On Singularities of Generic Projection Hypersurfaces

Davis C. Doherty

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Davis C. Doherty

and have found that it is complete and satisfactory in all respects, and that any and all revisions required by the final examining committee have been made.

Chair of the Supervisory Committee:

Sándor Kovács

Reading Committee:

Sándor Kovács

Rekha Thomas

James Morrow

Date: 08/10/06

University of Washington

Abstract

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Davis C. Doherty

Chair of the Supervisory Committee:
Professor Sándor Kovács
Mathematics

The present work studies singularities of hypersurfaces arising from generic projections of smooth projective varieties, in the context of Du Bois and semi log canonical singularities. It is demonstrated that Du Bois singularities are semi log canonical under certain reasonable assumptions, and that products of Du Bois singularities are also Du Bois. We then proceed to demonstrate that low-dimensional generic projection hypersurfaces are Du Bois, but general ones are not.

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Chapter 1
INTRODUCTION

Algebraic curves serve as the entry point into the study of algebraic varieties. The present understanding of curves is quite robust, and comes from various lines of study. *Moduli spaces* form the basis of the modern approach toward deeper understanding; however, we can learn a great deal from the more classical method of *generic projection.* In particular, we have the classical fact that any smooth projective curve C is birationally equivalent to a plane curve $C' \subset \mathbb{P}^2$ with at worst nodal singularities. C' is naturally obtained by first embedding C into $\mathbb{P}^3$, and then projecting from a generic point. Two salient features of such a plane curve are that it is a complete intersection, and nodal singularities are relatively simple.

Extending this idea, we can construct a general philosophy for studying smooth projective varieties: given a variety X, employ a sufficiently generic projection to obtain a hypersurface $X' \subset \mathbb{P}^{n+1}$ and hope that (a) the map is birational, and (b) the singularities of X' are reasonably nice. Birationality is straightforward enough, but what should we consider "reasonably nice"? Ultimately, this work is motivated by the exploration of what singularity classes can fill this role. To some extent, we shall be extending the work of Joel Roberts, Silvio Greco, and Carlo Traverso ([Rob71], [Rob75], [GT80]) by incorporating some modern notions of singularity.

For the purposes of this discussion, we shall explore whether the Minimal Model Program-related *semi log canonical* singularities can fill the "reasonably nice" role. Along the way, we shall also address the possibility that *Du Bois* singularities could fill this role. Consideration of semi log canonical singularities is not entirely arbitrary, however. On the classical side, nodal singularities of curves are (trivially) semi log canonical. On the modern side, semi log canonical varieties appear naturally on the boundaries of compactified moduli spaces. Thus there is reason to expect this is a possible generalization of nodal singularities in general.

Eventually we shall see that such speculation is only partially true – semi log canonical singularities are sufficient for low-dimensional varieties, but not in general. These two results form the heart of Chapter 8. Chapters 3, 4, 6, and 7 are devoted to the discussion and definition of the various classes of singularities employed, while Chapter 5 recalls some important results and definitions regarding generic projections. Chapter 2 covers notation and some key basic notions and theorems. The remainder of the present chapter illustrates connections between semi log canonical singularities and some major areas of current research.

1.1 Moduli Spaces

Moduli problems consider some class of objects for which it makes sense to speak of a family defined over a scheme B, together with an equivalence relation on the set $S(B)$ of all such families over B. The *moduli functor* of this problem is the map from schemes to sets defined by

$$F(B) = S(B)/\sim .$$

Ideally, F is *representable* in the category of schemes: there exists a scheme $\mathcal{M}$ and an isomorphism Ψ between F and the functor of points $\mathrm{Mor}_{\mathcal{M}}$ of $\mathcal{M}$. We call such an $\mathcal{M}$ a *fine moduli space* for the moduli problem F. Fine moduli spaces generally do not exist in the category of schemes; instead, we look for *coarse moduli spaces* (or work in a larger category, such as stacks).

Definition 1.1.1. A scheme $\mathcal{M}$ and a natural transformation $\Psi_{\mathcal{M}} : F \to \mathrm{Mor}_{\mathcal{M}}$ are a *coarse moduli space* for F if

(1) For k is algebraically closed, the map $\Psi : F(\mathrm{Spec}(k)) \to (k) = \mathrm{Mor}(\mathrm{Spec}(k), \mathcal{M})$ is a bijection of sets;

(2) If $\mathcal{M}'$ and $\Psi_{\mathcal{M}'}$ form another such pair, there is a unique morphism $\mathcal{M} \to \mathcal{M}'$ such that the associated natural transformation $\pi : \mathrm{Mor}_{\mathcal{M}} \to \mathrm{Mor}_{\mathcal{M}'}$ satisfies $\Psi_{\mathcal{M}'} = \pi \circ \Psi_{\mathcal{M}}$.

The most well-known example is $\mathcal{M}_g$, the coarse moduli space of smooth, complete, connected curves of genus g over $\mathbb{C}$. More generally we consider $\mathcal{M}_{g,n}$, which includes the additional data of n marked points. This is not a compact space, so one often constructs compactifications, denoted $\overline{\mathcal{M}_g}$ and $\overline{\mathcal{M}}_{g,n}$, respectively. The commonly considered compactification is itself a moduli space: the associated moduli problem is the parametrization of *stable curves* – complete connected curves with finite automorphism groups, with no worse than nodal singularities. Harris and Morrison provide an in-depth discussion of these moduli spaces in [HM98].

An obvious next step is to construct moduli spaces parametrizing surfaces. In practice, this turns out to be fraught with new difficulties; a full discussion is far beyond the scope of this paper, but we do address one aspect relevant to the present work. As noted above, we must extend the collection of parametrized objects to stable curves in order to compactify $\mathcal{M}_g$. Similarly, a surface moduli problem requires consideration of more than just smooth surfaces to obtain compactness – as before, we call the appropriate objects *stable surfaces.* The correct definition of "stable" seems to be the following:

Definition 1.1.2. Let S be a connected surface (a two-dimensional variety over $\mathbb{C}$). Call S *stable* if

(1) The canonical divisor K_S is ample;

(2) S has semi log canonical singularities.

Lest the reader think this definition is unrelated to the notion of stability for curves, it is worth noting that a stable surface does indeed have a finite automorphism group ([Iit81]). On the other hand, a curve having only nodal singularities is semi log canonical; this is easily seen from the definition (see Chapter 3). Moreover, requiring that a curve C has a finite automorphism group is equivalent to requiring that K_C be ample (this is left as an exercise in [HM98, Ex. 3.10]).

Thus the study of semi log canonical singularities is strongly connected to understanding the boundary of the moduli of surfaces. The same holds true for higher-dimensional varieties, where the correct notion of stability seems to be the same as the one given for surfaces.

1.2 The Minimal Model Program

One longstanding goal in algebraic geometry has been the completion of the Minimal Model Program (Mori's Program). The goal of the Program is to start with a projective $\mathbb{Q}$-factorial variety X over $\mathbb{C}$ which has terminal singularities, and in a finite number of steps produce a variety X', birational to X, which has some nice structure. In this case, "nice" means X' is either a minimal model (X' has terminal singularities and $K_{X'}$ is nef) or a fiber space, in which case we should be able to understand X' by studying lower-dimensional varieties.

There also exists a Log Minimal Model Program; the goal is similar, except the starting point is a pair (X, Δ) with log terminal singularities, where Δ is a $\mathbb{Q}$-divisor, and the resulting pair (X', Δ') is a minimal model if it is log terminal and has $K_{X'} + \Delta'$ nef. Thus the general Minimal Model Program can be thought of as the search for a minimal model birational to an initial variety.

Alternatively, we can look for other models; one example is the construction of canonical models – the program is essentially the same, but the result has canonical or log canonical singularities, together with an ample canonical bundle. Conceivably one could consider other variants as well, such as the construction of a hypersurface birational to a given projective variety (some sort of "hypersurface model"). The hope would be that the singularities of such a model could be reasonably constrained; in this case, semi log canonical or Du Bois singularities might be reasonable. As we shall see, however, this does not appear to be a fruitful line of study – the most natural method of constructing hypersurfaces, generic projections, do not in general produce varieties with either semi log canonical or Du Bois singularities.

Chapter 2
PRELIMINARY MATERIAL

2.1 *Basic Assumptions*

Throughout this work we assume that all rings are Noetherian and commutative, with a unity element 1. The image of 1 under any ring homomorphism is 1 in the target ring. Our primary focus will be on schemes; we shall assume that all schemes are separated and reduced. A variety is a Noetherian, separated, and reduced scheme of finite type over some algebraically closed field. In practice, almost all the schemes we consider will be varieties over $\mathbb{C}$, but we shall use the more general notion of scheme when applicable.

2.2 *The Derived Category*

Proper discussion of Du Bois singularities requires that the action take place in the bounded derived category of sheaves on a scheme X. In particular, we shall consider objects living in $D^b_{\mathrm{coh}}(X)$, the "bounded derived category of sheaves with coherent cohomology" on X. We first define the derived category $D(X)$. The objects of $D(X)$ are cochain complexes of sheaves

$$\dots \xrightarrow{d^{i-2}} \mathscr{F}^{i-1} \xrightarrow{d^{i-1}} \mathscr{F}^{i} \xrightarrow{d^{i}} \mathscr{F}^{i+1} \to \dots$$

such that the composition of two differential maps is zero; denote such a complex by $\mathscr{F}^{\cdot}$. The cohomology sheaves are denoted

$$h^i(\mathscr{F}^{\cdot}) = \frac{\operatorname{im} d^{i-1}}{\ker d^i}.$$

The morphisms in $D(X)$ are chain homotopy equivalences, "localized" so that quasi-isomorphisms of complexes are isomorphisms in $D(X)$. The bounded derived category, $D^b(X)$, is the full subcategory of $D(X)$ consisting of bounded cochain complexes (i.e., complexes such that $\mathscr{F}^i \neq 0$ only for finitely many i). $D^b_{\text{coh}}(X)$ is the subcategory consisting of bounded cochains $\mathscr{F}^\cdot$ such that $h^i(\mathscr{F}^\cdot)$ is a coherent sheaf for all i.

In studying objects and morphisms in the derived category, the role of exact sequences is filled by *exact triangles* – triples of morphisms (u, v, w)

$$\mathscr{A}^\cdot \xrightarrow{u} \mathscr{B}^\cdot \xrightarrow{v} \mathscr{C}^\cdot \xrightarrow{w} \mathscr{A}^\cdot[1] \to \dots$$

such that the induced morphisms on cohomology give rise to a long exact sequence

$$\dots \to h^i(\mathscr{A}^\cdot) \xrightarrow{u^*} h^i(\mathscr{B}^\cdot) \xrightarrow{v^*} h^i(\mathscr{C}^\cdot) \xrightarrow{w^*} h^{i+1}(\mathscr{A}^\cdot) \to \dots$$

There is a natural way to identify a sheaf $\mathscr{F}$ on X with an element of $D(X)$. Denote by $\mathscr{F}[n]$ the complex $\mathscr{F}^\cdot$ such that $\mathscr{F}^n = \mathscr{F}$ and $\mathscr{F}^i = 0$ for $i \neq n$. Abusing notation, $\mathscr{F}$ sometimes denotes $\mathscr{F}[0]$; the context should make it clear whether the sheaf or the complex is intended. For more details, [Wei94] provides a solid introduction to the derived category.

2.3 Gorenstein Schemes

Gorenstein schemes arise naturally from the contructions in this work, as most schemes we encounter will be complete intersections (more specifically, hypersurfaces). Various properties implied by the Gorenstein condition will come into play later on, so it behooves us to recall the relevant ones. As this is essentially a condition on rings translated over to schemes in the usual way, we begin with some commutative algebra. Many technicalities are glossed over here for the sake of brevity; [BH93] is a good reference for the detail-oriented reader.

Recall first that if $(R, \mathfrak{m})$ is a Noetherian local ring, a *regular sequence* is a sequence of elements $x_1, \dots, x_n$ such that x_i is not a zero-divisor in $R/(x_1, \dots, x_{i-1})$ and $R/(x_1, \dots, x_n) \neq 0$. The common length of all maximal regular sequences of elements in $\mathfrak{m}$ is called the *depth* of R. R is called *Cohen-Macaulay* if $\operatorname{depth} R = \dim R$. A general (not necessarily local) ring R is Cohen-Macaulay if $R_\mathfrak{p}$ is Cohen-Macaulay for all prime ideals $\mathfrak{p}$.

Again, we assume R is a Noetherian ring. The *injective dimension* of R is the smallest integer n for which there exists an injective resolution $I^\cdot$ of R (viewed as an R-module) with $I^m = 0$ for $m > n$; if no such integer exists, the injective dimension is infinite. When R is local, we say R is *Gorenstein* if its injective dimension is finite; otherwise, R is Gorenstein if its localization at every maximal ideal is Gorenstein. Gorenstein rings are Cohen-Macaulay, though this is not obvious from the definitions.

Example 2.3.1. *Regular rings* (local rings $(R, \mathfrak{m})$ such that $\mathfrak{m}$ can be generated by $\dim R$ elements) and complete intersection rings are both Gorenstein.

Sometimes it is useful to consider a weaker form of the Cohen-Macaulay condition; Serre's S_n condition fills this role. A Noetherian ring R satisfies condition S_n if

$$\operatorname{depth} R_\mathfrak{p} \geq \min(n, \dim R_\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. Equivalently, R satisfies S_n if $R_\mathfrak{p}$ is Cohen-Macaulay for all prime ideals such that $\operatorname{depth} R_\mathfrak{p} < n$. Thus we may think of condition S_n as requiring that R be "Cohen-Macaulay in codimension n."

Of course, we can translate the previous definitions to schemes in the usual manner – for example, a scheme X is Gorenstein if $\mathscr{O}_{X,x}$ is a Gorenstein local ring for every point $x \in X$ (and similarly for regular, Cohen-Macaulay, and S_n). However, the Gorenstein and Cohen-Macaulay properties can also be characterized using the dualizing complex, which is more appropriate for our purposes. We review the basics of the dualizing complex; [Har66] provides more detail.

Definition 2.3.2. Let X denote a scheme. A *dualizing complex* is a complex $\omega_X^\cdot \in D^b_{\text{coh}}(X)$ such that the natural map

$$\mathscr{F}^\cdot \to R\mathscr{H}om^\cdot(R\mathscr{H}om^\cdot(\mathscr{F}^\cdot, \omega_X^\cdot))$$

is an isomorphism for all $\mathscr{F}^{\cdot} \in D^{+}(X)$.

