The Anisotropic Gaussian Isoperimetric Inequality and Ehrhard Symmetrization

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Abstract

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In this thesis, we establish the isoperimetric inequality for the anisotropic Gaussian measure and characterize the cases of equality. Additionally, we present an example demonstrating that Ehrhard symmetrization fails to decrease for the anisotropic Gaussian perimeter and introduce a new inequality that includes an error term. This new inequality, in particular, provides a clue to a uniqueness result for the Ehrhard measure within the class of anisotropic Gaussian measures. Our final result, a collaboration with Sean McCurdy, expands the class of measures to which the previous uniqueness result applies.

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DEDICATION

To my beloved family, whose endless support and encouragement have been the guiding light throughout my academic journey.

Chapter 1
INTRODUCTION

1.1 Motivations

The isoperimetric problem is an ancient and fascinating problem in mathematics. The origins of it can be traced back to an ancient tale, Queen Dido's Problem. According to the story, Queen Dido used strips of bullhide to encircle the land that would become the city of Carthage [Ban17]. In doing so, she encountered the challenge of maximizing the enclosed area while adhering to a fixed amount of perimeter. Although the intuitive answer is a disk, the rigorous proof of this solution emerged only relatively recently. In the 19th century, Schwarz [Sch90] provided the first rigorous proof of the isoperimetric problem using a tool known as Steiner symmetrization. This technique allows for the transformation of a region in such a way that its perimeter decreases while its volume remains unchanged. The following figure demonstrates this transformation in $\mathbb{R}^2$ by redistributing area from set E vertically to the x-axis, resulting in a new set E^s symmetric with respect to the x-axis. Schwarz's work also highlighted an existence issue in Steiner's original proofs. Since then, symmetrization has become a pivotal tool in the field of geometric analysis.

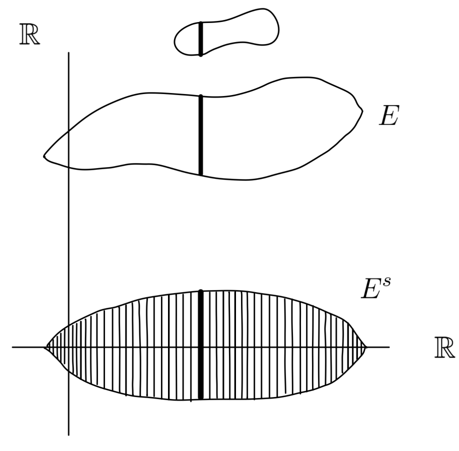

Figure 1.1: E^s is the Steiner symmetrization of E in direction $-e_2$ in $\mathbb{R}^2$.

To address the isoperimetric problem in higher dimensions, a refined notion of perimeter becomes important. Through the efforts of Cesari [Ces36] and De Giorgi [DG53], who utilized the concept of Divergence Theorem, we have a more precise definition for the perimeter. Their innovation bridged techniques from analysis, encompassing derivatives, integrals, measures, and more. These tools enable us to analyze sets that exhibit unconventional behavior, such as sharp corners, cusps, or tentacle-like structures, they are part of a fruitful field known as Geometric Measure Theory.

An analogous problem to the classical isoperimetric problem, wherein Gaussian measure is employed in lieu of Lebesgue measure, is termed the Gaussian isoperimetric problem. This intriguing problem was initially established by Sudakov, Tsirelson [ST78] and Borell [Bor75] via Paul Levy's spherical isoperimetric inequality. Furthermore, Carlen and Kerce [CK01] characterized half-spaces as the unique minimizers in the Gaussian isoperimetric problem, utilizing a Gaussian analog of the Laplacian known as the Ornstein-Uhlenbeck operator.

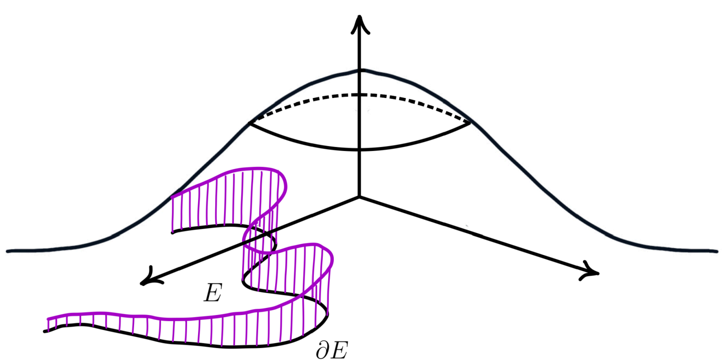

Figure 1.2: This purple drawing visualizes the perimeter under the Gaussian weight.

In Gaussian space, a counterpart to Steiner symmetrization was introduced by Ehrhard. This technique, now known as Ehrhard symmetrization, transforms a region to decrease its Gaussian perimeter while preserving its Gaussian mass [Ehr82, Ehr83, Ehr84]. A geometric approach to the Gaussian isoperimetric problem using Ehrhard symmetrization has been developed by Cianchi, Fusco, Maggi, and Pratelli [CFMP11]. In their work, they also established a fascinating result of the quantitative stability for Gaussian isoperimetric problem through symmetrization.

In this thesis, our goal is to extend the Gaussian isoperimetric problem and Ehrhard symmetrization to encompass anisotropic Gaussian measures. In this context, "anisotropic" denotes the presence of varying values in different directions, meaning the measure lacks rotational invariance and may involve cross terms. There are two major parts in this thesis. The first part, Chapters 2 and 3, characterizes minimizers in the anisotropic Gaussian isoperimetric problem and analyzes the relationship between anisotropic Gaussian measures and Ehrhard symmetrization. The second part of the thesis, Chapters 4 and 5, focuses on a compelling question: Does the isotropic Gaussian measure stand alone as the only weight for which Ehrhard symmetrization yields a decrease in perimeter? We will show that within some classes of measures, the isotropic Gaussian measure is indeed the only measure that satisfies this property. Chapters 2 to 4 are based on the author's work in [Yeh23], and Chapter 5 is a joint work with Sean McCurdy from [MY23].

1.2 Main results

1.2.1 Anisotropic Gaussian isoperimetric problems and Ehrhard symmetrization

Isoperimetric inequalities have many applications in the concentration of measure phenomena, inequalities in probability theory, Optimal Transport, Poincaré inequalities, Sobolev inequalities, and Machine Learning, etc (see, for example, [Bar01][Eld13][FMP10][BL06][GNP17][Wai19]). One important inequality that appears in isoperimetric problems is called the Gaussian isoperimetric inequality. It says that the minimizers of Gaussian perimeter among sets with fixed Gaussian mass are half-spaces. A natural generalization of it is to study the following optimization problem about the anisotropic Gaussian perimeter:

$$\inf \left\{ P_{\gamma_A}(E) : \gamma_A(E) = r \right\}, \tag{1.1}$$

where A is a positive definite matrix, γ_A is called the **A-anisotropic Gaussian measure** and the perimeter with respect to γ_A is called the **A-anisotropic Gaussian perimeter** which are defined as

$$\gamma_A(E) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\langle Ax,x\rangle/2}\, dx, \quad P_{\gamma_A}(E) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial^M E} e^{-\langle Ax,x\rangle/2}\, d\mathcal{H}^{n-1}(x).$$

Here $\partial^M E$ is the $(n-1)$-dimensional essential boundary of E (see Section 2.1). Note that we may assume without loss of generality that our positive definite matrix A is symmetric as

$$\langle Ax, x\rangle = \left\langle \frac{1}{2}\left(A + A^{\mathsf{T}}\right) x, x \right\rangle$$

and $\frac{1}{2}\left(A + A^{\mathsf{T}}\right)$ is a symmetric positive definite matrix. We will assume that A is symmetric throughout the rest of the thesis. Recall the case $A = I_n$ corresponds to the Gaussian perimeter.

In Chapter 2, we demonstrate some basic properties of the anisotropic Gaussian perimeter and show that the unique minimizers of (1.1) are half-spaces with eigenvector directions of A corresponding to the smallest eigenvalue. More precisely, we have the following.

Theorem 1.2.1. [Yeh23] *Let A be a symmetric positive definite matrix and E be a measurable set in $\mathbb{R}^n$. Then*

$$P_{\gamma_A}(E) \geq e^{-[\phi^{-1}(\gamma_A(E))]^2/2} \frac{1}{\|(\sqrt{A})^{-1}\|}. \tag{1.2}$$

Here $\|\cdot\|$ is the matrix norm induced by the Euclidean norm. Moreover,

(1) if $n = 1$, equality holds if and only if either $\gamma_A(E) = 0$ or $\gamma_A(E) = 1$, or E is equivalent to a half-line of the form

$$\left(-\infty, \frac{\phi^{-1}(\gamma_A(E))}{\sqrt{A}}\right) \quad \text{or} \quad \left(-\frac{\phi^{-1}(\gamma_A(E))}{\sqrt{A}}, +\infty\right).$$

(2) if $n \geq 2$, equality holds if and only if either $\gamma_A(E) = 0$ or $\gamma_A(E) = 1$, or E is equivalent to a half-space of the form

$$H\left(\omega, \frac{\phi^{-1}(\gamma_A(E))}{d_{\min}}\right) \qquad \text{for some unit vector } \omega \in V_{d_{\min}}(\sqrt{A}),$$

where $d_{\min}$ is the smallest eigenvalue of $\sqrt{A}$ and $V_{d_{\min}}(\sqrt{A})$ is the eigenspace of $\sqrt{A}$ associated with $d_{\min}$.

Notice that the anisotropic Gaussian isoperimetric inequality (1.2) is a special case of the Bakry-Ledoux isoperimetric inequality for log-concave measures if we consider the measure $e^{-\langle Ax,x\rangle/2}dx$ and use the ε-enlargement definition for the perimeter [LB96, Led99]. The main contribution here is to characterize the cases of equality in (1.2) and prove the uniqueness of the minimizer of (1.1).

As mentioned earlier, Ehrhard symmetrization has played a pivotal role in establishing Gaussian isoperimetric problems. This versatile tool has also been used in many pieces of literature (see, for example, [Lif95][Bob96][CFMP11][BCM12][CCDPM17]). Roughly speaking, the process of Ehrhard symmetrization consists in pushing the set E to infinity along the direction u in such a way that the mass distribution remains unchanged. The resulting set E^s is called the Ehrhard symmetrization of E in direction u.

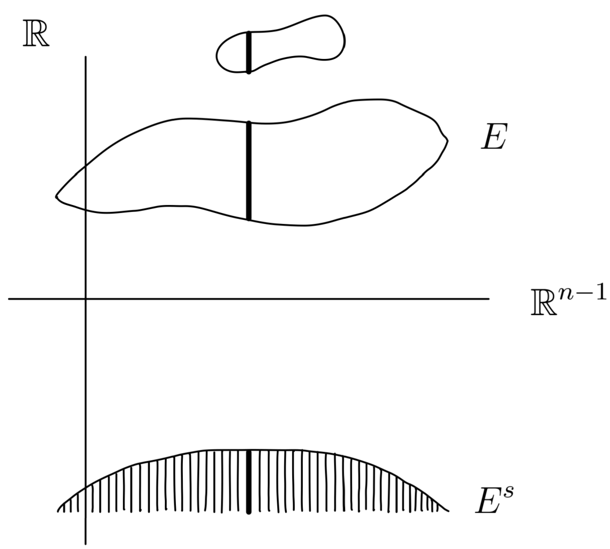

Figure 1.3: E^s is the Ehrhard symmetrization of E in direction $-e_n$ in $\mathbb{R}^n$.

Ehrhard showed that for the isotropic Gaussian measure, the Gaussian perimeter will always decrease after performing Ehrhard symmetrization in any direction. One might also expect the same for the anisotropic Gaussian measure γ_A, since γ_A exhibits exponential decay at infinity. However, we constructed a counterexample that shows that the perimeter could actually increase after symmetrization (see Example 3.3.1).

One of the key properties of the Gaussian measure is that it has a product structure, so that the problem can be reduced to the one-dimensional case. The main difficulty in the setting of anisotropic Gaussian measure is that the measure γ_A doesn't have the product structure, i.e., it has cross terms, which means new ideas will need to be developed to address this issue.

In Chapter 3, we study some regularity results related to the Ehrhard symmetrization set and present the counterexample mentioned above. Moreover, using the tools in [CCF05] and [CFMP11], we are able to find an upper bound for the perimeter of Ehrhard symmetrization set in terms of the original perimeter plus a term involving the deviation of A from the identity in the direction of the symmetrization times a term involving the differences of the anisotropic Gaussian barycenters. Our result can be stated as follows.

Theorem 1.2.2. [Yeh23] *Let $n \geq 2$, A be a symmetric positive definite matrix, and let E be a set of finite A-anisotropic Gaussian perimeter in $\mathbb{R}^n$ and $u \in \mathbb{S}^{n-1}$. Then, $E^s_{A,u}$ is a set of locally finite perimeter in $\mathbb{R}^n$. Moreover, for every Borel set $B \subseteq \langle u \rangle^\perp$ with $|u| = 1$ we have*

$$P_{\gamma_A}\left(E^s_{A,u}; B \oplus \langle u \rangle\right) \leq P_{\gamma_A}\left(E; B \oplus \langle u \rangle\right) + \sqrt{2\pi}\|Au - \langle Au, u\rangle u\| \langle b_{\gamma_A}(E^s_{A,u} \cap (B \oplus \langle u \rangle)) - b_{\gamma_A}(E \cap (B \oplus \langle u \rangle)), u\rangle.$$

Here $B \oplus \langle u \rangle := \{z + tu : z \in B, t \in \mathbb{R}\}$ and $E^s_{A,u}$ is the **Ehrhard symmetrization of** *E with respect to the u-direction and matrix A (see definition (3.32)) and*

$$b_{\gamma_A}(E) = \int_E x \; d\gamma_A(x)$$

is called the A**-anisotropic Gaussian barycenter** *of E.*

1.2.2 Characterization of Ehrhard measures

An immediate consequence of Theorem 1.2.2 tells us that the anisotropic Gaussian perimeter decreases if we do the Ehrhard symmetrization with respect to any eigenvector direction of A.

In Chapter 4, we show that the converse of the above statement is also true, i.e., the only situation in which the anisotropic Gaussian perimeter decreases is when the Ehrhard symmetrization occurs along an eigendirection of A.

Theorem 1.2.3. [Yeh23] *Let $n \geq 2$, A be a symmetric positive definite matrix, and $u \in \mathbb{S}^{n-1}$.*

$$P_{\gamma_A}(E^s_{A,u}) \leq P_{\gamma_A}(E) \text{ for all finite } A\text{-anisotropic Gaussian perimeter set } E \text{ in } \mathbb{R}^n$$
$$\iff u \in V_\lambda(A) \cap \mathbb{S}^{n-1} \text{ for some } \lambda > 0$$

where $V_\lambda(A)$ is the eigenspace of A associated with eigenvalue λ. Moreover,

$$\gamma_A \text{ is Ehrhard symmetrizable} \iff A = aI_n \text{ for some constant } a > 0.$$

Here γ_A is called **Ehrhard symmetrizable** *(or simply an* **Ehrhard measure***) if*

$$P_{\gamma_A}(E^s_{A,u}) \leq P_{\gamma_A}(E)$$

for all $u \in \mathbb{S}^{n-1}$, and for all finite A-anisotropic Gaussian perimeter set E in $\mathbb{R}^n$.

Let $\mathcal{A}(\mathbb{R}^n)$ be the class that contains all the anisotropic Gaussian measures. Theorem 1.2.3 says that $\gamma_A \in \mathcal{A}(\mathbb{R}^n)$ is an Ehrhard measure if and only if A is a multiple of the identity matrix. That is, among the sets of anisotropic Gaussian measures, the only measure that will decrease the anisotropic Gaussian perimeter under Ehrhard symmetrization in every direction is the isotropic Gaussian measure. This leads to a compelling question: Does Theorem 1.2.3 still hold true for a broader class of finite measures?

In Chapter 5, we will expand our class $\mathcal{A}(\mathbb{R}^n)$ to include a broader collection of finite measures. This work is a collaboration with Sean McCurdy. Consider the following class

$$\mathcal{W}(\mathbb{R}^n) = \left\{ \mu : \frac{d\mu}{d\mathcal{L}^n} = f \in C^1(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n), f > 0 \right\}.$$

We first establish theorems for weighted BV functions, demonstrating that half-spaces serve as minimizers for Ehrhard measures (see Definition 5.1.2). We then leverage the framework developed by Brock-Chiacchio-Mercaldo [BCM16] to infer that the Ehrhard measures in class $\mathcal{W}(\mathbb{R}^n)$ enjoys a product structure. Ultimately, this product structure yields the following uniqueness result.

Theorem 1.2.4. [MY23] *Let* $n \geq 2$ *and* $\mu \in \mathcal{W}(\mathbb{R}^n)$ *with a positive distribution function* f. *Then* μ *is Ehrhard symmetrizable if and only if* $f(x) = Ce^{-c|x-a|^2}$ *for some* $0 < c < \infty$, $0 < C < \infty$, *and* $a \in \mathbb{R}^n$.

Unlike much of the literature on Gaussian-type isoperimetric problems, it is important to note that we do not assume μ to be log-concave.

Chapter 2

ANISOTROPIC GAUSSIAN ISOPERIMETRIC PROBLEMS

2.1 Sets of locally finite perimeter

In this section, we will recall some useful definitions and theorems from Maggi's book [Mag12] and Evans-Gariepy's book [EG92].

Let U be an open subset in $\mathbb{R}^n$. A function $f \in L^1(U)$ has **bounded variation** in U if

$$\sup\left\{ \int_U f \operatorname{div} \varphi \, dx : \varphi \in C_c^1(U;\mathbb{R}^n),\ |\varphi| \le 1 \right\} < \infty.$$

We write $BV(U)$ to denote the space of such functions. A function $f \in L^1_{\text{loc}}(U)$ has **locally bounded variation** in U if for every open set $V \subset\subset U$,

$$\sup\left\{ \int_V f \operatorname{div} \varphi \, dx : \varphi \in C_c^1\left(V;\mathbb{R}^n\right),\ |\varphi| \le 1 \right\} < \infty.$$

We write $BV_{\text{loc}}(U)$ to denote the space of such functions. A $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has **finite perimeter** in U if $\chi_E \in BV(U)$. A $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has **locally finite perimeter** in U if $\chi_E \in BV_{\text{loc}}(U)$.

We recall the following theorem from [EG92], Chapter 5, Theorem 1.

Theorem 2.1.1. *Let $f \in BV_{loc}(U)$. Then there exists a Radon measure μ on U and a μ-measurable function $\sigma : U \to \mathbb{R}^n$ such that*

(1) $|\sigma(x)| = 1$ μ-a.e. on U.

(2) For any $\varphi \in C_c^1(U;\mathbb{R}^n)$,

$$\int_U f \operatorname{div} \varphi \, dx = -\int_U \varphi \cdot \sigma \, d\mu.$$

We write $|Df|$ *for* μ, $Df := \sigma|Df|$, *and* $D_i f := \sigma_i |Df|$, *where* $\sigma = (\sigma_1, \cdots, \sigma_n)$. *Moreover,*

$$|Df|(V) = \sup\left\{\int_V f \operatorname{div}\varphi \, dx : \varphi \in C_c^1(V;\mathbb{R}^n), \ |\varphi| \le 1\right\}$$
$$= \sup\left\{\int_V \varphi \cdot dDf : \varphi \in C_c^1(V;\mathbb{R}^n), \ |\varphi| \le 1\right\}$$

for any $V \subset\subset U$, *i.e., the total variation of* Df *is* $|Df|$. *Also,*

E *is a set of locally finite perimeter* $\Longleftrightarrow$ $|D\chi_E|(K) < \infty$ *for every compact set* $K \subset \mathbb{R}^n$.

Remark: If $f = \chi_E$, and E is a set of locally finite perimeter in U, we will write

$$\nu^E := \sigma, \quad \nu_E := -\sigma, \quad \mu_E := \nu_E |D\chi_E|,$$

where $\nu^E(x)$ ($\nu_E(x)$) is called the **generalized inner (outer) unit normal** of E at x and the $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ is called the **Gauss-Green measure** of E. Let E be a set of locally finite perimeter. The **reduced boundary** $\partial^* E$ of E is the set of those $x \in \operatorname{spt}\mu_E$ such that

$$\nu_E(x) = \frac{d\mu_E}{d|\mu_E|}(x) := \lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} \text{ exists and is in } \mathbb{S}^{n-1}, \tag{2.1}$$

where $\operatorname{spt}\mu_E := \{x : |\mu_E|(B(x,r)) > 0 \text{ for all } r > 0\}$. In fact, we have

$$\partial^* E \subset \operatorname{spt}\mu_E \subset \partial E$$

and $\operatorname{spt}\mu_E = \{x : 0 < |E \cap B(x,r)| < |B(x,r)| \text{ for all } r > 0\}$. Moreover, the De Giorgi structure theorem states that $\partial^* E$ is $(n-1)$-rectifiable and that

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E, \tag{2.2}$$

or equivalently,

$$D\chi_E = \nu^E \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad |D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E, \tag{2.3}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Hence, we have the divergence theorem in the following form:

$$\int_{\partial^* E} F \cdot \nu_E \, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^n} F \cdot d\mu_E = \int_E \operatorname{div} F, \qquad \text{for any } F \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), \tag{2.4}$$

where $\cdot$ is the Euclidean dot product (see [Mag12], Proposition 12.19, Theorem 15.9).

Let E be any measurable subset in $\mathbb{R}^n$ and $0 \leq d \leq 1$. The set of points of **density** d **of** E is defined as

$$E^{(d)} = \left\{ x \in \mathbb{R}^n : \theta_n(E)(x) := \lim_{\rho \to 0} \frac{\mathcal{L}^n\left(E \cap Q_\rho(x)\right)}{\mathcal{L}^n\left(Q_\rho(x)\right)} = d \right\}$$

where $Q_\rho(x)$ is the cube centered at x, whose sides are parallel to the coordinate axes with length 2ρ. We will use $|\cdot|$ or $\mathcal{L}^n$ for Lebesgue measure on $\mathbb{R}^n$. By Lebesgue points theorem,

$$\theta_n(E) = 1 \quad \text{a.e. on } E, \qquad \theta_n(E) = 0 \quad \text{a.e. on } \mathbb{R}^n \setminus E.$$

Therefore, $|E \Delta E^{(1)}| = 0$, i.e., every Lebesgue measurable set E is equivalent to $E^{(1)}$. Now we introduce the **essential boundary** $\partial^M E$ of a measurable set E defined as

$$\partial^M E := \mathbb{R}^n \setminus \left(E^{(0)} \cup E^{(1)} \right).$$

Then Federer's theorem tells us that for any set of locally finite perimeter E,

$$\partial^* E \subset E^{(1/2)} \subset \partial^M E, \qquad \mathcal{H}^{n-1}(\partial^M E \setminus \partial^* E) = 0.$$

Moreover,

$$E \text{ is a set of locally finite perimeter } \iff \mathcal{H}^{n-1}(\partial^M E \cap K) < \infty, \ \forall K \text{ compact.} \tag{2.5}$$

We also define the **(relative) perimeter of** E **in** F as

$$P(E; F) = \mathcal{H}^{n-1}(\partial^M E \cap F),$$

for any Borel set $F \subset \mathbb{R}^n$ (see [Mag12], Corollary 15.8, Theorem 16.2 and [Fed69], Theorem 4.5.11).

2.2 Important background results and notation

In this subsection, we collect some significant results that will be used in the later sections.

Proposition 2.2.1 ([Mag12], Proposition 17.1)**.**
If E is a set of locally finite perimeter in $\mathbb{R}^n$ and f is a diffeomorphism of $\mathbb{R}^n$ with $g = f^{-1}$, then $f(E)$ is a set of locally finite perimeter in $\mathbb{R}^n$ with $\mathcal{H}^{n-1}\left(f\left(\partial^* E\right) \Delta \partial^* f(E)\right) = 0$, and

$$\int_{\partial^* f(E)} \varphi \nu_{f(E)} \, d\mathcal{H}^{n-1} = \int_{\partial^* E} (\varphi \circ f) Jf (Dg \circ f)^* \nu_E \, d\mathcal{H}^{n-1}$$

for every $\varphi \in C_c\left(\mathbb{R}^n\right)$, where $Jf = |\det(Df)|$ is the Jacobian of f on $\mathbb{R}^n$.

Theorem 2.2.2 (Ehrhard's Inequality, [Bor03], Theorem 1.1)**.**
If A, B are Borel sets in $\mathbb{R}^n$, then

$$\phi^{-1}(\gamma(\lambda A + (1-\lambda)B)) \geq \lambda \phi^{-1}(\gamma(A)) + (1-\lambda)\phi^{-1}(\gamma(B)), \qquad \textit{for } \lambda \in (0,1),$$

where

$$\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{-|x|^2/2} \, dx, \qquad \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt.$$

Recall that we define $\mathbb{R}^m$-valued Radon measure μ on $\mathbb{R}^n$ as the bounded linear functional on $C_c(\mathbb{R}^n, \mathbb{R}^m)$, in the sense that, for every compact set $K \subset \mathbb{R}^n$,

$$\sup\left\{\langle \mu, \varphi \rangle : \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m), \operatorname{spt} \varphi \subset K, |\varphi| \leq 1\right\} < \infty,$$

and set

$$\langle \mu, \varphi \rangle := \int_{\mathbb{R}^n} \varphi \cdot d\mu, \qquad \varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m).$$

The following three propositions are useful when we calculate the total variation. The first one can be found in [Mag12], Remark 4.8, and the rest are straightforward applications of the results in [Mag12], Chapter 4 (see also [AFP00], Proposition 1.47).

Proposition 2.2.3 ([Mag12], Remark 4.6, 4.8)**.**
Let μ be a Radon measure on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a $\mathbb{R}^m$-valued function with $f \in L^1_{loc}(\mathbb{R}^n, \mu; \mathbb{R}^m)$. Then we may define a bounded linear functional $f\mu : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ as

$$\langle f\mu, \varphi \rangle := \int_{\mathbb{R}^n} f \cdot \varphi \, d\mu$$

for any $\varphi \in C_c(\mathbb{R}^n;\mathbb{R}^m)$*, i.e.,* $f\mu$ *is a* $\mathbb{R}^m$*-valued Radon measure on* $\mathbb{R}^n$*. Moreover, the total variation of* $f\mu$ *is* $|f\mu| = |f|\mu$*, where*

$$|f|\mu(E) := \int_E |f| \, d\mu, \qquad E \in \mathcal{B}(\mathbb{R}^n).$$

Proposition 2.2.4. *Let* μ *be a* $\mathbb{R}^m$*-valued Radon measure on* $\mathbb{R}^n$ *and let* $h : \mathbb{R}^n \to \mathbb{R}$ *be a real-valued locally bounded Borel function. Then we may define a bounded linear functional* $h\mu : C_c(\mathbb{R}^n;\mathbb{R}^m) \to \mathbb{R}$ *as*

$$\langle h\mu, \varphi \rangle := \int_{\mathbb{R}^n} h\varphi \cdot d\mu$$

for any $\varphi \in C_c(\mathbb{R}^n;\mathbb{R}^m)$*, i.e.,* $h\mu$ *is a* $\mathbb{R}^m$*-valued Radon measure on* $\mathbb{R}^n$*. Moreover, the total variation of* $h\mu$ *is* $|h\mu| = |h||\mu|$*, where*

$$|h||\mu|(E) := \int_E |h| \, d|\mu|, \qquad E \in \mathcal{B}(\mathbb{R}^n).$$

Proof. **Step 1**: For any compact set K and $\varphi \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ with $\operatorname{spt}\varphi \subset K$, $|\varphi| \le 1$, we have $|h| \le M_K$ on K for some $M_K > 0$ and

$$\langle h\mu, \varphi \rangle = \int_{\mathbb{R}^n} h\varphi \cdot d\mu \le \int_K |h||\varphi| \, d|\mu| \le M_K|\mu|(K) < \infty,$$

since $|\mu|$ is a Radon measure (see [Mag12], Lemma 4.17). This implies that $h\mu$ is a bounded linear functional, i.e., $h\mu$ is a $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$.

Step 2: By [Mag12], Chapter 4, equation (4.7), we just need to show that

$$|h\mu|(G) = |h||\mu|(G) \qquad \text{for any open set } G \subset \mathbb{R}^n.$$

Furthermore, we may assume that G is bounded since we can always intersect G with open balls. Recall that: ([Mag12], equation (4.35))

$$|\mu|(G) = \sup\left\{ \int_{\mathbb{R}^n} \varphi \cdot d\mu : \varphi \in C_c\left(G;\mathbb{R}^m\right), |\varphi| \le 1 \right\}.$$

By the polar decomposition ([Mag12], Remark 4.12), $\mu = \sigma|\mu|$ for some $|\mu|$-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that $|\sigma| = 1$ $|\mu|$-a.e. on $\mathbb{R}^n$. Therefore,

$$\begin{aligned} |h\mu|(G) &= \sup\left\{\int_{\mathbb{R}^n} \varphi \cdot d(h\mu) : \varphi \in C_{\mathrm{c}}\left(G;\mathbb{R}^m\right), |\varphi| \leq 1\right\} \\ &= \sup\left\{\int_{\mathbb{R}^n} h\varphi \cdot d\mu : \varphi \in C_{\mathrm{c}}\left(G;\mathbb{R}^m\right), |\varphi| \leq 1\right\} \\ &= \sup\left\{\int_{\mathbb{R}^n} h\varphi \cdot \sigma \; d|\mu| : \varphi \in C_{\mathrm{c}}\left(G;\mathbb{R}^m\right), |\varphi| \leq 1\right\} \\ &\leq \sup\left\{\int_{\mathbb{R}^n} |h||\varphi \cdot \sigma| \; d|\mu| : \varphi \in C_{\mathrm{c}}\left(G;\mathbb{R}^m\right), |\varphi| \leq 1\right\} \\ &\leq \int_G |h| \; d|\mu| = |h||\mu|(G). \end{aligned}$$

On the other hand, since $C_c(G;\mathbb{R}^m)$ is dense in $L^1(G, |\mu|; \mathbb{R}^m)$ and G is bounded, there exists $\varphi_k \in C_c(G;\mathbb{R}^m)$ such that

$$\varphi_k \to \operatorname{sgn}(h)\sigma \quad \text{in } L^1(G, |\mu|; \mathbb{R}^m).$$

By using a truncation argument, we may assume that $|\varphi_k| \leq 1$ for all k. Since $\varphi_k \to \operatorname{sgn}(h)\sigma\chi_G$ in $L^1(\mathbb{R}^n, |\mu|; \mathbb{R}^m)$, we have

$$\begin{aligned} |h\mu|(G) \geq \lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi_k \cdot d(h\mu) &= \lim_{k\to\infty} \int_{\mathbb{R}^n} h\varphi_k \cdot \sigma \; d|\mu| \\ &= \int_{\mathbb{R}^n} h\left(\operatorname{sgn}(h)\sigma\chi_G\right) \cdot \sigma \; d|\mu| = \int_G |h| \; d|\mu| = |h||\mu|(G). \end{aligned}$$

□

Proposition 2.2.5. *Let μ be a $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. Then we may define a bounded linear functional $f_{\#}\mu : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ as*

$$\langle f_{\#}\mu, \varphi\rangle := \int_{\mathbb{R}^n} (\varphi \circ f) \cdot d\mu$$

for any $\varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m)$, i.e., $f_{\#}\mu$ is a $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$. Moreover, the total variation of $f_{\#}\mu$ is $|f_{\#}\mu| = f_{\#}|\mu|$, where

$$f_{\#}|\mu|(E) := |\mu|(f^{-1}(E)), \qquad E \in \mathcal{B}(\mathbb{R}^n)$$

is called the **push-forward of $|\mu|$ through** f.

Proof. **Step 1**: For any compact set K and $\varphi \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ with $\operatorname{spt}\varphi \subset K$, $|\varphi| \leq 1$,

$$\langle f_{\#}\mu, \varphi \rangle = \int_{\mathbb{R}^n} (\varphi \circ f) \cdot d\mu \leq \int_K 1 \; d|\mu| \leq |\mu|(K) < \infty,$$

since $|\mu|$ is a Radon measure (see [Mag12], Lemma 4.17). This implies that $f_{\#}\mu$ is a bounded linear functional, i.e., $f_{\#}\mu$ is a $\mathbb{R}^m$-valued Radon measure on $\mathbb{R}^n$.

Step 2: By [Mag12], Proposition 2.14, $f_{\#}|\mu|$ is a Radon measure on $\mathbb{R}^n$ and for every Borel function $u : \mathbb{R}^n \to [0, \infty]$, we have

$$\int_{\mathbb{R}^n} u \; d(f_{\#}|\mu|) = \int_{\mathbb{R}^n} (u \circ f) \; d|\mu|.$$

Similar to Proposition 2.2.4, we just need to show that

$$|f_{\#}\mu|(G) = f_{\#}|\mu|(G) \qquad \text{for any bounded open set } G \subset \mathbb{R}^n.$$

By the polar decomposition, $\mu = \sigma|\mu|$, for some $|\mu|$-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that $|\sigma| = 1$ $|\mu|$-a.e. on $\mathbb{R}^n$. Therefore,

$$\begin{aligned} |f_{\#}\mu|(G) &= \sup \left\{ \int_{\mathbb{R}^n} \varphi \cdot d(f_{\#}\mu) : \varphi \in C_{\mathrm{c}}\left(G; \mathbb{R}^m\right), |\varphi| \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{R}^n} (\varphi \circ f) \cdot d\mu : \varphi \in C_{\mathrm{c}}\left(G; \mathbb{R}^m\right), |\varphi| \leq 1 \right\} \\ &\leq \sup \left\{ \int_{f^{-1}(G)} 1 \; d|\mu| : \varphi \in C_{\mathrm{c}}\left(G; \mathbb{R}^m\right), |\varphi| \leq 1 \right\} \\ &= |\mu|(f^{-1}(G)) = f_{\#}|\mu|(G). \end{aligned}$$

On the other hand, since $C_c(G; \mathbb{R}^m)$ is dense in $L^1(G, f_{\#}|\mu|; \mathbb{R}^m)$ and G is bounded, there exists $\varphi_k \in C_c(G; \mathbb{R}^m)$ such that

$$\varphi_k \to \sigma \circ f^{-1} \quad \text{in } L^1(G, f_{\#}|\mu|; \mathbb{R}^m).$$

By using a truncation argument, we may again assume that $|\varphi_k| \leq 1$ for all k. Moreover,

$$\begin{aligned} \int_{f^{-1}(G)} |\varphi_k \circ f - \sigma| \, d|\mu| &= \int_{\mathbb{R}^n} |\varphi_k \circ f - \sigma| \, \chi_{f^{-1}(G)} \; d|\mu| \\ &= \int_{\mathbb{R}^n} \left(\left| \varphi_k - \sigma \circ f^{-1} \right| \chi_G \right) \circ f \; d|\mu| = \int_G \left| \varphi_k - \sigma \circ f^{-1} \right| d(f_{\#}|\mu|) \to 0 \end{aligned}$$

since f is a homeomorphism. That is,

$$\varphi_k \circ f \to \sigma \quad \text{in } L^1(f^{-1}(G), |\mu|; \mathbb{R}^m).$$

Finally, we have

$$\begin{aligned}|f_\# \mu|(G) \geq \lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi_k \cdot d(f_\# \mu) &= \lim_{k\to\infty} \int_{\mathbb{R}^n} (\varphi_k \circ f) \cdot d\mu = \lim_{k\to\infty} \int_{\mathbb{R}^n} (\varphi_k \circ f) \cdot \sigma \; d|\mu| \\ &= \lim_{k\to\infty} \int_{f^{-1}(G)} (\varphi_k \circ f) \cdot \sigma \; d|\mu| = \int_{f^{-1}(G)} 1 \; d|\mu| = f_\# |\mu|(G).\end{aligned}$$

□

2.3 Anisotropic Gaussian perimeter

Let $A \in M_n(\mathbb{R})$ be a symmetric positive definite matrix. There exists a unique symmetric positive definite matrix $\sqrt{A}$ such that

$$A = (\sqrt{A})^2$$

(see [HJ12], Theorem 7.2.6). We will use the notation $A \succ 0$ ($A \succeq 0$) to mean the matrix A is symmetric positive definite (symmetric positive semi-definite). Notice that we have the following equalities:

$$e^{-\langle Ax,x\rangle/2} = e^{-\langle \sqrt{A}x, \sqrt{A}x\rangle/2} = e^{-|\sqrt{A}x|^2/2}, \qquad \sqrt{\det A} = \det \sqrt{A}.$$

The **matrix norm induced by the Euclidean norm** is defined as

$$\|A\| := \max_{\|x\|_2=1} \|Ax\|_2 = \max_{\|x\|_2=\|y\|_2=1} \langle Ax, y\rangle.$$

Notice that

$$\|A\| = \|\sqrt{A}\sqrt{A}\| = \left\| \sqrt{A}^{\mathsf{T}} \sqrt{A} \right\| = \|\sqrt{A}\|^2 \implies \|A\|^{\frac{1}{2}} = \|\sqrt{A}\|.$$

In addition, $\sqrt{A^{-1}} = (\sqrt{A})^{-1}$ and hence

$$\|A^{-1}\|^{\frac{1}{2}} = \|(\sqrt{A})^{-1}\|.$$

Now we define the **A-anisotropic Gaussian measure (mass)** as

$$\gamma_A(E) = \fint_E e^{-\langle Ax,x\rangle/2}\, dx = \frac{\sqrt{\det A}}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\langle Ax,x\rangle/2}\, dx,$$

for any (Lebesgue) measurable set $E \subset \mathbb{R}^n$. The connection between γ_A and γ is

$$\gamma_A(E) = \gamma(\sqrt{A}(E)),$$

where $\gamma := \gamma_{I_n}$ is the (standard) Gaussian measure on $\mathbb{R}^n$, i.e.,

$$\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{-|x|^2/2}\, dx.$$

Given any $k \in \mathbb{N}$ with $0 \leq k \leq n$, we define the **k-dimensional A-anisotropic Gaussian Hausdorff measure $\mathcal{H}^k_{\gamma_A}$** by

$$\mathcal{H}^k_{\gamma_A}(B) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{k}{2}}} \int_B e^{-\langle Ax,x\rangle/2}\, d\mathcal{H}^k(x), \quad \text{for any Borel set } B. \tag{2.6}$$

Let E be a measurable set in $\mathbb{R}^n$ and F be a Borel set in $\mathbb{R}^n$. The **(relative) A-anisotropic Gaussian perimeter of E in F** is defined by

$$P_{\gamma_A}(E; F) = \mathcal{H}^{n-1}_{\gamma_A}\left(\partial^M E \cap F\right),$$

and we say E is a set of **locally finite A-anisotropic Gaussian perimeter** if $P_{\gamma_A}(E; K) = \mathcal{H}^{n-1}_{\gamma_A}(\partial^M E \cap K) < \infty$ for every compact set $K \subset \mathbb{R}^n$. In particular, E is a set of **finite A-anisotropic Gaussian perimeter** if $P_{\gamma_A}(E) < \infty$. We will omit the notation A and simply say E is a set of finite anisotropic Gaussian perimeter if there is no confusion.

Proposition 2.3.1. *E is a set of locally finite perimeter if and only if E is a set of locally finite anisotropic Gaussian perimeter.*

Proof. ($\Rightarrow$) For any compact set K,

$$P_{\gamma_A}(E; K) \leq \frac{\sqrt{\det A}}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial^M E \cap K} 1\, d\mathcal{H}^{n-1}(x) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{n-1}{2}}} P(E; K) < \infty.$$

(⇐) For any compact set K, let

$$m_K := \min_{x\in K} e^{-|\sqrt{A}x|^2/2} \geq \min_{x\in K} e^{-d_{\max}^2|x|^2/2} > 0,$$

where $d_{\max}$ is the largest eigenvalue of $\sqrt{A}$. Then

$$\infty > P_{\gamma_A}(E;K) = \frac{\sqrt{\det A}}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial^M E\cap K} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1}(x) \geq \frac{\sqrt{\det A}}{(2\pi)^{\frac{n-1}{2}}} m_K P(E;K).$$

□

Remark: It is clear that

E is a set of finite perimeter $\implies$ E is a set of finite anisotropic Gaussian perimeter.

However, the converse is not true. For example, let $n = 2$, $E_\alpha = [-\alpha, \alpha] \times (0, \infty)$, and

$$A = 2\begin{bmatrix} a & b \\ b & c \end{bmatrix} \succ 0$$

with $a, c > 0$ and $b > 0$. Then

$$e^{-ax^2-2bxy-cy^2} = e^{-\langle A(x,y),(x,y)\rangle/2} = e^{-|\sqrt{A}(x,y)|^2/2} \leq e^{-\|(\sqrt{A})^{-1}\|^{-2}(x^2+y^2)/2} \leq e^{-\|(\sqrt{A})^{-1}\|^{-2}y^2/2}$$

and

$$\begin{aligned} P_{\gamma_A}(E_\alpha) &= \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\int_0^\infty e^{-a\alpha^2-2b\alpha y-cy^2}\, dy + \int_0^\infty e^{-a\alpha^2+2b\alpha y-cy^2}\, dy + \int_{-\alpha}^{\alpha} e^{-ax^2}\, dx \right) \\ &\leq \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(2\int_0^\infty e^{-\|(\sqrt{A})^{-1}\|^{-2}y^2/2} dy + 2\int_0^\alpha e^{-ax^2}\, dx \right) < \infty. \end{aligned}$$

That is, E_α is a set of finite anisotropic Gaussian perimeter and clearly $P(E_\alpha) = \infty$, i.e., E_α is not a set of finite perimeter.

Now we establish the lower semicontinuity, locality, complementation, and subadditivity for the anisotropic Gaussian perimeter. Proposition 2.3.2 is a straightforward consequence of Proposition 2.3.1. The locality, complementation, and subadditivity can be deduced by Proposition 2.3.2 with results in [Mag12], Chapter 12 and 16.

Proposition 2.3.2 (Lower semicontinuity)**.**

If E is a set of locally finite anisotropic Gaussian perimeter and $U \subseteq \mathbb{R}^n$ is an open set, then

$$P_{\gamma_A}(E;U) = \sqrt{2\pi} \sup \left\{ \int_E \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle \; d\gamma_A(x) : \varphi \in C_c^1\left(U;\mathbb{R}^n\right), \sup_U |\varphi| \leq 1 \right\}.$$

Moreover, for any sequence of sets of locally finite perimeter E_k with $\chi_{E_k} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n, \gamma_A)$,

$$P_{\gamma_A}(E;U) \leq \liminf_{k\to\infty} P_{\gamma_A}\left(E_k;U\right).$$

Conversely, if E is a measurable set, U is an open set, and for any open set $V \subset\subset U$,

$$\sup \left\{ \int_E \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle \; d\gamma_A(x) : \varphi \in C_c^1\left(V;\mathbb{R}^n\right), \sup_V |\varphi| \leq 1 \right\} < \infty,$$

then E is a set of locally finite anisotropic Gaussian perimeter in U.

Proof. Notice that for any $\varphi \in C_c^1\left(U;\mathbb{R}^n\right)$,

$$\begin{aligned}\int_E (\operatorname{div} \varphi)\; e^{-|\sqrt{A}x|^2/2}\; dx &= \int_E \operatorname{div}(\varphi\; e^{-|\sqrt{A}x|^2/2})\; dx - \int_E \varphi \cdot (\nabla e^{-|\sqrt{A}x|^2/2})\; dx \\ &= \int_{\partial^* E} e^{-|\sqrt{A}x|^2/2}\; \varphi \cdot \nu_E \; d\mathcal{H}^{n-1}(x) + \int_E \langle \varphi, Ax \rangle \; e^{-|\sqrt{A}x|^2/2}\; dx,\end{aligned}$$

since $\sqrt{A}$ is symmetric, E is a set of locally finite perimeter (see Proposition 2.3.1), and

$$\nabla e^{-|\sqrt{A}x|^2/2} = -\frac{1}{2} e^{-|\sqrt{A}x|^2/2} \nabla |\sqrt{A}x|^2 = -e^{-|\sqrt{A}x|^2/2} Ax.$$

Thus, for any $\varphi \in C_c^1\left(U;\mathbb{R}^n\right)$ with $\sup_U |\varphi| \leq 1$,

$$\begin{aligned}\sqrt{2\pi} \int_E \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle \; d\gamma_A(x) &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial^* E} e^{-|\sqrt{A}x|^2/2}\; \varphi \cdot \nu_E \; d\mathcal{H}^{n-1}(x) \\ &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_U e^{-|\sqrt{A}x|^2/2}\; \varphi \cdot d\mu_E(x).\end{aligned}$$

Consider

$$h(x) = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} e^{-|\sqrt{A}x|^2/2} > 0.$$

Taking the sup on both sides and applying total variation on $h\mu_E$ with Proposition 2.2.4, we have

$$\begin{aligned}
&\sqrt{2\pi}\sup\left\{\int_E \operatorname{div}\varphi(x) - \langle\varphi(x), Ax\rangle\ d\gamma_A(x) : \varphi\in C_c^1\left(U;\mathbb{R}^n\right), \sup_U|\varphi|\le 1\right\}\\
&= \sup\left\{\int_U h\varphi\cdot d\mu_E(x) : \varphi\in C_c^1\left(U;\mathbb{R}^n\right), \sup_U|\varphi|\le 1\right\}\\
&= \sup\left\{\int_U \varphi\cdot d(h\mu_E)(x) : \varphi\in C_c^1\left(U;\mathbb{R}^n\right), \sup_U|\varphi|\le 1\right\}\\
&= |h\mu_E|(U) = h|\mu_E|(U) = \int_U h\ d|\mu_E|\\
&= \int_U \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} e^{-|\sqrt{A}x|^2/2}\ d\mathcal{H}^{n-1}\llcorner\partial^* E = P_{\gamma_A}(E;U).
\end{aligned}$$

Now we show the lower semicontinuity. For any $\chi_{E_k}\to\chi_E$ in $L^1_{\text{loc}}\left(\mathbb{R}^n,\gamma_A\right)$ and $\varphi\in C_c^1\left(U;\mathbb{R}^n\right)$ with $\sup_U|\varphi|\le 1$,

$$\begin{aligned}
\sqrt{2\pi}\int_E \operatorname{div}\varphi(x) - \langle\varphi(x), Ax\rangle\ d\gamma_A(x) &= \lim_{k\to\infty}\sqrt{2\pi}\int_{E_k} \operatorname{div}\varphi(x) - \langle\varphi(x), Ax\rangle\ d\gamma_A(x)\\
&\le \liminf_{k\to\infty} P_{\gamma_A}(E_k;U).
\end{aligned}$$

Taking the sup over φ on left hand side, we have

$$P_{\gamma_A}(E;U)\le \liminf_{k\to\infty} P_{\gamma_A}(E_k;U).$$

Finally, we prove the converse. By Proposition 2.3.1, it is enough to show that E is a set of locally finite perimeter in U, i.e., for any $V\subset\subset U$,

$$\sup\left\{\int_E \operatorname{div}\varphi\ dx : \varphi\in C_c^1\left(V;\mathbb{R}^n\right),\ |\varphi|\le 1\right\}<\infty.$$

Given any $V\subset\subset U$, set

$$\alpha_V := \sup\left\{\int_E \operatorname{div}\varphi(x) - \langle\varphi(x), Ax\rangle\ d\gamma_A(x) : \varphi\in C_c^1\left(V;\mathbb{R}^n\right), \sup_V|\varphi|\le 1\right\}<\infty.$$

For any $\varphi\in C_c^1(V;\mathbb{R}^n)$ with $|\varphi|\le 1$, let

$$M := \max_{x\in\overline{V}} e^{|\sqrt{A}x|^2/2} > 0, \qquad \psi := \frac{e^{|\sqrt{A}x|^2/2}}{M}\varphi.$$

Then $\psi \in C_c^1(V;\mathbb{R}^n)$, $|\psi| \le 1$ and

$$\operatorname{div}\psi = \frac{1}{M}\left((\operatorname{div}\varphi)e^{|\sqrt{A}x|^2/2} + \varphi\cdot\nabla e^{|\sqrt{A}x|^2/2}\right).$$

Hence

$$\begin{aligned}
\alpha_V &\ge \int_E \operatorname{div}\psi - \langle\psi, Ax\rangle\ d\gamma_A(x)\\
&= \frac{\sqrt{\det A}}{(2\pi)^{n/2}}\int_E (\operatorname{div}\psi - \langle\psi, Ax\rangle)\ e^{-|\sqrt{A}x|^2/2}\ dx\\
&= \frac{\sqrt{\det A}}{(2\pi)^{n/2}}\frac{1}{M}\int_E \left(\operatorname{div}\varphi + \varphi\cdot\left(\nabla e^{|\sqrt{A}x|^2/2}\right)e^{-|\sqrt{A}x|^2/2} - \langle\varphi(x), Ax\rangle\right)dx.
\end{aligned}$$

That is,

$$\begin{aligned}
\int_E \operatorname{div}\varphi\ dx &\le \frac{(2\pi)^{n/2}}{\sqrt{\det A}}M\alpha_V - \int_E \varphi\cdot\left(\nabla e^{|\sqrt{A}x|^2/2}\right)e^{-|\sqrt{A}x|^2/2}\ dx + \int_E \langle\varphi(x), Ax\rangle\ dx\\
&= \frac{(2\pi)^{n/2}}{\sqrt{\det A}}M\alpha_V - \int_E \langle\varphi(x), Ax\rangle\ dx + \int_E \langle\varphi(x), Ax\rangle\ dx = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}M\alpha_V < \infty.
\end{aligned}$$

□

Proposition 2.3.3 (Properties of anisotropic Gaussian perimeters)**.**

(1) (Locality) *Let E be a set of locally finite anisotropic Gaussian perimeter. If F is equivalent to E in some open set $U \subset \mathbb{R}^n$, i.e., $|(E\Delta F)\cap U| = 0$, then F is also a set of locally finite anisotropic Gaussian perimeter and*

$$P_{\gamma_A}(E;U) = P_{\gamma_A}(F;U). \tag{2.7}$$

(2) (Complementation) *Let E be a set of locally finite anisotropic Gaussian perimeter. Then E^c is also a set of locally finite anisotropic Gaussian perimeter, and for any open set U in $\mathbb{R}^n$,*

$$P_{\gamma_A}(E;U) = P_{\gamma_A}(E^c;U). \tag{2.8}$$

(3) (Subadditivity) *If E, F are sets of locally finite anisotropic Gaussian perimeter. Then $E \cup F$, $E \cap F$ are also sets of locally finite anisotropic Gaussian perimeter, and for any open set U in $\mathbb{R}^n$,*

$$P_{\gamma_A}(E \cup F; U) + P_{\gamma_A}(E \cap F; U) \leq P_{\gamma_A}(E; U) + P_{\gamma_A}(F; U). \tag{2.9}$$

Proof. With the help of Proposition 2.3.1 and [Mag12], Theorem 16.3, our task reduces to demonstrating equations (2.7), (2.8), and (2.9).

(1) By Proposition 2.3.2,

$$\begin{aligned} P_{\gamma_A}(E; U) &= \sqrt{2\pi} \sup \left\{ \int_E \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle d\gamma_A(x) : \varphi \in C_c^1(U; \mathbb{R}^n), \sup_U |\varphi| \leq 1 \right\} \\ &= \sqrt{2\pi} \sup \left\{ \int_F \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle d\gamma_A(x) : \varphi \in C_c^1(U; \mathbb{R}^n), \sup_U |\varphi| \leq 1 \right\} \\ &= P_{\gamma_A}(F; U), \end{aligned}$$

since

$$\begin{aligned} &\left| \int_E \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle d\gamma_A(x) - \int_F \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle d\gamma_A(x) \right| \\ &\leq \int_{E \Delta F} \left| \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle \right| d\gamma_A(x) = \int_{(E \Delta F) \cap U} \left| \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle \right| d\gamma_A(x) = 0. \end{aligned}$$

(2) Let G be an open set in $\mathbb{R}^n$. For any $\varphi \in C_c^\infty(G; \mathbb{R}^n)$ with $|\varphi| \leq 1$,

$$0 = \int_{\mathbb{R}^n} \operatorname{div} \varphi \, dx = \int_E \operatorname{div} \varphi \, dx + \int_{\mathbb{R}^n \setminus E} \operatorname{div} \varphi \, dx.$$

Applying the divergence theorem (2.4), we have

$$\int_{\mathbb{R}^n} \varphi \cdot d\mu_{\mathbb{R}^n \setminus E} = - \int_{\mathbb{R}^n} \varphi \cdot d\mu_E.$$

Since $\varphi \in C_c^\infty(G; \mathbb{R}^n)$,

$$\int_G \varphi \cdot d\mu_{\mathbb{R}^n \setminus E} = - \int_G \varphi \cdot d\mu_E.$$

Taking the sup over $\varphi \in C_c^\infty(G; \mathbb{R}^n)$ with $|\varphi| \leq 1$,

$$|\mu_{\mathbb{R}^n \setminus E}|(G) = |\mu_E|(G).$$

Moreover, $|\mu_{\mathbb{R}^n\setminus E}| = |\mu_E|$ since G is an arbitrary open set and $|\mu_{\mathbb{R}^n\setminus E}|$, $|\mu_E|$ are Radon measures. By De Giorgi's theorem (2.2),

$$\mathcal{H}^{n-1} \llcorner \partial^*(\mathbb{R}^n \setminus E) = |\mu_{\mathbb{R}^n\setminus E}| = |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

In particular, $\mathcal{H}^{n-1}(\partial^*(\mathbb{R}^n \setminus E)\Delta\partial^* E) = 0$, and hence $\mathcal{H}^{n-1}_{\gamma_A}(\partial^*(\mathbb{R}^n \setminus E)\Delta\partial^* E) = 0$. Therefore,

$$P_{\gamma_A}(\mathbb{R}^n \setminus E; U) = \mathcal{H}^{n-1}_{\gamma_A}(\partial^*(\mathbb{R}^n \setminus E) \cap U) = \mathcal{H}^{n-1}_{\gamma_A}(\partial^* E \cap U) = P_{\gamma_A}(E; U).$$

(3) By [Mag12], Theorem 16.3 and Corollary 15.8, we have

$$\begin{aligned}
\partial^*(E \cap F) &\approx \left(F^{(1)} \cap \partial^* E\right) \cup \left(E^{(1)} \cap \partial^* F\right) \cup \{\nu_E = \nu_F\}, \\
\partial^*(E \cup F) &\approx \left(F^{(0)} \cap \partial^* E\right) \cup \left(E^{(0)} \cap \partial^* F\right) \cup \{\nu_E = \nu_F\},
\end{aligned}$$

where

$\{\nu_E = \nu_F\} := \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = \nu_F(x)\} \subset \partial^* E \cap \partial^* F \subset \partial^* E \cap F^{(1/2)}$ (or $E^{(1/2)} \cap \partial^* F$).

