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Essential spanning forests and electric networks in groups

by

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Abstract

Essential spanning forests and electric networks in groups

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Let Γ be a Cayley graph of a finitely generated group G. Subgraphs which contain all vertices of Γ, have no cycles, and no finite connected components are called *essential spanning forests.* The set $\mathcal{Y}$ of all such subgraphs being given a compact topology, G acts on $\mathcal{Y}$ continuously. We define a G-invariant measure μ on $\mathcal{Y}$ and investigate ergodic properties of the process $(\mathcal{Y}, \mu, G)$, called an ESF.

The case $G = \mathbf{Z}^d$ or their finite extensions was studied by Pemantle and Burton. For a general G we establish mixing and give a sufficient condition for *directional tail triviality* in terms of the transfer-current function ψ. For non-co-compact Fuchsian groups we establish the tail triviality of $(\mathcal{Y}, \mu, G)$ and describe ψ, which by a theorem of Burton and Pemantle determines the measure μ.

The second part of this thesis is concerned with the relation between ESF and algebraic dynamical systems. Burton nd Pemanle computed the entropy of $(\mathcal{Y}, \mathbf{Z}^d)$, where $\mathcal{Y}$ is the set of essential spanning forests in an arbitrary $\mathbf{Z}^d$-periodic graph. Their formula turned out to be the same as the one obtained by Lind, Schmidt and Ward for the entropy of $\mathbf{Z}^d$-actions by automorphisms on certain compact subgroups of $(\mathbf{R}/\mathbf{Z})^{\mathbf{Z}^d}$. We give a direct proof of equalities of entropies for ESF processes and corresponding algebraic systems, answering the question of the above authors.

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Chapter 1

INTRODUCTION

We study an Essential Spanning Forest Process (or ESF); it is a dynamical system which is naturally associated to any finitely generated group G relative to a fixed set of generators S. Such systems were first studied by Pemantle in [Pem90], and Burton and Pemantle in [BP93] for groups $\mathbf{Z}^d$ and their finite extensions. There are interesting and profound relations between ESF's and random walks on groups, potential theory, infinite electric networks, and actions by automorphisms of compact Abelian groups.

1.1 Groups and their Cayley graphs

Our exposition in the beginning follows [GH].

Let G be a finitely generated group. Choose a finite set S of generators for G. For simplicity, we will assume that S is *symmetric*, i.e. $s \in S$ if and only if $s^{-1} \in S$, and that $id \notin S$, where id is the group identity. The set of generators being fixed, G can be given the *word metric*, namely, define the length $|v|$ of any $v \in G$ to be the smallest integer n such that there exists a sequence $s_1, \ldots, s_n$ of generators in G for which $v = s_1 \ldots s_n$, and define the distance $d : G \times G \to \mathbf{R}_+$ by $d(v_1, v_2) = |v_1^{-1} v_2|$.

A convenient way to visualize G is to introduce the Cayley graph $\Gamma(G, S)$: this is a graph with vertex set $V = G$, and edge set E such that two vertices v_1, v_2 are joined by an edge if and only if $v_1^{-1} v_2 \in S$. This gives a non-directed graph, without any loops or multiple edges, which is infinite whenever G is infinite. There is an obvious left action by G on this graph which is transitive on the set of vertices.

Each edge of $\Gamma(G, S)$ can be made a metric space isometric to the line segment $[0, 1]$, in such a way that the left action of G produces isometries between the edges. One defines naturally the length of a path between two points (not necessarily vertices) of the graph, and the distance between two points is defined to be the infimum of the appropriate path-lengths. In this way $\Gamma(G, S)$ is made into a metric space (with the metric denoted by ρ), and the natural inclusion $G \subset \Gamma(G, S)$ is an isometry.

Examples of Cayley graphs can be found in [GH] or [C].

1.2 Essential spanning forests

By a *subgraph* of $\Gamma(G, S)$ we mean any subcollection of edges. One can view a subgraph γ as a map from E to $\{0, 1\}$ by assigning 1's to the edges in γ. Let $\mathcal{X}$ denote the set of all subgraphs in $\Gamma(G, S)$. It can be identified with $\{0, 1\}^E$. Endow $\mathcal{X}$ with the *product topology*, i.e. the topology generated by the *cylinder sets*:

$$T(e_1, \dots, e_k; f_1, \dots, f_r) = \left\{ \gamma \in \mathcal{X} \ \text{ such that } \begin{array}{c} \gamma(e_i) = 1, \ i = 1, \dots, k \\ \gamma(f_j) = 0, \ j = 1, \dots, r \end{array} \right\},$$

where $\{e_1, \dots, e_k\}$ and $\{f_1, \dots, f_r\}$ are finite collections of edges. A set of the form

$$T(e_1, \dots, e_k;) = \{\gamma \in \mathcal{X} \ \text{ such that } \ \gamma(e_i) = 1, \ i = 1, \dots, k\}$$

we call an *elementary cylinder set.* We will also write $T(A)$ for such a set, if $A = \{e_1, \dots, e_k\}$.

It is well known that $\mathcal{X}$ is compact. The action of G on $\Gamma(G, S)$ induces the left action of G on $\mathcal{X}$ by homeomorphisms.

By a *cycle* in a graph we mean a finite collection of distinct edges $e_1, e_2, \dots, e_m$, such that each pair e_m, e_1 and e_i, e_{i+1}, $1 \leq i < m$, has a vertex in common, but $e_i \cap e_j = \emptyset$ for all other pairs of edges. A subgraph of Γ is called an *essential spanning forest* if it contains at least one edge incident to each vertex of Γ, has no cycles, and if all its connected components are infinite. Let $\mathcal{Y}$ be the set of all essential spanning forests in Γ. Then $\mathcal{Y}$ is closed and therefore compact: indeed, if $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{Y}$ such that $\gamma_n \to \gamma$ as $n \to \infty$, then γ_n eventually coincides with γ on any finite collection of edges. Thus, neither a finite connected component, nor a cycle can appear in γ. Clearly, $\mathcal{Y}$ is G-invariant.

We now define a Borel probability measure μ on $\mathcal{Y}$.
Let O denote the vertex corresponding to *id* . For each n, let Γ_n be the ball of radius n, centered at O, with the graph structure. Recall that a *spanning tree* in a finite graph $\tilde{\Gamma}$ is a connected subgraph which contains all the vertices of $\tilde{\Gamma}$ and has no cycles.

The set of spanning trees in Γ_n being finite, one can put on it the uniform probability measure $\mu(\Gamma_n)(\cdot)$. Since any spanning tree is a subgraph of the whole graph Γ as well, we view $\mu(\Gamma_n)(\cdot)$ as a discrete measure on $\mathcal{X}$.
In chapter 3 we will prove, following [Pem90], that as $n \to \infty$, the measures $\mu(\Gamma_n)$ converge in the weak* topology to a measure μ, (i.e. $\mu(\Gamma_n)(C) \to \mu(C)$ for every cylinder set C); the limiting measure is concentrated on the set of essential spanning forests and is G-invariant.

Thus, we obtain the process $(\mathcal{Y}, \mu, G)$, called ESF, which is the main object of our investigation.

In the sequel we will use the following notation for cylinder sets:

$$\{e_1, \ldots, e_k \in \mathbf{T}; f_1, \ldots, f_r \notin \mathbf{T}\} \stackrel{\text{def}}{=} T(e_1, \ldots, e_k; f_1, \ldots, f_r),$$

$$\{e_1, \ldots, e_k \in \mathbf{T}\} \stackrel{\text{def}}{=} T(e_1, \ldots, e_k;\,)$$

There are other possibilities for defining similar limiting measures, see e.g. [LyPe]. Inspired by [LyPe], we introduce in Chapter 3 a new measure $\tilde{\mu}$ on $\mathcal{Y}$. Comparison of μ and $\tilde{\mu}$ is used in Chapter 4 to prove tail-triviality of μ in special cases.

1.3 Ergodic properties

We first recall some fundamental definitions:

Mixing. Let K be a compact Hausdorff space and μ be a Borel probability measure on K. Suppose that a finitely generated group G acts on K by homeomorphisms which preserve μ.

The dynamical system (K, μ, G) is *mixing* if for any Borel sets A and B, and any sequence $g_n \in G$ such that, in a word metric, $|g_n| \to \infty$ as $n \to \infty$, one has

$$|\mu(A \cap g_n B) - \mu(A)\mu(B)| \to 0 \quad \text{as } n \to \infty. \tag{1.1}$$

Tail triviality. Let $\Gamma = \Gamma(G, S)$ and Γ_n, $n = 1, 2 \ldots$, be as in Section 1.2. Denote by $\mathcal{T}_n$ the σ-algebra generated by cylinder sets $T(A)$ and $T(A)^c$, where $A \subset \Gamma_n^c$. We say that $(\mathcal{Y}, \mu, G)$ (or μ, for short) has *trivial tail* if the *tail σ-algebra*

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \mathcal{T}_n$$

contains only sets of measure 0 or 1.

Bernoullicity. Let X be compact and μ_0 be a Borel probability measure on X. Consider the direct product of copies of X, indexed by G

$$K = \prod_G (X, \mu_0) = \{\kappa : \ G \to (X, \mu_0)\}.$$

We assume that K is endowed with the product topology and with the product measure which we denote by μ. There is a natural left action by G on K by measure-preserving homeomorphisms: if $g, g_1 \in G$, then

$$g.\kappa(g_1) = \kappa(gg_1).$$

The dynamical system (K, μ, G) is called a Bernoulli process and any dynamical system metrically isomorphic to such is said to be *Bernoulli*. (See [OW] for the definition of metrically isomorphic dynamical systems.)

It is not hard to see that tail triviality implies mixing.

Given a finite subgraph $H \subset \Gamma$, the *boundary* ∂H of H is the set of vertices in H which are incident to a vertex in H^c. We say that Γ has *moderate growth,* if there exists a sequence of subgraphs H_n, $n = 1, 2, \dots$, which exhaust Γ, such that

$$\frac{|\partial \Gamma_n|}{|\Gamma_n|} \to 0, \text{ as } n \to \infty.$$

Here $|\tilde{\Gamma}|$ stands for the number of vertices in $\tilde{\Gamma}$. A finitely generated group whose Cayley graph has moderate growth is called *amenable*, (see [LyPe] and [Green]).
In Chapter 3 we will prove that $(\mathcal{Y}, \mu, G)$ is mixing for any (G, S). This was independently proved by Lyons (see [LyPe]).

Bernoullicity of $(\mathcal{Y}, \mu, G)$ for any amenable group G was proved by Burton and Steif in [BS96]. Tail triviality in case $G = \mathbf{Z}^d$ was proved by Pemantle (see [Pem90]); a proof for all amenable groups can be found in [LyPe]. In Chapter 4 we establish the tail triviality for a class of non-amenable groups, *viz* non-co-compact Fuchsian groups.

Rather than the whole tail $\mathcal{T}$, one can consider subalgebras $\mathcal{T}(\beta)$, which are tails in particular directions, β being infinite paths in Γ. More precisely, we say that a path $\beta = \beta_1, \beta_2 \cdots$ in $\Gamma(G)$ is *regular*, if for any n, $(\beta_1, \beta_2 \dots, \beta_n)$ is a shortest path between its endpoints.

Given a regular path $\beta = \beta_1, \beta_2, \dots$, denote by $Q_n(\beta)$ the set of all regular α, such that $\alpha_i = \beta_i$ for $1 \leq i \leq n$. Let $V_n(\beta)$ be the set of vertices encountered in such paths outside Γ_n, in other words, $V_n(\beta)$ is the set of all endpoints of edges in $\bigcup_{\alpha \in Q_n(\beta)} \{\alpha_i, i > n\}$. Next, let $E_n(\beta)$ be the set of edges with endpoints in $V_n(\beta)$. Denote by $\sigma_n(\beta)$ the σ-field generated by the cylinder sets $T(A)$, for $A \subset E_n(\beta)$. Finally, denote $\mathcal{T}(\beta) = \bigcap_{n=1}^{\infty} \sigma_n(\beta)$.

We say that *directional tail* is trivial, if $\mathcal{T}(\beta)$ is trivial for any regular path β. Clearly, directional tail triviality is stronger than mixing. We give a sufficient condition for $(\mathcal{Y}, \mu, G)$ to have trivial directional tail in terms of the transfer-current function which is discussed below.

1.4 Transfer-current function

The transfer-current function ψ is our main tool for investigating the measure μ. For a finite graph $\tilde{\Gamma}$, the function ψ is defined by viewing the graph as an electric network with 1-ohm resistances assigned to each edge. The domain of ψ is the set of all pairs of directed edges $(\vec{e}_1, \vec{e}_2)$, and $\psi(\vec{e}_1, \vec{e}_2)$ is the current through edge $\vec{e}_2$, given that a properly normalized battery is applied to $\vec{e}_1$.

Let $e_1, \ldots, e_k$ be edges in $\tilde{\Gamma}$. Pick an orientation arbitrarily. The Transfer-Impedance Theorem due to Burton and Pemantle [BP93] asserts that the measure of any elementary cylinder set is computed as

$$\mu(\{e_1, \ldots, e_k \in \mathbf{T}\}) = \det M(e_1, \ldots, e_k), \tag{1.2}$$

where μ is the uniform measure on spanning trees in $\tilde{\Gamma}$, and $M(e_1, \ldots, e_k)$ is the $k \times k$ matrix with entries $\psi(\vec{e}_i, \vec{e}_j)$. Using the theory of random walks, Burton and Pemantle proved for amenable groups that, if Γ_n are finite subgraphs of $\Gamma(G, S)$, as above, and ψ_n are the corresponding transfer-current functions, then ψ_n converge edge-wise to a limiting function ψ which determines μ by (1.2). Their proof relies on the triviality of the Poisson boundary and cannot be extended to the general case.

We provide an elementary observation that for any (G, S) the functions ψ_n^2 converge. Any limit point for the sequence $\{\psi_n\}_{n=1}^{\infty}$ would be sufficient for our considerations (i.e. will determine the limiting measure μ by (1.2)). It is true, however, that actually the ψ_n converge edge-wise and in a certain Hilbert space. A proof, relying on functional analysis can be found in [So94] or [LyPe].