Remark. For a ring R, the definition of a dualizing complex $\omega^{\cdot}_R \in D^b_{\mathrm{coh}}(R)$ is exactly the same. Note that the objects of $D^{+}(R)$ are bounded above complexes of R-modules.

Example 2.3.3 ([Har66, Ex. V.2.2]). Suppose X is regular, and has finite Krull dimension. Then $\mathscr{O}_X$ is a dualizing complex. Note that if X is (locally) Noetherian and admits a dualizing complex, then X has finite Krull dimension ([Har66, Cor. V.7.2]).

By assumption all our schemes are of finite type over $\mathbb{C}$. Since any scheme of finite type over a field has a dualizing complex ([Har66, §V.10]), we shall always assume existence of $\omega^{\cdot}_X$ without further comment. Moreover, $\omega^{\cdot}_X$ is unique up to tensor product, in the following sense: if $\mathscr{F}^{\cdot}$ is another dualizing complex, then by [Har66, Thm. V.3.1] there exists an invertible sheaf $\mathscr{L}$ on X and an integer n such that

$$\omega^{\cdot}_X \simeq_{\mathrm{qis}} \mathscr{F}^{\cdot} \otimes \mathscr{L}[n].$$

The dualizing complex provides more concise characterizations of Cohen-Macaulay and Gorenstein schemes.

Proposition 2.3.4 ([Har66, §V.6]). *A scheme X is Cohen-Macaulay if and only if $\omega^{\cdot}_X \simeq_{qis} \mathscr{F}[n]$ for some coherent sheaf $\mathscr{F}$ and some integer n.*

Proposition 2.3.5 ([Har66, Thm. V.9.1]). *A scheme X is Gorenstein if and only if it is Cohen-Macaulay, and the dualizing complex is (quasi-isomorphic to) a locally free rank one sheaf.*

2.4 Grothendieck Duality

Given a projective morphism of finite dimensional Noetherian schemes $f : X \to Y$, the functor Rf_* has a right adjoint

$$f^! : D^{+}_{\mathrm{coh}}(Y) \to D^{+}_{\mathrm{coh}}(X).$$

The reader interested in a constructive definition of $f^!$ should consult [Har66] and [Con00]. One especially nice feature of $f^!$ is the following proposition.

Proposition 2.4.1 ([Har66, §V.8]). *Let $f : X \to Y$ be as above, and suppose $\omega_Y^\cdot$ is a dualizing complex on Y. Then $f^!\omega_Y^\cdot$ is a dualizing complex on X.*

Remark. One interesting consequence of this result: if X is a projective scheme of finite type over $\mathbb{C}$ with $f : X \to \operatorname{Spec}\mathbb{C}$ the natural morphism, then $f^!\mathbb{C}$ is a dualizing complex on X, where we are viewing $\mathbb{C}$ as a sheaf on $\operatorname{Spec}\mathbb{C}$.

With the definition of $f^!$ in hand, we can state the Grothendieck duality theorem for projective morphisms (note that there are more general forms).

Theorem 2.4.2 ([Har66, Thm. III.11.1]). *Suppose $f : X \to Y$ is a projective morphism of finite dimensional Noetherian schemes. Assume $\mathscr{F}^\cdot \in D^-_{qcoh}(X)$ and $\mathscr{G}^\cdot \in D^+_{qcoh}(Y)$ (whose objects are complexes of quasi-coherent sheaves which are bounded below and bounded above, respectively). Then the duality morphism*

$$Rf_*R\mathscr{H}om^\cdot_X(\mathscr{F}^\cdot, f^!\mathscr{G}^\cdot) \to R\mathscr{H}om^\cdot_Y(Rf_*\mathscr{F}^\cdot, \mathscr{G}^\cdot)$$

is an isomorphism.

In the case where $\mathscr{G}^\cdot = \omega_Y^\cdot$, combining Grothendieck duality with Proposition 2.4.1 gives an isomorphism

$$Rf_*R\mathscr{H}om^\cdot_X(\mathscr{F}^\cdot, \omega_X^\cdot) \to R\mathscr{H}om^\cdot_Y(Rf_*\mathscr{F}^\cdot, \omega_Y^\cdot)$$

for all $\mathscr{F}^\cdot \in D^-_{\mathrm{qcoh}}(X)$. This special case of duality is the only version we shall employ.

2.5 Divisors

This work assumes familiarity with the language of divisors; however, we explicitly state the key assumptions and definitions to avoid ambiguity. If X is a scheme, a *prime divisor* is a reduced, irreducible codimension one subscheme. A *divisor* is a formal linear combination $D = \sum_i d_i D_i$ of (distinct) prime divisors, where $d_i \in \mathbb{Z}$. In general, we must consider $\mathbb{Q}$-*divisors*, where we allow $d_i \in \mathbb{Q}$; typically we shall abuse the terminology and simply call both "divisors." D has *simple normal crossings* if each D_i is smooth and intersects the other D_j transversely. D is called $\mathbb{Q}$-*Cartier* if there is a nonzero integer n such that nD is a Cartier divisor. The *exceptional set* of a birational map $f : X \to Y$ is the set where f fails to be a morphism; if the exceptional set is a divisor, we shall call it the *exceptional divisor.*

It is well-known that there is a one-to-one correspondence between Cartier divisors and line bundles (invertible sheaves) – see, for example, [Har77, Thm. II.6.13]. If X is Gorenstein, then $\omega^{\cdot}_X$ is a line bundle; the associated divisor is denoted K_X and called the *canonical divisor.* Suppose $\omega^{\cdot}_X$ is a sheaf (e.g., when X is Cohen-Macaulay); denote it ω_X. Denote by $\omega_X^{[n]}$ the nth reflexive power of ω_X, i.e., the double dual of the nth tensor power of ω_X. If this is a line bundle for some n, we say that X is $\mathbb{Q}$-Gorenstein; the associated divisor is denoted nK_X, and $K_X =_{\mathsf{def}} \frac{1}{n}(nK_X)$ is again called the canonical divisor. Thus a $\mathbb{Q}$-Gorenstein scheme has a $\mathbb{Q}$-Cartier canonical divisor. Conversely, a Cohen-Macaulay scheme with K_X $\mathbb{Q}$-Cartier is $\mathbb{Q}$-Gorenstein.

Since many divisors will only be $\mathbb{Q}$-Cartier, linear equivalence turns out to be the wrong notion of divisor equivalence. Instead, we use *numerical equivalence.* A divisor D is numerically equivalent to zero, denoted $D \equiv 0$, if $D \cdot C = 0$ for all irreducible curves $C \subset X$. Two divisors D and D' are numerically equivalent if $D - D' \equiv 0$. A $\mathbb{Q}$-Cartier divisor D is called *nef* (numerically effective) if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$.

Chapter 3

LOG CANONICAL AND SEMI LOG CANONICAL SINGULARITIES

3.1 Discrepancies

For this section, we assume X is a normal variety such that K_X is $\mathbb{Q}$-Cartier. More detailed discussion of the following topics can be found in chapter 2 of [KM98].

Let $f : Y \to X$ be a proper birational morphism from a normal variety Y, and suppose f has an irreducible exceptional divisor $E \subset Y$. Let e be a general point of E, let $\{y_i\}$ be a set of local coordinates such that E is given by $y_1 = 0$, and let σ be a local generator of $\mathscr{O}_X(mK_X)$. Then locally we have

$$f^*(\sigma) = y_1^{m \cdot a(E,X)} u(dy_1 \wedge \cdots \wedge dy_n),$$

where u is a unit. Call $a(E, X)$ the *discrepancy* of E with respect to X. In terms of divisors, this implies that

$$K_Y \equiv f^*(K_X) + \sum_i a(X, E_i)E_i,$$

where $\equiv$ denotes numerical equivalence and the E_i are the irreducible exceptional divisors of f. More generally, if $\Delta = \sum a_j D_j$ is a sum of distinct prime divisors such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then we can write

$$K_Y + f_*^{-1}\Delta \equiv f^*(K_X + \Delta) + \sum_i a(X, \Delta, E_i)E_i,$$

where the E_i are again the exceptional divisors of f, and f_*^{-1} is the usual notation for the map $(f^{-1})_*$. The number $a(E_i, X, \Delta)$ is called the *discrepancy* of E_i with respect to the pair (X, Δ).

Definition 3.1.1. The *discrepancy* of the pair (X, Δ) is

$$\operatorname{discrep}(X, \Delta) = \inf_E \{a(E, X, \Delta) \mid E \text{ is an exceptional divisor over } X\}.$$

If $\Delta = 0$ we usually write $\operatorname{discrep}(X)$, rather than $\operatorname{discrep}(X, 0)$.

Example 3.1.2. *An upper bound on the discrepancy:* Assume that $\dim X \geq 2$. Let $f : Y \to X$ be the blow-up of X along a smooth codimension 2 locus, with exceptional divisor E. By [Har77, §II, Ex 8.5] we have $K_Y \equiv f^*K_X + E$. Hence $\operatorname{discrep}(X, \Delta) \leq 1$ for any pair. In particular, for smooth X we have $\operatorname{discrep}(X) = 1$.

Example 3.1.3. *A lower bound on the discrepancy:* Let X be a surface. Suppose that $f : Y \to X$ is a birational map with an exceptional component E_k such that $K_Y \equiv f^*K_X - (1+\epsilon)E_k + E$, where $E = \sum_{i \neq k} a_i E_i$ comprises the remaining exceptional components of f and $\epsilon > 0$. Let $f_1 : Y_1 \to Y$ be the blow-up of Y at a general point of E_k, with exceptional divisor D_1. Then

$$\begin{aligned} K_{Y_1} &\equiv f_1^* K_Y + D_1 \\ &\equiv f_1^*(f^*K_X - (1+\epsilon)E_k + E) + D_1 \\ &\equiv f_1^* f^* K_X - (1+\epsilon) f_1^* E_k + f_1^* E + D_1 \\ &\equiv f_1^* f^* K_X - (1+\epsilon) E_k' + E' - \epsilon D_1, \end{aligned}$$

where E_k' and E' and the proper transforms of E_k and E, respectively. Construct a birational map $f_2 : Y_2 \to Y_1$ by blowing up Y_1 at a point $x \in E_k' \cap D_1$. A similar computation to the one above shows that the composite map $f_2 \circ f_1 \circ f$ has an exceptional divisor D_2 with coefficient -2ϵ. More generally, we can construct a map f_N such that the composite map to X has an exceptional component D_N with coefficient $-N\epsilon$. Hence $\operatorname{discrep}(X) = -\infty$.

Example 3.1.3 can be generalized to arbitrary dimension by replacing "point" with "codimension two locus." Thus we can combine the previous two examples to obtain the following:

Theorem 3.1.4. *Given a pair* (X, Δ) *as above, either* $-1 \leq \operatorname{discrep}(X, \Delta) \leq 1$ *or* $\operatorname{discrep}(X, \Delta) = -\infty$. *If* X *is smooth, then* $\operatorname{discrep}(X) = 1$.

3.2 Singularities of Pairs

Theorem 3.1.4 illustrates the fact that the discrepancy naturally sorts varieties into two classes – those with $\operatorname{discrep}(X) \geq -1$, and those with $\operatorname{discrep}(X) = -\infty$. However, this distinction is too coarse in practice. Refining finite discrepancy varieties into several "flavors" seems to be a more productive approach.

Definition 3.2.1. We say that (X, Δ) has *terminal* singularities if $\operatorname{discrep}(X, \Delta) > 0$, *canonical* singularities if $\operatorname{discrep}(X, \Delta) \geq 0$, *(purely) log terminal* singularities if $\operatorname{discrep}(X, \Delta) > -1$, and *log canonical* singularities if $\operatorname{discrep}(X, \Delta) \geq -1$.

Note that these are inclusive classes. Hence a pair $(X, 0)$ with X smooth falls into every class, log terminal singularities are log canonical, etc.

Minimal model-type problems provide the motivation for defining and studying discrepancies. In the case where $\Delta = 0$, the class of terminal singularities is the smallest necessary to run the Minimal Model Program for smooth varieties (i.e., one must allow intermediate varieties to have terminal singularities in order to successfully run the Program). Canonical singularities appear on the canonical models of varieties of general type. Purely log terminal, and its brethren Kawamata and divisorial log terminal, serve mostly technical purposes. Log canonical is the largest class of singularities where the discrepancy is a useful notion – in essence, log canonical versus not log canonical is the coarse distinction discussed above.

The discrepancy of a pair can be computed using only a log resolution, i.e., a resolution of singularities $f : Y \to X$ such that $\Delta \cup E$ is a simple normal crossing divisor (here E denotes the union of all the exceptional divisors of f). This allows us to avoid the complication of considering *all* possible birational maps and their exceptional divisors. [KM98, Cor 2.32] provides a rigorous treatment of this useful fact.

Example 3.2.2. *The standard example of a strictly log canonical surface singularity:* Let C be an elliptic curve, and let X be the cone over C. It is well-known that the cone point can be resolved with a single blow-up, with exceptional divisor $E \cong C$. Denote the blow-up by Y.

Now on the one hand, we have $K_Y = f^*K_X + aE$ for some a. Intersecting with E, we find that $E.K_Y = aE^2$. On the other hand, the adjunction formula gives $2g(E) - 2 = E.(E + K_Y)$. Noting that $g(E) = 1$, we can rearrange this equation to obtain $E.K_Y = -E^2$, whence $a = -1$. Thus we conclude that X is log canonical, but not purely log terminal.

3.3 Semi Log Canonical Singularities

Assuming normality simplifies the study of singularities, but at the cost of narrowing the scope. We could consider non-normal varieties with the tools developed thus far by first normalizing. However, the process of normalization destroys any information about codimension one singularities, and in certain cases actually results in a smooth variety. So we shall instead expand our definitions to encompass certain non-normal singularities – specifically, we shall consider schemes that have at worst double normal crossings in codimension one.

We begin with some definitions that specify what is meant by a resolution of singularities for a non-normal scheme. For the remainder of this section, X will denote a scheme over $\mathbb{C}$ with only double normal crossing singularities in codimension one.

Definition 3.3.1. An n-dimensional scheme X is called *semismooth* if every point $x \in X$ is either a smooth point, a double normal crossing point (analytically isomorphic to $\mathbb{C}[x_0, \dots, x_n]/(x_0x_1)$), or a pinch point (analytically isomorphic to $\mathbb{C}[x_0, \dots, x_n]/(x_0x_1^2 - x_2^2)$).