Then

$$\begin{aligned}
P_{\gamma_A}(E \cup F; U) &= \mathcal{H}_{\gamma_A}(\partial^*(E \cup F) \cap U) \\
&= \mathcal{H}_{\gamma_A}\left(F^{(1)} \cap \partial^* E \cap U\right) + \mathcal{H}_{\gamma_A}\left(E^{(1)} \cap \partial^* F \cap U\right) + \mathcal{H}_{\gamma_A}\left(\{\nu_E = \nu_F\} \cap U\right),
\end{aligned}$$

and

$$\begin{aligned}
P_{\gamma_A}(E \cap F; U) &= H_{\gamma_A}(\partial^*(E \cap F) \cap U) \\
&= \mathcal{H}_{\gamma_A}\left(F^{(0)} \cap \partial^* E \cap U\right) + \mathcal{H}_{\gamma_A}\left(E^{(0)} \cap \partial^* F \cap U\right) + \mathcal{H}_{\gamma_A}\left(\{\nu_E = \nu_F\} \cap U\right).
\end{aligned}$$

Summing them together, we have

$$\begin{aligned}
&P_{\gamma_A}(E \cup F; U) + P_{\gamma_A}(E \cap F; U) \\
&= \mathcal{H}_{\gamma_A}\left((F^{(1)} \cup F^{(0)}) \cap \partial^* E \cap U\right) + \mathcal{H}_{\gamma_A}\left((E^{(1)} \cup E^{(0)}) \cap \partial^* F \cap U\right) \\
&\quad + 2\mathcal{H}_{\gamma_A}\left(\{\nu_E = \nu_F\} \cap U\right) \\
&\leq \mathcal{H}_{\gamma_A}\left((F^{(1)} \cup F^{(0)} \cup F^{(1/2)}) \cap \partial^* E \cap U\right) + \mathcal{H}_{\gamma_A}\left((E^{(1)} \cup E^{(0)} \cup E^{(1/2)}) \cap \partial^* F \cap U\right) \\
&\leq \mathcal{H}_{\gamma_A}\left(\partial^* E \cap U\right) + \mathcal{H}_{\gamma_A}\left(\partial^* F \cap U\right) = P_{\gamma_A}(E; U) + P_{\gamma_A}(F; U).
\end{aligned}$$

□

Although the anisotropic Gaussian measure satisfies $\gamma_A(E) = \gamma_{I_n}(\sqrt{A}E)$, this kind of relation doesn't hold in the anisotropic Gaussian perimeter, i.e.,

$$P_{\gamma_A}(E) \neq P_{\gamma_{I_n}}(\sqrt{A}E).$$

In fact, we have the following formula:

Proposition 2.3.4. *If E is a set of locally finite perimeter, then*

$$\int_{F\cap\partial^*(\sqrt{A}E)} \nu_{\sqrt{A}E}(y)\ d\mathcal{H}^{n-1}_\gamma(y) = \int_{\left((\sqrt{A})^{-1}F\right)\cap\partial^*E} \left[(\sqrt{A})^{-1}\nu_E(x)\right]\ d\mathcal{H}^{n-1}_{\gamma_A}(x),$$

and hence

$$P_\gamma(\sqrt{A}E; F) = \int_{\left((\sqrt{A})^{-1}F\right)\cap\partial^*E} \left|(\sqrt{A})^{-1}\nu_E(x)\right|\ d\mathcal{H}^{n-1}_{\gamma_A}(x),$$

for any Borel set $F \subset \mathbb{R}^n$. Moreover,

(1) if $F = \mathbb{R}^n$, then

$$\|\sqrt{A}\|^{-1} P_{\gamma_A}(E) \leq P_\gamma(\sqrt{A}E) \leq \|(\sqrt{A})^{-1}\| P_{\gamma_A}(E).$$

(2) if O is an orthogonal matrix, then

$$P_{\gamma_A}(E; OF) = P_{\gamma_{O^\intercal AO}}(O^{-1}E; F).$$

In particular, E is a set of finite A-anisotropic Gaussian perimeter if and only if $O^{-1}E$ is a set of finite $O^\intercal AO$-anisotropic Gaussian perimeter.

Proof. Since E is a set of locally finite perimeter and $x \mapsto \sqrt{A}x$ is a diffeomorphism, by Proposition 2.2.1, for any $\varphi \in C_c(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\partial^*(\sqrt{A}E)} \varphi(y)\nu_{\sqrt{A}E}(y)d\mathcal{H}^{n-1}(y) &= |\det(\sqrt{A})| \int_{\partial^*E} \varphi(\sqrt{A}x)\left[[(\sqrt{A})^{-1}]^\intercal \nu_E(x)\right] d\mathcal{H}^{n-1}(x)\\ &= |\det(\sqrt{A})| \int_{\partial^*E} \varphi(\sqrt{A}x)\left[(\sqrt{A})^{-1}\nu_E(x)\right] d\mathcal{H}^{n-1}(x), \quad (2.10)\end{aligned}$$

where $\sqrt{A}$ is symmetric. Let F be a Borel set. We can set

$$\varphi(y) = \frac{1}{(2\pi)^{(n-1)/2}} e^{-|y|^2/2} \chi_F(y)$$

in (2.10) since we can first approximate open sets then Borel sets. Hence,

$$\int_{F\cap\partial^*(\sqrt{A}E)} \nu_{\sqrt{A}E}(y) d\mathcal{H}^{n-1}_\gamma(y) = \int_{\left((\sqrt{A})^{-1}F\right)\cap\partial^*E} \left[(\sqrt{A})^{-1}\nu_E(x)\right] d\mathcal{H}^{n-1}_{\gamma_A}(x). \tag{2.11}$$

Since F is an arbitrary Borel set, the following two measures are the same:

$$\nu_{\sqrt{A}E}\mathcal{H}^{n-1}_\gamma \llcorner \partial^*(\sqrt{A}E) = (\sqrt{A})_\# \left(\left[(\sqrt{A})^{-1}\nu_E\right]\mathcal{H}^{n-1}_{\gamma_A} \llcorner \partial^*E\right).$$

Taking total variation on both sides, by Proposition 2.2.3 and 2.2.5, we have

$$\mathcal{H}^{n-1}_\gamma \llcorner \partial^*(\sqrt{A}E) = (\sqrt{A})_\# \left(\left|(\sqrt{A})^{-1}\nu_E\right|\mathcal{H}^{n-1}_{\gamma_A} \llcorner \partial^*E\right).$$

That is,

$$P_\gamma(\sqrt{A}E; F) = \int_{\left((\sqrt{A})^{-1}F\right)\cap\partial^*E} \left|(\sqrt{A})^{-1}\nu_E(x)\right| \, d\mathcal{H}^{n-1}_{\gamma_A}(x). \tag{2.12}$$

(1) Let $F = \mathbb{R}^n$ in (2.12). Then

$$P_\gamma(\sqrt{A}E) = \int_{\partial^*E} \left|(\sqrt{A})^{-1}\nu_E(x)\right| \, d\mathcal{H}^{n-1}_{\gamma_A}(x) \le \|(\sqrt{A})^{-1}\| P_{\gamma_A}(E).$$

On the other hand,

$$1 = \left|\nu_E(x)\right| = \left|\sqrt{A}(\sqrt{A})^{-1}\nu_E(x)\right| \le \|\sqrt{A}\|\left|(\sqrt{A})^{-1}\nu_E(x)\right| \implies \left|(\sqrt{A})^{-1}\nu_E(x)\right| \ge \|\sqrt{A}\|^{-1}.$$

Thus,

$$P_\gamma(\sqrt{A}E) = \int_{\partial^*E} \left|(\sqrt{A})^{-1}\nu_E(x)\right| \, d\mathcal{H}^{n-1}_{\gamma_A}(x) \ge \|\sqrt{A}\|^{-1} P_{\gamma_A}(E).$$

(2) Let O be an orthogonal matrix. Notice that $O^\mathsf{T}AO$ is positive definite since A is positive definite. By using the same argument in (2.11) with $f : x \mapsto O^{-1}x$ and

$$\varphi(y) = \frac{\det O^\mathsf{T}\sqrt{A}O}{(2\pi)^{(n-1)/2}} e^{-|O^\mathsf{T}\sqrt{A}Oy|^2/2} \chi_F(y)$$

in Proposition 2.2.1, we have

$$\begin{aligned}
&\frac{\det O^{\mathsf{T}}\sqrt{A}O}{(2\pi)^{(n-1)/2}}\int_{F\cap\partial^*(O^{-1}E)} e^{-|O^{\mathsf{T}}\sqrt{A}Oy|^2/2}\nu_{O^{-1}E}(y)\ d\mathcal{H}^{n-1}(y)\\
&=\frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_{OF\cap\partial^*E} e^{-|\sqrt{A}x|^2/2}\left[O^{\mathsf{T}}\nu_E(x)\right]\ d\mathcal{H}^{n-1}(x),\\
&=\int_{OF\cap\partial^*E} O^{\mathsf{T}}\nu_E(x)\ d\mathcal{H}^{n-1}_{\gamma_A}(x).
\end{aligned} \tag{2.13}$$

Taking total variation again, we conclude that

$$\begin{aligned}
P_{\gamma_{O^{\mathsf{T}}AO}}(O^{-1}E;F) &= \frac{\det O^{\mathsf{T}}\sqrt{A}O}{(2\pi)^{(n-1)/2}}\int_{F\cap\partial^*(O^{-1}E)} e^{-|O^{\mathsf{T}}\sqrt{A}Oy|^2/2}\left|\nu_{O^{-1}E}(y)\right| d\mathcal{H}^{n-1}(y)\\
&=\int_{OF\cap\partial^*E}\left|O^{\mathsf{T}}\nu_E(x)\right| d\mathcal{H}^{n-1}_{\gamma_A}(x) = P_{\gamma_A}(E;OF).
\end{aligned}$$

□

2.4 Approximation for the finite anisotropic Gaussian perimeter

Proposition 2.4.1. *For any measurable set E with $P_{\gamma_A}(E)<\infty$, there exists a sequence $\{E_k\}$ of bounded open sets with smooth boundary such that*

$$\chi_{E_k}\to\chi_E \text{ in } L^1(\mathbb{R}^n,\gamma_A) \quad \text{and} \quad P_{\gamma_A}(E_k)\to P_{\gamma_A}(E).$$

Proof. For any measurable set E with $P_{\gamma_A}(E)<\infty$, by Proposition 2.3.1, E is a set of locally finite perimeter. Hence, $E\cap B_R$ is a set of finite perimeter, where $B_R := B(0,R)$ is an open ball (see [Mag12], Lemma 15.12).

Step 1: We first claim that, as $R\to\infty$,

$$\chi_{E\cap B_R}\to\chi_E \text{ in } L^1(\mathbb{R}^n,\gamma_A) \quad \text{and} \quad P_{\gamma_A}(E\cap B_R)\to P_{\gamma_A}(E).$$

To begin, by the dominated convergence theorem and $\gamma_A(\mathbb{R}^n)=1<\infty$,

$$\int_{\mathbb{R}^n}\left|\chi_E-\chi_{E\cap B_R}\right| d\gamma_A\to 0 \quad \text{as } R\to\infty.$$

For the second part, since $P_{\gamma_A}(E)<\infty$,

$$\lim_{R\to\infty}P_{\gamma_A}(E;\mathbb{R}^n\setminus B_R)=0.$$

Moreover, by [Mag12], Lemma 15.12, for a.e. $R > 0$,

$$|\mu_{E\cap B_R}| = \mathcal{H}^{n-1} \llcorner (E \cap \partial B_R) + |\mu_E| \llcorner B_R.$$

That is,

$$\begin{aligned} P_{\gamma_A}(E \cap B_R) = \mathcal{H}^{n-1}_{\gamma_A}(\partial^*(E \cap B_R)) &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial^*(E\cap B_R)} e^{-|\sqrt{A}x|^2/2} \, d\mathcal{H}^{n-1} \\ &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^n} e^{-|\sqrt{A}x|^2/2} \, d|\mu_{E\cap B_R}| \\ &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \left(\int_{B_R} e^{-|\sqrt{A}x|^2/2} \, d|\mu_E| + \int_{E\cap\partial B_R} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1}(x) \right) \\ &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{(\partial^* E)\cap B_R} e^{-|\sqrt{A}x|^2/2} \, d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}_{\gamma_A}(E \cap \partial B_R) \\ &= \mathcal{H}^{n-1}_{\gamma_A}((\partial^* E) \cap B_R) + \mathcal{H}^{n-1}_{\gamma_A}(E \cap \partial B_R) = P_{\gamma_A}(E; B_R) + \mathcal{H}^{n-1}_{\gamma_A}(E \cap \partial B_R). \end{aligned} \tag{2.14}$$

Notice that since

$$|x| = |(\sqrt{A})^{-1}\sqrt{A}x| \le \|(\sqrt{A})^{-1}\| |\sqrt{A}x| \implies |\sqrt{A}x| \ge \frac{1}{\|(\sqrt{A})^{-1}\|}|x|,$$

we have

$$\begin{aligned} \mathcal{H}^{n-1}_{\gamma_A}(E \cap \partial B_R) \le \mathcal{H}^{n-1}_{\gamma_A}(\partial B_R) &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial B_R} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1}(x) \\ &\le \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial B_R} e^{-\frac{1}{\|(\sqrt{A})^{-1}\|^2}|x|^2/2} d\mathcal{H}^{n-1}(x) \\ &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} e^{-\frac{1}{\|(\sqrt{A})^{-1}\|^2}R^2/2} \mathcal{H}^{n-1}(\partial B_R) \\ &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} e^{-\frac{1}{\|(\sqrt{A})^{-1}\|^2}R^2/2} \alpha_n R^{n-1} \to 0 \quad \text{as } R \to \infty, \end{aligned} \tag{2.15}$$

where α_n is the surface area of the unit ball in $\mathbb{R}^n$. Combining (2.14) and (2.15) together,

$$\begin{aligned} P_{\gamma_A}(E) &= P_{\gamma_A}(E; \mathbb{R}^n \setminus B_R) + P_{\gamma_A}(E; B_R) \\ &= P_{\gamma_A}(E; \mathbb{R}^n \setminus B_R) + (P_{\gamma_A}(E \cap B_R) - \mathcal{H}^{n-1}_{\gamma_A}(E \cap \partial B_R)), \end{aligned}$$

and hence, as $R \to \infty$,

$$|P_{\gamma_A}(E) - P_{\gamma_A}(E \cap B_R)| \le P_{\gamma_A}(E; \mathbb{R}^n \setminus B_R) + \mathcal{H}^{n-1}_{\gamma_A}(E \cap \partial B_R) \to 0.$$

Step 2: Consider

$$E^R := E \cap B_R.$$

Fix $R > 0$. Applying [Mag12], Theorem 13.8, on E^R, there exists a sequence $\{E^R_k\}_{k=1}^\infty$ of bounded open sets with smooth boundary such that $E^R_k \subset B_{R+1}$ for all k,

$$\chi_{E^R_k} \to \chi_{E^R} \quad \text{as } k \to \infty \text{ in } L^1(\mathbb{R}^n) \text{ (and hence in } L^1(\mathbb{R}^n, \gamma_A)), \tag{2.16}$$

and

$$|\mu_{E^R_k}| \overset{*}{\rightharpoonup} |\mu_{E^R}|\,.$$

Let $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be a smooth cutoff function with $\eta_\varepsilon = 1$ on B_{R+1} and $\eta_\varepsilon \to \chi_{B_L}$ with $L > R + 1$. Applying the weak-star convergence, as $k \to \infty$,

$$\begin{aligned} P_{\gamma_A}(E^R_k) &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial E^R_k} e^{-|\sqrt{A}x|^2/2}\, d\mathcal{H}^{n-1} = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial E^R_k} e^{-|\sqrt{A}x|^2/2} \eta_\varepsilon(x)\, d\mathcal{H}^{n-1} \\ &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^n} e^{-|\sqrt{A}x|^2/2} \eta_\varepsilon(x) d|\mu_{E^R_k}| \underset{k\to\infty}{\to} \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial^* E^R} e^{-|\sqrt{A}x|^2/2} \eta_\varepsilon(x) d\mathcal{H}^{n-1}. \end{aligned}$$

Taking $\varepsilon \to 0^+$ and $L \to \infty$ on both sides, we have

$$\lim_{k\to\infty} P_{\gamma_A}(E^R_k) = \lim_{L\to\infty} \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial^* E^R} e^{-|\sqrt{A}x|^2/2} \chi_{B_L}(x) d\mathcal{H}^{n-1} = P_{\gamma_A}(E^R). \tag{2.17}$$

By Step 1, we can let $\{R_k\}$ be a sequence with $R_k \nearrow \infty$ such that

$$\chi_{E^{R_k}} \to \chi_E \text{ in } L^1(\mathbb{R}^n, \gamma_A) \quad \text{and} \quad \lim_{k\to\infty} P_{\gamma_A}(E^{R_k}) = P_{\gamma_A}(E).$$

By a diagonal argument with (2.16), (2.17), there exists a sequence $\{E^{R_k}_{N_k}\}_{k=1}^\infty$ of bounded open sets with smooth boundary such that

$$\chi_{E^{R_k}_{N_k}} \to \chi_E \text{ in } L^1(\mathbb{R}^n, \gamma_A) \quad \text{and} \quad P_{\gamma_A}(E^{R_k}_{N_k}) \to P_{\gamma_A}(E).$$

$\square$

2.5 Anisotropic Gaussian isoperimetric inequalities

Define the function $\phi : \mathbb{R} \to (0, 1)$ as

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

and notice that ϕ is strictly increasing from 0 to 1. The inverse function $\phi^{-1} : [0, 1] \to [-\infty, +\infty]$ is also strictly increasing with $\phi^{-1}(0) = -\infty$ and $\phi^{-1}(1) = +\infty$. Moreover,

$$\lim_{t\to\infty} \frac{\phi^{-1}(\gamma(tB(0, 1)))}{t} = 1 \tag{2.18}$$

(See [Liv21], Section 3.3 and [Nay17], Lemma 9). We define the (non-renormalized) A-**anisotropic Gaussian barycenter** of the set E as

$$b_{\gamma_A}(E) := \int_E x \, d\gamma_A(x).$$

Let $H(\omega, t)$ be the half-space of the form $H(\omega, t) = \{x : \langle x, \omega\rangle < t\}$, where $\omega \in \mathbb{S}^{n-1}$, $t \in \mathbb{R}$, and $\langle\cdot, \cdot\rangle$ is the Euclidean inner product. Since $\gamma = \gamma_{I_n}$ is rotation invariant, we can compute the following quantities directly:

$$\gamma(H(\omega, t)) = \phi(t), \qquad P_\gamma(H(\omega, t)) = e^{-t^2/2}, \qquad b_\gamma(H(\omega, t)) = \frac{-1}{\sqrt{2\pi}} e^{-t^2/2}\omega. \tag{2.19}$$

Moreover, we have the following for half-spaces under γ_A:

Proposition 2.5.1. *Let $H(\omega, t)$ be the half-space with $\omega \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$.*

(1) If M is an invertible $n \times n$ matrix, then

$$M\left(H(\omega, t)\right) = H\left(\frac{(M^\intercal)^{-1}\omega}{|(M^\intercal)^{-1}\omega|}, \frac{t}{|(M^\intercal)^{-1}\omega|}\right).$$

(2) The anisotropic Gaussian mass of the half-space is

$$\gamma_A(H(\omega, t)) = \phi\left(\frac{t}{|(\sqrt{A})^{-1}\omega|}\right).$$

(3) The anisotropic Gaussian perimeter of the half-space is

$$P_{\gamma_A}(H(\omega,t)) = e^{-\frac{1}{2}\frac{t^2}{|(\sqrt{A})^{-1}\omega|^2}} \frac{1}{|(\sqrt{A})^{-1}\omega|}.$$

Moreover,

$$P_{\gamma_A}(H_1) = P_{\gamma_A}(H_2) \text{ for any half-spaces } H_1,\ H_2 \text{ with } \gamma_A(H_1) = \gamma_A(H_2)$$
$$\iff A = aI_n \text{ for some constant } a > 0.$$

(4) The anisotropic Gaussian barycenter of the half-space is

$$b_{\gamma_A}(H(\omega,t)) = \frac{-1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{t^2}{|(\sqrt{A})^{-1}\omega|^2}} \left(\frac{A^{-1}\omega}{|(\sqrt{A})^{-1}\omega|}\right).$$

Moreover,

$$\left|b_{\gamma_A}(H_1)\right| = \left|b_{\gamma_A}(H_2)\right| \text{ for any half-spaces } H_1,\ H_2 \text{ with } \gamma_A(H_1) = \gamma_A(H_2)$$
$$\iff A = aI_n \text{ for some constant } a > 0.$$

Proof. (1) Given any point $y = Mx \in M(H(\omega,t))$, we have $\langle x, \omega\rangle < t$. Then

$$\left\langle y, \frac{(M^\intercal)^{-1}\omega}{|(M^\intercal)^{-1}\omega|}\right\rangle = \left\langle Mx, \frac{(M^\intercal)^{-1}\omega}{|(M^\intercal)^{-1}\omega|}\right\rangle = \left\langle x, \frac{\omega}{|(M^\intercal)^{-1}\omega|}\right\rangle < \frac{t}{|(M^\intercal)^{-1}\omega|}.$$

Conversely, for any $y \in H\left(\frac{(M^\intercal)^{-1}\omega}{|(M^\intercal)^{-1}\omega|}, \frac{t}{|(M^T)^{-1}\omega|}\right)$, let $x = M^{-1}y$. Notice that

$$\left\langle x, \frac{\omega}{|(M^\intercal)^{-1}\omega|}\right\rangle = \left\langle M^{-1}y, \frac{\omega}{|(M^\intercal)^{-1}\omega|}\right\rangle = \left\langle y, \frac{(M^\intercal)^{-1}\omega}{|(M^\intercal)^{-1}\omega|}\right\rangle < \frac{t}{|(M^\intercal)^{-1}\omega|}.$$

Thus, $\langle x, \omega\rangle < t$, i.e., $x \in H(\omega,t)$.

(2) Observe that $\gamma_A(E) = \gamma(\sqrt{A}(E))$ for any Borel set E, and by (1),

$$\sqrt{A}(H(\omega,t)) = H\left(\frac{((\sqrt{A})^\intercal)^{-1}\omega}{|((\sqrt{A})^\intercal)^{-1}\omega|}, \frac{t}{|((\sqrt{A})^\intercal)^{-1}\omega|}\right) = H\left(\frac{(\sqrt{A})^{-1}\omega}{|(\sqrt{A})^{-1}\omega|}, \frac{t}{|(\sqrt{A})^{-1}\omega|}\right),$$

since $\sqrt{A}$ is symmetric. Applying equation (2.19), we have

$$\gamma_A(H(\omega,t)) = \phi\left(\frac{t}{|(\sqrt{A})^{-1}\omega|}\right).$$

(3) Notice that

$$
\begin{aligned}
P_{\gamma_A}(H(\omega,t)) &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{\partial^* H(\omega,t)} \left(e^{-|\sqrt{A}x|^2/2}\omega\right)\cdot \omega \, d\mathcal{H}^{n-1}(x) \\
&= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{H(\omega,t)} \operatorname{div}\left(e^{-|\sqrt{A}x|^2/2}\omega\right) dx \\
&= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{H(\omega,t)} -\langle Ax,\omega\rangle\, e^{-|\sqrt{A}x|^2/2} dx \\
&= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{H\left(\frac{(\sqrt{A})^{-1}\omega}{|(\sqrt{A})^{-1}\omega|}, \frac{t}{|(\sqrt{A})^{-1}\omega|}\right)} -\left\langle y, \sqrt{A}\omega\right\rangle e^{-|y|^2/2}\, \frac{1}{|\det\sqrt{A}|} dy \\
&= -\sqrt{2\pi}\left\langle \int_{H\left(\frac{(\sqrt{A})^{-1}\omega}{|(\sqrt{A})^{-1}\omega|}, \frac{t}{|(\sqrt{A})^{-1}\omega|}\right)} y d\gamma(y), \sqrt{A}\omega\right\rangle \\
&= -\sqrt{2\pi}\left\langle b_\gamma\left(H\left(\frac{(\sqrt{A})^{-1}\omega}{|(\sqrt{A})^{-1}\omega|}, \frac{t}{|(\sqrt{A})^{-1}\omega|}\right)\right), \sqrt{A}\omega\right\rangle \\
&= -\sqrt{2\pi}\left\langle \frac{-1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{t^2}{|(\sqrt{A})^{-1}\omega|^2}} \left(\frac{(\sqrt{A})^{-1}\omega}{|(\sqrt{A})^{-1}\omega|}\right), \sqrt{A}\omega\right\rangle = e^{-\frac{1}{2}\frac{t^2}{|(\sqrt{A})^{-1}\omega|^2}} \frac{1}{|(\sqrt{A})^{-1}\omega|},
\end{aligned}
$$

where we used the change of variables $y = \sqrt{A}x$ and the fact that the outer unit normal of $H(\omega,t)$ is $\nu_{H(\omega,t)} = \omega$. Next, we prove the second part. Let $H_1 = H(\omega_1,t_1)$ and $H_2 = H(\omega_2,t_2)$. Suppose that $A = aI_n$. Then

$$\gamma_A(H_1) = \gamma_A(H_2) \implies t_1 = t_2.$$

Therefore,

$$P_{\gamma_A}(H(\omega_1,t_1)) = e^{-at_1^2}\sqrt{a} = e^{-at_2^2}\sqrt{a} = P_{\gamma_A}(H(\omega_2,t_2)).$$

Conversely, for any $H_1 = H(\omega_1,t_1)$ and $H_2 = H(\omega_2,t_2)$ with $\gamma_A(H_1) = \gamma_A(H_2)$, i.e., $\frac{t_1}{|(\sqrt{A})^{-1}\omega_1|} = \frac{t_2}{|(\sqrt{A})^{-1}\omega_2|}$, we have

$$P_{\gamma_A}(H(\omega_1,t_1)) = P_{\gamma_A}(H(\omega_2,t_2)) \implies \frac{1}{|(\sqrt{A})^{-1}\omega_1|} = \frac{1}{|(\sqrt{A})^{-1}\omega_2|}.$$

That is, $|(\sqrt{A})^{-1}\omega|$ is a constant for all $\omega \in \mathbb{S}^{n-1}$. Since $\sqrt{A}$ is orthogonally diagonalizable, say $\sqrt{A} = ODO^{-1}$, where the orthogonal matrix $O = (v_1\ v_2 \cdots v_n)$ and $D =$

$\mathrm{diag}(d_1, d_2, \cdots d_n)$, i.e., for $j = 1, \cdots, n$, we have $|v_j| = 1$ and

$$\sqrt{A}v_j = d_j v_j \quad (\text{i.e. } (\sqrt{A})^{-1}v_j = d_j^{-1}v_j).$$

Therefore, $d_1 = d_2 \cdots = d_n > 0$ and hence $A = OD^2O^{-1} = O(d_1^2 I_n)O^{-1} = d_1^2 I_n$.

(4) Notice that

$$b_{\gamma_A}(E) = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} \int_E x e^{-|\sqrt{A}x|^2/2}\, dx = \frac{1}{(2\pi)^{n/2}} \int_{\sqrt{A}E} (\sqrt{A})^{-1} y e^{-|y|^2/2}\, dy = (\sqrt{A})^{-1} b_\gamma(\sqrt{A}E),$$

where we used the change of variables $y = \sqrt{A}x$. In particular, using the calculation above and (2.19), we have

$$\begin{aligned} b_{\gamma_A}(H(\omega, t)) = (\sqrt{A})^{-1} b_\gamma(\sqrt{A}H(\omega, t)) &= (\sqrt{A})^{-1} \frac{-1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{t^2}{|(\sqrt{A})^{-1}\omega|^2}} \left(\frac{(\sqrt{A})^{-1}\omega}{|(\sqrt{A})^{-1}\omega|} \right) \\ &= \frac{-1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{t^2}{|(\sqrt{A})^{-1}\omega|^2}} \left(\frac{A^{-1}\omega}{|(\sqrt{A})^{-1}\omega|} \right). \end{aligned}$$

The proof of the second part is the same as (3); hence, we omit the proof. □

We are ready to prove the anisotropic Gaussian isoperimetric inequality (ε-enlargement version). As mentioned in the introduction, Bakry and Ledoux proved a general result about isoperimetric inequality for log-concave measures (see [Led99], Theorem 1.1 and [LB96]). However, for completeness, we provide a proof for the case of anisotropic Gaussian measure using an argument inspired by Latała's paper [Lat03], Section 3. The proof is based on Ehrhard's inequality and the regularity of Radon measures.

Theorem 2.5.2 (Anisotropic Gaussian Isoperimetric Inequality (ε-enlargement version))**.**

(1) For any measurable set E in $\mathbb{R}^n$,

$$\phi^{-1}(\gamma_A(E_\varepsilon)) \geq \phi^{-1}(\gamma_A(E)) + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|},$$

where $\|\cdot\|$ is the matrix norm induced by the Euclidean norm. The set

$$E_\varepsilon := E + \varepsilon\overline{B}(0,1) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, E) \leq \varepsilon\}$$

*is called the ε-***(Minkowski) enlargement** *of E. Here $\overline{B}(0,1)$ is the closed unit ball in $\mathbb{R}^n$.*

(2) Let E be a measurable set in $\mathbb{R}^n$ and let $H(\omega,t)$ be a half-space such that

$$\gamma_A(E) \geq \gamma_A(H(\omega,t)).$$

Then, for every $\varepsilon > 0$,

$$\gamma_A\left(E_\varepsilon\right) \geq \gamma_A\left(H\left(\omega, t+\varepsilon\frac{|(\sqrt{A})^{-1}\omega|}{\|(\sqrt{A})^{-1}\|}\right)\right).$$

In particular, if we assume that $\omega \in V_{d_{\min}}(\sqrt{A})$,

$$\gamma_A(E_\varepsilon) \geq \gamma_A(H(\omega, t+\varepsilon)),$$

where $d_{\min}$ is the smallest eigenvalue of $\sqrt{A}$ and $V_{d_{\min}}(\sqrt{A})$ is the eigenspace of $\sqrt{A}$ associated with $d_{\min}$.

Proof. (1) We first assume that E is a Borel set. Applying Ehrhard's inequality (see Theorem 2.2.2) with $\gamma_A(E) = \gamma(\sqrt{A}(E))$, we have the following: if B, C are Borel sets in $\mathbb{R}^n$, then

$$\phi^{-1}(\gamma_A(\lambda C + (1-\lambda)B)) \geq \lambda\phi^{-1}(\gamma_A(C)) + (1-\lambda)\phi^{-1}(\gamma_A(B)), \qquad \text{for } \lambda \in (0,1),$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

Let $C = E$ and $B = \overline{B}(0,1)$. Then we have

$$\begin{aligned}
&\phi^{-1}(\gamma_A(E_\varepsilon)) = \phi^{-1}(\gamma_A(E+\varepsilon B)) = \phi^{-1}\Big[\gamma_A\Big(\lambda\Big(\frac{E}{\lambda}\Big) + (1-\lambda)\Big(\frac{\varepsilon}{1-\lambda}B\Big)\Big)\Big] \\
&\geq \lambda\phi^{-1}\Big[\gamma_A\Big(\frac{E}{\lambda}\Big)\Big] + (1-\lambda)\phi^{-1}\Big[\gamma_A\Big(\frac{\varepsilon}{1-\lambda}B\Big)\Big] \\
&= \lambda\phi^{-1}\Big[\gamma_A\Big(\frac{E}{\lambda}\Big)\Big] + (1-\lambda)\phi^{-1}\Big[\gamma\Big(\frac{\varepsilon}{1-\lambda}\sqrt{A}(B)\Big)\Big] \\
&\geq \lambda\phi^{-1}\Big[\gamma_A\Big(\frac{E}{\lambda}\Big)\Big] + (1-\lambda)\phi^{-1}\Big[\gamma\Big(\frac{\varepsilon}{1-\lambda}\frac{1}{\|(\sqrt{A})^{-1}\|}B\Big)\Big],
\end{aligned}$$

using that ϕ^{-1} is increasing and $\sqrt{A}(B) \supset \frac{1}{\|(\sqrt{A})^{-1}\|}B$. Taking $\lambda \to 1^-$, by (2.18), we have

$$(1-\lambda)\phi^{-1}\Big[\gamma\Big(\frac{\varepsilon}{1-\lambda}\frac{1}{\|(\sqrt{A})^{-1}\|}B\Big)\Big] = \left(\frac{\phi^{-1}\Big[\gamma\Big(\frac{\varepsilon}{1-\lambda}\frac{1}{\|(\sqrt{A})^{-1}\|}B\Big)\Big]}{\frac{\varepsilon}{1-\lambda}\frac{1}{\|(\sqrt{A})^{-1}\|}}\right)\frac{\varepsilon}{\|(\sqrt{A})^{-1}\|} \to \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|}.$$

That is, for any Borel set E,

$$\phi^{-1}(\gamma_A(E_\varepsilon)) \geq \phi^{-1}(\gamma_A(E)) + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|}. \tag{2.20}$$

Now we assume that E is a measurable set. For any $\delta > 0$, by the regularity of the Radon measure, there exists a compact set $K \subset E$ such that

$$\gamma_A(K) > \gamma_A(E) - \delta.$$

Applying (2.20) on the compact set K, we deduce that

$$\gamma_A(E_\varepsilon) \geq \gamma_A(K_\varepsilon) \geq \phi^{-1}(\gamma_A(K)) + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|} > \phi^{-1}(\gamma_A(E) - \delta) + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|},$$

since ϕ^{-1} is continuous and strictly increasing. Taking $\delta \to 0^+$, we have finished the proof.

(2) Using (1) with our assumption, we have

$$\begin{aligned}\phi^{-1}(\gamma_A(E_\varepsilon)) \geq \phi^{-1}(\gamma_A(E)) + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|} &\geq \phi^{-1}(\gamma_A(H(\omega,t))) + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|}\\ &= \frac{t}{|(\sqrt{A})^{-1}\omega|} + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|},\end{aligned}$$

since ϕ^{-1} is increasing. Applying ϕ on both sides and using Proposition 2.5.1,

$$\gamma_A(E_\varepsilon) \geq \phi\left(\frac{t}{|(\sqrt{A})^{-1}\omega|} + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|}\right) = \phi\left(\frac{t + \varepsilon\frac{|(\sqrt{A})^{-1}\omega|}{\|(\sqrt{A})^{-1}\|}}{|(\sqrt{A})^{-1}\omega|}\right) = \gamma_A\left(H\left(\omega, t + \varepsilon\frac{|(\sqrt{A})^{-1}\omega|}{\|(\sqrt{A})^{-1}\|}\right)\right).$$

In particular, if $\omega \in V_{d_{\min}}(\sqrt{A})$, then

$$\sqrt{A}\omega = d_{\min}\omega \implies (\sqrt{A})^{-1}\omega = \frac{1}{d_{\min}}\omega = \|(\sqrt{A})^{-1}\|\omega \implies \frac{|(\sqrt{A})^{-1}\omega|}{\|(\sqrt{A})^{-1}\|} = 1.$$

That is, $\gamma_A(E_\varepsilon) \geq \gamma_A(H(\omega, t+\varepsilon))$. □

Suppose the boundary of E is "nice" enough. Intuitively, we have the following

$$\frac{\gamma_A(E_\varepsilon)-\gamma_A(E)}{\varepsilon} \to \frac{1}{\sqrt{2\pi}} P_{\gamma_A}(E),$$

where the extra factor $\frac{1}{\sqrt{2\pi}}$ appears in front of P_{γ_A} since we define P_{γ_A} with coefficient $\frac{1}{(2\pi)^{(n-1)/2}}$. In order to use this idea, we need to introduce the signed distance function. Let E be a subset of $\mathbb{R}^n$. Define $d_E : \mathbb{R}^n \to \mathbb{R}$ to be the **signed distance function** from E:

$$d_E(x) := \operatorname{dist}(x,E) - \operatorname{dist}(x,E^c) = \begin{cases} -\operatorname{dist}(x,\partial E), & x \in E \\ \operatorname{dist}(x,\partial E), & x \notin E \end{cases}.$$

Moreover, $d_{E^c}(x) = -d_E(x)$,

$$d_E(x) = 0 \iff x \in \partial E,$$

and

$$\left\{x \in \mathbb{R}^n : x \in (\partial E)_\varepsilon\right\} = \left\{x \in \mathbb{R}^n : \left|d_E(x)\right| \le \varepsilon\right\},$$

with $\left|\nabla d_E(x)\right| = 1$ for any point $x \in \mathbb{R}^n \setminus E$ where it is differentiable if E is closed. In particular, d_E is Lipschitz, by Rademacher's theorem, ∇d_E is differentiable a.e. in $\mathbb{R}^n$ (see [Amb00], Section 4, Theorem 1 and Remark 3).

Theorem 2.5.3 (Anisotropic Gaussian Isoperimetric Inequality (perimeter version))**.**
Let E be a measurable set in $\mathbb{R}^n$. Then

$$P_{\gamma_A}(E) \ge e^{-[\phi^{-1}(\gamma_A(E))]^2/2} \frac{1}{\|(\sqrt{A})^{-1}\|}.$$

In particular, if $A = I_n$, we have the standard Gaussian isoperimetric inequality,

$$P_{\gamma_{I_n}}(E) \ge e^{-[\phi^{-1}(\gamma_{I_n}(E))]^2/2}.$$

Proof. **Step 1**: We first assume that E is a bounded open set with smooth boundary in $\mathbb{R}^n$ and $P_{\gamma_A}(E) < \infty$. Since E is a set of finite anisotropic Gaussian perimeter, by Proposition 2.3.1, E is a set of locally finite perimeter. Moreover, since E is bounded and open with smooth boundary,

$$d_E \text{ is smooth in a tubular neighborhood of } \partial E \text{ and } \nu_E = \nabla d_E \text{ on } \partial E$$

(see [Amb00], Theorem 2). By co-area formula, for any Borel function $g : \mathbb{R}^n \to [0, \infty]$ and Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$, we have

$$\int_{\mathbb{R}^n} g(x) \left|\nabla u(x)\right| \, dx = \int_{-\infty}^{+\infty} \left(\int_{\{u=t\}} g(y) \, d\mathcal{H}^{n-1}(y) \right) \, dt$$

(see [AFP00], Remark 2.97). Consider $u = d_E$,

$$g = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} \, e^{-|\sqrt{A}x|^2/2} \chi_{\{0 \le d_E \le \varepsilon\}},$$

and define the smooth function ψ_t as

$$\psi_t(x) = x + t\nabla d_E(x) \quad \text{in the tubular neighborhood of } \partial E.$$

Then we have

$$\begin{aligned}
\frac{\gamma_A(E_\varepsilon) - \gamma_A(E)}{\varepsilon} &= \frac{1}{\varepsilon} \int_{E_\varepsilon \setminus E} \frac{\sqrt{\det A}}{(2\pi)^{n/2}} e^{-|\sqrt{A}x|^2/2} \, dx \\
&= \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\{d_E = t\}} \frac{\sqrt{\det A}}{(2\pi)^{n/2}} e^{-|\sqrt{A}x|^2/2} \, d\mathcal{H}^{n-1}(x) \, dt \\
&= \frac{\sqrt{\det A}}{(2\pi)^{n/2}} \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\psi_t(\partial E)} e^{-|\sqrt{A}x|^2/2} \, d\mathcal{H}^{n-1}(x) \, dt.
\end{aligned}$$

Taking $\varepsilon \to 0^+$ on both sides, by the fundamental theorem of calculus,

$$\lim_{\varepsilon \to 0^+} \frac{\gamma_A(E_\varepsilon) - \gamma_A(E)}{\varepsilon} = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} \int_{\partial E} e^{-|\sqrt{A}x|^2/2} \, d\mathcal{H}^{n-1}(x) = \frac{1}{\sqrt{2\pi}} P_{\gamma_A}(E),$$

where $\psi_0(\partial E) = \partial E$. On the other hand, by Theorem 2.5.2, we have

$$\begin{aligned}
\frac{\gamma_A(E_\varepsilon) - \gamma_A(E)}{\varepsilon} &\ge \frac{\phi\left(\phi^{-1}(\gamma_A(E)) + \frac{\varepsilon}{\|(\sqrt{A})^{-1}\|}\right) - \gamma_A(E)}{\varepsilon} \\
&\to \phi'\left(\phi^{-1}(\gamma_A(E))\right) \frac{1}{\|(\sqrt{A})^{-1}\|} = \frac{1}{\sqrt{2\pi}} e^{-[\phi^{-1}(\gamma_A(E))]^2/2} \frac{1}{\|(\sqrt{A})^{-1}\|},
\end{aligned}$$

as $\varepsilon \to 0^+$. That is,

$$P_{\gamma_A}(E) \ge e^{-[\phi^{-1}(\gamma_A(E))]^2/2} \frac{1}{\|(\sqrt{A})^{-1}\|}.$$

Step 2: Now for any measurable set E, we may again assume that $P_{\gamma_A}(E) < \infty$. By Proposition 2.4.1, there exists a sequence $\{E_k\}$ of bounded open sets with smooth boundary such that

$$\chi_{E_k} \to \chi_E \text{ in } L^1(\mathbb{R}^n, \gamma_A), \quad P_{\gamma_A}(E_k) \to P_{\gamma_A}(E).$$

Applying (1) on E_k, we have

$$P_{\gamma_A}(E_k) \geq e^{-[\phi^{-1}(\gamma_A(E_k))]^2/2} \frac{1}{\|(\sqrt{A})^{-1}\|}.$$

Taking $k \to \infty$, we have finished the proof. □

In the paper of Cianchi-Fusco-Maggi-Pratelli [CFMP11], Proposition 3.1 and Theorem 4.1, they have characterized the equality cases for the standard Gaussian measure γ_{I_n}. The result reads as follows: let E be a measurable subset of $\mathbb{R}^n$. Then

$$P_{\gamma_{I_n}}(E) \geq e^{-[\phi^{-1}(\gamma_{I_n}(E))]^2/2}. \tag{2.21}$$

Moreover,

(1) if $n = 1$, equality holds if and only if either $\gamma_1(E) = 0$ or $\gamma_1(E) = 1$, or up to a set of measure zero and for some $\sigma \in \mathbb{R}$, $E = (-\infty, -\sigma)$ or $E = (\sigma, \infty)$.

(2) if $n \geq 2$, equality holds if and only if either $\gamma_{I_n}(E) = 0$ or $\gamma_{I_n}(E) = 1$, or E is equivalent to a half-space.

Notice that we can also derive Theorem 2.5.3 from Proposition 2.3.4 and equation (2.21),

$$\begin{aligned} P_{\gamma_A}(E) \geq P_{\gamma_{I_n}}(\sqrt{A}E) \frac{1}{\|(\sqrt{A})^{-1}\|} &\geq e^{-[\phi^{-1}(\gamma_{I_n}(\sqrt{A}E))]^2/2} \frac{1}{\|(\sqrt{A})^{-1}\|} \\ &= e^{-[\phi^{-1}(\gamma_A(E))]^2/2} \frac{1}{\|(\sqrt{A})^{-1}\|}. \end{aligned} \tag{2.22}$$

2.6 Proof of Theorem 1.2.1 (cases of equality)

Notice that (1) follows directly from Cianchi-Fusco-Maggi-Pratelli [CFMP11], Proposition 3.1. We just need to prove (2) here. Suppose the equality holds and assume that $\gamma_A(E) = \gamma_{I_n}(\sqrt{A}E) \in (0,1)$. By equation (2.22), we have

$$P_{\gamma_A}(E) = P_{\gamma_{I_n}}(\sqrt{A}E)\frac{1}{\|(\sqrt{A})^{-1}\|} = e^{-[\phi^{-1}(\gamma_{I_n}(\sqrt{A}E))]^2/2}\frac{1}{\|(\sqrt{A})^{-1}\|} = e^{-[\phi^{-1}(\gamma_A(E))]^2/2}\frac{1}{\|(\sqrt{A})^{-1}\|}.$$

That is,

$$P_{\gamma_{I_n}}(\sqrt{A}E) = e^{-[\phi^{-1}(\gamma_{I_n}(\sqrt{A}E))]^2/2}.$$

By [CFMP11], Theorem 4.1, $\sqrt{A}E$ is equivalent to a half-space, say $H(\omega,t)$, where $\omega \in \mathbb{S}^{n-1}$. Then

$$E \text{ is equivalent to } (\sqrt{A})^{-1}(H(\omega,t)) = H\left(\frac{\sqrt{A}\omega}{|\sqrt{A}\omega|}, \frac{t}{|\sqrt{A}\omega|}\right).$$

Moreover, by Proposition 2.5.1,

$$\gamma_A(E) = \gamma_A\left(H\left(\frac{\sqrt{A}\omega}{|\sqrt{A}\omega|}, \frac{t}{|\sqrt{A}\omega|}\right)\right) = \phi\left(t\right) \implies t = \phi^{-1}(\gamma_A(E))$$

and

$$P_{\gamma_A}(E) = P_{\gamma_A}\left(H\left(\frac{\sqrt{A}\omega}{|\sqrt{A}\omega|}, \frac{t}{|\sqrt{A}\omega|}\right)\right) = e^{-\frac{1}{2}t^2}|\sqrt{A}\omega| = e^{-[\phi^{-1}(\gamma_A(E))]^2/2}|\sqrt{A}\omega|.$$

By our assumption, we have

$$e^{-[\phi^{-1}(\gamma_A(E))]^2/2}\frac{1}{\|(\sqrt{A})^{-1}\|} = P_{\gamma_A}(E) = e^{-[\phi^{-1}(\gamma_A(E))]^2/2}|\sqrt{A}\omega| \implies |\sqrt{A}\omega| = d_{\min}, \quad (2.23)$$

where $d_{\min}$ is the smallest eigenvalue of $\sqrt{A}$ and we have used

$$\|(\sqrt{A})^{-1}\| = \text{ the largest eigenvalue of } (\sqrt{A})^{-1} = \frac{1}{d_{\min}}.$$

Now we claim that

$$\omega \in V_{d_{\min}}(\sqrt{A}) \cap \mathbb{S}^{n-1}.$$

Notice that we can decompose A into

$$A = O^\intercal D O,$$

with an orthogonal matrix O and a diagonal matrix D. If all eignevalues of D are the same, i.e., $D = d_{\min}^2 I_n$, then $\sqrt{A} = d_{\min} I_n$ and $\omega \in V_{d_{\min}}(\sqrt{A}) \cap \mathbb{S}^{n-1}$. Hence, we may assume that D has the following form:

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad D_1 = \lambda_{\min} I_1, \quad \text{and} \quad D_2 \text{ has eigenvalues strictly greater than } \lambda_{\min},$$

where $\lambda_{\min} = d_{\min}^2$ is the smallest eigenvalue of A. Let

$$O = \begin{bmatrix} O_1 & O_2 \\ O_3 & O_4 \end{bmatrix},$$

then $\sqrt{A} = O^\intercal D^{1/2} O$ and

$$|\sqrt{A}\omega|^2 = |O^\intercal D^{1/2} O\omega|^2 = |D^{1/2} y|^2 = |D_1^{1/2} y_1|^2 + |D_2^{1/2} y_2|^2 = d_{\min}^2 |y_1|^2 + |D_2^{1/2} y_2|^2 \quad (2.24)$$

where

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y := O\omega.$$

On the other hand,

$$d_{\min}^2 = d_{\min}^2 |\omega|^2 = d_{\min}^2 |y|^2 = d_{\min}^2 |y_1|^2 + d_{\min}^2 |y_2|^2. \quad (2.25)$$

Therefore, by (2.23), (2.24), and (2.25),

$$|D_2^{1/2} y_2|^2 = d_{\min}^2 |y_2|^2 \implies y_2 = 0$$

since D_2 has eigenvalues strictly greater than $\lambda_{\min} = d_{\min}^2$. Thus,

$$\sqrt{A}\omega = O^\intercal D^{1/2} O\omega = O^\intercal D^{1/2} y = \begin{bmatrix} O_1^\intercal & O_3^\intercal \\ O_2^\intercal & O_4^\intercal \end{bmatrix} \begin{bmatrix} d_{\min} I_1 & 0 \\ 0 & D_2^{1/2} \end{bmatrix} \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} d_{\min} O_1^\intercal y_1 \\ d_{\min} O_2^\intercal y_1 \end{bmatrix},$$

and

$$d_{\min}\omega = d_{\min}O^{\mathsf{T}}y = d_{\min}\begin{bmatrix} O_1^{\mathsf{T}} & O_3^{\mathsf{T}} \\ O_2^{\mathsf{T}} & O_4^{\mathsf{T}} \end{bmatrix}\begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} d_{\min}O_1^{\mathsf{T}}y_1 \\ d_{\min}O_2^{\mathsf{T}}y_1 \end{bmatrix} = \sqrt{A}\omega.$$

Hence,

$$\sqrt{A}\omega - d_{\min}\omega = 0 \implies \omega \in V_{d_{\min}}(\sqrt{A}).$$

We conclude that

$$E \text{ is equivalent to } (\sqrt{A})^{-1}(H(\omega, t)) = H\left(\frac{\sqrt{A}\omega}{|\sqrt{A}\omega|}, \frac{t}{|\sqrt{A}\omega|}\right) = H\left(\omega, \frac{\phi^{-1}(\gamma_A(E))}{d_{\min}}\right).$$

Now we prove the converse of (2) in Theorem 1.2.1. It is clear that the equality holds when $\gamma_A(E) = 0$ or $\gamma_A(E) = 1$. Hence, we may assume that $\gamma_A(E) \in (0, 1)$, i.e., $\phi^{-1}(\gamma_A(E)) \in \mathbb{R}$. Since $\omega \in V_{d_{\min}}(\sqrt{A}) \cap \mathbb{S}^{n-1}$, $\sqrt{A}\omega = d_{\min}\omega$. By Proposition 2.5.1, we have

$$\begin{aligned} P_{\gamma_A}(E) = P_{\gamma_A}\left(H\left(\omega, \frac{\phi^{-1}(\gamma_A(E))}{d_{\min}}\right)\right) &= e^{-\frac{1}{2}\frac{\frac{[\phi^{-1}(\gamma_A(E))]^2}{d_{\min}^2}}{|(\sqrt{A})^{-1}\omega|^2}} \frac{1}{|(\sqrt{A})^{-1}\omega|} \\ &= e^{-[\phi^{-1}(\gamma_A(E))]^2/2} d_{\min} = e^{-[\phi^{-1}(\gamma_A(E))]^2/2} \frac{1}{\|(\sqrt{A})^{-1}\|}. \end{aligned}$$

□

Chapter 3

ANISOTROPIC GAUSSIAN PERIMETER INEQUALITY UNDER EHRHARD SYMMETRIZATION

3.1 Ehrhard symmetrization

In this section, we will use the following notations:

$$x = (z, y) \text{ for } x \in \mathbb{R}^n, z \in \mathbb{R}^{n-1} \text{ and } y \in \mathbb{R}.$$

Similar to (2.6), we define two (outer) measures μ_z and $\mathcal{H}_z^0$ on $\mathbb{R}^1$ such that

$$\mu_z(F_1) = \int_{F_1} e^{-|\sqrt{A}x|^2/2} dy,\ \forall F_1 \in \mathcal{L}(\mathbb{R}^1), \quad \mathcal{H}_z^0(F_2) = \int_{F_2} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^0(y),\ \forall F_2 \subset \mathbb{R}^1, \tag{3.1}$$

where $\mathcal{H}^0$ is the counting measure. Moreover, we define

$$P_z(F) = \mathcal{H}_z^0(\partial^M F), \quad F \subset \mathbb{R}^1,$$

where $\partial^M F$ is the essential boundary of F. We also define an auxiliary function ϕ_z as

$$\phi_z(t) = \int_{-\infty}^{t} e^{-|\sqrt{A}x|^2/2} dy, \quad \phi_z(\infty) = \int_{-\infty}^{\infty} e^{-|\sqrt{A}x|^2/2} dy, \tag{3.2}$$

and $\phi_z(-\infty) = 0$, where $z \in \mathbb{R}^{n-1}$. Let E be a measurable set in $\mathbb{R}^n$ with $n \geq 2$. The **section** $E_z \subseteq \mathbb{R}$ of E is defined as

$$E_z = \{y \in \mathbb{R} : (z, y) \in E\}, \quad \text{where } z \in \mathbb{R}^{n-1}.$$

Define $v_E : \mathbb{R}^{n-1} \to \mathbb{R}$ as

$$v_E(z) = \mu_z(E_z), \quad \forall z \in \mathbb{R}^{n-1}.$$

Notice that $e^{-|\sqrt{A}x|^2/2} \leq e^{-\|(\sqrt{A})^{-1}\|^{-2}|x|^2/2}$ (see Lemma 3.2.1) and $x \mapsto e^{-\|(\sqrt{A})^{-1}\|^{-2}|x|^2/2} \in L^1(\mathbb{R}^n)$. By Fubini's theorem, we have

$$v_E \in L^1(\mathbb{R}^{n-1}).$$

The **Ehrhard symmetrization** of E with respect to the y-direction is defined as

$$E^s := E^s_{A,-e_n} := \left\{ (z,y) \in \mathbb{R}^n : y < \phi_z^{-1} \left(v_E(z) \right) \right\}, \tag{3.3}$$

and the **essential projection** of E with respect to the y-direction is defined as

$$\pi_+(E) := \pi_{+,A,-e_n}(E) := \left\{ z \in \mathbb{R}^{n-1} : v_E(z) = \mu_z \left(E_z \right) > 0 \right\}.$$

We now define

$$p_E(z) = \mathcal{H}^0_z \left[(\partial^M E)_z \right].$$

Roughly speaking, the set $\pi_+(E)$ captures the set in $\mathbb{R}^{n-1}$ over which the one-dimensional vertical slices in E have positive mass. We recall the co-area formula for sets of locally finite perimeter (see [CFMP11], equation (4.1) and [Mag12], Theorem 18.8): for any non-negative Borel function $g : \mathbb{R}^n \to [0, \infty]$, we have

$$\int_{\partial^M E} g(x) \left| \nu_n^E(x) \right| d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} \int_{\left(\partial^M E\right)_z} g(z,y) \; d\mathcal{H}^0(y) \; dz, \tag{3.4}$$

where ν_n^E means $\langle \nu^E, e_n \rangle$. We also recall the following theorem by Vol'pert from [Vol67] (see also [AFP00], Theorem 3.108 and [CCF05], Theorem G):

Theorem 3.1.1 (Vol'pert Theorem)**.**
Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter with $n \geq 2$. Then there exists a Borel set $B_E \subseteq \pi_+(E)$ with $\mathcal{L}^{n-1} \left(\pi_+(E) \setminus B_E \right) = 0$ such that for every $z \in B_E$,

(i) E_z is a set of locally finite perimeter in $\mathbb{R}$;

(ii) $\left(\partial^M E\right)_z = \partial^M \left(E_z\right) = \partial^* \left(E_z\right) = \left(\partial^* E\right)_z$;

(iii) $\nu_n^E(z,y) \neq 0$ for every y such that $(z,y) \in \partial^* E$.

