Essays on Macroeconomic Growth: The Role of Human Capital

Aditi Mitra

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Reading Committee:

Stephen J. Turnovsky, Chair

Philip L. Brock

Yu-chin Chen

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Abstract

This paper analyzes the effects of technological change on growth and inequality in a two-sector endogenous growth model. The first two chapters consider two variations of the time path of the shock – discrete and gradual. We find that the effects on inequality depend upon: (i) whether the underlying source of inequality stems from differential initial endowments of human capital or physical capital, (ii) the time horizon over which the productivity increase occurs. Our results suggest that an increase in the growth rate resulting from productivity enhancement in the human capital sector will increase inequality. Productivity enhancement in the final output sector, though does not have permanent growth effects, will reduce inequality. In either case the responses of inequality increase, the more gradually the productivity increase takes place. In the third chapter, we study this tradeoff in the context of fiscal policy. Where-as a subsidy to the human capital sector unambiguously increases growth and reduces inequality, the magnitude of the tradeoff depends on whether this subsidy is financed by taxes on income from physical capital, or from human capital. We find that, in general, a tax on human capital is preferable to one on physical capital, since it generates a more favorable tradeoff. Once again, the results eventually depend on the initial source of heterogeneity. The model can generate a positive or negative relationship between inequality and growth, depending upon the relative size of these effects, consistent with the diverse empirical evidence.

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Aditi Mitra

Chair of the Supervisory Committee:

Professor Stephen J. Turnovsky

Department of Economics

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Chapter I: Growth Inequality Trade-off in the presence of Instantaneous Shocks

Abstract

This paper analyzes the effects of technological change on growth and inequality in a two-sector endogenous growth model. It focuses on the technological aspects rather than the demographic aspects, highlighting the role of the sectoral production characteristics – which need not be uniform across the economy – as potentially important determinants of long run growth and associated inequality. We find that the effects on inequality depend upon: (i) whether the underlying source of inequality stems from differential initial endowments of human capital or physical capital, (ii) the sector that experiences the productivity shock. Our results suggest that an increase in the growth rate resulting from productivity enhancement in the human capital sector will increase inequality. Productivity enhancement in the final output sector, although not having permanent growth effects, will reduce inequality. The model can generate positive or negative trade-offs, depending on these effects, consistent with the diverse empirical evidence.

1 Introduction

Most developed economies have seen a trend of rising earnings inequality in the last two decades of the 20th century (Atkinson 1999). Evidence of this can be seen in the rising skill premium that has been emerging, an increase in the returns to skilled versus unskilled labor (Mitchell 2005). As a result of these developments, the role of human capital has received increasing attention both as a source of economic growth and of the observed rising income inequality.

Several specific channels linking the accumulation of human capital and inequality have been identified. It is well established in literature that wage differentials (Mincer Wage Equation)[1] arise from differentials in skill attainment, via human capital accumulation. Disparities in educational attainment have been seen as one important cause of greater income inequality. [see Becker and Tomes (1986), Galor and Zeira (1993), Fernandez and Rogerson (1998), Viaene and Zilcha (2001)]. Saint-Paul and Verdier (1991) analyze the relationship between public education, growth and income distribution in an OLG model. Durlauf (1991), also uses human capital formation and earnings to show the persistence of inequality in an OLG framework. Ehrlich and Liu(1991) and Ehrlic and Kim(2007) focus on the role of fertility difeferences as a source of income inequality. Hang and Mulligan (2000) use human capital heterogeneity to study intergenerational mobility.

We attempt to study the growth-inequality tradeoff in a framework of heterogeneity in both physical and human capital endowment. Our model follows closely on the lines of the seminal Lucas (1988) and Uzawa (1965) representative agent model, with externalities in human capital. We employ a Romer model to consider the impact of one of the major determinants of the growth-inequality relationship, namely an increase in productivity[2]. The key element of our model is that an individual's personal disposable income depends on her income from the accumulation of human capital and her income from physical capital. Hence, inequality in initial endowments of both physical as well as human capital, affect final levels of income inequality.

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[1] Human capital has positive and significant effects on growth, estimated to be as much as 6%, which coincides with the microeconomic evidence via the Mincer Wage Equation. In the class of regressions called the Macro Mincer equations, the growth of per capita GDP is regressed on the growth and level of the stock of human capital. Our results are largely consistent with the Mincerian earnings function literature in labor economics which assumes that growth is driven by the accumulation of human capital.

[2] This is one of the key factors influencing the growth-inequality relationship identified by Solimano (1998)

We also assume that physical capital is specific to the production of final output, in which case the human capital production sector becomes the fundamental engine of growth, consistent with much of the empirical evidence. This model focuses on the technological aspects rather than the demographic aspects, highlighting the role of the sectoral production characteristics – which need not be uniform across the economy – as potentially important determinants of long run growth and associated inequality. As Lucas argued, this extension of the Uzawa (1965) model and the introduction of intersectoral factor mobility provides important insights and yields a substantial improvement over the standard one-sector neoclassical model in explaining the process of economic development.

We show the existence of a sharp contrast between the effects of a productivity increase in the final output sector on the one hand, and a corresponding increase in the human capital producing sector, on the other, driven by differences in initial endowments of physical and human capital. In the former case, there is no impact on growth where-as income inequality falls. The opposite occurs when the initial productivity increase is in the education sector. While growth rate increases unambiguously, its impact on income inequality depends on relative sectoral intensities. In the more plausible case where the human capital sector is relatively more intensive in human capital, there is an increase in income inequality reflecting a positive trade-off. The key determinant of wealth inequality, is the behaviour of the consumption-wage ratio along the transitional path and its implications for the differential savings rate across agents. This may be important in explaining why otherwise similar economies differ largely in their levels of inequality. The evolution of income inequality, is then determined by the interaction of this changing wealth inequality along with the changing fraction of income from assets as a share of personal income.

In related literature, Galor and Moav (2004) use a one sector model to study the effect of inequality on the process of development, via accumulation of physical and human capital. Viaene and Zilcha (2009), consider a two sector model with both physical as well as human capital, and the impact on inequality in an overlapping generations model. In contrast to our paper though, they consider heterogeneity only in human capital and not in physical capital. They find that the effect of income inequality on growth is ambiguous and depends on the nature of education technology. In contrast to our model, they claim that if the technology shock is neutral, there is no impact on growth. Eckstein and Zilcha (1994) study inequality using a one sector OLG model with both human and physical capital, where heterogeneity in human capital is generated via heterogeneity in parental investment in their children's education(or the home component). They build on the idea that heterogeneity in this 'home component' is a major source of the observed income distribution [see Becker and Tomes (1979), Loury (1981)].

The relationship between growth and inequality has been subject to a great deal of controversy. It has been difficult to reconcile the different theories, especially since empirical evidence has been inconclusive. Viaene and Zilcha (2007) claim that the relationship between growth and income inequality is ambiguous and depends on the source of change in the human capital formation process. Our results show, that the growth-inequality trade-off depends crucially on the initial endowment of human capital vs. that of physical capital, in the context of productivity shocks.

There is a class of literature that studies the properties of alternative numerical techniques to model issues of heterogeneity. Infact, the focus is shifting on the accuracy of these numerical algorithms in their ability to solve complex and analytically intractable models. Krussel and Smith (1998), include heterogeneity in income and wealth arising from partially uninsurable idiosyncratic income shocks. Den Haan, Judd and Juilliard (2010) build on the work by Krussel and Smith (1998), and use DSGE modeling to describe the very first model that considers different numerical algorithms. In the face of highly complex non linear models, one needs to be more careful in the choice of the solution algorithm used and in particular in the evaluation of the accuracy of the numerical solution so generated. Krussel and Smith (2006) focus on the methodological aspects of inequality determination in the presence of heterogeneity. The key insight is that approximate aggregation allows one to solve the problems of forward looking agents with a very small set of state variables (aggregate capital only) and attain nonetheless a very high degree of accuracy[3]. All the same, they go on to show using a two period model, that, in the presence of additional frictions like credit market imperfections used to model heterogeneity, approximate aggregations of the representative agent model might not be robust.

When the equilibrium growth rate and income distribution are mutually dependent, their joint determination and the analysis of their relationship becomes intractable; see e.g. Sorger

[3] Quantitative Macro Models with Heterogeneous Agents', Per Krussel, Anthony Smith (2006)

(2000). Caselli and Ventura (2000), and more recently García Peñalosa and Turnovsky (2006, 2007, 2008), provide a systematic analysis of the distributional consequences of the representative agent model, where there exists a source of heterogeneity. They exploit the fact that if the utility function is homogeneous in its relevant arguments (which includes the constant elasticity utility function that seemingly dominates contemporary macrodynamic theory), the aggregate economy can be summarized by a representative agent. As a result, aggregate behavior becomes independent of the economy's distributional characteristics, and the analysis becomes tractable. But the tractability of the aggregation also depends upon the source of the heterogeneity. The model here sets out a framework in which both growth and distribution are jointly determined and where the heterogeneity of agents, which is the source of income inequality, stems from an their initial distribution of endowments of physical as well as human capital.

In the model we present a sequential equilibrium structure that may be extended to analyze alternative inequality measures. Although the various inequality measures are structurally related in this straightforward way, we conduct our analysis by numerically simulating a two-sector, three-factor production model.

2 Analytical Framework

The analysis extends the traditional two sector model to introduce agents who are heterogeneous in their initial endowments of physical as well as human capital.

2.1 Technology and Factor Payments

There is a single representative firm producing final output X, using the aggregate neo-classical production function (1).

$$X = F\left(K, H_X, L_X\right) = A\left(K^{\alpha}\right)\left(H_X\right)^{\beta}\left(L_X H\right)^{\eta} \qquad \alpha+\beta+\eta=1 \tag{1}$$

where all factors have positive but diminishing marginal physical products and are subject to constant returns to scale. At any point of time, the economy has accumulated $\boldsymbol{K}$ units of physical capital and $\boldsymbol{H}$ units of human capital. Both human capital and labor can move instantaneously and costlessly from one sector to another. Following Lucas and Uzawa (1988), we assume that physical capital is employed only in the final output sector[4].The sectoral allocations of **H** and **L** are constrained by:

$$L_X + L_Y = 1 \tag{2}$$

$$H_X + H_Y = H \tag{3}$$

We define the following variables which are used throughout the rest of the paper :

$$\phi_X = L_X / (H_X / H) \quad and \qquad \phi_Y = L_Y / (H_Y / H)$$

And, $$\phi_{X,i} = L_{X,i} H / H_{X,i} \qquad and \qquad \phi_{Y,i} = L_{Y,i} H / H_{Y,i}$$

We will prove in (a3) later that

$$\phi_{z,i} = \phi_{z,j} = \phi_z \text{ for all z} = X, Y$$

Our procedure is to follow Bond, Wang, and Yip (1996) and to express the macroeconomic dynamic equilibrium in terms of the ratio of physical capital to human capital, $k \equiv K/H$, the ratio of consumption to human capital, $c \equiv C/H$, and the ratio of the shadow value, *q*, augmented by the dynamics of *A*, *B*. To do so it is convenient to let $u \equiv H_X/H$ denote the allocation of human capital to the production of final output.

The aggregate firm makes its productive decisions to maximize profit, so that the returns to physical capital, r_K, human capital, r_H, and labor, W, are as below:

$$r_K = A\alpha \left(\frac{k}{u}\right)^{\alpha-1} \phi_X^{\eta} \tag{4a}$$

$$r_H = A\beta \left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta} \tag{4b}$$

$$\frac{W}{H} = w = A\eta \left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta-1} \tag{4c}$$

From (4a-c) we see that the equilibrium real wage is proportional to the economy-wide stock of human capital and a decreasing function of the ratio of raw labor to human capital employed in the

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[4] Mechanics of Economic Development in an optimal model of economic growth', H. Uzawa (1965)

final output sector, while the rates of return on human and physical capital are both increasing functions of that same ratio. In contrast, the rate of return on physical capital (human capital) is a decreasing (increasing) function of the ratio of physical capital to human capital employed in the final output sector.

2.2 Heterogeneous Consumers

There is a continuum of agents, each indexed by i, identical in all respects except their initial endowments of physical capital $K_{i,0}$ and human capital $H_{i,0}$. Since we consider a growing economy, we are interested in the shares of individual I in the accumulating total stocks of human capital, $h_i^*(t)=\dfrac{H_i(t)}{H(t)}$ and physical capital, $k_i^*(t)=\dfrac{K_i(t)}{K(t)}$, where $H(t)$ and $K(t)$ denote the corresponding economy-wide average quantities. The initial relative endowments are distributed with mean one and standard deviations $\sigma_{h,0},\sigma_{k,0}$ and may or may not be correlated[5]. The analysis makes two important assumptions regarding the underlying source and nature of the heterogeneity. First, there are clearly many sources of heterogeneity of which initial endowments are arguably the most significant. Compelling empirical evidence supporting this, is provided by Piketty (2010)[6] Second, the initial distribution of endowments can be perfectly arbitrary and therefore consistent with any required non-negativity constraints[7]. As will become apparent in due course, the distribution of the initial endowments will be reflected in the evolving distributions of wealth and income.

Each agent is endowed with a unit of raw labor that can be allocated either to employment in the final output sector, $L_{X,i}$ or to acquiring more human capital, $L_{Y,i}$ thus,

$$L_{X,i}+L_{Y,i}=1 \tag{6a}$$

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[5] Being the standard deviation of the relative capital stock, σ_k is in fact the coefficient of variation.

[6] On the Long Run Evolution of Inheritance - France 1820-2050', Thomas Piketty (2010)

[7]There is an extensive literature generating heterogeneity from agents' initial endowments of wealth; see e.g. Chatterjee (1994), Chatterjee and Ravikumar (1999), Sorger (2000), Caselli and Ventura (2000), Maliar and Maliar (2001), Alvarez-Pel a' ez and Diaz (2005) and Obiols-Homs and Urrutia (2005). An alternative source of heterogeneity in the earlier literature was the rate of time preference as e.g. in Becker (1980) where the most patient agent ends up holding all capital.

At any point of time, the agent has accumulated a stock of human capital, H_i that similarly can be allocated either to the final output sector or to the further accumulation of human capital

$$H_{X,i} + H_{Y,i} = H_i \tag{6b}$$

Physical capital, however, is employed only in the final output sector and therefore does not involve an allocation decision.

We assume that the wage rate earned by raw (unskilled) labor is W, while the returns to human capital and human capital are r_H, r_K, respectively, all three being expressed in units of final output. Each agent produces human capital using the following Cobb-Douglas production function:

$$\dot{H}_i = B\left(H_{Y,i}\right)^{\varepsilon}\left(L_{Y,i}H\right)^{1-\varepsilon} \tag{6c}$$

We can think of B, as a combination of both technological change and the average number of years of education in the labor force[8]. This function is of the standard Romer (1986) form, in that it is homogeneous of degree one in the accumulating asset (human capital), with the economy-wide average stock of human capital providing an externality that raises the productivity of each individual's raw labor. The only way an agent can accumulate human capital is by devoting his own physical resources to this activity; there is no market where human capital can be purchased.

The agent's budget constraint, describing his market activities, asserts that the income earned from his three productive factors may be spent either on consumption or on accumulating physical capital

$$\dot{K}_i = r_K K_i + r_H H_{X,i} + WL_{X,i} - C_i \tag{6d}$$

Each agent has a lifetime utility that is assumed to be an isoelastic function of the single consumption good C_i:

$$Max U_i \equiv \int_0^{\infty} \frac{1}{\gamma} C_i^{\gamma} e^{-\rho t} \, dt \qquad with -\infty < \gamma < 1 \tag{6}$$

[8]'Inequality and Growth', Cecelia Garc i' a Pe $\tilde{n}$ alosa (2008)

where $1/(1-\gamma)$ represents the intertemporal elasticity of substitution. The instantaneous utility function is taken to be concave, the consumption good is assumed to be a normal good, while the agent's rate of time preference ρ is taken to be constant for the analysis. Performing the maximization of (6) s.t. (6a) through (6d) yields the following first order optimality conditions:

$$C_i^{\gamma-1} = \lambda_i \tag{7a}$$

$$r_H = \frac{\nu_{2,i}}{\lambda_i} = A\beta\left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta} \tag{7b}$$

$$B\varepsilon\phi_{Y,i}^{1-\varepsilon} = \frac{\nu_{2,i}}{\mu_i} \tag{7c}$$

$$W = HA\eta\left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta-1} \tag{7d}$$

$$B(1-\varepsilon)H\phi_{Y,i}^{-\varepsilon} = \frac{\nu_{1,i}}{\mu_i} \tag{7e}$$

$$\frac{\dot{\mu}_i}{\mu_i} = \rho - \frac{\nu_{2,i}}{\mu_i} \tag{7f}$$

$$\frac{\dot{\lambda}_i}{\lambda_i} = \rho - r_K \tag{7g}$$

where λ_i is the agent's shadow value of wealth associated with K_i and μ_i is agent's shadow value of wealth associated with H_i. Equations (7f) and (7g) when combined give the intertemporal no arbitrage condition which ensures that the return on education (in terms of units of final goods) and the return on physical capital must be equalized at each point in time. In addition, the following transversality conditions must hold to ensure that the individual agent i's intertemporal budget constraint is met

$$\lim_{t\to\infty} \lambda_i K_i e^{-\rho t} = 0$$

$$\lim_{t\to\infty} \mu_i H_i e^{-\rho t} = 0$$

2.3 Aggregation

We now have the basic components, and the next task is to aggregate them to derive the economy-wide equilibrium. Equating the $MRTS_{LH}$ across sectors by dividing (7e) by (7c) and (7d) by (7b) we get:

$$\frac{v_{1,i}}{v_{2,i}} = \left(\frac{1-\varepsilon}{\varepsilon}\right)\left(\frac{H}{\phi_{Y,i}}\right) \tag{a1}$$

$$\frac{v_{1,i}}{v_{2,i}} = \left(\frac{\eta}{\beta}\right)\left(\frac{H}{\phi_X}\right) \tag{a2}$$

Equating (a1) and (a2)

$$\frac{v_{1,i}}{v_{2,i}} = \frac{1-\varepsilon}{\varepsilon}\frac{1}{\phi_{Y,i}} = \frac{\eta}{\beta}\frac{1}{\phi_X}$$

$$\Rightarrow \frac{\beta\phi_X}{\eta} = \frac{\varepsilon\phi_{Y,i}}{1-\varepsilon}$$

This implies $\phi_{Y,i} = \phi_{Y,j} = \phi_Y$ and so,

$$\frac{v_{1,i}}{v_{2,i}} = \frac{v_{1,j}}{v_{2,j}}$$

Also, from the Euler condition (7g) we know

$$\rho - \frac{\dot{\mu}_i}{\mu_i} = \frac{v_{2,i}}{\mu_i} = B\varepsilon\phi_Y^{1-\varepsilon}$$

And so $\frac{\dot{\mu}_i}{\mu_i}$ must be the same for all individuals.

$$\frac{v_{1,i}}{\mu_i} = \frac{v_{1,j}}{\mu_j} \quad and$$

$$\frac{v_{2,i}}{\mu_i} = \frac{v_{2,j}}{\mu_j} \quad and \quad so,$$

$$\frac{\lambda_i}{\mu_i} = \frac{\lambda_j}{\mu_j}$$

Defining $q_i \equiv \dfrac{\mu_i}{\lambda_i}$, the above implies that $q_i = q_j = q$ where $q \equiv \dfrac{\mu}{\lambda}$ is the unit price of human capital defined in terms of units of goods. The relevant point here is, that with all individuals following the same Euler equation, the aggregate economy evolves independently of distributional considerations. Under quite general conditions, the economy proceeds as if there is a single representative agent[9]. This is the case as long as the production function has the standard neoclassical properties, and agents have the same tastes represented by a utility function homogeneous in its single argument, consumption. This condition is crucial in facilitating the aggregation that permits us to derive the macroeconomic equilibrium.

Using q thus defined, the optimality conditions (5b), (5c) and (7c), (7e) give us the following results:

$$r_H = qB\varepsilon\phi_Y^{1-\varepsilon} \tag{8a}$$

$$w = qB(1-\varepsilon)\phi_Y^{-\varepsilon} \tag{8b}$$

$$\Rightarrow \frac{\dot{q}}{q} = r_K - \frac{r_H}{q} \tag{8c}$$

Taking time derivative of (7a)

$$\frac{\dot{C}_i}{C_i} = \frac{1}{1-\gamma}\frac{-\dot{\lambda}_i}{\lambda_i} = \frac{r_K - \rho}{1-\gamma} \tag{9}$$

Therefore, each agent will choose the same growth rate for consumption regardless of initial endowment of assets. In particular,

$$\frac{\dot{C}_i}{C_i} = \frac{\dot{C}}{C} = \psi_C \forall i \tag{9'}$$

and therefore equal to the average. We may then write $C_i = \theta_i C$ where $\int_0^1 \theta_i di$ =1 and θ_i is constant for each i and will be determined in the system. Thus defined, θ_i denotes the agent's total consumption relative to economy wide average consumption, and, as we will show later, will

[9] Gorman (1953)

determine relative welfare.

Since real rates of return are equalized across sectors for both labor as well as human capital, at every point of time, we get the following relationship between ϕ_X and ϕ_Y, which can be expressed in terms of q, A and B as below

$$\phi_X = \left[q \left(\frac{k}{u} \right)^{-\alpha} m \right]^{\frac{1}{\varepsilon+\eta-1}} \tag{10}$$

where $m = \frac{B}{A}\left(\frac{\varepsilon}{\beta}\right)^{\varepsilon}\left(\frac{(1-\varepsilon)}{\eta}\right)^{1-\varepsilon}$ and m is some constant for given values of A, B, ε, β and η.

Then,

$$\phi_X = \Gamma \phi_Y \quad where \quad \Gamma = \frac{\varepsilon\eta}{\beta(1-\varepsilon)}$$

which is the standard result that for given productive elasticities, β, η, ε, the relative intensities of raw labor to human capital will move proportionately in the two sectors. The quantity Γ measures the relative intensities of skilled to unskilled labor in the two sectors. If $\Gamma > 1$, human capital is relatively more intensive than unskilled labor in the human capital sector and vice versa. So we have $\phi_X = \Gamma\phi_Y = \phi(q,k)$ and hence, $r_K = r_K(q,k)$, $r_H = r_H(q,k)$ and $w = w(q,k)$

From the full employment condition (3), we can also solve for u which gets determined within the system, as a function of k and q

$$u = \frac{1-\phi_Y}{\phi_X - \phi_Y} \tag{11}$$

$$\Rightarrow u = u(q,k)$$

Equations (10) and (11) are critical in determining the impact effects of productivity changes.

$$\frac{d\phi_X}{\phi_X} = \frac{d\phi_Y}{\phi_Y} = \left[\varepsilon - \beta + \frac{\alpha}{u[\Gamma-1]}\right]^{-1}\left[\frac{dq}{q} - \alpha\frac{dk}{k} + \frac{dA}{A} - \frac{dB}{B}\right] \tag{10'}$$

$$\frac{du}{u} = -\frac{d\phi_Y}{\phi_Y}\left[1 + \frac{1}{u[\Gamma-1]}\right] \tag{11'}$$

One variable of particular interest is the skill premium. The reason for this is that the supply of raw labor is fixed, whereas human capital grows indefinitely over time, as a result of which in the long run the return to human capital is constant, while the return to raw labor grows at the equilibrium rate. We therefore define the skill premium as the ratio of income earned by human capital to the income earned by raw labor. Thus, a crucial determinant is the equilibrium ratio of raw to skilled labor employed in the human capital sector.

$$S = \frac{r_H}{W/H} = \frac{r_H}{w} = \frac{1-\varepsilon}{\varepsilon}\phi_Y = \frac{\beta}{\eta}\phi_X$$

The standard effect of relative labor supplies is captured by the inverse relationship between skill premium and the ratio of unskilled to skilled labor.

2.4 Macro Economic Equilibrium

The linear homogeneity of the production functions in the private factors allows us to express relations in terms of effective labor units. Defining $\frac{K}{H} = k$ to be the physical capital per effective labor unit and $\frac{C}{H} = c$ to be the consumption per effective labor unit, we can describe the dynamics in terms of these variables. Combining (9) and (9’), the equilibrium dynamics of aggregate consumption is:

$$\dot{C} = \frac{C}{1-\gamma}\left(r_K - \rho\right) \tag{12a}$$

$$\frac{\dot{C}}{C} = \psi_C = \frac{r_K - \rho}{1-\gamma} \tag{12a’}$$

Aggregating over individuals, final goods market equilibrium implies that the final output in excess of total consumption will be accumulated as capital:

$$\dot{K} = A(H_X)\left(\frac{k}{u}\right)^{\alpha}\phi_X^{\eta} - C \tag{12b}$$

$$\frac{\dot{K}}{K} = \psi_K = A\left(\frac{k}{u}\right)^{\alpha-1}\phi_X^{\eta} - \frac{c}{k} \tag{12b’}$$

Similarly, aggregating over each agent's human capital accumulation equation, (2),

$$\dot{H} = B(1-u)H\phi_Y^{1-\varepsilon} \tag{12c}$$

$$\frac{\dot{H}}{H} = \psi_H = B(1-u)\phi_Y^{1-\varepsilon} \tag{12c'}$$

The arbitrage equations (7f) and (7g) give us the final aggregate dynamic equation (8c)

$$\frac{\dot{q}}{q} = r_K - \frac{r_H}{q} \tag{12d}$$

The macro dynamic equilibrium is a modification of that analyzed by Bond et al. (1996), the differences being (i) the distinction between skilled and unskilled labor, and (ii) the assumption that physical capital is specific to the final output sector. But the key observation is that the evolution of the aggregate economy is independent of any distributional measures. As k, c, and q evolve, this will generate adjustment paths for the rates of return and the sectoral allocation of resources. The aggregation shown in equations (12) above allows us to represent the equilibrium dynamics of the economy as a whole as follows

(i) Labor Market Clearance $L_X + L_Y = 1$ (13a)

(ii) Human capital market clearance $H_X + H_Y = H$ (13b)

(iii) Goods market clearance $\dot{K} = A(uH)\left(\frac{k}{u}\right)^{\alpha}\phi_X^{\eta} - C$ (13c)

(iv) Human capital accumulation $\dot{H} = B(1-u)H\phi_Y^{1-\varepsilon}$ (13d)

This macroeconomic equilibrium has a simple structure. Equations (13) determine the sectoral factor allocations and the dynamic evolution of the system. Due to the homogeneity of the underlying utility function, equation (6), the macro-economic equilibrium is independent of any distributional aspects.

Transitional Dynamics

In terms of our previously defined variables, core dynamics can be represented by the following differential equations in k, c and q

$$\dot{k} = Au\left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta} - c - k\psi_H(q,k) \tag{14a}$$

$$\dot{c} = c\left(\frac{r_K(q,k)-\rho}{1-\gamma} - \psi_H(q,k)\right) \tag{14b}$$

$$\dot{q} = qr_K(q,k) - r_H(q,k) \tag{14c}$$

This system has many of the characteristics of a typical two sector economy as pioneered by Lucas (1988) and extended by Bond, Wang and Yip (1996)[10], where the (local) dynamics are discussed in detail by linearising this system around the steady state. There is no need to pursue that discussion here, except to note that the dynamics is a saddle point, with a one dimensional stable manifold. Steady state equilibrium will have the characteristics $\dot{k} = \dot{c} = \dot{q} = 0$.

We can show that the system has a saddle point if and only if the expression:

$$D \equiv [\epsilon(1-\epsilon)(1-\Gamma) + (1-\epsilon)(1-\gamma)\Gamma]\tilde{\phi}_Y + \epsilon(1-\gamma) > 0$$

This ensures stability (non indeterminacy) of the system. The system has two one sluggish variable $(\dot{k})$ and two jump variables. Thus, the above conditions ensure the presence of two negative and one positive eigen values, hence assuring stability of the saddle path.We show numerically that the system is saddle path stable with one negative root $\xi < 0$. Along the stable path, physical capital per effective labor unit evolves gradually, while consumption per effective labor unit and the relative price q, may jump in response to new information as it comes available. Details of the linearization and the associated jacobian have been given in Appendix A1 and A2.

10 'A General Two-Sector Model of Endogenous Growth with Human and Physical Capital: Balanced Growth and Transitional Dynamics', Eric W. Bond, Ping Wang, Chong K Yip (1996)

Balanced growth equilibrium

The steady state balanced growth equilibrium is reached when $\dot{k} = \dot{c} = \dot{q} = 0$ and is summarized by the following conditions

Sectoral allocation of raw labor $\quad \frac{\tilde{W}}{\tilde{H}} \equiv A\eta \left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha} \tilde{\phi}_X^{(\eta-1)} = \tilde{q}B(1-\epsilon)\left(\tilde{\phi}_Y\right)^{-\epsilon}$

Sectoral allocation of human capital: $\quad A\beta \left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha} \tilde{\phi}_X^{\eta} = \tilde{q}B\epsilon\left(\tilde{\phi}_Y\right)^{1-\epsilon}$

Full Employment $\quad \tilde{u}\tilde{\phi}_X + (1-\tilde{u})\tilde{\phi}_Y = 1$

Equilibrium Growth: $\quad \tilde{\psi} \equiv A\alpha \left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1} \tilde{\phi}_X^{\eta} = B(1-\tilde{u})\left(\tilde{\phi}_Y\right)^{1-\epsilon}$

$$\frac{1}{1-\gamma}\left[A\alpha \left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1} \tilde{\phi}_X^{\eta} - \rho\right] = B(1-\tilde{u})\left(\tilde{\phi}_Y\right)^{1-\epsilon}$$

Equilibrium rates of return: $\quad \tilde{r}_K \equiv A\alpha \left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1} \tilde{\phi}_X^{\eta} = A\beta \left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha} \tilde{\phi}_X^{\eta} \equiv \frac{1}{\tilde{q}}\tilde{r}_H$

Given the final values of A, B these equations determine the equilibrium values of $\tilde{u}, \tilde{\phi}_X, \tilde{\phi}_Y, \tilde{k}, \tilde{c}, \tilde{q},$ which then imply the corresponding equilibrium factor rates of return, $\frac{\tilde{W}}{\tilde{H}}, \tilde{r}_H,, \tilde{r}_K,$ skill premium, $\tilde{s}$, and equilibrium growth rate $\tilde{\psi}$. Two critical conditions constrain the equilibrium value of $\phi_Y$and ϕ_X. The first is the transversality condition, that each agent must satisfy. This will be met if and only if,

$$\lim_{t\to\infty}\left[\frac{\dot{\mu}_i}{\mu_i} + \frac{\dot{H}_i}{H_i} - \rho\right] < 0 \equiv \lim_{t\to\infty}\left[\frac{\dot{\mu}}{\mu} + \frac{\dot{H}}{H} - \rho\right] < 0$$

Which in steady state is equivalent to $\tilde{r}_K > \tilde{\psi}$. This is turn is equivalent to $\tilde{u} > 1 - \epsilon$. The other condition is that with no depreciation to human capital, the equilibrium growth rate is always positive, which in turn implies that $1 > \tilde{u}$. Combining these two conditions, together with the full employment condition, yields the following bounds on φ_Y for a feasible solution to exist:

If $\Gamma > 1$: $\quad \frac{1}{(1-\epsilon)\Gamma+\epsilon} > \tilde{\phi}_Y > \frac{1}{\Gamma}$

If $\Gamma < 1$: $\quad \frac{1}{(1-\epsilon)\Gamma+\epsilon} < \tilde{\phi}_Y < \frac{1}{\Gamma}$

2.4.1 Long run effects of productivity increases on aggregate equilibrium

The productivity increases in the two sectors have dramatically different long-run effects on the aggregate economy [Equations (10') and (11')]. These results are summarized in Table 1.

