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Towards a non- \mathbb{Q} -Gorenstein Minimal Model Program

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Abstract

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In this thesis we do the first steps towards a non- \mathbb{Q} -Gorenstein Minimal Model Program. We extensively study non- \mathbb{Q} -factorial singularities, using the techniques introduced by [dFH09]. We introduce a new class of singularities, *log terminal*⁺, which we show satisfies several nice properties; we investigate the finite generation of the canonical algebra of local sections, we relate log terminal⁺ singularities with existing classes, we show a Bertini-type theorem, and small deformation invariance. We also provide a list of examples of the pathologies that can occur when working in the non- \mathbb{Q} -factorial setting. We subsequently focus on defining and studying positivity for Weil divisors. We define nefness/amplitude/bigness/pseudo-effectivity for Weil divisors; we show various characterizations of this notions, and we prove vanishing and non-vanishing theorems. We conclude with a proposal of a non- \mathbb{Q} -Gorenstein MMP, we prove it for toric varieties, and we discuss where the obstacles lay in the general case. As application of our techniques, we prove the existence on non- \mathbb{Q} -factorial log terminal flips.

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Finally, I owe a great debt to Kollár and Mori for writing [KM98], which made me fall in love with birational geometry, and to Lazarsfeld for writing one of the best books on modern algebraic geometry, [Laz04]. Together with [Har77], these are the books from which I learned the most, and still am learning, and that shaped my research. As Urbinati put it, “a PhD to understand three books”.

DEDICATION

A Camil e Beatrice

INTRODUCTION

Put thyself into the trick of singularity.

– Shakespeare, *The twelfth night*

Singularity theory is one of the foundations of the Minimal Model Program, and non- \mathbb{Q} -factorial singularities, in particular, naturally appear in the program. Yet, non- \mathbb{Q} -factorial singularities have only recently started to be fully studied (especially in characteristic zero)¹. Until recently, it was not even clear how to define singularities in characteristic zero for a variety X , or for a pair (X, Δ) , without assuming that K_X , respectively $K_X + \Delta$, be \mathbb{Q} -Cartier. This work is part of a recent trend, started by de Fernex and Hacon, in [dFH09], and continued by de Fernex, Urbinati, and others. The two main focuses of this thesis will be the theory of singularities *per sé* and how to define positivity in this new setting. These are the first steps towards the creation of a *non- \mathbb{Q} -Gorenstein Minimal Model Program*. After explaining the context of the problem, we will discuss the problem itself, and the structure of the thesis.

Non- \mathbb{Q} -factorial singularities and the Minimal Model Program

A Weil divisor on a normal scheme X is called *\mathbb{Q} -Cartier* if some multiple is an integral Cartier divisor. A normal scheme is called *\mathbb{Q} -factorial* if every Weil divisor is \mathbb{Q} -Cartier. If the canonical divisor² K_X is \mathbb{Q} -Cartier, the scheme is called *\mathbb{Q} -Gorenstein*³.

The Minimal Model Program is an algorithmic way of choosing a “good” representative in each birational class. There are plenty of good references on the Minimal Model Program,

¹I have heard the question “Why?” more times after saying my research topic, than after stating my intention to move to a different country to do a PhD in Mathematics. However, this skepticism was mostly confined to the characteristic zero world. The reason for this difference is that singularities in positive characteristic are traditionally defined using Frobenius or tight closure, and thus not requiring the hypothesis of Gorenstein (or log Gorenstein) in the definition

²See §1.2 for the definition of the canonical divisor on a normal scheme

³Originally, \mathbb{Q} -Gorenstein meant that K_X was \mathbb{Q} -Cartier and X Cohen-Macaulay. However, we follow the most recent and common usage of the term

for example [KM98]. In §1.6 we will recall some of the relevant definitions and results. Let us briefly recall the main steps of the program.

Step 0 Let $X = X_i$ be a projective variety with \mathbb{Q} -factorial and terminal singularities.

Step 1 If K_X is nef (i.e. the intersection with every curve is non-negative), there is nothing to do and the program terminates. Otherwise, there is a contraction of K_X -negative curves: $\varphi : X \rightarrow Y$.

Step 2 There are three possibilities for the contraction above.

- (i) (Fano fiber space) We have $\dim Y < \dim X$. In this case the general fiber f is Fano (K_f is anti-ample), and we can restart the program with $X_0 = Y$ in smaller dimension.
- (ii) (Divisorial contraction) We have $\dim X = \dim Y$ and $\text{Exc}(\varphi)$ is an irreducible divisor. We set $X_{i+1} = Y$ and we restart the algorithm.
- (iii) (Small or flipping contraction) We have that $\dim X = \dim Y$ and $\text{codim}(\text{Exc}(\varphi)) \geq 2$. In this case there is a flip $\varphi^+ : X^+ \rightarrow Y$; we set $X_{i+1} = X^+$ and we restart the algorithm.

Let us first remark that, even if the original variety were smooth, the outcome of this algorithm would most likely be singular. We also point out that, if X_i has terminal (or klt for some boundary) and \mathbb{Q} -factorial singularities, so does X_{i+1} in each of the cases of Step 2. In the way the program is written, the fact that the singularities are \mathbb{Q} -factorial (or at least \mathbb{Q} -Gorenstein) seems inescapable, as Step 1 relies on intersecting K_X with curves. To guarantee this, in Step 2.(iii) we have to use a flip. The reason is that the target Y of the contraction does not have \mathbb{Q} -factorial singularities. This is easy to see. Let us assume, by contradiction, that K_Y is \mathbb{Q} -Cartier. Let mK_Y be Cartier, and let us look at $mK_X - \varphi^*(mK_Y)$. This is a divisor on X , supported on the exceptional locus of φ (as long as we chose $K_Y = \varphi_*(K_X)$). Since the exceptional locus of φ has codimension at least 2, it must be $mK_X = \varphi^*(mK_Y)$. Let C be a curve contracted by φ . By construction of the contraction, $0 > (mK_X).C = \varphi^*(mK_Y).C = 0$.

Towards a non- \mathbb{Q} -Gorenstein MMP

The basic idea is elementary. If we had a good way of handling non- \mathbb{Q} -factorial singularities and to define positivity for K_X even when K_X is not \mathbb{Q} -Cartier, in Step 2.(iii) we could proceed with the Minimal Model Program by setting $X_{i+1} = Y$, instead of using the flip. The main reason why this would be interesting is that, even in the cases where we know that the Minimal Model Program terminates, we only know it for a specific program, not for a generic one. And the problem resides precisely in Step 2.(iii). Indeed, we only know that a particular sequence of flips terminates, but it is still conjectural that *every* sequence of flips terminates. A *non- \mathbb{Q} -Gorenstein MMP* could provide a way of approaching the problem. Moreover, since non- \mathbb{Q} -factorial singularities naturally appear in the MMP, it is important to understand them.

In this work there are the first steps towards a non- \mathbb{Q} -Gorenstein MMP. We introduce a new class of singularities, *log terminal*⁺, study it, and relate it with the largest class of singularities where the MMP is known (namely, klt singularities, [BCHM10]). We try to identify for which class of singularities our program is feasible. Secondly, we define positivity for Weil divisors (nefness, amplitude, bigness, and pseudo-effectivity), thus potentially individuating when the program should stop.

The next natural result in this direction would be a variation of Step 1 in this context: a contraction theorem. We prove it in the particular context of toric varieties, but a proof in full generality is still missing, and we investigate some of the issues.

The structure of the thesis

In the first chapter we recall some of the basic tools that we will use in our work.

In the second chapter we start our study of singularities in the non- \mathbb{Q} -Gorenstein setting. We also define *log terminal*⁺ singularities. The third chapter is focused on examples. None of the results of this chapter will be used elsewhere, but it will help the reader be more acquainted with some of the difficulties and pathologies that happen in the non- \mathbb{Q} -factorial world. In the fourth chapter we study *log terminal*⁺ singularities more in detail.

In the fifth chapter we define and study positivity for Weil divisors. Finally, in the last

chapter we focus on a non- \mathbb{Q} -Gorenstein Minimal Model Program: what can be done, what is yet to be done, and what cannot be done.

Chapter 1

WEIL DIVISORS (AND OTHER SORROWS)

In this chapter we introduce the main notations for our work. We start by introducing some basic results about Weil divisors in §1.1. In §1.2 we recall how to define singularities in the \mathbb{Q} -Gorenstein setting, including how to define the canonical divisor on normal schemes. We recall some basic results about valuations in §1.3, which we will use to define the pullback of Weil divisors *via* birational morphisms in §1.4.1. In §1.4.2 we start proving some technical results which we will need later. These results relate the pullback of a Weil divisor D with the pullback of $D + \Delta$, for a suitable Δ such that $D + \Delta$ is \mathbb{Q} -Cartier, and they are generalizations of results of [dFH09]. Finally, in §1.5 we introduce b -divisors.

In §1.6 we focus on the Minimal Model Program. We will not go into all the details of the Minimal Model Program; there are numerous references on the subject, see [KM98], [Kol08], or [BCHM10]. We only recall here the results that are needed to have a good picture of the program. As mentioned in the introduction, the Minimal Model Program is an algorithmic way of choosing a good representative in each birational class. The basic idea of the algorithm is to perform a series of modifications of the original variety, until either the canonical divisor is “positive”, or there is a map to another variety of smaller dimension, with general fiber being a Fano variety (which are varieties with “negative” canonical divisor). In §1.6.1 we discuss positivity, and in §1.6.2 we recall some of the results of the Minimal Model Program.

The discussions in §1.2 and §1.6.2 are from [KM98]. The presentation in §1.3 is from [BdFFU13]. All the definitions and results in §1.4.1 are from [dFH09], while the notation in §1.5 is from [BdFF12]. Section §1.6.1 is based on [KM98] and [Laz04].

1.0 Notation and terminology

All our schemes will be normal. If X is a normal scheme, by a \mathbb{Q} -Weil divisor (resp. \mathbb{R} -Weil divisor) we mean an element of $\text{Div}_{\mathbb{Q}}(X) := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\text{Div}_{\mathbb{R}}(X) := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$). Similarly, a \mathbb{Q} -Cartier divisor (resp. \mathbb{R} -Cartier divisor) will be an element of $\text{CDiv}_{\mathbb{Q}}(X) := \text{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\text{CDiv}_{\mathbb{R}}(X) := \text{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{R}$). One word of caution. An \mathbb{R} -Weil divisor D is \mathbb{Q} -Cartier if and only if there exists $t \in \mathbb{Q}_{>0}$ such that tD is Cartier. The same might not be true for \mathbb{R} -Cartier divisors; explicitly, even if an \mathbb{R} -Weil divisor D is \mathbb{R} -Cartier, there might not exist $t \in \mathbb{R}_{>0}$ such that tD is Cartier. However, if D is a \mathbb{Q} -Weil divisor which is \mathbb{R} -Cartier, then D is \mathbb{Q} -Cartier. We use the terminologies *Cartier divisor*, *invertible sheaf*, or *line bundle* interchangeably.

If $D = \sum n_E E$ is an \mathbb{R} -Weil divisor, we will use the notations $[D] := \sum [n_E] E$ and $[D] := \sum [n_E] E$.

If X is a normal scheme, we denote by $k(X)$ its quotient field. Two Weil divisors D_1 and D_2 are *linearly equivalent* if there exists a \mathbb{Z} -linear combination $D_1 = D_2 + a_1(f_1) + \dots + a_n(f_n)$, where $f_i \in k(X)$, and we write $D_1 \sim D_2$. This is equivalent to say that there exists $f \in k(X)$ such that $D_1 = D_2 + (f)$. If there exist $a, b \in \mathbb{Q}$ such that $aD_1 = bD_2 + a_1(f_1) + \dots + a_n(f_n)$, with $a_i \in \mathbb{Q}$ and $f_i \in k(X)$, then we say that D_1 and D_2 are \mathbb{Q} -linearly equivalent, and we write $D_1 \sim_{\mathbb{Q}} D_2$. This is equivalent to saying that there exists $a, b \in \mathbb{Z}$ such that $aD_1 \sim bD_2$. Finally, if there exist $a, b \in \mathbb{R}$ such that $aD_1 = bD_2 + a_1(f_1) + \dots + a_n(f_n)$, with $a_i \in \mathbb{R}$ and $f_i \in k(X)$, then we say that D_1 and D_2 are \mathbb{R} -linearly equivalent, and we write $D_1 \sim_{\mathbb{R}} D_2$. This is *not* equivalent to saying that there exists $a, b \in \mathbb{R}$ such that $aD_1 \sim bD_2$.

If $f : Y \rightarrow X$ is a morphism of normal scheme, and D_1 and D_2 are Weil divisors on Y which can be restricted¹ to the generic fiber of f , we write $D_1 \sim_f D_2$ if D_1 and D_2 , restricted to the generic fiber of f , are linearly equivalent. As above, we can also work with $D_1 \sim_{f, \mathbb{Q}} D_2$ and $D_1 \sim_{f, \mathbb{R}} D_2$.

If $f : Y \rightarrow X$ is a proper, birational morphism of normal schemes and E is a prime divisor contained in the exceptional locus $\text{Exc}(f)$ of f , we say that E is *exceptional over* X . A *log resolution of* X , is a proper birational morphism $f : Y \rightarrow X$, with Y regular

¹Restricting a Weil divisor is not always possible. We will discuss some of the issues in §1.1

and such that the exceptional locus of f is a simple normal crossing divisor, or *snc*. If, moreover, Δ is a Weil divisor on X , a *log resolution of (X, Δ)* is a log resolution such that $\text{Exc}(f) \cup \text{supp } f_*^{-1}\Delta$ is simple normal crossing (where $f_*^{-1}\Delta$ is the strict transform of Δ on Y). A proper, birational morphism $f : Y \rightarrow X$ between normal varieties is *small* if $\text{codim}_Y \text{Exc}(f) \geq 2$.

A *variety* is an integral, separated scheme of finite type over a noetherian ring k .

On a normal projective variety over an algebraically closed field, two Cartier divisors D_1 and D_2 are *numerically equivalent*, denoted by $D_1 \equiv D_2$, if $D_1.C = D_2.C$ for every curve C (reduced and irreducible). If $f : Y \rightarrow X$ is a projective morphism of normal varieties (over an algebraically closed field), and D_1 and D_2 are Cartier divisors on Y , if D_1 and D_2 are numerically equivalent over the generic fiber of f we write $D_1 \equiv_f D_2$.

1.1 Divisorial sheaves

In this section X will denote a normal scheme and $k(X)$ will be its quotient field.

Definition 1.1.1. *A divisorial sheaf on X is a reflexive coherent sheaf on X of rank 1.*

Let us remark that a reflexive sheaf is necessarily torsion-free. Moreover, if \mathcal{L} is a rank 1, torsion free sheaf on X , the generic stalk $\mathcal{L} \otimes k(X)$ will be a 1-dimensional $k(X)$ -vector space, so choosing a basis will allow us to identify \mathcal{L} with a subsheaf of $k(X)$.

To each \mathbb{R} -Weil divisor $D = \sum n_E E$ on X we can associate the subsheaf $\mathcal{O}_X(D) \subset k(X)$ defined by

$$\Gamma(U, \mathcal{O}_X(D)) := \{f \in k(X) \mid \nu_E(f) \geq -n_E \text{ for all } E \subset U\}$$

(where ν_E is the valuation of the discrete valuation ring with $\mathcal{O}_{X,E}$); or, in short,

$$\mathcal{O}_X(D) = \{f \in k(X) \mid (f) \geq -D\}.$$

With this notation, $\mathcal{O}_X(D) = \mathcal{O}_X([D])$.

The next result is well known, see for example [Bou65, §4, no. 2, theorem 2 and §1, no. 3, theorem 2]

Lemma 1.1.2. *Let \mathcal{L} be a coherent sheaf of rank 1 on a normal scheme X . The following are equivalent:*

(a) \mathcal{L} is a reflexive sheaf;

(b) \mathcal{L} is torsion-free and, for any open subscheme $U \subset X$, with $\text{codim}_X(X \setminus U) \geq 2$,
 $\mathcal{L} = i_* \mathcal{L}|_U$ (where $i : U \rightarrow X$ is the inclusion);

(c) $\mathcal{L} \cong \mathcal{O}_X(D)$ for some Weil divisor D on X .

These are just some of the possible characterizations of divisorial sheaves. More properties can be found on [Bou65] and on [Har80].

Remark 1.1.3. If $\mathcal{L} \subset k(X)$, there exists a unique Weil divisor D in (c) such that $\mathcal{L} = \mathcal{O}_X(D)$.

Remark 1.1.4. The correspondence in (c) is not a morphism. Indeed, let $X = \{x^2 - yz\} \subseteq \mathbf{A}_{\mathbb{C}}^3$ and $D = \{x = y = 0\}$ in X . Then $\mathcal{O}_X(-D) \cong \mathcal{I}_D = I^\sim$, where $I = (x, y)$ in $R = \mathbb{C}[x, y, z]/(x^2 - yz)$. Then $\mathcal{O}_X(-D) \otimes \mathcal{O}_X(-D) \cong (I \otimes I)^\sim$. However, $\mathcal{O}_X(-2D) \cong \mathcal{I}_{2D} \cong J^\sim$, where $J = (y)$ in R .

The above correspondence is a morphism on the non-singular locus X^{reg} of X , whose complement has codimension at least 2, thus by the above proposition we have that

$$(\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee} \cong \mathcal{O}_X(D_1 + D_2),$$

where D_1 and D_2 are Weil divisors on X .

Lemma 1.1.5 ([Har80, 1.9]). *If X is a locally factorial normal scheme, any divisorial sheaf is invertible.*

We can use the divisorial sheaf associated to a Weil divisor to talk about restriction. Let D be a Weil divisor. Let $S \subseteq X$ be a normal (effective) Cartier divisor, and let us assume that $\mathcal{O}_X(D)$ is invertible at the generic point of S . Then $(\mathcal{O}_X(D) \otimes \mathcal{O}_S)^{\vee\vee}$ is a rank 1 reflexive sheaf on S , that is, a divisorial sheaf. Hence, there exists a Weil divisor, denoted by $D|_S$, such that

$$(\mathcal{O}_X(D) \otimes \mathcal{O}_S)^{\vee\vee} = \mathcal{O}_S(D|_S). \quad (1.1.1)$$

We call such divisor the *restriction* of D to S . In general, this is the only case in which we can restrict Weil divisors to subschemes. We point out that it is always possible to restrict K_X to normal (effective) Cartier divisors.

Finally, let us recall a very useful (and standard) tool of the MMP. To each \mathbb{R} -Weil divisor D , we associated the \mathcal{O}_X -algebra

$$\mathcal{R}(X, D) := \bigoplus_{m \geq 0} \mathcal{O}_X(mD). \quad (1.1.2)$$

This is called the *algebra of local sections* of D . Note that, if $\mathcal{R}(X, D)$ is finitely generated, D must be a \mathbb{Q} -Weil divisor, [Kol08, 94].

Remark 1.1.6. If $\mathcal{R}(X, D)$ is finitely generated, then $\mathcal{O}_X(mqD) \cdot \mathcal{O}_X(nqD) \cong \mathcal{O}_X((m+n)qD)$ for $q \gg 0$ and $m, n \geq 1$. Indeed, $\mathcal{O}_X(mqD) \cdot \mathcal{O}_X(nqD)$ is the torsion free quotient of $\mathcal{O}_X(mqD) \otimes \mathcal{O}_X(nqD)$; thus there is a natural injection

$$\mathcal{O}_X(mqD) \cdot \mathcal{O}_X(nqD) \hookrightarrow (\mathcal{O}_X(mqD) \cdot \mathcal{O}_X(nqD))^{\vee \vee} \cong \mathcal{O}_X((m+n)qD).$$

If this map were never surjective, we would have infinitely many new generators for $\mathcal{R}(X, D)$.

The importance of this algebra is contained in the following lemma.

Lemma 1.1.7 ([KM98, 6.2]). *Let X be a normal variety, D a Weil divisor on X . The following are equivalent:*

- (a) $\mathcal{R}(X, D)$ is a finitely generated sheaf of \mathcal{O}_X -algebras;
- (b) there exists a small projective birational morphism $g : Y \rightarrow X$, with Y normal, such that $g_*^{-1}D$ is \mathbb{Q} -Cartier and g -ample.

In this case, g is the natural map $g : Y \cong \text{Proj}_X \mathcal{R}(X, D) \rightarrow X$, and it is called the \mathbb{Q} -Cartierization of D .

If $f : Y \rightarrow X$ is a proper, birational morphism and D is an \mathbb{R} -Weil divisor on Y , we also have the algebra

$$\mathcal{R}(f, D) := \bigoplus_{m \geq 0} f_* \mathcal{O}_Y(mD), \quad (1.1.3)$$

which is naturally a $f_*\mathcal{O}_Y = \mathcal{O}_X$ -algebra.

We have the following fundamental consequence of [BCHM10].

Theorem 1.1.8 ([Kol08, 92]). *Let X be a normal variety over a field of characteristic zero, and let (X, Δ) be a pair with at most klt singularities. Then, for any \mathbb{Q} -Weil divisor D on X , the algebra $\mathcal{R}(X, D)$ is finitely generated.*

1.2 (Log)- \mathbb{Q} -Gorenstein singularities

Let X be a normal scheme. Let $i : X^{\text{reg}} \hookrightarrow X$ be the inclusion of the regular locus in X . By assumption, the complement of X^{reg} has codimension at least 2. The *canonical sheaf* ω_X on X is defined to be

$$\omega_X := i_*\omega_{X^{\text{reg}}}. \quad (1.2.4)$$

Since this is the push-forward of a line bundle (defined on an open subset whose complement has codimension at least 2), this is a divisorial sheaf on X . We define a *canonical divisor* K_X to be any Weil divisor such that $\mathcal{O}_X(K_X) \cong \omega_X$. Note that, as Weil divisor, K_X is not unique; it is however unique as linear class. Thus, with slight abuse of terminology, we still talk of *the* canonical divisor.

If $f : Y \rightarrow X$ is a proper birational morphism (of normal varieties) and if we choose a canonical divisor K_X on X , we always assume that the canonical divisor K_Y on Y be chosen such that $f_*K_Y = K_X$ (as Weil divisors).

We say that a variety has *\mathbb{Q} -Gorenstein* singularities if K_X is \mathbb{Q} -Cartier. In this we follow the most common usage of the term, although the earlier definitions of the concept also ask that X be Cohen-Macaulay (which we will not). This is the traditional setting for the definition of singularities in characteristic 0.

Let $f : Y \rightarrow X$ be any proper, birational morphism of normal varieties, with mK_X Cartier. The sheaf $f^*\mathcal{O}_X(mK_X)$ is an invertible sheaf on Y , thus corresponding to a divisor, which we denote $f^*(mK_X)$. We can define $f^*(K_X) := \frac{1}{m}f^*(mK_X)$. If also nK_Y is Cartier for some n (for example in the case of a resolution), then the sheaf $\mathcal{O}_Y(nmK_Y - nf^*(mK_X))$ is an invertible sheaf supported on the exceptional locus $\text{Exc}(f)$ of f . Thus, if $\{E_i\}$ are the

prime divisors contained in $\text{Exc}(f)$, there exist unique b_i s such that

$$mnK_Y \sim nf^*(mK_X) + \sum b_i E_i. \quad (1.2.5)$$

Dividing by m , we write

$$K_Y \sim_{\mathbb{Q}} f^*(K_X) + \sum a_i E_i, \quad (1.2.6)$$

with $a_i \in \mathbb{Q}$. Notice that the a_i s are independent on the choice of the canonical divisors K_X and K_Y (of course as long as $f_*K_Y = K_X$). The divisor $\sum a_i E_i$ is called the *discrepancy* of f . The constants a_i do not depend on f , but only on E_i ; thus they are denoted by $a(E_i, X) := a_i$. Even though both K_X and K_Y are Weil divisors, the a_i s are generally not integers.

If K_X is not \mathbb{Q} -Cartier, we can consider the *pair* (X, Δ) , that is, we consider X with a fixed divisor Δ on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Such Δ is called a *boundary*. Notice that we make no assumption on the effectiveness of Δ . The same reasoning as before gives that there exist unique a_i s such that

$$K_Y + f_*^{-1}\Delta \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum a_i E_i, \quad (1.2.7)$$

where $f_*^{-1}\Delta$ is the strict transform of Δ on Y (and we are assuming that $K_Y + f_*^{-1}\Delta$ is \mathbb{Q} -Cartier). The same could be done assuming that $K_X + \Delta$ were \mathbb{R} -Cartier, with the appropriate modifications. The divisor $\sum a_i E_i$ is called the *discrepancy* of f . As before, the a_i s do not depend on f , and they are denoted by $a(E_i, X, \Delta) := a_i$.

Definition 1.2.1 ([KM98, 2.28]). *Let (X, Δ) be a pair (possibly with $\Delta = 0$). The discrepancy of (X, Δ) is*

$$\text{discrep}(X, \Delta) = \inf \{a(E, X, \Delta) \mid E \text{ is exceptional over } X\}$$

and the total discrepancy of (X, Δ) is

$$\text{totaldiscrep}(X, \Delta) = \inf \{a(E, X, \Delta)\}.$$

If $\Delta = 0$, we use the notation $\text{discrep}(X)$ and $\text{totaldiscrep}(X)$.

We define singularities using discrepancies.

Lemma 1.2.2 ([KM98, 2.31]). *We have $-1 \leq \text{totaldiscrep}(X, \Delta) \leq \text{discrep}(X, \Delta) \leq 1$ or $\text{discrep}(X, \Delta) = -\infty$.*

Definition 1.2.3 ([KM98, 2.34]). *The singularities of a normal scheme X are called*

$$\left. \begin{array}{l} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{array} \right\} \text{ if } \text{discrep}(X, \Delta) \left\{ \begin{array}{l} > 0, \\ \geq 0, \\ > -1 \text{ and } [\Delta] = 0, \\ > -1, \\ \geq -1. \end{array} \right.$$

Klt is the short for Kawamata log terminal, plt for purely log terminal, and lc for log canonical. If $\Delta = 0$, klt and plt coincide and are called lt (log terminal).

Discrepancies can be checked on a unique resolution.

Proposition 1.2.4 ([KM98, 2.32]). *Let (X, Δ) be a pair, with $\Delta = \sum a_j D_j$, $a_j \leq 1$. Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) , with $\text{Exc}(f) = \bigcup E_i$. If $a(E_i, X, \Delta) \geq -1$ for every i ,*

$$\text{discrep}(X, \Delta) = \min \left\{ \min_i \{a(E_i, X, \Delta)\}, \min_j \{1 - a_j\}, 1 \right\};$$

otherwise $\text{discrep}(X, \Delta) = -\infty$.

Recall that log resolutions always exist in characteristic 0.

1.3 Valuations

From now on, X will be a normal *variety*, that is, a normal separated scheme of finite type over a noetherian ring k . Most of the results in this section are true in more generality, see [JM10].

A *divisorial valuation* on X is a discrete valuation of the function field $k(X)$ of X of the form $\nu = q \text{val}_F$ where $q \in \mathbb{R}_{>0}$ and F is a prime divisor over X , that is, a prime divisor on some normal variety birational to X . By a *real valuation* on X , we mean a real-valued valuation ν on the function field of X that is trivial on the base field and admits a center on X (which is the unique scheme point $c_X(\nu) = c_\nu$ such that $\nu > 0$ on the maximal ideal of

\mathcal{O}_{X, c_ν}). We denote by Val_X the space of real valuations on X , endowed with the topology of pointwise convergence, and by $\text{DivVal}_X \subset \text{Val}_X$ the space of divisorial valuations. There is a natural map $c_X : \text{Val}_X \rightarrow X$, which associates to each valuation ν its center.

If \mathcal{I} is a *coherent fractional ideal* of X (that is, a finitely generated sub- \mathcal{O}_X -module of the constant field $k(X)$ of rational functions on X) and ν is a valuation on X (be it real or discrete), we set

$$\nu(\mathcal{I}) := \min\{\nu(f) \mid f \in \mathcal{I}(U), c_X(\nu) \in U\}.$$

Definition 1.3.1 ([BdFFU13, 2.1]). *A normalizing subscheme is a (non-trivial) closed subscheme N of X containing $\text{Sing}(X)$. The normalized valuation space defined by N is*

$$\text{Val}_X^N := \{\nu \in \text{Val}_X \mid \nu(\mathcal{I}_N) = 1\},$$

where \mathcal{I}_N is the ideal sheaf defining N .

Note that

$$\mathbb{R}_{>0} \cdot \text{Val}_X^N = \{\nu \in \text{Val}_X \mid \nu(\mathcal{I}_N) > 0\} = c_X^{-1}(N).$$

We also have that

$$\text{Val}_X = \bigcup_N \mathbb{R}_{>0} \cdot \text{Val}_X^N, \tag{1.3.8}$$

with N ranging over all normalizing subschemes. The theory of valuation spaces can be found on [Ber90].

For the remaining of this section, let $k = \mathbb{C}$.

Definition 1.3.2 ([BdFFU13, 2.2]). *A good resolution of a normalizing subscheme $N \subset X$ is a proper birational morphism $\pi : X_\pi \rightarrow X$ such that:*

- (a) X is smooth;
- (b) π is an isomorphism over $X \setminus N$;
- (c) $\text{Exc}(\pi)$ and $\pi^{-1}(N)$ have pure codimension 1; and
- (d) $\pi^{-1}(N)_{\text{red}} = \sum_{i \in I} E_i$ is a simple normal crossing divisor such that, for each $J \subseteq I$, $\bigcap_{j \in J} E_j$ is irreducible (or empty).

To a given a good resolution $\pi : X_\pi \rightarrow X$ of N we can associate $\Delta_\pi^N \subset \text{Val}_X^N$, which is a geometric realization of the dual complex of $\pi^{-1}(N)_{\text{red}}$ (see [JM10, §3-4]). Let us recall that the dual complex of $\sum_{i \in I} E_i$ is the simplicial complex whose vertices are in bijection with I and that contains one simplicial face σ_J joining all vertices $j \in J$ for any subset $J \subset I$ such that $\bigcap_{j \in J} E_j \neq \emptyset$. Moreover, the divisorial valuations in Δ_π^N are the rational points, which implies

$$\overline{\text{DivVal}_X} \cap \Delta_X^N = \Delta_X^N. \quad (1.3.9)$$

A subset $\sigma \subset \text{Val}_X^N$ is said to be a *face* if σ is a face of Δ_π^N for some π . There is a natural continuous retraction

$$r_\pi^N : \text{Val}_X^N \rightarrow \Delta_\pi^N.$$

If π' factors through π , then there is a natural inclusion $\Delta_\pi^N \hookrightarrow \Delta_{\pi'}^N$. The next results says that we can think of Val_X^N as a limit of simplicial complexes.

Theorem 1.3.3 ([BdFFU13, 2.3]). *We have*

$$\text{Val}_X^N = \overline{\bigcup_{\pi} \Delta_\pi^N}, \quad (1.3.10)$$

where π runs over all good resolutions on N . More precisely, $\lim_{\pi} r_\pi^N(\nu) = \nu$ for each $\nu \in \text{Val}_X$.

1.4 Pullback of Weil divisors

1.4.1 Natural pullback

Definition 1.4.1. *To a given fractional ideal \mathcal{I} we can associate a Weil divisor, called the divisorial part of \mathcal{I} , as*

$$\text{div}(\mathcal{I}) := \sum_{E \subset X} \text{val}_E(\mathcal{I})E,$$

where the sum runs over all the prime divisors on X ; equivalently, $\text{div}(\mathcal{I})$ is such that

$$\mathcal{O}_X(-\text{div}(\mathcal{I})) = \mathcal{I}^{\vee \vee}.$$

Definition 1.4.2. *Let ν be a divisorial valuation on X . The \natural -valuation or natural valuation along ν of a divisor F on X is*

$$\nu^{\natural}(F) := \nu(\mathcal{O}_X(-F)).$$

De Fernex and Hacon showed ([dFH09, 2.8]) that, for every \mathbb{R} -Weil divisor D on X and every $m \in \mathbb{Z}_{>0}$, $m\nu^{\natural}(D) \geq \nu^{\natural}(mD)$ and

$$\inf_{m \geq 1} \frac{\nu^{\natural}(mD)}{m} = \liminf_{m \rightarrow \infty} \frac{\nu^{\natural}(mD)}{m} = \lim_{m \rightarrow \infty} \frac{\nu^{\natural}(m!D)}{m!} \in \mathbb{R}.$$

Definition 1.4.3. Let D be an \mathbb{R} -Weil divisor on X and ν a divisorial valuation. The valuation along ν of D is

$$\nu^*(D) := \lim_{m \rightarrow \infty} \frac{\nu^{\natural}(m!D)}{m!}.$$

Remark 1.4.4. Even when D is a \mathbb{Z} -Weil divisor (i.e. a Weil divisor), the above valuation could be irrational, see [Urb12, 3.3] with $D = K_X$.