We will call B_E the **Vol'pert set***.*

In order to understand the Ehrhard symmetrization set E^s, our first goal is to analyze the regularities of the mappings $z \mapsto v_E(z)$ and $z \mapsto \phi_z^{-1}(v_E(z))$. For the isotropic Gaussian case, the mapping $z \mapsto \phi^{-1}(\gamma_1(E_z))$ is in $BV_{\text{loc}}(\mathbb{R}^{n-1})$ since $z \mapsto \gamma_1(E_z)$ is in $BV(\mathbb{R}^{n-1})$ and $\omega \mapsto \phi^{-1}(\omega)$ is $C^1(\mathbb{R})$. Here we have used a fact proven by Vol'pert [Vol67] that the composition of a C^1 map with a BV function is again a BV function. In fact, Ambrosio-Dal Maso [ADM90] showed that this is also true if we compose a BV function with a Lipschitz map. However, in our setting, the function ϕ_z^{-1} is also depending on the variable $z \in \mathbb{R}^{n-1}$ which required a different proof for the regularity of $z \mapsto \phi_z^{-1}(v_E(z))$.

3.2 A regularity result for the map $z \mapsto \phi_z^{-1}(v_E(z))$

Our first goal is to show that $v_E \in BV(\mathbb{R}^{n-1})$ if E is a set of finite anisotropic Gaussian perimeter. The proof is similar to Chlebík-Cianchi-Fusco's paper [CCF05], Lemma 3.1 and Lemma 3.2. Before doing that, we need the following preliminary result for the integrand $e^{-|\sqrt{A}|^2/2}$. The cross term $a_{ij}x_ix_j$ in $\langle Ax, x\rangle = |\sqrt{A}x|^2$ also plays an important role in the calculation. We will need those estimates throughout this section.

Lemma 3.2.1 (Computational lemma)**.**
Let $n \geq 2$ and let $\sqrt{A}$ be a symmetric positive definite matrix. Then

(1) **(Derivative for the integrand)**
Let $\nabla' = (\partial_1, \dots, \partial_{n-1})$ and $x = (z, y)$. Then

$$\partial_{z_k} e^{-|\sqrt{A}x|^2/2} = -e^{-|\sqrt{A}x|^2/2}\langle \text{row}_k(A), x\rangle, \quad \partial_{yy}^2 |\sqrt{A}x|^2 = 2\sum_{i=1}^{n} a_{in}^2,$$

$$\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) = -e^{-|\sqrt{A}x|^2/2}A'x, \quad \textit{and} \quad \nabla\left(e^{-|\sqrt{A}x|^2/2}\right) = -e^{-|\sqrt{A}x|^2/2}Ax,$$

where $\sqrt{A} = (a_{ij})$ and $A' \in M_{(n-1)\times n}(\mathbb{R})$ is the first $n-1$ rows of matrix from A.

(2) **(Regularity estimates)**

(a) *For any* $z_0 \in \mathbb{R}^{n-1}$ *and for any measurable set* F,

$$\lim_{z \to z_0} \int_F \left(e^{-|\sqrt{A}(z,y)|^2/2} - e^{-|\sqrt{A}(z_0,y)|^2/2} \right) dy = 0.$$

In particular, the mapping

$$v : z \mapsto \int_F e^{-|\sqrt{A}(z,y)|^2/2}\, dy \text{ is continuous.}$$

(b) *For any* $z \in \mathbb{R}^{n-1}$ *and for any measurable set* F,

$$\lim_{k \to 0} \int_F \left(\frac{e^{-|\sqrt{A}(z+k,y)|^2/2} - e^{-|\sqrt{A}(z,y)|^2/2} - \nabla' \left(e^{-|\sqrt{A}(z,y)|^2/2} \right) \cdot k}{|k|} \right) dy = 0. \quad (3.5)$$

In particular, the mapping v *in (a) is differentiable and*

$$\nabla' v(z) = \int_F \nabla' \left(e^{-|\sqrt{A}(z,y)|^2/2} \right) dy.$$

Furthermore, $\nabla' v$ *is continuous, i.e.,* $v \in C^1(\mathbb{R}^{n-1})$.

(c) *Let* K *be a convex compact set in* $\mathbb{R}^{n-1}$ *and* $h \in C^1(K)$. *Then for any* $z_0, z \in K$,

$$\left| e^{|\sqrt{A}(z,h(z))|^2/2} - e^{|\sqrt{A}(z_0,h(z_0))|^2/2} \right| \leq C(K,h,A)\, |z - z_0|$$

for some constant $C(K,h,A) > 0$.

(3) **(Integral bounds)**

(a) *There exists a constant* $C_1(A) > 0$ *such that*

$$\sup_{1 \leq k \leq n-1} \sup_{z \in \mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \left| \frac{\partial}{\partial z_k} \left(e^{-|\sqrt{A}x|^2/2} \right) \right| dy \right) \leq C_1(A) < \infty.$$

(b) *There exists a constant* $C_2(A) > 0$ *such that*

$$\sup_{1 \leq k \leq n-1} \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial z_k} \left(e^{-|\sqrt{A}x|^2/2} \right) \right| dx \leq C_2(A) < \infty.$$

Proof. We will denote $x = (z, y)$ in the following calculations. When we do matrix multiplication, the notation $Ax = A(z, y)$ means

$$A(z,y)^{\mathsf{T}} \in M_{n\times 1}(\mathbb{R}).$$

(1) **(Derivative for the integrand)**

Let $A = (A_{ij})$, $\sqrt{A} = (a_{ij})$. Then

$$|\sqrt{A}x|^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right)^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n-1} a_{ij}z_j + a_{in}y\right)^2,$$

$$\partial_{yy}^2|\sqrt{A}x|^2 = \partial_y\left(\sum_{i=1}^{n} 2\left(\sum_{j=1}^{n-1} a_{ij}z_j + a_{in}y\right)a_{in}\right) = 2\sum_{i=1}^{n} a_{in}^2,$$

and

$$\partial_{z_k}|\sqrt{A}x|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n-1} 2a_{ik}a_{ij}z_j + \sum_{i=1}^{n} 2a_{ik}a_{in}y$$

for $k = 1, 2, \cdots, n-1$. Since $\sqrt{A}$ is symmetric, i.e., $a_{ik} = a_{ki}$, we have

$$\begin{aligned}\partial_{z_k}\left(e^{-|\sqrt{A}x|^2/2}\right) &= -e^{-|\sqrt{A}x|^2/2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n-1} a_{ik}a_{ij}z_j + \sum_{i=1}^{n} a_{ik}a_{in}y\right)\\ &= -e^{-|\sqrt{A}x|^2/2}\left(\sum_{j=1}^{n-1} A_{kj}z_j + A_{kn}y\right) = -e^{-|\sqrt{A}x|^2/2}\langle \mathrm{row}_k(A), x\rangle\end{aligned}$$

Therefore,

$$\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) = -e^{-|\sqrt{A}x|^2/2}\begin{pmatrix}\langle \mathrm{row}_1(A), x\rangle\\ \vdots\\ \langle \mathrm{row}_{n-1}(A), x\rangle\end{pmatrix} = -e^{-|\sqrt{A}x|^2/2}A'x$$

and $\nabla\left(e^{-|\sqrt{A}x|^2/2}\right) = -e^{-|\sqrt{A}x|^2/2}Ax.$

(2) **(Regularity estimates)**

(a) Let K be a compact set with $z_0, z \in K$. By using (1) and the mean value theorem,

$$\begin{aligned}
&\left|e^{-|\sqrt{A}(z,y)|^2/2} - e^{-|\sqrt{A}(z_0,y)|^2/2}\right| \leq |z - z_0| \left|e^{-|\sqrt{A}(\zeta,y)|^2/2} A'(\zeta, y)\right| \\
&\leq \left|A'(\zeta, y)\right| e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} |z - z_0| \\
&\leq \sqrt{\lambda_{\max}(A'^{\mathsf{T}} A')}\Big(|\zeta| + |y|\Big) e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} |z - z_0| \\
&\leq \sqrt{\lambda_{\max}(A'^{\mathsf{T}} A')}\Big(r(K) + |y|\Big) e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} |z - z_0| \qquad (3.6)
\end{aligned}$$

where ζ lies between z and z_0, $\lambda_{\max}(A'^{\mathsf{T}} A')$ is the largest eigenvalue of $A'^{\mathsf{T}} A'$, and $r(K) = \sup_{\zeta \in K} |\zeta|$. We now claim that

$$\lim_{z \to z_0} \int_F e^{-|\sqrt{A}(z,y)|^2/2} - e^{-|\sqrt{A}(z_0,y)|^2/2} dy = 0.$$

Since $z \to z_0$, we may assume that $z \in K := \overline{B}(z_0, 1)$. Thus, as $z \to z_0$,

$$\begin{aligned}
&\int_F \left|e^{-|\sqrt{A}(z,y)|^2/2} - e^{-|\sqrt{A}(z_0,y)|^2/2}\right| dy \\
&\leq \int_F \sqrt{\lambda_{\max}(A'^{\mathsf{T}} A')}\Big(r(K) + |y|\Big) e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} |z - z_0| \; dy \\
&= |z - z_0| \sqrt{\lambda_{\max}(A'^{\mathsf{T}} A')} \int_F \Big(r(K) + |y|\Big) e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} dy \to 0.
\end{aligned}$$

(b) By Taylor expansion, if $f \in C^1(\mathbb{R}^{n-1})$ and $x_1, x_2 \in \mathbb{R}^{n-1}$,

$$\begin{aligned}
f(x_2) &= f(x_1) + \langle \nabla' f(x_1), x_2 - x_1 \rangle \\
&\quad + \int_0^1 \langle \nabla' f(x_1 + t(x_2 - x_1)) - \nabla' f(x_1), x_2 - x_1 \rangle \; dt
\end{aligned}$$

(see, for example, [Dru20], Theorem 1.14). Fix $y \in \mathbb{R}$, $k \in \mathbb{R}^{n-1}$, and set

$$f(z) = e^{-|\sqrt{A}(z,y)|^2/2}.$$

Let K be a convex compact set with $z, z + k \in K$. Then by a similar argument as (3.6),

$$\begin{aligned}
&\left| e^{-|\sqrt{A}(z+k,y)|^2/2} - e^{-|\sqrt{A}(z,y)|^2/2} - \nabla' \left(e^{-|\sqrt{A}(z,y)|^2/2} \right) \cdot k \right| \\
&= \left| \int_0^1 \left(-e^{-|\sqrt{A}(z+tk,y)|^2/2} A'(z+tk,y) + e^{-|\sqrt{A}(z,y)|^2/2} A'(z,y) \right) \cdot k \, dt \right| \\
&\leq \int_0^1 \left| \left(-e^{-|\sqrt{A}(z+tk,y)|^2/2} A'(z+tk,y) + e^{-|\sqrt{A}(z+tk,y)|^2/2} A'(z,y) \right) \cdot k \right| dt \\
&\quad + \int_0^1 \left| \left(-e^{-|\sqrt{A}(z+tk,y)|^2/2} A'(z,y) + e^{-|\sqrt{A}(z,y)|^2/2} A'(z,y) \right) \cdot k \right| dt \\
&\leq \frac{1}{2} |k|^2 e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} \sqrt{\lambda_{\max}(A'^{\mathsf{T}} A')} \\
&\quad + \frac{1}{2} |k|^2 e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} \lambda_{\max}(A'^{\mathsf{T}} A') \Big(r(K) + |y| \Big) \Big(|z| + |y| \Big)
\end{aligned}$$

where $z + tk \in K$ since K is convex. In particular, for any $z \in \mathbb{R}^{n-1}$ and for any measurable set F, we have the following

$$\lim_{k \to 0} \int_F \left(\frac{e^{-|\sqrt{A}(z+k,y)|^2/2} - e^{-|\sqrt{A}(z,y)|^2/2} - \nabla' \left(e^{-|\sqrt{A}(z,y)|^2/2} \right) \cdot k}{|k|} \right) dy = 0.$$

Finally, using the estimates in (4.1), we can see that $\nabla' v$ is continuous.

(c) Recall that

$$|e^x - 1| \leq e^{|x|} - 1 \leq |x| e^{|x|} \quad \text{for all } x \in \mathbb{R}.$$

Then by $h \in C^1(K)$, the convexity of K, and a similar argument as (3.6),

$$\begin{aligned}
&\left| e^{|\sqrt{A}(z,h(z))|^2/2} - e^{|\sqrt{A}(z_0,h(z_0))|^2/2} \right| \\
&= e^{\frac{|\sqrt{A}(z_0,h(z_0))|^2}{2}} \left| e^{\frac{|\sqrt{A}(z,h(z))|^2 - |\sqrt{A}(z_0,h(z_0))|^2}{2}} - 1 \right| \\
&\leq e^{\frac{|\sqrt{A}(z_0,h(z_0))|^2}{2}} \left| \frac{|\sqrt{A}(z,h(z))|^2 - |\sqrt{A}(z_0,h(z_0))|^2}{2} \right| e^{\left| \frac{|\sqrt{A}(z,h(z))|^2 - |\sqrt{A}(z_0,h(z_0))|^2}{2} \right|} \\
&\leq C(K,h,A) |z - z_0|
\end{aligned}$$

for some constant $C(K,h,A) > 0$.

(3) **(Integral bounds)**

We will only prove (3)(a) since the proof of (3)(b) is similar. By (1), we have

$$\begin{aligned}\left|\frac{\partial}{\partial z_k}\left(e^{-|\sqrt{A}x|^2/2}\right)\right| &= \left|e^{-|\sqrt{A}x|^2/2}\langle \mathrm{row}_k(A), x\rangle\right| = \left|e^{-|\sqrt{A}x|^2/2}\langle A^{\mathsf{T}}e_k, x\rangle\right| \\ &\leq e^{-|\sqrt{A}x|^2/2}\|A\||x| \leq \|A\|e^{-\|(\sqrt{A})^{-1}\|^{-2}|x|^2/2}|x| \\ &\leq \|A\|e^{-\|(\sqrt{A})^{-1}\|^{-2}|z|^2/2}e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2}\left(|z|+|y|\right).\end{aligned}$$

Hence,

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial z_k}\left(e^{-|\sqrt{A}x|^2/2}\right)\right|dy &\leq \|A\|e^{-\|(\sqrt{A})^{-1}\|^{-2}|z|^2/2}|z|\int_{-\infty}^{\infty}e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2}dy \\ &\quad + \|A\|e^{-\|(\sqrt{A})^{-1}\|^{-2}|z|^2/2}\int_{-\infty}^{\infty}e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2}|y|dy.\end{aligned}$$

Therefore, there exists a constant $C_1(A) > 0$ such that

$$\sup_{1\leq k\leq n-1}\sup_{z\in\mathbb{R}^{n-1}}\left(\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial z_k}\left(e^{-|\sqrt{A}x|^2/2}\right)\right|dy\right) \leq C_1(A) < \infty.$$

□

We are now ready to show that v_E is in $BV(\mathbb{R}^{n-1})$. Moreover, we prove a relation between $|Dv_E|$ and $P_{\gamma_A}(E; G\times\mathbb{R})$ and a weak derivative formula for $D_i v_E$. These two ingredients play an important role in proving our main result (see Theorem 3.4.4).

Lemma 3.2.2 (Regularity of v_E and its distributional derivative formula)**.**
Let $n \geq 2$ and let E be a set of finite anisotropic Gaussian perimeter in $\mathbb{R}^n$. Then $v_E \in BV\left(\mathbb{R}^{n-1}\right)$, i.e., $|Dv_E|(\mathbb{R}^{n-1}) < \infty$, and

$$\frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}|Dv_E|(G) \leq P_{\gamma_A}(E; G\times\mathbb{R}) + \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_G\left|\int_{E_z}\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)dy\right|dz$$

for every open set $G \subseteq \mathbb{R}^{n-1}$. Moreover, let $D_i v_E(z) := \frac{dD_i v_E \llcorner B_E}{d\mathcal{L}^{n-1}\llcorner B_E}(z)$,

$$D_i v_E(z) = \int_{(\partial^* E)_z}\frac{\nu_i^E(z,y)}{\left|\nu_n^E(z,y)\right|}d\mathcal{H}^0_z(y) + \int_{E_z}\frac{\partial}{\partial x_i}\left(e^{-|\sqrt{A}x|^2/2}\right)dy \qquad \text{for } i = 1, 2, ..., n-1,$$

for $\mathcal{L}^{n-1}$-a.e. $z \in B_E$, where B_E is the set appearing in Vol'pert Theorem.

Proof. Since $v_E \in L^1(\mathbb{R}^{n-1})$, our goal is to show that $v_E \in BV(\mathbb{R}^{n-1})$.

Step 1: Let $\varphi \in C_c^1\left(\mathbb{R}^{n-1}\right)$ and $\psi_j \in C_c^1(\mathbb{R})$ with $0 \leq \psi_j(y) \leq 1$ for $y \in \mathbb{R}$, $j \in \mathbb{N}$, and such that $\lim_{j\to\infty} \psi_j(y) = 1$ for every $y \in \mathbb{R}$. For any $i = 1, \ldots, n-1$, by the dominated convergence theorem,

$$\int_{\mathbb{R}^{n-1}} \frac{\partial \varphi}{\partial z_i}(z)\, v_E(z)\, dz = \int_{\mathbb{R}^{n-1}} \frac{\partial \varphi}{\partial z_i}(z) \int_{E_z} e^{-|\sqrt{A}x|^2/2} dy\ dz$$

$$= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial \varphi}{\partial z_i}(z)\, \chi_E(z,y)\, e^{-|\sqrt{A}x|^2/2} dy \right) dz$$

$$= \lim_{j\to\infty} \int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial z_i}(z)\, \psi_j(y) \chi_E(z,y)\, e^{-|\sqrt{A}x|^2/2} dz dy \qquad (\text{insert } \lim_j \psi_j = 1)$$

$$= \lim_{j\to\infty} \left\{ \int_{\mathbb{R}^n} \operatorname{div}\left(\varphi(z)\, \psi_j(y) e^{-|\sqrt{A}x|^2/2} e_i \right) \chi_E(z,y)\, dz dy \right.$$
$$\left. - \int_{\mathbb{R}^n} \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2} \right) \varphi(z) \psi_j(y) \chi_E(z,y)\, dy dz \right\}$$

$$= \lim_{j\to\infty} \left\{ \int_E \operatorname{div}\left(\varphi(z)\, \psi_j(y) e^{-|\sqrt{A}x|^2/2} e_i \right) dz dy - \int_E \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2} \right) \varphi(z) \psi_j(y) dy dz \right\}$$

$$= -\lim_{j\to\infty} \left\{ \int_{\partial^* E} \varphi(z)\, \psi_j(y) e^{-|\sqrt{A}x|^2/2} e_i \cdot \nu^E\ d\mathcal{H}^{n-1}(z,y) + \int_E \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2} \right) \varphi(z) \psi_j(y) dy dz \right\}$$

$$= -\lim_{j\to\infty} \left\{ \int_{\mathbb{R}^n} \varphi(z)\, \psi_j(y) e^{-|\sqrt{A}x|^2/2} dD_i \chi_E + \int_E \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2} \right) \varphi(z) \psi_j(y) dy dz \right\} \qquad (\text{by } (2.3))$$

$$= -\int_{\mathbb{R}^n} \varphi(z)\, e^{-|\sqrt{A}x|^2/2} dD_i \chi_E - \int_E \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2} \right) \varphi(z) dy dz.$$

(a) Notice that for $|\varphi| \leq 1$,

$$\int_{\mathbb{R}^n} \left(-\varphi(z)\, \psi_j(y) \right) e^{-|\sqrt{A}x|^2/2} dD_i \chi_E \leq \int_{\partial^* E} e^{-|\sqrt{A}x|^2/2}\ d\mathcal{H}^{n-1}(x) = \frac{(2\pi)^{(n-1)/2}}{\det \sqrt{A}} P_{\gamma_A}(E),$$

and hence

$$-\lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi(z)\, \psi_j(y) e^{-|\sqrt{A}x|^2/2} dD_i \chi_E \leq \frac{(2\pi)^{(n-1)/2}}{\det \sqrt{A}} P_{\gamma_A}(E) < \infty.$$

(b) By Lemma 3.2.1, there exists a constant $C > 0$ such that

$$-\int_E \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2} \right) \varphi(z) \psi_j(y)\ dy dz \leq \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2} \right) \right| dy dz \leq C$$

and hence

$$-\lim_{j\to\infty}\int_E \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2}\right)\varphi(z)\psi_j(y)\ dydz \le C < \infty.$$

Taking the sup over $\varphi \in C_c^1\left(\mathbb{R}^{n-1}\right)$ with $|\varphi| \le 1$, (a), and (b), we conclude that $v_E \in BV\left(\mathbb{R}^{n-1}\right)$ since for any vector function $\varphi = (\varphi_1, .., \varphi_{n-1}) \in C_c^1\left(\mathbb{R}^{n-1};\mathbb{R}^{n-1}\right)$ with $|\varphi| \le 1$, the above argument works for each φ_i, $i = 1, ..., n-1$. Moreover, we have

$$\int_{\mathbb{R}^{n-1}} \varphi(z)\ dD_i v_E(z) = \int_{\mathbb{R}^n} \varphi(z)\, e^{-|\sqrt{A}x|^2/2} dD_i\chi_E + \int_E \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2}\right)\varphi(z)\ dydz \tag{3.7}$$

for every $\varphi \in C_c^1(\mathbb{R}^{n-1})$.

Now we consider $\eta = (\eta_1, ..., \eta_{n-1}) \in C_c^1(G;\mathbb{R}^{n-1})$ with $|\eta| \le 1$, where G is open in $\mathbb{R}^{n-1}$. By an approximation argument, we may set φ in (3.7) as

$$\varphi(z) = \eta_i(z)\chi_G(z).$$

That is, for $i = 1, ..., n-1$,

$$\int_G \eta_i(z)\ dD_i v_E\,(z) = \int_{G\times\mathbb{R}} e^{-|\sqrt{A}x|^2/2}\eta_i(z)\ dD_i\chi_E(x) + \int_{E\cap(G\times\mathbb{R})} \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2}\right)\eta_i(z)\ dydz.$$

Therefore,

$$\begin{aligned}\int_G \eta(z)\cdot dDv_E(z) = &\int_{G\times\mathbb{R}} e^{-|\sqrt{A}x|^2/2}\eta(z)\cdot d\left(D_1\chi_E, D_2\chi_E, ..., D_{n-1}\chi_E\right)(x) \\ &+ \int_{E\cap(G\times\mathbb{R})} \eta(z)\cdot\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)\ dydz\end{aligned} \tag{3.8}$$

for any $\eta \in C_c^1(G;\mathbb{R}^{n-1})$ with $|\eta| \le 1$. Let $\psi_j \in C_c^1(\mathbb{R})$ with $0 \le \psi_j(y) \le 1$ for $y \in \mathbb{R}$, $j \in \mathbb{N}$, and such that $\lim_{j\to\infty}\psi_j(y) = 1$ for every $y \in \mathbb{R}$. By the dominated convergence theorem and Proposition 2.2.4,

$$\begin{aligned}&\int_G \eta(z)\cdot dDv_E(z) - \int_{E\cap(G\times\mathbb{R})} \eta(z)\cdot\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)\ dydz \\ &= \lim_{j\to\infty}\int_{G\times\mathbb{R}} e^{-|\sqrt{A}x|^2/2}\left(\eta(z)\psi_j(y)\right)\cdot d\left(D_1\chi_E, D_2\chi_E, ..., D_{n-1}\chi_E\right)(x) \\ &\le \sup\left\{\int_{G\times\mathbb{R}} e^{-|\sqrt{A}x|^2/2}\tilde{\eta}(x)\cdot d\left(D_1\chi_E, D_2\chi_E, ..., D_{n-1}\chi_E\right)(x) : \tilde{\eta} \in C_c(G\times\mathbb{R};\mathbb{R}^{n-1}), |\tilde{\eta}| \le 1\right\}\end{aligned}$$

$$= \int_{G\times\mathbb{R}} e^{-|\sqrt{A}x|^2/2}\, d\left|(D_1\chi_E, D_2\chi_E, ..., D_{n-1}\chi_E)\right|(x)$$
$$\leq \int_{G\times\mathbb{R}} e^{-|\sqrt{A}x|^2/2}\, d\left|D\chi_E\right|(x) = \int_{G\times\mathbb{R}} e^{-|\sqrt{A}x|^2/2}\, d\mathcal{H}^{n-1} \llcorner \partial^* E(x) = \frac{(2\pi)^{(n-1)/2}}{\det\sqrt{A}} P_{\gamma_A}(E; G\times\mathbb{R}).$$

Thus,

$$\begin{aligned}\int_G \eta(z)\cdot dDv_E(z) &\leq \frac{(2\pi)^{(n-1)/2}}{\det\sqrt{A}} P_{\gamma_A}(E; G\times\mathbb{R}) + \int_{E\cap(G\times\mathbb{R})} \eta(z)\cdot\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)\, dydz \\ &= \frac{(2\pi)^{(n-1)/2}}{\det\sqrt{A}} P_{\gamma_A}(E; G\times\mathbb{R}) + \int_G \eta(z)\cdot\int_{E_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)\, dy\, dz \\ &\leq \frac{(2\pi)^{(n-1)/2}}{\det\sqrt{A}} P_{\gamma_A}(E; G\times\mathbb{R}) + \int_G \left|\int_{E_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)\, dy\right| dz.\end{aligned}$$

Taking the sup over η on both sides, we have

$$|Dv_E|(G) \leq \frac{(2\pi)^{(n-1)/2}}{\det\sqrt{A}} P_{\gamma_A}(E; G\times\mathbb{R}) + \int_G \left|\int_{E_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)\, dy\right| dz$$

for every open set $G\subseteq\mathbb{R}^{n-1}$.

Step 2: Let B_E be the set given by Vol'pert (Theorem 3.1.1). Applying equation (2.1), we have

$$\frac{\nu_i^E(z,y)}{\left|\nu_n^E(z,y)\right|} = \lim_{r\to 0} \frac{D_i\chi_E\left(B_r(z,y)\right)}{\left|D_n\chi_E\right|\left(B_r(z,y)\right)},$$

for every $z\in B_E$ and every y such that $(z,y)\in\partial^* E$. By the Besicovitch differentiation theorem,

$$D_i\chi_E \llcorner (B_E\times\mathbb{R}) = \frac{\nu_i^E}{|\nu_n^E|}\left|D_n\chi_E\right| \llcorner (B_E\times\mathbb{R}).$$

Now, let g be any function in $C_c\left(\mathbb{R}^{n-1}\right)$. We can set $\varphi(z) = g(z)\,\chi_{B_E}(z)$ in (3.7) since we can first approximate opens sets then Borel sets. Therefore,

$$\begin{aligned}\int_{B_E} g(z)\, dD_i v_E &= \int_{B_E\times\mathbb{R}} g(z)\, e^{-|\sqrt{A}x|^2/2} dD_i\chi_E(x) + \int_{E\cap(B_E\times\mathbb{R})} \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2}\right) g(z)\, dydz \\ &= \int_{B_E\times\mathbb{R}} \frac{\nu_i^E(z,y)}{\left|\nu_n^E(z,y)\right|} g(z)\, e^{-|\sqrt{A}x|^2/2} d\left|D_n\chi_E\right| \\ &\quad + \int_{B_E} g(z)\left(\int_{E_z} \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right) dz. \qquad (3.9)\end{aligned}$$

Moreover, by $|D_n\chi_E| = |\nu_n^E|\mathcal{H}^{n-1} \llcorner \partial^* E$ and co-area formula (3.4),

$$\begin{aligned}\int_{B_E\times\mathbb{R}} \frac{\nu_i^E(z,y)}{|\nu_n^E(z,y)|} g(z)\, e^{-|\sqrt{A}x|^2/2} d\,|D_n\chi_E| &= \int_{\partial^* E\cap(B_E\times\mathbb{R})} g(z)\, e^{-|\sqrt{A}x|^2/2}\nu_i^E(z,y)\, d\mathcal{H}^{n-1}\\ &= \int_{B_E} g(z) \int_{(\partial^* E)_z} \frac{\nu_i^E(z,y)}{|\nu_n^E(z,y)|} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^0(y)\ dz\\ &= \int_{B_E} g(z) \int_{(\partial^* E)_z} \frac{\nu_i^E(z,y)}{|\nu_n^E(z,y)|}\ d\mathcal{H}_z^0(y)\ dz \qquad (3.10)\end{aligned}$$

(see (3.1) for the definition of $\mathcal{H}_z^0$). Combining (3.9) and (3.10) together,

$$\int_{B_E} g(z)\, dD_i v_E = \int_{B_E} g(z)\left(\int_{(\partial^* E)_z} \frac{\nu_i^E(z,y)}{|\nu_n^E(z,y)|}\ d\mathcal{H}_z^0(y) + \int_{E_z} \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right) dz.$$

Since g is arbitrary, we have

$$D_i v_E \llcorner B_E = \left(\int_{(\partial^* E)_z} \frac{\nu_i^E(z,y)}{|\nu_n^E(z,y)|}\ d\mathcal{H}_z^0(y) + \int_{E_z} \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right) \mathcal{L}^{n-1} \llcorner B_E.$$

That is,

$$D_i v_E(z) = \int_{(\partial^* E)_z} \frac{\nu_i^E(z,y)}{|\nu_n^E(z,y)|}\ d\mathcal{H}_z^0(y) + \int_{E_z} \frac{\partial}{\partial z_i}\left(e^{-|\sqrt{A}x|^2/2}\right) dy, \qquad (3.11)$$

for $\mathcal{L}^{n-1}$-a.e. $z \in B_E$. □

With Lemma 3.2.1 in hand, we first show that $h : z \mapsto \phi_z^{-1}(v_E(z))$ is C^1 when v_E is C^1. The key idea is to use the integral equation (3.12) and the lower bound estimate of $e^{-|\sqrt{A}x|^2/2}$. We will prove the general case of v_E in Theorem 3.2.4.

Lemma 3.2.3 (Regularity Estimates for the map $z \mapsto \phi_z^{-1}(v(z))$)**.**
Let $x = (z, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and

$$\phi_z(t) = \int_{-\infty}^{t} e^{-|\sqrt{A}x|^2/2} dy.$$

Let Ω be an open set in $\mathbb{R}^{n-1}$ and $v \in C^1(\Omega)$ with $0 < v < g$, where $g(z) = \phi_z(\infty)$. Then the map $h : z \mapsto \phi_z^{-1}(v(z))$ is also $C^1(\Omega)$, and for all $z \in \Omega$, we have

$$\nabla' h(z) = e^{\frac{|\sqrt{A}(z,h(z))|^2}{2}}\left(\nabla' v(z) - \int_{-\infty}^{h(z)} \nabla'\left(e^{-|\sqrt{A}(z,y)|^2/2}\right) dy\right).$$

Moreover, if we assume that $\nabla' v$ is locally Lipschitz on Ω. Then

$$z \mapsto \nabla' h(z) \text{ is also locally Lipschitz on } \Omega.$$

Proof. Since $h(z) = \phi_z^{-1}(v(z))$, i.e., $\phi_z(h(z)) = v(z)$, we have

$$\int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy = v(z). \tag{3.12}$$

Step 1: Assume that $v \in C^0(\Omega)$. We first show that $h \in C^0(\Omega)$. For any $z_0 \in \Omega$,

$$\begin{aligned} v(z) - v(z_0) &= \int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy - \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z_0,y)|^2/2} dy \\ &= \int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy - \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z,y)|^2/2} dy \\ &\quad + \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z,y)|^2/2} dy - \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z_0,y)|^2/2} dy \\ &= \int_{h(z_0)}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy + \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z,y)|^2/2} - e^{-|\sqrt{A}(z_0,y)|^2/2} dy. \end{aligned}$$

By Lemma 3.2.1 (2)(a) and $v \in C^0(\Omega)$, as $z \to z_0$ in Ω,

$$\lim_{z \to z_0} \int_{h(z_0)}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy = 0$$

Now we claim that

$$h(z) \to h(z_0) \quad \text{as } z \to z_0.$$

Suppose not, there exists $\varepsilon_0 > 0$ and a sequence $z_k \to z_0$ in Ω such that

$$|h(z_k) - h(z_0)| \geq \varepsilon_0.$$

Then there exists a subsequence z_{k_n} such that

$$(1)\ h(z_{k_n}) \geq h(z_0) \text{ for all } n, \text{ or } (2)\ h(z_{k_n}) < h(z_0) \text{ for all } n.$$

We will only prove (1) since the proof of (2) is similar. For (1), we have

$$|h(z_{k_n}) - h(z_0)| \geq \varepsilon_0 \implies h(z_{k_n}) \geq h(z_0) + \varepsilon_0.$$

Hence, by Lemma 3.2.1 (1)(a),

$$\int_{h(z_0)}^{h(z_{k_n})} e^{-|\sqrt{A}(z,y)|^2/2}dy \geq e^{-\|\sqrt{A}\|^2|z_{k_n}|^2/2}\int_{h(z_0)}^{h(z_{k_n})} e^{-\|\sqrt{A}\|^2|y|^2/2}dy$$
$$\geq e^{-\|\sqrt{A}\|^2|z_{k_n}|^2/2}\int_{h(z_0)}^{h(z_0)+\varepsilon_0} e^{-\|\sqrt{A}\|^2|y|^2/2}dy.$$

Taking $n \to \infty$,

$$0 = \lim_{n\to\infty}\int_{h(z_0)}^{h(z_{k_n})} e^{-|\sqrt{A}(z,y)|^2/2}dy \geq \lim_{n\to\infty} e^{-\|\sqrt{A}\|^2|z_{k_n}|^2/2}\int_{h(z_0)}^{h(z_0)+\varepsilon_0} e^{-\|\sqrt{A}\|^2|y|^2/2}dy$$
$$\geq e^{-\|\sqrt{A}\|^2|z_0|^2/2}\int_{h(z_0)}^{h(z_0)+\varepsilon_0} e^{-\|\sqrt{A}\|^2|y|^2/2}dy > 0.$$

This gives us a contradiction. Therefore, $h \in C^0(\Omega)$.

Step 2: Now we assume that $v \in C^1(\Omega)$. Our goal is to show that $h \in C^1(\Omega)$.

Define

$$\ell(z) = e^{\frac{|\sqrt{A}(z,h(z))|^2}{2}}\left(\nabla' v(z) - \int_{-\infty}^{h(z)} \nabla'\left(e^{-|\sqrt{A}(z,y)|^2/2}\right)dy\right).$$

We show that h is differentiable and

$$\nabla' h(z) = \ell(z).$$

Using the mean value theorem, we have

$$\frac{1}{|k|}\left\{v(z+k) - v(z) - \nabla' v(z)\cdot k\right\}$$
$$= \frac{1}{|k|}\left\{\int_{-\infty}^{h(z+k)} e^{-|\sqrt{A}(z+k,y)|^2/2}dy - \int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2}dy - \nabla' v(z)\cdot k\right\}$$
$$= \frac{1}{|k|}\left\{\int_{h(z)}^{h(z+k)} e^{-|\sqrt{A}(z+k,y)|^2/2}dy + \int_{-\infty}^{h(z)}\left(e^{-|\sqrt{A}(z+k,y)|^2/2} - e^{-|\sqrt{A}(z,y)|^2/2}\right)dy - \nabla' v(z)\cdot k\right\}$$
$$= \frac{1}{|k|}\left\{\left(h(z+k) - h(z)\right)e^{-|\sqrt{A}(z+k,y(k))|^2/2}\right.$$
$$\left.+\int_{-\infty}^{h(z)}\left(e^{-|\sqrt{A}(z+k,y)|^2/2} - e^{-|\sqrt{A}(z,y)|^2/2}\right)dy - \nabla' v(z)\cdot k\right\}$$
$$= e^{-|\sqrt{A}(z+k,y(k))|^2/2}\frac{1}{|k|}\left(h(z+k) - h(z) - \ell(z)\cdot k\right) + \frac{1}{|k|}e^{-|\sqrt{A}(z+k,y(k))|^2/2}\ell(z)\cdot k$$

$$+\frac{1}{|k|}\left\{\int_{-\infty}^{h(z)}\left(e^{-|\sqrt{A}(z+k,y)|^2/2}-e^{-|\sqrt{A}(z,y)|^2/2}\right)dy-\nabla' v(z)\cdot k\right\}$$

where $y(k)$ lies between $h(z)$ and $h(z+k)$, and the continuity of h (Step 1) implies that

$$|y(k)-h(z)|\le|h(z+k)-h(z)| \implies y(k)\to h(z) \text{ as } k\to 0.$$

By using the definition of $\ell(z)$ and (3.5), we have

$$\begin{aligned}
&\frac{h(z+k)-h(z)-\ell(z)\cdot k}{|k|}\\
&=e^{|\sqrt{A}(z+k,y(k))|^2/2}\left(\frac{v(z+k)-v(z)-\nabla' v(z)\cdot k}{|k|}\right)-\frac{1}{|k|}\ell(z)\cdot k\\
&\quad -e^{|\sqrt{A}(z+k,y(k))|^2/2}\frac{1}{|k|}\left\{\int_{-\infty}^{h(z)}\left(e^{-|\sqrt{A}(z+k,y)|^2/2}-e^{-|\sqrt{A}(z,y)|^2/2}\right)dy-\nabla' v(z)\cdot k\right\}\\
&=e^{|\sqrt{A}(z+k,y(k))|^2/2}\left(\frac{v(z+k)-v(z)-\nabla' v(z)\cdot k}{|k|}\right)-\frac{1}{|k|}e^{|\sqrt{A}(z,h(z))|^2/2}\nabla' v(z)\cdot k\\
&\quad +\frac{1}{|k|}e^{|\sqrt{A}(z,h(z))|^2/2}\int_{-\infty}^{h(z)}\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)dy\cdot k\\
&\quad -e^{|\sqrt{A}(z+k,y(k))|^2/2}\frac{1}{|k|}\left\{\int_{-\infty}^{h(z)}\left(e^{-|\sqrt{A}(z+k,y)|^2/2}-e^{-|\sqrt{A}(z,y)|^2/2}\right)dy-\nabla' v(z)\cdot k\right\}\\
&=e^{|\sqrt{A}(z+k,y(k))|^2/2}\left(\frac{v(z+k)-v(z)-\nabla' v(z)\cdot k}{|k|}\right)\\
&\quad +\frac{1}{|k|}\left(e^{|\sqrt{A}(z+k,y(k))|^2/2}-e^{|\sqrt{A}(z,h(z))|^2/2}\right)\nabla' v(z)\cdot k\\
&\quad -e^{|\sqrt{A}(z+k,y(k))|^2/2}\int_{-\infty}^{h(z)}\left(\frac{e^{-|\sqrt{A}(z+k,y)|^2/2}-e^{-|\sqrt{A}(z,y)|^2/2}-\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)\cdot k}{|k|}\right)dy\\
&\quad -\left(e^{|\sqrt{A}(z+k,y(k))|^2/2}-e^{|\sqrt{A}(z,h(z))|^2/2}\right)\frac{1}{|k|}\int_{-\infty}^{h(z)}\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)dy\cdot k\to 0
\end{aligned}$$

as $k\to 0$. Therefore, h is differentiable and

$$\nabla' h(z)=e^{\frac{|\sqrt{A}(z,h(z))|^2}{2}}\left(\nabla' v(z)-\int_{-\infty}^{h(z)}\nabla'\left(e^{-|\sqrt{A}(z,y)|^2/2}\right)dy\right).$$

Now we claim that $\nabla' h\in C^0(\Omega)$. Since $\nabla' v\in C^0(\Omega)$ and $h\in C^0(\Omega)$, we just need to show that

$$z\mapsto\int_{-\infty}^{h(z)}\nabla'\left(e^{-|\sqrt{A}(z,y)|^2/2}\right)dy \quad \text{is in } C^0(\Omega). \tag{3.13}$$

Without loss of generality, we may assume that $|z-z_0| \leq 1$ and let $K = \overline{B}(z_0, 1)$.

$$\begin{aligned}
&\int_{-\infty}^{h(z)} \nabla' \left(e^{-|\sqrt{A}(z,y)|^2/2}\right) dy - \int_{-\infty}^{h(z_0)} \nabla' \left(e^{-|\sqrt{A}(z,y)|^2/2}\right) \Big|_{z=z_0} dy \qquad (3.14)\\
&= -\int_{-\infty}^{h(z)} e^{-|\sqrt{A}x|^2/2} A'(z,y)\ dy + \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z_0,y)|^2/2} A'(z_0,y)\ dy \\
&= -A' \left(\int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2}(z,y)\ dy - \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z_0,y)|^2/2}(z_0,y)\ dy \right) \\
&= -A' \Bigg[\left(\int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2}(z,y)\ dy - \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z,y)|^2/2}(z,y)\ dy \right) \\
&\quad + \left(\int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z,y)|^2/2}(z,y)\ dy - \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z_0,y)|^2/2}(z_0,y)\ dy \right) \Bigg] := (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

(i) We first estimate (II).

$$\begin{aligned}
&\left| \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z,y)|^2/2} A'(z,y)\ dy - \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z_0,y)|^2/2} A'(z_0,y)\ dy \right| \\
&\leq \int_{-\infty}^{h(z_0)} \left| e^{-|\sqrt{A}(z,y)|^2/2} - e^{-|\sqrt{A}(z_0,y)|^2/2} \right| \left| A'(z,y) \right| dy \\
&\quad + \int_{-\infty}^{h(z_0)} e^{-|\sqrt{A}(z_0,y)|^2/2} \left| A'(z-z_0, 0) \right| dy \\
&\leq \lambda_{\max}(A'^{\mathsf{T}} A') \int_{-\infty}^{\infty} \Big(r(K) + |y| \Big)^2 e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} |z-z_0| dy \\
&\quad + \sqrt{\lambda_{\max}(A'^{\mathsf{T}} A')} \int_{-\infty}^{\infty} e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} \, |z-z_0|\, dy \\
&= C(K,A)\, |z-z_0|
\end{aligned}$$

where we have used (3.6) and recall that $r(K) = \sup_{\zeta \in K} |\zeta|$.

(ii) We now estimate (I). By the mean value theorem,

$$\begin{aligned}
&\left| \int_{h(z_0)}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2}(z,y)\ dy \right| \\
&= \left| \left(\int_{h(z_0)}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} z\ dy, \int_{h(z_0)}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} y\ dy \right) \right| \\
&\leq \left| \int_{h(z_0)}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} z\ dy \right| + \left| \int_{h(z_0)}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} y\ dy \right|
\end{aligned}$$

$$= \left| e^{-|\sqrt{A}(z,y_z)|^2/2} z \right| |h(z) - h(z_0)| + \left| e^{-|\sqrt{A}(z,\tilde{y}_z)|^2/2} \tilde{y}_z \right| |h(z) - h(z_0)| \to 0 \tag{3.15}$$

where y_z and $\tilde{y}_z$ lies between $h(z)$ and $h(z_0)$, and by the continuity of h, $y_z, \tilde{y}_z \to h(z_0)$ as $z \to z_0$. Thus, (I) $\to 0$ and (II) $\to 0$. Therefore, $h \in C^1(\Omega)$.

Step 3: Finally, we claim that if $\nabla' v$ is locally Lipschitz on Ω,

$$\nabla' h \text{ is also locally Lipschitz on } \Omega.$$

By Lemma 3.2.1 (2)(c), we just need to show that (3.13) is locally Lipschitz on Ω. Let $z^* \in \Omega$. Since Ω is open, there exists $B(z^*, r) \subset \Omega$ s.t. $\overline{B}(z^*, r) \subset \Omega$. Then $K := \overline{B}(z^*, r)$ is a convex compact set. Thanks to the estimates in Step 2, we are left to estimate (3.15) with $z, z_0 \in B(z^*, r) \subset K$. Since $h \in C^1(\Omega)$ and $|\tilde{y}_z| \leq \|h\|_{L^\infty(K)}$,

$$(3.15) \leq r(K)\|\nabla h\|_{L^\infty(K)}|z - z_0| + \|h\|_{L^\infty(K)}\|\nabla h\|_{L^\infty(K)}|z - z_0|,$$

i.e., (3.13) is Lipschitz on $B(z^*, r)$. □

By Lemma 3.2.2, we know that $v_E \in BV(\mathbb{R}^{n-1})$. Now we are ready to show that $h(z) = \phi_z^{-1}(v_E(z))$ is $\mathcal{L}^{n-1}$-measurable and E^s is a set of locally finite perimeter in $\mathbb{R}^n$. The key idea is to approximate $v_E \in BV(\mathbb{R}^{n-1})$ by C^1 functions. Then we can apply Lemma 3.2.3 on each C^1 function.

Theorem 3.2.4 (Approximation theorem for the map $z \mapsto \phi_z^{-1}(v_E(z))$).
Let $n \geq 2$ and let E be a set of finite A-anisotropic Gaussian perimeter in $\mathbb{R}^n$. Define $h : \mathbb{R}^{n-1} \to [-\infty, \infty]$ as $h(z) = \phi_z^{-1}(v_E(z))$ and let $g(z) = \phi_z(\infty)$. Then

(1) for any bounded open set $\Omega \subset \mathbb{R}^{n-1}$ with smooth boundary, there exists a sequence of functions $v_k \in C^1(\Omega) \cap BV(\Omega)$ with $0 < v_k < g$, such that $v_k \to v_E$ in $L^1(\Omega)$ and a.e. in Ω, $Dv_k \overset{*}{\rightharpoonup} Dv_E$ in Ω, and

$$\limsup_{k\to\infty} \int_\Omega |\nabla' v_k| \, dz \leq |Dv_E|(\Omega) + 2\int_\Omega \left(\frac{|\nabla' g|}{g}\right) v_E \, dz.$$

Moreover,

$$h_k \to h \text{ a.e. in } \Omega, \quad \chi_{F_k} \to \chi_{E^s} \text{ a.e. in } \Omega \times \mathbb{R}$$

where $h_k(z) := \phi_z^{-1}(v_k(z))$ *and*

$$F_k := \left\{(z,y) \in \Omega \times \mathbb{R} : y < h_k(z) := \phi_z^{-1}(v_k(z))\right\}.$$

In particular, h *is* $\mathcal{L}^{n-1}$*-measurable and the set* E^s *is* $\mathcal{L}^n$*-measurable.*

(2) $E^s = E^s_{A,-e_n}$ *is a set of locally finite perimeter in* $\mathbb{R}^n$*.*

Proof. (1) **Step 1**: Since E is a set of finite anisotropic Gaussian perimeter, according to Lemma 3.2.2, $v_E \in BV(\mathbb{R}^{n-1})$. Let $g(z) = \phi_z(\infty) > 0$ (see definition (3.2)) and define

$$\widetilde{\phi}_z(t) = \frac{1}{g(z)}\phi_z(t), \quad \widetilde{v}_E(z) = \frac{1}{g(z)}v_E(z).$$

Notice that $0 \le \widetilde{v}_E(z) \le 1$ and $\widetilde{\phi}_z^{-1} : (0,1) \to \mathbb{R}$. By Lemma 3.2.1 (2), $g \in C^1(\mathbb{R}^{n-1})$. Therefore, $\frac{1}{g} \in C^1(\mathbb{R}^{n-1})$ since $\nabla'(\frac{1}{g}) = -\frac{\nabla' g}{g^2} \in C^0$ and $g > 0$. In particular, both g and $\frac{1}{g}$ are locally Lipschitz on $\mathbb{R}^{n-1}$. Thus, $\widetilde{v}_E = \frac{1}{g}v_E \in BV_{\mathrm{loc}}(\mathbb{R}^{n-1})$ and

$$Dv_E = D(g\widetilde{v}_E) = gD\widetilde{v}_E + (\nabla' g)\widetilde{v}_E\mathcal{L}^{n-1} \tag{3.16}$$

(see [AFP00], Proposition 3.2). Let Ω be a bounded open set in $\mathbb{R}^{n-1}$. We have $\widetilde{v}_E \in BV(\Omega)$ and $0 \le \widetilde{v}_E \le 1$. Hence there exists a sequence of functions $\widetilde{v}_k \in C^1(\Omega) \cap BV(\Omega)$ with $0 < \widetilde{v}_k < 1$, such that $\widetilde{v}_k \to \widetilde{v}_E$ in $L^1(\Omega)$ and a.e. in Ω, $D\widetilde{v}_k \overset{*}{\rightharpoonup} D\widetilde{v}_E$ in Ω, and

$$\lim_{k\to\infty} \int_\Omega \left|\nabla'\widetilde{v}_k(z)\right| dz = \left|D\widetilde{v}_E\right|(\Omega) \tag{3.17}$$

(see [AFP00], Theorem 3.9 and Proposition 3.13). Now we let

$$v_k = g\widetilde{v}_k, \quad h_k = \widetilde{\phi}_z^{-1}(\widetilde{v}_k) = \phi_z^{-1}(v_k),$$

where $0 < v_k(z) < g(z) = \phi_z(\infty)$ and $v_k \in C^1(\Omega) \cap BV(\Omega)$. In particular, by the definition of v_k and $\|g\|_{L^\infty(\mathbb{R}^{n-1})} < \infty$, we have $v_k \to v_E$ in $L^1(\Omega)$ and a.e. in Ω. Moreover,

$$Dv_k = D(g\widetilde{v}_k) = gD\widetilde{v}_k + (\nabla' g)\widetilde{v}_k\mathcal{L}^{n-1}. \tag{3.18}$$

By (3.16), (3.18), and the fact that $\widetilde{v}_k \to \widetilde{v}_E$ in $L^1(\Omega)$ and $D\widetilde{v}_k \overset{*}{\rightharpoonup} D\widetilde{v}_E$ in Ω, we obtain

$$Dv_k \overset{*}{\rightharpoonup} Dv_E \text{ in } \Omega, \tag{3.19}$$

$|Dv_k|(\Omega) \leq g|D\widetilde{v}_k|(\Omega) + |\nabla' g|\widetilde{v}_k \mathcal{L}^{n-1}(\Omega)$, and $g|D\widetilde{v}_E|(\Omega) \leq |Dv_E|(\Omega) + |\nabla' g|\widetilde{v}_E \mathcal{L}^{n-1}(\Omega)$. Applying [AFP00], Theorem 2.39 (Reshetnyak continuity) with the bounded continuous function g and (3.17),

$$\begin{aligned}
\limsup_{k\to\infty} \int_\Omega |\nabla' v_k|\, dz &\leq \limsup_{k\to\infty} \int_\Omega g\, d|D\widetilde{v}_k| + \limsup_{k\to\infty} \int_\Omega |\nabla' g|\widetilde{v}_k\, dz \\
&= \int_\Omega g\, d|D\widetilde{v}_E| + \int_\Omega |\nabla' g|\widetilde{v}_E\, dz \\
&\leq |Dv_E|(\Omega) + 2\int_\Omega |\nabla' g|\widetilde{v}_E\, dz = |Dv_E|(\Omega) + 2\int_\Omega |\nabla' g|\frac{v_E}{g}\, dz,
\end{aligned}$$

where we have used $\widetilde{v}_k \to \widetilde{v}_E$ in $L^1(\Omega)$ and $\|\nabla' g\|_{L^\infty(\mathbb{R}^{n-1})} < \infty$.