2.4.1. a Productivity increase in final output sector, *A*

Proposition 1: A productivity increase in the final output sector:

I. Leads to a equi-proportionate increases in (i) the ratio of physical to human capital, (ii) the price of human capital, (iii) the rates of return to human capital, (iv) the return to raw labor, and (v) the consumption to human capital ratio. This proportionate increase exceeds unity by an amount that reflects the productivity of physical capital in final output.

II. It has no effect on (i) the sectoral ratios of skilled to unskilled labor, (ii) human capital across sectors, (iii) the return to physical capital, (iv) the equilibrium growth rate, or (v) the skill premium.

The intuition underlying these responses is straightforward. An increase in productivity, *A*, of the final output sector attracts resources to that sector. This raises the productivity of raw labor and human capital proportionately in that sector, increasing their relative rates of return as measured in terms of final output. With linear homogeneous production functions, for proportionate increases in factor prices, there is no incentive to substitute; thus there is no movement of human capital and raw labor. Accordingly factor market equilibrium is maintained by a proportionate rise in the relative price of human capital, so that the skill premium, measured in terms of education remains unchanged.

2.4.1. b Productivity increase in the human capital sector, *B*

In this case the productivity increase is in the growth-generating sector, and as a result the effects are more complex and depend upon the relative sectoral factor intensities.

Proposition 2: A productivity increase in the human capital sector has the following long-run effects:

(i) It leads to an unambiguous increase in the equilibrium growth rate and a likely less than proportionate decline in the price of human capital.

(ii) If the human capital sector is relatively more intensive in skilled labor than is the final output sector $(\Gamma > 1)$, the rate of return on human capital will fall and that on raw labor will decrease too, though by relatively more, raising the skill premium. If $\Gamma < 1$, the responses are reversed.

The following intuition applies. An increase in productivity of human capital *B* attracts resources to that sector. Suppose $\Gamma > 1$, so that the human capital sector is relatively more intensive in human capital. As a result of the shock, the return to human capital will tend to rise and that of raw labor fall thus resulting in an increase in the skill premium. There is, however, another effect in operation. The productivity of raw labor and human capital are enhanced by their interaction with physical capital in the final output sector. As resources move away from this sector this effect is reduced, as represented by the term $-\alpha/(1-\alpha)$ for the corresponding expressions in Table 1. This reinforces the decline in the wage of raw labor and offsets the increase in the return to human capital, causing skill premium to eventually fall, though it remains higher than its pre-shock levels.

The most striking contrast between the productivity increases in the two sectors is in the impact on the long-run growth rate. To see the reason behind this it is constructive to substitute (19b), (19c), and (19f) into (19e), rewriting it as

$$\frac{1}{1-\gamma}\left[\tilde{B}\varepsilon\left(\tilde{\varphi}_Y\right)^{1-\varepsilon}-\rho\right]=\frac{\tilde{B}(\Gamma\tilde{\varphi}_Y-1)}{\Gamma-1}\left(\tilde{\varphi}_Y\right)^{-\varepsilon}=\tilde{\psi} \tag{19e'}$$

From the left-hand side equality, we may solve for long-run ratio of raw labor to human capital in the form $\tilde{\varphi}_Y=\tilde{\varphi}_Y(B)$. We may then write $\tilde{\psi}=\tilde{\psi}(B,\tilde{\varphi}_Y(B))$, from the right hand side equality, highlighting how, both directly and through $\tilde{\varphi}_Y$, the productivity of the human capital sector is the crucial long-run determinant of growth, which by the same token is independent of the productivity in the final output sector.[11] The reason that *B* will increase the growth rate is because of the limited substitution possibilities in the production function for human capital.

[11] Through Γ, the productive elasticities in the final output sector play a role in determining the equilibrium growth rate.

3 Evolution of Wealth, Income and Welfare Inequality

We now proceed to consider the consequences of changes in productivity for the evolution of wealth and income inequality, as well as the overall level of intertemporal welfare inequality.

3.1 Wealth Inequality

The wealth of agent i is defined by

$$V_i(t) = K_i(t) + q(t)H_i(t) \tag{15}$$

where we assume that $V_i > 0$ so that the agent has net positive wealth and is therefore solvent. Taking time derivative of (15) and using (14c) we have:

$$\dot{V}_i(t) = r_K(t)V_i(t) + W(t) - C_i(t) \tag{15'}$$

Both (15) and (15') indicate that in the absence of any market impediments, the shadow value of human capital, $q(t)$, behaves like a price in a competitive market. Summing (15') over all agents in the economy, gives us the aggregate wealth equation:

$$\dot{V}(t) = r_K(t)V(t) + W(t) - C(t) \tag{16}$$

where $V_i > 0$ ensures the solvency of the aggregate economy $V > 0$. We define individual i's share of aggregate wealth as $v_i(t) = \dfrac{V_i(t)}{V(t)}$. Taking time derivative of v_i and combining (15') together with (16), along with the fact that $C_i = \theta_i C$ we get:

$$\dot{v}_i(t) = \frac{1}{V(t)}\Big[\big(W(t) - C(t)\big)(1 - v_i(t)) + C(t)(1 - \theta_i)\Big] \tag{17}$$

Equation (17) highlights how the evolution of an individual agent's share of relative wealth depends on the evolution of aggregate consumption, wage, as well as his own specific endowment as reflected in v_i and θ_i.

Before solving for $v_i(t)$, we see from (17) that agent i's steady state share of wealth satisfies

$$\frac{\tilde{C}_i-\tilde{C}}{\tilde{C}}=\left[1-\frac{W(\tilde{k},\tilde{q})}{\tilde{C}}\right]\left(\frac{\tilde{V}_i-\tilde{V}}{\tilde{V}}\right) \tag{18}$$

$$\frac{\tilde{C}_i-\tilde{C}}{\tilde{V}}=\frac{(\rho-\gamma r_K)}{1-\gamma}\left(\tilde{v}_i-1\right)=\frac{(\rho-\gamma r_K)}{1-\gamma}\left(\tilde{V}_i-\tilde{V}\right) \tag{18'}$$

Thus if individual i's wealth places him above average, s.t. $v_i>1$, then his long run marginal propensity to consume out of this above average wealth is given, by $\frac{(\rho-\gamma r_K)}{1-\gamma}>0$ [12].

To analyze the evolution of relative wealth, we linearise (17) around the steady state, imposing $c=\frac{C}{H}, w=\frac{W}{H}$ and $v=\frac{V}{H}$. This leads to an equation of the form

$$\dot{v}_i=\frac{(\rho-\gamma r_K)}{1-\gamma}(v_i(t)-\tilde{v}_i)+(\tilde{v}_i-1)\left(\frac{\tilde{c}}{\tilde{v}}-\frac{(\rho-\gamma r_K)}{1-\gamma}\right)\left(\frac{c(t)-\tilde{c}}{\tilde{c}}\right)-\left(\frac{\tilde{v}_i-1}{\tilde{v}}\right)(w(t)-\tilde{w}) \tag{19}$$

Since the coefficient of $v_i(t)$ is positive, (19) highlights how the dynamics of $v_i(t)$ are driven by the (forward-looking) transitional time path of $c(t)/w(t)$. For notational convenience we shall denote $C(t)/W(t)$ by $z(t)$. We can then express (17) in the form:

$$\dot{v}_i(t)=\frac{W(t)}{V(t)}\left[(z(t)-1)(v_i(t)-1)+z(t)\left(\frac{1-\tilde{z}}{\tilde{z}}\right)(\tilde{v}_i-1)\right] \tag{17'}$$

Linearized around steady state, we can express (19) as:

$$\dot{v}_i(t)=\frac{\widetilde{W}}{\tilde{V}}\left[(\tilde{z}-1)(v_i(t)-\tilde{v}_i)+(\tilde{v}_i-1)\left(\frac{z(t)-\tilde{z}}{\tilde{z}}\right)\right] \tag{19'}$$

Suppose that the economy is initially in steady state and at time $t=0$ experiences a permanent increase in the productivity of the output sector A. The immediate effect of this is to generate a jump in q as part of the adjustment to ensure that the economy lies on its new stable saddle path.

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[12] From (18') we can compute $\theta_i=1+\frac{(\rho-\gamma r_K)\tilde{V}}{(1-\gamma)\tilde{C}}(\tilde{v}_i-1)$. This implies that the agent's relative consumption is constant throughout the transition.

This in turn causes a jump in agent i's initial relative wealth $v_i(0)=(K_{i,0}+q(0)H_{i,0})/(K_0+q(0)H_0)$

$$\frac{dv_i(0)}{v_i(0)}=\left[\frac{q(0)H_{i,0}}{K_{i,0}+q(0)H_{i,0}}-\frac{q(0)H(0)}{K_0+q(0)H_0}\right]\frac{dq(0)}{q(0)} \tag{20}$$

The nature of the jump depends upon (i) the deviation in the agent's initial relative asset endowment from the economy-wide average, and (ii) the relative price : $\operatorname{sgn}(dq(0))$. Thereafter, $v_i(t)$ will evolve in accordance with (19), given the initial jumps and consequent transitional paths of both $c(t)$ as well as $w(t)$.

The key point to observe about (19') is that the coefficient of $v_i(t)>0$. In order for the relative wealth to remain bounded the solution for $v_i(t)$ is given by the forward looking expression

$$v_i(t)-1=(\tilde{v}_i-1)\left[1-\left(\frac{\widetilde{W}}{\widetilde{V}}\right)\int_t^\infty\left(\frac{z(\tau)-\tilde{z}}{\tilde{z}}\right)e^{-\frac{\widetilde{W}}{\widetilde{V}}(z-1)(\tau-t)}\,d\tau\right] \tag{19'a}$$

and has the property that $\lim_{t\to\infty}\kappa(t)=1$ thus ensuring that $\lim_{t\to\infty}v_i(t)=\tilde{v}_i$. Setting $t=0$ in (19'a)

$$v_i(0)-1=(\tilde{v}_i-1)\left[1-\left(\frac{\widetilde{W}}{\widetilde{V}}\right)\int_0^\infty\left(\frac{z(\tau)-\tilde{z}}{\tilde{z}}\right)e^{-\frac{\widetilde{W}}{\widetilde{V}}(z-1)(\tau-t)}\,d\tau\right] \tag{19'b}$$

We can show that the solution is of the qualitative form

$$\begin{gathered} v_i(t)-1=\kappa(t)(\tilde{v}_i-1) \\ \Rightarrow \sigma_v(t)=\kappa(t)\tilde{\sigma}_v \end{gathered} \tag{21}$$

where $\kappa(t)$ represents the time paths of aggregate variables, $z(t)-\tilde{z}$ and $k(t)-\tilde{k}$.

Setting $t=0$ in (21),

$$\begin{gathered} v_i(0)-1=\kappa(0)(\tilde{v}_i-1) \\ \Rightarrow \sigma_{v,0}=\kappa(0)\tilde{\sigma}_v \end{gathered} \tag{22}$$

The above equation, (22), gives the steady state distribution of agent i's relative wealth $(\tilde{v}_i-1)$, given agent i's distribution of initial wealth $(v_i(0)-1)$ along with the initial jump in $z(t)$ This, together with (19'a) and (19'b) then gives the time path for agent i's relative wealth $v_i(t)$. In the simulations we conduct, the jump in initial wealth (20) is small and has negligible distributional consequences. For simplicity we shall assume that all agents initially hold the same relative shares of physical to human capital which are equal to the average. That is, $\frac{K_{i,0}}{H_{i,0}} = \frac{K_{j,0}}{H_{j,0}} = \frac{K_0}{H_0} = k(0) = k_0$ in which case $dv_i(0) = 0$. On impact, the initial distribution of wealth remains unchanged, so that

$$v_i(0) - 1 = v_{i,0} - 1 = k^*_{i,0} - 1 = h^*_{i,0} - 1$$

$$\textit{where} \quad h^*_{i,0} = \frac{H_{i,0}}{H_0} \quad \textit{and} \quad k^*_{i,0} = \frac{K_{i,0}}{K_0}$$

Because of the linearity of (20) and (21) we can readily transform these equations describing a specific agent's relative asset position into corresponding relationships for the standard deviation of the distribution of wealth across agents, which therefore serves as a convenient measure of wealth inequality.

$$\sigma_v(t) = \kappa(t)\tilde{\sigma}_v \tag{21'}$$

$$\sigma_v(0) = \kappa(0)\tilde{\sigma}_v \tag{22'}$$

Given $\sigma_{v,0}$, the steady state wealth distribution is determined by $\kappa(0)$, which in turn depends on the expected changes in $z(0) - \tilde{z}$, along the subsequent transitional adjustment path.

The solutions (19') and (20) highlight how agent *i*'s relative wealth at each point of time, *t*, and therefore the entire distribution of wealth is driven by the (expected) future time path of the consumption to wage ratio from time *t* forward, as these respond to the underlying structural change, in this case the increase in the level of technology. As a result, the path followed by $z(t)$ will have a permanent effect on the relative stock of wealth and therefore on its distribution across agents.

In the present case, following its initial jump, the evolution of wealth inequality during the transition will depend upon the relative speeds of adjustment of consumption and wage. At this point we can state the following proposition:

Proposition 3: If consumption adjusts more (less) rapidly than do wages along the transitional path, so that z approaches $\tilde{z}$ from below (above) , then wealth inequality will decline (increase) during the transition.

The intuition for this result is straightforward. If consumption grows faster than do wages on raw labor, savings grow at a slower rate. Since wealthier people tend to save more, their relative rate of wealth accumulation declines and wealth inequality declines as well.

3.2 Income Inequality: Distribution of Income

Using (13f) of the macro economic equilibrium as derived from the arbitrage conditions, individual i's personal income is

$$Y_i(t) = r_K(t)V_i(t) + W(t) \tag{23}$$

and summing over all agents gives us the average economy wide personal income

$$Y(t) = r_K(t)V(t) + W(t) \tag{23'}$$

Dividing (23) by (23') gives us the relative income of agent $i, y_i \equiv \frac{Y_i}{Y}$

$$y_i(t) = \frac{r_K(t)V_i(t) + W(t)}{r_K(t)V(t) + W(t)} \tag{24}$$

The linearity of (24) allows us to express the relationship between relative income and relative wealth in terms of the corresponding standard deviations of their respective distributions, $\sigma_v(t)$ and $\sigma_y(t)$, namely

$$\Rightarrow \sigma_y(t) = \frac{r_K(t)V(t)}{r_K(t)V(t) + W(t)}\sigma_v(t) \equiv s(t)\sigma_v(t); s(t) < 1 \tag{25}$$

$$\Rightarrow \sigma_y(t) < \sigma_v(t)$$

So, income is more equally distributed than wealth at any point of time. The time path of income inequality reflects that of wealth inequality and the share of income from wealth $s(t)$.

$$\frac{\dot{\sigma}_y(t)}{\sigma_y(t)} = \frac{\dot{s}(t)}{s(t)} + \frac{\dot{\sigma}_v(t)}{\sigma_v(t)}$$

$$= \left[\frac{\dot{r}_K(t)}{r_K(t)} + \frac{\dot{V}(t)}{V(t)} - \frac{\dot{W}(t)}{W(t)} \right] \frac{W(t)}{r_K(t)V(t) + W(t)} + \frac{\dot{\sigma}_v(t)}{\sigma_v(t)} \tag{25'}$$

where the right hand side depends upon the evolution of $V(t), r_K(t)$ and $W(t)$ and $\sigma_v(t)$, as they respond to the shock.

Using the simplifying assumption that $k^*_{i,0} - 1 = h^*_{i,0} - 1$ such that $\sigma_{v,0} = \sigma^*_{k,0} = \sigma^*_{h,0}$ and initial values are given, we see that the initial pre shock steady state income inequality is

$$\sigma_y(0) = \frac{r_K(0)(q(0)H_0 + K_0)}{r_K(0)(q(0)H_0 + K_0) + W_0} \sigma_{v,0} \tag{26}$$

while in the new steady state,

$$\tilde{\sigma}_y = \frac{\tilde{r}_K(\tilde{q} + \tilde{k})}{\tilde{r}_K(\tilde{q} + \tilde{k}) + \tilde{w}} \tilde{\sigma}_v \tag{26'}$$

From equation (26') we see that income inequality will rise relative to its initial equilibrium level, if: (i) the discounted value of effective wealth, v, rises; (ii) the real return to labor, w, decreases. Whether this happens will depend on whether the shock to productivity impacts the output or the education sector.

3.3 Welfare Inequality

Recalling (6) agent i's welfare at time t is

$$Z_i(t) \equiv \frac{1}{\gamma} C_i^{\gamma} \tag{29}$$

Substituting $C_i = \theta_i C$ into this expression yields

$$Z_i(t) = \frac{1}{\gamma}(\theta_i C(t))^{\gamma} \equiv (\theta_i)^{\gamma} Z(t) \tag{30}$$

where $Z(t)$ is the average welfare level at time t. Substituting (29) into (6) yields an analogous relationship for the relative intertemporal welfare evaluated along the equilibrium growth path.

$$\frac{U_i}{U} = \frac{Z_i(t)}{Z(t)} = \theta_i^{\gamma} \tag{31}$$

At each instant of time, agent i's relative welfare remains constant, so that his intertemporal relative welfare, $z_i \equiv Z_i / Z$ remains constant as well. Using (18') and the fact that $\tilde{c} \equiv \left(\frac{C}{H}\right)$ and $\tilde{v} \equiv \left(\frac{V}{H}\right)$ we can express relative welfare in the form

$$\Omega_i \equiv \frac{U_i}{U} = \left[1 + \left(\frac{\frac{(\rho - \gamma \bar{r}_K)}{1-\gamma}\tilde{v}}{\tilde{c}} \right)(\tilde{v}_i - 1) \right]^{\gamma} = z_i(t) \equiv \frac{Z_i(t)}{Z(t)} \tag{32}$$

We can now compute a measure of welfare inequality. A natural metric for this is obtained by applying the following monotonic transformation of relative utility. This enables us to express the relative utility of individual i as

$$\Omega_i^{1/\gamma} = \chi(\Omega_i) = \chi(z_i(t)) = \left[1 + \frac{\frac{(\rho - \gamma \bar{r}_K)}{1-\gamma}\tilde{v}}{\tilde{c}}(\tilde{v}_i - 1) \right] \tag{32'}$$

Both instantaneous and intertemporal welfare inequality, expressed in terms of equivalent units of wealth, can be measured by the standard deviation of relative utility and are constant and identical to the steady state level of income inequality

$$\tilde{\sigma}_\chi = \left(1 - \frac{\tilde{W}}{\tilde{C}}\right)\tilde{\sigma}_v = \left(1 - \frac{\tilde{W}}{\tilde{C}}\right)\tilde{\sigma}_y\, s(t) \tag{33}$$

4 Numerical Simulations

Given the complexity of the model, to analyze the consequences of a productivity shock, on the dynamics of wealth and income distribution it is necessary to employ numerical simulations. We begin by calibrating a benchmark economy using the following standard parameter values representing a typical economy.

Parameter Values

Preference Parameters	$\gamma = -1.5$ (IES $= 0.4$), $\rho = 0.04$
Production Parameters	$\alpha = 1/3,\ \beta = 1/3,\ \eta = 1/3,\ \varepsilon = 0.6$ $\Rightarrow \Gamma > 1$
Cases	$k_{i,0} = h_{i,0}; K_{i,0} = K_{j,0}; H_{i,0} = H_{j,0}$
Productivity Parameters	$A_0 = 0.2,\ B_0 = 0.2, \widetilde{A} = 0.22, \tilde{B} = 0.22$

First the preference parameters corresponding to a rate of time preference of 4% and an intertemporal elasticity of substitution of 0.4 are standard and noncontroversial. The exponents $\alpha = \beta = \eta = 1/3$ in the production function for final output approximates the empirical estimates of the generalized Solow production function obtained by Mankiw, Romer and Weil (1992)[13]. Empirical evidence on the production function for human capital is far more sparse. We feel that the most important input in augmenting the stock of human capital is human capital, followed by raw labor, with physical capital being the least important, and which we have set to zero. Thus we set $\varepsilon = 0.60$, as a plausible benchmark, which we may note is very close to the calibrated value of 0.62 obtained by Manuelli and Seshadri (2010). This configuration of productive exponents yields $\Gamma = 1.5$, implying that the production of human capital is more intensive in skilled, rather than unskilled, labor relative to the production of final output.

The first row of Table 2.A summarizes the key aggregate variables in this initial equilibrium. This equilibrium is associated with a capital-output ratio of 3.23, a growth rate of 2.56%, a rate of return on physical capital (ignoring depreciation) of 10.4%, and a return on human capital of 5.47%, with the skill premium of 105%. Nearly 90% of raw labor and 85% of human capital is allocated to the final output sector.

Also, as stated earlier, we impose the fact that the initial distribution of physical capital and human capital is the same. This leads to the fact that, $dv_i(0) = 0$ that is, the initial distribution of relative wealth remains unchanged on impact, in response to a shock. For the purpose of the simulations, this puts $\lambda = 1$ where λ is defined as $\dfrac{k_{i,0}^{*} - 1}{h_{i,0}^{*} - 1}$. The results for this have been tabulated in Table 2(b). In Table 2(b) we focus on three polar cases, (i) equi-proportionate

[13] They obtain estimates of $\alpha = 0.43, \beta = 0.31, \eta = 0.28$.

inequality across initial asset holdings, $\sigma_{k,0} = \sigma_{h,0} = \sigma_{kh,0} = 1$, (ii) no initial physical capital inequality, $\sigma_{k,0} = \sigma_{kh,0} = 0$, (iii) no initial human capital inequality, $\sigma_{h,0} = \sigma_{kh,0} = 0$.

Rather than calibrate to a specific economy, we prefer to ensure that our calibration lies within the observed range, so that it may be viewed as being typical of a class of economies rather than a specific economy. In this respect, the implied capital-output ratio, consumption-output ratio and the ratio of the value of human capital to physical capital, summarized for the basic calibrations are generally within a plausible range and thus suggest that the benchmark represents a plausible starting point.

5 Analysis of Results

5.1 Instantaneous shock to the final output sector

The long run effects of the aggregate economy are described in line 2 of Table 2.A. An increase in productivity specified as a instantaneous increase in A by 10%, leads to a long-run increase in the ratio of physical to human capital, the relative price of human capital,the consumption to human capital ratio, the real return on human capital and the real wage by 15.4%. The sectoral allocations of skilled and unskilled labor and the return on physical capital, along with the growth rate, remain unchanged. [see Panel A, Figure 1.1-1.9]

The productivity increase in the final output sector draws resources towards this sector. Given the relative sectoral intensities (Γ >1), we see from equations (10') and (11') that factor market equilibrium will require a decline in the relative demand for skilled labor, so that ϕ_X and ϕ_Y both fall on impact. At the same time, forward looking agents predict that this increase in productivity will increase $\tilde{k}$. The relative scarcity of human capital causes $q(0)$ to jump up on impact. The shift in resources toward the production of final output (which is intensive in raw labor) results in a short term decline in skill premium. On impact, the productivity increase results in an increase in consumption, since consumers adjust their consumption to the new higher permanent income. Wage too jumps, though by a greater proportionate amount, causing the consumption-wage ratio to decline. The productivity increase causes return to physical capital to increase more than the return to human capital. While this stimulates the growth of physical capital, that of human capital declines. The K/H ratio thus begins to rise.

The evolution of wealth inequality depends critically on the initial distribution of relative endowments. If the distribution is proportional across agents ($k_{i,0} = k_{j,0}$), initial wealth inequality remains unchanged. If they are entirely due to differences in human capital, the initial increase in $q(0)$ will raise short run wealth inequality, while if they are due to physical capital, wealth inequality will immediately decline [see (20)]. The effect on income inequality depends on the size of the initial positive effect from the share of income from capital ($s(0)$), relative to the initial response of wealth inequality($\sigma_v(0)$) . These two effects may be either re-enforcing or offsetting, depending on the source of heterogeneity.

These initial responses trigger the transitional dynamics. Since r_K exceeds r_H, this results in a reversal of the initial decline in ϕ_X, ϕ_Y. The increasing K/H ratio offsets this to some extent, though on balance, ϕ_X, ϕ_Y increase to the pre-shock levels. As resources move back toward the education sector (ϕ_Y increases), r_K declines while r_H increases. Growth rates gradually converge to their original steady state levels.

Dynamics of distribution are driven by the transition of the consumption to wage ratio. As the ratio of consumption to wage (C/W) increases in transition to its new higher steady state, savings decline [see (19')]. This is key in explaining the decline of wealth inequality in transition. Whether wealth inequality ends up being more or less equal than its pre-shock level depends on the source of the heterogeneity. In the case that wealth inequality is due to variations in the intial endowment of human capital, long run wealth inequality will increase by 2.8%. If instead it is due to differential endowments in physical capital, it will decline by 5.6%. [see Table 2.B.].

The transitional dynamics of $\sigma_y(t)$ mirrors that of wealth inequality and reflects a number of conflicting influences. In the short term, the increase in productivity increases capital income more than it does wage income. The increasing effective wealth thus dominates and income inequality rises on impact. Overtime, capital accumulation raises the wage rate, whereas the return to capital stays the same. Since labor is more equally distributed than capital, the return to the more equally distributed factor increases. This, coupled with the falling wealth inequality causes long-run income inequality to fall by 0.85% [Panel A, Figure 1.10 (a)]. It may be noted that with the return to physical capital unchanged in the long run, the long run income inequality will change by the same proportionate amount as does wealth inequality $(\dot{s}(t) = 0)$. The same is true for welfare inequality.

5.2 Instantaneous Shock to the Human Capital Sector

An increase in productivity is specified as an increase in B by 10%. These contrast from the responses in A above, in that the productivity increases in the human capital sector has a permanent growth effect, which for these simulations rises from 2.56% to 2.99%. There is empirical evidence that supports this result. Denison (1985) found that growth in years of schooling between 1929 and 1982 explained about 25% of the growth in the U.S. per capita income during the period. The experiences of nearly one hundred countries since 1960 suggest that education investments in 1960 are an important variable explaining subsequent growth in per capita incomes (Barro 1989).

The long run responses have been summarized in line 3 of Table 2.A. This productivity increase draws resources into the human capital sector, shifting both skilled and unskilled labor toward that sector. Thus, the share of both human capital as well as unskilled labor in the output sector fall. Since the education sector is more intensive in human capital, equilibrium in both markets requires small permanent increases in ϕ_X and ϕ_Y in the long run. Also, the increase in the long run ratio of human to physical capital (fall in $\tilde{k}$) results in a reduction in relative scarcity of human capital, thus a fall in $q(0)$.

In the long run, the relative increase in skilled labor in the final output sector (ϕ_X) raises the return on physical capital and reduces the real wage (return to unskilled labor). While the increase in ϕ_X increases the return to human capital, the fall in $\tilde{k}$ tends to lower it. On balance, the latter dominates and return to human capital falls. But the return to unskilled labor falls by more, causing skill premium to increase. The response of the return to human capital is also consistent with empirical evidence. Teulings and Rens (2009) show a clear negative relationship between the private return to education and the average level of schooling in the country, when workers have different skill levels. This is reflected in our analysis as a fall in the return to education with increasing stocks of human capital. Goldin and Katz (2001) use U.S. data from 1915-1999 to show that greater and universally higher levels of education reduced the rate of return to years of education relative to their extremely high level at the turn of the 20th century.

The short run transitional responses to an increase in B are approximately the mirror images of those to A (in section 5.1.1. above), and may be explained analogously. Much of the explanation may be explained by equations (10') and (11') where dB operates in the reverse direction, and $q(0)$ falls on impact.

With respect to distribution, since the technology increase now occurs in the human capital sector, the immediate rise in wage rate is offset. The consumption-wage ratio now rises, but declines through the subsequent transition. This causes wealth inequality to increase during transition as discussed above (through the savings channel). Once again, the long-run level of inequality depends on the initial endowments. When this is uniform, wealth inequality increases by 1% but will decline by 2.5% if the initial heterogeneity is due to human capital endowments alone.

This mechanism could partly explain the recent increase in income inequality in the US. Between 1975 and 1995 the Gini coefficient in the US increased by 6.4 Gini points. The period also witnessed the widespread adoption of IT technologies. Adoption of IT technologies typically requires a higher level of human capital/skill. It has been argued that the resulting productivity increase leads to greater wage inequality, and hence greater income inequality. Our analysis implies that an increase in the technology parameter B would also affect the distribution of income through its effect on the returns to capital – physical as well as human and labor.

In the long run, real wages of unskilled labour fall. Acemoglu (2002) shows that real returns to low-skilled (10^{th} percentile) have either stagnated or fallen in the last 30 years. Capital income meanwhile, falls by much more. This, coupled with the fall in effective wealth units, causes income inequality to fall on impact. In the long run though, the increasing wealth inequality dominates, causing income inequality to also rise, by 1.0% [Panel B, Figure 2.10 (a)].

In the case where the initial distribution of physical capital is the same for all agents ($\lambda = 0$) and income inequality dynamics are driven by the initial endowments of human capital, inequality falls on impact and then rises towards its new lower steady state, a fall of 2.5% [Panel B, Figure 2.10 (b)]. In contrast, when the initial distribution of human capital is the same for all agents ($\lambda = \infty$), inequality shows a marked increase of 5.2% [Panel B, Figure 2.10 (c)].

6 Conclusion

Human capital, particularly that attained through education, has replaced physical capital as the prime engine of growth [see Ehrlic (2007), Galor and Moav (2002)]. Greater education increases the skill and productivity of workers, which increases the output of goods and services. Evidence provided by Goldin and Katz (2001) and Abramovitz and David (2000) suggest that over the period 1890-1999 in the United States the contribution of human capital accumulation to the growth process has nearly doubled whereas the contribution of physical capital has declined significantly.

The relationship between growth and inequality remains largely unresolved, despite the intensive research devoted to it over the past 50 years. Empirical evidence is inconclusive, some authors find a negative relationship between these variables, while others obtain a positive relationship. Alesina and Rodrik (1993) find a negative relationship between inequality and growth. Persson and Tabellini (1994) conduct empirical analysis for a sample of 56 countries, post World War II, from 1960-1985 and also find a negative relationship between growth and inequality. Castelló and Doménech (2002) find a robust negative relationship between growth and inequality using initial human capital inequality rather than income inequality. K.J.Forbes (2000), uses empirical analysis to predict a positive relationship between economic growth and income inequality, especially when the short term impact is considered. Given that these variables are simultaneously and endogenously determined, they need to be specified within the context of a complete growth model.

There are some differences between earlier work and related literature, and our model. Whereas earlier work has focused on the effect of *wealth* inequality on growth, in our framework, the distribution of wealth has no impact on growth. We look at the impact of inter-sectoral movements in resources on the growth-inequality tradeoff, rather than demographics. We find that the initial source of heterogeneity and the sectoral location of the shocks are the primary determinants of the final level of inequality.

There are important policy implications that may be drawn from this. A shock to the output sector, while reducing inequality, does not raise growth rates. Thus, some sort of a policy combination that increases the productivity of the output sector while encouraging growth in the education sector could give a more favorable tradeoff between inequality and growth. I explore this further in my third paper.

Chapter II: Effect of Time Path on the Growth Inequality Trade-off

Abstract

This paper studies the interaction between human and physical capital, in the presence of a gradual change in productivity, and its consequent impact on the growth-inequality relationship. Our results suggest, that otherwise similar economies may have very different outcomes for income and wealth inequality depending on the evolutionary nature of a given structural change. We find that the effect of a productivity shock depends on (i) the sectoral location of that shock (ii) the initial source of heterogeneity and (iii) time path of the shock. Whereas the time path of productivity impacts only the transitional path of the aggregate variables, it has both transitional as well as permanent consequences for wealth and income distribution.

