On Cross Sections to the Horocycle and Geodesic Flows on Quotients by Hecke Triangle Groups

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Abstract

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The study of continuous dynamical systems via surfaces of section is one of the standard techniques in nonlinear mathematics. This is done by considering the intersections of trajectories in a phase space with a subspace of codimension one. The sought for goal is simplifying the study of the original dynamical system. In this manuscript thesis, we consider cross sections to the horocycle and geodesic flows on quotients of $\mathrm{SL}(2, \mathbb{R})$ by the Hecke triangle groups, and applications to Farey statistics and symbolic dynamics.

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Contents

The Boca-Cobeli-Zaharescu Map Analogue for the Hecke Triangle Groups G_q

Abstract

The Farey sequence $\mathcal{F}(Q)$ at level Q is the sequence of irreducible fractions in $[0,1]$ with denominators not exceeding Q, arranged in increasing order of magnitude. A simple "next-term" algorithm exists for generating the elements of $\mathcal{F}(Q)$ in increasing or decreasing order. That algorithm, along with a number of other properties of the Farey sequence, was encoded by F. Boca, C. Cobeli, and A. Zaharescu into what is now known as the Boca-Cobeli-Zaharescu (BCZ) map, and used to attack several problems that can be described using the statistics of subsets of the Farey sequence. In this paper, we derive the Boca-Cobeli-Zaharescu map analogue for the discrete orbits $\Lambda_q = G_q(1,0)^T$ of the linear action of the Hecke triangle groups G_q on the plane $\mathbb{R}^2$ starting with a Stern-Brocot tree analogue for the said orbits (theorem 2.2). We derive the next-term algorithm for generating the elements of Λ_q in vertical strips in increasing order of slope, and present a number of applications to the statistics of Λ_q.

Contents

1 Introduction

1 Introduction

For any integer $Q \geq 1$, the *Farey sequence at level* Q is the set

$$\mathcal{F}(Q) = \{a/q \mid a, q \in \mathbb{Z},\ 0 \leq a \leq q \leq Q,\ \gcd(a,q) = 1\}$$

of irreducible fractions in the interval $[0,1]$ with denominators not exceeding Q, arranged in increasing order. The Farey sequence is one of the famous enumerations of the rationals, and its applications permeate mathematics. Some of the fundamental properties of $\mathcal{F}(Q)$ are the following:

1. If $a_1/q_1, a_2/q_2 \in \mathcal{F}(Q)$ are two consecutive fractions, then $0 < q_1, q_2 \leq Q$, and $q_1 + q_2 \geq Q$.
2. If $a_1/q_1, a_2/q_2 \in \mathcal{F}(Q)$ are two consecutive fractions, then they satisfy the *Farey neighbor* identity $a_2 q_1 - a_1 q_2 = 1$.
3. If $a_1/q_1, a_2/q_2, a_3/q_3 \in \mathcal{F}(Q)$ are three consecutive fractions, then they satisfy the *next-term* identities
$$a_3 = k a_2 - a_1,$$
and
$$q_3 = k q_2 - q_1$$
where $k = \left\lfloor \frac{Q+q_1}{q_2} \right\rfloor$.

Around the turn of the new millennium, F. P. Boca, C. Cobeli, and A. Zaharescu [6] encoded the above properties of the Farey sequence as the *Farey triangle*

$$\mathscr{T} := \{(a,b) \mid 0 < a, b \leq 1,\ a+b > 1\},$$

and what is now increasingly known as the Boca-Cobeli-Zaharescu (BCZ) map $T : \mathscr{T} \to \mathscr{T}$[1]

$$T(a,b) := \left(b, -a + \left\lfloor \frac{1+a}{b} \right\rfloor b\right)$$

which satisfies the property that

$$T\left(\frac{q_1}{Q}, \frac{q_2}{Q}\right) = \left(\frac{q_2}{Q}, \frac{q_3}{Q}\right)$$

for any three consecutive fractions $a_1/q_1, a_2/q_2, a_3/q_3 \in \mathcal{F}(Q)$. Since then, quoting R. R. Hall, and P. Shiu [12], the aforementioned trio have "made some very interesting applications" of $\mathscr{T}$ and T (and the weak convergence of particular measures on $\mathscr{T}$

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[1] In the remainder of this paper, we will denote the BCZ maps we compute for the Hecke triangle groups G_q by BCZ$_q$, reserving the symbols T_q for particular generators of G_q.

supported on the orbits of T to the Lebesgue probability measure $dm = 2dadb$) to the study of distributions related to Farey fractions.

Earlier in the current decade, J. Athreya, and Y. Cheung [3] showed that the Farey triangle $\mathscr{T}$, the BCZ map $T : \mathscr{T} \to \mathscr{T}$, and the Lebesgue probability measure $dm = 2\,da\,db$ on $\mathscr{T}$ form a Poincaré section with roof function $R(a,b) = \frac{1}{ab}$ to the horocycle flow $h_s = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$, $s \in \mathbb{R}$, on $X_2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$, with (a scalar multiple of) the Haar probability measure μ_2 inherited from $\mathrm{SL}(2,\mathbb{R})$. Following that, analogues of the BCZ map have been computed for the golden L translation surface (whose $\mathrm{SL}(2,\mathbb{R})$ orbit corresponds to $\mathrm{SL}(2,\mathbb{R})/G_5$, where G_5 is the Hecke triangle group $(2,5,\infty)$) by J. Athreya, J. Chaika, and S. Lelievre[2] in [1], and later for the regular octagon by C. Uyanick, and G. Work in [18]. In both cases, the sought for application was determining the slope gap distributions for the holonomy vectors of the golden L and the regular octagon. Soon after, B. Heersink [14] computed the BCZ map analogues for finite covers of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ using a process developed by A. M. Fisher, and T. A. Schmidt [10] for lifting Poincaré sections of the geodesic flow on $\mathrm{SL}(2,\mathbb{R})/SL(2,\mathbb{Z})$ to covers of thereof. In that case, the sought for application was studying statistics of various subsets of the Farey sequence.

In this paper, we derive the BCZ map analogue for the Hecke triangle groups G_q, $q \geq 3$, which are the subgroups of $\mathrm{SL}(2,\mathbb{R})$ with generators

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } T_q := \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix},$$

where $\lambda_q := 2\cos\left(\frac{\pi}{q}\right) \geq 1$. Along the way, we investigate the discrete orbits

$$\Lambda_q = G_q(1,0)^T$$

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[2] By theorem 2.2, we get for $q = 5$ the indices $k_2(a,b) = \left\lfloor \frac{1-(a+\varphi b)}{\varphi^2(a+b)} \right\rfloor$, $k_3(a,b) = \left\lfloor \frac{1-b}{\varphi(a+\varphi b)} \right\rfloor$, and $k_4(a,b) = \left\lfloor \frac{1+a}{\varphi b} \right\rfloor$. This corrects the indices given in theorem 3.1 of [1].

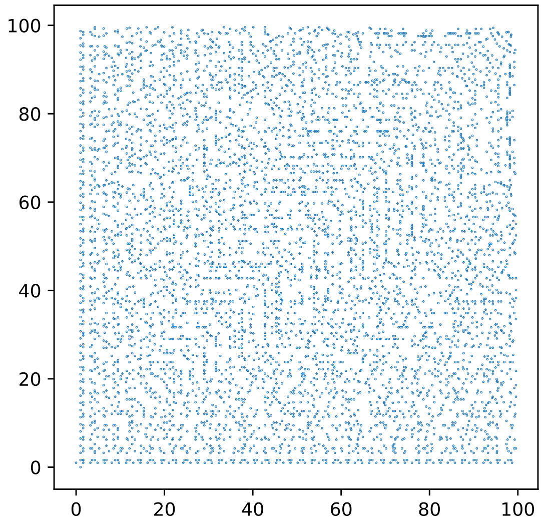

Figure 1: The elements of Λ_5 in the square $[0, 100]^2$ generated using theorem 2.3 and remark 2.1.

of the linear action of G_q on the plane $\mathbb{R}^2$, and present some results on the geometry of numbers, Diophantine properties, and statistics of Λ_q. Our starting point is showing that the orbits Λ_q have a tree structure that extend the famous Stern-Brocot trees for the rationals. The said trees were studied a bit earlier by C. L. Lang and M. L. Lang in [16], though their focus was on the Möbius action of G_q on the hyperbolic plane.

An earlier version of this paper was announced in October 2018 under the title "The Golden L Ford Circles", which only considered G_5 and its Ford circles. An excellent paper [8] by D. Davis and S. Lelievre that investigates the G_5-Stern-Brocot tree as a tool for studying the periodic paths on the pentagon, double pentagon, and golden L surfaces was announced at the same time. We strongly recommend the aforementioned paper as a more geometrically flavored application of the said trees.

1.1 Organization

This paper is organized as follows:

- In section 2, we characterize and study the discrete orbits of the linear action of G_q on the plane $\mathbb{R}^2$ (proposition 2.1), show that those discrete orbits have a tree structure analogous to the Stern-Brocot trees for the rationals (theorem 2.1), and derive the Boca-Cobeli-Zaharescu map analogues for G_q (theorem 2.2). We also characterize the periodic points for the G_q-BCZ map analogues (corollary 2.2), and present an algorithm for generating the elements of Λ_q in increasing order of slope (theorem 2.3). We also collect some consequences of the existence of G_q-Stern-Brocot trees that we use throughout the paper in corollary 2.1.

- In section 3, we give the Poincaré cross sections to the horocycle flow on the quotients $\mathrm{SL}(2,\mathbb{R})/G_q$ corresponding to the G_q BCZ map analogues we have in section 2 (theorem 3.1). As a consequence, we get an equidistribution result (theorem 3.2) that we use for the applications in section 4.
- In section 4, we present a number of applications of the results in this paper to the statistics of subsets of Λ_q. In particular, we give the main asymptotic term for the number of elements of Λ_q in homothetic dilations of triangles (proposition 4.1), equidistribution of homothetic dilations $\frac{1}{\tau}\Lambda_q$ in the square $[-1,1]^2$ as $\tau \to \infty$ (corollary 4.1), the slope gap distribution for the elements of Λ_q (corollary 4.2), and the distribution of the Euclidean distance between successive G_q-Ford circles (corollary 4.3). We also get a weak form of the Dirichelet approximation theorem for Λ_q for free (proposition 4.3).

1.2 Notation

As is customary when working with the groups G_q, we write

$$U_q := T_q S = \begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix U_q is conjugate to a rotation with angle π/q, and can be easily seen to preserve the quadratic form

$$Q_q((x,y)^T) = x^2 - \lambda_q xy + y^2$$

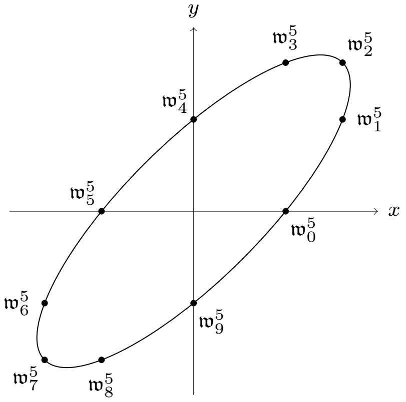

Figure 2: The vectors $\{\mathfrak{w}_i^5\}_{i=0}^9$ along with the ellipse $Q_5((x,y)^T) = x^2 - \lambda_5 xy + y^2 = 1$. Note that λ_5 is the golden ratio $\varphi = (1+\sqrt{5})/2$.

when U_q acts linearly on the plane $\mathbb{R}^2$.

The main object that we study in this paper is the orbit of the vector $(1,0)^T \in \mathbb{R}^2$ under the linear action of G_q on the plane

$$\Lambda_q = G_q(1,0)^T.$$

The set Λ_q is symmetric against the lines $y = \pm x$, $x = 0$, and $y = 0$ since G_q contains $S^3 T_q^{-1} S = T_q^T$, $S^3 = S^T$, and $(T_q S)^q = -\operatorname{Id}_2$.

Of special significance to us are the elements

$$\mathfrak{w}_i^q = (x_i^q, y_i^q) = U_q^i (1,0)^T,$$

where $i = 0, 1, \cdots, 2q-1$. Note that $\mathfrak{w}_0^q = (1,0)^T$, $\mathfrak{w}_1^q = (\lambda_q, 1)^T$, $\mathfrak{w}_{q-2}^q = (1, \lambda_q)^T$, $\mathfrak{w}_{q-1}^q = (0,1)^T$, and $\mathfrak{w}_q^q = (-1,0)^T$. (Since U_q is conjugate to a π/q-rotation, $U_q^q = -\operatorname{Id}_2$. This gives the last equality.) Moreover, the vectors $\{\mathfrak{w}_i^q\}_{i=0}^{2q-1}$ lie on the ellipse $Q_q((x,y)^T) = x^2 - \lambda_q xy + y^2 = 1$.

Given two vectors $\mathbf{u}_0 = (x_0, y_0)^T, \mathbf{u}_1 = (x_1, y_1)^T \in \mathbb{R}^2$, we denote their *(scalar) wedge product* by

$$\mathbf{u}_0 \wedge \mathbf{u}_1 = x_0 y_1 - x_1 y_0,$$

and their *dot product* by

$$\mathbf{u}_0 \cdot \mathbf{u}_1 = x_0 x_1 + y_0 y_1.$$

One useful inequality that we use more than once in this paper is that if $\mathbf{u}_0, \mathbf{u}_1, \mathbf{v}$ are non-zero vectors in $\mathbb{R}^2$, with the angle $\angle \mathbf{u}_0 \mathbf{u}_1$ not exceeding $\pi/2$, and $\mathbf{v}$ belonging to the sector $(0,\infty)\mathbf{u}_0 + (0,\infty)\mathbf{u}_1 = \{\alpha \mathbf{u}_0 + \beta \mathbf{u}_1 \mid \alpha, \beta > 0\}$, then

$$0 < \frac{\mathbf{u}_0 \wedge \mathbf{v}}{\|\mathbf{u}_0\|\|\mathbf{v}\|}, \frac{\mathbf{v} \wedge \mathbf{u}_1}{\|\mathbf{v}\|\|\mathbf{u}_1\|} < \frac{\mathbf{u}_0 \wedge \mathbf{u}_1}{\|\mathbf{u}_0\|\|\mathbf{u}_1\|}. \tag{1}$$

This follows from the identities $\mathbf{u}_0 \wedge \mathbf{v} = \|u_0\|\|\mathbf{v}\| \sin(\angle \mathbf{u}_0 \mathbf{v})$, $\mathbf{v} \wedge \mathbf{u}_1 = \|\mathbf{v}\|\|\mathbf{u}_1\| \sin(\angle \mathbf{v}\mathbf{u}_1)$, and $\mathbf{u}_0 \wedge \mathbf{u}_1 = \|\mathbf{u}_0\|\|\mathbf{u}_1\| \sin(\angle \mathbf{u}_0 \mathbf{u}_1)$, in addition to the inequalities $\sin(\angle \mathbf{u}_0 \mathbf{v}), \sin(\angle \mathbf{v}\mathbf{u}_1) < \sin(\angle \mathbf{u}_0 \mathbf{u}_1)$. Finally, we say that the two vectors $\mathbf{u}_0, \mathbf{u}_1 \in \mathbb{R}^2$ are *unimodular* if $\mathbf{u}_0 \wedge \mathbf{u}_1 = 1$. For readability, we sometimes will denote the usual product on $\mathbb{R}$ by $\times$. So $2 \times 3 = 6$, and so on.

Finally, we write

$$h_s := \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix},$$

for $s \in \mathbb{R}$,

$$s_\tau = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix},$$

for $\tau > 0$, and

$$g_{a,b} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$

for $a > 0$, and $b \in \mathbb{R}$. The above matrices satisfy the identities $g_{\tau,0} = s_\tau$, $h_s h_t = h_{s+t}$, and $h_s s_\tau = s_\tau h_{s\tau^2}$.

2 The Discrete Orbits, Stern-Brocot Trees, and Boca-Cobeli-Zaharescu Map Analogue for $\Lambda_q = G_q(1,0)^T$

2.1 The Discrete Orbits of the Linear Action of G_q on the Plane $\mathbb{R}^2$

Proposition 2.1. *The following are true.*

1. *If the orbit of* $\mathbf{u} \in \mathbb{R}^2$ *under the linear action of* G_q *is a discrete subset of* $\mathbb{R}^2$*, then either* $\mathbf{u} = (0,0)^T$*, or* $G_q\mathbf{u}$ *is a homothetic dilation of* $\Lambda_q = G_q(1,0)^T$*.*
2. *The ellipse* $Q_q((x,y)^T) = x^2 - \lambda_q xy + y^2 = 1$ *does not contain any elements of* Λ_q *in its interior.*
3. *The elements* $\mathfrak{w}_0^q, \mathfrak{w}_1^q, \cdots, \mathfrak{w}_{q-1}^q$ *of* Λ_q *satisfy the* Farey neighbor identities
$$\mathfrak{w}_i^q \wedge \mathfrak{w}_{i+1}^q = 1,$$
for $i = 0, 1, \cdots, q-2$*, in addition to*
$$\mathfrak{w}_0^q \wedge \mathfrak{w}_{q-1}^q = 1.$$
4. *If* $\mathbf{u}_0, \mathbf{u}_1 \in \Lambda_q$ *are two unimodular vectors (i.e* $\mathbf{u}_0 \wedge \mathbf{u}_1 = 1$*), then there exists* $A \in G_q$ *such that* $A\mathbf{u}_0 = \mathfrak{w}_0^q = (1,0)^T$ *and* $A\mathbf{u}_1 = \mathfrak{w}_{q-1}^q = (0,1)^T$*. That is, the pairs of unimodular vectors of* Λ_q *are in a one-to-one correspondence with the columns of the matrices in* G_q*.*

Proof. For the first claim: Assume without loss of generality that $\mathbf{u} \neq (0,0)^T$. Let $\Sigma_i^q = (0,\infty)\mathfrak{w}_i^q + [0,\infty)\mathfrak{w}_{i+1 \mod 2q}^q = \{\alpha\mathfrak{w}_i^q + \beta\mathfrak{w}_{i+1 \mod 2q}^q \mid \alpha, \beta > 0\}$, $i = 0, 1, \cdots, 2q-1$, be the radial sectors of $\mathbb{R}^2 \setminus \{(0,0)^T\}$ defined by the directions $\{\mathfrak{w}_i^q\}_{i=0}^{2q-1}$. Note that the matrix U_q bijectively maps each sector Σ_i^q to the sector $\Sigma_{i+1 \mod 2q}^q$ for $i = 0, 1, \cdots, 2q-$

1, and maintains the values of the quadratic form Q_q at each point. Also, T_q^{-1} maps the sector Σ_0^q to $\cup_{i=0}^{q-2}\Sigma_i = [0,\infty)(1,0)^T + (0,\infty)(1,0)^T$, decreasing the Q_q-values of all the points in the interior of Σ_0, and fixing all the points on the ray in the direction of $\mathfrak{w}_0^q = (1,0)^T$. (This follows from $T_q^{-1}\mathfrak{w}_0^q = \mathfrak{w}_0^q = (1,0)^T$, and $T_q^{-1}\mathfrak{w}_1^q = \mathfrak{w}_{q-1}^q = (0,1)^T$.) Starting with the vector $\mathbf{u}$ whose G_q-orbit is being considered, we repeatedly apply the following process:

1. If $\mathbf{u} \in \Sigma_i^q$ for some $1 \le i \le 2q-1$, then replace $\mathbf{u}$ with $U_q^{-i}\mathbf{u} \in \Sigma_0^q$. This maintains the Q_q-value of $\mathbf{u}$.
2. Replace $\mathbf{u} \in \Sigma_0^q$ with $T_q^{-1}\mathbf{u} \in \cup_{i=0}^{q-2}\Sigma_i^q$. This fixes $\mathbf{u}$ if it lies on the ray in the direction of $\mathfrak{w}_0^q = (1,0)^T$, and otherwise reduces the Q_q-value of $\mathbf{u}$.

After each iteration of this process, either the point $\mathbf{u}$ lands on the line $y = 0$ and is fixed by further applications of the process, or is mapped to another point in $G_q\mathbf{u}$ with a strictly smaller Q_q-value. By the discreteness of $G_q\mathbf{u}$, the point $\mathbf{u}$ will eventually land on the line $y = 0$. This implies that there exists a non-zero $\alpha \in \mathbb{R}$ such that $\mathbf{u} \in \alpha G_q(1,0)^T$, from which follows that $G_q\mathbf{u} = \alpha\Lambda_q$. This proves the first claim.

The second claim follows from the fact that $(1,0)^T \in \Lambda_q$ lies on the ellipse $Q_q((x,y)^T) = x^2 - \lambda_q xy + y^2 = 1$. No point in Λ_q can have a Q_q-value smaller than 1, as the iterative process used above will produce an element of Λ_q that is parallel to $(1,0)$ and shorter than it, which cannot happen by the discreteness of Λ_q.

For the third claim: we have for all $i = 0, 1, \cdots, q-2$ that

$$\mathfrak{w}_i^q \wedge \mathfrak{w}_{i+1}^q = (U_q^i\mathfrak{w}_0^q) \wedge (U_q^i\mathfrak{w}_1^q) = \det(U_q^i) \times \mathfrak{w}_0^q \wedge \mathfrak{w}_1^q = 1 \times (1,0)^T \wedge (\lambda_q, 1)^T = 1.$$

We also have that

$$\mathfrak{w}_0^q \wedge \mathfrak{w}_{q-1}^q = (1,0)^T \wedge (0,1)^T = 1.$$

For the fourth claim: By definition, there exists $B \in G_q$ such that $B\mathbf{u}_0 = (1,0)^T$.

Acting by B^{-1}, the two vectors $\widetilde{\mathbf{u}}_0 = B^{-1}\mathbf{u}_0 = (1,0)^T$ and $\widetilde{\mathbf{u}}_1 = B^{-1}\mathbf{u}_1$ satisfy

$$\widetilde{\mathbf{u}}_0 \wedge \widetilde{\mathbf{u}}_1 = \det(B^{-1}) \times \mathbf{u}_0 \wedge \mathbf{u}_1 = 1.$$

If $\widetilde{\mathbf{u}}_1 = (x, y)$, then $y = 1$. Shearing by T_q, we have $T_q^n\widetilde{\mathbf{u}}_0 = \widetilde{\mathbf{u}}_0$, and $T_q^n\widetilde{\mathbf{u}}_1 = (x + n\lambda_q, 1)^T$ for all $n \in \mathbb{Z}$. Since $\mathfrak{w}_1^q = (\lambda_q, 1)^T$ and $\mathfrak{w}_{q-1}^q = (0,1)^T$ are two elements of Λ_q on the ellipse $Q_q = 1$, are at height $y = 1$, are a horizontal distance λ_q away from each other, and $T_q^{-1}\mathfrak{w}_0^q = \mathfrak{w}_{q-1}^q$, then there exists $n_0 \in \mathbb{Z}$ such that $T_q^{n_0}\widetilde{\mathbf{u}}_1 = \mathfrak{w}_{q-1}^q$. Now, taking $A = T_q^{n_0}B^{-1}$ proves the claim. □

2.2 The Stern-Brocot Trees for $\Lambda_q = G_q(1,0)^T$

Definition 2.1. *We refer to the process of iteratively replacing a pair of vectors* $\mathbf{u}_0, \mathbf{u}_1 \in \Lambda_q$ *that are unimodular (i.e.* $\mathbf{u}_0 \wedge \mathbf{u}_1 = 1$*) with the vectors*

$$x_0^q\mathbf{u}_0 + y_0^q\mathbf{u}_1 = \mathbf{u}_0, x_1^q\mathbf{u}_0 + y_1^q\mathbf{u}_1, \cdots, x_{q-2}^q\mathbf{u}_0 + y_{q-2}^q\mathbf{u}_1, x_{q-1}^q\mathbf{u}_0 + y_{q-1}^q\mathbf{u}_1 = \mathbf{u}_1$$

as the G_q-Stern-Brocot process*. We refer to the vectors* $\{x_i^q\mathbf{u}_0 + y_i^q\mathbf{u}_1\}_{i=1}^{q-2}$ *as the* (G_q-Stern-Brocot) children *of* $\mathbf{u}_0, \mathbf{u}_1$*, and successive children of the children of* $\mathbf{u}_0, \mathbf{u}_1$ *as the* (G_q-Stern-Brocot) grandchildren *of* $\mathbf{u}_0, \mathbf{u}_1$*.*

Theorem 2.1. *Let* $\mathbf{u}_0, \mathbf{u}_1 \in \Lambda_q$ *be two unimodular vectors (i.e.* $\mathbf{u}_0 \wedge \mathbf{u}_1 = 1$*). The* G_q*-Stern-Brocot process applied to* $\mathbf{u}_0$ *and* $\mathbf{u}_1$ *generates a well-defined tree of elements of* Λ_q*, and exhausts the elements of* Λ_q *in the sector* $[0,\infty)\mathbf{u}_0 + [0,\infty)\mathbf{u}_1 = \{\alpha\mathbf{u}_0 + \beta\mathbf{u}_1 \mid \alpha, \beta \geq 0\}$*.*

Proof. That the Stern-Brocot process is well-defined for any two unimodular elements $\mathbf{u}_0$ and $\mathbf{u}_1$ of Λ_q follows from proposition 2.1. In particular, since $\mathbf{u}_0$ and $\mathbf{u}_1$ are unimodular, then there exists $A \in G_q$ whose columns are $\mathbf{u}_0$ and $\mathbf{u}_1$ (i.e. $A(1,0)^T = \mathbf{u}_0$ and $A(0,1)^T = \mathbf{u}_1$). The vectors $\mathfrak{w}_i^q = (x_i^q, y_i^q)^T = x_i^q(1,0)^T + y_i^q(0,1)^T$, with $i = 0, 1, \cdots, q-1$, are unimodular in pairs (by the Farey neighbor identities from proposition 2.1), and so their images $A\mathfrak{w}_i^q = x_i^q\mathbf{u}_0 + y_i^q\mathbf{u}_1$, $i = 0, 1, \cdots, q-1$, satisfy the same Farey neighbor identities, are all elements of Λ_q, and all belong to the sector $[0,\infty)\mathbf{u}_0 + [0,\infty)\mathbf{u}_1$. It remains to prove that the Stern-Brocot process is exhaustive, and our proof is similar to that of the classical proof for Farey fractions.