If X is semismooth, then the smooth codimension one subscheme where X is nonsmooth is denoted D_X (the "double locus" of X). The normalization $\nu : \widetilde{X} \to X$ is smooth, and the map $\widetilde{D}_X = \nu^1(D_X) \to D_X$ is a double cover ramified at the pinch points.

Definition 3.3.2. A proper birational map $f : Y \to X$ is called a *semiresolution* if Y is semismooth, no component of its double locus D_Y is in the exceptional locus of f, and there is a codimension two subset $S \subset X$ such that the restriction map $f^{-1}(X \setminus S) \to X \setminus S$ is an isomorphism.

Definition 3.3.3. A map $f : Y \to X$ is a *good semiresolution* if in addition to being a semiresolution, $D_Y \cup E$ is a simple normal crossing divisor, where E denotes the exceptional divisor of f.

We do not define semi log canonical singularities by considering all possible semiresolutions, as in the normal case, though this is perfectly reasonable (and equivalent). Instead, we shall only need to take a good semiresolution.

Definition 3.3.4. Let X be a reduced S_2 scheme which is semismooth in codimension one, and let Δ be a boundary, i.e., a $\mathbb{Q}$-Weil divisor $\sum d_j D_j$ with $0 \leq d_j \leq 1$. We say that (X, Δ) is *semi log canonical* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and there is a good semiresolution $f : Y \to X$ such that

$$K_Y + f_*^{-1}\Delta \equiv f^*(K_X + \Delta) + \sum_i a_i E_i,$$

with all $a_i \geq -1$, where E_i are the exceptional divisors. The notions of semi terminal, semi canonical, and semi log terminal are defined similarly. In practice Δ will typically be 0.

The following proposition demonstrates the connection between semi log canonical and log canonical singularities.

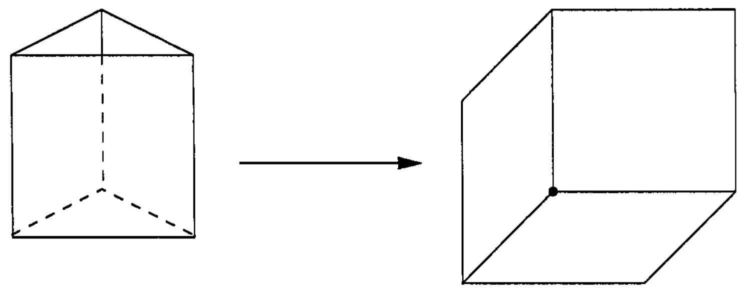

Figure 3.1: Blowing up X

Proposition 3.3.5 ([Kol92, Prop 12.2.4]). *Let $\pi : \widetilde{X} \to X$ be the normalization of a pair (X, Δ) satisfying the hypotheses of Definition 3.3.4. Set $\Theta = \pi^{-1}\Delta + \widetilde{D}_X$. Then (X, Δ) is semi log canonical if and only if $(\widetilde{X}, \Theta)$ is log canonical.*

Remark. Defining semi log canonical by requiring that the normalization be log canonical is a tempting idea, but it has the difficulty that $K_X + \Delta$ may not be $\mathbb{Q}$-Cartier even when $(\widetilde{X}, \Theta)$ is log canonical.

Example 3.3.6. *The standard example of a strictly semi log canonical surface singularity:* Let $X \subset \mathbb{P}^3$ be the union of the planes H_1, H_2, H_3, with $H_1 \cap H_2 \cap H_3 = P$ for some point P. X only fails to be semismooth at P, so we set $\widetilde{X} = \mathrm{Bl}_P X \subset \mathrm{Bl}_P \mathbb{P}^3 = B$. Figure 3.1 gives the geometric picture.

We first note that we can write

$$K_{\widetilde{X}} = (K_B + \widetilde{X})|_{\widetilde{X}},$$

where we are identifying $\widetilde{X}$ with its associated divisor in B. Now we have (via [Har77, II, Ex. 8.5]) that $K_B = f^*K_{\mathbb{P}^3} + 2E$. Also, $f^*X = \widetilde{X} + 3E$, as we pick up one copy of E for each of the H_i. Thus we can rewrite the above equation:

$$\begin{aligned} K_{\tilde{X}} &= (f^*K_{\mathbb{P}^3} + f^*X - E)|_{\tilde{X}} \\ &= (f^*(K_{\mathbb{P}^3} + X) - E)|_{\tilde{X}} \\ &= f^*(K_{\mathbb{P}^3} + X)|_X - E|_{\tilde{X}} \\ &= f^*K_X - E|_{\tilde{X}}. \end{aligned}$$

Finally, $E|_{\tilde{X}}$ is just the union of three curves $E_i \subset \widetilde{X}$, with E_i corresponding to the intersection of E with the proper transform of H_i. Hence $K_{\tilde{X}} = f^*K_X - E_1 - E_2 - E_3$, and we conclude that X is semi log canonical.

Remark. Note that the scheme X in the example has smooth normalization, yet is strictly semi log canonical. This is a good argument for using our definition of semismooth, rather than the naive definition of having smooth normalization. In this case, the naive definition would give a scheme that is semismooth, but not, say, semi canonical. In the normal case, smooth varieties fall into every class of singularities; the "correct" non-normal version should also have this property, as our definition of semismooth does. Later we shall find that there exist schemes with smooth normalization which are not even semi log canonical.

3.4 Existence of semiresolutions

One weakness of the definition of semi log canonical is that it only makes sense when there is a semiresolution of X. Hironaka's celebrated result demonstrated that resolutions of singularities always exist over fields of characteristic zero ([Hir64]); we require an analagous result for semiresolutions. As we shall see, in the category of varieties semiresolutions are not guaranteed to exist; instead, we must extend to the category of *algebraic spaces*. Our definition is that of [Art71].

Definition 3.4.1. An *algebraic space* X consists of an affine scheme U and a closed subscheme $R \subset U \times U$ such that:

(i) R is an equivalence relation;

(ii) The projection maps $p_i : R \to U$ $(i = 1, 2)$ are étale.

If in addition, R is the trivial equivalence relation on each component of U, then X is a scheme.

Essentially, algebraic spaces give algebraic structures to analytic spaces.

Example 3.4.2. To view a scheme X as an algebraic space, let $\{U_i\}$ be a collection of affine open sets covering X. Set $U = \amalg_i U_i$, which is also affine. Let $R \subset U \times U$ be the equivalence relation representing the gluing data on the U_i, i.e., $R = U \times_X U$, where $U \to X$ is the obvious morphism. Then X is equivalent to the algebraic space represented by U and R. Note that the choice of $\{U_i\}$ is not unique in general, so there may be multiple equivalent representations of the same algebraic space.

Theorem 3.4.3 ([Kol90, Prop 4.2]). *Let X be a pure-dimensional scheme over $\mathbb{C}$. Then there exists a semiresolution $f : Y \to X$, where Y is an algebraic space.*

We reproduce here the proof given in [Kol90].

Proof. By [Hir64], performing a series of blow-ups centered in the locus of points of multiplicity at least three allows us to obtain $g : X_1 \to X$ such that the singularities of X_1 are only double points, and the double locus $D_1 \subset X_1$ is smooth. Let $X_2 \to X_1$ be the normalization map, with D_2 the preimage of D_1. Then we have a natural involution τ on D_2 – if $x, y \in D_2$ are the preimages of $z \in D_1$, then τ sends x to y and vice-versa. Using a local computation, one can show that D_2 is a Cartier divisor.

By the equivariant version of Hironaka's result, we can resolve the singularities of D_2 in a τ-equivariant manner using a series of blow-ups to obtain X_3 and D_3, with D_3 a smooth Cartier divisor. X_3 is smooth along D_3, so we can resolve the singularities of X_3 to obtain X_4 which still contains (an isomorphic copy of) D_3. The set of fixed points of τ is smooth on D_3. Blow up that set to obtain X_5 and D_4; the fixed point set of τ on D_4 now has codimension one. Identifying points of D_4 via τ ("pinching" D_4) gives an algebraic space Y. Y is smooth outside the image of D_4, has simple normal crossings on the image of the non-fixed locus of τ, and has pinch points on the image of the τ-fixed locus. Thus it follows that Y is a semiresolution of X. $\square$

3.5 Semi Log Canonical Surface Singularities

Semi log canonical singularities of curves provide no new objects of study – the only non-smooth points must be nodes. This class becomes more complex for surfaces; however, semi log canonical surface singularities have been classified by Kollár and Shepherd-Barron ([KSB88]). We reproduce their classification for its instructive value, and to provide some intuition for what semi log canonical singularities "look like." Note that for surfaces, good semiresolutions are in fact schemes ([vS87] provides a proof of the existence of semiresolutions within the category of schemes), and so there is no need to extend to the category of algebraic spaces in this section.

Definition 3.5.1. A resolution or semiresolution (of a surface) f is called *minimal* if no -1 curve is contracted by f.

Given a resolution (or semiresolution) f, we can obtain a minimal one by contracting all the -1 curves that are contracted by f.

Recall that if X is a scheme, a point $x \in X$ is Gorenstein if $\mathcal{O}_{x,X}$ is Gorenstein.

Definition 3.5.2. (i) A normal Gorenstein surface singularity is called *simple elliptic* if the exceptional divisor of the minimal resolution is a smooth elliptic curve.

(ii) A normal Gorenstein surface singularity is called a *cusp* if the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve.

(iii) A non-normal Gorenstein surface singularity is called a *degenerate cusp* if and the exceptional divisor of the minimal semiresolution is a cycle of smooth rational curves or a rational nodal curve. In this case the pre-image Y has no pinch points and the irreducible components of X have cyclic quotient singularities.

Remark. Example 3.2.2 is a simple elliptic singularity. Example 3.3.6 is a degenerate cusp.

Theorem 3.5.3 ([KSB88, Thm 4.21]). *Let $x \in X$ be a Gorenstein surface singularity such that $X - x$ is semismooth. Then*

(i) *X is semi canonical if and only if x is either smooth, a double normal crossing point, a pinch point, or a rational double point (see [Dur79] for more information on rational double points).*

(ii) *X is semi log canonical if and only if x is either a simple elliptic singularity, a cusp, a degenerate cusp, or semi canonical.*

Theorem 3.5.4 ([KSB88, Thm 4.23]). *The semi log terminal surface singularities are as follows:*

(i) *Quotients of $\mathbb{C}^2$ (enumerated by Brieskorn in [Bri68]);*

(ii) *Double normal crossing or pinch points;*

(iii) *$xy = 0$ modulo the group action $x \mapsto \mu^a x; y \mapsto \mu^b y; z \mapsto \mu z$, where μ is a primitive rth root of unity, $(a, r) = 1$ and $(b, r) = 1$;*

(iv) *$xy = 0$ modulo the group action $x \mapsto \mu^a y; y \mapsto x; z \mapsto \mu z$, where μ is a primitive rth root of unity, $4|\mu$ and $(a, r) = 2$;*

(v) *$x^2 = y^2 z$ modulo the group action $x \mapsto \mu^{1+a} x; y \mapsto \mu^a y; z \mapsto \mu^2 z$, where μ is a primitive rth root of unity, r odd, $(a, r) = 1$.*

Theorem 3.5.5 ([KSB88, Thm 4.24]). *The semi log canonical singularities are as follows:*

(i) *The semi log terminal ones of Theorem 3.5.4;*

(ii) *The Gorenstein ones of Theorem 3.5.3;*

(iii) $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ *and* $\mathbb{Z}_6$ *quotients of simple elliptic ones as enumerated by Kawamata ([Kaw79]);*

(iv) $\mathbb{Z}_2$ *quotients of cusps and degenerate cusps.*

Chapter 4

DU BOIS SINGULARITIES AND THE DU BOIS COMPLEX

Du Bois singularities are defined in terms of the Du Bois complex of a scheme – an object in the filtered derived category analogous to the de Rham complex. Various methods exist for constructing the Du Bois complex. The traditional approach has been to use some sort of "hyperresolution" – simplicial resolutions ([DB81], [Ste85]), cubic hyperresolutions ([GNPP88]), and polyhedral resolutions ([Car85]) can all fill this role. Using a hyperresolution has the benefit that the various functorial properties of the Du Bois complex have trivial proofs. The principle downside to this approach is that it is extremely impractical to use for explicitly calculating the Du Bois complex of most interesting (i.e., singular) schemes.

Alternatively, recent work by Karl Schwede ([Sch06]) provides a "hyperresolution-free" method for calculating the zeroth graded piece of the Du Bois complex, which is sufficient for determining whether a scheme has Du Bois singularities. Using this approach to determine when a scheme has Du Bois singularities is far more practical, though it makes some of the functorial properties we use non-obvious. For the sake of completeness we shall sketch the simplicial resolution method of computing the Du Bois complex (though we omit some of the technical details), and then state Schwede's simpler characterization of Du Bois singularities.

4.1 Simplicial Resolutions

Definition 4.1.1. Let X be an n-dimensional scheme. A *simplicial space of level k* over X is a sequence $X_0, \dots, X_k$ of schemes together with morphisms $\varepsilon_{i,j} : X_i \to X_{i-1}$,

$0 \leq j \leq i \leq k$ (we set $X_{-1} = X$) such that for all $j < i$ we have

$$\varepsilon_{i,j} \circ \varepsilon_{i+1,j+1} = \varepsilon_{i,j} \circ \varepsilon_{i+1,j},$$

i.e., the maps obtained via composition of any two successive morphisms are equivalent.

The condition on the $\varepsilon_{i,j}$ maps allows us to obtain a well-defined map $\varepsilon_i : X_i \to X$ by composing any sequence of maps.

Denote by Δ^p the standard p-simplex

$$\{(t_0, \ldots, t_p) \in \mathbb{R}^{p+1} \mid \sum_{i=0}^{p} t_i = 1, t_i \geq 0\}.$$

Let $\varepsilon^{i,j} : \Delta^{i-1} \to \Delta^i$ be the map obtained by inserting a zero as the jth entry:

$$\varepsilon^{i,j}(t_0, \ldots, t_{i-1}) = (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-1}).$$

Let $S = \coprod X_i \times \Delta^i$, and let $\sim$ be the equivalence relation

$$(x, \varepsilon^{i,j}(t)) \sim (\varepsilon_{i,j}(x), t)$$

for $x \in X_i, t \in \Delta^{i-1}, i \geq 1$ and $0 \leq j \leq i$. We define the *geometric realization* of the simplicial space $X.$ to be $|X.| = S/\sim$. The maps ε_i give us a map $|\varepsilon| : |X.| \to X$.