In Chapter 4 we focus our attention on the class of groups isomorphic to non-co-compact Fuchsian groups. The Cayley graph of any such group can be embedded quasi-isometrically into the hyperbolic plane (see [GH] for the definition of quasi-isometry). We establish the exponential behavior of ψ and obtain, as a corollary, that the rate of convergence to mixing is exponential. We present the system of equations, in terms of generators and relations in the group, which uniquely determines ψ. This is used to prove that the tail σ-algebra is trivial.

In the case $G = \mathbf{Z}^d$ the transfer-current function ψ was computed explicitly by Burton and Pemantle [BP93] by means of harmonic analysis. We compute ψ explicitly for the Modular and Hecke groups by geometric methods.

1.5 Relation with algebraic dynamical systems

There is a profound and not totally understood relation between ESF's and group actions by automorphisms on compact Abelian groups.

Let Γ be a periodic graph with the vertex set $\mathbf{Z}^d$. (This is a slightly more general situation than before, since multiple edges are allowed). The set of essential spanning forests $\mathcal{Y}$ and measure μ are defined in the same way. Burton and Pemantle computed the topological entropy of $(\mathcal{Y}, \mathbf{Z}^d)$, and their formula turned out to be the same as one obtained previously by Lind, Schmidt, and Ward for $\mathbf{Z}^d$-actions on certain compact subgroups of $(\mathbf{R}/\mathbf{Z})^{\mathbf{Z}^d}$. The question was to explain the coincidence. In Chapter 5 we give a direct proof of equality of entropies.

Moreover, if Γ is a $\mathbf{Z}^d$-periodic graph, and $\mathcal{K}$ is the corresponding compact abelian group, endowed with the Haar measure λ, then $(\mathcal{Y}, \mu, \mathbf{Z}^d)$ and $(\mathcal{K}, \lambda, \mathbf{Z}^d)$ are both Bernoulli, ([BS96] and [Sch]), and have the same value of measure-theoretical entropy, because μ and λ are the measures of maximal entropy in each system, respectively. Hence, by Ornstein and Weiss's theorem [OW], they are metrically isomorphic. Constructing such an isomorphism explicitly is an open problem.

1.6 Summary

In Chapter 2 we explain the relation between spanning trees and electric networks in finite graphs.

In Chapter 3 we construct measures μ and $\widetilde{\mu}$, define the transfer-current function ψ on $\Gamma(G, S)$, prove that $(\mathcal{Y}, \mu, G)$ is mixing, and give a sufficient condition for directional tail triviality in terms of ψ.

Chapter 4 is devoted to the case of non-co-compact Fuchsian groups.

In Chapter 5 we give a direct proof of coincidence of entropies of ESF's and algebraic dynamical systems.

Chapter 2

SPANNING TREES AND FINITE ELECTRIC NETWORKS

In this chapter we will explain the relation between uniform measures on spanning trees and electric networks, and state some basic results which will be used afterwards.

We assume throughout this section that $\tilde{\Gamma}$ is a finite connected graph, which does not contain self-loops, i.e. edges connecting a vertex to itself. Let V, E denote the set of vertices and of edges, respectively. For distinct $x, y \in V$, write $x \sim y$ if they are joined by at least one edge, and set $k(x, y)$ to be the number of edges joining x, y. By a function on the graph we mean a function on its vertices. A function $\nu : \tilde{\Gamma} \to \mathbf{R}$ is called harmonic at x, if

$$\sum_{y \sim x} (\nu(y) - \nu(x)) = 0,$$

where the summation goes over all y adjacent to x, counting multiplicities.

If $A \subset V$, and $\nu : \tilde{\Gamma} \to \mathbf{R}$ is harmonic at each $x \in V \backslash A$, we say that ν is *harmonic with boundary conditions* $\nu|_A$.

Theorem 2.1 *For any subset $A \subset V$, and any function $\nu_A : A \to \mathbf{R}$, there exists a unique function $\nu : \tilde{\Gamma} \to \mathbf{R}$ which is harmonic with boundary conditions ν_A.*

See [DS] for the proof.

Given $\tilde{\Gamma}$, we can view it as an electric network by assigning to each edge a 1-ohm resistance. Choose two vertices a and b and put a one-volt battery across these points, establishing a voltage $\nu(a) = 1$, $\nu(b) = 0$. This induces a voltage $\nu(x)$ at each vertex x, and currents i_{xy} in the circuit. By Ohm's Law, the currents through resistors are determined by the voltages by

$$i_{xy} = \nu(x) - \nu(y). \tag{2.1}$$

Kirchhoff's Current Law requires that the total current flowing from any point other than a or b is 0. That is, for $x \neq a, b$

$$i_x = \sum_y i_{xy} = 0.$$

Hence by (2.1) the voltage function is harmonic with boundary conditions: $\nu(a) = 1$, $\nu(b) = 0$.

It is easy to check that $i_a = -i_b$. We call i_a the *total current* through the network, (it comes into the circuit from the outside source). It is natural, in view of Ohm's Law, to call

$$B = \frac{\nu(a) - \nu(b)}{i_a} = \frac{1}{i_a}$$

the *effective resistance* between the nodes a, b. If, rather than a unit battery, we apply a battery of value B across a, b, all currents are multiplied by B, and the total current through the network is one. The same consideration applies to any edge $e \in E$. We denote by $B(e)$ the effective resistance of e. In other words, a battery $B(e)$ applied to e induces the unit current into the circuit.

More generally, one can impose voltages $\nu_1, \ldots, \nu_k$ on vertices $a_1, \ldots, a_k$. The resulting voltage at other vertices is harmonic with boundary conditions: $\nu(a_1) = \nu_1, \ldots, \nu(a_k) = \nu_k$.

Let $\mu = \mu(\tilde{\Gamma})$ be the uniform probability measure on the set of spanning trees in $\tilde{\Gamma}$. Recall that $\{e_1, \ldots, e_k \in \mathbf{T}\}$ is the set of all subgraphs of $\tilde{\Gamma}$, containing edges e_1 through e_k. To make notations consistent, regard μ as a measure on the set of all subgraphs of $\tilde{\Gamma}$.

Theorem 2.2 (Kirchhoff, 1847) *For any edge e*

$$\mu(\{e \in \mathbf{T}\}) = B(e).$$

See [Pem90] for the proof.

Let $E_d = E \times \{0,1\}$ be the set of directed edges. Given $\vec{e}, \vec{f} \in E_d$, set $\psi(\vec{e}, \vec{f})$ to be the current through the resistor f, resulting from applying a battery $B(e)$ to e. The function $\psi : E_d \times E_d \to \mathbf{R}$ thus defined is called the *transfer-current function.*

Theorem 2.3 *(Matrix-Impedance). Let $e_1, \ldots, e_k$ be any edges in $\tilde{\Gamma}$. Pick an orientation for each e_i, $1 \leq i \leq k$ arbitrarily. Then*

$$\mu(\tilde{\Gamma})(\{e_1, \ldots e_k \in \mathbf{T}\}) = \begin{vmatrix} \psi(\vec{e}_1, \vec{e}_1) & \ldots & \psi(\vec{e}_1, \vec{e}_k) \\ & \ldots & \\ \psi(\vec{e}_k, \vec{e}_1) & \ldots & \psi(\vec{e}_k, \vec{e}_k) \end{vmatrix} \tag{2.2}$$

This theorem is due to Burton and Pemantle. See [LyPe] for the proof.

REMARK: A change of the orientation of any e_i would result in multiplying the i^{th} column and the i^{th} row by -1 and would not effect $\det M$.

We will need the following notions of *deletions* and *contractions* of graphs: The deletion of an edge e from $\tilde{\Gamma}$ is the graph $\tilde{\Gamma} - e$ consisting of all edges of $\tilde{\Gamma}$ except e. The contraction of $\tilde{\Gamma}$ by the edge $e = xy$ is the graph $\tilde{\Gamma}/e$ obtained by removing all edges between x and y, and identifying x and y. This may result in parallel edges, which must still be regarded as distinct.

Contraction commutes and associates with deletion, so it makes sense to speak of the graph $\tilde{\Gamma}$ with $e_1, \ldots, e_r$ contracted and $e'_1 \ldots e'_s$ deleted.

Electrically, deletion of an edge e amounts to making its resistance in the network $\tilde{\Gamma}$ infinite , while contraction by e amounts to making its resistance in $\tilde{\Gamma}$ equal zero and leaving all other resistances the same.

More generally, let $R \subset V \times V$ be an equivalence relation on the set of vertices. Denote by $[v]$ the equivalence class of v. Define the contraction $\tilde{\Gamma}/R$ to be the graph with the vertex set V/R and the number of edges connecting $[v_1]$, $[v_2]$ equal to the total number of edges in $\tilde{\Gamma}$ connecting a vertex in $[v_1]$ to a vertex in $[v_2]$. All self-loops must be erased.

Proposition 2.4 *Let* $\{e_1, \ldots, e_m\}$ *be a finite collection of edges in* $\tilde{\Gamma}$.
If $\mu(\tilde{\Gamma})(\{e_2, \ldots, e_m \in \mathbf{T}\}) \neq 0$, *then*

$$\mu(\tilde{\Gamma})(\{e_1 \in \mathbf{T}\}|\{e_2, \ldots, e_m \in \mathbf{T}\}) = \mu(\tilde{\Gamma}/\{e_2, \ldots, e_m\})(\{e_1 \in \mathbf{T}\}).$$

If $\mu(\tilde{\Gamma})(\{e_2, \ldots, e_m \notin \mathbf{T}\}) \neq 0$, *then*

$$\mu(\tilde{\Gamma})(\{e_1 \in \mathbf{T}\}|\{e_2, \ldots, e_m \notin \mathbf{T}\}) = \mu(\tilde{\Gamma} - \{e_2, \ldots, e_m\})(\{e_1 \in \mathbf{T}\}).$$

Here $\mu(\tilde{\Gamma})(C_1|C_2)$ stands for the conditional measure on C_2.

See [Pem90], p.1562.

An important tool is the following

Theorem 2.5 (Rayleigh's Monotonicity Law) *If the resistances of a circuit are increased, the effective resistance between any two points can only increase. If they are decreased, it can only decrease.*

See [DS] for a proof.

An application of Rayleigh's Monotonicity Law, which we will now demonstrate, will be repeatedly used in this work.

Example. Let e, e_1 be distinct edges in $\tilde{\Gamma}$. By Rayleigh's Monotonicity Law, the effective resistance of e_1 in $\tilde{\Gamma}$ is less than or equal to the effective resistance of e_1 in $\tilde{\Gamma} - e$. Hence, by Theorem 2.2,

$$\mu(\tilde{\Gamma})(\{e_1 \in T\}) \leq \mu(\tilde{\Gamma} - e)(\{e_1 \in T\}).$$

The latter inequality has a geometric explanation, as well: the more choices one has to pick edges for a spanning tree, the smaller is the chance of a particular edge to be chosen.

Chapter 3

ESSENTIAL SPANNING FORESTS PROCESSES: GENERAL CASE

3.1 Invariant measures

Let $\Gamma = \Gamma(G, S)$ be the Cayley graph of a finitely generated group G relative to a set of generators S. Recall that $\mathcal{X} = \{0,1\}^E$ is the set of all subgraphs of Γ, and $\Gamma_n \subset \Gamma$ is the ball of radius n centered at O. For a finite connected graph $\tilde{\Gamma}$ we write $\mathcal{S}(\tilde{\Gamma})$ for the set of all spanning trees in $\tilde{\Gamma}$.

Clearly, Γ_n is connected, thus $\mathcal{S}(\Gamma_n)$ is finite and non-empty. Therefore, the uniform probability measure on $\mathcal{S}(\Gamma_n)$, denoted by $\mu(\Gamma_n)$, is well-defined. Each spanning tree being a subgraph of Γ, we regard $\mu(\Gamma_n)$ as a discrete measure on $\mathcal{X}$.

Theorem 3.1 *As $n \to \infty$, $\mu(\Gamma_n)$ converges in the weak* topology. The limiting measure μ is concentrated on the set of essential spanning forests and is G-invariant.*

PROOF: First, observe that if A' is a finite collection of edges in Γ_n, then

$$\Gamma_n/A' \subset \Gamma_{n+1}/A'.$$

Hence, by Rayleigh's Monotonicity Law, for any edge e_0

$$\mu(\Gamma_n/A')(\{e_0 \in \mathbf{T}\} \geq \mu(\Gamma_{n+1}/A')(\{e_0 \in \mathbf{T}\}. \tag{3.1}$$

Given a finite set $A = \{e_1, \ldots, e_m\} \subset \Gamma_n$, we have

$$\mu(\Gamma_n)(\{e_1, \ldots, e_m \in \mathbf{T}\}) =$$

$$\mu(\Gamma_n)(\{e_1 \in \mathbf{T}\}|\{e_2,\ldots,e_m \in \mathbf{T}\}) \cdot \mu(\Gamma_n)(\{e_2,\ldots,e_m \in \mathbf{T}\}) =$$

$$\mu(\Gamma_n/\{e_2,\ldots,e_m\})(\{e_1 \in \mathbf{T}\}) \cdot \mu(\Gamma_n)(\{e_2,\ldots,e_m \in \mathbf{T}\}),$$

by Proposition 2.4.

It is easy to show by induction, using (3.1), that $\mu(\Gamma_n)(T(A))$ is decreasing in n for any elementary cylinder set $T(A)$). Hence, $\mu(\Gamma_n)$ converges on such sets and on all cylinder sets as well, by inclusion-exclusion principle. This uniquely defines a probability measure μ on $\mathcal{X}$.

Next, observe that μ is carried by the set of essential spanning forests: for instance, if $e_1,\ldots,e_m$ form a cycle, then $\mu(\Gamma_n)(\{e_1,\ldots,e_m \in \mathbf{T}\}) = 0$ for all large enough n. Hence, $\mu(\{e_1,\ldots,e_m \in \mathbf{T}\}) = 0$. Similarly, any particular finite connected component occurs with zero probability at each finite stage, and therefore, will not occur μ.-a.s.