Definition 1.4.5. Let $f : Y \rightarrow X$ be a proper birational morphism from a normal variety Y . For any \mathbb{R} -Weil divisor D on X , the \natural -pullback of D along f is

$$f^{\natural}D := \operatorname{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y); \quad (1.4.11)$$

equivalently, $f^{\natural}D$ is the divisor on Y such that

$$\mathcal{O}_Y(-f^{\natural}D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee \vee}. \quad (1.4.12)$$

Definition 1.4.6. We define the pullback of D along f as

$$f^*D := \sum_{E \subset Y} \operatorname{val}_E^*(D)E,$$

where the sum runs over all prime divisors on Y . Equivalently,

$$f^*D = \liminf_m \frac{f^{\natural}(mD)}{m} \text{ coefficient-wise.}$$

Remark 1.4.7. When we talk about an infimum of divisors, we have to be very careful if we are considering the divisors as Weil divisors, or as numerical (or as linear) classes. In the latter case, the infimum is not well defined².

²For example, on $X = \mathbf{P}^2$, let C_d be the divisor of a degree $d \geq 1$ curve. For each d , $C_d \sim dC_1$. Then, $\inf\{C_d/d\} = 0$ as Weil divisors: if a divisor appears in the sequence, is with coefficient $1/d$, and then does not appear anymore. If we had chosen $C_1 \sim_{\mathbb{Q}} C_d/d$ as representatives of the terms of our sequence, the infimum would have been C_1 , and clearly C_1 is not \mathbb{Q} -linearly equivalent (or even numerically equivalent) with 0

We notice that the evaluation along ν and the pullback above defined agree with corresponding notions in the case that the divisor D be \mathbb{Q} -Cartier.

Proposition 1.4.8 ([dFH09, 2.4 and 2.10]). *Let ν be a divisorial valuation on X and let $f : Y \rightarrow X$ be a birational morphism from a normal variety Y . Let D be any \mathbb{R} -Weil divisor let C be any \mathbb{R} -Cartier divisor, with $t \in \mathbb{R}_{>0}$ such that tC is Cartier.*

(a) *The definitions of $\nu^*(C)$ and $f^*(C)$ given above coincides with the usual ones. More precisely,*

$$\nu^*(C) = \frac{1}{t}\nu(tC) \quad \text{and} \quad f^*(C) = \frac{1}{t}f^*(tC).$$

Moreover,

$$\nu^*(tC) = \nu^\natural(tC) \quad \text{and} \quad f^\natural(tC) = f^*(tC).$$

(b) *The pullback is almost linear, in the sense that*

$$f^\natural(D + tC) = f^\natural(D) + f^\natural(tC) \quad \text{and} \quad f^*(D + tC) = f^*(D) + f^*(tC). \quad (1.4.13)$$

Remark 1.4.9. We observe that when working with natural valuation and natural pullback, the properties in (b) are no longer true for \mathbb{R} -Cartier divisors which are not Cartier. For example, if D is the divisor in 1.1.4, then D is not Cartier while $2D$ is. However, if ν is the valuation corresponding to the exceptional locus of the blow-up of the vertex, $\nu^\natural(D) = \nu^\natural(2D) = 1$.

Remark 1.4.10. The pullback is far from being linear if the divisors are not \mathbb{R} -Cartier. For example, in general $f^*(-D) \neq -f^*(D)$. A list of examples of the pathologies that can occur is in §3.2.

Lemma 1.4.11 ([dFH09, 2.7]). *Let $f : Y \rightarrow X$ and $g : V \rightarrow Y$ be two birational morphisms of normal varieties, and let D be an \mathbb{R} -Weil divisor on X . The divisor $(fg)^\natural(D) - g^\natural f^\natural(D)$ is effective and g -exceptional. Moreover, if $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is an invertible sheaf, $(fg)^\natural(D) = g^\natural f^\natural(D)$.*

Corollary 1.4.12 ([dFH09, 2.13]). *Let $f : Y \rightarrow X$ and $g : V \rightarrow Y$ be two birational morphisms of normal varieties, and let D be an \mathbb{R} -Weil divisor on X . The divisor $(f \circ g)^*(D) - g^*f^*(D)$ is effective and g -exceptional. Moreover, if $\mathcal{O}_X(-mD) \cdot \mathcal{O}_Y$ is an invertible sheaf for all sufficiently divisible m , $(f \circ g)^*(D) = g^*f^*(D)$.*

Proof. The result comes from the fact that

$$f^*(D) = \liminf_m \frac{1}{m} f^{\natural}(mD)$$

coefficient-wise. □

1.4.2 Pullback and boundaries

Let X be a normal variety over an algebraically closed field. In [dFH09, 5.4] de Fernex and Hacon observed that the natural pullback of K_X could be related with the pullback of $K_X + \Delta$ for a suitable boundary Δ (i.e., $K_X + \Delta$ is \mathbb{Q} -Cartier). In [CU12, 3.6], we generalized the result of de Fernex and Hacon to any divisor D . To make the notations more consistent, in [Chi13], we gave the following definition.

Definition 1.4.13. *Let X be a normal variety, and let $f : Y \rightarrow X$ be a proper birational map, with Y normal. Let D be a Weil divisor on X . For each $m \geq 2$, a divisor Δ_m is called a weak m -compatible D -boundary with respect to f if*

- (a) $\Delta_m \geq 0$;
- (b) $D + \Delta_m$ is \mathbb{Q} -Cartier;
- (c) $[\Delta_m] = 0$ and $m\Delta_m$ is integral; and
- (d) $\frac{f^{\natural}(mD)}{m} = f^*(D + \Delta_m) - f_*^{-1}\Delta_m$.

Moreover, if there exist a proper birational morphism $g : Z \rightarrow X$ and a D -boundary Δ_m as above such that

- (e) g is a resolution of X , and

(f) $\text{Exc}(g) \cup f_*^{-1}\Delta_m$ has simple normal crossing support,

such divisor will be called an m -compatible D -boundary.

With this terminology, we have the following lemma.

Lemma 1.4.14 ([dFH09, 5.4], [CU12, 3.6]). *For each proper birational morphism $f : Y \rightarrow X$ between normal varieties over an algebraically closed field k , any $m \geq 2$ and any divisor D on X , there exists a weak m -compatible D -boundary with respect to f . Moreover, if $\text{char } k = 0$, for any $m \geq 2$, there are m -compatible D -boundaries (with respect to some resolution).*

Proof. Let T be an effective divisor such that $D - T$ is Cartier. Let \mathcal{L} be a line bundle such that $\mathcal{L} \otimes \mathcal{O}_X(-mT)$ is generated by global sections, and let G be a general element in the linear system $\{H \in |\mathcal{L}|, H - mT \geq 0\}$. Let $M = G - mT$ and

$$\Delta_m = \frac{M}{m}.$$

We notice right away that $\Delta_m \geq 0$ and $[\Delta_m] = 0$. Moreover, $m(D + \Delta_m) = mD + M = mD - mT + G = m(D - T) + G$ is Cartier. Let E be an irreducible exceptional divisor of f . Since G is general enough, $\text{ord}_E G = \text{ord}_E(mT)$. Thus, $\text{ord}_E(-M) = \text{ord}_E(-G + mT) = -\text{ord}_E(G) + \text{ord}_E(mT) = 0$, and therefore $f^{\natural}(-M) = f_*^{-1}(-M)$. Then,

$$\begin{aligned} f^{\natural}(mD) &= f^{\natural}(m(D + \Delta_m) - m\Delta_m) = mf^*(D + \Delta_m) + f^{\natural}(-m\Delta_m) = \\ &= mf^*(D + \Delta_m) + f^{\natural}(-M) = mf^*(D + \Delta_m) - f_*^{-1}(M) = \\ &= mf^*(D + \Delta_m) - mf_*^{-1}(\Delta_m). \end{aligned}$$

If f is a resolution of $\mathcal{O}_X(-mD) + \mathcal{O}_X(-mT)$, then Δ_m can be chosen general enough so that $\text{Exc}(f) \cup f_*^{-1}\Delta_m$ has simple normal crossing support. \square

Corollary 1.4.15 ([Chi13, 2.27]). *Let X be a normal variety, and let $f : Y \rightarrow X$ be a proper birational map, with Y normal. Let D be a Weil divisor on X . Let*

$$\mathcal{D}_D = \{\Delta \geq 0, D + \Delta \text{ is } \mathbb{Q}\text{-Cartier}, [\Delta] = 0\}.$$

Then

$$f^*(D) = \inf_{\Delta \in \mathcal{D}_D} (f^*(D + \Delta) - f_*^{-1}\Delta), \quad (1.4.14)$$

coefficient-wise.

Proof. By definition,

$$f^*(D) = \inf_m \frac{f^\natural(mD)}{m},$$

coefficient-wise. By lemma 1.4.14, the left hand side is bigger or equal than the right one, since

$$f^*(D) = \inf_m \frac{f^\natural(mD)}{m} = \inf_m (f^*(D + \Delta_m) - f_*^{-1}\Delta_m). \quad (1.4.15)$$

To finish the proof, let $\Delta \in \mathcal{D}_D$ such that $m(D + \Delta)$ is Cartier. As in remark 2.1.3 for the particular case of the relative canonical divisor,

$$\frac{f^\natural(mD)}{m} \leq f^*(D + \Delta) - f_*^{-1}\Delta.$$

□

The same procedure as above can be done for simultaneous resolutions of pairs (X, S) , where S is an integral normal Cartier divisor in X , and when considering a divisor D in X and its restriction to S (if this is possible). Here we will do it for $D = -K_X$ (and on [Cho13] it can be found for $D = K_X$). The following lemma is a technical result, needed for the proof of 4.4.1.

Lemma 1.4.16 ([Chi13, 2.23]). *Let X be a normal variety over an algebraically closed field k and let $S \subseteq X$ be a normal (effective) Cartier divisor in X , and let $f : Y \rightarrow X$ be a proper birational morphism. Then there exists a weak m -compatible $(-K_X)$ -boundary Δ on X such that $\Delta|_S$ is a weak m -compatible $(-K_S)$ -boundary. Moreover, if $\text{char } k = 0$, f and Δ can be chosen so that Δ and $\Delta|_S$ are m -compatible.*

Proof. Let us fix some notation. Let $T := f_*^{-1}S$ and let $g := f|_T : T \rightarrow S$, which is still proper birational.

Let D be an effective divisor such that $-K_X - D$ is Cartier. By adjunction, $(-K_X - S - D)|_S = -K_S - D|_S$ is still Cartier. Let \mathcal{L} be a line bundle such that $\mathcal{L} \otimes \mathcal{O}_X(-mD)$ and $\mathcal{L}|_S \otimes \mathcal{O}_S(-mD|_S)$ are generated by global sections, and let G be a general element in the linear system $\{H \in |\mathcal{L}|, H - mD \geq 0\}$, which we can assume reduced and having no common components with D and S . Let $M = G - mD$ and

$$\Delta_m = \frac{M}{m}.$$

As in the proof of 1.4.14, the generality of the choice of G guarantees that Δ_m is a weak m -compatible $(-K_X)$ -boundary with respect to f , and that $\Delta_m|_S$ is a weak m -compatible $(-K_S)$ -boundary with respect to g .

If f is a resolution of $\mathcal{O}_X(mK_X) + \mathcal{O}_X(-mD)$ such that $g : T \rightarrow S$ is a resolution of $\mathcal{O}_S(mK_S) + \mathcal{O}_S(-mD|_S)$, then Δ_m can be chosen general enough so that $\text{Exc}(f) \cup f_*^{-1}\Delta_m$ and $\text{Exc}(g) \cup g_*^{-1}(\Delta_m|_S)$ have simple normal crossing support. \square

1.5 b -divisors

Let U be a scheme, and let X be a scheme over U . The *relative Riemann-Zariski space* of X over U is the projective limit

$$\mathfrak{X}/\mathfrak{U} := \varprojlim_{\pi} X_{\pi}$$

where $\pi : X_{\pi} \rightarrow X$ is a proper birational morphism over U . When $U = \text{Spec } k$, for a field k , we write \mathfrak{X} for $\mathfrak{X}/\mathfrak{U}$. This limit is defined in the category of locally ringed spaces, thus \mathfrak{X} is a locally ringed space. It is not a scheme, in general (unless $\dim X = 1$).

If X is a normal scheme over a field k , the group of Weil and Cartier b -divisors over X are respectively defined as

$$\begin{aligned} \text{Div}(\mathfrak{X}) &:= \varprojlim_{\pi} \text{Div}(X_{\pi}) \quad \text{and} \\ \text{CDiv}(\mathfrak{X}) &:= \varinjlim_{\pi} \text{CDiv}(X_{\pi}), \end{aligned}$$

where the first limit is taken with respect to the pushforward and the second with respect to the pullback. We say that a Cartier b -divisor C is *determined* on X_{π} for a model X_{π} over X such that $C_{\pi'} = f^*C_{\pi}$ for every $f : X_{\pi'} \rightarrow X_{\pi}$. Since the space \mathfrak{X} is a normal

locally ringed space, we can define Weil and Cartier divisors directly on it; these definitions coincide with the above one. The set of \mathbb{R} -Weil b -divisors is $\text{Div}_{\mathbb{R}}(\mathfrak{X}) := \text{Div}(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{R}$, and the set of \mathbb{R} -Cartier b -divisors is $\text{CDiv}_{\mathbb{R}}(\mathfrak{X}) := \text{CDiv}(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{R}$. Similarly, we can define \mathbb{Q} -Weil b -divisors and \mathbb{Q} -Cartier b -divisors.

Remark 1.5.1. If we assume the existence of resolution of singularities (if the characteristic of the field is 0, for example, or if X is toric), the set of \mathbb{R} -Cartier b -divisors is dense in the set of \mathbb{R} -Weil b -divisors.

From now on, let X be a normal projective variety over a field k . Let \mathfrak{a} be a coherent fractional ideal sheaf on X . If $X_{\pi} \rightarrow X$ is the normalized blow-up of X along \mathfrak{a} , we denote $Z(\mathfrak{a})$ the Cartier b -divisor determined by $\mathfrak{a} \cdot \mathcal{O}_{X_{\pi}}$.

Lemma 1.5.2 ([BdFF12, 1.7]). *A Cartier b -divisor $C \in \text{CDiv}(\mathfrak{X})$ is of the form $Z(\mathfrak{a})$ if and only if C is relatively globally generated over X .*

Definition 1.5.3 ([BdFF12, 2.3]). *Let D be an \mathbb{R} -Weil divisor on X . The nef envelope $\text{Env}_X(D)$ is the \mathbb{R} -Weil b -divisor associated to the graded sequence $\mathfrak{a}_{\bullet} = \{\pi_* \mathcal{O}_X(mD)\}_{m \geq 1}$, where*

$$Z(\mathfrak{a}_{\bullet}) := \lim_{m \rightarrow \infty} \frac{1}{m} Z(\mathfrak{a}_m).$$

Lemma 1.5.4 ([BdFF12, 2.10 and 2.12]). *We have that*

(a) $Z(\mathfrak{a}_{\bullet})$ is X -nef, and

(b) $\text{Env}_X(D)$ is the largest X -nef \mathbb{R} -Weil b -divisor W such that $\pi_*(W_{\pi}) \leq D$.

The relation of this construction with the pullback is the following.

Lemma 1.5.5 ([BdFF12, 2.4]). *If $\pi : X_{\pi} \rightarrow X$ is a birational model of X , proper over X , and D is an \mathbb{R} -Weil divisor on X , then $\text{Env}_X(D)_{\pi} = -\pi^*(-D)$.*

Boucksom, de Fernex and Favre gave the following definition, identifying the Weil divisors which behave “Cartier-like” with respect to the pullback.

Definition 1.5.6 ([BdFF12, 2.26]). *We say that an \mathbb{R} -Weil divisor D on X is numerically Cartier if $\text{Env}_X(-D) = -\text{Env}_X(D)$. In the special case where $D = K_X$, we say that X is numerically Gorenstein if K_X is numerically Cartier.*

For example, we have already discussed how the pullback fails to be linear when considering general Weil divisors. We have, however, the following nice result.

Lemma 1.5.7 ([BdFF12, 2.28]). *Let D be an \mathbb{R} -Weil divisor on X . Then D is numerically Cartier if and only if*

$$\text{Env}_X(D + D') = \text{Env}_X(D) + \text{Env}_X(D')$$

for every \mathbb{R} -Weil divisor D' on X .

1.6 Minimal Model Program

In this section X will be a projective variety over an algebraically closed field. Most of the theory works in the relative setting as well, that is, for projective morphisms of normal varieties, with the obvious modifications.

1.6.1 Positivity

A 1-cycle is a formal \mathbb{Z} -linear combination of irreducible, reduced and proper curves $C = \sum a_i C_i$. We say that such cycle is *effective* if $a_i \geq 0$. Two cycles C and C' are *numerically equivalent* if $C.D = C'.D$ for every Cartier divisor D . We will denote by $N_1(X)$ the vector space of 1-cycles modulo linear equivalence. By $\text{NS}(X)$ we denote the Néron-Severi group of X , that is, the group of Cartier divisors modulo numerical equivalence. We use the notations $N_1(X)_{\mathbb{R}} := N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\text{NS}(X)_{\mathbb{R}} := \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Intersection of curves and Cartier divisors gives a perfect pairing

$$N_1(X)_{\mathbb{R}} \times \text{NS}(X)_{\mathbb{R}} \rightarrow \mathbb{R}. \tag{1.6.16}$$

The theorem of the base of Néron-Severi implies that both spaces are finitely dimensional; their dimension is the *Picard number*, denoted by $\rho(X)$.

To X we associate the subset of $N_1(X)_{\mathbb{R}}$

$$\text{NE}(X) := \left\{ \sum a_i [C_i] \mid C_i \subset X, a_i \in \mathbb{R}_{\geq 0} \right\},$$

and we will denote by $\overline{\text{NE}}(X)$ its closure (in $N_1(X)_{\mathbb{R}}$).

The above sets are *cones* (i.e., they are closed under multiplication by non-negative scalars). If D is a Cartier divisor on X , we can consider the following subsets:

- (a) $D_{>0} := \{x \in N_1(X)_{\mathbb{R}} \mid x.D > 0\}$ (and similarly for ≥ 0 , < 0 , ≤ 0);
- (b) $D^{\perp} := \{x \in N_1(X)_{\mathbb{R}} \mid x.D = 0\}$;
- (c) $\overline{\text{NE}}(X)_{D>0} := \overline{\text{NE}}(X) \cap D_{>0}$ (and similarly for ≥ 0 , < 0 , ≤ 0).

Kleiman's ampleness criterion can be restated using the above notation.

Theorem 1.6.1 ([Kle66, III.1, Theorem 1]). *Let D be a Cartier divisor on X . Then D is ample if and only if $\overline{\text{NE}}(X) \setminus \{0\} \subseteq D_{>0}$.*

As immediate corollary of the previous result, we obtain that ampleness is numerically invariant. Thus we can define the set

$$\text{Amp}(X) := \{D \in \text{NS}(X)_{\mathbb{R}} \mid \overline{\text{NE}}(X) \setminus \{0\} \subseteq D_{>0}\}.$$

This is the cone generated by ample classes and is referred to as the *ample cone*.

A Cartier divisor D is *numerically effective*, or *nef*, if $D.C \geq 0$ for every curve C . This condition is clearly numerically invariant, so we can consider the cone generated by nef classes, which is called the *nef cone*

$$\text{Nef}(X) := \{D \in \text{NS}(X)_{\mathbb{R}} \mid D.C \geq 0, \text{ for all curves } C \subset X\}.$$

As a consequence of Kleiman's criterion we have that

$$\text{Nef}(X) = \overline{\text{Amp}(X)}. \tag{1.6.17}$$

Moreover, the ample cone is the interior of the nef cone, [Laz04, I.1.4.23]. Via the perfect pairing of (1.6.16), $\text{Nef}(X)$ and $\overline{\text{NE}}(X)$ are dual to each other, [Laz04, I.1.4.28]. Notice

that, since each Cartier divisor can be written as the difference of two ample divisors, the above cones are full dimensional.

A Cartier divisor D is called *big* if there exist an ample Cartier divisor A and an effective Cartier divisor E such that $D \sim_{\mathbb{Q}} A + E$. We have the *big cone* $\text{Big}(X) \subset \text{NS}(X)_{\mathbb{R}}$, which is the cone generated by big classes. A Cartier divisor is called *pseudo effective*, or *pseff*, if it is \mathbb{Q} -linearly equivalent to an effective divisor; we denote by $\text{Pseff}(X) \subset \text{NS}(X)_{\mathbb{R}}$ the cone of pseudo effective classes, the *pseudo effective cone*. This cone is often denoted also by $\overline{\text{Eff}}(X)$. Recall that $\overline{\text{Big}}(X) = \text{Pseff}(X)$, and that the big cone is the interior of the pseudo effective cone, [Laz04, I.2.2.26].

1.6.2 Minimal Model Program

Let us start with the cone theorem, which is one of the crucial steps of the Minimal Model Program. This version is from [KM98, 3.7].

Theorem 1.6.2 (Cone theorem). *Let (X, Δ) be a klt pair. Then*

- (1) *There are (countably many) rational curves $C_j \subset X$ such that $-2 \dim X \leq (K_X + \Delta) \cdot C_j < 0$ and*

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X + \Delta \geq 0} + \sum \mathbb{R}_{\geq 0} \cdot C_j.$$

- (2) *For each $\varepsilon > 0$ and ample \mathbb{Q} -Cartier \mathbb{Q} -divisor H , then*

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X + \Delta + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0} \cdot C_j.$$

- (3) *(Contraction theorem) Let $F \subset \overline{\text{NE}}(X)$ be a $(K_X + \Delta)$ -negative extremal face (i.e., a sub-cone such that, if $v, w \in \overline{\text{NE}}(X)$ with $v + w \in F$, then $v, w \in F$). Then there is a unique morphism $\varphi_F : X \rightarrow Y$ to a projective variety Y such that $(\varphi_F)_* \mathcal{O}_X = \mathcal{O}_Y$ and an irreducible curve $C \subset X$ is contracted to a point by φ_F if and only if $[C] \in F$.*

- (4) *Let F and φ_F as in the previous point. Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in F$. Then there exists a line bundle L_Y on Y such that $L \cong \varphi_F^* L_Y$.*

Part (3) is called the *contraction theorem*, and φ_F is called the *contraction* of F . One of the main ingredients of the proof of this theorem is the basepoint-free theorem, see [KM98, 3.3].

Theorem 1.6.3 (Basepoint-free theorem). *Let (X, Δ) be a projective klt pair. Let D be a nef Cartier divisor such that $aD - (K_X + \Delta)$ is nef and big for some $a > 0$. Then, for all $b \gg 0$, $|bD|$ is basepoint-free.*

To prove 1.6.2(3) one produces a Cartier divisor D satisfying the hypothesis of the Basepoint-free theorem. By said theorem, $|bD|$ is basepoint-free and thus it induces a morphism $f : X \rightarrow Z$. The Stein factorization of f gives the desired contraction.

If the face F in 1.6.2(3) is a *ray* (it has dimension 1), there are three possibilities, [KM98, 2.5]:

- (a) (*Fano contraction*) $\dim X > \dim Y$,
- (b) (*divisorial contraction*) φ_F is birational and $\text{Exc}(\varphi_F)$ is an irreducible divisor, and
- (c) (*small or flipping contraction*) φ_F is small birational.

Proposition 1.6.4 ([KM98, 3.18 and 3.36]). *Let (X, Δ) be a projective klt pair, with X \mathbb{Q} -factorial, and let $\varphi : X \rightarrow Y$ be a Fano contraction. Then $\rho(X/Y) = 1$, Y is \mathbb{Q} -factorial, and the generic fiber is a \mathbb{Q} -factorial log Fano variety.*

Definition 1.6.5. *A projective variety Z is log Fano if there exists a boundary Δ on Z such that (Z, Δ) is a klt pair and $-(K_Z + \Delta)$ is ample.*

Proposition 1.6.6 ([KM98, 3.36 and 3.43]). *Let (X, Δ) be a projective klt pair, with X \mathbb{Q} -factorial, and let $\varphi : X \rightarrow Y$ be a divisorial contraction. Then $\rho(X/Y) = 1$, Y is \mathbb{Q} -factorial, and $(Y, f_*\Delta)$ is a klt pair.*

Definition 1.6.7 ([KM98, 3.33]). *Let (X, Δ) be a pair. Let $f : X \rightarrow Y$ be a proper, small birational morphism such that $-(K_X + \Delta)$ is f -ample. A pair (X^+, Δ^+) , together with a proper, small birational morphism $f^+ : X^+ \rightarrow Y$ is called a flip if*

- (a) $K_{X^+} + \Delta^+$ is \mathbb{Q} -Cartier,
- (b) $K_{X^+} + \Delta^+$ is f^+ -ample, and
- (c) Δ^+ is the strict transform of Δ via the birational map $X \dashrightarrow X^+$ induced by f and (f^+) .

Proposition 1.6.8 ([KM98, 3.37 and 3.42]). *Let (X, Δ) be a projective klt pair, with X \mathbb{Q} -factorial, let $\varphi : X \rightarrow Y$ be a flipping contraction, and let (X^+, Δ^+) be the flip of (X, Δ) . Then $\rho(X/Y) = 1$ and $\rho(X) = \rho(X^+)$, Y is \mathbb{Q} -factorial, and (X^+, Δ^+) is a klt pair.*

Let us now recall the klt Minimal Model Program.

Step 0 Let $(X, \Delta) := (X_i, \Delta_i)$ be a \mathbb{Q} -factorial, projective klt pair.

Step 1 If $K_X + \Delta$ is nef, there is nothing to do and the program terminates. Otherwise let R be a $(K_X + \Delta)$ -negative extremal ray, and let $\varphi : X \rightarrow Y$ be the corresponding contraction.

Step 2 There are three possibilities for the contraction above.

- (i) If φ is a Fano contraction, we set $(X_0, \Delta_0) := (Y, \varphi_*\Delta)$ and we can restart the program in smaller dimension.
- (ii) If φ is a divisorial contraction, we set $(X_{i+1}, \Delta_{i+1}) := (Y, \varphi_*\Delta)$ and we restart the algorithm.
- (iii) If φ is a flipping contraction, we set $(X_{i+1}, \Delta_{i+1}) := (X^+, \Delta^+)$ (the flip), and we restart the algorithm.

The most important result in algebraic geometry in recent times is the following.

Theorem 1.6.9 ([BCHM10]). *Klt flips exist. Moreover, a specific program terminates.*

Remark 1.6.10. In the cases of Fano and divisorial contractions, the Picard number decreases by 1, and we can use this observation to deduce the termination of the program. However, in the case of flips, the Picard number remains constant. It is conjectured that every sequence of flips terminates.

Chapter 2

DISCREPANCIES

In this chapter we will start our study of non- \mathbb{Q} -Gorenstein singularities. In §2.1 we define the various relative canonical divisors and we see how they relate to each other. In §2.2 we use the notions introduced in the previous section to define singularities and the discrepancy. In particular, we define lt^+ singularities in 2.2.7. Finally in §2.3 we show some relations between the different singularities.

Most of the results and notations of this chapter are from [CU12] (until, and included, 2.3.5).

2.1 Relative canonical divisors

Following [dFH09], in [CU12] we introduce the following definitions (with a slightly different notation¹).

Definition 2.1.1. *Let $f : Y \rightarrow X$ be a proper birational map of normal varieties. The m -limiting relative canonical \mathbb{Q} -divisors are*

$$\begin{aligned} K_{m,Y/X}^- &:= K_Y - \frac{1}{m} f^{\natural}(mK_X), & K_{Y/X}^- &:= K_Y - f^*(K_X) \\ K_{m,Y/X}^+ &:= K_Y + \frac{1}{m} f^{\natural}(-mK_X), & K_{Y/X}^+ &:= K_Y + f^*(-K_X). \end{aligned}$$

As shown by [dFH09], for all $m, q \geq 1$,

$$K_{m,Y/X}^- \leq K_{qm,Y/X}^- \leq K_{Y/X}^- \leq K_{Y/X}^+ \leq K_{mq,Y/X}^+ \leq K_{m,Y/X}^+ \quad (2.1.1)$$

and

$$K_{Y/X}^- = \limsup K_{m,Y/X}^-, \quad K_{Y/X}^+ = \liminf K_{m,Y/X}^+ \quad (2.1.2)$$

¹De Fernex and Hacon denote by $K_{m,Y/X}$ what we denote by $K_{m,Y/X}^-$ and by $K_{Y/X}$ what we denote by $K_{Y/X}^+$. Our notation is simply meant to help us keep track of the different relative canonical divisors

(coefficient-wise).

In general, the inequalities of (2.1.1) are strict. Thus, we could use any of these notions to define the various flavors of singularities, and there is no reason to suspect that one behaves better than the others. We will study this different types of singularities in the context of the MMP, starting from section §2.2.

Definition 2.1.2. *Let Δ be a boundary on X and let $f : Y \rightarrow X$ be a proper birational morphism of normal schemes. The relative canonical \mathbb{Q} -divisor of $(Y, f_*^{-1}\Delta)$ over (X, Δ) is given by*

$$K_{Y/X}^\Delta := K_Y + f_*^{-1}\Delta - f^*(K_X + \Delta) = K_Y + f_*^{-1}\Delta + f^*(-K_X - \Delta)$$

where $f_*^{-1}\Delta$ is the strict transform of Δ on Y .

This is the discrepancy of f , see §1.2.

Remark 2.1.3. With the same computation as [dFH09, 3.9], we find that, if Δ is a boundary for X and $m \geq 1$ is such that $m(K_X + \Delta)$ is Cartier,

$$\begin{aligned} K_{m,Y/X}^- &= K_{Y/X}^\Delta - \frac{1}{m}f^{\natural}(-m\Delta) - f_*^{-1}\Delta, & K_{Y/X}^- &= K_{Y/X}^\Delta - f^*(-\Delta) - f_*^{-1}\Delta, \\ K_{m,Y/X}^+ &= K_{Y/X}^\Delta + \frac{1}{m}f^{\natural}(m\Delta) - f_*^{-1}\Delta, & K_{Y/X}^+ &= K_{Y/X}^\Delta + f^*(\Delta) - f_*^{-1}\Delta. \end{aligned}$$

Notice that, if m is such that $m(K_X + \Delta)$ is Cartier and Δ is *effective*, we have

$$K_{Y/X}^\Delta \leq K_{m,Y/X}^- \leq K_{Y/X}^- \leq K_{Y/X}^+ \leq K_{m,Y/X}^+.$$

On the other hand, if Δ is *anti-effective* and $m(K_X + \Delta)$ is Cartier,

$$K_{Y/X}^\Delta \leq K_{m,Y/X}^+.$$

As immediate consequence of 1.4.14 and the above inequalities, Urbinati noticed the following, [Urb12, 2.12].