Definition 4.1.2. $X. \to X$ is a *simplicial resolution* of X if the X_i are smooth, the morphisms $\varepsilon_{i,j}$ are proper, and $|\varepsilon|$ has contractible fibers.

Since simplicial resolutions will be used to define the Du Bois complex, it is worth noting that they do indeed always exist:

Proposition 4.1.3 ([Car85, Thm. 6.1]). *Let X be an n-dimensional scheme. Then there exists a simplicial resolution $X.$ of X with* $\dim X_i \leq n - i$.

Given any hyperresolution $\varepsilon : X_{\cdot} \to X$, set

$$\underline{\Omega}^{\cdot}_X = \lim R\varepsilon_* \Omega^{\cdot}_{X_{\cdot}};$$

we call $\underline{\Omega}^{\cdot}_X$ the *Du Bois complex* of X. It follows from [DB81, Thm 2.2] that our definition is independent of the choice of simplicial resolution.

Remark. The limit appearing above is neither a direct nor an inverse limit, as there is no associated direct or inverse system. We shall omit the technical details involved in defining this limit, and refer the reader to [DB81] for a full exposition.

4.2 Properties of the Du Bois Complex

The Du Bois complex may best be understood by considering some of its more useful properties. We start by making explicit the connection between the Du Bois and de Rham complexes.

4.2.1 Du Bois versus de Rham

$\underline{\Omega}^{\cdot}_X$ is an object in the derived category of X. If we give $\Omega_{X_{\cdot}}$ the "filtration bête,"

$$\sigma_r(\Omega^j_{X_{\cdot}}) = \begin{cases} 0 & \text{if } j < r, \\ \Omega^j_{X_{\cdot}} & \text{if } j \geq r, \end{cases}$$

then $\underline{\Omega}^{\cdot}_X$ inherits a filtration F, and is thus an object in the filtered derived category of X. Denote by $\underline{\Omega}^p_X$ the complex $\mathrm{Gr}^p_F \underline{\Omega}^{\cdot}_X = F^p\underline{\Omega}^{\cdot}_X / F^{p+1}\underline{\Omega}^{\cdot}_X[p]$.

There is a natural map $\Omega_X \to R\varepsilon_*\Omega_{X_{\cdot}}$ for any hyperresolution. This gives a map from Ω_X to each term of the limit in the definition, whence we get a natural map $\Omega^{\cdot}_X \to \underline{\Omega}^{\cdot}_X$. Moreover, if X is smooth this map is clearly a quasi-isomorphism as $X = X_0 \to X$ is then a simplicial resolution. That is, for smooth schemes the Du Bois complex is the de Rham complex, at least in the derived category.

4.3 *Functoriality*

Let $f : Y \to X$ be a morphism of schemes. Following [DB81, 3.2], we can find simplicial hyperresolutions of X and Y and a morphism $f.$ between them "resolving" f such that the following diagram commutes:

$$\begin{array}{ccc} Y. & \xrightarrow{f.} & X. \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ Y & \xrightarrow[f]{} & X \end{array} \tag{4.1}$$

This gives us a morphism $\Omega^{\cdot}_{X.} \to Rf._*(\Omega^{\cdot}_{Y.})$. Applying $R\varepsilon_*$, this gives rise to a morphism $R\varepsilon_*(\Omega^{\cdot}_{X.}) \to Rf_*R\eta_*(\Omega^{\cdot}_{Y.})$. Finally, passing to the limit we obtain a morphism

$$f^* : \underline{\Omega}^{\cdot}_X \to Rf_*\underline{\Omega}^{\cdot}_Y.$$

In fact, f^* is a morphism in the filtered derived category, i.e., it respects the filtrations on the Du Bois complexes.

4.3.1 Alternative Methods of Computation

In practice, hyperresolutions are used to compute the Du Bois complex only in fairly simple cases. Using the following theorem, one can often reduce the problem of computing $\underline{\Omega}^{\cdot}_X$ to the computation of $\underline{\Omega}^{\cdot}_Y$ for smooth or normal crossing schemes Y.

Proposition 4.3.1 ([DB81, Prop 4.11]). *Let $f : Y \to X$ be a proper morphism of separated schemes of finite type over $\mathbb{C}$. Assume that f is an isomorphism outside a closed subscheme $\Sigma \subset X$, with reduced preimage $f^{-1}(\Sigma) = E$. Then there exists an exact triangle in the filtered derived category*

$$\underline{\Omega}^{\cdot}_X \to \underline{\Omega}^{\cdot}_\Sigma \oplus Rf_*\underline{\Omega}^{\cdot}_Y \to Rf_*\underline{\Omega}^{\cdot}_E \xrightarrow{+1} .$$

4.3.2 Local Nature

Let $f : U \to X$ be an étale morphism of separated schemes of finite type. Let $\eta : X. \to X$ be a simplicial resolution of U, and let $\varepsilon : U. = X. \times U \to U$ be the simplicial resolution of U obtained from η via base change. Finally, let $f_. : U_. \to X_.$ be the induced map.

By [DB81, §1.11], there is an isomorphism $f^* R\eta_* \Omega^{\cdot}_{X_.} \to R\varepsilon_* f_.^* \Omega^{\cdot}_{X_.}$; since $f_.$ is étale, $f_.^* \Omega^{\cdot}_{U_.} \to \Omega^{\cdot}_{U_.}$ is also an isomorphism. Composing these gives an isomorphism $f^* \underline{\Omega}^{\cdot}_X \to \underline{\Omega}^{\cdot}_U$. Hence the Du Bois complex is local in the étale topology (see also [DB81, §4.3]).

4.4 Du Bois Singularities

The Du Bois complex is, at least intuitively, intended as a generalization of the de Rham complex. One salient feature of the de Rham complex is the fact that, for a scheme X, $\Omega^0_X \cong \mathscr{O}_X$. This fact provides some motivation for the following definition:

Definition 4.4.1. Let X be a scheme, with $\underline{\Omega}^{\cdot}_X$ its Du Bois complex. Then we say that X has *Du Bois singularities* if $\underline{\Omega}^0_X \simeq_{\mathrm{qis}} \mathscr{O}_X$.

So rather than trying to compute the entire Du Bois complex, we shall restrict ourselves to the slightly easier problem of determining when a scheme has Du Bois singularities. To that end, the exact triangle of Proposition 4.3.1 will serve as one of our primary computational tools. The following theorem demonstrates its use, and is also extremely useful in its own right.

Theorem 4.4.2 ([Ste81, Thm 3]). *Let X be a variety, with $\pi : \widetilde{X} \to X$ its normalization and $\mathscr{C} = \mathrm{Ann}_{\mathscr{O}_X}(\pi_* \mathscr{O}_{\widetilde{X}} / \mathscr{O}_X)$ the conductor ideal sheaf of the map π. Define $\Sigma \subseteq X$ to be the subvariety defined by $\mathscr{C}$, and let $E = \pi^{-1}(\Sigma)$. Suppose $\widetilde{X}, E$ and Σ all have Du Bois singularities. Then X has Du Bois singularities.*

Proof. We have a diagram of exact triangles

$$\begin{array}{ccccccc}
\mathscr{O}_X & \xrightarrow{u} & \mathscr{O}_\Sigma \oplus \pi_* \mathscr{O}_{\widetilde{X}} & \xrightarrow{v} & \pi_* \mathscr{O}_E & \xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow & \\
\underline{\Omega}^0_X & \xrightarrow[u]{} & \underline{\Omega}^0_\Sigma \oplus \pi_* \underline{\Omega}^0_{\widetilde{X}} & \xrightarrow[v]{} & \pi_* \underline{\Omega}^0_E & \xrightarrow{+1}
\end{array}$$

where u is given by $u(f) = (f|_\Sigma, \pi^* f)$ and $v(g,h) = \pi^*(g) - h|_E$ (we're abusing notation slightly by writing π instead of $\pi|_E$ in the obvious places). Note that in the top exact triangle, we are identifying a sheaf $\mathscr{F}$ with the associated complex $\mathscr{F}[0]$.

The top triangle is just a restatement of the short exact sequence obtained from the normalization map, while the bottom triangle is the degree zero version of Proposition 4.3.1. The vertical maps come from the natural filtered morphisms between the de Rham and Du Bois complexes. The hypotheses imply that the last two vertical maps are quasi-isomorphisms, hence the first is by the Short Five Lemma. □

Theorem 4.4.2 is extremely useful in demonstrating that a scheme has Du Bois singularities. The following examples illustrate the application of this theorem.

Example 4.4.3. The pinch point, defined as the singular point at the origin of $k[x_1, \ldots, x_n]/(x_1^2 - x_2^2 x_3)$, is a Du Bois singularity. The normalization of this ring is $k[y_1, y_2, x_4, \ldots, x_n]$, which defines a smooth (and therefore Du Bois) scheme. The normalization map is given by

$$\begin{aligned} x_1 &\mapsto y_1 y_2 \\ x_2 &\mapsto y_2 \\ x_3 &\mapsto y_1^2 \\ x_i &\mapsto x_i,\ i \geq 4. \end{aligned}$$

The conductor is the ideal (x_1, x_2), which defines a smooth subscheme. To obtain the preimage of this subscheme, we take the image of the conductor in the normalization, which is the ideal (y_2); this also defines a smooth subscheme. Thus the hypotheses of the theorem are satisfied, so we conclude that the pinch point is Du Bois.

Example 4.4.4. Double normal crossing singularities, analytically isomorphic to $k[x_1, \ldots, x_n]/(x_1 x_2)$, are Du Bois. The normalization of this ring is $k[x_1, \ldots, x_n]/(x_1) \oplus k[x_1, \ldots, x_n]/(x_2)$, which defines a smooth variety. The conductor is given by the ideal

(x_1, x_2), which defines a smooth subscheme; the preimage of this subscheme is two copies of $k[x_1, \ldots, x_n]/(x_1, x_2)$, which is again smooth.

Example 4.4.5. Rational singularities are Du Bois, by [Kov99, Cor. 2.6].

Proposition 4.4.6. *If X is semismooth, then X is Du Bois.*

Proof. Every point of a semismooth scheme is either a pinch point or a double normal crossing point. Thus by the previous examples, X is Du Bois. □

Using this result, we obtain a more useful version of Theorem 4.3.1.

Proposition 4.4.7. *Let $f : Y \to X$ be a good semiresolution. Assume that π is an isomorphism outside a closed subscheme $\Sigma \subset X$, with preimage $f^{-1}(\Sigma) = E$. Then there exists an exact triangle*

$$\underline{\Omega}^0_X \to \underline{\Omega}^0_\Sigma \oplus Rf_*\mathcal{O}_Y \to Rf_*\mathcal{O}_E \xrightarrow{+1} .$$

Proof. Since Y is semismooth, $\underline{\Omega}^0_Y \simeq_{\text{qis}} \mathcal{O}_Y$. The assumption that f is a good semiresolution means that E has only normal crossing singularities, so that $\underline{\Omega}^0_E \simeq_{\text{qis}} \mathcal{O}_E$. The result is then just an application of Proposition 4.3.1. □

Karl Schwede's recent theorem characterizing $\underline{\Omega}^0_X$ is also exceedingly useful in determining when a scheme has Du Bois singularities.

Theorem 4.4.8 ([Sch06, Thm. 5.3.4])**.** *Let X be a variety over $\mathbb{C}$, and assume X is embedded in a variety Y which has only rational singularities. Assume one of the following:*

(i) *Y is smooth, and $\pi : \widetilde{Y} \to Y$ is a strong log resolution (this means that π is log resolution which is an isomorphism outside of X);*

(ii) *Y is smooth outside of X, $\widetilde{Y} \to Y$ is a proper birational map which is an isomorphism away from X, with $\widetilde{Y}$ smooth, the exceptional divisor E' is reduced* *and has simple normal crossings, the strict transform $\widetilde{X}$ of X is a log resolution, and E' meets $\widetilde{X}$ transversely in a simple normal crossing divisor;*

(iii) *There is a proper birational map $\pi : \widetilde{Y} \to Y$ such that $\widetilde{Y}$ has rational singularities, the reduced preimage of X has Du Bois singularities, and π is an isomorphism outside X.*

Then if E denotes the reduced pre-image of X, we have $\underline{\Omega}^0_X \simeq_{qis} R\pi_* \mathscr{O}_E$.

The following theorem, due to Kovács ([Kov99, Thm 3.6]) is the model for a key result we shall prove later.

Theorem 4.4.9. *Let X be a normal variety, and assume that K_X is Cartier and X has Du Bois singularities. Then X is log canonical.*

Proof. Let $f : Y \to X$ be a log resolution of X, with Σ and $E = f^{-1}(\Sigma)$ defined as in Proposition 4.3.1. Then we obtain the exact triangle

$$\underline{\Omega}^0_X \to \underline{\Omega}^0_\Sigma \oplus Rf_* \mathscr{O}_Y \to Rf_* \mathscr{O}_E \xrightarrow{+1} .$$

We also have a short exact sequence

$$0 \to \mathscr{O}_Y(-E) \to \mathscr{O}_Y \to \mathscr{O}_E \to 0;$$

applying Rf_* gives a natural morphism $\phi : Rf_* \mathscr{O}_Y(-E) \to Rf_* \mathscr{O}_Y$. Since the composition $Rf_* \mathscr{O}_Y(-E) \to Rf_* \mathscr{O}_E$ is zero, ϕ factors through $\underline{\Omega}^0_X \simeq_{\text{qis}} \mathscr{O}_X$, giving us a morphism $Rf_* \mathscr{O}_Y(-E) \to \mathscr{O}_X$ which is a quasi-isomorphism on $X \backslash \Sigma$. Apply the functor $R\mathscr{H}om_{\mathscr{O}_X}(-, \omega^{\cdot}_X)$ to this morphism to obtain $\omega^{\cdot}_X \to Rf_* \omega_Y(E)[n]$. Taking the $-n$th cohomology, this gives a morphism $\omega_X \to f_* \omega_Y(E)$ which is an isomorphism on $X \backslash \Sigma$. By adjointness, we then have a (nonzero) morphism $f^* \omega_X \to \omega_Y(E)$; since $f^* \omega_X$ is a line bundle, this implies that $f^* \omega_X \subseteq \omega_Y(E)$ (equivalently, $f^* K_X \leq K_Y + E$), hence X is log canonical. □

Remark. There are two obvious ways to weaken this theorem. One might ask for some condition weaker than normality, which would force some modification to the log canonicity condition; we shall give an explicit example of this modification later. Alternatively, requiring only that K_X is $\mathbb{Q}$-Cartier is tempting, as this is the condition assumed in the definition of log canonical. However, there exist normal Du Bois schemes (in fact, they have rational singularities) satisfying this condition, but which are not log canonical.