We first need to show that the wedge products of pairs of non-parallel elements of Λ_q are bounded away from zero.[3] Given two elements $\mathbf{w}_0, \mathbf{w}_1$ of Λ_q, we assume that if $0 < \mathbf{w}_0 \wedge \mathbf{w}_1 < \epsilon$, then ϵ cannot be arbitrarily small. Pick any $A \in G_q$ with $A\mathbf{u}_0 = (1,0)^T$. Writing $A\mathbf{u}_1 = (x,y)^T$, then $0 < A\mathbf{u}_0 \wedge A\mathbf{u}_1 = y < \epsilon$. Shearing by $T_q^{\pm} = \begin{pmatrix} 1 & \pm\lambda_q \\ 0 & 1 \end{pmatrix}$, we can find $n \in \mathbb{Z}$ such that $T_q^n A\mathbf{u}_1 = (x + n\lambda_q y, y)^T$ has an x-component $0 \le x + n\lambda_q y < \lambda_q \epsilon$. From this follows that $\|T_q^n A\mathbf{u}_1\| \le \epsilon\sqrt{1+\lambda_q^2}$, and so ϵ cannot be arbitrarily small by the discreteness of Λ_q. It thus follows that for all $q \ge 3$, there exists ϵ_q such that the wedge product of any non-parallel pair of elements of Λ_q is bounded below by ϵ_q in absolute value.

Now, we write $\mathbf{u}_0 = (q_0, a_0)^T$, and $\mathbf{u}_1 = (q_1, a_1)^T$, and assume that $\mathbf{u}_0, \mathbf{u}_1$ belong to the first quadrant. (We can safely do that by the last claim of proposition 2.1.) If $(x,y)^T \in \Lambda_q$ belongs to the sector $(0,\infty)\mathbf{u}_0 + (0,\infty)\mathbf{u}_1$, the orientation of the vectors gives $\mathbf{u}_0 \wedge (x,y)^T, (x,y)^T \wedge \mathbf{u}_1 > 0$, and so $\mathbf{u}_0 \wedge (x,y)^T, (x,y)^T \wedge \mathbf{u}_1 \ge \epsilon_q$. We define the component sum function $\varsigma : \mathbb{R}^2 \to \mathbb{R}$ by $\varsigma(r,s)^T = r + s$ for all $(r,s)^T \in \mathbb{R}^2$. We thus get

$$\begin{aligned}
\varsigma(\mathbf{u}_1)\left(\mathbf{u}_0 \wedge (x,y)^T\right) + \varsigma(\mathbf{u}_0)\left((x,y)^T \wedge \mathbf{u}_1\right) &= \begin{array}{l}(a_1+q_1)(yq_0 - xa_0) \\ \quad + (a_0+q_0)(a_1x - q_1y)\end{array} \\
&= (a_1q_0 - a_0q_1)(x+y) \\
&= \mathbf{u}_0 \wedge \mathbf{u}_1 \times \varsigma(x,y)^T \\
&= \varsigma(x,y)^T,
\end{aligned}$$

and so

$$\varsigma(x,y)^T \ge \epsilon_q \left(\varsigma(\mathbf{u}_0) + \varsigma(\mathbf{u}_1)\right). \tag{2}$$

---

[3] This can be trivially extended into a proof that the set of wedge products of the elements of Λ_q is discrete, similar to a characterization of lattice surfaces from [19].

Assuming without loss of generality that we are starting the Stern-Brocot process with $(1,0)^T$ and $(0,1)^T$, we have that the ς value of any vector that is generated at the nth step, $n \geq 0$, is bounded below by $n+1$. (We demonstrate this fact at the end of this proof.) At any step, if $(x,y)^T$ is not one of the $q-2$ Stern-Brocot children of $\mathbf{u}_0$ and $\mathbf{u}_1$, then it belongs to a sector defined by one of the $q-1$ pairs of successive unimodular vectors that have been generated at this step. This cannot take place forever as each step of Stern-Brocot increases the right hand side of eq. (2) by at least ϵ_q. This implies that $(x,y)^T$ eventually shows up as a child, and we are done.

Now we prove the lower bound on the ς value. If $\mathbf{c}$ is the G_q-Stern-Brocot child of two vectors $\mathbf{p}_1, \mathbf{p}_2$ in the first quadrant, then $\mathbf{c} = x^q_{i_0}\mathbf{p}_1 + y^q_{i_0}\mathbf{p}_2$ for some $1 \leq i_0 \leq q-2$, and so $\varsigma(\mathbf{c}) = x^q_{i_0}\varsigma(\mathbf{p}_1) + y^q_{i_0}\varsigma(\mathbf{p}_2) \geq \varsigma(\mathbf{p}_1) + \varsigma(\mathbf{p}_2)$, since $x^q_{i_0}, y^q_{i_0} \geq 1$. It is easy to see that each of the vectors that are generated at one stage must have at least one parent that was generated at the previous stage. Since $\varsigma((1,0)^T), \varsigma((0,1)^T) = 1$, it now follows by induction that the $\varsigma \geq n+1$ for all the vectors that are generated at the nth stage for $n \geq 0$. $\square$

In the following corollary, we collect some consequences of the existence of Stern-Brocot tree for Λ_q that we use in the remainder of this paper.

Corollary 2.1. *The following are true.*

1. *If* $\mathbf{v}_0, \mathbf{v}_1 \in \Lambda_q$ *are such that* $\mathbf{v}_0 \neq \pm\mathbf{v}_1$*, then* $|\mathbf{v}_0 \wedge \mathbf{v}_1| \geq 1$.
2. *Let* $\mathbf{v} \in \mathbb{R}^2 \setminus \{(0,0)^T\}$ *be an arbitrary non-zero vector in the plane. Then either* $\mathbf{v}$ *is parallel to a vector in* Λ_q*, or for any unimodular pair* $\mathbf{u}_0, \mathbf{u}_1 \in \Lambda_q$*, if* $\mathbf{v}$ *belongs to the sector* $(0,\infty)\mathbf{u}_0 + (0,\infty)\mathbf{u}_1$*, then there exists a pair of unimodular* G_q*-Stern-Brocot grandchildren* $\mathbf{w}_0, \mathbf{w}_1$ *of* $\mathbf{u}_0, \mathbf{u}_1$ *such that* $\mathbf{v}$ *belongs to the sector* $(0,\infty)\mathbf{w}_0 + (0,\infty)\mathbf{w}_1$*, and* $\mathbf{w}_0, \mathbf{w}_1$ *are different from* $\mathbf{u}_0, \mathbf{u}_1$.
3. *Let* $\mathbf{u}_0, \mathbf{u}_1 \in \Lambda_q$ *be two unimodular vectors, and* $\{\mathbf{w}_n\}_{n=1}^{\infty}$ *be any sequence of elements of* Λ_q *such that for each* $n \geq 1$, $\mathbf{w}_n$ *is generated at the* n*th iteration of the* G_q*-Stern-Brocot process applied to the two unimodular vectors* $\mathbf{u}_0, \mathbf{u}_1$*. Then* $\lim_{n\to\infty} \|\mathbf{w}_n\| = \infty$.

4. The slopes of the non-vertical vectors in Λ_q are dense in $\mathbb{R}$.

Proof. We first prove the following: If $\mathbf{u}_0, \mathbf{u}_1 \in \Lambda_q$ are unimodular (i.e. $\mathbf{u}_0 \wedge \mathbf{u}_1 = 1$), then after $n \geq 1$ applications of the G_q-Stern-Brocot process, the two vectors $\mathbf{w}^r_n = n\lambda_q \mathbf{u}_0 + \mathbf{u}_1 = (n\lambda_q, 1)^T$ and $\mathbf{w}^l_n = \mathbf{u}_0 + n\lambda_q \mathbf{u}_1 = (1, n\lambda_q)^T$ are G_q-Stern-Brocot grandchildren of $\mathbf{u}_0$ and $\mathbf{u}_1$, and all the grandchildren of $\mathbf{u}_0$ and $\mathbf{u}_1$ that have been generated by the nth step belong to the sector $[0,\infty)\mathbf{w}^r_n + [0,\infty)\mathbf{w}^l_n$. Now, since $(x^q_1, y^q_1)^T = \mathfrak{w}^q_1 = U_q(1,0)^T = (\lambda_q, 1)^T$, and $(x^q_{q-2}, y^q_{q-2})^T = \mathfrak{w}^q_{q-2} = U_q^{-1}(0,1)^T = (1, \lambda_q)^T$, it follows from theorem 2.1 that the two vectors $\mathbf{w}^r_1 = x^q_1 \mathbf{u}_0 + y^q_1 \mathbf{u}_1 = \lambda_q \mathbf{u}_0 + \mathbf{u}_1$ and $\mathbf{w}^l_1 = x^q_{q-2}\mathbf{u}_0 + y^q_{q-2}\mathbf{u}_1 = \mathbf{u}_0 + \lambda_q \mathbf{u}_1$ are Stern-Brocot children of $\mathbf{u}_0$ and $\mathbf{u}_1$, and that all the children of $\mathbf{u}_0$ and $\mathbf{u}_1$ that were generated after one iteration are contained in the sector corresponding to $\mathbf{w}^r_1$ and $\mathbf{w}^l_1$. The remainder of the claim follows by repeatedly applying the Stern-Brocot process to the unimodular pair $\mathbf{u}_0$ and $\mathbf{w}^r_n$, and the unimodular pair $\mathbf{w}^l_n$ and $\mathbf{u}_1$, for all $n \geq 2$.

For the first claim: Since $-\mathrm{Id}_2 = U_q^q$ is in G_q, we can assume that the angle between $\mathbf{v}_0$ and $\mathbf{v}_1$ does not exceed $\pi/2$. We also permute $\mathbf{v}_0$ and $\mathbf{v}_1$ if need be so that $\mathbf{v}_0 \wedge \mathbf{v}_1 > 0$. Furthermore, we can assume that $\mathbf{v}_0 = (1,0)^T$. (There exists $A \in G_q$ such that $\mathbf{v}_0 = A(1,0)^T$, and so we can replace $\mathbf{v}_0$ and $\mathbf{v}_1$ with $\widetilde{\mathbf{v}}_0 = A^{-1}\mathbf{v}_0$ and $\widetilde{\mathbf{v}}_1 = A^{-1}\mathbf{v}_1$, and preserve the wedge product $\widetilde{\mathbf{v}}_0 \wedge \widetilde{\mathbf{v}}_1 = \det(A^{-1}) \times \mathbf{v}_0 \wedge \mathbf{v}_1 = \mathbf{v}_0 \wedge \mathbf{v}_1$.) We now have that $\mathbf{v}_0 = (1,0)^T$, and that $\mathbf{v}_1$ is in the first quadrant. That is, $\mathbf{v}_1$ is either $\mathfrak{w}^q_{q-1} = (0,1)^T$, or a G_q-Stern-Brocot grandchild of $\mathfrak{w}^q_0 = (1,0)^T$ and $\mathfrak{w}^q_{q-1} = (0,1)^T$. Writing $\mathbf{v}_1 = (x_{\mathbf{v}_1}, y_{\mathbf{v}_1})^T$, we have $\mathbf{v}_0 \wedge \mathbf{v}_1 = y_{\mathbf{v}_1}$. The y components of the vectors $\{\mathbf{w}^r_n\}_{n=1}^{\infty}$ from the first claim in the corollary all are all $y = 1$, and so $y_{\mathbf{v}_1} = \mathbf{v}_0 \wedge \mathbf{v}_1 \geq 1$ as required.

For the second claim: The unit vectors in the directions of $\{\mathbf{w}^r_n\}_{n=1}^{\infty}$ and $\{\mathbf{w}^l_n\}_{n=1}^{\infty}$ converge to $\mathbf{u}_0$ and $\mathbf{u}_1$ as $n \to \infty$. As such, if the vector $\mathbf{v}$ is not in Λ_q, then it will eventually be contained in the sector bounded by $\mathbf{w}^l_{n_0}$ and $\mathbf{w}^l_{n_0}$ for some $n_0 \geq 1$, and consequently belongs to the sector bounded by a pair of unimodular grandchildren of $\mathbf{u}_0$ and $\mathbf{u}_1$.

For the third claim: By the fourth claim in proposition 2.1, and the boundedness of the elements of G_q as linear operators on $\mathbb{R}^2$, we can assume without loss of generality that $\mathbf{u}_0 = (1,0)^T$ and $\mathbf{u}_1 = (0,1)^T$. At the end of the proof of theorem 2.1, we showed that if $\mathbf{w}_n$ is generated at the nth stage of the G_q-Stern-Brocot process applies to $(1,0)^T$ and $(0,1)^T$, then $\mathbf{w}_n \geq n+1$. If $\mathbf{w}_n = (r,s)^T$, then $\varsigma(\mathbf{w}_n) = r + s \leq \sqrt{2}\sqrt{r^2+s^2} \leq \sqrt{2}\|\mathbf{w}_n\|$, which proves the claim.

For the fourth claim: It suffices to show that if $\alpha \geq 0$ is not the slope of a vector in Λ_q, then α can be approximated by slopes of vectors in Λ_q. Writing $\mathbf{v} = (1,\alpha)^T$, we note that if $\mathbf{u}_0, \mathbf{u}_1 \in \Lambda_q$ are two unimodular vectors in the first quadrant whose sector contains $\mathbf{v}$, then by eq. (1) we have

$$0 < \mathbf{u}_0 \wedge \mathbf{v} < \frac{\|\mathbf{v}\|}{\|\mathbf{u}_1\|}\mathbf{u}_0 \wedge \mathbf{u}_1 = \mathbf{v} < \frac{\|\mathbf{v}\|}{\|\mathbf{u}_1\|}.$$

Writing $\mathbf{u}_0 = (x,y)^T$, and assuming that $x > 0$, we thus get

$$0 \leq \alpha - \frac{y}{x} \leq \frac{\sqrt{1+\alpha^2}}{x\|\mathbf{u}_1\|}. \tag{3}$$

Now, we can start with $\mathbf{u}_0 = (1,0)^T$ and $\mathbf{u}_1 = (0,1)^T$ as two vectors in the first quadrant whose sector contains $\mathbf{v}$, and by the second claim in this corollary, we can repeatedly replace $\mathbf{u}_0$ and $\mathbf{u}_1$ with unimodular pairs that are generated at later stages of the Stern-Brocot process. In eq. (3), $x \geq 1$, and $\lim_{n\to\infty} \|\mathbf{u}_1\| = \infty$, and we are done. □

2.3 The Boca-Cobeli-Zaharescu Map Analogue for $\Lambda_q = G_q(1,0)^T$

In the following theorem, we present the BCZ map analogue for Λ_q. In essence, this theorem along with the next-term algorithm (theorem 2.3) extend the properties of the Farey sequence alluded to in the introduction using the BCZ map formalization.

Theorem 2.2. *The following are true.*

1. *For any $A \in \mathrm{SL}(2,\mathbb{R})$, if $A\Lambda_q$ has a horizontal vector of length not exceeding* 1

(i.e. a horizontal vector in $A\Lambda_q \cap S_1$*), then* $A\Lambda_q$ *can be uniquely identified with a point* (a_A, b_A) *in the* G_q-Farey triangle

$$\mathscr{T}^q = \{(a,b) \in \mathbb{R}^2 \mid 0 < a \leq 1,\ 1 - \lambda_q a < b \leq 1\}$$

through $B\Lambda_q = g_{a_A,b_A}\Lambda_q$*. Moreover, the value* a_A *agrees with the length of the horizontal vector in* $A\Lambda_q \cap S_1$*.*

2. *Let* $(a,b) \in \mathscr{T}^q$ *be any point in the* G_q*-Farey triangle. The set* $g_{a,b}\Lambda_q \cap S_1$ *has a vector with smallest positive slope. Consequently, there exists a smallest* $s = R_q(a,b) > 0$ *such that* $h_s g_{a,b}\Lambda_q$ *has a horizontal vector of length not exceeding* 1*, and hence* $h_s g_{a,b}\Lambda_q$ *corresponds to a unique point* $\mathrm{BCZ}_q(a,b) \in \mathscr{T}^q$ *in the* G_q*-Farey triangle. The function* $R_q : \mathscr{T}^q \to \mathbb{R}_+$ *is referred to as the* G_q-roof function*, and the map* $\mathrm{BCZ}_q(a,b) : \mathscr{T}^q \to \mathscr{T}^q$ *is referred to as the* G_q-BCZ map*.*

3. *The* G_q*-Farey triangle* $\mathscr{T}^q$ *can be partitioned into the union of*

$$\mathscr{T}_i^q := \{(a,b) \in \mathscr{T}^q \mid (a,b)^T \cdot \mathfrak{w}_{i-1} > 1,\ (a,b)^T \cdot \mathfrak{w}_i \leq 1\},$$

with $i = 2, 3, \cdots, q-1$*, such that if* $(a,b) \in \mathscr{T}_i^q$*, then* $g_{a,b}\mathfrak{w}_i^q$ *is the vector of least positive slope in* $g_{a,b}\Lambda_q \cap S_1$*, and*

   - *the value of the roof function* $R_q(a,b)$ *is given by*

$$R_q(a,b) = \frac{y_i^q}{a \times (a,b)^T \cdot \mathfrak{w}_i^q}, \text{ and}$$

   - *the value of the BCZ map* $\mathrm{BCZ}_q(a,b)$ *is given by*

$$\mathrm{BCZ}_q(a,b) = \left((a,b)^T \cdot \mathfrak{w}_i^q, (a,b)^T \cdot \mathfrak{w}_{i+1}^q + k_i^q(a,b) \times \lambda_q \times (a,b)^T \cdot \mathfrak{w}_i^q\right),$$

   where the G_q-index $k_i^q(a,b)$ *is given by*

$$k_i^q(a,b) = \left\lfloor \frac{1 - (a,b)^T \cdot \mathfrak{w}_{i+1}^q}{\lambda_q \times (a,b)^T \cdot \mathfrak{w}_i^q} \right\rfloor.$$

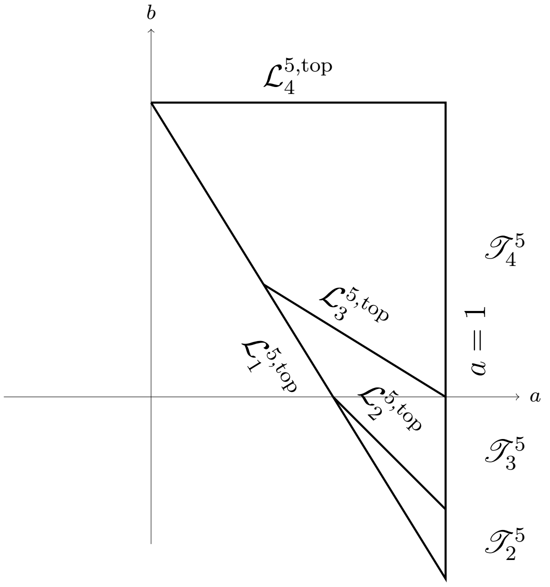

Figure 3: The G_5-Farey triangle $\mathscr{T}^5$ with the subregions $\mathscr{T}_2^5$, $\mathscr{T}_3^5$, and $\mathscr{T}_4^5$ from theorem 2.2 indicated. The figure also shows the lines $\mathcal{L}_i^{5,\mathrm{top}} = \{(a,b) \in \mathscr{T}^5 \mid (a,b)^T \cdot \mathfrak{w}_i^q = 1\}$, $i = 1,2,3,4$, that bound the aforementioned subregions in the proof of theorem 2.2.

We first need the following lemma.

Lemma 2.1. *Given $A \in \mathrm{SL}(2,\mathbb{R})$, if $A\Lambda_q$ contains both $(1,0)^T$ and $(0,1)^T$, then $A\Lambda_q = \Lambda_q$. In particular, the following are true.*

1. *For any $B \in \mathrm{SL}(2,\mathbb{R})$, $B\Lambda_q = \Lambda_q$ if and only if $B \in G_q$. From this follows that the sets $C\Lambda_q$, with C varying over $\mathrm{SL}(2,\mathbb{R})$, can be identified with the elements of $\mathrm{SL}(2,\mathbb{R})/G_q$.*
2. *For any $B \in \mathrm{SL}(2,\mathbb{R})$, if $B\Lambda_q$ contains a horizontal vector of length $a > 0$, then there exists $b \in \mathbb{R}$ such that $B\Lambda_q = g_{a,b}\Lambda_q$.*

Proof. We first prove the main claim. Let $A \in \mathrm{SL}(2,\mathbb{R})$ be such that $A\Lambda_q$ contains both $(1,0)^T$ and $(0,1)^T$. Then there exists $\mathbf{u}_0, \mathbf{u}_1 \in \Lambda_q$ such that $\mathbf{u}_0 = A^{-1}(1,0)^T$ and $\mathbf{u}_1 = A^{-1}(0,1)^T$, and $\mathbf{u}_0 \wedge \mathbf{u}_1 = \det(A^{-1}) \times (1,0)^T \times (0,1)^T = 1$. The columns of A^{-1} thus form a unimodular pair of elements of Λ_q, and so by the last claim of proposition 2.1, the matrix A^{-1}, and by necessity A, belong to the group G_q.

The first claim now follows from the fact that if $B \in \mathrm{SL}(2,\mathbb{R})$ is such that $B\Lambda_q = \Lambda_q$, then $B\Lambda_q$ contains both $(1,0)^T$ and $(0,1)^T$.