Step 2: Since $v_k \to v_E$ a.e. in Ω, there exists a measure zero set $Z \subset \Omega$ such that for any $z \in \Omega \setminus Z$, $v_k(z) \to v_E(z)$. Notice that

$$v_k(z) - v_E(z) = \int_{-\infty}^{h_k(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy - \int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy = \int_{h(z)}^{h_k(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy.$$

As $k \to \infty$, we have

$$\lim_{k\to\infty} \int_{h(z)}^{h_k(z)} e^{-|\sqrt{A}(z,y)|^2/2} dy = 0 \quad \text{for any } z \in \Omega \setminus Z.$$

By using the same argument as in Lemma 3.2.3 Step 1, we have for any $z \in \Omega \setminus Z$, $h_k(z) \to h(z)$ as $k \to \infty$. By Lemma 3.2.3, $h_k \in C^1(\Omega)$ since $v_k \in C^1(\Omega)$. In particular, h_k is $\mathcal{L}^{n-1} \llcorner \Omega$ -measurable on Ω and hence the limit function $h|_\Omega$ is also $\mathcal{L}^{n-1} \llcorner \Omega$ -measurable on Ω. Since Ω is arbitrary, h is also $\mathcal{L}^{n-1}$-measurable. Next we show that

$$\chi_{F_k} \to \chi_{E^s} \text{ a.e. in } \Omega \times \mathbb{R}.$$

Let $\Gamma(h)$ be the graph of h, let $Z' = (Z \times \mathbb{R}) \cup \Gamma(h;\Omega) \subset \Omega \times \mathbb{R}$, where Z is the measure zero set from above and $\Gamma(h;\Omega)$ is the graph of h over Ω, i.e.,

$$\Gamma(h;\Omega) = \{(z,y) \in \Omega \times \mathbb{R} : y = h(z)\} = \Gamma(h) \cap (\Omega \times \mathbb{R}).$$

We first check that $\mathcal{L}^n(Z') = 0$. Notice that $\mathcal{L}^n(Z \times \mathbb{R}) = 0$. It is enough to show that $\mathcal{L}^n(\Gamma(h)) = 0$. Since h is $\mathcal{L}^{n-1}$-measurable, we define

$$g(z, y) := f_2 \circ f_1(z, y) = h(z) - y$$

where $f_1 : (z, y) \mapsto (h(z), y)$ is $\mathcal{L}^n$-measurable, $f_2 : (x, y) \mapsto x - y$ is continuous. Therefore, g is also $\mathcal{L}^n$-measurable and hence $\Gamma(h) = g^{-1}(\{0\})$ is $\mathcal{L}^n$-measurable. By Fubini's theorem, $\mathcal{L}^n(\Gamma(h)) = 0$. Next, for any $x = (z, y) \in \Omega \times \mathbb{R} \setminus Z'$, we have $z \notin Z$. If $\chi_{E^s}(x) = 1$, then $y < h(z)$ and hence there exists k such that $y < h_k(z)$ since $h_k(z) \to h(z)$. That is, $x = (z, y) \in F_k$, i.e., $\chi_{F_k}(x) = 1$ and $\chi_{F_k} \to \chi_{E^s}$. If $\chi_{E^s}(x) = 0$, then $y \geq h(z)$. However, $x \notin \Gamma(h; \Omega)$, so $y \neq h(z)$ and there exists k such that $y > h_k(z)$. That is, $x = (z, y) \notin F_k$, i.e., $\chi_{F_k}(x) = 0$ and $\chi_{F_k} \to \chi_{E^s}$. Therefore, $\chi_{F_k} \to \chi_{E^s}$ a.e. in $\Omega \times \mathbb{R}$. In particular, $E^s = \{g > 0\}$ is $\mathcal{L}^n$-measurable.

(2) Now we claim that E^s is a set of locally finite perimeter in $\mathbb{R}^n$. Since E^s is $\mathcal{L}^n$-measurable, we have $\chi_{E^s} \in L^1_{loc}(\mathbb{R}^n)$. In order to show $\chi_{E^s} \in BV_{loc}(\mathbb{R}^{n-1})$, by Proposition 2.3.2 and 2.3.1, we just need to prove that for any open set $V \subset\subset \mathbb{R}^n$,

$$\sup \left\{ \int_{E^s} \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle \, d\gamma_A(x) : \varphi \in C^1_c\left(V; \mathbb{R}^n\right), |\varphi| \leq 1 \right\} < \infty.$$

Since $\overline{V} \subset \mathbb{R}^n$ is a compact set, there exists an open bounded set $\Omega \subset \mathbb{R}^{n-1}$ such that $V \subset \Omega \times \mathbb{R}$. In fact, we claim that

$$\sup \left\{ \int_{E^s} \operatorname{div} \varphi(x) - \langle \varphi(x), Ax \rangle \, d\gamma_A(x) : \varphi \in C^1_c\left(\Omega \times \mathbb{R}; \mathbb{R}^n\right), |\varphi| \leq 1 \right\} < \infty. \tag{3.20}$$

For convenience, let $\varphi_z = (\varphi_1, \varphi_2, ..., \varphi_{n-1})$ and

$$\begin{aligned}
&\int_{E^s} \operatorname{div} \varphi - \langle \varphi, Ax \rangle \, d\gamma_A(x) \\
&= \int_{E^s} \operatorname{div} \varphi + \langle \varphi, \nabla(-|\sqrt{A}x|^2/2) \rangle \, d\gamma_A(x) \\
&= \sum_{i=1}^{n-1} \int_{E^s} \frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i} \, d\gamma_A(x) + \int_{E^s} \frac{\partial \varphi_n}{\partial y} + \varphi_n \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial y} \, d\gamma_A(x) \\
&:= \text{(I)} + \text{(II)}.
\end{aligned} \tag{3.21}$$

Step 1: We estimate (I) using approximation of v_E by $C^1(\Omega)$ functions. Consider $\widetilde{v} \in C^1(\Omega)$ with $0 < \widetilde{v} < g$ and let $\varphi \in C_c^1(\Omega \times \mathbb{R}; \mathbb{R}^n)$ such that $|\varphi| \leq 1$. Let

$$F := \{x = (z, y) \in \Omega \times \mathbb{R} : y < y(z)\}, \quad y(z) = \phi_z^{-1}(\widetilde{v}(z)).$$

Since $y(z) = \phi_z^{-1}(\widetilde{v}(z))$, i.e.,

$$\tilde{v}(z) = \phi_z(y(z)) = \int_{-\infty}^{y(z)} e^{-|\sqrt{A}x|^2/2} dy,$$

and by Lemma 3.2.3, the mapping $z \mapsto y(z)$ is in $C^1(\Omega)$. Hence

$$\nabla' \widetilde{v}(z) = e^{-\frac{|\sqrt{A}(z,y(z))|^2}{2}} \nabla' y(z) + \int_{-\infty}^{y(z)} \nabla' \left(e^{-|\sqrt{A}x|^2/2}\right) dy,$$

$$\frac{\partial \widetilde{v}}{\partial z_i} = e^{-\frac{|\sqrt{A}(z,y(z))|^2}{2}} \frac{\partial y(z)}{\partial z_i} + \int_{-\infty}^{y(z)} \frac{\partial}{\partial z_i} \left(e^{-|\sqrt{A}(z,y)|^2/2}\right) dy.$$

We now compute the equivalent of (I).

$$\begin{aligned}
&\sum_{i=1}^{n-1} \int_F \frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i} \, d\gamma_A(x) \\
&= \sum_{i=1}^{n-1} \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_\Omega \int_{-\infty}^{y(z)} \left(\frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i} \right) e^{-|\sqrt{A}x|^2/2} dy \, dz \\
&= \sum_{i=1}^{n-1} \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_\Omega \left(\frac{\partial}{\partial z_i} \int_{-\infty}^{y(z)} \varphi_i e^{-|\sqrt{A}x|^2/2} dy - \frac{\partial y(z)}{\partial z_i} \varphi_i(z, y(z)) e^{-|\sqrt{A}(z,y(z)|^2/2} \right) dz \\
&= -\sum_{i=1}^{n-1} \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_\Omega \frac{\partial y(z)}{\partial z_i} \varphi_i(z, y(z)) e^{-|\sqrt{A}(z,y(z)|^2/2} \, dz \\
&= -\sum_{i=1}^{n-1} \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_\Omega \varphi_i(z, y(z)) \left(\frac{\partial \widetilde{v}}{\partial z_i} - \int_{-\infty}^{y(z)} \frac{\partial}{\partial z_i} \left(e^{-|\sqrt{A}(z,y)|^2/2}\right) dy \right) dz \\
&= -\frac{\det \sqrt{A}}{(2\pi)^{n/2}} \left(\int_\Omega \varphi_z(z, y(z)) \cdot \nabla' \widetilde{v}(z) \, dz \right. \\
&\quad \left. - \int_\Omega \int_{-\infty}^{y(z)} \varphi_z(z, y(z)) \cdot \nabla' \left(e^{-|\sqrt{A}(z,y)|^2/2}\right) dy dz \right) \qquad (3.22)
\end{aligned}$$

where we have used the divergence theorem and φ has compact support in $\Omega \times \mathbb{R}$.

Now we approximate v_E by Theorem 3.2.4 (1), i.e., there exists a sequence of functions $v_k \in C^1(\Omega)\cap BV(\Omega)$ with $0 < v_k < g$, such that $v_k \to v_E$ in $L^1(\Omega)$ and a.e. in Ω, $Dv_k \overset{*}{\rightharpoonup} Dv_E$ in Ω, and

$$\limsup_{k\to\infty} \int_\Omega |\nabla' v_k|\, dz \le |Dv_E|(\Omega) + 2\int_\Omega \left(\frac{|\nabla' g|}{g}\right) v_E\, dz.$$

Moreover, $\chi_{F_k} \to \chi_{E^s}$ a.e. in $\Omega \times \mathbb{R}$ and $h_k(z) \to h(z) := \phi_z^{-1}(v_E(z))$ a.e. in Ω, where

$$F_k := \left\{(z,y) \in \Omega \times \mathbb{R} : y < h_k(z) := \phi_z^{-1}(v_k(z))\right\}.$$

Replacing F as F_k and $y(z)$ as $h_k(z)$ in equation (3.22), we have the following estimate

$$\begin{aligned}
\text{(I)} &= \sum_{i=1}^{n-1} \int_{E^s} \frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i}\, d\gamma_A(x) \\
&= \lim_{k\to\infty} \Big(\sum_{i=1}^{n-1} \int_{F_k} \frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i}\, d\gamma_A(x) \Big) \\
&= \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \Bigg(\int_\Omega \big(-\varphi_z(z,h_k(z))\big)\cdot \nabla' v_k(z) dz \\
&\quad - \int_{\Omega_j} \int_{-\infty}^{h_k(z)} \big(-\varphi_z(z,h_k(z))\big) \cdot \nabla' \left(e^{-|\sqrt{A}x|^2/2}\right) dy dz \Bigg) \\
&\le \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \limsup_{k\to\infty} \int_\Omega \left|\nabla' v_k(z)\right| dz + \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_\Omega \int_{-\infty}^{h(z)} \varphi_z(z,h(z)) \cdot \nabla' \left(e^{-|\sqrt{A}x|^2/2}\right) dy dz \\
&\le \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \left|Dv_E\right|(\Omega) + \frac{2\det\sqrt{A}}{(2\pi)^{n/2}} \int_\Omega \left(\frac{|\nabla' g|}{g}\right) v_E\, dz + \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \left|e^{-|\sqrt{A}x|^2/2} A' x\right| dy dz \\
&\le \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \left|Dv_E\right|(\Omega) + \frac{2\det\sqrt{A}}{(2\pi)^{n/2}} \int_\Omega |\nabla' g|\, dz + \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \left|e^{-|\sqrt{A}x|^2/2} A' x\right| dy dz \\
&\le \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \left|Dv_E\right|(\Omega) + \frac{3\det\sqrt{A}}{(2\pi)^{n/2}} \sqrt{\lambda_{\max}(A'^{\mathsf{T}} A')} \int_{\mathbb{R}^n} e^{-\|\sqrt{A}\|^{-2}|x|^2/2} |x|\, dx < \infty,
\end{aligned}$$

where we have used $0 \le v_E \le g$ and Lemma 3.2.1.

Step 2: Now we estimate (II).

$$\begin{aligned}
\text{(II)} &= \int_{E^s} \frac{\partial \varphi_n}{\partial y} + \varphi_n \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial y} d\gamma_A(x) \\
&= \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{E^s} \frac{\partial}{\partial y} \left(\varphi_n e^{-|\sqrt{A}(z,y)|^2/2}\right) dx
\end{aligned}$$

$$= \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \int_{F_k} \frac{\partial}{\partial y} \left(\varphi_n e^{-|\sqrt{A}(z,y)|^2/2} \right) dx$$
$$= \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \int_{\Omega} \int_{-\infty}^{h_k(z)} \frac{\partial}{\partial y} \left(\varphi_n e^{-|\sqrt{A}(z,y)|^2/2} \right) dydz$$
$$= \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \int_{\Omega} \varphi_n(z, h_k(z)) e^{-|\sqrt{A}(z,h_k(z))|^2/2} \, dz$$
$$= \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega} \varphi_n(z, h(z)) e^{-|\sqrt{A}(z,h(z))|^2/2} \, dz \le \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \mathcal{L}^{n-1}(\Omega) < \infty$$

since φ is C^1 with compact support, $|\varphi| \le 1$, and recall that $h_k \to h$ a.e. in Ω by Theorem 3.2.4 (1). □

3.3 Ehrhard symmetrization in 1D and an example in 2D

In this section, we show that the Ehrhard symmetrization preserves the mass under any one-dimensional slices in the y-direction, and hence it preserves the total mass. Moreover, the anisotropic Gaussian perimeters of the one-dimensional sections decrease under Ehrhard symmetrization. Geometrically, we will rearrange the mass in each one-dimensional section of E to a half-line with the same mass. The resulting new shape E^s is the Ehrhard symmetrization of E.

Proposition 3.3.1. *Let $n \ge 2$ and let E be a set of finite anisotropic Gaussian perimeter in $\mathbb{R}^n$.*

(1) **(Properties for E^s)**

$$v_{E^s}(z) = v_E(z), \qquad \mu_z\left((E^s)_z\right) = \mu_z\left(E_z\right),$$

for all $z \in \mathbb{R}^{n-1}$. Hence $\pi_+(E^s) = \pi_+(E)$ and

$$\nabla' v_E(z) = \nabla' v_{E^s}(z)$$

for a.e. $z \in B_E \cap B_{E^s}$, where

$$\nabla' v_E(z) := \left(D_1 v_E(z), \cdots, D_{n-1} v_E(z)\right), \quad D_i v_E := \frac{dD_i v_E \llcorner B_E}{d\mathcal{L}^{n-1} \llcorner B_E}.$$

Moreover, $\gamma_A(E) = \gamma_A(E^s)$ *and*

$$\gamma_A(E_1 \Delta E_2) \geq \gamma_A(E_1^s \Delta E_2^s).$$

In particular, for any sequence of sets of finite anisotropic Gaussian perimeter E_k *with* $\chi_{E_k} \to \chi_E$ *in* $L^1(\mathbb{R}^n, \gamma_A)$*, we have*

$$\chi_{E_k^s} \to \chi_{E^s} \text{ in } L^1(\mathbb{R}^n, \gamma_A),$$

and

$$P_{\gamma_A}(E^s; U) \leq \liminf_{k \to \infty} P_{\gamma_A}(E_k^s; U) \quad \text{for any open set } U \subset \mathbb{R}^n.$$

(2) **(Cross terms estimate)**

For any $z \in \mathbb{R}^{n-1}$,

$$\int_{E_z} y \; d\mu_z(y) \geq \int_{E_z^s} y \; d\mu_z(y),$$

and

$$\begin{aligned} &\left| \int_{E_z} \nabla' e^{-|\sqrt{A}x|^2/2} dy - \int_{E_z^s} \nabla' e^{-|\sqrt{A}x|^2/2} dy \right| \\ &= \left(\int_{E_z} y \; d\mu_z(y) - \int_{E_z^s} y \; d\mu_z(y) \right) \|Ae_n - \langle Ae_n, e_n \rangle e_n\|. \end{aligned}$$

(3) **(Ehrhard symmetrization in 1D)**

The anisotropic Gaussian perimeter of almost every one-dimensional section in y-direction decreases under Ehrhard symmetrization, i.e.,

$$p_{E^s}(z) \leq p_E(z),$$

for all $z \in B_E \cap B_{E^s}$*, where* $p_E : \mathbb{R}^{n-1} \to [0, \infty]$ *is defined as*

$$p_E(z) = \mathcal{H}_z^0 \left[(\partial^M E)_z \right].$$

Proof. (1) Notice that

$$(E^s)_z = \left\{ y : y < \phi_z^{-1}\left(v_E(z)\right) \right\} = \left(-\infty, \phi_z^{-1}\left(v_E(z)\right) \right) \text{ is a measurable set.}$$

Therefore, for any $z \in \mathbb{R}^{n-1}$,

$$v_{E^s}(z) = \mu_z\left((E^s)_z\right) = \mu_z\left(-\infty, \phi_z^{-1}\left(v_E(z)\right)\right)$$
$$= \int_{-\infty}^{\phi_z^{-1}\left(v_E(z)\right)} e^{-|Ax|^2/2}\, dy = \phi_z\left(\phi_z^{-1}\left(v_E(z)\right)\right) = v_E(z).$$

Thus,

$$\pi_+(E) = \{z \in \mathbb{R}^{n-1} : v_E(z) > 0\} = \{z \in \mathbb{R}^{n-1} : v_{E^s}(z) > 0\} = \pi_+(E^s).$$

Moreover, for any measurable set A,

$$D_i v_E \llcorner B_E(A) = \int_A D_i v_E(z)\, d\mathcal{L}^{n-1} \llcorner B_E(z),$$

and

$$D_i v_{E^s} \llcorner B_{E^s}(A) = \int_A D_i v_{E^s}(z)\, d\mathcal{L}^{n-1} \llcorner B_{E^s}(z).$$

Therefore, for any measurable set $B \subset B_E \cap B_{E^s}$, setting $A = B$ in the above equations, we have

$$\int_B D_i v_E(z)\, d\mathcal{L}^{n-1}(z) = D_i v_E(B) = D_i v_{E^s}(B) = \int_B D_i v_{E^s}(z)\, d\mathcal{L}^{n-1}(z)$$

where the second equality holds since $v_E = v_{E^s}$. By the arbitrariness of B, for a.e. $z \in B_E \cap B_{E^s}$,

$$D_i v_E(z) = D_i v_{E^s}(z).$$

By Fubini's theorem,

$$\gamma_A(E) = \frac{|\det\sqrt{A}|}{(2\pi)^{n/2}} \int_E e^{-|\sqrt{A}x|^2/2}\, dx = \frac{|\det\sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{E_z} e^{-|\sqrt{A}x|^2/2}\, dy\, dz$$
$$= \frac{|\det\sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} v_E(z)\, dz = \frac{|\det\sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} v_{E^s}(z)\, dz$$
$$= \frac{|\det\sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{E_z^s} e^{-|\sqrt{A}x|^2/2}\, dy\, dz = \gamma_A(E^s).$$

Similarly, we have

$$\gamma_A(E_1 \Delta E_2) = \frac{|\det\sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{(E_1 \Delta E_2)_z} e^{-|\sqrt{A}x|^2/2}\, dy\, dz$$

$$= \frac{|\det \sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{(E_1)_z \Delta (E_2)_z} e^{-|\sqrt{A}x|^2/2}\, dy\, dz$$
$$\geq \frac{|\det \sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \left| \int_{(E_1)_z} e^{-|\sqrt{A}x|^2/2}\, dy - \int_{(E_2)_z} e^{-|\sqrt{A}x|^2/2}\, dy \right| dz$$
$$= \frac{|\det \sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \left| v_{E_1}(z) - v_{E_2}(z) \right| \, dz$$
$$= \frac{|\det \sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \left| v_{(E_1)^s}(z) - v_{(E_2)^s}(z) \right| \, dz$$
$$= \frac{|\det \sqrt{A}|}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \int_{(E_1^s)_z \Delta (E_2^s)_z} e^{-|\sqrt{A}x|^2/2}\, dy\, dz = \gamma_A(E_1^s \Delta E_2^s).$$

For any sequence of measurable sets E_k with $\chi_{E_k} \to \chi_E$ in $L^1(\mathbb{R}^n, \gamma_A)$, we have

$$\int_{\mathbb{R}^n} \left| \chi_{E_k^s} - \chi_{E^s} \right| d\gamma_A = \gamma_A(E_k^s \Delta E^s) \leq \gamma_A(E_k \Delta E) = \int_{\mathbb{R}^n} \left| \chi_{E_k} - \chi_E \right| d\gamma_A \to 0.$$

By Theorem 3.2.4 and Proposition 2.3.2, E_k^s and E^s are sets of locally finite perimeter and hence

$$P_{\gamma_A}(E^s; U) \leq \liminf_{k\to\infty} P_{\gamma_A}(E_k^s; U) \quad \text{for any open set } U \subset \mathbb{R}^n.$$

(2) Notice that

$$\mu_z(E_z^s \setminus E_z) + \mu_z(E_z^s \cap E_z) = \mu_z(E_z^s) = v_{E^s}(z) = v_E(z) = \mu_z(E_z) = \mu_z(E_z \setminus E_z^s) + \mu_z(E_z \cap E_z^s).$$

That is, $\mu_z(E_z^s \setminus E_z) = \mu_z(E_z \setminus E_z^s)$. Let $y(z) = \phi_z^{-1}(v_E(z))$, we have

$$y - y(z) < 0 \quad \text{if } y \in E_z^s, \quad y - y(z) \geq 0 \quad \text{if } y \notin E_z^s.$$

Now we are ready to show that

$$\int_{E_z^s} y\, d\mu_z(y) \leq \int_{E_z} y\, d\mu_z(y).$$

Notice that

$$\int_{E_z^s} y\, d\mu_z(y) - \int_{E_z} y\, d\mu_z(y)$$
$$= \left(\int_{E_z^s \setminus E_z} y\, d\mu_z(y) + \int_{E_z^s \cap E_z} y\, d\mu_z(y) \right) - \left(\int_{E_z \setminus E_z^s} y\, d\mu_z(y) + \int_{E_z \cap E_z^s} y\, d\mu_z(y) \right)$$

$$
\begin{aligned}
&= \int_{E_z^s \setminus E_z} y \; d\mu_z(y) - \int_{E_z \setminus E_z^s} y \; d\mu_z(y) \\
&= \left(\int_{E_z^s \setminus E_z} y - y(z) \; d\mu_z(y) + \int_{E_z^s \setminus E_z} y(z) \; d\mu_z(y) \right) \\
&\quad + \left(\int_{E_z \setminus E_z^s} y(z) - y \; d\mu_z(y) - \int_{E_z \setminus E_z^s} y(z) \; d\mu_z(y) \right) \\
&\leq \left(\int_{E_z^s \setminus E_z} y(z) \; d\mu_z(y) - \int_{E_z \setminus E_z^s} y(z) \; d\mu_z(y) \right) = y(z) \Big(\mu_z(E_z^s \setminus E_z) - \mu_z(E_z \setminus E_z^s) \Big) = 0.
\end{aligned}
$$

On the other hand, by Lemma 3.2.1 (1), we have

$$
\partial_{z_k} \left(e^{-|\sqrt{A}x|^2/2} \right) = -e^{-|\sqrt{A}x|^2/2} \langle \mathrm{row}_k(A), x \rangle,
$$

and hence

$$
\begin{aligned}
\int_{E_z} \partial_{z_k} \left(e^{-|\sqrt{A}x|^2/2} \right) dy &= - \int_{E_z} e^{-|\sqrt{A}x|^2/2} \left(\sum_{j=1}^{n-1} A_{kj} z_j + A_{kn} y \right) dy \\
&= - \sum_{j=1}^{n-1} A_{kj} z_j v_E(z) - A_{kn} \left(\int_{E_z} y \; d\mu_z(y) \right).
\end{aligned}
$$

Similarly, we have

$$
\int_{E_z^s} \partial_{z_k} \left(e^{-|\sqrt{A}x|^2/2} \right) dy = - \sum_{j=1}^{n-1} A_{kj} z_j v_{E^s}(z) - A_{kn} \left(\int_{E_z^s} y \; d\mu_z(y) \right).
$$

Since $v_E(z) = v_{E^s}(z)$,

$$
\int_{E_z} \partial_{z_k} \left(e^{-|\sqrt{A}x|^2/2} \right) dy - \int_{E_z^s} \partial_{z_k} \left(e^{-|\sqrt{A}x|^2/2} \right) dy = \left(\int_{E_z^s} y \; d\mu_z(y) - \int_{E_z} y \; d\mu_z(y) \right) A_{kn.}
$$

Therefore,

$$
\begin{aligned}
&\left| \int_{E_z} \nabla' e^{-|\sqrt{A}x|^2/2} dy - \int_{E_z^s} \nabla' e^{-|\sqrt{A}x|^2/2} dy \right| \\
&= \left| \int_{E_z} y \; d\mu_z(y) - \int_{E_z^s} y \; d\mu_z(y) \right| \|(A_{1n}, A_{2n}, \cdots, A_{(n-1)n})\|
\end{aligned}
$$

$$= \left| \int_{E_z} y \, d\mu_z(y) - \int_{E_z^s} y \, d\mu_z(y) \right| \|Ae_n - \langle Ae_n, e_n \rangle e_n\|$$
$$= \left(\int_{E_z} y \, d\mu_z(y) - \int_{E_z^s} y \, d\mu_z(y) \right) \|Ae_n - \langle Ae_n, e_n \rangle e_n\|.$$

(3) In order to work with probability measures, we first normalize our definitions of μ_z and ϕ_z as

$$\widetilde{\mu}_z(F) := \frac{1}{\mu_z(\mathbb{R})}\mu_z(F) = \int_F \frac{1}{\mu_z(\mathbb{R})} e^{-|\sqrt{A}(z,y)|^2/2} dy \quad \text{and} \quad \widetilde{\phi}_z(s) := \frac{1}{\mu_z(\mathbb{R})} \int_{-\infty}^{s} e^{-|\sqrt{A}(z,y)|^2/2} dy.$$

Notice that

$$(\widetilde{\phi}_z)^{-1}(\widetilde{\mu}_z(F)) = \phi_z^{-1}(\mu_z(F)).$$

Now we claim that $\widetilde{\mu}_z$ is a log-concave measure on $\mathbb{R}$. We just need to show that

$$\log \left(\frac{1}{\mu_z(\mathbb{R})} e^{-|\sqrt{A}(z,y)|^2/2} \right) = -\log(\mu_z(\mathbb{R})) - \frac{1}{2}|\sqrt{A}(z,y)|^2 \quad \text{is a concave function in } y.$$

Applying Lemma 3.2.1 (1), we have

$$\partial_{yy}^2 \left(-\log(\mu_z(\mathbb{R})) - \frac{1}{2}|\sqrt{A}(z,y)|^2 \right) \leq 0.$$

Thus, $\widetilde{\mu}_z$ is a log-concave measure on $\mathbb{R}$. Let F be a Borel set in $\mathbb{R}$ with $p := \widetilde{\mu}_z(F) \in (0,1)$. Since $\widetilde{\mu}_z(F^s) = \widetilde{\mu}_z(F) = p$, and $F^s = (-\infty, \phi_z^{-1}(\mu_z(F)))$ is a half-line, with the help of the one-dimensional log-concave isoperimetric inequality (see [Bob96], Proposition 2.1), we have

$$\inf_{\widetilde{\mu}_z(A) \geq p} \widetilde{\mu}_z(A + B_h) = \widetilde{\mu}_z(F^s + B_h)$$

for all $h > 0$, where $B_h = [-h, h]$. In particular, for any Borel set $F \subset \mathbb{R}$,

$$\mu_z(F + B_h) \geq \mu_z(F^s + B_h) \quad \text{for all } h > 0. \tag{3.23}$$

Let $z \in B_E \cap B_{E^s}$, by Vol'pert Theorem (Theorem 3.1.1), E_z is a set of locally finite perimeter in $\mathbb{R}$. Moreover, by [Mag12], Lemma 15.12, $E_z \cap (-R, R)$ is also a set of locally finite perimeter in $\mathbb{R}$ for any $R > 0$. Applying [Mag12], Proposition 12.13 on $E_z \cap (-R, R)$, we may assume that $E_z \cap (-R,R)$ is a disjoint union of open intervals with positive distance, i.e., $a_i < b_i < a_{i+1} < b_{i+1}$ for all i and

$$E_z \cap (-R,R) = \bigcup_{i \in S_R} (a_i, b_i), \tag{3.24}$$

where S_R is a countable set. First we claim that S_R is a finite set. By using equation (3.24),

$$\begin{aligned}
\infty > p_E(z) + e^{-|\sqrt{A}(z,R)|^2/2} + e^{-|\sqrt{A}(z,-R)|^2/2} &= P_z(E_z) + e^{-|\sqrt{A}(z,R)|^2/2} + e^{-|\sqrt{A}(z,-R)|^2/2} \\
&\geq P_z(E_z \cap (-R,R)) = P_z\left(\bigcup_{i \in S_R} (a_i, b_i)\right) = \sum_{i \in S_R} P_z\big((a_i, b_i)\big) \\
&= \sum_{i \in S_R} e^{-|\sqrt{A}(z,a_i)|^2/2} + e^{-|\sqrt{A}(z,b_i)|^2/2} \\
&\geq \sum_{i \in S_R} e^{-\|\sqrt{A}\|^2(|z|^2+R^2)/2} + e^{-\|\sqrt{A}\|^2(|z|^2+R^2)/2} \geq 2e^{-\|\sqrt{A}\|^2(|z|^2+R^2)/2}|S_R|,
\end{aligned}$$

i.e., $|S_R| < \infty$, where we have used

$$|\sqrt{A}(z,a_i)|^2 \leq \|\sqrt{A}\|^2(|z|^2 + a_i^2) \leq \|\sqrt{A}\|^2(|z|^2 + R^2).$$

Moreover, by the definition of μ_z and the fundamental theorem of calculus, for any $-\infty \leq a < b \leq \infty$,

$$\lim_{h \to 0} \frac{\mu_z((a,b) + B_h) - \mu_z((a,b))}{h} = e^{-|\sqrt{A}(z,a)|^2/2} + e^{-|\sqrt{A}(z,b)|^2/2}. \tag{3.25}$$

Next we claim that

$$P_z(E_z \cap (-R,R)) \geq e^{-|\sqrt{A}(z,y_R(z))|^2/2},$$

where

$$y_R(z) := \phi_z^{-1}\Big(\mu_z\Big(\big(E \cap (\mathbb{R}^{n-1} \times (-R,R))\big)_z\Big)\Big).$$

Notice that

$$\big(E \cap (\mathbb{R}^{n-1} \times (-R,R))\big)_z = E_z \cap (-R,R), \quad \big(E \cap (\mathbb{R}^{n-1} \times (-R,R))\big)_z^s = (-\infty, y_R(z)),$$

$$\mu_z\Big(\big(E \cap (\mathbb{R}^{n-1} \times (-R,R))\big)_z^s\Big) = \mu_z\Big(\big(E \cap (\mathbb{R}^{n-1} \times (-R,R))\big)_z\Big),$$

$$y_R(z) = \phi_z^{-1}\Big(\mu_z\Big(E_z \cap (-R,R)\Big)\Big) \to y(z) := \phi_z^{-1}(\mu_z(E_z)), \quad \text{as } R \to \infty,$$

and

$$\limsup_{R\to\infty} P_z(E_z \cap (-R,R)) \le \limsup_{R\to\infty} \Big(P_z(E_z) + e^{-|\sqrt{A}(z,R)|^2/2} + e^{-|\sqrt{A}(z,-R)|^2/2}\Big) = p_E(z).$$

By using equation (3.25), and the fact that the intervals (a_i, b_i) are disjoint,

$$\begin{aligned}
&P_z(E_z \cap (-R,R)) = \sum_{i\in S_R} e^{-|\sqrt{A}(z,a_i)|^2/2} + e^{-|\sqrt{A}(z,b_i)|^2/2} \\
&= \sum_{i\in S_R} \lim_{h\to 0^+} \frac{\mu_z((a_i,b_i)+B_h) - \mu_z((a_i,b_i))}{h} \\
&= \lim_{h\to 0^+} \sum_{i\in S_R} \frac{\mu_z((a_i,b_i)+B_h) - \mu_z((a_i,b_i))}{h} \qquad (S_R \text{ is finite}) \\
&\ge \lim_{h\to 0^+} \frac{1}{h}\left(\mu_z\Big(\bigcup_{i\in S_R}((a_i,b_i)+B_h)\Big) - \mu_z\Big(\bigcup_{i\in S_R}(a_i,b_i)\Big)\right) \\
&\ge \lim_{h\to 0^+} \frac{1}{h}\Big(\mu_z\Big(E_z \cap (-R,R) + B_h\Big) - \mu_z\Big(E_z \cap (-R,R)\Big)\Big) \\
&= \lim_{h\to 0^+} \frac{1}{h}\Big(\mu_z\Big(\big(E \cap (\mathbb{R}^{n-1}\times(-R,R))\big)_z + B_h\Big) - \mu_z\Big(\big(E \cap (\mathbb{R}^{n-1}\times(-R,R))\big)_z\Big)\Big) \\
&\ge \lim_{h\to 0^+} \frac{1}{h}\Big(\mu_z\Big(\big(E \cap (\mathbb{R}^{n-1}\times(-R,R))\big)_z^s + B_h\Big) - \mu_z\Big(\big(E \cap (\mathbb{R}^{n-1}\times(-R,R))\big)_z^s\Big)\Big) \\
&= \lim_{h\to 0^+} \frac{1}{h}\int_{y_R(z)}^{y_R(z)+h} e^{-|\sqrt{A}(z,y)|^2/2}dy = e^{-|\sqrt{A}(z,y_R(z))|^2/2}
\end{aligned}$$

in which we have used (3.23) where $F = \big(E \cap (\mathbb{R}^{n-1}\times(-R,R))\big)_z$ in the last inequality. Taking $R \to \infty$, we have $p_E(z) \ge e^{-|\sqrt{A}(z,y(z))|^2/2} = p_{E^s}(z)$. □

We have seen that the anisotropic Gaussian perimeter of the one-dimensional section decreases by Ehrhard symmetrization. Intuitively, this gives us hope that the higher dimensional anisotropic Gaussian perimeter might also decrease after doing the Ehrhard symmetrization. However, our next example shows that this is not true in general. The main idea is to understand the asymptotic behavior of the quantity $h(z) = \phi_z^{-1}(v_E(z))$ via the equation $v_{E^s}(z) = v_E(z)$.

Example 3.3.1. Let $n = 2$ and

$$A = 2\begin{bmatrix} a & b \\ b & c \end{bmatrix} \succ 0$$

with $b \neq 0$. Consider

$$E_\alpha = [-\alpha, \alpha] \times (0, \infty).$$

Then there exists some $\delta > 0$ such that for any $0 < \alpha < \delta$, we have

$$P_{\gamma_A}(E_\alpha) < P_{\gamma_A}(E_\alpha^s)$$

and

$$P_{\gamma_A}(E_\alpha^s) < P_{\gamma_A}(E_\alpha) + \sqrt{2\pi}\|Ae_2 - \langle Ae_2, e_2\rangle e_2\|\langle b_{\gamma_A}(E) - b_{\gamma_A}(E^s), e_2\rangle.$$

Proof. Notice that both a and c are positive since A is a positive definite matrix. Additionally, notice that

$$e^{-|\sqrt{A}(x,y)|^2/2} = e^{-\langle A(x,y),(x,y)\rangle/2} = e^{-ax^2-2bxy-cy^2}.$$

Let $K = [-1, 1]$ and Ω be an open set in $\mathbb{R}^1$ such that $K \subset \Omega$. Let $E = \Omega \times (0, \infty)$. Then

$$E_x = (0, \infty)$$

for all $x \in \Omega$. By Lemma 3.2.1 (2)(b),

$$v_E(x) = \int_{E_x} e^{-|\sqrt{A}(x,y)|^2/2}\, dy = \int_0^\infty e^{-ax^2-2bxy-cy^2}\, dy \quad \text{is differentiable on } \Omega.$$

By Lemma 3.2.3, $h : x \mapsto \phi_x^{-1}(v_E(x))$ is also differentiable on Ω. Notice that

$$\int_0^\infty e^{-ax^2-2bxy-cy^2}\, dy = v_E(x) = v_{E^s}(x) = \int_{-\infty}^{h(x)} e^{-ax^2-2bxy-cy^2}\, dy. \tag{3.26}$$

Setting $x = 0$ in equation (3.26), we have

$$v_E(0) = v_{E^s}(0) \implies \int_0^\infty e^{-cy^2}\, dy = \int_{-\infty}^{h(0)} e^{-cy^2}\, dy \implies h(0) = 0.$$

Taking derivative on equation (3.26), we also have

$$\begin{aligned}&\int_0^\infty e^{-ax^2-2bxy-cy^2}(-2ax-2by)dy\\&=e^{-ax^2-2bxh(x)-ch^2(x)}h'(x)+\int_{-\infty}^{h(x)} e^{-ax^2-2bxy-cy^2}(-2ax-2by)dy.\end{aligned}$$

In particular for $x=0$,

$$h'(0)=\int_0^\infty e^{-cy^2}(-2by)dy-\int_{-\infty}^0 e^{-cy^2}(-2by)dy=-4b\int_0^\infty e^{-cy^2}ydy=-\frac{2b}{c}.$$

That is,

$$h(0)=0,\qquad h'(0)=-\frac{2b}{c}. \tag{3.27}$$

Notice that the Ehrhard symmetrization of E_α has the form

$$E_\alpha^s=\{(x,y):x\in[-\alpha,\alpha],y<h(x)\}\quad\text{for } 0<\alpha<1.$$

We now claim that

$$P_{\gamma_A}(E_\alpha)-P_{\gamma_A}(E_\alpha^s)=2\frac{\sqrt{\det A}}{\sqrt{2\pi}}\left(1-\sqrt{1+\frac{4b^2}{c^2}}\right)\alpha+o(\alpha). \tag{3.28}$$

By using the Taylor expansion, we have

$$\begin{aligned}\frac{\sqrt{2\pi}}{\sqrt{\det A}}P_{\gamma_A}(E_\alpha)&=\int_0^\infty e^{-a\alpha^2-2b\alpha y-cy^2}\,dy+\int_0^\infty e^{-a\alpha^2+2b\alpha y-cy^2}\,dy+\int_{-\alpha}^\alpha e^{-ax^2}\,dx\\&=\left(\int_0^\infty e^{-cy^2}\,dy+\frac{d}{d\alpha}\int_0^\infty e^{-a\alpha^2-2b\alpha y-cy^2}\,dy\Big|_{\alpha=0}\alpha+o(\alpha)\right)\\&\quad+\left(\int_0^\infty e^{-cy^2}\,dy+\frac{d}{d\alpha}\int_0^\infty e^{-a\alpha^2+2b\alpha y-cy^2}\,dy\Big|_{\alpha=0}\alpha+o(\alpha)\right)+2\int_0^\alpha e^{-ax^2}\,dx\\&=\left(\frac{\sqrt{\pi}}{2\sqrt{c}}+\left(\frac{-b}{c}\right)\alpha+o(\alpha)\right)+\left(\frac{\sqrt{\pi}}{2\sqrt{c}}+\left(\frac{b}{c}\right)\alpha+o(\alpha)\right)+\big(2\alpha+o(\alpha)\big)\\&=\frac{\sqrt{\pi}}{\sqrt{c}}+2\alpha+o(\alpha).\end{aligned}$$

Moreover, by (3.27), we have

$$\begin{aligned}
\frac{\sqrt{2\pi}}{\sqrt{\det A}}P_{\gamma_A}(E^s_\alpha) &= \int_{-\infty}^{h(\alpha)} e^{-a\alpha^2-2b\alpha y-cy^2}\ dy + \int_{-\infty}^{h(-\alpha)} e^{-a\alpha^2+2b\alpha y-cy^2}\ dy \\
&\quad + \int_{-\alpha}^{\alpha} e^{-at^2-2bth(t)-ch^2(t)}\sqrt{1+h'(t)^2}\ dt \\
&= \left(\int_{-\infty}^{h(0)} e^{-cy^2}\ dy + \frac{d}{d\alpha}\int_{-\infty}^{h(\alpha)} e^{-a\alpha^2-2b\alpha y-cy^2}\ dy\bigg|_{\alpha=0} \alpha + o(\alpha)\right) \\
&\quad + \left(\int_{-\infty}^{h(0)} e^{-cy^2}\ dy + \frac{d}{d\alpha}\int_{-\infty}^{h(-\alpha)} e^{-a\alpha^2+2b\alpha y-cy^2}\ dy\bigg|_{\alpha=0} \alpha + o(\alpha)\right) \\
&\quad + \left(\frac{d}{d\alpha}\int_{-\alpha}^{\alpha} e^{-at^2-2bth(t)-ch^2(t)}\sqrt{1+h'(t)^2}\ dt\bigg|_{\alpha=0} \alpha + o(\alpha)\right) \\
&= \left(\frac{\sqrt{\pi}}{2\sqrt{c}} + \alpha\left[-\frac{2b}{c}+\frac{b}{c}\right] + o(\alpha)\right) + \left(\frac{\sqrt{\pi}}{2\sqrt{c}} + \alpha\left[\frac{2b}{c}-\frac{b}{c}\right] + o(\alpha)\right) \\
&\quad + \left(2\alpha\sqrt{1+\frac{4b^2}{c^2}} + o(\alpha)\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{c}} + 2\alpha\sqrt{1+\frac{4b^2}{c^2}} + o(\alpha),
\end{aligned}$$

where the third term is the anisotropic Gaussian perimeter of the graph of h. Therefore,

$$P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E^s_\alpha) = 2\frac{\sqrt{\det A}}{\sqrt{2\pi}}\left(1-\sqrt{1+\frac{4b^2}{c^2}}\right)\alpha + o(\alpha).$$

Next we show that

$$\sqrt{2\pi}\|Ae_2 - \langle Ae_2, e_2\rangle e_n\| \langle b_{\gamma_A}(E) - b_{\gamma_A}(E^s), e_2\rangle = \frac{\sqrt{\det A}}{\sqrt{2\pi}} 2\left(\frac{2b}{c}\right)\alpha + o(\alpha). \tag{3.29}$$

Notice that

$$\|Ae_2 - \langle Ae_2, e_2\rangle e_2\| = 2|b|.$$

Now we compute the barycenter of E_α and E^s_α, i.e.,

$$b_{\gamma_A}(E_\alpha) = \frac{\sqrt{\det A}}{2\pi}\int_{-\alpha}^{\alpha}\int_0^{\infty}(x,y)e^{-|\sqrt{A}(x,y)|^2/2}dydx,$$

and

$$b_{\gamma_A}(E^s_\alpha) = \frac{\sqrt{\det A}}{2\pi} \int_{-\alpha}^{\alpha} \int_{-\infty}^{h(x)} (x,y) e^{-|\sqrt{A}(x,y)|^2/2} dy dx.$$

Then

$$b_{\gamma_A}(E_\alpha) - b_{\gamma_A}(E^s_\alpha) = \frac{\sqrt{\det A}}{2\pi} \int_{-\alpha}^{\alpha} \left(\int_0^\infty (x,y) e^{-|\sqrt{A}(x,y)|^2/2} dy - \int_{-\infty}^{h(x)} (x,y) e^{-|\sqrt{A}(x,y)|^2/2} dy \right) dx$$

and hence

$$\begin{aligned}
\langle b_{\gamma_A}(E_\alpha) - b_{\gamma_A}(E^s_\alpha), e_2 \rangle &= \frac{\sqrt{\det A}}{2\pi} \int_{-\alpha}^{\alpha} \left(\int_0^\infty y e^{-|\sqrt{A}(x,y)|^2/2} dy - \int_{-\infty}^{h(x)} y e^{-|\sqrt{A}(x,y)|^2/2} dy \right) dx \\
&= \frac{\sqrt{\det A}}{2\pi} \left[2 \left(\int_0^\infty y e^{-|\sqrt{A}(0,y)|^2/2} dy - \int_{-\infty}^{h(0)} y e^{-|\sqrt{A}(0,y)|^2/2} dy \right) \alpha + o(\alpha) \right] \\
&= \frac{\sqrt{\det A}}{2\pi} \left[4 \left(\int_0^\infty y e^{-cy^2} dy \right) \alpha + o(\alpha) \right] = \frac{\sqrt{\det A}}{2\pi} \left(\frac{2}{c} \right) \alpha + o(\alpha).
\end{aligned}$$

Thus,

$$\sqrt{2\pi} \| A e_2 - \langle A e_2, e_2 \rangle e_2 \| \langle b_{\gamma_A}(E) - b_{\gamma_A}(E^s), e_2 \rangle = 2 \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\frac{2|b|}{c} \right) \alpha + o(\alpha).$$

Thanks to (3.28) and (3.29), we have

$$\lim_{\alpha \to 0^+} \frac{P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E^s_\alpha)}{\alpha} = 2 \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(1 - \sqrt{1 + \frac{4b^2}{c^2}} \right) < 0$$

and

$$\begin{aligned}
&\lim_{\alpha \to 0^+} \frac{\sqrt{2\pi} \| A e_2 - \langle A e_2, e_2 \rangle e_2 \| \langle b_{\gamma_A}(E) - b_{\gamma_A}(E^s), e_2 \rangle + P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E^s_\alpha)}{\alpha} \\
&= 2 \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\left(1 + \frac{2|b|}{c} \right) - \sqrt{1 + \frac{4b^2}{c^2}} \right) > 0.
\end{aligned}$$

Since $b \neq 0$, let

$$\varepsilon = \min \left\{ \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\sqrt{1 + \frac{4b^2}{c^2}} - 1 \right), \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\left(1 + \frac{2|b|}{c} \right) - \sqrt{1 + \frac{4b^2}{c^2}} \right) \right\} > 0.$$

There exists $\delta > 0$ such that for all $0 < \alpha < \delta$,

$$\left| \frac{P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s)}{\alpha} - 2\frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(1 - \sqrt{1 + \frac{4b^2}{c^2}} \right) \right| < \varepsilon \leq \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\sqrt{1 + \frac{4b^2}{c^2}} - 1 \right)$$

and

$$\left| \frac{\sqrt{2\pi}\|Ae_2 - \langle Ae_2, e_2\rangle e_2\| \langle b_{\gamma_A}(E) - b_{\gamma_A}(E^s), e_2\rangle + P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s)}{\alpha} \right.$$
$$\left. - 2\frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\left(1 + \frac{2|b|}{c} \right) - \sqrt{1 + \frac{4b^2}{c^2}} \right) \right|$$
$$< \varepsilon \leq \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\left(1 + \frac{2|b|}{c} \right) - \sqrt{1 + \frac{4b^2}{c^2}} \right).$$

Therefore, for any $0 < \alpha < \delta$,

$$P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s) < \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(1 - \sqrt{1 + \frac{4b^2}{c^2}} \right) \alpha < 0$$

and

$$\sqrt{2\pi}\|Ae_2 - \langle Ae_2, e_2\rangle e_2\| \langle b_{\gamma_A}(E) - b_{\gamma_A}(E^s), e_2\rangle + P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s)$$
$$> \frac{\sqrt{\det A}}{\sqrt{2\pi}} \left(\left(1 + \frac{2|b|}{c} \right) - \sqrt{1 + \frac{4b^2}{c^2}} \right) \alpha > 0,$$

i.e.,

$$P_{\gamma_A}(E_\alpha^s) < P_{\gamma_A}(E_\alpha) + \sqrt{2\pi}\|Ae_2 - \langle Ae_2, e_2\rangle e_2\| \langle b_{\gamma_A}(E) - b_{\gamma_A}(E^s), e_2\rangle.$$

□

Remark: From Example 3.3.1, we see that there exists some E such that

$$P_{\gamma_A}(E) < P_{\gamma_A}(E^s).$$

Although the anisotropic Gaussian perimeter does not always decrease under Ehrhard symmetrization, a natural question to ask here is whether there still exists an upper bound for

$P_{\gamma_A}(E^s)$ in terms of $P_{\gamma_A}(E)$. In our next subsection, we will show that there exists an upper bound and

$$P_{\gamma_A}(E^s) \leq P_{\gamma_A}(E) + \sqrt{2\pi}\|Ae_n - \langle Ae_n, e_n\rangle e_n\| \langle b_{\gamma_A}(E) - b_{\gamma_A}(E^s), e_n\rangle,$$

for any set of finite anisotropic Gaussian perimeter E in $\mathbb{R}^n$ (see Theorem 3.4.4).

3.4 Ehrhard symmetrization on anisotropic Gaussian measures

Our next goal is to show that the perimeter of a Ehrhard symmetrization set E^s can still be controlled by the perimeter of E plus an error term with a form like $A - \lambda I_n$. In particular, the Ehrhard symmetrization along the eigendirections decreases the anisotropic Gaussian perimeter. We will break this into several lemmas. Our next three lemmas are modifications of Cianchi-Fusco-Maggi-Pratelli's paper [CFMP11] Lemma 4.5 and Lemma 4.6. We will prove the "dust estimate", "cylindrical estimate", and "graphical estimate". Starting from this section, the notation C means a constant that depends only on n and A, which may change from line to line.

Lemma 3.4.1 (Dust estimate for E)**.**
Let E be a set of finite anisotropic Gaussian perimeter in $\mathbb{R}^n$ and B be a Borel set such that

$$v_E(z) = 0 \quad \textit{for all } z \in B.$$

Then

$$P_{\gamma_A}(E^s; B \times \mathbb{R}) \leq P_{\gamma_A}(E; B \times \mathbb{R}).$$

In particular, if we assume that B is open with smooth boundary, then

$$P_{\gamma_A}(E^s; B \times \mathbb{R}) = 0.$$

Proof. Without loss of generality, we may assume that B is bounded since we can consider $B \cap B(0, R)$. Given $\varepsilon > 0$ and let Ω be an open set with $\Omega \supset B$ such that

$$\mathcal{L}^{n-1}(\Omega \setminus B) < \varepsilon.$$

By [Dan08], Proposition 8.2.1, there exists a sequence of bounded open smooth sets $\Omega_j \nearrow \Omega$, i.e., $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$ and $\bigcup_j \Omega_j = \Omega$. Then

$$P_{\gamma_A}(E^s; \Omega_j \times \mathbb{R}) \nearrow P_{\gamma_A}(E^s; \Omega \times \mathbb{R})$$

and

$$P_{\gamma_A}(E^s; B \times \mathbb{R}) \leq P_{\gamma_A}(E^s; \Omega \times \mathbb{R})$$

since $B \subset \Omega$. Recall from Proposition 2.3.2 that in order to compute $P_{\gamma_A}(E^s, \Omega_j \times \mathbb{R})$ it is enough to look at (3.20) and equation (3.21) as we have seen in Theorem 3.2.4, i.e., let $\varphi \in C^1_c(\Omega_j \times \mathbb{R}; \mathbb{R}^n)$ with $|\varphi| \leq 1$ and let $\varphi_z = (\varphi_1, \varphi_2, ..., \varphi_{n-1})$, we estimate integrals (I) and (II) from (3.21).