1 Introduction

This paper studies the impact of the time path of a productivity change, on the growth inequality tradeoff. Whereas the increase in productivity was a one time instantaneous shock in our previous paper, here the shock is an anticipated shock, and it is the the growth rate of productivity that increases.

In the case where the location of the shock is the output sector, there is no impact on growth whereas income inequality falls. The opposite occurs when the initial productivity increase is in the education sector. Thus, in the aggregate steady state, variables behave much in the same way as when the shock is instantaneous. At the same time, in transition, the path followed by many of these aggregate variables contrast sharply with the case where the productivity shock is instantaneous. The impact on distribution though varies largely, both in transition as well as the final steady state. More specifically, income inequality shows non-monotonic changes during transition to the new steady state, something that contrasts starkly in case of the instantaneous shock, where all changes are monotonic. The presence of two sources of heterogeneity makes possible this non-monotonicty.

In the more plausible case where the human capital sector is relatively more intensive in human capital, there is an increase in income inequality reflecting a positive trade-off. The key determinant of wealth inequality, is the behaviour of the consumption-wage ratio along the transitional path and its implications for the differential savings rate across agents. In the case of an anticipated shock, the consumption to wage ratio also shows non-monotonicity, staying below (above) the steady state for a longer time than in the instantaneous case, when the sectoral location of the shock is the output (education) sector. This may be important in explaining why otherwise similar economies differ largely in their levels of inequality.

We find that a change in the rate of growth of productivity may either increase or decrease wealth inequality, depending upon whether the final output sector or the education sector experiences the shock to growth rate, and the source of initial heterogeneity. In our earlier analysis, we adopt the conventional procedure of specifying the structural change (productivity) in the form of an unanticipated instantaneous permanent increase. Thus, a instantaneous increase of 10% in the output sector, induces an instantaneous increase in consumption. However, if the increase is acquired gradually, over a known time path, and is therefore anticipated after the first instant, the initial response of consumption is exactly opposite. This result has different distributional consequences, relative to the first case, when the productivity shock is implemented instantaneously. Thus, two identical economies that experience a shock that results in exactly the same level of final productivity, behave very differently as regards their income distribution, depending on whether the shock was instantaneous or if it was along a known time path. We find, that a gradual productivity increase re-enforces the growth-inequality relationship, making it more negative. Thus, an economy that experiences a gradual change in productivity in the output sector ends up with a more equal distribution of income than an economy that experiences a one time permanent increase. Although growth remains unchanged (and equal) in both, the magnitude of the fall in income inequality is much higher for the former.

Besides these steady-state differences, the gradual implementation of a productivity increase leads to several differences along the transitional paths of all variables. Income inequality is profoundly impacted in transition, first increasing for about 5 periods and then falling from above its (lower) final steady state, thus following an inverted U-shaped trajectory.

The model here sets out a framework, to further study the growth-inequality trade-off, in which both growth and distribution are jointly determined and where the heterogeneity of agents, which is the source of income inequality, stems from an their initial distribution of endowments of physical as well as human capital.

2 Analytical Framework

The analysis extends the traditional two sector model to introduce agents who are heterogeneous in their initial endowments of physical as well as human capital.

2.1 Technology and Factor Payments

Aggregate production takes place in the output sectors by a single representative firm. This sector produces the final output X, using physical capital K, H_X , and raw labour L_X as inputs. The production function is neoclassical (Cobb-Douglas) where all factors have positive but diminishing marginal physical products are are subject to constant returns to scale.

$$X = F(K, uH, L_X) = A(t)(K^{\alpha})(H_X)^{\beta}(L_X H)^{\eta} \quad ; \alpha + \beta + \eta = 1 \tag{1}$$

The economy is populated by L individuals, each of whom is endowed with one unit of labor, so that L is also the total labour supply. We normalise the total labor supply to one unit such that $L = 1$. In addition, $A(t)$ represents the level of producitvity, which is exogenous to the firm's decisions. Following Lucas and Uzawa (1988), we assume that physical capital is employed only in the final output sector[14]. In this case human capital production structure becomes the engine of growth, consistent with empirical evidence.

They key feature of our analysis is that the level of productivity is assumed to increase gradually from its initial level A_0 to its enhanced long-run level $\tilde{A}$ both of which are known to the firm. This is specified by the (known) deterministic growth path

$$A(t) = \tilde{A} + (A_0 - \tilde{A})e^{-\theta_A t}\,; \;\; \theta_A \geq 0 \tag{2a}$$

Or equivalently,

$$\dot{A} = \theta_A(\tilde{A} - A(t)) \tag{2b}$$

The parameter θ_A thus defines the time path followed by the increase in productivity. The conventional approach to modeling productivity increases, is to assume that they occur instantaneously. This is approximated by letting $\theta_A \to \infty$, so that the new productivity level is acheived virtually instantaneously.

However, the more general specification being introduced in (2a) is important for two reasons. First, there is a sharp contrast between how θ_A affects aggregate behaviour of the economy and its consequences for wealth and income distribution. While it affects the transitional path of the aggregate economy, it has no effect on the aggregate steady state. In contrast, the choice of θ_A has profound impacts on both the time path, as well as the steady-state equlibria of wealth and income distributions. The sectoral allocations of H and L are constrained by:

$$L_X + L_Y = 1 \tag{3}$$

$$H_X + H_Y = H \tag{4}$$

We define the following variables which are used throughout the rest of the paper :

$$\phi_{X,i} = \frac{L_{X,i}}{H_{X,i}/H} \quad and \quad \phi_{Y,i} = \frac{L_{Y,i}}{H_{Y,i}/H}$$

[14] On the Mechanics of Economic Development', R. E. Lucas (1988) and 'Optimum technical change in an aggregate model of economic growth', H. Uzawa (1965)

We will show in (a3) later that $\phi_{Z,i} = \phi_{Z,j} = \phi_Z \quad \forall \quad Z = X, Y.$

We also assume that a proportion u of human capital is allocated to the output sector for production, whereas a proportion $(1-u)$ is allocated by the inidivudla in his accumulation of human capital. So,

$$H_X = uH \; ; \; H_Y = (1-u)\,H$$

The aggregate firm makes its productive decisions to maximize profit. Using the variables defined above, the returns to physical capital, r_K, human capital, r_H, and labour, W, are as below:

$$r_K = A(t)\alpha\left(\frac{k}{u}\right)^{\alpha-1}\phi_X^{\eta} \tag{5a}$$

$$r_H = A(t)\beta\left(\frac{k}{u}\right)^{\alpha}\phi_X^{\eta} \tag{5b}$$

$$\frac{W}{H} = w = A(t)\eta\left(\frac{k}{u}\right)^{\alpha}\phi_X^{\eta-1} \tag{5c}$$

Note, that wage rate, as well as the returns to both physical and human capital reflect the current level of productivity, $A(t)$. Whereas the return to human capital and the return to raw labor are increasing functions of the physical to human capital ratio in the output sector, the return to physical capital is a decreasing function of that same ratio. The returns to both types of capital (raw labor) though are increasing(decreasing) functions of the ratio of unskilled to skilled labour in the output sector (ϕ_X) .

2.2 Heterogeneous Consumers

There is a continuum of agents, each indexed by i, identical in all respects except their initial endowments of physical capital $K_{i,0}$ and human capital $H_{i,0}$. Each agent has one unit of labor that it supplies inelastically. We shall define the relative shares of individual i's holding of physical and human capital as $k_i^*(t) = \frac{K_i(t)}{K(t)}$ and $h_i^*(t) = \frac{H_i(t)}{H(t)}$ respectively, where $K(t)$ and $H(t)$ denote the corresponding economy-wide average quantities. The initial relative endowments are distributed with mean one and standard deviations $\sigma_{k,0}$, $\sigma_{h,0}$ across agents. The analysis makes two important assumptions regarding the underlying source and nature of the heterogeneity.

First, there are clearly many sources of heterogeneity of which initial endowments are arguably the most significant. Compelling empirical evidence supporting this, is provided by Piketty (2010)[15]. Second, the initial distributions of endowments can be perfectly arbitrary and therefore consistent with any required non-negativity constraints[16]. Each agent has a lifetime utility that is assumed to be an isoelastic function of the single consumption good C_i:

$$MaxU_i \equiv \int_0^\infty \frac{1}{\gamma} C_i^\gamma e^{-\rho t} \, dt \qquad ; -\infty < \gamma < 1 \tag{6}$$

$1/(1-\gamma)$ represents the intertemporal elasticity of substitution. The instantaneous utility function is taken to be concave, the consumption good is assumed to be a normal good, while the agent's rate of time preference ρ is taken to be constant for the analysis. The agent chooses C_i and his rates of accumulation of physical capital and human capital are subject to the following instantaneous budget constraints

$$\dot{K}_i = r_K K_i + r_H(uH_i) + WL_{X,i} - C_i \tag{6a}$$

$$\dot{H}_i = B(t)((1-u)H_i)^\varepsilon (L_{Y,i}H)^{1-\varepsilon} \tag{6b}$$

given initial stocks of $K_i(0) = K_{i,0}$ and $H_i(0) = H_{i,0}$. (6a) then assumes that physical capital is accumulated in the output sector, whereas (6b) states that human capital is accumulated by the agent in the education sector. We can think of $B(t)$, as a combination of both technological change and the average number of years of education in the labour force[17]. Both human capital and labour can move instantaneously and costlessly from one sector to another.

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[15]'On the Long Run Evolution of Inheritance - France 1820-2050', Thomas Piketty (2010). Becker and Tomes (1986) make an argument for inheritance in human capital as a source of heterogeneity. Hang and Mulligan(2000) use initial heterogeneity in human capital to study intergenerational mobility.

[16]There is an extensive literature generating heterogeneity from agents' initial endowments of wealth; see e.g. Chatterjee (1994), Chatterjee and Ravikumar (1999), Sorger (2000), Caselli and Ventura (2000), Maliar and Maliar (2001), Alvarez-Pela'ez and Diaz (2005) and Obiols-Homs and Urrutia (2005). An alternative source of heterogeneity in the earlier literature was the rate of time preference as e.g. in Becker (1980) where the most patient agent ends up holding all capital.

[17] 'Inequality and Growth', Cecelia Garci'a Pe$\tilde{n}$alosa (2008)

The following equation decides the evolution of $B(t)$,

$$B(t) = \tilde{B} + \left(B_0 - \tilde{B}\right)e^{-\theta_B t};\ \theta_B \geq 0 \tag{2b}$$

Or equivalently,

$$\dot{B} = \theta_B(\tilde{B} - B(t)) \tag{2b'}$$

where $\theta_B > 0$ and $\theta_B \to \infty$ is a special case where productivity reaches its new level instantaneously. This is analogous to the evolution for $A(t)$ in equations 2a, 2a'. Performing the maximisation of (6) s.t.(6a) and (6b) yields the following first order optimality conditions:

$$C_i^{\gamma-1} = \lambda_i \tag{7a}$$

$$r_H = \frac{\nu_{2,i}}{\lambda_i} = A(t)\beta\left(\frac{k}{u}\right)^{\alpha}\phi_X^{\eta} \tag{7b}$$

$$B(t)\varepsilon\phi_{Y,i}^{\varepsilon-1} = \frac{\nu_{2,i}}{\mu_i} \tag{7c}$$

$$W = \frac{\nu_{1,i}}{\lambda_i} = HA(t)\eta\left(\frac{k}{u}\right)^{\alpha}\phi_X^{\eta-1} \tag{7d}$$

$$B(t)(1-\varepsilon)H\phi_{Y,i}^{-\varepsilon} = \frac{\nu_{1,i}}{\mu_i} \tag{7e}$$

$$\frac{\dot{\mu}_i}{\mu_i} = \rho - \frac{\nu_{2,i}}{\mu_i} \tag{7f}$$

$$\frac{\dot{\lambda}_i}{\lambda_i} = \rho - r_K \tag{7g}$$

where λ_i is agent i's shadow value of wealth associated with K_i and μ_i is agent i's shadow value of wealth associated with H_i. Equations (7f) and (7g) when combined give the intertemporal no arbitrage condition which ensures that the return on education (in terms of units of final goods) and the return on physical capital must be equalized at each point in time. In addition, the following transversality conditions must hold to ensure that the individual agent i's intertemporal budget constraint is met

$$\lim_{t\to\infty}\lambda_i K_i e^{-\rho t} = 0$$

$$\lim_{t\to\infty}\mu_i H_i e^{-\rho t} = 0$$

2.3 Aggregation

The aggregation presented below is virtually identical to that in our first paper.Dividing (7e) by (7c) and (7d) by (7b) we get $MRTS_{LH}$ for each sector

$$\frac{\nu_{1,i}}{\nu_{2,i}} = \left(\frac{1-\varepsilon}{\varepsilon}\right)\left(\frac{H}{\phi_{Y,i}}\right) \tag{a1}$$

$$\frac{\nu_{1,i}}{\nu_{2,i}} = \left(\frac{\eta}{\beta}\right)\left(\frac{H}{\phi_X}\right) \tag{a2}$$

Equating $MRTS_{LH}$ [(a1) and (a2)] across sectors:

$$\frac{\nu_{1,i}}{\nu_{2,i}} = \frac{1-\varepsilon}{\varepsilon}\frac{1}{\phi_{Y,i}} = \frac{\eta}{\beta}\frac{1}{\phi_X}$$

$$\Rightarrow \frac{\beta\phi_X}{\eta} = \frac{\varepsilon\phi_{Y,i}}{1-\varepsilon}$$

This implies $\phi_{Z,i} = \phi_{Z,j} = \phi_Z \; \forall \; Z = X, Y$ (a3)

and so, $\dfrac{\nu_{1,i}}{\nu_{2,i}} = \dfrac{\nu_{1,j}}{\nu_{2,j}}$

Also, from the Euler condition (7g) we know

$$\rho - \frac{\dot{\mu}_i}{\mu_i} = \frac{\nu_{2,i}}{\mu_i} = B\varepsilon\phi_Y^{1-\varepsilon}$$

And so $\frac{\dot{\mu}_i}{\mu_i}$ must be the same for all individuals

$$\frac{\nu_{1,i}}{\mu_i} = \frac{\nu_{1,j}}{\mu_j} \quad and \quad \frac{\nu_{2,i}}{\mu_i} = \frac{\nu_{2,j}}{\mu_j}$$

$$\Rightarrow \frac{\lambda_i}{\mu_i} = \frac{\lambda_j}{\mu_j}$$

Defining $q_i \equiv \frac{\mu_i}{\lambda_i}$, the above implies that $q_i = q_j = q$ where $q \equiv \frac{\mu}{\lambda}$ is the unit price of human capital defined in terms of units of goods. The relevant point here is that all agents choose the same growth rate for the ratio of their respective shadow values of the two types of capital (i.e. q), independent of their relative levels. This condition, arising from the homothetic utility function, is crucial in facilitating the aggregation that permits us to derive the macroeconomic equilibrium.

Using q thus defined, and using the fact that returns across sectors must be equalised for both labour and human capital, the optimality conditions (5b), (5c) and (7c), (7e) give us the following results:

$$r_H = qB(t)\varepsilon\phi_Y^{1-\varepsilon} \tag{8a}$$

$$w = qB(t)(1-\varepsilon)\phi_Y^{-\varepsilon} \tag{8b}$$

$$\frac{\dot{q}}{q} = \frac{\dot{\mu}}{\mu} - \frac{\dot{\lambda}}{\lambda}$$

$$\Rightarrow \frac{\dot{q}}{q} = r_K - \frac{r_H}{q} \tag{8c}$$

Taking time derivative of (7a)

$$\frac{\dot{C}_i}{C_i} = \frac{1}{1-\gamma}\frac{-\dot{\lambda}_i}{\lambda_i} = \frac{r_K - \rho}{1-\gamma} \tag{9}$$

Therefore, each agent will choose the same *growth rate* for consumption regardless of initial endowment of assets, even though the individual *level* of consumption will depend on initial endowments In particular,

$$\frac{\dot{C}_i}{C_i} = \frac{\dot{C}}{C} = \psi_C \forall i$$

and therefore equal to the average. We may then write $C_i = \theta_i C$ where $\int_0^1 \theta_i di = 1$, θ_i is constant for each i and will be determined in the system. Thus defined, θ_i denotes the agent's total consumption relative to economy wide average consumption, and, as we will show later, will determine relative welfare.

Since real rates of return are equalized across sectors for both labour as well as human capital, at every point of time, we get the following relationship between ϕ_X and ϕ_Y, which can be expressed in terms of k, q, A and B as below

$$\phi_X = \left[q\left(\frac{k}{u}\right)^{-\alpha} m\right]^{\frac{1}{\varepsilon+\eta-1}} \tag{9′}$$

$$m = \frac{B(t)}{A(t)}\left(\frac{\varepsilon}{\beta}\right)^{\varepsilon}\left(\frac{1-\varepsilon}{\eta}\right)^{1-\varepsilon} \tag{9″}$$

where m evolves according to the time paths of $A(t)$ and $B(t)$ for given values of ε, β and η. This is the crucial difference between this model and our earlier paper. Transitional dynamics of aggregate variables are driven by the transitional dynamics of $\phi_X(t)$ and hence of $m(t)$. In the aggregate though, $\tilde{\phi}_X$ settles to the same steady state value as in the first paper since $A(t)$ and $B(t)$ approach the same final level of productivity as $t \to \infty$. As before we define,

$$\phi_X = \Gamma\phi_Y \quad where \;\; \Gamma = \frac{\varepsilon\eta}{\beta(1-\varepsilon)}$$

So we have $\phi_X = \Gamma\phi_Y = \phi(q, k, A(t), B(t))$ and hence, $r_K = r_K(q, k, A(t), B(t))$, $r_H = r_H(q, k, A(t), B(t))$and $w = w(q, k, A(t), B(t))$

From the full employment condition (3), we can also solve for u which gets determined within the system, as a function of k and q

$$u = \frac{1-\phi_Y}{\phi_X - \phi_Y}$$

$$\Rightarrow u = u(q, k, A(t), B(t)) \tag{11}$$

Using the relationships so defined in (10) and (11) we can then look at the impact of a change in $A\ (B)$ on ϕ_X as follows,

$$\frac{d\phi_X}{\phi_X} = \frac{1}{\epsilon+\eta-1}\left[\frac{dq}{q} + \frac{dB}{B} - \frac{dA}{A} - \alpha\frac{dk}{k} + \alpha\frac{du}{u}\right]$$

Where each of the variables within parenthesis depend on the time path followed by $A\ (B)$.

2.4 Macro Economic Equilibrium

The linear homogeneity of the production functions in the private factors allows us to express relations in terms of effective labour units. Defining $\frac{K}{H} = k$ to be the physical capital per effective labour unit and $\frac{C}{H} = c$ to be the consumption per effective labour unit, we can describe the dynamics in terms of these variables. Combining (9) and (9'), the equilibrium dynamics of aggregate consumption is:

$$\dot{C} = \frac{C}{1-\gamma}(r_K - \rho) \tag{12a}$$

$$\frac{\dot{C}}{C} = \psi_C = \frac{r_K - \rho}{1-\gamma} \tag{12a'}$$

Aggregating over individuals, final goods market equilibrium implies that the final output in excess of total consumption will be accumulated as capital:

$$\dot{K} = A(uH)\left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta} - C \tag{12b}$$

$$\frac{\dot{K}}{K} = \psi_K = \left(\frac{k}{u}\right)^{\alpha-1} \phi_X{}^{\eta} - \frac{c}{k} \tag{12b'}$$

Similarly, aggregating over each agent's human capital accumulation equation,

$$\dot{H} = B(1-u)H\phi_Y^{1-\varepsilon} \tag{12c}$$

$$\frac{\dot{H}}{H} = \psi_H = B(1-u)\phi_Y^{1-\varepsilon} \tag{12c'}$$

The arbitrage equations (7f) and (7g) give us (8c)

$$\frac{\dot{q}}{q} = r_K - \frac{r_H}{q} \tag{12d}$$

The evolution of A gives us the aggregate dynamic equation (2a)

$$\dot{A}(t) = \theta_A(\tilde{A} - A(t)) \tag{12e}$$

The evolution of B gives us the final aggregate dynamic equation (2b)

$$\dot{B}(t) = \theta_B(\tilde{B} - B(t)) \tag{12f}$$

The aggregation shown in equations (12) above allows us to represent the equilibrium dynamics of the economy as a whole as follows

$$L_X + L_Y = 1 \tag{13a}$$

$$H_X + H_Y = H \tag{13b}$$

$$\dot{K} = A(\mathrm{u}H)\left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta} - C \tag{13c}$$

$$\dot{H} = B(1-u)H\phi_Y^{1-\varepsilon} \tag{13d}$$

$$\dot{C} = \frac{C}{1-\gamma}(r_K - \rho) \tag{13e}$$

$$\frac{\dot{q}}{q} = r_K - \frac{r_H}{q} \tag{13f}$$

$$\dot{A}(t) = \theta_A(\tilde{A} - A(t)) \tag{13g}$$

$$\dot{B}(t) = \theta_B(\tilde{B} - B(t)) \tag{13h}$$

This macroeconomic equilibrium has a simple structure. Equations (13a,b) determine the sectoral factor allocations and equations (13 c-h), the dynamic evolution of the system. Due to the homogeneity of the underlying utility function, equation (6), the macro-economic equilibrium is independent of any distributional aspects.

Transitional Dynamics

In terms of our previously defined variables, core dynamics can be represented by the following differential equations in $A, B, k,$ c and q

$$\dot{k} = Au\left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta} - c - k\psi_H(q,k) \tag{14a}$$

$$\dot{c} = c\left(\frac{r_K(q,k) - \rho}{1-\gamma} - \psi_H(q,k)\right) \tag{14b}$$

$$\dot{q} = qr_K(q,k) - r_H(q,k) \tag{14c}$$

$$\dot{A}(t) = \theta_A(\tilde{A} - A(t)) \tag{14d}$$

$$\dot{B}(t) = \theta_B(\tilde{B} - B(t)) \tag{14e}$$

The three major differences from their model are (i) Physical capital is used exclusively in the output sector and; (ii) Production depends on both skilled as well as raw labor and (iii) we introduce a gradual change in productivity as part of the dynamical system so that technology parameters are time dependent. Going forward we will consider a change in productivity of the output sector only for ease of analytics. Results for the education sector will follow on similar lines. We thus exclude (14e) from the system above, which reduces it to a $4X4$ system. In this situation, it is important to note that the dynamics is a saddle point, with a two dimensional stable manifold. Steady state equilibrium will have the characteristics $\dot{k} = \dot{c} = \dot{q} = \dot{A}(= \dot{B}) = 0$. We show numerically that the system is saddle path stable with two negative roots, $\xi < 0$ and $-\theta_A < 0$ $(-\theta_B < 0)$. Along the stable path, K/H evolves gradually, as does productivity, while C/H and the relative price, q, may jump in response to new information as it comes available. Details of the linearisation and the associated jacobian have been given in Appendix A1 and A2.

3 Evolution of Wealth, Income and Welfare Inequality

We proceed to consider the consequences of the time path of a productivity change on the evolution of wealth and income inequality, as well as the overall level of intertemporal welfare inequality. We consider changes in A alone for the analytics presented below. Changes in B follow the same pattern.

3.1 Wealth Inequality

The wealth of agent i is defined by

$$V_i = K_i + qH_i \tag{15}$$

where we assume that $V_i > 0$ so that the agent has net positive wealth and is therefore solvent. Taking time derivative of (15) and using (14c) we have:

$$\dot{V}_i = r_K V_i + W - C_i \tag{15'}$$

Summing (15') over all agents in the economy, gives us the aggregate wealth equation:

$$\dot{V} = r_K V + W - C \tag{16}$$

where $V_i > 0$ ensures the solvency of the aggregate economy $V > 0$. We define individual $i's$ share of aggregate wealth as $v_i = \frac{V_i}{V}$. Taking time derivative of v_i and combining (15') together with (16), along with the fact that $C_i = \theta_i C$ we get:

$$\dot{v}_i = \frac{1}{V(t)}[(W(t) - C(t))(1 - v_i(t)) + C(t)(1 - \theta_i)] \tag{17}$$

Equation (17) highlights how the evolution of an individual agent's share of relative wealth depends on the evolution of aggregate consumption, wage, as well as his own specific endowment as reflected in v_i and θ_i. Before solving for $v_i(t)$, we see from (17) that agent $i's$ steady state share of wealth $(\dot{v}_i = 0)$ satisfies

$$\tilde{C}_i - \tilde{C} = \left[\tilde{C} - W(\tilde{k}, \tilde{q})\right]\left(\frac{\tilde{V}_i - \tilde{V}}{\tilde{V}}\right) \tag{18}$$

Noting from (7g) and (16) that in steady state, $\frac{\rho - \gamma r_K}{1-\gamma}\tilde{V} = \tilde{C} - \widetilde{W}$, (18) can be re-written as:

$$\tilde{C}_i - \tilde{C} = \frac{\rho - \gamma r_K}{1 - \gamma}\tilde{V}(\tilde{v}_i - 1) = \frac{\rho - \gamma r_K}{1 - \gamma}\left(\tilde{V}_i - \tilde{V}\right) \tag{18'}$$

Thus if individual i's wealth places him above average, s.t. $v_i > 1$ $(\tilde{V}_i > \tilde{V})$, then his long run marginal propensity to consume out of this above average wealth is given by $\frac{\rho-\gamma r_K}{1-\gamma}$[18]. Also, given that $C > \rho\tilde{V}$, average propensity to consume out of wealth exceeds ρ. Since average propensity exceeds marginal propensity, that implies, wealthier agents have a tendency to save proportionately more than their less wealthy counterparts and consume proprtionately less.

To analyze the evolution of relative wealth, we linearize (17) around the steady state (considering a productivity shock to A alone), imposing $c = \frac{C}{H}, w = \frac{W}{H}$ and $v = \frac{V}{H}$. This leads to an equation of the form:

$$\begin{aligned}\dot{v_i} &= \frac{1}{\tilde{k}+\tilde{q}}(w'_k(1-\tilde{v}_i)(k(t)-\tilde{k}) + w'_q(1-\tilde{v}_i)(q(t)-\tilde{q}) - 1(\theta_i-\tilde{v}_i)(c(t)-\tilde{c}) \\ &\quad + (w'_A(1-\tilde{v}_i))(A(t)-(\tilde{A})) - (\tilde{w}-\tilde{c})(v_i(t)-\tilde{v}_i))\end{aligned} \tag{19}$$

$$\dot{v_i} = \frac{\rho-\gamma r_K}{1-\gamma}(v_i(t)-\tilde{v}_i) + (\tilde{v}_i - 1)\left(\frac{\tilde{c}}{\tilde{v}} - \frac{\rho-\gamma r_K}{1-\gamma}\right)\left(\frac{c(t)-\tilde{c}}{\tilde{c}}\right) - \left(\frac{\tilde{v}_i-1}{\tilde{v}}\right)(w(t)-\tilde{w}) \tag{19'}$$

Since the coefficient of $v_i(t)$ is positive, the stable solution to the linearized equation (19) is given by (19') highlights how the dynamics of $v_i(t)$ are driven by the (forward-looking) transitional time paths of $c(t)$ and $w(t)$. (19) and its stable solution (19') may be re-written in terms of the consumption-wage ratio as:

$$\dot{v}_i(t) = \frac{\tilde{W}}{\tilde{V}}\left[(\tilde{z}-1)(v_i(t)-\tilde{v}_i) + (\tilde{v}_i-1)\left(\frac{z(t)-\tilde{z}}{\tilde{z}}\right)\right] \tag{19a}$$

$$v_i(t) - 1 = (\tilde{v}_i - 1)\left[1 - \left(\frac{\tilde{W}}{\tilde{V}}\right)\int_t^\infty \left(\frac{z(\tau)-\tilde{z}}{\tilde{z}}\right) e^{-\frac{\tilde{W}}{\tilde{V}}(\tilde{z}-1)(\tau-t)} d\tau\right] \tag{19a'}$$

To compute the evolution of agent i's equilibrium relative wealth we must derive the stable (bounded) solution to (19'). The solution is of the qualitative form

$$v_i(t) - 1 = \kappa(t)(\tilde{v}_i - 1) \tag{21}$$

where $\kappa(t)$ represents the time paths of aggregate variables, $c(t)-\tilde{c}$ and $w(t)-\tilde{w}$ in terms of

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[18] From (18') we can compute $\theta_i = 1 + \frac{(\rho-\gamma r_K)\tilde{v}}{(1-\gamma)\tilde{c}}(\tilde{v}_i - 1)$

the time path of $k, k(t) - \tilde{k}$ and $A, A(t) - \tilde{A}$. (Similarly for B).

Suppose that the economy is initially in steady state. At time $t = 0$ there is an anticipation of a permanent increase in the productivity of the output sector, $A(t)$. At $t = 0$, though, this change has not yet come into affect; $A(t)$ only increases in the next instant of time. The immediate effect of this is to generate a jump in $q(0)$ as part of the adjustment to ensure that the economy lies on its new stable saddle path. This in turn causes a jump in agent i's initial relative wealth, $v_i(0) = (K_{i,0} + q(0)H_{i,0})/(K_0 + q(0)H_0)$

$$\frac{dv_i(0)}{v_i(0)} = \left(\frac{q(0)H_{i,0}}{K_{i,0} + q(0)H_{i,0}} - \frac{q(0)H(0)}{K_0 + q(0)H_0}\right)\frac{dq(0)}{q(0)} \tag{20}$$

The nature of the jump depends upon (i) the deviation in the agent's initial relative asset endowment from the economy-wide average, and (ii) the relative price : sgn$(dq(0))$. Thereafter, $v_i(t)$ will evolve in accordance with (19'), given the initial jumps and consequent transitional paths of both $c(t)$ as well as $w(t)$. The smaller is the initial jump in $c(0)$ and $w(0)$, the further they are from their respective steady states, larger is the subsequent adjustment during transition, and hence larger is the overall change in agent i's relative wealth. Because of the linearity of (21) in $v_i\ (t)$ we can readily transform the equation describing a specific agent's relative asset position into corresponding relationships for the standard deviation of the distribution of wealth across agents, which therefore serves as a convenient measure of wealth inequality.

Setting $t = 0$ in (21)

$$\Rightarrow \sigma_v(t) = \kappa(t)\tilde{\sigma}_v \tag{22}$$

$$\Rightarrow \sigma_{v,0} = \kappa(0)\tilde{\sigma}_v \tag{23}$$

The above equation (23) gives the steady state distribution of agent i's relative wealth $(\tilde{v}_i - 1)$, given agent i's distribution of initial wealth $(v_i(0) - 1)$ along with the initial jump in $w(0)$ and $c(0)$ in response to a shock. This, together with (20) and (21) then gives the time path for agent i's relative wealth $v_i(t)$.

For simplicity we shall assume that all agents initially hold the same relative shares of physical to human capital which are equal to the average. That is, $\frac{K_{i,0}}{H_{i,0}} = \frac{K_{i,0}}{H_{i,0}} = \frac{K_0}{H_0} = k(0) = k_0$ in which case $dv_i(0) = 0$. On impact, the initial distribution of wealth remains unchanged, so that

$$v_i(0) - 1 = v_{i,0} - 1 = k^*_{i,0} - 1 = h^*_{i,0} - 1$$

$$where \qquad h^*_{i,0} = \frac{H_{i,0}}{H_0} \quad and \quad k^*_{i,0} = \frac{K_{i,0}}{K_0}$$

Given $\sigma_{v,0}$, the steady state wealth distribution is determined by $\kappa(0)$, which in turn depends on the expected changes in $c(0) - \tilde{c}$ and $w(0) - \widetilde{w}$, along the subsequent transitional adjustment path. These in turn, depend on the evolution of relative capital which itself depends on the time path of productivity. So, we can solve for agent i's relative wealth as:

$$v_i(t) - 1 = \kappa(t)(\tilde{v}_i - 1) = \left(1 + \int_t^\infty \left[P.\left(1 - \frac{w(\tau)}{\widetilde{w}}\right) + Q.\left(1 - \frac{c(\tau)}{\tilde{c}}\right)\right] e^{-\rho\tau} d\tau\right)(\tilde{v}_i - 1)$$

Where P, Q are constants in steady state and depend on the steady state values , $\tilde{A}, \tilde{k}, \tilde{c}$ etc.