We now prove the second claim. Let $\mathbf{u}_0 \in \Lambda_q$ be such that $B\mathbf{u}_0 = (a,0)^T \in B\Lambda_q$ is parallel to the horizontal vector in question. If $A \in \mathrm{SL}(2,\mathbb{R})$ is such that $\mathbf{u}_0 = A(1,0)^T$, then $\mathbf{u}_1 = A(0,1)^T$ is an element of Λ_q with $\mathbf{u}_0 \wedge \mathbf{u}_1 = 1$. Writing $\widetilde{\mathbf{u}}_0 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} B\mathbf{u}_0 = (1,0)^T$, and $\widetilde{\mathbf{u}}_1 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} B\mathbf{u}_1 = (x,y)^T$, we have that $\widetilde{\mathbf{u}}_0 \wedge \widetilde{\mathbf{u}}_1 = 1$, and so $y = 1$. Shearing by $T_{-x} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$, we have that $T_{-x}\widetilde{\mathbf{u}}_0 = (1,0)^T$, and $T_{-x}\widetilde{\mathbf{u}}_1 = (0,1)^T$. That is, the set $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} B\Lambda_q$ contains both $(1,0)^T$, and $(0,1)^T$, and so is equal to Λ_q. From this follows that

$$\begin{aligned} B\Lambda_q &= \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}^{-1} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}^{-1} \Lambda_q \\ &= \begin{pmatrix} a & ax \\ 0 & a^{-1} \end{pmatrix} \Lambda_q \\ &= g_{a,ax}\Lambda_q, \end{aligned}$$

and taking $b = ax$ proves the claim. $\square$

We now proceed to prove theorem 2.2.

Proof. We first derive the explicit values of the roof function $R_q(a,b)$ and BCZ map $\mathrm{BCZ}_q(a,b)$ in the second half of the third claim for a given point $(a,b) \in \mathscr{T}_i^q$, $i = 2, 3, \cdots, q-1$, assuming the remainder of the theorem, and then resume the proof of the theorem from the beginning.

If $(a,b) \in \mathscr{T}_i^q$, with $2 \leq i \leq q-1$, then $g_{a,b}\mathfrak{w}_i^q$ has the smallest positive slope in

$g_{a,b}\Lambda_q \cap S_1$ by our (yet to be proven) assumption. This gives

$$\begin{aligned}
R_q(a,b) &= \operatorname{slope}\left(g_{a,b}\mathfrak{w}_i^q\right) \\
&= \operatorname{slope}\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} x_i^q \\ y_i^q \end{pmatrix}\right) \\
&= \frac{y_i^q}{a(ax_i^q + by_i^q)} \\
&= \frac{y_i^q}{a \times (a,b)^T \cdot \mathfrak{w}_i^q}.
\end{aligned}$$

Now, let $[\mathfrak{w}_i^q \ \mathfrak{w}_{i+1}^q]$ be the matrix whose columns are $\mathfrak{w}_i^q$ and $\mathfrak{w}_{i+1}^q$. The matrix $[\mathfrak{w}_i^q \ \mathfrak{w}_{i+1}^q]$ is in G_q by proposition 2.1 since its columns are two unimodular elements of Λ_q. We show that $h_{R_q(a,b)}g_{a,b}[\mathfrak{w}_i^q \ \mathfrak{w}_{i+1}^q] = g_{(a,b)^T\cdot\mathfrak{w}_i^q,(a,b)^T\cdot\mathfrak{w}_{i+1}^q}$, and follow that by finding the representative of $g_{(a,b)^T\cdot\mathfrak{w}_i^q,(a,b)^T\cdot\mathfrak{w}_{i+1}^q}\Lambda_q$ in $\mathscr{F}^q$. (Note that $h_{R_q(a,b)}g_{a,b}\Lambda_q = g_{(a,b)^T\cdot\mathfrak{w}_i^q,(a,b)^T\cdot\mathfrak{w}_{i+1}^q}\Lambda_q$ by lemma 2.1.) Keeping the Farey neighbor identity $\mathfrak{w}_i^q \wedge \mathfrak{w}_{i+1}^q = x_i^q y_{i+1}^q - x_{i+1}^q y_i^q = 1$ in mind, we have

$$\begin{aligned}
h_{R_q(a,b)}g_{a,b}[\mathfrak{w}_i^q \ \mathfrak{w}_{i+1}^q] &= \begin{pmatrix} 1 & 0 \\ -\frac{y_i^q}{a(ax_i^q+by_i^q)} & 1 \end{pmatrix}\begin{pmatrix} ax_i^q + by_i^q & ax_{i+1}^q + by_{i+1}^q \\ a^{-1}y_i^q & a^{-1}y_{i+1}^q \end{pmatrix} \\
&= \begin{pmatrix} ax_i^q + by_i^q & ax_{i+1}^q + by_{i+1}^q \\ 0 & \frac{1}{ax_i^q+by_i^q} \end{pmatrix} \\
&= g_{ax_i^q+by_i^q,\,ax_{i+1}^q+by_{i+1}^q} \\
&= g_{(a,b)^T\cdot\mathfrak{w}_i^q,(a,b)^T\cdot\mathfrak{w}_{i+1}^q}.
\end{aligned}$$

Write $\alpha = (a,b)^T \cdot \mathfrak{w}_i^q$, and $\beta = (a,b)^T \cdot \mathfrak{w}_{i+1}^a$. Since $T_q = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}$ is in G_q, and $g_{\alpha,\beta}T_q^k = g_{\alpha,\beta+k\lambda_q\alpha}$ for all $k \in \mathbb{Z}$, then $g_{\alpha,\beta}\Lambda_q = g_{\alpha,\beta+k\lambda_q\alpha}\Lambda$ for all $k \in \mathbb{Z}$ by lemma 2.1. Taking $k_0 = \left\lfloor \frac{1-\beta}{\lambda_q\alpha} \right\rfloor$, we get $1 - \lambda_q\alpha < \beta + k_0\lambda_q\alpha \leq 1$. We will also see in a bit that $0 < \alpha \leq 1$ (which is equivalent to (a,b) lying between the lines $\mathscr{L}_i^{q,\mathrm{bot}}$ and $\mathscr{L}_i^{q,\mathrm{top}}$ that we will be working with for the remainder of the proof). We thus have $h_{R_q(a,b)}g_{a,b}\Lambda_q =$

$g_{\alpha,\beta+k_0\lambda_q\alpha}\Lambda_q$, with $(\alpha, \beta + k_0\lambda_q\alpha) \in \mathscr{T}^q$, and so

$$\begin{aligned}\mathrm{BCZ}_q(a,b) &= (\alpha, \beta + k_0\lambda_q\alpha) \\ &= \left((a,b)^T \cdot \mathfrak{w}_i^q, (a,b)^T \cdot \mathfrak{w}_{i+1}^q + k_i^q(a,b) \times \lambda_q \times (a,b)^T \cdot \mathfrak{w}_i^q\right)\end{aligned}$$

as required.

Now, for the first claim of the theorem: If $B\Lambda_q$ has a horizontal vector of length $a \in (0,1]$, then there exists $b \in \mathbb{R}$ such that $B\Lambda_q = g_{a,b}\Lambda_q$ by lemma 2.1. Since $T_q \in G_q$, and $g_{a,b}T_q^{\pm 1} = g_{a,b\pm\lambda_q a}$, then $B\Lambda_q = g_{a,b}\Lambda_q = g_{a,b+n\lambda_q a}\Lambda_q$ for all $n \in \mathbb{Z}$. From this follows that $B\Lambda_q = g_{(a_B,b_B)}\Lambda_q$, with $(a_B, b_B) = \left(a, b + \left\lfloor \frac{1-b}{\lambda_q a} \right\rfloor \lambda_q a\right) \in \mathscr{T}^q$ as required. It now remains to show that this identification is unique. That is, given $(a,b), (c,d) \in \mathscr{T}^q$, if $g_{a,b}\Lambda_q = g_{c,d}\Lambda_q$, then $(a,b) = (c,d)$. Now,

$$g_{c,d}^{-1}g_{a,b} = \begin{pmatrix} a/c & b/c - d/a \\ 0 & c/a \end{pmatrix} \in G_q.$$

By the identification in proposition 2.1, we thus have $(a/c, 0)^T = g_{c,d}^{-1}g_{a,b}(1,0)^T \in \Lambda_q$, and so $a/c = \pm 1$, from which $a = c$. We also have $(b/c - d/a, 1)^T = g_{c,d}^{-1}g_{a,b}(0,1)^T \in \Lambda_q$. It can be easily seen from the second claim in proposition 2.1 that all the points in Λ_q at height $y = 1$ are of the form $(n\lambda_q, 1)^T = T_q^n(0,1)^T$ with $n \in \mathbb{Z}$, and so $b/c - d/a = n\lambda_q$ for some $n_0 \in \mathbb{Z}$. That is, $b - d = n_0\lambda_q a$. At the same time, $b, d \in (1 - \lambda_q a, 1]$, and so, since $a > 0$, we get that $b - d \in (-\lambda_q, \lambda_q)$. It now follows that $n_0 = 0$, and $b = d$.

Finally, for the second claim, and the beginning of the third claim of the theorem, we consider the lines

$$\mathcal{L}_i^{q,\mathrm{bot}} := \{(a,b) \in \mathbb{R}^2 \mid (a,b)^T \cdot \mathfrak{w}_i^q = 0\},$$

and

$$\mathcal{L}_i^{q,\mathrm{top}} := \{(a,b) \in \mathbb{R}^2 \mid (a,b)^T \cdot \mathfrak{w}_i^q = 1\}$$

for $i = 1, 2, \cdots, q-1$. Note that the lines $\mathcal{L}_1^{q,\mathrm{top}}$ and $\mathcal{L}_{q-1}^{q,\mathrm{top}}$ agree with the sides

$\lambda_q a + b = 1$ and $b = 1$ of $\mathscr{T}^q$. We now show that for $i = 2, 3, \cdots, q-1$, if $(a, b) \in \mathscr{T}^q$ is in $\mathscr{T}_i^q$ (i.e. above the line $\mathcal{L}_{i-1}^{q,\mathrm{top}}$ and below, or on the line $\mathcal{L}_i^{q,\mathrm{top}}$), then $g_{a,b}\mathfrak{w}_i^q$ belongs to the strip S_1, and has the smallest positive slope among the elements of $g_{a,b}\Lambda_q \cap S_1$. For any $i = 2, 3, \cdots, q-1$, if $(a, b) \in \mathscr{T}^q$ lies in the region above the line $\mathcal{L}_i^{q,\mathrm{bot}}$, and below or on the line $\mathcal{L}_i^{q,\mathrm{top}}$, then the x-component $(a, b)^T \cdot \mathfrak{w}_i^q$ of $g_{a,b}\mathfrak{w}_i^q$ satisfies $0 < (a, b)^T \cdot \mathfrak{w}_i^q \leq 1$, and so $g_{a,b}\mathfrak{w}_i^q$ belongs to $g_{a,b}\Lambda_q \cap S_1$. As we will see in a bit, the regions $\mathscr{T}_i^q$, $i = 2, 3, \cdots, q-1$, cover $\mathscr{T}^q$, and so $g_{a,b}\Lambda_q \cap S_1 \neq \emptyset$ for all $(a, b) \in \mathscr{T}^q$. Moreover, for any $(a, b) \in \mathscr{T}^q$, the elements of $g_{a,b}\Lambda_q \cap S_1$ do not accumulate by the discreteness of Λ_q, and so there must exist an element of $g_{a,b}\Lambda_q \cap S_1$ with smallest positive slope. This proves the second claim.

Finally, we prove that the regions in question cover the triangle $\mathscr{T}^q$, along with the first half of the third claim of the theorem. I.e., that for $i = 2, 3, \cdots, q-1$, if $(a, b) \in \mathscr{T}^q$, then $g_{a,b}\mathfrak{w}_i^q$ has the smallest positive slope in $g_{a,b}\Lambda_q$. We break this down into three steps:

1. For $i = 1, 2, \cdots, q-1$, the line segments $\mathcal{L}_i^{q,\mathrm{top}} \cap \mathscr{T}^q$ lie above each other, and have increasing (non-positive) slopes. (That is, if $1 \leq i_1 < i_2 \leq q-1$, then the line segment $\mathcal{L}_{i_1}^{q,\mathrm{top}} \cap \mathscr{T}^q$ lies below the line segment $\mathcal{L}_{i_2}^{q,\mathrm{top}} \cap \mathscr{T}^q$, and $\mathrm{slope}(\mathcal{L}_{i_2}^{q,\mathrm{top}}) > \mathrm{slope}(\mathcal{L}_{i_1}^{q,\mathrm{top}})$.)

2. For each $i = 2, 3, \cdots, q-1$, the line segment $\mathcal{L}_i^{q,\mathrm{bot}} \cap \mathscr{T}^q$ lies below the line segment $\mathcal{L}_{i-1}^{q,\mathrm{top}}$. (This proves the claim that the regions $\mathscr{T}_i^q$, $i = 2, 3, \cdots, q-1$, cover $\mathscr{T}^q$.)

3. For $i = 1, 2, \cdots, q-2$, if $(a, b) \in \mathscr{T}^q$ lies above the line $\mathcal{L}_i^{q,\mathrm{top}}$, the the $g_{a,b}$ images of $\mathfrak{w}_i^q$ along with its G_q-Stern-Brocot children with $\mathfrak{w}_{i+1}^q$ have x-components that exceed 1, and so are not in $g_{a,b}\Lambda_q \cap S_1$.

The third step follows immediately from the fact that the G_q-Stern-Brocot children of $\mathfrak{w}_i^q$ and $\mathfrak{w}_{i+1}^q$, $i = 1, 2, \cdots, q-2$, are all linear combinations of $\mathfrak{w}_i^q$ and $\mathfrak{w}_{i+1}^q$ with coefficients that are at least 1. It thus remains to prove the first two steps.

For the first step: The lines $\mathcal{L}_i^{q,\mathrm{top}}$, $i = 2, 3, \cdots, q-1$, intersect the right side $a = 1$

of the Farey triangle $\mathscr{T}^q$ at $(1, b_i)^T$, where $b_i = \frac{1-x_i}{y_i}$ (recall that $\mathfrak{w}_i^q = (x_i, y_i)^T$). It is easy to see that the heights b_i increase as i increases. (For instance, by acting on the vectors $\{\mathfrak{w}_i^q\}_{i=2}^{q-1}$, which go around the ellipse $Q_q((x,y)^T) = x^2 - \lambda_q xy + y^2 = 1$, by the linear function $T : (x,y)^T \mapsto (1-x, y)^T$, and considering the inverse slopes of the images.) It now suffices to show that for $i = 2, 3, \cdots, q-2$, the lines $\mathcal{L}_i^{q,\mathrm{top}}$ and $\mathcal{L}_{i+1}^{q,\mathrm{top}}$ intersect below on the left side $\lambda_q a + b = 1$ of the triangle $\mathscr{T}^q$ to show that the segment $\mathcal{L}_{i+1}^{q,\mathrm{top}} \cap \mathscr{T}^q$ lies entirely above the segment $\mathcal{L}_i^{q,\mathrm{top}} \cap \mathscr{T}^q$, and that the former has a bigger slope than the latter. (Recall that the side $\lambda_q a + b = 1$ does *not* belong to the set $\mathscr{T}^q$.) To find the sought for intersection, we solve the simultaneous system of equations $(a_0, b_0) \cdot \mathfrak{w}_i^q = 1$ and $(a_0, b_0) \cdot \mathfrak{w}_{i+1}^q = 1$, or equivalently $\begin{pmatrix} x_i^q & y_i^q \\ x_{i+1}^q & y_{i+1}^q \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = 1$, for $(a_0, b_0)^T$. Since $\mathfrak{w}_i^q \wedge \mathfrak{w}_{i+1}^q = x_i^q y_{i+1}^q - x_{i+1}^q y_i^q = 1$, we have $\begin{pmatrix} a_o \\ b_0 \end{pmatrix} = \begin{pmatrix} y_{i+1}^q & -y_i^q \\ -x_{i+1}^q & x_i^q \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} y_{i+1}^q - y_i^q \\ -x_{i+1}^q + x_i^q \end{pmatrix}$. Recalling that $(x_{i+1}^q, y_{i+1}^q)^T = \mathfrak{w}_{i+1}^q = U_q \mathfrak{w}_i^q = (\lambda_q x_i^q - y_i^q, x_i^q)^T$, we have

$$\begin{aligned} \lambda_q a_0 + b_0 &= \lambda_q (y_{i+1}^q - y_i^q) + (-x_{i+1}^q + x_i^q) \\ &= \lambda_q (x_i^q - y_i^q) + (-\lambda_q x_i^q + y_i^q + x_i^q) \\ &= (1 - \lambda_q) y_i^q + x_i^q. \end{aligned}$$

The intersection $(a_0, b_0)^T$ thus lies on or below $\lambda_q a + b = 1$ if $(1-\lambda_q) y_i^q + x_i^q \leq 1$, or equivalently $\frac{x_i^q - 1}{\lambda_q - 1} \leq y_i^q$. (Recall that $\lambda_q = 2\cos(\pi/q) \geq 1$ for $q \geq 3$.) Now we consider the ellipse $x^2 - \lambda_q xy + y^2 = 1$ and the line $y = \frac{x-1}{\lambda_q - 1}$. The two points $\mathfrak{w}_0^q = (1, 0)^T$ and $\mathfrak{w}_1^q = (\lambda_q, 1)^T$ lie at the intersection of the aforementioned ellipse and line, and so the remaining points $\{\mathfrak{w}_i^q\}_{i=2}^{q-1}$ lie above the line $y = \frac{x-1}{\lambda_q - 1}$, thus proving the inequality $y_i \geq \frac{x_i - 1}{\lambda_q - 1}$ for all $i = 2, 3, \cdots, q-2$.

For the second step: If $i = 2, 3, \cdots, q-1$, the slope of the line segment $\mathcal{L}_i^{q,\mathrm{bot}}$ agrees with that of $\mathcal{L}_i^{q,\mathrm{top}}$, and so exceeds that of $\mathcal{L}_{i-1}^{q,\mathrm{top}}$. It thus suffices to show that for $i = 2, 3, \cdots, q-1$, the lines $\mathcal{L}_i^{q,\mathrm{bot}}$ and $\mathcal{L}_{i-1}^{q,\mathrm{top}}$ intersect at a point on the right of the side $a = 1$ of the triangle $\mathscr{T}^q$. Towards that end, we compare the heights $b_{i-1} = \frac{1-x^q_{i-1}}{y^q_{i-1}}$ and $b'_i = \frac{-x^q_1}{y^q_1}$ at which the lines $\mathcal{L}^{q,\text{top}}_{i-1}$ and $\mathcal{L}^{q,\text{bot}}_{i}$ intersect the side $a = 1$ of $\mathcal{T}^q$. We have that $b_{i-1} \geq b'_i$ if and only if $y^q_i \geq x^q_{i-1} y^q_i - x^q_i y^q_{i-1} = \mathfrak{w}^q_{i-1} \wedge \mathfrak{w}^q_i = 1$, which is true for all $i = 2, 3, \cdots, q-1$ by proposition 2.1. This ends the proof. $\square$

2.4 The h.-Periodic Points in $\mathrm{SL}(2, \mathbb{R})/G_q$, and the BCZ_q-Periodic Points in $\mathscr{T}^q$

Lemma 2.2. *For any $A \in \mathrm{SL}(2, \mathbb{R})$, the following are equivalent.*

1. *The set $A\Lambda_q$ contains a vertical vector.*
2. *There exists $s_0 > 0$ such that $h_{s_0}(A\Lambda_q) = A\Lambda_q$. That is, $A\Lambda_q$ is h.-periodic.*
3. *There exists $\tau_0 > 0$ such that $A\Lambda_q \cap S_{\tau_0} = \emptyset$.*

Moreover, if $A\Lambda_q$ contains a vertical vector of length a, then the h.-period of $A\Lambda_q$ is $\lambda_q a^2$.

Proof. We prove (1) ⇒ (2) ⇒ (3) directly, and (3) ⇒ (1) by contradiction.

First, we note that Λ_q is h.-periodic since $h_\lambda \in G_q$, and so $h_\lambda \Lambda_q = \Lambda_q$. We also note that for any $s, t \in \mathbb{R}$, $\tau > 0$, and $B \in \mathrm{SL}(2, \mathbb{R})$ we have $h_t(s_\tau h_s B\Lambda_q) = s_\tau h_s(h_{t\tau^2} B\Lambda_q)$, and so $B\Lambda_q$ is h.-periodic iff $s_\tau h_s B\Lambda_q$ is h.-periodic.

For (1) ⇒ (2): Let $A\Lambda_q$ contain a vertical vector $(0, a)^T$, with $a > 0$. Then $s_a A\Lambda_q$ contains the vertical vector $(0, 1)^T$. Pick any vector $\mathbf{u}_0 \in s_a A\Lambda_q$ such that $\mathbf{u}_0 \wedge (0, 1)^T = 1$ (and so the x-component of $\mathbf{u}_0$ is 1). If $s = \mathrm{slope}(\mathbf{u}_0) \neq 0$, then $h_s \mathbf{u}_0$ is a horizontal vector with the same x-component as $\mathbf{u}_0$, i.e. 1, and $h_s(0, 1)^T = (0, 1)^T$. By lemma 2.1, $h_s s_a A\Lambda_q = \Lambda_q$, and so $A\Lambda_q = s_{\frac{1}{a}} h_{-s} \Lambda_q$, from which $A\Lambda_q$ is h.-periodic.

For (2) ⇒ (3): Let $\tau_1 > 0$ be such that $A\Lambda_q \cap S_{\tau_1} \neq \emptyset$. For any vector $\mathbf{u}_0 \in A\Lambda_q \cap S_{\tau_1}$, and any $s > 0$, the vector $h_s \mathbf{u}_0$ has the same x-component as $\mathbf{u}_0$, and $\mathrm{slope}(h_s \mathbf{u}_0) = \mathrm{slope}(\mathbf{u}_0) - s$. If s_0 is an h.-period of $A\Lambda_q$, then the set of lengths of the finitely many horizontal vectors that appear in $h_s(A\Lambda_q \cap S_{\tau_1})$ as s goes from 0 to s_0 agrees with the set of x-components of the vectors in $A\Lambda_q \cap S_{\tau_1}$. This implies that the

X-components of vectors in $A\Lambda_q$ are bounded from below, and so there must exist a $\tau_0 > 0$ such that $A\Lambda_q \cap S_{\tau_0} = \emptyset$.

Finally, we prove (3) $\Rightarrow$ (1) by contradiction. If $A\Lambda_q$ contains no vertical vectors, then $\mathbf{v} = (0,1)^T$ is not parallel to any vector in $A\Lambda_q$. By corollary 2.1, there exists sequences of unimodular pairs $\{\mathbf{u}_{0,n}, \mathbf{u}_{1,n}\}_{n=1}^{\infty}$ such that for each $n \geq 2$, the vector $\mathbf{v}$ belongs to the sector $(0,\infty)\mathbf{u}_{0,n} + (0,\infty)\mathbf{u}_{1,n}$, and $\mathbf{u}_{0,n}, \mathbf{u}_{1,n}$ are G_q-Stern-Brocot children of $\mathbf{u}_{0,n-1}, \mathbf{u}_{1,n-1}$. From eq. (1), we get

$$0 < \mathbf{u}_{0,n} \wedge (0,1)^T < \frac{1}{\|\mathbf{u}_{1,n}\|} \to 0.$$

That is, $A\Lambda_q$ contains vectors with arbitrarily small positive x-components.

Finally, if $A\Lambda_q$ contains a vertical vector of length a, we showed earlier in this proof that $A\Lambda_q$ must be of the form $s_{\frac{1}{a}} h_{-s}\Lambda_q$ for some $s \in \mathbb{R}$. For any $t \in \mathbb{R}$, we have $h_t\left(s_{\frac{1}{a}} h_{-s}\Lambda_q\right) = s_{\frac{1}{a}} h_{-s}\left(h_{\frac{t}{a^2}}\Lambda_q\right)$, which implies that the h.-period of $A\Lambda_q$ is a^2 times that of Λ_q. ☐

Corollary 2.2. *For any $(a,b) \in \mathscr{T}^q$, the following are equivalent.*

1. *The point (a,b) is* BCZ_q*-periodic.*
2. *The set $g_{a,b}\Lambda_q$ is h.-periodic.*
3. *The ratio b/a is the (inverse) slope of a vector in Λ_q.*

Proof. That the first two claims are equivalent is obvious, and so we proceed to characterize the points $(a,b) \in \mathscr{T}^q$ for which $g_{a,b}\Lambda_q$ is h.-periodic.