Lemma 2.1.4. *If $f : Y \rightarrow X$ is a proper birational morphism of normal varieties over an algebraically closed field, $K_{Y/X}^- = \sup K_{Y/X}^\Delta$, where Δ runs among the effective boundaries with $[\Delta] = 0$ (or among the compatible boundaries).*

Proof. For each $m \geq 2$, there exists an effective boundary Δ_m such that $K_{m,Y/X}^- = K_{Y/X}^{\Delta_m}$. Thus $\sup K_{Y/X}^\Delta \geq \sup K_{Y/X}^{\Delta_m} = \sup K_{m,Y/X}^- = K_{Y/X}^-$. On the other hand, since $K_{Y/X}^\Delta \leq K_{Y/X}^-$, $\sup K_{Y/X}^\Delta \leq K_{Y/X}^-$. \square

Since $K_{Y/X}^+ = \inf K_{m,Y/X}^+$ and $K_{Y/X}^\Delta \leq K_{m,Y/X}^+$, *a priori* there is no clear relation between $K_{Y/X}^\Delta$ and $K_{Y/X}^+$. Investigating such relation, in [CU12] we showed the following.

Lemma 2.1.5. *Let X be a normal variety, and let $f : Y \rightarrow X$ be a proper birational morphism of normal varieties. Let Δ be an effective divisor such that $K_X - \Delta$ is \mathbb{Q} -Cartier. Then*

$$K_{Y/X}^+ \leq K_{Y/X}^{-\Delta}.$$

Proof. We have to compare

$$K_{Y/X}^+ = K_Y + f^*(-K_X)$$

and

$$K_{Y/X}^{-\Delta} = K_Y + f_*^{-1}(-\Delta) - f^*(K_X - \Delta).$$

We have

$$\begin{aligned} K_{Y/X}^+ &= K_Y + f^*(-K_X) = K_Y + f^*(-K_X + \Delta - \Delta) = \\ &= K_Y + f^*(-K_X + \Delta) + f^*(-\Delta) = \\ &= K_Y - f^*(K_X - \Delta) + f^*(-\Delta). \end{aligned}$$

In order to obtain the desired inequality, we need to show that

$$f^*(-\Delta) \leq f_*^{-1}(-\Delta).$$

We claim that, if $-D$ is anti-effective \mathbb{Q} -divisor such that mD is integral,

$$f^*(-D) \leq \frac{f^{\natural}(-mD)}{m} \leq f_*^{-1}(-D).$$

The first inequality is by definition of pullback. To prove the second inequality, we can assume directly that D is an integral divisor, and show that $f^{\natural}(-D) \leq f_*^{-1}(-D)$. Notice

that, if D is a Weil divisor on a normal variety, $D \geq 0$ if and only if $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$. By duality, $-D \leq 0$ if and only if $\mathcal{O}_X(D) \supseteq \mathcal{O}_X$. Then, since $-D \leq 0$,

$$\mathcal{O}_Y \subseteq \mathcal{O}_X(D) \cdot \mathcal{O}_Y \subseteq (\mathcal{O}_X(D) \cdot \mathcal{O}_Y)^{\vee\vee} = \mathcal{O}_X(-f^{\natural}(-D)).$$

Therefore, $f^{\natural}(-D) = f_*^{-1}(-D) + F \leq 0$, where F is the exceptional component of $f^{\natural}(-D)$.

Therefore, it must be $F \leq 0$, and thus $f^{\natural}(-D) \leq f_*^{-1}(-D)$. \square

Corollary 2.1.6. *If $f : Y \rightarrow X$ is a proper birational morphism of normal varieties over an algebraically closed field,*

$$K_{Y/X}^+ = \inf K_{Y/X}^{-\Delta},$$

where Δ run among the effective divisors such that $K_X - \Delta$ is \mathbb{Q} -Cartier and $[\Delta] = 0$ (or among the compatible $(-K_X)$ -boundaries).

2.2 Discrepancies and singularities: definitions

In [CU12], following the work of [dFH09], and at the same time generalizing it, we introduced the following notation for the discrepancies.

Definition 2.2.1. *Let $Y \rightarrow X$ be a proper birational morphism with Y normal, and let F be a prime divisor on Y .*

(a) *For each integer $m \geq 1$, the m -limiting discrepancy of F with respect to X is*

$$a_m(F, X) := \text{ord}_F(K_{m,Y/X}^-).$$

(b) *The discrepancy of F with respect to X is*

$$a(F, X) := \text{ord}_F(K_{Y/X}^-).$$

(c) *The discrepancy⁺ of F with respect to X is*

$$a^+(F, X) := \text{ord}_F(K_{Y/X}^+).$$

We recall that, §1.2, if Δ is a boundary for K_X , we have

$$a(F, X, \Delta) := \text{ord}_F(K_{Y/X}^\Delta).$$

Remark 2.2.2. Lemma 2.1.4 can be rephrased as follows. Let $f : Y \rightarrow X$ be a proper birational morphism between normal varieties over an algebraically closed field, and let F be a divisor on Y . For any $m \geq 2$ there is an m -compatible boundary such that

$$a_m(F, X) = a(F, X, \Delta_m).$$

In particular

$$\begin{aligned} a(F, X) &= \sup\{a_m(F, X)\} = \\ &= \sup\{a(F, X, \Delta) \mid (X, \Delta) \text{ is a log pair, } \Delta \geq 0\} \end{aligned} \tag{2.2.3}$$

In [dFH09], de Fernex and Hacon used the m -limiting discrepancy to define log canonical and log terminal singularities, while they used the discrepancy⁺ to define canonical and terminal singularities. In [CU12] we gave the following definition.

Definition 2.2.3. *The variety X is said to satisfy condition $M_{\geq -1}$ (resp. $M_{> -1}$, resp. $M_{\geq 0}$, resp. $M_{> 0}$) if there is an integer m_0 such that $a_m(F, X) \geq -1$ (resp. > -1 , resp. ≥ 0 , resp. > 0) for every prime divisor F over X and for any positive multiple m of m_0 .*

Definition 2.2.4 ([dFH09, 7.1]). *The variety X has log canonical (resp. log terminal) singularities if it satisfies condition $M_{\geq -1}$ (resp., $M_{> -1}$).*

Definition 2.2.5 ([dFH09, 8.1]). *We say that X has canonical (resp. terminal) singularities if $a^+(F, X) \geq 0$ (resp. > 0) for all prime divisors F , exceptional over X .*

Remark 2.2.6. Since log terminal and log canonical singularities are defined using $K_{m,Y/X}^-$ and canonical and terminal with $K_{Y/X}^+$, there is no immediate relation between them. In [Urb12, 4.1], Urbinati gave an example of a singularity with canonical, but not log terminal singularities.

Trying to find the largest class of singularities where to apply results of the MMP, see §4.2, in [CU12] we gave the following definition.

Definition 2.2.7. *Let X be a normal variety. We say that X has log terminal⁺, or simply lt^+ , singularities if $a^+(F, X) > -1$ for all prime divisors F , exceptional over X .*

Finally, in [CU12], we introduced the notion of discrepancy.

Definition 2.2.8. *Let X be a normal variety. The discrepancy of X is given by*

$$\text{discrep}(X) := \inf_E \{a(E, X), E \text{ is an exceptional divisor over } X\}$$

(where E runs through all the irreducible exceptional divisors of all (proper) birational morphisms $Y \rightarrow X$).

The standard theory of discrepancies can be extended to this setting. One result that is different in this case is the following (see [KM98, 2.30]).

Lemma 2.2.9. *Let X be a normal variety, $f : Y \rightarrow X$ be a resolution of singularities and let Δ_Y be the \mathbb{Q} -divisor on Y such that*

$$K_Y + \Delta_Y = f^*(K_X), \quad f_*\Delta_Y = 0.$$

For any prime divisor F over X ,

$$a(F, X) \leq a(F, Y, \Delta_Y),$$

with equality if

- (a) F is a divisor on Y ,
- (b) K_X is \mathbb{Q} -Cartier, or
- (c) $\mathcal{O}_X(-m(K_X)) \cdot \mathcal{O}_Y$ is invertible for all sufficiently divisible m .

The above lemma can be formulated more generally in the sense of [dFH09], for pairs (X, Z) and assuming that $f : Y \rightarrow X$ is just a proper birational morphism of normal varieties.

Proof. The proof proceeds as in the log \mathbb{Q} -Gorenstein case. Let $g : V \rightarrow Y$ be a resolution with F a divisor on V , and let $h = f \circ g$. We have the diagram

$$\begin{array}{ccc} V & & \\ g \downarrow & \searrow h & \\ Y & \xrightarrow{f} & X. \end{array}$$

By 1.4.12,

$$(f \circ g)^*(K_X) = g^*f^*(K_X) + B = g^*(K_Y + \Delta_Y) + B,$$

where B is effective and g -exceptional. Therefore, if A' is the discrepancy of h with respect to X and A is the discrepancy of g with respect to (Y, Δ_Y) , then

$$A' = K_V - (f \circ g)^*(K_X) = K_V - g^*(K_Y + \Delta_Y) - B = A - B,$$

that is $A = A' + B$, with B effective and g -exceptional. Since the same computation holds for any resolution $g : V \rightarrow Y$, this proves the inequality in general. Since B is g -exceptional, (a) is immediate. Case (b) is a particular case of (c), and (c) follows again from 1.4.12 ($K_Y + \Delta_Y$ is \mathbb{Q} -Cartier, since Y is smooth). \square

Remark 2.2.10. In the \mathbb{Q} -Gorenstein case, we have the following ([KM98, 2.30]). If Δ_X is a boundary on X , $f : Y \rightarrow X$ is a proper birational morphism of normal varieties, and Δ_Y is a divisor on Y such that $K_Y + \Delta_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta_X)$ and $f_*\Delta_Y = \Delta_X$, then $\text{discrep}(X, \Delta_X) = \text{totaldiscrep}(Y, \Delta_Y)$, and this is the standard way of computing discrepancies. In our case, if $(Y, \Delta_Y) \rightarrow X$ are as in 2.2.9, we only have

$$\text{discrep}(X) \leq \text{totaldiscrep}(Y, \Delta_Y).$$

As in the \mathbb{Q} -Gorenstein case, we have the following result (see [KM98, 2.31(a)]).

Corollary 2.2.11. *Let X be a normal variety. Then $\text{discrep}(X) = -\infty$ or $-1 \leq \text{discrep}(X) \leq 1$.*

Proof. By blowing up a point of codimension 2 contained in the regular locus we have $\text{discrep}(X) \leq 1$. If $\text{discrep}(X) < -1$, let E be a prime exceptional divisor on a resolution $f : Y \rightarrow X$ such that $a(E, X) < -1$. Let Δ_Y be as in 2.2.9. Then E appears in Δ_Y with coefficient $-a(E, X)$, and $\text{discrep}(X) \leq \text{totaldiscrep}(Y, \Delta_Y) = -\infty$. \square

This implies that the case $\text{discrep}(X) = -1$ is the broadest class where it makes sense to define singularities using the discrepancies.

2.3 Discrepancies and singularities: relations

Firstly, let us point out that all these different notions agree with the usual ones in the \mathbb{Q} -Gorenstein case.

Lemma 2.3.1. *Let X be a normal \mathbb{Q} -Gorenstein variety. Then X satisfies condition $M_{\geq -1}$ (resp. $M_{> -1}$, resp. $M_{\geq 0}$, resp. $M_{> 0}$) if and only if X is log canonical (resp. Kawamata log terminal, resp. canonical, resp. terminal), if and only if $\text{discrep}(X) \geq -1$ (resp. > -1 , resp. ≥ 0 , resp. > 0).*

Secondly, some implications are immediate.

Lemma 2.3.2. *Let X be a normal variety. If it satisfies $M_{> 0}$, then it satisfies $M_{\geq 0}$. If it satisfies $M_{\geq 0}$, then it satisfies $M_{> -1}$. If it satisfies $M_{> -1}$, then it satisfies $M_{\geq -1}$.*

Essentially, we have the chain of implications

$$M_{> 0} \Rightarrow M_{\geq 0} \Rightarrow M_{> -1} \Rightarrow M_{\geq -1}.$$

Lemma 2.3.3. *Since $K_{m,Y/X}^- \leq K_{Y/X}^-$ (see 2.1.3), if X is a normal variety satisfying $M_{\geq -1}$ (log canonical), then $\text{discrep}(X) \geq -1$. Similarly if X satisfies $M_{> -1}$ (log terminal), then $\text{discrep}(X) > -1$; if it satisfies $M_{\geq 0}$, then $\text{discrep}(X) \geq 0$; if it satisfies $M_{> 0}$, then $\text{discrep}(X) > 0$.*

Lemma 2.3.4. *Let X be a normal variety. If X has (Kawamata) log terminal singularities, then it is lt^+ (the converse is true if X is \mathbb{Q} -Gorenstein).*

The first important theorem relating these different singularities was proven by de Fernex and Hacon. The version below is from [CU12].

Theorem 2.3.5. *Let X be a normal variety defined over an algebraically closed field of characteristic 0. Then X satisfies condition $M_{\geq -1}$ (resp. $M_{> -1}$, resp. $M_{\geq 0}$, resp. $M_{> 0}$) if and only if there is an effective boundary Δ such that (X, Δ) is log canonical (resp. Kawamata log terminal, resp. canonical, resp. terminal).*

Proof. For the $M_{\geq -1}$ and $M_{> -1}$ cases, this is [dFH09, 7.2]. The same proof given there proves the other cases. We repeat it for the convenience of the reader.

Let us first assume that there exists a boundary Δ on X giving the singularity type. Let m be an integer, such that $m(K_X + \Delta)$ is Cartier. By 2.1.3, for any proper birational morphism $f : Y \rightarrow X$, with Y normal, and any divisor F on Y , $K_{Y/X}^\Delta \leq K_{m,Y/X}^-$. Hence X will have singularities no worse than those given by the pair.

Conversely, let X satisfy condition $M_{\geq -1}$ (resp. $M_{> -1}$, resp. $M_{\geq 0}$, resp. $M_{> 0}$) and let m_0 be chosen as in 2.2.3. Let Δ_{m_0} be an m_0 -compatible boundary for a log resolution $f : Y_{m_0} \rightarrow X$ of $((X, \Delta_{m_0}), \mathcal{O}_X(-m_0K_X))$, see 1.4.14. For each exceptional divisor F on Y_{m_0} , $a_{m_0}(F, X) = a(F, X, \Delta_{m_0})$, which proves that (X, Δ_{m_0}) is log canonical (resp. log terminal, resp. canonical, resp. terminal). \square

Remark 2.3.6. Given the example of a canonical singularity not log terminal, see 2.2.6, one could ask the following. If X has canonical and log terminal singularities does there exists a boundary Δ on X such that (X, Δ) is canonical? Equivalently, if X has canonical singularities and satisfies $M_{> -1}$, will X satisfy $M_{\geq 0}$? Unfortunately, the answer to this question is negative, see 3.2.9.

De Fernex and Hacon asked the following purposely vague question.

Question 2.3.7. Is it possible to detect the log terminal singularities from $K_{Y/X}^-$?

An answer to question 2.3.7 over \mathbb{C} was given in [BdFFU13]. Once we fix X , we can think of the discrepancies $a_m(F, X)$ and $a(F, X)$ as functions on the space of divisorial valuations. Then, the question can be rephrased in the following terms: if $\inf_F a(F, X) > -1$ for all divisors F exceptional over X , is there an m such that $a_m(X, F) > -1$ for all F exceptional over X ? The basic idea of the proof is to extend the discrepancies on the space of real valuations, and to use the compactness of the space of real valuations (see [Ber90]) and a Dini-type theorem. Thus the point-wise convergence will imply an uniform convergence. Notice that such approach works for $a(F, X) > -1$ (and $a(F, X) > 0$), but will not work for $a(X, F) \geq 1$ (or $a(F, X) \geq 0$).

Let us state the aforementioned result. We can define the *log discrepancies* $A_m(\text{ord}_F) := a_m(F, X) + 1$ and $A(\text{ord}_F) := a(F, X) + 1$.

Theorem 2.3.8 ([BdFFU13, 3.1]). *Let X be a normal variety over \mathbb{C} , and let Val_X be the space of real valuations on X . There is a way to extend A and A_m ($m \geq 1$) to functions $A_m, A : \text{Val}_X \rightarrow \mathbb{R} \cup \{+\infty\}$ such that*

- (a) *A and A_m are homogeneous, lower semicontinuous functions on Val_X ;*
- (b) *$A = \sup_{m \geq 1} A_m$ on Val_X ;*
- (c) *A and A_m are convex and continuous on all the faces of Val_X^N ;*
- (d) *for each m there exists π such that A_m is affine on the faces of $\Delta_{\pi'}^N$, for each π' factoring through π ;*
- (e) *on Val_X^N , $A_m = \sup_{\pi} A_m \circ r_{\pi}^N$ and $A = \lim_{\pi} A \circ r_{\pi}^N$;*
- (f) *for each $\lambda \in \mathbb{R}$, $\{A_1 \leq \lambda\} \cap \text{Val}_X^N$ is compact, and A_m converges uniformly to A on this set.*

Moreover, such extensions are unique under the conditions (c)-(f).

Using the above extensions, Boucksom, de Fernex, Favre, and Urbinati prove the following theorems.

Theorem 2.3.9 ([BdFFU13, 4.1]). *Let X be a normal variety over \mathbb{C} . Then X has log terminal singularities (in the sense of [dFH09]) if and only if $A(\nu) > 0$ for all real valuations ν on X .*

In the particular case, when X is numerically Gorenstein, this implies the following.

Theorem 2.3.10 ([BdFFU13, 1.4 and 5.17]). *Let X be a normal variety over \mathbb{C} . Then X is \mathbb{Q} -Gorenstein and log terminal if and only if X is numerically Gorenstein and $a(X, F) > -1$ for all prime divisors F on some (equivalently, any) log resolution $Y \rightarrow X$.*

As corollary of 2.3.9 and 2.3.10, we can handle the cases $A \geq 1$ and $A > 1$.

Proposition 2.3.11. *Let X be a normal variety over \mathbb{C} . Then $A(\nu) > 0$ (resp. ≥ 1 , resp. > 1) for all $\nu \in \text{Val}_X$, if and only if there exists a boundary Δ on X , with $[\Delta] = 0$, such that (X, Δ) is klt (resp. canonical, resp. terminal).*

In the particular case of X numerically Gorenstein, then X is \mathbb{Q} -Gorenstein and log terminal (resp. canonical, resp. terminal) if and only if $a(X, F) > -1$ (resp. ≥ 0 , resp. > 0) for all prime divisors F on some (equivalently, any) log resolution $Y \rightarrow X$.

Proof. One direction is clear. If $A(\nu) > 0$, then by 2.3.9, X is log terminal, that is, it satisfies condition $M_{>-1}$. By 2.3.5, X is klt. By 1.1.8, $\mathcal{R}(X, -K_X) = \bigoplus_{m \geq 1} \mathcal{O}_X(-mK_X)$ is finitely generated. Thus, for $q \gg 0$ and any $m, n \geq 1$, $\mathcal{O}_X(-mqK_X) \cdot \mathcal{O}_X(-nqK_X) = \mathcal{O}_X(-q(m+n)K_X)$, see 1.1.6. This implies that, for $q \gg 0$ and $m \geq 1$, $a_{qm}(X, F) = a_q(F, X)$, which in turns implies that, for m sufficiently divisible, $a_m(X, F) = a(X, F)$. Hence, if $A(\nu) \geq 1 > 0$ (resp. $A(\nu) > 1 > 0$), X satisfies condition $M_{\geq 0}$ (resp. $M_{> 0}$). Again by 2.3.5 we have the statement. \square

What about the discrepancy $\text{discrep}(X)$? In light of 2.3.3, and in the spirit of a more traditional approach to singularity theory, Urbinati and I posed the following.

Question 2.3.12. *If X is a normal variety (defined over a field of characteristic 0), does $\text{discrep}(X) > -1$ imply that X is log terminal, in the sense of [dFH09]?*

Notice that the same proof as 2.3.11 gives the following.

Proposition 2.3.13. *Let us assume that question 2.3.12 has a positive answer, and let X be a normal variety over a field of characteristic 0. If $\text{discrep}(X) > -1$ (resp. ≥ 0 , resp. > 0), then there exists a boundary Δ on X , with $[\Delta] = 0$, such that (X, Δ) is klt (resp. canonical, resp. terminal).*

The only case left would be $\text{discrep}(X) = -1$. However, [Zha13] found an example of a cone singularity such that $\text{discrep}(X) = -1$ but X is not log canonical.

Unfortunately, 2.3.9 does not answer 2.3.12. Indeed, if $\text{discrep}(X) > -1$, then there exists $\varepsilon > 0$ such that, for each prime divisor F exceptional over X , $a(F, X) > -1 + \varepsilon$. Every divisorial valuation of X will be of the form $q\text{val}_F$, for some F and $q \in \mathbb{R}_{>0}$. This

a priori simply means that $A(\nu) > 0$ for all $\nu \in \text{DivVal}_X$. But then, by 1.3.9, this implies that $A(\nu) \geq 0$ for every $\nu \in \Delta_\pi^N$ (for every N, π), which in turn implies that $A(\nu) \geq 0$ on Val_X . Moreover, on each simplicial complex Δ_π^N , A is *convex*, by 2.3.8(c). Thus, even though $A(\nu) > 0$ on the vertices of Δ_π^N (which correspond to val_F for some F exceptional), $A(\nu)$ could very well be 0 at some point of the interior.

This suggests that a purely convex-geometric approach will not allow us to deduce 2.3.12 from 2.3.9. An alternative approach was the first reason why Urbinati and I introduced lt^+ singularities. Indeed, clearly, if $\text{discrep}(X) > -1$ then X has lt^+ singularities (since $K_{Y/X}^- \leq K_{Y/X}^+$). With Urbinati we showed, see 4.1.6, that if X has lt^+ singularities, then $\mathcal{R}(X, K_X)$ is finitely generated. This does not imply that $\mathcal{R}(X, -K_X)$ be finitely generated as well, but we were hoping that it would be enough to deduce the “stabilization” of the valuations of $\mathcal{O}_X(-mK_X)$, that is, that $a_m(F, X) = a(F, X)$ for m sufficiently divisible. However, the example of [Zha13] has lt^+ singularities; yet, if E is the exceptional locus of the blow-up of the vertex, $a_m(E, X) < -1$ for all m and $a(E, X) = 1$.

Chapter 3

EXAMPLES

That bad behaviors were to be expected when working with non- \mathbb{Q} -Gorenstein singularities had always been clear, but even more so after [Urb12], where the author showed an example of a canonical singularity which was not log terminal, and of a singularity where the canonical class had irrational valuations (in the latter case, the jumping numbers were also irrational). In this chapter we will focus on examples, to illustrate some of the pathologies that can happen in the study of non- \mathbb{Q} -Gorenstein singularities. We start with a series of questions about the behavior of resolutions and boundaries, and on how to detect the singularities. As it happens, all the questions will have negative answer. It is worth noticing that almost all the pathologies which we will discuss happen already on lt^+ singularities. Finally, we discuss in general the behavior of cone singularities.

The discussion in this chapter is based on [Chi13].

3.1 What could go wrong?

3.1.1 Computing discrepancies directly

If X is a \mathbb{Q} -Gorenstein variety, $f : Y \rightarrow X$ is a resolution, E is one of the components of the exceptional divisor which is mapped to a point, there is a standard technique to compute the discrepancy. For example, in the case that X is a cone over some smooth polarized variety (V, L) , and $f : Y \rightarrow X$ is the blow-up of the vertex, we can write $K_Y = f^*(K_X) + A$ (as \mathbb{Q} -divisors), where $A = aE$ is the discrepancy. Thus, $K_Y + E = f^*(K_X) + aE + E$. Restricting this identity to E , $(K_Y + E)|_E \sim K_E$ and $f^*(K_X)|_E \sim_{\mathbb{Q}} 0$ (E is a fiber), and we find $K_E \sim_{\mathbb{Q}} (a + 1)E|_E \sim_{\mathbb{Q}} -(a + 1)L$. We can use this to determine a , and thus $A = aE$. This procedure can fail in several steps if we use the pullback of [dFH09].

Let D be a Weil divisor on a normal variety X , and F be the fiber of a birational morphism $f : Y \rightarrow X$. Recall that the pullback is defined as a limit $f^*(D) = \inf f^{\natural}(mD)/m$.

If D is not \mathbb{Q} -Cartier, and F is a fiber of a (birational) morphism $f : Y \rightarrow X$, in general we will not be able to restrict $f^*(D)$ to F . Even if we can (for example if F is a normal reduced Cartier divisor) there is no reason to suspect that $f^*(D)|_F \sim_{\mathbb{Q}} 0$.

(F1) If each $f^{\natural}(mD)/m|_F \sim_{\mathbb{Q}} \Gamma_m$, for some divisor Γ_m , it does not make sense to talk about an infimum of the Γ_m s, as they are defined as \mathbb{Q} -linear classes (see remark 1.4.7). So, we cannot compute the restriction of the pullback as a limit. We point out, that it will not be, in general, $f^{\natural}(mD)/m|_F \sim_{\mathbb{Q}} 0$.

(F2) We should not expect that, in general, $f^*(D)|_F \equiv 0$ or $f^*(D)|_F \sim_{\mathbb{R}} 0$.

(F3) Since we are working with a limiting process, and linear equivalence is not preserved in the limiting process, we should expect that $f^*(D)|_F \sim_{\mathbb{R}} \Gamma$ or $f^*(D)|_F \equiv \Gamma$ will encode very different meanings.

We give an example, 3.2.1, of a cone X , where, if $f : Y \rightarrow X$ is the blow-up of the vertex, with exceptional divisor E , $f^*(K_X)|_E$ is not numerically trivial (thus, $f^*(K_X)|_E \not\sim_{\mathbb{R}} 0$).

We also give an example, 3.2.8, of a cone X and a divisor T' on X , such that, if $f : Y \rightarrow X$ is the blow-up of the vertex, with exceptional divisor E , $f^*(T')|_E \equiv 0$ but $f^*(T')|_E \not\sim_{\mathbb{R}} 0$.

3.1.2 Computing discrepancies with boundaries

Another approach to the non- \mathbb{Q} -Gorenstein case is to use boundaries, and work in a log setting. This approach is the most commonly used, and we saw that, for each m , there exists a compatible boundary (X, Δ_m) for some resolution f of (X, Δ_m) , 1.4.14 (let us assume characteristic 0). Let us recall that an effective boundary Δ_m , with $[\Delta_m] = 0$, is *weakly compatible* for f if

$$K_Y - \frac{1}{m} f^{\natural}(mK_X) = K_{m,Y/X}^- = K_{Y/X}^{\Delta_m} = K_Y + f_*^{-1} \Delta_m - f^*(K_X + \Delta_m),$$

and it is *compatible* if, moreover, f is a log resolution of the pair (X, Δ_m) , 1.4.13. If Δ_m is compatible, then the singularities of X can be detected from the discrepancy of (X, Δ_m) with respect to f . More precisely, since f is a log resolution of (X, Δ_m) , the discrepancy

$\text{discrep}(X, \Delta_m)$ can be computed directly, 1.2.4. In turn, this will tell us if X satisfies condition $M_{\geq -1}$ (or $M_{> -1}$, or $M_{\geq 0}$, or $M_{> 0}$), 2.3.5.

In the proof of 1.4.14, the boundary and the resolution depend on each other. Practically, if we are simply given a boundary Δ_m weakly compatible for some map g , and we take a log resolution f of (X, Δ_m) , the boundary Δ_m might not be weakly compatible for f . What if we are given a resolution $f : Y \rightarrow X$? For each m (or for some m), can we find a boundary Δ_m such that $K_{m, Y/X}^- = K_{Y/X}^{\Delta_m}$ and f is a log resolution of (X, Δ_m) ? In addition, if X is a cone, can we choose such Δ_m to be a cone as well?

In §3.2.3, we give an example of a blow-up of a vertex of a cone $f : Y \rightarrow X$, with Y smooth, where $K_{m, Y/X}^- > -1$ (so as to suggest that X has log terminal singularities), but where for any choice of compatible boundary Δ on X (with Δ a cone), f is not a resolution of the pair (X, Δ) . In particular, we can conclude that X has lt^+ singularities, but we cannot conclude that it has log terminal singularities (in the sense of [dFH09]).

If X is a cone, it is still not known whether the compatible boundaries of 1.4.14 can be chosen to be cones.

3.1.3 Boundaries and lt^+

We saw that log terminal singularities can be detected as a “limit” for the singularities of all the pairs (X, Δ) , with Δ a boundary. Indeed, X has log terminal singularities if and only if there is a boundary Δ on X such that (X, Δ) is klt, 2.3.5. Moreover, log terminal singularities can be detected by valuations of $K_{Y/X}^-$, for all possible resolutions $Y \rightarrow X$, 2.3.9, and, for each $Y \rightarrow X$, $K_{Y/X}^- = \sup K_{Y/X}^{\Delta}$, 2.1.4. Finally, as we will see in §4.3, there is a multiplier ideal $\mathcal{J}(X)$ on X such that X has log terminal singularities if and only if $\mathcal{J}(X) = \mathcal{O}_X$ and $\mathcal{J}(X) = \sum_{\Delta} \mathcal{J}(X, \Delta)$ (see 4.3.3, 4.3.4, and 4.3.5). It is a natural question whether similar results hold for lt^+ singularities.

We give an example, 3.2.7, where $K_{m, Y/X}^+ > -1$ for each m , but $K_{Y/X}^+ = -1$. In particular this implies that the multiplier ideal in 4.3.6 had to be constructed by directly looking at $K_{Y/X}^+$ and not by doing a limiting process on the $K_{m, Y/X}^+$, see 4.3.9.

3.2 Examples

In this section and the next, for simplicity, we work over a field of characteristic 0.

For the first example, we write the entire computation. The other examples use the same ideas.

3.2.1 First example

Example 3.2.1. Let us consider the $(1, 2)$ embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in \mathbf{P}^5 ; let us call it Z . Let $X' = C(Z) \subseteq \mathbf{A}^6$ be the affine cone over it, and let $X = \overline{C(Z)} \subseteq \mathbf{P}^6$ be the projective cone. Notice that the singularities of affine and projective cones are the same.

Let us fix some notation. Let $f_1 = \mathbf{P}^1 \times \{pt\}$ and $f_2 = \{pt\} \times \mathbf{P}^1$ and let $L \equiv f_1 + 2f_2$ be the ample divisor giving the embedding. The divisor group of $\mathbf{P}^1 \times \mathbf{P}^1$ is generated by f_1, f_2 and we have $f_1 \cdot f_2 = 1, f_1^2 = f_2^2 = 0$.

Let $f : Y \rightarrow X$ be the blow-up of the vertex of the cone, which is a log resolution of X , and let E be the exceptional divisor.

Lemma 3.2.2. *The cone X in 3.2.1 is normal and non- \mathbb{Q} -Gorenstein.*

Proposition 3.2.3. *The cone X in 3.2.1 has $K_{Y/X}^- = 0$. Moreover, for all $m \geq 1$, $K_{m, Y/X}^- = 0$. In particular, X has canonical singularities.*

Lemma 3.2.4 ([Urb12, proof of 3.3]). *Let V be a smooth variety, and let L be a very ample line bundle on V determining an embedding $V \hookrightarrow \mathbf{P}^n$, and let X be the projective cone over it. Let us assume that X is normal. Let $f : Y \rightarrow X$ be the blow-up of the vertex, with exceptional divisor E . Then*

$$\text{ord}_E(K_{Y/X}^-) = -(1 + t), \quad (3.2.1)$$

where

$$t = \inf\{s \in \mathbb{Q} \mid sL - K_V \sim_{\mathbb{Q}} \Delta, \Delta \text{ is effective}\}.$$

Proof. This is a particular case of 3.3.5, for $D = K_V - L$, as $K_X \sim C_{K_V} - C_L$. \square

Proof of 3.2.3. Notice that, since $Z \cong \mathbf{P}^1 \times \mathbf{P}^1$, $K_Z \sim -2f_1 - 2f_2$. For each $s \in \mathbb{R}$,

$$sL - K_Z \sim s(f_1 + 2f_2) - (-2f_1 - 2f_2) = (s + 2)f_1 + (2s + 2)f_2,$$

which is effective as long as $s \geq -1$. Therefore, with the notation of the lemma, $t = -1$ and $\text{ord}_E(K_{Y/X}^-) = -(1 - 1) = 0$. Thus $K_{Y/X}^- = 0$, and X has limiting canonical singularities.