Our final goal for this section is to show that products of Du Bois singularities are Du Bois. We introduce the following new class of singularity, similar to normal crossings but more general, to facilitate the proof of this result.

Definition 4.4.10. If X is a variety, we say that X has *generalized simple normal crossings* if for each singular point $x \in X$, we have

$$\widehat{\mathscr{O}}_{X,x} \cong \mathbb{C}[[x_1, \ldots, x_n]]/(I_1 \cap \cdots \cap I_k),$$

where each I_k is generated by coordinate functions, i.e., $I_k = (x_{k_1}, \ldots, x_{k_j})$.

Lemma 4.4.11. *Suppose that X has generalized simple normal crossings. Then X has Du Bois singularities.*

Proof. Note the following "gluing" fact of Du Bois singularities: if X is a variety over $\mathbb{C}$ with components X_1 and X_2 such that X_1, X_2 and $X_1 \cap X_2$ have Du Bois singularities, then X has Du Bois singularities ([Sch06, Thm. 5.2.1]).

We proceed by induction on the maximum dimension of a component and the number of components. If X is the union of two components of any dimension, it has Du Bois singularities by the gluing property (here the intersection is actually smooth). Now assume that the conclusion holds when all components are less than dimension d, and that it holds for n components of dimension d. Suppose that we have $n+1$ components of dimension at most d. Then X can be expressed as the union of the components X_1 and X_2, where X_1 is the union of the first n components and X_2 is the last component. By the induction hypothesis X_1 is Du Bois, and X_2 is smooth. $X_1 \cap X_2$ has generalized simple normal crossing singularities and components of dimension $d-1$ or less, so by the induction hypothesis it also has Du Bois singularities. By the gluing property, we conclude that X has Du Bois singularities. ☐

Lemma 4.4.12. *Suppose X and Y have generalized simple normal crossings. Then $X \times Y$ also has generalized simple normal crossings.*

Proof. Let $z = (x, y) \in X \times Y$. By assumption, $\widehat{\mathscr{O}}_{X,x} \cong \mathbb{C}[[x_1, \dots, x_n]]/I$ and $\widehat{\mathscr{O}}_{Y,y} \cong \mathbb{C}[[y_1, \dots, y_m]]/J$ where I and J are intersections of ideals generated by coordinate functions. Thus

$$\begin{aligned}\widehat{\mathscr{O}}_{X\times Y,z} &\cong \mathbb{C}[[x_1, \dots, x_n]]/I \otimes_k \mathbb{C}[[y_1, \dots, y_m]]/J \\ &\cong \mathbb{C}[[x_1, \dots, x_n, y_1, \dots, y_m]]/IJ \\ &= \mathbb{C}[[x_1, \dots, x_n, y_1, \dots, y_m]]/(I \cap J),\end{aligned}$$

where the last equality follows from the fact that I and J are ideals in disjoint polynomial rings. Thus $X \times Y$ has generalized simple normal crossings. ☐

Theorem 4.4.13. *Suppose X_1 and X_2 are varieties (over $\mathbb{C}$) which have Du Bois singularities. Then $X_1 \times X_2$ also has Du Bois singularities.*

Proof. The property of having Du Bois singularities is local (in the étale topology), so by restricting to affine open sets we may assume that X_1 and X_2 are affine. Embed each X_i into a smooth variety Y_i, and let $f_i : \widetilde{Y}_i \to Y_i$ be a strong log resolution of X_i. Denote by E_i the (reduced) pre-image of X_i. By case (i) of Theorem 4.4.8, we have $\underline{\Omega}^0_{X_i} \simeq_{\text{qis}} Rf_{i*}\mathscr{O}_{E_i}$. Since X_i was assumed to have Du Bois singularities, we have $\mathscr{O}_{X_i} \simeq_{\text{qis}} Rf_{i*}\mathscr{O}_{E_i}$.

We first claim that $g : \widetilde{Y}_1 \times \widetilde{Y}_2 \to Y_1 \times Y_2$ satisfies case (iii) of the Theorem 4.4.8. Clearly, $\widetilde{Y}_1 \times \widetilde{Y}_2$ is smooth (hence rational) and g is an isomorphism outside of $X_1 \times X_2$.

It remains to show that $E_1 \times E_2$ has Du Bois singularities. By assumption, E_1 and E_2 have generalized simple normal crossings. Hence $E_1 \times E_2$ has generalized simple normal crossings by Lemma 4.4.12, and so has Du Bois singularities by Lemma 4.4.11. In particular, this means that $Rg_* \mathcal{O}_{E_1 \times E_2} \simeq_{\text{qis}} \underline{\Omega}^0_{X_1 \times X_2}$.

Now consider the following diagram:

$$\begin{array}{ccccc}
E_1 & \xleftarrow{\widetilde{\pi}_1} & E_1 \times E_2 & & \\
{\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle g_1} & & \\
X_1 & \xleftarrow{\pi_1} & X_1 \times E_2 & \xrightarrow{\widetilde{\pi}_2} & E_2 \\
& & \big\downarrow{\scriptstyle g_2} & & \big\downarrow{\scriptstyle f_2} \\
& & X_1 \times X_2 & \xrightarrow[\pi_2]{} & X_2
\end{array}$$

(Note that g_1 and g_2 are the obvious maps such that $g_2 \circ g_1 = g|_{E_1 \times E_2}$, and $\widetilde{\pi}_i$ and π_i are the usual projection maps.) Applying [Har77, Thm. III.9.3] to the upper left square in the diagram, we find that $\pi_1^* Rf_{1*} \mathcal{O}_{E_1} \cong Rg_{1*}(\widetilde{\pi}_1^* \mathcal{O}_{E_1})$. Since $\widetilde{\pi}_1$ is projection onto a factor, we have $\widetilde{\pi}_1^* \mathcal{O}_{E_1} \cong \mathcal{O}_{E_1 \times E_2}$; by assumption, $Rf_{1*} \mathcal{O}_{E_1} \simeq_{\text{qis}} \mathcal{O}_{X_1}$, so we obtain $Rg_{1*}(\mathcal{O}_{E_1 \times E_2}) \simeq_{\text{qis}} \pi_1^* \mathcal{O}_{X_1} \cong \mathcal{O}_{X_1 \times E_2}$ (the last isomorphism is again due to the fact that π_1 is a projection map). Applying this same argument again to the lower right square gives $Rg_{2*} \mathcal{O}_{X_1 \times E_2} \simeq_{\text{qis}} \mathcal{O}_{X_1 \times X_2}$. Thus we see that $Rg_* \mathcal{O}_{E_1 \times E_2} \simeq_{\text{qis}} \mathcal{O}_{X_1 \times X_2}$; combined with the above, we see that $\mathcal{O}_{X_1 \times X_2} \simeq_{\text{qis}} \underline{\Omega}^0_{X_1 \times X_2}$, so that $X_1 \times X_2$ is Du Bois. $\square$

Chapter 5

GENERIC PROJECTIONS

5.1 Linear Projections

Let Y be a smooth projective variety of dimension n, so that Y can be embedded in $\mathbb{P}^m$ for some $m > n$. Ideally we would like to have $m = n + 1$ so that Y is defined locally by a single equation, as this simplifies many computations. However, there is no reason we should expect that Y is isomorphic to a hypersurface. On the other hand, once Y is appropriately embedded in projective space there is a natural way to obtain a hypersurface which is *birational* to Y, via generic linear projections.

First let us consider what precisely is meant by the notion of linear projection between projective spaces (following [Mum70]). A linear subspace of $\mathbb{P}^m$ is determined by linear forms

$$l_i = \sum_{j=0}^{m} a_{ij}x_j \in k[x_0, \ldots, x_m], i = 0, \ldots, r.$$

Define W_i to be the open set where $l_i \neq 0$, given by the affine scheme

$$\operatorname{Spec} k \left[\frac{x_0}{l_i}, \ldots, \frac{x_m}{l_i} \right].$$

Let $L \subset \mathbb{P}^m$ be the linear subspace $l_0 = \ldots = l_r = 0$. If $\mathbb{P}^r$ is covered by affine open sets

$$V_i = \operatorname{Spec} k \left[\frac{y_0}{y_i}, \ldots, \frac{y_r}{y_i} \right], i = 0, \ldots, r,$$

then define a morphism $\pi_L : \mathbb{P}^m - L \to \mathbb{P}^r$ by requiring $\pi_L(W_i) \subset V_i$ and $\pi_L^*(y_j/y_i) = l_j/l_i$.

The following is a basic fact about linear projections which we shall later generalize.

Theorem 5.1.1 ([Mum70, §2.7, Prop. 6]). *Let Z be a closed subscheme of $\mathbb{P}^m$ disjoint from L. Then*

$$\pi_L|_Z : Z \to \mathbb{P}^r$$

is a finite morphism.

The set of linear projections $\mathbb{P}^m \to \mathbb{P}^r$ is in bijection with the $(m-r)$-dimensional linear subspaces of $\mathbb{P}^m$, which are parametrized by the Grassmannian variety $G(m, m-r-1)$. Thus we will say that a *generic projection* has a property P if the linear projections corresponding to some dense Zariski-open subset of $G(m, m-r-1)$ have property P. In practice, this entails showing that the projections which do not have property P live in some closed subset of $G(m, m-r-1)$.

5.2 Birational Linear Projections

We would like to find some useful properties of linear projections which can be said to be generic in the above sense. Following the work of Roberts ([Rob71]), we shall see that under some mild conditions we can guarantee that generic projections of smooth projective varieties are birational finite morphisms.

Let $\sigma_d : P^m \to P^N$ be the d-uple embedding, where $N = \binom{m+d}{d} - 1$. (The map σ_d is defined by the global sections of the sheaf $\mathscr{O}(d)$.) The geometric properties of this map will make it useful later on. Let $Y \subset \mathbb{P}^N$ be some embedding of a projective variety, and for $x \in Y$ denote by $t_x Y$ the subspace of $\mathbb{P}^N$ tangent to V at x.

Proposition 5.2.1 ([Rob71, Cor 1]). *If $x_1, \ldots, x_r$ are distinct closed points of $\mathbb{P}^m$ and $d \geq 3r$, then the tangent spaces $t_{\sigma_d(x_i)}\sigma_d(\mathbb{P}^m)$, $i = 1, \ldots, r$ span a linear subspace of $\mathbb{P}^N$ of the largest possible dimension, i.e., $r(m+1) - 1$.*

If $Y \subset \mathbb{P}^m$ is a smooth variety then $\sigma_d(Y) \cong Y$. Thus without loss of generality we shall always assume that Y has undergone an appropriate d-uple embedding, so that we can avoid the possibility of a geometrically "bad" embedding.

The following theorem motivates our consideration of generic projections of smooth projective varieties: ultimately, we obtain a variety birational to the originating one.

Theorem 5.2.2 ([Rob71, Thm 1]). *Let $Y \subset \mathbb{P}^m$ be a closed subvariety of dimension $n \geq 1$, and assume $d \geq 3n$. Let $\sigma_d : Y \to \mathbb{P}^N$, N as above, be the d-uple embedding. Then there is a non-empty open subset $U \subset G(N, N-n-2)$ such that for any closed point $L \in U$, (the restriction of) the linear projection $\pi_L : \sigma_d(Y) \to X \subset \mathbb{P}^{n+1}$ is a finite birational morphism.*

One nice consequence of Theorem 5.2.2 is that the normalization of the image hypersurface is isomorphic to the originating variety; in particular, such a hypersurface has smooth normalization.

Proposition 5.2.3. *Assume Y and X are as above. Let $\psi : \widetilde{X} \to X$ be the normalization of the image hypersurface. Then $\widetilde{X} \cong Y$.*

Proof. By the universal property of normalization, there exists a unique morphism $\phi : Y \to \widetilde{X}$. Since π and ψ are both finite birational maps, ϕ must also be a finite birational map. By Zariski's Main Theorem ([Har77, §III, Cor 11.4]), ϕ is an isomorphism. □

5.3 Singularities of Generic Projections

We would like to apply the notions of Du Bois and semi log canonical singularities to the study of hypersurfaces obatined via generic projection of smooth varieties. This requires a better understanding of the local equations of the singularities arising on such hypersurfaces; Joel Roberts ([Rob75]) provides this with his work on singularity subschemes.

For the purposes of this section, we assume that all varieties are irreducible over $\mathbb{C}$, and that all morphisms are separated. Moreover, we shall frequently ask that a variety satisfy the following property:

Let V be a smooth projective variety of dimension n, with some embedding $V \subset \mathbb{P}^m$ If $r \leq n+1$ and $x_1, \ldots, x_r$ are distinct closed points of V, the embedding satisfies condition $(*)$ if the tangent spaces $t_{x_1}V, \ldots, t_{x_r}V$ span a subspace of $\mathbb{P}^N$ of the largest possible dimension, i.e., $r(n+1)-1$.

By Proposition 5.2.1, given any embedding of V, the d-uple embedding $\sigma_d(V)$ with $d \geq 3(n+1)$ satisfies this condition.

5.3.1 Singularity Subschemes

Following Roberts, our approach will be to define the first order singularity subschemes, and then refine one such subscheme to obtain the higher order singularity subschemes.

First Order Singularity Subschemes

Let $f : Y \to X$ be a morphism of varieties, where Y is nonsingular; then for each $y \in Y$ there is an induced morphism of tangent spaces, $df_y : T_yY \to T_{f(y)}X$. Define the *first order singularity subschemes* of f to be the closed subschemes defined by

$$S_i(f) = \{y \in Y \mid \operatorname{rank} df_y \leq \dim Y - i\},$$

i.e., $S_i(f)$ is the locus where df drops rank by at least i.

Alternatively, the $S_i(f)$ can be defined using *Fitting ideals.* Let R be a ring, and M a finitely generated R-module. Suppose we have a presentation

$$R^m \xrightarrow{\phi} R^n \to M \to 0,$$

with $\phi(e_j) = \sum_{k=1}^n a_{jk} f_k$, where $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ are the standard bases. The *ith Fitting ideal* $\Delta_i(M) \subset R$ is the ideal generated by the $(m-i) \times (m-i)$ minors of the matrix (a_{jk}) for $0 \leq i < r$ (if $i \geq r$, set this equal to R). The $\Delta_i(M)$ are independent of the choice of presentation, and $\Delta_i(M) \subset \Delta_{i+1}(M)$ for all i.