Note that $g_{a,b} = s_a g_{1,b/a}$, and that for any $s \in \mathbb{R}$, $h_s(s_a g_{1,b/a}) = s_a(h_{a^2 s} g_{1,b/a})$. That is, $g_{a,b}\Lambda_q$ is h.-periodic if $g_{1,b/a}\Lambda_q$ is h.-periodic. By lemma 2.2, $g_{1,b/a}\Lambda_q$ is h.-periodic iff it contains a vertical vector. Since $g_{1,b/a}$ is a horizontal shear, the set $g_{1,b/a}\Lambda_q$ contains a vertical vector exactly when b/a is the inverse slope of a vector in Λ_q. The claim now follows from the symmetry of Λ_q against the line $y = x$. ☐

2.5 The G_q-Next-Term Algorithm

Theorem 2.3. *Let* $A \in \mathrm{SL}(2,\mathbb{R})$, $\tau > 0$ *be such that* $A\Lambda_q \cap S_\tau \neq \emptyset$, *and* $\{\mathbf{u}_n = (q_n, a_n)^T\}_{n=0}^{\infty}$ *be elements of* $A\Lambda_q \cap S_\tau$ *with successive slopes. The set* $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_0)} A\Lambda_q$ *has a horizontal vector* $(q_0/\tau, 0)^T \in s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_0)} A\Lambda_q \cap S_1$, *and hence corresponds to a unique point* $(a,b) \in \mathscr{T}^q$ *(i.e.* $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_0)} A\Lambda_q = g_{a,b}\Lambda_q$*). The following are then true.*

1. *For each* $n \geq 0$, *the set* $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)} A\Lambda_q$ *has a horizontal vector* $(q_n/\tau, 0)^T \in s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)} A\Lambda_q \cap S_1$, *and corresponds to* $\mathrm{BCZ}_q^n(a,b)$.
2. *If we denote the* x*-component of* $\mathrm{BCZ}^n(a,b)$ *by* $L_n^q(a,b)$ *for all* $n \geq 0$, *then the* x*-components of the vectors* $\{\mathbf{u}_n\}_{n=0}^{\infty}$ *are equal to*

$$q_n = \tau L_n^q(a,b) = \tau L_0^q(\mathrm{BCZ}_q^n(a,b)).$$

Moreover, the y*-components of the vectors* $\{\mathbf{u}_n\}_{n=0}^{\infty}$ *can be recursively generated using the formula*

$$a_{n+1} = q_{n+1}\left(\frac{a_n}{q_n} + \frac{1}{\tau^2} R_q(\mathrm{BCZ}_q^n(a,b))\right)$$

for all $n \geq 0$.

This motivates the following definition.

Definition 2.2. *For any* $A \in \mathrm{SL}(2,\mathbb{R})$, $\tau > 0$ *with* $A\Lambda_q \cap S_\tau \neq \emptyset$, *and* $\mathbf{u} \in A\Lambda_q \cap S_\tau$, *we refer to the unique point in the Farey triangle* $\mathscr{T}^q$ *corresponding to* $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u})} A\Lambda_q$ *from theorem 2.3 as the* G_q-Farey triangle representatitve *of the triple* $(A, \tau, \mathbf{u})$, *and denote it by* $\mathrm{FTR}_q(A, \tau, \mathbf{u})$.

Using this notation, we can succinctly rewrite the first claim in theorem 2.3 as

$$\mathrm{FTR}_q(A, \tau, \mathbf{u}_n) = \mathrm{BCZ}_q^n\left(\mathrm{FTR}_q(A, \tau, \mathbf{u}_0)\right)$$

for all $n \geq 0$.

Remark 2.1. *For any* $\tau \geq 1$, *the vector* $\mathbf{u}_0 = (1,0)^T$ *belongs to* $\Lambda_q \cap S_\tau$, *and so* $\mathrm{FTR}_q(I_2, \tau, \mathbf{u}_0)$ *is well defined. We have*

$$s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u_0})}\Lambda_q = s_{\frac{1}{\tau}}\Lambda_q = s_{\frac{1}{\tau}} T_q^{\left\lfloor \frac{\tau}{\lambda_q}\right\rfloor}\Lambda_q = g_{\frac{1}{\tau},\left\lfloor \frac{\tau}{\lambda_q}\right\rfloor \frac{\lambda_q}{\tau}}\Lambda_q,$$

with $0 < \frac{1}{\tau} \leq 1$, *and* $1 - \lambda_q\left(\frac{1}{\tau}\right) < \left\lfloor \frac{\tau}{\lambda_q}\right\rfloor \frac{\lambda_q}{\tau} \leq 1$, *from which* $\left(\frac{1}{\tau}, \left\lfloor \frac{\tau}{\lambda_q}\right\rfloor \frac{\lambda_q}{\tau}\right) \in \mathscr{T}^q$ *is the* G_q*-Farey triangle representative of the triple* $(\Lambda_q, \tau, \mathbf{u}_0)$. *By the symmetry of* Λ_q *against the lines* $y = \pm x$, $x = 0$, *and* $y = 0$, *it suffices to generate the vectors in* $\Lambda_q \cap S_\tau$ *with slopes in* $[0,1]$ *to get all the vectors in* $\Lambda_q \cap [-\tau, \tau]^2$.

Proof of theorem 2.3. For each $n \geq 0$, a direct calculation gives $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)}\mathbf{u}_n = (q_n/\tau, 0)$, which is a horizontal vector of length not exceeding 1 in $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)} A\Lambda_q$. By the first claim in theorem 2.2, the set $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)} A\Lambda_q$ corresponds to a unique point $(c_n, d_n) \in \mathscr{T}^q$ with $q_n/\tau = a_n$, and $(c_0, d_0) = (a, b)$. The vectors $\mathbf{u}_n$ and $\mathbf{u}_{n+1}$ have consecutive slopes in $A\Lambda_q \cap S_\tau$, and so the two vectors $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)}\mathbf{u}_n$ and $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)}\mathbf{u}_{n+1}$ have consecutive slopes in $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)} A\Lambda_q \cap S_1$. In other words, the vector $s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)}\mathbf{u}_{n+1}$ is the vector of smallest positive slope in $g_{c_n,d_n}\Lambda_q \cap S_1$, from which

$$\begin{aligned} R_q(c_n, d_n) &= \mathrm{slope}\left(s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_n)}\mathbf{u}_{n+1}\right) \\ &= \tau^2 \left(\mathrm{slope}(\mathbf{u}_{n+1}) - \mathrm{slope}(\mathbf{u}_n)\right) \\ &= \tau^2 \left(\frac{a_{n+1}}{q_{n+1}} - \frac{a_n}{q_n}\right), \end{aligned}$$

and

$$h_{R_q(c_n,d_n)} g_{c_n,d_n}\Lambda_q = s_{\frac{1}{\tau}} h_{\mathrm{slope}(\mathbf{u}_{n+1})} A\Lambda_q = g_{c_{n+1},d_{n+1}}\Lambda_q,$$

and so

$$\mathrm{BCZ}_q(c_n, d_n) = (c_{n+1}, d_{n+1})$$

by the second claim in theorem 2.2. By induction, we get $(c_n, d_n) = \mathrm{BCZ}_q^n(c_0, d_0)$, $q_n = \tau c_n = \tau L_n^q(c_0, d_0)$, and the sought for recursive expression for a_{n+1}.

□

3 A Poincaré Cross Section for the Horocycle Flow on the Quotient $\mathrm{SL}(2,\mathbb{R})/G_q$

Let X_q be the homogeneous space $\mathrm{SL}(2,\mathbb{R})/G_q$, μ_q be the probability Haar measure on X_q (i.e. $\mu_q(X_q) = 1$), and Ω_q be the subset of X_q corresponding to sets $A\Lambda_q$, $A \in \mathrm{SL}(2,\mathbb{R})$, with a horizontal vector of length not exceeding 1. Note that Ω_q can be identified with the Farey triangle $\mathscr{T}^q$ via $((a,b) \in \mathscr{T}^q) \mapsto (g_{a,b}G_q \in \Omega_q)$ by lemma 2.1 and theorem 2.2. Finally, let $m_q = \frac{2}{\lambda_q} dadb$ be the Lebesgue probability measure on $\mathscr{T}^q$. Following [3], we have the following.

Theorem 3.1. *The triple* $(\mathscr{T}^q, m_q, \mathrm{BCZ}_q)$*, with* $\mathscr{T}^q$ *identified with* Ω_q*, is a cross section to* $(X_q, \mu_q, h.)$*, with roof function* R_q*.*

Proof. Consider the suspension space

$$S_{R_q}\mathscr{T}^q := \{((a,b),s) \in \mathscr{T}^q \times \mathbb{R} \mid 0 \le s \le R_q(a,b)\}/\sim_q,$$

with $((a,b), R_q(a,b)) \sim_q (\mathrm{BCZ}_q(a,b), 0)$ for all $(a,b) \in \mathscr{T}^q$, as a subset of X_q. The suspension flow of $S_{R_q}\mathscr{T}^q$ can be identified with the horocycle flow $h.$ on $S_{R_q}\mathscr{T}^q$ as a subset of X_q by theorem 2.2. The probability measure $dm_q^{R_q} = \frac{1}{m_q(R_q)} dm_q ds$ is $h.$-invariant, and the suspension space $S_{R_q}\mathscr{T}^q$ contains non-closed horocycles (e.g by lemma 2.2 and corollary 2.2). By Dani-Smillie [7], the subset $S_{R_q}\mathscr{T}^q$ has full μ_q measure in X_q, and the probability measures $dm_q^{R_q}$ and μ_q can be identified. This proves the claim.

□

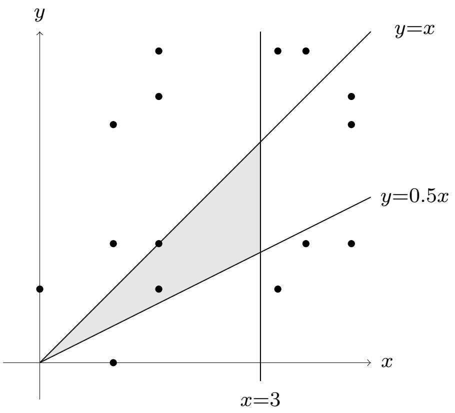

Figure 4: The set $\mathcal{F}_{[0.5,1]}(\Lambda_5, 3)$ is the collection of points from Λ_5 inside the shaded region bounded by the lines $y = 0.5x$, $y = x$, and $x = 3$.

3.1 Limiting Distributions of Farey Triangle Representatives, and Equidistribution of the Slopes of Λ_q

For any $A \in \mathrm{SL}(2, \mathbb{R})$, $\tau > 0$, and interval $I \subseteq \mathbb{R}$, we denote by

$$\mathcal{F}_I(A\Lambda_q, \tau) := \{\mathbf{u} \in A\Lambda_q \cap S_\tau \mid \mathrm{slope}(\mathbf{u}) \in I\}$$

the set of vectors in $A\Lambda_q$ with positive x-components not exceeding τ, and slopes in I. If $I \subset \mathbb{R}$ is a finite interval, we write

$$N_I(A\Lambda_q, \tau) := \#\mathcal{F}_I(A\Lambda_q, \tau)$$

for the number of elements of $\mathcal{F}_I(A\Lambda_q, \tau)$. Note that if I is a non-degenerate interval, then $\lim_{\tau\to\infty} N_I(A\Lambda_q, \tau) = \infty$ by the density of the slopes of $A\Lambda_q$ from corollary 2.1.

For any $A \in \mathrm{SL}(2, \mathbb{R})$, finite, non-empty, non-degenerate interval $I \subset \mathbb{R}$, and $\tau > 0$ with $\mathcal{F}_I(A\Lambda_q, \tau) = \{\mathbf{u}_i\}_{i=0}^{N_I(A\Lambda_q,\tau)-1} \neq \emptyset$, we define the following probability measure on the Farey triangle $\mathscr{T}^q$

$$\begin{aligned}\rho_{A\Lambda_q,I,\tau} &:= \frac{1}{N_I(A\Lambda_q,\tau)}\sum_{i=0}^{N_I(A\Lambda_q,\tau)-1}\delta_{\mathrm{FTR}_q(A,\tau,\mathbf{u}_i)}\\ &= \frac{1}{N_I(A\Lambda_q,\tau)}\sum_{i=0}^{N_I(A\Lambda_q,\tau)-1}\delta_{\mathrm{BCZ}_q^i(\mathrm{FTR}_q(A,\tau,\mathbf{u}_0))}.\end{aligned}$$

Theorem 3.2. *Let $A \in \mathrm{SL}(2,\mathbb{R})$, and $I \subset \mathbb{R}$ be a finite, non-empty, non-degenerate interval in $\mathbb{R}$. Then as $\tau \to \infty$, the number of elements of $\mathcal{F}_I(A\Lambda_q,\tau)$ has the asymptotic growth*

$$N_I(A\Lambda_q,\tau) \sim \frac{|I|}{m_q(R_q)}\tau^2,$$

and the measures $\rho_{A\Lambda_q,I,\tau}$ converge weakly

$$\rho_{A\Lambda_q,I,\tau} \rightharpoonup m_q$$

to the probability Lebesgue measure m_q.

Corollary 3.1. *For any $A \in \mathrm{SL}(2,\mathbb{R})$, and any finite interval $\emptyset \neq I \subset \mathbb{R}$, the slopes of the vectors $\mathcal{F}_I(A\Lambda_q,\tau)$ equidistribute in I as $\tau \to \infty$.*

Proof of theorem 3.2 and corollary 3.1. For $\tau > 0$ with $\mathcal{F}_I(A\Lambda_q,\tau) \neq \emptyset$, we define the measures

$$\sigma_{A\Lambda_q,I,\tau} = \frac{N_I(A\Lambda_q,\tau)}{\tau^2}\rho_{A\Lambda_q,I,\tau}$$

on the Farey triangle $\mathscr{T}^q$. Denote by $d\sigma^{R_q}_{A\Lambda_q,I,\tau}$ the measure $d\sigma_{g\Lambda_q,I,\tau}ds$ on the suspension space $S_{R_q}\mathscr{T}^q$ (which can identified with X_q by theorem 3.1). In what follows, we denote the elements of $\mathcal{F}_I(A\Lambda_q,\tau)$ by $\{\mathbf{u}_i = \mathbf{u}_i(A\Lambda_q,I,\tau)\}_{i=0}^{N_I(A\Lambda_q,\tau)-1}$, and write $\mathbf{u}_{N_I(A\Lambda_q,\tau)} = \mathbf{u}_{N_I(A\Lambda_q,\tau)}(A\Lambda_q,I,\tau)$ for the element of $A\Lambda_q \cap S_\tau$ of smallest slope bigger than any value in I. By the density of the slopes of $A\Lambda_q$ from corollary 2.1, we have that slope$(\mathbf{u}_0)$ and slope$(\mathbf{u}_{N_I(A\Lambda_q,\tau)})$ converge to the end points of the interval I, which we denote α and β (i.e. $|I| = \beta - \alpha$). We show the convergence $\sigma_{A\Lambda_q,I,\tau} \rightharpoonup |I|/m_q(R_q)m_q$ by proving the convergence $\sigma^{R_q}_{A\Lambda_q,I,\tau} \rightharpoonup |I|\mu_q$. Given any continuous, bounded function

$f : X_q \to \mathbb{R}$, we have

$$\begin{aligned}
\sigma^{R_q}_{g\Lambda_q,I,\tau}(f) &= \frac{1}{\tau^2}\int_{\tau^2\,\mathrm{slope}(\mathbf{u}_0)}^{\tau^2\,\mathrm{slope}(\mathbf{u}_{N_I(A\Lambda_q,\tau)})} f\left(h_s\left(s_{\frac{1}{\tau}}AG_q\right)\right)\,ds \\
&= \frac{1}{\tau^2}\int_{\tau^2\,\mathrm{slope}(\mathbf{u}_0)}^{\tau^2\,\mathrm{slope}(\mathbf{u}_{N_I(A\Lambda_q,\tau)})} f\left(s_{\frac{1}{\tau}}h_{\frac{s}{\tau^2}}AG_q\right)\,ds \\
&= \int_{\mathrm{slope}(\mathbf{u}_0)}^{\mathrm{slope}(\mathbf{u}_{N_I(A\Lambda_q,\tau)})} f\left(s_{\frac{1}{\tau}}h_tAG_q\right)\,dt \\
&= \int_{\alpha}^{\beta} f\left(s_{\frac{1}{\tau}}h_tAG_q\right)\,dt + o(1) \\
&\to (b-a)\mu_q(f)
\end{aligned}$$

as $\tau \to \infty$ (with the convergence of the measures supported on horocycles following from, for example, [15, 2.2.1]). This proves the weak convergence $\sigma^{R_q}_{g\Lambda_q,I,\tau} \rightharpoonup |I|\mu_q$. Denoting by $\pi_q : S_{R_q}\mathscr{T}^q \to \mathscr{T}^q$ the projection map $\big(((a,b),s) \in S_{R_q}\mathscr{T}^q\big) \mapsto ((a,b) \in \mathscr{T}^q)$, we thus have

$$\sigma_{g\Lambda_q,I,\tau} = \frac{1}{R_q}(\pi_q)_*\sigma^{R_q}_{g\Lambda_q,I,\tau} \rightharpoonup \frac{|I|}{R_q}(\pi_q)_*\mu_q = \frac{|I|}{m_q(R_q)}m_q.$$

From $\rho_{A\Lambda_q,I,\tau}(\mathscr{T}^q) = 1$, we get

$$\lim_{\tau\to\infty}\frac{N_I(A\Lambda_q,\tau)}{\tau^2} = \lim_{\tau\to\infty}\sigma_{A\Lambda_q,I,\tau}(\mathscr{T}^q) = \frac{|I|}{m_q(R_q)}m_q(\mathscr{T}^q) = \frac{|I|}{m_q(R_q)},$$

which is the asymptotic growth from theorem 3.2. This also gives the weak limit $\rho_{A\Lambda_q,I,\tau} \rightharpoonup m_q$.

As for corollary 3.1, if $\emptyset \neq J \subseteq I$ is any non-empty subinterval of I, we have

$$\lim_{\tau\to\infty}\frac{N_J(A\Lambda_q,\tau)}{N_I(A\Lambda_q,\tau)} = \frac{|J|}{|I|},$$

which proves the sought for equidistribution. $\square$

4 Applications

In this section, we give a few applications of the G_q-BCZ maps to the statistics of subsets of Λ_q. In section 4.1, we derive the main asymptotic term for the number of vectors of Λ_q in homothetic dilations of triangles. In section 4.2.1, we derive the distribution of the slope gaps of Λ_q. Finally, in section 4.2.2, we derive the distribution of the Euclidean distances between the centers of G_q -Ford circles. Several other applications of the G_3-BCZ map to the statistics of the visible lattice points $\Lambda_3 = \mathbb{Z}^2_{\text{prim}} = \{(x, y) \in \mathbb{Z}^2 \mid \gcd(x, y) = 1\}$ can be similarly extended–almost verbatim–to general Λ_q. This list includes, but is not limited to, an old Diophantine approximation problem of [9] Erdös, P., Szüsz, P., & Turán solved independently by Xiong and Zaharescu [20], and Boca [5] for $G_3 = \mathrm{SL}(2, \mathbb{Z})$, and Heersink [14] for finite index subgroups of $G_3 = \mathrm{SL}(2, \mathbb{Z})$; the average depth of cusp excursions of the horocycle flow on $X_2 = \mathrm{SL}(2, \mathbb{R})/G_3$ by Athreya and Cheung [3]; and the statistics of weighted Farey sequences by Panti [17].

4.1 Asymptotic Growth of the Number of Elements of Λ_q in Homothetic Dilations of Triangles

For any $A \in \mathrm{SL}(2, \mathbb{R})$, $\tau > 0$, and finite interval $I \subset \mathbb{R}$, the set $\mathcal{F}_I(A\Lambda_q, \tau)$ introduced in section 3.1 is the collection of points of $A\Lambda_q$ which belong to the triangle $\{(x, y)^T \in \mathbb{R}^2 \mid y/x \in I, 0 < x \leq \tau\}$. We have the main term for the asymptotic growth rate of the number of aforementioned vectors $N_I(A\Lambda_q, \tau)$ as $\tau \to \infty$, which can be immediately interpreted as a statement on the asymptotic growth of the number of vectors of $A\Lambda_q$ in homothetic dilations of triangles that have a vertex at the origin as we do in proposition 4.1. In corollary 4.1, we show the equidistribution of the homothetic dilations $\frac{1}{\tau}\Lambda_q$ in the square $[-1, 1]^2$ as $\tau \to \infty$. In what follows, we write $f(\tau) \sim g(\tau)$ as $\tau \to \infty$ for any two functions f, g to indicate that $\lim_{\tau \to \infty} f(\tau)/g(\tau) = 1$.

Proposition 4.1. *Let Δ be a triangle in the plane $\mathbb{R}^2$ with one vertex at the origin. Then for any $A \in \mathrm{SL}(2, \mathbb{R})$, and any $\tau > 0$ the number of elements $\#\left(A\Lambda_q \cap \tau\Delta\right)$ has* *the asymptotic growth rate*

$$\#\left(A\Lambda_q \cap \tau\Delta\right) \sim \left(\frac{2}{m_q(R_q)}\operatorname{area}(\Delta)\right)\tau^2$$

as $\tau \to \infty$.

We also get the following.

Corollary 4.1. *For any* $A \in \mathrm{SL}(2,\mathbb{R})$*, and* $\tau \geq 1$*, let* $\lambda_\tau^{A\Lambda_q}$ *be the probability measure defined for any Borel subset* C *of the square* $[-1,1]^2$ *by*

$$\lambda_\tau^{A\Lambda_q}(C) = \frac{\#(\frac{1}{\tau}A\Lambda_q \cap C)}{\#\left(\frac{1}{\tau}A\Lambda_q \cap [-1,1]^2\right)}.$$

Then the measures $\lambda_\tau^{A\Lambda_q}$ *converge weakly to the Lebesgue probability measure* $\mathrm{Unif}_{[-1,1]^2}$ *on* $[-1,1]^2$ *as* $\tau \to \infty$.

Proof of proposition 4.1. We first prove the theorem assuming that the side L of the triangle Δ opposite to the origin is included in Δ. Let $\mathrm{rot}_\Delta \in \mathrm{SL}(2,\mathbb{R})$ be the rotation that rotates the side of Δ opposite to the vertex at the origin onto a vertical line segment. (That is, the side of $\mathrm{rot}_\Delta \Delta$ opposite to the vertex at the origin is vertical.) Denote by $d_\Delta > 0$ the perpendicular distance from the vertex at the origin to the side of $\mathrm{rot}_\Delta \Delta$ opposite to the aforementioned vertex, and by $I_\Delta \subset \mathbb{R}$ the interval of slopes of the points in $\mathrm{rot}_\Delta \Delta$. For any $\tau > 0$, we have that $\tau(\mathrm{rot}_\Delta \Delta) \cap (\mathrm{rot}_\Delta A\Lambda_q) = \mathcal{F}_I(\mathrm{rot}_\Delta A\Lambda_q, \tau d)$, and that the rotation rot_Δ is a bijection from $\tau\Delta \cap A\Lambda_q$ to $\tau(\mathrm{rot}_\Delta \Delta) \cap (\mathrm{rot}_\Delta A\Lambda_q)$. From this and theorem 3.2 follows that

$$\#\left(A\Lambda_q \cap \tau\Delta\right) = N_{I_\Delta}(\mathrm{rot}_\Delta A\Lambda_q, \tau d) \sim \frac{|I_\Delta|}{m_q(R_q)}(\tau d_\Delta)^2 = \frac{2}{m_q(R_q)}\operatorname{area}(\Delta)\tau^2$$

which proves the claim.