In the case where the productivity increase occurs instantaneously, the above expression reduces to one where $\widetilde{w} - w(\tau) = \big(\widetilde{w} - w(0)\big)e^{\mu\tau}$ and similarly for $\tilde{c} - c(\tau)$. In such a case only the current allocation of wages (consumption) relative to long run wages (consumption) is relevant in determining current wealth inequality. When the productivity increase occurs gradually over time, the entire time profile of $A(t)$ as reflected in $w(t)$ and $c(t)$ also needs to be taken into account.

3.2 Income Inequality: Distribution of Income

Using (13f) of the macro economic equilibrium as derived from the arbitrage conditions, individual i's personal income is

$$Y_i(t) = r_K(t)V_i(t) + W(t) \tag{24}$$

and summing over all agents gives us the average economy wide personal income

$$Y(t) = r_K(t)V(t) + W(t) \tag{25}$$

Dividing (24) by (25) gives us the relative income of agent $i, y_i \equiv \frac{Y_i}{Y}$

$$y_i(t) = \frac{r_K(t)V_i(t) + W(t)}{r_K(t)V(t) + W(t)} \tag{26}$$

The linearity of (26) permits us to express the relationship between relative income and relative wealth in terms of the corresponding standard deviations of their respective distributions, $\sigma_v(t)$ and $\sigma_y(t)$, namely

$$\Rightarrow \sigma_y(t) = \frac{r_K(t)V(t)}{r_K(t)V(t) + W(t)}\sigma_v(t) \equiv \nu(t)\sigma_v(t); \nu(t) < 1 \tag{27}$$

$$\Rightarrow \sigma_y(t) < \sigma_v(t)$$

So, income is more equally distributed than wealth at any point of time. Income inequality can be decomposed into the product of wealth inequality and the income from wealth as a share of total income, $\mathrm{v}(t)$ at each instant of time. The time path of income inequality reflects that of wealth inequality and the share of income from wealth.

$$\frac{\dot{\sigma}_y(t)}{\sigma_y(t)} = \frac{\dot{\mathrm{v}}(t)}{\mathrm{v}(t)} + \frac{\dot{\sigma}_v(t)}{\sigma_v(t)} \tag{28}$$

$$= \left(\frac{\dot{r}_K(t)}{r_K(t)} + \frac{\dot{v}(t)}{\dot{v}(t)} - \frac{\dot{w}(t)}{w(t)}\right)\frac{w(t)}{r_K(t)v(t) + w(t)} + \frac{\dot{\sigma}_v(t)}{\sigma_v(t)}$$

where the right hand side depends upon the evolution of $v(t), r_K(t)$ and $w(t)$ and $\sigma_v(t)$, as they respond to the shock.

From equation (28) we see that income inequality depend on wealth inequality and also factor returns. Which dominates will depend on whether the shock to productivity impacts the output or the education sector.

3.3 Welfare Inequality

Recalling (6) agent i's welfare at time t is

$$Z_i(t) \equiv \frac{1}{\gamma}C_i^{\gamma} \tag{29}$$

Substituting $C_i = \theta_i C$ into this expression yields

$$Z_i(t) = \frac{1}{\gamma}(\theta_i C(t))^{\gamma} \equiv (\theta_i)^{\gamma} Z(t) \tag{30}$$

where $Z(t)$ is the average welfare level at time t. Substituting (29) into (6) yields an analogous relationship for the relative intertemporal welfare evaluated along the equilibrium growth path.

$$\frac{U_i}{U} = \frac{Z_i(t)}{Z(t)} = \theta_i^{\gamma} \tag{31}$$

$$\Omega_i \equiv \frac{U_i}{U} = \left[1 + \left(\frac{\frac{\rho - \gamma \tilde{r}_K}{1-\gamma}\tilde{V}}{\tilde{C}}\right)(\tilde{v}_i - 1)\right]^{\gamma} = z_i(t) \equiv \frac{Z_i(t)}{Z(t)} \tag{32}$$

At each instant of time, agent i's relative welfare remains constant, so that his intertemporal relative welfare, $z_i \equiv Z_i/Z$ remains constant as well. Using (18') we can express relative welfare in the form of (32).

We can now compute a measure of welfare inequality. A natural metric for this is obtained by applying the following monotonic transformation of relative utility, enabling us to express the relative utility of individual i as

$$\Omega_i^{1/\gamma} = \chi(\Omega_i) = \chi(z_i(t)) = \left[1 + \frac{\frac{\rho - \gamma \tilde{r}_K}{1-\gamma}\tilde{v}}{\tilde{c}}(\tilde{v}_i - 1)\right] \tag{32'}$$

Both instantaneous and intertemporal welfare inequality, expressed in terms of equivalent units of wealth, can be measured by the standard deviation of relative utility and are constant and identical to the steady state level of income inequality

$$\tilde{\sigma}_{\chi} = \frac{\frac{\rho - \gamma \tilde{r}_K}{1-\gamma}\tilde{V}}{\tilde{C}}\tilde{\sigma}_v < \tilde{\sigma}_y \tag{33}$$

Then, the change in income inequality exceeds the change in wealth inequality if and only if the long run change in wealth exceeds the long run change in consumption. If this is true, then given that wealthier agents have a smaller propensity to consume, percentage change in wealth inequality is high but the percentage change in income inequality is even higher. In our simulations, when productivity of the output sector increases by 10 percent, then $\frac{d\tilde{V}}{\tilde{V}} = \frac{d\tilde{C}}{\tilde{C}}$ and so, $\frac{d\tilde{\sigma}_y}{\tilde{\sigma}_y} = \frac{d\tilde{\sigma}_v}{\tilde{\sigma}_v}$ as stated above.

4 Numerical Simulations

We emply numerical simulations similar to the paper above. Given the complexity of the model, we resort to numerical simulations to attain results.

Parameter values

Preference parameters	$\gamma = -1.5, \rho = 0.04$
Production parameters	$\alpha = 1/3, \beta = 1/3, \eta = 1/3, \varepsilon = 0.6$
Productivity levels	$A_0 = 0.20, \tilde{A} = 0.22$; $B_0 = 0.20, \tilde{B} = 0.22$
Productivity growth	$\theta_A = 0.10$; $\theta_B = 0.10$

As before, the preference parameters are the same. We also simulate the three cases representing differences in initial heterogeneity. One, when all agents have the same amount of physical capital as average and hence distributional dynamics are generated due to differences in human capital endowments. Second, when distributional dynamics are a result of heterogeneity in initial endowmens of human capital alone. And finally, distributional dynamics are generated due to proprotional differences in initial endowment of both physical and human capital.

Starting from the initial benchmark, we consider in turn the aggregate and distributional consequences of 10% increases in productivity in the two sectors. The increase in productivity is from 0.2 to 0.22 and takes place gradually, adjusting at the known rate of 10% per year. The higher accumulated productivity level is achieved only asymptotically, although it is straightforward to impose a finite time horizon. The key point is that the moment the productivity starts to increase, its subsequent levels along the transitional path become fully anticipated. This is explained in more detail is Section 5 below.

The resulting macroeconomic equilibrium is summarized in Table 2, line 1. The implied output-physical capital ratio is 0.31, almost 90% of raw labor is employed in the final out sector, while over 85% of human capital is allocated to the human capital sector. In addition, the ratio of human capital to physical capital is around 1.2, the skill premium, as measured by the ratio of the income earned by human capital to the raw wage is 1.05, and the equilibrium growth rate is 2.56%. The equilibrium rate of return of both forms of capital, measured in terms of their respective own units is 10.4%.[19]It is important to note that this steady state (for aggregate variables) exactly

[19] This implies that the ratio of earnings by skilled workers to unskilled workers is approximately 2. There is an

replicates the steady state observed in my first paper. It is the transition and the steady state for the distributions that will differ. Regarding the specification of the productivity increase, we adopt the following strategy.

Rather than calibrate to a specific economy, we prefer to ensure that our calibration lies within the observed range, so that it may be viewed as being typical of a class of economies rather than a specific economy. In this respect, the implied capital-output ratio and consumption-output ratio summarized for the basic calibrations are generally within a plausible range and thus suggest that the benchmark represents a reasonable starting point.

The graphs in Figure 1-5 show a comparative analysis of dynamics in two cases (i) where the shock is instantaneous and (ii) where the shock is gradual. It is clear to see that the dynamics contrast sharply for these cases. Although the aggregate variables reach the same steady state, the steady state of the distributions are impacted by the type of shock (gradual or instantaneous) in addition to the sectoral location of the shock. In section 5 below, I analyze the results in the case where the shock is gradual. An analogous analysis for instantaneous shocks has been dealth with comprehensively in Chapter 1 of my thesis.

5 Analysis of Results

5.1 Gradual Productivity Shock to the Output Sector, A

We simulate an increase in productivity along a predefined time path, specified as an increase in the level of A by 10%, adjusting at a known rate of 10% per year. This leads to the same long run changes in the aggregate variables, as when the impact is instantaneous. There is a long-run increase in physical capital per unit human capital by 15% [see Panel A, Figure 1.1]. Consumption per unit human capital, as expected, increases by 15% also in the long run. These responses are typical of a sector which uses physical capital intensively in production. These changes are no different from the changes when the shock to A is instantaneous.

The time path of productivity, though, results in changes in the transitional path of these aggregate variables, thereby resulting in differences in the distribution of income and wealth, both

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extensive literature measuring the skill premium, which is seen to vary widely with the measure adopted, the group under consideration, and different periods of time. Our equilibrium value of 2 is broadly compatible with the estimates reported in the comprehensive study of Autor, Katz, and Kearny (2008). In assessing the rate of return on physical capital, we should bear in mind that we are abstracting from depreciation.

during transition as well as in the steady state. In contrast to the previous case, when the full productivity increase takes effect at time $t=0$, the level of productivity, $A(0)$, now remains unchanged; instead, $\dot{A}(0)$ begins to increase. Agents now anticipate an increase in the future productivity level, as a result of the accumulation of the growth rate of A along the transitional path.

The anticipation that the productivity will eventually increase by 10% and that human capital will eventually increase in relative scarcity relative to physical capital, causes $q(0)$ to increase, though not by as much as when the productivity increase occurs fully on impact; [see Panel A, 1.2]. Thus, with the level of productivity remaining unchanged on impact, the total effect on initial labor allocation, ϕ_X, is now only the expectational effect which operates through the relative price, and this remains positive. Consequently, with the level of technology fixed in the short run, φ_X, φ_Y now increase, rather than decrease on impact. Hence, the instantaneous impact of this announcement at $t = 0$, is a movement of labor from the output to the human capital sector. This causes an initial fall in final output, and since capital continues to accumulate (albeit at a slower rate), initial consumption also declines. This decrease in output is reflected in an initial decline in the wage and in the return to physical capital.

Over time, as the productivity growth continues, the accumulated increase in the productivity level A begins to dominate and the decline is gradually reversed. In the case of the declining ratio of physical cpaital to human capital, this reversal occurs after about 5 periods, and thereafter it increases monotonically to its new higher steady-state level. This reversal is also reflected in the the real wage and the factor returns, as well as in the C/H ratio.

The response of the consumption to wage ratio is again key to explaining the distribution of wealth. It initially falls, continuing to do so for the next 5 periods, before the trend reverses and C/W increases towards its original steady state. Compared to the instantaneous productivity increase, the decline in wealth inequality is much larger when the productivity change is gradual. This is because of the slowing down of consumption to wage ratio during the first five periods, which causes the time spent in transition below the steady state to be much longer than when the shock is instantaneous. This is seen in Table 2.B, and in fact in the case where the heterogeneity is due to initial endowment of human capital, long-run wealth inequality declines by 1.4%, in contast to the increase of 2.8% when the productivity increase occurs instantaneously.

In contrast to the monotonic time path followed by wealth inequality, income inequality reflects the non-monotonic adjustment of the aggregate variables. The initial decline in income inequality due primarily to the decline in income from capital is followed by a rise for about 5 periods as the return on capital increases, after which it falls towards its new lower steady state, as the return on capital reverts to its original equilibrium level. As for the case where the productivity increase occurs instantaneously, the change in long-run income inequality is proportionate to that of wealth inequality, as can be seen from Table 2.B.

The elements driving the evolution of wealth inequality are the increase in effective consumption (and hence aggregate consumption) along the transitional path along with a corresponding increase in effective wage income. Since consumption-wage ratio increases towards its new steady state, savings fall. Equation (18') shows that wealthier agents save more. Together with (19'), this implies a fall in wealth inequality and indeed over time $\tilde{\sigma}_v$ falls by 4.9% [Panel A, Figure 1.4 (a)].

The transitional dynamics of $\sigma_y(t)$ is given by (25) and reflects a number of conflicting influences. In the short term, the increase in productivity results in a greater increase in wage income relative to capital income. Since wage income is more equally distributed in the short run, income inequality falls on impact. As wages dip, while capital income continues to rise, inequality reverses the initial fall, also increasing for about 10 periods. In the long run however, capital accumulation raises the wage rate (marginal productivity of raw labour), whereas the return to physical capital stays the same. This, coupled with the falling wealth inequality causes long-run income inequality to fall by almost 5% [Panel A, Figure 1.5 (a)].

Relative to the case of an instantaneous productivity shock, the magnitude of the fall in inequality is much higher when the productivity change is gradual. Why? It is the slowing down of consumption and wage in the first five periods that can account for this difference in magnitude. When consumption and wages fall for the first five periods, the time spent in transition below the steady state is much longer than when the shock is instantaneous. From the expression for $\kappa(t)$ we can see that the longer $c(\tau)/w(\tau) < \tilde{c}/\widetilde{w}$, the greater the impact on inequality.

In the case where physical capital is equally distributed amongst all agents (regardless of human capital endowments), that is, $K_{i,0} = K_{j,0} = K_0$. The output sector uses physical capital exclusively, and so, inequality is greatly impacted by this equal endowment of physical capital. Inequality, in fact, falls by 7.54% as a result of this initial condition [Panel A, Figure 1.5 (b)].

On the other hand, a more equal distribution of human capital, that is, $H_{i,0} = H_{j,0} = H_0$, results in a fall of 2.8% in inequality [Panel A, Figure 1.5 (c)].

5.2 Gradual Productivity Shock to the Human Capital Sector, B

We now consider an increase in productivity of the human capital sector, specified as a 10% increase in *B*. These contrast from the long-run responses to an increase in *A* due to the fact that the productivity increase in the human sector has a permanent growth effect, which for these simulations rises from 2.56% to 2.99%. There is empirical evidence that supports this result. Denison (1985) found that the growth in years of schooling between 1929 and 1982 explained about 25 percent of the growth in U.S. per capita income during the period. The experiences of nearly one hundred countries since 1960 suggest that education investments in 1960 are important in explaining subsequent growth in per capita incomes (Barro 1991).

The long-run aggregate responses are summarized in line 3 of Table 2.A and are independent of the time path over which they occur. The productivity increase in the human capital sector shifts both skilled and unskilled labor toward the human capital sector so that both the share of unskilled labor and human capital employed in the final output sector are reduced. With the human capital sector being relatively intensive in skilled labor, equilibrium in both markets requires small permanent increases in $\tilde{\varphi}_X, \tilde{\varphi}_Y$ at $t = 0$.. With the increase in *B* raising the long-run ratio of human capital to physical capital (i.e. reducing $\tilde{k}$) this reduces the relative scarcity of human capital (relative to physical capital) and its relative price, $\tilde{q}$ declines by around 13.2%. The relative long run increase in skilled labor in the final output sector raises the return on physical capital, and reduces the real wage. While the increase in $\tilde{\varphi}_X$ tends to raise the return on human capital, the reduction in $\tilde{k}$ tends to lower it. On balance this latter effect dominates and the rate of return on human capital declines, though not by as much the unskilled wage and the skill premium increases.

The response of the return to human capital is also consistent with empirical evidence. Teulings and Rens (2009), show a clear negative relationship between the private return to education and the average level of schooling in the country, when workers have different skill levels. This is reflected in our analysis as a fall in the return to education with increasing stocks of human capital. Goldin and Katz (2001) use U.S. data from 1915-1999 to show that greater and universally higher levels of education reduced the rate of return to years of education relative to their extremely high level at the turn of the 20th century.

The short-run transitional responses illustrated in Panel B of Figures 1-5 are approximately the mirror images of the responses to Panel A and can be explained analogously.

With respect to distribution, since the technology increase now occurs in the human capital sector, the immediate rise in the wage rate is mitigated. The consumption-wage ratio now rises rather than falls, but declines through the subsequent transition. For precisely the same reason as discussed earlier, this causes wealth inequality to increase during the transition. The overall change must also take account of the initial jump, through $q(0)$, and via the relative initial endowments. In the case where this is uniform, wealth inequality increases by 1%, but it will decline by 2.5%, if the initial heterogeneity is due entirely to variations in the endowment of human capital. As in the case of a shock to A, by slowing down the adjustment of the consumption-wage ratio, the gradual introduction of the increase in technology greatly exacerbates the increase in wealth inequality.

This mechanism may help explain the recent increase in income inequality in the US. Between 1975 and 1995 the Gini coefficient in the US increased by 6.4 Gini points. The period also witnessed the widespread adoption of IT technologies, the adoption of which typically requires a higher level of human capital/skill. It has been argued that the resulting productivity increase led to greater wage inequality, and hence greater income inequality. Our analysis implies that an increase in the technology parameter B would also impact the distribution of income through its effect on the returns to capital – both physical as well as human - and labor.

In the long run, real wages of unskilled labor fall. The increasing wealth inequality dominates, causing income inequality to also rise, by 1.0% [Panel B, Fig 4 (i), Fig 5(i)].
In the case where the initial distribution of physical capital is the same for all agents and income inequality dynamics are driven by the initial endowments of human capital, inequality falls on impact and then rises towards its new lower steady state, to an eventual decline of 2.5% [Panel B, Figure 4,5 (ii)]. In contrast, when the initial distribution of human capital is uniform across agents, inequality shows a marked increase of 2.5% [Panel B, Figure 4,5 (iii)].

On impact, labour – skilled and unskilled - moves into the output sector in response to a fall in the return to education and wages. This causes decumulation of education output. In the next instant of time the shock gets realised, and this trend is reversed as workers start to move back toward the education sector. Although income inequality rises on impact due to the fall in wages, it dips for about five periods in response to wages overshooting the pre-shock steady state level, before falling toward their new lower post-shock level.

6 Conclusion

With forward looking agents, the initial response of consumption is dependent upon the anticipation of structural changes, making the long run effects of these initial responses path dependent. This paper has examined the path dependence of income distributions generated by the time path of a change in productivity. There are marked differences in both transition as well as the final steady state of distributions when the change is gradual relative to when the change is instantaneous. This has important implications for government policy. It suggests that if the government introduces a productivity enhancing investment in the output sector, it should do so gradually since the impact on lowering inequality is much larger. On the other hand, an increase in productivity of the education sector, should come about instantaneously, since that results in a smaller increase in inequality. A gradual approach taken here, results in a worsening of the income distribution in the long run.

Although this paper focusses on the time path of productivity, this analysis would hold true for any structural change, as long as the time path and steady state of income distribution depends on the initial response of consumption to the potential time path of the structural change.

Chapter III: Growth Inequality Tradeoff: Fiscal Policy Mix

Abstract

This paper explores the impact of fiscal policy on the growth-inequality tradeoff. A subsidy to return on human capital employed in the output sector is financed by a tax either on income from physical capital, or on returns to human capital in the output sector, such that the government budget is balanced. The growth increasing effect of the subsidy is dampened by the presence of taxes. The tax on human capital has a larger dampening effect than does the tax on physical capital. At the same time, when initial endowments of human and physical capital are distributed proportionally, a tax on human capital reduces income inequality, thus providing a favorable tradeoff. In sharp contrast, a physical capital tax increases inequality. The factor most affecting this tradeoff is the initial heterogeneity in human and physical capital endowments.

1 Introduction

In this paper, we extend the Uzawa-Lucas model to account for taxes and subsidies. Fiscal policy not only has growth effects on the steady state, but also determines the transitional path along which the economy converges to the balanced growth equilibrium. In related literature, Bond, Wang and Yip (1996) study the impact of taxes and subsidies on the growth rates. They find that whereas a subsidy increases growth, a tax on human capital reduces it. Mino (1996) analyzes both the long run and transitional effects of a change in capital taxation in the presence of lump-sum transfers. Stokey and Rebelo (1995) focus on how the effect of taxes depends on model features and parameters. Both Turnovsky (1996) and Stokey and Rebelo (1995) show that an increase in marginal tax rates on physical and human capital, lower the steady state rates of growth. Garcia-Castrillo and Sanso (1997) study taxation using an analytical approach that leads the economy to an optimal steady state using a path that is optimal too. Jaime-Alonso Carrera (2000) considers the effect of a subsidy to human capital on a two-sector endogenous growth model. He finds that a subsidy unambiguously increases growth rates in the steady state. In transition, it has both income and substitution effects, and which effect dominates depends on the intertemporal elasticity of substitution vs. the marginal productivity of labor to capital. Gomez (2000) considers an optimal fiscal policy where a subsidy to investment in human capital is financed by time varying distortionary taxes on physical and human capital. In an empirical study, Kneller et al (1999) look at the impact of distortionary taxes financed by non-productive government expenditure, on growth rates, and find that a fall in taxes by one percent will increase growth rates by 0.1 to 0.2 percentage points.

While we do not attempt to look for an optimal fiscal policy, we study the impact of a subsidy financed by taxes, on the growth-inequality tradeoff. Thus, there is a distortionary effect of both taxes as well as subsidies, on the returns to factors. In the model, there is a subsidy to the return on human capital employed in the human capital sector. This is financed either by a tax on the return to human capital employed in the output sector or by a tax on the return to physical capital. We find that in general, a subsidy's positive growth effects outweigh the negative effects of taxes.

On distribution, wealth inequality falls regardless of the type of tax, when initial inequality in the distribution of human and physical capital is proportional. Income inequality, on the other hand, rises when the subsidy is financed by a tax on physical capital. In sharp contrast, when the tax is on human capital, it falls. When initial inequality is due to differences in physical capital endowment, both wealth and income inequality increase, regardless of the type of tax. The magnitude of this increase is more in the case of a human capital tax. When initial inequality is due to differences in endowment of human capital, both wealth and income inequality fall. Once again, the magnitude of this fall is greater in the case where the subsidy is financed by a tax on human capital. Thus, there are favorable tradeoffs to be achieved using the appropriate fiscal policy mix.

2 The Analytical Framework

The analytical framework we employ is an adaptation of the two-sector production economy pioneered by Lucas, augmented to include a subsidy to the human capital sector that is funded by taxes on physical capital, or human capital. We assume a balanced government budget. While the return to human capital in the output sector is taxed, the return to skilled labor in the human capital sector is subsidized.

2.1 Technology and Factor Payments

There is a single representative firm producing the final output, Y, using the aggregate production function:

$$X = F(K, H_X, L_X) = AK^{\alpha}(H_X)^{\beta}(L_X H)^{\eta}, \quad \alpha + \beta + \eta = 1 \tag{1}$$

Where A is the level of technology in the final output sector, L_X is the raw labor, and H_X is the human capital, both employed in the final output sector. Economy-wide human capital provides a productivity enhancing externality for raw labor. The firm faces a standard profit maximization problem, where it chooses raw labor, physical capital and human capital.

$$\Pi = AK^{\alpha}(H_X)^{\beta}(L_X H)^{\eta} - WL_X - r_K K - r_H H \tag{2}$$

Yielding the following optimality conditions:

$$r_K = A\alpha\left(\frac{K}{H_X}\right)^{\alpha-1}\left(\frac{L_X}{H_X/H}\right)^{\eta} \tag{2a}$$

$$r_H = A\beta\left(\frac{K}{H_X}\right)^{\alpha}\left(\frac{L_X}{H_X/H}\right)^{\eta} \tag{2b}$$

$$\frac{W}{H} = A\eta\left(\frac{K}{H_X}\right)^{\alpha}\left(\frac{L_X}{H_X/H}\right)^{\eta-1} \tag{2c}$$

The focus of our analysis is, to consider the effect of a subsidy to the return on skilled labor employed in the education sector. This is financed either by a tax on physical capital, or by a tax on the return to human capital employed in the output sector. Although the subsidy itself has a large positive impact on growth, the tax that finances the subsidy offsets the effect of the subsidy. The degree to which it is offset, depends on whether the tax is on physical capital or on human capital. We find, as shown subsequently, that financing the subsidy with a tax on human capital has a greater negative effective on the economy.

2.2 Heterogeneous Consumers

As before, the economy consists of a continuum of infinitely lived agents. Agents are indexed by i and are identical in all respects except for their initial endowments of human capital, $H_{i,0}$ and physical capital, $K_{i,0}$. $H(t)$ and $K(t)$ denote the corresponding economy-wide average quantities. The initial relative endowments across agents are assumed to have mean 1 and standard deviations $\sigma_{h,0}, \sigma_{k,0}$. The distribution of the initial endowments is reflected in the evolving distributions of wealth and income. Each individual is endowed with a unit of raw labor that can be allocated to the final output sector, $L_{X,i}$, or to acquiring more human capital, $L_{Y,i}$.

$$L_{X,i} + L_{Y,i} = 1 \tag{3}$$

The agent also accumulates a stock of human capital, H_i, that similarly can be allocated either to final output sector or to the further accumulation of human capital.

$$H_{X,i} + H_{Y,i} = H_i \tag{4}$$

Physical capital is employed only in the production of final output, and therefore does not involve a sectoral allocation decision.

The wage rate earned by raw labor (unskilled labor) is W, while the return to physical capital is r_K. The agent's budget constraint, describing his market activities, asserts that post tax and post subsidy income earned from his productive factors can be spent either on consumption, or accumulating physical capital.

$$\dot{K}_i = r_K(1-\tau_k)K_i + r_H(1-\tau_H)\left(H_{X,i}\right) + WL_{X,i} - C_i + s_H r_H H_{Y,i} \tag{5}$$

The post-tax budget constraint (5) incorporates the tax on the return to physical capital r_K at the rate τ_K, and the tax on return to human capital in the output sector, at the rate τ_H. It also consists of a subsidy on the return to human capital in the human capital sector, at the rate s_H.
In addition, the agent accumulates human capital in accordance with the Cobb-Douglas production function (common to all agents)

$$\dot{H}_i = BH_{Y,i}{}^{\epsilon}\left(L_{Y,i}H\right)^{1-\epsilon} \tag{6}$$

where H denotes the average (aggregate) stock of human capital, and B denotes the level of technology in the production of new human capital. This function is homogenous of degree one in the accumulating asset (human capital), with the economy-wide average stock of human capital providing an externality that enhances the productivity of each individual's raw labor. The only way an agent can accumulate human capital is by devoting his own physical resources to this activity; there is no market for the purchase of human capital. The representative agent chooses consumption, C_i, raw labor allocations, $L_{X,i}$, $L_{Y,i}$, human capital allocations, $H_{X,i}$, $H_{Y,i}$ and the rates of human capital accumulation, $\dot{H}_i$, and physical capital accumulation, $\dot{K}_i$, to maximize the iso-elastic utility function subject to the constraints (3), (4), (5) and (6)

$$U_i = \int \frac{1}{\gamma} C_i^{\gamma} e^{-\rho t} dt \tag{7}$$

where $-\infty < \gamma < 1$

The optimality conditions respectively are:

$$C_i^{\gamma-1} = \lambda_i \tag{7a}$$

$$r_H(1-\tau_H-s_H) = \frac{\nu_{2,i}}{\lambda_i} = \frac{\mu_i}{\lambda_i} B\epsilon\left(\frac{L_{Y,i}H}{H_{Y,i}}\right)^{1-\epsilon} \tag{7b}$$

$$W = \frac{\nu_{1,i}}{\lambda_i} = \frac{\mu_i}{\lambda_i} B(1-\epsilon)\left(\frac{L_{Y,i}H}{H_{Y,i}}\right)^{-\epsilon} H \tag{7c}$$

$$\frac{\dot{\lambda}_i}{\lambda_i} = \rho - (1 - \tau_K) r_K \tag{7d}$$

$$\frac{\dot{\mu}_i}{\mu_i} = \rho - \frac{\nu_{2,i}}{\mu_i} - \frac{\lambda_i}{\mu_i} s_H r_H = \rho - (1 - \tau_H) r_H \frac{\lambda_i}{\mu_i} \tag{7e}$$

Together with the transversality conditions

$$\lim_{t \to \infty} \mu_i H_i e^{-\rho t} = 0; \ \lim_{t \to \infty} \lambda_i K_i e^{-\rho t} = 0$$

where λ_i is the agent's shadow value of physical capital, μ_i is his shadow value of an extra unit of investment in human capital, $\nu_{1,i}$ is his shadow value of raw labor, and $\nu_{2,i}$ is his shadow value of human capital.

The equations above equate the post-tax marginal product of raw labor and skilled labor in human capital production, expressed in terms of final output, to the wage rate and to the rate of return on human capital, respectively, in the final output sector. The last equation equates the rate of return on consumption to the post tax rate of return on human capital, both expressed in terms of units of human capital. It is to be noted here, that the effect of the subsidy on the return to consumption drops out in equation (7e.). (7d) describes an analogous relationship expressed in terms of physical capital.

2.3 Government

We assume that the government balances its budget. Thus, the subsidy to the education sector is financed entirely by taxes, on physical capital as well as on human capital. The government budget constraint is as follows:

$$s_H r_H (1 - u) H = r_K \tau_K K + r_H \tau_H (uH) \tag{8}$$

Aggregation

We now aggregate the basic components, laid out above, to derive the economy wide equilibrium. To do this we first note the following aggregation relationships and market clearing conditions, describing the raw labor and human capital markets:

$$\sum_i L_{X,i} + \sum_i L_{Y,i} = L_X + L_Y = 1 \tag{3'}$$

$$\sum_i H_{X,i} + \sum_i H_{Y,i} = H_X + H_Y = 1 \tag{4'}$$

For notational convenience, let

$$\phi_X = \frac{L_X}{H_X/H};\ \phi_{Y,i} = \frac{L_{Y,i}}{H_{Y,i}/H}$$

denote the ratio of raw labor to human capital employed in the final output sector, and the ratio allocated by individual agent i in the production of human capital respectively. Using this notation enables us to express the individual optimality conditions in the form

$$C_i^{\gamma-1} = \lambda_i \tag{7a'}$$

$$r_H(1-\tau_H-s_H)\lambda_i = B\epsilon\phi_{Y,i}^{1-\epsilon} = \nu_{2,i} \tag{7b'}$$

$$\lambda_i W = \mu_i B(1-\epsilon)\phi_{Y,i}^{-\epsilon} H = \nu_{1,i} \tag{7c'}$$

$$\frac{\dot{\lambda}_i}{\lambda_i} = \rho - (1-\tau_K)r_K \tag{7d'}$$

$$\frac{\dot{\mu}_i}{\mu_i} = \rho - \frac{\nu_{2,i}}{\mu_i} - \frac{\lambda_i}{\mu_i}s_H r_H = \rho - (1-\tau_H)r_H\frac{\lambda_i}{\mu_i} \tag{7e'}$$

Aggregation follows exactly as in previous papers. We see that $\phi_{Y,i}$ and $\nu_{2,i}$ and $\nu_{1,i}$ are common across all agents. This is a consequence of the assumption that all agents face a common wage rate , W, and return on human capital r_H. Thus we have

$$\phi_{Y,i} = \phi_Y = \frac{L_Y}{H_Y/H}$$

Here ϕ_Y denotes the factor intensity in the production of human capital. Another consequence of this is that the growth rate of the shadow value of human capital $\frac{\dot{\mu}_i}{\mu_i}$ is the same across agents. Consequently $\frac{\dot{\lambda}_i}{\lambda_i}$ is also the same across agents. Differentiating (7a') w.r.t time,

$$(\gamma-1)\frac{\dot{C}_i}{C_i} = \frac{\dot{\lambda}_i}{\lambda_i} = (\gamma-1)\frac{\dot{C}}{C}$$

so that by choosing a common growth rate for the marginal utility of income, agents choose a common growth rate of consumption, which therefore coincides with the aggregate (average) economy-wide consumption growth rate. Each agent i thus maintans a constant ratio of consumption to average consumption

$$C_i = \theta_i C \qquad \text{for each } i \text{ and } \sum_i \theta_i = 1$$

where the constant θ_i is to be determined by his relative steady-state wealth.

