x}i\frac{k_1x_1}{z}\Gamma_2^0 + \hat{y}i\frac{k_2y_1}{z}\Gamma_2^0\right)\cdot\left(\hat{x}\frac{\partial\Gamma_s}{\partial x_1} + \hat{y}\frac{\partial\Gamma_s}{\partial y_1}\right)$$

$$= i\frac{k_1 x_1}{z}\Gamma_2^0\frac{\partial\Gamma_s}{\partial x_1} + i\frac{k_2 y_1}{z}\Gamma_2^0\frac{\partial\Gamma_s}{\partial x_1} \tag{5.12}$$

Combining equations (5.5) and (5.7) gives

$$\nabla_2\Gamma_2^0 = \left(\hat{x}\frac{\partial}{\partial x_2} + \hat{y}\frac{\partial}{\partial y_2}\right)\left[\frac{1}{z^2}exp\left(i(k_1 - k_2)z + i\frac{k_1\left(x_1^2+y_1^2\right)}{2z} - i\frac{k_2\left(x_2^2+y_2^2\right)}{2z}\right)\right]$$

$$= \hat{x}\Gamma_2^0\frac{\partial}{\partial x_2}\left(-i\frac{k_2x_2^2+k_2y_2^2}{2z}\right) + \hat{y}\Gamma_2^0\frac{\partial}{\partial y_2}\left(-i\frac{k_2x_2^2+k_2y_2^2}{2z}\right)$$

$$= -\hat{x}i\frac{k_2x_2}{z}\Gamma_2^0 - \hat{y}i\frac{k_2y_2}{z}\Gamma_2^0 \tag{5.13}$$

Combining equations (5.5) and (5.7) gives

$$\nabla_2\Gamma_2^0\cdot\nabla_2\Gamma_s = \left(-\hat{x}i\frac{k_2x_2}{z}\Gamma_2^0 - \hat{y}i\frac{k_2y_2}{z}\Gamma_2^0\right)\cdot\left(\hat{x}\frac{\partial\Gamma_s}{\partial x_2} + \hat{y}\frac{\partial\Gamma_s}{\partial y_2}\right)$$

$$= -i\frac{k_2x_2}{z}\Gamma_2^0\frac{\partial\Gamma_s}{\partial x_2} - i\frac{k_2y_2}{z}\Gamma_2^0\frac{\partial\Gamma_s}{\partial y_2} \tag{5.14}$$

Combining equations (5.11), (5.12) and (5.14) gives

$$\frac{1}{k_1}\nabla_1\Gamma_2^0\cdot\nabla_1\Gamma_s - \frac{1}{k_2}\nabla_2\Gamma_2^0\cdot\nabla_2\Gamma_s$$

$$= \frac{1}{k_1}\left(i\frac{k_1x_1}{z}\Gamma_2^0\frac{\partial\Gamma_s}{\partial x_1} + i\frac{k_2y_1}{z}\Gamma_2^0\frac{\partial\Gamma_s}{\partial x_1}\right) - \frac{1}{k_2}\left(-i\frac{k_2x_2}{z}\Gamma_2^0\frac{\partial\Gamma_s}{\partial x_2} - i\frac{k_2y_2}{z}\Gamma_2^0\frac{\partial\Gamma_s}{\partial y_2}\right)$$

$$= i\Gamma_2^0\left(\frac{x_1}{z}\frac{\partial\Gamma_s}{\partial x_1} + i\frac{y_1}{z}\frac{\partial\Gamma_s}{\partial y_1}\right) + i\Gamma_2^0\left(\frac{x_2}{z}\frac{\partial\Gamma_s}{\partial x_2} + i\frac{y_2}{z}\frac{\partial\Gamma_s}{\partial y_2}\right) \tag{5.15}$$

Combining equations (5.9) and (5.15) gives

$$\Gamma_2^0\frac{\partial\Gamma_s}{\partial z} - i\left(i\Gamma_2^0\left(\frac{x_1}{z}\frac{\partial\Gamma_s}{\partial x_1} + i\frac{y_1}{z}\frac{\partial\Gamma_s}{\partial y_1}\right) + i\Gamma_2^0\left(\frac{x_2}{z}\frac{\partial\Gamma_s}{\partial x_2} + i\frac{y_2}{z}\frac{\partial\Gamma_s}{\partial y_2}\right)\right)$$

$$-\frac{i}{2}\left(\frac{1}{k_1}\nabla_1^2\Gamma_s - \frac{1}{k_2}\nabla_2^2\Gamma_s\right) + P(\bar{\rho}_1 - \bar{\rho}_2)\Gamma_2^0\Gamma_s = 0$$

$$\frac{\partial \Gamma_s}{\partial z} + \frac{1}{z}\left(x_1 \frac{\partial \Gamma_s}{\partial x_1} + y_1 \frac{\partial \Gamma_s}{\partial y_1} + x_2 \frac{\partial \Gamma_s}{\partial x_2} + y_2 \frac{\partial \Gamma_s}{\partial y_2}\right) - \frac{i}{2}\left(\frac{1}{k_1}\nabla_1^2 \Gamma_s - \frac{1}{k_2}\nabla_2^2 \Gamma_s\right)$$

$$+P(\bar{\rho}_1 - \bar{\rho}_2)\Gamma_s = 0 \qquad (5.16)$$

It is convenient to make the change of variables

$$\bar{\rho}_c = \frac{1}{2}(\bar{\rho}_1 + \bar{\rho}_2), \; \bar{\rho}_d = \bar{\rho}_1 - \bar{\rho}_2 \qquad (5.17)$$

The partial derivatives can be written as

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_c}\frac{\partial x_c}{\partial x_1} + \frac{\partial}{\partial x_d}\frac{\partial x_d}{\partial x_1} = \frac{1}{2}\frac{\partial}{\partial x_c} + \frac{\partial}{\partial x_d} \qquad (5.18a)$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_c}\frac{\partial x_c}{\partial x_2} + \frac{\partial}{\partial x_d}\frac{\partial x_d}{\partial x_1} = \frac{1}{2}\frac{\partial}{\partial x_c} - \frac{\partial}{\partial x_d} \qquad (5.18b)$$

$$\frac{\partial}{\partial y_1} = \frac{\partial}{\partial y_c}\frac{\partial y_c}{\partial y_1} + \frac{\partial}{\partial y_d}\frac{\partial y_d}{\partial y_1} = \frac{1}{2}\frac{\partial}{\partial y_c} + \frac{\partial}{\partial y_d} \qquad (5.18c)$$

$$\frac{\partial}{\partial y_2} = \frac{\partial}{\partial y_c}\frac{\partial y_c}{\partial y_2} + \frac{\partial}{\partial y_d}\frac{\partial y_d}{\partial x_1} = \frac{1}{2}\frac{\partial}{\partial y_c} - \frac{\partial}{\partial y_d} \qquad (5.18d)$$

Combining equations (5.16) and (5.18) gives

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} = \left(x_c + \frac{x_d}{2}\right)\left(\frac{1}{2}\frac{\partial}{\partial x_c} + \frac{\partial}{\partial x_d}\right) + \left(x_c - \frac{x_d}{2}\right)\left(\frac{1}{2}\frac{\partial}{\partial x_c} - \frac{\partial}{\partial x_d}\right)$$

$$= x_c \frac{\partial}{\partial x_c} + x_d \frac{\partial}{\partial x_d} \qquad (5.19a)$$

$$y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} = \left(y_c + \frac{y_d}{2}\right)\left(\frac{1}{2}\frac{\partial}{\partial y_c} + \frac{\partial}{\partial y_d}\right) + \left(y_c - \frac{y_d}{2}\right)\left(\frac{1}{2}\frac{\partial}{\partial y_c} - \frac{\partial}{\partial y_d}\right)$$

$$= y_c \frac{\partial}{\partial y_c} + y_d \frac{\partial}{\partial y_d} \qquad (5.19b)$$

$$\nabla_1 \cdot \nabla_1 = \left(\hat{x}\frac{\partial}{\partial x_1} + \hat{y}\frac{\partial}{\partial y_1}\right) \cdot \left(\hat{x}\frac{\partial}{\partial x_1} + \hat{y}\frac{\partial}{\partial y_1}\right)$$

$$= \left(\hat{x}\left(\frac{1}{2}\frac{\partial}{\partial x_c} + \frac{\partial}{\partial x_d}\right) + \hat{y}\left(\frac{1}{2}\frac{\partial}{\partial y_c} + \frac{\partial}{\partial y_d}\right)\right) \cdot \left(\hat{x}\left(\frac{1}{2}\frac{\partial}{\partial x_c} + \frac{\partial}{\partial x_d}\right) + \hat{y}\left(\frac{1}{2}\frac{\partial}{\partial y_c} + \frac{\partial}{\partial y_d}\right)\right)$$

$$= \left(\frac{1}{2}\frac{\partial}{\partial x_c} + \frac{\partial}{\partial x_d}\right)^2 + \left(\frac{1}{2}\frac{\partial}{\partial y_c} + \frac{\partial}{\partial y_d}\right)^2 \tag{5.19c}$$

$$\nabla_2 \cdot \nabla_2 = \left(\hat{x}\frac{\partial}{\partial x_2} + \hat{y}\frac{\partial}{\partial y_2}\right) \cdot \left(\hat{x}\frac{\partial}{\partial x_2} + \hat{y}\frac{\partial}{\partial y_2}\right)$$

$$= \left(\hat{x}\left(\frac{1}{2}\frac{\partial}{\partial x_c} - \frac{\partial}{\partial x_d}\right) + \hat{y}\left(\frac{1}{2}\frac{\partial}{\partial y_c} - \frac{\partial}{\partial y_d}\right)\right) \cdot \left(\hat{x}\left(\frac{1}{2}\frac{\partial}{\partial x_c} - \frac{\partial}{\partial x_d}\right) + \hat{y}\left(\frac{1}{2}\frac{\partial}{\partial y_c} - \frac{\partial}{\partial y_d}\right)\right)$$

$$= \left(\frac{1}{2}\frac{\partial}{\partial x_c} - \frac{\partial}{\partial x_d}\right)^2 + \left(\frac{1}{2}\frac{\partial}{\partial y_c} - \frac{\partial}{\partial y_d}\right)^2 \tag{5.19d}$$

Combining equations (5.16) and (5.19) gives

$$x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} = x_c\frac{\partial}{\partial x_c} + x_d\frac{\partial}{\partial x_d} + y_c\frac{\partial}{\partial y_c} + y_d\frac{\partial}{\partial y_d}$$

$$= \bar{\rho}_c \cdot \nabla_c + \bar{\rho}_d \cdot \nabla_d \tag{5.20a}$$

$$\frac{1}{k_1}\nabla_1^2 - \frac{1}{k_2}\nabla_2^2 = \frac{1}{k_1}\left[\left(\frac{1}{2}\frac{\partial}{\partial x_c} + \frac{\partial}{\partial x_d}\right)^2 + \left(\frac{1}{2}\frac{\partial}{\partial y_c} + \frac{\partial}{\partial y_d}\right)^2\right]$$

$$-\frac{1}{k_2}\left[\left(\frac{1}{2}\frac{\partial}{\partial x_c} - \frac{\partial}{\partial x_d}\right)^2 + \left(\frac{1}{2}\frac{\partial}{\partial y_c} - \frac{\partial}{\partial y_d}\right)^2\right] \tag{5.20b}$$

For statistically homogeneous and isotropic random medium, the function Γ_s should be independent of variable $\bar{\rho}_c$

$$\frac{\partial \Gamma_s}{\partial x_c} = \frac{\partial \Gamma_s}{\partial y_c} = 0 \tag{5.21}$$

Combining equations (5.20) and (5.21) gives

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial \Gamma_s}{\partial x_2} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right)\Gamma_s = \bar{\rho}_d \cdot \nabla_d \Gamma_s \tag{5.22a}$$

$$\left(\frac{1}{k_1}\nabla_1^2 - \frac{1}{k_2}\nabla_2^2\right)\Gamma_s = \left(\frac{1}{k_1}\nabla_d^2 - \frac{1}{k_2}\nabla_d^2\right)\Gamma_s \tag{5.22b}$$

Combining (5.16) and (5.22) gives

$$\left(\frac{\partial}{\partial z} + \frac{1}{z}\bar{\rho}_d \cdot \nabla_d - \frac{i}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\nabla_d^2 + P(\bar{\rho}_d)\right)\Gamma_s(z, \bar{\rho}_d, \omega_d) = 0 \tag{5.23}$$

The asymptotic forms for Γ_s as $\rho_d \to 0$ and $\rho_d \to \infty$ are

$$\lim_{\rho_d \to 0}\{\Gamma_s(z, \rho_d)\} = exp(-az) \tag{5.24a}$$

$$\lim_{\rho_d \to \infty}\{\Gamma_s(z, \rho_d)\} = exp(-(a+b)z) \tag{5.24b}$$

where a is the absorption coefficient and b is the scattering coefficient.

5.2 Gaussian Volume Scattering Function

We assume the volume scattering function can be approximated by a Gaussian function

$$\beta(s) = b\frac{\gamma}{\pi}exp(-\gamma s^2) \tag{5.24}$$

where $s = 2sin\left(\frac{\theta}{2}\right)$ and θ is the scattering angle.