Notice that, for all $m \geq 1$, and each Δ such that $m(K_X + \Delta)$ is Cartier,

$$K_{Y/X}^\Delta \leq K_{m,Y/X}^- \leq K_{Y/X}^- = 0.$$

If $\Delta = C_{f_1}$, the cone over f_1 , then $K_Z + f_1 \sim -2f_1 - 2f_2 + f_1 = -f_1 - 2f_2 = -L$. By [Kol08, 70.(1)], $K_X + \Delta$ is Cartier, and therefore

$$K_{Y/X}^\Delta \leq K_{m,Y/X}^-,$$

for all m . As discussed in the proof of the lemma, we have that $f_1 = (-1)L - K_Z$ and $K_{Y/X}^\Delta = -(1 - 1)E = 0$. Thus, for all $m \geq 1$, we have the chain of inequalities

$$0 = K_{Y/X}^\Delta \leq K_{m,Y/X}^- \leq K_{Y/X}^- = 0,$$

which concludes the computation. \square

Corollary 3.2.5. *The cone in 3.2.1 satisfies the condition $M_{\geq 0}$, that is, there exists an effective boundary on X such that (X, Δ) is canonical.*

Proof. The basis of the cone is $\mathbf{P}^1 \times \mathbf{P}^1$, and each effective divisor on $\mathbf{P}^1 \times \mathbf{P}^1$ is \mathbb{Q} -linearly equivalent to a smooth one. \square

Corollary 3.2.6. *With the notation of 3.2.1, $f^*(K_X)|_E \sim_{\mathbb{Q}} -f_1$, which is neither \mathbb{R} -linearly equivalent nor numerically equivalent to 0.*

Proof. We have that $K_{Y/X}^- = 0$, so that $K_Y = f^*(K_X)$. Then,

$$(K_Y + E)|_E \sim_{\mathbb{Q}} K_E \sim -2f_1 - 2f_2$$

($E \cong Z \cong \mathbf{P}^1 \times \mathbf{P}^1$) and

$$(f^*(K_X) + E)|_E \sim_{\mathbb{Q}} f^*(K_X)|_E + E|_E \sim_{\mathbb{Q}} f^*(K_X)|_E - L \sim f^*(K_X)|_E - f_1 - 2f_2.$$

Therefore, $f^*(K_X)|_E \sim_{\mathbb{Q}} -f_1$. \square

3.2.2 Three more examples

With the techniques of the previous example, we can study three more examples. The computations are the same as before, and thus left to the reader.

Example 3.2.7. Let $\mathcal{L} = \mathcal{O}_{\mathbf{P}^1}(3)$, and let $\pi : V = \text{Spec}_{\mathbf{P}^1} S(\mathcal{L}) \rightarrow \mathbf{P}^1$. Then V is an elliptic surface, and $K_V \sim f$, where f denotes the fiber [Mir89, III.1.1]. Since the sheaf of (local) sections of π is \mathcal{L} [Har77, Ex II.5.18(2)], which has global sections, there exists a section $s : \mathbf{P}^1 \rightarrow V$. Then $\text{Pic}(V)/\text{Pic}^0(V) \cong \text{NS}(V)$, $\text{Pic}^0(V) \cong \text{Pic}^0(\mathbf{P}^1)$, $\text{Pic}(V)/\text{Pic}(C) \cong \text{NS}(V)/\mathbb{Z}\bar{f}$ [Mir89, VII.1.1 and VII.1.3]. Moreover, if V_η is the general fiber of π , $\text{Pic}(V) \rightarrow \text{Pic}(V_\eta)$ is surjective, with kernel generated by classes of vertical divisors [Mir89, VII.1.5]. Notice that, in particular, since $\text{Pic}(\mathbf{P}^1) \cong \mathbb{Z}$, the kernel of the restriction $\text{Pic}(V) \rightarrow \text{Pic}(V_\eta)$ is generated by the class of f .

If we choose L on V to be $L \sim f + \sum e_i P_i$, where $\sum e_i P_i$, $e_i > 0$, is ample on the elliptic curve V_η , L will be ample on V . Then, as long as $s \geq 0$, $sL + K_V \sim_{\mathbb{Q}} D \geq 0$. However, as soon as $s < 0$, $sL + K_V \sim_{\mathbb{Q}} (s+1)f - \sum e_i P_i$ which cannot be linearly equivalent to an effective divisor. Thus,

$$K_{Y/X}^+ = -1.$$

On the other hand, with similar computations to the ones in [Zha13, 4.9], $K_{m,Y/X}^+ > -1$ for all m .

Example 3.2.8. Let (\mathcal{E}, O) be an elliptic curve with an embedding in \mathbf{P}^n such that the cone over it is normal, and let X be the projective cone over \mathcal{E} . Let P be a point on \mathcal{E} with infinite order, and let $T = P - O$. Then T is non-torsion, but $T \in \text{Pic}^0(\mathcal{E})$, $T \equiv 0$. Let $f : Y \rightarrow X$ be the blow-up of the vertex of the cone, with exceptional divisor E , and C_T be the cone over T in X . The computation in 3.3.7 shows that $f^*C_T = f_*^{-1}C_T$, so that $f^*C_T|_E = T \equiv 0$ but $f^*(C_T)|_E \not\sim_{\mathbb{R}} 0$.

The cone over an elliptic curve is not log terminal, however, we can use the above idea to give an example of such a phenomenon on an lt^+ variety. For example, if X is the cone over $\mathcal{E} \times \mathbf{P}^1$ in [Urb12, 4.1], then we can consider the cone over T' , where $\mathcal{O}_X(T') = \mathcal{O}_{\mathcal{E}}(T) \boxtimes \mathcal{O}_{\mathbf{P}^1}$ and $T \in \text{Pic}^0(\mathcal{E})$. If $f : Y \rightarrow X$ is the blowup of the vertex of X , with exceptional divisor

E , we have an example of a divisor $C_{T'}$ on a variety with lt^+ singularities, such that $f^*C_{T'}|_E = T' \equiv 0$ but $f^*(C_{T'})|_E \not\sim_{\mathbb{R}} 0$.

Example 3.2.9. The formula in 3.3.5 gives us a nice way of creating examples. Let V be the $(2, 3)$ embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in \mathbf{P}^n , and let X be the projective cone over it. It can be checked that X is normal. The above mentioned formula gives

$$K_{Y/X}^- = -\frac{1}{3}E, \quad K_{Y/X}^+ = 0,$$

giving an example of a singularity log terminal and canonical (in the sense of [dFH09]), but with $K_{Y/X}^-$ less than 0, hence not admitting an effective boundary Δ such that (X, Δ) is canonical (in the usual sense).

3.2.3 Last example

Finally, we give an example of a cone over a smooth variety where by simply looking at boundaries that are cones and at the blow-up of the vertex, we cannot determine the singularity.

First we identify the hypothesis that we need.

Example 3.2.10. Let V be a variety, Δ be a divisor on V , and n a positive integer such that

- (a) V is smooth Fano;
- (b) Δ is effective and, for each $m \geq 0$, all the divisors in the linear system $|m\Delta|$ have singularities worst than normal crossing;
- (c) $-nK_V - \Delta$ is very ample and, for each $\varepsilon > 0$,

$$\left| -\left(\frac{1}{n} + \varepsilon\right)\Delta + n\varepsilon K_V \right| = \emptyset.$$

Finding such a Fano variety in low dimension seems a hard task. We could not find any such examples in dimension 3 or less. The example we found is in dimension 4. We will discuss this later. Now we show why the above hypothesis are useful.

Proposition 3.2.11. *Let V , Δ and n as in 3.2.10, and let $L := -nK_V - \Delta$, which is very ample. Let X be the cone over (V, L) , and let $f : Y \rightarrow X$ be the blow-up of the vertex. Then $K_{Y/X}^- > -1$, and for $1 \nmid m$, for each Δ_m such that Δ_m is a cone over a divisor in V and $K_{m,Y/X}^- = K_{Y/X}^{\Delta_m}$, f is not a resolution of the pair (X, Δ_m) .*

Corollary 3.2.12. *If X is as in the above proposition, although $K_{Y/X}^- > -1$, we can not say what are the singularities of the variety X .*

Proof. Notice that condition (a) is used to guarantee the existence of a positive n such that $-nK_V - \Delta$ is very ample.

As in 3.2.4, $K_{Y/X}^- = (-1 - t)E$, where $t = \inf\{s \mid sL - K_V \sim_{\mathbb{Q}} \Gamma \geq 0\}$. Condition (c) implies that $t = -\frac{1}{n}$, so that $K_{Y/X}^- = (-1 + \frac{1}{n})E > -E$.

If we consider $s = -\frac{1}{n}$, then $sL - K_V = \frac{1}{n}\Delta$, so that, for $1 \nmid m$, $m(K_V + \frac{1}{n}\Delta)$ is Cartier, and the cone over Δ is a boundary. Thus, for $1 \nmid m$, $K_{m,Y/X}^- = K_{Y/X}^-$. Let C_{Γ_m} be an effective boundary computing the discrepancy, and a cone over a divisor Γ_m in V . The divisor Γ_m will have to be \mathbb{Q} -linearly equivalent to Δ (or to some multiple), and thus a singular non-nc divisor, by condition (b). But then the blow-up of the vertex is not a log resolution of the pair (X, C_{Γ_m}) . \square

Now we discuss of how to realize such a Fano variety (and such Δ). We give two examples. The first one is a simplified version of the second one, which is from [AW98].

Example 3.2.13. Let $V_1 \rightarrow \mathbf{P}^4$ be the blow-up of a point. Let $S_0 \cong \mathbf{P}^3$ be the exceptional divisor. Notice that V_1 has a natural structure of \mathbf{P}^1 -bundle over \mathbf{P}^3 , $V_1 \cong \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1))$. Let H be the pullback on V_1 of an hyperplane section in \mathbf{P}^3 . Then $L = 2H + S_0$ is basepoint-free. Let $B \in |2L|$ be a smooth general member, and let $V \rightarrow V_1$ be the double cover ramified over B . There is a natural map $V \rightarrow \mathbf{P}^4$, which admits Stein factorization $V \rightarrow Z \rightarrow \mathbf{P}^4$. The first map in the Stein factorization is a divisorial contraction, while the second map is generically $2 : 1$. Let E be the exceptional divisor of $V \rightarrow Z$.

It is just a computation to check that V is a smooth Fano variety. Explicitly, K_V is the pullback of $-3H - 2S_0$.

We claim that we can choose a smooth B such that E is singular. Indeed, E is the double cover of S_0 ramified over $B \cap S_0$. The divisor $B \cap S_0 \in |\mathcal{O}_{\mathbf{P}^3}(2)|$, and we can choose it to be a degree 2 cone of equations $x_0^2 + x_1^2 = x_2^2$. By [AW98, 3.5.2], this can be extended to a smooth divisor B on V . By direct computation, as in [EV92, 3.13], we see that E is still singular (with equation $t^2 + x_2^2 = x_0^2 + x_1^2$).

Since E is an exceptional divisor, for any $m \geq 0$, $|mE| = \{mE\}$.

Condition of (c) is the last thing to be verified. This again can be directly checked. We have that $L = -nK_V - E$ is very ample for $n \geq 2$. For the last condition of (c), the key point is to notice that, if $D \sim_{\mathbb{Q}} -(\frac{1}{n} + \varepsilon)E + n\varepsilon K_V$, then $f_*D = -\alpha S_0 - \beta H$, with $\alpha, \beta > 0$, and thus f_*D can not be effective. But then D cannot be effective.

In conclusion, V and $\Delta := E$ satisfy the conditions of 3.2.10.

Example 3.2.14 ([AW98, 3.5.4]). Let $V_1 \rightarrow \mathbf{P}^3$ be the blow-up of \mathbf{P}^3 at one point, and let S_0 be the exceptional divisor of the blow-up. The variety V_1 has a \mathbf{P}^1 -bundle structure $V_1 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(1)) \rightarrow \mathbf{P}^2$; let H be the pullback via this map of the line. Let Y be the product $V_1 \times \mathbf{P}^1$, with projections p_1 and p_2 . Let $L := p_1^*(S_0 + 2H)$, and let B be a smooth divisor in $|2L|$. As above, we can consider the $2 : 1$ cover $V \rightarrow Y$, ramified over B , and consequently the Stein factorization $V \rightarrow Z \rightarrow \mathbf{P}^3 \times \mathbf{P}^1$. The morphism $V \rightarrow Z$ is a divisorial contraction of a divisor E which is mapped to \mathbf{P}^1 . From the theory of 3-dimensional good contractions, [Mor82], we know that the generic fiber of $E \rightarrow \mathbf{P}^1$ is either a smooth quadric or a quadric cone. In the latter case, with a similar computation to the one above, it can be checked that the pair $(V, \Delta := E)$ satisfies the conditions of 3.2.10.

3.3 Cones

Let us recall that, for simplicity, we are working over a field of characteristic 0.

Lemma 3.3.1. *Let $f : Y \rightarrow X$ be a proper birational morphism between normal varieties, with Y smooth. Let D be a Weil divisor on X , and let E be any normal component of the exceptional divisor which is also a fiber (if it exists). The restriction of $f^*(D)$ to E is numerically antieffective.*

Proof. By 1.4.14,

$$f^*(D)|_E = \left(\inf_m (f^*(D + \Delta_m) - f_*^{-1}\Delta_m) \right)|_E,$$

where Δ_m are weak m -compatible D -boundaries, which are effective by construction. For each m ,

$$(f^*(D + \Delta_m) - f_*^{-1}\Delta_m)|_E \sim_{\mathbb{Q}} -f_*^{-1}\Delta_m|_E.$$

Since $\Delta_m \geq 0$ and E is not contained in the support of $f_*^{-1}\Delta_m$, $-f_*^{-1}\Delta_m|_E \leq 0$. Taking the limit, we get the desired result. \square

Remark 3.3.2. As discussed in (F3), we do not know that the restriction to a fiber of $f^*(D)$ will be \mathbb{R} -linearly equivalent to an antieffective divisor.

Example 3.3.3 ([dFH09, 2.3]). Let $X \subseteq \mathbf{A}^3$ be the cone over a conic, $X = \{x^2 + y^2 = z^2\}$, let $f : Y \rightarrow X$ be the blow-up of the vertex, with exceptional divisor E . Let L be a line passing through the origin. Notice that L is \mathbb{Q} -Cartier, but not Cartier. In this case $f^{\natural}(L) = f_*^{-1}L + E$, so that $f^{\natural}(L)|_E \sim_{\mathbb{Q}} -pt$, which is not zero (even though L is \mathbb{Q} -Cartier).

A word of caution is required. The behavior in general of the restriction of the pullback of a Weil divisor to a fiber is extremely unclear. Even assuming that we are working in the case of a resolution, the positivity of the restriction is not a given. In the previous 3.3.1, for example, a key point is that E being a divisor, it cannot be contained in the support of $f_*^{-1}\Delta_m$. In this particular case we see that the restriction of the fibers of the pullback is “negative”. More evidence of this is given by the construction of the *nef envelope* $\text{Env}_X(-D)$ by [BdFF12], see 1.5.3. Such b -divisor is X -nef, 1.5.4, and $\text{Env}_X(-D)_{\pi} = -\pi^*(D)$, 1.5.5.

However, this “negativity” of the pullback does not happen when we consider the restriction to fibers which are not themselves divisors. Let D be a Weil divisor on X such that $\mathcal{R}(X, D)$ is finitely generated, and let $f : Y = \text{Proj}_X \mathcal{R}(X, D) \rightarrow X$. This is the situation of flipping contractions, for example. In this case f is small and $f_*^{-1}(D) = f^{\natural}(D) = f^*(D)$ is \mathbb{Q} -Cartier and f -ample. Thus, for any curve C in the fiber of f , $f^*(D) \cdot C > 0$.

Theorem 3.3.4. *Let X be a (projective) cone over a smooth projective variety, and let us assume that X is normal; let $f : Y \rightarrow X$ be the blow-up of the vertex, and let E be the*

exceptional divisor of f . Let D be a Weil divisor on X . Then $f^*(D)|_E \sim_{\mathbb{R}} 0$ if and only if D is \mathbb{Q} -Cartier.

Proof. Let us fix the notation. Let V be a smooth variety, let L be a very ample line bundle on V giving an embedding $V \hookrightarrow \mathbf{P}^n$, and let $X \subseteq \mathbf{P}^{n+1}$ be the projective cone over it. We are assuming that X is normal. We have that $V \subseteq X$ as section at infinity.

Let D be a Weil divisor in X . Since X is a projective cone, by [Har77, Ex. II.6.3], $D \sim C_{D|_V}$ the cone over the restriction of D to V . The pullback of [dFH09] coincides with the usual pullback for Cartier divisors, thus it preserves linear equivalence. Hence $f^*(D) \sim_{\mathbb{Q}} f^*(C_{D|_V})$, and $f^*(D)|_E \sim_{\mathbb{R}} f^*(C_{D|_V})|_E$. Since the property of being \mathbb{R} -Cartier is also preserved by linear equivalence, it is enough to show that, if D is a divisor on V , $f^*(C_D)|_E \sim_{\mathbb{R}} 0$ if and only if $D \sim_{\mathbb{Q}} rL$, for some $r \in \mathbb{Q}$ (which is the only case in which C_D can be \mathbb{Q} -Cartier, by [Kol08, 70.(1)]).

Claim 3.3.5. *Let D be a divisor on V and let C_D be the cone over it in X . Then*

$$f^*(C_D) = f_*^{-1}C_D + tE, \quad (3.3.2)$$

where

$$t = \inf\{s \mid sL - D \sim_{\mathbb{Q}} \Delta, [\Delta] = 0\}.$$

Proof of claim 3.3.5. Let D be a divisor on V and let C_D be the cone over it in X . Then

$$\begin{aligned} f^*(C_D) &= \inf \left\{ (f^*(C_D + \Gamma) - f_*^{-1}\Gamma) \mid C_D + \Gamma \text{ is } \mathbb{Q}\text{-Cartier}, \Gamma \geq 0, [\Gamma] = 0 \right\} \leq \\ &\leq \inf \left\{ (f^*(C_D + C_\Delta) - f_*^{-1}C_\Delta) \mid C_D + C_\Delta \text{ is } \mathbb{Q}\text{-Cartier}, \right. \\ &\quad \left. C_\Delta \geq 0, [C_\Delta] = 0 \right\}. \end{aligned}$$

Let C_Δ be as above. Then $\Delta \geq 0$ and $[\Delta] = 0$. Since $C_D + C_\Delta$ is \mathbb{Q} -Cartier, there exists $s \in \mathbb{Q}$ such that $\Delta \sim_{\mathbb{Q}} sL - D$. If $\Delta \sim_{\mathbb{Q}} sL - D$, then

$$\begin{aligned} f^*(C_D + C_\Delta) - f_*^{-1}C_\Delta &\sim_{\mathbb{Q}} f^*(C_{sL}) - f_*^{-1}C_{sL-D} = f_*^{-1}C_{sL} + sE - f_*^{-1}C_{sL-D} = \\ &= f_*^{-1}C_D + sE. \end{aligned}$$

Therefore,

$$\text{val}_E(f^*(C_D)) \leq \inf\{s \mid sL - D \sim_{\mathbb{Q}} \Delta, \Delta \geq 0\}.$$

Let Γ be a divisor on X with $\Gamma \geq 0$, $C_D + \Gamma$ \mathbb{Q} -Cartier, and

$$f^*(C_D + \Gamma) - f_*^{-1}\Gamma = f_*^{-1}C_D + kE,$$

with $k \in \mathbb{Q}$. Let $\Delta = f_*^{-1}\Gamma|_E$, which we can think of as a divisor on V , since $E \cong V$. Then, $\Delta \geq 0$ and

$$\Delta = f_*^{-1}\Gamma|_E \sim_{\mathbb{Q}} (-kE - f_*^{-1}C_D)|_E \sim_{\mathbb{Q}} kL - D,$$

which means that $C_D + C_\Delta$ is \mathbb{Q} -Cartier. With the same computation of above,

$$f^*(C_D + C_\Delta) - f_*^{-1}C_\Delta = f_*^{-1}C_D + kE = f^*(C_D + \Gamma) - f_*^{-1}\Gamma.$$

Therefore, if

$$t = \inf\{s \mid sL - D \sim_{\mathbb{Q}} \Delta, [\Delta] = 0\},$$

then

$$\text{val}_E(f^*(C_D)) = t,$$

which means that

$$f^*(C_D) = f_*^{-1}C_D + tE.$$

□

Restricting the above identity to E , we find

$$f^*(C_D)|_E \sim_{\mathbb{R}} -tL + D = -(tL - D), \quad (3.3.3)$$

which is 0 if and only if $D \sim_{\mathbb{R}} tL$. Since D and L are a integral Weil divisors, it means that $t \in \mathbb{Q}$, and that the condition is equivalent with C_D being \mathbb{Q} -Cartier. □

The next result is similar to [BdFF12, 2.29].

Corollary 3.3.6. *Let X be a (projective) cone over a smooth projective polarized variety (V, L) , and let us assume that X is normal; let $f : Y \rightarrow X$ be the blow-up of the vertex, with exceptional divisor E . Let D be a Weil divisor on X . The following are equivalent*

- (a) $f^*(-D) = -f^*(D)$ (as \mathbb{R} -Weil divisors);

(b) $f^*(D)|_E \equiv 0$;

(c) $D \sim_{\mathbb{Q}} C_{\Delta}$ is the cone over a divisor Δ on V such that $\Delta \equiv rL$, for some $r \in \mathbb{Q}$.

In particular, in any of the above cases, $f^*(D)$ is a \mathbb{Q} -divisor.

Proof. Notice that, if $\Delta = D|_V$, then $D \sim C_{\Delta}$, by [Har77, Ex. II.6.3]. Since all the conditions in the above statement are invariants under \mathbb{Q} -linear equivalence, we can directly assume that $D = C_{\Delta}$, with Δ a divisor on V .

(a) \Rightarrow (b) By 3.3.1, $f^*(D)|_E$ is numerically antieffective. On the other hand, $f^*(D)|_E \equiv -f^*(-D)|_E$ is numerically effective. Thus, it must be numerically trivial.

(b) \Rightarrow (c) Let $D = C_{\Delta}$ and $f^*(D) = f_*^{-1}C_{\Delta} + tE$. Then, $0 \equiv f^*(D)|_E \equiv \Delta - tL$. This means that $\Delta \equiv tL$. Since they are both divisors, $t \in \mathbb{Q}$.

(c) \Rightarrow (a) Let $D = C_{\Delta}$ and $\Delta \equiv rL$, with $r \in \mathbb{Q}$. It is enough to prove that

$$f^*(D) = f_*^{-1}D + rE \tag{3.3.4}$$

(the same r). If this is true, then $-\Delta \equiv -rL$ and

$$f^*(-D) = f_*^{-1}(-D) - rE = -f_*^{-1}D - rE = -(f_*^{-1}D + rE) = -f^*(D).$$

Notice that the formula (3.3.4) shows that, in this case, $f^*(D)$ is a \mathbb{Q} -divisor. In order to prove (3.3.4), we can show that, if $\Delta \equiv 0$, then

$$f^*(C_{\Delta}) = f_*^{-1}C_{\Delta}.$$

Let $\Delta \equiv rL$ and $T = \Delta - rL \equiv 0$, and let us assume that $f^*(C_T) = f_*^{-1}C_T$, as claimed above. Then

$$f^*(C_{\Delta}) = f^*(C_{T+rL}) = f^*(C_T + C_{rL}) = f^*(C_T) + f^*(C_{rL}),$$

since C_{rL} is \mathbb{Q} -Cartier. In turn,

$$f^*(C_T) + f^*(C_{rL}) = f_*^{-1}C_T + f_*^{-1}(C_{rL}) + rE = f_*^{-1}C_{\Delta} + rE,$$

as desired (the strict transform is linear). It remains to show the following claim.

Claim 3.3.7. *Let $T \equiv 0$. Then $f^*(C_T) = f_*^{-1}C_T$.*

Proof of claim 3.3.7. By 3.3.5, $f^*(C_T) = f_*^{-1}(C_T) + tE$ with

$$t = \inf\{s \mid sL - T \sim_{\mathbb{Q}} \Delta, [\Delta] = 0\}.$$

For each $s \in \mathbb{Q}$, $s > 0$, since L is very ample and $T \equiv 0$, $sL - T \equiv sL$ is ample. Therefore $sL - T \sim_{\mathbb{Q}} \Delta$, $\Delta \geq 0$. On the other hand, for each $s \in \mathbb{Q}$, $s < 0$, $sL - T \equiv sL$ is antiample, and thus, $sL - T \sim_{\mathbb{Q}} \Delta$, $\Delta \leq 0$. Therefore,

$$t = \inf\{s \mid sL - T \sim_{\mathbb{Q}} \Delta, \Delta \geq 0\} = 0.$$

□

□

Remark 3.3.8. The claim 3.3.7 does not give a necessary condition, as example 3.2.1 shows. With the notation of that example, we have a divisor $T = -2f_2$ numerically non-trivial such that $f^*(C_T) = f_*^{-1}C_T$. This can be easily checked by direct computation.

The previous corollary and [BdFF12, 2.29] give

Corollary 3.3.9. *Let X be a (projective) cone over a smooth projective polarized variety (V, L) , and let us assume that X is normal; let $f : Y \rightarrow X$ be the blow-up of the vertex. Let D be a Weil divisor on X . Then $f^*(-D) = -f^*(D)$ if and only if D is numerically Cartier, i.e., $g^*(-D) = -g^*(D)$ for any proper birational morphism of normal varieties $g : Z \rightarrow X$.*

Chapter 4

LOG TERMINAL⁺ SINGULARITIES

In this chapter we study lt^+ singularities. In §4.1 we prove that the canonical sheaf of algebras on a lt^+ singularity is finitely generated, 4.1.6. Subsequently, in §4.2 we relate lt^+ singularities with klt singularities, §4.2.1, and with \mathbb{Q} -Gorenstein log terminal singularities, §4.2.2. In §4.3 we show the existence of a multiplier ideal detecting lt^+ singularities. In §4.4 we prove that a small deformation of an lt^+ singularity is still lt^+ . We do this by showing that the generic hyperplane section of an lt^+ singularity is still lt^+ , §4.4.1, and, conversely, that if a hyperplane section of a variety has lt^+ singularities, so will the ambient variety in a neighborhood of the hyperplane section, §4.4.2. Finally, in §4.5 we prove a relation between lt^+ singularities and rational singularities.

The results in §4.1 and §4.2 are from [CU12], while the remaining is from [Chi13].

In this chapter, all varieties are normal over a field of characteristic 0.

4.1 The canonical sheaf of algebras

Let us recall that X has *log terminal*⁺, or lt^+ , singularities if $a^+(F, X) > -1$ for every prime divisor F , exceptional over X . The first result shows that this can be checked with a single log resolution. By *log resolution* of X , we will mean a resolution $f : Y \rightarrow X$ with exceptional locus a simple normal crossing divisor. We will need this to prove the existence of a sheaf criterion for lt^+ singularities, 4.1.3. Notice that since $(fg)^* \neq g^*f^*$, 1.4.12, this result is interesting *per sé*.

Lemma 4.1.1. *Let X be a normal variety, and let $f : Y \rightarrow X$ be a log resolution, i.e. the exceptional locus of f is a simple normal crossing divisor. If $a^+(E, X) > -1 + \varepsilon$ for all prime exceptional divisors E on Y , for some $\varepsilon > 0$, then $a^+(F, X) > -1 + \varepsilon$ for all prime divisors F exceptional over X . In particular, then, X is lt^+ .*

Remark 4.1.2. Notice that the hypothesis of this lemma is satisfied if $a^+(E, X) > -1$ for all prime exceptional divisors E on Y , since there are only finitely many such divisors.

Proof. Let $h : Z \rightarrow X$ be any other resolution, that we can assume is a log resolution factoring through f , and let $g : Z \rightarrow Y$ so that $h = f \circ g$. Let F_Y be the reduced exceptional divisor of f , and F_Z be the one of h . By [dFH09, 2.13],

$$h^*(-K_X) - g^*f^*(-K_X) = E^{+,g} \geq 0,$$

which is effective and g -exceptional. Then

$$\begin{aligned} K_{Z/X}^+ + (1 - \varepsilon)F_Z &= K_Z + h^*(-K_X) + (1 - \varepsilon)F_Z = \\ &= K_Z + g^*f^*(-K_X) + (1 - \varepsilon)F_Z + E^{+,g} = \\ &= K_Z - g^*(K_Y + (1 - \varepsilon)F_Y) + (1 - \varepsilon)F_Z + \\ &\quad + g^*(K_Y + f^*(-K_X) + (1 - \varepsilon)F_Y) + \\ &\quad + E^{+,g} = \\ &= K_Z - g^*(K_Y + (1 - \varepsilon)F_Y) + (1 - \varepsilon)F_Z + \\ &\quad + g^*(K_{Y/X}^+ + (1 - \varepsilon)F_Y) + \\ &\quad + E^{+,g} \end{aligned}$$

We only need to check this expression on the prime divisors on Z which are exceptional over Y . Then, $K_Z - g^*(K_Y + (1 - \varepsilon)F_Y) + (1 - \varepsilon)F_Z$ has positive valuations, since $(Y, (1 - \varepsilon)F_Y)$ has log terminal singularities; the term $K_{Y/X}^+ + (1 - \varepsilon)F_Y$ is effective by assumption, and thus so is its pullback; $E^{+,g}$ is effective, as noticed above. \square

When studying with \mathbb{Q} -factorial singularities, we can work with Weil divisors or with sheaves. However, in the non- \mathbb{Q} -Gorenstein case, the two approaches will lead to different singularities. In a certain sense, the results in §2.3 say that log terminal singularities correspond to working with the Weil divisor K_X . In the same spirit as [dFH09, 8.2], we have the sheaf criterion, which suggests that log terminal⁺ singularities correspond to working with sheaves.

Proposition 4.1.3. *Let X be a normal variety. Then X is lt^+ if and only if there exists $\varepsilon \in \mathbb{Q}$, $\varepsilon > 0$, such that, for all sufficiently divisible $m \geq 1$, and for all resolutions of X ,*

$$\mathcal{O}_X(mK_X) \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(m(K_Y + (1 - \varepsilon)F_Y)),$$

where F_Y is the reduced exceptional divisor of f .

Proof. Let X be lt^+ . By lemma 4.1.1, there exists ε such that $a^+(E, X) > -1 + \varepsilon$ for all prime divisors E , exceptional over X (choose an arbitrary log resolution, and choose the value of ε that works for that resolution: it will work for all). We can choose $\varepsilon \in \mathbb{Q}$. If f and m are chosen as in the statement, with m divisible enough so that $m\varepsilon \in \mathbb{N}$,

$$\begin{aligned} m(K_Y + (1 - \varepsilon)F_Y) + f^\sharp(-mK_X) &\geq m(K_Y + (1 - \varepsilon)F_Y) + f^*(-mK_X) = \\ &= m(K_Y + (1 - \varepsilon)F_Y + f^*(-K_X)) \geq \\ &\geq 0. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{O}_X(mK_X) \cdot \mathcal{O}_Y &\subseteq (\mathcal{O}_X(mK_X) \cdot \mathcal{O}_Y)^{\vee\vee} = \\ &= \mathcal{O}_Y(-f^\sharp(-mK_X)) \subseteq \mathcal{O}_Y(m(K_Y + (1 - \varepsilon)F_Y)). \end{aligned}$$

Conversely, suppose that X is not lt^+ . Let E be a prime divisor, exceptional over X , such that $a^+(E, X) \leq -1$. Let $f : Y \rightarrow X$ be a resolution, with E as one of the components of the exceptional divisor. Since $a^+(E, X) \leq -1$, for all $\varepsilon \in \mathbb{Q}$, $\varepsilon > 0$, there exists m such that $\text{val}_E(K_Y + \frac{1}{m}f^\sharp(-mK_X)) < -1 + \varepsilon$. The same will be true for all $m' \geq m$, so we can assume that $m\varepsilon \in \mathbb{N}$. By further blowing up, we can assume that f is a resolution of $\mathcal{O}_X(mK_X)$, since the value of $\text{val}_E(K_Y + \frac{1}{m}f^\sharp(-mK_X))$ does not actually depend on the smooth model where we compute it. But then,

$$\mathcal{O}_X(mK_X) \cdot \mathcal{O}_Y = (\mathcal{O}_X(mK_X) \cdot \mathcal{O}_Y)^{\vee\vee} \not\subseteq \mathcal{O}_Y(m(K_Y + (1 - \varepsilon)F_Y)).$$

□

Remark 4.1.4. The necessary condition is still true if we substitute ε with any other $0 < \varepsilon' \leq \varepsilon$, or if $Y \rightarrow X$ is just a proper birational map between normal varieties. Moreover, the divisibility condition on m is only to ensure that $m\varepsilon \in \mathbb{Z}$.