Let $q = \frac{\mu}{\lambda} \left(= \frac{\mu_i}{\lambda_i}\right)$ denote the common relative shadow price of human capital to income. With this notation, we can use the conditions above to express economy-wide equilibrium allocation of raw labor and human capital. Equating return to human capital across sectors we get and imposing $q_i = q, \phi_{Y,i} = \phi_Y$, we can express economy wide equilibrium allocation of raw labor and human capital:

$$r_H = A\beta \left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta} = (qB\epsilon\phi_Y^{1-\epsilon}) \frac{1}{1-\tau_H - s_H} \tag{9}$$

$$w = A\eta \left(\frac{k}{u}\right)^{\alpha} \phi_X^{\eta-1} = qB(1-\epsilon)\phi_Y^{-\epsilon} \tag{10}$$

Aggregating the individual accumulation equation (5), while imposing the government budget constraint (8) gives us the following:

$$\dot{K} = r_K K + r_H H + W - C \tag{11}$$

$$\dot{K} = A\left(\frac{K}{H_X}\right)^{\alpha} \phi_X^{\eta} K - C$$

Similarly aggregating over the individuals' accumulation equation for human capital (6) gives:

$$\dot{H} = BH_Y\phi_Y^{1-\epsilon} \tag{12}$$

2.4 Macroeconomic Equilibrium

The aggregate equilibrium has many of the characteristics of a standard two sector production economy. The macro equilibrium includes the following conditions:

(i) Labor Market Clearance: $L_X + L_Y = 1$

(ii) Human capital market clearance: $H_X + H_Y = H$

(iii) Goods market clearance: $\dot{K} = A\left(\frac{K}{H_X}\right)^{\alpha} \phi_X^{\eta} K - C$

(iv) Human capital accumulation: $\dot{H} = BH_Y\phi_Y^{1-\epsilon}$

As before, we express the macroeconomic equilibrium in terms of ratios of physical capital to human capital, $k \equiv K/H$, the ratio of consumption to human capital, $c \equiv C/H$, and the ratio of shadow values, q. We also express the allocation of human capital to the production of final output, H_X/H , by u and its allocation to the production of human capital, H_Y/H, by $1-u$. Using these definitions, we can solve for the sectoral allocations of raw and skilled labor, ϕ_X, ϕ_Y, u in terms of $k, q, \tau_K, \tau_H, s_H$ and all the parameters.

$$\phi_X = \phi(k,q,\tau_H,s_H), \phi_Y = \phi(k,q,\tau_H,s_H), u = u(k,q,\tau_H,s_H)$$

Using these conditions, we can now solve for the sectoral allocations of raw and skilled labor. Dividing (9) by (10) we get:

$$\phi_X = \frac{\epsilon\eta}{\beta(1-\epsilon)(1-\tau_H-s_H)}\ \phi_Y, \mathit{\Gamma} = \frac{\epsilon\eta}{\beta(1-\epsilon)}, S = \frac{1}{1-\tau_H-s_H} \tag{13}$$

$$=> \ \phi_X = \Gamma S \phi_Y$$

Taking total differentials of (13)

$$\frac{d\phi_X}{\phi_X} = \frac{d\phi_Y}{\phi_Y} + \frac{dS}{S} = \frac{d\phi_Y}{\phi_Y} + \frac{ds_H}{1-\tau_H-s_H} \tag{13'}$$

Replacing for ϕ_Y in (9):

$$\phi_X = \left[q\left(\frac{k}{u}\right)^{-\alpha} m\right]^{\frac{1}{\epsilon+\eta-1}}, m = \frac{B}{A}\left(\frac{\epsilon}{\beta}\right)^{\epsilon}\left(\frac{1-\epsilon}{\eta}\right)^{1-\epsilon}\frac{1}{(1-\tau_H-s_H)^{\epsilon}} \tag{14}$$

Taking total differentials of (14) we get the following key result:

$$\frac{d\phi_X}{\phi_X} = \frac{1}{\epsilon+\eta-1}\left[\frac{dq}{q} - \alpha\frac{dk}{k} + \alpha\frac{du}{u} + \epsilon\left(\frac{ds_H}{1-\tau_H-s_H}\right)\right] \tag{14'}$$

From the full employment condition (3) and using the definitions of ϕ_X and ϕ_Y:

$$u\phi_X + (1-u)\phi_Y = 1$$

$$u = \frac{1-\phi_Y}{\phi_X-\phi_Y} = \frac{\epsilon\eta - \beta(1-\epsilon)(1-\tau_H-s_H)\phi_X}{\phi_X\,(\epsilon\eta - \beta(1-\epsilon)(1-\tau_H-s_H))}$$

$$\frac{du}{u} = -\frac{1-\phi_Y}{(\phi_X-\phi_Y)^2}\left(\frac{d\phi_X}{\phi_X} - \frac{d\phi_Y}{\phi_Y}\right) - \frac{1}{(\phi_X-\phi_Y)}\frac{d\phi_Y}{\phi_Y}$$

Re-writing u in terms of S:

$$u = \frac{1-\phi_Y}{\phi_X - \phi_Y} = \frac{1-\phi_Y}{(\Gamma S - 1)\phi_Y} \tag{15}$$

$$\frac{du}{u} = (\phi_Y - 1)\frac{d\phi_Y}{\phi_Y} - \frac{\Gamma S}{\Gamma S - 1}\frac{dS}{S} \tag{15'}$$

We then have the following aggregate conditions:

$$\frac{\dot{k}}{k} = A\left(\frac{k}{u}\right)^{\alpha-1}\phi_X^{\eta} - B(1-u)\phi_Y^{1-\epsilon} - \frac{c}{k} \tag{16a}$$

$$\frac{\dot{c}}{c} = \frac{1}{1-\gamma}[r_K(1-\tau_K) - \rho] - B(1-u)\phi_Y^{1-\epsilon} \tag{16b}$$

$$\dot{q} = q\left[r_K(1-\tau_K) - \frac{r_H}{q}(1-\tau_H)\right] \tag{16c}$$

Substituting the solutions for ϕ_X, ϕ_Y, u from equations (13) and (15), the above equations describe the evolution of the macroeconomic equilibrium in the presence of tax financed subsidy. As $k, c\ and\ q$ evolve, this will generate adjustment paths for the rates of returns and the sectoral allocation of resources. The key feature of this equilibrium is that it is independent of any distributional measures.

Skill premium is defined as the ratio of income earned by human capital to the income earned by raw labor. The crucial determinant is the equilibrium ratio of raw to skilled labor employed in the human capital sector, ϕ_Y. Thus it depends on the tax on human capital, as well as the subsidy to the investment in human capital, and not directly on the tax on physical capital.

3 Evolution of Wealth, Income and Welfare Inequality

We now turn to the distributional implications. We will start with relative wealth dynamics, the factor that drives the macroeconomic dynamics.

3.1 Wealth Inequality

At any instant of time, the total wealth of agent i is defined as:

$$V_i(t) = K_i(t) + q(t)H_i(t) \tag{17}$$

Differentiating with respect to time, the agent's rate of wealth accumulation is

$$\dot{V}_\iota(t) = \dot{K}_\iota(t) + \dot{q}(t)H_i(t) + q(t)H_i\dot{(}t) \quad (17')$$

where $\dot{K}_\iota(t), \dot{H}_\iota(t)$ and $\dot{q}(t)$ are as defined earlier.

Combining with (17) and (17') we get:

$$\dot{V}_\iota(t) = r_K(t)(1-\tau_k)V_i(t) + W(t) - C_i(t) \quad (18)$$

In the aggregate economy, this becomes (summing the above equation over all agents):

$$V\dot{(}t) = r_K(t)(1-\tau_k)V(t) + W(t) - C(t) \quad (18')$$

Now define the agent's relative wealth as, $v_i(t) \equiv V_i(t)/V(t)$. Taking the time derivative of v_i and combining (18) and (18') above along with $C_i = \theta_i C$:

$$\frac{\dot{v}_\iota}{v_i} = \frac{(r_K(1-\tau_K)V_i + W - C_i)}{V_i} - \frac{r_K(1-\tau_k)V + W - C}{V}$$

$$=> \dot{v}_\iota(t) = \frac{1}{k(t)+q(t)}[(w(t))(1-v_i(t)) - c(t)(\theta_i - v_i(t))] \quad (19)$$

In steady state when all aggregate quantities grow at the same rate, $\frac{\dot{v}}{v} = \frac{\dot{V}}{V} - \frac{\dot{H}}{H} = 0$

$$\frac{\dot{\tilde{V}}}{\tilde{V}} = \frac{\dot{\tilde{C}}}{\tilde{C}} = \frac{\widetilde{r_K}(1-\tau_K) - \rho}{1-\gamma} = \widetilde{r_K}(1-\tau_K) + \frac{\widetilde{W}}{\tilde{V}} - \frac{\tilde{C}}{\tilde{V}} \quad (20)$$

Then,

$$\frac{\tilde{C}}{\tilde{V}} - \frac{\widetilde{W}}{\tilde{V}} = \frac{\rho - \gamma\tilde{r}_K(1-\tau_K)}{1-\gamma} > 0 \quad (20')$$

Setting $\dot{v}_\iota(t) = 0$ in equation (19) implies, that the agent's relative consumption, which is constant throughout transition is:

$$\theta_i - 1 = \left(\frac{\rho - \gamma\tilde{r}_K(1-\tau_K)}{1-\gamma}\right)\left(\frac{\widetilde{V}_\iota - \tilde{V}}{\tilde{C}}\right) \quad (21)$$

This implies that if the agent has above average long run wealth, then he has above average long run consumption. As before, the dynamics of relative wealth are driven by the aggregate consumption-wage ratio. Let $C(t)/W(t)$ be denoted by $z(t)$. We can then write (15') as:

$$\theta_i - 1 = \frac{\tilde{z}-1}{\tilde{z}}(\tilde{v}_i - 1); \tilde{z} > 1 \; since \; \left(\frac{\rho - \gamma \tilde{r}_K(1-\tau_K)}{1-\gamma}\right) > 0$$

Then we can express equation (19) in the following form:

$$\dot{v}_i(t) = \frac{W(t)}{V(t)}\left[(z(t)-1)(v_i(t)-1) + z(t)\left(\frac{1-\tilde{z}}{\tilde{z}}\right)(\tilde{v}_i - 1)\right] \tag{19'}$$

Linearizing (19) around steady state and substituting for $(\tilde{c} - \widetilde{w})$ from (20'):

$$\dot{v}_i = \frac{\rho - \gamma \tilde{r}_K(1-\tau_K)}{1-\gamma}(v_i - \tilde{v}_i) + \frac{1}{\tilde{k}+\tilde{q}}\Big[\left(\widetilde{w}_k'(1-\tilde{v}_i)\right)\left(k-\tilde{k}\right) + \left(w_q'(1-\tilde{v}_i)\right)(q-\tilde{q}) \\ - (\theta_i - \tilde{v}_i)(c-\tilde{c})\Big] \tag{22}$$

Since the coefficient on $v_i(t)$ in the linearized equation is positive, in order for the agent's wealth to remain bounded, the solution for $v_i(t)$ is given by the forward-looking solution:

$$v_i\,(t) = \tilde{v}_i - (\tilde{v}_i - 1)\left(\frac{1}{\left(\frac{\rho-\gamma\tilde{r}_K(1-\tau_K)}{1-\gamma} - \mu\right)} * \frac{1}{(\tilde{k}+\tilde{q}\,)}\left(\widetilde{w}_k' + P\widetilde{w}_q' - Q\frac{\widetilde{w}}{\tilde{c}}\right)\left(k_0 - \tilde{k}\right)e^{\mu t}\right) \tag{23}$$

where P and Q are elements of the jacobian.

In view of the linearity of $v_i(t) - 1$ in (23), we can sum over the agents and transform these expressions into standard deviations across agents, as a convenient measure of inequality.

$$\sigma_v(t) = v_i(t) - 1 = \kappa(t)\tilde{\sigma}_v$$

Which at time $t = 0$ is

$$=> \; \sigma_v(0) = \kappa(0)\tilde{\sigma}_v$$

The following relationship may then be obtained between initial and final steady state values of income distribution:

$$\sigma_v(t) = \frac{\kappa(t)}{\kappa(0)}\sigma_v(0) \tag{24}$$

3.2 Income Inequality

3.2.1 Pre-tax income inequality

Let us express pre-tax income as the disposable income available to agents at time t, before the direct effect of taxes.

$$Y_i = r_K V_i + W \tag{25}$$

Then, relative income, defined as $y_i = \frac{Y_i}{Y}$, may be expressed as in terms of its standard deviation:

$$\sigma_y = \frac{r_K V(v_i - 1)}{r_K V + W} = \frac{r_K V}{r_K V + w}\sigma_v = v(t)\sigma_v(t) \tag{25'}$$

Where $v(t)$ is the share of income from wealth over total income.

3.2.2 Post tax income inequality

We consider an additional measure of income that takes into account the direct effect of taxes on personal disposable income.

$$Y_i = (1-\tau_K)r_K K_i + (1-\tau_H - s_H)r_H H_{X,i} + s_H r_H H_i + W \tag{26}$$

Such that,

$$\sigma_y = y_i - 1 = \frac{(1-\tau_K)r_K k(k_i - 1) + (1-\tau_H - s_H)r_H u(h_i - 1) + s_H r_H (h_i - 1)}{r_K k + r_H u + s_H r_H + W} \tag{27}$$

where $k_i = \frac{K_i}{K}$ *and* $h_i = \frac{H_i}{H}$. We assume initial distributions for $k_{i,0}, h_{i,0}$. We can obtain distributions for $\tilde{k}_i, \tilde{h}_i$ in the new steady states (See Appendix D).

3.3 Welfare Inequality

Computation of welfare inequality follows closely on the lines of the above chapters. The difference now is the presence of the physical capital tax. Recalling (6) agent i' s welfare at time t is

$$Z_i(t) \equiv \frac{1}{\gamma} C_i^{\gamma} \tag{27}$$

Substituting $C_i = \theta_i C$ into this expression yields

$$Z_i(t) = \frac{1}{\gamma}(\theta_i C(t))^{\gamma} \equiv (\theta_i)^{\gamma} Z(t) \tag{28}$$

where $Z(t)$ is the average welfare level at time t. Substituting (27) into (6) yields an analogous relationship for the relative intertemporal welfare evaluated along the equilibrium growth path.

$$\frac{U_i}{U} = \frac{Z_i(t)}{Z(t)} = \theta_i^{\gamma} \tag{29}$$

At each instant of time, agent i's relative welfare remains constant, so that his intertemporal relative welfare, $z_i \equiv Z_i / Z$ remains constant as well. Using (18') and the fact that $\tilde{c} \equiv \left(\frac{C}{H}\right)$ and $\tilde{v} \equiv \left(\frac{V}{H}\right)$ we can express relative welfare in the form

$$\Omega_i \equiv \frac{U_i}{U} = \left[1 + \left(\frac{\frac{(\rho - \gamma \bar{r}_K(1-\tau_K)}{1-\gamma}\tilde{v}}{\tilde{c}}\right)(\tilde{v}_i - 1)\right]^{\gamma} = z_i(t) \equiv \frac{Z_i(t)}{Z(t)} \tag{30}$$

We can now compute a measure of welfare inequality. A natural metric for this is obtained by applying the following monotonic transformation of relative utility. This enables us to express the relative utility of individual i as

$$\Omega_i^{1/\gamma} = \chi(\Omega_i) = \chi(z_i(t)) = \left[1 + \frac{\frac{(\rho - \gamma \bar{r}_K(1-\tau_K)}{1-\gamma}\tilde{v}}{\tilde{c}}(\tilde{v}_i - 1)\right] \tag{31}$$

Both instantaneous and intertemporal welfare inequality, expressed in terms of equivalent units of wealth, can be measured by the standard deviation of relative utility and are constant and identical to the steady state level of income inequality

$$\tilde{\sigma}_{\chi} = \left(1 - \frac{\widetilde{W}}{\tilde{C}}\right)\tilde{\sigma}_v = \left(1 - \frac{\widetilde{W}}{\tilde{C}}\right)\tilde{\sigma}_y\, \nu(t) \tag{32}$$

4 Numerical Simulations

Given the complexity of the model, it is necessary to employ numerical simulations to analyze the dynamics. We begin by calibrating a benchmark economy using the following standard parameter values representing a typical economy**.** The preference parameters corresponding to a rate of time preference of 4% and an intertemporal elasticity of substitution of 0.4 are standard and noncontroversial. The exponents in the production function for final output approximate the empirical estimates of the generalized Solow production function obtained by Mankiw, Romer and Weil (1992). Empirical evidence on the production function for human capital is far sparser. Thus we use $\epsilon = 0.6$ as a plausible benchmark, which is very close to the calibrated value of 0.62 obtained by Manuelli and Seshadri (2010).

The resulting pre-tax macroeconomic equilibrium is summarized in Table 4A, Line 1. The implied output-physical capital ratio is 0.31; almost 90 percent of raw labor and 85 percent of skilled labor is employed in the final output sector. The ratio of human capital to physical capital is around 1.2, the skill premium, as measured by the ratio of income earned by human capital to the raw wage is 1.05 and the equilibrium growth rate is 2.56 percent. The equilibrium rate of return of both forms of capital, measured in terms of their respective own units is 10.4 percent.

The equilibrium distribution measures (and their evolution) depend upon the distributions of the initial endowments. These results have been tabulated in Table 4B for the three extreme cases considered.

While we do not attempt to calibrate to a specific economy, we view these as providing plausible benchmarks that will facilitate our understanding of the difference mechanisms as the economy evolves over time in response to government intervention.

Regarding the specification of fiscal policy, we report the numerical effects of financing a fixed (arbitrary) investment subsidy of 10% through a tax on physical capital income, and a tax on human capital income. In Figures 12 through 15 below, the left hand column represents subsidy financed by a tax on income from physical capital (12A-15A), whereas the right hand column represents a subsidy financed by tax on return to human capital (12B-15B). Using Appendix A.3. we also study the effect of tax policy on the steady state effects on post-tax income inequality. Fournier and Koske (2012) show that a rise in educational attainment, results in a lower level of inequality, while also increasing growth rate. Thus policies aimed at promoting equity in education, reduce wage dispersion. Our results below, point to similar conclusions.

5 Analysis of Results

5.1 Investment Subsidy Financed by Tax on Income from Physical Capital

We now consider a 10 percent increase in subsidy financed by a tax on physical capital. The return on physical capital, r_K, gets taxed at the rate τ_K which is 2.07 percent so that the government budget is balanced. Figure 12 shows the response of the aggregate variables. The increase in stock of human capital results in a decline in the long run ratio of physical to human capital by 14.5 percent. Consumption to human capital ratio falls by about 6 percent attributable to the increase in the stock of human capital. The relative price of human capital, q, falls by 10.2 percent, due to the fall in its relative scarcity. The increased accumulation of human capital, also results in a fall in the return to human capital by 3.5 percent and an increase in return to physical capital by almost 10 percent.

There are two effects in play in the case outlined here. The tax on income from physical capital causes substitution toward the use of human capital in the output sector, relative to physical capital. The subsidy to human capital in the education sector re-enforces this effect, by causing substitution toward use of skilled labor in the human capital sector. Both these result in the accumulation of human capital.

The increase in taxes on physical capital income, combined with the increase in income due to subsidization has two immediate effects on factor allocation. Given the relative sectoral labor intensities, $\Gamma > 1$, the factor market equilibrium will require a fall in the relative demand for skilled labor in the output sector, so that ϕ_X increases on impact. ϕ_Y on the other hand jumps down on impact, reflecting the greater increase in skilled relative to unskilled labor in the human capital sector, since the return to skill in the human capital sector is subsidized. This is shown in Figures 12.4 A and 12.5 A

At the same time, the fact that a subsidy on human capital investment will reduce the relative scarcity of human capital, its price, $q(0)$, falls on impact. With the movement of resources out of the output sector, there is an increase in the demand for skilled labor, causing skill premium to jump up in the short-run. Both consumption and wages overshoot their final steady states in the short run. This is a direct consequence of the income effect of the subsidy. Since taxes on physical capital reduce the opportunity cost of consumption, consumption overshoots in the short-run. On impact, the tax on physical capital results in a drop in the productivity of physical capital at $t = 0$. $r_K$thus falls. This, coupled with the high consumption and fall in output reduces the rate of accumulation of physical capital (per its growth rate). Return on human capital though jumps up at $t = 0$, owing to higher productivity due to the subsidy on skill.

The introduction of a physical capital tax acts as a negative productivity shock, causing an instant fall in the growth rate of physical capital, which gradually increases to a higher rate, but spends most of the time below its original steady state. On the other hand, the subsidy acts as a positive shock to the human capital sector, causing its growth rate to jump up at $t = 0$, and stay above its final steady state throughout its transition, eventually falling to its new (and higher) steady state. Thus, the ratio of human to physical capital begins to rise (k falls). The rising ratio of human to physical capital puts further downward pressure on its relative price.

The above jumps give rise to the following effects in transition. Since r_K falls on impact, while r_H rises, $\dot{q}(0) < 0$ in transition, so that returns are equated in equilibrium. This fall in the relative price of human capital, reverses the initial jump in ϕ_X. As both skilled as well as unskilled labor employed in the output sector start to increase (resources move back to the output sector), ϕ_X falls in transition. The subsidy to return to skill in the human capital sector attracts relatively more skilled resources in that sector, such that while $(1 - u)$ increases by almost 16 percent in the new steady state, unskilled labor in the human capital sector increases by only about 6 percent causing ϕ_Y to fall steadily towards its new lower value. Eventually, due to the accumulation of human capital and the higher return on physical capital, growth rates rise to a level higher than the pre-shock value, by 12.11 percent.

On impact, consumption rises by more than wages. In the long-run though, the fall in consumption is higher than the fall in wages. For this, consumption must fall at a faster rate than wages, as reflected in the falling C/W ratio (Figure 12.6A). Since savings fall, and wealthier individuals save more, wealth inequality falls in transition. In the new steady state, wealth inequality declines by 0.375 percent. Figures 15A describe the dynamics of wealth inequality. When the initial inequality in endowments of human and physical capital is not proportional, the results are starkly contrasting. Whereas wealth inequality falls by 1.75 percent when the initial endowment of human capital varies, (Figure 15.2A), it rises by 1.25 percent when this difference is due to differences in physical capital endowment. These results are driven by the effect of initial heterogeneity that affect the way changes in relative price impact final distribution of wealth.

Pre-tax income inequality falls at $t = 0$, in response to the higher wages and falling returns to physical capital. As this is reversed wages eventually fall and returns to capital increase, income inequality increases to a level higher than the benchmark, eventually rising by 0.27 percent, thus worsening the growth-pre tax inequality tradeoff.

When initial endowment of physical capital differs, pre-tax income inequality rises by 1.90 percent, but it falls by 1.12 percent, when initial endowment of human capital differs. This is evident in Figures 16A. Post tax income inequality though falls in all cases.

5.2 Investment Subsidy Financed by Tax on Income from Human Capital

We now consider a 10 percent increase in subsidy financed by a tax on return to human capital. The return to human capital r_H is taxed at the rate τ_H, 2.06 percent in order that the government budget stays balanced. Since returns to human capital in the output sector are taxed, the responses are very similar to those obtained above. Since a tax on human capital directly offsets the subsidy on its investment, growth rate increases by less than in the case above, whereas pre-tax income inequality now increases in case of proportional initial endowments. Thus, a tax on human capital offsets the positive effect of the subsidy by more than a tax on physical capital.

In the long-run the physical to human capital ratio falls by 10.8 percent, less than in the case above. This is because physical capital does not experience a direct negative shock in way of taxes. The subsidy has a dominating effect resulting in a shift of resources from the output sector to the human capital sector.

Consumption to human capital ratio falls by five percent, less than in the previous case, since the output sector is relatively less affected. The relative price of human capital, q, falls by 10.2 percent, due to the fall in its relative scarcity. The increased accumulation of human capital relative to physical capital results in a fall in the return to human capital by two percent and an increase in return to physical capital by 6.7 percent.

Taxing the resources foregone in accumulating human capital, combined with the increase in income due to subsidization of investment in human capital has the following short term effects on sectoral factor allocation.

Sectoral factor intensities respond very similarly to the case above. On impact ϕ_X and ϕ_Y move in opposite directions due to the presence of both subsidy and tax. The proportion of human capital in the output sector (u) falls by more than the raw labor allocated to it (L_X), resulting in an increase in ϕ_X. For similar reasons, ϕ_Y falls. The fall in ϕ_Y is more than in the case above, since the tax on returns to human capital accumulation dampens the magnitude of resource re-allocation, such that both skilled and unskilled labor increase by a smaller amount, on impact. Due to the tax, return to human capital is now lower than in the case above, so in order that factor returns are equated $q(0)$ falls by more (five percent as opposed to three percent in the above case). With the movement of resources out of the output sector, there is an increase in the demand for skilled labor, causing skill premium to jump up on impact. On impact, the tax on human capital along with the offsetting subsidy, results in an increase in the rate of growth of human capital due to an inflow of resources and the increase in its productivity. At the same time, the fall in the return to physical capital, results in a slower growth rate of physical capital.

Turning to distribution, the distribution of wealth does not change on impact, when the initial endowments are proportionately distributed. Wealth inequality depends on the initial jump, which is a function of initial heterogeneity, and the impact of $q(0)$ due to this heterogeneity. Though on impact, wages rise by more than consumption, in the long run, the fall in consumption is higher than the fall in wages. For this, consumption must fall at a faster rate than wages, as reflected in the falling C/W ratio. This implies that wealth inequality declines in transition eventually falling by 0.28 percent. If the initial endowments are not distributed proportionally, these results differ. In the case that initial endowments of human capital are proportionally distributed, wealth inequality is affected by differences in endowments of physical capital, and eventually rises by 2.48 percent. If instead, the initial endowments differ in human capital, wealth inequality falls by 2.65 percent, again more than the case above.

Pre-tax income inequality when initial endowments are proportionally distributed, falls on impact. Since wages are equally distributed, and these overshoot their initial higher steady state at $t = 0$, income inequality falls. It then gradually rises in response to the falling wage rates, rising returns to physical capital and the higher returns to human capital. The combination of changing shares from wealth, and wealth inequality, eventually causes it to fall to a lower than pre-shock level by 0.32 percent. It rises by 2.45 percent when initial endowment differs in physical capital, and falls by 2.69 percent, when it differs in human capital. This is evident in Figures 16B. Post tax income inequality though, once again, falls in all cases. It falls more than in the case above when initial endowments are proportional, and less when agents are endowed with proportional amounts of human capital.

6 Conclusion

A subsidy by itself unambiguously increases growth rates. When the subsidy is financed by taxes, the type of tax determines the impact on growth. In our model, when the tax is on physical capital income, it reduces growth by less than when return to human capital is taxed. At the same time, a tax on physical capital raises income inequality, due to the relative scarcity of physical capital so generated, resulting in higher returns to capital exacerbating income inequality.

A fiscal policy mix that consists of a subsidy to human capital, financed by a tax on its returns, seems to give the most favorable tradeoff between growth and pre-tax income inequality, increasing the former while reducing the latter. It is important to note though, that in certain cases, fiscal policy may have opposite effects on gross and net income inequality. Where the initial endowments of human and physical capital are proportional, a physical capital tax increases gross income inequality but reduces net income inequality. In those cases where initial endowments are not proportional, both gross and net income inequality respond in the same direction, albeit differ in magnitude. The impact, whether positive or negative, is more in the case when the human capital tax finances the subsidy.