To show that μ is G-invariant, observe that for any $g \in G$ and any cylinder set C,

$$\mu(\Gamma_n)(gC) = \mu(g^{-1}\Gamma_n)(C).$$

If K is such that $g^{-1}\Gamma_n \in \Gamma_K$, then

$$\mu(\Gamma_n)(gC) = \mu(g^{-1}\Gamma_n)(C) \geq \mu(\Gamma_K)(C).$$

Passing to the limit yields

$$\mu(gC) \geq \mu(C).$$

Since g and C were arbitrary, the result follows.

REMARK: This proof follows the proof in [Pem90] for the case $G = \mathbf{Z}^d$, see also [LyPe].

Next we define another Borel probability measure on $\mathcal{Y}$. Recall that, given a finite subgraph $H \subset \Gamma$, by the *boundary* ∂H of H we mean the set of vertices in H which are adjacent to a vertex in H^c.

Given n, say that v_1, $v_2 \in \partial\Gamma_n$ are equivalent, if they can be connected by a path in Γ, which lies outside B_{n-1}. Let R_n be the equivalence relation just defined, with equivalence classes denoted by $[v]$, $v \in B_n$. (If $v \in B_{n-1}$, then $[v]$ consists of just one element). Consider

$$\Gamma^{(n)} = \Gamma_n / R_n.$$

(See Chapter 2 for the definition of contraction.)

For each n, $\Gamma^{(n)}$ is a finite graph, so $\mu(\Gamma^{(n)})(\cdot)$ is well-defined. However, $\Gamma^{(n)}$ is not a subgraph of Γ, and, therefore, $\mu(\Gamma^{(n)})$ cannot be regarded as a measure on $\mathcal{X}$. In order to make sense of a limiting measure for $\mu(\Gamma^{(n)})$, $n = 1, 2, \ldots$, we consider projections

$$\pi_n : S(\Gamma^{(n)}) \rightarrow F(\Gamma_{n-1}),$$

where $F(\Gamma_{n-1})$ is the set of spanning forests in Γ_{n-1}. More precisely, if $\tau \in S(\Gamma^{(n)})$, then $\pi_n(\tau)$ is defined to be the subgraph of Γ_{n-1} obtained by erasing all vertices and edges outside of Γ_{n-1}.

Now, $\tilde{\mu}(\Gamma_{n-1}) \overset{\text{def}}{=} \pi_n(\mu(\Gamma^{(n)}))$ is a measure on $\mathcal{X}$, and the following is true:

Theorem 3.2 *As $n \rightarrow \infty$, $\tilde{\mu}(\Gamma_n)$ converges in the weak* topology. The limiting measure $\tilde{\mu}$ is concentrated on the set of essential spanning forests and is G-invariant.*

PROOF: Observe that $\Gamma_n \subset \Gamma^{(n+1)}$, and two vertices v_1, $v_2 \in \partial\Gamma_n$ can be connected by a path in $\Gamma^{(n+1)}$ which lies outside of Γ_{n-1} if, and only if the same two vertices can be connected by a path in Γ which lies outside of Γ_{n-1}. Moreover, it is easy to see that

$$\Gamma^{(n)} = \Gamma^{(n+1)} / \{e_1, \ldots, e_m\},$$

where $\{e_1, \ldots, e_m\}$ is the set of edges xy such that $x \in \partial\Gamma_n$, and $y \in \partial\Gamma_{n+1}$. Thus, by Rayleigh's Monotonicity Law, for any elementary cylinder set $T(A))$, and n large enough,

$$\mu(\Gamma^{(n)})(T(A)) \leq \mu(\Gamma^{(n+1)})(T(A)).$$

By definition of $\tilde{\mu}(\Gamma_n)$,

$$\tilde{\mu}(\Gamma_{n-1})(T(A)) \leq \tilde{\mu}(\Gamma_n)(T(A)).$$

Hence, $\tilde{\mu}(\Gamma_n)$ converges on elementary cylinder sets, and therefore determines a probability measure $\tilde{\mu}$ on $\mathcal{X}$. As in proof of Theorem 3.1, $\tilde{\mu}(\mathcal{Y}) = 1$.
To show G-invariance of $\tilde{\mu}$, let

$$g^{-1}\Gamma^{(n)} = g^{-1}(\Gamma_n)/g^{-1}R_n \quad \text{for } g \in G,$$

where R_n is the equivalence relation defined above.

Fix g, and let K be such that $g^{-1}\Gamma_n \subset \Gamma_K$. It is easy to see that $g^{-1}\Gamma^{(n)}$ can be obtained from $\Gamma^{(K)}$ by contracting edges in $\Gamma^{(K)} \setminus g^{-1}\Gamma_n$. Hence, for any cylinder set C, any $g \in G$, and large enough n and K,

$$\tilde{\mu}(\Gamma_{n-1})(gC) = \mu(g^{-1}\Gamma^{(n)})(C) \leq \mu(\Gamma^{(K)}(C) = \tilde{\mu}(\Gamma_{K-1})(C).$$

Passing to the limit implies $\tilde{\mu}(gC) \leq \tilde{\mu}(C)$. Since g and C were arbitrary, the result follows.

Remark: Another way to define a similar limiting measure, say $\tilde{\tilde{\mu}}$, is to set for each n all vertices in the boundary $\partial\Gamma_n$ to be equivalent. Such a measure is considered in [LyPe]; it is closely related to random walks on G. We have the following relation :

$$\tilde{\tilde{\mu}} \preceq \tilde{\mu} \preceq \mu,$$

where '$\preceq$' means *stochastically dominated*, (see[LyPe] for the definition). In many cases two or even all tree measures μ, $\tilde{\mu}$ and $\tilde{\tilde{\mu}}$ coincide.

Let $\mathcal{T}_n$ be the Borel σ-algebra generated by cylinder sets depending only on edges outside Γ_n. Recall that the tail σ-algebra is

$$\mathcal{T} = \cap_{n=1}^{\infty} \mathcal{T}_n.$$

It was proved in [LyPe] that if $\tilde{\tilde{\mu}} = \mu$, then the tail σ-algebra is trivial. By a similar argument we prove the following

Proposition 3.3 *If* $\tilde{\mu} = \mu$, *then the tail σ-algebra is trivial, i.e. contains only sets of μ-measure* 0 or 1.

Proof: Let $C \in \mathcal{T}$ be a tail event such that $\mu(C) \neq 0$. Approximate C by the cylinder sets C_n, $n = 1, 2, \ldots$, so that

$$\mu(C_n \triangle C) \to 0, \text{ as } n \to \infty,$$

and C_n depends only on edges outside Γ_n. By Rayleigh's Monotonicity Law, given an elementary cylinder set $T(A)$ such that $A \subset \Gamma_{n-1}$, we have

$$\tilde{\mu}(\Gamma_{n-1})(T(A)) = \mu(\Gamma^{(n)})(T(A)) \leq \mu(T(A)|C_n) \leq \mu(\Gamma_n)(T(A)).$$

By Theorems 3.1 and 3.2, $\mu(\Gamma_n)(T(A))$ and $\tilde{\mu}(\Gamma_{n-1})(T(A))$ converge as $n \to \infty$. Hence,

$$\mu(T(A)|C_n) \to \mu(T(A)), \text{ as } n \to \infty,$$

implying

$$\mu(T(A)|C) = \mu(T(A)). \tag{3.2}$$

By inclusion-exclusion principle, (3.2) is true for any cylinder set B replacing $T(A)$. Choosing cylinder sets approximating C, and passing to the limit, we obtain

$$\mu(C|C) = \mu(C).$$

Hence, $\mu(C) = 1$, q.e.d.

3.2 Transfer-current function.

Let $E_d = E \times \{0, 1\}$ be the set of edges with a chosen direction. If $\vec{e} \in E_d$, we denote its endpoints by $s(\vec{e})$, $t(\vec{e})$, so that $\vec{e}$ is directed from $s(\vec{e})$ to $t(\vec{e})$, and we write e for the underlying non-directed edge.

Lemma 3.4 *Let $\tilde{\Gamma}$ be a finite graph, and ψ be the transfer-current function, (see Chapter 2). Then ψ is symmetric, i.e., for any directed edges $\vec{e}_1$ and $\vec{e}_2$, one has $\psi(\vec{e}_1, \vec{e}_2) = \psi(\vec{e}_2, \vec{e}_1)$.*

PROOF: For vertices v, u denote by $k(v,u)$ the number of edges joining v to u.

Enumerate all vertices $v_1, \ldots, v_n$ of $\tilde{\Gamma}$, and define the Kirchhoff matrix of $\tilde{\Gamma}$ to be $n \times n$ matrix K with entries :

$$K(i,i) = deg(v_i, \tilde{\Gamma})$$

$$K(i,j) = -k(v_i, v_j) \text{ if } i \neq j$$

If we impose voltages ν_i, $i = 1, \ldots, n$ on the vertices, and denote $\nu = (\nu_1, \ldots, \nu_n)^T$, then

$$I = K\nu$$

is the vector of currents through the vertices. Indeed, for any $1 \leq i \leq n$,

$$I_i = (K\nu)_i = \sum_{v_i \sim v_j} (\nu(v_i) - \nu(v_j)).$$

As usual, the summation is over all vertices v_j adjacent to v_i, counting multiplicities.

Let $\bar{\nu} = (1, 1, \ldots 1)^T$. Then $K\bar{\nu} = 0$, and, conversely, if $K\nu' = 0$ for some ν', then ν' is a scalar multiple of $\bar{\nu}$. (This amounts to the fact that all harmonic functions on $\tilde{\Gamma}$ are constant). Let $H \subset \mathbf{R}^N$ be the hyperplane orthogonal to $\bar{\nu}$. Clearly, K is one-to-one on H.

Define an $N \times N$ matrix L by

$$L(i,j) = K(i,j) + \frac{1}{N}, \qquad 1 \leq i,j \leq N.$$

Then $L\nu = K\nu$ for any $\nu \in H$, and $L(\bar{\nu}) = \bar{\nu}$, so L is invertible.

Fix edges $\vec{e}_1$ with the endpoints v_i, v_j, and $\vec{e}_2$ with the endpoints v_r, v_s. Applying a battery $B(e_1)$ to e_1 will induce current 1 through v_i, current -1 through v_j, and zero currents through all other vertices, by Kirchhoff's Law.

Denote by $\eta = (\eta_1, \ldots, \eta_N)^T$ the corresponding currents vector, i.e.

$$\eta_i = 1,$$

$$\eta_j = -1,$$

$$\eta_k = 0 \quad \text{if } k \neq i, j.$$

The voltage ν induced by the battery above is defined up to a constant and can be chosen $\nu = L^{-1}\eta$. Observe that ν equals the difference of the i^{th} and j^{th} columns of L^{-1}. Accordingly, the current through $\vec{e}_2$ is

$$\psi(\vec{e}_1, \vec{e}_2) = \nu_r - \nu_s = L^{-1}(r,i) - L^{-1}(r,j) - L^{-1}(s,i) + L^{-1}(s,j).$$

But L^{-1} is symmetric, so $\psi(\vec{e}_1, \vec{e}_2) = \psi(\vec{e}_2, \vec{e}_1)$. *q.e.d.*

REMARK: The symmetry of the transfer-current function can be proved similarly for networks with arbitrary resistances.

Now let $\Gamma = \Gamma(G,S)$, and Γ_n be as before. Pick edges $\vec{e}_1$ and $\vec{e}_2$. Denote by ψ_n the transfer-current function in Γ_n, $n = 1, 2, \ldots$.

By the Matrix-Impedance Theorem,

$$\mu(\Gamma_n)(\{e_1, e_2 \in \mathbf{T}\}) = \begin{vmatrix} \psi_n(\vec{e}_1, \vec{e}_1) & \psi_n(\vec{e}_1, \vec{e}_2) \\ \psi_n(\vec{e}_2, \vec{e}_1) & \psi_n(\vec{e}_2, \vec{e}_2) \end{vmatrix},$$

and

$$\mu(\Gamma_n)(\{e_i \in \mathbf{T}\}) = \psi(\vec{e}_i, \vec{e}_i) \text{ for } i = 1, 2. \tag{3.3}$$

By Theorem 3.1, the determinant and diagonal entries converge, hence, by symmetry, $\psi_n(\vec{e}_1, \vec{e}_2)^2$ converges, as $n \to \infty$.

[So94] asserts that, actually, $\psi_n(\vec{e}_1, \vec{e}_2)$ converges, so it makes sense to talk about infinite electric networks. Let ψ be the limiting function. Clearly, for any edges $e_1, \ldots e_k$, one can apply (2.2) to Γ_n and ψ_n and pass to the limit as $n \to \infty$ to obtain

$$\mu(\Gamma)(\{e_1,\dots e_k \in \mathbf{T}\}) = \begin{vmatrix} \psi(\vec{e}_1,\vec{e}_1) & \dots & \psi(\vec{e}_1,\vec{e}_k) \\ & \dots & \\ \psi(\vec{e}_k,\vec{e}_1) & \dots & \psi(\vec{e}_k,\vec{e}_k) \end{vmatrix},$$

The same argument applies to $\Gamma^{(n)}$, $n = 1, 2, \dots$. Denote the transfer current function in $\Gamma^{(n)}$ by $\tilde{\psi}_n$. Then for any $\vec{e}_1$ and $\vec{e}_2$ the sequence $\tilde{\psi}_n^2(\vec{e}_1,\vec{e}_2)$ converges. One can chose a subsequence n_j, $j = 1, 2, \dots$, and a function $\tilde{\psi}$, such that for any $\vec{e}_1$ and $\vec{e}_2$

$$\tilde{\psi}_{n_j}(\vec{e}_1,\vec{e}_2) \to \tilde{\psi}(\vec{e}_1,\vec{e}_2).$$

Applying he Transfer-Impedance Theorem to graphs $\Gamma^{(n_j)}$ and passing to the limit gives for any edges $e_1, e_2, \dots, e_k$

$$\tilde{\mu}(\Gamma)(\{e_1,\dots e_k \in \mathbf{T}\}) = \begin{vmatrix} \tilde{\psi}(\vec{e}_1,\vec{e}_1) & \dots & \tilde{\psi}(\vec{e}_1,\vec{e}_k) \\ & \dots & \\ \tilde{\psi}(\vec{e}_k,\vec{e}_1) & \dots & \tilde{\psi}(\vec{e}_k,\vec{e}_k) \end{vmatrix}.$$

We will use $\tilde{\psi}$ in Theorem 4.3 below.