Applying Theorem 3.2.4 on Ω_j and $y(z) := \phi_z^{-1}(v_E(z))$, there exists a sequence of functions $v_k^j \in C^1(\Omega_j)$ with $0 < v_k^j < g$ and $g(z) := \phi_z(\infty)$, such that $v_k^j \to v_E$ in $L^1(\Omega_j)$ and a.e. in Ω_j, $Dv_k^j \overset{*}{\rightharpoonup} Dv_E$ in Ω_j, and

$$\lim_{k\to\infty} \int_{\Omega_j} \left|\nabla' v_k^j(z)\right| dz \leq |Dv_E|(\Omega_j) + 2\int_{\Omega_j} \left(\frac{|\nabla' g|}{g}\right) v_E \, dz.$$

Moreover,

$$\chi_{F_k^j} \to \chi_{E^s} \text{ a.e. in } \Omega_j \times \mathbb{R} \text{ and } y_k^j(z) \to y(z) = \phi_z^{-1}(v_E(z)) \text{ a.e. in } \Omega_j$$

where $y_k^j(z) := \phi_z^{-1}(v_k^j(z))$ and

$$F_k^j := \left\{(z, y) \in \Omega_j \times \mathbb{R} : y < y_k^j(z) := \phi_z^{-1}(v_k^j(z))\right\}.$$

Since $v_E(z) = 0$ for all $z \in B$, we have

$$v_k^j(z) \to v_E(z) = 0 \text{ a.e. in } \Omega_j \cap B,$$

$$y_k^j(z) \to y(z) = \phi_z^{-1}(v_E(z)) = -\infty \text{ a.e. in } \Omega_j \cap B,$$

and

$$p_{E^s}(z) = e^{-[\phi_z^{-1}(v_E(z))]^2/2} = 0 \text{ on } B.$$

Therefore, by equation (3.22) where F as F_k^j, $y(z)$ as $y_k^j(z)$, and Ω as Ω_j,

$$
\begin{aligned}
(\mathrm{I}) &= \sum_{i=1}^{n-1} \int_{E^s} \frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i}\ d\gamma_A(x) \\
&= \lim_{k\to\infty} \Big(\sum_{i=1}^{n-1} \int_{F_k^j} \frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i}\ d\gamma_A(x) \Big) \\
&= \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \left(\int_{\Omega_j} \Big(-\varphi_z(z, y_k^j(z)) \Big) \cdot \nabla' v_k^j(z) dz \right. \\
&\quad \left. - \int_{\Omega_j} \int_{-\infty}^{y_k^j(z)} \Big(-\varphi_z(z, y_k^j(z)) \Big) \cdot \nabla' \Big(e^{-|\sqrt{A}x|^2/2} \Big)\, dydz \right) \\
&\le \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \limsup_{k\to\infty} \int_{\Omega_j} \Big| \nabla' v_k^j(z) \Big|\, dz \\
&\quad + \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \int_{\Omega_j \setminus B} \int_{-\infty}^{y_k^j(z)} \varphi_z(z, y_k^j(z)) \cdot \nabla' \Big(e^{-|\sqrt{A}x|^2/2} \Big)\, dydz \\
&\quad + \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \int_{\Omega_j \cap B} \int_{-\infty}^{y_k^j(z)} \varphi_z(z, y_k^j(z)) \cdot \nabla' \Big(e^{-|\sqrt{A}x|^2/2} \Big)\, dydz \\
&\le \frac{\det \sqrt{A}}{(2\pi)^{n/2}} |Dv_E|(\Omega_j) + \frac{2\det \sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j} \left(\frac{|\nabla' g|}{g} \right) v_E\, dz \\
&\quad + \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j \setminus B} \int_{-\infty}^{y(z)} \varphi_z(z, y(z)) \cdot \nabla' \Big(e^{-|\sqrt{A}x|^2/2} \Big)\, dydz \\
&\le \frac{\det \sqrt{A}}{(2\pi)^{n/2}} |Dv_E|(\Omega_j) + \frac{2\det \sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j \setminus B} \left(\frac{|\nabla' g|}{g} \right) v_E\, dz \\
&\quad + \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j \setminus B} \int_{-\infty}^{\infty} \left| \nabla' \Big(e^{-|\sqrt{A}x|^2/2} \Big) \right| dydz \\
&\le \frac{\det \sqrt{A}}{(2\pi)^{n/2}} |Dv_E|(\Omega_j) + \frac{2\det \sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j \setminus B} |\nabla' g|\, dz + C\varepsilon \le \frac{\det \sqrt{A}}{(2\pi)^{n/2}} |Dv_E|(\Omega_j) + C\varepsilon
\end{aligned}
$$

where we have used $\mathcal{L}^{n-1}(\Omega_j \setminus B) < \varepsilon$, $v_E = 0$ on B, $0 \le v_E \le g$, and Lemma 3.2.1 (3), i.e.,

$$
\sup_{z\in\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \left| \nabla' \Big(e^{-|\sqrt{A}x|^2/2} \Big) \right| dy \le (n-1)C.
$$

Therefore,

$$
(\mathrm{I}) \le \frac{\det \sqrt{A}}{(2\pi)^{n/2}} |Dv_E|(\Omega_j) + C\varepsilon.
$$

Now we estimate (II). Recalling that $\mathcal{H}^{n-1}(\partial^M E^s \setminus \partial^* E^s) = 0$ and $(\partial^M E^s)_z = \partial^M((E^s)_z)$ for $\mathcal{L}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$, we get

$$\begin{aligned}
(\mathrm{II}) &= \int_{E^s} \frac{\partial \varphi_n}{\partial y} + \varphi_n \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial y} \; d\gamma_A(x) = \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{E^s} \frac{\partial}{\partial y}\left(\varphi_n e^{-|\sqrt{A}x|^2/2}\right) dz \\
&= -\frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\partial^M E^s} \varphi_n e^{-|\sqrt{A}x|^2/2} \nu_n^{E^s} \; d\mathcal{H}^{n-1}(x) \\
&\le \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\partial^M E^s \cap (\Omega_j \times \mathbb{R})} \left|\nu_n^{E^s}\right| e^{-|\sqrt{A}x|^2/2} \; d\mathcal{H}^{n-1}(x) \\
&= \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j} \int_{\left(\partial^M E^s\right)_z} e^{-|\sqrt{A}x|^2/2} \; d\mathcal{H}^0(y) \; dz \\
&= \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j} p_{E^s}(z) \; dz = \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j \setminus B} p_{E^s}(z) \; dz \le C\varepsilon
\end{aligned}$$

where we have used the co-area formula (3.4), the definition of $p_{E^s}(z)$, $p_{E^s}(z) = 0$ if $z \in B$, and

$$p_{E^s}(z) = e^{-[\phi_z^{-1}(v_E(z))]^2/2} \le 1.$$

Combining (I) and (II) together,

$$\int_{E^s} \operatorname{div}\varphi - \langle \varphi, Ax\rangle \; d\gamma_A(x) = (\mathrm{I}) + (\mathrm{II}) \le \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \left|Dv_E\right|(\Omega_j) + C\varepsilon.$$

Taking the sup over φ gives us

$$\frac{1}{\sqrt{2\pi}} P_{\gamma_A}\left(E^s; \Omega_j \times \mathbb{R}\right) \le \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \left|Dv_E\right|(\Omega_j) + C\varepsilon.$$

Applying Lemma 3.2.2 on Ω_j,

$$\begin{aligned}
P_{\gamma_A}\left(E^s; \Omega_j \times \mathbb{R}\right) &\le \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}} \left|Dv_E\right|(\Omega_j) + C\varepsilon \\
&\le P_{\gamma_A}\left(E; \Omega_j \times \mathbb{R}\right) + \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}} \int_{\Omega_j} \left| \int_{E_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy \right| dz + C\varepsilon \\
&\le P_{\gamma_A}\left(E; \Omega_j \times \mathbb{R}\right) + \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}} \int_{B} \left| \int_{E_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy \right| dz \\
&\quad + \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}} \int_{\Omega_j \setminus B} \left| \int_{E_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy \right| dz + C\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&\le P_{\gamma_A}\left(E;\Omega_j\times\mathbb{R}\right)+\frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B\left|\int_{E_z}\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)dy\right|dz+C\varepsilon\\
&=P_{\gamma_A}\left(E;\Omega_j\times\mathbb{R}\right)+C\varepsilon
\end{aligned}
$$

where we have used $v_E=0$ on B, i.e., $\mathcal{L}(E_z)=0$ if $z\in B$, and Lemma 3.2.1 (3)(a), i.e.,

$$
\int_{\Omega_j\setminus B}\left|\int_{E_z}\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)dy\right|dz\le\int_{\Omega_j\setminus B}\int_{-\infty}^{\infty}\left|\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)\right|dydz\le(n-1)C\varepsilon.
$$

Taking $j\to\infty$ on both sides,

$$
P_{\gamma_A}\left(E^s;B\times\mathbb{R}\right)\le P_{\gamma_A}\left(E^s;\Omega\times\mathbb{R}\right)\le P_{\gamma_A}\left(E;\Omega\times\mathbb{R}\right)+C\varepsilon
$$

Taking the inf over $\Omega\supset B$ with $\mathcal{L}^{n-1}(\Omega\setminus B)<\varepsilon$,

$$
P_{\gamma_A}\left(E^s;B\times\mathbb{R}\right)\le P_{\gamma_A}\left(E;B\times\mathbb{R}\right)+C\varepsilon.
$$

Taking $\varepsilon\to 0$, we have $P_{\gamma_A}\left(E^s;B\times\mathbb{R}\right)\le P_{\gamma_A}\left(E;B\times\mathbb{R}\right)$. Now we claim that

$$
P_{\gamma_A}\left(E^s;B\times\mathbb{R}\right)=0,\ \text{if } B \text{ is open with smooth boundary and } v_E(z)=0 \text{ for all } z\in B.
$$

In this situation, we can just apply all the previous estimates on B instead of Ω_j, which means that we can replace v_k^j as v_k, y_k^j as y_k, and F_k^j as F_k in the previous calculation and notice that

$$
y_k(z)\to y(z)=\phi_z^{-1}\left(v_E(z)\right)=-\infty \text{ a.e. in } B,
$$

and

$$
\varphi_z\left(z,y_k(z)\right)=(0,\dots,0)\text{ for a.e. } z\in B \text{ and large enough } k,
$$

since φ has compact support and $y_k(z)\to-\infty$. Now (I) and (II) become

$$
\begin{aligned}
\text{(I)}&=\sum_{i=1}^{n-1}\int_{E^s}\frac{\partial\varphi_i}{\partial z_i}+\varphi_i\frac{\partial((-|\sqrt{A}x|^2/2))}{\partial z_i}d\gamma_A(x)\\
&=\lim_{k\to\infty}\left(\sum_{i=1}^{n-1}\int_{F_k}\frac{\partial\varphi_i}{\partial z_i}+\varphi_i\frac{\partial((-|\sqrt{A}x|^2/2))}{\partial z_i}d\gamma_A(x)\right)
\end{aligned}
$$

$$= \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \Bigg(\int_B \Big(-\varphi_z \left(z, y_k(z)\right) \Big) \cdot \nabla' v_k(z) dz$$
$$- \int_B \int_{-\infty}^{y_k(z)} \Big(-\varphi_z \left(z, y_k(z)\right) \Big) \cdot \Big(\nabla' e^{-|\sqrt{A}x|^2/2} \Big)\, dydz \Bigg) = 0$$

and

$$(\mathrm{II}) \leq \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_B p_{E^s}(z)\; dz = \frac{\det \sqrt{A}}{(2\pi)^{n/2}} \int_B e^{-[\phi_z^{-1}(v_E(z))]^2/2}\; dz = 0.$$

Thus, we conclude that

$$P_{\gamma_A}(E^s; B \times \mathbb{R}) = 0.$$

□

Lemma 3.4.2 (Cylindrical estimate for E)**.**
Let E be a set of finite anisotropic Gaussian perimeter in $\mathbb{R}^n$ and B be a Borel set in $\mathbb{R}^{n-1}$ with $\mathcal{L}^{n-1}(B) = 0$. Then

$$P_{\gamma_A}(E^s; B \times \mathbb{R}) \leq P_{\gamma_A}(E; B \times \mathbb{R}).$$

Proof. Given $\varepsilon > 0$ and let Ω be an open set with $\Omega \supset B$ such that

$$\mathcal{L}^{n-1}(\Omega) < \varepsilon.$$

The proof of this lemma is similar to Lemma 3.4.1, i.e., we approximate our set Ω by a sequence of bounded open subsets Ω_j. Hence, we keep the same notation as we have seen in Lemma 3.4.1, Ω_j, v_k^j, and y_k^j etc. The only difference here is the estimates of (I) and (II) from Lemma 3.4.1.

$$(\mathrm{I}) = \sum_{i=1}^{n-1} \int_{E^s} \frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i}\; d\gamma_A(x)$$
$$= \lim_{k\to\infty} \Big(\sum_{i=1}^{n-1} \int_{F_k} \frac{\partial \varphi_i}{\partial z_i} + \varphi_i \frac{\partial(-|\sqrt{A}x|^2/2)}{\partial z_i}\; d\gamma_A(x) \Big)$$
$$= -\frac{\det \sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \Bigg(\int_{\Omega_j} \varphi_z(z, y_k^j(z)) \cdot \nabla' v_k^j(z) dz - \int_{\Omega_j} \int_{-\infty}^{y_k^j(z)} \varphi_z(z, y_k^j(z)) \cdot \nabla' \Big(e^{-|\sqrt{A}x|^2/2} \Big)\, dydz \Bigg)$$

$$\leq \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \lim_{k\to\infty} \int_{\Omega_j} \left|\nabla' v_k^j(z)\right| dz + \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j} \int_{-\infty}^{y(z)} \varphi_z(z, y(z)) \cdot \nabla' \left(e^{-|\sqrt{A}x|^2/2}\right) dydz$$
$$\leq \frac{\det\sqrt{A}}{(2\pi)^{n/2}} |Dv_E|(\Omega_j) + \frac{2\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j} \left(\frac{|\nabla' g|}{g}\right) v_E \, dz$$
$$+ \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j} \int_{-\infty}^{\infty} \left|\nabla' \left(e^{-|\sqrt{A}x|^2/2}\right)\right| dydz$$
$$\leq \frac{\det\sqrt{A}}{(2\pi)^{n/2}} |Dv_E|(\Omega_j) + C\mathcal{L}^{n-1}(\Omega_j) \leq \frac{\det\sqrt{A}}{(2\pi)^{n/2}} |Dv_E|(\Omega_j) + C\varepsilon$$

where we have used $\mathcal{L}^{n-1}(\Omega_j) < \varepsilon$, $0 \leq v_E \leq g$, and Lemma 3.2.1 (3)(a). Moreover, applying Lemma 3.2.2 on Ω_j and Lemma 3.2.1 (3)(a),

$$\frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}} |Dv_E|(\Omega_j) \leq P_{\gamma_A}(E; \Omega_j \times \mathbb{R}) + C\varepsilon.$$

Therefore,

$$\text{(I)} \leq \frac{1}{\sqrt{2\pi}} P_{\gamma_A}(E; \Omega_j \times \mathbb{R}) + C\varepsilon.$$

The estimate of (II) is the same as Lemma 3.4.1, i.e.,

$$\text{(II)} \leq \frac{\det\sqrt{A}}{(2\pi)^{n/2}} \int_{\Omega_j} p_{E^s}(z) \; dz \leq C\varepsilon$$

where we have used the definition of $p_{E^s}(z)$, and $\mathcal{L}^{n-1}(\Omega_j) < \varepsilon$. Combining (I) and (II) together,

$$\int_{E^s} \operatorname{div}\varphi - \langle \varphi, Ax \rangle \; d\gamma_A(x) = \text{(I)} + \text{(II)} \leq \frac{1}{\sqrt{2\pi}} P_{\gamma_A}(E; \Omega_j \times \mathbb{R}) + C\varepsilon.$$

Taking the sup over φ, $j \to \infty$, inf over $\Omega \supset B$ with $\mathcal{L}^{n-1}(\Omega) < \varepsilon$, and $\varepsilon \to 0$, we get

$$P_{\gamma_A}(E^s; B \times \mathbb{R}) \leq P_{\gamma_A}(E; B \times \mathbb{R}).$$

□

Lemma 3.4.3 (Graphical estimate for E)**.**
Let E be a set of finite anisotropic Gaussian perimeter in $\mathbb{R}^n, n \geq 2$. Then

$$P_{\gamma_A}(E; B \times \mathbb{R}) \geq \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \sqrt{p_E(z)^2 + \left|\nabla' v_E(z) - \int_{E_z} \nabla' \left(e^{-|\sqrt{A}x|^2/2}\right) dy\right|^2} \; dz,$$

for every Borel set $B \subset B_E$,

$$P_{\gamma_A}(E^s; B\times\mathbb{R}) = \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B \sqrt{p_{E^s}(z)^2 + \left|\nabla' v_{E^s}(z) - \int_{E^s_z}\nabla'\left(e^{-|\sqrt{A}x|^2/2}\right)dy\right|^2}\,dz,$$

for every Borel set $B \subset B_{E^s}$, *and*

$$P_{\gamma_A}(E^s; B\times\mathbb{R}) \le P_{\gamma_A}(E; B\times\mathbb{R}) + \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B \left|\int_{E_z}\nabla' e^{-|\sqrt{A}x|^2/2}dy - \int_{E^s_z}\nabla' e^{-|\sqrt{A}x|^2/2}dy\right| dz,$$

for every Borel set $B \subseteq B_E \cap B_{E^s}$.

Proof. By the co-area formula (3.4) and Vol'pert Theorem (Theorem 3.1.1),

$$\begin{aligned} P_{\gamma_A}(E; B\times\mathbb{R}) &= \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_{\partial^* E\cap(B\times\mathbb{R})}\frac{|\nu_n^E|}{|\nu_n^E|}e^{-|\sqrt{A}x|^2/2}\,d\mathcal{H}^{n-1} \\ &= \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B\int_{\partial^* E_z}\frac{1}{|\nu_n^E|}\,d\mathcal{H}^0_z\,dz. \end{aligned}$$

Applying Jensen's inequality to the convex function $\varphi : (x_1, ..., x_{n-1}) = x \mapsto \sqrt{1+|x|^2}$, we have

$$\begin{aligned} &\sqrt{1+\sum_{i=1}^{n-1}\left(\fint_{\partial^* E_z}\frac{\nu_i^E}{|\nu_n^E|}\,d\mathcal{H}^0_z\right)^2} = \varphi\left(\fint_{\partial^* E_z}\frac{\nu_1^E}{|\nu_n^E|}\,d\mathcal{H}^0_z, \cdots, \fint_{\partial^* E_z}\frac{\nu_{n-1}^E}{|\nu_n^E|}\,d\mathcal{H}^0_z\right) \qquad (3.30) \\ &= \varphi\left(\fint_{\partial^* E_z}\left(\frac{\nu_1^E}{|\nu_n^E|}, \cdots, \frac{\nu_{n-1}^E}{|\nu_n^E|}\right)d\mathcal{H}^0_z\right) \le \fint_{\partial^* E_z}\varphi\left(\frac{\nu_1^E}{|\nu_n^E|}, \cdots, \frac{\nu_{n-1}^E}{|\nu_n^E|}\right)d\mathcal{H}^0_z \\ &= \fint_{\partial^* E_z}\sqrt{1+\sum_{i=1}^{n-1}\frac{|\nu_i^E|^2}{|\nu_n^E|^2}}\,d\mathcal{H}^0_z. \end{aligned}$$

By Lemma 3.2.2, (3.30), and the definition of $p_E(z)$,

$$\begin{aligned} P_{\gamma_A}(E; B\times\mathbb{R}) &= \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B\int_{\partial^* E_z}\sqrt{1+\frac{\sum_{i=1}^{n-1}|\nu_i^E|^2}{|\nu_n^E|^2}}\,d\mathcal{H}^0_z\,dz \\ &= \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B p_E(z)\fint_{\partial^* E_z}\sqrt{1+\frac{\sum_{i=1}^{n-1}|\nu_i^E|^2}{|\nu_n^E|^2}}\,d\mathcal{H}^0_z\,dz \\ &\ge \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B p_E(z)\sqrt{1+\sum_{i=1}^{n-1}\left(\fint_{\partial^* E_z}\frac{\nu_i^E}{|\nu_n^E|}\,d\mathcal{H}^0_z\right)^2}\,dz \end{aligned}$$

$$= \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B \sqrt{p_E(z)^2 + \sum_{i=1}^{n-1}\left(\fint_{\partial^* E_z} \frac{\nu_i^E}{|\nu_n^E|}\, d\mathcal{H}_z^0\right)^2 p_E(z)^2}\; dz$$

$$= \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B \sqrt{p_E(z)^2 + \sum_{i=1}^{n-1}\left(\int_{\partial^* E_z} \frac{\nu_i^E}{|\nu_n^E|}\, d\mathcal{H}_z^0\right)^2}\; dz$$

$$= \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B \sqrt{p_E(z)^2 + \left|\nabla' v_E(z) - \int_{E_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right|^2}\; dz,$$

where $\nabla' v_E(z) = (D_1 v_E(z), \cdots, D_{n-1} v_E(z))$, $B \subset B_E$ and

$$D_i v_E(z) = \int_{(\partial^* E)_z} \frac{\nu_i^E(z,y)}{|\nu_n^E(z,y)|} d\mathcal{H}_z^0(y) + \int_{E_z} \frac{\partial}{\partial x_i}\left(e^{-|\sqrt{A}x|^2/2}\right) dy \qquad \text{for } i = 1, 2, ..., n-1.$$

Applying the same calculation on E^s with $E_z^s = (-\infty, y(z))$, $y(z) = \phi_z^{-1}(v_E(z))$, and

$$p_{E^s}(z) = e^{-|\sqrt{A}(z,y(z))|^2/2},$$

we have for any $B \subset B_{E^s}$,

$$P_{\gamma_A}(E^s; B\times\mathbb{R}) = \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B \sqrt{p_{E^s}(z)^2 + \left|\nabla' v_{E^s}(z) - \int_{E_z^s} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right|^2}\; dz.$$

Notice that we have the following inequality

$$\sqrt{a^2+b^2} - \sqrt{a^2+c^2} \le |b-c|, \quad \text{if } b, c \ge 0. \tag{3.31}$$

Plugging

$$a = p_E(z),\ b = \left|\nabla' v_E(z) - \int_{E_z^s} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right|, \text{ and } c = \left|\nabla' v_E(z) - \int_{E_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right|$$

into (3.31), we have for any $B \subset B_E \cap B_{E^s}$,

$$P_{\gamma_A}(E^s; B\times\mathbb{R}) - P_{\gamma_A}(E; B\times\mathbb{R})$$
$$\le \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B \sqrt{p_{E^s}(z)^2 + \left|\nabla' v_{E^s}(z) - \int_{E_z^s} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right|^2}\; dz$$

$$
\begin{aligned}
&- \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \sqrt{p_E(z)^2 + \left| \nabla' v_E(z) - \int_{E_z} \nabla' \left(e^{-|\sqrt{A}x|^2/2} \right) dy \right|^2} \, dz \\
&\leq \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \sqrt{p_E(z)^2 + \left| \nabla' v_E(z) - \int_{E_z^s} \nabla' \left(e^{-|\sqrt{A}x|^2/2} \right) dy \right|^2} \, dz \\
&\quad - \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \sqrt{p_E(z)^2 + \left| \nabla' v_E(z) - \int_{E_z} \nabla' \left(e^{-|\sqrt{A}x|^2/2} \right) dy \right|^2} \, dz \\
&\leq \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \Bigg| \left| \nabla' v_E(z) - \int_{E_z} \nabla' \left(e^{-|\sqrt{A}x|^2/2} \right) dy \right| \\
&\quad - \left| \nabla' v_E(z) - \int_{E_z^s} \nabla' \left(e^{-|\sqrt{A}x|^2/2} \right) dy \right| \Bigg| \, dz \\
&\leq \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \left| \int_{E_z} \nabla' \left(e^{-|\sqrt{A}x|^2/2} \right) - \int_{E_z^s} \nabla' \left(e^{-|\sqrt{A}x|^2/2} \right) dy \right| dz,
\end{aligned}
$$

where we have used Proposition 3.3.1 (1) and (3), i.e., $\nabla' v_E(z) = \nabla' v_{E^s}(z)$ and $p_E(z) \geq p_{E^s}(z)$ for a.e. $z \in B \subset B_E \cap B_{E^s}$. □

Although the perimeter might not decrease in every direction when we do the Ehrhard Symmetrization, we are still able to give an upper bound for the perimeter of the Ehrhard Symmetrization with an error term involving $A - \lambda I_n$ and barycenters. Combining Lemma 3.4.1, Lemma 3.4.2, and Lemma 3.4.3, we have the following estimate which tells us how the direction of Ehrhard Symmetrization affects the anisotropic Gaussian perimeter.

Theorem 3.4.4 (Anisotropic Gaussian Perimeter Inequality).
Let $n \geq 2$ and let E be a set of finite A-anisotropic Gaussian perimeter in $\mathbb{R}^n$. Then, for every Borel set $B \subseteq \mathbb{R}^{n-1}$ we have

$$
\begin{aligned}
P_{\gamma_A}(E^s; B \times \mathbb{R}) &\leq P_{\gamma_A}(E; B \times \mathbb{R}) \\
&\quad + \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \left| \int_{E_z} \nabla' e^{-|\sqrt{A}x|^2/2} dy - \int_{E_z^s} \nabla' e^{-|\sqrt{A}x|^2/2} dy \right| dz.
\end{aligned}
$$

Moreover,

$$P_{\gamma_A}(E^s; B\times\mathbb{R}) \le P_{\gamma_A}(E; B\times\mathbb{R}) + \sqrt{2\pi}\|Ae_n - \langle Ae_n, e_n\rangle e_n\|\langle b_{\gamma_A}(E\cap(B\times\mathbb{R})) - b_{\gamma_A}(E^s\cap(B\times\mathbb{R})), e_n\rangle,$$

where

$$b_{\gamma_A}(E) := \int_E x\ d\gamma_A(x).$$

Proof. **Step 1**: For any Borel set B,

$$\begin{aligned} B &= (B\cap(B_E\cap B_{E^s}))\cup(B\cap\pi_+(E)\setminus(B_E\cap B_{E^s}))\cup(B\setminus\pi_+(E)) \\ &:= B_1\cup B_2\cup B_3. \end{aligned}$$

Recall that: $\pi_+(E)=\pi_+(E^s)$ and

$$B_E\subset\pi_+(E),\quad B_{E^s}\subset\pi_+(E^s)=\pi_+(E),\quad \mathcal{L}^{n-1}(\pi_+(E)\setminus B_E)=0,\quad \mathcal{L}^{n-1}(\pi_+(E)\setminus B_{E^s})=0.$$

Then

$$B_2\subset\pi_+(E)\setminus(B_E\cap B_{E^s}) = (\pi_+(E)\setminus B_E)\cup(\pi_+(E)\setminus B_{E^s}) \implies \mathcal{L}^{n-1}(B_2)=0.$$

Moreover, for any $z\in B_3$, $v_E(z)=0$. Thus, applying the dust estimate (Lemma 3.4.1) on B_3, the cylindrical estimate (Lemma 3.4.2) on B_2, the graphical estimate (Lemma 3.4.3) on $B_1\subset B_E\cap B_{E^s}$, we have

$$\begin{aligned} P_{\gamma_A}(E^s; B\times\mathbb{R}) &= P_{\gamma_A}(E^s; B_1\times\mathbb{R}) + P_{\gamma_A}(E^s; B_2\times\mathbb{R}) + P_{\gamma_A}(E; B_3\times\mathbb{R}) \\ &\le P_{\gamma_A}(E; B_1\times\mathbb{R}) + P_{\gamma_A}(E; B_2\times\mathbb{R}) + P_{\gamma_A}(E; B_3\times\mathbb{R}) \\ &\quad + \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_{B_1\cup B_2\cup B_3}\left|\int_{E_z}\nabla' e^{-|\sqrt{A}x|^2/2}dy - \int_{E_z^s}\nabla' e^{-|\sqrt{A}x|^2/2}dy\right| dz \\ &= P_{\gamma_A}(E; B\times\mathbb{R}) + \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B\left|\int_{E_z}\nabla' e^{-|\sqrt{A}x|^2/2}dy - \int_{E_z^s}\nabla' e^{-|\sqrt{A}x|^2/2}dy\right| dz. \end{aligned}$$

Step 2: Now we claim that

$$\begin{aligned} &\frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}}\int_B\left|\int_{E_z}\nabla' e^{-|\sqrt{A}x|^2/2}dy - \int_{E_z^s}\nabla' e^{-|\sqrt{A}x|^2/2}dy\right| dz \\ &= \sqrt{2\pi}\|Ae_n - \langle Ae_n, e_n\rangle e_n\|\langle b_{\gamma_A}(E\cap(B\times\mathbb{R})) - b_{\gamma_A}(E^s\cap(B\times\mathbb{R})), e_n\rangle. \end{aligned}$$

By Proposition 3.3.1 (2) and recall that $x = (z, y)$, i.e., $y = \langle x, e_n \rangle$, we have

$$\begin{aligned}
&\frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \left| \int_{E_z} \nabla' e^{-|\sqrt{A}x|^2/2} dy - \int_{E_z^s} \nabla' e^{-|\sqrt{A}x|^2/2} dy \right| dz \\
&= \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_B \left(\int_{E_z} y \, d\mu_z(y) - \int_{E_z^s} y \, d\mu_z(y) \right) \|Ae_n - \langle Ae_n, e_n \rangle e_n\| \, dz \\
&= \|Ae_n - \langle Ae_n, e_n \rangle e_n\| \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \left(\int_B \int_{E_z} y \, d\mu_z(y) dz - \int_B \int_{E_z^s} y \, d\mu_z(y) dz \right) \\
&= \|Ae_n - \langle Ae_n, e_n \rangle e_n\| \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \left(\int_{E \cap (E \times \mathbb{R})} y e^{-|\sqrt{A}x|^2/2} \, dx - \int_{E^s \cap (B \times \mathbb{R})} y e^{-|\sqrt{A}x|^2/2} \, dx \right) \\
&= \sqrt{2\pi} \|Ae_n - \langle Ae_n, e_n \rangle e_n\| \left\langle \left(\int_{E \cap (B \times \mathbb{R})} x \, d\gamma_A(x) - \int_{E^s \cap (B \times \mathbb{R})} x \, d\gamma_A(x) \right), e_n \right\rangle \\
&= \sqrt{2\pi} \|Ae_n - \langle Ae_n, e_n \rangle e_n\| \langle b_{\gamma_A}(E \cap (B \times \mathbb{R})) - b_{\gamma_A}(E^s \cap (B \times \mathbb{R})), e_n \rangle.
\end{aligned}$$

□

Our main goal here is to define the Ehrhard symmetrization to any direction $u \in \mathbb{S}^{n-1}$ and then extend the result of Theorem 3.4.4 to this new definition. We will start by recalling that Vol'pert theorem (Theorem 3.1.1) actually holds for every direction (see [AFP00], Theorem 3.108 and [Fus04], Theorem 3.21). That is, if E is a set of locally finite perimeter, the **one-dimensional slice of E through z in direction u** defined as

$$E_{z,u} := \{x = z + tu \in E : \; t \in \mathbb{R}\}$$

is also a set of locally finite perimeter. Moreover, $\left(\partial^M E\right)_{z,u} = \partial^M \left(E_{z,u}\right) = \partial^* \left(E_{z,u}\right) = \left(\partial^* E\right)_{z,u}$ and $\nu_u^E(x) := \langle \nu^E(x), u \rangle \neq 0$ for every t such that $x = z + tu \in \partial^* E$ where

$$x = z + tu \in \langle u \rangle^\perp \oplus \langle u \rangle.$$

The **Ehrhard symmetrization $E^s_{A,u}$ of E with respect to the u-direction and matrix A** is defined as

$$E^s_{A,u} := \left\{ x = z + tu \in \mathbb{R}^n : t > \Phi^{-1}_{A,z,u}\left(\mu_A(E_{z,u})\right) \right\}, \tag{3.32}$$

and the **essential projection of E with respect to the u-direction and matrix A** is defined as

$$\pi_{+,A,u}(E) := \left\{ z \in \langle u \rangle^{\perp} : \mu_A\left(E_{z,u}\right) > 0 \right\},$$

where

$$\mu_A(F) := \int_F e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^1(x), \quad \Phi_{A,z,u}(s) := \int_s^{\infty} e^{-|\sqrt{A}(z+tu)|^2/2} dt.$$

It is not hard to see that $\mu_A(E_{z,u}) = \mu_A((E^s_{A,u})_{z,u})$ and $\pi_{+,A,u}(E) = \pi_{+,A,u}(E^s_{A,u})$. Notice that the definition (3.32) agrees with the definition (3.3) in Section 3.1, i.e., if $u = -e_n$, we have

$$\pi_{+,A,-e_n}(E) = \pi_+(E), \quad E^s_{A,-e_n} = E^s.$$

Moreover, Theorem 3.4.4 says that

$$\begin{aligned} P_{\gamma_A}\left(E^s_{A,-e_n}; B \times \mathbb{R}\right) \leq P_{\gamma_A}(E; B \times \mathbb{R}) \\ + \sqrt{2\pi}\|A(-e_n) - \langle A(-e_n), (-e_n)\rangle(-e_n)\| \big\langle b_{\gamma_A}(E^s_{A,-e_n} \cap (B \times \mathbb{R})) \\ - b_{\gamma_A}(E \cap (B \times \mathbb{R})), -e_n \big\rangle. \end{aligned}$$

Our next goal is to extend this result to the Ehrhard symmetrization $E^s_{A,u}$. Before doing that, we need a lemma that helps us handle the rotation of the Ehrhard symmetrization $E^s_{A,u}$. The proof of it can be easily deduced by the change of variables and set theory and hence we omit the verification.

Lemma 3.4.5. *Let O be an orthogonal matrix such that $u = O(-e_n)$. Then*

$$(O^{-1}E)^s_{O^{\intercal}AO,-e_n} = O^{-1}E^s_{A,u}. \tag{3.33}$$

3.5 *Proof of Theorem 1.2.2*

We can always find an orthogonal matrix O such that $u = O(-e_n)$. By equation (3.33), we have

$$(O^{-1}E)^s_{O^{\intercal}AO,-e_n} = O^{-1}E^s_{A,u}.$$

Now we claim that

$$E^s_{A,u} = O[(O^{-1}E)^s_{O^\intercal AO,-e_n}] \quad \text{is a set of locally finite perimeter in } \mathbb{R}^n. \tag{3.34}$$

By Proposition 2.3.1, E is a set of locally finite perimeter. Applying Proposition 2.3.4 (3),

$$P_{\gamma_{O^\intercal AO}}(O^{-1}E) = P_{\gamma_A}(E) < \infty,$$

i.e., $O^{-1}E$ is a set of finite $O^\intercal AO$-anisotropic Gaussian perimeter. Then Theorem 3.2.4 tells us that $(O^{-1}E)^s_{O^\intercal AO,-e_n}$ is also a set of locally finite perimeter. By [Mag12], Exercise 15.10, $E^s_{A,u} = O[(O^{-1}E)^s_{O^\intercal AO,-e_n}]$ is a set of locally finite perimeter.

Next we prove the second part of the theorem. By equation (3.33), Proposition 2.3.4 (3), and $O^{-1}B \subset \mathbb{R}^{n-1}$, we have

$$\begin{aligned}
&P_{\gamma_A}\left(E^s_{A,u}; B \oplus \langle u\rangle\right) = P_{\gamma_A}\left(E^s_{A,u}; O(O^{-1}B) \oplus O\langle -e_n\rangle\right) = P_{\gamma_A}\left(E^s_{A,u}; O(O^{-1}B \times \mathbb{R})\right) \\
&= P_{\gamma_{O^\intercal AO}}\left(O^{-1}E^s_{A,u}; O^{-1}B \times \mathbb{R}\right) = P_{\gamma_{O^\intercal AO}}\left((O^{-1}E)^s_{O^\intercal AO,-e_n}; O^{-1}B \times \mathbb{R}\right).
\end{aligned}$$

Since $O^{-1}E$ is a set of finite $O^\intercal AO$-anisotropic Gaussian perimeter, we can apply Theorem 3.4.4 with E as $O^{-1}E$, B as $O^{-1}B$, and A as $O^\intercal AO$. Hence,

$$\begin{aligned}
&P_{\gamma_A}\left(E^s_{A,u}; B \oplus \langle u\rangle\right) = P_{\gamma_{O^\intercal AO}}\left((O^{-1}E)^s_{O^\intercal AO,-e_n}; O^{-1}B \times \mathbb{R}\right) \\
&\leq P_{\gamma_{O^\intercal AO}}(O^{-1}E; O^{-1}B \times \mathbb{R}) \\
&\quad + \sqrt{2\pi}\|O^\intercal AOe_n - \langle O^\intercal AOe_n, e_n\rangle e_n\| \Big\langle b_{\gamma_{O^\intercal AO}}((O^{-1}E)^s_{O^\intercal AO,-e_n} \cap (O^{-1}B \times \mathbb{R})) \\
&\quad - b_{\gamma_{O^\intercal AO}}(O^{-1}E \cap (O^{-1}B \times \mathbb{R})), -e_n\Big\rangle \\
&= P_{\gamma_A}\left(E; B \oplus \langle u\rangle\right) + \sqrt{2\pi}\|Au - \langle Au, u\rangle u\| \langle b_{\gamma_A}(E^s_{A,u} \cap (B \oplus \langle u\rangle)) - b_{\gamma_A}(E \cap (B \oplus \langle u\rangle)), u\rangle
\end{aligned}$$

where we have used $(O^{-1}E)^s_{O^\intercal AO,-e_n} = O^{-1}E^s_{A,u}$ and $b_{\gamma_{O^\intercal AO}}(E) = O^{-1}b_{\gamma_A}(OE)$. □

Our next result tells us that the anisotropic Gaussian perimeter decreases when the Ehrhard symmetrization is done along an eigenvector direction of A.

Corollary 3.5.1. *Let $n \geq 2$ and let E be a set of finite A-anisotropic Gaussian perimeter in $\mathbb{R}^n$. Assume that*

$$u \in V_\lambda(A) \cap \mathbb{S}^{n-1}$$

where $V_\lambda(A)$ is the eigenspace of A associated with eigenvalue λ. Then, for every Borel set $B \subseteq \langle u \rangle^\perp$, we have

$$P_{\gamma_A}\left(E^s_{A,u}; B \oplus \langle u \rangle\right) \leq P_{\gamma_A}\left(E; B \oplus \langle u \rangle\right), \tag{3.35}$$

and in particular $P_{\gamma_A}(E^s_{A,u}) \leq P_{\gamma_A}(E)$. Moreover, if $P_{\gamma_A}(E) = P_{\gamma_A}(E^s_{A,u})$, then

for $\mathcal{H}^{n-1}$-a.e. $z \in \langle u \rangle^\perp$, the slice $E_{z,u}$ is $\mathcal{H}^1$-equivalent to either $\varnothing$ or $\langle u \rangle$ or a half-line.

Proof. Since for $u \in V_\lambda(A) \cap \mathbb{S}^{n-1}$,

$$Au = \lambda u \implies \langle Au, u \rangle = \lambda \implies Au - \langle Au, u \rangle u = 0.$$

Then Theorem 1.2.2 shows that $P_{\gamma_A}\left(E^s_{A,u}; B \oplus \langle u \rangle\right) \leq P_{\gamma_A}\left(E; B \oplus \langle u \rangle\right)$. For the second part, we may assume that $u = -e_n$ since we can always rotate the coordinate system. Thus, by assumption,

$$P_{\gamma_A}(E) = P_{\gamma_A}(E^s), \qquad u = -e_n \in V_\lambda(A) \cap \mathbb{S}^{n-1}.$$

By (3.35), we have

$$P_{\gamma_A}(E^s; B \times \mathbb{R}) \leq P_{\gamma_A}(E; B \times \mathbb{R})$$

for every Borel set $B \subset \mathbb{R}^{n-1}$. We claim that

$$P_{\gamma_A}(E; B \times \mathbb{R}) = P_{\gamma_A}(E^s; B \times \mathbb{R}) \quad \text{for every Borel set } B \subseteq \mathbb{R}^{n-1}.$$

Suppose not, $P_{\gamma_A}(E; B \times \mathbb{R}) > P_{\gamma_A}(E^s; B \times \mathbb{R})$ for some Borel set B. Then

$$P_{\gamma_A}(E) = P_{\gamma_A}(E; B \times \mathbb{R}) + P_{\gamma_A}(E; B^c \times \mathbb{R}) > P_{\gamma_A}(E^s; B \times \mathbb{R}) + P_{\gamma_A}(E^s; B^c \times \mathbb{R}) = P_{\gamma_A}(E^s),$$

which contradicts our assumption. Now we plug in the Borel set $B_E \cap B_{E^s}$ from Vol'pert Theorem (Theorem 3.1.1), i.e., we have

$$P_{\gamma_A}\left(E; (B_E \cap B_{E^s}) \times \mathbb{R}\right) = P_{\gamma_A}\left(E^s; (B_E \cap B_{E^s}) \times \mathbb{R}\right). \tag{3.36}$$

Notice that $Ae_n = \lambda e_n$ and let $\lambda = d^2$ with $d > 0$, $x = (z, y)$, and $z = (z_1, \ldots, z_{n-1})$. Then

$$
\begin{aligned}
|\sqrt{A}x|^2 = \langle Ax, x\rangle &= \left\langle \sum_{i=1}^{n-1} z_i A e_i + y\lambda e_n, \sum_{j=1}^{n-1} z_j e_j + y e_n \right\rangle \\
&= \left\langle \sum_{i=1}^{n-1} z_i A e_i, \sum_{j=1}^{n-1} z_j e_j \right\rangle + \left\langle \sum_{i=1}^{n-1} z_i A e_i, y e_n \right\rangle + \left\langle y\lambda e_n, \sum_{j=1}^{n-1} z_j e_j \right\rangle + \lambda y^2 \\
&= \left\langle A \sum_{i=1}^{n-1} z_i e_i, \sum_{i=1}^{n-1} z_i e_i \right\rangle + \left\langle \sum_{i=1}^{n-1} z_i e_i, y A e_n \right\rangle + 0 + \lambda y^2 \\
&= \left| \sum_{i=1}^{n-1} z_i \sqrt{A} e_i \right|^2 + d^2 y^2 \qquad (3.37)
\end{aligned}
$$

since e_n is an eigenvector of A and A is symmetric. Therefore,

$$
\begin{aligned}
v_E(z) &= \int_{E_z} e^{-|\sqrt{A}x|^2/2} dy = e^{-\left|\sum_{i=1}^{n-1} z_i \sqrt{A} e_i\right|^2/2} \int_{E_z} e^{-d^2y^2/2}\, dy, \\
v_{E^s}(z) &= \int_{E_z^s} e^{-|\sqrt{A}x|^2/2} dy = e^{-\left|\sum_{i=1}^{n-1} z_i \sqrt{A} e_i\right|^2/2} \int_{E_z^s} e^{-d^2y^2/2}\, dy,
\end{aligned}
$$

and hence

$$
\int_{E_z} e^{-d^2y^2/2}\, dy = \int_{E_z^s} e^{-d^2y^2/2}\, dy
$$

since $v_E(z) = v_{E^s}(z)$. Moreover,

$$
\begin{aligned}
\int_{E_z} \nabla' \left(e^{-|\sqrt{A}x|^2/2}\right) dy &= \int_{E_z} \nabla' \left(e^{-\left|\sum_{i=1}^{n-1} z_i \sqrt{A} e_i\right|^2/2}\right) e^{-d^2y^2/2} dy \\
&= \nabla' \left(e^{-\left|\sum_{i=1}^{n-1} z_i \sqrt{A} e_i\right|^2/2}\right) \int_{E_z} e^{-d^2y^2/2} dy \\
&= \nabla' \left(e^{-\left|\sum_{i=1}^{n-1} z_i \sqrt{A} e_i\right|^2/2}\right) \int_{E_z^s} e^{-d^2y^2/2} dy \\
&= \int_{E_z^s} \nabla' \left(e^{-|\sqrt{A}x|^2/2}\right) dy. \qquad (3.38)
\end{aligned}
$$

By Lemma 3.4.3, Proposition 3.3.1 (1)(3), (3.36), and (3.38),

$$
\begin{aligned}
&P_{\gamma_A}(E; (B_E \cap B_{E^s}) \times \mathbb{R}) \\
&\geq \frac{\det \sqrt{A}}{(2\pi)^{(n-1)/2}} \int_{B_E \cap B_{E^s}} \sqrt{p_E(z)^2 + \left|\nabla' v_E(z) - \int_{E_z} \nabla' \left(e^{-|\sqrt{A}x|^2/2}\right) dy\right|^2}\, dz
\end{aligned}
$$

$$\geq \frac{\det\sqrt{A}}{(2\pi)^{(n-1)/2}} \int_{B_E\cap B_{E^s}} \sqrt{p_{E^s}(z)^2 + \left|\nabla' v_{E^s}(z) - \int_{E^s_z} \nabla'\left(e^{-|\sqrt{A}x|^2/2}\right) dy\right|^2}\, dz \tag{3.39}$$
$$= P_{\gamma_A}\left(E^s; (B_E\cap B_{E^s})\times\mathbb{R}\right) = P_{\gamma_A}(E; (B_E\cap B_{E^s})\times\mathbb{R}).$$

Therefore, (3.39) implies that

$$p_E(z) = p_{E^s}(z) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } z\in B_E\cap B_{E^s}. \tag{3.40}$$

Moreover, let $z\in B_E\cap B_{E^s}$,

$$p_E(z) = \int_{\partial^* E_z} e^{-|\sqrt{A}x|^2/2}\, d\mathcal{H}^0(y) = e^{-\left|\sum_{i=1}^{n-1} z_i\sqrt{A}e_i\right|^2/2} \int_{\partial^* E_z} e^{-d^2y^2/2}\, d\mathcal{H}^0(y), \tag{3.41}$$

$$\phi_z(t) = \int_{-\infty}^t e^{-|\sqrt{A}x|^2/2}\, dy = e^{-\left|\sum_{i=1}^{n-1} z_i\sqrt{A}e_i\right|^2/2}\int_{-\infty}^t e^{-d^2y^2/2}\, dy = \frac{\sqrt{2\pi}}{d} e^{-\left|\sum_{i=1}^{n-1} z_i\sqrt{A}e_i\right|^2/2}\phi(dt),$$

and hence

$$\phi_z^{-1}(v_E(z)) = \frac{1}{d}\phi^{-1}\left(\frac{d}{\sqrt{2\pi}}\int_{E_z} e^{-d^2y^2/2}dy\right) = \frac{1}{d}\phi^{-1}(\gamma_{d^2}(E_z)), \tag{3.42}$$

where γ_{d^2} is the d^2-anisotropic Gaussian measure, i.e.,

$$\gamma_{d^2}(F) = \frac{d}{\sqrt{2\pi}}\int_F e^{-d^2y^2/2}dy.$$

By equation (3.42),

$$p_{E^s}(z) = e^{-\left|\sum_{i=1}^{n-1} z_i\sqrt{A}e_i\right|^2/2}\int_{\partial^* E^s_z} e^{-d^2y^2/2}\, d\mathcal{H}^0(y) = e^{-\left|\sum_{i=1}^{n-1} z_i\sqrt{A}e_i\right|^2/2}\, e^{-d^2[\phi_z^{-1}(v_E(z))]^2/2}$$
$$= e^{-\left|\sum_{i=1}^{n-1} z_i\sqrt{A}e_i\right|^2/2}\, e^{-[\phi^{-1}(\gamma_{d^2}(E_z))]^2/2} \tag{3.43}$$

where $E^s_z = \left(-\infty, \phi_z^{-1}(v_E(z))\right)$. Therefore, (3.40), (3.41), and (3.43) implies that

$$\int_{\partial^* E_z} e^{-d^2y^2/2}\, d\mathcal{H}^0(y) = e^{-[\phi^{-1}(\gamma_{d^2}(E_z))]^2/2}$$

for $\mathcal{H}^{n-1}$-a.e. $z\in B_E\cap B_{E^s}$. Since $\mathcal{H}^{n-1}(\pi_+(E)\setminus B_E\cap B_{E^s}) = 0$, we have

$$P_{\gamma_{d^2}}(E_z) = d\int_{\partial^* E_z} e^{-d^2y^2/2}d\mathcal{H}^0(y) = e^{-[\phi^{-1}(\gamma_{d^2}(E_z))]^2/2}d$$

for $\mathcal{H}^{n-1}$-a.e. $z \in \pi_+(E)$. Thanks to the equality case of the one-dimensional anisotropic Gaussian isoperimetric inequality (see Theorem 1.2.1), for $\mathcal{H}^{n-1}$-a.e. $z \in \pi_+(E)$, E_z is either $\mathcal{H}^1$-equivalent to $\varnothing$ or $\mathbb{R}$ or a half-line. Notice that for any $z \in \pi_+(E)^c$,

$$v_E(z) = 0 \implies \mathcal{H}^1(E_z) = 0 \implies E_z \text{ is } \mathcal{H}^1\text{-equivalent to } \varnothing.$$

In other words,

for $\mathcal{H}^{n-1}$-a.e. $z \in \langle -e_n \rangle^\perp$, the slice $E_{z,-e_n}$ is $\mathcal{H}^1$-equivalent to either $\varnothing$ or $\mathbb{R}$ or a half-line.

□

From Theorem 1.2.2, we see that

$$P_{\gamma_A}(E^s_{A,u}) \leq P_{\gamma_A}(E) + \sqrt{2\pi}\|Au - \langle Au, u\rangle u\| \langle b_{\gamma_A}(E^s_{A,u}) - b_{\gamma_A}(E), u\rangle,$$

for any set of finite anisotropic Gaussian perimeter in $\mathbb{R}^n$. A natural question here is whether

$$\left| P_{\gamma_A}(E^s_{A,u}) - P_{\gamma_A}(E) \right| \leq M\|Au - \langle Au, u\rangle u\| \langle b_{\gamma_A}(E^s_{A,u}) - b_{\gamma_A}(E), u\rangle$$

for some constant M. Our final example shows that this is not the case.

Example 3.5.1. We give an example to show that the following statement is not true: for any $0 < \lambda_1 < \lambda_2$, there exists $M > 0$ such that for any $\lambda(A) \subset [\lambda_1, \lambda_2]$, for any $u \in \mathbb{S}^{n-1}$, and for any set of finite anisotropic Gaussian perimeter E in $\mathbb{R}^n$,

$$\left| P_{\gamma_A}(E^s_{A,u}) - P_{\gamma_A}(E) \right| \leq M\|Au - \langle Au, u\rangle u\| \langle b_{\gamma_A}(E^s_{A,u}) - b_{\gamma_A}(E), u\rangle,$$

where $\lambda(A)$ is the set of all eigenvalues of A, i.e., the **spectrum** of A.

Proof. Consider $\lambda_1 = \frac{1}{2} < \frac{3}{2} = \lambda_2$. Suppose there exists $M > 0$ such that for any $\lambda(A) \subset [\lambda_1, \lambda_2]$, for any $u \in \mathbb{S}^{n-1}$, and for any set of finite anisotropic Gaussian perimeter E in $\mathbb{R}^n$,

$$\left| P_{\gamma_A}(E^s_{A,u}) - P_{\gamma_A}(E) \right| \leq M\|Au - \langle Au, u\rangle u\| \langle b_{\gamma_A}(E^s_{A,u}) - b_{\gamma_A}(E), u\rangle.$$

Take

$$n = 2, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad u = -e_2, \quad E = [-1,1]^2.$$

Then clearly, $\lambda(A) \subset [\lambda_1, \lambda_2]$ and $Au - \langle Au, u \rangle u = 0$. That is,

$$P_{\gamma_A}(E^s) = P_{\gamma_A}(E),$$

where $E^s = E^s_{A,-e_n}$. By Corollary 3.5.1, for $\mathcal{H}^1$-a.e. $z \in \mathbb{R}^1$, the slice E_z is $\mathcal{H}^1$-equivalent to either $\varnothing$ or $\mathbb{R}$ or a half-line. However, for all $z \in [-1,1]$, the slice E_z is an interval $[-1,1]$. This gives us a contradiction. $\square$

Chapter 4

CHARACTERIZATION OF EHRHARD MEASURES IN THE CLASS $\mathcal{A}(\mathbb{R}^n)$

The anisotropic Gaussian perimeter always decreases if $u \in V_\lambda(A) \cap \mathbb{S}^{n-1}$, where $V_\lambda(A)$ is the eigenspace of A associated with eigenvalue λ (see Corollary 3.5.1). In fact, this is a necessary and sufficient condition for the anisotropic Gaussian perimeter to be decreasing in a given direction. A measure γ_A is called **Ehrhard symmetrizable** (or simply an **Ehrhard measure**) if

$$P_{\gamma_A}(E^s_{A,u}) \le P_{\gamma_A}(E)$$

for all $u \in \mathbb{S}^{n-1}$, and for all finite A-anisotropic Gaussian perimeter set E in $\mathbb{R}^n$. Let

$$\begin{aligned}\mathcal{A}(\mathbb{R}^n) &= \{\text{anisotropic Gaussian measures}\}\\ &= \left\{\mu : \frac{d\mu}{d\mathcal{L}^n}(x) = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} e^{-\langle Ax,x\rangle/2},\ A \text{ is a symmetric positive definite matrix}\right\}.\end{aligned}$$

We show that $\mu = \gamma_A \in \mathcal{A}(\mathbb{R}^n)$ is Ehrhard symmetrizable if and only if A is a multiple of the identity matrix (see Theorem 1.2.3).