Let us recall that $\mathcal{R}(X, K_X) = \bigoplus_{m \geq 0} \mathcal{O}_X(mK_X)$. As a consequence of the Minimal Model Program, if (X, Δ) has klt singularities, $\mathcal{R}(X, K_X)$ is finitely generated, 1.1.8. For which singularities the canonical sheaf of algebras finitely generated is an interesting question. In [Kol91] Kollár conjectured the following.

Conjecture 4.1.5 ([Kol91, 6.2.1]). *Let $x \in X$ be a 3-dimensional normal singularity and $x \in H \subset X$ be a Cartier divisor. Assume that $x \in H$ is a (normal) rational surface singularity. Then $\mathcal{R}(X, K_X)$ is a finitely generated sheaf of \mathcal{O}_X -algebras.*

Up to this point, klt was the largest (general) class of singularities for which the finite generation of $\mathcal{R}(X, K_X)$ was known.

Theorem 4.1.6. *If X is a lt^+ normal variety, then $\mathcal{R}(X, K_X)$ is finitely generated.*

Proof. We may assume that X is affine. Let $p : \tilde{X} \rightarrow X$ be a resolution (which we can assume projective, [KM98, 0.2]). By [BCHM10], $\mathcal{R}(p, K_{\tilde{X}})$ is finitely generated. Running the relative Minimal Model Program for \tilde{X} over X , we obtain the model $X^c = \text{Proj}_X \mathcal{R}(p, K_{\tilde{X}})$, with induced morphism $f : X^c \rightarrow X$. Let F_{X^c} be the reduced exceptional divisor of f . The variety X^c has Kawamata log terminal singularities.

Since X is lt^+ , there is a rational $\varepsilon > 0$ such that, for any $m > 0$ sufficiently divisible, there is an inclusion

$$\mathcal{O}_{X^c} \cdot \mathcal{O}_X(mK_X) \subseteq \mathcal{O}_{X^c}(m(K_{X^c} + (1 - \varepsilon)F_{X^c})).$$

To be precise, if d is a positive integer such that $d\varepsilon \in \mathbb{Z}$, then for any $m \geq 1$,

$$\mathcal{O}_{X^c} \cdot \mathcal{O}_X(dmK_X) \subseteq \mathcal{O}_{X^c}(dm(K_{X^c} + (1 - \varepsilon)F_{X^c}))$$

(see proposition 4.1.3 and remark 4.1.4). Pushing this forward we obtain inclusions

$$f_*(\mathcal{O}_{X^c} \cdot \mathcal{O}_X(dmK_X)) \subseteq f_*\mathcal{O}_{X^c}(dm(K_{X^c} + (1 - \varepsilon)F_{X^c})) \subseteq \mathcal{O}_X(dmK_X).$$

Since the left and right hand sides have isomorphic global sections, then

$$H^0(f_*\mathcal{O}_{X^c}(m(dK_{X^c} + d(1 - \varepsilon)F_{X^c}))) \cong H^0(\mathcal{O}_X(mdK_X)).$$

Since X is affine, $\mathcal{O}_X(mK_X)$ is globally generated and hence

$$f_*\mathcal{O}_{X^c}(m(dK_{X^c} + d(1 - \varepsilon)F_{X^c})) = \mathcal{O}_X(mK_X).$$

But then $\mathcal{R}(X, dK_X) \cong \mathcal{R}(f, dK_{X^c} + d(1 - \varepsilon)F_{X^c})$ is finitely generated over \mathcal{O}_X , since X^c has Kawamata log terminal singularities (again, by 1.1.8). Finally, since $\mathcal{R}(X, dK_X) = \mathcal{R}(X, K_X)^{(d)}$ is finitely generated, then $\mathcal{R}(X, K_X)$ is finitely generated over \mathcal{O}_X ([HK10, 5.68]). \square

Remark 4.1.7. Note that we have seen that

$$\mathcal{R}(X, dK_X) \cong \mathcal{R}(f, dK_{X^c} + d(1 - \varepsilon)F_{X^c}) \cong \mathcal{R}(p, dK_{\tilde{X}}).$$

Hence

$$\begin{aligned} X^c &= \text{Proj}_X \mathcal{R}(p, K_{\tilde{X}}) \cong \text{Proj}_X \mathcal{R}(p, K_{\tilde{X}})^{(d)} = \text{Proj}_X \mathcal{R}(p, dK_{\tilde{X}}) \cong \\ &\cong \text{Proj}_X \mathcal{R}(X, dK_X) = \text{Proj}_X \mathcal{R}(X, K_X)^{(d)} \cong \text{Proj}_X \mathcal{R}(X, K_X), \end{aligned}$$

and with this chain of isomorphisms $\mathcal{O}_{X^c}(1)$ is preserved ([Har77, Ex II.5.13]), and so $X^c \rightarrow X$ is a small morphism. *A posteriori* then, we do not have an exceptional divisor for f , and we have inclusions

$$\mathcal{O}_{X^c} \cdot \mathcal{O}_X(mK_X) \subseteq \mathcal{O}_{X^c}(mK_{X^c})$$

for all $m \geq 1$. Moreover, we have isomorphisms

$$\mathcal{R}(X, K_X) \cong \mathcal{R}(f, K_{X^c}) \cong \mathcal{R}(p, K_{\tilde{X}}).$$

Remark 4.1.8. Notice that if (X, Δ) is a klt pair, then X has lt^+ singularities, 2.3.4, while there are examples of singularities which are lt^+ , but cannot be klt ([Urb12, 4.1]). Thus, lt^+ is a class broader than klt for which we have finite generation of the algebra $\mathcal{R}(X, K_X)$. Of course, our result relies heavily on [BCHM10].

As a consequence of 4.1.6, we have the existence of a relative canonical model for X .

Corollary 4.1.9. *If X is lt^+ , then the relative canonical model $X_{\text{can}} = \text{Proj}_X \mathcal{R}(X, K_X)$ exists and has log terminal singularities.*

Proof. From the proof of 4.1.6 and 4.1.7, we have the existence of the relative canonical model $X_{\text{can}} = \text{Proj}_X \mathcal{R}(X, K_X) = X^c$, with log terminal singularities. \square

4.2 Log terminal singularities

Continuing the work done in §2.3, we will now relate log terminal⁺ singularities with other singularities.

4.2.1 lt^+ and klt singularities

Let us start with a remark.

Remark 4.2.1. Let X be a variety with lt^+ singularities. One of the steps of the proof of 4.1.6, was to use that the relative model $X^c \rightarrow X$ has log terminal singularities. Then, as a corollary of the theorem, we deduced that $X^c \cong \text{Proj}_X \mathcal{R}(X, K_X)$, 4.1.9.

Notice that, using the methods of [Urb12, 4.4], and 4.1.1 and 4.1.2, we can show directly that $\text{Proj}_X \mathcal{R}(X, K_X)$ has log terminal singularities (assuming the finite generation of $\mathcal{R}(X, K_X)$). We reproduce the argument here, since is similar to the one we will use to prove 4.2.2.

Since X is lt^+ , for any sufficiently high log resolution $f : Y \rightarrow X$, we have $K_Y + \frac{1}{m} f^{\natural}(-mK_X) > -1$. By 1.1.7, there exists a small birational morphism $\pi : X^+ \rightarrow X$ such that K_{X^+} is a π -ample \mathbb{Q} -Cartier divisor. Also, for this morphism we have that $\pi^{\natural}(-mK_X) = -mK_{X^+}$. Let us consider a common log resolution of X and X^+ , $f : Y \rightarrow X$ and $g : Y \rightarrow X^+$.

Let us consider the map $\mathcal{O}_{X^+} \cdot \mathcal{O}_X(mK_X) \rightarrow \mathcal{O}_{X^+}(mK_{X^+})$. Since $\pi_*^{-1}(K_X) = K_{X^+}$ is π -ample, $\mathcal{O}_{X^+}(mK_{X^+})$ is globally generated over X for m sufficiently divisible, hence we have an isomorphism of sheaves. Thus

$$K_Y - g^*(K_{X^+}) = K_Y + \frac{1}{m} g^*(-mK_{X^+}) = K_Y + \frac{1}{m} f^{\natural}(-mK_X) > -1,$$

where the last equality holds by [dFH09, 2.7]. By 4.1.1 and 4.1.2, the canonical model X^+ has lt^+ singularities, that is, log terminal singularities since it is \mathbb{Q} -Gorenstein.

Using the techniques of the above remark, we can show the following relation between lt^+ and klt singularities.

Theorem 4.2.2. *Let X be a normal variety with lt^+ singularities. Then, $\mathcal{R}(X, -K_X)$ is a finitely generated \mathcal{O}_X -algebra if and only if X is log terminal.*

Proof. If X is log terminal, then $\mathcal{R}(X, -K_X)$ is finitely generated by 1.1.8.

Let $\mathcal{R}(X, -K_X)$ be finitely generated. By 1.1.7, there is a small map $\pi : X^- \rightarrow X$ such that $\pi_*^{-1}(-K_X)$ is \mathbb{Q} -Cartier and π -ample. For any m sufficiently divisible, consider the natural map $\mathcal{O}_{X^-} \cdot \mathcal{O}_X(-mK_X) \rightarrow \mathcal{O}_{X^-}(-mK_{X^-})$, which, since the map is small, is an isomorphism of sheaves. Thus, considering $f : Y \rightarrow X$ and $g : Y \rightarrow X^-$, a common log resolution, we have

$$K_Y + \frac{1}{m}g^*(-mK_{X^-}) = K_Y + \frac{1}{m}g^*(\pi^{\natural}(-mK_X)) = K_Y + \frac{1}{m}f^{\natural}(-mK_X) > -1.$$

This means that X^- has at most lt^+ singularities. Since K_{X^-} is \mathbb{Q} -Cartier, X^- is log terminal.

Choosing a general ample \mathbb{Q} -divisor $H^- \sim_{\mathbb{Q}, X^-} -K_{X^-}$, let $1 \parallel m$, and $G^- \in |mH^-|$ be a general irreducible divisor. Then, setting $\Delta^- := \frac{G^-}{m}$, we have that $K_{X^-} + \Delta^- \sim_{\mathbb{Q}, X^-} 0$ is still log terminal and $\pi^*(K_X + \pi_*\Delta^-) =_{\mathbb{Q}} K_{X^-} + \Delta^-$. Hence $(X, \Delta = \pi_*\Delta^-)$ is log terminal. \square

Corollary 4.2.3. *Let X be a lt^+ normal variety. Then X is log terminal if and only if $\mathcal{R}(X, D)$ is finitely generated for any Weil divisor D .*

Proof. If X is log terminal, then this is 1.1.8. Conversely, in particular $\mathcal{R}(X, -K_X)$ is finitely generated, and by the previous theorem, X is log terminal. \square

Remark 4.2.4. In the spirit of corollary 4.1.9, if X has klt singularities, the birational map

$$\begin{array}{ccc} \text{Proj}_X \mathcal{R}(X, -K_X) & \dashrightarrow & \text{Proj}_X \mathcal{R}(X, K_X) \\ & \searrow & \swarrow \\ & X & \end{array}$$

is a flip. In particular, if searching for minimal (or canonical) models, the finite generation of the algebra $\mathcal{R}(X, -K_X)$ is not essential.

4.2.2 Lt^+ and (\mathbb{Q} -Gorenstein) log terminal singularities

We introduce the notion of defect ideals, which is a variation of the one used by Boucksom, de Fernex and Favre in [BdFF12] in the context of log terminal singularities (according to [dFH09]). As we will show in 4.2.7, these ideals will measure the failure of a divisor D to be \mathbb{Q} -Cartier.

Definition 4.2.5. Let X be a normal variety and D any divisor on X and m any positive natural number. Let $f_m : Y_m \rightarrow X$ be a log resolution of $(X, \mathcal{O}_X(mD) \otimes \mathcal{O}_X(-mD))$. The m -th defect ideal of $D \subseteq X$ is given by

$$\mathfrak{d}^m(D) := f_{m,*} \mathcal{O}_Y \left(\lfloor \frac{1}{m} (-f_m^\natural(-mD) - f_m^\natural(mD)) \rfloor \right).$$

Notice that $-f_m^\natural(-mD) - f_m^\natural(mD) \leq 0$, so that indeed $\mathfrak{d}^m(D)$ is an ideal.

It is in general very hard to control the behavior of the defect ideals when m changes. We will prove that, when the divisorial ring associated to D is finitely generated (with no assumptions on the ring associated to $-D$), then the sequence of ideals stabilizes when m is big enough.

Proposition 4.2.6. Let X be a normal variety and D a Weil divisor such that $\mathcal{R}(X, D)$ is finitely generated. Then the limit of the defect ideals exists and it is an ideal:

$$\mathfrak{D}(D) := \limsup_{m \rightarrow \infty} f_{m,*} \mathcal{O}_{Y_m} \left(\lfloor \frac{1}{m} (-f_m^\natural(-mD) - f_m^\natural(mD)) \rfloor \right).$$

Proof. For every map $f : Y \rightarrow X$, we already know that $f^\natural(D) \geq \frac{f^\natural(mD)}{m}$. Even more, if $\mathcal{R}(X, D)$ is finitely generated, there exists an m_0 such that $f^\natural(-mkD) = kf^\natural(-mD)$ for every $k > 0$. Hence we have (choosing a resolution f_{mk} high enough)

$$\begin{aligned} \mathfrak{d}^{mk}(D) &= f_{km,*} \mathcal{O}_Y \left(\lfloor \frac{1}{mk} (-f_{mk}^\natural(-mkD) - f_{mk}^\natural(mkD)) \rfloor \right) = \\ &= f_{km,*} \mathcal{O}_Y \left(\lfloor \frac{-f_{mk}^\natural(-mD)}{m} - \frac{f_{mk}^\natural(mkD)}{mk} \rfloor \right) \supseteq \\ &\supseteq f_{km,*} \mathcal{O}_Y \left(\lfloor \frac{1}{m} (-f_{mk}^\natural(-mD) - f_{mk}^\natural(mD)) \rfloor \right) = \\ &= \mathfrak{d}^m(D). \end{aligned}$$

By noetherianity these ideals will stabilize for $k \gg 0$. □

Theorem 4.2.7. If $\mathfrak{D}(D) = \mathcal{O}_X$, then D is \mathbb{Q} -Cartier.

Proof. We will prove that D is numerically Cartier in the sense of [BdFF12], and that $\mathcal{R}(-D, X)$ is finitely generated. Then the strict transforms of D must be trivial on the

fibers of $\text{Proj}_X \mathcal{R}(D, X)$ and $\text{Proj}_X \mathcal{R}(-D, X)$, since D is numerically Cartier, and ample and anti-ample as well (respectively). Thus D must be \mathbb{Q} -Cartier.

Let $\mathfrak{D}(D) = \mathfrak{d}^m(D) = \mathfrak{d}^{mk}(D) = \mathcal{O}_X$, for some m and all $k > 0$. Since $\mathfrak{d}^{km}(D) = \mathcal{O}_X$, then, for any resolution f high enough

$$\lfloor -\frac{1}{mk}(-f^{\natural}(-mkD) - f^{\natural}(mkD)) \rfloor = 0.$$

Since the quantity inside the round down is non-positive, it must be

$$f^{\natural}(-mkD) = -f^{\natural}(mkD). \quad (4.2.1)$$

More precisely, we have the above identity for a given resolution f_{mk} and for any other higher resolution. But since this is an equality of Weil divisors, by pushforward we have the same identity on any model.

The first consequence of (4.2.1) is that, for any resolution f of X , $f^*(-D) = -f^*(D)$, that is, D is numerically Cartier.

The second consequence is the finite generation of $\mathcal{R}(X, -D)$, and this comes from observing the relation between finite generation and numerically Cartier (see, for example, [dFH09, 2.2]). In this case, we have that, since D is numerically Cartier, $\mathcal{O}_X(mD) \cdot \mathcal{O}_X(-mD) = \mathcal{O}_X$ for m sufficiently divisible, [dFH09, 2.22 and 2.24]. But then, for any $q > 0$ and m sufficiently divisible,

$$\begin{aligned} \mathcal{O}_X(-mD)^q &\cong \mathcal{O}_X(-mD)^q \cdot \mathcal{O}_X(mqD) \cdot \mathcal{O}_X(-mqD) \cong \\ &\cong \mathcal{O}_X(-mD)^q \cdot \mathcal{O}_X(mD)^q \cdot \mathcal{O}_X(-mqD) \cong \\ &\cong \mathcal{O}_X(-mqD). \end{aligned}$$

The first isomorphism comes from what observed earlier, while the second one from the finite generation of $\mathcal{R}(X, D)$. Thus $\mathcal{R}(X, -mD) = \mathcal{R}(X, -D)^{(m)}$ is finitely generated, which implies that $\mathcal{R}(X, -D)$ is finitely generated. \square

Corollary 4.2.8. *Let X be a normal variety with lt^+ singularities and such that $\mathfrak{D}(K_X) = \mathcal{O}_X$. Then X is \mathbb{Q} -Gorenstein log terminal.*

4.3 Multiplier ideal

A standard construction in singularity theory are multiplier ideals. In [dFH09], the authors construct a multiplier ideal to detect log terminal singularities. Let us recall their construction.

Let X be a normal variety. The first step is to construct an m -limiting multiplier ideal, for each m . Fix a canonical divisor K_X on X , and consider a log resolution $f : Y \rightarrow X$ of X and the ideal $\mathcal{O}_X(-mK_X)$, i.e. a log resolution $f : Y \rightarrow X$ such that $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y = \mathcal{O}_Y(D)$ is invertible and the support of $D \cup \text{Exc}(f)$ is a simple normal crossing. Then, let us define

$$\mathcal{I}_m(X) := f_*\mathcal{O}_Y([K_{m,Y/X}^-]) = f_*\mathcal{O}_Y([K_Y - \frac{1}{m}f^{\natural}(mK_X)]). \quad (4.3.2)$$

Proposition 4.3.1 ([dFH09, 4.4]). *The sheaf $\mathcal{I}_m(X)$ is a (coherent) sheaf of ideals on X and its definition is independent of the choice of the log resolution f .*

The proof of this result proceeds in the usual way. The key point is that, since $f : Y \rightarrow X$ is a log resolution of $\mathcal{O}_X(-mK_X)$, for any other $g : Z \rightarrow Y$, $(fg)^{\natural}(mK_X) = g^*f^{\natural}(mK_X)$, 2.2.9(c).

The second step is to pick a maximal element among the collection $\{\mathcal{I}_m(X)\}_{m \geq 1}$, which exists by noetherianity.

Proposition 4.3.2 ([dFH09, 4.7]). *The set of ideals $\{\mathcal{I}_m(X)\}_{m \geq 1}$ has a unique maximal element.*

Definition 4.3.3 ([dFH09, 4.8]). *The unique maximal element of $\{\mathcal{I}_m(X)\}_{m \geq 1}$ is called the multiplier ideal sheaf of X and is denoted by $\mathcal{I}(X)$.*

One consequence of 1.4.14 is the following.

Corollary 4.3.4 ([dFH09, 5.5]). *The set $\{\mathcal{I}(X, \Delta)\}_{\Delta}$, where Δ is an effective boundary, has a unique maximal element, namely $\mathcal{I}(X)$.*

We can restate some of the results of §2.2 and §2.3 as the following.

Lemma 4.3.5 ([dFH09]). *We have that X has log terminal singularities if and only if $\mathcal{I}(X) = \mathcal{O}_X$ (if and only if there exists a boundary Δ such that (X, Δ) has klt singularities).*

If $g : Z \rightarrow Y$ and $f : Y \rightarrow X$ are two proper birational morphisms, there is no way in general to guarantee that $(fg)^*(-K_X) = g^*f^*(-K_X)$. However, it turns that we can still associate to each normal variety an ideal sheaf detecting lt^+ singularities.

Definition 4.3.6. *Let X be a normal variety, and let $f : Y \rightarrow X$ be a log resolution of X . We can define the ideal*

$$\mathcal{J}^+(X) := f_*\mathcal{O}_X([K_{Y/X}^+]) = f_*\mathcal{O}_Y([K_Y + f^*(-K_X)]). \quad (4.3.3)$$

Lemma 4.3.7. *The above ideal is independent of the resolution.*

Proof. It is enough to prove this result for two log resolutions $f : Y \rightarrow X$ and $h : W \rightarrow X$ with $h = f \circ g$, for some $g : W \rightarrow Y$. It is known that $K_{W/X}^+ - g^*K_{Y/X}^+$ is effective and g -exceptional, 1.4.11; thus, the result follows as in the usual setting of multiplier ideals.

Let $h^*(-K_X) - g^*f^*(-K_X) = E_g^+$, which is effective and g -exceptional, 1.4.11. We have the equalities

$$\begin{aligned} h_*\mathcal{O}_W([K_W + h^*(-K_X)]) &= h_*\mathcal{O}_W(K_{W/Y} + g^*(K_Y) + [h^*(-K_X)]) = \\ &= f_*g_*\mathcal{O}_W(K_{W/Y} + [h^*(-K_X)] + g^*(K_Y)) = \\ &= f_*\left(g_*\mathcal{O}_W(K_{W/Y} + [h^*(-K_X)]) \otimes \mathcal{O}_Y(K_Y)\right) \end{aligned}$$

by the projection formula. We are done if we prove that

$$g_*\mathcal{O}_W(K_{W/Y} + [h^*(-K_X)]) = \mathcal{O}_Y([f^*(-K_X)]).$$

Notice that we reduced to considering a map between smooth varieties and $\text{supp } f^*(-K_X)$ is snc. Hence,

$$\begin{aligned} g_*\mathcal{O}_W(K_{W/Y} + [h^*(-K_X)]) &= g_*\mathcal{O}_W(K_{W/Y} + [g^*f^*(-K_X) + E_g^+]) = \\ &= g_*\mathcal{O}_W(K_{W/Y} + [g^*f^*(-K_X)] + F_g^+), \end{aligned}$$

where F_g^+ is effective and g -exceptional. Indeed, since E_g^+ is effective and g -exceptional,

$[g^* f^*(-K_X) + E_g^+] \geq [g^* f^*(-K_X)]$ and the difference is g -exceptional. Thus,

$$\begin{aligned}
g_* \mathcal{O}_W(K_{W/Y} + [h^*(-K_X)]) &= g_* \mathcal{O}_W(K_{W/Y} + [g^* f^*(-K_X)] + F_g^+) = \\
&= g_* \mathcal{O}_W(K_{W/Y} + F_g^+ - [-g^* f^*(-K_X)]) = \\
&= g_* \mathcal{O}_W(K_{W/Y} + F_g^+ - [g^*(-f^*(-K_X))]) = \\
&= \mathcal{O}_Y(-[-f^*(-K_X)]) = \\
&= \mathcal{O}_Y([f^*(-K_X)])
\end{aligned}$$

by [Laz04, II.9.2.19]. □

Corollary 4.3.8. *A normal variety X has lt^+ singularities if and only if $\mathcal{J}^+(X) = \mathcal{O}_X$.*

Remark 4.3.9. As shown in 1.4.14, for each $m \geq 2$, we can find an anti-effective compatible boundary Δ_m on a log resolution $f : Y \rightarrow X$ of (X, Δ_m) such that $K_{m,Y/X}^+ = K_{Y/X}^{\Delta_m}$. So, *a priori*, we could construct multiplier fractional ideals for each m for the pair (X, Δ_m) and then do a limiting process on these multiplier ideals, as it was done in [dFH09] for $\mathcal{J}(X)$. However, there are two issues with this approach. The first one is that, since $K_{mq,Y/X}^+ \leq K_{m,Y/X}^+$, such limiting multiplier ideal should be a minimal element, and not a maximal one (in the collection of multiplier ideals for each m). Hence, there is no guarantee that it exists. The second problem is that, even when such minimal ideal exists, it may not detect the singularities, as 3.2.7 shows. Indeed, in that case, for each m the multiplier ideal associated to (X, Δ_m) would be \mathcal{O}_X since, for any log resolution $Y \rightarrow X$, $K_{m,Y/X}^+ > -1$. Indeed, any log resolution would factor through the blow-up of the vertex of the cone $Z \rightarrow X$ and $g : Y \rightarrow Z$, and $K_{m,Y/X}^+ \geq K_{Y/Z} + g^*(K_{m,Z/X}^+)$. However, $\mathcal{J}^+(X) \neq \mathcal{O}_X$ since X does not have lt^+ singularities.

4.4 Small deformations

In this section we will prove the invariance of lt^+ singularities, following the standard approach of proving a Bertini-type theorem, 4.4.1, and inversion of adjunction, 4.4.4.

4.4.1 Bertini-type theorem

Theorem 4.4.1. *Let X be a normal variety having lt^+ singularities. Then the generic hyperplane section of X has lt^+ singularities.*

Proof. Let $f : Y \rightarrow X$ be any log resolution of X . Then, for a generic hyperplane section S of X , S will be normal and $T := f_*^{-1}S = f^*S$ will be smooth. Moreover, the map $g := f|_T : T \rightarrow S$ will be a log resolution. By 1.4.16, for each $m \geq 2$, we can find $(-K_X)$ -boundaries Δ_m on X such that

- (a) $-K_X + \Delta_m$ and $-K_S + \Delta_{m,S}$ are \mathbb{Q} -Cartier, where $\Delta_{m,S} := \Delta_m|_S$,
- (b) $K_{m,Y/X}^+ = f^*(-K_X + \Delta_m) - f_*^{-1}\Delta_m = K_{Y/X}^{-\Delta_m}$, and
- (c) $K_{m,T/S}^+ = g^*(-K_S + \Delta_{m,S}) - g_*^{-1}\Delta_{m,S} = K_{T/S}^{-\Delta_{m,S}}$.

As in [Kol97, 7.7], (for example), $\text{discrep}(X, -\Delta_m) \leq \text{discrep}(S, -\Delta_{m,S})$. Therefore,

$$-1 < \inf_m \text{discrep}(X, -\Delta_m) \leq \inf_m \text{discrep}(S, -\Delta_{m,S}).$$

Notice that, since f and g are log resolution, the above discrepancies are computed directly by looking at the orders of $K_{Y/X}^{-\Delta_m}$ and $K_{T/S}^{-\Delta_{m,S}}$ along exceptional divisors over X and S respectively. As

$$K_{T/S}^+ = \inf_m K_{m,T/S}^+ = \inf_m K_{T/S}^{-\Delta_{m,S}},$$

S has lt^+ singularities, 4.1.1. □

Notice that the same technique proves Bertini-type theorems for canonical and terminal singularities.

Proposition 4.4.2. *Let X be a normal variety having canonical (resp. terminal) singularities. Then, the generic hyperplane section of X has canonical (resp. terminal) singularities.*

As corollary of the above, we have the following, that, in particular, implies that a 2-dimensional variety with lt^+ singularities has \mathbb{Q} -Gorenstein log terminal singularities.

Corollary 4.4.3. *Let X be a normal variety having terminal singularities; then X is regular in codimension 2. Let X be a normal variety having lt^+ singularities; then X is \mathbb{Q} -factorial in codimension 2.*

Proof. By taking hyperplane sections (and using induction on dimension), we reduce to the case $\dim X = 2$. Let $f : Y \rightarrow X$ be any proper birational map. For surfaces the pullback corresponds to the numerical pullback ([BdFF12, 2.20]) which is linear, and thus $K_{Y/X}^- = K_{Y/X}^+$. But then, as in [dFH09, 7.13], if X is terminal (resp. lt^+) this is equivalent to satisfying condition $M_{>0}$ (resp. $M_{>-1}$). Therefore X is smooth (resp. \mathbb{Q} -factorial). \square

4.4.2 Inversion of adjunction and small deformations

Theorem 4.4.4. *Let X be a normal variety, and let S be an effective (normal and reduced) Cartier divisor in X having lt^+ singularities. Then X has lt^+ singularities in a neighborhood of S .*

This version of inversion of adjunction is a nice feature of lt^+ singularities, and it is not known for log terminal singularities (see 4.4.7 for a discussion of the issues). As in the usual inversion of adjunction for klt singularities, [KM98, 5.50], this result relies on the following connectedness theorem.

Theorem 4.4.5 ([KM98, 5.48]). *Let $f : Y \rightarrow X$ be a proper and birational morphism, Y smooth, X normal. Let $D = \sum d_i D_i$ be an snc \mathbb{Q} -divisor on Y such that $f_* D$ is effective and $-(K_Y + D)$ is f -nef. Write $F = \sum_{i: d_i \geq 1} D_i$; then $\text{supp } F$ is connected.*

Now we will use this theorem to prove the above inversion of adjunction. The first result is a version of [KM98, 5.50] for pairs with an anti-effective boundary.

Lemma 4.4.6. *Let X be a normal variety, and let S be an effective (normal and reduced) Cartier divisor in X . Let $f : Y \rightarrow X$ be a log resolution of (X, S) , which restricts to a log resolution $g := f|_T : T \rightarrow S$, $T := f_*^{-1} S$. Let Δ be an effective divisor such that $K_X - \Delta$ is \mathbb{Q} -Cartier, and let $\Delta_S = \Delta|_S$. If the discrepancy of $(S, -\Delta_S)$ relative to g is bigger than -1 , then so is the one of $(X, -\Delta + S)$ with respect to f in a neighborhood of S .*

Proof. Let $\Delta^Y = f_*^{-1}\Delta$. Let us write

$$K_Y - \Delta^Y + T + F = f^*(K_X + S - \Delta) + A,$$

where all the coefficient in F are bigger or equal than -1 , and all the coefficients of A are strictly bigger than -1 . Moreover, $T + F - A$ is snc, and F is effective. Restricting the above identity to T , we obtain

$$K_T - \Delta_S^T = g^*(K_S - \Delta_S) + (A - F)|_T.$$

The discrepancy of $(S, -\Delta_S)$ (with respect to g) is bigger than -1 if and only if $F \cap T = \emptyset$, while the discrepancy of $(X, -S + \Delta)$ (with respect to f) is bigger than -1 if and only if $F \cap f^{-1}(S) = \emptyset$. Let $D = T + F - A$; notice that $f_*D = S \geq 0$ and

$$-(K_Y + D) = \Delta^Y + f^*(K_X + S - \Delta)$$

is f -nef. By 4.4.5, each $x \in S$ has an open neighborhood $x \in U_x \subseteq X$ such that $(T \cup F) \cap f^{-1}(U_x)$ is connected, hence $F \cap f^{-1}(U_x) = \emptyset$. Moving $x \in S$, we obtain the claim. \square

Remark 4.4.7. Notice that in the statement we do not assume that the maps f and g are resolutions of the pairs $(X, -\Delta + S)$ and $(S, -\Delta_S)$, so that the discrepancies relative to f and g are not necessarily the discrepancies of the pairs (i.e., non necessarily those pairs are log terminal). For this reason, the inversion of adjunction as stated here is not known in the log terminal case (in the sense of [dFH09]).

Remark 4.4.8. The above remark, and the consequent inversion of adjunction, is known to be false for a non-effective boundary Δ . However, in all the counterexamples, you need the boundary to have both an effective and an anti-effective component.