3.3 Mixing

Lemma 3.5 *Let $e_1, e_2, \dots$ be a sequence of edges such that $\rho(e_1, e_n) \to \infty$, as $n \to \infty$. Then $\psi(\vec{e}_1, \vec{e}_n) \to 0$ as $n \to \infty$.*

PROOF:

We may assume that $\rho(O, e_n) = n$, so that for any $K > n + 1$ we have $e_n \in \Gamma_K \setminus \Gamma_{n-1}$. Since $\Gamma_{n-1} \subset \Gamma_K - e_n$, by Rayleigh's Monotonicity Law

$$\mu(\{e_1 \in \mathbf{T}\}) \leq \mu(\Gamma_K - e_n)(\{e_1 \in \mathbf{T}\}) \leq \mu(\Gamma_{n-1})(\{e_1 \in \mathbf{T}\}).$$

By Proposition 2.4,

$$\mu(\Gamma_K - e_n)(\{e_1 \in \mathbf{T}\}) = \mu(\Gamma_K)(e_1 \in \mathbf{T} \mid e_n \notin \mathbf{T}).$$

Pick an $\epsilon > 0$. One can find N, such that for any $n \geq N$, $K > n+1$,

$$|\mu(\Gamma_K)(e_1 \in \mathbf{T} | e_n \notin \mathbf{T}) - \mu(e_1 \in \mathbf{T}))| < \epsilon$$

Then

$$|\mu(\Gamma_K)(e_1 \in \mathbf{T} | e_n \in \mathbf{T}) - \mu(e_1 \in \mathbf{T})| < \epsilon \tag{3.4}$$

Let

$$M_{K,n} = \begin{vmatrix} \psi_K(\vec{e}_1, \vec{e}_1) & \psi_K(\vec{e}_1, \vec{e}_n) \\ \psi_K(\vec{e}_n, \vec{e}_1) & \psi_K(\vec{e}_n, \vec{e}_n) \end{vmatrix}$$

By Theorem 2.3,

$$\det M_{K,n} = \mu(\Gamma_K)(\{e_1 \in \mathbf{T}\} | \{e_n \in \mathbf{T}\}) \cdot \mu(\Gamma_K)(\{e_n \in \mathbf{T}\}).$$

Hence,

$$|\det M_{K,n} - \psi_K(\vec{e}_1, \vec{e}_1) \cdot \psi_K(\vec{e}_n, \vec{e}_n)| < \epsilon \cdot \mu(\{e_n \in \mathbf{T}\}) < \epsilon$$

for any $K > n+1$. In other words, $\psi_K(\vec{e}_1, \vec{e}_n) \to 0$, as $n \to \infty$, uniformly in K. Passing to the limit in K first, implies that $\lim_{n \to \infty} \psi(\vec{e}_1, \vec{e}_n) = 0$, *q.e.d.*

Theorem 3.6 *If G is a finitely generated group with the set of generators S, and Γ is the Cayley graph, then ESF on $\Gamma(G, S)$ is mixing.*

PROOF: Let $A = \{e_1, \dots, e_k\}$, $B = \{e_{k+1}, \dots, e_r\}$ be finite sets of edges. Recall that $|g|$ stands for the length of $g \in G$ in the word metric. By Corollary 4.4, ([BP93], p. 1343), it suffices to show that if $g_n \in G$, $n = 1, \dots$, and $|g_n| \to \infty$, then

$$\mu(\mathbf{T}(A) \cap \mathbf{T}(g_n B)) \to \mu(\mathbf{T}(A)) \cdot \mu(\mathbf{T}(B)).$$

Let M_n be the $r \times r$ transfer-current matrix for edges

$$e_1, e_2, \ldots, e_k, g_n e_{k+1}, \ldots, g_n e_r.$$

Write M_n as a block matrix

$$M_n = \begin{bmatrix} A_n & C_n \\ C_n^T & B_n \end{bmatrix}, \text{ where } A_n \text{ is } k \times k \text{ matrix,}$$

and B_n is $(r-k) \times (r-k)$, and observe that, because of translation invariance of μ, for any n

$$\begin{aligned} \det A_n &= \mu(\mathbf{T}(A)) \\ \det B_n &= \mu(\mathbf{T}(B)). \end{aligned}$$

By Lemma 3.4, $C_n \to 0$. Thus $\det M_n \to \mu(\mathbf{T}(A)\mu(\mathbf{T}(B))$, implying the result.

3.4 *Directional tail triviality*

Let G be finitely generated, $\Gamma = \Gamma(G, S)$, and ψ be the transfer-current function. A path $\alpha = \alpha_1, \alpha_2 \cdots$ in $\Gamma(G)$ we will call *regular*, if for any n, $(\alpha_1, \alpha_2 \ldots, \alpha_n)$ is a shortest path between its endpoints.

Given a regular path $\beta = \beta_1, \beta_2, \ldots$, denote by $Q_n(\beta)$ the set of all regular α such that $\alpha_i = \beta_i$ for $1 \le i \le n$. Let $V_n(\beta)$ be the set of all vertices in $Q_n(\beta) \cap \Gamma_n^c$, or, more precisely, the set of endpoints for edges in $\bigcup_{\alpha \in Q_n(\beta)} \{\alpha_i, i > n\}$. Next, let $E_n(\beta)$ be the set of edges with endpoints in $V_n(\beta)$.

Denote by $\sigma_n(\beta)$ the σ-field generated by the cylinder sets $\mathbf{T}(A)$, for A finite subsets of $E_n(\beta)$. Finally, denote $\mathcal{T}(\beta) = \bigcap_{n=1}^{\infty} \sigma_n(\beta)$.

Theorem 3.7 *Let* $c_n = \sup |\psi(\vec{e}_1, \vec{e}_2)|$, *where supremum is taken over all* e_1, e_2 *with* $\rho(e_1, e_2) = n$. *If*

$$\sum_{n=1}^{\infty} c_n < \infty,$$

then for any regular path β, $\mathcal{T}(\beta)$ *is trivial, i.e.* $\mu(C) = 0$ *or 1 for every* $C \in \mathcal{T}(\beta)$.

PROOF: Fix a regular path β. It suffices to show that for any cylinder set Y and any $C \in \mathcal{T}(\beta)$, such that $\mu(C) \neq 0$,

$$\mu(Y|C) = \mu(Y), \tag{3.5}$$

where $\mu(Y|C)$ is the conditional measure. By inclusion-exclusion principle, it suffices to check (3.5) for elementary cylinder sets $Y = T(A)$, where A is a finite collection of edges. First, assume that A consists of a single edge, say e_0. Let $C_n \in \sigma_n(\beta)$, $n = 1, \ldots,$ be cylinder sets such that $\mu(C_n \Delta C) \to 0$ as $n \to \infty$. Each C_n is determined by a finite set of edges $F_n = \{f_1, \ldots, f_r, f_{r+1}, \ldots, f_s\}$, so that

$$C_n = \{f_1, \ldots, f_r \in \mathbf{T}\} \cap \{f_{r+1}, \ldots, f_s \notin \mathbf{T}\}.$$

We want to show that $\mu(T(A)|C_n) \to \mu(A)$, as $n \to \infty$. Fix n. For K big enough, (more precisely, $K > \max_{1 \leq i \leq s} \rho(O, f_i)$), denote

$$\Gamma_K^{-} = \Gamma_K / F_n,$$

and

$$\Gamma_K^{\sim} = \Gamma_K / \{f_1, \ldots, f_r\} - \{f_{r+1}, \ldots, f_s\},$$

(see Chapter 2 for definitions of contraction and deletion). By Proposition 2.4,

$$\mu(\Gamma_K)(T(A)|C_n) = \mu(\Gamma_K^{\sim})(T(A)). \tag{3.6}$$

Since $\mu(\Gamma_K) \to \mu$, as $K \to \infty$, it suffices to show that

$$\mu(\Gamma_K^{\sim})(T(A)) - \mu(\Gamma_K)(T(A))| \to 0 \quad \text{as} \quad n, K \to \infty.$$

Consider the electrical network $\Gamma_K^{\sim}$. We can think of deleted edges f_j, $r < j \leq s$ as infinite resistors, and of contracted edges f_i, $i \leq j \leq r$ as zero resistors. Changing the resistances of f_j, $r < j \leq s$ from ∞ to 0 in $\Gamma_K^{\sim}$ results in a network equivalent to Γ_k^{-}. On the other hand, changing the resistances of all $e \in \Gamma_K^{\sim} \setminus \Gamma_n$ to infinite, would result in the network Γ_n. Hence, by Rayleigh's Monotonicity Law,

$$\mu(\Gamma_K^{-})(\mathbf{T}(A)) \leq \mu(\Gamma_K^{\sim})(\mathbf{T}(A)) \leq \mu(\Gamma_n)(\mathbf{T}(A)) \tag{3.7}$$

A similar observation applies for Γ_K, therefore

$$\mu(\Gamma_K^-)(\mathbf{T}(A)) \leq \mu(\Gamma_K)(\mathbf{T}(A)) \leq \mu(\Gamma_n)(\mathbf{T}(A)). \tag{3.8}$$

Keeping in mind that $\mu(\Gamma_K)(\mathbf{T}(A))$ and $\mu(\Gamma_n)(\mathbf{T}(A))$ can be made arbitrarily close to $\mu(\mathbf{T}(A))$ by choosing large n, we will now estimate $|\mu(\Gamma_K^-)(\mathbf{T}(A)) - \mu(\Gamma_K)(\mathbf{T}(A))|$.

For a set of edges F we denote by V_F the set of vertices in F. Let $\nu : \Gamma \to \mathbf{R}$, $\nu_K : \Gamma_K \to \mathbf{R}$, $\nu_K^- : \Gamma_K^- \to \mathbf{R}$ be the potentials induced by batteries B, B_K, B_K^- applied to e_0 in the respective networks, so that the induced total current is one. Such potentials (or voltages), can be computed from transfer-current functions, in particular, $\nu_K \to \nu$ pointwise, as $K \to \infty$. Let $\epsilon(n) = \sum_{i=n}^{\infty} c_i$. We can assume that K is chosen so that, for any $x \in V_{F_n}$,

$$|\nu(x) - \nu_K(x)| < \epsilon(n).$$

Since $|\nu(x) - \nu(t(\beta_n))| \leq \epsilon(n)$ for any $x \in V_n(\beta)$, one has

$$|\nu_K(x) - \nu_K(y)| \leq 4\epsilon(n), \tag{3.9}$$

for any $x, y \in V_{F_n}$.

Denote $k_1 = \nu_K(t(\beta_n))$. Let $\varphi : \Gamma_K \to \mathbf{R}$ be such that

(i) $\varphi(x) = k_1 - \nu_K(x)$ if $x \in V_{F_n}$,

(ii) $\varphi(x)$ is harmonic at any $x \in \Gamma_K \setminus V_{F_n}$.

By Theorem 2.1, φ is well defined. By the Maximum Principle and (3.9), for any $u, v \in \Gamma_K$,

$$|\varphi(u) - \varphi(v)| \leq \max_{x \in V_{F_n}} \varphi(x) - \min_{y \in V_{F_n}} \varphi(y) \leq 4\epsilon(n).$$

Set

$$\eta = \varphi + \nu_K.$$

Recall that ν_K is harmonic at any $x \in \Gamma_K \setminus \{s(e_0), t(e_0)\}$; accordingly η is harmonic at any $x \in \Gamma_K \setminus V_K \cup \{s(e_0), t(e_0)\}$. Also, by (i), η is constant on V_{F_n}, and, by (ii), η induces current 1 into $s(e_0)$, and current -1 into $t(e_0)$.

It is easy to see that total current into V_{F_n} induced by the potential η must be zero, i.e.

$$\sum_{x \in V_{F_n}} \sum_{x \sim y} (\eta(x) - \eta(y)) = 0.$$

Thus, η can be regarded as potential on $\Gamma_K^{\sim}$, induced by exactly the right battery applied to e_0.

By the Matrix Impedance Theorem,

$$\mu(\Gamma_K^{\sim})(\mathbf{T}(A)) = \eta(s(e_0)) - \eta(t(e_0)),$$

and

$$\mu(\Gamma_K)(\mathbf{T}(A)) = \nu_K(s(e_0)) - \nu_K(t(e_0)).$$

Thus, $|\mu(\Gamma_K^{\sim})(\mathbf{T}(A)) - \mu(\Gamma_K)(\mathbf{T}(A))| \leq 4\epsilon(n)$. By (3.7) and (3.8),

$$\mu(\Gamma_K^{\sim})(T(A)) - \mu(\Gamma_K)(T(A))| \to 0 \quad \text{as} \quad n, K \to \infty.$$

This and (3.5) implies the result in case $A = \{e_0\}$.

Now, let $A = \{e_0, e_1, \ldots, e_m\}$. Then

$$\mu(\Gamma)(T(A)) = \mu(\Gamma)(\{e_0, e_1, \ldots, e_m \in \mathbf{T}\})$$

$$= \mu(\Gamma)(\{e_0 \in \mathbf{T}\} | \{e_1, \ldots, e_m \in \mathbf{T}\}) \cdot \mu(\Gamma)(\{e_1, \ldots, e_m \in \mathbf{T}\})$$

$$= \mu(\Gamma / e_1, \ldots, e_m)(\{e_0 \in \mathbf{T}\}) \cdot \mu(\Gamma)(\{e_1, \ldots, e_m \in \mathbf{T}\}),$$

again, by Proposition 2.4. Here the contraction is defined in the same way as for finite graphs, see Chapter 2. The argument for the case $A = \{e_0\}$ can be applied to graphs $\Gamma / e_i, \ldots, e_m$ replacing Γ, for $i = 1, 2, \ldots, m$; the result follows by induction.

Chapter 4

CASE OF A NON-CO-COMPACT FUCHSIAN GROUP

We are citing [Ser] for the relevant facts about Fuchsian groups acting in the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Such a group is by definition a discrete subgroup of the group

$$\left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbf{C}, |a|^2 - |b|^2 = 1 \right\}$$

of conformal automorphisms of the disk.