The Gaussian function satisfies the normalization condition

$$2\pi\int_0^\infty \beta(s)sds = b\int_0^\infty exp(-\gamma s^2)d(\gamma s^2) = b \tag{5.25}$$

Combining (5.2), (5.24) and (5.25) gives

$$\int_0^\infty \beta(s)\big(1 - J_0(ks\rho_d)\big)sds = \int_0^\infty \beta(s)sds - \int_0^\infty \beta(s)J_0(ks\rho_d)sds$$

$$= \frac{b}{2\pi} - \frac{b\gamma}{\pi}\int_0^\infty exp(-\gamma s^2)J_0(ks\rho_d)sds \tag{5.26}$$

Applying the integral identity

$$\int_0^\infty J_v(at)exp(-p^2t^2)t^{v+1}dt = \frac{a^v}{(2p^2)^{v+1}}exp\left(-\frac{a^2}{4p^2}\right) \tag{5.27}$$

We obtain

$$\int_0^\infty \beta(s)\big(1-J_0(ks\rho_d)\big)sds = \frac{b}{2\pi} - \frac{b\gamma}{\pi}\int_0^\infty exp(-\gamma s^2)J_0(ks\rho_d)sds$$

$$= \frac{b}{2\pi} - \frac{b}{2\pi}exp\left(-\frac{k^2\rho_d^2}{4\gamma}\right) \tag{5.28}$$

Combining (5.2), (5.25) and (5.28) gives

$$P(\rho_d) = a + b\frac{\int_0^\infty \beta_s(s)\big(1-J_0(ksr_d)\big)sds}{\int_0^\infty \beta_s(s)sds} = a + b\left(1 - exp\left(-\frac{k^2\rho_d^2}{4\gamma}\right)\right) \tag{5.29}$$

Since the scattering pattern $\beta(s)$ is peaked in the forward direction, then

$$P(\rho_d) \approx a + b\frac{k^2\rho_d^2}{4\gamma}, \gamma \gg 1 \tag{5.30}$$

Combining (5.23) and (5.30) gives

$$\frac{\partial\Gamma_s}{\partial z} + \frac{1}{z}\bar{\rho}_d\cdot\nabla_d\Gamma_s - \frac{i}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\nabla_d^2\Gamma_s + \left(a + b\frac{k^2\rho_d^2}{4\gamma}\right)\Gamma_s = 0 \tag{5.31}$$

Let us assume the function Γ_s takes the form

$$\Gamma_s(z,\bar{\rho}_d,\omega_d) = \Gamma_{s0}(z)\Gamma_{s1}(z,\bar{\rho}_d,\omega_d) = exp(-az)\Gamma_{s1}(z,\bar{\rho}_d,\omega_d) \tag{5.32}$$

Combining (5.31) and (5.32)

$$\left(\frac{\partial}{\partial z} + \frac{1}{z}\bar{\rho}_d\cdot\nabla_d - \frac{i}{2}\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\nabla_d^2 + b\frac{k^2\rho_d^2}{4\gamma}\right)\Gamma_{s1}(z,\bar{\rho}_d,\omega_d) = 0 \tag{5.33}$$

We have the differential equation for the function Γ_{s1}. However, it is noticed that equation (5.32) does not predict the correct asymptotic form for Γ_s as the separation ρ_d approaches infinite

$$\lim_{\rho_d \to \infty} \{\Gamma_s(z, \rho_d)\} = exp(-az - bz) \tag{5.34}$$

In order to get the correct asymptotic forms given by (5.24), we use the approximate formula

$$\Gamma_s(z, \rho_d) = exp(-\tau_0 z) + exp(-\tau_a)\Gamma_{s1}(z, \rho_d)\big(1 - exp(-\tau_s)\big) \tag{5.35}$$

where $\tau_0 = (a + b)z$ is the optical depth, $\tau_a = az$ is the optical absorption depth and $\tau_s = bz$ is the optical scattering depth.

The Γ_{s1} has the correct asymptotic forms

$$\lim_{\rho_d \to 0} \{\Gamma_{s1}(z, \rho_d)\} = 1 \tag{5.36a}$$

$$\lim_{\rho_d \to \infty} \{\Gamma_{s1}(z, \rho_d)\} = 0 \tag{5.36b}$$

It is convenient to define the normalized distance $z' = \frac{z}{L}$.

Thus we have

$$\left(\frac{\partial}{\partial z'} + \frac{1}{z'}\bar{\rho}_d \cdot \nabla_d + \frac{i}{2}\frac{k_d L}{k^2}\nabla_d^2 + bL\frac{k^2 \rho_d^2}{4\gamma}\right)\Gamma_{s1}(z, \bar{\rho}_d, \omega_d) = 0 \tag{5.37}$$

where $k_d = k_1 - k_2$ and k is the wavenumber at carrier frequency.

For statistically homogenous and isotropic random medium, it prefers to use the cylindrical coordinates because of the azimuthal symmetry.

Making change of variable

$$\rho_d' = \frac{\rho_d}{\rho_0} \tag{5.38}$$

where $\rho_0 = \frac{2}{k}\sqrt{\frac{\gamma}{bL}}$.

Combining (5.37) and (5.38) gives

$$\nabla_d = \hat{\rho}\frac{\partial}{\partial\rho_d} = \hat{\rho}\frac{\partial}{\partial\rho'_d}\frac{\partial\rho'_d}{\partial\rho_d} = \hat{\rho}\frac{1}{\rho_0}\frac{\partial}{\partial\rho'_d} \tag{5.39a}$$

$$\bar{\rho}_d \cdot \nabla_d = (\hat{\rho}\rho'_d\rho_0)\cdot\left(\hat{\rho}\frac{1}{\rho_0}\frac{\partial}{\partial\rho'_d}\right) = \rho'_d\frac{\partial}{\partial\rho'_d} \tag{5.39b}$$

$$\nabla_d^2 = \frac{1}{\rho_d}\frac{\partial}{\partial\rho_d}\left(\rho_d\frac{\partial}{\partial\rho_d}\right) = \frac{1}{\rho_0^2\rho'_d}\frac{\partial}{\partial\rho'_d}\left(\rho'_d\frac{\partial}{\partial\rho'_d}\right) \tag{5.39c}$$

Combining (5.37) and (5.39) gives

$$\left(\frac{\partial}{\partial z'} + \frac{\rho'_d}{z'}\frac{\partial}{\partial\rho'_d} + i\frac{k_d L}{2k^2\rho_0^2}\left(\frac{1}{\rho'_d}\frac{\partial}{\partial\rho'_d}\left(\rho'_d\frac{\partial}{\partial\rho'_d}\right)\right) + \rho_d'^2\right)\Gamma_{s1}(z',\rho'_d,\omega_d) = 0 \tag{5.40}$$

It is convenient to define variables

$$\alpha = \frac{k_d}{k_{coh}} = \frac{\omega_d}{\omega_{coh}} \tag{5.41a}$$

$$k_{coh} = \frac{2k^2\rho_0^2}{L} \tag{5.41b}$$

$$\omega_{coh} = k_{coh}c = 2k^2\rho_0^2\frac{c}{L} \tag{5.41c}$$

where c is the free-space light speed.

Combining (5.40) and (5.41) gives

$$\left(\frac{\partial}{\partial z'} + \frac{\rho'_d}{z'}\frac{\partial}{\partial\rho'_d} + i\alpha\frac{1}{\rho'_d}\frac{\partial}{\partial\rho'_d}\left(\rho'_d\frac{\partial}{\partial\rho'_d}\right) + \rho_d'^2\right)\Gamma_{s1}(z',\rho'_d,\omega_d) = 0 \tag{5.42}$$

The boundary condition is given by

$$\Gamma_{s1}(z'=0,\rho'_d,\omega_d) = 1 \tag{5.43}$$

We let the function Γ_{s1} take the form

$$\Gamma_{s1}(z',\rho'_d,\omega_d)=\frac{exp\left(g(z')\rho'^2_d\right)}{f(z')} \tag{5.44}$$

Combining (5.40) and (5.44) gives

$$\frac{\partial}{\partial z'}\left(\frac{exp\left(g(z')\rho'^2_d\right)}{f(z')}\right)=\frac{f(z')g'(z')\rho'^2_d exp\left(g(z')\rho'^2_d\right)-f'(z')exp\left(g(z')\rho'^2_d\right)}{f^2(z')}$$

$$=exp\left(g(z')\rho'^2_d\right)\frac{f(z')g'(z')\rho'^2_d-f'(z')}{f^2(z')} \tag{5.45a}$$

$$\frac{\partial}{\partial\rho'_d}\left(\frac{exp\left(g(z')\rho'^2_d\right)}{f(z')}\right)=\frac{2g(z')}{f(z')}\rho'_d exp\left(g(z')\rho'^2_d\right) \tag{5.45b}$$

$$\frac{\rho'_d}{z'}\frac{\partial}{\partial\rho'_d}\left(\frac{exp\left(g(z')\rho'^2_d\right)}{f(z')}\right)=\frac{2g(z')}{z'f(z')}\rho'^2_d exp\left(g(z')\rho'^2_d\right) \tag{5.45c}$$

$$\frac{1}{\rho'_d}\frac{\partial}{\partial\rho'_d}\left(\rho'_d\frac{\partial}{\partial\rho'_d}\left(\frac{exp\left(g(z')\rho'^2_d\right)}{f(z')}\right)\right)=\frac{1}{\rho'_d}\frac{\partial}{\partial\rho'_d}\left(\frac{2g(z')}{f(z')}\rho'^2_d exp\left(g(z')\rho'^2_d\right)\right)$$

$$=\frac{2g(z')}{f(z')\rho'_d}\left(2\rho'_d exp\left(g(z')\rho'^2_d\right)+2g(z')\rho'^3_d exp\left(g(z')\rho'^2_d\right)\right)$$

$$=\frac{4g(z')exp\left(g(z')\rho'^2_d\right)}{f(z')}\left(1+g(z')\rho'^2_d\right) \tag{5.45d}$$

Combining (5.40) and (5.45) gives

$$\frac{f(z')g'(z')\rho'^2_d-f'(z')}{f^2(z')}+\frac{2g(z')}{z'f(z')}\rho'^2_d+i\alpha\frac{4g(z')}{f(z')}\left(1+g(z')\rho'^2_d\right)+\frac{\rho'^2_d}{f(z')}=0$$

$$-\frac{f'(z')}{f(z')}+i4\alpha g(z')+\left(g'(z')+\frac{2g(z')}{z'}+i4\alpha g^2(z')+1\right)\rho'^2_d=0 \tag{5.46}$$

It is noticed that first two terms in (5.46) only depend on z' regardless of ρ'_d.

We get two equations

$$-\frac{f'(z')}{f(z')}+i4\alpha g(z')=0 \tag{5.47a}$$

$$g'(z')+\frac{2g(z')}{z'}+i4\alpha g^2(z')+1=0 \tag{5.47b}$$

Combining (5.43) and (5.44) gives

$$g(z'=0)=0 \tag{5.48a}$$

$$f(z'=0)=1 \tag{5.48b}$$

The Riccati equation takes the form

$$\frac{dy}{dx}+P(x)y+Q(x)y^2=R(x) \tag{5.49}$$

The Riccati equation can be reduced to a 2nd order linear ordinary differential equation by substituting

$$y(x)=\frac{u'(x)}{Q(x)u(x)} \tag{5.50}$$

Combining (5.49) and (5.50) gives

$$\frac{d}{dx}\left(\frac{u'(x)}{Q(x)u(x)}\right)+P(x)\left(\frac{u'(x)}{Q(x)u(x)}\right)+Q(x)\left(\frac{u'(x)}{Q(x)u(x)}\right)^2=R(x)$$

$$\frac{u''(x)Q(x)u(x)-u'(x)\big(Q'(x)u(x)+Q(x)u'(x)\big)}{Q^2(x)u^2(x)}+\frac{P(x)u'(x)}{Q(x)u(x)}+\frac{Q(x)u'^2(x)}{Q^2(x)u^2(x)}=R(x)$$

$$u''(x)-\frac{Q'(x)}{Q(x)}u'(x)+P(x)u'(x)=R(x)Q(x)u(x)$$

$$u''(x)+\left(P(x)-\frac{Q'(x)}{Q(x)}\right)u'(x)-R(x)Q(x)u(x)=0 \tag{5.51}$$

Combining (5.47) and (5.51) gives

$$\frac{d^2u(z')}{dz'^2} + \frac{2}{z'}\frac{du(z')}{dz'} + i4\alpha u(z') = 0 \tag{5.52}$$

From ordinary differential equation handbook

$$xy''_{xx} + ay'_x + bxy = 0 \tag{5.53a}$$

$$y = C_1\left(\frac{1}{x}\frac{d}{dx}\right)^n cos\left(x\sqrt{b}\right) + C_2\left(\frac{1}{x}\frac{d}{dx}\right)^n sin\left(x\sqrt{b}\right)$$

$$b > 0,\ a = 2n,\ n = 1, 2,\ \cdots \tag{5.53b}$$

The general solution is

$$u(z') = C_1\frac{1}{z'}\frac{d}{dz'}\left(cos\left(z'\sqrt{i4\alpha}\right)\right) + C_2\frac{1}{z'}\frac{d}{dz'}\left(sin\left(z'\sqrt{i4\alpha}\right)\right)$$

$$= -\sqrt{i4\alpha}C_1\frac{sin\left(z'\sqrt{i4\alpha}\right)}{z'} + \sqrt{i4\alpha}C_2\frac{cos\left(z'\sqrt{i4\alpha}\right)}{z'} \tag{5.54}$$

The general solution is

$$g(z') = \frac{1}{i4\alpha}\frac{u'(z')}{u(z')} \tag{5.55}$$

Combining (5.54) and (5.55) gives

$$u'(z') = \frac{d}{dz'}\left(-\sqrt{i4\alpha}C_1\frac{sin\left(z'\sqrt{i4\alpha}\right)}{z'} + \sqrt{i4\alpha}C_2\frac{cos\left(z'\sqrt{i4\alpha}\right)}{z'}\right)$$

$$= \frac{\sqrt{i4\alpha}}{z'^2}C_1\left(sin\left(z'\sqrt{i4\alpha}\right) - \sqrt{i4\alpha}z'cos\left(z'\sqrt{i4\alpha}\right)\right)$$

$$-\frac{\sqrt{i4\alpha}}{z'^2}C_2\left(cos\left(z'\sqrt{i4\alpha}\right) + \sqrt{i4\alpha}z'sin\left(z'\sqrt{i4\alpha}\right)\right) \tag{5.56a}$$

$$g(z') = \frac{1}{i4\alpha}\frac{u'(z')}{u(z')}$$

$$= \frac{C_1 sin\left(z'\sqrt{i4\alpha}\right) - C_2 cos\left(z'\sqrt{i4\alpha}\right) - C_1\sqrt{i4\alpha}z' cos\left(z'\sqrt{i4\alpha}\right) - C_2\sqrt{i4\alpha}z' sin\left(z'\sqrt{i4\alpha}\right)}{i4\alpha z'\left(-C_1 sin\left(z'\sqrt{i4\alpha}\right) + C_2 cos\left(z'\sqrt{i4\alpha}\right)\right)}$$

$$= -\frac{1}{i4\alpha z'} + \frac{C_1 + C_2 tan\left(z'\sqrt{i4\alpha}\right)}{\sqrt{i4\alpha}\left(C_1 tan\left(z'\sqrt{i4\alpha}\right) - C_2\right)} \tag{5.56b}$$

Combining (5.48) and (5.56) gives

$$\lim_{z'\to 0}\{g(z')\} = 0$$

$$\lim_{z'\to 0}\left\{-\frac{1}{i4\alpha z'} + \frac{C_1 + C_2 z'\sqrt{i4\alpha}}{\sqrt{i4\alpha}\left(C_1 z'\sqrt{i4\alpha} - C_2\right)}\right\} = 0$$

$$C_2 = 0 \tag{5.57}$$

The general solution is

$$g(z') = -\frac{1}{i4\alpha z'} + \frac{cot\left(z'\sqrt{i4\alpha}\right)}{\sqrt{i4\alpha}} \tag{5.58}$$