Lemma 5.3.1 ([Rob75, Lemma 2.1]). *If (R, m) is local, and a minimal generating set for M contains n elements, then $\Delta_{n-1}(M) \neq R$, and $\Delta_i(M) \subset m^{n-i}$ for $0 \leq i \leq n-1$.*

Since the Fitting ideals are independent of the choice of presentation, we can extend the definition to coherent sheaves on Noetherian varieties. For a coherent sheaf $\mathscr{F}$ on Y and $i \geq 0$, the ith Fitting ideal is the unique sheaf of ideals $\Delta_i(\mathscr{F}) \subset \mathscr{O}_Y$ such that $\Gamma(U, \Delta_i(\mathscr{F})) = \Delta_i(\Gamma(U, \mathscr{F}))$ for every affine open set $U \subset Y$. Assuming the sheaf $\Omega^1_{Y/X}$ is coherent, we have $S_i(f) = \Delta_{i-1}(\Omega^1_{Y/X})$. The equivalence of the two definitions follows from the following proposition, which illuminates the nature of these definitions somewhat.

Proposition 5.3.2 ([Rob75, Prop 2.4]). *If $y \in Y$, then $y \in S_i$ if and only if $\dim_{k(y)} \Omega^1_{Y/X}(y) \geq i$; equality holds when $y \in S_i \backslash S_{i+1}$. If y is a closed point, then $y \in S_i$ if and only if the map of Zariski tangent spaces $df_y : T_yY \to T_{f(y)}X$ has rank $\leq r - i$, where $r = \dim_k(T_yY)$.*

Corollary 5.3.3. $S_{i+1} \subset \mathrm{Sing}(S_i)$.

Finally, we note the following:

Proposition 5.3.4 ([Rob75, Cor. 7.5]). *If $Y \subset \mathbb{P}^N$ is a smooth projective variety of dimension n embedded so that it satisfies $(*)$, and $f : Y \to \mathbb{P}^m$ a generic projection, then $S_i(f)$ is either empty, or has pure codimension $i(m - n + i)$ in Y. In particular, if f projects Y to a hypersurface in $\mathbb{P}^{n+1}$, then $S_i(f)$ has codimension $i^2 + i$ in Y.*

Higher Order Singularity Subschemes

Assume $f : Y \to X$ is a morphism of Noetherian varieties such that $\Omega^1_{X/Y}$ is coherent. Set $S = Y \times_X Y$, and let $\mathscr{J} \subset \mathscr{O}_S$ be the ideal sheaf of the diagonal. Define the $\mathscr{O}_Y$-algebra of *relative principal parts corresponding to* f by $P^q_{Y/X} = p_{1*}(\mathscr{O}_S/\mathscr{J}^{q+1})$, where $p_1 : S \to Y$ is the projection onto the first factor. Considered as an $\mathscr{O}_Y$-module, $P^q_{Y/X}$

has a finite filtration by the factors $p_{1*}(\mathscr{J}^q/\mathscr{J}^{q+1})$. Note that $p_{1*}(\mathscr{J}/\mathscr{J}^2) = \Omega^1_{Y/X}$, so that $P^q_{Y/X}$ is coherent.

Definition 5.3.5. If $q \geq 0$, the *qth order singularity subscheme* $S_1^{(q)}(f) \subset X$ is the closed subscheme of $X \backslash S_2(f)$ corresponding to the ideal sheaf $\Delta_q(P^q_{Y/X}) \subset \mathscr{O}_X$.

Note that $P^0_{Y/X} = \mathscr{O}_X$ and $P^1_{Y/X} \cong \mathscr{O}_X \oplus \Omega^1_{Y/X}$, so that $S_1^{(0)}(f) = X \backslash S_2(f)$ and $S_1^{(1)}(f) = S_1(f) \backslash S_2(f)$.

The following two propositions provide some intuition about the higher order singularity subschemes.

Proposition 5.3.6 ([Rob75, Prop 3.2]). *Assume $y \in Y \backslash S_2(f)$. Then $y \in S_1^{(q)}(f)$ if and only if* $\dim_{k(y)} P^q_{Y/X}(y) \geq q+1$. *In particular,* $S_1^{(q)}(f) \supset S_1^{(q+1)}(f)$.

Proposition 5.3.7 ([Rob75, Prop 3.5]). *Let $q \geq 1$, and assume that X and Y are varieties. Let $y \in Y$ be a closed point. Then $y \in S_i^{(q)}(f)$ if and only if the following hold:*

(1) $\dim_{\mathbb{C}} m_y/(f^*(m_x)\mathscr{O}_{Y,y} + m_y^2) = 1$, *and*

(2) $\dim_{\mathbb{C}} \mathscr{O}_{Y,y}/(f^*(m_x)\mathscr{O}_{Y,y}) \geq q+1$.

We require one further definition before proceeding to the key results.

Definition 5.3.8. Given an integer $d \geq 1$, let $S = Y \times_X \ldots \times_X Y$ be the d-fold product. Define $\Sigma_d(f) \subset S$ to be the complement of the union of all diagonals. The closed points of $\Sigma_d(f)$ are called the *d-fold points of f*.

If $q = (q_1, \ldots, q_d)$ is a d-tuple of nonnegative integers, we refine $\Sigma_d(f)$ with the following construction:

$$\Sigma_d(f;q) = \Sigma_d(f) \cap \left(S_1^{(q_1)}(f) \times_Y \ldots \times_Y S_1^{(q_d)}(f)\right).$$

$\Sigma_d(f;q)$ can be thought of as the subscheme of S consisting of d-tuples $(x_1,\ldots,x_d)$ of distinct closed points of Y such that $x_i \in S_1^{(q_i)}$ for each i, and $f(x_1) = \ldots f(x_d)$. The dimensions of both subschemes can be easily computed, as the following theorems demonstrate.

Theorem 5.3.9 ([Rob75, Thm 7.6]). *Let $V \subset \mathbb{P}^N$ be a smooth variety of dimension n embedded so that it satisfies property $(*)$. If $f : V \longrightarrow \mathbb{P}^m$ is a generic projection, then for every $d \geq 0$ $\Sigma_d(f)$ is either empty, or smooth and of pure dimension $dn-(d-1)m$.*

Remark. In particular, if f is a generic projection to a hypersurface in $\mathbb{P}^{n+1}$, then $\Sigma_d(f)$ has pure dimension $n-d+1$, and is empty when $d \geq n+1$.

Theorem 5.3.10 ([Rob75, Thm A]). *Let V be a variety of dimension n, with $V \subset \mathbb{P}^N$ an embedding satisfying $(*)$. If $f : V \longrightarrow \mathbb{P}^m$ be a generic projection, $n \leq m \leq 2n$, then for every $d \geq 1$ and every d-tuple $q = (q_1,\ldots,q_d)$, $\Sigma_d(f;q)$ is either empty or of pure dimension $dn-(d-1)m-\sum_{j=1}^{d} q_j(m-n+1)$.*

Theorem 5.3.11. *If f is as in the previous theorem, then $S_1^{(q)}(f)$ is smooth, and every irreducible component is either empty or has pure codimension $q(m-r+1)$ in V. In particular, if f is the projection of V to a hypersurface in $\mathbb{P}^{n+1}$, then $S_1^{(q)}$ is either empty or has codimension $2q$ in V.*

With this machinery in place, we can describe the analytic isomorphism classes of the singularities occurring in the higher order singularity subschemes.

Theorem 5.3.12 ([Rob75, Thm 4.1]). *Let $f : V \longrightarrow W$ be a finite morphism of smooth varieties, with $\dim V = n$ and $\dim W = m, m \geq n$. Suppose $x \in S_1^{(q)}(f) \backslash S_1^{(q+1)}(f)$, and assume $S_1^{(q)}(f)$ is smooth at x and locally of codimension $q(m-n+1)$ in V (e.g., f is the generic projection of a smooth projective scheme satisfying condition $(*)$). Then there are isomorphisms $\phi : \hat{\mathcal{O}}_{V,x} \longrightarrow B = k[[t_1,\ldots,t_n]]$ and $\psi : \hat{\mathcal{O}}_{W,f(x)} \longrightarrow A = k[[s_1,\ldots,s_m]]$ such that $g = \phi \circ f^* \circ \psi^{-1} : A \rightarrow B$ satisfies*

(i) $g(s_i) = t_i, i = 1, \ldots, n-1$;

(ii) $g(s_n) = t_n^{q+1} + \sum_{j=1}^{q-1} t_n^j t_{q(m-n)+j}$;

(iii) $g(s_{n+i}) = \sum_{j=1}^{q} t_n^j t_{q(i-1)+j}$, $i = 1, \ldots, m-n$.

Now let $\pi : V \longrightarrow \mathbb{P}^{n+1}$ be a generic projection, where V is a smooth n-dimensional variety, with the embedding $V \subset \mathbb{P}^N$ satisfying (*). Let $V' = \pi(V)$, and assume that $n \leq 5$, so that $\dim S_2(\pi) = n - 6 < 0$, i.e., $S_2(\pi) = \emptyset$.

Proposition 5.3.13. *Let $y \in V'$ be a nonsmooth point. The following are the possible analytic isomorphism classes of $\widehat{\mathscr{O}}_{V',y}$:*

(1) $\widehat{\mathscr{O}}_{V',y} \cong \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1 \cdots x_d)$, *which can occur along a subset of dimension* $n - d + 1$, *for* $d \leq n + 1$;

(2) $\widehat{\mathscr{O}}_{V',y} \cong \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_n^2 - x_2^2 x_{n+1})$, *which can occur along a subset of dimension* $n - 2$;

(3) $\widehat{\mathscr{O}}_{V',y} \cong \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_n^3 + \Phi_4 + \Phi_5)$, *where*

$$\Phi_4 = x_1^2 x_3 x_n - x_1^3 x_{n+1} + 2x_2 x_3 x_n^2 - 3x_1 x_2 x_n x_{n+1},$$
$$\Phi_5 = x_2^2 x_3^2 x_n - x_1 x_2^2 x_3 x_{n+1} - x_2^3 x_{n+1}^2;$$

this can occur along a subset of dimension $n - 4$;

(4) $\widehat{\mathscr{O}}_{V',y} \cong \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1(x_n^2 - x_2^2 x_{n+1}))$, *which can occur along a subset of dimension* $n - 3$;

(5) $\widehat{\mathscr{O}}_{V',y} \cong \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1(x_n^3 + \Psi_4 + \Psi_5))$, *where*

$$\Psi_i = \Phi_i(x_2, x_3, x_4, x_n, x_{n+1});$$

this can occur along a subset of dimension $n - 5$;

(6) $\widehat{\mathscr{O}}_{V',y} \cong \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1x_2(x_n^2 - x_3^2x_{n+1}))$, *which can occur along a subset of dimension* $n-4$*;*

(7) $\widehat{\mathscr{O}}_{V',y} \cong \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1x_2x_3(x_n^2 - x_4^2x_{n+1}))$, *which can occur along a subset of dimension* $n-5$*.*

Proof. By Theorem 5.3.9, the subscheme $\Sigma_d(\pi)$ is nonempty only for $d \leq n+1$. This gives case (1). Similarly, by Theorem 5.3.10 $\Sigma_d(\pi; q)$ is nonempty only when $2(q_1 + \ldots + q_d) \leq n - d + 1$. Hence there is a finite list of d-tuples q to consider; applying Theorem 5.3.12 gives the remaining classes. □

Chapter 6
SEMINORMALITY

The class of non-normal singular schemes is extremely broad, making it unsuitable for study without some additional condition to replace normality. *Seminormality* is one condition that can serve as a useful alternative restriction on the schemes being studied. Like the other "semi" conditions, its properties are analogous to those of plain-old-normality. Our principle motivation here is the connection between seminormality and generic projections.

6.1 Seminormal Rings

The normality condition on schemes is essentially a requirement on rings (integral closure) that has geometric consequences when applied to the structure sheaf of a scheme. The same holds true for the seminormal condition, so we begin by studying the algebraic context of the definition.

Definition 6.1.1. Call a ring homomorphism $f : A \to B$ *subintegral* if $(B \otimes_A k(\mathfrak{p}))_{\text{red}} = k(\mathfrak{p})$ for all $\mathfrak{p} \in \text{Spec}(A)$. Equivalently, f is subintegral if the induced map $\text{Spec}(B) \to \text{Spec}(A)$ is bijective and the associated extensions of the residue fields are trivial.

Definition 6.1.2. Suppose A is a subring of B such that B is a finitely generated A-module. Then we define the *seminormalization* of A in B to be the largest subring of B containing A which is a subintegral extension; denote it by ${}^+_BA$. If $A = {}^+_BA$, say that A is *seminormal* in B.

Note that the seminormalization of A in B always exists, by [Tra70].

Definition 6.1.3. Let A be a reduced ring such that its integral closure $\overline{A}$ is finite over A (we call such a ring a *Mori ring* – this will be true of all rings we shall consider). Then we call ${}^{+}_{\overline{A}}A$ the *seminormalization* of A. If $A = {}^{+}_{\overline{A}}A$, we say that A is *seminormal.*

An alternative characterization of seminormality is given by setting

$$ {}^{+}_{B}A = \{b \in B \mid b_{\mathfrak{p}} \in A_{\mathfrak{p}} + R(B_{\mathfrak{p}}),\ \forall \mathfrak{p} \in \operatorname{Spec}(A)\}. $$

The equivalence of the two definitions is shown in [Tra70]. However, this definition still makes sense even if A is not a Mori ring.

Ultimately we are interested in defining seminormality for schemes, but let us consider some useful properties of the seminormalization.

Proposition 6.1.4 ([GT80, Cor 2.7]). *Suppose A is a Mori ring, and $\overline{A}$ is its integral closure. Then the following are equivalent:*

(i) *A is seminormal;*

(ii) *$A_{\mathfrak{m}}$ is seminormal for any maximal ideal $\mathfrak{m}$ of A;*

(iii) *$A_{\mathfrak{p}}$ is seminormal for any $\mathfrak{p} \in \operatorname{Spec}(A)$;*

(iv) *$A_{\mathfrak{p}}$ is seminormal for any $\mathfrak{p} \in \operatorname{Ass}_A(\overline{A}/A)$;*

(v) *$A_{\mathfrak{p}}$ is seminormal for any $\mathfrak{p}$ such that $\operatorname{depth} A_{\mathfrak{p}} = 1$.*
If in addition A is S_2, the above are also equivalent to:

(vi) *$A_{\mathfrak{p}}$ is seminormal for any $\mathfrak{p}$ such that $\dim A_{\mathfrak{p}} = 1$;*

(vii) *$\overline{A}/\mathfrak{c}$ is reduced, where $\mathfrak{c}$ is the conductor.*

6.2 Seminormal Schemes

Now that we have a definition of seminormality for rings, we make the analogous definition for schemes by requiring the condition hold for all local rings of the structure sheaf. The Mori condition must still hold, which for schemes means that X is reduced and the normalization $\widetilde{X}$ is finite over X. Since X is assumed to be a scheme over $\mathbb{C}$, this is equivalent to simply assuming X is reduced.