Proof of theorem 4.4.4. Let us fix a log resolution $f : Y \rightarrow X$ of (X, S) , which restricts to a log resolution $g := f|_T : T \rightarrow S$, $T := f_*^{-1}S$. For each $m \geq 2$, we can find a weak m -compatible $(-K_X)$ -boundary Δ_m for f such that $\Delta_{m,S} := \Delta_m|_S$ is a weak m -compatible $(-K_S)$ -boundary for g , 1.4.16. As $K_{T/S}^- \leq K_{T/S}^{-\Delta_m}$ and S has lt^+ singularities, the discrepancy of $(S, -\Delta_{m,S})$ with respect to g is bigger than -1 . By 4.4.6, the discrepancy

of $(X, -\Delta_m + S)$ with respect to f is bigger than -1 . As S is Cartier and effective, the discrepancy of $(X, -\Delta_m)$ with respect to f is bigger than -1 . More precisely, we have

$$\begin{aligned}
K_Y - \Delta_m^Y + T &= f^*(K_X - \Delta_m + S) + \sum a_i E_i = \\
&= f^*(K_X - \Delta_m) + f^*(S) + \sum a_i E_i = \\
&= f^*(K_X - \Delta_m) + T + \sum b_i E_i + \sum a_i E_i = \\
&= f^*(K_X - \Delta_m) + T + \sum (a_i + b_i) E_i,
\end{aligned}$$

where $f^*(S) - T = \sum b_i E_i \geq 0$, so that

$$K_Y - \Delta_m^Y = f^*(K_X - \Delta_m) + \sum (a_i + b_i) E_i \geq f^*(K_X - \Delta_m) + \sum a_i E_i.$$

As in the proof of 4.4.6, in this case the coefficients of the discrepancy of $(S, -\Delta_{m,S})$ (with respect to g) are exactly the ones of the discrepancy of $(X, -\Delta_m + S)$ in a neighborhood of S (with respect to f). As the

$$\inf_m K_{T/S}^{-\Delta_{m,S}} = K_{T/S}^+$$

has all coefficient strictly bigger than -1 , so has

$$\inf_m K_{Y/X}^{-\Delta_m} = K_{Y/X}^+$$

(in a neighborhood of S). Thus X has lt^+ singularities in a neighborhood of S , 4.1.1. \square

As immediate corollary of 4.4.4 we have deformation invariance.

Corollary 4.4.9. *Let $f : X \rightarrow T$ be a proper flat family of varieties over a smooth curve T and $t \in T$ a closed point. If the fiber X_t has lt^+ singularities, then so do the other fibers near t .*

Proof. The technique of this proof is standard (see, for example, [KS11, 4.2]). By theorem 4.4.4, X has lt^+ singularities near X_t . Let Z be the non- lt^+ locus of X , which is closed in X . Since f is proper, $f(Z)$ is a closed subset of T not containing t . By restricting T to an open set, we may assume that X has lt^+ singularities. By theorem 4.4.1, all fibers over nearby points of $t \in T$ have lt^+ singularities. \square

4.5 Rational singularities

In this section we will prove that Cohen-Macaulay lt^+ singularities are rational. This is one of various results that relate log terminal singularities and rational singularities. The first one is Elkik's theorem, [Elk81], which says that, \mathbb{Q} -Gorenstein log terminal singularities are rational. More generally, [KM98, 5.22], dlt singularities are rational. We recall that also log terminal singularities in the sense of [dFH09] are rational, [dFH09, 7.7].

Of particular interest is also the following result of de Fernex and Docampo Álvarez, [dFDA11, 7.2], which bears a resemblance with ours (see 4.5.4). Let X be a (normal) variety such that rK_X is Cartier, and let \mathfrak{d}_X denote the lci defect ideal of X . Then if X has rational singularities, (X, \mathfrak{d}_X^{-1}) is J -canonical. The converse holds if X is Cohen-Macaulay. We refer to [dFDA11] for all the relevant definitions.

We start by recalling the definition of rational singularities.

Definition 4.5.1. *Let X be a variety. X is said to have rational singularities if for a resolution $f : Y \rightarrow X$ (or, equivalently, for all resolutions) $Rf_*\mathcal{O}_Y \simeq_{q.is.} \mathcal{O}_X$.*

We have the following useful characterization of rational singularities (see for example [Kol97, 11.9])

Theorem 4.5.2 (Kempf's criterion). *Let X be a normal variety and $f : Y \rightarrow X$ a resolution. Then X has rational singularities if and only if $f_*\omega_Y = \omega_X$ and X is Cohen-Macaulay.*

Theorem 4.5.3. *If X has Cohen-Macaulay and lt^+ singularities, then it has rational singularities.*

Proof. Let Y be any resolution. For each prime divisor E in Y exceptional over X ,

$$-1 < \text{ord}_E(K_{Y/X}^+) \leq \text{ord}_E(K_Y + f^{\natural}(-K_X)).$$

Since $K_Y + f^{\natural}(-K_X)$ is an integral divisor, we have that

$$K_Y + f^{\natural}(-K_X) \geq 0.$$

As in the proof of 4.1.3, this is equivalent with the condition $\mathcal{O}_X(K_X) \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(K_Y)$, that is

$$\omega_X \cdot \mathcal{O}_Y \subseteq \omega_Y.$$

Pushing forward on X this inclusion (f_* is left exact) we obtain

$$f_*(\omega_X \cdot \mathcal{O}_Y) \subseteq f_*\omega_Y.$$

Notice that, if \mathcal{F} is a torsion-free sheaf on a normal variety X subsheaf of $k(X)$ and $f : Y \rightarrow X$ is any proper birational morphism, then

$$\mathcal{F} \subseteq f_*(\mathcal{F} \cdot \mathcal{O}_Y).$$

Indeed $\mathcal{F} \cdot \mathcal{O}_Y$ is the image of $f^*\mathcal{F}$ in $k(Y)$. We have natural maps

$$\mathcal{F} \rightarrow f_*f^*\mathcal{F} \rightarrow f_*(\mathcal{F} \cdot \mathcal{O}_Y).$$

The composition above is an isomorphism at the generic point of X , and thus the kernel must be torsion. As \mathcal{F} is torsion-free, the above composition is injective. In our case, we have the natural inclusion $\omega_X \subseteq f_*(\omega_X \cdot \mathcal{O}_Y)$.

We have the chain of inclusions

$$\omega_X \subseteq f_*(\omega_X \cdot \mathcal{O}_Y) \subseteq f_*\omega_Y \subseteq \omega_X,$$

which proves

$$f_*\omega_Y = \omega_X.$$

As X is Cohen-Macaulay, X has rational singularities. □

Remark 4.5.4. The assumption of Cohen-Macaulay is necessary. The example [Urb12, 4.1] has lt^+ singularities but is not Cohen-Macaulay. This phenomenon is similar to the one happening in the case of Jacobian singularities. In the result [dFDA11, 7.2] the assumption of Cohen-Macaulay for the converse direction is essential as well.

Chapter 5

POSITIVE WEIL DIVISORS

In this chapter we define positivity for Weil divisors, which is the first step towards a non- \mathbb{Q} -Gorenstein Minimal Model Program, generalizing the notions of §1.6.1. Our definitions are in extreme generality, and we show some properties that are interesting even outside the Minimal Model Program. In §5.1 we define positivity and prove some basic properties. In §5.2 we prove some useful characterizations, which will constitute more evidence that our definitions are in the correct direction. Finally, in §5.3 we prove a version of Fujita vanishing and Kawamata-Viehweg vanishing for Weil divisors.

All the definitions and results (with the exception of 5.3.6) are from [CU13].

5.1 Definitions and first properties

Throughout this section and the next one we use the following convention.

Convention 5.1.1. Whenever we consider a projective morphism of normal quasi-projective varieties $f : X \rightarrow Y$, and a Weil divisor D on X , we will assume that the generic fiber of f is normal (or at least, so is each irreducible component), and that D is invertible at their generic point. This will guarantee that we can restrict D to the generic fiber, see (1.1.1).

Remark 5.1.2. When we will work over algebraically closed fields, the above condition is automatically verified. However, when we are simply working over a noetherian ring k , there is no guarantee that 5.1.1 holds. Of course it holds if $Y = \text{Spec } k$.

5.1.1 Definitions

Let us start with a key observation.

Remark 5.1.3. Positivity for Cartier divisors is sometimes defined in terms of global generation or basepoint-freeness. However, for linear systems associated to Weil divisors,

basepoint-freeness and global generation are not equivalent. For example, let us consider the case of an affine cone and a non- \mathbb{Q} -Cartier divisor on it. Since the variety is affine, every coherent sheaf is globally generated; on the other hand, this divisor (and all its powers) must go through the vertex of the cone. In this case, every basepoint-free Weil divisor is necessarily \mathbb{Q} -Cartier. When thinking of how to define positivity for Weil divisors, presented with the choice between basepoint-freeness and global generation, we chose the latter, which is more natural from the point of view of singularity theory. Basepoint-freeness seems also a much stronger requirement. As the reader will see, our choice leads more naturally to cohomological statements, §5.3, but it might be less natural when applied in the context of the MMP, §6.4.

Let us recall the definition of relative global generation.

Definition 5.1.4. *Let $f : X \rightarrow U$ be a projective morphism of schemes, and let \mathcal{F} be a coherent sheaf on X . We say that \mathcal{F} is relatively globally generated if the natural map $f^*f_*\mathcal{F} \rightarrow \mathcal{F}$ is surjective.*

Remark 5.1.5. This condition is local on the base, that is, it can be checked on open affine subschemes of U , where it reduces to the usual global generation.

As typical application of this observation, we can see that the product of two relatively globally generated coherent sheaves is still relatively globally generated.

From now on we will fix a projective morphism of quasi-projective normal varieties $X \rightarrow U$.

Definition 5.1.6. *Let $f : X \rightarrow U$ be a projective morphism of quasi-projective normal varieties. A Weil \mathbb{Q} -divisor D on X is relatively asymptotically globally generated, in short relatively agg or f -agg, if, for every positive $1 \parallel m$, $\mathcal{O}_X(mD)$ is relatively globally generated.*

Generalizing [Urb11, 4.1], in [CU13] we gave the following definition.

Definition 5.1.7. *Let $f : X \rightarrow U$ be a projective morphism of quasi-projective normal varieties over a noetherian ring k . A Weil \mathbb{Q} -divisor D on X is relatively nef, or f -nef, if for every relatively ample \mathbb{Q} -Cartier divisor A , $\mathcal{O}_X(D + A)$ is f -agg.*

If $U = \text{Spec } k$ (so that X is projective) we will simply say that D is nef.

For each of the positivity notions in §1.6.1, in [CU13] we have defined the corresponding one for Weil divisors.

Definition 5.1.8. *Let $f : X \rightarrow U$ be a projective morphism of quasi-projective normal varieties over a noetherian ring k . A Weil \mathbb{Q} -divisor D on X is called relatively almost ample, or almost f -ample, if, for every relatively ample \mathbb{Q} -Cartier divisor A there exists a $b > 0$ such that $bD - A$ is f -nef. It is called relatively ample, or f -ample, if, in addition, $\mathcal{R}(X, D)$ is finitely generated.*

If $U = \text{Spec } k$, in the two cases above we will simply say almost ample or ample.

Remark 5.1.9. If X is projective over \mathbb{C} with klt singularities, for all \mathbb{Q} -divisors, the algebras $\mathcal{R}(X, D)$ are finitely generated, 1.1.8, thus the notions of almost ample and ample coincide. In general almost ample is not equivalent to ample, as 5.1.14 shows.

Definition 5.1.10. *Let $f : X \rightarrow U$ be a projective morphism of quasi-projective normal varieties over a noetherian ring k . A Weil \mathbb{Q} -divisor D is called relatively big, or f -big, if there exist an f -ample \mathbb{Q} -Cartier divisor A and an effective Weil \mathbb{Q} -divisor E such that $D \sim_{f, \mathbb{Q}} A + E$.*

If $U = \text{Spec } k$, we will simply say that D is big.

Definition 5.1.11. *Let $f : X \rightarrow U$ be a projective morphism of quasi-projective normal varieties over a noetherian ring k . A Weil \mathbb{Q} -divisor D is called relatively pseudo-effective, or f -pseff, if for every f -ample \mathbb{Q} -Cartier divisor A , $D + A$ is f -big.*

If $U = \text{Spec } k$, we will simply say that D is pseudo-effective.

5.1.2 First properties

We have the following immediate properties, which say that these notions of positivity behave like their Cartier counterparts.

Lemma 5.1.12. *Let $f : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties over a noetherian ring k . Let D be a Weil \mathbb{Q} -divisor on X . The following are true.*

- (a) Assume that D is a \mathbb{Q} -Cartier divisor and k is an algebraically closed field; then the above notions of (relative) nefness, bigness, pseudo-effectiveness and amplitude coincide with the usual ones.
- (b) Relative nefness, bigness, pseudo-effectiveness and (almost) amplitude are asymptotic conditions, that is, D is relatively nef / big / pseff / (almost) ample if and only if for any, or all, $m > 0$, mD is relatively nef / big / pseff / (almost) ample.
- (c) Let D be an f -nef divisor, and let L be a relatively agg \mathbb{Q} -Cartier divisor; then $D + L$ is relatively nef.
- (d) Let D be a Weil \mathbb{Q} -divisor; then D is relatively almost ample if and only if, for any \mathbb{Q} -Cartier divisor L , there exists $b > 0$ such that $bD - L$ is relatively nef.
- (e) Let D be relatively almost ample; then D is relatively nef.
- (f) Let D be relatively almost ample; then D is relatively agg.
- (g) Let D be relatively almost ample and k be an algebraically closed field; then D is f -big.
- (h) Let D be relatively big; then it is relatively pseudo-effective.
- (i) Let D be a relatively agg Weil \mathbb{Q} -divisor, and let A be a relatively ample \mathbb{Q} -Cartier divisor; then $D + A$ is relatively almost ample. If moreover $\mathcal{R}(X, D)$ is finitely generated, $D + A$ is relatively ample.
- (j) Let D be a relatively (almost) ample Weil \mathbb{Q} -divisor and let A be a relatively agg \mathbb{Q} -Cartier divisor; then $D + A$ is relatively (almost) ample.
- (k) Let D be a Weil \mathbb{Q} -divisor; then D is relatively (almost) ample if and only if, for any \mathbb{Q} -Cartier divisor L , there exists $b > 0$ such that $bD - L$ is relatively (almost) ample.

Proof. (a) See the discussion in §1.6.1

- (b) Clear. Notice that, if D is a Weil divisor, $\mathcal{R}(X, D)$ is finitely generated if and only if, for some $a > 0$, $\mathcal{R}(X, aD)$ is finitely generated, [HK10, 5.68].
- (c) Let A be an ample Cartier divisor. By hypothesis $\mathcal{O}_X(m(D + A))$ is relatively globally generated for all $1 \parallel m$. Since L is relatively agg, $\mathcal{O}_X(m(D + A)) \otimes \mathcal{O}_X(mL) \cong \mathcal{O}_X(m(D + L + A))$ is relatively agg.
- (d) The sufficient condition is immediate. Let us show the necessary one. Let L be any \mathbb{Q} -Cartier divisor, which without loss of generality we can assume Cartier. Let A be a relatively ample \mathbb{Q} -Cartier divisor such that $A + L$ is relatively ample. By definition there exists $b > 0$ such that $bD - A - L$ is relatively nef. Since A is relatively agg, $bD - L = (bD - A - L) + A$ is relatively nef.
- (e) In the previous part, choose $L = 0$.
- (f) Let A be relatively ample \mathbb{Q} -Cartier divisor, and let $b > 0$ such that $bD - A$ is relatively nef. By definition for every relatively ample \mathbb{Q} -Cartier divisor L , $bD - A + L$ is relatively agg. Setting $A = L$, we get that bD is relatively agg, that is, D is relatively agg.
- (g) Let A be a \mathbb{Q} -Cartier f -ample divisor, and let $b > 0$ such that $bD - A$ is f -agg. Let $m > 0$ such that $m(bD - A)$ is relatively globally generated. Let $V \subseteq U$ be an open affine subscheme, such that the complement of $f^{-1}V$ has codimension at least 2 in X . Then, $m(bD - A)$ has a section on $f^{-1}(V)$, which extends to a global section (the sheaf $\mathcal{O}_X(mbD - mA)$ is S_2). Let $M \in |\mathcal{O}_X(mbD - mA)|$ (general enough), constructed as explained before. Then $M \geq 0$ and $D \sim_{f, \mathbb{Q}} \frac{1}{mb}A + \frac{1}{m}M$.
- (h) Clear, since if you add a relatively ample Cartier divisor to a relatively big divisor, you still have a relatively big Weil divisor.
- (i) Since all the conditions are asymptotic, it is enough to work with Cartier divisors. In particular, we can assume that A is Cartier. Let L_1 be a relatively ample Cartier divisor, and let $b > 0$ such that $bA - L_1$ is relatively very ample. Then, for all every

relatively ample Cartier divisor L_2 , $bA - L_1 - L_2$ is relatively agg. Thus $bD + (bA - L_1 + L_2) = b(D + A) - L_1 + L_2$ is relatively agg. By definition, this means that $bD + bA - L_1$ is relatively nef. Therefore, $D + A$ is relatively almost ample. For the second part of the statement, notice that $\mathcal{O}_X(m(D + A)) = \mathcal{O}_X(mD) \otimes \mathcal{O}_X(mA)$ since A is Cartier.

- (j) Let L be a relatively ample \mathbb{Q} -Cartier divisor and let $b > 0$ such that $bD - L$ is relatively nef. Thus $bD - L + bA = b(D + A) - L$ is relatively nef, which implies that $D + A$ is relatively almost ample. As in the previous part, the finite generation of the sheaf algebra is preserved by adding a Cartier divisor, and this concludes the proof.
- (k) The sufficient condition is clear. Let D be (almost) relatively ample and let L be \mathbb{Q} -Cartier relatively ample. Then there exists $b > 0$ such that $bD - 3L$ is relatively nef. Hence $bD - 2L = bD - 3L + L$ is relatively agg (by definition of relative nefness), and thus $bD - L = bD - 2L + L$ is relatively (almost) ample.

□

Example 5.1.13. Let us fix an algebraically closed field k . Let X be the quadric cone $X = \{xy - wz = 0\}$ in \mathbf{P}^4 , which is a cone over an embedding of type $(1, 1)$ of $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$. Let C_D be a divisor on X , which is a cone over a divisors D in Q , of type (a, b) . Using the above lemma it is not hard to see that C_D is almost ample if and only if $a, b > 0$, that is, if and only if D is ample, while C_D is nef if and only if $a, b \geq 0$, that is, if and only if D is nef. If $k = \mathbb{C}$, then X is log terminal, and thus all algebras of local sections are finitely generated, and hence we can say that C_D is ample if and only if D is ample.

Example 5.1.14. We will give an example of a divisor which is almost ample, but not ample. Let X be a normal variety with isolated singularities and let D be a Weil divisor such that $\mathcal{R}(X, D)$ is not finitely generated. For example, X can be the cone of [Urb12, §3], and $D = K_X$. Let A be any Cartier ample divisor on X . By the above lemma and [Urb12, 5.5], for b sufficiently large $bA + D$ is almost ample. On the other hand, since A is Cartier and $\mathcal{R}(X, D)$ is not finitely generated, $\mathcal{R}(X, bA + D)$ is not finitely generated, showing that $bA + D$ is not ample.

After the good news of 5.1.12, let us point out two issues: the first one regarding additivity of nefness, and the second regarding the restriction of ample divisors. Both properties are trivially verified for Cartier divisors.

Question 5.1.15. Under which conditions on the singularities of X is nefness additive?

Remark 5.1.16. Nefness is probably not additive in general. Let D_1 and D_2 be two nef Weil divisors such that for all $m \geq 1$, $\mathcal{O}_X(mD_1) \otimes \mathcal{O}_X(mD_2) \neq \mathcal{O}_X(mD_1 + mD_2)$. Then, for any ample Cartier divisor A and for any $m \gg 0$, $\mathcal{O}_X(mA + mD_1)$ and $\mathcal{O}_X(mA + mD_2)$ are globally generated; thus so is $\mathcal{O}_X(mA + mD_1) \otimes \mathcal{O}_X(mA + mD_2)$. However, $\mathcal{O}_X(mA + mD_1) \otimes \mathcal{O}_X(mA + mD_2) \neq \mathcal{O}_X(m2A + m(D_1 + D_2))$, and the sheaf on the right is the reflexive hull of the sheaf on the left. Thus it is not immediately possible to deduce the global generation of $\mathcal{O}_X(m2A + m(D_1 + D_2))$. See 5.1.21 and §6.2 for more on the additivity (and why we care).

Question 5.1.17. Let D be an (almost) ample Weil divisor on X and let $S \subset X$ be a normal effective Cartier divisor such that D is invertible at the generic point of S . Is the restriction $D|_S$ (almost) ample?

5.1.3 *b*-nefness

Let us conclude by discussing an alternative approach to nefness, which was suggested to us by de Fernex, using *b*-divisors. In [CU13], we called this notion *b*-nefness. Let us recall that to each Weil divisor D on a projective variety X in characteristic 0 we can associate an X -nef *b*-divisor, the nef envelope $\text{Env}_X(D)$, 1.5.3 and 1.5.4. Moreover, in this case the \mathbb{R} -Cartier *b*-divisors are dense in the \mathbb{R} -Weil *b*-divisors, 1.5.1.

Definition 5.1.18. We say that D is *b*-nef if $\text{Env}(D)$ is a nef *b*-divisor, in the sense that is a limit of \mathbb{R} -Cartier *b*-divisors determined by ample divisors.

Proposition 5.1.19. Let X be a normal variety and D be a Weil \mathbb{R} -divisor. If $\mathcal{R}(X, D)$ is a finitely generated \mathcal{O}_X -algebra, then D is nef if and only if D is *b*-nef.

Proof. Since $\text{Env}_X(D)$ is determined at the level of the small map $\text{Proj}_X \mathcal{R}(X, D) \rightarrow X$, it is an easy consequence of 5.2.2. □

Proposition 5.1.20. *If D is nef, then D is b -nef.*

Proof. Let us consider a birational map $f : Y \rightarrow X$. We have that $\mathcal{O}_X(m(D+A))$ is globally generated for some ample divisor A . In particular $\mathcal{O}_Y \cdot \mathcal{O}_X(m(D+A))$ is globally generated.

If Y is a log resolution of $\mathcal{O}_X(mD)$ then we have that

$$\mathcal{O}_Y \cdot \mathcal{O}_X(m(D+A)) = [\mathcal{O}_Y \cdot \mathcal{O}_X(m(D+A))]^{\vee \vee}$$

and in particular $\text{div}(\mathcal{O}_Y \cdot \mathcal{O}_X(m(D+A)))$ is nef.

Since for m sufficiently divisible we can choose A to be as small as possible, we have a sequence of nef b -divisors whose limit is exactly $\text{Env}(D)$. \square

We do not know if the converse implication holds in general.

Remark 5.1.21. Since the sum of limits is the limit of the sum, the sum of nef b -divisors is still nef. However, if D_1 and D_2 are general Weil divisors, $\text{Env}_X(D_1) + \text{Env}_X(D_2) \neq \text{Env}_X(D_1 + D_2)$, thus it is not clear if b -nefness is additive.

5.2 Characterizations

The characterizations of this section are of two types. Some of them are the equivalent of well-known results for \mathbb{Q} -Cartier divisors, while others investigate the behavior of positivity of Weil divisors under \mathbb{Q} -Cartierizations. Let us recall that, if D is a Weil divisor on X , the \mathbb{Q} -Cartierization of D is the small birational morphism $f : Y \rightarrow X$ such that $f_*^{-1}D$ is \mathbb{Q} -Cartier and f -ample. This map exists if and only if $\mathcal{R}(X, D)$ is finitely generated, and the \mathbb{Q} -Cartierization is $Y \cong \text{Proj}_X \mathcal{R}(X, D) \rightarrow X$, see 1.1.7.

5.2.1 Nef Weil divisors

Proposition 5.2.1 (see [Laz04, 1.4.10]). *Let $f : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties over a noetherian ring k . Let D be a divisor on X and let H be an f -ample divisor. Let us assume that either D or H is \mathbb{Q} -Cartier. Then D is f -nef if and only if, for all sufficiently small $\varepsilon \in \mathbb{Q}$, $0 < \varepsilon \ll 1$, $D + \varepsilon H$ is f -ample.*

Proof. Let us assume that $D + \varepsilon H$ is f -ample for all $\varepsilon \in \mathbb{Q}$, $0 < \varepsilon \ll 1$. Let A be an f -ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. For rational positive ε sufficiently small, $A - \varepsilon H$ is f -ample. Since $\mathcal{R}(X, H)$ is finitely generated and either D or H is \mathbb{Q} -Cartier, for all positive $1 \parallel m$,

$$\mathcal{O}_X(m(D + A)) = \mathcal{O}_X(m(D + \varepsilon H) + (A - \varepsilon H)) \cong \mathcal{O}_X(m(D + \varepsilon H)) \cdot \mathcal{O}_X(m(A - \varepsilon H)).$$

Since both $D + \varepsilon H$ and $A - \varepsilon H$ are f -ample, they are f -agg, and thus $\mathcal{O}_X(m(D + \varepsilon H)) \otimes \mathcal{O}_X(m(A - \varepsilon H))$ is globally generated; as this sheaf surjects on $\mathcal{O}_X(m(D + \varepsilon H)) \cdot \mathcal{O}_X(m(A - \varepsilon H))$, $D + A$ is f -agg. By definition, D is f -nef.

Conversely, let D be f -nef. By substituting H with εH , we reduce to prove that $D + H$ is f -ample. Let A be \mathbb{Q} -Cartier and f -ample. For $1 \gg \delta > 0$, $H - \delta A$ is f -ample; moreover $D + \delta A$ is f -agg. By the hypothesis, either $D + \delta A$ or $H - \delta A$ is \mathbb{Q} -Cartier. Hence $D + H = (D + \delta A) + (H - \delta A)$ is f -ample. \square

Theorem 5.2.2. *Let $X \rightarrow U$ be a projective morphism of normal projective varieties over an algebraically closed field k . If $g : Y \rightarrow X$ is a small projective birational map over U such that $\bar{D} := g_*^{-1}D$ is \mathbb{Q} -Cartier and g -ample, then D is nef over U if and only if \bar{D} is nef over U .*

Remark 5.2.3. This is the relative and characteristic free version of [Urb11, 4.3]. The strategy of our proof is to reduce to an affine base and then to use [Urb11, 4.3]. We point out that, although the referred result is only stated in characteristic 0, its proof works over any algebraically closed field.

Proof. Let D be nef over U . By definition, for every Cartier divisor A ample over U , $\mathcal{O}_X(m(D + A))$ is relatively globally generated (over U) for $1 \parallel m$. Since \bar{D} is g -ample, we also have that $\mathcal{O}_Y(m\bar{D})$ is globally generated over X for $1 \parallel m$. We claim that this implies that so is $\mathcal{O}_Y(m(\bar{D} + g^*A))$. Since both the assumption and the claim (and the properties of the map g and the divisor \bar{D}) are local on the base U , we can assume U to be affine. Thus we can assume that $\mathcal{O}_X(m(D + A))$ is globally generated and we will show that so is $\mathcal{O}_Y(m(\bar{D} + g^*A))$. This is proven in [Urb11, 4.3]. Going back to the relative setting, we showed that $\mathcal{O}_Y(m(\bar{D} + g^*A))$ is relatively globally generated over U ; thus it is relatively nef. Since nefness is a closed property, \bar{D} is relatively nef.

Now let us suppose that \bar{D} is relatively nef over U , and let A be a Cartier divisor on X , ample over U . The claim is that, for $1||m$, $\mathcal{O}_X(m(D+A))$ is globally generated over U . As in the previous part, we can assume that U is affine. Because of [Kee03, 2.15], nefness is an absolute notion, so \bar{D} is nef. Then, the same proof of [Urb11, 4.3] shows that $\mathcal{O}_X(m(D+A))$ is globally generated, thus concluding our proof. \square

5.2.2 Ample Weil divisors

Proposition 5.2.4 (see [Har77, II.7]). *Let $X \rightarrow U$ be a projective morphism of normal quasi-projective varieties and let D be a relatively almost ample Weil \mathbb{Q} -divisor. For any coherent sheaf \mathcal{F} and for all positive $1||m$, $\mathcal{O}_X(mD) \otimes \mathcal{F}$ is relatively globally generated.*

Conversely, let D be a Weil \mathbb{Q} -divisor such that $\mathcal{R}(X, D)$ is finitely generated. If for any coherent sheaf \mathcal{F} and for all positive $1||m$ the sheaf $\mathcal{F} \otimes \mathcal{O}_X(mD)$ is relatively globally generated, then D is relatively ample.

Proof. Notice that, for any relatively ample \mathbb{Q} -Cartier divisor A , there exists $b > 0$ such that $bD - A$ is relatively agg. Indeed, there exists $b > 0$ such that $bD - 2A$ is relatively nef (by definition). But then $bD - A = (bD - 2A) + A$ is relatively agg. Since A is relatively ample, $\mathcal{F} \otimes \mathcal{O}_X(mA)$ is relatively globally generated for positive $1||m$. Thus

$$\mathcal{F} \otimes \mathcal{O}_X(mbD) \cong \mathcal{F} \otimes \mathcal{O}_X(mA) \otimes \mathcal{O}_X(mbD - mA)$$

is relatively globally generated for positive $1||m$.

Now let us prove the converse. Since we are working with asymptotic conditions, we can substitute D with aD for a positive $1||a$, so that $\mathcal{R}(X, D)$ is generated in degree 1, that is, $\mathcal{O}_X(mD) \cong \mathcal{O}_X(D)^m$ for all $m \geq 1$. Let L be any relatively ample Cartier divisor, and let $b > 0$ such that $\mathcal{O}_X(bD - L)$ is relatively globally generated. Since $\mathcal{O}_X(bD - L)^m \cong \mathcal{O}_X(m(bD - L))$, $bD - L$ is relatively agg. Thus, $bD = (bD - L) + L$ is relatively almost ample, by 5.1.12(i). Since $\mathcal{R}(X, bD - L)$ is finitely generated, because L is Cartier and $\mathcal{R}(X, bD)$ is finitely generated, then bD is relatively ample. Hence, D is relatively ample. \square

Theorem 5.2.5. *Let $X \rightarrow U$ be a projective morphism of normal projective varieties over an algebraically closed field k . If $g : Y \rightarrow X$ is a small projective birational map over U*

such that $\bar{D} := g_*^{-1}D$ is \mathbb{Q} -Cartier and g -ample, then D is ample over U if and only if \bar{D} is ample over U .

Proof. Let D be ample over U . By definition, for every Cartier divisor H ample over U , there exists a rational $\varepsilon > 0$ such that $D - \varepsilon H$ is relatively nef. Moreover, $g_*^{-1}(D - \varepsilon H) = \bar{D} - \varepsilon g^*H$ is Cartier and g -ample for $0 < \varepsilon \ll 1$. By 5.2.2 $g_*^{-1}(D - \varepsilon H)$ is nef over U . Hence, there exists $m \gg 0$ such that $g_*^{-1}(mD) = mg_*^{-1}(D - \varepsilon H) + g^*(m\varepsilon H)$ is ample over U .

Similarly, if $\bar{D} = g_*^{-1}D$ is ample over U (and g -ample), for every Cartier divisor H on X ample over U , there exists a rational $1 \gg \varepsilon > 0$ such that $g_*^{-1}(D) - g^*(\varepsilon H) = g_*^{-1}(D - \varepsilon H)$ is nef over U and g -ample. Then by 5.2.2, $D - \varepsilon H$ is nef over U , and this concludes the proof. \square

Remark 5.2.6. As observed before, if D is ample, such a map $g : Y \rightarrow X$ must necessarily exist.