4.1 A regularity lemma for Ehrhard symmetrization sets

Lemma 4.1.1 (A regularity lemma for E^s)**.**
Let $A = (A_{ij}) \in M_n(\mathbb{R})$ be a symmetric positive definite matrix. Suppose $E = \Omega \times (0,\infty)$ with an open set $\Omega \subset \mathbb{R}^{n-1}$ that contains the origin. Then

$$E^s = E^s_{A,-e_n} = \{x = (z,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : z \in \Omega, y < h(z)\},$$

where $h(z) := \phi_z^{-1}(v_E(z))$ is $C^1(\Omega)$ and $\nabla' h$ is locally Lipschitz on Ω. In particular, $h(0) = 0$,

$$\nabla' h(0) = -2\left(\int_0^\infty y e^{-|\sqrt{A}(0,y)|^2/2}\, dy\right) A' e_n,$$

where $\nabla' = (\partial_1, \ldots, \partial_{n-1})$ *and* $A' \in M_{(n-1)\times n}(\mathbb{R})$ *is the first* $n-1$ *rows of matrix from* A. *Also,*

$$\nabla' h(0) = 0 \iff A_{1n} = A_{2n} = \ldots = A_{n-1,n} = 0 \iff e_n \in V_{A_{nn}}(A).$$

Moreover,

$$h(z) = \ell(z) + \int_0^1 \langle \nabla' h(tz) - \nabla' h(0), z\rangle \, dt \quad \textit{for all } z \in \Omega$$

where

$$\ell(z) := h(0) + \langle \nabla' h(0), z\rangle = -2\left(\int_0^\infty y e^{-|\sqrt{A}(0,y)|^2/2} \, dy\right) A' e_n \cdot z$$

Proof. Recall that

$$E^s = \left\{(z,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : z \in \Omega, y < h(z) = \phi_z^{-1}\left(v_E(z)\right)\right\}$$

where

$$v_E(z) = \int_{E_z} e^{-|\sqrt{A}x|^2/2} dy, \qquad \phi_z(t) = \int_{-\infty}^t e^{-|\sqrt{A}x|^2/2} dy.$$

Since $E = \Omega \times (0,\infty)$, $E_z = (0,\infty)$ for all $z \in \Omega$. By Lemma 3.2.1 (2)(a)(b),

$$v_E(z) = \int_{E_z} e^{-|\sqrt{A}x|^2/2} dy = \int_0^\infty e^{-|\sqrt{A}x|^2/2} dy \quad \text{is differentiable on } \Omega$$

and

$$\nabla' v_E(z) = -\int_0^\infty e^{-|\sqrt{A}x|^2/2} A' x \, dy.$$

Now we claim that $\nabla' v_E$ is locally Lipschitz on Ω and hence, by Lemma 3.2.3,

$$h : z \mapsto \phi_z^{-1}(v_E(z)) \text{ is in } C^1(\Omega)$$

and $\nabla' h$ is locally Lipschitz on Ω. Let K be any compact set in Ω and let $z_1, z_2 \in K$. Then

$$\begin{aligned}
|\nabla' v_E(z_1) - \nabla' v_E(z_2)| &= \left|\int_0^\infty e^{-|\sqrt{A}(z_1,y)|^2/2} A'(z_1,y) dy - \int_0^\infty e^{-|\sqrt{A}(z_2,y)|^2/2} A'(z_2,y) dy\right| \\
&\le \left|\int_0^\infty e^{-|\sqrt{A}(z_1,y)|^2/2} A'(z_1,y) dy - \int_0^\infty e^{-|\sqrt{A}(z_1,y)|^2/2} A'(z_2,y) dy\right| \\
&\quad + \left|\int_0^\infty e^{-|\sqrt{A}(z_1,y)|^2/2} A'(z_2,y) dy - \int_0^\infty e^{-|\sqrt{A}(z_2,y)|^2/2} A'(z_2,y) dy\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left| \int_0^\infty e^{-|\sqrt{A}(z_1,y)|^2/2} A'(z_1 - z_2, 0) dy \right| \\
&\quad + \int_0^\infty \left| e^{-|\sqrt{A}(z_1,y)|^2/2} - e^{-|\sqrt{A}(z_2,y)|^2/2} \right| |A'(z_2, y)| dy \\
&\leq \int_0^\infty e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} \sqrt{\lambda_{\max}(A'^\mathsf{T} A')} |z_1 - z_2| dy \\
&\quad + \int_0^\infty \lambda_{\max}(A'^\mathsf{T} A') \Big(r(K) + |y| \Big)^2 e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} |z_1 - z_2| \, dy \\
&= C(K, A') |z_1 - z_2|, \qquad (4.1)
\end{aligned}
$$

where we have used the estimate (3.6), $r(K) = \sup_{\zeta \in K} |\zeta|$, and

$$
C(K, A') := \sqrt{\lambda_{\max}(A'^\mathsf{T} A')} \left(\int_0^\infty \left(1 + \sqrt{\lambda_{\max}(A'^\mathsf{T} A')} \Big(r(K) + |y| \Big)^2 \right) e^{-\|(\sqrt{A})^{-1}\|^{-2}|y|^2/2} \, dy \right).
$$

Next, notice that

$$
\int_0^\infty e^{-|\sqrt{A}x|^2/2} dy = v_E(z) = v_{E^s}(z) = \int_{-\infty}^{h(z)} e^{-|\sqrt{A}x|^2/2} dy. \qquad (4.2)
$$

Setting $z = 0$, we have

$$
\int_{-\infty}^0 e^{-|\sqrt{A}(0,y)|^2/2} dy = \int_0^\infty e^{-|\sqrt{A}(0,y)|^2/2} dy = \int_{-\infty}^{h(0)} e^{-|\sqrt{A}(0,y)|^2/2} dy.
$$

Therefore,

$$
h(0) = 0. \qquad (4.3)
$$

Taking the derivative on both sides with respect to z of equation (4.2), by Lemma 3.2.1 (1),

$$
- \int_0^\infty e^{-|\sqrt{A}x|^2/2} A' x \, dy = (\nabla' h(z)) e^{-|\sqrt{A}(z, h(z))|^2/2} - \int_{-\infty}^{h(z)} e^{-|\sqrt{A}x|^2/2} A' x \, dy.
$$

Setting $z = 0$ again,

$$
\begin{aligned}
\nabla' h(0) &= - \int_0^\infty e^{-|\sqrt{A}(0,y)|^2/2} A'(0, y) \, dy + \int_{-\infty}^0 e^{-|\sqrt{A}(0,y)|^2/2} A'(0, y) \, dy \\
&= -2 \int_0^\infty e^{-|\sqrt{A}(0,y)|^2/2} A'(0, y) \, dy = -2 \left(\int_0^\infty y e^{-|\sqrt{A}(0,y)|^2/2} \, dy \right) A' e_n. \qquad (4.4)
\end{aligned}
$$

Thus,

$$\nabla' h(0) = 0 \iff A' e_n = 0 \iff A_{1n} = A_{2n} = \ldots = A_{n-1,n} = 0 \iff e_n \in V_{A_{nn}}(A)$$

since A is symmetric and

$$\int_0^\infty y e^{-|\sqrt{A}(0,y)|^2/2}\, dy \geq \int_0^\infty y e^{-\|\sqrt{A}\|^2 |y|^2/2} dy = \frac{1}{\|\sqrt{A}\|^2} > 0.$$

Applying [Dru20], Theorem 1.14 with (4.3) and (4.4), we have

$$h(z) = \ell(z) + \int_0^1 \langle \nabla' h(tz) - \nabla' h(0), z \rangle \, dt$$

where

$$\ell(z) := h(0) + \langle \nabla' h(0), z \rangle = -2 \left(\int_0^\infty y e^{-|\sqrt{A}(0,y)|^2/2}\, dy \right) A' e_n \cdot z.$$

□

4.2 *Proof of Theorem 1.2.3*

For the first part, we just need to show that

$$P_{\gamma_A}(E^s_{A,u}) \leq P_{\gamma_A}(E) \text{ for all finite } A\text{-anisotropic Gaussian perimeter set } E \text{ in } \mathbb{R}^n$$
$$\implies u \in V_\lambda(A) \cap \mathbb{S}^{n-1} \text{ for some } \lambda > 0 \tag{4.5}$$

since Corollary 3.5.1 gives us the converse of the statement.

Step 1: Assume that $u = -e_n$ and we have $P_{\gamma_A}(E^s) \leq P_{\gamma_A}(E)$ for all finite A-anisotropic Gaussian perimeter set E in $\mathbb{R}^n$, where $E^s = E^s_{A,-e_n}$. Our goal is to show that

$$e_n \in V_\lambda(A) \cap \mathbb{S}^{n-1}$$

for some $\lambda > 0$. Let $K = [-1,1]^{n-1}$ and Ω be an open convex set that contains K. Consider

$$E = \Omega \times (0, \infty), \quad E_\alpha = [-\alpha, \alpha]^{n-1} \times (0, \infty)$$

for $\alpha \in (0,1)$. By Lemma 4.1.1, the Ehrhard symmetrization of E has the form

$$E^s = \{x = (z,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : z \in \Omega, y < h(z)\}$$

where $h(z) := \phi_z^{-1}(v_E(z))$ is $C^1(\Omega)$ and $\nabla' h$ is locally Lipschitz on Ω. Hence $\nabla' h$ is Lipschitz on K. Also, the Ehrhard symmetrization of E_α has the form

$$E_\alpha^s = \{x = (z, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : z \in [-\alpha, \alpha]^{n-1}, y < h(z)\}.$$

We claim that

$$P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s) = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \left(1 - \sqrt{1 + [\nabla' h(0)]^2}\right) \alpha + o(\alpha).$$

Let $S_k = C_k \times (0, \infty)$ $(k \geq 1)$ be hypersurfaces in $\mathbb{R}^n$, where $\{C_k\}_{k=1}^{2(n-1)}$ are faces of the $(n-1)$ dimension cube $[-\alpha, \alpha]^{n-1} \subset \mathbb{R}^{n-1}$, and

$$S_0 = [-\alpha, \alpha]^{n-1} \times \{0\}.$$

For example,

$$\begin{aligned}
S_1 &= \left([-\alpha, \alpha]^{n-2} \times \{-\alpha\}\right) \times (0, \infty), \\
S_2 &= \left([-\alpha, \alpha]^{n-2} \times \{\alpha\}\right) \times (0, \infty), \\
S_3 &= \left([-\alpha, \alpha]^{n-3} \times \{-\alpha\} \times [-\alpha, \alpha]\right) \times (0, \infty), \\
S_4 &= \left([-\alpha, \alpha]^{n-3} \times \{\alpha\} \times [-\alpha, \alpha]\right) \times (0, \infty), \\
S_5 &= \left([-\alpha, \alpha]^{n-4} \times \{-\alpha\} \times [-\alpha, \alpha]^2\right) \times (0, \infty), \\
S_6 &= \left([-\alpha, \alpha]^{n-4} \times \{\alpha\} \times [-\alpha, \alpha]^2\right) \times (0, \infty), \\
&\vdots
\end{aligned}$$

and

$$\partial^* E_\alpha = \bigcup_{k=1}^{2(n-1)} S_k \cup S_0 \implies P_{\gamma_A}(E_\alpha) = \sum_{k=1}^{2(n-1)} \mathcal{H}_{\gamma_A}^{n-1}(S_k) + \mathcal{H}_{\gamma_A}^{n-1}(S_0).$$

For E_α^s, we also have

$$\begin{aligned}
S_0^s &= \{x = (z, y) \in [-\alpha, \alpha]^{n-1} \times \mathbb{R} : y = h(z)\}, \\
S_1^s &= \left([-\alpha, \alpha]^{n-2} \times \{-\alpha\}\right) \times \left(-\infty, h\left([-\alpha, \alpha]^{n-2} \times \{-\alpha\}\right)\right), \\
S_2^s &= \left([-\alpha, \alpha]^{n-2} \times \{\alpha\}\right) \times \left(-\infty, h\left([-\alpha, \alpha]^{n-2} \times \{\alpha\}\right)\right),
\end{aligned}$$

$$
\begin{aligned}
S_3^s &= \left([-\alpha,\alpha]^{n-3}\times\{-\alpha\}\times[-\alpha,\alpha]\right)\times\left(-\infty, h\left([-\alpha,\alpha]^{n-3}\times\{-\alpha\}\times[-\alpha,\alpha]\right)\right),\\
S_4^s &= \left([-\alpha,\alpha]^{n-3}\times\{\alpha\}\times[-\alpha,\alpha]\right)\times\left(-\infty, h\left([-\alpha,\alpha]^{n-3}\times\{\alpha\}\times[-\alpha,\alpha]\right)\right),\\
S_5^s &= \left([-\alpha,\alpha]^{n-4}\times\{-\alpha\}\times[-\alpha,\alpha]^2\right)\times\left(-\infty, h\left([-\alpha,\alpha]^{n-4}\times\{-\alpha\}\times[-\alpha,\alpha]^2\right)\right),\\
S_6^s &= \left([-\alpha,\alpha]^{n-4}\times\{\alpha\}\times[-\alpha,\alpha]^2\right)\times\left(-\infty, h\left([-\alpha,\alpha]^{n-4}\times\{\alpha\}\times[-\alpha,\alpha]^2\right)\right),\\
&\vdots
\end{aligned}
$$

and

$$\partial^* E_\alpha^s = \bigcup_{k=1}^{2(n-1)} S_k^s \cup S_0^s \implies P_{\gamma_A}(E_\alpha^s) = \sum_{k=1}^{2(n-1)} \mathcal{H}_{\gamma_A}^{n-1}(S_k^s) + \mathcal{H}_{\gamma_A}^{n-1}(S_0^s).$$

Therefore,

$$P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s) = \sum_{k=1}^{2(n-1)} \left(\mathcal{H}_{\gamma_A}^{n-1}(S_k) - \mathcal{H}_{\gamma_A}^{n-1}(S_k^s)\right) + \left(\mathcal{H}_{\gamma_A}^{n-1}(S_0) - \mathcal{H}_{\gamma_A}^{n-1}(S_0^s)\right). \tag{4.6}$$

(a) First we claim that

$$\left|\mathcal{H}_{\gamma_A}^{n-1}(S_k) - \mathcal{H}_{\gamma_A}^{n-1}(S_k^s)\right| = o(\alpha^{n-1}) \quad \text{for all } k \geq 1.$$

For $k = 1$, we have $S_1 = [-\alpha,\alpha]^{n-2}\times\{-\alpha\}\times(0,\infty)$ and

$$S_1^s = \left([-\alpha,\alpha]^{n-2}\times\{-\alpha\}\right)\times\left(-\infty, h\left([-\alpha,\alpha]^{n-2}\times\{-\alpha\}\right)\right).$$

Let $r(u,v) = (u,-\alpha,v) \in [-\alpha,\alpha]^{n-2}\times\{-\alpha\}\times(0,\infty)$. Then $J(r) = \sqrt{\det(Dr)^{\mathsf{T}}(Dr)} = 1$ and

$$\mathcal{H}_{\gamma_A}^{n-1}(S_1) = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{S_1} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{[-\alpha,\alpha]^{n-2}} \int_0^\infty e^{-|\sqrt{A}(u,-\alpha,v)|^2/2} dv du.$$

Similarly, we have

$$\mathcal{H}_{\gamma_A}^{n-1}(S_1^s) = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{S_1^s} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{[-\alpha,\alpha]^{n-2}} \int_{-\infty}^{h(u,-\alpha)} e^{-|\sqrt{A}(u,-\alpha,v)|^2/2} dv du.$$

Let $z = (u, -\alpha)$. We now estimate the following two quantities:

$$\text{(i)} \int_0^\infty e^{-|\sqrt{A}(z,v)|^2/2} dv, \qquad \text{(ii)} \int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,v)|^2/2} dv.$$

For (i), using the Taylor expansion on the map

$$z \mapsto \int_0^\infty e^{-|\sqrt{A}(z,v)|^2/2} dv,$$

we have

$$\nabla' \int_0^\infty e^{-|\sqrt{A}(z,v)|^2/2} dv = \int_0^\infty e^{-|\sqrt{A}(z,v)|^2/2} \big(-A'(z,v)\big)\, dv$$

and

$$\int_0^\infty e^{-|\sqrt{A}(z,v)|^2/2} dv = \int_0^\infty e^{-|\sqrt{A}(0,v)|^2/2} dv + \int_0^\infty e^{-|\sqrt{A}(0,v)|^2/2} (-A'(0,v))\ dv \cdot z + o(|z|).$$

For (ii), using the Taylor expansion on the map

$$z \mapsto \int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,v)|^2/2} dv,$$

we have

$$\nabla' \int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,v)|^2/2} dv = \nabla' h(z) e^{-|\sqrt{A}(z,h(z))|^2/2} + \int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,v)|^2/2} \big(-A'(z,v)\big)\, dv$$

and

$$\begin{aligned}
\int_{-\infty}^{h(z)} e^{-|\sqrt{A}(z,v)|^2/2} dv &= \int_{-\infty}^{0} e^{-|\sqrt{A}(0,v)|^2/2} dv \\
&\quad + \left(\nabla' h(0) + \int_{-\infty}^{0} e^{-|\sqrt{A}(0,v)|^2/2} (-A'(0,v))\ dv\right) \cdot z + o(|z|) \\
&= \int_0^\infty e^{-|\sqrt{A}(0,v)|^2/2} dv \\
&\quad + \left(\nabla' h(0) + \int_0^\infty e^{-|\sqrt{A}(0,v)|^2/2} A'(0,v)\ dv\right) \cdot z + o(|z|).
\end{aligned}$$

since $h(0) = 0$. Therefore,

$$\begin{aligned}
\text{(i)} - \text{(ii)} &= -\nabla' h(0) \cdot z + \left(-2 \int_0^\infty e^{-|\sqrt{A}(0,v)|^2/2} A'(0,v)\ dv\right) \cdot z + o(|z|) \\
&= -\nabla' h(0) \cdot z + \left(-2 \int_0^\infty v e^{-|\sqrt{A}(0,v)|^2/2}\ dv\right) A' e_n \cdot z + o(|z|) \\
&= o(|z|) \qquad (4.7)
\end{aligned}$$

where we have used Lemma 4.1.1, i.e.,

$$\nabla' h(0) = -2 \left(\int_0^\infty y e^{-|\sqrt{A}(0,y)|^2/2} \, dy \right) A' e_n.$$

Since $u \in [-\alpha, \alpha]^{n-2}$, $|z| = |(u, -\alpha)| \leq \sqrt{n-1}\alpha$, by equation (4.7),

$$\int_0^\infty e^{-|\sqrt{A}(u,-\alpha,v)|^2/2} dv - \int_{-\infty}^{h(u,-\alpha)} e^{-|\sqrt{A}(u,-\alpha,v)|^2/2} dv = o(\alpha)$$

and hence

$$\begin{aligned}
&\mathcal{H}^{n-1}_{\gamma_A}(S_1) - \mathcal{H}^{n-1}_{\gamma_A}(S_1^s) \\
&= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{[-\alpha,\alpha]^{n-2}} \left(\int_0^\infty e^{-|\sqrt{A}(u,-\alpha,v)|^2/2} dv - \int_{-\infty}^{h(u,-\alpha)} e^{-|\sqrt{A}(u,-\alpha,v)|^2/2} dv \right) du \\
&= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{[-\alpha,\alpha]^{n-2}} o(\alpha) \, du = o(\alpha^{n-1}).
\end{aligned}$$

That is,

$$\left| \mathcal{H}^{n-1}_{\gamma_A}(S_1) - \mathcal{H}^{n-1}_{\gamma_A}(S_1^s) \right| = o(\alpha^{n-1}).$$

Similarly for all $k \geq 1$, we have

$$\left| \mathcal{H}^{n-1}_{\gamma_A}(S_k) - \mathcal{H}^{n-1}_{\gamma_A}(S_k^s) \right| = o(\alpha^{n-1}).$$

(b) Next we claim that

$$\begin{aligned}
P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s) &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \left(\int_{S_0} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} - \int_{S_0^s} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} \right) + o(\alpha^{n-1}) \\
&= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \left(1 - \sqrt{1 + [\nabla' h(0)]^2} \right) (2\alpha)^{n-1} + o(\alpha^{n-1}).
\end{aligned}$$

Let $r(z) = (z, h(z))$. Then $J(r) = \sqrt{\det (Dr)^\intercal (Dr)} = \sqrt{1 + |\nabla' h(z)|^2}$ and

$$\begin{aligned}
\mathcal{H}^{n-1}_{\gamma_A}(S_0^s) &= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{S_0^s} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} \\
&= \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \int_{[-\alpha,\alpha]^{n-1}} e^{-|\sqrt{A}(z,h(z))|^2/2} \sqrt{1 + |\nabla' h(z)|^2} \, dz.
\end{aligned}$$

Define $f : \Omega \to \mathbb{R}$ as

$$f(z) = e^{-|\sqrt{A}(z,0)|^2/2} - e^{-|\sqrt{A}(z,h(z))|^2/2}\sqrt{1 + [\nabla' h(z)]^2}.$$

Since $h \in C^1(\Omega)$, f is continuous on Ω and hence

$$\lim_{\alpha \to 0^+} \frac{1}{(2\alpha)^{n-1}} \int_{[-\alpha,\alpha]^{n-1}} f(z)dz = f(0).$$

Therefore,

$$\begin{aligned}
&\int_{S_0} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} - \int_{S_0^s} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} \\
&= \int_{[-\alpha,\alpha]^{n-1}} e^{-|\sqrt{A}(z,0)|^2/2} dz - \int_{[-\alpha,\alpha]^{n-1}} e^{-|\sqrt{A}(z,h(z))|^2/2}\sqrt{1 + [\nabla' h(z)]^2}\ dz \\
&= \int_{[-\alpha,\alpha]^{n-1}} \left(e^{-|\sqrt{A}(z,0)|^2/2} - e^{-|\sqrt{A}(z,h(z))|^2/2}\sqrt{1 + [\nabla' h(z)]^2}\right)\ dz \\
&= \int_{[-\alpha,\alpha]^{n-1}} f(0) + \big(f(z) - f(0)\big)\ dz \\
&= \int_{[-\alpha,\alpha]^{n-1}} \left(1 - \sqrt{1 + [\nabla' h(0)]^2}\right) dz + \int_{[-\alpha,\alpha]^{n-1}} \big(f(z) - f(0)\big)\, dz \\
&= \left(1 - \sqrt{1 + [\nabla' h(0)]^2}\right)(2\alpha)^{n-1} + o(\alpha^{n-1}).
\end{aligned} \tag{4.8}$$

By equation (4.6) and (a),

$$\begin{aligned}
&\left| P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s) - \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \left(\int_{S_0} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} - \int_{S_0^s} e^{-|\sqrt{A}x|^2/2} d\mathcal{H}^{n-1} \right) \right| \\
&\leq \sum_{k=1}^{2(n-1)} \left| \mathcal{H}^{n-1}_{\gamma_A}(S_k) - \mathcal{H}^{n-1}_{\gamma_A}(S_k^s) \right| = o(\alpha^{n-1}).
\end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), we have

$$P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s) = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} \left(1 - \sqrt{1 + [\nabla' h(0)]^2}\right)(2\alpha)^{n-1} + o(\alpha^{n-1}).$$

(c) We claim that

$$A_{1n} = A_{2n} = \ldots = A_{n-1,n} = 0.$$

By our assumption and (b),

$$0 \le P_{\gamma_A}(E_\alpha) - P_{\gamma_A}(E_\alpha^s) = \frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}}\left(1 - \sqrt{1 + [\nabla' h(0)]^2}\right)(2\alpha)^{n-1} + o(\alpha^{n-1}).$$

Dividing $(2\alpha)^{n-1}$ on both sides and taking $\alpha \to 0^+$, by Lemma 4.1.1,

$$0 \le \left(1 - \sqrt{1 + [\nabla' h(0)]^2}\right) \implies \nabla' h(0) = 0 \implies A_{1n} = A_{2n} = \ldots = A_{n-1,n} = 0.$$

Hence, $e_n \in V_\lambda(A) \cap \mathbb{S}^{n-1}$ for some $\lambda > 0$.

Step 2: For general $u \in \mathbb{S}^{n-1}$, there exists an orthogonal matrix O such that $O(-e_n) = u$. Let $B = O^\intercal AO$. Given any finite B-anisotropic Gaussian perimeter set $\widetilde{E}$ and let $E = O\widetilde{E}$. By Proposition 2.3.1, $\widetilde{E}$ is a set of locally finite perimeter. Applying Proposition 2.3.4 (3) with E as $\widetilde{E}$, A as B, and O as O^{-1},

$$P_{\gamma_A}(E) = P_{\gamma_{OBO^\intercal}}(O\widetilde{E}) = P_{\gamma_B}(\widetilde{E}) < \infty,$$

i.e., E is a set of finite A-anisotropic Gaussian perimeter. Then Theorem 1.2.2 tells us that $E^s_{A,u}$ is also a set of locally finite perimeter. Since γ_A is Ehrhard symmetrizable and E is a set of finite A-anisotropic Gaussian perimeter, Proposition 2.3.4 (3) and equation (3.33) give us

$$\begin{aligned} P_{\gamma_B}(\widetilde{E}^s_{B,-e_n}) = P_{\gamma_{O^\intercal AO}}\left((O^{-1}E)^s_{O^\intercal AO,-e_n}\right) &= P_{\gamma_{O^\intercal AO}}(O^{-1}E^s_{A,u}) \\ &= P_{\gamma_A}(E^s_{A,u}) \le P_{\gamma_A}(E) = P_{\gamma_{O^\intercal AO}}(O^{-1}E) = P_{\gamma_B}(\widetilde{E}). \end{aligned}$$

Applying Step 1 on γ_B and $\widetilde{E}$, we conclude that $e_n \in V_\lambda(B) \cap \mathbb{S}^{n-1}$ for some eigenvalue λ, i.e., $Be_n = \lambda e_n$ and hence

$$Au = AO(-e_n) = OB(-e_n) = -O\lambda e_n = \lambda O(-e_n) = \lambda u.$$

Thus, if $P_{\gamma_A}(E^s_{A,u}) \le P_{\gamma_A}(E)$ for all finite A-anisotropic Gaussian perimeter set E in $\mathbb{R}^n$, we have

$$u \in V_\lambda(A) \quad \text{for some } \lambda > 0. \tag{4.10}$$

This finishes the first part of the theorem.

For the second part, it is enough to prove that

$$\gamma_A \text{ is Ehrhard symmetrizable } \implies A = aI_n \text{ for some constant } a > 0$$

since we can apply Corollary 3.5.1 again, and conclude the converse of the statement. Suppose now we have two distinct eigenvalues λ_1, λ_2 of A with eigenvectors u_1, u_2 in $\mathbb{S}^{n-1}$. Notice that $\langle u_1, u_2 \rangle = 0$ since A is symmetric and $\lambda_1 \neq \lambda_2$. Consider

$$u := \frac{u_1 + u_2}{\sqrt{2}} \in \mathbb{S}^{n-1}.$$

Since γ_A is Ehrhard symmetrizable, by (4.5), we have $u \in V_\lambda(A) \cap \mathbb{S}^{n-1}$ for some $\lambda > 0$. However,

$$\lambda \left(\frac{u_1 + u_2}{\sqrt{2}} \right) = \lambda u = Au = \frac{\lambda_1 u_1 + \lambda_2 u_2}{\sqrt{2}} \implies (\lambda_1 - \lambda) u_1 + (\lambda_2 - \lambda) u_2 = 0 \implies \lambda_1 = \lambda = \lambda_2.$$

This contradicts the assumption that $\lambda_1 \neq \lambda_2$. Therefore, all the eigenvalues of A are the same and hence $A = aI_n$ for some $a > 0$. $\square$

Chapter 5

CHARACTERIZATION OF EHRHARD MEASURES IN THE CLASS $\mathcal{W}(\mathbb{R}^n)$

In this chapter, we extend the uniqueness result, Theorem 1.2.3, to a larger class of measures. This work is a collaboration with Sean McCurdy [MY23].

Let

$$\mathcal{W}(\mathbb{R}^n) = \left\{ \mu : \frac{d\mu}{d\mathcal{L}^n} = f \in C^1(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n), f > 0 \right\}. \tag{5.1}$$

In particular, $\mathcal{A}(\mathbb{R}^n) = \{\text{anisotropic Gaussian measures}\} \subset \mathcal{W}(\mathbb{R}^n)$. In Section 5.1.1, we will first develop theorems for weighted BV functions with the measure $\mu = f\mathcal{L}^n$ where f is a positive almost everywhere distribution function in $W^{1,1}(\mathbb{R}^n)$. Then, in Section 5.1.2, we will further assume that $f \in C^1(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)$ when dealing with generalized Ehrhard symmetrization.

5.1 *Weighted BV functions and generalized Ehrhard symmetrization*

5.1.1 Weighted BV functions

Let $\mu = f\mathcal{L}^n$ with a positive almost everywhere distribution function $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ and let U be an open set in $\mathbb{R}^n$. A function $g \in L^\infty(U)$ has **bounded μ-variation** in $U \subset \mathbb{R}^n$ if

$$\sup\left\{ \int_U g\Big(f \operatorname{div}\varphi + \varphi \cdot \nabla f\Big) dx : \varphi \in C^1_c(U;\mathbb{R}^n), |\varphi| \le 1 \right\} < \infty.$$

We will denote the collection of all functions with bounded μ-variation in U by $BV_\mu(U)$. A $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has **finite μ-perimeter** in U if $\chi_E \in BV_\mu(U)$. A function $g \in L^\infty_{\text{loc}}(U)$ has **locally bounded μ-variation** in $U \subset \mathbb{R}^n$ if for each open set $V \subset\subset U$,

$$\sup\left\{ \int_V g\Big(f \operatorname{div}\varphi + \varphi \cdot \nabla f\Big) dx : \varphi \in C^1_c(V;\mathbb{R}^n), |\varphi| \le 1 \right\} < \infty.$$

We will denote the collection of all functions with locally bounded μ-variation in U by $BV_{\mathrm{loc},\mu}(U)$. A $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has **locally finite μ-perimeter** in U if $\chi_E \in BV_{\mathrm{loc},\mu}(U)$. We will also use the following notations

$$\mathrm{Per}_\mu(E;U) := \sup\left\{\int_E \Big(f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big)dx : \varphi \in C_c^1(U;\mathbb{R}^n), |\varphi| \le 1\right\} \tag{5.2}$$

and $\mathrm{Per}_\mu(E) := \mathrm{Per}_\mu(E;\mathbb{R}^n)$. It is clear that

$$E \text{ is a set of finite } \mu\text{-perimeter in } U \iff \mathrm{Per}_\mu(E;U) < \infty.$$

Lemma 5.1.1 (Riesz Representation Theorem)**.** *Let $\mu = f\mathcal{L}^n$ with a positive almost everywhere distribution function $f \in W^{1,1}_{loc}(\mathbb{R}^n)$ and $g \in BV_{loc,\mu}(U)$, where $U \subset \mathbb{R}^n$ is an open set. Then there exists a Radon measure ν on U and a ν-measurable function $\sigma : U \to \mathbb{R}^n$ s.t.*

(1) $|\sigma(x)| = 1$ for ν-a.e. on U.

(2) For all $\varphi \in C_c^1(U;\mathbb{R}^n)$,

$$\int_U g\Big(f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big)dx = -\int_U \varphi\cdot\sigma \, d\nu.$$

We write $|Dg|_\mu$ for ν, $(Dg)_\mu := \sigma|Dg|_\mu$, and $(D_i g)_\mu = \sigma_i|Dg|_\mu$, where $\sigma = (\sigma_1,\cdots,\sigma_n)$. Moreover,

$$\begin{aligned}|Dg|_\mu(V) &= \sup\left\{\int_V g\Big(f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big)dx : \varphi \in C_c^1(V;\mathbb{R}^n), |\varphi| \le 1\right\}\\ &= \sup\left\{\int_V \varphi\cdot d(Dg)_\mu : \varphi \in C_c^1(V;\mathbb{R}^n),\ |\varphi| \le 1\right\}.\end{aligned}$$

for any $V \subset\subset U$, i.e., the total variation of $(Dg)_\mu$ is $|Dg|_\mu$. Also, $\mathrm{Per}_\mu(E;U) = |D\chi_E|_\mu(U)$.

Proof. Consider the linear functional $L : C_c^1(U;\mathbb{R}^n) \to \mathbb{R}$ defined as

$$L(\varphi) = -\int_U g\Big(f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big)dx.$$

Notice that $L(\varphi) \in \mathbb{R}$ since $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, $g \in BV_{\mathrm{loc},\mu}(U)$, and hence $g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\left|\int_U g\Big(f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big)dx\right| \le \|g\|_{L^\infty(K)}\left(\|\operatorname{div}\varphi\|_{L^\infty(K)}\|f\|_{L^1(K)} + \|\varphi\|_{L^\infty(K)}\|\nabla f\|_{L^1(K)}\right) < \infty,$$

where $K := \operatorname{spt}(\varphi)$. The rest of the proof of extending L to a linear functional on $C_c(U;\mathbb{R}^n)$ follows from [EG92], Section 5.1, Theorem 1. $\square$

Remark: In particular,

(1) if $\mu = \mathcal{L}^n$, i.e., $f \equiv 1 \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$, then $(D\chi_E)_{\mathcal{L}^n} = D\chi_E$ and

$$\begin{aligned}\operatorname{Per}_{\mathcal{L}^n}(E;U) = |D\chi_E|_{\mathcal{L}^n}(U) = \sup\left\{\int_E \operatorname{div}\varphi\, dx : \varphi \in C^1_c(U;\mathbb{R}^n), |\varphi| \le 1\right\}\\ = |D\chi_E|(U) = P(E;U),\end{aligned}$$

where the right hand side is the usual definition from Section 2.1.

(2) if $\mu = \gamma_A$, i.e., $f = \frac{\sqrt{\det A}}{(2\pi)^{n/2}} e^{-\langle Ax,x\rangle/2}$, by Proposition 2.3.2,

$$\begin{aligned}\operatorname{Per}_{\gamma_A}(E;U) &= \sup\left\{\int_E \operatorname{div}\varphi(x) - \langle \varphi(x), Ax\rangle \; d\gamma_A(x) : \varphi \in C^1_c\left(U;\mathbb{R}^n\right), |\varphi| \le 1\right\}\\ &= \frac{1}{\sqrt{2\pi}} P_{\gamma_A}(E;U)\end{aligned}$$

where the right hand side is the definition from Section 2.3. The reason for the extra factor $\frac{1}{\sqrt{2\pi}}$ in front of P_{γ_A} is that we define P_{γ_A} using $\frac{\sqrt{\det A}}{(2\pi)^{(n-1)/2}} e^{-\langle Ax,x\rangle/2}$ instead of $\frac{\sqrt{\det A}}{(2\pi)^{n/2}} e^{-\langle Ax,x\rangle/2}$.

(3) if $g \in C^1(U)$, then

$$0 = \int_U \operatorname{div}(fg\varphi)\, dx = \int_U (\nabla f)\cdot(g\varphi) + f(\nabla g)\cdot\varphi + fg\operatorname{div}\varphi\; dx$$

where ∇f is the weak gradient. That is,

$$\int_U \varphi \cdot d(Dg)_\mu = -\int_U g\Big(f\operatorname{div}\varphi + \varphi\cdot\nabla f\Big)dx = \int_U \varphi\cdot\nabla g\; d\mu.$$

Therefore, $(Dg)_\mu = (\nabla g)\mu$ and

$$|Dg|_\mu(U) = \sup\left\{\int_U \varphi\cdot\nabla g\; d\mu : \varphi \in C^1_c\left(U;\mathbb{R}^n\right),\ |\varphi| \le 1\right\} = \|\nabla g\|_{L^1(U,\mu)}, \tag{5.3}$$

since we may choose a sequence of smooth functions that approximate $\frac{\nabla g}{|\nabla g|}$ to obtain the second equality (see, for example, [Fol13], Theorem 6.13).

Our next lemma is an analogue of [EG92], Section 5.2, Theorem 1 and [Mag12], Proposition 12.15. Notice that the assumption of $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$ and $f > 0$ a.e. implies that f is finite and positive almost everywhere, enabling us to consider $1/f$.

Lemma 5.1.2 (Lower Semicontinuity of bounded μ-variation)**.** *Let $\mu = f\mathcal{L}^n$ with a positive almost everywhere distribution function $f \in W^{1,1}_{loc}(\mathbb{R}^n)$ and $U \subset \mathbb{R}^n$ be an open set. Let $g_k \in BV_\mu(U)$ be a sequence that satisfies the following two conditions.*

(1) $\sup_k \|g_k\|_{L^\infty(U)} < \infty$ *and* $\liminf_{k\to\infty} |Dg_k|_\mu(U) < \infty$.

(2) There exists a function $g \in L^1_{loc}(U, \mu)$ such that

$$g_k \to g \quad \textit{in } L^1_{loc}(U, \mu).$$

Then $g \in BV_\mu(U)$ and $|Dg|_\mu(U) \leq \liminf_{k\to\infty} |Dg_k|_\mu(U)$.

Proof. Let $B_R = B(0, R)$ be the open ball centered at 0 with radius $R > 0$.

Step 1: We first claim that

$$\|g\|_{L^\infty(U)} \leq \sup_k \|g_k\|_{L^\infty(U)} < \infty.$$

It is enough to show that for all $\varepsilon > 0$, $|g| \leq \sup_k \|g_k\|_{L^\infty(U)} + \varepsilon$ for $\mathcal{L}^n$-a.e. $x \in U$. That is, we just need to show that for all $R > 0$,

$$\left| \left\{ |g| > \sup_k \|g_k\|_{L^\infty(U)} + \varepsilon \right\} \cap B_R \right| = 0.$$

We argue this by contradiction. Suppose that $|\{|g| > \sup_k \|g_k\|_{L^\infty(U)} + \varepsilon\} \cap B_R| > 0$ for some $\varepsilon > 0$ and $R > 0$. Since $f > 0$ a.e., we also have $\mu\left(\{|g| > \sup_k \|g_k\|_\infty + \varepsilon\} \cap B_R\right) > 0$ and

$$|g - g_k| \geq |g| - \|g_k\|_{L^\infty(U)} > \sup_k \|g_k\|_{L^\infty(U)} + \varepsilon - \|g_k\|_{L^\infty(U)} \geq \varepsilon$$

for all $x \in \{|g| > \sup_k \|g_k\|_{L^\infty(U)} + \varepsilon\} \cap B_R$. Therefore, using assumption (2),

$$\begin{aligned} \varepsilon\mu\left(\{|g| > \sup_k \|g_k\|_{L^\infty(U)} + \varepsilon\} \cap B_R \right) &\leq \int_{\{|g| > \sup_k \|g_k\|_{L^\infty(U)} + \varepsilon\} \cap B_R} |g - g_k| d\mu \\ &\leq \int_{B_R} |g - g_k| d\mu \to 0. \end{aligned}$$

Taking $k \to 0$, we conclude that $\mu\left(\{|g| > \sup_k \|g_k\|_{L^\infty(U)} + \varepsilon\} \cap B_R\right) = 0$. However, this gives us a contradiction.

Step 2: We claim that $g \in BV_\mu(U)$ and

$$|Dg|_\mu(U) \le \liminf_{k\to\infty} |Dg_k|_\mu(U).$$

Since $\mu = f\mathcal{L}^n$ and $f > 0$ a.e. in $W^{1,1}_{\text{loc}}(\mathbb{R}^n)$, we have $\|\nabla f\|_{L^1(V)} < \infty$ and

$$h := \frac{\nabla f}{f} \in L^1_{\text{loc}}(\mathbb{R}^n, \mu).$$

Our first goal is to show that for $\varphi \in C^1_c(U; \mathbb{R}^n)$ with $|\varphi| \le 1$,

$$\int_U g\left(f \operatorname{div}\varphi + \varphi \cdot \nabla f\right) dx = \lim_{k\to\infty} \int_U g_k(f \operatorname{div}\varphi + \varphi \cdot \nabla f) dx.$$

We will break this integral into two pieces. Observe that for any $\varphi \in C^1_c(U; \mathbb{R}^n)$ with $|\varphi| \le 1$, divφ is bounded. Let $\operatorname{spt}\varphi \subset V \subset\subset U$,

$$\left|\int_U (g - g_k) f \operatorname{div}\varphi dx\right| \le \|\operatorname{div}\varphi\|_{L^\infty(U)} \|g - g_k\|_{L^1(V,\mu)} \to 0,$$

as $k \to \infty$, and for any $M > 0$,

$$\begin{aligned}
&\left|\int_U (g - g_k)(\varphi \cdot \nabla f) dx\right| = \left|\int_V (g - g_k)(\varphi \cdot h) d\mu\right| \\
&\le \|g - g_k\|_{L^\infty(V)} \int_{V\cap\{|h|>M\}} |h| d\mu + \int_{V\cap\{|h|\le M\}} |g_k - g||h| d\mu \\
&\le \left(\|g\|_{L^\infty(U)} + \|g_k\|_{L^\infty(U)}\right) \|h\|_{L^1(V\cap\{|h|>M\},\mu)} + M\|g_k - g\|_{L^1(V,\mu)} \\
&\le 2\sup_k \|g_k\|_{L^\infty(U)} \|h\|_{L^1(V\cap\{|h|>M\},\mu)} + M\|g_k - g\|_{L^1(V,\mu)},
\end{aligned}$$

where we have used Step 1, i.e., $\|g\|_{L^\infty(U)} \le \sup_k \|g_k\|_{L^\infty(U)}$. Given any $\varepsilon > 0$, we can take M sufficiently large such that

$$\|h\|_{L^1(V\cap\{|h|>M\},\mu)} \le \varepsilon,$$

since $\|h\|_{L^1(V,\mu)} < \infty$. Taking $k \to \infty$,

$$\lim_{k\to\infty} \left|\int_U (g - g_k)(\varphi \cdot \nabla f) dx\right| \le 2\sup_k \|g_k\|_{L^\infty(U)} \varepsilon.$$

Letting $\varepsilon \to 0$, we see that

$$\int_U g\,(f\operatorname{div}\varphi + \varphi\cdot\nabla f)\,dx = \lim_{k\to\infty}\int_U g_k(f\operatorname{div}\varphi + \varphi\cdot\nabla f)dx$$
$$\leq \liminf_{k\to\infty}|Dg_k|_\mu(U),$$

where we have applied Lemma 5.1.1 on g_k. Taking the sup in $\varphi \in C_c^1(U;\mathbb{R}^n)$ with $|\varphi| \leq 1$,

$$|Dg|_\mu(U) \leq \liminf_{k\to\infty}|Dg_k|_\mu(U) < \infty,$$

and hence $g \in BV_\mu(U)$. □

Lemma 5.1.3 (Weak Approximation by Smooth Functions)**.** *Let $\mu = f\mathcal{L}^n$ with a positive almost everywhere distribution function $f \in W^{1,1}(\mathbb{R}^n)$. If $U \subset \mathbb{R}^n$ is an open set and $g \in BV_\mu(U)$, then there is a sequence of smooth functions $g_\varepsilon \in BV_\mu(U)$ such that*

(1) $g_\varepsilon \to g$ in $L^1(U,\mu)$ and $\sup_{\varepsilon>0}\|g_\varepsilon\|_{L^\infty(U)} \leq 3\|g\|_{L^\infty(U)}$.

(2) $\limsup_{\varepsilon\to 0}|Dg_\varepsilon|_\mu(U) \leq |Dg|_\mu(U) + C(n)\|g\|_{L^\infty(U)}\|\nabla f\|_{L^1(U)}$.

Proof. We follow the proof from Evans and Gariepy [EG92], Section 5.2, Theorem 2. Fix $\varepsilon > 0$, and let $m \in \mathbb{N}$. For $k = 1, 2, ...$, we define the open sets

$$U_k := \left\{x \in U : \operatorname{dist}(x,\partial U) > \frac{1}{m+k}\right\} \cap B(0, m+k).$$

Now, choose m sufficiently large such that $|Dg|_\mu(U \setminus U_1) \leq \varepsilon$. Define $U_0 = \varnothing$ and

$$V_k = U_{k+1} \setminus \overline{U}_{k-1}$$

for $k \in \mathbb{N}$. Let $\{\xi_k\}$ be a partition of unity subordinate to $\{V_k\}_k$, i.e., $\xi_k \in C_c^\infty(V_k)$, $0 \leq \xi_k \leq 1$, and $\sum_{k=1}^\infty \xi_k = 1$ on U.

Now, let η be a standard positive, smooth, radial mollifier with $\operatorname{spt}(\eta) \subset \overline{B}(0,1)$, $\|\eta\|_{L^1(\mathbb{R}^n)} = 1$, and let $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$. For each $k \in \mathbb{N}$, there exists $\varepsilon_k > 0$ such that

$$\operatorname{spt}(\xi_k) + \overline{B}(0,\varepsilon_k) \subset V_k, \quad \operatorname{spt}(\eta_{\varepsilon_k} \star (g\xi_k)) \subset V_k, \tag{5.4}$$

$$\int_U |\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k| dx \le \frac{\varepsilon}{2^k}, \tag{5.5}$$

and

$$\int_U |\eta_{\varepsilon_k} \star (fg\nabla\xi_k) - fg\nabla\xi_k| dx \le \frac{\varepsilon}{2^k}. \tag{5.6}$$

Define

$$g_\varepsilon := \sum_{k=1}^{\infty} \eta_{\varepsilon_k} \star (g\xi_k).$$

Note that ε_k depends on ε and k. Hence, $\sum_{k=1}^{\infty} \eta_{\varepsilon_k} \star (g\xi_k)$ depends only on ε. Furthermore, $g_\varepsilon \in C^\infty(U)$ and each point in U belongs to at most three of the sets $\{V_k\}_{k=1}^{\infty}$. That is, for any $x \in U$,

$$\sum_{k=1}^{\infty} \chi_{V_k}(x) \le 3. \tag{5.7}$$

We first claim that

$$\|\eta_{\varepsilon_k} \star (g\xi_k)\|_{L^\infty(U)} \le \|g\|_{L^\infty(U)}, \quad \|\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k\|_{L^\infty(U)} \le 2\|g\|_{L^\infty(U)}. \tag{5.8}$$

For any $x \in \operatorname{spt}(\eta_{\varepsilon_k} \star (g\xi_k))$, by (5.4), if $y \in B(0, \varepsilon_k)$,

$$x - y \in \operatorname{spt}(\xi_k) + B(0, \varepsilon_k) \subset V_k \subset U,$$

and hence

$$\begin{aligned} |\eta_{\varepsilon_k} \star (g\xi_k)| &\le \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y) \left|(g\xi_k)(x-y)\right| dy \\ &\le \|\eta_{\varepsilon_k}\|_{L^1(\mathbb{R}^n)} \|g\xi_k\|_{L^\infty(U)} \le \|g\|_{L^\infty(U)}. \end{aligned}$$

Then $\|\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k\|_{L^\infty(U)} \le \|\eta_{\varepsilon_k} \star (g\xi_k)\|_{L^\infty(U)} + \|g\xi_k\|_{L^\infty(U)} \le 2\|g\|_{L^\infty(U)}$. A direct consequence of (5.8) yields the following

$$|g_\varepsilon(x)| \le \sum_{k=1}^{\infty} \left|\eta_{\varepsilon_k} \star (g\xi_k)(x)\right| \chi_{V_k}(x) \le 3\|g\|_{L^\infty(U)}$$

for all $x \in U$, where we have used (5.4) and (5.7). Thus,

$$\sup_{\varepsilon>0} \|g_\varepsilon\|_{L^\infty(U)} \leq 3\|g\|_{L^\infty(U)}. \tag{5.9}$$

We now claim that

$$g_\varepsilon \to g \text{ in } L^1(U,\mu).$$

Given $\delta > 0$, there exists $K = K(\delta) > 0$ such that $\int_{\{f\geq K\}} f dx \leq \delta$, since $f \in L^1(\mathbb{R}^n)$. By using (5.5), (5.7), (5.8), and the fact that $g = \sum_{k=1}^\infty g\xi_k$, we can estimate $\|g - g_\varepsilon\|_{L^1(U,\mu)}$ as the following.

$$\begin{aligned}
\|g - g_\varepsilon\|_{L^1(U,\mu)} &\leq \int_U \sum_{k=1}^\infty |\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k| f dx \\
&= \int_{U\cap\{f<K\}} \sum_{k=1}^\infty |\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k| f dx + \int_{U\cap\{f\geq K\}} \sum_{k=1}^\infty |\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k| f dx \\
&\leq K \sum_{k=1}^\infty \int_{U\cap\{f<K\}} |\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k| dx + \int_{U\cap\{f\geq K\}} \sum_{k=1}^\infty |\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k| \chi_{V_k} f dx \\
&\leq K \sum_{k=1}^\infty \int_U |\eta_{\varepsilon_k} \star (g\xi_k) - g\xi_k| dx + \int_{U\cap\{f\geq K\}} 2\|g\|_{L^\infty(U)} \Big(\sum_{k=1}^\infty \chi_{V_k}\Big) f dx \\
&\leq K \sum_{k=1}^\infty \frac{\varepsilon}{2^k} + 2\|g\|_{L^\infty(U)} \int_{U\cap\{f\geq K\}} \Big(\sum_{k=1}^\infty \chi_{V_k}\Big) f dx \leq K\varepsilon + 6\|g\|_{L^\infty(U)}\delta.
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$ and then $\delta \to 0$, the claim has been proved. Finally, we show that

$$\limsup_{\varepsilon\to 0} |Dg_\varepsilon|_\mu(U) \leq |Dg|_\mu(U) + C(n)\|g\|_{L^\infty(U)}\|\nabla f\|_{L^1(U)}.$$

Step 1: Notice that

$$\begin{aligned}
\int_U g_\varepsilon(f \operatorname{div}\varphi + \varphi\cdot\nabla f) dx &= \sum_{k=1}^\infty \int_U \eta_{\varepsilon_k} \star (g\xi_k) \Big[f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big] dx \\
&= \sum_{k=1}^\infty \int_U \int_U \eta_{\varepsilon_k}(x-y)(g\xi_k)(y) \Big[f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big](x) dy dx \\
&= \sum_{k=1}^\infty \int_U (g\xi_k)(y) \int_U \eta_{\varepsilon_k}(y-x) \Big[f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big](x) dx dy
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g\xi_k \Big[\eta_{\varepsilon_k} \star (f \operatorname{div} \varphi + \varphi \cdot \nabla f) \Big] dx, \tag{5.10}$$

since $\eta_{\varepsilon_k}(z) = \eta_{\varepsilon_k}(-z)$, where $\varphi \in C_c^1(U; \mathbb{R}^n)$ and $|\varphi| \leq 1$. Now, we focus on the convolution in (5.10). Notice that $x - y \in \mathrm{spt}(\xi_k) + B(0, \varepsilon_k) \subset V_k \subset U$ for all $x \in \mathrm{spt}(\xi_k)$, $y \in B(0, \varepsilon_k)$. Hence, for any $x \in \mathrm{spt}(\xi_k)$,

$$\begin{aligned}
\eta_{\varepsilon_k} \star (f \operatorname{div} \varphi + \varphi \cdot \nabla f)(x) &= \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y) \Big[f(x-y)(\operatorname{div} \varphi)(x-y) + \varphi(x-y) \cdot \nabla f(x-y) \Big] dy \\
&= \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y) \Big[f(x)(\operatorname{div} \varphi)(x-y) + \varphi(x-y) \cdot \nabla f(x) \Big] dy \\
&\quad + \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y)(\operatorname{div} \varphi)(x-y) \Big[f(x-y) - f(x) \Big] dy \\
&\quad + \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y) \varphi(x-y) \cdot \Big[\nabla f(x-y) - \nabla f(x) \Big] dy \\
&= \Big[(\eta_{\varepsilon_k} \star \operatorname{div} \varphi)(x) f(x) + (\eta_{\varepsilon_k} \star \varphi)(x) \cdot \nabla f(x) \Big] \\
&\quad + \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y)(\operatorname{div} \varphi)(x-y) \Big[f(x-y) - f(x) \Big] dy \\
&\quad + \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y) \varphi(x-y) \cdot \Big[\nabla f(x-y) - \nabla f(x) \Big] dy \\
&:= \text{(I)} + \text{(II)} + \text{(III)}.
\end{aligned}$$

Step 2: Assume that $f \in C^1(U) \cap W^{1,1}(U)$ and define the function h as

$$h_x(y) := \int_0^1 |\nabla f(x - ty)| dt.$$

We claim that

$$|\text{(II)}| \leq C_2 \varepsilon_k^{-n} \int_{B(0,\varepsilon_k)} h_x(y) dy + C_1 \varepsilon_k^{-n} \int_{B(0,\varepsilon_k)} |\nabla f(x-y)| dy,$$

where $C_1 := \|\eta\|_{L^\infty(\mathbb{R}^n)}$ and $C_2 := \|\nabla \eta\|_{L^\infty(\mathbb{R}^n)}$. By integration by parts and estimates on the mollifier η, we may estimate the middle term (II) as follows.

$$|\text{(II)}| = \left| \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y)(\operatorname{div} \varphi)(x-y) \Big[f(x-y) - f(x) \Big] dy \right|$$

$$
\begin{aligned}
&= \bigg| - \int_{B(0,\varepsilon_k)} \nabla_y \cdot \Big(\eta_{\varepsilon_k}(y)\varphi(x-y)\Big[f(x-y)-f(x)\Big]\Big) dy \\
&\quad + \int_{B(0,\varepsilon_k)} \nabla\eta_{\varepsilon_k}(y) \cdot \Big(\varphi(x-y)\Big[f(x-y)-f(x)\Big]\Big) dy \\
&\quad + \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y)\varphi(x-y)\cdot\nabla_y\Big[f(x-y)-f(x)\Big]\Big) dy \bigg| \\
&\leq \int_{B(0,\varepsilon_k)} |\nabla\eta_{\varepsilon_k}(y)||f(x-y)-f(x)|dy + \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y)|\nabla f(x-y)|dy \\
&\leq C_2\varepsilon_k^{-n-1}\int_{B(0,\varepsilon_k)} |f(x-y)-f(x)|dy + C_1\varepsilon_k^{-n}\int_{B(0,\varepsilon_k)} |\nabla f(x-y)|dy \\
&= C_2\varepsilon_k^{-n-1}\int_{B(0,\varepsilon_k)} \left|\int_0^1 \frac{d}{dt}f(x-ty)dt\right| dy + C_1\varepsilon_k^{-n}\int_{B(0,\varepsilon_k)} |\nabla f(x-y)|dy \qquad (5.11)\\
&\leq C_2\varepsilon_k^{-n}\int_{B(0,\varepsilon_k)} h_x(y)dy + C_1\varepsilon_k^{-n}\int_{B(0,\varepsilon_k)} |\nabla f(x-y)|dy,
\end{aligned}
$$

where we have used $\left|\frac{d}{dt}f(x-ty)\right| \leq |\nabla f(x-ty)||y| \leq |\nabla f(x-ty)|\varepsilon_k$ and

$$
|\nabla\eta_{\varepsilon_k}(y)| = \frac{1}{\varepsilon_k^{n+1}}\left|\nabla\eta\left(\frac{y}{\varepsilon_k}\right)\right| \leq \frac{1}{\varepsilon_k^{n+1}}C_2, \quad |\eta_{\varepsilon_k}(y)| = \frac{1}{\varepsilon_k^{n}}\left|\eta\left(\frac{y}{\varepsilon_k}\right)\right| \leq \frac{1}{\varepsilon_k^{n}}C_1.
$$

Step 3: We estimate the last term (III) by

$$
|(\text{III})| \leq C_1\varepsilon_k^{-n}\int_{B(0,\varepsilon_k)} |\nabla f(x-y)|dy + |\nabla f(x)|.
$$

Notice that

$$
\begin{aligned}
|(\text{III})| &= \left| \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y)\varphi(x-y)\cdot\Big[\nabla f(x-y)-\nabla f(x)\Big]dy \right| \\
&\leq \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y)|\nabla f(x-y)|dy + \int_{B(0,\varepsilon_k)} \eta_{\varepsilon_k}(y)|\nabla f(x)|dy \\
&\leq C_1\varepsilon_k^{-n}\int_{B(0,\varepsilon_k)} |\nabla f(x-y)|dy + |\nabla f(x)|.
\end{aligned}
$$