Combining (5.47) and (5.58) gives

$$\frac{1}{f(z')}\frac{df(z')}{dz'} - i4\alpha\left(-\frac{1}{i4\alpha z'} + \frac{cot\left(z'\sqrt{i4\alpha}\right)}{\sqrt{i4\alpha}}\right) = 0 \tag{5.59}$$

Integrating both sides

$$\int \frac{df(z')}{f(z')} - i4\alpha \int \left(-\frac{1}{i4\alpha z'} + \frac{cot\left(z'\sqrt{i4\alpha}\right)}{\sqrt{i4\alpha}}\right) dz' = 0$$

$$ln\left(f(z')\right) + C + ln(z') - \frac{i4\alpha}{\sqrt{i4\alpha}} \int cot\left(z'\sqrt{i4\alpha}\right) dz' = 0 \tag{5.60}$$

Applying the integral identity

$$\int \frac{cos(ax)}{sin(ax)} dx = \frac{1}{a} ln\left(sin(ax)\right) \tag{5.61}$$

Combining (5.60) and (5.61) gives

$$ln(f(z')z') + C - ln\left(sin\left(\sqrt{i4\alpha}z'\right)\right) = 0$$

$$ln\left(\frac{f(z')z'}{sin\left(\sqrt{i4\alpha}z'\right)}\right) + C = 0 \tag{5.62}$$

Combining (5.48) and (5.62) gives

$$\lim_{z' \to 0}\left\{ln\left(\frac{f(z')z'}{sin\left(\sqrt{i4\alpha}z'\right)}\right)\right\} + C = 0$$

$$\lim_{z' \to 0}\left\{ln\left(\frac{f(z')}{\sqrt{i4\alpha}}\right)\right\} + C = 0$$

$$C = ln\left(\sqrt{i4\alpha}\right) \tag{5.63}$$

Combining (5.62) and (5.63) gives

$$ln\left(\frac{f(z')z'}{sin\left(\sqrt{i4\alpha}z'\right)}\right) + ln\left(\sqrt{i4\alpha}\right) = 0$$

$$f(z') = \frac{sin\left(\sqrt{i4\alpha}z'\right)}{\sqrt{i4\alpha}z'} \tag{5.64}$$

Combining (5.44), (5.58) and (5.64) gives

$$\Gamma_{s1}(z', \rho'_d, \omega_d) = \frac{\sqrt{i4\alpha}z'}{sin\left(\sqrt{i4\alpha}z'\right)} exp\left(\left(-\frac{1}{i4\alpha z'} + \frac{cot\left(z'\sqrt{i4\alpha}\right)}{\sqrt{i4\alpha}}\right)\rho'^2_d\right) \tag{5.65}$$

It is convenient to define variables

$$g_0 = g(z' = 1) = -\frac{1}{i4\alpha} + \frac{cot\left(\sqrt{i4\alpha}\right)}{\sqrt{i4\alpha}} \tag{5.66}$$

$$f_0 = f(z' = 1) = \frac{sin\left(\sqrt{i4\alpha}\right)}{\sqrt{i4\alpha}} \tag{5.67}$$

The limits of $k_d \rightarrow 0$ is

$$\lim_{\alpha\to 0}\{\Gamma_{s1}(z' = 1, \bar{\rho}'_d, \omega_d)\} = \lim_{\alpha\to 0}\left\{\frac{-sin(\sqrt{i4\alpha})+\sqrt{i4\alpha}cos(\sqrt{i4\alpha})}{i4\alpha sin(\sqrt{i4\alpha})}\right\}$$

$$= \frac{-(\sqrt{i4\alpha}-i2/3\alpha\sqrt{i4\alpha})+\sqrt{i4\alpha}(1-i2\alpha)}{i4\alpha\sqrt{i4\alpha}} = -\frac{1}{3} \qquad (5.68)$$

The function is

$$\Gamma_{s1}(z' = 1, \rho'_d, \omega_d) = \frac{exp(g_0\rho'^2_d)}{f_0} \qquad (5.69)$$

Combining (5.35) and (5.68) gives

$$\Gamma_s(z, \rho_d) = exp(-\tau_0 z) + \frac{exp(-\tau_a + g_0\rho'^2_d)}{f_0}\left(1 - exp(-\tau_s)\right) \qquad (5.70)$$

It is convenient to define the center-of-mass and difference coordinates

$$k_c = \frac{k_1+k_2}{2},\ k_d = k_1 - k_2 \qquad (5.71)$$

Combining (5.5) and (5.71) gives

$$k_1\rho_1^2 - k_2\rho_2^2 = 2k_c\bar{\rho}_c \cdot \bar{\rho}_d + k_d\left(\bar{\rho}_c \cdot \bar{\rho}_c + \frac{1}{4}\bar{\rho}_d \cdot \bar{\rho}_d\right) \qquad (5.72)$$

We have

$$k_1 z_1 - k_2 z_2 = k_c(z_1 - z_2) + \frac{1}{2}k_d(z_1 + z_2) \qquad (5.73)$$

Combining (2.117), (2.118) and (2.119) gives

$$\Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, k_1, k_2) = \frac{1}{(4\pi L)^2} exp\left(\frac{i}{2L}(k_1\rho_1^2 - k_2\rho_2^2) + i(k_1 z_1 - k_2 z_2)\right)$$

$$= \frac{1}{(4\pi L)^2} exp\left(ik_c \frac{\bar{\rho}_c \cdot \bar{\rho}_d}{L} + i\frac{k_d}{2L}\left(\bar{\rho}_c \cdot \bar{\rho}_c + \frac{1}{4}\bar{\rho}_d \cdot \bar{\rho}_d\right) + ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2)\right)$$

$$\approx \frac{1}{(4\pi L)^2} exp\left(ik_c(z_1 - z_2) + i\frac{k_d}{2}(z_1 + z_2)\right) \qquad (5.74)$$

The function is

$$\Gamma_2^0(\bar{\rho}_1, \bar{\rho}_2, L, \omega_1, \omega_2) = \langle U(\bar{\rho}_1, L, \omega_1) U^*(\bar{\rho}_2, L, \omega_2)\rangle$$

$$= \frac{1}{(4\pi L)^2} exp\left(i\frac{\omega_c}{c}(z_1 - z_2) + i\frac{\omega_d}{c}(z_1 + z_2)\right) \qquad (5.75)$$

Chapter 6 Time-Domain Radar Cross Section Simulation

6.1 Time-Domain Radar Cross Section of a Dirichlet Plate

6.1.1 Simulation Setup

In this section, we will apply equation (4.77) - (4.80) to calculate the time-domain RCS from a large Dirichlet plate. The geometric configuration is shown in Figure 6.1.

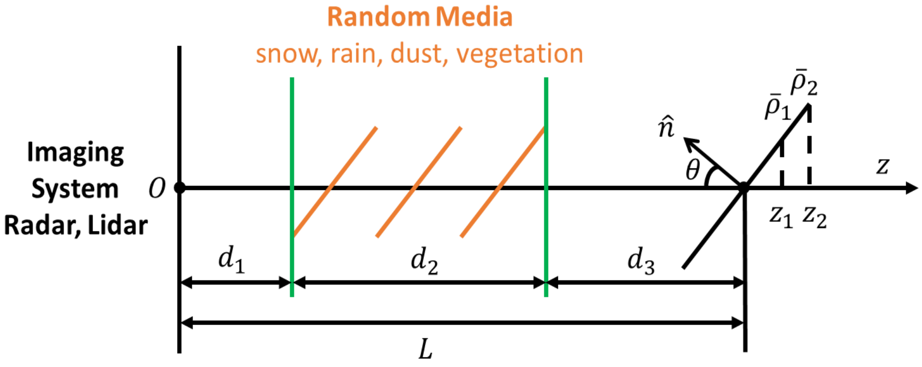

Figure 6.1 The monostatic RCS of an inclined square conducting plate is simulated. The random media with thickness d_2 is located between the imaging system and the plate. The plate size is $2a \times 2a$. The coordinate is $\bar{\rho}_1 = x_1\hat{x} + y_1\hat{y}$ and $\bar{\rho}_2 = x_2\hat{x} + y_2\hat{y}$.

As shown in the figure, for the square conducting plate, we have $z(\bar{\rho}) = xtan(\theta)$. The geometry related terms of the time-domain RCS expression can be written as

$$\iint d\hat{\rho}_1 d\hat{\rho}_2 = \int_{-acos(\theta)}^{acos(\theta)} dx_1 \int_{-a}^{a} dy_1 \int_{-acos(\theta)}^{acos(\theta)} dx_2 \int_{-a}^{a} dy_2 \qquad (6.1a)$$

$$z_1 + z_2 = (x_1 + x_2)tan(\theta) \qquad (6.1b)$$

$$z_1 - z_2 = (x_1 - x_2)tan(\theta) \qquad (6.1c)$$

$$\rho_d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad (6.1d)$$

Here the size of the square plate is $2a \times 2a$. The 2-dimensional surface integral (6.1a) can be evaluated using the numerical quadrature. A sampling rate of 4 points per wavelength should provide good accuracy [12].

It is also noticed that for a large Dirichlet plate, the closed-form expression for the frequency-domain monostatic RCS can be found by using the physical optics method (Kirchhoff approximation). Therefore, we can compare the monostatic RCS given by the time-domain formulas with that predicted from the frequency-domain formulas. This will help us to verify the correctness of the time-domain RCS equations.

6.1.2 Shower Curtain Effects

The shower curtain effect is the degradation of image quality due to the presence of the random media. It is known that the image of a person close to the shower curtain can be seen clearly, whereas a person farther from the shower curtain is difficult to see. The shower curtain effect has been studied extensively in the past [2-3, 8]. Normally, the shower curtain effect is one-way imaging [13]. RCS measurement, however, is the round trip case and it involves the time-domain technique. In this paper the simulations were conducted to show the time-domain shower curtain effect. The degradation of the image quality can be explained by the characteristics of the incoherent component $I_3(t)$. Both the pulse arrival time and pulse broadening increase significantly when the random medium is placed far from the target. This will be discussed in the following section.

6.2 Time-Domain vs. Frequency-Domain Radar Cross Sections

We will calculate both time-domain and frequency domain monostatic RCS of a large Dirichlet plate. The geometric configuration is demonstrated in Figure 6.2.

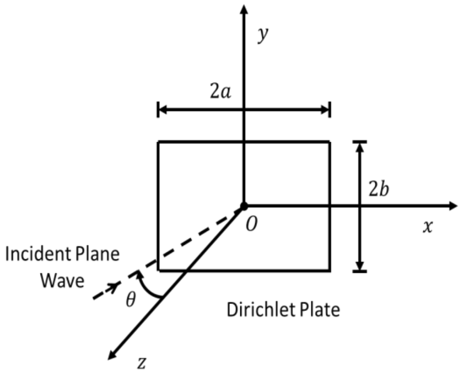

Figure 6.2 A large Dirichlet plate is illuminate by a plane wave source. The plate size is $2a \times 2b$. The incident (scattering) angle is θ.

The simulation parameters are given in the following paragraph. The bandwidth of the transmitted waveform is $\Delta f = 5GHz$ and the carrier frequency is $f_0 = 100GHz$. The surface area of the conducting plate is $2\lambda_0 \times 2\lambda_0$, where λ_0 is the carrier wavelength. As mentioned in the reference [14], $2\lambda \times 2\lambda$ is the minimum surface area of the square plate that the Kirchhoff approximation is applicable.

The frequency-domain monostatic RCS is given by

$$\sigma_b(\theta) = \frac{64\pi a^2 b^2}{\lambda_0^2} cos^2(\theta) \left(\frac{sin(2k_0 asin(\theta))}{2k_0 asin(\theta)}\right)^2 \tag{6.2}$$

where θ is the incident (scattering) angle and $k_0 = \frac{2\pi}{\lambda_0}$ is the carrier wavenumber. All other geometry related symbols are defined in the figure.

The time-domain monostatic RCS is given by

$$\sigma_b(\theta) = peak(|I_1(t, \theta, \tau_0 = 0)|) \tag{6.3}$$

where I_1 is the coherent component when the optical depth is $\tau_0 = 0$ and $peak$ means taking the peak value. By setting the optical depth $\tau_0 = 0$, the coherent component will not be attenuated and the effects of the random medium are excluded.

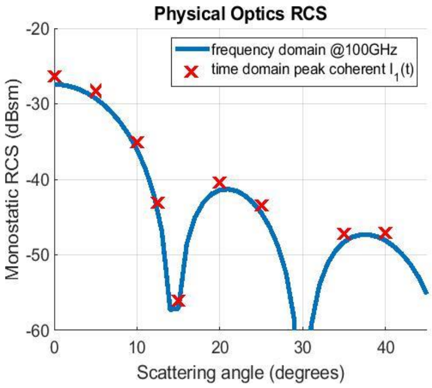

Figure 6.3 The time-domain and frequency-domain monostatic RCS versus incident (scattering) angle θ.

Figure 6.3 shows that the agreement between the time-domain RCS and the frequency domain RCS is fairly good for scattering (incident) angle from 0° to 40°. It should be noticed that the physical optics method (Kirchhoff approximation) may not be accurate for scattering (incident) angles larger than 40°.

6.3 Time-Domain Radar Cross Section Simulation

The distance between the imaging system and the square conducting plate is $L = 10m$. The thickness of the random media is $d_2 = 1m$. The random media contains discrete scatters with the Gaussian phase function. The parameter α_p of the Gaussian phase function is 9.2 which gives the half-power beam width approximately 30°. The single-scattering albedo is 0.9. The bandwidth of the transmitted waveform is $\Delta\omega = 5GHz$ and the carrier frequency is $100GHz$. The surface area of the conducting plate is $2\lambda_0 \times 2\lambda_0$, where λ_0 is the wavelength at $100GHz$. As mentioned in the reference [14], $2\lambda \times 2\lambda$ is the minimum surface area of the square plate that the Kirchhoff approximation is valid.

In the first case, we choose the optical depth $\tau_0 = 1$ and compare the time-domain RCS for the two cases, $d_1 = 4.5m$ and $d_1 = 8m$, where d_1 is the distance between the imaging system and the random media. Different colours are used to represent the contribution from different components. $I(t)$ is the total time-domain RCS, $I_1(t)$ is the coherent component of the time-domain RCS, $I_2(t)$ is the mixed component of the time-domain RCS and $I_3(t)$ is the incoherent component of the time-domain RCS. Time zero corresponds to the round-trip time $t_r = \frac{2L}{c}$.