5.2.3 Big Weil divisors

Proposition 5.2.7 (see [KM98, 3.23]). *Let $f : X \rightarrow U$ be a projective morphism of quasi-projective normal varieties over an algebraically closed field k (U irreducible), and let D be a Weil \mathbb{Q} -divisor on X . The following are equivalent:*

- (a) *there exist an almost f -ample Weil \mathbb{Q} -divisor A and an effective divisor E such that $D \sim_{f, \mathbb{Q}} A + E$;*
- (b) *there exist an f -ample \mathbb{Q} -Cartier divisor and an effective divisor E such that $D \sim_{f, \mathbb{Q}} A + E$ (D is f -big).*

They both imply

- (c) *there exists a $c > 0$ such that $\text{rank } f_*\mathcal{O}_X(mD) > cm^n$ for all $m > 0$, $1 \parallel m$, where n is the dimension of the general fiber of f .*

Moreover, if $\mathcal{R}(X, D)$ is finitely generated, then they are all equivalent.

Proof. Clearly (b) \Rightarrow (a) and (a) \Rightarrow (b) is a consequence of 5.1.12(g). Indeed, you can write $A \sim_{\mathbb{Q}} A' + E'$, where A' is f -ample and \mathbb{Q} -Cartier and E' is effective, and thus $D \sim_{\mathbb{Q}} A' + (E' + E)$. It is immediate that (b) \Rightarrow (c).

So let us assume (c) and $\mathcal{R}(X, D)$ finitely generated, and prove (b). Since both (b) and (c) are asymptotic conditions, we can assume that $\mathcal{R}(X, D)$ is finitely generated in degree 1, that is, $\mathcal{O}_X(mD) \cong \mathcal{O}_X(D)^m$. Moreover, using the same technique as in the proof of 5.1.12(g), we can assume that $U = \text{Spec } R$ is affine. Thus, condition (c) becomes

$$\text{rank}_R H^0(X, \mathcal{O}_X(mD)) > cm^n,$$

for some $c > 0$ and all $m > 0, 1 \parallel m$. Let \mathcal{A} be an f -ample \mathbb{Q} -Cartier line bundle, let $A \in |\mathcal{A}|$ be a general enough element, reduced and irreducible (by generality of \mathcal{A} and $A \in |\mathcal{A}|$), and let $i : A \rightarrow X$ be the inclusion.

The first step is to notice that $i^*\mathcal{O}_X(D)$ is torsion-free and invertible at the generic point of A . Both properties can be checked locally on X , so that we can assume that X is affine. Let $X = \text{Spec } B$, $A = \text{Spec } B/h$, for some irreducible element $h \in B$. By hypothesis, h is not a unit. Let $K = \text{Frac}(B)$ and $L = \text{Frac}(B/h)$. Let $\mathcal{O}_X(D) \cong M^{\sim}$. Moreover, we can always twist $\mathcal{O}_X(D)$ with a Cartier divisor and assume that $M \subseteq B$. Under these assumptions, $i^*\mathcal{O}_X(D) \cong M \otimes_B B/h$. Let $m \in M \otimes_B B/h$ be torsion, $m \neq 0$. Then, without loss of generality, m is the image of an element of M , which by abuse of notation we can still denote by m . Since the image of m in $M \otimes_B B/h$ is torsion, there exists $b \in B/h, b \neq 0$, such that $bm = 0$ in $M \otimes_B B/h$. Again, by abuse of notation, we can think of b as the class of an element $b \in B$. Thus, we have that $h \nmid m$ ($m \in M \subseteq B$) and $h \nmid b$. However, $h \mid (bm)$. Since h is irreducible, this is a contradiction. Now, let us compute the rank of $M \otimes_B B/h$. Let us assume, by contradiction, that

$$\text{rank } M \otimes_B B/h = \dim_L M \otimes_B B/h \otimes_{B/h} L \geq 2.$$

Let $s_1, s_2 \in M \otimes_B B/h$ be two L -linearly independent elements. As before, we can assume that they are the image of two elements (again denoted by s_1 and s_2) in M . Since M has rank 1, there exists a non-trivial combination $\kappa_1 s_1 + \kappa_2 s_2 = 0$, with $\kappa_i \in K, i = 1, 2$, and at least one of the $\kappa_i \neq 0$. Since K is the fraction field of A , we can assume that $\kappa_i \in B, i = 1, 2$.

This relation induces a relation in $M \otimes_B B/h$. Since s_1 and s_2 are linearly independent in $M \otimes_B B/h$, h must divide both κ_1 and κ_2 . Substituting κ_i with κ_i/h ($i = 1, 2$), we obtain a new relation $\kappa_1 s_1 + \kappa_2 s_2 = 0$, with $k_i \in B$, $i = 1, 2$. Since we can repeat this process infinitely many times, this implies that each κ_i is infinitely divisible by h . Since h is not a unit and at least one of the κ_i is non-zero, this is a contradiction.

The second step is to restrict to A , and use induction. We have the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-A) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_A \rightarrow 0.$$

Tensoring with $\mathcal{O}_X(mD)$ with get

$$\mathcal{O}_X(mD - A) \rightarrow \mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD)|_A \rightarrow 0.$$

Let \mathcal{K}_m be the kernel of $\mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD)|_A$, so that we have the short exact sequence

$$0 \rightarrow \mathcal{K}_m \rightarrow \mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD)|_A \rightarrow 0$$

and $\mathcal{K}_m \subseteq \mathcal{O}_X(mD - A)$. The above short exact sequence induces a long exact sequence at the level of cohomology:

$$0 \rightarrow H^0(X, \mathcal{K}_m) \rightarrow H^0(X, \mathcal{O}_X(mD)) \rightarrow H^0(A, \mathcal{O}_X(mD)|_A).$$

As observed before, $\mathcal{O}_X(D)|_A$ is a torsion-free sheaf of rank 1. Moreover, $\mathcal{O}_X(mD)|_A \cong \mathcal{O}_X(D)^m|_A \cong (\mathcal{O}_X(D)|_A)^m$. Since $\mathcal{O}_X(D)|_A$ is torsion-free of rank 1, there exists a very ample Cartier divisor H on A over $U = \text{Spec } R$ such that $\mathcal{O}_X(D)|_A \subseteq \mathcal{O}_A(H)$ (as in [KM98, 2.58]). Then, for any m ,

$$\text{rank}_R H^0(A, \mathcal{O}_X(mD)|_A) \leq \text{rank}_R H^0(A, \mathcal{O}_A(mH)) < C m^{n-1},$$

for some $C > 0$. Since, by hypothesis

$$\text{rank}_R H^0(X, \mathcal{O}_X(mD)) \geq c m^n,$$

for some $c > 0$, for a positive $m \gg 0$, $H^0(X, \mathcal{K}_m)$ has a section. Since $\mathcal{K}_m \subseteq \mathcal{O}_X(mD - A)$, $H^0(X, \mathcal{K}_m) \subseteq H^0(X, \mathcal{O}_X(mD - A))$. Thus, this section determines an effective divisor $E \sim mD - A$, which concludes the proof. \square

Lemma 5.2.8. *Let $f : Y \rightarrow X$ be a proper birational morphism between normal varieties over U and let D be a Weil divisor on X . If D is big over U , so are $f^{\natural}(mD)/m$, $-f^{\natural}(-mD)/m$, $f^*(D)$ and $-f^*(-D)$, for all $m \geq 1$.*

Proof. For simplicity we will only show that $f^{\natural}(D)$ is big over U . The proof proceeds in the same way for all pullbacks. Let $p : X \rightarrow U$ and let $q : Y \rightarrow U$. Since D is p -big, we can find a p -ample \mathbb{Q} -Cartier divisor A and an effective Weil \mathbb{Q} -divisor E such that $D \sim_{\mathbb{Q}} A + E$. Then, $f^{\natural}(D) = f^{\natural}(A + E) = f^*(A) + f^{\natural}(E)$. Since E is effective, so is $f^{\natural}(E)$. Since A is a p -ample \mathbb{Q} -Cartier and f is birational, $f^*(A)$ is q -big; thus there exist a q -ample \mathbb{Q} -Cartier divisor B and an effective divisor E' such that $f^*(A) \sim_{\mathbb{Q}} B + E'$. Hence $f^{\natural}(D) \sim_{\mathbb{Q}} B + (E' + f^{\natural}(E))$, with B q -ample and $E' + f^{\natural}(E) \geq 0$. \square

Theorem 5.2.9. *Let $X \rightarrow U$ be a projective morphism of normal projective varieties over an algebraically closed field k . If $g : Y \rightarrow X$ is a small projective birational map over U such that $\bar{D} := g_*^{-1}D$ is \mathbb{Q} -Cartier and g -ample, then D is big over U if and only if \bar{D} is big over U .*

Proof. Since $\bar{D} = g^{\natural}(D)$ (g is small), if D is big over U , so is \bar{D} by 5.2.8.

Let \bar{D} be big over U . Let $f : X \rightarrow U$ and let $h = f \circ g : Y \rightarrow U$. For each $m \geq 0$, $g_*\mathcal{O}_Y(m\bar{D}) = \mathcal{O}_X(mD)$. Pushing forward the sheaves, $h_*\mathcal{O}_Y(m\bar{D}) = f_*g_*\mathcal{O}_Y(m\bar{D}) = f_*\mathcal{O}_X(mD)$. Since \bar{D} is h -big over U , there exists $c > 0$ such that

$$cm^n < \text{rank } h_*\mathcal{O}_Y(m\bar{D}) = \text{rank } f_*\mathcal{O}_X(mD)$$

for all $m > 0$ sufficiently divisible. By 5.2.7, D is f -big. \square

5.2.4 Pseudo-effective Weil divisors

Lemma 5.2.10. *Let $f : X \rightarrow U$ be a projective morphism of quasi-projective normal varieties over an algebraically closed field k (U irreducible), and let D be a Weil \mathbb{Q} -divisor on X . Then D is f -pseff if and only if, for every f -ample \mathbb{Q} -Cartier divisor A , $A + D \sim_{f, \mathbb{Q}} E$, for some E effective.*

Proof. If D is f -pseff, for each f -ample \mathbb{Q} -Cartier A , $D + A$ is f -big, and thus clearly \mathbb{Q} -linearly equivalent over U to an effective divisor.

Conversely, let D verify the above condition. Let A be any f -ample \mathbb{Q} -Cartier divisor. Then $D + A/2 \sim_{f, \mathbb{Q}} E$, for some E effective. Thus, since A is \mathbb{Q} -Cartier, $A + D = A/2 + (A/2 + D) \sim_{f, \mathbb{Q}} A/2 + E$, which is f -big. \square

Theorem 5.2.11. *Let $f : X \rightarrow U$ be a projective morphism of normal projective varieties over an algebraically closed field k . If $g : Y \rightarrow X$ is a small projective birational map over U such that $\bar{D} := g_*^{-1}D$ is \mathbb{Q} -Cartier and g -ample, then D is pseudo-effective over U if and only if \bar{D} is pseudo-effective over U .*

Proof. Let D be f -pseff. Let A be any \mathbb{Q} -Cartier f -ample divisor. The map g is small, so $\bar{D} + g^*A = g_*^{-1}(D + A)$; thus the divisor $g_*^{-1}(D + A)$ is still g -ample since g^*A is g -trivial and \bar{D} is g -ample. Moreover, by assumption $D + A$ is f -big. Hence, by 5.2.9, $g_*^{-1}(D + A) = \bar{D} + g^*A$ is big over U . Since the pseudo-effective cone is the closure of the big cone, \bar{D} is pseudo-effective over U .

Conversely, let \bar{D} be pseudo-effective over U , and let A be an f -ample Cartier divisor on X . Since \bar{D} is ample over X and A is ample over U , there exists $k \geq 2$ such that $\bar{D} + kg^*A$ is ample over U . Since \bar{D} is pseudo-effective over U and $\bar{D} + kg^*A$ is ample over U , $k(\bar{D} + g^*A) = (k-1)\bar{D} + \bar{D} + kg^*A$ is big over U . Thus $\bar{D} + g^*A$ is big over U . Since $\bar{D} + g^*A = g_*^{-1}(D + A)$ is g -ample, by 5.2.9 $D + A$ is big over U . Since this is true for every A ample over U , D is f -pseff. \square

5.3 Vanishing theorems

From now on, all of our varieties we will be defined over an algebraically closed field k .

When talking about positivity of divisors on projective varieties, it is natural to talk about vanishing theorems. Two of the main theorems are Fujita (or Serre) in arbitrary characteristic and Kawamata-Viehweg (or Kodaira) in characteristic 0. It is unlikely for Weil divisors to have vanishing theorems as strong as those for Cartier divisors.

5.3.1 Serre and Fujita vanishing

Theorem 5.3.1 (Relative Fujita vanishing for locally free sheaves). *Let $f : X \rightarrow U$ be a projective morphism of quasi projective varieties, let H be an f -ample Weil divisor on X*

and let \mathcal{F} be a locally free coherent sheaf on X . There exists an integer $m(\mathcal{F}, H)$ such that

$$R^i f_* (\mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0, \quad \text{for all } i > 0,$$

for all positive m such that $m(\mathcal{F}, H) | m$ and for any relatively nef Cartier divisor D on X .

Proof. Let $g : Y \rightarrow X$ be the \mathbb{Q} -Cartierization of H (which exists since H is f -ample). Let $\bar{H} := f_*^{-1}H$ and let $h = f \circ g : Y \rightarrow U$; by theorem 5.2.5, \bar{H} is h -ample, and f -ample. By lemma 1.1.7, for all $m \geq 0$,

$$g_* \mathcal{O}_Y(m\bar{H}) = \mathcal{O}_X(mH).$$

Let D be any f -nef Cartier divisor; let us denote $\mathcal{F}(D) = \mathcal{F} \otimes \mathcal{O}_X(D)$. By the projection formula,

$$\begin{aligned} g_*(g^* \mathcal{F}(D) \otimes \mathcal{O}_Y(m\bar{H})) &\cong \mathcal{F} \otimes \mathcal{O}_X(D) \otimes g_* \mathcal{O}_Y(m\bar{H}) \cong \\ &\cong \mathcal{F} \otimes \mathcal{O}_X(D + mH) \end{aligned} \quad (5.3.1)$$

for all $m \geq 0$. The sheaf $g^* \mathcal{O}_X(D)$ is g -trivial while $g^* \mathcal{F}$ is still coherent and $\mathcal{O}_Y(\bar{H})$ is g -ample; thus there exists $m_1 := m_1(H)$ such that, for all $m_1 | m$,

$$R^j g_*(g^* \mathcal{F}(D) \otimes \mathcal{O}_Y(m\bar{H})) = 0, \quad \text{for all } j > 0 \quad (5.3.2)$$

(this is the relative version of Serre's vanishing, [Kee03, 1.5]). We have the Grothendieck spectral sequence

$$R^i f_* \left(R^j g_*(g^* \mathcal{F}(D) \otimes \mathcal{O}_Y(m\bar{H})) \right) \Rightarrow R^{i+j} h_*(g^* \mathcal{F}(D) \otimes \mathcal{O}_Y(m\bar{H})).$$

By (5.3.2), the Grothendieck spectral sequence converges immediately, and we have the equalities

$$R^i f_* \left(g_*(g^* \mathcal{F}(D) \otimes \mathcal{O}_Y(m\bar{H})) \right) = R^i h_*(g^* \mathcal{F}(D) \otimes \mathcal{O}_Y(m\bar{H}))$$

for every $m_1 | m$ and every $i \geq 0$. By (5.3.1), the cohomology on the left-hand of the equality is

$$R^i f_* \left(g_*(g^* \mathcal{F}(D) \otimes \mathcal{O}_Y(m\bar{H})) \right) = R^i f_*(\mathcal{F} \otimes \mathcal{O}_X(D + mH)),$$

for all $m_0|m$. On the other hand, since g^*D is h -nef on Y (it is the pullback of an f -nef divisor), $g^*\mathcal{F}$ is coherent and $\mathcal{O}_Y(\bar{H})$ is h -ample, by the relative Fujita's vanishing theorem, [Kee03, 1.5], there exists $m_2(H, g^*\mathcal{F}) = m_2(H, \mathcal{F})$ such that, for all $m_2 := m_2(H, \mathcal{F})|m$ and for all $i > 0$,

$$R^i h_*(g^*\mathcal{F}(D) \otimes \mathcal{O}_Y(m\bar{H})) = R^i h_*(g^*\mathcal{F} \otimes \mathcal{O}_Y(m\bar{H} + g^*D)) = 0.$$

Thus, if $m(H, \mathcal{F}) = \text{lcm}(m_1, m_2)$, for all $i > 0$ and positive $m(H, \mathcal{F})|m$,

$$R^i f_*(\mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0.$$

□

Corollary 5.3.2 (Fujita vanishing for locally free sheaves). *Let X be a projective variety, let H be an ample Weil divisor and let \mathcal{F} be any locally free coherent sheaf on X . There exists an integer $m(\mathcal{F}, H)$ such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = 0, \quad \text{for all } i > 0,$$

for all positive m such that $m(\mathcal{F}, H)|m$ and for any nef Cartier divisor D on X .

There are three possible ways of generalizing the previous result.

Question 5.3.3. Is Fujita vanishing true if \mathcal{F} is not necessarily locally free?

Let us point out that such generalization is not tremendously important, since, in the applications, \mathcal{F} is an invertible sheaf.

Question 5.3.4. Is Fujita vanishing true if we ask for H to be \mathbb{Q} -Cartier, but we allow D to be Weil?

Question 5.3.5. Is Fujita vanishing true if both H and D are allowed to be Weil?

Note that 5.3.5 is, at a first glance, too much to ask, while there might be some hope for 5.3.4. Moreover, 5.3.4 appears to be what is needed in many applications (especially in positive characteristic)¹. The good/bad news is that 5.3.5 and 5.3.4 are equivalent when nefness is additive (see 5.1.16 and 6.2.1).

¹This statement is based on communications with K. Schwede and K. Tucker

Lemma 5.3.6. *Let X be such that nefness is additive. Then, the two questions 5.3.4 and 5.3.5 are equivalent.*

Proof. Let us assume that \mathcal{F} satisfies whichever hypothesis are needed for the version of Fujita vanishing that we are considering (for example locally free).

Clearly if 5.3.5 is true, so is 5.3.4. Conversely, let us assume that 5.3.4 is true, and let H be any ample Weil divisors. Let L be a fixed ample Cartier divisor and let $b > 0$ such that $bH - L$ is Weil nef. Then, for every nef Weil divisor D and every m divisible by some $m_0 = m(\mathcal{F}, \mathcal{L}, b)$,

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(mH + D)) = H^i\left(X, \mathcal{F} \otimes \mathcal{O}_X\left(\frac{m}{b}L + (m(H - L/b) + D)\right)\right) = 0$$

since $m(H - L/b) + D$ is nef, by the additivity of nefness. \square

5.3.2 Kawamata-Viehweg vanishing

With a similar technique to the one used in the proof of the relative Fujita vanishing, we can prove the relative Kawamata-Viehweg.

Theorem 5.3.7 (Relative Kawamata-Viehweg vanishing). *Let $f : X \rightarrow U$ be a projective morphism of complex quasi-projective varieties. Let (X, Δ) be a klt pair and let D be a Weil divisor on X such that $D - (K_X + \Delta)$ is f -big and f -nef. Then*

$$R^i f_* \mathcal{O}_X(D) = 0, \quad \text{for } i > 0.$$

Proof. Since (X, Δ) is klt, both $\mathcal{R}(X, D)$ and $\mathcal{R}(X, D - K_X - \Delta)$ are finitely generated. Let $g : Y = \text{Proj}_X \mathcal{R}(X, D) \rightarrow X$, let $\bar{D} = g_*^{-1}D$ and let $h = f \circ g : Y \rightarrow U$. Let $\Delta_Y = g_*^{-1}\Delta$. Since g is small $K_Y + \Delta_Y = g^*(K_X + \Delta)$ and $g_*\Delta_Y = \Delta$. By [KM98, 2.30], (Y, Δ_Y) is a klt pair. Moreover, since \bar{D} is g -ample and $K_Y + \Delta_Y = g^*(K_X + \Delta)$ is g -trivial, $\bar{D} - (K_Y + \Delta_Y) = \bar{D} - g^*(K_X + \Delta)$ is still g -ample and $\bar{D} - (K_Y + \Delta_Y) = g_*^{-1}(D - (K_X + \Delta))$. By 5.2.2 and 5.2.9, $\bar{D} - (K_Y + \Delta_Y)$ is \mathbb{Q} -Cartier, h -nef and h -big. By the relative Kawamata-Viehweg vanishing,

$$R^i h_* \mathcal{O}_Y(\bar{D}) = 0, \quad \text{for } i > 0.$$

On the other hand, by 1.1.7, $g_*\mathcal{O}_Y(\bar{D}) = \mathcal{O}_X(D)$, and by the relative Kawamata-Viehweg vanishing

$$R^i g_*\mathcal{O}_Y(\bar{D}) = 0, \quad \text{for } i > 0,$$

since $\bar{D} - (K_Y + \Delta_Y)$ is g -ample. Thus the Grothendieck spectral sequence

$$R^i f_* R^j g_*\mathcal{O}_Y(\bar{D}) \Rightarrow R^{i+j} h_*\mathcal{O}_Y(\bar{D})$$

converges immediately, and we have

$$R^i f_*\mathcal{O}_X(D) = R^i f_*(g_*\mathcal{O}_Y(\bar{D})) \cong R^i h_*\mathcal{O}_Y(\bar{D}) = 0, \quad \text{for } i > 0.$$

□

Corollary 5.3.8 (Kawamata-Viehweg vanishing). *Let X be a complex projective variety; let (X, Δ) be a klt pair. Let D be a Weil divisor such that $D - (K_X + \Delta)$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(D)) = 0, \quad \text{for } i > 0.$$

Chapter 6

A NON- \mathbb{Q} -GORENSTEIN MMP

In this chapter we discuss a non- \mathbb{Q} -Gorenstein Minimal Model Program. We outline such program in §6.1. In §6.2 we construct the space $\text{NS}(X)_W$ (which we show is finitely dimensional in 6.2.9) and we define the nef and ample cones and the cone of curves. We also prove a non- \mathbb{Q} -Cartier version of Kleiman's ampleness criterion, 6.2.15. In §6.3 we prove a result related to Fano varieties, 6.3.1, and a (weak) global generation statement, 6.3.4. In §6.4 we focus on possible contraction theorems. In §6.4.1 we discuss two approaches that work in the toric case, but seem not to work in the general case, and in §6.4.2 and §6.4.3 we discuss two more techniques. In particular, in §6.4.3 we prove a stronger global generation statement, 6.4.9, and a special rational contraction theorem, 6.4.11. As an application of our techniques, we prove the existence of non- \mathbb{Q} -factorial log terminal flips, 6.4.7.

The results in §6.3 and 6.4.4 are from [CU13].

In this chapter we work over an algebraically closed field k of characteristic 0.

6.1 The program

We start by presenting what should be a non- \mathbb{Q} -Gorenstein Minimal Model Program.

Step 0 Let X be a projective variety with log terminal singularities.

Step 1 If K_X is nef, there is nothing to do and the program terminates. Otherwise, let $\varphi : X \dashrightarrow Y$ be a contraction.

Step 2 There are two possibilities for the contraction above.

- (i) If $\dim Y < \dim X$, the generic fiber of φ is a Fano variety. We set $X_0 := Y$ and we can restart the program in smaller dimension.

(ii) Otherwise, we set $X_{i+1} := Y$, and we restart the algorithm.

The biggest problem, at the moment, is how to obtain the contraction in Step 1. For the program to be well defined, we need to show that at each step we are remaining in the same class of singularities, that is, that the variety Y of Step 2 has log terminal singularities, and, finally, there is the problem of termination. Depending on the approach, we will discuss some of these issues.

6.2 The nef cone

In this section, X will always be a (normal) projective variety with log terminal singularities.

Lemma 6.2.1. *Let X be a normal projective variety with log terminal singularities. Then nefness is additive (for \mathbb{Q} -divisors).*

Proof. By 1.1.8, all the algebras $\mathcal{R}(X, D)$ are finitely generated (for \mathbb{Q} -divisors). Let D_1 and D_2 be nef Weil divisors and let A be any ample \mathbb{Q} -Cartier divisor. By definition, $A/2 + D_1$ and $A/2 + D_2$ are agg. Let m_1 be such that, for every positive m divisible by m_1 (and by 2), $\mathcal{O}_X(m(A/2 + D_1))$ is globally generated. Let m_2 be similarly defined for D_2 and let $m_0 = \text{lcm}(m_1, m_2)$. Then, for each positive m divisible by m_0 , $\mathcal{O}_X(m(A/2 + D_1))$ and $\mathcal{O}_X(m(A/2 + D_2))$ are globally generated. Hence, so is their tensor product

$$\mathcal{O}_X(m(A/2 + D_1)) \otimes \mathcal{O}_X(m(A/2 + D_2)).$$

This sheaf naturally surjects onto

$$\mathcal{O}_X(m(A/2 + D_1)) \cdot \mathcal{O}_X(m(A/2 + D_2)),$$

which is therefore also globally generated. Since all the algebras of local sections are finitely generated,

$$\mathcal{O}_X(m(A/2 + D_1)) \cdot \mathcal{O}_X(m(A/2 + D_2)) \cong \mathcal{O}_X(m(A + D_1 + D_2)).$$

Thus $D_1 + D_2$ is nef. □

Definition 6.2.2. *We say that a \mathbb{Q} -Weil divisor D on a normal projective variety is numerically trivial if it is nef and anti-nef (i.e., $-D$ is nef).*

Remark 6.2.3. We do not need nefness to be additive to give the above definition. However, when it is, the set of numerically trivial \mathbb{Q} -Weil divisors is a subgroup of $\text{Div}(X)_{\mathbb{Q}}$.

Definition 6.2.4. Let X be a projective variety with log terminal singularities. We define $\text{NS}(X)_{W,\mathbb{Q}}$ to be the space of \mathbb{Q} -Weil divisors modulo the numerically trivial ones, and by $\text{NS}(X)_W := \text{NS}(X)_{W,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.

It is well known that $\text{NS}(X)_{\mathbb{R}}$ is a finitely generated vector space. The standard proof, see [Kle66], uses the fact that, if $f : X \rightarrow Y$ is a proper, birational morphism, then the pullback induces an injection $f^* : \text{NS}(Y)_{\mathbb{R}} \hookrightarrow \text{NS}(X)_{\mathbb{R}}$. Indeed, clearly the pullback of a nef \mathbb{Q} -Cartier divisor is nef. Moreover, if C is a curve on Y , there is a curve C' on X such that $f(C') = C$, [Kle66, lemma 4.1], and $D.C = f^*D.C'$ for every \mathbb{Q} -Cartier divisor D on Y .

Using this, we can reduce to the smooth case (using a resolution), where the result is well-known. Since the pullback of Weil divisors is not numerically trivial on the fibers (see the discussion in §3.1), a proper, birational morphism $f : X \rightarrow Y$ will not induce a map $\text{NS}(Y)_W \rightarrow \text{NS}(X)_W$ in general, and, even when such map exists, it will not necessarily be injective. Moreover, we do not define nefness for Weil divisors with intersection with curves, so the technique of [Kle66] does not work in this case. We will prove that $\text{NS}(X)_W$ is finitely generated in 6.2.9, but we need some preliminary results.

The same argument of 6.2.1 shows that, when the singularities are log terminal, ampleness is additive. Moreover, we can now generalize 5.2.1, without assuming that either D or H be \mathbb{Q} -Cartier.

Lemma 6.2.5. Let X be a projective variety with log terminal singularities. Let D be a divisor on X and let H be an ample divisor. Then D is nef if and only if, for all sufficiently small $\varepsilon \in \mathbb{Q}$, $0 < \varepsilon \ll 1$, $D + \varepsilon H$ is ample.

This in particular implies that ampleness is numerically invariant also for Weil divisors.

Definition 6.2.6. Let X be a projective variety with log terminal singularities. We define the ample cone (resp. the nef cone, resp. the big cone, resp. the pseudo-effective one) to be the subset of $\text{NS}(X)_W$ generated by the ample (resp. nef, resp. big, resp. pseff) \mathbb{Q} -

Weil divisors, and it will be denoted by $\text{Amp}(X)_W$ (resp. $\text{Nef}(X)_W$, resp. $\text{Big}(X)_W$, resp. $\text{Pseff}(X)_W$).

By the above discussions, these sets are indeed cones,

$$\text{Nef}(X)_W = \overline{\text{Amp}(X)_W} \quad (6.2.1)$$

and $\text{Amp}(X)_W = \text{Int}(\text{Nef}(X)_W)$ (and $\text{Pseff}(X)_W = \overline{\text{Big}(X)_W}$). They are also full dimensional, in the sense that every \mathbb{R} -Weil divisor is the difference of two nef/ample \mathbb{R} -Weil divisors. This is an immediate consequence of the next lemma.

Lemma 6.2.7. *Let X be a projective variety with log terminal singularities. For any Weil divisor D and ample Cartier divisor A , there exists $b > 0$ such that $bA + D$ is ample.*

Proof. Let b' such that $m(b'A + D)$ is globally generated for some $m \gg 0$ (it is enough for m to be divisible by the all the degrees of the generators of $\mathcal{R}(X, D)$). Since all the algebras of local sections are finitely generated, this implies that $b'A + D$ is agg. But then $(b' + 1)A + D = (b'A + D) + A$ is ample. \square

Now we can prove that the space $\text{NS}(X)_W$ is finitely generated. The following lemma is a non- \mathbb{Q} -Cartier version of [Kle66, Proposition 4.1].

Lemma 6.2.8. *Let $f : Y \rightarrow X$ be a small, projective, birational morphism of normal projective varieties and let A be an ample \mathbb{R} -Cartier divisor on Y . Then f_*A is nef.*

Proof. It is enough to assume that A is \mathbb{Q} -Cartier.

Since A is ample and f is small, $f_*\mathcal{O}_Y(mA) = \mathcal{O}_X(mf_*A)$ for m sufficiently divisible. Indeed, since A is ample, for m sufficiently divisible $\mathcal{O}_Y(mA)$ is globally generated. Moreover, since f is small and A is relatively asymptotically globally generated, $f_*\mathcal{O}_Y(mA) = \mathcal{O}_X(mf_*A)$ for m sufficiently divisible.

Let B be an ample \mathbb{Q} -Cartier divisor on X ; then f^*B is a nef \mathbb{Q} -Cartier divisor on Y . Thus $f^*B + A$ is ample. By what observed before, $f_*\mathcal{O}_Y(m(A + f^*B)) = \mathcal{O}_X(m(f_*A + B))$, which is also globally generated for m sufficiently divisible. By definition, f_*A is nef. \square

Theorem 6.2.9. *Let X be a projective normal variety with log terminal singularities. The space $\text{NS}(X)_W$ is finitely dimensional.*

Proof. Let $f : Y \rightarrow X$ be a \mathbb{Q} -factorialization of X , that is, f is a small, projective, birational morphism with Y normal and \mathbb{Q} -factorial. This map exists since the singularities are log terminal, see [Kol08, 108]. Then, f^* is an isomorphism $f^* : \text{Div}(X)_{\mathbb{R}} \rightarrow \text{Div}(Y)_{\mathbb{R}} \cong \text{CDiv}(Y)_{\mathbb{R}}$. Let N_Y be the subspace of numerically trivial \mathbb{R} -Weil divisors on Y , and similarly N_X the subspace of numerically trivial \mathbb{R} -Weil divisors on X . Then $\text{NS}(X)_W = \text{Div}(X)_{\mathbb{R}}/N_X$ and $\text{NS}(Y)_W = \text{NS}(Y)_{\mathbb{R}} = \text{Div}(Y)_{\mathbb{R}}/N_Y$. Let $N = N_X \cap (f^*)^{-1}N_Y$. The pullback does not induce a morphism $\text{NS}(X)_W \rightarrow \text{NS}(Y)_W$; however, it induces a morphism $\text{Div}(X)_{\mathbb{R}}/N \rightarrow \text{NS}(Y)_{\mathbb{R}}$, and $\text{Div}(X)_{\mathbb{R}}/N$ naturally surjects onto $\text{NS}(X)_W$. Since $\text{NS}(Y)_{\mathbb{R}}$ is finitely generated, it is enough to prove that the map $\text{Div}(X)_{\mathbb{R}}/N \rightarrow \text{NS}(Y)_{\mathbb{R}}$ is injective:

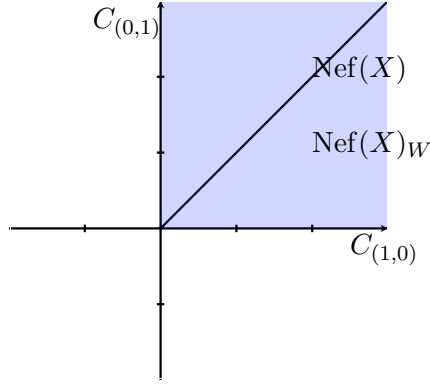
$$\begin{array}{ccc}
 \text{Div}(X)_{\mathbb{R}} & \xrightarrow[\cong]{f^*} & \text{Div}(Y)_{\mathbb{R}} \\
 \downarrow & & \downarrow \\
 \text{Div}(X)_{\mathbb{R}}/N & \xrightarrow{f^*} & \text{NS}(Y)_{\mathbb{R}} \\
 \downarrow & & \\
 \text{NS}(X)_W & &
 \end{array}$$

Let D be a class of Weil divisor such that f^*D is numerically trivial on Y . We will show that D is nef. Since the same reasoning on $-D$ shows that D is anti-nef, we will be done. Let B be any ample \mathbb{Q} -Cartier divisor on Y . Since f^*D is nef, $f^*D + B$ is ample. By 6.2.8, $D + f_*B$ is nef. Since nefness is a closed property, D is nef. \square

Remark 6.2.10. Since $\text{Div}(Y)_{\mathbb{R}} \rightarrow \text{NS}(Y)_{\mathbb{R}}$ is surjective, $\text{Div}(X)_{\mathbb{R}}/N \rightarrow \text{NS}(Y)_{\mathbb{R}}$ is also surjective, which implies that $\text{Div}(X)_{\mathbb{R}}/N \cong \text{NS}(Y)_{\mathbb{R}}$. So, in particular, there is no natural map $\text{NS}(X)_W \rightarrow \text{NS}(Y)_W$, but the pushforward induces a natural surjective map $\text{NS}(Y)_W \rightarrow \text{NS}(X)_W$.