With the metric $ds = 2|dz|/(1 - |z|^2)$, $\mathbf{D}$ becomes a model for non-Euclidean geometry in which the straight lines are circular arcs orthogonal to S^1. By a fundamental region for G we mean a geodesically convex polygon $R \subset \bar{\mathbf{D}}$ with a finite number of sides (possibly including arcs of S^1), such that no two interior points of R are conjugate under G and every point in $\mathbf{D}$ is conjugate to a point in $\bar{R}$. Many such fundamental regions exist for any given G. Each side s of $R \cap \mathbf{D}$ is identified with another side s', by an element $g(s) \in G$. The set $S = \{g(s) : s \in \partial R\}$ forms a symmetric set of generators for G.

If $\bar{R} \subset \mathbf{D}$, then G is called *co-compact.* Otherwise, $\bar{R} \cap S^1 \neq \emptyset$, and G is called *non-co-compact.* The property of being co-compact does not depend on the choice of R.

Throughout Chapter 4 we assume that G is non-co-compact, S is the set of generators as above, and $\Gamma = \Gamma(G, S)$.

4.1 *Exponential behavior of* ψ

The graph $\Gamma = \Gamma(G, S)$ may be represented as a net in the hyperbolic disk $\mathbf{D}$ (see [Ser]). Regions bounded by edges of Γ with no edges intersecting the interior we call polygons. If P is a polygon, we write ∂P for the boundary of P.

By a *chain* in Γ we will mean a sequence of distinct polygons P_j, $1 \leq j \leq n$, such that P_i, P_{i+1} have a common edge for all $1 \leq i < n$. The number of polygons in a chain is called its *length.*

Let E_d be the set of directed edges, and $\psi : E_d \times E_d \rightarrow \mathbf{R}$ be the transfer-current function, defined as in Section 3.2.

Theorem 4.1 *There exists* $0 < \lambda < 1$ *such that* $|\psi(e, e')| \leq \lambda^n$, *whenever* $\rho(e, e') \geq n$.

PROOF: We use the fact that since G is non-co-compact, there is at least one polygon with infinitely many sides, attached to each vertex.

Fix an edge e_0. Assume that $e \neq e_0$ is a common edge of two finite polygons. Let F_i be the endpoints of e, and P_i be infinite polygons adjacent to F_i, $i = 1, 2$. One can find points F_i' on the boundary $\partial\mathbf{D}$ and curves α_i connecting F_i to F_i' , $i = 1, 2$, so that $\alpha_i \setminus \{F_i, F_i'\}$ lies in the interior of P_i. Then the curve $\alpha_1 \cup e \cup \alpha_2$ divides $\mathbf{D}$ into two parts, say $\mathcal{P}$ and $\mathcal{P}'$, so that $H(e) = \mathcal{P} \cap \Gamma$ and $H(e)' = \mathcal{P}'$ are the connected subgraphs of Γ which satisfy the conditions:

$$H(e) \cap H'(e) = e$$

$$H(e) \cup H'(e) = \Gamma$$

We may assume that $e_0 \in H'(e)$.

Lemma 4.2 *Let* e_0, e *be edges,* $\nu : \Gamma \rightarrow \mathbf{R}$ *be the potential function resulting from a battery of some value* B *applied to* e_0, *and* $\eta : \Gamma \rightarrow \mathbf{R}$ *be the potential function* *resulting from imposing voltages* $\nu(s(e))$ *at* $s(e)$*, and* $\nu(t(e))$ *at* $t(e)$*. Then for any* $x \in H(e)$, $\eta(x) = \nu(x)$.

PROOF: Let $\nu_n : \Gamma_n \to \mathbf{R}$ be the voltages which arise in Γ_n, if a battery B is applied to e_0, and $\eta_n : \Gamma_n \to \mathbf{R}$ be the voltages which arise in Γ_n, if the voltages $\nu_n(s(e))$, $\nu_n(t(e))$ are applied to $s(e), t(e)$, respectively. Then ν_n and η_n are both harmonic on $H \cap \Gamma_n$ with boundary conditions $\nu_n(s(e)) = \eta_n(s(e))$, $\nu_n(t(e)) = \eta_n(t(e))$. By Theorem 2.1, $\nu_n(x) = \eta_n(x)$ for all $x \in H \cap \Gamma_n$. Passing to the limit yields the result.

To proceed with the proof of Theorem 4.2 we consider two cases.

Case (a). There is a unique polygon with infinitely many sides, adjacent to each vertex.

Lemma 4.3 *Given distinct edges* e, e'*, there exists a unique chain* $P_1, \ldots, P_n$*, such that* $e \in \partial P_1$, $e' \in \partial P_n$.

PROOF: If e, e' have a vertex in common, say A, then, enumerating all finite polygons adjacent to A in a clockwise order, we will encounter only finite polygons either between e, e', or between e', e. These polygons form a chain connecting e to e'.

If $e \cap e' = \emptyset$, then e and e' can be connected by a path $e = f_1, f_2, \ldots, f_s = e'$ in Γ. Each pair of consecutive edges can be connected by a chain, so one can find a chain connecting e to e'.

Finally, if there were two distinct chains connecting e to e', then one could find a chain $Q_1, Q_2, \ldots, Q_m$, such that $Q_m = Q_1$. There would be inner vertices, which are not adjacent to any infinite polygon. *q.e.d.*

Let $f_1, \ldots, f_N$ be all edges which belong to finite polygons adjacent to O. We enumerate them so that edges adjacent to O are listed first. For any i, j denote by λ_{ij} the absolute value of current through f_j, if a **unit** battery is applied to f_i, $1 \leq i \leq d$, $1 \leq j \leq N$. Here d equals the number of edges adjacent to O, (also equals the number of generators). It follows from the Maximum Principle, (see[DS], sec.2.4) that $\lambda_{ij} < 1$ if $i \neq j$. Let $\lambda_0 = \max\limits_{i \neq j} \lambda_{ij}$. Suppose that edges e, e' are connected by a chain $P_1, P_2, \ldots, P_k$, of length k, and let $p_i = \partial P_i \cap \partial P_{i+1}$, $i = 1, \ldots, k-1$. In particular, p_i and p_{i+1} are edges of the polygon P_{i+1}, so there exists $\gamma_i \in G$, such that $\gamma_i p_i = f_k$ and $\gamma_i p_{i+1} = f_m$ for some k and m. Since $\psi(e, p_i)$ is just the difference of voltages at the endpoints of p_i, it follows from the group symmetry and Lemma 4.2 that

$$|\psi(e_1, p_{i+1})| = |\psi(e_1, p_i)| \cdot \lambda_{k,m}.$$

(Abusing notation, we do not indicate a choice of directions, which can be arbitrary.) It follows that $|\psi(e, e')| \leq \lambda_0^k$. Now, if $\rho(e, e') = n$, then $n \leq ks$, where s is the maximal number of sides in a finite polygon. Set $\lambda = \lambda_0^{1/s}$. Then $\psi(e_1, e_2) \leq \lambda^n$, whenever $\rho(e_1, e_2) \geq n$. This implies the result in case (a).

Case (b). There are at least two infinite polygons adjacent to each vertex. Let v be a vertex and Q_1, Q_2 be infinite polygons adjacent to v. One can find curves α_1, α_2 joining v to $\partial\mathbf{D}$, such that $\alpha_i \setminus \{v\}$ lies completely in the interior of P_i, $i = 1, 2$. Then $\alpha_1 \cup \alpha_2$ divides Γ into two connected parts, say, H and H', such that $H \cup H' = \Gamma$, and $H \cap H' = v$.

Lemma 4.4 *A battery applied to an edge in H induces zero currents through edges in H'.*

PROOF: Let $e_1 \in H$. Choose n such that $e_1 \in \Gamma_n$. Let $\nu_n : \Gamma_n \to \mathbf{R}$ be the potential induced by a unit battery applied to e_1. Then ν_n is harmonic at any $x \in H' \cap \Gamma_n$ such that $x \neq v$. Therefore, $\nu_n|_{H' \cap \Gamma_n}$ is constant. Passing to the limit implies the result.

Now, given edges e, e', such that $\rho(e, e') = n$, one can find a path $e = f_0, f_1, \cdots, f_{n+1}$ in Γ connecting e to e'. Let $m \leq n$ be the maximal number such that e can be connected to f_m by a chain. If $m = n+1$, we are in the situation of case (a). If $m < n+1$, then f_m, f_{m+1} cannot be connected by a chain. Thus, there must be infinite polygons adjacent to $v = f_m \cap f_{m+1}$, lying to the right and to the left of the path $\cdots f_{m-1}, f_m, f_{m+1}, \cdots$. By Lemma 4.4, $\psi(e, f_k) = 0$ for $m < k \leq n$. This completes the proof of case (b) and of the Theorem 4.1.

Corollary. *If G is a non-co-compact Fuchsian group, the rate of convergence to mixing is exponential.*

PROOF: Let C_1 and C_2 be cylinder sets determined by the collections of edges E_1 and E_2, respectively. If $\rho(E_1, E_2)$ is at least n, then, by inclusion-exclusion principle and the Matrix-Impedance Theorem,

$$|\mu(C_1 \cap C_2) - \mu(C_1) \cdot \mu(C_2)| \leq K\lambda^{-n},$$

where λ is as in Theorem 4.1, and K is a constant which depends on the cardinalities of E_1 and E_2.

4.2 Tail triviality

In this section we obtain a result which implies the tail triviality:

Theorem 4.5 *If Γ is as above, then $\mu = \tilde{\mu}$.*

PROOF: It suffices to show that $\psi = \tilde{\psi}$. Given a directed edge $\vec{e}$, set $P(\vec{e})$ to be the polygon to the right of $\vec{e}$, if such a polygon is finite, and $P(\vec{e}) = \emptyset$ otherwise. Observe that, as in the proof of Theorem 4.1, $\vec{e}$ divides Γ into two subgraphs, $H(\vec{e}), H'(\vec{e})$. Assume that $P(\vec{e}) \subset H(\vec{e})$.

Let $\vec{f_1}, \vec{f_2}, \ldots, \vec{f_k}$ be the other edges of $P(\vec{e})$, directed in such a way that the path $\vec{f_1}\, \vec{f_2}, \ldots \vec{f_k}$ goes around $P(\vec{e})$ counterclockwise, connecting $s(\vec{e})$ to $t(\vec{e})$. Then, since an infinite polygon is adjacent to each vertex of $P(\vec{e})$,

$$H(\vec{e}) = \{e\} \cup \bigcup_{i=1}^{k} H(\vec{f_i}).$$

Also, if $j = i + 1$, then

$$H(\vec{f_i}) \cap H(\vec{f_j}) = t(\vec{f_i}) = s(\vec{f_j});$$

Otherwise,

$$H(\vec{f_i}) \cap H(\vec{f_j}) = \emptyset.$$

Let Γ_n, $\Gamma^{(n)}$ be defined as in section 3.1. For big enough n, set $r(\vec{e}, n)$, $\tilde{r}(\vec{e}, n)$ to be the effective resistances across the endpoints of $\vec{e}$ in the networks $\Gamma_n \cap H(\vec{e})$, $\Gamma^{(n)} \cap H(\vec{e})$, respectively (see Chapter 2 for the definition).

For each $i = 1, 2, \ldots, k$, the finite piece $H(\vec{f_i}) \cap \Gamma_n$ of the network $H(\vec{e}) \cap \Gamma_n$ can be replaced by a single resistor of value $r(\vec{f_i}, n)$ between the endpoints $s(\vec{f_i})$ and $t(\vec{f_i})$. Applying the laws of resistors in series and parallel to $P(\vec{e})$ with edge resistances $r(\vec{f_i}, n)$, $i = 1, 2, \ldots, k$, we obtain

$$r(\vec{e}, n) = \frac{1}{1 + (\sum_{i=1}^{k} r(\vec{f_i}, n))^{-1}}. \tag{4.1}$$

Here we treat the sum as ∞, if the set $\{\vec{f_1}, \ldots, \vec{f_k}\}$ is empty.

Repeating the same argument for $H(\vec{e}) \cap \Gamma^{(n)}$, we obtain

$$\tilde{r}(\vec{e}, n) = \frac{1}{1 + (\sum_{i=1}^{k} \tilde{r}(\vec{f_i}, n))^{-1}}. \tag{4.2}$$

By Rayleigh's Monotonicity Law, $\tilde{r}(\vec{e}, n)$ is increasing, and $r(\vec{e}, n)$ is decreasing in n, and for each n

$$\tilde{r}(\vec{e}, n) \leq r(\vec{e}, n).$$

Let $r(\vec{e}) = \lim_{n\to\infty} r(\vec{e}, n)$, and $\tilde{r}(\vec{e}) = \lim_{n\to\infty} \tilde{r}(\vec{e}, n)$. The intuitive meaning of $r(\vec{e})$, [respectively, $\tilde{r}(\vec{e})$] is that for computing the values of the transfer-current function $\psi(\vec{e'}, \vec{e})$ [respectively, $\tilde{\psi}(\vec{e'}, \vec{e})$,] the infinite piece $H(\vec{e})$ not containing the battery may be replaced by a single resistor of value $r(\vec{e})$, [respectively, $\tilde{r}(\vec{e})$] across the endpoints of $\vec{e}$.

If $s_0 \in S$ is a generator, we say that the edge $\vec{e}$ is *labeled* by s_0 if

$$s(\vec{e})^{-1} \cdot t(\vec{e}) = s_0.$$

Directed edges labeled by the same generator we call *congruent.* Clearly, if $\vec{e}_1$ and $\vec{e}_2$ are congruent, then $r(\vec{e}_1) = r(\vec{e}_2)$, and $\tilde{r}(\vec{e}_1) = \tilde{r}(\vec{e}_2)$, by the group symmetry. (Thus, equivalent resistances may be assigned to the generators of G).

Let $E_0 = \{\vec{e}_1, \ldots, \vec{e}_d\}$ be the set of directed edges coming out of O. Here d is the degree of each vertex. For each i denote by $\vec{f}^i_1, \ldots, \vec{f}^i_{k(i)}$ the edges of $P(\vec{e}_i)$ directed as was described above. Any edge $\vec{f}^i_j$ is congruent to some $\vec{e}_r$.