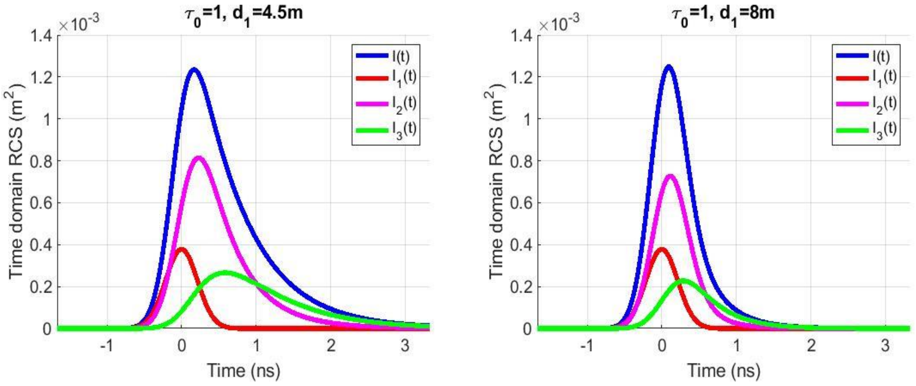

Figure 6.4 Time-domain Radar Cross Section for (Left) the optical depth is $\tau_0 = 1.0$ and the distance from the imaging system to the random media is $d_1 = 4.5m$ and (Right) the optical depth $\tau_0 = 1.0$ and the distance from the imaging system to the random media is $d_1 = 8m$

Figure 6.4 shows that the mixed component $I_2(t)$ gives the largest contribution to the time-domain RCS. The coherent component $I_1(t)$ has the time delay which is equal to the round-trip time t_r. The mixed component $I_2(t)$ and the incoherent component $I_3(t)$ show a larger time delay and a pulse broadening than the coherent component $I_1(t)$. It is also observed that if the random media is placed close to the object, the time-domain RCS waveform shows a lower dispersion. This demonstrates the time-domain shower curtain effect.

In the second case, we increase the optical depth to $\tau_0 = 5$ and compare the time-domain RCS for two cases $d_1 = 4.5m$ and $d_1 = 8m$.

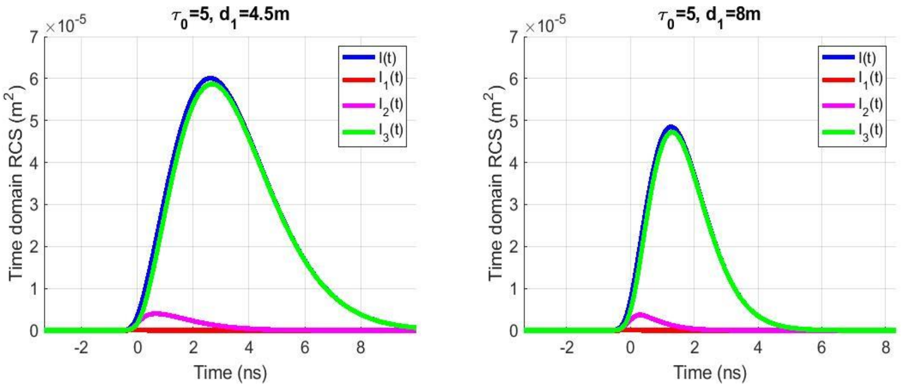

Figure 6.5 Time-domain Radar Cross Section for (Left) the optical depth is $\tau_0 = 5$ and the distance from the imaging system to the random media is $d_1 = 4.5m$ and (Right) the optical depth $\tau_0 = 5$ and the distance from the imaging system to the random media is $d_1 = 8m$.

From the simulation results shown in Figure 6.5, the incoherent component $I_2(t)$ gives the largest contribution to the total RCS for this optical depth. The contributions from the coherent component $I_1(t)$ and the mixed component $I_2(t)$ are negligible. This is due to the fact that when the optical depth increases, the multiple scattering effects will become dominant. It is also observed that if the random media is placed close to the object, the time-domain RCS waveform shows a lower dispersion. This demonstrates the time-domain shower curtain effect.

In the third case, we increase the optical depth to $\tau_0 = 10$ and compare the time-domain RCS for two cases $d_1 = 4.5m$ and $d_1 = 8m$. In the figure, time zero corresponds to the round-trip time $t_r = \frac{2L}{c}$.

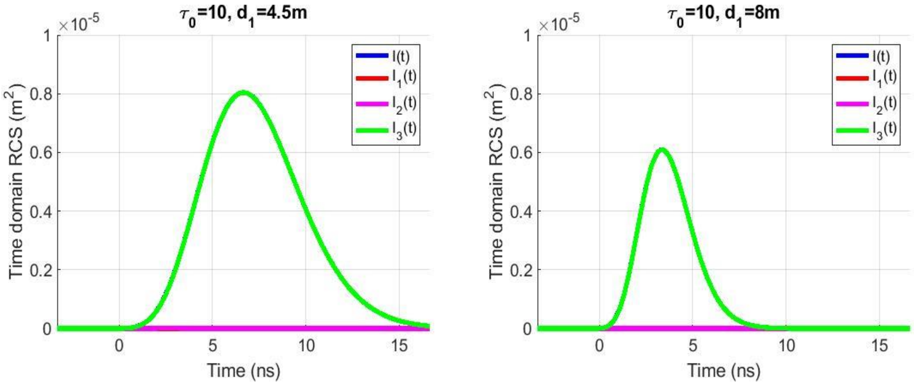

Figure 6.6 Time-domain Radar Cross Section for (Left) the optical depth is $\tau_0 = 10$ and the distance from the imaging system to the random media is $d_1 = 4.5m$ and (Right) the optical depth $\tau_0 = 10$ and the distance from the imaging system to the random media is $d_1 = 8m$.

From the simulation results shown in Figure 6.6, the incoherent component $I_3(t)$ gives the largest contribution to the total RCS for this optical depth. Again, the shower curtain effect in time-domain can be observed.

Comparing the time-domain RCS results for the different optical depth and the different locations of the random media as shown in Figure 6.4-6.6, we can see that when the optical depth increases, the magnitude of the time-domain RCS will decrease. This is expected, because the higher attenuation is introduced when the optical depth becomes larger. Moreover, the incoherent component of the time-domain RCS will dominate when the optical depth becomes larger than 5. The time-domain shower curtain effect is also clearly observed in these figures which show the time-domain resolution will be improved when the random media is placed closer to the object. To emphasis these observations, we plot the time-domain RCS results using dB scale as shown in Figure 6.7.

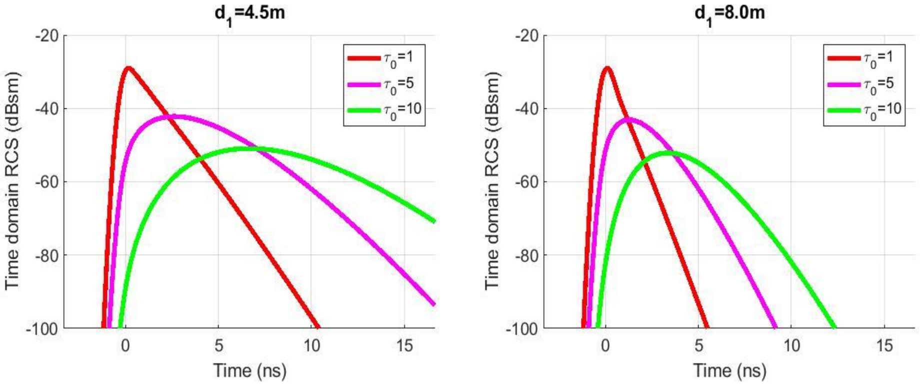

Figure 6.7 Time-domain Radar Cross Section when optical depth $\tau_0 = 1, 5, 10$ for (Left) the distance from the imaging system to the random media is $d_1 = 4.5m$ and (Right) the distance from the imaging system to the random media is $d_1 = 8m$.

Figure 6.8 demonstrates the time-domain shower curtain effect. The normalized propagation delay is defined by $\left|\frac{t_p - t_r}{t_r}\right|$ where t_p is the time instance corresponding to the peak value of the time-domain RCS and $t_r = \frac{2L}{c}$ is the round-trip time. The normalized pulse broadening is given by $\left|\frac{FWHM(I(t)) - FWHM(I_1(t))}{FWHM(I_1(t))}\right|$ where $FWHM$ is full width at half maximum of a waveform, $I(t)$ total time-domain RCS and $I_1(t)$ is the coherent component of the time-domain RCS. From the figure, it can be seen that both the propagation delay and the pulse broadening increase when the optical depth of the random media becomes larger. However, the increasing rate is much faster when the random media is placed closer to the imaging system. This suggests that the better resolution could be achieved for the imaging system if the random media is placed closer to the object.

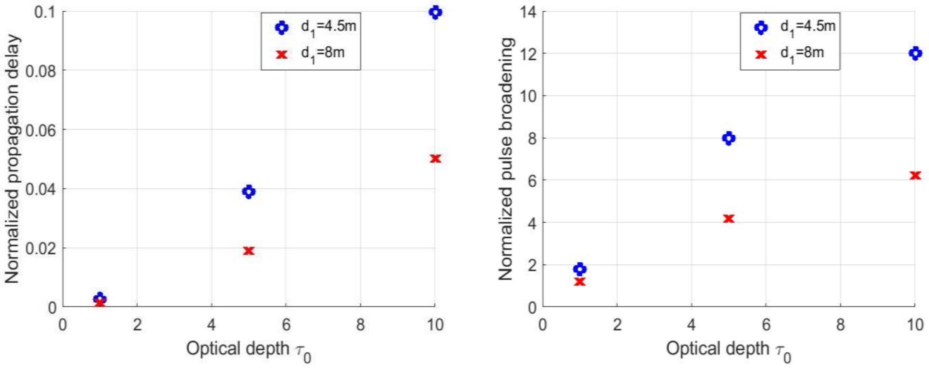

Figure 6.8 Time-domain shower curtain effect when optical depth $\tau_0 = 1, 5, 10$ for (Left) normalized time delay and (Right) normalized pulse broadening

Figure 6.9 compares the time-domain RCS calculated from equations (28) - (32) which includes the backscattering enhancement with those computed from equations (33) - (37) which does not include the backscattering enhancement. It can be seen that the enhancement approaches 2 when the optical depth is large. In the figure, time zero corresponds to the round-trip time $t_r = \frac{2L}{c}$.

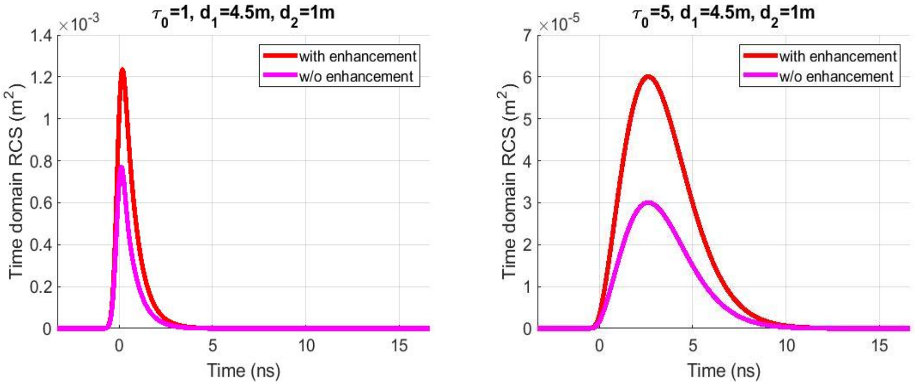

Figure 6.9 Time-domain Radar Cross Section comparison for the cases with and without backscattering enhancement when the distance from the imaging system to the random media is $d_1 = 4.5m$ and the thickness of the random media is $d_2 = 1m$.

We also calculate the time-domain RCS using both the 2$^{\text{nd}}$ order Rytov approximation and the strong fluctuation theory. Since the strong fluctuation theory is only valid when the optical depth of the random medium is sufficiently large, we choose $\tau_0 = 10$ in the simulation. The thickness of the random medium is $d_2 = 10m$, so the space between the imaging system and the object is entirely filled with the random medium. In the figure, time zero corresponds to the round-trip time $t_r = \frac{2L}{c}$.

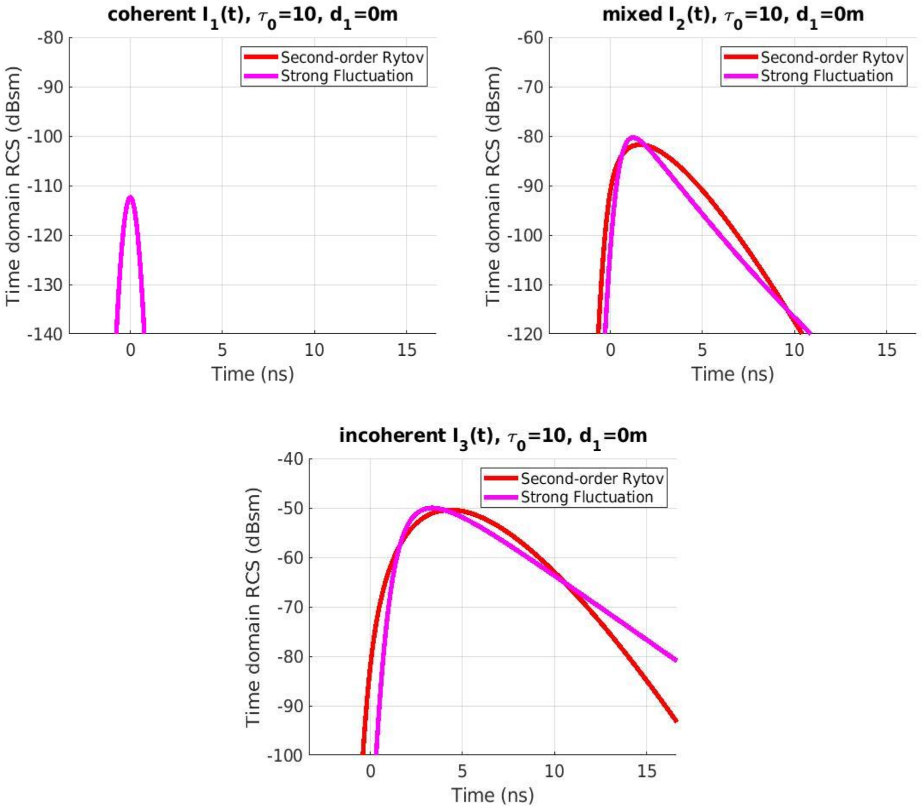

Figure 6.10 Time-domain Radar Cross Section when the optical depth $\tau_0 = 10$,$d_2 = 10m$ for (Top Left) the coherent component (Top Right) the mixed component and (Button) the incoherent component

Figure 6.10 compares the time-domain RCS calculated by both methods. It shows that both methods give the same results for the coherent component $I_1(t)$ which is expected. The mixed component $I_2(t)$ and the incoherent components $I_3(t)$ predicted by both methods are in good agreement.