Step 4: Applying equation (5.10), Step 2, and Step 3, we have

$$
\left|\int_U g_\varepsilon(f\operatorname{div}\varphi + \varphi\cdot\nabla f)dx\right| = \left|\sum_{k=1}^{\infty}\int_{\operatorname{spt}(\xi_k)} g\xi_k\Big[\eta_{\varepsilon_k}\star(f\operatorname{div}\varphi+\varphi\cdot\nabla f)\Big]dx\right|
$$

$$
\begin{aligned}
&= \left| \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g\xi_k \Big[(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) \Big] dx \right| \\
&\leq \left| \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g\xi_k \Big[(\eta_{\varepsilon_k} \star \operatorname{div} \varphi)(x) f(x) + (\eta_{\varepsilon_k} \star \varphi)(x) \cdot \nabla f(x) \Big] dx \right| \\
&\quad + C_2 \|g\|_{L^\infty(U)} \sum_{k=1}^{\infty} \varepsilon_k^{-n} \int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} h_x(y) dy dx \\
&\quad + 2C_1 \|g\|_{L^\infty(U)} \sum_{k=1}^{\infty} \varepsilon_k^{-n} \int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} |\nabla f(x-y)| dy dx \\
&\quad + \|g\|_{L^\infty(U)} \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} |\nabla f(x)| dx := I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where we have used that fact that $|\xi_k| \leq 1$. Moreover,

$$
\begin{aligned}
I_1 &= \left| \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g\xi_k \Big[(\eta_{\varepsilon_k} \star \operatorname{div} \varphi)(x) f(x) + (\eta_{\varepsilon_k} \star \varphi)(x) \cdot \nabla f(x) \Big] dx \right| \\
&= \left| \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g\xi_k \Big[f \operatorname{div}(\eta_{\varepsilon_k} \star \varphi) + (\eta_{\varepsilon_k} \star \varphi) \cdot \nabla f \Big] dx \right| \\
&= \left| \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g \Big[f \operatorname{div} \big(\xi_k (\eta_{\varepsilon_k} \star \varphi) \big) + \xi_k (\eta_{\varepsilon_k} \star \varphi) \cdot \nabla f \Big] dx - \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g f \nabla \xi_k \cdot (\eta_{\varepsilon_k} \star \phi) dx \right| \\
&\leq \left| \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g \Big[f \operatorname{div} \big(\xi_k (\eta_{\varepsilon_k} \star \varphi) \big) + \xi_k (\eta_{\varepsilon_k} \star \varphi) \cdot \nabla f \Big] dx \right| + \left| \sum_{k=1}^{\infty} \int_{\mathrm{spt}(\xi_k)} g f \nabla \xi_k \cdot (\eta_{\varepsilon_k} \star \phi) dx \right| \\
&:= J_1 + J_2.
\end{aligned}
$$

We claim that

$$
I_1 \leq |Dg|_\mu(U) + 4\varepsilon.
$$

Because $\xi_k(\eta_{\varepsilon_k} \star \varphi) \in C_c^1(V_k)$, $|\xi_k(\eta_{\varepsilon_k} \star \varphi)| \leq 1$, and $\sum_{k=1}^{\infty} \chi_{V_k}(x) \leq 3$, we have

$$
\begin{aligned}
J_1 &\leq \left| \int_{\mathrm{spt}(\xi_1)} g \Big[f \operatorname{div} \big(\xi_1 (\eta_{\varepsilon_1} \star \varphi) \big) + \xi_1 (\eta_{\varepsilon_1} \star \varphi) \cdot \nabla f) \Big] dx \right| \\
&\quad + \sum_{k=2}^{\infty} \left| \int_{\mathrm{spt}(\xi_k)} g \Big[f \operatorname{div} \big(\xi_k (\eta_{\varepsilon_k} \star \varphi) \big) + \xi_k (\eta_{\varepsilon_k} \star \varphi) \cdot \nabla f) \Big] dx \right| \\
&\leq |Dg|_\mu(U) + \sum_{k=2}^{\infty} |Dg|_\mu(V_k) = |Dg|_\mu(U) + \sum_{k=2}^{\infty} \int_{U \setminus U_1} \chi_{V_k} d|Dg|_\mu
\end{aligned}
$$

$$\leq |Dg|_\mu(U) + 3|Dg|_\mu(U - U_1) \leq |Dg|_\mu(U) + 3\varepsilon.$$

Similarly, because $\sum_{k=1}^\infty \nabla\xi_k(x) \equiv 0$ in U, we may use (5.6) to obtain

$$\begin{aligned} J_2 &= \left| \sum_{k=1}^\infty \int_U gf\nabla\xi_k \cdot (\eta_{\varepsilon_k} \star \varphi)dx \right| = \left| \sum_{k=1}^\infty \int_U \varphi \cdot \Big[\eta_{\varepsilon_k} \star (gf\nabla\xi_k)\Big] dx \right| \\ &= \left| \sum_{k=1}^\infty \int_U \varphi \cdot \Big[\eta_{\varepsilon_k} \star (gf\nabla\xi_k)\Big] dx - \sum_{k=1}^\infty \int_U \varphi \cdot \Big[gf\nabla\xi_k\Big] dx \right| \\ &\leq \sum_{k=1}^\infty \int_U |\eta_{\varepsilon_k} \star (gf\nabla\xi_k) - gf\nabla\xi_k| dx \leq \varepsilon, \end{aligned}$$

where we have employed a calculation similar to (5.10) in the first equality. Thus, $I_1 = J_1 + J_2 \leq |Dg|_\mu(U) + 4\varepsilon$. Similarly, by (5.7), it is clear that

$$\begin{aligned} I_4 &= \|g\|_{L^\infty(U)} \sum_{k=1}^\infty \int_{\mathrm{spt}(\xi_k)} |\nabla f(x)| dx \leq \|g\|_{L^\infty(U)} \sum_{k=1}^\infty \int_{V_k} |\nabla f(x)| dx \\ &= \|g\|_{L^\infty(U)} \sum_{k=1}^\infty \int_U \chi_{V_k}(x) |\nabla f(x)| dx \leq 3\|g\|_{L^\infty(U)} \int_U |\nabla f(x)| dx. \end{aligned}$$

Next, using our assumption that $\mathrm{spt}(\xi_k) \subset V_k$, $\sum_k \chi_{V_k}(x) \leq 3$, and the fact that for any $y \in B(0, \varepsilon_k)$ and $t \in (0,1)$,

$$\mathrm{spt}(\xi_k) - ty \subset \mathrm{spt}(\xi_k) + B(0, \varepsilon_k) \subset V_k,$$

i.e., equation (5.4), we estimate I_2 as follows.

$$\begin{aligned} I_2 &= C_2 \|g\|_{L^\infty(U)} \sum_{k=1}^\infty \varepsilon_k^{-n} \int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} h_x(y) dy dx \\ &= C_2 \|g\|_{L^\infty(U)} \sum_{k=1}^\infty \varepsilon_k^{-n} \int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} \left(\int_0^1 |\nabla f(x - ty)| dt \right) dy dx \\ &\leq C_2 \|g\|_{L^\infty(U)} \sum_{k=1}^\infty \varepsilon_k^{-n} \int_0^1 \int_{B(0,\varepsilon_k)} \int_{\mathrm{spt}(\xi_k)} |\nabla f(x - ty)| \, dx dy dt \\ &= C_2 \|g\|_{L^\infty(U)} \sum_{k=1}^\infty \varepsilon_k^{-n} \int_0^1 \int_{B(0,\varepsilon_k)} \int_{\mathrm{spt}(\xi_k) - ty} |\nabla f(w)| \, dw dy dt \\ &\leq C_2 \|g\|_{L^\infty(U)} \sum_{k=1}^\infty \varepsilon_k^{-n} \int_0^1 \int_{B(0,\varepsilon_k)} \int_{V_k} |\nabla f(w)| \, dw dy dt \end{aligned}$$

$$= C_2|B(0,1)|\|g\|_{L^\infty(U)} \sum_{k=1}^{\infty} \int_{V_k} |\nabla f| dx \le 3C_2|B(0,1)|\|g\|_{L^\infty(U)} \int_U |\nabla f| dx,$$

where we have used Fubini's theorem. Similarly, we have the following estimate for I_3.

$$I_3 = 2C_1\|g\|_{L^\infty(U)} \sum_{k=1}^{\infty} \varepsilon_k^{-n} \int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} |\nabla f(x-y)| dy dx \le 6C_1\|g\|_{L^\infty(U)} \int_U |\nabla f| dx.$$

Putting it all together, we obtain

$$\left| \int_U g_\varepsilon (f \operatorname{div} \varphi + \varphi \cdot \nabla f) dx \right| \le |Dg|_\mu(U) + 4\varepsilon + C\|g\|_{L^\infty(U)} \int_U |\nabla f| dx,$$

where $C = 3 + 6C_1 + 3C_2|B(0,1)| > 0$. Taking the sup over $\varphi \in C_c^1(U;\mathbb{R}^n)$ with $|\varphi| \le 1$ and $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon\to 0} |Dg_\varepsilon|_\mu(U) \le |Dg|_\mu(U) + C(n)\|g\|_{L^\infty(U)} \|\nabla f\|_{L^1(U)},$$

for $f \in C^1(U) \cap W^{1,1}(U)$.

Step 5: To address general $f \in W^{1,1}(\mathbb{R}^n)$, we observe that the only instance where we require $f \in C^1(U) \cap W^{1,1}(U)$ is in equation (5.11) when estimating (II) and hence I_2. In fact, we just need to show that for $k \in \mathbb{N}$,

$$\int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} |f(x-y) - f(x)|\, dy\, dx \le |B(0,1)|\varepsilon_k^{n+1} \int_{V_k} |\nabla f|\, dx$$

for $f \in W^{1,1}(U)$. Then the lemma holds true for $f \in W^{1,1}(\mathbb{R}^n)$. We will prove this by an approximation argument. Applying [EG92], Section 4.2, Theorem 2 on $f \in W^{1,1}(U)$, there exists a sequence $f_j \in C^\infty(U) \cap W^{1,1}(U)$ such that $f_j \to f$ in $W^{1,1}(U)$. Then

$$\begin{aligned}
\int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} |f(x-y) - f(x)|\, dy\, dx &\le \int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} |f(x-y) - f_j(x-y)|\, dy\, dx \\
&\quad + \int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} |f_j(x-y) - f_j(x)|\, dy\, dx \\
&\quad + \int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} |f_j(x) - f(x)|\, dy\, dx \\
&\le \int_{B(0,\varepsilon_k)} \int_{\mathrm{spt}(\xi_k)+y} |f(z) - f_j(z)|\, dz\, dy
\end{aligned}$$

$$+ |B(0,1)|\varepsilon_k^{n+1} \int_{V_k} |\nabla f_j| dx$$
$$+ |B(0,1)|\varepsilon_k^{n} \int_{V_k} |f_j(x) - f(x)|\, dx,$$

where we have applied the arguments in (5.11), (II), and I_2 to obtain

$$\int_{\mathrm{spt}(\xi_k)} \int_{B(0,\varepsilon_k)} |f_j(x-y) - f_j(x)|\, dy\, dx \leq |B(0,1)|\varepsilon_k^{n+1} \int_{V_k} |\nabla f_j| dx.$$

Finally, noticing that

$$\int_{B(0,\varepsilon_k)} \int_{\mathrm{spt}(\xi_k)+y} |f(z) - f_j(z)|\, dz\, dy \leq |B(0,1)|\varepsilon_k^{n} \int_{V_k} |f(z) - f_j(z)|\, dz,$$

and taking $j \to \infty$, we have finished the proof. □

Remark: In particular,

(1) if we only assume that our distribution function $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, then the open set U in Lemma 5.1.3 needs to be bounded.

(2) if we set $f \equiv 1 \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and let U be a bounded open set, by Lemam 5.1.2 and Lemma 5.1.3, there exists a sequence of smooth functions $g_k \in BV_\mu(U)$ such that

$$g_k \to g \text{ in } L^1(U,\mu), \qquad \lim_{k\to\infty} |Dg_k|_\mu(U) = |Dg|_\mu(U),$$

where $\mu = \mathcal{L}^n$ and $|Dg|_\mu(U) = |Dg|(U)$ since $g \in L^\infty(U) \subset L^1(U)$.

Lemma 5.1.4 (Weak Compactness for BV_μ)**.** *Let $\mu = f\mathcal{L}^n$ with a positive almost everywhere distribution function $f \in W^{1,1}(\mathbb{R}^n)$. Let $U \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. For any sequence of functions $h_j \in BV_\mu(U)$ such that*

$$\sup_j |Dh_j|_\mu(U) < \infty, \qquad \sup_j \|h_j\|_{L^\infty(U)} < \infty,$$

there exists a subsequence $\{h_{j_k}\}_{k=1}^\infty$ and a function $h \in BV_\mu(U)$ such that

$$h_{j_k} \to h \text{ in } L^1_{loc}(U,\mu), \qquad |Dh|_\mu(U) \leq \liminf_{k\to\infty} |Dh_{j_k}|_\mu(U).$$

In particular, for any sequence of sets of finite μ-perimeter $\{E_j\}_{j=1}^{\infty}$ in U such that

$$\sup_j \operatorname{Per}_\mu(E_j;U) < \infty,$$

there exists a subsequence $\{E_{j_k}\}_{k=1}^{\infty}$ and a set of finite μ-perimeter E such that

$$\chi_{E j_k} \to \chi_E \text{ in } L^1_{loc}(U,\mu), \qquad \operatorname{Per}_\mu(E;U) \le \liminf_{k\to\infty} \operatorname{Per}_\mu(E_{j_k};U).$$

Proof. **Step 1**: Applying Lemma 5.1.3 to h_j on the open set U, we obtain smooth functions $g_j \in BV_\mu(U)$ such that $\|g_j\|_{L^\infty(U)} \le 3\|h_j\|_{L^\infty(U)}$ and

$$\begin{cases} \displaystyle\int_U |h_j - g_j| d\mu \le \frac{1}{j}, \\ |Dg_j|_\mu(U) \le |Dh_j|_\mu(U) + C(n)\|h_j\|_{L^\infty(U)}\|\nabla f\|_{L^1(U)} + 1. \end{cases} \tag{5.12}$$

Since we are assuming $\sup_j |Dh_j|_\mu(U) < \infty$, $\sup_j \|h_j\|_{L^\infty(U)} < \infty$, we have

$$M_1 := \sup_j \|g_j\|_{L^\infty(U)} < \infty, \qquad M_2 := \sup_j |Dg_j|_\mu(U) < \infty.$$

That is, by (5.3) and g_j is smooth,

$$\sup_j \|\nabla g_j\|_{L^1(U,\mu)} = M_2 < \infty.$$

We claim that

$$\nabla(g_j f) = (\nabla g_j) f + g_j(\nabla f).$$

Notice that for any $\varphi \in C^1_c(\mathbb{R}^n)$, we have $g_j\varphi \in C^1_c(\mathbb{R}^n)$. By the definition of the weak gradient of f,

$$\int_{\mathbb{R}^n} (\nabla f)(g_j\varphi)\, dx = -\int_{\mathbb{R}^n} f\nabla(g_j\varphi)\, dx = -\int_{\mathbb{R}^n} f\varphi\nabla g_j + f g_j \nabla\varphi\, dx.$$

Equivalently,

$$\int_{\mathbb{R}^n} (g_j f)\nabla\varphi\, dx = -\int_{\mathbb{R}^n} \Big[(\nabla g_j) f + g_j(\nabla f)\Big]\varphi\, dx.$$

That is, we have the Leibniz's rule $\nabla(g_j f) = (\nabla g_j)f + g_j(\nabla f)$, and hence the assumptions that $f \in W^{1,1}(\mathbb{R}^n)$ and $\sup_j \|g_j\|_{L^\infty(U)} < \infty$ imply that

$$\begin{aligned}\|\nabla(g_j f)\|_{L^1(U)} &\le \|\nabla g_j\|_{L^1(U,\mu)} + \|g_j\|_{L^\infty(U)}\|\nabla f\|_{L^1(U)}\\ &\le \sup_j \|\nabla g_j\|_{L^1(U,\mu)} + \sup_j \|g_j\|_{L^\infty(U)}\|\nabla f\|_{L^1(U)}\\ &= M_2 + M_1\|\nabla f\|_{L^1(U)} < \infty.\end{aligned}$$

Thus, the sequence $\{g_j f\}_{j=1}^\infty$ is uniformly bounded in $W^{1,1}(U)$. Now we consider

$$U_k := U \cap B(0,k).$$

Applying the unweighted Rellich-Kondrashov compactness theorem with on U_k (see, for example, [EG92], Section 4.6, Theorem 1, Remark), there exists a function $\ell^k \in L^1(U_k)$ and a subsequence $\{g_j^k f\}_{j=1}^\infty \subset \{g_j f\}_{j=1}^\infty$ such that

$$g_j^k f \to \ell^k \text{ in } L^1(U_k) \quad \text{as } j \to \infty$$

and $\{g_j f\}_j \supset \{g_j^1 f\}_j \supset \{g_j^2 f\}_j \supset \cdots$. In particular, $\ell^m = \ell^k$ on U_k for all $m \ge k$. Thus, we may define ℓ to be ℓ^k on U_k for all $k \ge 1$. It is clear that $\ell \in L^1_{\mathrm{loc}}(U)$. By a diagonal argument, there exists $\{g_k^k f\}_k \subset \{g_j f\}_j$ such that

$$g_k^k f \to \ell \text{ in } L^1_{\mathrm{loc}}(U).$$

We can write $\ell = hf$, where $h := \ell/f \in L^1_{\mathrm{loc}}(U,\mu)$. Therefore, $g_k^k f \to hf$ in $L^1_{\mathrm{loc}}(U_k)$ which is the definition of $g_k^k \to h$ in $L^1_{\mathrm{loc}}(U,\mu)$.

Step 2: Let $\{g_{j_k}\}_k := \{g_k^k\}_k$. By Step 1, we have $g_{j_k} \to h$ in $L^1_{\mathrm{loc}}(U,\mu)$. Recalling (5.12), there exists a subsequence $\{h_{j_k}\} \subset \{h_j\}$ such that

$$\int_U |g_{j_k} - h_{j_k}|\, d\mu < \frac{1}{j_k}.$$

We claim that

$$h_{j_k} \to h \text{ in } L^1_{\mathrm{loc}}(U,\mu). \tag{5.13}$$

For any compact set $K \subset U$, there exists N such that $U_k \supset K$ for all $k > N$. Therefore,

$$\int_K |h - h_{j_k}|\, d\mu \le \int_K |h - g_{j_k}|\, d\mu + \int_K |g_{j_k} - h_{j_k}|\, d\mu \le \int_K |h - g_{j_k}|\, d\mu + \frac{1}{j_k} \to 0$$

as $k \to \infty$. Thus, we may invoke Lemma 5.1.2 to obtain

$$|Dh|_\mu(U) \le \liminf_{k\to\infty} |Dh_{j_k}|_\mu(U).$$

Step 3: Now, it remains to show that if $h_j = \chi_{E_j}$ then there exists a subsequence such that

$$\chi_{E_{j_k}} \to h \text{ in } L^1_{\text{loc}}(U, \mu),$$

where $h = \chi_E$ for some finite μ-perimeter set E. Hence, by Lemma 5.1.2,

$$\text{Per}_\mu(E; U) \le \liminf_{k\to\infty} \text{Per}_\mu(E_{j_k}; U).$$

By (5.13), $\chi_{E_{j_k}} = h_{j_k} \to h$ in $L^1_{\text{loc}}(U, \mu)$. Then there exists a subsequence (again indexed by j_k) such that $h_{j_k} \to h$ μ-a.e. and hence $h_{j_k} \to h$ $\mathcal{L}^n$-a.e., since $f > 0$ $\mathcal{L}^n$-a.e. (see, for example, [Fol13], Section 6.1, Exercise 9). Therefore, $|\{h \ne 0, 1\}| = 0$, i.e., $h = \chi_E$ $\mathcal{L}^n$-a.e., where $E = \{h = 1\}$. This gives the claim. □

Our last lemma in this section demonstrates the equivalence between our definition of μ-perimeter (see (5.2)) and the distributional definition of μ-perimeter (see (5.14)) when we assume $\mu = f\mathcal{L}^n$ with $f \in C^1(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)$. This equivalence enables us to compute the μ-perimeter of graphs using the area formula.

Lemma 5.1.5. *Let $\mu = f\mathcal{L}^n$ with a positive distribution function $f \in C^1(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)$ and let U be an open set in $\mathbb{R}^n$.*

(1) Assume that E is a set of locally finite perimeter in U. Then

$$\text{Per}_\mu(E; U) = \int_{\partial^* E \cap U} f\, d\mathcal{H}^{n-1}. \tag{5.14}$$

(2) E is a set of locally finite perimeter in U if and only if E is a set of locally finite μ-perimeter in U.

Proof. (1) By the divergence theorem (2.4) and Proposition 2.2.4,

$$\begin{aligned}
\mathrm{Per}_\mu(E;U) &= \sup\left\{\int_E \Big(f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big)dx : \varphi\in C_c^1(U;\mathbb{R}^n), |\varphi|\le 1\right\}\\
&= \sup\left\{\int_E \operatorname{div}(f\varphi)\,dx : \varphi\in C_c^1(U;\mathbb{R}^n), |\varphi|\le 1\right\}\\
&= \sup\left\{\int_{\mathbb{R}^n} (f\varphi)\cdot d\mu_E : \varphi\in C_c^1(U;\mathbb{R}^n), |\varphi|\le 1\right\}\\
&= |f\mu_E|(U) = f|\mu_E|(U) = \int_{\partial^* E\cap U} f\mathcal{H}^{n-1},
\end{aligned}$$

where we have used the fact that $f\varphi\in C_c^1(U;\mathbb{R}^n)$.

(2) ($\Rightarrow$) For any $V\subset\subset U$, by (1),

$$\mathrm{Per}_\mu(E;V) = \int_{\partial^* E\cap V} f d\mathcal{H}^{n-1} \le \max_{x\in\overline{V}} f \int_{\partial^* E\cap V} 1\, d\mathcal{H}^{n-1} = \left(\max_{x\in\overline{V}} f\right) P(E;V) < \infty.$$

($\Leftarrow$) We follow the proof in Proposition 2.3.2. Given any $V\subset\subset U$, we have

$$\mathrm{Per}_\mu(E;V) = \sup\left\{\int_E \Big(f \operatorname{div}\varphi + \varphi\cdot\nabla f\Big)dx : \varphi\in C_c^1(V;\mathbb{R}^n), |\varphi|\le 1\right\} < \infty.$$

For any $\varphi\in C_c^1(V;\mathbb{R}^n)$ with $|\varphi|\le 1$, let

$$M := \max_{x\in\overline{V}} \frac{1}{f(x)} > 0, \qquad \psi := \left(\frac{1/f}{M}\right)\varphi.$$

Then $\psi\in C_c^1(V;\mathbb{R}^n)$, $|\psi|\le 1$, and

$$\operatorname{div}\psi = \frac{1}{M}\left((\operatorname{div}\varphi)\frac{1}{f} - \varphi\cdot\frac{\nabla f}{f^2}\right).$$

Hence,

$$\begin{aligned}
\infty > \mathrm{Per}_\mu(E;V) &\ge \int_E \Big(f\operatorname{div}\psi + \psi\cdot\nabla f\Big)dx\\
&= \frac{1}{M}\int_E\left(\operatorname{div}\varphi - \varphi\cdot\frac{\nabla f}{f} + \varphi\cdot\frac{\nabla f}{f}\right)dx = \frac{1}{M}\int_E \operatorname{div}\varphi\, dx.
\end{aligned}$$

□

5.1.2 Generalized Ehrhard symmetrization

In this section, we generalize the notion of Ehrhard symmetrization from γ_{I_n} to a large class of finite measures.

Definition 5.1.1 (Generalized Ehrhard Symmetrization)**.** Let $\mu = f\mathcal{L}$ with a positive distribution function $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$. For a vector $u \in \mathbb{S}^0 = \{-1, 1\}$, we define the **generalized** u**-Ehrhard symmetrization of** E **with respect to** μ to be the set $E^s_{\mu,u}$ described below. For any Borel set $E \subset \mathbb{R}$, we define

$$E^s_{\mu,u} := \{x : x \cdot u \geq c\},$$

where $c \in \mathbb{R}$ is chosen to be the largest constant such that $\mu(E) = \mu(E^s_{\mu,u})$.

Let $\mu = f\mathcal{L}^n$ with a positive distribution function $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. For a vector $u \in \mathbb{S}^{n-1}$, we define the **generalized** u**-Ehrhard symmetrization of** E **with respect to** μ to be the set $E^s_{\mu,u}$ described below. For any Borel set $E \subset \mathbb{R}^n$ and $z \in \langle u \rangle^\perp$, we define $E_z := E \cap \pi^{-1}_{\langle u \rangle^\perp}(z)$, where $\pi_{\langle u \rangle^\perp}$ is the orthogonal projection onto $\langle u \rangle^\perp$, $\langle u \rangle := \mathrm{span}(u)$, and $\pi^{-1}_{\langle u \rangle^\perp}(z) = z + \langle u \rangle$. We define

$$E^s_{\mu,u} := \bigcup_{z \in \langle u \rangle^\perp} (E_z)^s_{\mu_z,u},$$

where we identify $\pi^{-1}_{\langle u \rangle^\perp}(z)$ with $\mathbb{R}$ and use $\mu_z := f\mathcal{H}^1 \llcorner \pi^{-1}_{\langle u \rangle^\perp}(z)$ to define $(E_z)^s_{\mu_z,u}$. We will omit the notation μ and simply write E^s_u if there is no confusion. For a sequence $u_1, ..., u_N$ in $\mathbb{S}^{n-1}$, we will use the following notation

$$E^s_{\mu,u_1,\ldots,u_N} := (\cdots (E^s_{\mu,u_1})^s_{\mu,u_2} \cdots)^s_{\mu,u_N}. \tag{5.15}$$

Note that the definition of the generalized Ehrhard symmetrization presented here differs slightly from the one introduced in Chapter 3 for the anisotropic Gaussian measure (see (3.32)). Recall that we have shown the Ehrhard symmetrization set $E^s_{A,-e_n}$ is measurable in Theorem 3.2.4. The expectation is that the proof of the measurability of $E^s_{\mu,u}$ is similar to the arguments in Lemma 3.2.3 and Theorem 3.2.4.

Definition 5.1.2 (Ehrhard Measures)**.** Let $n \in \mathbb{N}$ and $\mu \in \mathcal{W}(\mathbb{R}^n)$ (see (5.1)). A measure μ is called **Ehrhard symmetrizable** (or simply an **Ehrhard measure**) if

$$\mathrm{Per}_\mu(E^s_{\mu,u}) \leq \mathrm{Per}_\mu(E) \tag{5.16}$$

for all $u \in \mathbb{S}^{n-1}$, and for all Borel set $E \subset \mathbb{R}^n$.

Definition 5.1.3. (Half spaces) For a vector $v \in \mathbb{S}^{n-1}$ and $r \in \mathbb{R}$, we define

$$H(v,r) := \{x \in \mathbb{R}^n : x \cdot v \geq r\}.$$

Let $\mu = f\mathcal{L}^n$ with a positive almost everywhere distribution function $f \in L^1(\mathbb{R}^n)$. For a non-empty measurable set $E \subset \mathbb{R}^n$, we define

$$H_\mu(E,v) := H(v, r(E,v)), \tag{5.17}$$

where $r(E,v)$ is chosen such that $\mu(H(v,r(E,v))) = \mu(E)$. Note that $H_\mu(E,v)$ depends only on v and $\mu(E)$, and in particular,

$$H_\mu(F,v) = H_\mu(E,v) \tag{5.18}$$

for all sets $F \subset \mathbb{R}^n$ such that $\mu(F) = \mu(E)$.

Remark: We claim that (5.17) is well-defined. Let

$$g(t) = \int_{H(v,t)} f dx.$$

Observe that $g(t) \to 0$ as $t \to \infty$ since $\chi_{H(v,t)} \to 0$ as $t \to \infty$. Similarly, $g(t) \to \mu(\mathbb{R}^n)$ as $t \to -\infty$ since $\chi_{H(v,t)} \to 1$ as $t \to -\infty$. If $\mu(E) = 0$ or $\mu(\mathbb{R}^n)$, we define $r(E,v)$ to be ∞ or $-\infty$. Thus, we may assume that $\mu(E) \in (0, \mu(\mathbb{R}^n))$. Since g is strictly decreasing, we now need to show that g is continuous to use the intermediate value theorem to conclude the existence of $r(E,v)$. Notice that $\chi_{H(v,t)} \to \chi_{H(v,t_0)}$ a.e. as $t \to t_0$ since $\mathcal{L}^n(\partial H(v,t_0)) = 0$. Therefore, $g(t) \to g(t_0)$ as $t \to t_0$.

Definition 5.1.4. Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is **symmetric (around x)** if for all $y_1 < x < y_2$ with $|y_1 - x| = |x - y_2|$, we have $f(y_1) = f(y_2)$. Similarly, we say that a measure μ on $\mathbb{R}$ is **symmetric (around x)** if for all $y_1 < x < y_2$ with $|y_1 - x| = |x - y_2|$,

$$\mu([y_1, x]) = \mu([x, y_2]).$$

Note that if $\mu = f\mathcal{L}$ with a positive distribution $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$, then f being symmetric (around x) implies μ is also symmetric (around x).

The connection between generalized Ehrhard symmetrization and symmetric measures is well-known. For example, in Bobkov's solution to the characterization problem in $n = 1$ for log-concave measures and generalized Ehrhard symmetrization, he proves that in this class, μ is an Ehrhard measure if and only if μ is symmetric ([Bob96], Proposition 2.2). Since we are not assuming that the distribution function is log-concave, we present the following related statement.

Lemma 5.1.6. *Let $\mu = f\mathcal{L} \in \mathcal{W}(\mathbb{R})$. If μ is an Ehrhard measure, then f is symmetric.*

Proof. **Step 1**: For any y_1, y_2 such that $\mu((-\infty, y_1]) = \mu([y_2, \infty))$, we have

$$(-\infty, y_1] = ([y_2, \infty))^s_{-e_1}, \quad ((-\infty, y_1])^s_{e_1} = [y_2, \infty),$$

since $f > 0$ and f is continuous. Therefore, if μ is an Ehrhard measure, then

$$\mathrm{Per}_\mu([y_2, \infty)) = \mathrm{Per}_\mu((-\infty, y_1]^s_{e_1}) \leq \mathrm{Per}_\mu((-\infty, y_1]),$$

and

$$\mathrm{Per}_\mu((-\infty, y_1]) = \mathrm{Per}_\mu([y_2, \infty)^s_{-e_1}) \leq \mathrm{Per}_\mu([y_2, \infty)).$$

That is, $\mathrm{Per}_\mu((-\infty, y_1]) = \mathrm{Per}_\mu([y_2, \infty))$. By Lemma 5.1.5, we obtain

$$\begin{aligned} f(y_1) = \int_{\partial^*(-\infty, y_1]} f \, d\mathcal{H}^0 &= \mathrm{Per}_\mu((-\infty, y_1]) \\ &= \mathrm{Per}_\mu([y_2, \infty)) = \int_{\partial^*[y_2, \infty)} f \, d\mathcal{H}^0 = f(y_2). \end{aligned} \tag{5.19}$$

Step 2: Let $F : \mathbb{R} \to (0, \mu(\mathbb{R}))$ be defined by $x \mapsto \mu((-\infty, x])$. Note that $F \in C^1(\mathbb{R})$ and F^{-1} is well-defined since $f > 0$. Moreover, $(F^{-1})'(p) = \frac{1}{f(F^{-1}(p))}$. For any $p \in (0, \mu(\mathbb{R}))$, let $y_1 = F^{-1}(p)$ and $y_2 = F^{-1}(\mu(\mathbb{R}) - p)$. Then $F(y_2) = \mu(\mathbb{R}) - F(y_1)$ and

$$\mu((-\infty, y_1]) = F(y_1) = \mu(\mathbb{R}) - F(y_2) = \mu([y_2, \infty)).$$

By (5.19) in Step 1, we conclude that

$$f(F^{-1}(p)) = f(y_1) = f(y_2) = f(F^{-1}(\mu(\mathbb{R}) - p)).$$

Thus,

$$(F^{-1})'(p) = \frac{1}{f(F^{-1}(p))} = \frac{1}{f(F^{-1}(\mu(\mathbb{R}) - p))} = (F^{-1})'(\mu(\mathbb{R}) - p) = -\left[(F^{-1})(\mu(\mathbb{R}) - p)\right]'.$$

Integrating, we see that there must exist a constant m such that for all $p \in (0, \mu(\mathbb{R}))$,

$$F^{-1}(p) + F^{-1}(\mu(\mathbb{R}) - p) = 2m. \tag{5.20}$$

Step 3: Finally, we claim that μ is symmetric around m. For any $y_1 < m < y_2$ with $|m - y_1| = |y_2 - m|$, i.e., $y_1 + y_2 = 2m$, we can write $y_1 = F^{-1}(p)$, where $p = \mu((-\infty, y_1]) \in (0, \mu(\mathbb{R}))$. Applying (5.20) on p, we have $y_2 = F^{-1}(\mu(\mathbb{R}) - p)$. That is,

$$\mu((-\infty, y_1]) = p = \mu(\mathbb{R}) - F(y_2) = \mu([y_2, \infty)).$$

Therefore, by (5.19), $f(y_1) = f(y_2)$. □

Now we turn our attention to the Ehrhard measure on $\mathbb{R}^n$. Our next lemma shows that if an Ehrhard measure possesses a product structure, indicating that it is a product measure $\mu_1 \times \cdots \times \mu_n$, then each component measure μ_i must be symmetric.

Lemma 5.1.7. *Let $\mu = \mu_1 \times \cdots \times \mu_n$ and let $\mu_i = f_i \mathcal{L} \in \mathcal{W}(\mathbb{R})$. If μ is an Ehrhard measure, then f_i is symmetric for all $i = 1, ..., n$.*

Proof. Let $f(x_1, \cdots, x_n) = f_1(x_1) \cdots f_n(x_n)$. The product measure μ can be expressed as

$$\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n).$$

For any y_1, y_2 such that $\mu\left((-\infty, y_1] \times \mathbb{R}^{n-1}\right) = \mu\left([y_2, \infty) \times \mathbb{R}^{n-1}\right)$, we have

$$(-\infty, y_1] \times \mathbb{R}^{n-1} = \left([y_2, \infty) \times \mathbb{R}^{n-1}\right)^s_{-e_1}, \quad \left((-\infty, y_1] \times \mathbb{R}^{n-1}\right)^s_{e_1} = [y_2, \infty) \times \mathbb{R}^{n-1},$$

since $\mu = \mu_1 \times \cdots \times \mu_n$. Therefore, if μ is an Ehrhard measure, then

$$\operatorname{Per}_\mu\left([y_2, \infty) \times \mathbb{R}^{n-1}\right) = \operatorname{Per}_\mu\left(\left((-\infty, y_1] \times \mathbb{R}^{n-1}\right)^s_{e_1}\right) \leq \operatorname{Per}_\mu\left((-\infty, y_1] \times \mathbb{R}^{n-1}\right),$$

and

$$\operatorname{Per}_\mu\left((-\infty, y_1] \times \mathbb{R}^{n-1}\right) = \operatorname{Per}_\mu\left(\left([y_2, \infty) \times \mathbb{R}^{n-1}\right)^s_{-e_1}\right) \leq \operatorname{Per}_\mu\left([y_2, \infty) \times \mathbb{R}^{n-1}\right).$$

That is, $\operatorname{Per}_\mu\left((-\infty, y_1] \times \mathbb{R}^{n-1}\right) = \operatorname{Per}_\mu\left([y_2, \infty) \times \mathbb{R}^{n-1}\right)$. Notice that

$$\begin{aligned}
\operatorname{Per}_\mu\left((-\infty, y_1] \times \mathbb{R}^{n-1}\right) &= \int_{\mathbb{R}^{n-1}} f(y_1, x_2, \cdots, x_n)\, dx_2 \cdots dx_n \\
&= \int_{\mathbb{R}^{n-1}} f_1(y_1) f_2(x_2) \cdots f_n(x_n)\, dx_2 \cdots dx_n \\
&= f_1(y_1)\mu_2(\mathbb{R}) \cdots \mu_n(\mathbb{R}),
\end{aligned}$$

where we have used Lemma 5.1.5 and the area formula. Similarly, $\operatorname{Per}\mu\left([y_2, \infty) \times \mathbb{R}^{n-1}\right) = f_1(y_2)\mu_2(\mathbb{R}) \cdots \mu_n(\mathbb{R})$. Hence, $f_1(y_1) = f_1(y_2)$. We can apply the rest of the proof in Lemma 5.1.6 to μ_1 to conclude that f_1 is symmetric. □

The following lemma tells us that shifting an Ehrhard measure results in another Ehrhard measure. Additionally, we can shift our Ehrhard measure with a product structure so that each component is symmetric around zero.

Lemma 5.1.8. *Let $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$ and let $a \in \mathbb{R}^n$. If μ is an Ehrhard measure, then*

$$\nu := \widetilde{f}\mathcal{L}^n$$

is also an Ehrhard measure, where $\widetilde{f}(x) := f(x+a)$. Moreover, if $f(x_1, ..., x_n) = f_1(x_1) \cdots f_n(x_n)$ with $f_i \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ satisfying $f_i > 0$, then there exists $m \in \mathbb{R}^n$ such that

$$\nu := \widetilde{f}\mathcal{L}^n$$

is an Ehrhard measure, where $\widetilde{f}(x) = f(x + m)$, $\widetilde{f}(x_1, ..., x_n) = \widetilde{f}_1(x_1) \cdots \widetilde{f}_n(x_n)$, $\widetilde{f}_i > 0$, $\widetilde{f}_i \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$, and $\widetilde{f}_i(-t) = \widetilde{f}_i(t)$ for all $i = 1, ..., n$.

Proof. Given any finite ν-perimeter set E in $\mathbb{R}^n$ and $u \in \mathbb{S}^{n-1}$,

$$\begin{aligned}\operatorname{Per}_\nu(E) &= \sup\left\{\int_E \left(\widetilde{f}\operatorname{div}\varphi + \varphi\cdot\nabla\widetilde{f}\right)dx : \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), |\varphi| \le 1\right\} \\ &= \sup\left\{\int_{E+a} \Big(f(x)(\operatorname{div}\varphi)(x-a) + \varphi(x-a)\cdot\nabla f(x)\Big)dx : \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), |\varphi| \le 1\right\} \\ &= \sup\left\{\int_{E+a} \Big(f(\operatorname{div}\widetilde{\varphi} + \widetilde{\varphi}\cdot\nabla f\Big)dx : \widetilde{\varphi} \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), |\widetilde{\varphi}| \le 1\right\} = \operatorname{Per}_\mu(E+a).\end{aligned}$$

Moreover, we have

$$E^s_{\nu,u} + a = (E+a)^s_{\mu,u}\,.$$

Combining them together,

$$\operatorname{Per}_\nu(E^s_{\nu,u}) = \operatorname{Per}_\mu\left(E^s_{\nu,u} + a\right) = \operatorname{Per}_\mu\left((E+a)^s_{\mu,u}\right) \le \operatorname{Per}_\mu(E+a) = \operatorname{Per}_\nu(E)$$

since μ is an Ehrhard measure. Thus, ν is also an Ehrhard measure. In particular, if $f(x_1,...,x_n) = f_1(x_1)\cdots f_n(x_n)$, by Lemma 5.1.7, there exist $m_1,...,m_n$ in $\mathbb{R}$ such that f_i is symmetric around m_i for all $i = 1,...,n$. Let $m = (m_1,...,m_n) \in \mathbb{R}^n$ and define $\widetilde{f}(x) = f(x+m)$. That is, $\widetilde{f}(x_1,...,x_n) = \widetilde{f}_1(x_1)\cdots\widetilde{f}_n(x_n)$, where $\widetilde{f}_i(t) = f_i(t+m_i)$. Since f_i is symmetric around m_i, we have

$$\widetilde{f}_i(-t) = f_i(m_i - t) = f_i(m_i + t) = \widetilde{f}_i(t)$$

for any $-t < 0 < t$. □

Remark: Note that the equality $(E+a)^s_{\mu,u} = E^s_{\mu,u} + a$ does not hold in general.

5.2 Half-planes are local isoperimetric sets

The main result in this section is the following lemma.

Lemma 5.2.1. *Let $n \ge 2$, and let $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$ be an Ehrhard measure. For every $v \in \mathbb{S}^{n-1}$ and every Borel set $E \subset \mathbb{R}^n$,*

$$\operatorname{Per}_\mu(H_\mu(E,v)) \le \operatorname{Per}_\mu(E).$$

The main idea of the proof is that for any Borel set $E \subset \mathbb{R}^n$, if μ is an Ehrhard measure, then generalized Ehrhard symmetrization allows us to move mass from $E \setminus H_\mu(E,u)$ to "fill in" $H_\mu(E,u) \setminus E$ while not increasing the perimeter. As noted in Section 1.2, Ehrhard provides a geometric proof of Lemma 5.2.1 for $\mu = \gamma_{I_n}$, [Ehr83], Proposition 1.5 (See Lifschits's book [Lif95], Chapter 11 for a detailed proof in English). However, the core of Ehrhard's proof is to produce a sequence of symmetrizations and show that a sequence of cones which widen to the half-space. This proof relies essentially upon the measure μ having a product structure for any orthonormal frame. While this is obviously true for the isotropic Gaussian, we cannot assume this for general μ. We begin by studying the one-dimensional case.

Lemma 5.2.2 (One-Dimensional Behavior)**.** *Let $\mu = f\mathcal{L}$ with a positive distribution function $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$. Suppose $E \subset \mathbb{R}$ is a Borel set and the interval (a,b) satisfies $\mu(E^c \cap (a,b)) = \alpha$ and $\mu(E \cap [b,\infty)) = \beta$. Then*

$$\mu\left(E^s_{-e_1} \cap [b,\infty)\right) \leq \max\{\beta - \alpha, 0\}.$$

Equivalently,

$$\mu\left(E^s_{-e_1} \setminus (-\infty,b]\right) \leq \mu\left(E \setminus (-\infty,b]\right) - \min\{\beta,\alpha\}. \tag{5.21}$$

In particular,

(1) $\mu\left(E^s_{-e_1} \setminus (-\infty,b]\right) \leq \mu\left(E \setminus (-\infty,b]\right)$ for any measurable set $E \subset \mathbb{R}$ and $b \in \mathbb{R}$.

(2) if there exists $x > y$, and $r > 0$ such that $(x-r,x+r) \cap (y-r,y+r) = \varnothing$ and

$$\mu(E \cap (x-r,x+r)) > 0, \quad \mu(E^c \cap (y-r,y+r)) > 0,$$

then $\mu\left(E^s_{-e_1} \setminus (-\infty,b]\right) < \mu\left(E \setminus (-\infty,b]\right)$ for any $y + r < b < x - r$.

Proof. It is clear that $\mu(\mathbb{R}) < \infty$ since $f \in L^1(\mathbb{R})$. Without loss of generality, we may assume that $E^s_{-e_1} \cap [b,\infty) \neq \varnothing$. Since $E^s_{-e_1}$ is an interval of the form $(-\infty,c]$ for some c, we have $E^s_{-e_1} \cup [b,\infty) = \mathbb{R}$. Notice that

$$\left(E^c \cap (a,b)\right) \cap \left(E \cup [b,\infty)\right) = \varnothing.$$

Then

$$\begin{aligned}\alpha + \mu\left(E \cup [b,\infty)\right) &= \mu\left(E^c \cap (a,b)\right) + \mu\left(E \cup [b,\infty)\right)\\ &= \mu\Big(\left(E^c \cap (a,b)\right) \cup \left(E \cup [b,\infty)\right)\Big) \leq \mu(\mathbb{R}),\end{aligned}$$

and hence

$$\begin{aligned}\mu\left(E^s_{-e_1} \cap [b,\infty)\right) &= \mu\left(E^s_{-e_1}\right) + \mu\left([b,\infty)\right) - \mu\left(E^s_{-e_1} \cup [b,\infty)\right)\\ &= \mu\left(E\right) + \mu\left([b,\infty)\right) - \mu(\mathbb{R})\\ &= \mu\left(E \cap [b,\infty)\right) + \mu\left(E \cup [b,\infty)\right) - \mu(\mathbb{R})\\ &= \beta - \Big(\mu(\mathbb{R}) - \mu\left(E \cup [b,\infty)\right)\Big) \leq \beta - \alpha.\end{aligned}$$

Observe that $\mu\left(E \setminus (-\infty,b]\right) = \mu\left(E \cap [b,\infty)\right) = \beta$ since $\mu(\{b\}) = 0$. This gives us,

$$\begin{aligned}\mu\left(E^s_{-e_1} \setminus (-\infty,b]\right) = \mu\left(E^s_{-e_1} \cap [b,\infty)\right) \leq \max\{\beta-\alpha,0\} &= \beta - \min\{\beta,\alpha\}\\ &= \mu(E \setminus (-\infty,b]) - \min\{\beta,\alpha\}.\end{aligned}$$

Since $\min\{\beta,\alpha\} \geq 0$ for any $b \in \mathbb{R}$, this yields (1). Now we prove (2). Suppose there exists $x > y$, and $r > 0$ such that $(x-r,x+r) \cap (y-r,y+r) = \varnothing$ and

$$\mu(E \cap (x-r,x+r)) > 0, \quad \mu(E^c \cap (y-r,y+r)) > 0.$$

Let $a = y - r$ and $y + r < b < x - r$. Then

$$\alpha = \mu(E^c \cap (a,b)) \geq \mu(E^c \cap (y-r,y+r)) > 0$$

and

$$\beta = \mu(E \cap [b,\infty)) \geq \mu(E \cap (x-r,x+r)) > 0.$$

In particular, $\min\{\beta,\alpha\} > 0$. By (5.21), we have finished the proof of (2). □

Lemma 5.2.2 has the following, immediate corollary in higher dimensions.

Lemma 5.2.3. *Let $n \geq 2$, and let $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$ be an Ehrhard measure, $E \subset \mathbb{R}^n$ be a Borel set, and $v \in \mathbb{S}^{n-1}$. Then for every $\eta \in \mathbb{S}^{n-1}$ with $v \cdot \eta > 0$,*

$$\mu(E^s_\eta \setminus H_\mu(E,v)) \leq \mu(E \setminus H_\mu(E,v)). \tag{5.22}$$

Proof. This follows from Fubini's theorem and Lemma 5.2.2 applied to each of the fibers $\pi^{-1}_{\langle\eta\rangle^\perp}(z)$ for $z \in \langle\eta\rangle^\perp$. Indeed, Fubini's theorem tells us that

$$\begin{aligned}\mu(E \setminus H_\mu(E,v)) &= \int_{\langle\eta\rangle^\perp} \int_{\pi^{-1}_{\langle\eta\rangle^\perp}(z) \cap (E \setminus H_\mu(E,v))} f(x)\, d\mathcal{H}^1(x) d\mathcal{H}^{n-1}(z) \\ &= \int_{\langle\eta\rangle^\perp} \int_{\left(E \cap \pi^{-1}_{\langle\eta\rangle^\perp}(z)\right) \setminus \left(\pi^{-1}_{\langle\eta\rangle^\perp}(z) \cap H_\mu(E,v)\right)} f|_{\pi^{-1}_{\langle\eta\rangle^\perp}(z)}(x)\, d\mathcal{H}^1(x) d\mathcal{H}^{n-1}(z),\end{aligned}$$

and for $\mathcal{H}^{n-1}$-a.e. z in $\langle\eta\rangle^\perp$, $f|_{\pi^{-1}_{\langle\eta\rangle^\perp}(z)}$ is positive, continuous, and $\mathcal{H}^1$-integrable. Define

$$(-\infty, b(z)] := \pi^{-1}_{\langle\eta\rangle^\perp}(z) \cap H_\mu(E,v).$$

Notice that

$$\left(E \cap \pi^{-1}_{\langle\eta\rangle^\perp}(z)\right)^s_{\mu_z,\eta} = E^s_\eta \cap \pi^{-1}_{\langle\eta\rangle^\perp}(z).$$

Applying Lemma 5.2.2 (1) with $\mu_z = f\mathcal{H}^1 \llcorner \pi^{-1}_{\langle\eta\rangle^\perp}(z)$, we have

$$\begin{aligned}\int_{(E^s_\eta \cap \pi^{-1}_{\langle\eta\rangle^\perp}(z)) \setminus (-\infty, b(z)]} f|_{\pi^{-1}_{\langle\eta\rangle^\perp}(z)}(x)\, d\mathcal{H}^1(x) &= \mu_z\left((E^s_\eta \cap \pi^{-1}_{\langle\eta\rangle^\perp}(z)) \setminus (-\infty, b(z)]\right) \\ &= \mu_z\left(\left(E \cap \pi^{-1}_{\langle\eta\rangle^\perp}(z)\right)^s_{\mu_z,\eta} \setminus (-\infty, b(z)]\right) \\ &\leq \mu_z\left((E \cap \pi^{-1}_{\langle\eta\rangle^\perp}(z)) \setminus (-\infty, b(z)]\right) \\ &= \int_{(E \cap \pi^{-1}_{\langle\eta\rangle^\perp}(z)) \setminus (-\infty, b(z)]} f|_{\pi^{-1}_{\langle\eta\rangle^\perp}(z)}(x)\, d\mathcal{H}^1(x).\end{aligned}$$

By integrating the above equation on both sides with respect to z, we complete the proof. ☐

However, Lemma 5.2.3 is not strong enough to prove Lemma 5.2.1. Instead, we need the following improvement.

Lemma 5.2.4. *Let $n \geq 2$, and let $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$ be an Ehrhard measure, $E \subset \mathbb{R}^n$ be a Borel set, $v \in \mathbb{S}^{n-1}$, and assume that*

$$\mu(E \Delta H_\mu(E,v)) > 0.$$

There exists a vector $\eta \in \mathbb{S}^{n-1}$ such that $v \cdot \eta > 0$ and

$$\mu(E^s_\eta \setminus H_\mu(E,v)) < \mu(E \setminus H_\mu(E,v)). \tag{5.23}$$

In particular, $\mu(E^s_\eta \Delta H_\mu(E,v)) < \mu(E \Delta H_\mu(E,v))$.

Proof. Notice that for any A, B with $\mu(A) = \mu(B)$,

$$\mu(A \setminus B) = \mu(A) - \mu(A \cap B) = \mu(B) - \mu(A \cap B) = \mu(B \setminus A).$$

In particular, we have $\mu(A \Delta B) = 2\mu(A \setminus B) = 2\mu(B \setminus A)$. Thus, the assumption gives us

$$2\mu(E \setminus H_\mu(E,v)) = 2\mu(H_\mu(E,v) \setminus E) = \mu(E \Delta H_\mu(E,v)) > 0.$$

It follows that both $\mathcal{L}^n(E \setminus H_\mu(E,v)) > 0$ and $\mathcal{L}^n(H_\mu(E,v) \setminus E) > 0$ since $f > 0$. Therefore, we can choose x and y such that

(a) $x \in \big(E \setminus H_\mu(E,v)\big)$ is a point of density 1 with respect to $E \setminus H_\mu(E,v)$,

(b) $y \in H_\mu(E,v) \setminus E$ is a point of density 1 with respect to $H_\mu(E,v) \setminus E$, and

(c) $v \cdot (y - x) > 0$.