Consider the $d \times d$ matrix M with entries $m_{i,j}$, where $m_{i,j}$ is the number of edges in the path $\vec{f}^i_1 \ldots \vec{f}^i_{k(i)}$, which are congruent to $\vec{e}_j$. Then, passing to the limit in (4.1) and (4.2), we obtain

$$\tilde{r}(\vec{e}_i) = \frac{1}{1 + (\sum_{j=1}^{d} m_{i,j} \tilde{r}(\vec{e}_j))^{-1}},$$

$$r(\vec{e}_i) = \frac{1}{1 + (\sum_{j=1}^{d} m_{i,j} r(\vec{e}_j))^{-1}}.$$

Lemma 4.6 *For any i,*

$$1/2 \le \tilde{r}(\vec{e}_i) \le r(\vec{e}_i) \le 1.$$

PROOF: The only inequality which needs to be checked is

$$\min_{i \in [1,d]} \tilde{r}(\vec{e}_i) \ge 1/2.$$

Let $r_0 = \min_{i \in [1,d]} \tilde{r}(\vec{e}_i)$. Since a non-empty $P(\vec{e}_i)$ contains at least two edges besides e_i, we have for each i

$$r_i \ge \frac{1}{1 + (2r_0)^{-1}}.$$

Thus,

$$r_0 \ge \frac{1}{1 + (2r_0)^{-1}},$$

implying $r_0 \ge 1/2$. *q.e.d.*

Let $\mathcal{R} = [1/2, 1]^d$, and consider the map $\Omega : \mathcal{R} \to \mathbf{R}^d$, defined by

$$\Omega_i(x_1, \dots, x_d) = \frac{1}{1 + (\sum_{j=1}^{d} m_{i,j} x_j)^{-1}}, \qquad i = 1, \dots d$$

(If, for some i, $m_{i,j} = 0$ for all j, we set $\Omega_i((x_1, \dots, x_d)) = 1$). It is easily checked that $\Omega(\mathcal{R}) \subset \mathcal{R}$.

Consider the following metric $\delta : \mathcal{R} \times \mathcal{R} \to \mathbf{R}^+$: if $x = (x_1, x_2, \dots, x_d)$ and $y = (y_1, y_2, \dots, y_d)$, then

$$\delta(x, y) = \max_{i=1,2,\dots,d} |x_i - y_i|.$$

Lemma 4.7 *There exists* $0 < \beta < 1$ *such that for any* $x, y \in \mathcal{R}$

$$\delta(\Omega(x), \Omega(y)) \leq \beta \delta(x, y).$$

PROOF: We have for any i, j

$$\frac{\partial \Omega_i}{\partial x_j}|_{(x_1,\dots,x_d)} = \frac{1}{(1 + (\sum_{j=1}^{d} m_{i,j} x_j)^{-1})^2} \cdot \frac{1}{(\sum_{j=1}^{d} m_{i,j} x_j)^2} \cdot m_{i,j} = \frac{m_{i,j}}{(1 + \sum_{j=1}^{d} m_{i,j} x_j)^2}$$

$$\leq \frac{m_{i,j}}{(1 + \frac{1}{2} \sum_{j=1}^{d} m_{i,j})^2},$$

since $x_j \geq 1/2$.

Now, if $x = (x_1, x_2, \dots, x_d)$ and $y = (y_1, y_2, \dots, y_d)$, set

$$p_0 = x, \quad p_1 = (y_1, x_2, \dots, x_d),$$

$$p_2 = (y_1, y_2, \dots, x_d),$$

$$p_d = (y_1, y_2, \dots, y_d) = y,$$

so that the segment $[p_{i-1}, p_i]$ is parallel to the i^{th} axis.

For any i,

$$|\Omega_i(x) - \Omega_i(y)| \leq$$

$$\int_{[p_0,p_1]} |\frac{\partial \Omega_i}{\partial x_1}| + \int_{[p_1,p_2]} |\frac{\partial \Omega_i}{\partial x_2}| + \ldots + \int_{[p_{d-1},p_d]} |\frac{\partial \Omega_i}{\partial x_d}|$$

$$\leq \max_i |y_i - x_i| \cdot \frac{\sum_{j=1}^d m_{i,j}}{(1 + \frac{1}{2}\sum_{j=1}^d m_{i,j})^2} \leq \beta \cdot \max_i |y_i - x_i|,$$

where the choice of β is clear.

Now we observe that $(\tilde{r}(\vec{e}_1), \tilde{r}(\vec{e}_2), \ldots, \tilde{r}(\vec{e}_d))$ and $(r(\vec{e}_1), r(\vec{e}_2), \ldots, r(\vec{e}_d))$ are fixed points for Ω. By Lemma 4.7, Ω is a contraction, therefore $\tilde{r}(\vec{e}_i) = r(\vec{e}_i)$ $i = 1, 2, \ldots, d$.

Finally, the latter resistances uniquely determine ψ as follows:
First,

$$\psi(\vec{e}_i, \vec{f}_j^i) = \frac{r(\vec{f}_j^i)}{1 + \sum_{j=1}^k r(\vec{f}_j^i)}. \tag{4.3}$$

Next, if e_1 and e_2 are not in the same polygon, then either $\psi(\vec{e}_1, \vec{e}_2) = 0$, or e_1 and e_2 can be connected by a unique chain of polygons, say of length m. By Lemma 4.2, $\psi(\vec{e}_1, \vec{e}_2)$ is the product of m terms of the form(4.3) with suitable choice of i, j. Since (4.3) is true with $\tilde{\psi}$ replacing ψ, and $\tilde{r}$ replacing r, we conclude that $\tilde{\psi} = \psi$. This completes the proof of Theorem 4.5.

The following is immediate from Proposition 3.3.

Corollary. *If $\Gamma = \Gamma(G, S)$ is as above, then the tail σ-algebra is trivial with respect to μ.*

4.3 Examples

In this section we compute explicitly the transfer-current function for the Modular and Hecke groups, (both being examples of non-co-compact Fuchian groups) and, automatically, for all groups isomorphic to them.

1. Modular group

Let $M = \langle a, b \mid (ab)^3 = id, b^2 = id \rangle$. See Fig.1 for $\Gamma(M)$.

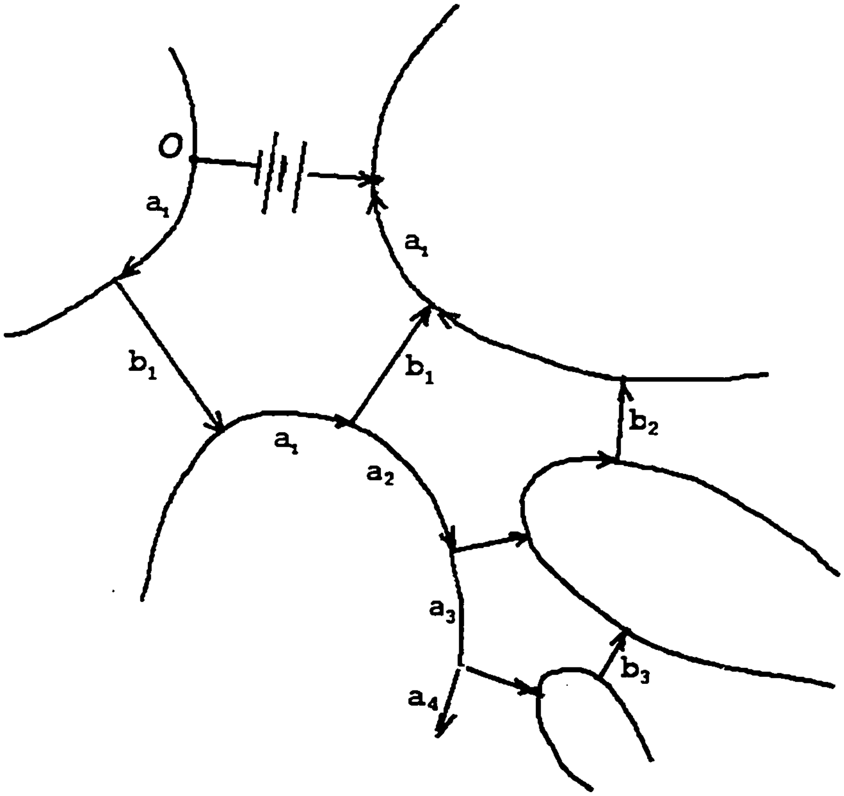

Figure 4.1: Modular Group

Throughout this example fix a directed edge $\vec{e}_0$. Suppose that a battery is applied to $\vec{e}_0$, so that current goes from $s(\vec{e}_0)$ to $t(\vec{e}_0)$. We are going to choose a particular direction for all other edges, and then check that it is the same as the direction of current.

Let P be a polygon containing e_0. If P is to the right of $\vec{e}_0$ (i.e., $\vec{e}_0$ goes clockwise around P), we direct all other edges of P counterclockwise. If P is to the left of $\vec{e}_0$, all other edges of P are directed clockwise. Now suppose, a direction is chosen for all edges in a finite collection of polygons $\mathcal{P} = \{P_1, \dots, P_N\}$. Let Q be a polygon such that $Q \notin \mathcal{P}$, and $\partial Q \cap \partial P_i = e_1$ for some $1 \le i \le N$, and $e_1 \in E$. Then, if $\vec{e}_1$ goes clockwise around Q, we direct all other edges counterclockwise, and vice versa. Since, by Lemma 4.3, any two edges are connected by a unique chain of polygons, this defines the direction of all edges in Γ.

We claim that this direction coincides with the direction of current. Let $\vec{f}_1, \vec{f}_2, \dots, \vec{f}_k$ be the edges of P other than e_0, going either clockwise or counterclockwise around P, according to the chosen direction. Denote by ν the induced potential in Γ. Recall that any directed edge $\vec{e}$ divides Γ into parts $H'(\vec{e})$ and $H(\vec{e})$; we denote by $H(\vec{e})$ the one not containing e_0 (see Section 4.1).

Replace each $H(\vec{f}_i)$, $i = 1, 2, \dots, k$ by an equivalent resistor, (see Section 4.2) to obtain a finite network P, which we call *reduced.* Observe that the voltages at each vertex of P in the reduced and original networks are the same. Clearly, in the reduced network the current will flow along $\vec{f}_i$, therefore,

$$\nu(s(\vec{f}_i) - \nu(t(\vec{f}_i)) \ > 0.$$

Hence, the current will flow along $\vec{f}_i$ in Γ, as well. Proceeding by induction, we obtain the result. q.e.d.

Let $E_1 \subset E_d$ be the set of edges directed as above. Define the *number function* $N : E_1 \to \mathbf{N}$, by $N(\vec{e}) = n$, if e can be connected to e_0 by a chain of length n.

Define also the *label function* $L : E \longrightarrow \{a, a^{-1}, b\}$, by $L(\vec{e}) = s(\vec{e})^{-1} \cdot t(\vec{e})$. We remark that function N depends on the choice of e_0.

Lemma 4.8 *If* $\vec{e}_1, \vec{e}_2 \in E_1$ *are such that* $N(\vec{e}) = N(\vec{e}_2)$, *and* $L(\vec{e}_1) = L(\vec{e}_2)$ *or* $L(\vec{e}_1) = L(\vec{e}_2)^{-1}$, *then*

$$\psi(\vec{e}_0, \vec{e}_1) = \psi(\vec{e}_0, \vec{e}_2).$$

PROOF:

By symmetry, we may assume that e_1, e_2 are both to the right of $\vec{e}_0$. It is easy to see that in such case e_1 and e_2 are congruent, i.e. $L(\vec{e}_1) = L(\vec{e}_2)$. Again, let $\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k$ be the edges of P other than e_0. Reduce $H(\vec{e}_0)$ to the finite circuit P; by Ohm's Law, $\psi(\vec{e}_0, \vec{f}_i)$ is constant on each set of congruent edges. This implies the result in case $\vec{e}_1, \vec{e}_2 \in \partial P$. Moreover, by Lemma 4.2, the networks $H(\vec{f}_i)$ and $H(\vec{f}_j)$ are 'identical', whenever $\vec{f}_i$ and $\vec{f}_j$ are congruent, therefore, currents through corresponding edges in $H(\vec{f}_i)$ and $H(\vec{f}_j)$ are the same. The result follows.

Pick any edges $\vec{e}$, $\vec{f} \in E_1$, such that $L(\vec{e}) = a$, $L(\vec{f}) = b$, $N(\vec{e}) = N(\vec{f}) = n$. Set

$$a_n \overset{\text{def}}{=} \psi(\vec{e}_0, \vec{e}),$$

$$b_n \overset{\text{def}}{=} \psi(\vec{e}_0, \vec{f}).$$

There are two cases:

Case 1: $L(e_0) = b$.

By Kirchhoff's Law, for any $k > 0$,

$$a_k = a_{k+1} + b_k.$$

Using the fact that the sum of currents through edges forming a cycle is zero, we also have

$$b_k = 3a_{k+1} + 2b_{k+1}$$

It follows that

$$\begin{aligned} a_k &= 4a_{k+1} + 2b_{k+1} \\ b_k &= 3a_{k+1} + 2b_{k+1} \end{aligned} \tag{4.4}$$

or,

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = A^n \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}.$$

Let $\mathbf{u}$ be an eigenvector, corresponding to the largest eigenvalue of A, namely, $3+\sqrt{7}$. By Perron-Frobenius Theorem (see [Sen]), we can find such a $\mathbf{u}$ with strictly positive coordinates; assume also that $\|\mathbf{u}\| = 1$. Then

$$\frac{A^n\mathbf{v}}{\|A^n\mathbf{v}\|} - \mathbf{u} \to 0, \quad \text{as } n \to \infty$$

uniformly on the positive cone

$$\mathbf{R}^2_+ = \{\mathbf{v} = (v_1, v_2)^T : v_1 \geq 0, v_2 \geq 0, v_1 + v_2 > 0.\}$$

Because of the choice of directions, $(a_n, b_n)^T \in \mathbf{R}^2_+$, $n = 1, 2, \ldots$. Therefore,

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = c\mathbf{u},$$

where c is some constant. Next, $b_0 = 3a_1 + 2b_1$, and the total current into the circuit is $2a_1 + b_0 = 1$, so, we have the normalization

$$5a_1 + 2b_1 = 1.$$

Further,

$$a_n = a_1\lambda^{1-n}, \quad b_n = b_1\lambda^{1-n}. \tag{4.5}$$

Case 2: $L(e_0) = a$, or a^{-1}. Define N, L, a_k, b_k, $k = 1, 2, \ldots$ as above. It is easy to see that the equations (4.5) hold, (even though a_k, b_k stand for the currents through different edges).