Since the time-domain RCS requires computation at N_f discrete frequency points, the simulation time of the time-domain RCS will be N_f times longer than that of the conventional RCS. Here N_f is the number of frequency points which depends on both the imaging system bandwidth and the imaging system-target distance. If the parallelization is applied and the computations at different frequencies are carried out simultaneously, the time-domain RCS could be calculated in an efficient way.

Chapter 7 Future Work

In this study, we present a theory of the time-domain radar cross section (RCS) of large conducting objects in discrete random media. The time-domain formula is obtained by applying the inverse Fourier transform of the two-frequency mutual coherence function(MCF). The extended Huygens-Fresnel principle and the 2nd order Rytov approximation are used to derive the two-frequency MCF. Numerical examples of the time-domain RCS of a conducting square plate in a discrete random medium characterized by the Gaussian phase function is shown to highlight the random media effects on the time-domain waveforms such as time delay and pulse broadening in terms of optical depth and random medium location. The degradation of the image quality, known as the shower curtain effect, is observed from the simulation results and can be explained by the characteristics of the incoherent component of the time-domain RCS.

In future, we will continue to improve our time-domain RCS model in following aspects. First of all, we will replace our current isotropic antenna pattern with more realistic antenna pattern. So the effects of the antenna pattern on the time-domain RCS can be studied [4, 41]. Secondly, we will run simulation with different phase functions other than Gaussian. By doing this, the influence of the scattering characteristics of the scatters on the time-domain RCS can be understood [49-50]. Thirdly, different waveforms will be used in the simulation. By comparing the time-domain RCS results, we could able to determine which waveform is best for minimizing the random medium effects [51]. Fourthly, we will apply the theory to study the Radar/Lidar range profile in the presence of the random media [52]. By including the backscattering enhancement and the time-domain shower-curtain effect, our theory should have higher accuracy comparing with the multiple-scattering lidar equation proposed in previous studies. Fifthly, we will increase the simulation speed using parallel programming. Therefore, the time-domain RCS of realistic target can be simulated within reasonable amount of time. Finally, we will conduct controlled experiments and validate our model using experimental data.

Appendix A Circular Complex Gaussian Random Variable

In previous chapters, we have derived the expressions for the time-domain RCS. The formula of the time-domain RCS contains the 4th order moment which can be reduced to the summation of the 2nd order moments by assuming that both incident and scattered fields are circular complex Gaussian random variables. Obviously, the moment theorem of circular complex Gaussian random variables plays a crucial role in this study. However, the commonly used moment theorem [46] is only applicable for the zero-mean random variables. Only few references [9] discuss nonzero mean case and the derivations in those references are very concise. In this appendix, we will follow the procedure in [9] and give a detailed derivation of the 4th order moment of the complex Gaussian random variables.

A.1 The Real Gaussian Process

Define real Gaussian distributed random vector $\bar{X}_{n\times1}$

$$\bar{X}_{n\times1} = [x_1 \quad x_2 \quad \cdots \quad x_n]^T \tag{A-1}$$

where T represents the transpose.

Taking the average, we have the mean vector $\bar{A}_{n\times1}$

$$\bar{A}_{n\times1} = E[\bar{X}_{n\times1}] = [E[x_1] \quad E[x_2] \quad \cdots \quad E[x_n]]^T \tag{A-2}$$

where E denotes the expectation.

The covariance matrix $\bar{\bar{R}}_{n\times n}$ is

$$\bar{\bar{R}}_{n\times n} = \begin{bmatrix} COV(x_1,x_1) & COV(x_1,x_2) & \cdots & COV(x_1,x_n) \\ COV(x_2,x_1) & COV(x_2,x_2) & \cdots & COV(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ COV(x_n,x_1) & COV(x_n,x_2) & \cdots & COV(x_n,x_n) \end{bmatrix} \tag{A-3}$$

The moment generating function is defined as

$$E[exp(\bar{T}_{1\times n}^{T}\bar{X}_{n\times 1})] = exp\left(\frac{1}{2}\bar{T}_{1\times n}^{T}\bar{\bar{R}}_{n\times n}\bar{T}_{n\times 1} + \bar{T}_{1\times n}^{T}\bar{A}_{n\times 1}\right) \qquad \text{(A-5a)}$$

where the real parameter vector $\bar{T}_{n\times 1}$, is given by

$$\bar{T}_{n\times 1} = [t_1 \quad t_2 \quad \cdots \quad t_n]^T \qquad \text{(A-5b)}$$

The moments of $\bar{X}$ is given by the formula

$$E\left[x_1^{m_1} \cdots x_n^{m_n}\right] = \left\{\frac{\partial^{m_1+\cdots+m_n}}{\partial t_1^{m_1}\cdots t_n^{m_n}}\left(E\left[e^{\bar{T}_{1\times n}^{T}\bar{X}_{n\times 1}}\right]\right)\right\}_{\bar{T}_{n\times 1}=\bar{0}} \qquad \text{(A-6)}$$

A.2 The Complex Gaussian Process

Introduce the complex Gaussian random process $\{z_t: -\infty < t < +\infty\}$ with

$$E[z_t] = c_t \qquad \text{(A-7a)}$$

$$COV(z_t, z_s^*) = E[(z_t - c_t)(z_s^* - c_s^*)] = \psi(t,s) \qquad \text{(A-7b)}$$

$$COV(z_s, z_t^*) = E[(z_s - c_s)(z_t^* - c_t^*)] = \psi(s,t) \qquad \text{(A-7c)}$$

$$\psi(t,s) = \psi^*(s,t) \qquad \text{(A-7d)}$$

where $*$ denotes the complex conjugate.

If $\{z_t: -\infty < t < +\infty\}$ is a circular complex Gaussian process, we have

$$COV(z_t, z_s) = COV(z_s, z_t) = 0 \qquad \text{(A-8a)}$$

$$COV(z_t^*, z_s^*) = COV(z_s^*, z_t^*) = 0 \qquad \text{(A-8b)}$$

This expression can be best explained by observing that z_t represents the field in the random medium and its amplitude and phase are determined by both the scattering characteristics of the scatters and the total path lengths. When we calculate the covariance function $COV(z_t, z_s)$ by summing up the contributions from a large volume containing many randomly distributed scatters, the contributions from different scatters tend to cancel each other.

Define complex Gaussian random vector $\bar{Z}_{2n\times1}$

$$\bar{Z}_{2n\times1} = [z_1 \quad \cdots \quad z_n \quad z_1^* \quad \cdots \quad z_n^*]^T \tag{A-9}$$

where T represents the transpose.

Taking the average, we have the mean vector $\bar{C}_{2n\times1}$

$$\bar{C}_{2n\times1} = E[\bar{Z}_{2n\times1}] = [c_1 \quad \cdots \quad c_n \quad c_1^* \quad \cdots \quad c_n^*]^T \tag{A-10}$$

It is convenient to define vectors

$$\bar{Z}'_{n\times1} = [z_1 \quad \cdots \quad z_n]^T \tag{A-11a}$$

$$\bar{Z}'^*_{n\times1} = [z_1^* \quad \cdots \quad z_n^*]^T \tag{A-11b}$$

$$\bar{C}'_{n\times1} = [c_1 \quad \cdots \quad c_n]^T \tag{A-11c}$$

$$\bar{C}'^*_{n\times1} = [c_1^* \quad \cdots \quad c_n^*]^T \tag{A-11d}$$

Combining (A-9), (A-10) and (A-11) gives

$$E[(\bar{Z}'_{n\times1} - \bar{C}'_{n\times1})(\bar{Z}'^T_{1\times n} - \bar{C}'^T_{1\times n})]$$

$$= E\left[\begin{pmatrix} z_1 - c_1 \\ \vdots \\ z_n - c_n \end{pmatrix}_{n\times1} (z_1 - c_1 \quad \cdots \quad z_n - c_n)_{1\times n}\right]$$

$$= \begin{bmatrix} E[(z_1 - c_1)(z_1 - c_1)] & \cdots & E[(z_1 - c_1)(z_n - c_n)] \\ \vdots & \ddots & \vdots \\ E[(z_n - c_n)(z_1 - c_1)] & \cdots & E[(z_n - c_n)(z_n - c_n)] \end{bmatrix}$$

$$= \begin{bmatrix} COV(z_1, z_1) & \cdots & COV(z_1, z_n) \\ \vdots & \ddots & \vdots \\ COV(z_n, z_1) & \cdots & COV(z_n, z_n) \end{bmatrix} = \bar{\bar{0}}_{n\times n} \qquad \text{(A-12a)}$$

$$E[(\bar{Z}'_{n\times 1} - \bar{C}'_{n\times 1})(\bar{Z}''^{T}_{1\times n} - \bar{C}''^{T}_{1\times n})]$$

$$= E\left[\begin{pmatrix} z_1 - c_1 \\ \vdots \\ z_n - c_n \end{pmatrix}_{n\times 1} (z_1^* - c_1^* \quad \cdots \quad z_n^* - c_n^*)_{1\times n}\right]$$

$$= \begin{bmatrix} E[(z_1 - c_1)(z_1^* - c_1^*)] & \cdots & E[(z_1 - c_1)(z_n^* - c_n^*)] \\ \vdots & \ddots & \vdots \\ E[(z_n - c_n)(z_1^* - c_1^*)] & \cdots & E[(z_n - c_n)(z_n^* - c_n^*)] \end{bmatrix}$$

$$= \begin{bmatrix} COV(z_1, z_1^*) & \cdots & COV(z_1, z_n^*) \\ \vdots & \ddots & \vdots \\ COV(z_n, z_1^*) & \cdots & COV(z_n, z_n^*) \end{bmatrix} = \bar{\bar{\bar{\Psi}}}_{n\times n} \qquad \text{(A-12b)}$$

$$E[(\bar{Z}'^{*}_{n\times 1} - \bar{C}'^{*}_{n\times 1})(\bar{Z}'^{T}_{1\times n} - \bar{C}'^{T}_{1\times n})]$$

$$= E\left[\begin{pmatrix} z_1^* - c_1^* \\ \vdots \\ z_n^* - c_n^* \end{pmatrix}_{n\times 1} (z_1 - c_1 \quad \cdots \quad z_n - c_n)_{1\times n}\right]$$

$$= \begin{bmatrix} E[(z_1^* - c_1^*)(z_1 - c_1)] & \cdots & E[(z_1^* - c_1^*)(z_n - c_n)] \\ \vdots & \ddots & \vdots \\ E[(z_n^* - c_n^*)(z_1 - c_1)] & \cdots & E[(z_n^* - c_n^*)(z_n - c_n)] \end{bmatrix}$$

$$= \begin{bmatrix} COV(z_1^*, z_1) & \cdots & COV(z_n^*, z_1) \\ \vdots & \ddots & \vdots \\ COV(z_1^*, z_n) & \cdots & COV(z_n^*, z_n) \end{bmatrix} = \bar{\bar{\bar{\Psi}}}^{*}_{n\times n} \qquad \text{(A-12c)}$$

$$E[(\bar{Z}'^{*}_{n\times 1} - \bar{C}'^{*}_{n\times 1})(\bar{Z}'^{*T}_{1\times n} - \bar{C}'^{*T}_{1\times n})]$$

$$= E\left[\begin{pmatrix} z_1^* - c_1^* \\ \vdots \\ z_n^* - c_n^* \end{pmatrix}_{n\times 1} (z_1^* - c_1^* \quad \cdots \quad z_n^* - c_n^*)_{1\times n}\right]$$

$$= \begin{bmatrix} E[(z_1^* - c_1^*)(z_1^* - c_1^*)] & \cdots & E[(z_1^* - c_1^*)(z_n^* - c_n^*)] \\ \vdots & \ddots & \vdots \\ E[(z_n^* - c_n^*)(z_1^* - c_1^*)] & \cdots & E[(z_n^* - c_n^*)(z_n^* - c_n^*)] \end{bmatrix}$$

$$= \begin{bmatrix} COV(z_1^*, z_1^*) & \cdots & COV(z_1^*, z_n^*) \\ \vdots & \ddots & \vdots \\ COV(z_n^*, z_1^*) & \cdots & COV(z_n^*, z_n^*) \end{bmatrix} = \bar{\bar{0}}_{n\times n} \qquad \text{(A-12d)}$$

Combining (A-9), (A-10) and (A-12) gives

$$\bar{\bar{\Sigma}}_{2n\times 2n} = E[(\bar{Z}_{2n\times 1} - \bar{C}_{2n\times 1})(\bar{Z}_{2n\times 1} - \bar{C}_{2n\times 1})^T]$$

$$= E\left[\begin{pmatrix} \bar{Z}'_{n\times 1} - \bar{C}'_{n\times 1} \\ \bar{Z}'^*_{n\times 1} - \bar{C}'^*_{n\times 1} \end{pmatrix}_{2n\times 1} (\bar{Z}'^T_{1\times n} - \bar{C}'^T_{1\times n} \quad \bar{Z}'^{*T}_{1\times n} - \bar{C}'^{*T}_{1\times n})_{1\times 2n}\right]$$

$$= \begin{bmatrix} E[(\bar{Z}'_{n\times 1} - \bar{C}'_{n\times 1})(\bar{Z}'^T_{1\times n} - \bar{C}'^T_{1\times n})] & E[(\bar{Z}'_{n\times 1} - \bar{C}'_{n\times 1})(\bar{Z}'^{*T}_{1\times n} - \bar{C}'^{*T}_{1\times n})] \\ E[(\bar{Z}'^*_{n\times 1} - \bar{C}'^*_{n\times 1})(\bar{Z}'^T_{1\times n} - \bar{C}'^T_{1\times n})] & E[(\bar{Z}'^*_{n\times 1} - \bar{C}'^*_{n\times 1})(\bar{Z}'^{*T}_{1\times n} - \bar{C}'^{*T}_{1\times n})] \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\bar{0}}_{n\times n} & \bar{\bar{\Psi}}_{n\times n} \\ \bar{\bar{\Psi}}^*_{n\times n} & \bar{\bar{0}}_{n\times n} \end{bmatrix} \qquad \text{(A-13)}$$