Table 1.1 Structural Changes

Steady State Variable	$\frac{d\tilde{A}}{\tilde{A}}$	$\frac{d\tilde{B}}{\tilde{B}}$
$\frac{d\tilde{\phi}_Y}{\tilde{\phi}_Y}$	0	$\frac{(\Gamma-1)\rho}{\tilde{B}\tilde{\phi}_Y^{-\epsilon}\,D} = sgn(\Gamma-1)$
$\frac{d\tilde{q}}{\tilde{q}}$	$\frac{1}{1-\alpha}$	$-\frac{1}{1-\alpha}\left[1+\frac{(1-\epsilon-\eta)(\Gamma-1)\rho}{\tilde{B}\tilde{\phi}_Y^{-\epsilon}\,D}\right]$
$\frac{d\tilde{\psi}}{\tilde{\psi}}$	0	$\frac{\tilde{B}\tilde{\phi}_Y^{-\epsilon}}{D} > 0$
$\frac{d\tilde{u}}{u}$	0	$-\frac{1}{1-\tilde{\phi}_Y}\frac{(\Gamma-1)\rho}{\tilde{B}\tilde{\phi}_Y^{-\epsilon}\,D}$
$\frac{d\tilde{k}}{\tilde{k}}$	$\frac{1}{1-\alpha}$	$\left[-\frac{1}{1-\alpha}+\left(\frac{(1-\epsilon-\eta)}{1-\alpha}+\frac{1}{1-\tilde{\phi}_Y}\right)\frac{(\Gamma-1)\rho}{\tilde{B}\tilde{\phi}_Y^{-\epsilon}\,D}\right]$
$\frac{d\tilde{c}}{\tilde{c}}$	$\frac{1}{1-\alpha}$	$d\hat{k}+\frac{\tilde{k}}{\tilde{c}}A\left(\frac{\alpha}{\beta}\tilde{q}\right)^{\alpha-1}\left(1-\frac{\alpha}{1-\gamma}\right)\tilde{\phi}_X^{\eta}d\hat{r}_K$
$\frac{d\tilde{w}}{\tilde{w}}$	$\frac{1}{1-\alpha}$	$-\frac{\alpha}{1-\alpha}+\frac{\alpha\epsilon+\eta-1}{1-\alpha}\frac{(\Gamma-1)\rho}{\tilde{B}\tilde{\phi}_Y^{-\epsilon}\,D}$
$\frac{d\tilde{r}_H}{\tilde{r}_H}$	$\frac{1}{1-\alpha}$	$-\frac{\alpha}{1-\alpha}+\frac{\alpha(\epsilon-1)+\eta}{1-\alpha}\frac{(\Gamma-1)\rho}{\tilde{B}\tilde{\phi}_Y^{-\epsilon}\,D}$
$\frac{d\tilde{r}_K}{\tilde{r}_K}$	0	$1+(1-\epsilon)\frac{(\Gamma-1)\rho}{\tilde{B}\tilde{\phi}_Y^{-\epsilon}\,D}$
$\frac{d\tilde{s}}{\tilde{s}}$	0	$\frac{(\Gamma-1)\rho}{\tilde{B}\tilde{\phi}_Y^{-\epsilon}\,D}$

Table 1. 2.A. Aggregate Variables

Variable	**k**	**q**	**c**	**u**	L_X	r_K	r_H	**w**	$S = r_H$/**w**	ψ_H
Original	0.448	0.526	0.128	0.852	0.896	0.104	0.055	0.052	1.052	0.026
↑A by 10 %	0.517	0.606	0.148	0.852	0.896	0.104	0.063	0.060	1.052	0.026
↑B by 10 %	0.384	0.455	0.121	0.844	0.890	0.115	0.052	0.049	1.055	0.029
% ∆: A	15.4	15.4	15.4	0.00	0.00	0.00	15.4	15.4	0.00	0.00
% ∆: B	-14.2	-13.4	-5.9	-0.99	-0.7	10.2	-4.6	-4.8	0.29	16.8

Table 1.2.B. Distributions

Initial Endowment	Shocks	σ_v	σ_y	σ_w
Proportional Initial Endowment $k_{i,0} = h_{i,0}$	**Original**	1	0.6608	0.5949
	↑A by 10 % **(% change)**	0.9897 (-1.03)	0.654 (-1.03)	0.5887 (-1.03)
	↑B by 10 % **(% change)**	1.0107 (+1.07)	0.6675 (+1.014)	0.5962 (+0.218)
Uniform Initial Endowment of Physical Capital $k_{i,0} = 1$	**Original**	0.5399	0.3568	0.3212
	↑A by 10 % **(% change)**	0.5554 (+2.87)	0.3670 (+2.87)	0.3304 (+2.9)
	↑B by 10 % **(% change)**	0.5264 (-2.50)	0.3476 (-2.56)	0.3105 (-3.3)
Uniform Initial Endowment of Human Capital $h_{i,0} = 1$	**Original**	0.4601	0.3040	0.2737
	↑A by 10 % **(% change)**	0.4343 (-5.63)	0.2869 (-5.63)	0.2582 (-6.1)
	↑B by 10 % **(% change)**	0.4844 (+5.27)	0.3199 (+5.23)	0.2857 (+4.3)

Table 2.2A: Steady State Responses of Aggregate Variables to a Gradual shock

Variable	**K**	**q**	**c**	**u**	L_X	r_K	r_H	**w**	$S = r_H$/w	ψ_H
Original	0.448	0.526	0.128	0.852	0.896	0.104	0.055	0.052	1.052	0.026
↑A by 10 %	0.517	0.606	0.148	0.852	0.896	0.104	0.063	0.060	1.052	0.026
↑B by 10 %	0.384	0.455	0.121	0.844	0.890	0.115	0.052	0.049	1.055	0.029
% ∆: A	15.4	15.4	15.4	0.00	0.00	0.00	15.4	15.4	0.00	0.00
% ∆: B	-14.2	-13.4	-5.9	-0.99	-0.7	10.2	-4.6	-4.8	0.29	16.8

Table 2.2B: Response of distributions to a Gradual Shock

	Shocks	$\boldsymbol{\sigma_v}$	$\boldsymbol{\sigma_y}$	$\boldsymbol{\sigma_w}$
Proportional initial endowments $k_{i,0} = h_{i,0}$	**Benchmark**	1	0.6608	0.5949
	Gradual ↑A by 10 % (% change)	0.96293 (-3.707)	0.63631 (-3.706)	0.5728 (-3.715)
	Gradual ↑B by 10 % (% change)	1.0659 (+6.59)	0.70397 (+6.53)	0.6288 (+5.69)
Uniform initial endowments of physical capital $k_{i,0} = 1$	**Benchmark**	0.5399	0.35677	0.3212
	Gradual ↑A by 10 % (% change)	0.53187 (-1.487)	0.35147 (-1.493)	0.3164 (-1.494)
	Gradual ↑B by 10 % (% change)	0.55895 (+3.52)	0.36916 (+3.46)	0.3297 (+2.646)
Uniform initial endowments of human capital $h_{i,0} = 1$	**Benchmark**	0.4601	0.30404	0.2739
	Gradual ↑A by 10 % (% change)	0.43106 (-6.312)	0.28484 (-6.315)	0.2564 (-6.38)
	Gradual ↑B by 10 % (% change)	0.50695 (+10.18)	0.33481 (+10.12)	0.299 (+9.16)

Table 3.2B: Aggregates: Steady State Response to Fiscal Policy Mix

Policy		$\tilde{k}$	$\tilde{q}$	$\tilde{c}$	$\tilde{u}$	$\tilde{L}_X$	$\tilde{\varphi}_X$	$\tilde{r}_K$	$\tilde{r}_H$	$\tilde{W}/\tilde{H}$	$\tilde{s}$	$\tilde{\psi}$
No govt.	$s_H = 0.0$ $\tau_K = 0.0$ $\tau_H = 0.0$	0.4478	0.5255	0.1284	0.8522	0.8964	1.0519	0.1041	0.0547	0.0520	1.0518	0.0256
Tax on *K*	$s_H = 0.10$ $\tau_K = 0.0206$ $\tau_H = 0.0$ **(% change)**	0.3831 (-14.5)	0.4720 (-10.18)	0.1202 (-6.39)	0.8287 (-2.76)	0.8897 (-0.75)	1.0736 (2.07)	0.1142 (9.70)	0.0528 (-3.47)	0.0492 (-5.41)	1.0736 (2.07)	0.0287 (12.11)
Tax on *H*	$s_H = 0.10$ $\tau_K = 0.0$ $\tau_H = 0.0207$ **(% change)**	0.3994 (-10.8)	0.4719 (-10.20)	0.1218 (-5.14)	0.8289 (-2.73)	0.8920 (-0.49)	1.0762 (2.32)	0.1111 (6.72)	0.0536 (-2.01)	0.0498 (-4.26)	1.0762 (2.32)	0.0284 (10.94)

Table 3.2B: Distributions

	Shocks	σ_v	σ_y^p(pre-tax)	σ_y^a (after-tax)	σ_w
Proportional initial endowments $k_{i,0} = h_{i,0}$	**Benchmark**	1	0.6608	0.6608	0.5949
	Subsidy financed by τ_K (% change)	0.9963 (-0.375)	0.6626 (+0.27)	0.5792 (-12.34)	0.5888 (-1.02)
	Subsidy financed by τ_H (% change)	0.9971 (-0.288)	0.6587 (-0.32)	0.57885 (-12.40)	0.5898 (-0.86)
Uniform initial endowments of physical capital $k_{i,0} = 1$	**Benchmark**	0.5399	0.35677	0.35677	0.3212
	Subsidy financed by τ_K (% change)	0.5304 (-1.76)	0.35277 (-1.12)	0.3298 (-7.53)	0.3135 (-2.40)
	Subsidy financed by τ_H (% change)	0.5256 (-2.65)	0.34718 (-2.69)	0.3269 (-8.35)	0.3109 (-3.21)
Uniform initial endowments of human capital $h_{i,0} = 1$	**Benchmark**	0.4601	0.3040	0.3040	0.2737
	Subsidy financed by τ_K (% change)	0.4658 (+1.25)	0.3098 (+1.90)	0.2953 (-2.86)	0.2753 (+0.60)
	Subsidy financed by τ_H (% change)	0.4715 (+2.48)	0.3115 (+2.45)	0.2981 (-1.95)	0.2789 (+1.92)

Shock to A | **Shock to B**

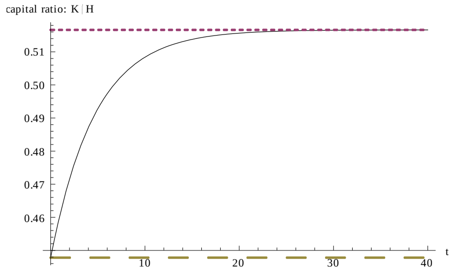

Figure 1.1A

Dynamics of capital $\frac{\boldsymbol{K}}{\boldsymbol{H}}$ for changes in $\boldsymbol{A}$

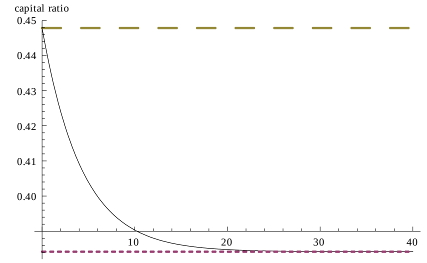

Figure 1.1B

Dynamics of capital $\frac{\boldsymbol{K}}{\boldsymbol{H}}$ for changes in $\boldsymbol{B}$

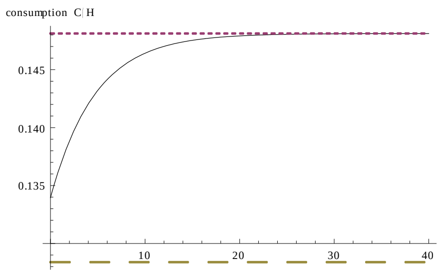

Figure 1.2A

Dynamics of consumption $\frac{\boldsymbol{C}}{\boldsymbol{H}}$ for changes in $\boldsymbol{A}$

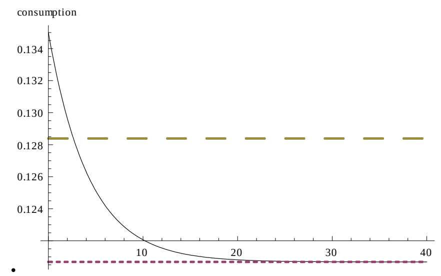

Figure 1.2B

Dynamics of consumption $\frac{\boldsymbol{C}}{\boldsymbol{H}}$ for changes in $\boldsymbol{B}$

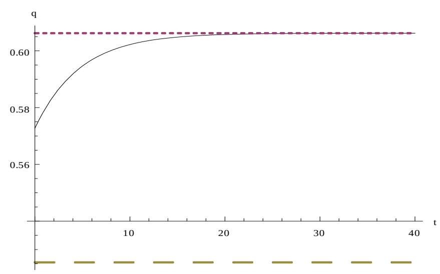

Figure 1.3A

Relative price of human capital: $\boldsymbol{q}$

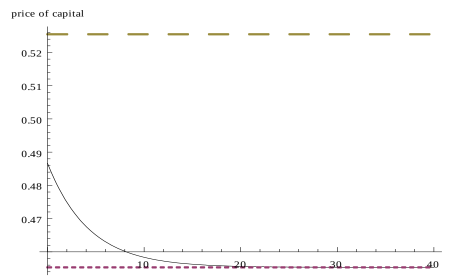

Figure 1.3B

Relative price of human capital: $\boldsymbol{q}$

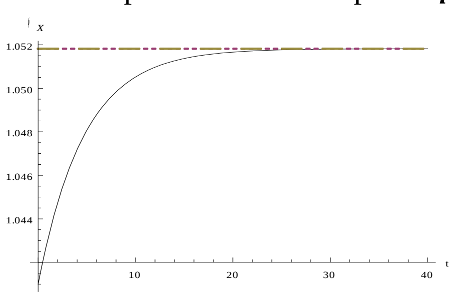

Figure 1.4A

Dynamics of $\boldsymbol{\phi_X}$ for changes in $\boldsymbol{A}$

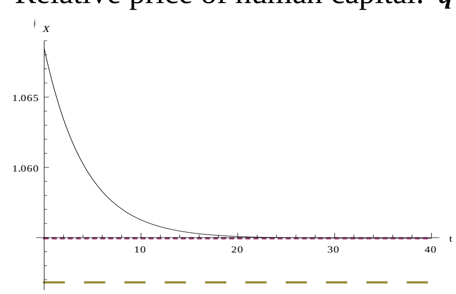

Figure 1.4B

Dynamics of $\boldsymbol{\phi_X}$ for changes in $\boldsymbol{B}$

Figure 1: Aggregates: Transitional Dynamics: Responses to Instantaneous Shocks to A, B

Shock to A	**Shock to B**

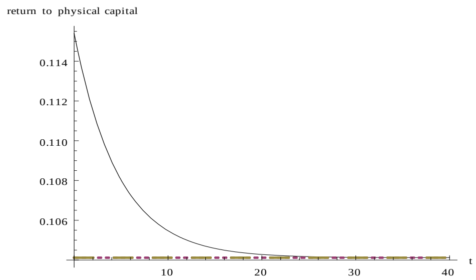

Figure 2.1A
Dynamics of return $\boldsymbol{r_K}$ for changes in $\boldsymbol{A}$

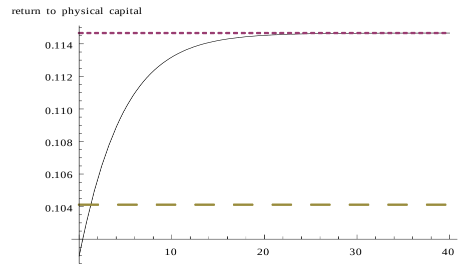

Figure 2.1B
Dynamics of return $\boldsymbol{r_K}$ for changes in $\boldsymbol{B}$

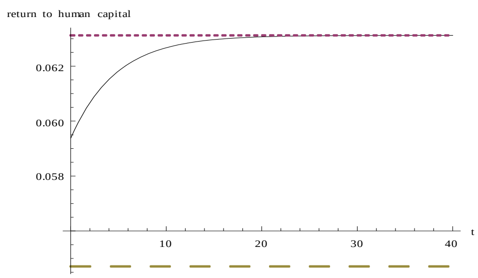

Figure 2.2A
Dynamics of return $\boldsymbol{r_H}$ for changes in $\boldsymbol{A}$

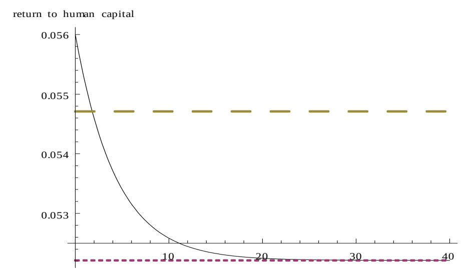

Figure 2.2B
Dynamics of return $\boldsymbol{r_H}$ for changes in $\boldsymbol{B}$

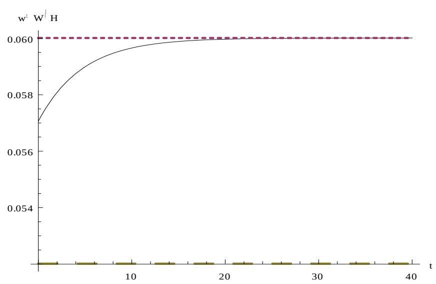

Figure 2.3A
Dynamics of $\boldsymbol{w}$ for changes in $\boldsymbol{A}$

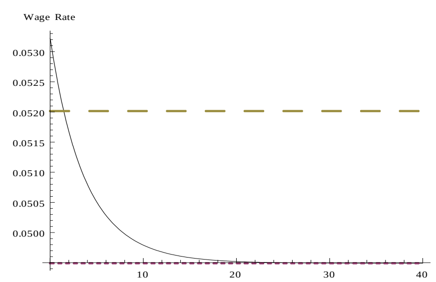

Figure 2.3B
Dynamics of $\boldsymbol{w}$ for changes in $\boldsymbol{B}$

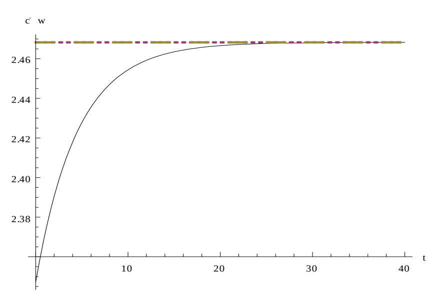

Figure 2.4A
Dynamics of $\frac{\mathbf{C}}{\mathbf{W}}$

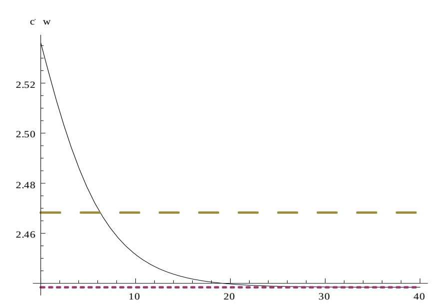

Figure 2.4B
Dynamics of $\left(\frac{C}{W}\right) : (\boldsymbol{B})$

Figure 2: Aggregates: Transitional Dynamics: Responses to Instantaneous Shocks to A, B

Shock to A | **Shock to B**

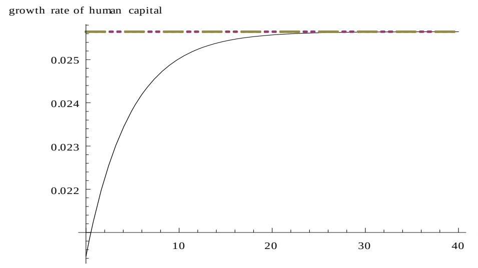

Figure 3.1A

Dynamics of return $\boldsymbol{\psi_H}$ for changes in $\boldsymbol{A}$

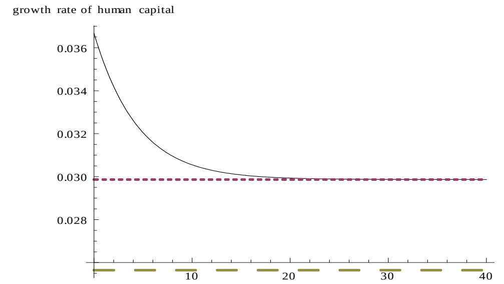

Figure 3.1B

Dynamics of return $\boldsymbol{\psi_H}$ for changes in $\boldsymbol{B}$

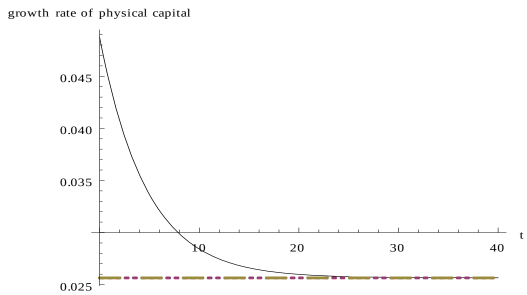

Figure 3.2A

Dynamics of return $\boldsymbol{\psi_K}$ for changes in $\boldsymbol{A}$

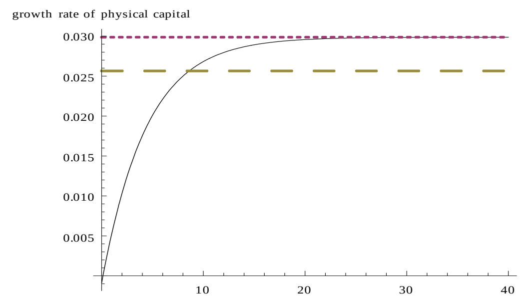

Figure 3.2B

Dynamics of return $\boldsymbol{\psi_K}$ for changes in $\boldsymbol{B}$

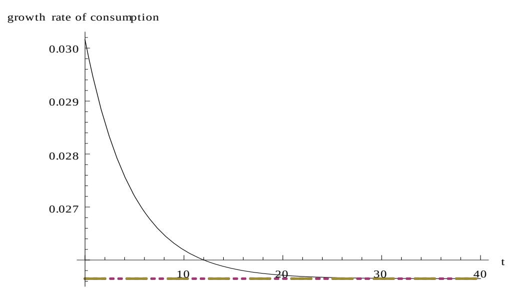

Figure 3.3A

Dynamics of return $\boldsymbol{\psi_C}$ for changes in $\boldsymbol{A}$

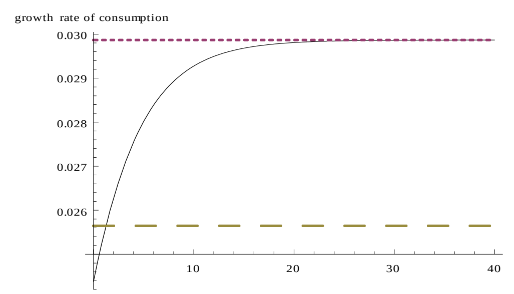

Figure 3.3B

Dynamics of return $\boldsymbol{\psi_C}$ for changes in $\boldsymbol{B}$

Figure 3: Aggregate growth rates: Transitional Dynamics: Responses to Instantaneous Shocks to A, B

Shock to A

Shock to B

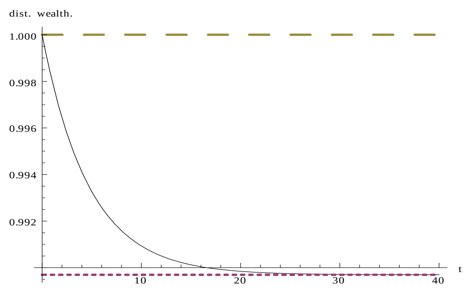

Figure 4.1A

Relative Wealth Distribution:$k_{i,0} = h_{i,0}$

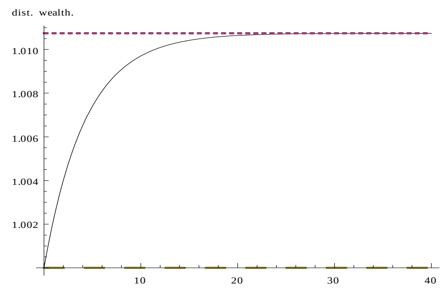

Figure 4.1B

Relative Wealth Distribution $k_{i,0} = h_{i,0}$

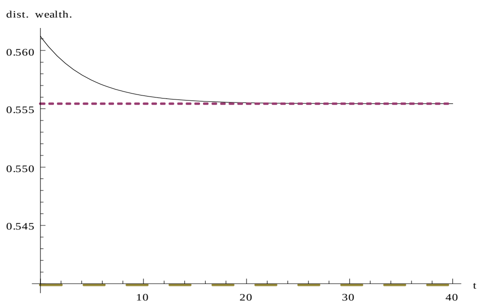

Figure 4.2A

Relative Wealth Distribution :$k_{i,0} = \mathbf{1}$

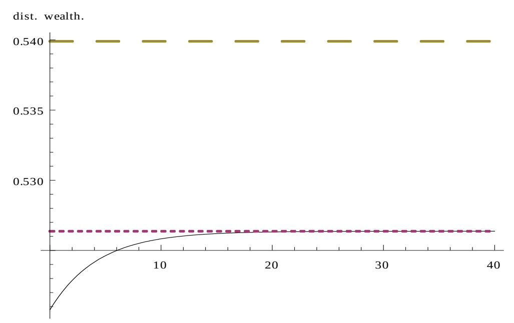

Figure 4.2B

Relative Wealth Distribution :$k_{i,0} = \mathbf{1}$

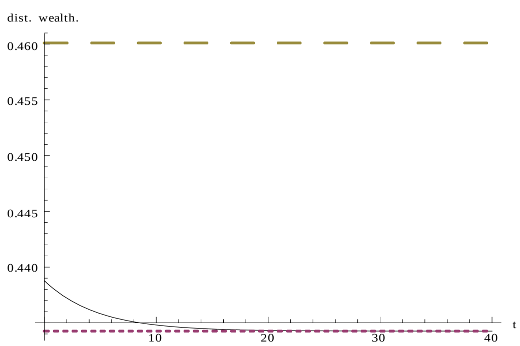

Figure 4.3A

Relative Wealth Distribution:$h_{i,0} = \mathbf{1}$

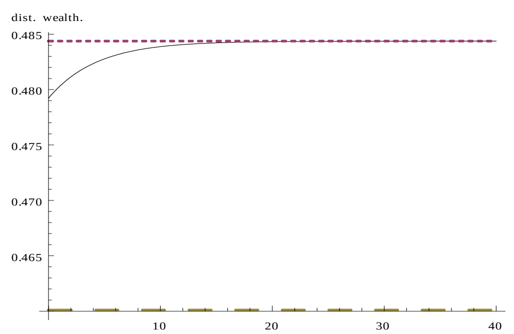

Figure 4.3B

Relative Wealth Distribution $h_{i,0} = \mathbf{1}$

Figure 4: Distributionof Relative Wealth: Responses to Instantaneous Shocks to A, B

Shock to A

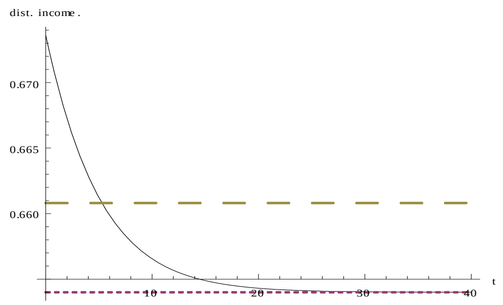

Figure 5.1A

Relative Income Distribution: $k_{i,0} = h_{i,0}$

Shock to B

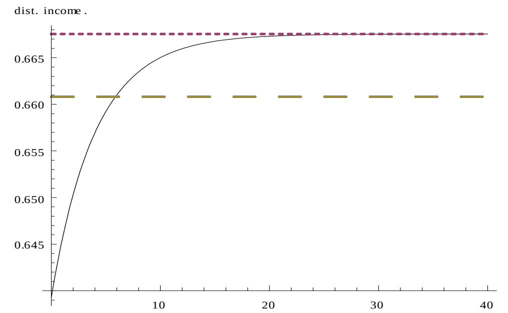

Figure 5.1B

Relative Income Distribution: $k_{i,0} = h_{i,0}$

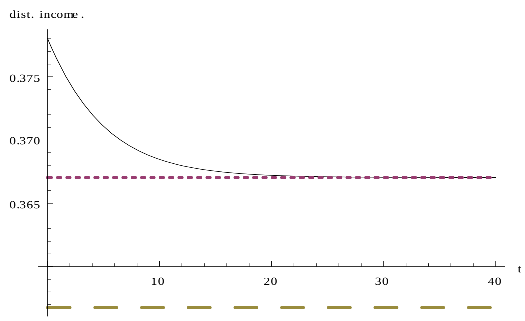

Figure 5.2A

Relative Income Distribution: $k_{i,0} = \mathbf{1}$

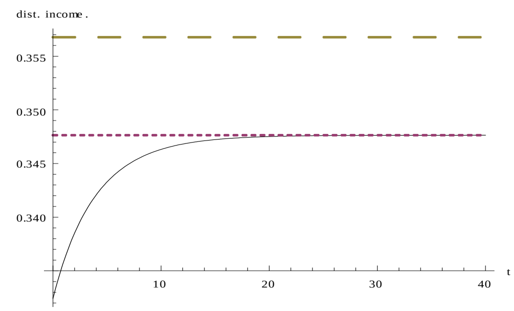

Figure 5.2B

Relative Income Distribution: $k_{i,0} = \mathbf{1}$

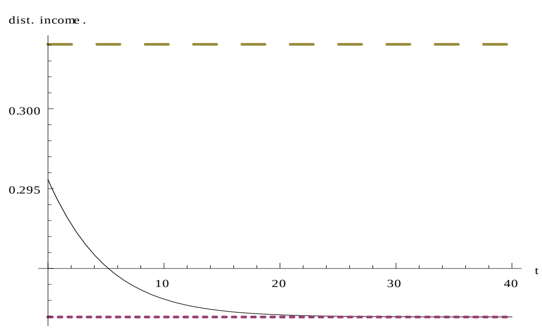

Figure 5.3A

Relative Wealth Distribution: $h_{i,0} = \mathbf{1}$

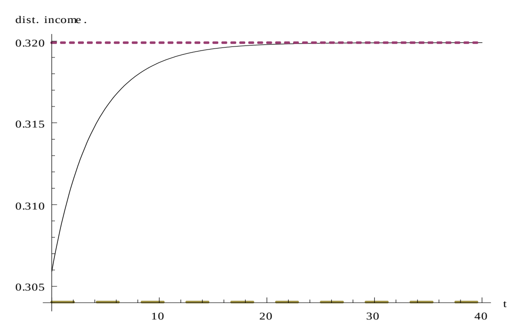

Figure 5.3B

Relative Wealth Distribution: $h_{i,0} = \mathbf{1}$

Figure 5: Distributionof Relative Income: Responses to Instantaneous Shocks to A, B

Shock to A | **Shock to B**

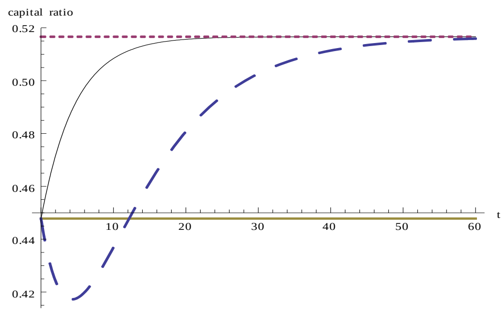

Figure 6.1A

Capital Ratio K/H

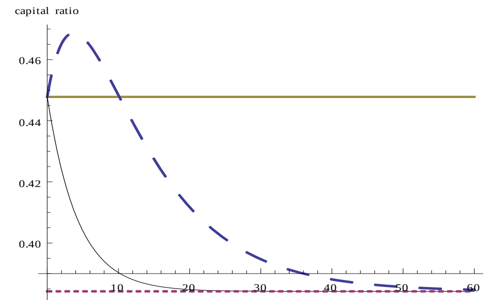

Figure 6.1B

Capital Ratio K/H

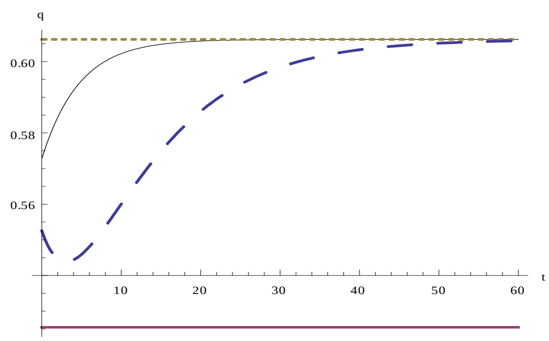

Figure 6.2A

Relative price of Human capital: q

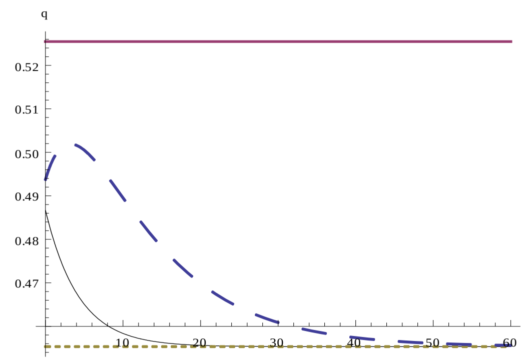

Figure 6.2B

Relative price of Human capital: q

Pre shock Steady State

Post Shock Steady State (Instantaneous Shock)

Time path when shock is realised instantaneously

Time Path when shock is gradual

Post Shock Steady State (Gradual Shock)

Figure 6: Aggregates: Comparison of Instantaneous vs. Gradual Shocks

Shock to A

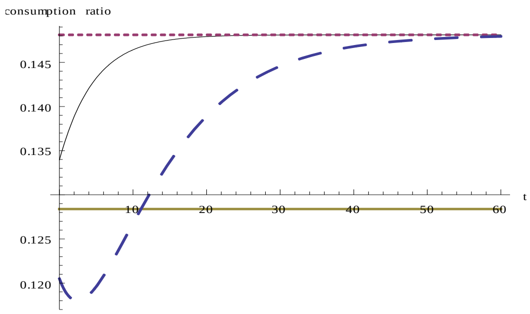

Figure 7.1A
Consumption Ratio C/H

Shock to B

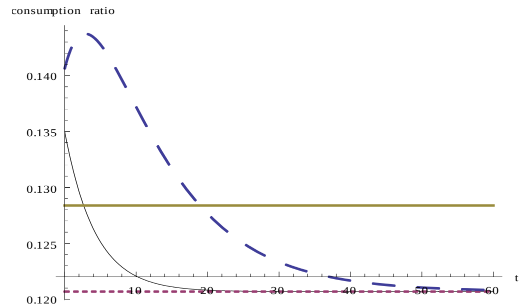

Figure 7.1B
Consumption Ratio C/H

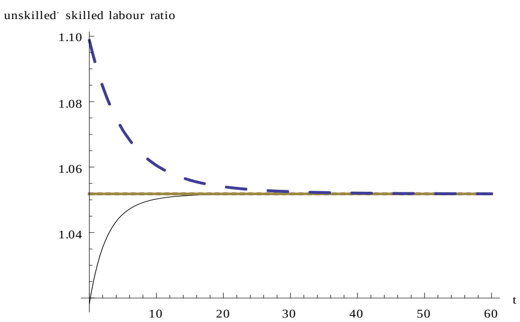

Figure 7.2A
Skill Premium: /
Unskilled-Skilled Labor Ratio : $\phi_X = \frac{L_X}{u}$