Including or excluding any of the two sides of the triangle Δ that pass through the origin does not change $|I_\Delta|$, and hence the main term for the asymptotic growth in question remains the same. We now show that the main term does not change when the side

L of Δ opposite to the origin is removed as well. For any $\delta > 0$, denote by $\Delta' = \Delta'(\Delta, \delta)$ the homothetic dilation of Δ such that $0 < \text{area}(\Delta) - \text{area}(\Delta') \leq \delta$. The line segment L belongs to $\Delta \setminus \Delta'$. By the above, $\lim_{\tau\to\infty} \left(\#(A\Lambda_q \cap \tau\Delta) - \#(A\Lambda_q \cap \tau\Delta')\right)/\tau^2 = 2\left(\text{area}(\Delta) - \text{area}(\Delta')\right)/m_q(R_q) \leq 2\delta/m_q(R_q)$. It thus follows that for all $\epsilon > 0$, there exists $\tau_0 = \tau_0(A\Lambda_q, \Delta, \delta, \epsilon)$ such that for all $\tau > \tau_0$ we have

$$\frac{\#\left(A\Lambda_q \cap \tau L\right)}{\tau^2} \leq \frac{\#(A\Lambda_q \cap \tau\Delta) - \#(A\Lambda_q \cap \tau\Delta')}{\tau^2} \leq \frac{2\delta}{m_q(R_q)} + \epsilon.$$

By the arbitrariness of δ and ϵ, we get $\lim_{\tau\to\infty} \frac{\#(A\Lambda_q \cap \tau L)}{\tau^2} = 0$. This proves that adding or removing a finite number of line segments does not affect the main term for the asymptotic growth of the number of elements of $A\Lambda_q$ in homothetic dilations of triangles. $\square$

Proof of corollary 4.1. That the set functions $\lambda_\tau^{A\Lambda_q}$ are probability measures on $[-1,1]^2$ is clear. We proceed to prove that they converge weakly to $\text{Unif}_{[-1,1]}$.

First, we note that given any rectangle $\mathcal{R}$ in the plane, we can express $\mathcal{R}$ using the union and/or difference of four triangles each having a vertex at the origin. From this follows that $\lim_{\tau\to\infty} \frac{\#(A\Lambda_q \cap \tau\mathcal{R})}{\tau^2} = \frac{2}{m_q(R_q)}\,\text{area}(\mathcal{R})$. Consequently, if $\mathcal{R}$ belongs to $[-1,1]^2$, then $\lim_{\tau\to\infty} \lambda_\tau^{A\Lambda_q}(\mathcal{R}) = \text{Unif}_{[-1,1]^2}(\mathcal{R})$.

Fix a continuous function $f : [-1,1]^2 \to \mathbb{R}$. Given a $\delta > 0$, there exists a finite partition $\mathscr{P} = \mathscr{P}(f, \delta)$ of the square $[-1,1]^2$ into rectangles such that the difference between the supremum and infimum of f over each of the rectangles in the partition does not exceed δ. That is, $\sup_{\mathcal{R}}(f) \leq \inf_{\mathcal{R}}(f) + \delta$ for all $\mathcal{R} \in \mathscr{P}$. (This is possible by the uniform continuity of f over $[-1,1]^2$.) Given $\epsilon > 0$, there exists $\tau_0 = \tau_0(A\Lambda_q, \mathscr{P}(f,\delta), \epsilon)$

such that $\left|\lambda_\tau^{A\Lambda_q}(\mathcal{R}) - \mathrm{Unif}_{[-1,1]^2}(\mathcal{R})\right| \leq \epsilon$ for all $\tau > \tau_0$, and all $\mathcal{R} \in \mathscr{P}$. We thus have

$$\begin{aligned}
\lambda_\tau^{A\Lambda_q}(f) &\leq \sum_{\mathcal{R}\in\mathscr{P}} \sup_{\mathcal{R}}(f)\ \lambda_\tau^{A\Lambda_q}(\mathcal{R}) \\
&\leq \epsilon \max_{[-1,1]^2}(|f|)\#\mathscr{P} + \sum_{\mathcal{R}\in\mathscr{P}} \sup_{\mathcal{R}}(f)\,\mathrm{Unif}_{[-1,1]^2}(\mathcal{R}) \\
&\leq \epsilon \max_{[-1,1]^2}(|f|)\#\mathscr{P} + \delta + \sum_{\mathcal{R}\in\mathscr{P}} \inf_{\mathcal{R}}(f)\,\mathrm{Unif}_{[-1,1]^2}(\mathcal{R}) \\
&\leq \epsilon \max_{[-1,1]^2}(|f|)\#\mathscr{P}(f,\delta) + \delta + \mathrm{Unif}_{[-1,1]^2}(f).
\end{aligned}$$

Similarly

$$\begin{aligned}
\lambda_\tau^{A\Lambda_q}(f) &\geq \sum_{\mathcal{R}\in\mathscr{P}} \inf_{\mathcal{R}}(f)\ \lambda_\tau^{A\Lambda_q}(\mathcal{R}) \\
&\geq -\epsilon \max_{[-1,1]^2}(|f|)\#\mathscr{P} + \sum_{\mathcal{R}\in\mathscr{P}} \inf_{\mathcal{R}}(f)\,\mathrm{Unif}_{[-1,1]^2}(\mathcal{R}) \\
&\geq -\epsilon \max_{[-1,1]^2}(|f|)\#\mathscr{P} - \delta + \sum_{\mathcal{R}\in\mathscr{P}} \sup_{\mathcal{R}}(f)\,\mathrm{Unif}_{[-1,1]^2}(\mathcal{R}) \\
&\geq -\epsilon \max_{[-1,1]^2}(|f|)\#\mathscr{P}(f,\delta) - \delta + \mathrm{Unif}_{[-1,1]^2}(f).
\end{aligned}$$

That is, $\left|\lambda_\tau^{A\Lambda_q}(f) - \mathrm{Unif}_{[-1,1]^2}(f)\right| \leq \epsilon \max_{[-1,1]^2}(|f|)\#\mathscr{P}(f,\delta) + \delta$. By the arbitrariness of δ and ϵ, we get $\lim_{\tau\to\infty} \lambda_\tau^{A\Lambda_q}(f) = \mathrm{Unif}_{[-1,1]^2}(f)$. This proves the claim. $\square$

4.2 G_q-Farey Statistics

In proposition 4.2 below we derive the limiting distribution of quantities that can be expressed as functions in the G_q-Farey triangle representatives (section 2.5) of the elements of the sets $\mathcal{F}_I(A\Lambda_q, \tau)$ (section 3.1) as $\tau \to \infty$. As examples of said distributions, we consider the slope gap distribution of Λ_q in section 4.2.1, and the distribution of the Euclidean distance between G_q-Ford circles in section 4.2.2.

Proposition 4.2. *Let $F : \mathscr{T}^q \to \mathbb{R}$ be a function, continuous on the Farey triangle $\mathscr{T}^q$ except perhaps on the image of finitely many C^1 curves $\{c_i : I_i \to \mathscr{T}^q\}_{i=1}^m$, with $I_i \subset \mathbb{R}$ being finite, closed intervals of $\mathbb{R}$. For any $A \in \mathrm{SL}(2,\mathbb{R})$, and any finite interval*

$I \subset \mathbb{R}$, the limit of the distribution

$$\mathrm{FareyStat}_{F,\tau}^{A\Lambda_q,I}(t) := \frac{\#\left\{\mathbf{u} \in \mathcal{F}_I(A\Lambda_q,\tau) \mid F\left(\mathrm{FTR}_q(A\Lambda_q,\tau,\mathbf{u})\right) \geq t\right\}}{N_I(A\Lambda_q,\tau)}$$

as $\tau \to \infty$ exists for all $t \in \mathbb{R}$, and is equal to

$$\mathrm{FareyStat}_F(t) = m_q\left(\mathbb{1}_{F \geq t}\right),$$

where $\mathbb{1}_{F \geq t}$ is the indicator function of the subset

$$\{(a,b) \in \mathscr{T}^q \mid F(a,b) \geq t\}$$

of $\mathscr{T}^q$, and m_q is the Lebesgue probability measure $dm_q = \frac{2}{\lambda_q} dadb$ on $\mathscr{T}^q$.

Proof. Fix $t \in \mathbb{R}$. We then have

$$\begin{aligned}
\mathrm{FareyStat}_{F,\tau}^{A\Lambda_q,I}(t) &= \frac{\#\left\{\mathbf{u} \in \mathcal{F}_I(A\Lambda_q,\tau) \mid F\left(\mathrm{FTR}_q(A,\tau,\mathbf{u})\right) \geq t\right\}}{N_I(A\Lambda_q,\tau)} \\
&= \frac{1}{N_I(A\Lambda_q,\tau)} \sum_{i=0}^{N_I(A\Lambda_q,\tau)-1} \mathbb{1}_{F \geq t}\left(\mathrm{FTR}_q(A,\tau,\mathbf{u}_i)\right) \\
&= \rho_{A\Lambda_q,I,\tau}\left(\mathbb{1}_{F \geq t}\right),
\end{aligned}$$

and so we proceed to show that $\lim_{\tau\to\infty} \rho_{A\Lambda_q,I,\tau}\left(\mathbb{1}_{F \geq t}\right) = m_q\left(\mathbb{1}_{F \geq t}\right)$.

Consider the following sets

$$\begin{aligned}
A_t &= \{(a,b) \in \mathscr{T}^q \mid F(a,b) \geq t\}, \\
B_t &= \{(a,b) \in \mathscr{T}^q \mid F(a,b) \geq t\} \cup \bigcup_{i=1}^{m} c_i(I_i), \text{ and} \\
C &= \bigcup_{i=1}^{m} c_i(I_i).
\end{aligned}$$

The set C_t is null with respect to the measure m_q, and $A_t \Delta B_t \subseteq C$, and so $m_q(\mathbb{1}_{A_t}) = m_q(\mathbb{1}_{B_t})$. The sets B_t and C are closed, and so their indicator functions $\mathbb{1}_{B_t}$ and $\mathbb{1}_C$ are bounded, and upper semi-continuous. Theorem 3.2 gives $\lim_{\tau\to\infty} \rho_{A\Lambda_q,I,\tau}(\mathbb{1}_{B_t}) =$

$m_q(\mathbb{1}_{B_t})$, and $\lim_{\tau\to\infty} \rho_{A\Lambda_q,I,\tau}(\mathbb{1}_{C_t}) = m_q(\mathbb{1}_{C_t}) = 0$. Since $|\mathbb{1}_{A_t} - \mathbb{1}_{B_t}| \leq \mathbb{1}_C$ on all of $\mathscr{T}^q$, we get $\lim_{\tau\to\infty} \rho_{A\Lambda_q,I,\tau}(\mathbb{1}_{A_t}) = m_q(\mathbb{1}_{A_t})$. □

4.2.1 Slope Gap Distribution

Let $A \in \mathrm{SL}(2,\mathbb{R})$, and $\tau > 0$ be such that $\mathcal{F}(A\Lambda_q, \tau) \neq \emptyset$. Given two vectors $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{F}(A\Lambda_q, \tau)$ with consecutive slopes, we denote the difference between the slopes of $\mathbf{u}_0$ and $\mathbf{u}_1$ by $\mathrm{slopegap}(A\Lambda_q, \tau, \mathbf{u}_0) = \mathrm{slope}(\mathbf{u}_1) - \mathrm{slope}(\mathbf{u}_0)$. We have the following on the limiting distribution of slopegap.

Corollary 4.2. *Let $A \in \mathrm{SL}(2,\mathbb{R})$, $I \subset \mathbb{R}$ be a finite interval. The limit of*

$$\mathrm{SlopeGap}_{\tau}^{A\Lambda_q,I}(t) := \frac{\#\left\{\mathbf{u} \in \mathcal{F}_I(A\Lambda_q, \tau) \mid \tau^2\, \mathrm{slopegap}(A\Lambda_q, \tau, \mathbf{u}) \geq t\right\}}{N_I(A\Lambda_q, \tau)}$$

as $\tau \to \infty$ exists for all $t \in \mathbb{R}$, and is equal to $m_q(\mathbb{1}_{R_q \geq t})$, where $m_q = \frac{2}{\lambda_q} da db$ is the Lebesgue probability measure on the G_q-Farey triangle $\mathscr{T}^q$.

Proof. Let $\tau > 0$ be such that $\mathcal{F}_I(A\Lambda_q, \tau) = \{\mathbf{u}_n = (q_n, a_n)^T\}_{n=0}^{N_I(A\Lambda_q,\tau)-1} \neq \emptyset$. For any $0 \leq n \leq N_I(A\Lambda_q, \tau) - 2$, we have by theorem 2.3 that

$$\mathrm{slopegap}(A\Lambda_q, \tau, \mathbf{u}_n) = \frac{a_{n+1}}{q_{n+1}} - \frac{a_n}{q_n} = \frac{1}{\tau^2} R_q(\mathrm{FTR}_q(A\Lambda_q, \tau, \mathbf{u}_n)).$$

This implies that $\mathrm{SlopeGap}_{\tau}^{A\Lambda_q,I}(t) = \mathrm{FareyStat}_{R_q,\tau}^{A\Lambda_q,I}(t)$, and the proposition then follows from proposition 4.2. □

4.2.2 The G_q-Ford Circles, and Their Geometric Statistics

For any point $\mathbf{w} = (r, s)^T \in \mathbb{R}^2$, the *Ford circle* $C[\mathbf{w}]$ [11] corresponding to $\mathbf{w}$ is defined to be either

- the circle with radius $\frac{1}{2r^2}$, and center at $\left(\frac{s}{r}, \frac{1}{2r^2}\right)$, if $r \neq 0$, or
- the straight line $y = s^2$, if $r = 0$.

It is well-known that for any two vectors $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^2$, the Ford circles $C[\mathbf{w}_1]$ and

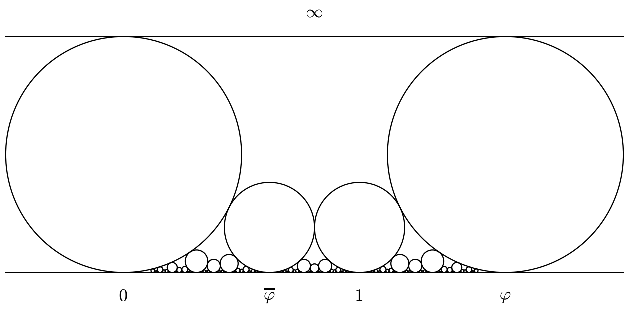

Figure 5: The G_5-Ford circles corresponding to the vectors in $\mathcal{F}_{[0,\varphi]}(\Lambda_5,\infty)$ along with the circle at infinity. The circles tangent to the line $y=0$ at $\overline{\varphi}$, 1, and φ are the G_5-Stern-Brocot children of the circles at 0 and infinity.

$C[\mathbf{w}_2]$ intersect if $|\mathbf{w}_0 \wedge \mathbf{w}_1| < 1$, are tangent if $|\mathbf{w}_0 \wedge \mathbf{w}_1| = 1$, and are wholly external if $|\mathbf{w}_0 \wedge \mathbf{w}_1| > 1$.

It follows from theorem 2.1 and corollary 2.1 that for any $A \in \mathrm{SL}(2,\mathbb{R})$, the Ford circles corresponding to any two distinct elements of $A\Lambda_q \cap S_\infty$ are either tangent or wholly external, and that the G_q-Stern-Brocot children of any two unimodular vectors of $A\Lambda_q \cap S_\infty$ correspond to a chain of $q-2$ tangent circles between the two circles corresponding to the "parents".

Let $A \in SL(2,\mathbb{R})$, and $\tau > 0$ be such that $\mathcal{F}(A\Lambda_q,\tau) \neq \emptyset$. Given two vectors $\mathbf{u}_0, \mathbf{u}_1 \in \mathcal{F}(A\Lambda_q,\tau)$ with consecutive slopes, we denote the distance between the centers of $C[\mathbf{u}_0]$ and $C[\mathbf{u}_1]$ by $\mathrm{centdist}(A\Lambda_q,\tau,\mathbf{u}_0)$. We have the following on the limiting distribution of centdist, extending a result from [2] for G_3-Ford circles.

Corollary 4.3. *Let $A \in \mathrm{SL}(2,\mathbb{R})$, and $I \subset \mathbb{R}$ be a finite interval. The limit of*

$$\mathrm{CentDist}_\tau^{A\Lambda_q,I}(t) := \frac{\#\left\{\mathbf{u} \in \mathcal{F}_I(A\Lambda_q,\tau) \mid \tau^2\,\mathrm{centdist}(A\Lambda_q,\tau,\mathbf{u}) \geq t\right\}}{N_I(A\Lambda_q,\tau)}$$

as $\tau \to \infty$ exists for all $t \in \mathbb{R}$, and is equal to $m_q(\mathbb{1}_{F_q \geq t})$, where $m_q = \frac{2}{\lambda_q} dadb$ is the Lebesgue probability measure on the G_q-Farey triangle $\mathscr{T}^q$, and $F_q : \mathscr{T}^q \to \mathbb{R}$ is the function defined by

$$F_q(a,b) = \sqrt{R_q(a,b)^2 + \frac{1}{4}\left(\frac{1}{L_1^q(a,b)^2} - \frac{1}{L_0^q(a,b)^2}\right)^2},$$

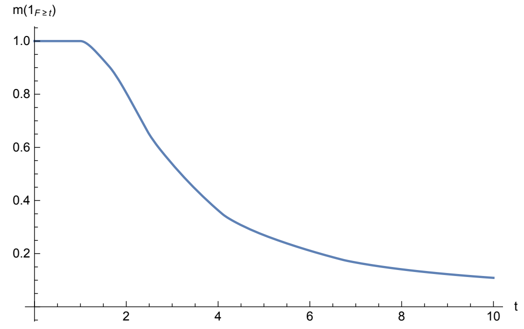

Figure 6: The graph of the limiting distribution $\lim_{t\to\infty} \mathrm{CentDist}_{\tau}^{\Lambda_5,I}(t) = m_5(\mathbb{1}_{F_5 \geq t})$ of the Euclidean distance between successive G_5-Ford circles from corollary 4.3.

where L_0^q and L_1^q are as in theorem 2.3.

As an immediate consequence of the second claim in corollary 2.1, we get the following weak form of Dirichelet's approximation theorem for Λ_q.

Proposition 4.3. *Let $A \in \mathrm{SL}(2,\mathbb{R})$, and $\alpha \in \mathbb{R}$. The line $x = \alpha$ either passes through the center of a G_q-Ford circle corresponding to a vector in $A\Lambda_q$, or there exist infinitely many vectors in $A\Lambda_q$ whose Ford circles intersect $x = \alpha$. In particular, α is either the slope of a vector in $A\Lambda_q$, or there exist infinitely many $(q,a)^T \in A\Lambda_q$ such that*

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{2q^2}.$$

Proof. Let $\tau > 0$ be such that $\mathcal{F}_I(A\Lambda_q,\tau) = \{\mathbf{u}_n = (q_n, a_n)^T\}_{n=0}^{N_I(A\Lambda_q,\tau)-1} \neq \emptyset$. For any $0 \leq n \leq N_I(A\Lambda_q,\tau) - 2$, we have by theorem 2.3 that

$$\frac{a_{n+1}}{q_{n+1}} - \frac{a_n}{q_n} = \frac{1}{\tau^2} R_q(\mathrm{FTR}_q(A\Lambda_q,\tau,\mathbf{u}_n)),$$

and

$$\frac{1}{2q_{n+1}^2} - \frac{1}{2q_n^2} = \frac{1}{2\tau^2 L_1^q(\mathrm{FTR}_q(A\Lambda_q,\tau,\mathbf{u}_n))^2} - \frac{1}{2\tau^2 L_0^q(\mathrm{FTR}_q(A\Lambda_q,\tau,\mathbf{u}_n))^2}.$$

From this follows that the distance between the centers of $C[\mathbf{u}_n]$ and $C[\mathbf{u}_{n+1}]$ is given by

$$\begin{aligned}\mathrm{centdist}(A\Lambda_q, \tau, \mathbf{u}_n) &= \sqrt{\left(\frac{a_{n+1}}{q_{n+1}} - \frac{a_n}{q_n}\right)^2 + \left(\frac{1}{2q_{n+1}^2} - \frac{1}{2q_n^2}\right)^2} \\ &= \frac{1}{\tau^2} F_q(\mathrm{FTR}_q(A\Lambda_q, \tau, \mathbf{u}_n)).\end{aligned}$$

This implies that $\mathrm{CentDist}_{\tau}^{A\Lambda_q, I}(t) = \mathrm{FareyStat}_{F_q,\tau}^{A\Lambda_q, I}(t)$, and the proposition then follows from proposition 4.2. □

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On Cross Sections to the Geodesic and Horocycle Flows on Quotients of $\mathrm{SL}(2,\mathbb{R})$ by Hecke Triangle Groups G_q

Abstract

In this paper, we provide a model for cross sections to the geodesic and horocycle flows on $\mathrm{SL}(2,\mathbb{R})/G_q$ using an extension of a heuristic of P. Arnoux and A. Nogueira. Our starting point is a continued fraction algorithm related to the group G_q, and a cross section to the horocycle flow on $\mathrm{SL}(2,\mathbb{R})/G_q$ from a previous paper. As an application, we get the natural extension and invariant measure for a symmetric G_q-Farey interval map resulting from projectivizing the aforementioned continued fraction algorithm.

Contents

1 Introduction

For each integer $q \geq 3$, the *Hecke triangle group* G_q is the discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ generated by

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } T_q := \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \tag{1.1}$$

where $\lambda_q := 2\cos\frac{\pi}{q}$. The modular group $\mathrm{SL}(2,\mathbb{Z})$ is the Hecke triangle group corresponding to $q = 3$. As such, the family of Hecke triangle groups provides a natural test ground for understanding how several classical constructs (e.g. the Farey sequence, Stern-Brocot tree, and so on) can be extended to other Fuchsian groups. One such set of objects that is of interest is the family of the discrete orbits

$$\Lambda_q := G_q(1,0)^T \tag{1.2}$$

of the linear action of G_q on the plane $\mathbb{R}^2$. (For $q = 3$, the orbit Λ_3 is the set of primitive pairs of integers $\mathbb{Z}^2_{\text{prim}} := \{(a,b) \in \mathbb{Z}^2 \mid \gcd(a,b) = 1\}$. Those are exactly the non-zero points of $\mathbb{Z}^2$ that are visible from the origin.)

In [14], we derived an analogue of the classical Stern-Brocot process for the set Λ_q of G_q-"visible lattice points" starting with a particular continued fraction algorithm that we refer to, following Janvresse, Rittaud, and De La Rue in [7], as the *λ_q-continued fraction algorithm.* Our sought for application for the G_q-Stern-Brocot tree in [14] was deriving an explicit cross section to the horocycle flow on $\mathrm{SL}(2,\mathbb{R})/G_q$. In this paper, we study the aforementioned continued fraction algorithm itself, with our focus being the derivation of an explicit cross section to the geodesic flow on $\mathrm{SL}(2,\mathbb{R})/G_q$ whose first return map is a natural extension to (a projective version of) the λ_q-continued fraction algoritm.

Our main tool is an extension of a heuristic introduced by P. Arnoux and A. Nogueira in [4] for deriving geometric models of natural extensions of multidimensional continued fraction algorithms. Our extension of the said heuristic provides a simple picture of the cross section to the horocycle flow $h_{\cdot} \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$ we derived in [14] (theorem 2.1), along with the cross section to the geodesic flow $g_{\cdot} \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$ we derive here (proposition 1.2). As a by product, we get an infinite invariant measure for the (projective) λ_q-continued fraction algorithm. We accelerate the continued fraction algorithm to get a map with a finite invariant measure, and a corresponding finite area geodesic cross section.

In the remainder of this section, we recall the λ_q-continued fraction algorithm, present our extension of the Arnoux-Nogueira heuristic, and present our main results.