Here we will prove the theorem : Let $\{z_t | -\infty < t < +\infty\}$ be a complex Gaussian random process with mean function c_t and covariance function $\psi(t,s)$. Let $\bar{\bar{C}}_{2n\times 1}$ and $\bar{\bar{\Sigma}}_{2n\times 2n}$ be as defined above and let $\bar{V}_{2n\times 1}$ be a real parameter vector. Then

$$E\left[z_1^{m_1} \cdots z_n^{m_n} z_1^{*m_{n+1}} \cdots z_1^{*m_{2n}}\right]$$

$$= \left\{\frac{\partial^{m_1+\cdots+m_{2n}}}{\partial v_1^{m_1} \cdots v_n^{m_{2n}}}\left(exp\left(\frac{1}{2}\bar{V}^T_{1\times 2n}\bar{\bar{\Sigma}}_{2n\times 2n}\bar{V}_{2n\times 1} + \bar{V}^T_{1\times 2n}\bar{\bar{C}}_{2n\times 1}\right)\right)\right\}_{\bar{V}_{n\times 1}=\bar{0}} \qquad \text{(A-14)}$$

We can separate the complex Gaussian process $\{z_t | -\infty < t < +\infty\}$ into the real and the imaginary parts

$$z_t = x_t + jy_t \qquad \text{(A-15)}$$

Then we can write the mean functions as

$$E[z_t] = E[x_t + jy_t] = E[x_t] + jE[y_t] = c_t \tag{A-16a}$$

$$E[z_s^*] = E[x_s - jy_s] = E[x_s] - jE[y_s] = c_s^* \tag{A-16b}$$

The covariance functions for the real and imaginary parts are

$$COV(x_t, x_s) = E[x_t x_s] - E[x_t]E[x_s] \tag{A-17a}$$

$$COV(y_t, y_s) = E[y_t y_s] - E[y_t]E[y_s] \tag{A-17b}$$

Combining (A-7), (A-16) and (A-17) gives

$$\psi(t, s) = COV(z_t, z_s^*) = E[z_t z_s^*] - c_t c_s^*$$

$$= E[(x_t + jy_t)(x_s - jy_s)] - (E[x_t] + jE[y_t])(E[x_s] - jE[y_s])$$

$$= COV(x_t, x_s) + COV(y_t, y_s) + jCOV(x_s, y_t) - jCOV(x_t, y_s) \tag{A-18a}$$

$$\psi(s, t) = COV(z_s, z_t^*) = E[z_s z_t^*] - c_s c_t^*$$

$$= E[(x_s + jy_s)(x_t - jy_t)] - (E[x_s] + jE[y_s])(E[x_t] - jE[y_t])$$

$$= COV(x_s, x_t) + COV(y_s, y_t) + jCOV(x_t, y_s) - jCOV(x_s, y_t) \tag{A-18b}$$

Combining (A-7) and (A-18) gives

$$COV(x_t, x_s) = COV(y_t, y_s) = \frac{1}{2} Re\{\psi(t, s)\} = \rho(t, s) \tag{A-19a}$$

$$COV(x_s, y_t) = \frac{1}{2} Im\{\psi(t, s)\} = \sigma(s, t) \tag{A-19b}$$

$$COV(x_t, y_s) = -\frac{1}{2} Im\{\psi(t, s)\} = -\sigma(t, s) \tag{A-19c}$$

Define real Gaussian random vector $\bar{\Xi}_{2n\times 1}$

$$\bar{\Xi}_{2n\times 1} = [x_1 \quad \cdots \quad x_n \quad y_1 \quad \cdots \quad y_n]^T \tag{A-20}$$

Taking the average, we have the mean vector $\bar{\Gamma}_{2n\times 1}$

$$\bar{\Gamma}_{2n\times 1} = [E[x_1] \quad \cdots \quad E[x_n] \quad E[y_1] \quad \cdots \quad E[y_n]]^T \tag{A-21}$$

It is convenient to define vectors

$$\bar{\Xi}'_{n\times 1} = [x_1 \quad \cdots \quad x_n]^T \tag{A-22a}$$

$$\bar{\Xi}''_{n\times 1} = [y_1 \quad \cdots \quad y_n]^T \tag{A-22b}$$

$$\bar{\Gamma}'_{n\times 1} = [E[x_1] \quad \cdots \quad E[x_n]]^T \tag{A-22c}$$

$$\bar{\Gamma}''_{n\times 1} = [E[y_1] \quad \cdots \quad E[y_n]]^T \tag{A-22d}$$

Combining (A-19), (A-21) and (A-22) gives

$$E[(\bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1})(\bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1})]$$

$$= E\left[\begin{pmatrix} x_1 - E[x_1] \\ \vdots \\ x_n - E[x_n] \end{pmatrix}_{n\times 1} (x_1 - E[x_1] \quad \cdots \quad x_n - E[x_n])_{1\times n}\right]$$

$$= \begin{bmatrix} E[(x_1 - E[x_1])(x_1 - E[x_1])] & \cdots & E[(x_1 - E[x_1])(x_n - E[x_n])] \\ \vdots & \ddots & \vdots \\ E[(x_n - E[x_n])(x_1 - E[x_1])] & \cdots & E[(x_n - E[x_n])(x_n - E[x_n])] \end{bmatrix}$$

$$= \begin{bmatrix} COV(x_1, x_1) & \cdots & COV(x_1, x_n) \\ \vdots & \ddots & \vdots \\ COV(x_n, x_1) & \cdots & COV(z_n, z_n) \end{bmatrix} = \bar{\bar{R}}_{n\times n} \tag{A-23a}$$

$$E[(\bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1})(\bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1})]$$

$$= E\left[\begin{pmatrix} x_1 - E[x_1] \\ \vdots \\ x_n - E[x_n] \end{pmatrix}_{n\times 1} \begin{pmatrix} y_1 - E[y_1] & \cdots & y_n - E[y_n] \end{pmatrix}_{1\times n}\right]$$

$$= \begin{bmatrix} E[(x_1 - E[x_1])(y_1 - E[y_1])] & \cdots & E[(x_1 - E[x_1])(y_n - E[y_n])] \\ \vdots & \ddots & \vdots \\ E[(x_n - E[x_n])(y_1 - E[y_1])] & \cdots & E[(x_n - E[x_n])(y_n - E[y_n])] \end{bmatrix}$$

$$= \begin{bmatrix} COV(x_1, y_1) & \cdots & COV(x_1, y_n) \\ \vdots & \ddots & \vdots \\ COV(x_n, y_1) & \cdots & COV(x_n, y_n) \end{bmatrix} = -\bar{\bar{\bar{\mathrm{S}}}}_{n\times n} \qquad \text{(A-23b)}$$

$$E[(\bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1})(\bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1})]$$

$$= E\left[\begin{pmatrix} y_1 - E[y_1] \\ \vdots \\ y_n - E[y_n] \end{pmatrix}_{n\times 1} \begin{pmatrix} x_1 - E[x_1] & \cdots & x_n - E[x_n] \end{pmatrix}_{1\times n}\right]$$

$$= \begin{bmatrix} E[(y_1 - E[y_1])(x_1 - E[x_1])] & \cdots & E[(y_1 - E[y_1])(x_n - E[x_n])] \\ \vdots & \ddots & \vdots \\ E[(y_n - E[y_n])(x_1 - E[x_1])] & \cdots & E[(y_n - E[y_n])(x_n - E[x_n])] \end{bmatrix}$$

$$= \begin{bmatrix} COV(y_1, x_1) & \cdots & COV(y_1, x_n) \\ \vdots & \ddots & \vdots \\ COV(y_n, x_1) & \cdots & COV(y_n, x_n) \end{bmatrix} = \bar{\bar{\bar{\mathrm{S}}}}_{n\times n} \qquad \text{(A-23c)}$$

$$E[(\bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1})(\bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1})]$$

$$= E\left[\begin{pmatrix} y_1 - E[y_1] \\ \vdots \\ y_n - E[y_n] \end{pmatrix}_{n\times 1} \begin{pmatrix} y_1 - E[y_1] & \cdots & y_n - E[y_n] \end{pmatrix}_{1\times n}\right]$$

$$= \begin{bmatrix} E[(y_1 - E[y_1])(y_1 - E[y_1])] & \cdots & E[(y_1 - E[y_1])(y_n - E[y_n])] \\ \vdots & \ddots & \vdots \\ E[(y_n - E[y_n])(y_1 - E[y_1])] & \cdots & E[(y_n - E[y_n])(y_n - E[y_n])] \end{bmatrix}$$

$$= \begin{bmatrix} COV(y_1, y_1) & \cdots & COV(y_1, y_n) \\ \vdots & \ddots & \vdots \\ COV(y_n, y_1) & \cdots & COV(y_n, y_n) \end{bmatrix} = \bar{\bar{\bar{R}}}_{n\times n} \qquad \text{(A-23d)}$$

Combining (A-19), (A-21) and (A-23) gives

$$\bar{\bar{M}}_{2n\times 2n} = E[(\bar{\Xi}_{2n\times 1} - \bar{\Gamma}_{2n\times 1})(\bar{\Xi}_{2n\times 1} - \bar{\Gamma}_{2n\times 1})^T]$$

$$= E\left[\begin{pmatrix} \bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1} \\ \bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1} \end{pmatrix}_{2n\times 1} \begin{pmatrix} \bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1} & \bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1} \end{pmatrix}_{1\times 2n}\right]$$

$$= \begin{bmatrix} E[(\bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1})(\bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1})] & E[(\bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1})(\bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1})] \\ E[(\bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1})(\bar{\Xi}'_{n\times 1} - \bar{\Gamma}'_{n\times 1})] & E[(\bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1})(\bar{\Xi}''_{n\times 1} - \bar{\Gamma}''_{n\times 1})] \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\bar{R}}_{n\times n} & -\bar{\bar{S}}_{n\times n} \\ \bar{\bar{S}}_{n\times n} & \bar{\bar{R}}_{n\times n} \end{bmatrix} \tag{A-24}$$

Combining (A-12), (A-18) and (A-24) gives

$$\bar{\bar{\Psi}}_{n\times n} = \begin{bmatrix} COV(z_1, z_1^*) & \cdots & COV(z_1, z_n^*) \\ \vdots & \ddots & \vdots \\ COV(z_n, z_1^*) & \cdots & COV(z_n, z_n^*) \end{bmatrix}$$

$$= \begin{bmatrix} COV(x_1, x_1) + COV(y_1, y_1) & \cdots & COV(x_1, x_n) + COV(y_1, y_n) \\ \vdots & \ddots & \vdots \\ COV(x_n, x_1) + COV(y_n, y_1) & \cdots & COV(x_n, x_n) + COV(y_n, y_n) \end{bmatrix}$$

$$+ j \begin{bmatrix} COV(x_1, y_1) - COV(y_1, y_1) & \cdots & COV(x_1, x_n) - COV(y_n, y_1) \\ \vdots & \ddots & \vdots \\ COV(x_n, x_1) - COV(y_1, y_n) & \cdots & COV(x_n, x_n) - COV(y_n, y_n) \end{bmatrix}$$

$$= 2\left(\bar{\bar{R}}_{n\times n} + j\bar{\bar{S}}_{n\times n}\right) \tag{A-25}$$

The parameter vector $\bar{V}_{2n\times 1}$ is

$$\bar{V}_{2n\times 1} = [v_1 \quad \cdots \quad v_n \quad v_{n+1} \quad \cdots \quad v_{2n}]^T \tag{A-26}$$

It is convenient to define matrix

$$\overline{\overline{\Delta}}_{2n\times 2n} = \begin{bmatrix} \overline{\overline{I}}_{n\times n} & \overline{\overline{I}}_{n\times n} \\ j\overline{\overline{I}}_{n\times n} & -j\overline{\overline{I}}_{n\times n} \end{bmatrix} \tag{A-27}$$

Combining (A-26) and (A-27) gives

$$\overline{\mathrm{U}}_{2n\times 1} = \overline{\overline{\Delta}}_{2n\times 2n}\overline{\mathrm{V}}_{2n\times 1} = \begin{bmatrix} 1 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 1 \\ j & \cdots & 0 & -j & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & j & 0 & \cdots & -j \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ v_{n+1} \\ \vdots \\ v_{2n} \end{bmatrix}$$

$$= [v_1 + v_n \quad \cdots \quad v_n + v_{2n} \quad j(v_1 - v_n) \quad \cdots \quad j(v_n - v_{2n})] \tag{A-28}$$

Combining (A-9), (A-27) and (A-28) gives

$$\overline{\mathrm{Z}}_{2n\times 1} = \overline{\overline{\Delta}}{}^{T}_{2n\times 2n}\overline{\Xi}_{2n\times 1} = \begin{bmatrix} 1 & \cdots & 0 & j & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & j \\ 1 & \cdots & 0 & -j & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & -j \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= [x_1 + jy_1 \quad \cdots \quad x_n + jy_n \quad x_1 - jy_1 \quad \cdots \quad x_n - jy_n] \tag{A-29}$$

Taking the average, the mean vector $\overline{\mathrm{C}}_{2n\times 1}$ is

$$\overline{\mathrm{C}}_{2n\times 1} = E[\overline{\mathrm{Z}}_{2n\times 1}] = \overline{\overline{\Delta}}{}^{T}_{2n\times 2n}E[\overline{\Xi}_{2n\times 1}] = \overline{\overline{\Delta}}{}^{T}_{2n\times 2n}\overline{\Gamma}_{2n\times 1} \tag{A-30}$$

Combining (A-26), (A-27) and (A-28) gives

$$\overline{\mathrm{U}}{}^{T}_{1\times 2n}\overline{\Xi}_{2n\times 1} = \left(\overline{\overline{\Delta}}_{2n\times 2n}\overline{\mathrm{V}}_{2n\times 1}\right)^{T}\overline{\Xi}_{2n\times 1} = \overline{\mathrm{V}}{}^{T}_{1\times 2n}\overline{\overline{\Delta}}{}^{T}_{2n\times 2n}\overline{\Xi}_{2n\times 1}$$

$$= \overline{\mathrm{V}}{}^{T}_{1\times 2n}\overline{\mathrm{Z}}_{2n\times 1} \tag{A-31}$$