We conclude that $(a_1, b_1)^T$ is the eigenvector of A corresponding to the eigenvalue $\lambda = 3 + \sqrt{7}$, which satisfies the equations: $\begin{cases} a_0 + b_1 + a_1 &= 1, \\ 3b_1 + 2a_1 &= a_0. \end{cases}$

This gives the normalization

$$3a_1 + 4b_1 = 1.$$

Next,

$$a_n = a_1 \lambda^{1-n}, \quad b_n = b_1 \lambda^{1-n}.$$

2. Hecke groups

For $k \geq 4$, let $H_k = \langle a, b | (ab)^k = \mathrm{Id},\ b^2 = \mathrm{Id} \rangle$.

Fix e_0. Define $a_n, b_n,\ n = 1, \ldots$ as for M. Then we have

$$\begin{aligned} a_n &= a_{n+1} + b_n \\ b_n &= k a_{n+1} + (k-1) b_{n+1} \end{aligned}$$

In other words, $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = A_k \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix}$, where $A_k = \begin{bmatrix} k+1 & k-1 \\ k & k-1 \end{bmatrix}$. Let λ_k be the maximal eigenvalue of A_k. By Perron-Frobenius Theorem, $(a_1, b_1)^T$ is the eigenvector of A_k corresponding to λ_k, normalized so that the total current into the circuit is one. As before, $a_n = \lambda_k^{1-n} a_1$, and $b_n = \lambda_k^{1-n} b_1$.

Chapter 5

ALGEBRAIC DYNAMICAL SYSTEMS

5.1 *Introduction*

Let $R_d = \mathbf{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in d commuting variables. We will use x, y, z to denote elements of $\mathbf{Z}^d$. For $x = (n_1, n_2, \dots, n_d)$ write $\mathbf{u}^x = u_1^{n_1} u_2^{n_2} \dots u_d^{n_d}$. A Laurent polynomial $f = \sum_{x \in \mathbf{Z}^d} c(x) \mathbf{u}^x$ can be visualized as the integral-valued configuration p_f on $\mathbf{Z}^d$, i.e. the function $p_f : \mathbf{Z}^d \to \mathbf{Z}$ such that $p_f(x) = c(x)$. Note that all but finitely many $c(x)$ are equal to zero. Let

$$X_f = \{\phi : \mathbf{Z}^d \to \mathbf{R}/\mathbf{Z} \quad \text{such that} \quad \sum_{x \in \mathbf{Z}^d} p_f(x+z) \cdot \phi(x) = 0 \ \text{ for any } \ z \in \mathbf{Z}^d\}. \tag{5.1}$$

Then X_f is a compact Abelian group, and $\mathbf{Z}^d$ acts on it by translations (see [LSW]). For $\mathbf{Z}^d$-actions on more general compact groups see [Sch]. The main result of [LSW] is that the topological entropy of $(X_f, \mathbf{Z}^d)$ is given by

$$h(X_f) = \int_0^1 \cdots \int_0^1 \log |f(\exp\{2\pi i t_1\}, \dots, \exp\{2\pi i t_d\})| dt_1 \dots dt_d \tag{5.2}$$

(The definition of topological entropy will be given below).

Now, let Γ be an infinite graph with the vertex set $\mathbf{Z}^d$, and the edge set E. For distinct vertices x and y, write $x \sim y$ if they are joined by at least one edge, and set $k(x, y)$ to be the number of edges joining x, y. We do not allow edges connecting a vertex to itself. We assume that Γ is periodic, i.e. for any $v \in \mathbf{Z}^d$, $k(x, y) = k(x + v, y + v)$.

The number of edges coming out of each vertex is the same, it is called the *degree* of a vertex and denoted by D.

Recall that a subgraph of Γ is *spanning* if it contains all the vertices of Γ, and is a *forest* if it has no cycles. A forest in Γ is called *essential* if all its connected components are infinite.

Let $\mathcal{X} = \{$ all subgraphs of Γ $\} = \{0,1\}^E$. Endowed with the product topology, $\mathcal{X}$ is compact, and $\mathbf{Z}^d$ acts on it by translations. Let $\mathcal{Y} \subset \mathcal{X}$ be the set of all essential spanning forests in Γ, which is a closed, therefore compact, $\mathbf{Z}^d$- invariant subset. The dynamical system $(\mathcal{Y}, \mathbf{Z}^d)$ is called an *essential spanning forests process.* Its topological entropy h_Γ was computed in [BP93]: If Γ is a periodic graph with vertex degree D, and $k(x,y)$ defined as above, then

$$h_\Gamma = \int_{\mathbf{T}^d} \log(D - \sum_{0 \neq x \in \mathbf{Z}^d} k(0,x) \exp\{2\pi i \alpha \cdot x\}) d\alpha \ , \tag{5.3}$$

where the integral is over the d-dimensional torus with respect to Haar measure.

Let us compare this formula with (5.2). Given a periodic graph Γ, let $f = f(\Gamma) \in R_d$ correspond to the configuration p_f given by

$$p_f(y) = -k(0,y) \text{ if } \ y \neq 0, \ \text{ and } \ p_f(0) = D. \tag{5.4}$$

If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{T}^d$, then

$$\begin{aligned} D - \sum_{0 \neq x \in \mathbf{Z}^d} k(0,x) \exp\{2\pi i \alpha \cdot x\} &= \sum_{x \in \mathbf{Z}^d} p_f(x) \exp\{2\pi i \alpha \cdot x\} \\ &= f(\exp\{2\pi i \alpha_1\}, \ldots, \exp\{2\pi i \alpha_d\}) \, , \end{aligned}$$

so that right-hand sides of (5.2) and (5.3) are the same, implying $h_\Gamma = h(X_f)$. The coincidence of entropies was observed by authors of [BP93] and [LSW], and they asked whether this can be shown directly. We will do this in sections 5.2 and 5.3.

Example. Let Γ be the nearest neighbor graph on $\mathbf{Z}^2$, i.e. each x is joined by one edge with each of its four neighbors. Then $f(\Gamma) = 4 - u_1 - u_1^{-1} - u_2 - u_2^{-1}$, and

$X_{f(\Gamma)} \subset (\mathbf{R}/\mathbf{Z})^{\mathbf{Z}^2}$ consists of those configurations ϕ for which

$$4\phi(x) - \phi(x+(0,1)) - \phi(x+(1,0)) - \phi(x-(0,1)) - \phi(x-(1,0)) = 0.$$

By (5.2) and (5.3),

$$h(X_{f(\Gamma)}) = h_\Gamma = \int_0^1 \int_0^1 \log(4 - 2\cos(2\pi\alpha_1) - 2\cos(2\pi\alpha_2))\, d\alpha_1 d\alpha_2 \approx 1.166\,.$$

Next we define the topological entropy. There are several equivalent definitions (see [LSW] or [Sch]); we will use the one dealing with separated sets. Let α be a $\mathbf{Z}^d$ action on a metrizable compact space X with metric ρ. We write $|A|$ for the cardinality of A.

Let $Q_n = \prod_{j=1}^d \{-n, -n+1, \ldots, n\}$ be the cube in $\mathbf{Z}^d$. A set $E \subset X$ is called (n,ε)-separated for α if for distinct $u, v \in X$ there exists $\mathbf{j} \in Q_n$ such that $\rho(\alpha^{\mathbf{j}}u, \alpha^{\mathbf{j}}v) \geq \varepsilon$. Let $s_n(\varepsilon)$ be the largest cardinality of an (n,ε)-separated set, and put $s(\varepsilon,\alpha) = \limsup_{n\to\infty} |Q_n|^{-1} \log s_n(\varepsilon)$. The *topological entropy* $h(\alpha)$ is defined by $h(\alpha) = \lim_{\varepsilon\to 0} s(\varepsilon,\alpha)$.

REMARK: Let $B_n \subset \mathbf{Z}^d$, $n = 1, 2 \ldots$, be a Følner sequence, i.e. a sequence satisfying the condition: for any finite $K \subset \mathbf{Z}^d$, one has $|B_n \Delta (K + B_n)| \cdot |B_n|^{-1} \to 0$, as $n \to \infty$. It is easy to check that the definition above with Q_n replaced by B_n yields the same value of $h(\alpha)$.

5.2 *Proof of coincidence of entropies*

If $\tilde{\Gamma}$ is a graph (either finite or infinite), we say that $\phi : \tilde{\Gamma} \to [0, 1)$ is *harmonic mod* 1 w.r.t. $\tilde{\Gamma}$, if for each vertex x

$$deg(x, \tilde{\Gamma})\phi(x) = \sum_{y \sim x} \phi(y) \quad mod\ 1,$$

where $deg(x, \tilde{\Gamma})$ is the degree of x in $\tilde{\Gamma}$, and the summation goes through all y adjacent to x, counting multiplicities. If this equation holds for a particular vertex x, we say that ϕ is *harmonic mod* 1 at x.

Recall that the *contraction of* $\tilde{\Gamma}$ by an edge e is the graph $\tilde{\Gamma}/e$, obtained from $\tilde{\Gamma}$ by identifying the endpoints of e and erasing all self-loops which may result.

Let Γ be a periodic graph on $\mathbf{Z}^d$ and let $f = f(\Gamma)$ and $X_{f(\Gamma)}$ be defined by (5.4) and (5.1) respectively. To simplify notation set $\mathcal{K} = X_{f(\Gamma)}$. We will identify $\mathbf{R}/\mathbf{Z}$ with $[0, 1)$, so for instance, $\mathcal{K}$ is identified with the set of functions harmonic *mod* 1 w.r.t. Γ.

Define $B_0 = \{(0, 0)\}$, and B_n to be the set of vertices which can be reached from the origin by a path of length at most n. The set $\partial B_n = B_n \setminus B_{n-1}$ is called the *boundary* of B_n. It is easy to see that B_n, $n = 1, 2 \cdots$, form a Følner sequence; they will be used in the definition of $h(\mathcal{K})$, (see REMARK in the end of Section 5.1).

The idea of the proof is to approximate $\mathcal{K}$ by finite groups of functions harmonic *mod* 1 on finite graphs.

By Γ_n we denote the graph, induced from Γ on B_n, $n = 1, 2, \ldots$, in other words, Γ_n is obtained from Γ by erasing all vertices outside B_n and all edges adjacent to them.

For each n, let $\Gamma^{(n)}$ be the contraction of Γ_{n+1} by all edges xy, with $x, y \in \partial B_{n+1}$. Thus, the vertices in ∂B_{n+1} are identified. Denote the new vertex by $\mathcal{O}_n$. Then $deg(\mathcal{O}_n, \Gamma^{(n)})$ equals the number of edges xy in Γ with $x \in \partial B_n$, $y \in \partial B_{n+1}$. Clearly, $deg(x, \Gamma^{(n)}) = D$, for any $x \in B_n$.

Lemma 5.1 *Let U_n be the set of all functions $\phi : \Gamma^{(n)} \to [0, 1)$ such that*

(i) ϕ *is harmonic mod* 1 *w.r.t.* $\Gamma^{(n)}$

(ii) $\phi(\mathcal{O}_n) = 0$.

Then $\limsup_{n\to\infty} \frac{1}{|B_n|} \log |U_n| = h(\mathcal{K})$.

Proof will be given in Section 5.3.

Now we are aiming to prove that the exponential growth rate of $|U_n|$ equals h_Γ. Fix $n \in \mathbf{Z}^+$ and denote $N = |B_n| + 1$. Enumerate the vertices $x_1, x_2, \ldots, x_N \in \Gamma^{(n)}$ starting from $\mathcal{O}_n$. The *Kirchhoff matrix* of $\Gamma^{(n)}$ is the matrix K_n of size $N \times N$ such that

$$K_n(i,j) = -k(x_i, x_j), \quad \text{if } i \neq j, \quad \text{and} \quad K_n(i,i) = deg(x_i, \Gamma^{(n)}).$$

Any function $\phi : \Gamma^{(n)} \to [0,1)$ can be viewed as a column vector (but we write it as a row vector $\phi = (\phi_1, \ldots, \phi_N)$). It is easy to see that ϕ is harmonic *mod* 1 w.r.t. $\Gamma^{(n)}$ if and only if $K_n\phi$ is an integral vector, i.e. $(K_n\phi)_i \in \mathbf{Z}$ for $i = 1, \ldots, N$. Observe that K_n is a singular matrix, since the sum of all rows is equal to zero. We define a new matrix K'_n of size $N \times N$ by

$$K'_n(i,j) = \begin{cases} K_n(i,j) & \text{if } i > 1, j > 1 \\ 1 & \text{if } i = 1, j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 5.2 *Let $\phi : \Gamma^{(n)} \to [0,1)$. Then $\phi \in U_n$ if and only if $K'_n\phi$ is an integral vector.*

PROOF. Suppose $(K'_n\phi)_i \in \mathbf{Z}$ for $i = 1, \ldots, N$. Since $\phi_1 = (K'_n\phi)_1 \in [0,1)$, we have $\phi_1 = 0$, so $(K_n\phi)_i = (K'_n\phi)_i \in \mathbf{Z}$ for $i > 1$. But $\sum_1^N (K_n\phi)_i = 0$, so $(K_n\phi)_1 \in \mathbf{Z}$ as well. Thus $\phi \in U_n$. The other direction is obvious.

Let T_n be the set of *spanning trees* in $\Gamma^{(n)}$.

Lemma 5.3

$$\det K'_n = |T_n| \; .$$

Proof. This is the Matrix Tree Theorem (see [Gib], Th 2.6 p.54).

In particular, K'_n is non-singular, and we conclude that $|U_n|$ equals the number of lattice points in the image of $[0,1)^N$ under the linear transformation K'_n.