1.1 The λ_q-continued fraction algorithm

As is customary when working with Hecke triangle groups, we write

$$U_q := T_q S = \begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

For $i = 0, 1, \cdots, q-2$, we consider the following vectors

$$\mathfrak{w}_i^q = (x_i^q, y_i^q)^T := U_q^i (1,0)^T \tag{1.4}$$

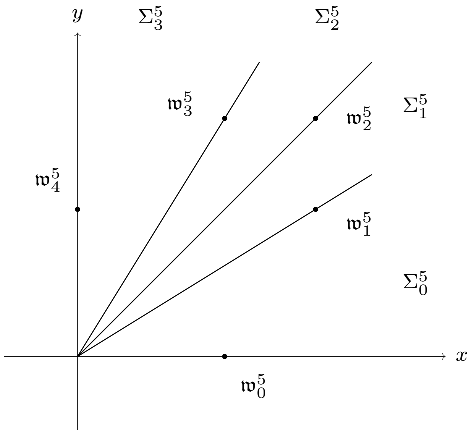

Figure 1: The vectors $\{\mathfrak{w}_i^5\}_{i=0}^4$, and the sectors $\{\Sigma_i^5\}_{i=0}^3$.

in Λ_q. (Note that $\mathfrak{w}_0^q = (1,0)^T$, $\mathfrak{w}_1^q = (\lambda_q, 1)^T$, $\mathfrak{w}_{q-2}^q = (1, \lambda_q)^T$, and $\mathfrak{w}_{q-1}^q = (0,1)^T$.) For $i = 0, 1, \cdots, q-2$, denote by

$$\Sigma_i^q := (0, \infty)\mathfrak{w}_i^q + [0, \infty)\mathfrak{w}_{i+1}^q \tag{1.5}$$

the sector in the first quadrant between $\mathfrak{w}_i^q$ (inclusive), and $\mathfrak{w}_{i+1}^q$ (exclusive).

Definition 1.1 ([6, 7, 14])**.** For any non-zero vector $\mathbf{u} \in \mathbb{R}^2$, an application of the λ_q-*continued fraction algorithm* is the following: If $\mathbf{u}$ is in the sector Σ_i^q with $0 \le i \le q-2$, then replace $\mathbf{u}$ with $(M_i^q)^{-1}\mathbf{u} \in \cup_{j=0}^{q-2}\Sigma_j^q$, where

$$M_i^q := U_q^i T_q = [\mathfrak{w}_i^q \ \mathfrak{w}_{i+1}^q] \tag{1.6}$$

for all $i = 0, 1, \cdots, q-2$ are the matrices in G_q whose columns are $\mathfrak{w}_i^q$ and $\mathfrak{w}_{i+1}^q$.

(Note that for every $i = 0, 1, \cdots, q-2$, the matrix M_i^q bijectively maps $\cup_{j=0}^{q-2}\Sigma_j^q$ to Σ_i^q.) In [14], we showed that the vector $\mathbf{u}$ eventually lands on the line $y = 0$ and is fixed by the algorithm if and only if $\mathbf{u}$ is parallel to a vector in Λ_q.

1.2 The Arnoux-Nogueira heuristic

We below describe a geometric model that can be used to describe cross sections to the horocycle and geodesic flows on $\mathrm{SL}(2, \mathbb{R})/G_q$. This model is based on a heuristic introduced by P. Arnoux and A. Nogueira [4] for deriving geometric models of the natural extensions, and invariant measures of multidimensional continued fractions. We extend the heuristic to provide, in our setting, a unified description of cross sections to the horocycle flow in addition to the geodesic flow on $\mathrm{SL}(2,\mathbb{R})/G_q$. An excellent presentation of the applications of this heuristic for finding cross sections to the geodesic flow can be found in [3], or the more classical [15].

Towards that end, we write

$$\mathsf{FQ} := \{(x,y)^T \in \mathbb{R}^2 \mid x > 0, y \geq 0\} \tag{1.7}$$

for the collection of vectors in the first quadrant, and

$$\widehat{\mathsf{FQ} \times \mathsf{FQ}} := \{(\mathbf{u},\mathbf{v}) \in \mathsf{FQ} \times \mathsf{FQ} \mid \mathbf{u}\cdot\mathbf{v} = 1\} \tag{1.8}$$

for the collection of pairs of vectors in $\mathsf{FQ} \times \mathsf{FQ}$ with dot product 1. We have the following elementary linear-algebraic properties for the horocycle and geodesic flows on $\widehat{\mathsf{FQ} \times \mathsf{FQ}}$. (The matrices $h_s, g_{a,b}, g_t$ are as in eq. (2.4), eq. (2.6), and eq. (2.5).)

Proposition 1.1. *Let* $\Phi : \widehat{\mathsf{FQ} \times \mathsf{FQ}} \to \mathrm{SL}(2,\mathbb{R})$ *be the map defined by*

$$\Phi\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) := \begin{pmatrix} a & b \\ -d & c \end{pmatrix}. \tag{1.9}$$

The map Φ *is well-defined, and injective. The following are also true.*

1. *For any pair* $(\mathbf{u},\mathbf{v}) \in \widehat{\mathsf{FQ} \times \mathsf{FQ}}$*, and matrix* $A \in M_{2\times 2}(\mathbb{R})$ *with* $A^{-1}\mathbf{u}, A^T\mathbf{v} \in \mathsf{FQ}$*, we have that* $(A^{-1}\mathbf{u}, A^T\mathbf{v}) \in \widehat{\mathsf{FQ} \times \mathsf{FQ}}$*, and*

$$\Phi(A^{-1}\mathbf{u}, A^T\mathbf{v}) = \Phi(\mathbf{u},\mathbf{v})(A^{-1})^T. \tag{1.10}$$

2. *For any* $((a,b)^T,(c,d)^T)\in \widehat{\mathsf{FQ}\times\mathsf{FQ}}$*, there exists* $s\in[0,\frac{1}{ab})$ *such that*

$$\begin{pmatrix} c \\ d \end{pmatrix} = (1-abs)\begin{pmatrix} 1/a \\ 0 \end{pmatrix} + abs\begin{pmatrix} 0 \\ 1/b \end{pmatrix}.$$

   In that case,

$$\Phi((a,b)^T,(c,d)^T) = h_s\Phi((a,b)^T,(1/a,0)^T) = h_s g_{a,b}. \tag{1.11}$$

3. *For any* $t\in\mathbb{R}$*, and* $(\mathbf{u},\mathbf{v})\in\widehat{\mathsf{FQ}\times\mathsf{FQ}}$*, we have that*

$$g_t\Phi(\mathbf{u},\mathbf{v}) = \Phi(e^t\mathbf{u},e^{-t}\mathbf{v}). \tag{1.12}$$

Moreover, the set $\widehat{\mathsf{FQ}\times\mathsf{FQ}}$ *can be parametrized by the set*

$$\left\{((a,b),s)\in\mathsf{FQ}\times\mathbb{R}\mid s\in\left[0,\frac{1}{ab}\right)\right\}. \tag{1.13}$$

We apply this heuristic in our setting as follows: The λ_q-continued fraction from definition 1.1 can be written as a map $\mathbf{A}:\mathsf{FQ}\to\mathsf{FQ}$, where each $\mathbf{u}\in\mathsf{FQ}$ is sent to $A(\mathbf{u})^{-1}\mathbf{u}$, with $A(\mathbf{u})=M_i^q$ if $\mathbf{u}$ belongs to the sector Σ_i^q. This extends to a map $\widehat{\mathbf{A}}:\widehat{\mathsf{FQ}\times\mathsf{FQ}}\to\widehat{\mathsf{FQ}\times\mathsf{FQ}}$ that sends each $(\mathbf{u},\mathbf{v})\in\widehat{\mathsf{FQ}\times\mathsf{FQ}}$ to $(A(\mathbf{u})^{-1}\mathbf{u},A(\mathbf{u})^T\mathbf{v})$. In proposition 1.2, we find a nice fundamental domain in $\widehat{\mathsf{FQ}\times\mathsf{FQ}}$ for the orbits $\widehat{\mathsf{FQ}\times\mathsf{FQ}}/\widehat{\mathbf{A}}$. That is, we find a region in $\widehat{\mathsf{FQ}\times\mathsf{FQ}}$ that contains a unique representative of each $\widehat{\mathbf{A}}$-orbit. Because of the particulars of the λ_q-continued fraction algorithm, the aforementioned fundamental domain turns out to be identifiable with the quotient $\mathrm{SL}(2,\mathbb{R})/G_q$ (up to a set of measure zero). Finally, we describe the return maps of the horocycle and geodesic flows to the boundary of the said domain, giving cross sections to the said flow.

1.3 Main results

We below overview the main results of this paper. To achieve our goal of presenting a useful, unified description of explicit cross sections to the geodesic and horocycle flows on $\mathrm{SL}(2,\mathbb{R})/G_q$, and providing invariant measures for the first return maps to the geodesic cross section, we go through the following three steps.

1. We start with the suspension $S_{R_q}\mathcal{T}^q$ of the horocycle flow on $\mathrm{SL}(2,\mathbb{R})/G_q$ over the G_q-Farey triangle $\mathcal{T}^q$ from theorem 2.1, and reparametrize it into another suspension $S\mathcal{P}^q$ over what we call the *G_q-Stern-Brocot polygon* $\mathcal{P}^q$. The rationale behind this reparametrization is that it provides a simple way to work with the geodesic flow and its coding. It is possible to work directly with $S_{R_q}\mathcal{T}^q$, but that obfuscates the coding, and is mechanically more demanding than going through the reparametrization, and then working with $S\mathcal{P}^q$.
2. Parametrise the sides of $S\mathcal{P}^q$. This gives an explicit cross section to the geodesic flow using a planar region with infinite area. As a by product, we get a natural extension to the λ_q-continued fraction algorithm, and an infinite invariant measure for a parametrization of the algorithm that we refer to as the *symmetric G_q-Farey map* $\mathcal{F}_q$.
3. We accelerate the map $\mathcal{F}_q$ into a another map $\mathcal{G}_q$ that has finite invariant measure. The corresponding natural extension gives a cross section to the geodesic flow using a planar region with finite area.

1.3.1 The suspension $S\mathcal{P}^q$ of the horocycle flow $h_\cdot \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$ over the Stern-Brocot polygon $\mathcal{P}^q$

The proposition stated below is follows from a previous result on a cross section to the horocycle flow on $\mathrm{SL}(2,\mathbb{R})/G_q$ (theorem 2.1), and the Arnoux-Nogueira heuristic (proposition 1.1). Motivated by the triangles $\mathcal{T}^q$ from theorem 2.1 being referred to as *G_q-Farey triangles*, we refer to the polygons $\mathcal{P}^q$ in proposition: polygon P is a cross section to the horocycle flow below as *G_q-Stern-Brocot polygons.* (In [14], the vertices

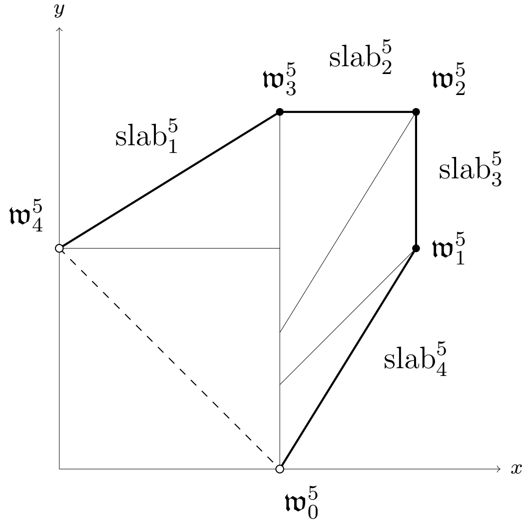

Figure 2: The polygon $\mathcal{P}^5$ from proposition 1.2, along with the slabs $\left\{\mathrm{slab}_i^5\right\}_{i=1}^4$ in the proof of the aforementioned proposition.

of the Stern-Brocot polygons are involved in the G_q-Stern-Brocot process for Λ_q.)

Proposition 1.2. *Let the polygon $\mathcal{P}^q \subset \mathsf{FQ}$ be the convex hull of the points $\mathfrak{w}_0^q, \cdots, \mathfrak{w}_{q-1}^q$, with the line segment joining $\mathfrak{w}_0^q$ and $\mathfrak{w}_{q-1}^q$ removed, and denote by*

$$S\mathcal{P}^q := \{(\mathbf{u}, \mathbf{v}) \in \widehat{\mathsf{FQ} \times \mathsf{FQ}} \mid \mathbf{u} \in \mathcal{P}^q\} \tag{1.14}$$

the portion of $\widehat{\mathsf{FQ} \times \mathsf{FQ}}$ lying above the polygon $\mathcal{P}^q$. Then the G_q-cosets corresponding to the matrices in $\Phi(\widehat{\mathsf{FQ} \times \mathsf{FQ}})$ can be bijectively identified with the points in the suspension $S_{R_q}\mathcal{T}^q$ of the horocycle flow $h. \curvearrowright \mathrm{SL}(2, \mathbb{R})/G_q$ over the Farey triangle $\mathcal{T}^q$. Consequently, the base $\mathcal{P}^q$ of $S\mathcal{P}^q$ is a cross section to the horocycle flow $h. \curvearrowright \mathrm{SL}(2, \mathbb{R})/G_q$, and the open side of $S\mathcal{P}^q$ lying above the line segment joining $\mathfrak{w}_0^q$ and $\mathfrak{w}_{q-1}^q$ is a cross section to the geodesic flow $g. \curvearrowright \mathrm{SL}(2, \mathbb{R})/G_q$.

In light of proposition 1.1, the suspension $S\mathcal{P}^q$ can be parametrised as

$$S\mathcal{P}^q = \left\{((a, b), s) \in \mathsf{FQ} \times \mathbb{R} \mid (a, b) \in \mathcal{P}^q, s \in \left[0, \frac{1}{ab}\right)\right\}.$$

An explicit identification of the top ($s = 1/(ab)$) and bottom ($s = 0$) of $S\mathcal{P}^q$ that involves the BCZ_q map from theorem 2.1 can be derived from the proof of proposition 1.2. (We have no use for this identification in the current paper, and we omit it.) In proposition 3.1, we parametrize the sides of the suspension $S\mathcal{P}^q$ lying above the sides of the polygon $\mathcal{P}^q$. This provids the sought for explicit cross section to the geodesic flow $g. \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$. Moreover, tracking the successive closed sides of $S\mathcal{P}^q$ that the geodesic orbit of a point hits gives a discrete coding of the geodesic flow $g. \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$ using $q-1$ symbols.

Finally, it should be noted that the side identification of the suspension $S\mathcal{P}^q$ in proposition 3.1 involves the matrices $\{M_i^q\}_{i=0}^{q-2}$ defining the λ_q-continued fraction algorithm, hence explaining its relevance to the current work.

1.3.2 The symmetric Farey and Gauss interval maps for the Hecke triangle group G_q and their natural extensions

It is immediate that the itineraries of the λ_q-continued fraction algorithm only depend on the slope of the given vector. We are thus motivated to projectivize the algorithm, and for that we choose the parametrized line segment $\{(a,1-a)^T \in \mathbb{R}^2 \mid a \in (0,1]\}$ joining the vertices $\mathfrak{w}_0^q = (1,0)^T$ and $\mathfrak{w}_{q-1}^q = (0,1)^T$ of the G_q-Stern-Brocot polygon $\mathcal{P}^q$. As we will see in theorem 4.1, this choice produces the *symmetric G_q-Farey map* $\mathcal{F}_q : (0,1] \to (0,1]$ given for any $a \in I_i^q := \left(\frac{1}{1+\mathrm{slope}(\mathfrak{w}_{i+1}^q)}, \frac{1}{1+\mathrm{slope}(\mathfrak{w}_i^q)}\right]$, with $i = 0, 1, \cdots, q-2$, by

$$\mathcal{F}_q(a) := \frac{1}{1+\mathrm{slope}((M_i^q)^{-1}(a,1-a)^T)}.$$

In theorem 4.1, we show that the side identification of the suspension $S\mathcal{P}^q$ over the G_q-Stern-Brocot polygon $\mathcal{P}^q$ is a natural extension of $\mathcal{F}_q$, giving the infinite $\mathcal{F}_q$-invariant measure with density

$$d\mu_{\mathcal{F}_q} = \frac{da}{a(1-a)}.$$

In theorem 4.2, we accelerate the symmetric G_q-Farey map and its natural extension to get the symmetric G_q-Gauss map $\mathcal{G}_q$ and a finite $\mathcal{G}_q$-invariant measure $d\mu_{\mathcal{G}_q}$.

Remark 1.1. As is usually the case with Farey-like maps, there is an ambiguity when it comes to defining the function at the points corresponding to the vectors in Λ_q. We resolve this ambiguity in what follows by adding the right end points to the sets S, V_i^q, and H_i^q for all $i \in \{0, 1, \cdots, q-2\}$ from proposition 3.1 when we use them in this section. This corresponds to the intervals $\{I_i^q\}_{i=0}^{q-2}$ in theorem 4.1 being closed from the right. We made this particular choice as it gives nice itineraries for the "ambiguous" points, and also because this agrees with the choice we made for the continued fraction algorithm in [14].

2 Preliminaries, the G_q-BCZ map, and the suspension $S_{R_q}\mathcal{T}^q$ of the horocycle flow $h_\cdot \curvearrowright \mathrm{SL}(2, \mathbb{R})/G_q$

We below review some notation, properties of vectors and matrices that are related to the groups G_q, and an explicit cross section to the horocycle flow $h_\cdot \curvearrowright \mathrm{SL}(2, \mathbb{R})/G_q$ from [14].

2.1 Notation

For any integer $n \geq 1$, we at some points write

$$[n] = \{0, 1, \cdots, n-1\} \tag{2.1}$$

for convenience.

Given two vectors $\mathbf{u}_0 = (x_0, y_0)^T, \mathbf{u}_1 = (x_1, y_1)^T \in \mathbb{R}^2$, we denote their *(scalar) wedge product* by

$$\mathbf{u}_0 \wedge \mathbf{u}_1 = x_0 y_1 - x_1 y_0, \tag{2.2}$$

and their *dot product* by

$$\mathbf{u}_0 \cdot \mathbf{u}_1 = x_0 x_1 + y_0 y_1. \tag{2.3}$$

Finally, we write

$$h_s := \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}, \tag{2.4}$$

for $s \in \mathbb{R}$,

$$g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \tag{2.5}$$

for $t \in \mathbb{R}$, and

$$g_{a,b} = g_{(a,b)^T} := \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \tag{2.6}$$

for $a > 0$, and $b \in \mathbb{R}$. The above matrices satisfy the identities $g_{e^t,0} = g_t$, $h_s h_t = h_{s+t}$, and $h_s g_t = g_t h_{se^{2t}}$. Left multplication on $\mathrm{SL}(2,\mathbb{R})/G_q$ by $(h_s)_{s\in\mathbb{R}}$ (resp. $(g_t)_{t\in\mathbb{R}}$) corresponds to the horocycle flow (resp. geodesic flow) on $\mathrm{SL}(2,\mathbb{R})/G_q$.

2.2 Some properties of the vectors $\{\mathfrak{w}_i^q\}_{i=0}^{q-1}$ and matrices $\{M_i^q\}_{i=0}^{q-2}$

We have the following elementary properties from [14] of the vectors $\{\mathfrak{w}_i^q\}_{i=0}^{q-1}$ and the matrices $\{M_i^q\}_{i=0}^{q-2}$ that we use throughout the paper.

Proposition 2.1 ([14]). *The following are true.*

1. *The vectors $\{\mathfrak{w}_i^q\}_{i=0}^{q-1}$ lie on the ellipse $Q_q(x,y) := x^2 - \lambda_q xy + y^2 = 1$.*
2. *For any $i = 0, 1, \cdots, q-2$, we have the Farey neighbor/unimodularity identitiy*

$$\mathfrak{w}_i^q \wedge \mathfrak{w}_{i+1}^q = x_i^q y_{i+1}^q - x_{i+1}^q y_i^q = 1, \tag{2.7}$$

   along with

$$\mathfrak{w}_0^q \wedge \mathfrak{w}_{q-1}^q = 1. \tag{2.8}$$

3. *The set $\Lambda_q = G_q(1,0)^T$ is symmetric against the line $y = x$, and so*

$$(M_i^q)^T = M_{q-2-i}^q \tag{2.9}$$

for all $i = 0, 1, \cdots, q-2$.

2.3 The G_q-BCZ maps, and cross sections to the horocycle flow $h. \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$

We here present our main result from [14] on a cross section to the horocycle flow $h. \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$. The result essentially says the following.

- The orbit of the "G_q-lattice" Λ_q under the linear action of $\mathrm{SL}(2,\mathbb{R})$ on the plane $\mathbb{R}^2$ can be identified with $\mathrm{SL}(2,\mathbb{R})/G_q$. This is an extension of the classical well-known fact that the space of unidmodular lattices can be identified with the cosets $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$.
- The G_q-Farey triangle $\mathcal{T}^q$, indentified as a subset of $\mathrm{SL}(2,\mathbb{R})/G_q$ via the matrices from eq. (2.6), is a cross section to the horocycle flow $h. \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$, with R_q as a roof function. (In [14], we show that the suspension $S_{R_q}\mathcal{T}^q$ is $\mathrm{SL}(2,\mathbb{R})/G_q$ minus particular closed horocycles, which constitute a null set with respect to the Haar measure on $\mathrm{SL}(2,\mathbb{R})/G_q$.)

In the proof of proposition 1.2, we show that the suspension $S_{R_q}\mathcal{T}^q$ can be chopped into "slabs" and rearranged as the suspension $S\mathcal{P}^q$ in proposition 1.2, hence revealing an intimate relationship between the definition of the roof function R_q and the coding of the geodesic flow $g. \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$. That, and the above two points, are the reasons we need theorem 2.1 here.

Theorem 2.1 ([14])**.** *For* $\tau > 0$*, we write* $S_\tau := \{(a,b) \in \mathbb{R}^2 \mid 0 < a \leq \tau\}$*. The following are true.*

1. *For any* $B \in \mathrm{SL}(2,\mathbb{R})$*,* $B\Lambda_q = \Lambda_q$ *if and only if* $B \in G_q$*. From this follows that the sets* $C\Lambda_q$*, with* C *varying over* $\mathrm{SL}(2,\mathbb{R})$*, can be identified with the elements of* $\mathrm{SL}(2,\mathbb{R})/G_q$*.*
2. *For any* $A \in \mathrm{SL}(2,\mathbb{R})$*, if* $A\Lambda_q$ *has a horizontal vector of length not exceeding* 1 *(i.e. a horizontal vector in* $A\Lambda_q \cap S_1$*), then* $A\Lambda_q$ *can be uniquely identified with a*

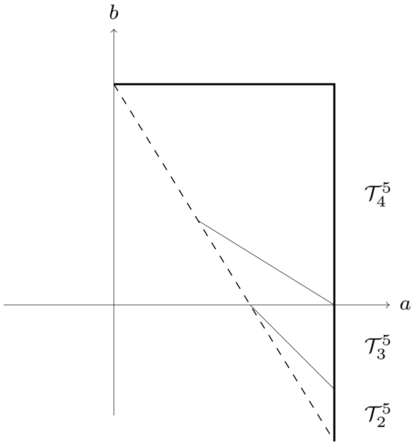

Figure 3: The G_5-Farey triangle $\mathcal{T}^5$ with the subregions $\mathcal{T}_2^5$, $\mathcal{T}_3^5$, and $\mathcal{T}_4^5$ from theorem 2.1 indicated.

point (a_A, b_A) *in the* G_q-Farey triangle

$$\mathcal{T}^q = \{(a,b) \in \mathbb{R}^2 \mid 0 < a \leq 1,\ 1 - \lambda_q a < b \leq 1\} \tag{2.10}$$

through $B\Lambda_q = g_{a_A,b_A}\Lambda_q$. *Moreover, the value* a_A *agrees with the length of the horizontal vector in* $A\Lambda_q \cap S_1$.

3. *Let* $(a,b) \in \mathcal{T}^q$ *be any point in the* G_q*-Farey triangle. The set* $g_{a,b}\Lambda_q \cap S_1$ *has a vector with smallest positive slope. Consequently, there exists a smallest* $s = R_q(a,b) > 0$ *such that* $h_s g_{a,b}\Lambda_q$ *has a horizontal vector of length not exceeding* 1, *and hence* $h_s g_{a,b}\Lambda_q$ *corresponds to a unique point* $\mathrm{BCZ}_q(a,b) \in \mathcal{T}^q$ *in the* G_q*-Farey triangle. The function* $R_q : \mathcal{T}^q \to \mathbb{R}_+$ *is referred to as the* G_q-roof function, *and the map* $\mathrm{BCZ}_q(a,b) : \mathcal{T}^q \to \mathcal{T}^q$ *is referred to as the* G_q-BCZ map.