Definition 6.2.1. A reduced scheme X is seminormal if $\mathscr{O}_{X,x}$ is seminormal for all $x \in X$.

Some useful properties of seminormal schemes are summarized in the following theorem. Most follow immediately from the analogous statements for rings in Proposition 6.1.4.

Proposition 6.2.2 ([GT80, Prop 3.3, 3.4]). *Let X be a scheme (defined over $\mathbb{C}$). Then the following are equivalent:*

(i) *X is seminormal;*

(ii) *every open subscheme of X is seminormal;*

(iii) *X can be covered by open noetherian affine seminormal subschemes;*

(iv) *for any open affine noetherian subscheme $U \subset X$ the ring $\Gamma(U, \mathscr{O}_X)$ is seminormal.*

If in addition X is S_2, then the above are also equivalent to:

(v) *there is a closed subscheme $Z \subset X$ with $\operatorname{codim} Z \geq 2$ such that $X \backslash Z$ is seminormal;*

(vi) *$\mathscr{O}_{X,x}$ is seminormal whenever it has dimension 1;*

(vii) *there is an open* $U \subset X$ *with* $\operatorname{codim}(X \backslash U) \geq 2$ *such that* $\mathscr{O}_{X,x}$ *is seminormal for all closed points of* X *which are in* U.

Continuing the analogy with normality, there is a universal property of seminormalization.

Proposition 6.2.3 (Universal Property of Seminormalization). *Let* $\pi : \widetilde{X} \to X$ *be the seminormalization of the scheme* X*. If* $f : Y \to X$ *is any morphism with* Y *seminormal, then* f *factors uniquely through* π.

6.2.1 Seminormal Gorenstein Schemes

Ultimately we will be focused on certain hypersurfaces in projective space, which are in particular Gorenstein. Thus we recall a useful result regarding seminormal Gorenstein schemes, which we shall give in a form slightly more general than we shall need.

Definition 6.2.4. A scheme X is said to be G1 if it is S_2 and Gorenstein in codimension 1, i.e., $\mathscr{O}_{X,x}$ is Gorenstein whenever it has dimension ≤ 1.

Example 6.2.5. Gorenstein schemes are G1, as $\mathscr{O}_{X,x}$ is Gorenstein regardless of its dimension.

Theorem 6.2.6 ([GT80, Thm 9.10]). *Let* X *be a reduced* S_2 *scheme. Then* X *is seminormal and G1 if and only if* X *has only double normal crossing singularities in codimension 1.*

6.2.2 Seminormality of Generic Projections

Let us consider the motivating case, where $Y \subset \mathbb{P}^N$ is a smooth projective scheme of dimension n, and $X \subset \mathbb{P}^{n+1}$ is the hypersurface obtained from Y by the generic projection π. As we have seen, $\pi : Y \to X$ is the normalization map, which gives rise to an exact sequence

$$0 \to \mathscr{O}_X \to \pi_*\mathscr{O}_Y \to \pi_*\mathscr{O}_Y/\mathscr{O}_X \to 0.$$

Recall that the conductor is defined as the ideal sheaf $\mathscr{C} = \mathrm{Ann}_{\mathscr{O}_X}(\pi_*\mathscr{O}_Y/\mathscr{O}_X)$; the subscheme $D \subset X$ defined by $\mathscr{C}$ is the locus where X is not normal. Since X has smooth normalization, D is the singular locus of X. (Sometimes this D is called the *double locus* of X, even though its non-generic points can be worse than double points.) Let $\Delta = \pi^{-1}(D)$ be the preimage of D in Y – it corresponds to the sheaf of ideals obtained by lifting $\mathscr{C}$ to $\mathscr{O}_Y$. We then have the following useful fact:

Theorem 6.2.7 ([RZN84, Prop 4.1]). *X is a seminormal scheme if and only if Δ is a reduced subscheme of Y.*

Proof. Both properties are local, so we consider an affine open subset $V = \mathrm{Spec}\, R \subset X$, and set $U = \pi^{-1}(V) = \mathrm{Spec}(\overline{R})$. If $\mathfrak{c}$ is the conductor of R, the statement is equivalent to saying R is seminormal if and only if $\overline{R}/\mathfrak{c}$ is reduced. This follows immediately from Proposition 6.1.4. □

We conclude by noting that generic projection hypersurfaces are seminormal.

Corollary 6.2.8 ([GT80, Thm 3.7]). *Let $Y \subset \mathbb{P}^N$ be a smooth irreducible scheme of dimension n. Then the generic projection X in $\mathbb{P}^{n+1}$ is seminormal.*

Proof. By the classification given in Proposition 5.3.13, the only singularities which can occur in codimension 1 have local analytic form $k[[x_0, \ldots, x_n]]/(x_0x_1)$ (if $n \geq 6$ then the higher S_i and $S_1^{(q)}$ are nonempty, but these always have higher codimension). The preimage of the conductor is reduced, as was demonstrated in Example 4.4.4, so such singularities are seminormal. The result then follows from Proposition 6.2.2 and the theorem. □

The following corollary is an immediate consequence of the preceding proof. It does not seem to appear in the literature, though it is probably known.

Corollary 6.2.9. *If X is a semi log canonical scheme, then X is seminormal.*

Chapter 7

F-PURE SINGULARITIES

Before proceeding to the main results, we consider one more type of singularity – *F-pure singularities.* The notion of F-purity fits into a larger class of "F-singularities;" these are defined in positive characteristic, but still make sense over fields of characteristic 0 via standard reduction methods. Our primary purpose for considering these particular singularities is that they provide an elegant way of determining when hypersurface singularities are Du Bois, following recent results of Karl Schwede connecting the two types of singularity. We shall first give the relevant definitions for rings, and then apply them to schemes in the natural way.

7.1 F-Pure Rings

Let R be a reduced ring of characteristic $p > 0$, and let $F : R \to R$ be the Frobenius map, $F(a) = a^p$. Let $q = p^e$ denote a power of p for the remainder of the present discussion. Given any R-module M, we can view R as acting on M via $r \cdot m = r^q m$; to distinguish this action, we will denote it by ${}^eM$. We shall primarily be interested in ${}^1R$, i.e., when R is considered as an R-module via the Frobenius map.

Since we are assuming R to be reduced, we can identify the following maps:

1. $F : R \to {}^1R$.

2. $R \to R^{1/p}$, where $R^{1/p}$ is the ring of pth roots of elements of R.

3. $R^p \to R$, where R^p is the ring of pth powers of elements of R.

Definition 7.1.1. R is *F-pure* if any one (hence all) of the equivalent maps above is *pure*, i.e., for every R-module M

$$0 \to {}^1R \otimes M \to R \otimes M$$

is exact.

This definition becomes more intuitive if we impose an additional assumption on R.

Definition 7.1.2. R is *F-finite* if ${}^1R$ is a finite R-module.

Throughout this section, we shall assume R is F-finite. As the following shows, however, this will be the case for any rings we shall come across.

Proposition 7.1.3 ([Fed83, Lemma 1.5]). *Let k be a perfect field of characteristic p, and suppose R is a finitely generated k-algebra. Then R is F-finite.*

Proof. Suppose $R = k[a_1, \dots, a_n]$. By assumption, $k^p = k$, so ${}^1R$ is generated by the finite collection of monomials $a_1^{i_1} \cdots a_n^{i_n}$, with $0 \leq i_j \leq p-1$. □

Definition 7.1.4. R is *F-split* if the natural map $R \to R^{1/p}$ (or any of the equivalent maps) splits.

Under the assumption that R is F-finite, R is F-pure if and only if it is F-split ([HR76]).

In general, showing that a ring R is F-pure can be nontrivial; however, the following results, due to Richard Fedder, give us an easily-checked criterion for certain special cases. Recall that if I is an ideal of R, then $I^{[p]}$ is the ideal generated by $\{a^p \mid a \in I\}$.

Proposition 7.1.5 (Fedder's Criterion, [Fed83, Prop 1.7]). *Let $(S, \mathfrak{m})$ be an F-finite regular local ring of characteristic p, and let $R = S/I$. Then R is F-pure if and only if $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$.*

For hypersurfaces, this characterization is especially nice.

Corollary 7.1.6 ([Fed83, Prop 2.1]). *Let $(S, \mathfrak{m})$ be a regular local ring of characteristic p, and suppose $f \in \mathfrak{m}$. Then $S/(f)$ is F-pure if and only if $f^{p-1} \notin \mathfrak{m}^{[p]}$.*

7.2 Reduction to Characteristic p

The natural requirement for a scheme X to have F-pure singularities is to require that all local rings of X are F-pure. However, this only makes sense if X is defined over a ring of characteristic $p > 0$, and we are primarily interested in schemes defined over $\mathbb{C}$. In order to define the notion of F-purity on general schemes, we need to "reduce to characteristic p."

Let $X = \operatorname{Spec} R$ be an affine scheme over $\mathbb{C}$, where

$$R = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)}.$$

Set $A = \mathbb{Z}[a_1, \ldots, a_m]$, where the a_j are the coefficients of the polynomials f_i, and define

$$R_A = \frac{A[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)}.$$

The induced map $X_A = \operatorname{Spec} R_A \to \operatorname{Spec} A$ is called a *family of models* for X. The generic fiber of the map is X, and a generic closed fiber is called a *characteristic p model* of X. If $\mathfrak{p} \in \operatorname{Spec} R_A$ is a maximal ideal, the fiber over $\mathfrak{p}$ is $\operatorname{Spec}(R_A \otimes_A k(\mathfrak{p}))$; since $k(\mathfrak{p})$ has characteristic p for some prime p, this really does define a scheme of characteristic p.

Reduction to characteristic p also makes sense on points. Suppose $x \in X$ is a closed point, corresponding to the maximal ideal $\mathfrak{m}_x \subset R$. Construct A as above, including the coefficients of a set of generators for $\mathfrak{m}_x$ to obtain an ideal $\mathfrak{m}_A \subset A$. This gives the exact sequence

$$0 \to \mathfrak{m}_A \to R_A \to R_A/\mathfrak{m}_A \to 0;$$

taking the tensor product $\otimes_A \mathbb{C}$ then gives the usual sequence

$$0 \to \mathfrak{m}_x \to R \to R/\mathfrak{m}_x \to 0,$$

where $R/\mathfrak{m}_x \cong \mathbb{C}$. Hence $R_A/\mathfrak{m}_A$ is a finite extension of A; so without loss of generality we can assume $R_A/\mathfrak{m}_A \cong A$.

Now let P be some property defined for rings of characteristic p (e.g., F-pure). Then the ring R has open (respectively, dense) P type if $\operatorname{Spec} R$ admits a family of models in which a Zariski open (respectively, dense) set of closed fibers have property P.

With the above notions in hand, the following definition is natural.

Definition 7.2.1. Let X be a scheme defined over $\mathbb{C}$, and let X_p be the reduction of X to characteristic p. Then X_p has *F-pure singularities* if all its local rings are F-pure. X is of *open F-pure type* if X_p has F-pure singularities for an open set of primes p (and similarly for *dense F-pure type*).

7.3 F-Purity and Du Bois

Our primary purpose in considering F-purity is that it serves as a useful tool for demonstrating that a scheme has Du Bois singularities. Two ingredients are necessary for completing this connection: the notion of F-injectivity, and results connecting these three types of singularities.

Definition 7.3.1. A local ring $(R, \mathfrak{m})$ of characteristic p is *weakly F-injective* if the Frobenius morphism induces an injective map $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}({}^1R)$. A ring R of characteristic p is *F-injective* if every localization at a maximal ideal is weakly F-injective.

It is fairly easy to show that F-purity implies F-injectivity:

Lemma 7.3.2 ([Fed83, Lemma 3.3]). *If $(R, \mathfrak{m})$ is a local ring of characteristic p and R is F-pure, then R is F-injective.*

Proof. Without loss of generality, we may assume that R is complete (since local cohomology remains unaffected), so that R has a canonical module ω_R. Since R is F-pure, the map $R \to {}^1R$ splits; this implies that

$$\mathscr{E}xt_R^{n-i}({}^1R, \omega_R) \to \mathscr{E}xt_R^{n-i}(R, \omega_R)$$

is surjective for $0 \leq i \leq n$, where $n = \dim R$. By local duality, it follows that $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}({}^1R)$ is injective. □

Now we can draw the connection to Du Bois singularities, and hence to the rest of our work.

Theorem 7.3.3 ([Sch06, Thm. 6.4.3]). *Suppose that X is a variety over $\mathbb{C}$ of dense F-injective type. Then X has Du Bois singularities.*

Taken together with Lemma 7.3.2, we see that a scheme of dense F-pure type has Du Bois singularities. This fact simplifies some otherwise unpleasant computations in the next chapter.

Chapter 8

SEMI LOG CANONICITY OF GENERIC PROJECTIONS

As we have seen, hypersurfaces obtained by generically projecting smooth projective schemes have smooth normalization, and are seminormal. In the context of the present work, the next logical question is whether such schemes have semi log canonical singularities. Thus the following conjecture seems reasonable:

Conjecture. *Let $Y \subset \mathbb{P}^N$ be a smooth projective variety of dimension n, suitably embedded (i.e., via an appropriate d-uple embedding). Then if $\pi : Y \to X \subset \mathbb{P}^{n+1}$ is a generic projection of Y, the resulting hypersurface X has semi log canonical singularities.*

Unfortunately, this conjecture is false in general, as we shall see. But the statement holds if Y is of sufficiently low dimension – we shall prove it up to dimension 5. Of course, the 1-dimensional statement is simply the classical result that the generic projection of a smooth curve to a plane curve has at worst nodal singularities. The remaining cases rely on the classification of generic projection singularities given in Proposition 5.3.13, and a result connecting semi log canonical singularities to Du Bois singularities, which is where we begin.