Example 6.2.11. Let us consider the quadric cone $X = \{xy - zw = 0\}$ in \mathbf{P}^4 . Every Weil divisor is \mathbb{Q} -linearly equivalent to a cone C_D over a divisor D in $\mathbf{P}^1 \times \mathbf{P}^1$. Thus $\text{NS}(X)_W \cong \mathbb{R}^2$. Moreover, if D is of type (a, b) , C_D is nef if and only if $a, b \geq 0$ and C_D is ample if and only if $a, b > 0$, 5.1.13. Thus the nef cone is $\text{Nef}(X)_W = \{(x, y) \mid x, y \geq 0\}$ and $\text{Amp}(X)_W = \{(x, y) \mid x, y > 0\}$. The only Cartier divisors on X are \mathbb{Q} -linearly equivalent to the cone over a divisor of type $(1, 1)$. Hence, in this case, $\text{Nef}(X) = \{(x, x) \mid x \geq 0\}$ and

$\text{Amp}(X) = \{(x, x) \mid x > 0\}$. We have the following picture:



Example 6.2.12. Continuing the previous example, we have $\text{Big}(X)_W = \text{Amp}(X)_W$ and $\text{Pseff}(X)_W = \text{Nef}(X)_W$. In particular, the divisor $C_{(0,-1)}$ is not nef. However, if a divisor Δ is effective, it must be numerically equivalent to a divisor $C_{(a,b)}$ with $a, b \geq 0$. Thus, for any effective Weil divisor Δ , such that $C_{(0,-1)} + \Delta$ is \mathbb{Q} -Cartier, $C_{(0,-1)} + \Delta$ is nef (the nearest \mathbb{Q} -Cartier divisor possible being $C_{(0,0)} \equiv 0$).

What is happening is very simple. It is true that a neighborhood of D in $\text{NS}(X)_W$ avoids the nef cone; however, the \mathbb{R} -Cartier divisors are not dense, so we hit the nef cone when we try to modify the Weil divisor with an effective boundary to make it \mathbb{Q} -Cartier.

Using duality, we can define the cone of curves.

Definition 6.2.13. Let X be a projective variety with log terminal singularities. We define the cone of effective curves $\overline{\text{NE}}(X)_W := \text{Nef}(X)_W^*$.

Remark 6.2.14. The cone $\overline{\text{NE}}(X)_W$ is still closed. Moreover, since these cones are finite dimensional, $\overline{\text{NE}}(X)_W^* \cong \text{Nef}(X)_W$.

If D is an \mathbb{R} -Weil divisor on X , we define the following subsets (as in the \mathbb{Q} -Cartier case):

- $D_{>0} := \{x \in \text{NS}(X)_W^* \mid x(D) > 0\}$ (and similarly for ≥ 0 , < 0 , ≤ 0);
- $D^\perp := \{x \in \text{NS}(X)_W^* \mid x(D) = 0\}$;

◦ $\overline{\text{NE}}(X)_{W, D_{>0}} := \overline{\text{NE}}(X)_W \cap D_{>0}$ (and similarly for ≥ 0 , < 0 , ≤ 0).

Theorem 6.2.15 (Kleiman criterion). *Let X be a projective variety having log terminal singularities and let D be an \mathbb{R} -Weil divisor on X . Then D is ample if and only if $\overline{\text{NE}}(X)_W \setminus \{0\} \subseteq D_{>0}$.*

Proof. The proof proceeds as in the \mathbb{Q} -Cartier case, and is taken from [Laz04]. We proceed in several steps.

(1) Since $\text{NS}(X)_W$ is finitely generated, 6.2.9, $\text{Nef}(X)_W \cong \text{NE}(X)_W^*$. Then, a divisor N is nef if and only if $x(N) \geq 0$ for every $x \in \overline{\text{NE}}(X)_W$.

(2) Let us fix an ample divisor H . By 6.2.5, D is ample if and only if there exists $\varepsilon > 0$ such that

$$\frac{x(D)}{x(H)} \geq \varepsilon, \quad (6.2.2)$$

for every $x \in \overline{\text{NE}}(X)_W$. Indeed, this equation is true if and only if $D - \varepsilon H$ is nef, by (1). If D is ample, then $D - \varepsilon H$ is ample for $0 < \varepsilon \ll 1$, 6.2.5. Conversely, if $D - \varepsilon H$ is nef, then $(D - \varepsilon H) + \varepsilon H = D$ is ample.

(3) Fix any norm $\|\cdot\|$ on $\text{NS}(X)_W^*$. The inclusion $\overline{\text{NE}}(X)_W \setminus \{0\} \subseteq D_{>0}$ is equivalent to the inclusion

$$(\overline{\text{NE}}(X)_W \cap S) \subseteq (D_{>0} \cap S), \quad (6.2.3)$$

where $S = \{\|x\| = 1\} \subset \text{NS}(X)_W^*$. Let us assume (6.2.3), and let us show that D is ample. Let $L_D : \text{NS}(X)_W^* \rightarrow \mathbb{R}$ be defined as $L_D(x) = x(D)$. This is a linear functional, which is positive on $\overline{\text{NE}}(X)_W \cap S$. Since S is compact, there is positive ε such that $x(D) = L_D(x) \geq \varepsilon > 0$ for all $x \in \overline{\text{NE}}(X)_W \cap S$. Thus $L_D(x) \geq \varepsilon \|x\|$ for all $x \in \overline{\text{NE}}(X)_W$.

(4) Let H_1, \dots, H_n be ample divisors generating $\text{NS}(X)_W$. Since all norms are equivalent on a finite dimensional Banach space, $\|\cdot\|$ is equivalent to the taxicab norm, $\|x\|_{\text{taxi}} = \sum |x(H_i)|$. Since $\sum |x(H_i)| \geq |x(H)|$, where $H = \sum H_i$, there exists $\varepsilon' > 0$ such that $x(D) \geq \varepsilon' x(H)$ for all $x \in \overline{\text{NE}}(X)_W$. By (2), this implies that D is ample.

(5) Let D be ample. By duality, for each $x \in \overline{\text{NE}}(X)_W^* \setminus \{0\}$ there exists $H_x \in \text{NS}(X)_W$ such that $x(H_x) > 0$. Since D is ample, there exists $b > 0$ such that $bD - H_x$ is nef. Then, $0 \leq x(bD - H_x) = bx(D) - x(H_x)$, which implies that $x(D) \geq x(H_x)/b > 0$. \square

Definition 6.2.16. *If F is a face of $\overline{\text{NE}}(X)_W$, we say that F is K_X -negative if $x(K_X) < 0$ for all $x \in F$.*

Question 6.2.17. What is the structure of $\overline{\text{NE}}(X)_W$? More precisely, is $\overline{\text{NE}}(X)_{W, K_X < 0}$ generated by countably many rays? For each ample Weil H and for each $\varepsilon > 0$, is $\overline{\text{NE}}(X)_{W, K_X + \varepsilon H < 0}$ generated by finitely many rays?

6.3 Fano varieties and (weak) global generation

6.3.1 Fano varieties

Let us recall that a projective variety X is called *log Fano* if there exists a boundary Δ such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample, 1.6.5.

Theorem 6.3.1. *Let X be a projective variety having lt^+ singularities and such that $-K_X$ is ample (not necessarily \mathbb{Q} -Cartier). Then X is log Fano.*

Proof. Since $\mathcal{R}(X, -K_X)$ is finitely generated, we can consider the \mathbb{Q} -Cartierization of $-K_X$, $g : Y = \text{Proj}_X \mathcal{R}(X, -K_X) \rightarrow X$, with $g_*^{-1}(-K_X)$ g -ample and ample. Moreover, X has klt singularities, 4.2.2. Notice that, the map g being small, $-K_Y = g_*^{-1}(-K_X)$ and $K_Y = g^\natural(K_X)$. As in 4.2.1, we can show that Y has log terminal singularities. Indeed, since X has klt singularities, there exists an $m > 0$ such that, for any resolution $f : Z \rightarrow X$, $K_Z - \frac{1}{m}f^\natural(mK_X) > -1$. Let Z be any such log resolution, which without loss of generality we can assume factors through g as $f = g \circ h$, for some $h : Z \rightarrow Y$. Since $g_*^{-1}(-K_X) = -K_Y$ is g -ample, $\mathcal{O}_Y(-mK_Y)$ is globally generated over X for m sufficiently divisible, hence we have an isomorphism at the level of sheaves, that is, $\mathcal{O}_X(-K_X) \cdot \mathcal{O}_Y \cong \mathcal{O}_Y(-K_Y)$. Thus

$$K_Z - h^*(K_Y) = K_Z - \frac{1}{m}h^*(mK_Y) = K_Y - \frac{1}{m}f^\natural(mK_X) > -1,$$

where the last equality holds by 1.4.11. Thus Y (which is \mathbb{Q} -Gorenstein) has log terminal singularities.

Let now A be any ample Cartier divisor on X . Since $-K_Y$ is ample, there exists $b > 1$ such that $-bK_Y - g^*A$ is globally generated. Let M be the general element in the system $M \in |\mathcal{O}_X(-bK_Y - g^*A)|$. Notice that $-bK_Y - bM \sim g^*A$, which is trivial on the fibers. Let

$\Delta := g_*M/b$. Since $K_Y + M/b \sim_{\mathbb{Q}} -g^*A$, $K_X + \Delta$ is \mathbb{Q} -Cartier, and $K_X + \Delta \sim_{\mathbb{Q}} -A$. Finally, since Y has log terminal singularities, and M was general, $(Y, M/b)$ has klt singularities. Notice that $g_*(M/b) = \Delta$ and $K_Y + M/b = g^*(K_X + \Delta)$ (by construction). Thus (X, Δ) has klt singularities [KM98, 2.30]. \square

Remark 6.3.2. Recall that lt^+ is a broader class than klt, 2.3.4. This results suggests that the positivity condition on $-K_X$ affects the type of singularity. See [SS10] for a phenomenon opposite in nature.

6.3.2 Non-vanishing and Global generation

We will start by proving a non- \mathbb{Q} -Cartier version of Shokurov's non-vanishing theorem, see [KM98, 3.4]. Next we will prove a generalization of the basepoint-free theorem, see [KM98, 3.3]. We collected these two results together as they show peculiar aspects of Weil divisors, which make the original results, in some sense, impossible to fully generalize. Moreover, the technique of the proofs is the same.

Proposition 6.3.3 (Non-vanishing). *Let $f : X \rightarrow U$ be a projective morphism of quasi-projective varieties over k . Let (X, Δ) be a klt pair and let D be an f -nef divisor such that $aD - (K_X + \Delta)$ is f -nef and f -big for some $a > 0$. Then, for all positive $1 \parallel m$, $mD + [-\Delta] \sim_{\mathbb{Q}, f} F \geq 0$.*

Proof. Without loss of generality, we can assume that $a = 1$. Since (X, Δ) is klt, the algebras $\mathcal{R}(X, D)$ and $\mathcal{R}(X, D - K_X - \Delta)$ are both finitely generated. Let $g : Y = \text{Proj}_X \mathcal{R}(X, D) \rightarrow X$, let $\bar{D} = g_*^{-1}D$ and let $h = f \circ g : Y \rightarrow U$. Let $\Delta_Y = g_*^{-1}\Delta$. Since g is small $K_Y + \Delta_Y = g^*(K_X + \Delta)$ and $g_*\Delta_Y = \Delta$. By [KM98, 2.30], (Y, Δ_Y) is a klt pair. Moreover, since \bar{D} is g -ample and $K_Y + \Delta_Y = g^*(K_X + \Delta)$ is g -trivial, $\bar{D} - (K_Y + \Delta_Y) = \bar{D} - g^*(K_X + \Delta)$ is still g -ample and $\bar{D} - (K_Y + \Delta_Y) = g_*^{-1}(D - (K_X + \Delta))$. By 5.2.2 and 5.2.9, $\bar{D} - (K_Y + \Delta_Y)$ is \mathbb{Q} -Cartier, h -nef and h -big and \bar{D} is \mathbb{Q} -Cartier h -nef. So we can apply the usual non-vanishing theorem and deduce that $m\bar{D} + [-\Delta_Y] \sim_{\mathbb{Q}, h} F_Y \geq 0$.

Let $G = [-\Delta]$ and $G_Y = [-\Delta_Y] = g_*^{-1}G$. Since $g_*\mathcal{O}_Y(m\bar{D}) = \mathcal{O}_X(mD)$ and G and G_Y are effective, $g_*\mathcal{O}_Y(m\bar{D} + G_Y) \subseteq \mathcal{O}_X(mD + G)$, which in turn implies that $h_*\mathcal{O}_Y(m\bar{D} + G_Y) \subseteq f_*\mathcal{O}_X(mD + G)$. Hence $mD + G = mD + [-\Delta] \sim_{f, \mathbb{Q}} F \geq 0$. \square

Proposition 6.3.4 (Global generation). *Let (X, Δ) be a klt pair, $f : X \rightarrow U$ be a projective morphism of quasi-projective normal varieties. Let D be an f -nef divisor such that $aD - (K_X + \Delta)$ is f -nef and f -big for some $a > 0$. Then D is f -agg (relatively asymptotically globally generated).*

Proof. Since (X, Δ) is klt, $\mathcal{R}(X, D)$ is finitely generated. Let $g : Y \rightarrow X$ be the \mathbb{Q} -Cartierization of D , $\bar{D} = g_*^{-1}D$, and $h = f \circ g$. As in the previous proof, if $\Delta_Y = g_*^{-1}\Delta$, (Y, Δ_Y) is klt, \bar{D} is g -ample and h -nef, while $a\bar{D} - (K_Y + \Delta_Y)$ is h -nef and h -big. By the basepoint-free theorem, $m\bar{D}$ is h -basepoint-free for positive $1 \ll m$, that is, \bar{D} is h -agg. Since $g_*\mathcal{O}_Y(m\bar{D}) = \mathcal{O}_X(mD)$, D is f -agg. \square

6.4 The contraction theorem

In this section we focus on the issue of a possible contraction theorem for our program. We first discuss two possible strategies, and why they will not probably produce a satisfactory answer. Then we discuss, separately, two more approaches. For these, we do not ask for the contraction map to be a morphism, but we allow it to be a rational map. The reason is based on the following observation.

Remark 6.4.1. In the usual Minimal Model Program, the contraction theorem is a consequence of the basepoint-free theorem (see 1.6.2 and the subsequent discussion). As discussed in 5.1.3, global generation and basepoint-freeness do not agree for Weil divisors. For this reason, 6.3.4 is only a statement about global generation, and not basepoint-freeness. Hence, the linear system associated with bD , for b sufficiently divisible, does not induce a morphism.

6.4.1 The contraction as a morphism

In this section we will discuss two possible approaches to a contraction theorem, with the contraction being a morphism: with boundaries or with \mathbb{Q} -Cartierizations.

The former is based on the following question.

Question 6.4.2. Let X be log terminal with K_X not nef. Can we find a boundary Δ such that the pair (X, Δ) is klt and $K_X + \Delta$ is not nef?

Approach 6.1 (Boundary). Let us assume that 6.4.2 has a positive answer. Then, there exists Δ such that $K_X + \Delta$ is \mathbb{Q} -Cartier and not nef, so we can reduce to the usual contraction theorem.

There are two issues with this approach. The first one is that such boundary might not exist, as discussed in 6.2.12. Indeed, it is possible that, for all $\Delta \geq 0$ such that $K_X + \Delta$ is \mathbb{Q} -Cartier, $K_X + \Delta$ falls in the nef cone. The second issue is the following.

Lemma 6.4.3. *The number of contractions with the boundary approach is less than or equal to the Picard number of the original variety. In particular, such a program would always terminate.*

Proof. If φ is a Fano contraction, then there is nothing to do. In either of the remaining cases, we have $\rho(Y) = \rho(X) - 1$. \square

If X is our original variety, it would seem that after at most $\rho(X)$ contractions, we would have a nef canonical class. This suggests that also the number of flips in the regular MMP for X should be bounded by $\rho(X)$, and this is extremely unlikely for a general choice of X .

The latter approach is the following. Since X is klt, $\mathcal{R}(X, K_X)$ is finitely generated. Let us consider the \mathbb{Q} -Cartierization of K_X , $g : \tilde{X} = \text{Proj}_X \mathcal{R}(X, K_X) \rightarrow X$. Since K_X is not nef, neither is $K_{\tilde{X}}$. As in 4.2.1, we have that \tilde{X} has \mathbb{Q} -Gorenstein log terminal singularities. By the usual contraction theorem, we have a contraction $\tilde{\varphi} : \tilde{X} \rightarrow Y$ of an extremal face F .

Approach 6.2 (\mathbb{Q} -Cartierization). Can we choose the face F such that $\tilde{\varphi}$ induces a map $\varphi : X \rightarrow Z$, as in the following diagram:

$$\begin{array}{ccc}
 \tilde{X} & & \\
 \downarrow g & \searrow \tilde{\varphi} & \\
 X & & Y \\
 \downarrow \varphi & & \searrow f \\
 & & Z?
 \end{array}$$

One of the main issues with this approach is that we have no control over the map f , besides the fact that it must exist, since $K_{\tilde{X}}$ is g -ample and $\tilde{\varphi}$ -antiample. For this reason we have no control on the singularities of Z and on a possible termination of this program.

Moreover, we suspect that a general choice of the face F will not work. For this approach to work we need to find a condition on the face F that will guarantee the existence of the map φ .

Although these two approaches seem not satisfactory for the general case, they are simultaneously true in the toric case.

Proposition 6.4.4. *Let $\pi : Z \rightarrow X$ be a small morphism of toric varieties such that K_Z is ample and \mathbb{Q} -Cartier over X . Then for any K_Z -negative curve C , there exists a boundary Δ on X such that $(K_X + \Delta) \cdot \pi_* C < 0$ i.e. every contraction on Z arising from the cone theorem, induces a contraction on X .*

Proof. Let $C \subset Z$ such that $K_Z \cdot C < 0$. Let D be an ample effective Cartier divisor on X , in particular $D \cdot \pi_* C > 0$. Also, writing $D = \sum d_\rho D_\rho$, we can assume that there exists an integer $k > 0$ such that $D' := D/k$ is a \mathbb{Q} -Cartier divisor such that $|d_\rho| < 1$ for every $\rho \in \Sigma(1)$. Let us choose $\Delta = -K_X - D'$, that by assumption is an effective divisor and $K_X + \Delta = -D'$ is \mathbb{Q} -Cartier and $(K_X + \Delta) \cdot \pi_* C < 0$. \square

As corollary we deduce the existence (and termination) of a toric non- \mathbb{Q} -Gorenstein MMP.

6.4.2 A rational contraction theorem, I

Lemma 6.4.5 (Rational contraction theorem, I). *Let X be a projective variety with log terminal singularities such that K_X is not nef, and let $g : \tilde{X} = \text{Proj}_X \mathcal{R}(X, K_X)$.*

(a) *For each K_X -negative extremal face F of $\overline{\text{NE}}(\tilde{X})$, the contraction $\tilde{\varphi} : \tilde{X} \rightarrow Y$ of F induces a rational contraction $\varphi : X \dashrightarrow Y$.*

(b) *If D is the divisor inducing $\tilde{\varphi}$, then $g_* \mathcal{O}_{\tilde{X}}(D) = \mathcal{O}_X(g_* D)$ and $\mathcal{O}_X(g_* D) \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(D)$, and $g_* D$ is agg.*

Proof. The first part is clear, since g and $\tilde{\varphi}$ are birational:

$$\begin{array}{ccc} \tilde{X} = \text{Proj}_X \mathcal{R}(X, K_X) & & \\ g \downarrow & \searrow \tilde{\varphi} & \\ X & \dashrightarrow_{\varphi} & Y. \end{array}$$

Let $\tilde{U} = \tilde{X} - \text{Exc}(g)$ and let $U = X - g(\text{Exc}(g))$, and let j' and j be the corresponding inclusions. Notice that both $\text{Exc}(g)$ and $g(\text{Exc}(g))$ have codimension at least 2, and $\mathcal{O}_{\tilde{X}}(D)$ and $\mathcal{O}_X(g_*D)$ are reflexive sheaves. Moreover, $g|_{\tilde{U}} : \tilde{U} \rightarrow U$ is an isomorphism. Then

$$\begin{aligned} g_*\mathcal{O}_{\tilde{X}}(D) &= g_*j'_*\mathcal{O}_{\tilde{U}}(D|_{\tilde{U}}) \cong j_*(g|_{\tilde{U}})_*\mathcal{O}_{\tilde{U}}(D|_{\tilde{U}}) \cong j_*\mathcal{O}_U((g|_{\tilde{U}})_*D|_{\tilde{U}}) = \\ &= j_*\mathcal{O}_U(g_*D|_U) = \mathcal{O}_X(g_*D). \end{aligned}$$

Let us see the equality of sheaves

$$g_*\mathcal{O}_{X'}(mD) \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(mD),$$

for all $m \gg 0$. The sheaf $g_*\mathcal{O}_{X'}(mD) \cdot \mathcal{O}_{X'}$ is the torsion free part of $g_*\mathcal{O}_{X'}(mD) \otimes \mathcal{O}_{X'}$, and since g is small clearly

$$(g_*\mathcal{O}_{X'}(mD) \cdot \mathcal{O}_{X'})^{\vee\vee} = \mathcal{O}_{X'}(g^{\sharp}(mg_*D)) = \mathcal{O}_{X'}(mD),$$

so that we have the natural inclusion

$$g_*\mathcal{O}_{X'}(mD) \cdot \mathcal{O}_{X'} \subseteq \mathcal{O}_{X'}(mD).$$

Since $\mathcal{O}_{X'}(mD)$ is globally generated for $m \gg 0$ (it is a basepoint-free Cartier divisor), it is enough to show that the two sheaves have the same global sections. And in fact, $H^0(X, g_*\mathcal{O}_{X'}(mD)) = H^0(X', \mathcal{O}_{X'}(mD))$, and these sections will also be in $g_*\mathcal{O}_{X'}(mD) \cdot \mathcal{O}_{X'}$. \square

Remark 6.4.6. The fact that g_*D is agg, and therefore it induces a rational map, can also be deduced by observing that it satisfies the hypothesis of 6.4.9 (see also 6.4.1).

Finally, we show that, with the rational contraction theorem we are not leaving our original class of singularities, that is, that the singularities of the target of the contraction are still log terminal. This is well known for Fano or divisorial contractions, see §1.6.2. The remaining case is a consequence of the following lemma.

Theorem 6.4.7. (*Existence of flips*) Let $f : X \rightarrow Y$ be a flipping contraction with respect to a klt pair (X, Δ) . Then the flip of f exists, and Y has log terminal singularities.

We give two proofs of this lemma. The first one assuming $\Delta = 0$, which uses lt^+ singularities and which is enough for our program, and the second one for the general case, using [BdFFU13].

Proof 1 (for $\Delta = 0$). Notice that $X \cong \text{Proj}_Y \mathcal{R}(Y, -K_Y)$, which implies that $\mathcal{R}(Y, -K_Y)$ is finitely generated. Since $-K_X$ is f -ample and X is log terminal, Y has lt^+ singularities (with a similar argument as the one in 6.3.1). Indeed, for any resolution $g : Z \rightarrow Y$ factoring through X , $-1 < K_{Z/Y}^- \leq K_{Z/Y}^+$. Since the algebra $\mathcal{R}(Y, K_Y)$ is finitely generated, 4.1.6, the flip $f^+ : \text{Proj}_Y \mathcal{R}(Y, K_X) \rightarrow Y$ exists. Moreover, by 4.2.2, Y is log terminal. \square

Proof 2. Let $\Delta_Y = f_*\Delta$. Since $K_X + \Delta$ is f -antiample, $f_*\mathcal{O}_X(-m(K_X + \Delta)) = \mathcal{O}_Y(-m(K_Y - \Delta_Y))$ for m sufficiently divisible, and $\mathcal{O}_Y(-m(K_Y + \Delta_Y)) \cdot \mathcal{O}_X \cong \mathcal{O}_X(-m(K_X + \Delta))$. Thus, we have $K_Z + g_*^{-1}\Delta_Y - g^*(K_Y + \Delta_Y) > -1$ (we are not saying that $K_Y + \Delta_Y$ is \mathbb{Q} -Cartier). If A_Δ is the log discrepancy associated with $K_Y + \Delta_Y$ (as a function on the real valuations), we have that $A_\Delta(\nu) > 0$ for every real valuation ν , 1.4.12. Since Δ is effective, $A_Y(\nu) \geq A_\Delta(\nu)$, which implies that $A_Y(\nu) > 0$ for every real valuation ν . By 2.3.9, Y is log terminal. By 1.1.8, the algebra $\mathcal{R}(Y, K_Y + \Delta)$ is finitely generated, and we obtain the flip of f . \square

Remark 6.4.8. With this approach, we are essentially recovering the usual MMP. Thus we do not know if it will always terminate, but a version *with scaling* probably will.

6.4.3 A rational contraction theorem, II

Another approach is using K_X -negative extremal faces of $\overline{\text{NE}}(X)_W$. Unfortunately, for such approach, 6.3.4 is not strong enough.

Theorem 6.4.9. (*Global generation, II*) Let X be a projective variety with log terminal singularities. Let D be a nef Weil divisor such that $aD - K_X$ is nef and big for some $a > 0$ (resp. for all $a > 0$ sufficiently divisible). Then D is agg.

Remark 6.4.10. We do not assume that K_X is \mathbb{Q} -Cartier.

Proof. First, let us assume that D is \mathbb{Q} -Cartier. Let $g : Z \rightarrow X$ be the \mathbb{Q} -Cartierization of $-K_X$. Since D is \mathbb{Q} -Cartier and nef, this will also be the \mathbb{Q} -Cartierization of $aD - K_X$ for $a > 0$. Then $ag^*D - K_Y = g_*^{-1}(aD - K_X)$ is big and nef (and \mathbb{Q} -Cartier). By the assumption on the singularities, Z still has \mathbb{Q} -Gorenstein log terminal singularities, as in the proof of 6.3.1. By the usual basepoint-free theorem, g^*D is agg. Since D is \mathbb{Q} -Cartier, by the projection formula D is also agg.

For the general case, let $f : Y \rightarrow X$ be the \mathbb{Q} -Cartierization of D . As in the proof of 6.3.4, Y still has log terminal singularities. Moreover, $D_Y := f_*^{-1}D$ is \mathbb{Q} -Cartier, nef and f -ample.

We will prove that, for a sufficiently divisible, $aD_Y - K_Y$ is nef. Let A be any ample Cartier divisor on X . For each m and a sufficiently divisible, $m(aD - K_X + A)$ is globally generated. Moreover, since D_Y is f -ample, $m(aD_Y - K_Y)$ is f -ample and relatively globally generated for m and a sufficiently divisible. Since f is small,

$$\varphi : \mathcal{O}_Y \cdot \mathcal{O}_X(m(aD - K_X + A)) \rightarrow \mathcal{O}_Y(m(aD_Y - K_Y + f^*A))$$

is an isomorphism at the level of global sections. Since $aD_Y - K_Y$ is f -ample, $\mathcal{O}_Y(m(aD_Y - K_Y + g^*A)) \otimes \mathcal{O}_Y(kg^*A)$ is also globally generated for k sufficiently divisible. But then φ must be surjective, and hence an isomorphism. Since $\mathcal{O}_Y \cdot \mathcal{O}_X(m(aD - K_X + A))$ is generated by global sections, so is $\mathcal{O}_Y(m(aD_Y - K_Y + f^*A))$. This implies that $aD_Y - K_Y + g^*A$ is nef, and since nefness is a closed property on varieties with log terminal singularities, (6.2.1), $aD_Y - K_Y$ is nef.

By 5.2.8, $aD_Y - K_Y$ is also big. By what we proved at the beginning, D_Y is agg. Since D_Y is g -ample, this implies that D is agg. \square

Theorem 6.4.11 (Almost rational contraction theorem, II). *Let X be a projective variety with log terminal singularities such that K_X is not nef. If $\overline{\text{NE}}(X)_{W, K_X < 0}$ is generated by extremal rays which do not accumulate on the half space $K_X > 0$, for each K_X -negative extremal face F , there is a rational contraction induced by F .*

Proof. This proof follows [KM98, §3.3, Steps 6 and 7]. Let $\langle F \rangle \subset \text{NS}(X)_W^*$ be the linear span of F . Let H be an ample divisor and $\varepsilon > 0$ such that $K_X + \varepsilon H$ is negative on F . Notice that, since F is extremal, $\langle F \rangle \cap \overline{\text{NE}}(X)_W = F$. Let

$$W_F := \overline{\text{NE}}(X)_{W, K_X + \varepsilon H \geq 0} + \sum_{\dim R=1, R \not\subset F} R,$$

where R are extremal rays of $\overline{\text{NE}}(X)_W$. Then W_F is a closed cone which intersects $\langle F \rangle$ only at the origin and $\overline{\text{NE}}(X)_W = W_F + F$. Thus, there is an \mathbb{R} -Weil divisor G such that $\langle F \rangle \subseteq G_{=0}$, but $G_{=0} \cap W_F = \{0\}$ (where $G_{=0} := G_{\geq 0} \cap G_{\leq 0}$). Moreover, such \mathbb{R} -Weil divisor D can be chosen to be positive on W_F . Notice that such G will necessarily be nef. The set of such divisors is open and non-empty, thus there must be one defined over \mathbb{Q} . By taking suitable multiples, we can assume that D is a nef Weil divisor which is positive on W_F and zero exactly on F .

Since $-K_X$ is positive on F , while D is positive on W_F and zero on F , for $b > 0$, $bD - K_X$ is positive on $\overline{\text{NE}}(X)_W$. By 6.2.15, $bD - K_X$ is ample. By 6.4.9, D is agg. Thus a high enough (reflexive) power will define a rational map, whose Stein factorization is the desired contraction. \square

There are two related questions, that we can ask immediately.

Question 6.4.12. Are these contractions the same ones that we induced from $\text{Proj}_X \mathcal{R}(X, K_X)$, as in §6.4.2?

Question 6.4.13. If $\varphi : X \dashrightarrow Y$ is the above rational contraction, does Y still have log terminal singularities?

A satisfactory program will probably use a combination of these approaches.

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