4. *The* G_q*-Farey triangle* $\mathcal{T}^q$ *can be partitioned into the union of*

$$\mathcal{T}_i^q := \{(a,b) \in \mathcal{T}^q \mid (a,b)^T \cdot \mathfrak{w}_{i-1} > 1,\ (a,b)^T \cdot \mathfrak{w}_i \leq 1\}, \tag{2.11}$$

with $i = 2, 3, \cdots, q-1$*, such that if* $(a,b) \in \mathcal{T}_i^q$*, then* $g_{a,b}\mathfrak{w}_i^q$ *is the vector of least positive slope in* $g_{a,b}\Lambda_q \cap S_1$*, and*

- *the value of the roof function* $R_q(a,b)$ *is given by*

$$R_q(a,b) = R_{q,i}(a,b) := \frac{y_i^q}{a \times (a,b)^T \cdot \mathfrak{w}_i^q}, \text{ and} \tag{2.12}$$

- *the value of the BCZ map* $\mathrm{BCZ}_q(a,b)$ *is given by*

$$\mathrm{BCZ}_q(a,b) := \left((a,b)^T \cdot \mathfrak{w}_i^q, (a,b)^T \cdot \mathfrak{w}_{i+1}^q + k_i^q(a,b) \times \lambda_q \times (a,b)^T \cdot \mathfrak{w}_i^q\right),$$

where the G_q-index $k_i^q(a,b)$ *is given by*

$$k_i^q(a,b) := \left\lfloor \frac{1-(a,b)^T \cdot \mathfrak{w}_{i+1}^q}{\lambda_q \times (a,b)^T \cdot \mathfrak{w}_i^q} \right\rfloor.$$

5. *Let* X_q *be the homogeneous space* $\mathrm{SL}(2,\mathbb{R})/G_q$*,* μ_q *be the probability Haar measure on* X_q *(i.e.* $\mu_q(X_q) = 1$*), and* Ω_q *be the subset of* X_q *corresponding to sets* $A\Lambda_q$*,* $A \in \mathrm{SL}(2,\mathbb{R})$*, with a horizontal vector of length not exceeding* 1*. (Note that* Ω_q *can be identified with the Farey triangle* $\mathcal{T}^q$ *via* $((a,b) \in \mathcal{T}^q) \mapsto (g_{a,b}G_q \in \Omega_q)$*.) Finally, let* $m_q = \frac{2}{\lambda_q} dadb$ *be the Lebesgue probability measure on* $\mathcal{T}^q$*. Then the triple* $(\mathcal{T}^q, m_q, \mathrm{BCZ}_q)$*, with* $\mathcal{T}^q$ *identified with* Ω_q*, is a cross section to* $(X_q, \mu_q, h_\cdot)$*, with roof function* R_q*.*

3 The suspension $S\mathcal{P}^q$ of the horocycle flow $h_\cdot \curvearrowright \mathrm{SL}(2,\mathbb{R})/G_q$

In this section, we prove proposition 1.2, and give an explicit parametrisation of the side identification of $S\mathcal{P}^q$ in proposition 3.1.

3.1 Proof of proposition 1.2

Proof. Consider the G_q-cosets of $h_s g_{a,b}$, with $(a,b)^T$ belonging to the triangle that is the convex hull of the three vectors $\mathfrak{w}_0^q = (1,0)^T, (1,1)^T, \mathfrak{w}_{q-1}^q = (0,1)^T$ with the line segment between $\mathfrak{w}_0^q$ and $\mathfrak{w}_{q-1}^q$ removed, and $s \in \left[0, \frac{1}{ab}\right)$. By proposition 1.1, the cosets in question can be bijectively identified with the portions of the suspensions $S_{R_q}\mathcal{T}^q$ and $S\mathcal{P}^q$ above the aforementioned triangle. (Note that for $S_{R_q}\mathcal{T}^q$, the roof function R_q satisfies $R_q(a,b) = \frac{1}{ab}$ when $(a,b)^T$ belongs to the aforementioned triangle that belongs to $\mathcal{T}_{q-1}^q$.) It thus remains to identify the remainder of the suspensions $S_{R_q}\mathcal{T}^q$ and $S\mathcal{P}^q$.

Let $\mathcal{T}^{q\prime}$ be the portion of the Farey triangle $\mathcal{T}^q$ outside the triangle from the previous paragraph. We now partition the suspension $S_{R_q}\mathcal{T}^{q\prime}$ into "slabs" as follows: Let

$$\mathrm{slab}_1^q := \{h_s g_{a,b} \mid (a,b)^T \in \mathcal{T}^{q\prime}, s \in [R_{q,0}(a,b), R_{q,1}(a,b))\},$$

and for $i = 2, \cdots, q-1$, let

$$\mathrm{slab}_i^q := \{h_s g_{a,b} \mid (a,b)^T \in \mathcal{T}^{q\prime}, (a,b)^T \cdot \mathfrak{w}_{i-1} > 1, s \in [R_{q,i-1}(a,b), R_{q,i}(a,b)\}.$$

Note that for any $i = 2, 3, \cdots, q-1$, the collection of points $(a,b)^T \in \mathcal{T}^{q\prime}$ satisfying $(a,b)^T \cdot \mathfrak{w}_{i-1} > 1$ is exactly the union $\cup_{j=i}^{q-1}\mathcal{T}_j^q$, and that for all $(a,b)^T \in \mathcal{T}^q$ with $(a,b)^T \cdot \mathfrak{w}_{i-1}^q > 1$ (and necessarily $(a,b)^T \cdot \mathfrak{w}_i^q > 0$) that

$$\begin{aligned} R_{q,i}(a,b) - R_{q,i-1}(a,b) &= \frac{y_i^q}{a(x_i^q a + y_i^q b)} - \frac{y_{i-1}^q}{a(x_{i-1}^q a + y_{i-1}^q b)} \\ &= \frac{1}{a}\left(\frac{(x_{i-1}^q y_i^q - x_i^q y_{i-1}^q)a}{(x_i^q a + y_{i-1}^q b)(x_{i-1}^q a + y_{i-1}^q b)}\right) \\ &= \frac{1}{((a,b)^T \cdot \mathfrak{w}_i)((a,b)^T \cdot \mathfrak{w}_{i-1}^q)} \\ &> 0, \end{aligned}$$

where we used the fact that $\mathfrak{w}_{i-1}^q \wedge \mathfrak{w}_i^q = x_{i-1}^q y_i^q - x_i^q y_{i-1}^q = 1$. For any $(a,b)^T \in \mathcal{T}^q$, we have $\lambda_q a + b > 1$, and so

$$R_{q,1}(a,b) - R_{q,0}(a,b) = \frac{1}{a(\lambda_q a + b)} - 0 > 0.$$

This implies that the slabs $\mathrm{slab}_1^q, \cdots, \mathrm{slab}_{q-1}^q$ form a partition of $S_{R_q}\mathcal{T}^{q\prime}$. For any

$i = 1, 2, \cdots, q-1$, if $h_s g_{a,b} \in \mathrm{slab}_i^q$, then by proposition 1.1 we have that

$$\begin{aligned} h_s g_{a,b} M_{i-1}^q &= \Phi\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \frac{1}{a} - bs \\ as \end{pmatrix}\right) M_{i-1}^q \\ &= \Phi\left((M_{i-1}^q)^T \begin{pmatrix} a \\ b \end{pmatrix}, (M_{i-1}^q)^{-1} \begin{pmatrix} \frac{1}{a} - bs \\ as \end{pmatrix}\right) \\ &= \Phi\left(\begin{pmatrix} (a,b)^T \cdot \mathfrak{w}_{i-1}^q \\ (a,b)^T \cdot \mathfrak{w}_i^q \end{pmatrix}, M_{i-1}^{-1} \begin{pmatrix} \frac{1}{a} - bs \\ as \end{pmatrix}\right). \end{aligned}$$

It is evident that $\begin{pmatrix} (a,b)^T \cdot \mathfrak{w}_{i-1}^q \\ (a,b)^T \cdot \mathfrak{w}_i^q \end{pmatrix} \in \mathsf{FQ}$. When $s = R_{q,i-1}(a,b) = \frac{y_{i-1}^q}{a(a,b)^T \cdot \mathfrak{w}_{i-1}^q}$ we have that

$$\begin{aligned} (M_{i-1}^q)^{-1} \begin{pmatrix} \frac{1}{a} - bs \\ as \end{pmatrix} &= \begin{pmatrix} y_i^q & -x_i^q \\ -y_{i-1}^q & x_{i-1}^q \end{pmatrix} \begin{pmatrix} \frac{x_{i-1}^q}{(a,b)^T \cdot \mathfrak{w}_{i-1}^q} \\ \frac{y_{i-1}^q}{(a,b)^T \cdot \mathfrak{w}_{i-1}^q} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{(a,b)^T \cdot \mathfrak{w}_{i-1}^q} \\ 0 \end{pmatrix}, \end{aligned}$$

and when $s = R_{q,i}(a,b) = \frac{y_i^q}{a(a,b)^T \cdot \mathfrak{w}_i^q}$ we have that

$$\begin{aligned} (M_{i-1}^q)-1 \begin{pmatrix} \frac{1}{a} - bs \\ as \end{pmatrix} &= \begin{pmatrix} y_i^q & -x_i^q \\ -y_{i-1}^q & x_{i-1}^q \end{pmatrix} \begin{pmatrix} \frac{x_i^q}{(a,b)^T \cdot \mathfrak{w}_i^q} \\ \frac{y_i^q}{(a,b)^T \cdot \mathfrak{w}_i^q} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \frac{1}{(a,b)^T \cdot \mathfrak{w}_i^q} \end{pmatrix}, \end{aligned}$$

where we used the fact that $\mathfrak{w}_{i-q}^q \wedge \mathfrak{w}_i^q = x_{i-1}^q y_i^q - x_i^q y_{i-1}^q = 1$. This implies by proposition 1.1 that as s varies in the interval $[R_{q,i-1}(a,b), R_{q,i}(a,b))$, then $h_s g_{a,b}$ bijectively identifies with the points $(\mathbf{u}, \mathbf{v}) \in S\mathcal{P}^q$ with $\mathbf{u} = ((a,b)^T \cdot \mathfrak{w}_{i-1}^q, (a,b)^T \cdot \mathfrak{w}_i^q)^T$. It thus remains to show that the bases of the slabs $\mathrm{slab}_1^q, \cdots, \mathrm{slab}_{q-1}^a$ are mapped by $(M_0^q)^T, \cdots, (M_{q-2}^q)^T$ into a partition of the remaining part of the polygon $\mathcal{P}^q$.

The base of the slab slab^q_1 is a triangle with vertices $(1, 1-\lambda_q)^T, (1,0)^T, (0,1)^T$ (with the line segment between $(1, 1-\lambda_q)$ and $(0,1)^T$ removed), and the vertices in question are mapped by $(M^q_0)^T = \begin{pmatrix} 1 & 0 \\ \lambda_q & 1 \end{pmatrix}$ to $(1,1)^T, (1,\lambda_q)^T = \mathfrak{w}^q_{q-2}, (0,1)^T = \mathfrak{w}^q_{q-1}$. For a fixed $i = 2, 3, \cdots, q-1$, the base of the slab slab^q_i is a quadrilateral[1] with vertices $A^q_i = \left(1, \frac{1-x^q_{i-1}}{y^q_{i-1}}\right), (1,0)^T, (0,1)^T, B^q_i = \left(\frac{y^q_{i-1}-1}{\lambda_q y^q_{i-1} - x^q_{i-1}}, \frac{\lambda_q - x^q_{i-1}}{\lambda_q y^q_{i-1} - x^q_{i-1}}\right)$ (with both the line segment between $(0,1)^T$ and B^q_i, and the line segment between B^q_i and A^q_i removed). (The point A^q_i is the solution of the two equations $a = 1$ and $(a,b)^T \cdot \mathfrak{w}^q_{i-1} = 1$, and B^q_i is the solution of $\lambda_q a + b = 1$ and $(a,b)^T \cdot \mathfrak{w}^q_{i-1} = 1$.) The matrix $(M^q_{i-1})^T = M^q_{q-1-i}$ maps the point $(1,0)^T$ to $\mathfrak{w}^T_{q-1-i}$, the point $(0,1)^T$ to $\mathfrak{w}^T_{q-i}$, the point A^q_i to

$$\begin{aligned}
(M^q_{i-1})^T A^q_i &= \begin{pmatrix} x^q_{i-1} & y^q_{i-1} \\ x^q_i & y^q_i \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1-x^q_{i-1}}{y^q_{i-1}} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \frac{y^q_i - 1}{y^q_{i-1}} \end{pmatrix},
\end{aligned}$$

and the point B^q_i to

$$\begin{aligned}
(M^q_{i-1})^T B^T_i &= \begin{pmatrix} x^q_{i-1} & y^q_{i-1} \\ x^q_i & y^q_i \end{pmatrix} \begin{pmatrix} \frac{y^q_{i-1}-1}{\lambda_q y^q_{i-1} - x^q_{i-1}} \\ \frac{\lambda_q - x^q_{i-1}}{\lambda_q y^q_{i-1} - x^q_{i-1}} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \frac{\lambda_q y^q_i - x^q_i + x^q_i y^q_{i-1} - x^q_{i-1} y^q_i}{\lambda_q y^q_{i-1} - x^q_{i-1}} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \frac{y^q_{i-1} - 1}{y^q_{i-2}} \end{pmatrix}
\end{aligned}$$

where we used the facts that $\mathfrak{w}^q_{i-1} \wedge \mathfrak{w}^q_i = x^q_{i-1} y^q_i - x^q_i y^q_{i-1} = 1$, $\begin{pmatrix} x^q_{i-1} \\ y^q_{i-1} \end{pmatrix} = \begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} x^q_i \\ y^q_i \end{pmatrix}$,

---

[1] It should be noted that for slab^q_2, the points B^q_2 and $(0,1)^T$ agree, and for slab^q_{q-1}, the points A^q_{q-1} and $(1,0)^T$ agree. We also go through our computations with the understanding that $0/0 = 1$.

and $\begin{pmatrix} x^q_{i-2} \\ y^q_{i-2} \end{pmatrix} = \begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} x^q_{i-1} \\ y^q_{i-1} \end{pmatrix}$. For $i = 3, \cdots, q-1$, we have that $M^T_{i-1} B^q_i = M^T_{i-2} A^q_{i-1}$ is a point on the line $a = 1$. Moreover, $(M^q_{q-2})^T A^T_{q-1} = (1, 0)^T = \mathfrak{w}^q_1$, and $(M^q_1)^T B^q_2 = (1, \lambda_q)^T = \mathfrak{w}^q_{q-2}$. This proves that bases of $\{\mathrm{slab}^q_i (M^q_{i-1})^T\}^{q-1}_{i=1}$ partition $\mathcal{P}^q$ as required.

That $\mathcal{P}^q$ forms a cross section to the horocycle flow follows from proposition 1.1 and the fact that $\mathcal{T}^q$ is such a cross section.

That the sides of $S\mathcal{P}^q$ follows from proposition 1.1, and the fact that any ray in FQ passing through the origin intersects the sides of $\mathcal{P}^q$. This is expanded on in proposition 3.1 and its proof. $\square$

3.2 Side identification of the suspension $S\mathcal{P}^q$

Proposition 3.1. *Consider the set*

$$\mathsf{S} := \left\{(a, s) \in \mathbb{R}^2 \mid a \in (0, 1), s \in \left[0, \frac{1}{a(1-a)}\right)\right\} \bigcup \left\{(1, s) \in \mathbb{R}^2 \mid s \in [0, \lambda_q)]\right\} \quad (3.1)$$

endowed with the Lebesgue measure $d\mu_{\mathsf{S}} = dads$, and understood, in light of proposition 1.1, to be parametrizing the portion of $\widehat{\mathsf{FQ} \times \mathsf{FQ}}$ lying above the line segment joining the vectors $\mathfrak{w}^q_0$ and $\mathfrak{w}^q_{q-1}$ via $(a, s) \mapsto h_s g_{a,1-a}$. Similarly, for $i = 0, 1, \cdots, q-2$, consider the set

$$\mathsf{S}^q_i := \begin{cases} \left\{(\alpha, \sigma) \in \mathbb{R}^2 \mid \alpha \in (0, 1), s \in \left[0, \frac{1}{(\alpha x^q_i + (1-\alpha) x^q_{i+1})(\alpha y^q_i + (1-\alpha) y^q_{i+1})}\right)\right\}, & i = 0 \\ \left\{(\alpha, \sigma) \in \mathbb{R}^2 \mid \alpha \in (0, 1], s \in \left[0, \frac{1}{(\alpha x^q_i + (1-\alpha) x^q_{i+1})(\alpha y^q_i + (1-\alpha) y^q_{i+1})}\right)\right\}, & i \neq 0 \end{cases} \quad (3.2)$$

endowed with the Lebesgue measure $d\mu_{\mathsf{S}^q_i} = d\alpha d\sigma$, and understood to be parametrising the portion of $\widehat{\mathsf{FQ} \times \mathsf{FQ}}$ lying above the line segment joining the vectors $\mathfrak{w}^q_i$ and $\mathfrak{w}^q_{i+1}$ via $(\alpha, \sigma) \mapsto h_\sigma g_{\alpha \mathfrak{w}^q_i + (1-\alpha) \mathfrak{w}^q_{i+1}}$. Futhermore, consider the partitioning of S into $q-1$

horizontal strips

$$\mathsf{H}_i^q := \{(a,s) \in \mathsf{S} \mid s \in [R_{q,i}(a,1-a), R_{q,i+1}(a,1-a))\} \tag{3.3}$$

with $i \in [q-1]$*, and* $q-1$ *vertical strips*

$$\mathsf{V}_i^q := \begin{cases} \left\{(a,s) \in \mathsf{S} \mid \mathrm{slope}((a,1-a)^T) \in \left(\mathrm{slope}(\mathfrak{w}_i^q), \mathrm{slope}(\mathfrak{w}_{i+1}^q)\right)\right\}, & i = 0 \\ \left\{(a,s) \in \mathsf{S} \mid \mathrm{slope}((a,1-a)^T) \in \left[\mathrm{slope}(\mathfrak{w}_i^q), \mathrm{slope}(\mathfrak{w}_{i+1}^q)\right)\right\}, & i \neq 0 \end{cases} \tag{3.4}$$

with $i \in [q-1]$*. The following are true.*

1. *For any* $i \in [q-1]$*, and for any* $(a,s) \in \mathsf{V}_i^q$*, the smallest* $t > 0$ *such that — in light of proposition 1.1 and the above parametrizations —* $g_t h_s g_{a,1-a}$ *is in* S_i^q *is* $t = -\log \rho_i^q(a)$*, where*

$$\rho_i^q(a) = (x_{i+1}^q - y_i^q)a + (x_i^q - x_{i+1}^q). \tag{3.5}$$

   The map $\mathsf{V}_i^q \to \mathsf{S}_i^q$ *induced by*

$$h_s g_{a,1-a} \mapsto h_\sigma g_{\alpha\mathfrak{w}_i^q + (1-\alpha)\mathfrak{w}_{i+1}^q} = g_{-\log \rho_i^q(a)} h_s g_{a,1-a} \tag{3.6}$$

   is bijective, and its Jacobian determinant is equal to 1*.*

2. *For any* $i \in [q-1]$*, and for any* $(\alpha,\sigma) \in \mathsf{S}_i^q$*, the map* $\mathsf{S}_i^q \to \mathsf{H}_{q-2-i}^q$ *induced by*

$$h_\sigma g_{\alpha\mathfrak{w}_i^q + (1-\alpha)\mathfrak{w}_{i+1}^q} \mapsto h_s g_{a,1-a} = h_\sigma g_{\alpha\mathfrak{w}_i^q + (1-\alpha)\mathfrak{w}_{i+1}^q} ((M_i^q)^{-1})^T \tag{3.7}$$

   is bijective, and its Jacobian determinant is equal to 1*.*

Proof. We begin by proving the first numbered claim. The bijectivity of the map $\mathsf{V}_i^q \to \mathsf{S}_i^q$ in question follows from proposition 1.1, and the fact that a point $((a,b)^T, (1/a,0)^T)$ on the "floor" of $\widehat{\mathsf{FQ} \times \mathsf{FQ}}$ is mapped by the geodesic flow to another point $((e^t a, e^t b)^T, (1/(e^t a), 0)^T)$ on the floor, and a point $((a,b)^T, (0,1/b)^T)$ on the "roof" of $\widehat{\mathsf{FQ} \times \mathsf{FQ}}$ is mapped by the geodesic flow to another point $((e^t a, e^t b)^T, (0, 1/(e^t b))^T)$ on the roof. It thus remains to compute the hit times and the Jacobian determinant of the map $\mathsf{V}_i^q \to \mathsf{S}_i^q$. The equation of the straight line L_i^q going through the points $\mathfrak{w}_i^q$ and $\mathfrak{w}_{i+1}^q$ is $(y_{i+1}^q - y_i^q)a + (-x_{i+1}^q + x_i^q)b = 1$. The line L_i^q and the line going through the origin $(0,0)^T$ and $(a, 1-a)^T$ intersect at the point

$$\left(\frac{a}{(y_{i+1}^q - y_i^q)a + (-x_{i+1}^q + x_i^q)(1-a)}, \frac{1-a}{(y_{i+1}^q - y_i^q)a + (-x_{i+1}^q + x_i^q)(1-a)}\right)^T = \left(\frac{a}{\rho_i^q(a)}, \frac{1-a}{\rho_i^q(a)}\right)^T,$$

and the hitting time is $t = \log \frac{1}{\rho_i^q(a)}$ as claimed. We have that $\begin{pmatrix} x_{i+1}^q \\ y_{i+1}^q \end{pmatrix} = \begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_i^q \\ y_i^q \end{pmatrix}$, and so we get the expression for $\rho_i^q(a)$ in the statement. We thus have

$$\begin{aligned}
h_\sigma g_{\alpha\mathfrak{w}_i^q + (1-\alpha)\mathfrak{w}_{i+1}^q} &= g_{-\log\rho_i^q(a)} h_s g_{a,1-a} \\
\begin{pmatrix} 1 & 0 \\ -\sigma & 1 \end{pmatrix} \begin{pmatrix} \alpha x_i^q + (1-\alpha)x_{i+1}^q & \alpha y_i^q + (1-\alpha)y_{i+1}^q \\ 0 & * \end{pmatrix} &= \begin{pmatrix} \frac{1}{\rho_i^q(a)} & 0 \\ 0 & \rho_i^q(a) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix} \begin{pmatrix} a & 1-a \\ 0 & * \end{pmatrix} \\
\begin{pmatrix} \alpha x_i^q + (1-\alpha)x_{i+1}^q & \alpha y_i^q + (1-\alpha)y_{i+1}^q \\ -\sigma(\alpha x_i^q + (1-\alpha)x_{i+1}^q) & * \end{pmatrix} &= \begin{pmatrix} \frac{a}{\rho_i^q(a)} & \frac{1-a}{\rho_i^q(a)} \\ -as\rho_i^q(a) & * \end{pmatrix}
\end{aligned}$$

If $x_i^q - x_{i+1}^q \neq 0$, we use the identity $\alpha x_i^q + (1-\alpha)x_{i+1}^q = \frac{a}{\rho_i^q(a)}$ to compute $\frac{d\alpha}{da}$ and $\frac{d\alpha}{ds}$. If $x_i^q - x_{i+1}^q = 0$, then $y_i^q - y_{i+1}^q \neq 0$, and we use the identity $\alpha y_i^q + (1-\alpha)y_{i+1}^q = \frac{1-a}{\rho_i^q(a)}$ instead to find the aforementioned derivatives. We proceed assuming that $x_i^q - x_{i+1}^q \neq 0$, with the other case being similar. We thus have that

$$\frac{d\alpha}{da} = \frac{1}{x_i^q - x_{i+1}^q} \frac{\rho_i^q(a) - (\rho_i^q)'(a)a}{\rho_i^q(a)^2} = \frac{1}{\rho_i^q(a)^2},$$

and

$$\frac{d\alpha}{ds} = 0,$$

and so we have that $\frac{\partial(\alpha,\sigma)}{\partial(a,s)} = \left|\frac{d\alpha}{da}\frac{d\sigma}{ds}\right|$. We also have that

$$\frac{d\sigma}{ds} = \frac{a\rho_i^q(a)}{\alpha x_i^q + (1-\alpha)x_{i+1}^q} = \rho_i^q(a)^2.$$

This proves that $\frac{\partial(\alpha,\sigma)}{\partial(a,s)} = 1$ as required.