Combining (A-24), (A-27) and (A-28) gives

$$\bar{\bar{\Delta}}^T_{2n\times 2n}\bar{\bar{\mathrm{M}}}_{2n\times 2n}\bar{\bar{\Delta}}_{2n\times 2n} = \begin{bmatrix} \bar{\bar{I}}_{n\times n} & i\bar{\bar{I}}_{n\times n} \\ \bar{\bar{I}}_{n\times n} & -i\bar{\bar{I}}_{n\times n} \end{bmatrix} \begin{bmatrix} \bar{\bar{R}}_{n\times n} & -\bar{\bar{\mathrm{S}}}_{n\times n} \\ \bar{\bar{\mathrm{S}}}_{n\times n} & \bar{\bar{R}}_{n\times n} \end{bmatrix} \begin{bmatrix} \bar{\bar{I}}_{n\times n} & \bar{\bar{I}}_{n\times n} \\ i\bar{\bar{I}}_{n\times n} & -i\bar{\bar{I}}_{n\times n} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\bar{R}}_{n\times n} + i\bar{\bar{\mathrm{S}}}_{n\times n} & -\bar{\bar{\mathrm{S}}}_{n\times n} + i\bar{\bar{R}}_{n\times n} \\ \bar{\bar{R}}_{n\times n} - i\bar{\bar{\mathrm{S}}}_{n\times n} & -\bar{\bar{\mathrm{S}}}_{n\times n} - i\bar{\bar{R}}_{n\times n} \end{bmatrix} \begin{bmatrix} \bar{\bar{I}}_{n\times n} & \bar{\bar{I}}_{n\times n} \\ i\bar{\bar{I}}_{n\times n} & -i\bar{\bar{I}}_{n\times n} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\bar{0}}_{n\times n} & 2\left(\bar{\bar{R}}_{n\times n} + i\bar{\bar{\mathrm{S}}}_{n\times n}\right) \\ 2\left(\bar{\bar{R}}_{n\times n} - i\bar{\bar{\mathrm{S}}}_{n\times n}\right) & \bar{\bar{0}}_{n\times n} \end{bmatrix} = \begin{bmatrix} \bar{\bar{0}}_{n\times n} & \bar{\bar{\Psi}}_{n\times n} \\ \bar{\bar{\Psi}}^*_{n\times n} & \bar{\bar{0}}_{n\times n} \end{bmatrix} \qquad \text{(A-32)}$$

Combining (A-13) and (A-32) gives

$$\bar{\bar{\Sigma}}_{2n\times 2n} = \bar{\bar{\Delta}}^T_{2n\times 2n}\bar{\bar{\mathrm{M}}}_{2n\times 2n}\bar{\bar{\Delta}}_{2n\times 2n} \qquad \text{(A-33)}$$

Combining (A-28) and (A-33) gives

$$\bar{\mathrm{U}}^T_{1\times 2n}\bar{\bar{\mathrm{M}}}_{2n\times 2n}\bar{\mathrm{U}}_{2n\times 1} = \left(\bar{\bar{\Delta}}_{2n\times 2n}\bar{\mathrm{V}}_{2n\times 1}\right)^T\bar{\bar{\mathrm{M}}}_{2n\times 2n}\left(\bar{\bar{\Delta}}_{2n\times 2n}\bar{\mathrm{V}}_{2n\times 1}\right)$$

$$= \bar{\mathrm{V}}^T_{1\times 2n}\bar{\bar{\Delta}}^T_{2n\times 2n}\bar{\bar{\mathrm{M}}}_{2n\times 2n}\bar{\bar{\Delta}}_{2n\times 2n}\bar{\mathrm{V}}_{2n\times 1} = \bar{\mathrm{V}}^T_{1\times 2n}\bar{\bar{\Sigma}}_{2n\times 2n}\bar{\mathrm{V}}_{2n\times 1} \qquad \text{(A-34)}$$

Combining (A-28), (A-30) and (A-31) gives

$$\bar{\mathrm{U}}^T_{1\times 2n}\bar{\Gamma}_{2n\times 1} = \left(\bar{\bar{\Delta}}_{2n\times 2n}\bar{\mathrm{V}}_{2n\times 1}\right)^T\bar{\Gamma}_{2n\times 1} = \bar{\mathrm{V}}^T_{2n\times 1}\bar{\bar{\Delta}}^T_{2n\times 2n}\bar{\Gamma}_{2n\times 1}$$

$$= \bar{\mathrm{V}}^T_{1\times 2n}\bar{\mathrm{C}}_{2n\times 1} \qquad \text{(A-35)}$$

Combining (A-5) and (A-35) gives

$$E[exp(\bar{\mathrm{U}}^T_{1\times 2n}\bar{\Xi}_{2n\times 1})] = exp\left(\frac{1}{2}\bar{\mathrm{U}}^T_{1\times 2n}\bar{\bar{M}}_{2n\times 2n}\bar{U}_{2n\times 1} + \bar{\mathrm{U}}^T_{1\times 2n}\bar{\Gamma}_{2n\times 1}\right) \quad \text{(A-36)}$$

Combining (A-5) and (A-35) gives

$$E[exp(\bar{\mathrm{V}}^T_{1\times 2n}\bar{\mathrm{Z}}_{2n\times 1})] = exp\left(\frac{1}{2}\bar{\mathrm{U}}^T_{1\times 2n}\bar{\bar{M}}_{2n\times 2n}\bar{U}_{2n\times 1} + \bar{\mathrm{U}}^T_{1\times 2n}\bar{\Gamma}_{2n\times 1}\right)$$

$$= exp\left(\frac{1}{2}\bar{\mathrm{V}}^T_{1\times 2n}\bar{\bar{\Sigma}}_{2n\times 2n}\bar{V}_{2n\times 1} + \bar{\mathrm{V}}^T_{1\times 2n}\bar{\mathrm{C}}_{2n\times 1}\right) \tag{A-37}$$

Here we will apply equation (A-37) to calculate the 4th order moment.

The parameter vector $\bar{V}_{8\times 1}$ is

$$\bar{V}_{8\times 1} = \begin{bmatrix} v_1 & \cdots & v_4 & v_5 & \cdots & v_8 \end{bmatrix}^T = \begin{bmatrix} \bar{V}'_{1\times 4} & \bar{V}''_{1\times 4} \end{bmatrix}^T \tag{A-38}$$

The covariance matrix $\bar{\bar{\Sigma}}_{8\times 8}$ is

$$\bar{\bar{\Sigma}}_{8\times 8} = \begin{bmatrix} \bar{\bar{0}}_{4\times 4} & \bar{\bar{\Psi}}_{4\times 4} \\ \bar{\bar{\Psi}}^*_{4\times 4} & \bar{\bar{0}}_{4\times 4} \end{bmatrix} \tag{A-39}$$

The moment generating function is

$$E[exp(\bar{\mathrm{V}}^T_{1\times 8}\bar{\mathrm{Z}}_{8\times 1})] = exp\left(\frac{1}{2}\bar{\mathrm{V}}^T_{1\times 8}\bar{\bar{\Sigma}}_{8\times 8}\bar{V}_{8\times 1} + \bar{\mathrm{V}}^T_{1\times 8}\bar{\mathrm{C}}_{8\times 1}\right)$$

$$= exp\left(\frac{1}{2}\begin{bmatrix} \bar{V}'^T_{1\times 4} & \bar{V}''^T_{1\times 4} \end{bmatrix}\begin{bmatrix} \bar{\bar{0}}_{4\times 4} & \bar{\bar{\Psi}}_{4\times 4} \\ \bar{\bar{\Psi}}^*_{4\times 4} & \bar{\bar{0}}_{4\times 4} \end{bmatrix}\begin{bmatrix} \bar{V}'_{1\times 4} \\ \bar{V}''_{1\times 4} \end{bmatrix} + \begin{bmatrix} \bar{V}'^T_{1\times 4} & \bar{V}''^T_{1\times 4} \end{bmatrix}\begin{bmatrix} \bar{C}'_{4\times 1} \\ \bar{C}'^*_{4\times 1} \end{bmatrix}\right)$$

$$= exp\left(\frac{1}{2}\left(\bar{V}'^T_{1\times 4}\bar{\bar{\Psi}}_{4\times 4}\bar{V}''_{1\times 4} + \bar{V}''^T_{1\times 4}\bar{\bar{\Psi}}^*_{4\times 4}\bar{V}'_{1\times 4}\right)\right)$$

$$\times\, exp(\bar{V}'^T_{1\times 4}\bar{C}'_{4\times 1} + \bar{V}''^T_{1\times 4}\bar{C}'^*_{4\times 1}) \tag{A-40}$$

We have

$$\bar{V}'^T_{1\times 4}\bar{\bar{\Psi}}_{4\times 4}\bar{V}''_{1\times 4} = \sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n \psi_{mn} \tag{A-41a}$$

$$\bar{V}''^T_{1\times 4}\bar{\bar{\Psi}}^*_{4\times 4}\bar{V}'_{1\times 4} = \sum_{m=1}^{4}\sum_{n=1}^{4} v''_m v'_n \psi^*_{mn} \tag{A-41b}$$

$$\bar{V}'^T_{1\times 4}\bar{C}'_{4\times 1} = \sum_{n=1}^{4} v'_n c_n \tag{A-41c}$$

$$\bar{V}''^T_{1\times 4}\bar{C}'^*_{4\times 1} = \sum_{n=1}^{4} v''_n c^*_n \tag{A-41d}$$

Combining (A-7), (A-40) and (A-41) gives

$$E[exp(\overline{\mathrm{V}}_{1\times 8}^{T}\overline{\mathrm{Z}}_{8\times 1})] = exp\left(\frac{1}{2}\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n'' \psi_{mn} + \sum_{m=1}^{4}\sum_{n=1}^{4} v_m'' v_n' \psi_{mn}^{*}\right)\right)$$

$$\times\, exp\left(\sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right)$$

$$= exp\left(\frac{1}{2}\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n''\left(\psi_{mn} + \psi_{nm}^{*}\right)\right) + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right)$$

$$= exp\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n''\, \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right) \qquad \text{(A-42)}$$

The random vector $\bar{Z}_{8\times 1}$ is

$$\bar{Z}_{8\times 1} = [z_t \quad z_s \quad z_\tau \quad z_\sigma \quad z_t^{*} \quad z_s^{*} \quad z_\tau^{*} \quad z_\sigma^{*}]^{T} \qquad \text{(A-43)}$$

The 4th order moment is

$$E[z_t z_\tau z_s^{*} z_\sigma^{*}] = \left\{\frac{\partial^4 exp\left(\frac{1}{2}\overline{\mathrm{V}}_{1\times 8}^{T}\overline{\overline{\Sigma}}_{8\times 8}\overline{V}_{8\times 1} + \overline{\mathrm{V}}_{1\times 8}^{T}\overline{\mathrm{C}}_{8\times 1}\right)}{\partial v_1 \partial v_3 \partial v_6 \partial v_8}\right\}_{\overline{V}_{8\times 1}=\overline{0}}$$

$$= \left\{\frac{\partial^4 exp\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n'' \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right)}{\partial v_1' \partial v_3' \partial v_2'' \partial v_4''}\right\}_{\overline{V}_{8\times 1}=\overline{0}} \qquad \text{(A-44)}$$

The Taylor expansion of the exponential term is

$$exp\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n''\, \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right) = 1$$

$$+\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n''\, \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right)$$

$$+\frac{1}{2!}\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n''\, \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right)^2$$

$$+\frac{1}{3!}\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n''\, \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right)^3$$

$$+\frac{1}{4!}\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n''\, \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^{*}\right)^4 + \cdots \qquad \text{(A-45)}$$

The 2nd order term is

$$\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn} + \sum_{n=1}^{4} v'_n c_n + \sum_{n=1}^{4} v''_n c_n^*\right)^2$$

$$= \left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn}\right)^2 + 2\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn}\right)\left(\sum_{n=1}^{4} v'_n c_n + \sum_{n=1}^{4} v''_n c_n^*\right)$$

$$+\left(\sum_{n=1}^{4} v'_n c_n + \sum_{n=1}^{4} v''_n c_n^*\right)^2 \qquad \text{(A-46)}$$

Collecting terms contain product $v'_1 v'_3 v''_2 v''_4$

$$\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn}\right)^2 = \left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn}\right)\left(\sum_{m'=1}^{4}\sum_{n'=1}^{4} v'_{m'} v''_{n'}\, \psi_{m'n'}\right)$$

$$= (\cdots + v'_1 v''_2 \psi_{12} + v'_1 v''_4 \psi_{14} + v'_3 v''_2 \psi_{32} + v'_3 v''_4 \psi_{34} + \cdots)$$

$$\times (\cdots + v'_1 v''_2 \psi_{12} + v'_1 v''_4 \psi_{14} + v'_3 v''_2 \psi_{32} + v'_3 v''_4 \psi_{34} + \cdots)$$

$$= v'_1 v'_3 v''_2 v''_4 \psi_{12}\psi_{34} + v'_1 v'_3 v''_2 v''_4 \psi_{14}\psi_{32} + v'_1 v'_3 v''_2 v''_4 \psi_{32}\psi_{14} + v'_1 v'_3 v''_2 v''_4 \psi_{12}\psi_{34}$$

$$= 2 v'_1 v'_3 v''_2 v''_4 (\psi_{12}\psi_{34} + \psi_{14}\psi_{32}) \qquad \text{(A-47)}$$

The 3rd order term is

$$\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn} + \sum_{n=1}^{4} v'_n c_n + \sum_{n=1}^{4} v''_n c_n^*\right)^3$$

$$= \left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn}\right)^3 + 3\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn}\right)^2\left(\sum_{n=1}^{4} v'_n c_n + \sum_{n=1}^{4} v''_n c_n^*\right)$$

$$+3\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn}\right)\left(\sum_{n=1}^{4} v'_n c_n + \sum_{n=1}^{4} v''_n c_n^*\right)^2$$

$$+\left(\sum_{n=1}^{4} v'_n c_n + \sum_{n=1}^{4} v''_n c_n^*\right)^3 \qquad \text{(A-48)}$$

Collecting terms contain product $v'_1 v'_3 v''_2 v''_4$

$$\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v'_m v''_n\, \psi_{mn}\right)\left(\sum_{n=1}^{4} v'_n c_n + \sum_{n=1}^{4} v''_n c_n^*\right)^2$$

$$= (\cdots + v_1'v_2''\psi_{12} + v_1'v_4''\psi_{14} + v_3'v_2''\psi_{32} + v_3'v_4''\psi_{34} + \cdots)$$

$$\times (\cdots + v_1'c_1 + v_3'c_3 + v_2''c_2^* + v_4''c_4^* + \cdots)$$

$$\times (\cdots + v_1'c_1 + v_3'c_3 + v_2''c_2^* + v_4''c_4^* + \cdots)$$

$$= (\cdots + v_1'v_2''\psi_{12} + v_1'v_4''\psi_{14} + v_3'v_2''\psi_{32} + v_3'v_4''\psi_{34} + \cdots)$$

$$\times (\cdots + 2v_1'v_2''c_1c_2^* + 2v_1'v_4''c_1c_4^* + 2v_3'v_2''c_3c_2^* + 2v_3'v_4''c_3c_4^* + \cdots)$$

$$= 2v_1'v_3'v_2''v_4''\psi_{12}c_3c_4^* + 2v_1'v_3'v_2''v_4''\psi_{14}c_2^*c_3$$

$$+2v_1'v_3'v_2''v_4''\psi_{32}c_1c_4^* + 2v_1'v_3'v_2''v_4''\psi_{34}c_1c_2^*$$

$$= 2v_1'v_3'v_2''v_4''(\psi_{12}c_3c_4^* + \psi_{14}c_2^*c_3 + \psi_{32}c_1c_4^* + \psi_{34}c_1c_2^*) \qquad \text{(A-49)}$$