Let $\eta = \frac{y-x}{|y-x|}$ and let $\eta_x^\perp := x + \langle \eta \rangle^\perp$. For $z \in \eta_x^\perp$, we denote $\mathcal{H}^1_z = \mathcal{H}^1 \llcorner \pi^{-1}_{\eta_x^\perp}(z)$ and $\mu_z = f\mathcal{H}^1 \llcorner \pi^{-1}_{\eta_x^\perp}(z)$, where $\pi_{\eta_x^\perp}$ is the orthogonal projection onto $\eta_x^\perp$. Consider $Q_\eta(x,r)$ as the cube centered at x with each side equal to $2r$ and whose sides are parallel to the η direction. From (a) and (b), we have

$$\lim_{r\to 0} \frac{\mathcal{L}^n\Big(Q_\eta(x,r) \cap \big(E \setminus H_\mu(E,v)\big)\Big)}{\mathcal{L}^n(Q_\eta(x,r))} = 1, \quad \lim_{r\to 0} \frac{\mathcal{L}^n\Big(Q_\eta(y,r) \cap (\mathring{H}_\mu(E,v) \setminus E)\Big)}{\mathcal{L}^n(Q_\eta(y,r))} = 1.$$

In particular, there exists $r > 0$ such that $Q_\eta(x,r) \cap Q_\eta(y,r) = \varnothing$ and

$$\frac{\mathcal{L}^n\left(Q_\eta(x,r)\cap E\right)}{\mathcal{L}^n(Q_\eta(x,r))} = \frac{\mathcal{L}^n\Big(Q_\eta(x,r)\cap\big(E\setminus H_\mu(E,v)\big)\Big)}{\mathcal{L}^n(Q_\eta(x,r))} > \frac{3}{4},$$

since $H_\mu(E,v)$ is a closed set. Moreover, by Fubini's theorem, we have

$$\begin{aligned}
\frac{3}{4}(2r)^n = \frac{3}{4}\mathcal{L}^n(Q_\eta(x,r)) &< \mathcal{L}^n\left(Q_\eta(x,r)\cap E\right)\\
&= \int_{\eta_x^\perp\cap Q_\eta(x,r)}\int_{\pi^{-1}_{\eta_x^\perp}(z)\cap\big(Q_\eta(x,r)\cap E\big)} 1\, d\mathcal{H}^1\, d\mathcal{H}^{n-1}(z)\\
&= \int_{\eta_x^\perp\cap Q_\eta(x,r)\cap\{z:\mathcal{H}^1_z(Q_\eta(x,r)\cap E)>0\}} \mathcal{H}^1_z\Big(Q_\eta(x,r)\cap E\Big)\, d\mathcal{H}^{n-1}(z)\\
&\le (2r)\mathcal{H}^{n-1}\left(\eta_x^\perp\cap Q_\eta(x,r)\cap\{z:\mathcal{H}^1_z(Q_\eta(x,r)\cap E)>0\}\right),
\end{aligned}$$

where we have used $\mathcal{H}^1_z(Q_\eta(x,r)\cap E)\le 2r$ in the last inequality. Note that $\{z:\mathcal{H}^1_z(Q_\eta(x,r)\cap E)>0\} = \{z:\mu_z(Q_\eta(x,r)\cap E)>0\}$ since $f>0$. Therefore,

$$\begin{aligned}
&\mathcal{H}^{n-1}\left(\{z\in\eta_x^\perp\cap Q_\eta(x,r):\mu_z(E\cap Q_\eta(y,r))>0\}\right)\\
&= \mathcal{H}^{n-1}\left(\eta_x^\perp\cap Q_\eta(x,r)\cap\{z:\mathcal{H}^1_z(Q_\eta(x,r)\cap E)>0\}\right) > \frac{3}{4}(2r)^{n-1}.
\end{aligned}$$

A similar argument shows that

$$\mathcal{H}^{n-1}\left(\{z\in\eta_x^\perp\cap Q_\eta(x,r):\mu_z(E^c\cap Q_\eta(y,r))>0\}\right) > \frac{3}{4}(2r)^{n-1}.$$

Combining them together, we obtain

$$\begin{aligned}
&\mathcal{H}^{n-1}\left(\Big\{z\in\eta_x^\perp\cap Q_\eta(x,r):\mu_z(E\cap Q_\eta(x,r))>0 \text{ and } \mu_z(E^c\cap Q_\eta(y,r))>0\Big\}\right)\\
&= \mathcal{H}^{n-1}\left(\{z\in\eta_x^\perp\cap Q_\eta(x,r):\mu_z(E\cap Q_\eta(y,r))>0\}\right)\\
&\quad + \mathcal{H}^{n-1}\left(\{z\in\eta_x^\perp\cap Q_\eta(x,r):\mu_z(E^c\cap Q_\eta(y,r))>0\}\right)\\
&\quad - \mathcal{H}^{n-1}\left(\Big\{z\in\eta_x^\perp\cap Q_\eta(x,r):\mu_z(E\cap Q_\eta(x,r))>0 \text{ or } \mu_z(E^c\cap Q_\eta(y,r))>0\Big\}\right)\\
&\ge \frac{3}{4}(2r)^{n-1} + \frac{3}{4}(2r)^{n-1} - (2r)^{n-1} = \frac{1}{2}(2r)^{n-1} > 0.
\end{aligned}$$

Therefore, applying Lemma 5.2.2 (2) to each of the fibers inside the tube based on the above set from $Q_\eta(x,r)$ to $Q_\eta(y,r)$, and Lemma 5.2.2 (1) to each of the fibers outside that tube, yields the claim of the lemma. $\square$

Lemma 5.2.5. *Let $n \geq 2$, $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$ be an Ehrhard measure, $v \in \mathbb{S}^{n-1}$, and assume that E is a Borel set with finite μ-perimeter. Let $\mathcal{S}(E)$ be the collection of all sets $F \subset \mathbb{R}^n$ such that*

$$F = E^s_{\eta_1,\ldots,\eta_k}$$

for some finite sequence of vectors $\{\eta_i\}_{i=1}^k$ such that $v \cdot \eta_i > 0$ for all $i = 1, ..., k$. Then

$$\inf_{F \in \mathcal{S}(E)} \mu(F \Delta H_\mu(E,v)) = 0.$$

Proof. We argue by contradiction. Suppose that $\inf_{F\in\mathcal{S}(E)} \mu(F\Delta H_\mu(E,v)) > 0$. Then, for a minimizing sequence F_j, we may invoke Lemma 5.1.4 to obtain a set of finite μ-perimeter $F_\infty \subset \mathbb{R}^n$ such that there exists a subsequence (still denoted as F_j)

$$\chi_{F_j} \to \chi_{F_\infty} \text{ in } L^1_{\text{loc}}(\mathbb{R}^n,\mu), \quad \operatorname{Per}_\mu(F_\infty) \leq \liminf_{j\to\infty} \operatorname{Per}_\mu(F_j),$$

where we have used $\operatorname{Per}_\mu(F_j) \leq \operatorname{Per}_\mu(E)$ since μ is an Ehrhard measure and hence

$$\sup_j |D\chi_{F_j}|_\mu(\mathbb{R}^n) = \sup_j \operatorname{Per}_\mu(F_j) \leq \operatorname{Per}_\mu(E) < \infty.$$

Moreover, by the strong convergence in $L^1_{\text{loc}}(\mathbb{R}^n,\mu)$, it also follows that

$$\mu(F_\infty \Delta H_\mu(E,v)) = \inf_{F\in\mathcal{S}(E)} \mu(F\Delta H_\mu(E,v)) > 0. \tag{5.24}$$

Indeed, since $f \in L^1(\mathbb{R}^n)$, for any $\varepsilon > 0$, there exists $B_R := B(0,R)$ such that $\mu(B_R^c) < \varepsilon$. Hence,

$$\begin{aligned}
&\left|\mu(F_j\Delta H_\mu(E,v)) - \mu(F_\infty\Delta H_\mu(E,v))\right| \\
&\leq \left|\mu((F_j\Delta H_\mu(E,v)) \cap B_R) - \mu((F_\infty\Delta H_\mu(E,v)) \cap B_R)\right| \\
&\quad + \left|\mu((F_j\Delta H_\mu(E,v)) \cap B_R^c) - \mu((F_\infty\Delta H_\mu(E,v)) \cap B_R^c)\right| \\
&\leq \mu((F_j\Delta F_\infty) \cap B_R) + 2\varepsilon,
\end{aligned}$$

where we have used the triangle inequality for symmetric differences. Taking $j \to \infty$ and $\varepsilon \to 0$, we have

$$0 < \inf_{F \in \mathcal{S}(E)} \mu(F \Delta H_\mu(E,v)) = \lim_{j\to\infty} \mu(F_j \Delta H_\mu(E,v)) = \mu(F_\infty \Delta H_\mu(E,v)).$$

By a similar argument, we also have $\mu(F_\infty) = \lim_{j\to\infty} \mu(F_j) = \mu(E)$. Therefore, we may apply Lemma 5.2.4 to μ, f, F_∞, v to obtain a vector $\eta \in \mathbb{S}^{n-1}$ with $v \cdot \eta > 0$ and

$$\begin{aligned} \mu\left((F_\infty)^s_\eta \Delta H_\mu(E,v)\right) &= \mu\left((F_\infty)^s_\eta \Delta H_\mu(F_\infty,v)\right) \\ &< \mu\left(F_\infty \Delta H_\mu(F_\infty,v)\right) \\ &= \mu\left(F_\infty \Delta H_\mu(E,v)\right) = \inf_{F\in\mathcal{S}(E)} \mu(F\Delta H_\mu(E,v)), \end{aligned} \tag{5.25}$$

where the first equality follows from (5.18). Therefore, the proof of the lemma will be accomplished if we can show that for sufficiently large j we have

$$\mu\left((F_j)^s_\eta \Delta H_\mu(E,v)\right) \to \mu\left((F_\infty)^s_\eta \Delta H_\mu(E,v)\right).$$

However, this is immediate from the strong $L^1_{\text{loc}}(\mathbb{R}^n,\mu)$ convergence. That is, for the fixed η above, define the marginal function $m_{F_j} : \langle\eta\rangle^\perp \to \mathbb{R}$ as

$$m_{F_j}(a) = \int_{\pi^{-1}_{\langle\eta\rangle^\perp}(a)\cap F_j} f d\mathcal{H}^1.$$

Then

$$\begin{aligned} \int_{\langle\eta\rangle^\perp\cap B_R} |m_{F_j}(a) - m_{F_\infty}(a)| d\mathcal{H}^{n-1}(a) &\le \int_{\langle\eta\rangle^\perp\cap B_R} \int_{\pi^{-1}_{\langle\eta\rangle^\perp}(a)\cap B_R} \left|\chi_{F_j} - \chi_{F_\infty}\right| f d\mathcal{H}^1 d\mathcal{H}^{n-1}(a) \\ &= \int_{B_R} \left|\chi_{F_j} - \chi_{F_\infty}\right| d\mu \to 0, \end{aligned}$$

i.e., $m_{F_j} \to m_{F_\infty}$ in $L^1_{\text{loc}}(\langle\eta\rangle^\perp)$, where we have used $F_j \to F_\infty$ in $L^1_{\text{loc}}(\mathbb{R}^n,\mu)$ and Fubini's theorem. Since $v \cdot \eta > 0$, we have $H_\mu(E,v) = \left(H_\mu(E,v)\right)^s_\eta$ and

$$\mu\left((F_j)^s_\eta \Delta H_\mu(E,v)\right) = \mu\left((F_j)^s_\eta \Delta \left(H_\mu(E,v)\right)^s_\eta\right) = \int_{\langle\eta\rangle^\perp} |m_{F_j}(a) - m_{H_\mu(E,v)}(a)| d\mathcal{H}^{n-1}(a)$$

$$\leq \int_{\langle\eta\rangle^\perp} |m_{F_j}(a) - m_{F_\infty}(a)| + |m_{F_\infty}(a) - m_{H_\mu(E,v)}(a)| d\mathcal{H}^{n-1}(a)$$
$$= \int_{\langle\eta\rangle^\perp} |m_{F_j}(a) - m_{F_\infty}(a)| d\mathcal{H}^{n-1}(a) + \mu\left((F_\infty)^s_\eta \Delta \left(H_\mu(E,v)\right)^s_\eta\right)$$
$$= \int_{\langle\eta\rangle^\perp} |m_{F_j}(a) - m_{F_\infty}(a)| d\mathcal{H}^{n-1}(a) + \mu\left((F_\infty)^s_\eta \Delta H_\mu(E,v)\right)$$
$$\leq \int_{\langle\eta\rangle^\perp \cap B_R} |m_{F_j}(a) - m_{F_\infty}(a)| d\mathcal{H}^{n-1}(a) + 2\varepsilon + \mu\left((F_\infty)^s_\eta \Delta H_\mu(E,v)\right).$$

Therefore, as $j \to \infty$ and $\varepsilon \to 0$, we obtain

$$\lim_{j' \to \infty} \mu\left((F_j)^s_\eta \Delta H_\mu(E,v)\right) \leq \mu\left((F_\infty)^s_\eta \Delta H_\mu(E,v)\right).$$

By (5.25), for sufficiently large j, there is a F_j such that

$$\mu\left((F_j)^s_\eta \Delta H_\mu(E,v)\right) < \inf_{F \in \mathcal{S}(E)} \mu(F \Delta H_\mu(E,v)).$$

But, since $(F_j)^s_\eta = (E^s_{v_1, v_2 \dots, v_N})^s_\eta \in \mathcal{S}(E)$, we obtain a contradiction. Therefore, the lemma holds. □

5.2.1 Proof of Lemma 5.2.1

Let E, μ, v be as in the hypotheses of Lemma 5.2.1. If $\mathrm{Per}_\mu(E) = \infty$ or $\mu(E \Delta H_\mu(E,v)) = 0$, there is nothing to prove. Therefore, we may assume that E is a set of finite μ-perimeter and $\mu(E \Delta H_\mu(E,v)) > 0$. By Lemma 5.2.5, we may take a minimizing sequence $F_j \in \mathcal{S}(E)$ such that

$$\lim_{j \to \infty} \mu\left(F_j \Delta H_\mu(E,v)\right) = 0.$$

Since $F_j \in \mathcal{S}(E)$ and μ is an Ehrhard measure, we have $\sup_j \mathrm{Per}_\mu(F_j) \leq \mathrm{Per}_\mu(E) < \infty$. By Lemma 5.1.4, we may extract a subsequence (still denoted as F_j) such that $\chi_{F_j} \to \chi_{F_\infty}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mu)$ for some Borel set F_∞ with

$$\mathrm{Per}_\mu(F_\infty) \leq \liminf_{j \to \infty} \mathrm{Per}_\mu(F_j) \leq \mathrm{Per}_\mu(E).$$

Moreover, similar to (5.24), we have

$$\mu(F_\infty \Delta H_\mu(E,v)) = \lim_{j\to\infty} \mu\left(F_j \Delta H_\mu(E,v)\right) = 0.$$

This implies $\mathrm{Per}_\mu(F_\infty) = \mathrm{Per}_\mu(H_\mu(E,v))$ and hence $\mathrm{Per}_\mu(H_\mu(E,v)) \le \mathrm{Per}_\mu(E)$. □

Lemma 5.2.1 has the following trivial consequence.

Lemma 5.2.6. *Let $n \ge 2$, and let $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$ be an Ehrhard measure. For any v, η in $\mathbb{S}^{n-1}$ and any Borel set $E \subset \mathbb{R}^n$,*

$$\mathrm{Per}_\mu(H_\mu(E,\eta)) = \mathrm{Per}_\mu(H_\mu(E,v)),$$

i.e., the half-spaces $\{H_\mu(E,\eta)\}_{\eta\in\mathbb{S}^{n-1}}$ are minimal for μ-perimeter among all Borel sets F with $\mu(F) = \mu(E)$. In particular, for any $u \in \mathbb{S}^{n-1}$ and $c \in \mathbb{R}$,

$$\mathrm{Per}_\mu(H(u,c)) \le \mathrm{Per}_\mu(F),$$

for every Borel set F with $\mu(F) = \mu(H(u,c))$, i.e., the half-space $H(u,c)$ is minimal for μ-perimeter among all Borel sets F with $\mu(F) = \mu(H(u,c))$.

Proof. Applying Lemma 5.2.1 to $H_\mu(E,v)$ and η in place of v shows that

$$\mathrm{Per}_\mu(H_\mu(E,\eta)) \le \mathrm{Per}_\mu(H_\mu(E,v)).$$

Applying it again gives the reverse inequality. Now, for any $u \in \mathbb{S}^{n-1}$, $c \in \mathbb{R}$, and F with $\mu(F) = \mu(H(u,c))$, we have

$$H(u,c) = H_\mu(F,u),$$

since $f > 0$. Then by Lemma 5.2.1 again,

$$\mathrm{Per}_\mu(H(u,c)) = \mathrm{Per}_\mu(H_\mu(F,u)) \le \mathrm{Per}_\mu(F).$$

□

5.3 The density enjoys a product structure

The main result in this section demonstrates that if $\mu \in \mathcal{W}(\mathbb{R}^n)$ with distribution function f is an Ehrhard measure, then the function f possesses a product structure subordinate to all orthogonal frames $\{e_i\}_{i=1}^n$ in $\mathbb{R}^n$. This assertion is formalized in Corollary 5.3.3.

Our first lemma serves as a technical prerequisite before proving the product structure. It ensures that we can apply the Implicit Function Theorem and obtain sufficient smoothness when perturbing our half-space. We include the additional assumption of $f \in W^{1,1}(\mathbb{R}^n)$ because the half-space is unbounded, necessitating some control to apply the dominated convergence theorem.

Lemma 5.3.1. *Let $n \geq 2$ and $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$. Write $x \in \mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Assume that $\mathrm{Per}_\mu(\{x : x_n > c\}) < \infty$ for some $c \in \mathbb{R}$, $h \in C_c^\infty(\mathbb{R}^{n-1})$, and define*

$$F(\varepsilon, \eta) := \mu\left(\left\{x : x_n > c + \varepsilon h(x') + \eta\right\}\right) - \mu\left(\{x : x_n > c\}\right).$$

Then $F \in C^1$, $F(0,0) = 0$ and $\frac{\partial F}{\partial \eta}(0,0) \neq 0$.

Proof. Clearly $F(0,0) = 0$. Now, we establish the existence of $\frac{\partial F}{\partial \eta}$.

$$\begin{aligned}
\frac{\partial}{\partial \eta} F(\varepsilon, \eta) &= \lim_{\delta \to 0} \frac{\mu\left(\left\{x : x_n > c + \varepsilon h(x') + \eta + \delta\right\}\right) - \mu\left(\left\{x : x_n > c + \varepsilon h(x') + \eta\right\}\right)}{\delta} \\
&= -\lim_{\delta \to 0} \int_{\mathbb{R}^{n-1}} \left(\frac{1}{\delta} \int_{c+\varepsilon h(x')+\eta}^{c+\varepsilon h(x')+\eta+\delta} f(x', x_n)\, dx_n \right) dx' \\
&= -\int_{\mathbb{R}^{n-1}} f(x', c + \varepsilon h(x') + \eta)\, dx',
\end{aligned}$$

where we have used the dominated convergence theorem with the following estimations.

$$\begin{aligned}
&\left| \frac{1}{\delta} \int_{c+\varepsilon h(x')+\eta}^{c+\varepsilon h(x')+\eta+\delta} f(x', x_n)\, dx_n - f(x', c + \varepsilon h(x') + \eta) \right| \\
&= \left| \frac{1}{\delta} \int_{c+\varepsilon h(x')+\eta}^{c+\varepsilon h(x')+\eta+\delta} f(x', x_n) - f(x', c + \varepsilon h(x') + \eta)\, dx_n \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{1}{\delta} \int_{c+\varepsilon h(x')+\eta}^{c+\varepsilon h(x')+\eta+\delta} \left(\int_{c+\varepsilon h(x')+\eta}^{x_n} \partial_{x_n} f(x',t)\, dt \right) dx_n \right| \\
&\leq \frac{1}{|\delta|} \int_{c+\varepsilon h(x')+\eta-|\delta|}^{c+\varepsilon h(x')+\eta+|\delta|} \left| \int_{c+\varepsilon h(x')+\eta}^{x_n} \partial_{x_n} f(x',t)\, dt \right| dx_n \\
&\leq \frac{1}{|\delta|} \int_{c+\varepsilon h(x')+\eta-|\delta|}^{c+\varepsilon h(x')+\eta+|\delta|} \int_{c+\varepsilon h(x')+\eta-|\delta|}^{c+\varepsilon h(x')+\eta+|\delta|} |\partial_{x_n} f(x',t)|\, dt\, dx_n \\
&= 2 \int_{c+\varepsilon h(x')+\eta-|\delta|}^{c+\varepsilon h(x')+\eta+|\delta|} |\partial_{x_n} f(x',t)|\, dt \leq 2 \int_{\mathbb{R}} |\nabla f(x',t)|\, dt \in L^1(\mathbb{R}^{n-1}).
\end{aligned}
$$

Furthermore, by Lemma 5.1.5 and the area formula,

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}} f(x', c+\varepsilon h(x')+\eta)\, dx' &= \int_{\mathbb{R}^{n-1}} \int_{c}^{c+\varepsilon h(x')+\eta} \partial_{x_n} f(x',t)\, dt dx' + \int_{\mathbb{R}^{n-1}} f(x',c)\, dx' \\
&\leq \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |\nabla f(x',t)|\, dt dx' + \operatorname{Per}_\mu(\{x : x_n > c\}) < \infty.
\end{aligned}
$$

Indeed, $\frac{\partial F}{\partial \eta}(0,0) \neq 0$ since $f > 0$. Next, we establish the continuity of $\frac{\partial F}{\partial \eta}$. It suffices to demonstrate that

$$
\lim_{\eta \to \eta_0} \int_{\mathbb{R}^{n-1}} f(x', c+\varepsilon h(x')+\eta)\, dx' = \int_{\mathbb{R}^{n-1}} f(x', c+\varepsilon h(x')+\eta_0)\, dx'.
$$

Observe that

$$
\begin{aligned}
|f(x', c+\varepsilon h(x')+\eta) - f(x', c+\varepsilon h(x')+\eta_0)| &= \left| \int_{c+\varepsilon h(x')+\eta_0}^{c+\varepsilon h(x')+\eta} \partial_{x_n} f(x',t)\, dt \right| \\
&\leq \int_{\mathbb{R}} |\nabla f(x',t)|\, dt \in L^1(\mathbb{R}^{n-1}). \qquad (5.26)
\end{aligned}
$$

Similarly, for the existence of $\frac{\partial F}{\partial \varepsilon}$,

$$
\begin{aligned}
\frac{\partial}{\partial \varepsilon} F(\varepsilon, \eta) &= \lim_{\delta \to 0} \frac{\mu\left(\left\{x : x_n > c + (\varepsilon+\delta) h(x') + \eta\right\}\right) - \mu\left(\left\{x : x_n > c + \varepsilon h(x') + \eta\right\}\right)}{\delta} \\
&= -\lim_{\delta \to 0} \int_{\mathbb{R}^{n-1}} \left(\frac{1}{\delta} \int_{c+\varepsilon h(x')+\eta}^{c+(\varepsilon+\delta)h(x')+\eta} f(x', x_n)\, dx_n \right) dx' \\
&= -\int_{\mathbb{R}^{n-1}} f(x', c+\varepsilon h(x')+\eta) h(x')\, dx',
\end{aligned}
$$

where we have again used the dominated convergence theorem with the following estimations.

$$\begin{aligned}
&\left|\frac{1}{\delta}\int_{c+\varepsilon h(x')+\eta}^{c+(\varepsilon+\delta)h(x')+\eta} f(x',x_n)\,dx_n - f(x',c+\varepsilon h(x')+\eta)h(x')\right| \\
&= \left|\frac{1}{\delta}\int_{c+\varepsilon h(x')+\eta}^{c+(\varepsilon+\delta)h(x')+\eta} f(x',x_n) - f(x',c+\varepsilon h(x')+\eta)\,dx_n\right| \\
&= \left|\frac{1}{\delta}\int_{c+\varepsilon h(x')+\eta}^{c+(\varepsilon+\delta)h(x')+\eta} \left(\int_{c+\varepsilon h(x')+\eta}^{x_n} \partial_{x_n} f(x',t)\,dt\right) dx_n\right| \\
&\le \frac{1}{|\delta|}\int_{c+\varepsilon h(x')+\eta-|\delta|\|h\|_{L^\infty}}^{c+\varepsilon h(x')+\eta+|\delta|\|h\|_{L^\infty}} \left|\int_{c+\varepsilon h(x')+\eta}^{x_n} \partial_{x_n} f(x',t)\,dt\right| dx_n \\
&\le \frac{1}{|\delta|}\int_{c+\varepsilon h(x')+\eta-|\delta|\|h\|_{L^\infty}}^{c+\varepsilon h(x')+\eta+|\delta|\|h\|_{L^\infty}} \int_{c+\varepsilon h(x')+\eta-|\delta|\|h\|_{L^\infty}}^{c+\varepsilon h(x')+\eta+|\delta|\|h\|_{L^\infty}} |\partial_{x_n} f(x',t)|\,dt\,dx_n \\
&= 2\|h\|_{L^\infty}\int_{c+\varepsilon h(x')+\eta-|\delta|\|h\|_{L^\infty}}^{c+\varepsilon h(x')+\eta+|\delta|\|h\|_{L^\infty}} |\partial_{x_n} f(x',t)|\,dt \le 2\|h\|_{L^\infty}\int_{\mathbb{R}} |\nabla f(x',t)|\,dt \in L^1(\mathbb{R}^{n-1}).
\end{aligned}$$

The continuity proof of $\frac{\partial F}{\partial \varepsilon}$ is analogous to (5.26), given that $\|h\|_{L^\infty} < \infty$. Since both $\frac{\partial F}{\partial \varepsilon}$ and $\frac{\partial F}{\partial \eta}$ exist and are continuous, according to [Rud76], Theorem 9.21, F belongs to C^1. $\square$

The key idea in our next lemma is to use variational techniques employed by [CÍ4] in the study of isoperimetric sets for perturbations of log-concave measures and by [BCM12] [BCM16] in the study of the structure of measures with foliations by isoperimetric sets.

Lemma 5.3.2. *Let $n \ge 2$ and $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$. Write $x \in \mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Assume that for all $c \in \mathbb{R}$, the set*

$$H(e_n, c) = \{x \in \mathbb{R}^n : x_n \ge c\}$$

is minimal for μ-perimeter among all Borel sets F with $\mu(F) = \mu(H(e_n, c))$. Then there exist positive functions $A : \mathbb{R}^{n-1} \to \mathbb{R}$, $B : \mathbb{R} \to \mathbb{R}$ such that

$$f(x', x_n) = A(x')B(x_n) \tag{5.27}$$

for all $(x', x_n) \in \mathbb{R}^n$ and $A \in C^1(\mathbb{R}^{n-1})$, $B \in C^1(\mathbb{R})$. In particular, by Lemma 5.2.6, if $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$ is an Ehrhard measure, then (5.27) holds for f.

Proof. **Step 1**: First, observe that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(x', c)\, dx' dc = \int_{\mathbb{R}^n} f(x)\, dx < \infty.$$

Then, by Fubini's theorem, Lemma 5.1.5, and the area formula, for $\mathcal{L}$-a.e. $c \in \mathbb{R}$,

$$\mathrm{Per}_\mu(H(e_n, c)) = \int_{\mathbb{R}^{n-1}} f(x', c)\, dx' < \infty. \tag{5.28}$$

Furthermore, by the Lebesgue differentiation theorem (see [EG92], Section 1.7, Theorem 1), for $\mathcal{L}$-a.e. $c \in \mathbb{R}$,

$$\lim_{r \to 0} \frac{1}{2r} \int_{c-r}^{c+r} \left(\int_{\mathbb{R}^{n-1}} |\nabla f(x', t)|\, dx' \right) dt = \int_{\mathbb{R}^{n-1}} |\nabla f(x', c)|\, dx'. \tag{5.29}$$

Given a function $h \in C_c^\infty(\mathbb{R}^{n-1})$, we construct a corresponding family of measure-preserving perturbations of $H(e_n, c)$ as follows. Define

$$F(\varepsilon, \eta) = \mu\left(\left\{ x : x_n > c + \varepsilon h(x') + \eta \right\} \right) - \mu\left(\{ x : x_n > c \} \right).$$

By Lemma 5.3.1, (5.28), and the Implicit Function Theorem, there exists a number $\varepsilon_0 > 0$ and a function $s_h \in C^1(-\varepsilon_0, \varepsilon_0)$ such that $s_h(0) = 0$ and $F(\varepsilon, s_h(\varepsilon)) = 0$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. That is,

$$G_\varepsilon := \left\{ x \in \mathbb{R}^n : x_n > c + \varepsilon h(x') + s_h(\varepsilon) \right\}$$

satisfies

$$\mu(G_\varepsilon) = \mu(H(e_n, c)). \tag{5.30}$$

Moreover, if we employ the notation

$$g(x', \varepsilon) := c + \varepsilon h(x') + s_h(\varepsilon),$$

then the condition (5.30) can be re-written as

$$\int_{\mathbb{R}^{n-1}} \int_{g(x', \varepsilon)}^{\infty} f(x', x_n)\, dx_n dx' = \int_{\mathbb{R}^{n-1}} \int_{c}^{\infty} f(x', x_n)\, dx_n dx'. \tag{5.31}$$

Now, let's consider the limit of the difference quotients.

$$0 = \lim_{\varepsilon\to 0} \frac{\mu(G_\varepsilon) - \mu(H(e_n, c))}{\varepsilon} = -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^{n-1}} \int_c^{g(x',\varepsilon)} f(x', x_n)\, dx_n dx'.$$

Applying the mean value theorem, there exists a $x_n^\varepsilon \in [\min\{c, g(x', \varepsilon)\}, \max\{c, g(x', \varepsilon)\}]$ such that

$$\int_c^{g(x',\varepsilon)} f(x', x_n)\, dx_n = (g(x', \varepsilon) - c) f(x', x_n^\varepsilon)$$

and $x_n^\varepsilon \to c$ as $\varepsilon \to 0$, since s_h is continuous at 0 and hence $g(x', \varepsilon) = c + \varepsilon h(x') + s_h(\varepsilon) \to c$. Therefore,

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^{n-1}} \int_c^{g(x',\varepsilon)} f(x', x_n)\, dx_n dx' \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^{n-1}} (\varepsilon h(x') + s_h(\varepsilon)) f(x', x_n^\varepsilon)\, dx' \\ &= \int_{\mathbb{R}^{n-1}} h(x') f(x', c) dx' + s_h'(0) \int_{\mathbb{R}^{n-1}} f(x', c)\, dx', \end{aligned}$$

where we have used $s_h(0) = 0$ and

$$\begin{aligned} f(x', x_n^\varepsilon) &= \int_c^{x_n^\varepsilon} \partial_{x_n} f(x', t)\, dt + f(x', c) \\ &\le \int_{\mathbb{R}} |\nabla f(x', t)|\, dt + f(x', c) \in L^1(\mathbb{R}^{n-1}). \end{aligned}$$

In particular, if

$$\int_{\mathbb{R}^{n-1}} f(x', c) h(x')\, dx' = 0,$$

then $s_h'(0) = 0$ since $f > 0$.

Step 2: Notice that G_ε has C^1-boundary since $g \in C^1$. In particular, G_ε is a set of locally finite perimeter in $\mathbb{R}^n$ (see, for example, [Mag12], Example 12.5). By Lemma 5.1.5 and the area formula, we have

$$\mathrm{Per}_\mu(G_\varepsilon) = \int_{\mathbb{R}^{n-1}} f(x', g(x', \varepsilon)) \sqrt{1 + |\nabla_{x'}(g(x', \varepsilon))|^2}\, dx'. \tag{5.32}$$

Our goal is to show that if $s_h'(0) = 0$, then

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{n-1}} \left(\frac{1}{\varepsilon} \int_c^{c+\varepsilon h(x') + s_h(\varepsilon)} \partial_{x_n} f(x', t)\, dt \right) dx' = \int_{\mathbb{R}^{n-1}} \partial_{x_n} f(x', c) h(x')\, dx'.$$

Notice that

$$\left|\frac{1}{\varepsilon}\int_c^{c+\varepsilon h(x')+s_h(\varepsilon)} \partial_{x_n} f(x',t)\,dt\right| \le \frac{1}{|\varepsilon|}\int_{c-|\varepsilon|\|h\|_{L^\infty}-|s_h(\varepsilon)|}^{c+|\varepsilon|\|h\|_{L^\infty}+|s_h(\varepsilon)|} |\nabla f(x',t)|\,dt := k_\varepsilon(x')$$

and

$$k_\varepsilon(x') \to 2\|h\|_{L^\infty}|\nabla f(x',c)| := k(x').$$

Moreover, if $s_h'(0)=0$, by (5.29), we obtain

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} k_\varepsilon(x')\,dx' &= \frac{1}{|\varepsilon|}\int_{c-|\varepsilon|\|h\|_{L^\infty}-|s_h(\varepsilon)|}^{c+|\varepsilon|\|h\|_{L^\infty}+|s_h(\varepsilon)|}\left(\int_{\mathbb{R}^{n-1}} |\nabla f(x',t)|\,dx'\right)dt \\ &\to 2\|h\|_{L^\infty}\int_{\mathbb{R}^{n-1}} |\nabla f(x',c)|\,dx' = \int_{\mathbb{R}^{n-1}} k(x')\,dx'\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_c^{c+\varepsilon h(x')+s_h(\varepsilon)} \partial_{x_n} f(x',t)\,dt &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(\int_c^{c+\varepsilon h(x')+s_h(\varepsilon)} \partial_{x_n} f(x',t)\,dt\right) \\ &= \partial_{x_n} f(x',c+\varepsilon h(x')+s_h(\varepsilon))\big(h(x')+s_h'(\varepsilon)\big)\Big|_{\varepsilon=0} \\ &= \partial_{x_n} f(x',c)h(x'),\end{aligned}$$

since $f\in C^1(\mathbb{R}^n)$ and $s_h(0)=0$. Applying the generalized dominated convergence theorem using k_ε (see, for example, [Fol13], Section 2.3, Exercise 20), we conclude that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-1}}\left(\frac{1}{\varepsilon}\int_c^{c+\varepsilon h(x')+s_h(\varepsilon)} \partial_{x_n} f(x',t)\,dt\right)dx' = \int_{\mathbb{R}^{n-1}} \partial_{x_n} f(x',c)h(x')\,dx'.$$

Step 3: Assume that

$$\int_{\mathbb{R}^{n-1}} f(x',c)h(x')\,dx' = 0.$$

By Step 1 and Step 2, we have $s_h'(0)=0$ and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\mathrm{Per}_\mu(G_\varepsilon)-\mathrm{Per}_\mu(H(e_n,c))}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_{\mathbb{R}^{n-1}} f(x',g(x',\varepsilon))\sqrt{1+\varepsilon^2|\nabla_{x'}h(x')|^2}dx' - \int_{\mathbb{R}^{n-1}} f(x',c)dx'\right) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}^{n-1}}\Big(f(x',c+\varepsilon h(x')+s_h(\varepsilon))-f(x',c)\Big)\sqrt{1+\varepsilon^2|\nabla_{x'}h(x')|^2}\,dx'\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-1}} f(x',c)\left(\frac{\sqrt{1+\varepsilon^2|\nabla_{x'}h(x')|^2}-1}{\varepsilon}\right)dx' \\
&=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-1}}\left(\frac{1}{\varepsilon}\int_c^{c+\varepsilon h(x')+s_h(\varepsilon)}\partial_{x_n}f(x',t)\,dt\right)\sqrt{1+\varepsilon^2|\nabla_{x'}h(x')|^2}\,dx' \\
&+\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-1}} f(x',c)|\nabla_{x'}h(x')|^2 o(1)\,dx' \\
&=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-1}}\left(\frac{1}{\varepsilon}\int_c^{c+\varepsilon h(x')+s_h(\varepsilon)}\partial_{x_n}f(x',t)\,dt\right)\left(1+|\nabla_{x'}h(x')|^2 o(\varepsilon)\right)dx' \\
&=\int_{\mathbb{R}^{n-1}}\partial_{x_n}f(x',c)h(x')\,dx'.
\end{aligned}
$$

Since $H(e_n,c)$ is minimal for μ-perimeter and $\mu(G_\varepsilon)=\mu(H(e_n,c))$, i.e., $\mathrm{Per}_\mu(G_\varepsilon)\geq \mathrm{Per}_\mu(H(e_n,c))$,

$$\int_{\mathbb{R}^{n-1}}\partial_{x_n}f(x',c)h(x')\,dx'=\lim_{\varepsilon\to 0^+}\frac{\mathrm{Per}_\mu(G_\varepsilon)-\mathrm{Per}_\mu(H(e_n,c))}{\varepsilon}\geq 0.$$

Taking $\varepsilon\to 0^-$, we obtain the reverse inequality, and thus

$$\int_{\mathbb{R}^{n-1}}\partial_{x_n}f(x',c)h(x')\,dx'=0.$$

Step 4: From Step 3, for $\mathcal{L}$-a.e. $c\in\mathbb{R}$, we have

$$\int_{\mathbb{R}^{n-1}}\partial_{x_n}f(x',c)h(x')\,dx'=0, \tag{5.33}$$

for any $h\in C_c^\infty(\mathbb{R}^{n-1})$ with $\int_{\mathbb{R}^{n-1}}f(x',c)h(x')\,dx'=0$. Now we claim that for $\mathcal{L}$-a.e. $c\in\mathbb{R}$, there exists a constant $K(c)$ such that for any $h\in C_c^\infty(\mathbb{R}^{n-1})$,

$$\int_{\mathbb{R}^{n-1}}\partial_{x_n}f(x',c)h(x')\,dx'=K(c)\int_{\mathbb{R}^{n-1}}f(x',c)h(x')\,dx'.$$

For any $h_1,h_2\in C_c^\infty(\mathbb{R}^{n-1})$ with $c_1:=\int_{\mathbb{R}^{n-1}}f(x',c)h_1(x')\,dx'\neq 0$ and $c_2:=\int_{\mathbb{R}^{n-1}}f(x',c)h_2(x')\,dx'\neq 0$, we define

$$\widetilde{h}(x')=h_1(x')-\left(\frac{c_1}{c_2}\right)h_2(x')\in C_c^\infty(\mathbb{R}^{n-1}).$$

Applying (5.33) on $\widetilde{h}$, we obtain

$$\int_{\mathbb{R}^{n-1}}\partial_{x_n}f(x',c)\widetilde{h}(x')\,dx'=0.$$

That is,

$$\frac{\displaystyle\int_{\mathbb{R}^{n-1}} \partial_{x_n} f(x',c)h_1(x')\,dx'}{\displaystyle\int_{\mathbb{R}^{n-1}} f(x',c)h_1(x')\,dx'} = \frac{\displaystyle\int_{\mathbb{R}^{n-1}} \partial_{x_n} f(x',c)h_2(x')\,dx'}{\displaystyle\int_{\mathbb{R}^{n-1}} f(x',c)h_2(x')\,dx'}.$$

Setting $K(c)$ as the quotient above proves the claim. By the Fundamental Lemma of Calculus of Variations (see [Mag12], Exercise 4.14), for $\mathcal{L}$-a.e. $c \in \mathbb{R}$,

$$\partial_{x_n} f(x',c) = K(c)f(x',c) \quad \mathcal{L}^{n-1}\text{-a.e. on } \mathbb{R}^{n-1}. \tag{5.34}$$

Since $\partial_{x_n} f$ and f are both in $C(\mathbb{R}^n)$, the function $K \in C(\mathbb{R})$. Integrating (5.34) on c from 0 to x_n, we have

$$\log\big(f(x',x_n)\big) - \log\big(f(x',0)\big) = \int_0^{x_n} \frac{\partial_{x_n} f(x',c)}{f(x',c)}\,dc = \int_0^{x_n} K(c)\,dc.$$

Therefore,

$$f(x',x_n) = f(x',0)e^{\int_0^{x_n} K(c)\,dc} := A(x')B(x_n).$$

It is not hard to see that $A \in C^1(\mathbb{R}^{n-1})$, $B \in C^1(\mathbb{R})$, and both A, B are strictly positive. ☐

With the assistance of Lemma 5.2.6, we can extend the argument from Lemma 5.3.2 to any direction $u \in \mathbb{S}^{n-1}$. Combining Lemma 5.3.2 with Lemma 5.2.6 yields the following corollary.

Corollary 5.3.3. *Let $n \geq 2$, and let $\mu = f\mathcal{L}^n \in \mathcal{W}(\mathbb{R}^n)$ be an Ehrhard measure. Then for any orthonormal frame $\{e_i\}_{i=1}^n$ and coordinates $(x_1,...,x_n)$ on $\mathbb{R}^n$ imposed by $\{e_i\}_{i=1}^n$,*

$$f\Big(\sum_{i=1}^n x_i e_i\Big) = \prod_{i=1}^n f_i(x_i) \tag{5.35}$$

for some functions $f_i \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$, depending upon $\{e_i\}_{i=1}^n$.

Proof. Without loss of generality, we may assume that $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. Utilizing Lemma 5.2.6, we can employ Lemma 5.3.2 in each direction e_i to establish a product structure for f. Specifically, applying Lemma 5.3.2 in e_n and e_{n-1} yields

$$f(x_1,...,x_n) = A(x_1,...,x_{n-1})B(x_n) = \widetilde{A}(x_1,...,x_{n-2},x_n)\widetilde{B}(x_{n-1}).$$

In particular, $\frac{B}{\widetilde{A}}$ and $\frac{\widetilde{B}}{A}$ are functions in $x_1, ..., x_{n-2}$, i.e.,

$$\left(\frac{B}{\widetilde{A}}\right)(x_1, ..., x_{n-2}, x_n) = \left(\frac{\widetilde{B}}{A}\right)(x_1, ..., x_{n-2}, x_{n-1}) := k(x_1, ..., x_{n-2}) > 0.$$

That is,

$$f(x_1, ..., x_n) = \frac{1}{k(x_1, ..., x_{n-2})}\widetilde{B}(x_{n-1})B(x_n).$$

Similarly, we can apply Lemma 5.3.2 in e_{n-2} to get

$$f(x_1, ..., x_n) = \widehat{A}(x_1, ..., x_{n-3}, x_{n-1}, x_n)\widehat{B}(x_{n-2})$$

and hence

$$k(x_1, ..., x_{n-3}, x_{n-2})\widehat{B}(x_{n-2}) = \frac{\widetilde{B}(x_{n-1})B(x_n)}{\widehat{A}(x_1, ..., x_{n-3}, x_{n-1}, x_n)} := \ell(x_1, ..., x_{n-3})$$

is a function in $x_1, ..., x_{n-3}$, i.e., $f(x_1, ..., x_n) = \frac{1}{\ell(x_1,...,x_{n-3})}\widehat{B}(x_{n-2})\widetilde{B}(x_{n-1})B(x_n)$. Proceeding in this manner, we can observe that f has a product structure., i.e.,

$$f(x_1, ..., x_n) = f_1(x_1) \cdots f_n(x_n),$$

where $f_i \in C^1(\mathbb{R})$. Given that $f \in L^1(\mathbb{R}^n)$, we deduce

$$\left(\int_{\mathbb{R}} f_1(x_1)dx_1\right) \cdots \left(\int_{\mathbb{R}} f_n(x_n)dx_n\right) = \int_{\mathbb{R}^n} f dx < \infty \implies f_i \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \text{ for all } i.$$

Similarly, $\nabla f \in L^1(\mathbb{R}^n)$ implies

$$\left(\int_{\mathbb{R}} \left|\partial_{x_1} f_1(x_1)\right| dx_1\right) \left(\int_{\mathbb{R}} f_2(x_2)dx_n\right) \cdots \left(\int_{\mathbb{R}} f_n(x_n)dx_n\right) \leq \int_{\mathbb{R}^n} |\nabla f| dx < \infty.$$

Therefore, $\partial_{x_1} f_1 \in L^1(\mathbb{R})$. The same argument applies to $\partial_{x_2} f_2, \cdots, \partial_{x_n} f_n$. Hence, for all $i = 1, \cdots n$, we have $f_i \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$. □

5.4 *Proof of Theorem 1.2.4*

Step 1: Our objective is to demonstrate the existence of constants $0 < c < \infty$, $0 < C < \infty$, and $a \in \mathbb{R}^n$ such that

$$f(x) = Ce^{-c|x-a|^2}.$$

With the aid of Corollary 5.3.3, we can assume that

$$f(x_1,...,x_n) = f_1(x_1)\cdots f_n(x_n).$$

Applying Lemma 5.1.8 to f, there exists $m \in \mathbb{R}$ such that

$$\nu = \widetilde{f}\mathcal{L}^n$$

is an Ehrhard measure, where $\widetilde{f}(x) = f(x+m)$, $\widetilde{f}(x_1,...,x_n) = \widetilde{f}_1(x_1)\cdots\widetilde{f}_n(x_n)$, $\widetilde{f}_i > 0$, and $\widetilde{f}_i(-t) = \widetilde{f}_i(t)$ for all $i = 1,...,n$. Let $a = m \in \mathbb{R}^n$. Our aim is to establish that

$$\widetilde{f}(x) = Ce^{-c|x|^2}$$

for some $0 < c < \infty$, $0 < C < \infty$. For ease of reference, we will continue to use f to denote $\widetilde{f}$ throughout the remainder of the proof.

Step 2: In Step 1, we established that $f(x_1,...,x_n) = f_1(x_1)\cdots f_n(x_n)$, $f_i \in C^1(\mathbb{R})\cap W^{1,1}(\mathbb{R})$, $f_i > 0$ and $f_i(-t) = f_i(t)$ for all $i = 1,...,n$. Consequently, each component function f_i can be expressed as

$$f_i(t) = f_i(0)\left(\frac{f_i(t)}{f_i(0)}\right) = c_i e^{g_i(t)} \tag{5.36}$$

for some continuous function g_i and $g_i(-t) = g_i(t)$, where $c_i = f_i(0)$ and $g_i(0) = 0$. That is,

$$f(x_1,...,x_n) = Ce^{g_1(x_1)+\cdots+g_n(x_n)},$$

where $C := f(0,...,0) = f_1(0)\cdots f_n(0) > 0$. We claim that

$$g_1 = g_2 = \cdots = g_n. \tag{5.37}$$

Consider

$$O = \begin{bmatrix} O_2 & \mathbf{0} \\ \mathbf{0} & I_{n-2} \end{bmatrix} \in M_n(\mathbb{R}), \quad O_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \in M_2(\mathbb{R}).$$

Then

$$\begin{aligned} f(Ox) &= f\left(\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2, \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2, x_3, \cdots x_n\right) \\ &= C\exp\left(g_1\left(\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2\right) + g_2\left(\frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2\right) + \cdots + g_n(x_n)\right). \end{aligned} \tag{5.38}$$

Utilizing Lemma 5.3.3 with the orthogonal frame $\{Oe_i\}_{i=1}^n$ and the argument in (5.36), we obtain

$$f(Ox) = f\left(\sum_{i=1}^n x_i O e_i\right) = Ce^{h_1(x_1)+\cdots+h_n(x_n)}, \tag{5.39}$$

where h_i depends on matrix O and satisfies $h_i(0) = 0$. To confirm the property of (5.37), we first check that

$$h_1(\alpha) = h_1(-\alpha), \quad h_1(\alpha) = h_2(\alpha), \tag{5.40}$$

for all $\alpha \in \mathbb{R}$. Setting $x_3 = x_4 = \cdots = x_n = 0$ in (5.38) and (5.39), we have

$$h_1(x_1) + h_2(x_2) = g_1\left(\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2\right) + g_2\left(\frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2\right). \tag{5.41}$$

In particular, letting $x_1 = \alpha$ and $x_2 = 0$, we find

$$h_1(\alpha) = g_1\left(\frac{1}{\sqrt{2}}\alpha\right) + g_2\left(\frac{1}{\sqrt{2}}\alpha\right) = h_1(-\alpha),$$

where we have used the fact that $g_i(-t) = g_i(t)$. Similarly, for $x_1 = 0$ and $x_2 = \alpha$, we have

$$h_2(\alpha) = g_1\left(\frac{1}{\sqrt{2}}\alpha\right) + g_2\left(-\frac{1}{\sqrt{2}}\alpha\right) = g_1\left(\frac{1}{\sqrt{2}}\alpha\right) + g_2\left(\frac{1}{\sqrt{2}}\alpha\right) = h_1(\alpha).$$

Consider $x_1 = x_2 = \frac{\alpha}{\sqrt{2}}$ in (5.41) and apply (5.40), we also have

$$g_1(\alpha) = h_1\left(\frac{\alpha}{\sqrt{2}}\right) + h_2\left(\frac{\alpha}{\sqrt{2}}\right) = 2h_1\left(\frac{\alpha}{\sqrt{2}}\right).$$

Similarly, for $x_1 = \frac{\alpha}{\sqrt{2}}$ and $x_2 = -\frac{\alpha}{\sqrt{2}}$, we apply (5.40) to obtain,

$$g_2(\alpha) = h_1\left(\frac{\alpha}{\sqrt{2}}\right) + h_2\left(-\frac{\alpha}{\sqrt{2}}\right) = h_1\left(\frac{\alpha}{\sqrt{2}}\right) + h_1\left(-\frac{\alpha}{\sqrt{2}}\right) = 2h_1\left(\frac{\alpha}{\sqrt{2}}\right).$$

Hence, $g_1 = g_2$. An identical argument yields $g_1 = g_2 = \cdots = g_n$.

Step 3: We claim that for all positive integers k, we have

$$2g_1(\sqrt{k}\alpha) + 2g_1(\alpha) = g_1((\sqrt{k}-1)\alpha) + g_1((\sqrt{k}+1)\alpha). \tag{5.42}$$

Consider

$$O = \begin{bmatrix} O_2 & \mathbf{0} \\ \mathbf{0} & I_{n-2} \end{bmatrix} \in M_n(\mathbb{R}), \quad O_2 = \begin{bmatrix} \frac{\sqrt{k+k^2}}{k+1} & -\frac{\sqrt{k+1}}{k+1} \\ \frac{\sqrt{k+1}}{k+1} & \frac{\sqrt{k+k^2}}{k+1} \end{bmatrix} \in M_2(\mathbb{R}).$$

Using the argument in Step 2 and (5.37), we have

$$f(Ox) = C\exp\left(g_1\left(\frac{\sqrt{k+k^2}}{k+1}x_1 - \frac{\sqrt{k+1}}{k+1}x_2\right) + g_1\left(\frac{\sqrt{k+k^2}}{k+1}x_1 + \frac{\sqrt{k+1}}{k+1}x_2\right) + \cdots + g_1(x_n)\right),$$

and

$$f(Ox) = f\left(\sum_{i=1}^{n} x_i O e_i\right) = Ce^{\ell_1(x_1)+\cdots+\ell_n(x_n)},$$

for some ℓ_i depending on matrix O and $\ell_i(0) = 0$. Setting $x_3 = x_4 = \cdots = x_n = 0$,

$$\ell_1(x_1) + \ell_2(x_2) = g_1\left(\frac{\sqrt{k+k^2}}{k+1}x_1 - \frac{\sqrt{k+1}}{k+1}x_2\right) + g_1\left(\frac{\sqrt{k+k^2}}{k+1}x_1 + \frac{\sqrt{k+1}}{k+1}x_2\right).$$

In particular, setting $x_2 = 0$, we find

$$\ell_1(x_1) = 2g_1\left(\frac{\sqrt{k}}{\sqrt{k+1}}x_1\right).$$

Similarly, when $x_1 = 0$, we get

$$\ell_2(x_2) = 2g_1\left(\frac{1}{\sqrt{k+1}}x_2\right),$$

where we have again utilized the fact that $g_i(-t) = g_i(t)$. Thus,

$$\begin{aligned} &2g_1\left(\frac{\sqrt{k}}{\sqrt{k+1}}x_1\right) + 2g_1\left(\frac{1}{\sqrt{k+1}}x_2\right) \\ &= g_1\left(\frac{\sqrt{k+k^2}}{k+1}x_1 - \frac{\sqrt{k+1}}{k+1}x_2\right) + g_1\left(\frac{\sqrt{k+k^2}}{k+1}x_1 + \frac{\sqrt{k+1}}{k+1}x_2\right). \end{aligned}$$

Finally, if we set $x_1 = x_2 = \sqrt{k+1}\alpha$, we conclude that

$$2g_1(\sqrt{k}\alpha) + 2g_1(\alpha) = g_1((\sqrt{k}-1)\alpha) + g_1((\sqrt{k}+1)\alpha).$$

Step 4: We claim that for all positive integers k, we have

$$g_1(k\alpha) = k^2 g_1(\alpha). \tag{5.43}$$

We will prove this by induction. Note that the base case $k = 1$ is trivial. Now, assume that the claim holds for all integers less than or equal to k. For any $m \in \mathbb{N}$, substituting $k = m^2$ into (5.42), we obtain

$$2g_1(m\alpha) + 2g_1(\alpha) = g_1((m-1)\alpha) + g_1((m+1)\alpha).$$

That is,

$$g_1(m\alpha) - g_1((m-1)\alpha) + 2g_1(\alpha) = g_1((m+1)\alpha) - g_1(m\alpha). \tag{5.44}$$

Summing m from 1 to k in (5.44), we have

$$\begin{aligned} g_1(k\alpha) + 2kg_1(\alpha) &= \sum_{m=1}^{k} \Big(g_1(m\alpha) - g_1((m-1)\alpha) + 2g_1(\alpha) \Big) \\ &= \sum_{k=1}^{k} \Big(g_1((m+1)\alpha) - g_1(m\alpha) \Big) = g_1((k+1)\alpha) - g_1(\alpha). \end{aligned}$$

Therefore,

$$g((k+1)\alpha) = g_1(k\alpha) + 2kg_1(\alpha) + g_1(\alpha) = k^2 g_1(\alpha) + 2kg_1(\alpha) + g_1(\alpha) = (k+1)^2 g_1(\alpha),$$

where we have used the induction hypothesis on k. This proves the claim.

Step 5: Finally, we define

$$h(\alpha) := \frac{g_1(\alpha)}{\alpha^2}$$

for all $\alpha > 0$. By (5.43), we know $g_1(k\alpha) = k^2 g_1(\alpha)$ and hence $h(k\alpha) = h(\alpha)$ for any positive integer k. In particular,

$$h(1) = h(1/q)$$

for all positive integers q. For any positive rational number p/q, we have

$$h(p/q) = h\left(p \left(\frac{1}{q} \right) \right) = h(1/q) = h(1).$$

For any $\alpha > 0$, there exists a sequence of rational numbers $\{r_k\}$ such that $r_k \to \alpha$. Since h is continuous at α, we see that

$$h(1) = \lim_{k\to\infty} h(r_k) = h(\alpha)$$

for all $\alpha > 0$. Letting $c = -h(1)$ and unwinding the definition of h, we have $g_1(\alpha) = -c\alpha^2$ for all $\alpha > 0$. By $g_1(\alpha) = g_1(-\alpha)$ and $g_1(0) = 0$,

$$g_1(\alpha) = -c\alpha^2$$

for all $a \in \mathbb{R}$. To complete the proof, we only need to prove that the constant c is positive. However, if $c \leq 0$, then $f \notin L^1(\mathbb{R}^n)$. Therefore,

$$f(x) = Ce^{g_1(x_1)+\cdots+g_n(x_n)} = Ce^{-c(x_1^2+x_2^2+\cdots+x_n^2)} = Ce^{-c|x|^2}.$$

That is, f is an isotropic Gaussian. □

Remark: The constant $c > 0$ also follows from [Cha19], where Chambers proves that balls are the unique perimeter minimizers of measures with distributions that are log-convex, rotationally symmetric functions. Given that Lemma 5.2.1 establishes all half-spaces as μ-perimeter minimizers, it follows that $c > 0$.

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