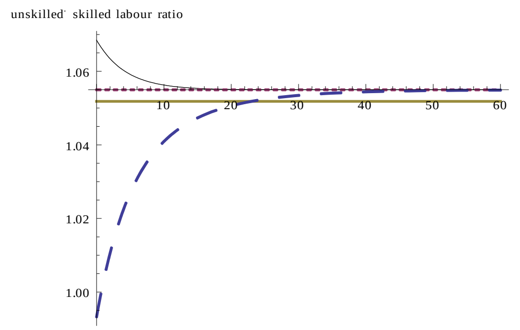

Figure 7.2B
Skill Premium: /
Unskilled-Skilled Labor Ratio : $\phi_X = \frac{L_X}{u}$

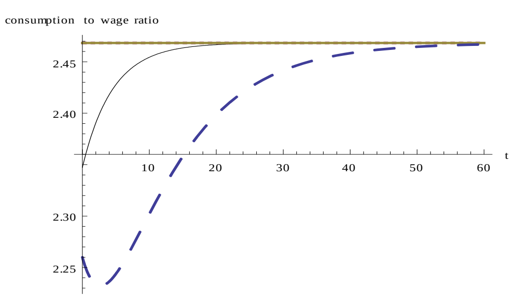

Figure 7.3A
Consumption-Wage Ratio (C/W)

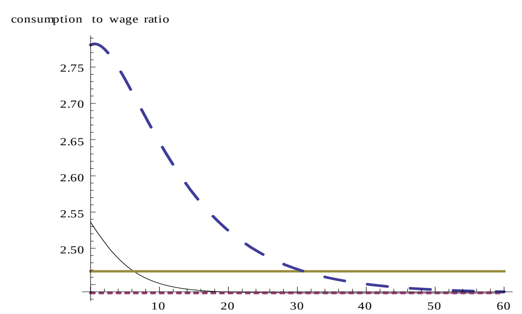

Figure 7.3B
Consumption-Wage Ratio (C/W)

Figure 7: Aggregates: Comparison of Instantaneous vs. Gradual Shocks

Shock to A

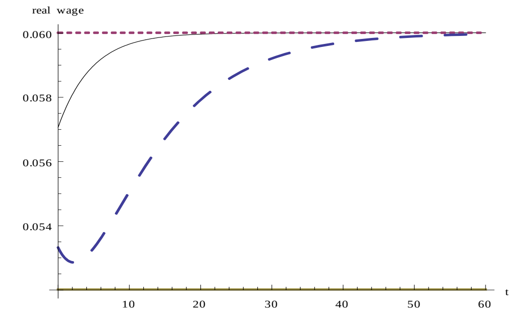

Figure 8.1A
Real Wage W/H

Shock to B

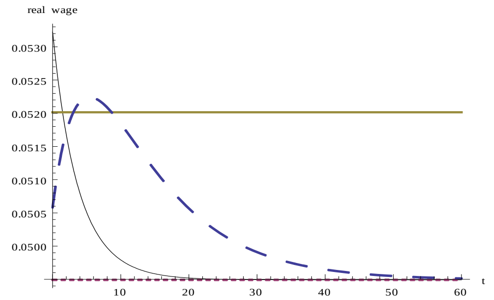

Figure 8.1B
Real Wage W/H

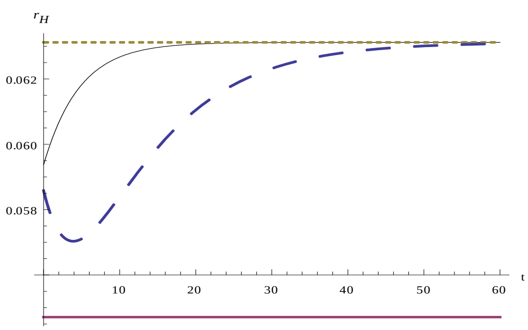

Figure 8.2A

Return to Human Capital r_H

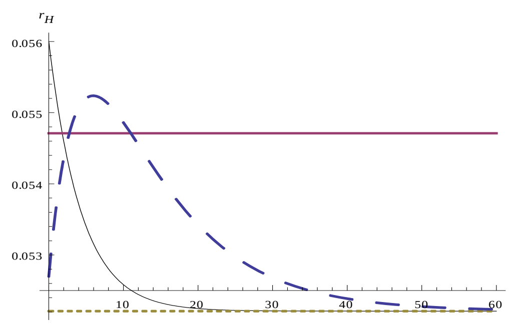

Figure 8.2B

Return to Human Capital r_H

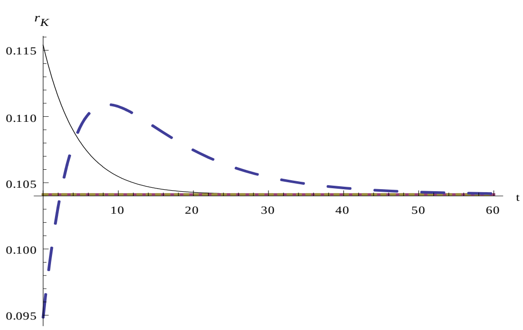

Figure 8.3A

Return to Physical Capital r_K

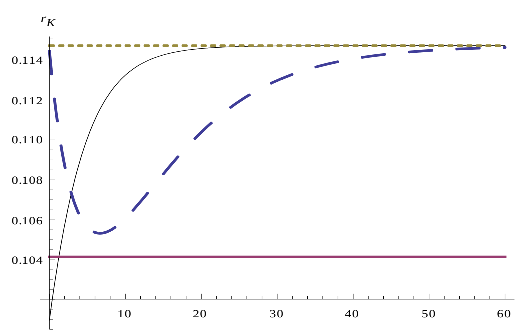

Figure 8.3B

Return to Physical Capital r_K

Figure 8: Aggregate prices: Comparison of Instantaneous vs. Gradual Shocks

Shock to A

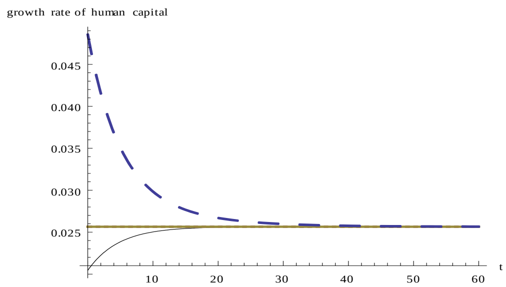

Figure 9.1A

Growth Rate of Human Capital: $\psi_H = \frac{\dot{H}}{H}$

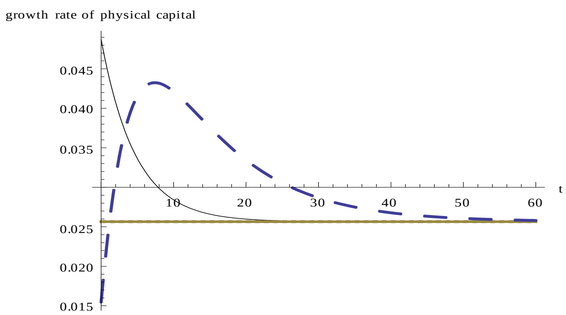

Figure 9.2A

Growth Rate of Physical Capital: $\psi_K = \frac{\dot{K}}{K}$

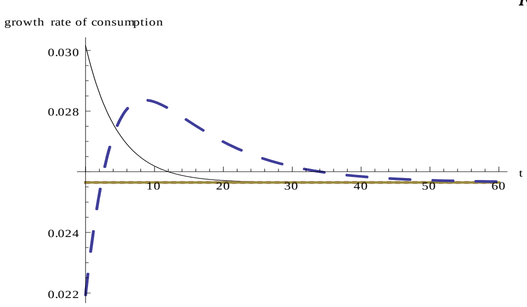

Figure 9.3A

Growth Rate of Consumption: $\psi_C = \frac{\dot{C}}{C}$

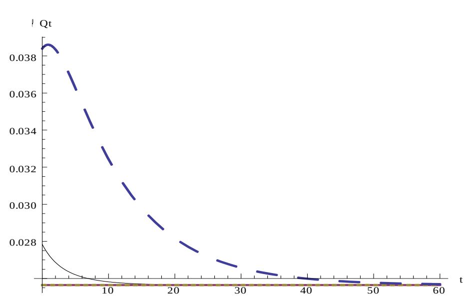

Figure 9.4A

Growth Rate of Total Output: $X + qY$

Shock to B

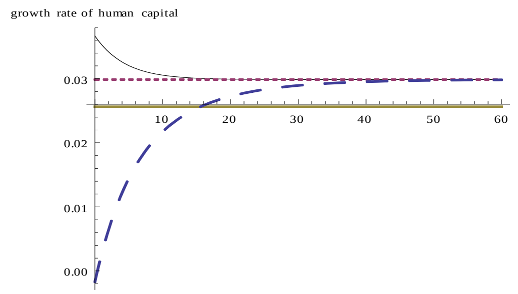

Figure 9.1B

Growth Rate of Human Capital: $\psi_H = \frac{\dot{H}}{H}$

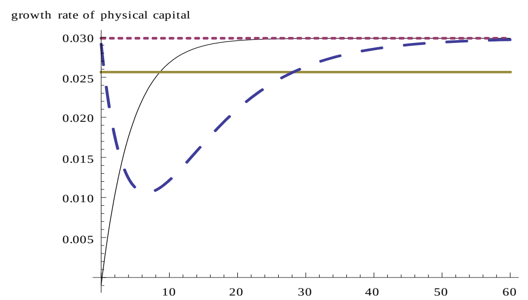

Figure 9.2B

Growth Rate of Physical Capital: $\psi_K = \frac{\dot{K}}{K}$

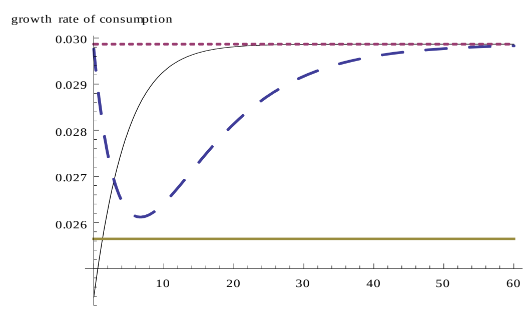

Figure 9.3B

Growth Rate of Consumption: $\psi_C = \frac{\dot{C}}{C}$

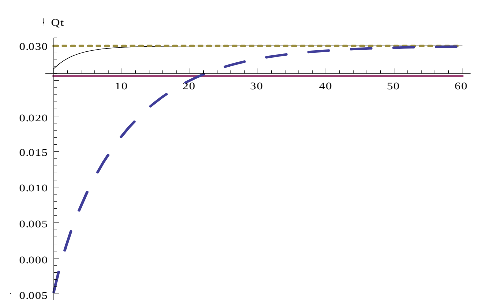

Figure 9.4B

Growth Rate of Total Output: $X + qY$

Figure 9: Aggregate Growth Rates: Comparison of Instantaneous vs. Gradual Shocks

Shock to A | **Shock to B**

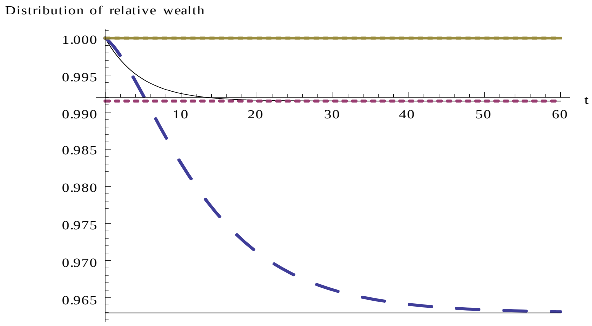

Figure 10.1A
Distribution of Relative Wealth:
$k_{i,0} = h_{i,0}$

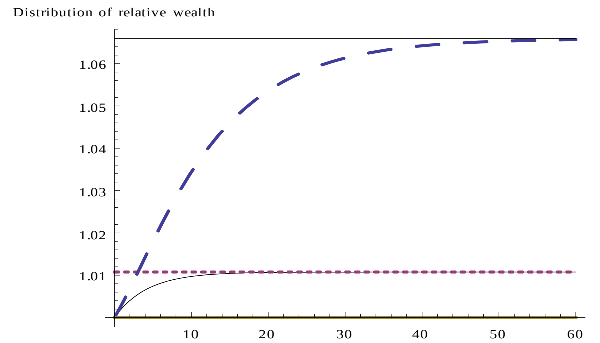

Figure 10.1B
Distribution of Relative Wealth:
$k_{i,0} = h_{i,0}$

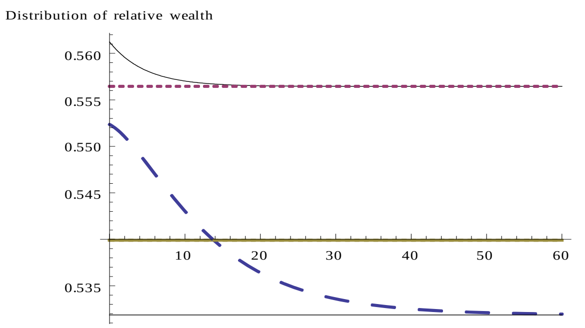

Figure 10.2A
Distribution of Relative Wealth: $k_{i,0} = 1$

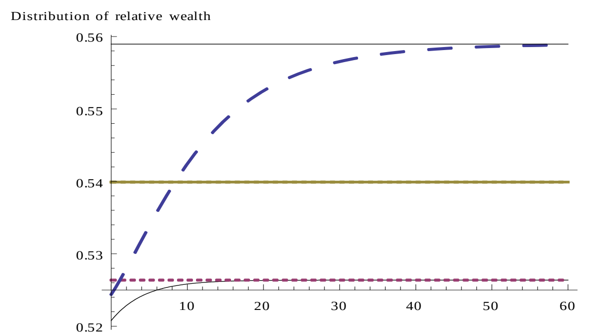

Figure 10.2B
Distribution of Relative Wealth: $k_{i,0} = 1$

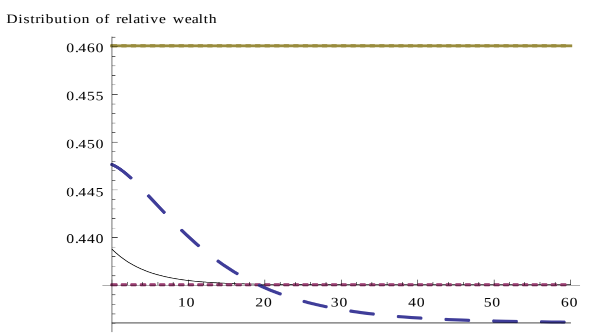

Figure 10.3A
Distribution of Relative Wealth: $h_{i,0} = 1$

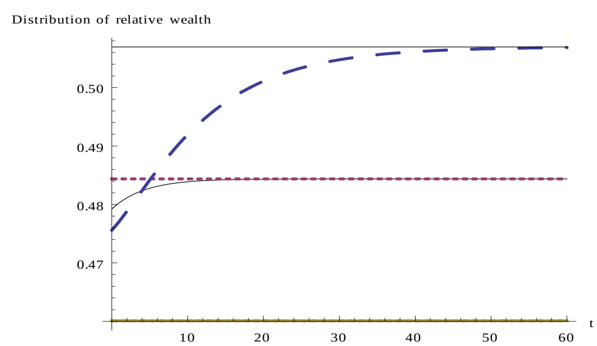

Figure 10.3B
Distribution of Relative Wealth: $h_{i,0} = 1$

Figure 10: Comparison: Distribution of Relative Wealth: Instantaneous vs. Gradual Shocks

Shock to A

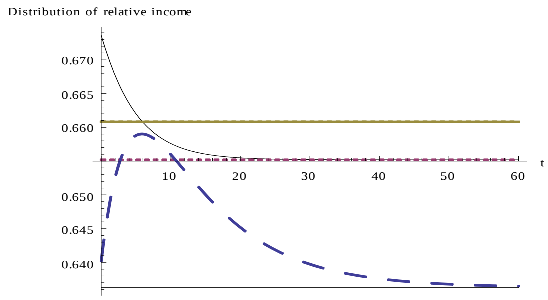

Figure 11.1A
Distribution of Relative Income:
$k_{i,0} = h_{i,0}$

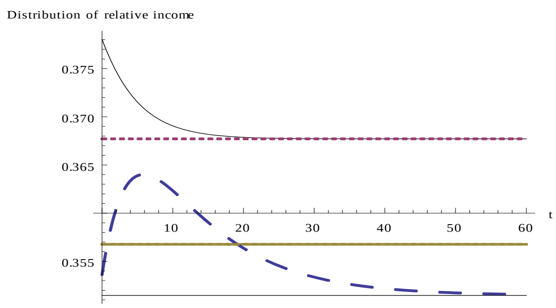

Figure 11.2A
Distribution of Relative Income: $k_{i,0} = 1$

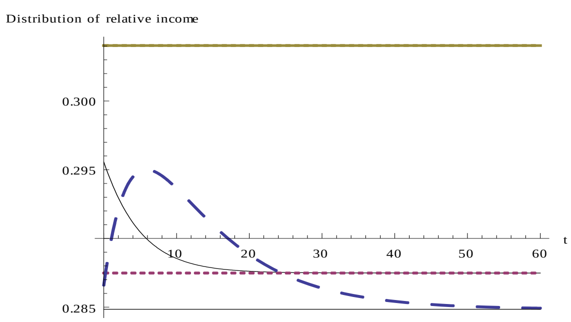

Figure 11.3A
Distribution of Relative Income: $h_{i,0} = \mathbf{1}$

Shock to B

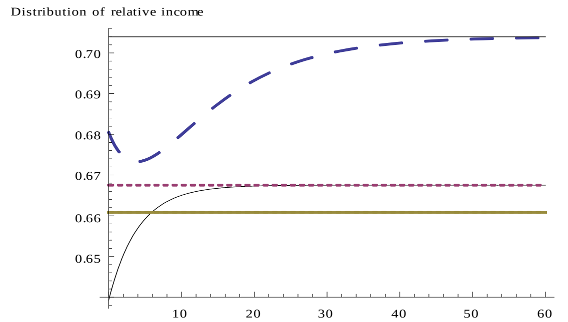

Figure 11.1B
Distribution of Relative Income:
$k_{i,0} = h_{i,0}$

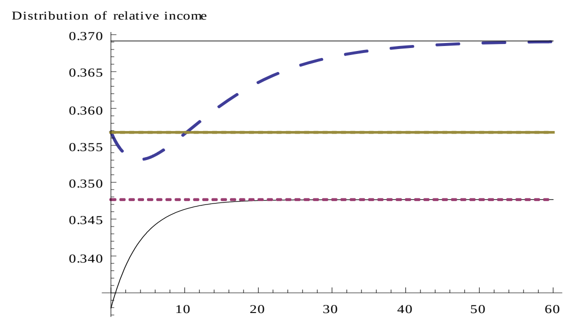

Figure 11.2B
Distribution of Relative Income: $k_{i,0} = 1$

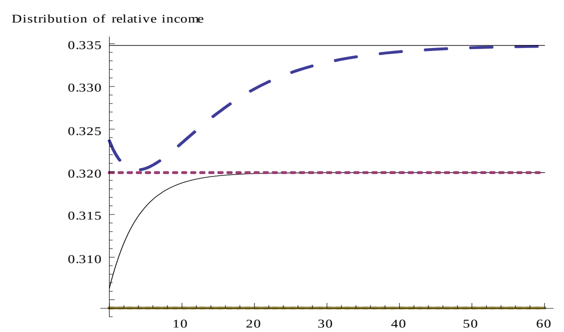

Figure 11.3B
Distribution of Relative Income: $h_{i,0} = 1$

Figure 11: Comparison: Distribution of Relative Income: Instantaneous vs. Gradual Shocks

Panel A: Financed by τ_K | **Panel B: Financed by τ_H**

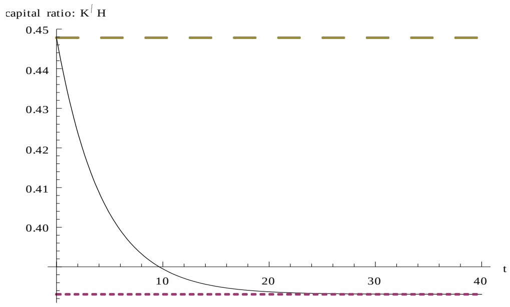

Figure 12.1A: Capital Ratio: K/H

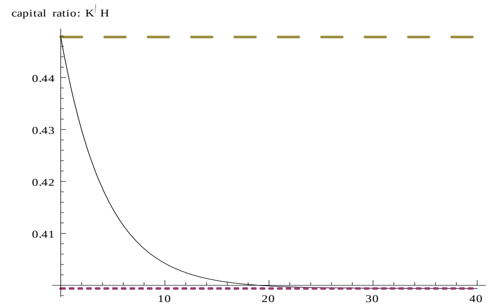

Figure 12.1B:Capital Ratio :K/H

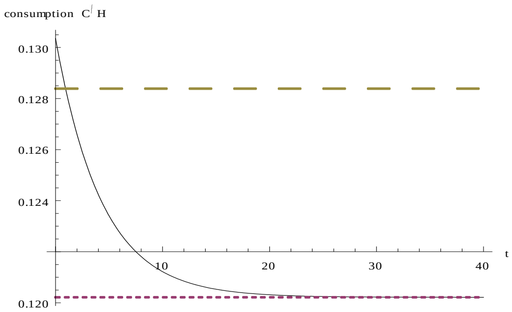

Figure 12.2A: Effective Consumption:C/H

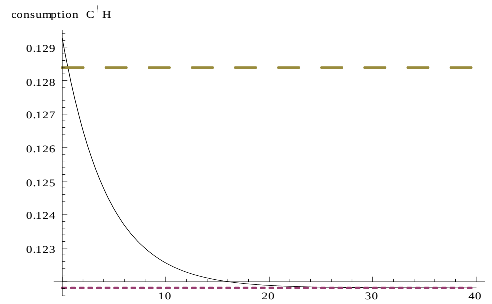

Figure 12.2B: Effective Consumption C/H

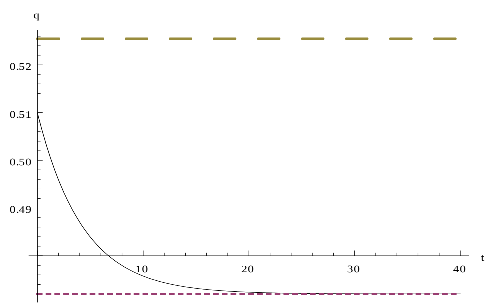

Figure 12.3A:Relative Price: q

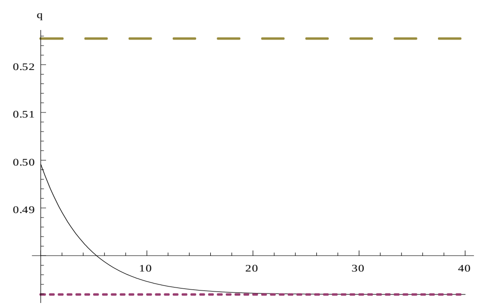

Figure 12.3B:Relative Price: q

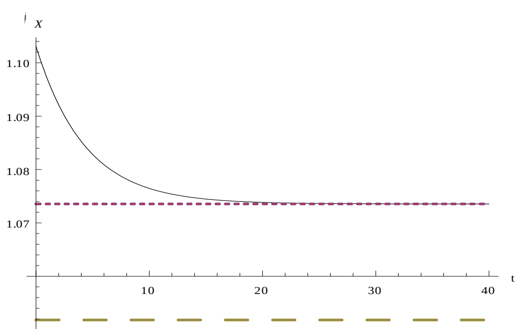

Figure 12.4A: Skill Premium: $\phi_X = \frac{L_X}{u}$

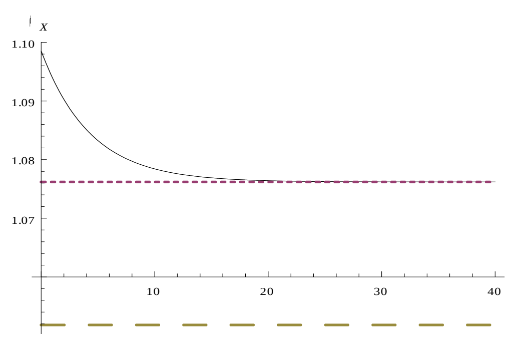

Figure 12.4B: Skill Premium:$\phi_X = \frac{L_X}{u}$

Figure 12: Aggregates: Fiscal Policy Mix

Panel A: Financed by τ_K

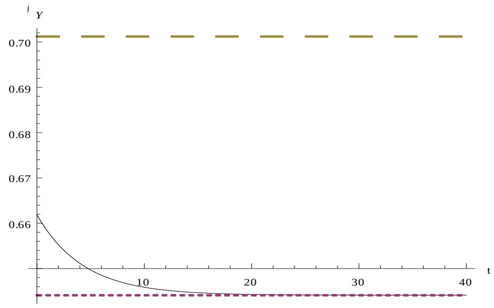

Figure 12.5A: Sectoral Factor Allocation (Human Capital): $\phi_Y = \frac{L_Y}{1-u}$

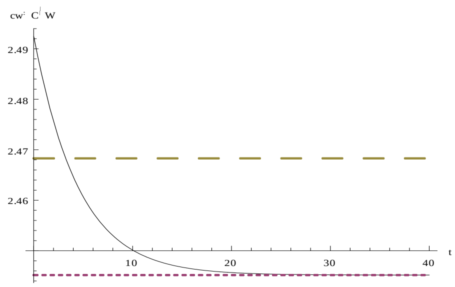

Figure 12.6A Consumption-Wage Ratio : C/W

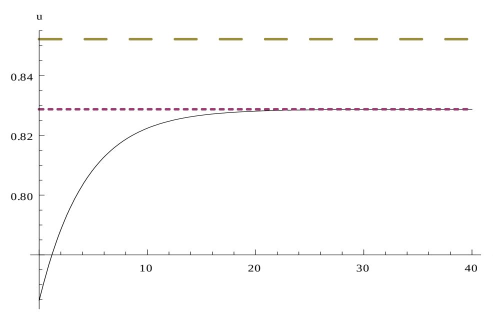

Figure 12.7 A: Skilled Labor: u

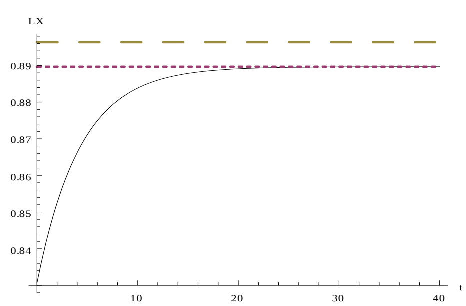

Figure 12.8 A: Unskilled labor: L_X

Panel B: Financed by τ_H

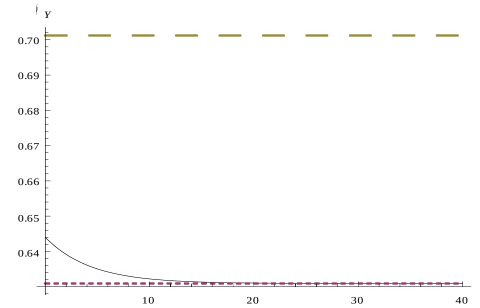

Figure 12.5A: Sectoral Factor Allocation (Human Capital) : $\phi_Y = \frac{L_Y}{1-u}$