The 4th order term is

$$(\textstyle\sum_{m=1}^4 \sum_{n=1}^4 v_m'v_n''\,\psi_{mn} + \sum_{n=1}^4 v_n'c_n + \sum_{n=1}^4 v_n''c_n^*)^4$$

$$= (\textstyle\sum_{m=1}^4 \sum_{n=1}^4 v_m'v_n''\,\psi_{mn})^4$$

$$+4(\textstyle\sum_{m=1}^4 \sum_{n=1}^4 v_m'v_n''\,\psi_{mn})^3(\sum_{n=1}^4 v_n'c_n + \sum_{n=1}^4 v_n''c_n^*)$$

$$+6(\textstyle\sum_{m=1}^4 \sum_{n=1}^4 v_m'v_n''\,\psi_{mn})^2(\sum_{n=1}^4 v_n'c_n + \sum_{n=1}^4 v_n''c_n^*)^2$$

$$+4(\textstyle\sum_{m=1}^4 \sum_{n=1}^4 v_m'v_n''\,\psi_{mn})^1(\sum_{n=1}^4 v_n'c_n + \sum_{n=1}^4 v_n''c_n^*)^2$$

$$+(\textstyle\sum_{n=1}^4 v_n'c_n + \sum_{n=1}^4 v_n''c_n^*)^4 \qquad \text{(A-50)}$$

Applying the multinomial formula

$$(x_1 + \cdots + x_m)^n = \textstyle\sum_{k_1+\cdots+k_m=n} \frac{n!}{k_1!\cdots k_m!} \prod_{t=1}^m x_t^{k_t} \qquad \text{(A-51)}$$

Collecting terms contain product $v_1' v_3' v_2'' v_4''$

$$\left(\sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^*\right)^4$$

$$= \left(\cdots + v_1' c_1 + v_3' c_3 + v_2'' c_2^* + v_4'' c_4^* + \cdots\right)^4$$

$$= \left(v_1' c_1 + v_3' c_3 + v_2'' c_2^* + v_4'' c_4^*\right)^4 + \cdots$$

$$= 24 v_1' v_3' v_2'' v_4'' c_1 c_2^* c_3 c_4^* + \cdots \tag{A-52}$$

The 4th order moment is

$$E[z_t z_\tau z_s^* z_\sigma^*] = \left\{\frac{\partial^4 \Phi}{\partial v_1' \partial v_3' \partial v_2'' \partial v_4''}\right\}_{\overline{V}_{8\times 1} = \overline{0}} \tag{A-53a}$$

$$\Phi = \left\{\begin{matrix} 1 + \left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n'' \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^*\right) \\ + \frac{1}{2!}\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n'' \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^*\right)^2 \\ + \frac{1}{3!}\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n'' \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^*\right)^3 \\ + \frac{1}{4!}\left(\sum_{m=1}^{4}\sum_{n=1}^{4} v_m' v_n'' \psi_{mn} + \sum_{n=1}^{4} v_n' c_n + \sum_{n=1}^{4} v_n'' c_n^*\right)^4 + \cdots \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \frac{1}{2!}\left[2 v_1' v_3' v_2'' v_4'' (\psi_{12}\psi_{34} + \psi_{14}\psi_{32})\right] \\ + \frac{1}{3!}\left[6 v_1' v_3' v_2'' v_4'' (\psi_{12} c_3 c_4^* + \psi_{14} c_2^* c_3 + \psi_{32} c_1 c_4^* + \psi_{34} c_1 c_2^*)\right] \\ + \frac{1}{4!}\left[24 v_1' v_3' v_2'' v_4'' c_1 c_2^* c_3 c_4^*\right] + \cdots \end{matrix}\right\}$$

$$= \left\{\begin{matrix} v_1' v_3' v_2'' v_4'' (\psi_{12}\psi_{34} + \psi_{14}\psi_{32}) \\ + v_1' v_3' v_2'' v_4'' (\psi_{12} c_3 c_4^* + \psi_{14} c_2^* c_3 + \psi_{32} c_1 c_4^* + \psi_{34} c_1 c_2^*) \\ + v_1' v_3' v_2'' v_4'' c_1 c_2^* c_3 c_4^* + \cdots \end{matrix}\right\}$$

$$= \left\{\begin{matrix} v_1' v_3' v_2'' v_4'' (\psi_{12}\psi_{34} + \psi_{14}\psi_{32}) \\ + v_1' v_3' v_2'' v_4'' (\psi_{12} c_3 c_4^* + \psi_{14} c_2^* c_3 + \psi_{32} c_1 c_4^* + \psi_{34} c_1 c_2^*) \\ + v_1' v_3' v_2'' v_4'' c_1 c_2^* c_3 c_4^* + \cdots \end{matrix}\right\} \tag{A-53b}$$

We obtain

$$E[z_t z_\tau z_s^* z_\sigma^*] = \left\{\begin{matrix} \psi_{12}\psi_{34} + \psi_{14}\psi_{32} + \psi_{12}c_3c_4^* + \psi_{14}c_2^*c_3 \\ +\psi_{32}c_1c_4^* + \psi_{34}c_1c_2^* + c_1c_2^*c_3c_4^* \end{matrix}\right\} \tag{A-54}$$

The 4th order moment is

$$E[z_t z_s^* z_\tau z_\sigma^*] = \left\{\begin{matrix} \psi(t,s)\psi(\tau,\sigma) + \psi(t,\sigma)\psi(\tau,s) + \psi(t,s)c_\tau c_\sigma^* \\ +\psi(t,\sigma)c_\tau c_s^* + \psi(\tau,s)c_t c_\sigma^* + c_t c_s^* c_\tau c_\sigma^* \end{matrix}\right\} \tag{A-55}$$

Combining (A-7) and (A-55) gives

$$E[z_t z_s^* z_\tau z_\sigma^*] = \left\{\begin{matrix} \psi(t,s)\psi(\tau,\sigma) + \psi(t,\sigma)\psi(\tau,s) + \psi(t,s)c_\tau c_\sigma^* \\ +\psi(t,\sigma)c_\tau c_s^* + \psi(\tau,s)c_t c_\sigma^* + c_t c_s^* c_\tau c_\sigma^* \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (\langle z_t z_s^*\rangle - c_t c_s^*)(\langle z_t z_s^*\rangle - c_t c_s^*) + (\langle z_t z_\sigma^*\rangle - c_t c_\sigma^*)(\langle z_\tau z_s^*\rangle - c_\tau c_s^*) \\ +(\langle z_t z_s^*\rangle - c_t c_s^*)c_\tau c_\sigma^* + (\langle z_t z_\sigma^*\rangle - c_t c_\sigma^*)c_\tau c_s^* + (\langle z_\tau z_s^*\rangle - c_\tau c_s^*)c_t c_\sigma^* \\ +(\langle z_\tau z_\sigma^*\rangle - c_\tau c_\sigma^*)c_t c_s^* + c_t c_s^* c_\tau c_\sigma^* \end{matrix}\right\}$$

$$= \langle z_t z_s^*\rangle\langle z_\tau z_\sigma^*\rangle + \langle z_t z_\sigma^*\rangle\langle z_\tau z_s^*\rangle - c_t c_s^* c_\tau c_\sigma^* \tag{A-56}$$

Appendix B Reciprocity Theorem for Green's Function in a Linear Inhomogeneous Medium

In previous chapters, we have derived the expressions for the time-domain RCS. The reciprocity theorem for the Green's function in an inhomogeneous medium is applied to calculate the 4th order moment within the time-domain RCS formula. In this appendix, we will follow the procedure in [47-48] and show that the Green's function satisfies the reciprocity relationship in a linear inhomogeneous medium. More specifically, the Green's function is symmetrical with the source point and the observation point interchanged.

In an inhomogeneous medium, the field $U(\bar{r})$ obeys the scalar Helmholtz equation

$$\nabla^2 U(\bar{r}) + k_0^2 \big(1 + n_1(\bar{r})\big)^2 U(\bar{r}) = 0 \qquad \text{(B-1)}$$

where k_0 is the free-space wavenumber and $n_1(\bar{r})$ is the refractive index fluctuation.

In the absence of the random fluctuations$(n_1(\bar{r}) = 0)$, the scalar Helmholtz equation becomes

$$\nabla^2 U_0(\bar{r}) + k_0^2 U_0(\bar{r}) = 0 \qquad \text{(B-2)}$$

where $U_0(\bar{r})$ represents the primary field.

The free-space Green's function satisfies the equation for a point source

$$\nabla^2 G_0(\bar{r} - \bar{r}') + k_0^2 G_0(\bar{r} - \bar{r}') = -\delta(\bar{r} - \bar{r}') \qquad \text{(B-3)}$$

where $\delta(\bar{r} - \bar{r}')$ is the Dirac delta function.

Solving the equation (B-3), the free-space Green's function is given by

$$G_0(\bar{r},\bar{r}') = \frac{exp(ik_0|\bar{r}-\bar{r}'|)}{4\pi|\bar{r}-\bar{r}'|} \tag{B-4}$$

Combining (B-2) and (B-3), the solution of (B-1) can be expressed by an integral equation for $U(\bar{r})$

$$U(\bar{r}) = U_0(\bar{r}) + k_0^2 \int G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)U(\bar{r}_1)\mathrm{d}\bar{r}_1 \tag{B-5a}$$

$$\delta\mathrm{n}(\bar{r}) = \left(1+n_1(\bar{r})\right)^2 - 1 = 2n_1(\bar{r}) + n_1^2(\bar{r}) \tag{B-5b}$$

where the integration is over the space occupied by the random medium.

Write the field at point $\bar{r} = \bar{r}_1$

$$U(\bar{r}_1) = U_0(\bar{r}_1) + k_0^2 \int G_0(\bar{r}_1-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)U(\bar{r}_2)\mathrm{d}\bar{r}_2 \tag{B-6}$$

Combining (B-5a) and (B-6) gives

$$U(\bar{r}) = U_0(\bar{r}) + k_0^2 \int G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)U_0(\bar{r}_1)\mathrm{d}\bar{r}_1$$

$$+k_0^4 \iint G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)U(\bar{r}_2)\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2 \tag{B-7}$$

Write the field at point $\bar{r} = \bar{r}_2$

$$U(\bar{r}_2) = U_0(\bar{r}_2) + k_0^2 \int G_0(\bar{r}_2-\bar{r}_3)\delta\mathrm{n}(\bar{r}_3)U_0(\bar{r}_3)\mathrm{d}\bar{r}_3$$

$$+k_0^4 \iint G_0(\bar{r}_2-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}_3)\delta\mathrm{n}(\bar{r}_3)U(\bar{r}_3)\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_3 \tag{B-8}$$

Combining (B-7) and (B-8) gives

$$U(\bar{r}) = U_0(\bar{r}) + k_0^2 \int G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)U_0(\bar{r}_1)\mathrm{d}\bar{r}_1$$

$$+k_0^4 \iint G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)U_0(\bar{r}_2)\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2$$

$$+k_0^6 \iiint G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)$$

$$G_0(\bar{r}_2-\bar{r}_3)\delta\mathrm{n}(\bar{r}_3)U_0(\bar{r}_3)\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2\mathrm{d}\bar{r}_3+\cdots \tag{B-9}$$

By repeating the process, we can obtain an infinite perturbation series for the field in an inhomogeneous medium. This series is known as the Neumann series for the integral equation and the Born expansion in physics [47].

In an inhomogeneous medium, the stochastic Green's function obeys the scalar Helmholtz equation

$$\nabla^2 G(\bar{r},\bar{r}')+k_0^2\big(1+\delta\mathrm{n}(\bar{r})\big)G(\bar{r},\bar{r}')=-\delta(\bar{r}-\bar{r}') \tag{B-10}$$

where k_0 is the free-space wavenumber. The stochastic Green's function also satisfies the radiation condition at infinite.

Substituting $U_0(\bar{r})=G_0(\bar{r},\bar{r}')$ and $U(\bar{r})=G(\bar{r},\bar{r}')$ into (B-9) gives

$$G(\bar{r},\bar{r}')=G_0(\bar{r},\bar{r}')+k_0^2\int G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}')\mathrm{d}\bar{r}_1$$

$$+k_0^4\iint G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)G_0(\bar{r}_2-\bar{r}')\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2$$

$$+k_0^6\iiint G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)$$

$$G_0(\bar{r}_2-\bar{r}_3)\delta\mathrm{n}(\bar{r}_3)G_0(\bar{r}_3-\bar{r}')\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2\mathrm{d}\bar{r}_3+\cdots \tag{B-11}$$

Applying the reciprocity relation for the free-space Green's function, that is, $G_0(\bar{r},\bar{r}')=G_0(\bar{r}',\bar{r})$, the second term in the series (B-11) can be written as

$$k_0^2\int G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}')\mathrm{d}\bar{r}_1=k_0^2\int G_0(\bar{r}_1-\bar{r})\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}'-\bar{r}_1)\mathrm{d}\bar{r}_1$$

$$=k_0^2\int G_0(\bar{r}'-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r})\mathrm{d}\bar{r}_1 \tag{B-12}$$

Following the same procedure and properly changing the integration variables, the third term in the infinite series (B-11) becomes

$$k_0^4 \iint G_0(\bar{r}-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)G_0(\bar{r}_2-\bar{r}')\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2$$

$$= k_0^4 \iint G_0(\bar{r}_1-\bar{r})\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_2-\bar{r}_1)\delta\mathrm{n}(\bar{r}_2)G_0(\bar{r}'-\bar{r}_2)\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2$$

$$= k_0^4 \iint G_0(\bar{r}'-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)G_0(\bar{r}_2-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r})\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2$$

$$= k_0^4 \iint G_0(\bar{r}'-\bar{r}_1)\delta\mathrm{n}(\bar{r}_1)G_0(\bar{r}_1-\bar{r}_2)\delta\mathrm{n}(\bar{r}_2)G_0(\bar{r}_2-\bar{r})\mathrm{d}\bar{r}_1\mathrm{d}\bar{r}_2 \qquad \text{(B-13)}$$

Repeating this process and transforming all terms in the series (B-11), we obtain the reciprocity relationship for the stochastic Green's function for a linear inhomogeneous medium

$$G(\bar{r},\bar{r}') = G(\bar{r}',\bar{r}) \qquad \text{(B-14)}$$

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