8.1 Semi Log Canonicity of Du Bois Singularities

In light of Theorem 4.4.9, one should expect an analogous statement for semi log canonical singularities on moral grounds – the "semi" part entails only a slight weakening of the original definition, after all. The following theorem confirms this expectation.

Theorem 8.1.1. *Let X be an S_2 scheme which is semismooth in codimension one, and assume that K_X is Cartier and X has Du Bois singularities. Then X is semi log canonical.*

Proof. Let $f : Y \to X$ be a good semiresolution of X, with $\Sigma \subseteq X$ the singular set of X, and $E = f^{-1}(\Sigma)$. There exists a natural morphism $\phi : Rf_*\mathscr{O}_Y(-E) \to Rf_*\mathscr{O}_Y$ arising from the short exact sequence

$$0 \to \mathscr{O}_Y(-E) \to \mathscr{O}_Y \to \mathscr{O}_E \to 0.$$

We note that $Rf_*\mathscr{O}_Y(-E) \to Rf_*\mathscr{O}_E$ is the zero map, from which it follows (via the exact triangle in Proposition 4.4.7) that ϕ factors through $\underline{\Omega}^0_X$. By assumption, $\underline{\Omega}^0_X \simeq_{\text{qis}} \mathscr{O}_X$, so we obtain a morphism $Rf_*\mathscr{O}_Y(-E) \to \mathscr{O}_X$ which is a quasi-isomorphism on $X \backslash \Sigma$. Applying $R\mathscr{H}om_{\mathscr{O}_X}(-, \omega^{\cdot}_X)$, we obtain a morphism $\omega^{\cdot}_X \to Rf_*\omega_Y(E)[n]$. Taking the $-n$th cohomology gives a morphism $\omega_X \to f_*\omega_Y(E)$ which is an isomorphism on $X \backslash \Sigma$. Adjointness produces a nonzero morphism $f^*\omega_X \to \omega_Y(E)$, which is still nonzero after taking the double dual and obtaining $[f^*\omega_X]^{**} \to [\omega_Y(E)]^{**}$. Since $[f^*\omega_X]^{**}$ is a line bundle, $[f^*\omega_X]^{**} \subseteq [\omega_Y(E)]^{**}$ and hence X is semi log canonical. □

The following corollary gives a more useful version of the theorem.

Corollary 8.1.2. *Let X be a seminormal Gorenstein scheme with Du Bois singularities. Then X is semi log canonical.*

Proof. The Gorenstein assumption is equivalent to the statement that K_X is Cartier, and also implies that X is G1. By Theorem 6.2.6, a seminormal G1 scheme is semismooth in codimension one. The result then follows immediately from the theorem. □

Theorem 4.4.9 can actually be viewed as a corollary to this statement, as normality implies seminormality, and a normal semi log canonical scheme is actually log canonical.

8.2 Du Bois Singularities and Generic Projections

Main Theorem. *Let $Y \subset \mathbb{P}^N$ be a smooth projective variety of dimension n, $n \leq 5$, embedded via the d-uple embedding with $d \geq 3n$. Let $X \subset \mathbb{P}^{n+1}$ be the image of Y under a generic projection $\pi : Y \to \mathbb{P}^{n+1}$. Then X has Du Bois singularities.*

Proof. Since the Du Bois complex is local in the étale toplogy, it will suffice to show that each of the singularities appearing in Proposition 5.3.13 is Du Bois. Let us address each case.

Case (1) Let $R = \mathbb{C}[[x_1, \dots, x_{n+1}]]/(x_1 \cdots x_d)$; applying Fedder's criterion (the version given in Corollary 7.1.6) is especially simple here. In this case, we have $f_1 = x_1 \cdots x_d$. Reducing to characteristic p, we see that $f_1^{p-1} = x_1^{p-1} \cdots x_n^{p-1}$, and $\mathfrak{m}^{[p]} = (x_1^p, \dots, x_{n+1}^p)$. We clearly have $f_1^{p-1} \notin \mathfrak{m}^{[p]}$, so R has dense F-pure type (it's actually F-pure for *all* p). Applying Lemma 7.3.2 and Theorem 7.3.3, this singularity is Du Bois.

Case (2) The pinch point was shown to be Du Bois in Example 4.4.3.

Case (3) Let $R = \mathbb{C}[[x_1, \dots, x_{n+1}]]/(x_n^3 + \Phi_4 + \Phi_5)$, with

$$\Phi_4 = x_1^2 x_3 x_n - x_1^3 x_{n+1} + 2x_2 x_3 x_n^2 - 3x_1 x_2 x_n x_{n+1},$$
$$\Phi_5 = x_2^2 x_3^2 x_n - x_1 x_2^2 x_3 x_{n+1} - x_2^3 x_{n+1}^2;$$

denote the full polynomial generating the ideal by f_3. Examination of the monomials occurring in f_3 shows that there is a term of the form $(-3x_1 x_2 x_n x_{n+1})^{p-1}$ in f_3^{p-1}. No other product of monomials in f_3 can generate a monomial of the form $(x_1 x_2 x_n x_{n+1})^k$, so the coefficient of $(x_1 x_2 x_n x_{n+1})^{p-1}$ is nonzero for $p \neq 3$. Since this monomial is not in $\mathfrak{m}^{[p]}$, it follows that $f_3 \notin \mathfrak{m}^{[p]}$. Using the same argument as above, this singularity is Du Bois.

Case (4) The ring $R = \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1(x_n^2 - x_2^2 x_{n+1}))$ is the coordinate ring of a product $X_1 \times X_2$ (where we take X_1 to be a pinch point and X_2 to be a hyperplane), hence defines a Du Bois singularity by Theorem 4.4.13.

Case (5) $R = \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1(x_n^3 + \Psi_4 + \Psi_5))$, where

$$\Psi_i = \Phi_i(x_2, x_3, x_4, x_n, x_{n+1}),$$

is also Du Bois by Theorem 4.4.13 and case (3).

Case (6) $R = \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1 x_2(x_n^2 - x_3^2 x_{n+1}))$ is Du Bois by the product theorem and case (4).

Case (7) Finally, $R = \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1 x_2 x_3(x_n^2 - x_4^2 x_{n+1}))$ is Du Bois by the product theorem and case (6).

□

Corollary 8.2.1. *X has semi log canonical singularities.*

Proof. Since X is a hypersurface, it is a complete intersection; in particular, X is Gorenstein. By [GT80, Thm 3.7], X is seminormal, so 8.1.2 implies that X is semi log canonical. □

8.3 Singularities in Higher Dimensions

As noted earlier, generic projection hypersurfaces do not generally have semi log canonical singularities; by Corollary 8.1.2 such hypersurfaces must also fail to have Du Bois singularities. In this section we demonstrate the failure of the conjecture for hypersurfaces of sufficiently large dimension. Our approach is to verify the existence of high-multiplicity points on generic projections; the following theorem demonstrates why such points are a problem for semi log canonicity.

Theorem 8.3.1. *Suppose $X \subset \mathbb{P}^{n+1}$ is a hypersurface, and that there exists some point $x \in X$ having multiplicity $\mu > n+1$. Then X is not semi log canonical.*

Proof. Let $f : Z \to \mathbb{P}^{n+1}$ be the blow-up at x, and let $g : X' \to X$ be the restriction of f to the strict transform of X. Since f is a blow-up at a point, we have (by [Har77, Ex. II.8.5])

$$K_Z \equiv f^* K_{\mathbb{P}^{n+1}} + nE.$$

Similarly, by the definition of X' and our choice of x we have

$$X' = g^* X - \mu E|_{X'},$$

where we are abusing notation and identifying X and X' with the corresponding divisors. Applying the adjunction formula, we obtain

$$\begin{aligned} K_{X'} &\equiv (K_Z + X')|_{X'} \\ &\equiv (f^* K_{\mathbb{P}} + X' + nE)|_{X'} \\ &\equiv (f^* K_{\mathbb{P}} + g^* X - \mu E|_{X'} + nE)|_{X'} \\ &\equiv f^*(K_{\mathbb{P}} + X) + (n - \mu) E|_{X'} \\ &\equiv g^* K_X + (n - \mu) E|_{X'}. \end{aligned}$$

Note that a good semiresolution of X' also produces a good semiresolution of X. Furthermore, semiresolving X' will not increase the coefficient of E. Since $n - \mu < -1$, we conclude that X is not semi log canonical. ☐

With this theorem in hand, it will suffice to demonstrate the existence of such points on generic projections. To this end, the following result is key:

Proposition 8.3.2 ([Laz04, Cor. 7.2.18]). *Let Y be a scheme of dimension n. If the cotangent bundle $T^*Y = \Omega^1_Y$ is nef, and if $f : Y \to \mathbb{P}^m$ is any finite morphism, then $S_i(f) \neq \emptyset$ provided that $n \geq i(m - n + i)$.*

Remark. Applying this to a generic projection $\pi : Y \to \mathbb{P}^{n+1}$, we have $S_i(\pi) \neq \emptyset$ provided that $n \geq i^2 + i$. Note that this only differs from Proposition 5.3.4 in that it guarantees nonemptiness of the S_i, whereas the proposition did not.

Before proceeding, we consider some examples where the proposition applies, i.e., where Ω_Y^1 is nef.

Example 8.3.3. If Y is a smooth projective scheme over $\mathbb{C}$ which is uniformized by $\mathfrak{B}^n \subset \mathbb{C}^n$, then Ω_Y^1 is ample, and thus nef (cf. [Laz04, 6.3.36], [ZL86]).

Example 8.3.4. Let $Y_1, \ldots, Y_m$ be smooth projective varieties of dimension $d \geq 1$, each with big cotangent bundle. If $Y \subset Y_1 \times \cdots \times Y_m$ is a general linear section, with

$$\dim Y \leq \frac{d(m+1)+1}{2(d+1)},$$

then Ω_Y^1 is ample (proven by Bogomolov, published in [Deb05]).

Example 8.3.5. If Y is the complete intersection of at least $n/2$ sufficiently ample general hypersurfaces in an abelian variety of dimension n, then Ω_Y^1 is ample ([Deb05]).

Example 8.3.6. If Y is a projective variety over $\mathbb{C}$ whose universal covering space is a bounded domain in $\mathbb{C}^n$ or a Stein manifold, then Ω_Y^1 is nef ([Kra97]).

Remark. It is conjectured that if Y is the complete intersection of at least $n/2$ hypersurfaces of sufficiently high degree in $\mathbb{P}^n$, then Ω_Y^1 is ample (e.g., see [Deb05]).

The last piece in falsifying the conjecture is establishing a lower bound on the multiplicity of the image of points in S_i. The following proposition was established via correspondence with Robert Lazarsfeld.

Proposition 8.3.7. *Let $f : Y \to X \subset \mathbb{P}^{n+1}$ be a finite morphism, and suppose $y \in S_i(f)$. Then the point $f(y) \in X$ has multiplicity at least 2^i.*

Proof. Since $y \in S_i(f)$, the map $df_y : T_yY \to T_{f(y)}X$ has rank at most $n - i$. To compute the multiplicity of $f(y)$ on X, we compute the intersection multiplicity of a general line L with X at $f(y)$. L is determined by n linear forms, say $l_1, \ldots, l_n$. We can compute the intersection multiplicity by pulling back the l_i to Y, where they generate n hypersurfaces. Since df drops rank by i at y, no more than $n-i$ of the equations defining these hypersurfaces have independent linear terms at y. Without loss of generality, we may assume that the remaining i equations have at least degree 2 at y; thus $(f^*L).Y$ has multiplicity at least 2^i at y. Therefore $f(x)$ also has multiplicity at least 2^i. $\square$

Corollary 8.3.8. *Let $X \subset \mathbb{P}^{31}$ be a generic projection hypersurface obtained via $\pi : Y \to \mathbb{P}^{31}$, where Ω^1_Y is nef. Then X is not semi log canonical.*

Proof. By Proposition 8.3.2, $S_5(\pi) \neq \emptyset$. Proposition 8.3.7 implies that for any $y \in S_5(\pi)$, the image $f(y)$ has multiplicity at least $2^5 = 32$. The result then follows by Theorem 8.3.1. $\square$

Chapter 9
FURTHER QUESTIONS

The original question regarding semi log canonicity of generic projections has been addressed, but there is a definite unsatisfying aspect to the answer. Namely, we have not established a clear dimensional cut-off for the failure of the conjecture.

Question. *Are generic projection hypersurfaces semi log canonical or Du Bois when the originating smooth projective variety X is such that $6 \leq \dim Y \leq 29$?*

One possible approach to this question is to obtain a better understanding of the singularity subschemes of the higher-dimensional generic projections. Specifically, do the "bad" points only arise in the subschemes S_i for large i, or are there points in $S_1^{(q)}$ which cause similar problems? Roberts provides a framework for considering the points of $S_1^{(q)}$, as we can explicitly write local equations of these singularities. However, such equations soon become intractable – in dimension 6, we encounter a singularity in $S_1^{(3)}$ whose local equation has 44 terms. If the conjecture fails in dimension 6, it is conceivable that the brute-force approach is sufficient for determining this fact. Otherwise, it seems that some additional insight may be necessary.

Alternatively, we might note that the only examples of the failure of the conjecture are when Ω_Y is nef. This condition implies that K_Y is nef, which means that Y is a minimal model.

Question. *Suppose Y is* not *a minimal model, so that K_Y is not nef. Is this sufficient to guarantee that the hypersurface obtained via generic projection is semi log canonical?*

An affirmative answer to this question has an interesting consequence: suppose X

is a generic projection hypersurface whose pre-image Y is not a minimal model, such that K_X is ample. Then X is a semi log canonical model for Y. This would provide a relatively simple method for constructing such models.

More general questions remain regarding semi log canonical and Du Bois singularities. Kollár has conjectured that log canonical singularities are Du Bois, so the following seems a natural generalization:

Conjecture. *Semi log canonical singularities are Du Bois.*

Confirmation of this statement would allow us to consider only Du Bois singularities in answering the first question. That is, we would need only demonstrate the conditions under which the generic projection hypersurface fails to be Du Bois. Of course, proof of this conjecture would likely be even more interesting in its own right.

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VITA

Davis Doherty was born in Albany, New York, and grew up in the small town of Round Lake, New York. He received his Bachelor of Arts degree in physics and mathematics from the University of Rochester in 2000, where he also completed a minor in philosophy. Continuing his studies in mathematics, he received his Doctorate of Philosophy from the University of Washington in 2006. During his time in Seattle he became an avid cyclist and a practitioner of chado, the Japanese Way of Tea.