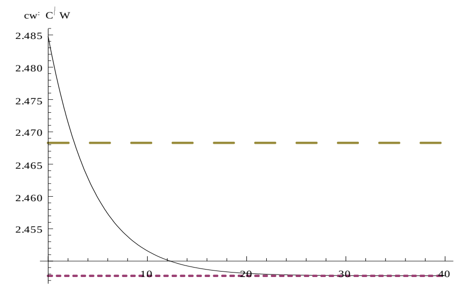

Figure 12.6B Consumption-Wage Ratio: C/W

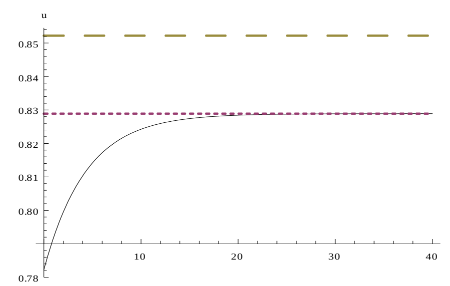

Figure 12.7 B: Skilled Labor: u

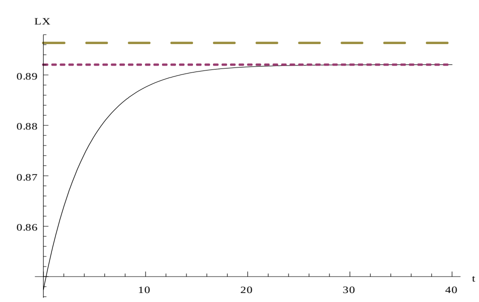

Figure 12.8 B: Unskilled Labor: L_X

Figure 12: Aggregates: Fiscal Policy Mix continued

Panel A: Financed by τ_K **Panel B: Financed by τ_H**

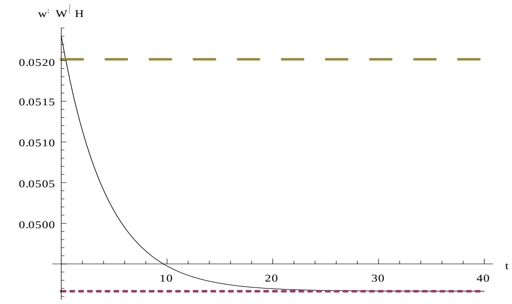

Figure 13.1A Real Wage W/H

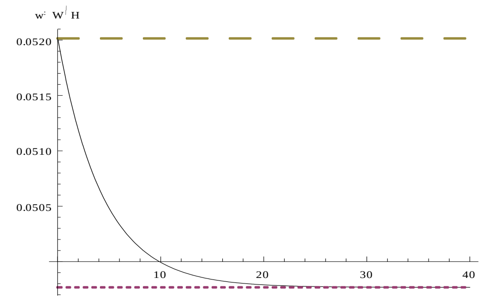

Figure 13.1 B: Real Wage W/H

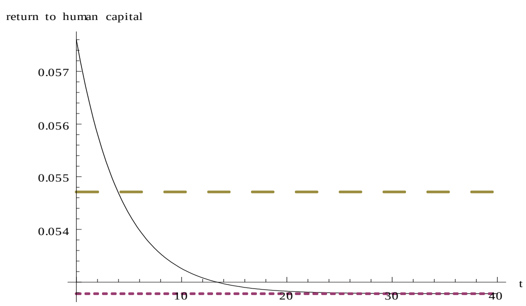

Figure 13.2 A Return to Human Capital r_H

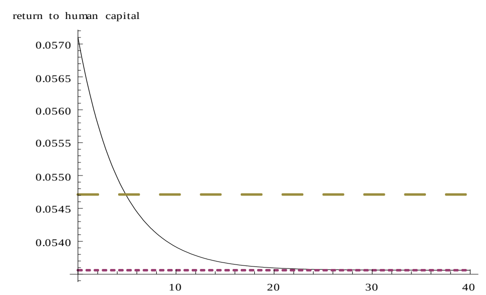

Figure 13.2 B: Return to Human Capital r_H

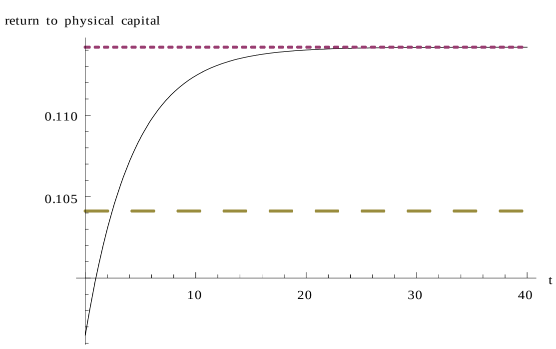

Figure 13.3 A: Return to Physical Capital r_K

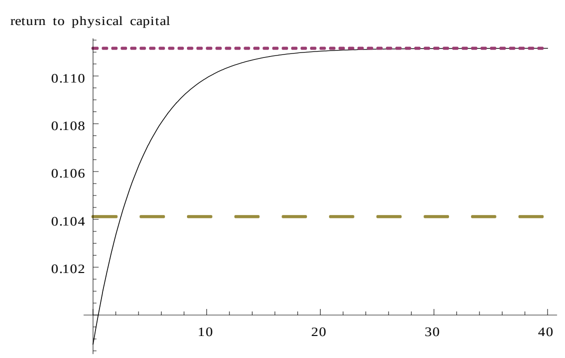

Figure 13.3 B: Return to Physical Capital r_K

Figure 13: Aggregate Prices: Fiscal Policy Mix

Panel A: Financed by τ_K | **Panel B: Financed by τ_H**

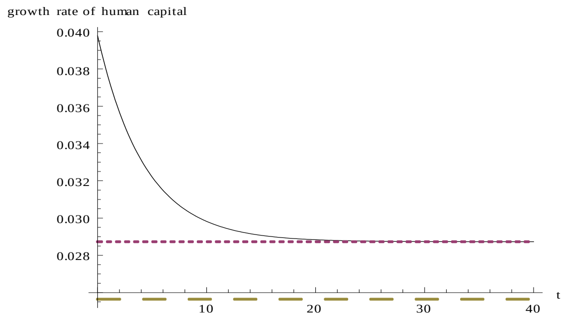

Figure 14.1.A

Growth Rate of Human Capital: $\psi_H = \frac{\dot{H}}{H}$

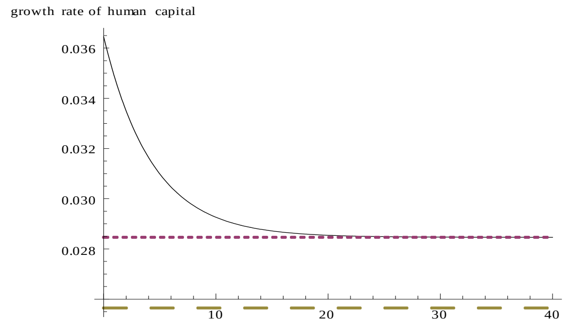

Figure 14.1. B

Growth Rate of Human Capital: $\psi_H = \frac{\dot{H}}{H}$

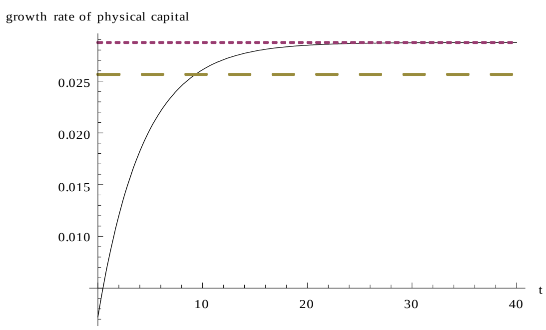

Figure 14.2A

Growth Rate of Physical Capital: $\psi_K = \frac{\dot{K}}{K}$

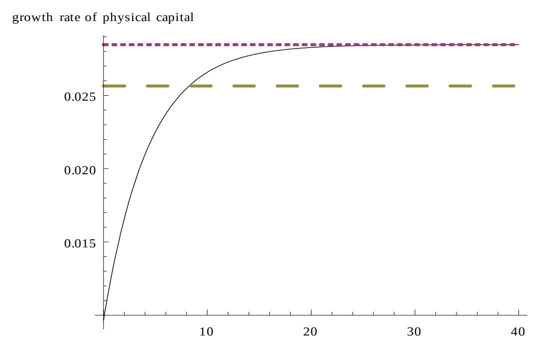

Figure 14.2.B

Growth Rate of Physical Capital: $\psi_K = \frac{\dot{K}}{K}$

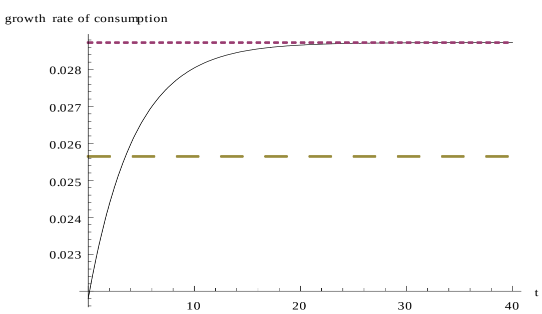

Figure 14.3A

Growth Rate of Consumption: $\psi_C = \frac{\dot{C}}{C}$

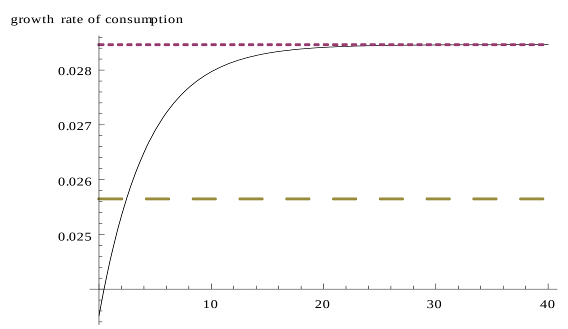

Figure 14.3B

Growth Rate of Consumption: $\psi_C = \frac{\dot{C}}{C}$

Figure 14: Aggregate Growth Rates: Fiscal Policy Mix

Panel A: Financed by τ_K | **Panel B: Financed by τ_H**

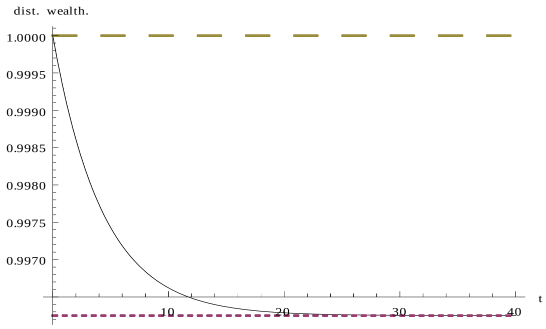

Figure 15.1.A

Distribution of Relative Wealth: $k_{i,0} = h_{i,0}$

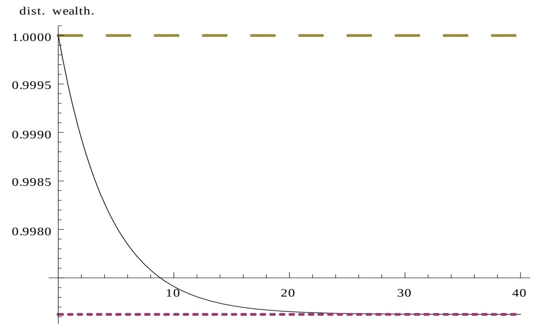

Figure 15.1.B

Distribution of Relative Wealth: $k_{i,0} = h_{i,0}$

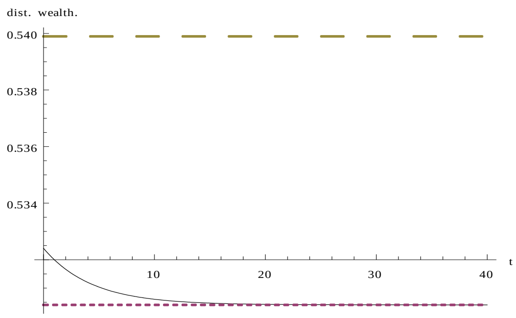

Figure 15.2.A

Distribution of Relative Wealth: $k_{i,0} = 1$

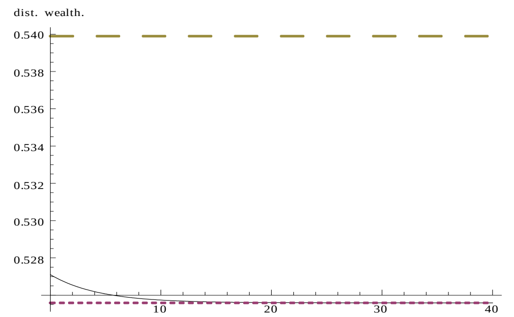

Figure 15.2 B

Distribution of Relative Wealth: $k_{i,0} = 1$

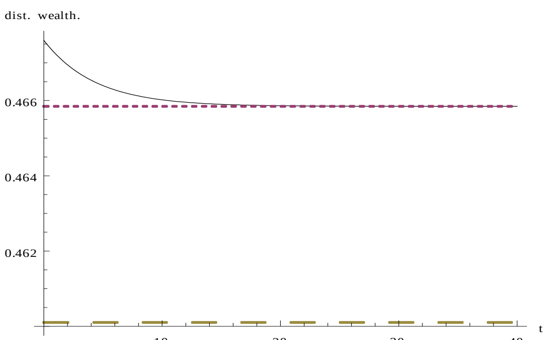

Figure 15.3A

Distribution of Relative Wealth: $h_{i,0} = 1$

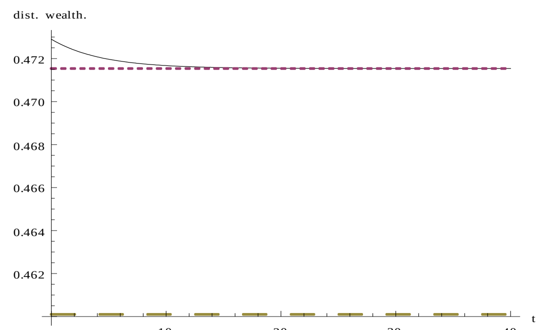

Figure 15.3B

Distribution of Relative Wealth: $h_{i,0} = 1$

Figure 15: Relative Wealth Distribution: Fiscal Policy Mix

Panel A: Financed by τ_K | **Panel B: Financed by τ_H**

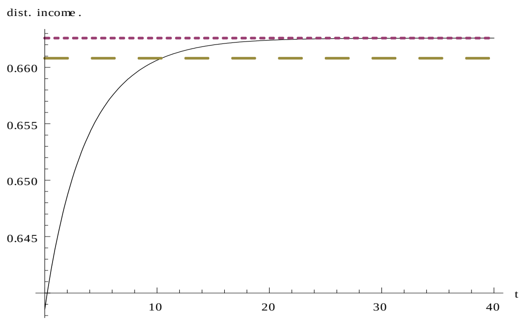

Figure 16.1A

Distribution of Relative Income: $k_{i,0} = h_{i,0}$

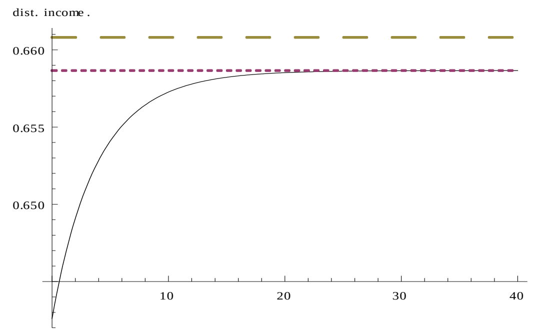

Figure 16.1B

Distribution of Relative Income: $k_{i,0} = h_{i,0}$

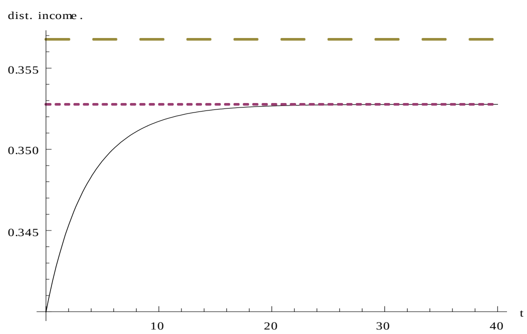

Figure 16.2 A

Distribution of Relative Income : $k_{i,0} = 1$

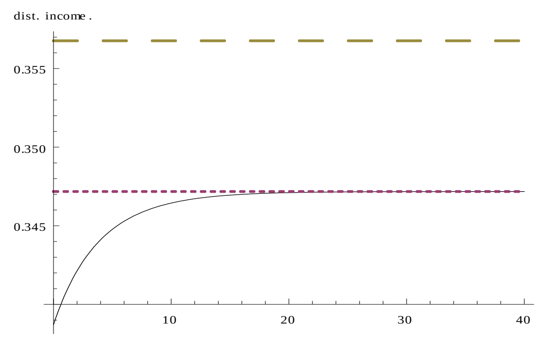

Figure 16.2 B

Distribution of Relative Income : $k_{i,0} = 1$

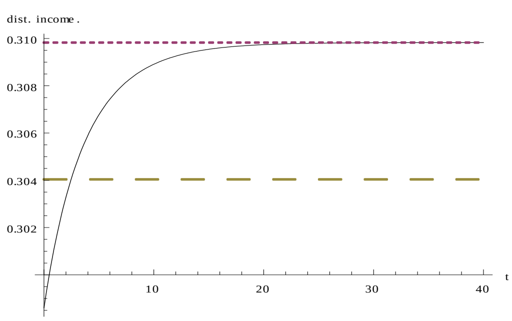

Figure 16.3A

Distribution of Relative Income $h_{i,0} = 1$

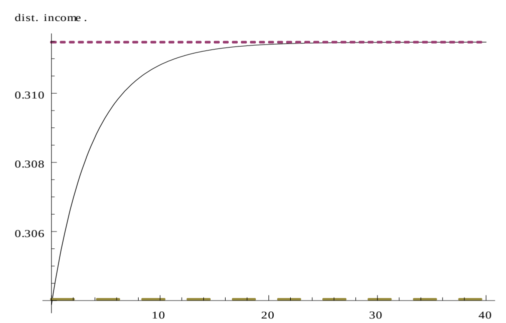

Figure 16.3B

Distribution of Relative Income $h_{i,0} = 1$

Figure 16: Pre-tax Relative Income Distribution: Fiscal Policy Mix

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Appendix

A.1 Dynamics of Aggregate Equilibrium

The aggregate macroeconomic dynamics is described by equations (17a)-(17c), together with the evolution of A, B specified by (6a) and (6b). To illustrate this structure we shall focus on the dynamics in response to A; B is analogous. To write the equilibrium system it is convenient to note that by substituting (15) into (5b), (5c), and into (12b'), we may write

$$r_K = A\alpha\left(\frac{k}{u}\right)^{\alpha-1}\varphi_X^{\eta} = r_K(k,q,A,B) \tag{A.1a}$$

$$r_H = A\beta\left(\frac{k}{u}\right)^{\alpha}\varphi_X^{\eta} = r_H(k,q,A,B) \tag{A.1b}$$

$$\psi_H = B(1-u)\varphi_Y^{1-\varepsilon} = \psi_H(k,q,A,B) \tag{A.1c}$$

the partial derivatives of which can be obtained from (16).

Using this notation, we can express the dynamics in the form

$$\dot{k}(t) = \left(\frac{r_K}{\alpha} - c - \psi_H\right)k(t) \tag{A.2a}$$

$$\dot{c}(t) = \left[\frac{1}{1-\gamma}\left(r_K - \rho\right)\right]c(t) \tag{A.2b}$$

$$\dot{q}(t) = r_H q - r_H \tag{A.2c}$$

$$\dot{A}(t) = \theta_A(\tilde{A} - A(t)) \tag{A.2d}$$

Linearizing (17a)-(17c), and (6a) around their steady state, we may write the dynamics as

$$\begin{pmatrix} \dot{k}(t) \\ \dot{c}(t) \\ \dot{q}(t) \\ \dot{A}(t) \end{pmatrix} = \begin{pmatrix} a_{11} & -1 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & -\theta_A \end{pmatrix} \begin{pmatrix} k(t) - \tilde{k} \\ c(t) - \tilde{c} \\ q(t) - \tilde{q} \\ A(t) - \tilde{A} \end{pmatrix} \tag{A.2}$$

where $a_{11}=\left(\frac{1}{\alpha}\frac{\partial r_K}{\partial k}-\frac{\partial \psi_H}{\partial k}\right)k$; $a_{13}=\left(\frac{1}{\alpha}\frac{\partial r_K}{\partial q}-\frac{\partial \psi_H}{\partial q}\right)k$; $a_{14}=\left(\frac{1}{\alpha}\frac{\partial r_K}{\partial A}-\frac{\partial \psi_H}{\partial A}\right)k$

$$a_{21}=\left(\frac{1}{1-\gamma}\frac{\partial r_K}{\partial k}-\frac{\partial \psi_H}{\partial k}\right)c;\ a_{23}=\left(\frac{1}{1-\gamma}\frac{\partial r_K}{\partial q}-\frac{\partial \psi_H}{\partial q}\right)c;\ a_{24}=\left(\frac{1}{1-\gamma}\frac{\partial r_K}{\partial A}-\frac{\partial \psi_H}{\partial A}\right)c$$

$$a_{31}=q\frac{\partial r_K}{\partial k}-\frac{\partial r_H}{\partial k};\ a_{33}=r_K+q\frac{\partial r_K}{\partial q}-\frac{\partial r_H}{\partial q};\ a_{34}=q\frac{\partial r_K}{\partial A}-\frac{\partial r_H}{\partial A}$$

This is a fourth order dynamic system with two sluggish variables, k, A, and two jump variables c, q. It will have a unique stable adjustment path if and only if there are two stable eigenvalues; (i) $-\theta_A$ and (ii) μ, where $\mu<0$ is the negative root to the cubic equation

$$\mu^3-(a_{11}+a_{33})\mu^2+(a_{11}a_{33}+a_{21}-a_{13}a_{31})\mu+(a_{31}a_{23}-a_{21}a_{33})=0$$

A necessary condition for this equation to have one negative root is that $a_{31}a_{23}-a_{21}a_{33}<0$, which can be shown be equivalent to $D>0$ in (18). With $\mu<0, \theta_A<0$, the system is a saddlepoint, the stable solution to which is given by

$$k(t)-\tilde{k} \quad = \quad B_1e^{\mu t}+B_2e^{-\theta_A t} \qquad \text{(A.4a)}$$

$$c(t)-\tilde{c} \quad = \quad B_1\delta_{21}e^{\mu t}+B_2\delta_{22}e^{-\theta_A t} \qquad \text{(A.4b)}$$

$$q(t)-\tilde{q} \quad = \quad B_1\delta_{31}e^{\mu t}+B_2\delta_{32}e^{-\theta_A t} \qquad \text{(A.4c)}$$

$$A(t)-\tilde{A} \quad = \quad (A_0-\tilde{A})e^{-\theta_A t}, \qquad \text{(A.4d)}$$

where $(1\ \kappa_{21}\ \kappa_{31}\ 0)'$, $(1\ \kappa_{22}\ \kappa_{32}\ (A_0-\tilde{A}))'$ are the normalized eigenvectors associated with the stable eigenvalues, $\mu, -\theta_A$. The arbitrary constants, B_1 and B_2, are obtained from initial conditions, given that the economy starts out with given initial stocks of capital, K_0 and technology level A_0. The system of equations (A.4) provides the basis for the analysis of the transitional dynamics. With B_1, B_2 determined by the initial conditions K_0 and A_0, the initial values of $c(0), q(0)$, following a jump are $c(0)=\tilde{c}+B_1\delta_{21}+B_2\delta_{22}$, $q(0)=\tilde{q}+B_1\delta_{31}+B_2\delta_{32}$.

A.2 Steady-State Distributions of physical and human capital

In the text we focus on the distribution of total wealth, v_i. In this Appendix, we indicate how we can derive the steady-state distributions of physical capital, $\tilde{k}_i$, and human capital, $\tilde{h}_i$. Beginning with

$$\tilde{V}_i = \tilde{K}_i + \tilde{q}H_i \tag{A.5a}$$

$$\tilde{V} = \tilde{K} + \tilde{q}H \tag{A.5b}$$

and dividing (A.5a) by (A.5b) yields

$$\tilde{v}_i = \frac{1}{\tilde{k} + \tilde{q}}\left[\tilde{k}\tilde{k}_i + \tilde{q}\tilde{h}_i\right] \tag{A.6}$$

Combining (12b) and (12b') we obtain the differential equation

$$\dot{h}_i = \frac{B\left(\varphi_Y\right)^{1-\varepsilon} H_Y}{H}\left[\frac{H_{Y,i}}{H_Y} - h_i\right] \tag{A.7}$$

which in steady state implies

$$\tilde{h}_i = \frac{\tilde{H}_{Y,i}}{\tilde{H}_Y} \tag{A.8}$$

Now consider capital accumulation equations (12a) and (12a'). We obtain

$$\frac{\dot{K}_i}{K_i} = A\left(\frac{K}{H_X}\right)^{\alpha-1}\left(\varphi_X\right)^{\eta}\left[\alpha + \beta\frac{K}{K_i}\frac{H_{X,i}}{H_X} + \eta\frac{K}{K_i}\frac{L_{X,i}}{L_X}\right] - \frac{\zeta C}{K_i}$$

$$\frac{\dot{K}}{K} = A\left(\frac{K}{H_X}\right)^{\alpha-1}\left(\varphi_X\right)^{\eta} - \frac{C}{K}$$

Using labor and human capital allocations, for individual and aggregate, we have the following

$$\frac{H_{X,i}}{H_X} = \frac{H_i - H_{Y,i}}{H_X} = \frac{H_i/H - (H_{Y,i}/H_Y)(H_Y/H)}{u} = \frac{h_i - (H_{Y,i}/H_Y)(1-u)}{u}$$

$$\frac{L_{X,i}}{L_X} = \frac{1-L_{Y,i}}{1-L_Y} = \frac{1-\varphi_Y(H_{Y,i}/H_Y)(H_Y/H)}{1-\varphi_Y(1-u)} = \frac{1-\varphi_Y(H_{Y,i}/H_Y)(1-u)}{1-\varphi_Y(1-u)}$$

Focusing on the steady state, when these two growth rates are equal implies

$$A\left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1}(\tilde{\varphi}_X)^{\eta}\left[\alpha + \frac{\beta}{\tilde{k}_i}\left(\frac{h_i - (H_{Y,i}/H_Y)(1-\tilde{u})}{\tilde{u}}\right) + \frac{\eta}{\tilde{k}_i}\left(\frac{1-\tilde{\varphi}_Y(H_{Y,i}/H_Y)(1-\tilde{u})}{1-\tilde{\varphi}_Y(1-\tilde{u})}\right)\right] - \frac{\zeta\tilde{c}}{\tilde{k}_i}$$

$$+= A\left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1}(\tilde{\varphi}_X)^{\eta}\left[\alpha+\beta+\gamma\right] - \tilde{c} \tag{A.9}$$

Using the steady-state condition (A.8) this equation simplifies to

$$A\left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1}(\tilde{\varphi}_X)^{\eta}\left[\beta(\tilde{h}_i - \tilde{k}_i) + \eta\left(\frac{1-\tilde{\varphi}_Y\tilde{h}_i(1-\tilde{u})}{1-\tilde{\varphi}_Y(1-\tilde{u})} - \tilde{k}_i\right)\right] = \tilde{c}(\zeta_i - \tilde{k}_i)$$

Thus given $\tilde{k}, \tilde{u}, \tilde{\varphi}_X, \tilde{\varphi}_Y, \tilde{v}_i, \tilde{c}, \tilde{\zeta}_i$ which have been determined previously, these two equations yield agent *i*'s steady state relative holdings of of human capital and physical capital, $\tilde{h}_i$ and $\tilde{k}_i$

$$\tilde{v}_i = \frac{1}{\tilde{k}+\tilde{q}}\left[\tilde{k}\tilde{k}_i + \tilde{q}\tilde{h}_i\right] \tag{A.10a}$$

$$A\left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1}(\tilde{\varphi}_X)^{\eta}\left[\beta(\tilde{h}_i - \tilde{k}_i) + \eta\left(\frac{1-\tilde{\varphi}_Y\tilde{h}_i(1-\tilde{u})}{1-\tilde{\varphi}_Y(1-\tilde{u})} - \tilde{k}_i\right)\right] = \tilde{c}(\zeta_i - \tilde{k}_i) \tag{A.10b}$$

A.3 Steady-State Distributions of physical and human capital under tax scenario

In this Appendix, we indicate how we can derive the steady-state distributions of physical capital, $\tilde{k}_i$, and human capital, $\tilde{h}_i$ under fiscal policy. Beginning with

$$\tilde{V}_i = \tilde{K}_i + \tilde{q}H_i \tag{A.11a}$$

$$\tilde{V} = \tilde{K} + \tilde{q}H \tag{A.11b}$$

and dividing (A.11a) by (A.11b) yields

$$\tilde{v}_i = \frac{1}{\tilde{k} + \tilde{q}}\left[\tilde{k}\tilde{k}_i + \tilde{q}\tilde{h}_i\right] \tag{A12}$$

Combining (12b) and (12b') we obtain the differential equation

$$\dot{h}_i = \frac{B\left(\varphi_Y\right)^{1-\varepsilon} H_Y}{H}\left[\frac{H_{Y,i}}{H_Y} - h_i\right] \tag{A.13}$$

which in steady state implies

$$\tilde{h}_i = \frac{\tilde{H}_{Y,i}}{\tilde{H}_Y} \tag{A.14}$$

Now consider capital accumulation equations (12a) and (12a'). We obtain

$$\frac{\dot{K}_i}{K_i} = A\left(\frac{K}{H_X}\right)^{\alpha-1}\left(\varphi_X\right)^{\eta}\left[\alpha(1-\tau_K) + \beta\frac{K}{K_i}\frac{H_{X,i}}{H_X}(1-\tau_H - s_H) + \eta\frac{K}{K_i}\frac{L_{X,i}}{L_X} + \beta\frac{K}{K_i}\frac{H_{X,i}}{H_X}\frac{H_{Y,i}}{H_{X,i}}(s_H)\right] - \frac{\zeta C}{K_i}$$

$$\frac{\dot{K}}{K} = A\left(\frac{K}{H_X}\right)^{\alpha-1}\left(\varphi_X\right)^{\eta} - \frac{C}{K}$$

Using labor and human capital allocations, for individual and aggregate as above, we have the following

$$\frac{H_{X,i}}{H_X} = \frac{H_i - H_{Y,i}}{H_X} = \frac{H_i/H - (H_{Y,i}/H_Y)(H_Y/H)}{u} = \frac{h_i - (H_{Y,i}/H_Y)(1-u)}{u}$$

$$\frac{L_{X,i}}{L_X} = \frac{1-L_{Y,i}}{1-L_Y} = \frac{1-\varphi_Y(H_{Y,i}/H_Y)(H_Y/H)}{1-\varphi_Y(1-u)} = \frac{1-\varphi_Y(H_{Y,i}/H_Y)(1-u)}{1-\varphi_Y(1-u)}$$

Focusing on the steady state, when these two growth rates are equal implies

$$A\left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1}(\tilde{\varphi}_X)^{\eta}\left[\begin{array}{l}\left[\left[\alpha(1-\tau_K)+\frac{\beta}{\tilde{k}_i}\left(\frac{h_i-(H_{Y,i}/H_Y)(1-\tilde{u})}{\tilde{u}}\right)(1-\tau_H-s_H)+\frac{\eta}{\tilde{k}_i}\left(\frac{1-\tilde{\varphi}_Y(H_{Y,i}/H_Y)(1-\tilde{u})}{1-\tilde{\varphi}_Y(1-\tilde{u})}\right)\right]\right] \\ +s_H\,\frac{\beta}{\tilde{k}_i}\left(\frac{h_i-(H_{Y,i}/H_Y)(1-\tilde{u})}{\tilde{u}}\right)\frac{(1-\tilde{u})}{\tilde{u}}\end{array}\right]$$

$$-\frac{\zeta\tilde{c}}{\tilde{k}_i} \quad = A\left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1}(\tilde{\varphi}_X)^{\eta}\left[\alpha+\beta+\eta\right]-\tilde{c}$$

(A.15)

Using the steady-state condition (A.14) this equation simplifies to

$$A\left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1}(\tilde{\varphi}_X)^{\eta}\left[\alpha[\tilde{k}_i(1-\tau_K)-1]+\beta(\tilde{h}_i(1-\tau_H-s_H)-\tilde{k}_i)+\eta\left(\frac{1-\tilde{\varphi}_Y\tilde{h}_i(1-\tilde{u})}{1-\tilde{\varphi}_Y(1-\tilde{u})}-\tilde{k}_i\right)+\beta s_H\,\frac{(1-\tilde{u})}{\tilde{u}}\tilde{h}_i\right]$$

$$=\tilde{c}(\zeta_i-\tilde{k}_i)$$

Thus given $\tilde{k},\tilde{u},\tilde{\varphi}_X,\tilde{\varphi}_Y,\tilde{v}_i,\tilde{c},\tilde{\zeta}_i$ which have been determined previously for given values of s_H,τ_H,τ_K, these two equations yield agent i's steady state relative holdings of of human capital and physical capital, $\tilde{h}_i$ and $\tilde{k}_i$

$$\tilde{v}_i = \frac{1}{\tilde{k}+\tilde{q}}\left[\tilde{k}\tilde{k}_i+\tilde{q}\tilde{h}_i\right] \tag{A.16a}$$

$$A\left(\frac{\tilde{k}}{\tilde{u}}\right)^{\alpha-1}(\tilde{\varphi}_X)^{\eta}\left[\alpha[\tilde{k}_i(1-\tau_K)-1]+\beta(\tilde{h}_i(1-\tau_H-s_H)-\tilde{k}_i)+\eta\left(\frac{1-\tilde{\varphi}_Y\tilde{h}_i(1-\tilde{u})}{1-\tilde{\varphi}_Y(1-\tilde{u})}-\tilde{k}_i\right)+\beta s_H\,\frac{(1-\tilde{u})}{\tilde{u}}\tilde{h}_i\right]$$

$$=\tilde{c}(\zeta_i-\tilde{k}_i)$$

(A.16b)

CURRICULUM VITAE

EDUCATION

Ph.D. (**Economics)**, University of Washington June 2012

- Macroeconomics, Growth and Development
- Econometrics/International Finance

M.B.A (PGDBM), Indian Institute of Management, Lucknow Mar 2002

- Finance

B.A. (Honors) Economics, Lady Sri Ram College, University of Delhi Jun 2000

DISSERTATION

Essays on Macroeconomic Growth: The Role of Human Capital

PROFESSIONAL EXPERIENCE

Consultant (Industry Division) Nov 2002- Nov 2004

National Council of Applied Economic Research [NCAER]

Project Manager, GENPACT LLC May 2002- Oct 2002

TEACHING

Teaching Assistant/ Instructor Winter 2007 - Winter 2011

FELLOWSHIPS & SCHOLARSHIPS

Graduate Student Scholarship 2006-2010

James O. York Scholarship Autumn 2006

Corkery Fellowship Spring 2011