Lemma 4. *Let $m \in \mathbf{Z}^+$, $K : \mathbf{R}^m \to \mathbf{R}^m$ be a linear transformation given by a matrix with integral entries. If M is the number of lattice points in $K([0,1)^m) \stackrel{\text{def}}{=} P$, then $M = Vol\, P$.*

PROOF. For $r \in \mathbf{N}$, let G_r be the grid of size $1/r$, that is $G_r = \{x = (x_1, x_2, \ldots, x_m), : \ rx_i \in \mathbf{Z}$ for all $i\}$. Let $M_r = |G_r \bigcap P|$. If we blow up the picture by the factor of r, we get also that $M_r = |\mathbf{Z}^m \bigcap K([0,r)^m)|$. Fix $\varepsilon > 0$. It follows from Lebesgue measure theory that we can approximate P by a union of cubes with vertices in G_r, in other words, we can find r such that

$$(1-\varepsilon)\, Vol\, P < M_r \frac{1}{r^m} < (1+\varepsilon)\, Vol\, P.$$

Note that K maps $[0,r)^m$ to the polyhedron which consists of r^m translations of P by integral vectors. All of these translations have the same number of lattice points with the total of M_r. Thus there are M_r/r^m lattice points in P . Since ε was arbitrary, $M = Vol\, P$. q.e.d.

By Lemmas 3 and 4,

$$|U_n| = |T_n|. \tag{5.5}$$

Given n, we say that a subgraph γ of Γ_n is *compatible*, if it can be extended to an essential spanning forest in Γ by only adding edges outside Γ_n. It is easy to see that compatible graphs are precisely those forests in Γ_n which contain all vertices of B_n, and such that every connected component touches ∂B_n. Let F_n be the set of compatible subgraphs in Γ_n. Recall that T_n stands for the set of spanning trees in $\Gamma^{(n)}$.

Consider the map $\alpha_n : T_n \to F_n$, which erases O_n and all edges adjacent to it. Then α_n is onto, for any $\gamma \in F_n$ can be extended to a tree $\tau \in T_n$ by adding some of the edges adjacent to $\mathcal{O}_n$. Since $deg(\mathcal{O}_n, \Gamma^{(n)}) \leq D \cdot |\partial B_n|$,

$$|F_n| \leq |T_n| \leq 2^{D\cdot|\partial B_n|}|F_n|.$$

But $|\partial B_n|/|B_n| \to 0$, as $n \to \infty$, so

$$\limsup_{n\to\infty} \frac{1}{| B_n |} \log |T_n| = \limsup_{n\to\infty} \frac{1}{| B_n |} \log |F_n| = h_\Gamma.$$

In view of (5.5), the equality of entropies results.

5.3 *Proof of Lemma 5.1*

Consider the set U_n^0 of all functions $\phi : \Gamma_n \to [0,1)$ such that

(i) $\phi(y) = 0$, if $y \in \partial B_n$,
(ii) $\phi(x)$ is harmonic *mod* 1 at all $x \in B_{n-1}$.

For any $\phi \in U_n^0$, consider $\psi : \Gamma^{(n-1)} \to [0,1)$ defined by $\psi(\mathcal{O}_{n-1}) = 0$, and $\psi(x) = \phi(x)$ for all $x \in B_{n-1}$. Observe that ψ is harmonic *mod* 1 at any $x \in B_{n-1}$, and, therefore, at $\mathcal{O}_{n-1}$ as well, since

$$\sum_{x\in\Gamma^{(n-1)}} \sum_{y\sim x} (\psi(y) - \psi(x)) = 0.$$

This defines a one-to-one correspondence between U_n^0 and U_{n-1}, so it suffices to prove Lemma 5.1 with U_n replaced by U_n^0.

Part (a).

$$h(\mathcal{K}) \leq \limsup_{n\to\infty} \frac{1}{| B_n |} \log |U_n^0|.$$

PROOF OF PART (A). For $s, t \in [0,1)$ we write, throughout this section, $|s - t| = dist(s + \mathbf{Z}, t + \mathbf{Z})$. Let $n \in \mathbf{Z}^+, \varepsilon > 0$. A set $F_n(\varepsilon) \subset \mathcal{K}$ is called (n, ε)-*coordinatewise* separated, if for any $f_1, f_2 \in F_n(\varepsilon)$ there exists $x \in B_n$ such that $|f_1(x) - f_2(x)| \geq \varepsilon$. It is shown in [LSW] (see the argument following Lemma 3.2, pp. 597 -598), that coordinatewise separated sets can be used rather than separated sets in the computation of entropy, i.e. $F_n(\varepsilon)$ being chosen of maximal cardinality,

$$h(\mathcal{K}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{1}{\mid B_n \mid} \log |F_n(\varepsilon)|.$$

Fix n and ε, and assume that $L = 3/\varepsilon$ is an integer. Partition $\mathbf{R}/\mathbf{Z}$ into intervals $[0,\varepsilon/3) = I_1, I_2, \ldots, I_L$ of length $\varepsilon/3$. We say that $f_1, f_2 \in F_n(\varepsilon)$ are equivalent, if $f_1(x), f_2(x)$ are in the same I_j for any $x \in \partial B_n$. There are at most $L^{|\partial B_n|}$ equivalence classes. Let $F_n'(\varepsilon)$ be one of maximal cardinality. Then

$$\mid F_n'(\varepsilon) \mid \leq \mid F_n(\varepsilon) \mid \leq L^{|\partial B_n|} \mid F_n'(\varepsilon) \mid .$$

Since $\mid \partial B_n \mid / \mid B_n \mid \to 0$ as $n \to \infty$,

$$h(\mathcal{K}) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{\log \mid F_n'(\varepsilon) \mid}{\mid B_n \mid} \tag{5.6}$$

Let $\nu : \partial B_n \to \mathbf{R}$. We say that $f : B_n \to \mathbf{R}$ is *harmonic with boundary condition* ν, if

$$\begin{aligned} f(x) &= \nu(x) && \text{for } x \in \partial B_n \\ deg(x,\Gamma_n) f(x) &= \textstyle\sum_{y\sim x} f(y) && \text{for } x \in B_n \setminus \partial B_n \ . \end{aligned}$$

Let $f \in F_n'(\varepsilon)$. It is known (see [DS], Sec.2.5), that there exists a unique harmonic function $g^f : B_n \to \mathbf{R}$ with boundary condition $f|_{\partial B_n}$. By the Maximum Principle (see [DS], Sec.2.4), $0 \leq g^f < 1$ on B_n. If $f_1, f_2 \in F_n'(\varepsilon)$ then

$$-\varepsilon/3 < (g^{f_1} - g^{f_2})(x) < \varepsilon/3 \tag{5.7}$$

for all $x \in \partial B_n$, and therefore for all $x \in B_n$, again by the Maximum Principle.

Consider $U_n^0(\varepsilon) = \{f|_{B_n} - g^f, f \in F_n'(\varepsilon)\}$, where differences are taken *mod* 1, so that elements of $U_n^0(\varepsilon)$ take values in $[0,1)$. Since $f|_{B_n}$ is harmonic *mod* 1 at all $x \in B_{n-1}$, and g^f is harmonic at such x, any function $\psi \in U_n^0(\varepsilon)$ is harmonic *mod* 1 with zero boundary conditions on ∂B_n.

By construction,

$$|F'_n(\varepsilon)| = |U_n^0(\varepsilon)| \ . \tag{5.8}$$

Since $F'_n(\varepsilon)$ is (n,ε)-coordinatewise separated, it follows from (5.7) that for any $\phi_1, \phi_2 \in U_n^0(\varepsilon)$ there exists $z \in B_n$ such that $|\phi_1(z) - \phi_2(z)| > \varepsilon/3$, so the elements of U_n^0 are distinct.

It follows that

$$|U_n^0(\varepsilon)| \leq |U_n^0|.$$

Combining this with (5.6),(5.8) yields the result.

Part (b).

$$h(\mathcal{K}) \geq \limsup_{n\to\infty} \frac{1}{|B_n|} \log |U_n^0|.$$

PROOF OF PART (b). It suffices to show that if $\Phi : \Gamma_n \to [0,1)$ is harmonic *mod* 1 at all $x \in B_{n-1}$ with some boundary conditions on ∂B_n, then Φ can be extended to a function $\Phi' : \Gamma_{n+1} \to [0,1)$ which is harmonic *mod* 1 at all $x \in B_n$. Let n and such a Φ be fixed.

Write $B_{n+1} = B_{n-1} \cup \partial B_n \cup \partial B_{n+1}$, and let $|B_{n-1}| = N$, $|\partial B_n| = L$, $|\partial B_{n+1}| = M$. Fix some enumeration of vertices in B_{n+1} so that B_{n-1} is listed first, then ∂B_n, and then ∂B_{n+1}. Write K_{n+1} in the block form :

$$K_{n+1} = \begin{pmatrix} A_{N\times N} & A_{N\times L} & A_{N\times M} \\ A_{L\times N} & A_{L\times L} & A_{L\times M} \\ A_{M\times N} & A_{M\times L} & A_{M\times M} \end{pmatrix},$$

where the subscripts indicate the size of each block. Write $\Phi = (\Phi_N, \Phi_L)$, where Φ_N denotes the first N coordinates, Φ_L denotes the next L coordinates of Φ. By assumption, Φ is harmonic *mod* 1 with boundary conditions Φ_L, which implies that

$$A_{N\times N}\Phi_N + A_{N\times L}\Phi_L \ \in \mathbf{Z}^N.$$

We would like to find M-dimensional vector Φ_M with coordinates in $[0,1)$ such that if

$$\Phi' = (\Phi_N, \Phi_L, \Phi_M)$$

and

$$K_{n+1}\Phi' = (\Psi_N, \Psi_L, \Psi_M),$$

then Ψ_N and Ψ_L are integral. By definition of ∂B_{n+1}, we have $A_{M\times N} = A^T_{N\times M} = 0$, so $\Psi_N = A_{N\times N}\Phi_N + A_{N\times L}\Phi_L$ is integral by above. Thus we only need to show that there exists Φ_M with coordinates in $[0,1)$ such that $\Psi_L = A_{L\times N}\Phi_N + A_{L\times L}\Phi_L + A_{L\times M}\Phi_M$ is integral.

Sublemma.

$$rank\ A_{L\times M} = L\ .$$

PROOF. Suppose $rank\ A_{L\times M} < L$, so that the rows of $A_{L\times M}$ are linearly dependent. Recall that $A_{L\times M}$ consists of non-positive entries. Reordering vertices of ∂B_n, we can assume that there exist positive $\alpha_1, \ldots, \alpha_k$ such that

$$\alpha_1\bar{v}_1 + \ldots + \alpha_m\bar{v}_m = \alpha_{m+1}\bar{v}_{m+1} + \ldots + \alpha_k\bar{v}_k\ ,$$

where $\bar{v}_i$ are rows of $A_{L\times M}$. This implies that the set of vertices corresponding to $\bar{v}_1, \ldots \bar{v}_m$ and the set of vertices corresponding to $\bar{v}_{m+1}, \ldots \bar{v}_k$ in ∂B_n are joined to exactly the same set of vertices in ∂B_{n+1}, which contradicts the periodicity of Γ. This proves the Sublemma.

By the Sublemma, there exists $\tilde{\Phi}_M : \partial B_{n+1} \to \mathbf{R}$ such that

$$A_{L\times M}\tilde{\Phi}_M = -A_{L\times L}\Phi_L - A_{L\times N}\Phi_N$$

It remains to choose

$$\Phi_M = \tilde{\Phi}_M\ \ mod\ 1$$

so that Φ_M has coordinates in $[0,1)$. Then $\Phi' = (\Phi_N, \Phi_L, \Phi_M)$ is the extension of (Φ_N, Φ_L) we were after.

By induction, one can extend any element of U_n^0 to an element of $\mathcal{K}$. Finally, we check that U_n^0 so extended forms a (n, ε)-coordinatewise separated set, provided $\varepsilon < 1/D$. Indeed, let $\phi_1, \phi_2 \in U_n^0$, and $\phi_0 = \phi_1 - \phi_2$. Then $\phi_0 \in U_n^0$. If $|\phi_1(z) - \phi_2(z)| = |\phi_0(z)| \leq \varepsilon$ for all $z \in B_n$, then ϕ_0 must be just harmonic on Γ_n, therefore identically zero.

It follows that $h(\mathcal{K}) \geq \limsup_{n \to \infty} |B_n|^{-1} \log |U_n^0|$. This completes the proof of Lemma 1 and of coincidence of entropies.

BIBLIOGRAPHY

[Green] F.P. Greenleaf, *Invariant means on topological groups.* New York, Van Nostrand ·Reinhold Co. 1969

[BP93] R. Burton and R. Pemantle, Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedance. *The Annals of Probability* 1993, Vol. 21, No. 3, 1329–1371.

[BS96] R. Burton and J. Steif, Some 2-d symbolic dynamical Systems: entropy and mixing, *Ergodic Theory of* $\mathbf{Z}^d$*-ctions,* LMS Lecture Notes Series **228** (1996), 297-305.

[C] J.W. Cannon, The theory of negatively curved spaces and groups, *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces.* Oxford Science Publications. Oxford University Press, 1991

[DS] G. Doyle and J. L. Snell, *Random Walks and Electric Networks.* The Carus Mathematical Monogrpahs #22. The Math. Association of America, Washington, D.C. 1984

[GH] E. Ghys and P. de la Harpe, Infinite groups as geometric objects (after Gromov). *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces.* Oxford Science Publications. Oxford University Press, 1991

[Gib] A. Gibbons, *Algorithmic graph theory.* Cambridge University Press, Cambridge; New York, 1985

[LSW] D. Lind, K. Schmidt, and T. Ward, Mahler measure and entropy for commuting automorphisms of compact groups. *Invent.math.* 101, 593-629 (1990)

[LyPe] R. Lyons and Yu. Peres, *Probability on Trees.* To appear.

[Pem90] R. Pemantle, Choose a spanning tree for the integer lattice uniformly, *The Annals of Probability.* (1991) Vol. 19, No. 4, 1559–1574.

[OW] D. Ornstein and B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Analyse Math.* (1987) Vol. 48, 1–141.

[Sch] K. Schmidt, *Dynamical Systems of Algebraic Origin.* Birkhäuser Verlag, Basel; Boston, 1995

[Sen] E. Seneta, *Non-negative Matrices and Markov Chains* (Second Ed.), Springer, New York,1980.

[Ser] C. Series, The infinite word problem and limit sets in Fuchsian groups. *Ergodic Th. & Dynam. Systems* (1981) 1, 337–360.

[So94] P. Soardi, *Potential Theory on infinite Networks.* Lectures notes in Mathematics 1590; Springer-Verlag, Berlin 1994

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