We now prove the second numbered claim. By proposition 1.1, and the proof of proposition 1.2, we get that the map $\mathsf{S}_i^q \to \mathsf{H}_{q-2-i}^q$ in question is bijective. The reason is that S_i^q parametrises the side of $\mathrm{slab}_{q-1-i}^q\, M_{q-2-i}^q$ lying above the line segment joining $\mathfrak{w}_i^q$ and $\mathfrak{w}_{i+1}^q$, and H_{q-2-i}^q parametrises the side of slab_{q-1-i}^q lying above the line segment joining $(1,0)^T$ and $(0,1)^T$, and right multiplication by $(M_{q-2-i}^q)^{-1} = ((M_i^q)^{-1})^T$ maps S_i^q bijectively back to H_{q-2-i}^q. It remains to show that its Jacobian determinant is 1. We have that

$$\begin{aligned}
h_s g_{a,1-a} &= h_\sigma g_{\alpha\mathfrak{w}_i^q+(1-\alpha)\mathfrak{w}_{i+1}^q}((M_i^q)^{-1})^T \\
\begin{pmatrix} a & * \\ -as & * \end{pmatrix} &= h_\sigma \begin{pmatrix} \alpha x_i^q + (1-\alpha)x_{i+1}^q & \alpha y_i^q + (1-\alpha)y_{i+1}^q \\ 0 & * \end{pmatrix} \begin{pmatrix} y_{i+1}^q & -y_i^q \\ -x_{i+1}^q & x_i^q \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -\sigma & 1 \end{pmatrix} \begin{pmatrix} \alpha & * \\ -\frac{x_{i+1}^q}{\alpha x_i^q+(1=\alpha)x_{i+1}^q} & * \end{pmatrix} \\
&= \begin{pmatrix} \alpha & * \\ -\frac{x_{i+1}^q}{\alpha x_i^q+(1=\alpha)x_{i+1}^q} - \sigma\alpha & * \end{pmatrix}
\end{aligned}$$

where we used the fact that $\mathfrak{w}_i^q \wedge \mathfrak{w}_{i+1}^q = x_i^q y_{i+1}^q - x_{i+1}^q y_i^q = 1$. We thus have that $\frac{da}{d\alpha} = 1$, $\frac{da}{d\sigma} = 0$, and $\frac{ds}{d\sigma} = 1$, and so $\frac{\partial(a,s)}{\partial(\alpha,\sigma)} = 1$ as required. $\square$

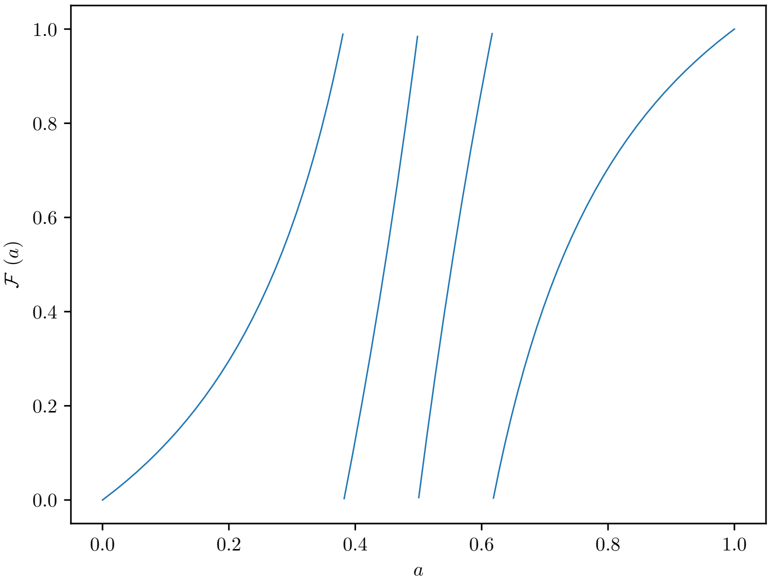

Figure 4: The symmetric G_5-Farey interval map $\mathfrak{F}_5$.

4 The symmetric Farey and Gauss interval maps for the Hecke triangle group G_q, and their natural extensions

In this section, we use the identification of the sides of $S\mathcal{P}^q$ from proposition 3.1 to produce in theorem 4.1 a model of the natural extension of the Farey map corresponding to our choice of projectivization for the λ_q-continued fraction algorithm, along with an infinite invariant measure for the Farey map. We also accelerate the Farey map and its natural extension, to get a Gauss map with finite invariant measure in theorem 4.2.

4.1 The symmetric G_q Farey interval map $\mathfrak{F}_q$, and its natural extension

Theorem 4.1. *Using the notation in proposition 3.1 and the arbitrary choice we make in remark 1.1, the map* $\widetilde{\mathfrak{F}}_q : \mathsf{S} \to \mathsf{S}$ *that for any* $i \in [q-1]$ *sends any point* $(a, s) \in \mathsf{V}_i^q \subset \mathsf{S}$

to the point (a', s') *given, according to proposition 1.1, by*

$$h_{s'} g_{a',1-a'} = g_{-\log \rho_i^q(a)} h_s g_{a,1-a} ((M_i^q)^{-1})^T \tag{4.1}$$

preserves the Lebesgue measure $d\mu_{\widetilde{F}_q} := d\mu_{\mathsf{S}} = dads$ *on* S*. Moreever, the map* $\widetilde{\mathfrak{F}}_q$ *satisfies the Markov condition* $\widetilde{\mathfrak{F}}_q(\mathsf{V}_i^q) = \mathsf{H}_{q-2-i}^q$ *for all* $i \in [q-1]$*.*

Consequently, $\widetilde{\mathfrak{F}}_q$ *is a model of the natural extension of the map* $\mathfrak{F}_q : (0,1] \to (0,1]$ *defined for all* $i \in [q-1]$ *and*

$$\begin{aligned} a \in I_i^q \quad &:= \quad \left(\frac{1}{1+\mathrm{slope}(\mathfrak{w}_{i+1}^q)}, \frac{1}{1+\mathrm{slope}(\mathfrak{w}_i^q)} \right] && (4.2) \\ &= \quad \left(\frac{x_{i+1}^q}{x_{i+1}^q + y_{i+1}^q}, \frac{x_i^q}{x_i^q + y_i^q} \right] && (4.3) \end{aligned}$$

by

$$\begin{aligned} \mathfrak{F}_q(a) \quad &:= \quad \frac{1}{1+\mathrm{slope}\left((M_i^q)^{-1}(a,1-a)^T\right)} && (4.4) \\ &= \quad \frac{(x_{i+1}^q + y_{i+1}^q)a - x_{i+1}^q}{(x_{i+1}^q - y_i^q)a + (x_i^q - x_{i+1}^q)}. && (4.5) \end{aligned}$$

Moreover, the map $\mathfrak{F}_q$ *preserves the measure*

$$d\mu_{\mathfrak{F}_q} := \frac{da}{a(1-a)} \tag{4.6}$$

on $(0,1]$*.*

Proof. The map $\widetilde{\mathfrak{F}}_q : \mathsf{S} \to \mathsf{S}$ is the composition of the maps from the two numbered claims in proposition 3.1. This proves that $\widetilde{\mathfrak{F}}_q$ preserves $d\mu_{\widetilde{F}_q}$ and satisfies the sought for Markov condition.

For any $i \in [q-1]$, if $(a, *)$ belongs to V_i^q, then $(a, 1-a)^T$ belongs to the sector Σ_i^q, and so the slope of $(a,1-a)^T$ is in $[\mathrm{slope}(\mathfrak{w}_i^q), \mathfrak{w}_{i+1}^q)$. This corresponds to $a \in I_i^q$ in the claim of the theorem. By proposition 1.1, the image $(a', *)$ of $(a, *)$ under $\widetilde{\mathfrak{F}}_q$ is given by

$$\begin{aligned}\Phi((a', 1-a')^T, *) &= g_{-\log \rho_i^q(a)}\left(\Phi((a, 1-a)^T, *)\right)((M_i^q)^{-1})^T \\ &= \Phi\left(\frac{1}{\rho_i^q(a)}(M_i^q)^{-1}(a, 1-a)^T, *\right),\end{aligned}$$

and so

$$\begin{aligned}\frac{1-a'}{a'} &= \text{slope}\left(\frac{1}{\rho_i^q(a)}(M_i^q)^{-1}(a, 1-a)^T\right) \\ &= \text{slope}\left(\begin{pmatrix} y_{i+1}^q & -x_{i+1}^q \\ -y_i^q & x_i^q \end{pmatrix}\begin{pmatrix} a \\ 1-a \end{pmatrix}\right) \\ &= \frac{-(x_i^q + y_i^q)a + x_i^q}{(x_{i+1}^q + y_{i+1}^q)a - x_{i+1}^q}.\end{aligned}$$

This, along with the fact that $\begin{pmatrix} x_{i+1}^q \\ y_{i+1}^q \end{pmatrix} = \begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_i^q \\ y_i^q \end{pmatrix}$ give the expression for $\mathcal{F}_q(a)$ in the theorem. Integrating the measure $d\mu_{\widetilde{\mathcal{F}}_q}$ along the s fibers gives $d\mu_{\mathcal{F}_q}$. This proves the claim. □

4.2 The symmetric G_q Gauss map $\mathcal{G}_q$, and its natural extension

Theorem 4.2. *Fix the notation as in remark 1.1 and theorem 4.1. Define the maps* $n_0^q, n_{q-2}^q : (0,1] \to \mathbb{N}$ *for any* $a \in (0,1]$ *by*

$$n_i^q(a) = \min\{n \in \mathbb{N} \mid \mathcal{F}_q^n(a) \notin I_i^q\} \tag{4.7}$$

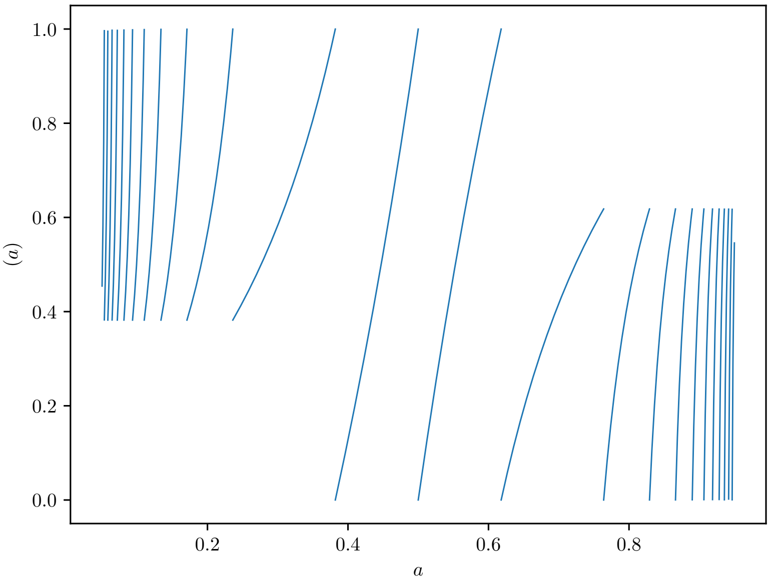

Figure 5: The symmetric G_5-Gauss interval map $\mathcal{G}_5$.

when $i = 0, q-2$, *and consider the set* $\mathsf{R}^q = \mathsf{S} \setminus \Big((\mathsf{H}_0^q \cap \mathsf{V}_{q-2}^q) \cup (\mathsf{H}_{q-2}^q \cap \mathsf{V}_0^q) \Big)$. *The map* $\widetilde{\mathcal{G}}_q : \mathsf{R}^q \to \mathsf{R}^q$ *given for any* $(a, s) \in \mathsf{R}^q$ *by*

$$\widetilde{\mathcal{G}}_q(a, s) := \begin{cases} \widetilde{\mathcal{F}}_q^{n_0^q(a)}(a, s), & a \in I_0^q = (\lambda_q/(\lambda_q + 1), 1] \\ \widetilde{\mathcal{F}}_q(a, s), & a \notin I_0^q \cup I_{q-2}^q \\ \widetilde{\mathcal{F}}_q^{n_{q-2}^q(a)}(a, s), & a \in I_{q-2}^q = (0, 1/(\lambda_q + 1]) \end{cases} \tag{4.8}$$

is a.e. bijective, and preserves the Lebesgue measure $d\mu_{\widetilde{\mathcal{G}}_q} := dads$ *on* R^q. *Consequently,* $\widetilde{\mathcal{G}}_q$ *is a model of the natural extension of the map* $\mathcal{G}_q : (0, 1] \to (0, 1]$ *defined for every* $a \in (0, 1]$ *by*

$$\mathcal{G}_q(a) := \begin{cases} \mathcal{F}_q^{n_0^q(a)}(a), & a \in I_0^q \\ \mathcal{F}_q(a), & a \notin I_0^q \cup I_{q-2}^q \\ \mathcal{F}_q^{n_{q-2}^q(a)}(a), & a \in I_{q-1}^q \end{cases}. \tag{4.9}$$

Moreover, the map $\mathcal{G}_q$ *preserves the finite measure*

$$d\mu_{\mathcal{G}_q} := \begin{cases} \Big(R^q_{q,q-2}(a,1-a) - R^q_{q,0}(a,1-a)\Big)\, da = \frac{\lambda_q\, da}{a(a+\lambda_q(1-a))}, & a \in I^q_0 = (\lambda_q/(\lambda_q+1), 1] \\ \frac{da}{a(1-a)}, & a \notin I^q_0 \cup I^q_{q-2} \\ \Big(R^q_{q,q-1}(a,1-a) - R^q_{q,1}(a,1-a)\Big)\, da = \frac{\lambda_q\, da}{(1-a)((1-a)+\lambda_q a)}, & a \in I^q_{q-2} = (0, 1/(\lambda_q+1)] \end{cases} \tag{4.10}$$

on $(0,1]$.

Proof. We first show that the map $\mathcal{F}_q : (0,1] \to (0,1]$ is intermittent, with exactly two indifferent fixed points at $a \to 0+0, 1-0$ by computing the derivative $\frac{d\mathcal{F}_q}{da}$. For any $i \in [q-1]$, and any $a \in I^q_i = \left(\frac{x^q_{i+1}}{x^q_{i+1}+y^q_{i+1}}, \frac{x^q_i}{x^q_i+y^q_i}\right]$, we write $\varsigma_i = x^q_i + y^q_i$, and $\varsigma^q_{i+1} = x^q_{i+1} + y^q_{i+1}$, which gives

$$\begin{aligned} \frac{d\mathcal{F}_q}{da}(a) &= \frac{\varsigma^q_{i+1}\left((\varsigma^q_{i+1}-\varsigma^q_i)a + (x^q_i - x^q_{i+1})\right) - (\varsigma^q_{i+1}-\varsigma^q_i)\left(\varsigma^q_{i+1}a - x^q_{i+q}\right)}{\left((x^q_{i+1}-y^q_i)a + (x^q_i - x^q_{i+1})\right)^2} \\ &= \frac{x^q_i\varsigma^q_{i+1} - \varsigma^q_i x^q_{i+1}}{\left((x^q_{i+1}-y^q_i)a + (x^q_i - x^q_{i+1})\right)^2} \\ &= \frac{1}{\left((x^q_{i+1}-y^q_i)a + (x^q_i - x^q_{i+1})\right)^2} \end{aligned}$$

where we use the fact that $\mathfrak{w}^q_i \wedge \mathfrak{w}^q_{i+1} = x^q_i y^q_{i+1} - x^q_{i+1} y^q_i = 1$. Similarly, we get

$$\lim_{a \to \frac{x^q_{i+1}}{x^q_{i+1}+y^q_{i+1}}+0} \frac{d\mathcal{F}_q}{da}(a) = (x^q_{i+1} + y^q_{i+1})^2,$$

and

$$\lim_{a \to \frac{x^q_i}{x^q_i+y^q_i}-0} \frac{d\mathcal{F}_q}{da}(a) = (x^q_i + y^q_i)^2$$

at the end points of I^q_i. Since $(x_i + y^q_i)^2$ is equal to 1 for $i = 0, q-1$, and is strictly greater than 1 for $i = 1, 2, \cdots, q-2$, we have that $\mathcal{F}_q$ is expanding everywhere except at $a \to 0+0, 1-0$. It is immediately true that $\lim_{a\to 0+0} \mathcal{F}_q(a) = 0$ and $\lim_{a \to 1-0} \mathcal{F}_q(a) = 1$, proving that $0, 1$ are the only indifferent fixed points of $\mathcal{F}_q$.

We now accelerate the branches of $\mathcal{F}_q$ that correspond to 0 and 1. In what follows, denote the branches of $\mathcal{F}_q$ over I_0^q and I_{q-2}^q by $\mathcal{F}_{q,0}$ and $\mathcal{F}_{q,q-2}$. It can be easily seen that $n_0^q(a) = 1$ iff $a \in I_{0,0}^q := \left(\frac{\lambda_q}{\lambda_q+1}, \mathcal{F}_{q,0}^{-1}\left(\frac{\lambda_q}{\lambda_q+1}\right)\right]$, and by induction, we get for $n \geq 1$ that $n_0^q(a) = n$ iff $a \in I_{0,n}^q := \left(\mathcal{F}_{q,0}^{-n+1}\left(\frac{\lambda_q}{\lambda_q+1}\right), \mathcal{F}_{q,0}^{-n}\left(\frac{\lambda_q}{\lambda_q+1}\right)\right]$. Similarly, we for $n \geq 1$ have that $n_{q-2}^q(a) = n$ iff $a \in I_{q-2,n}^q := \left(\mathcal{F}_q^{-n}\left(\frac{1}{\lambda_q+1}\right), \mathcal{F}_{q,0}^{-n+1}\left(\frac{1}{\lambda_q+1}\right)\right]$. For $n \geq 1$, we define

$$\mathsf{V}_{0,n}^q := \left\{(a,s) \in V_0^q, s < R_{q,q-2}(a, 1-a)\right\},$$

and

$$\mathsf{V}_{q-2,n}^q := \left\{(a,s) \in \mathsf{V}_{q-2}^q, s \geq R_{q,1}(a, 1-a)\right\}.$$

Note that $\{\mathsf{V}_{0,n}^q\}_{n=1}^\infty$ is a partition of $\mathsf{V}_0^q \setminus \mathsf{H}_{q-2}^q$, and that $\{\mathsf{V}_{q-2,n}^q\}_{n=1}^\infty$ is a partition of $\mathsf{V}_{q-2}^q \setminus \mathsf{H}_0^q$, and that $\mathsf{R}^q = \left(\cup_{n=1}^\infty \mathsf{V}_{0,n}^q\right) \cup \left(\cup_{i=1}^{q-3} \mathsf{V}_i^q\right) \cup \left(\cup_{n=1}^\infty \mathsf{V}_{q-2,n}^q\right)$. It is clear that the map $\widetilde{\mathcal{G}}_q$ when restricted to $\cup_{i=1}^{q-3} \mathsf{V}_i^q$ is equal to $\widetilde{\mathcal{F}}_q$, and that it sends $\cup_{i=1}^{q-3} \mathsf{V}_i^q$ bijectively to $\cup_{i=1}^{q-3} \mathsf{H}_i^q$. It thus remains to show that $\widetilde{\mathcal{G}}_q$ maps $\left(\cup_{n=1}^\infty \mathsf{V}_{0,n}^q\right) \cup \left(\cup_{n=1}^\infty \mathsf{V}_{q-2,n}^q\right)$ bijectively to $\left(\mathsf{H}_{q-2}^q \setminus \mathsf{V}_0^q\right) \cup \left(\mathsf{H}_0^q \setminus \mathsf{V}_{q-2}^q\right)$.

We prove that $\widetilde{\mathcal{G}}_q$ bijectively maps $\cup_{n=1} V_{q-2,n}^q$ to $\mathsf{H}_0^q \setminus \mathsf{V}_{q-2}^q$, and omit the (similar) proof that $\cup_{n=1}^\infty \mathsf{V}_{0,n}^q$ is bijectively mapped to $\mathsf{H}_{q-2}^q \setminus \mathsf{V}_0^q$. Towards this goal, we consider the curve

$$\mathcal{U} := \left\{(a,s) \in \mathbb{R}^2 \mid a \in (0,1), s = \frac{1}{a(1-a)}\right\}.$$

Note that $\mathcal{U}$ is the upper boundary of H_{q-2}^q (and necessarily S) in the plane. We define a sequence of curves $(\mathcal{U}_n^q)_{n=1}^\infty$ as follows: $\mathcal{U}_1^q := \mathcal{U}$, and for any $n \geq 1$, $\mathcal{U}_{n+1}^q := \widetilde{\mathcal{F}}_q\left(\mathcal{U}_n^q \cap \{(a,s) \in \mathbb{R}^2 \mid a \in I_{q-2}^q\}\right)$. That is, for every $n \geq 1$, we map the portion of $\mathcal{U}_n^q$ that lies in the V_{q-2}^q branch by $\widetilde{\mathcal{F}}_q$. By virtue of the definition of $\widetilde{\mathcal{F}}_q$ when it maps V_{q-2}^q to H_0^q, we have that $\mathcal{U}_2^q$ agrees with the upper boundary of H_0^q in the plane, and that for every $n \geq 1$, the curve $\mathcal{U}_{n+1}^q$ lies *below* $\mathcal{U}_n^q$ inside H_0^q. For convenience, we write for any $(a,s) \in \mathsf{S}$ and $n \geq 1$ that $(a,s) \in \left[\mathcal{U}_{n+1}^q, \mathcal{U}_n^q\right)$ to indicate that (a,s) lies between the curves $\mathcal{U}_{n+1}^q$ (inclusive), and $\mathcal{U}_n^q$ (exclusive). For any $n \geq 1$, we get from the definitions of $\mathsf{V}_{q-2,n}^q$, $I_{q-2,n}^q$, $\mathcal{U}_n^q$ and $\mathcal{U}_{n+1}^q$ that

$$\widetilde{\mathcal{F}}_q^k\left(\mathsf{V}_{q-2,n}^q\right) = \left\{(a,s) \in \mathsf{S} \mid a \in I_{q-2,n-k}^q, (a,s) \in \left[\mathcal{U}_{k+2}^q, \mathcal{U}_{k+1}^q\right)\right\}$$

for $k = 0, 1, \cdots, n-1$, and that

$$\widetilde{\mathfrak{F}}_q^n \left(\mathsf{V}_{q-2,n}^q \right) = \left\{ (a,s) \in \mathsf{S} \mid a \in \cup_{i=0}^{q-3} I_i^q, (a,s) \in \left[\mathcal{U}_{n+2}^q, \mathcal{U}_{n+1}^q \right) \right\}.$$

That is, for each $n \geq 1$, $\widetilde{\mathfrak{G}}_q$ bijectively map $\mathsf{V}_{q-2,n}^q$ to $\widetilde{\mathfrak{F}}_q^n \left(\mathsf{V}_{q-2,n}^q \right) \subset \mathsf{H}_0^q \setminus \mathsf{V}_{q-2}^q$, with the left and right boundaries of $\widetilde{\mathfrak{F}}_q^n \left(\mathsf{V}_{q-2,n}^q \right)$ agreeing with those of $\mathsf{H}_0^q \setminus \mathsf{V}_{q-2}^q$, and the bottom boundary of $\widetilde{\mathfrak{F}}_q^n \left(\mathsf{V}_{q-2,n}^q \right)$ agreeing with the top boundary of $\widetilde{\mathfrak{F}}_q^{n+1} \left(\mathsf{V}_{q-2,n+1}^q \right)$. The function $\widetilde{\mathfrak{G}}_q$ is a composition of $\widetilde{\mathfrak{F}}_q$, and so the mapping in question preserves the measure $d\mu_{\widetilde{F}_q}$.

Finally, integrating the measure $d_{\widetilde{\mathfrak{G}}_q}$ over the s fibers gives the measure $d_{\mathfrak{G}_q}$ in the statement of the theorem. $\square$

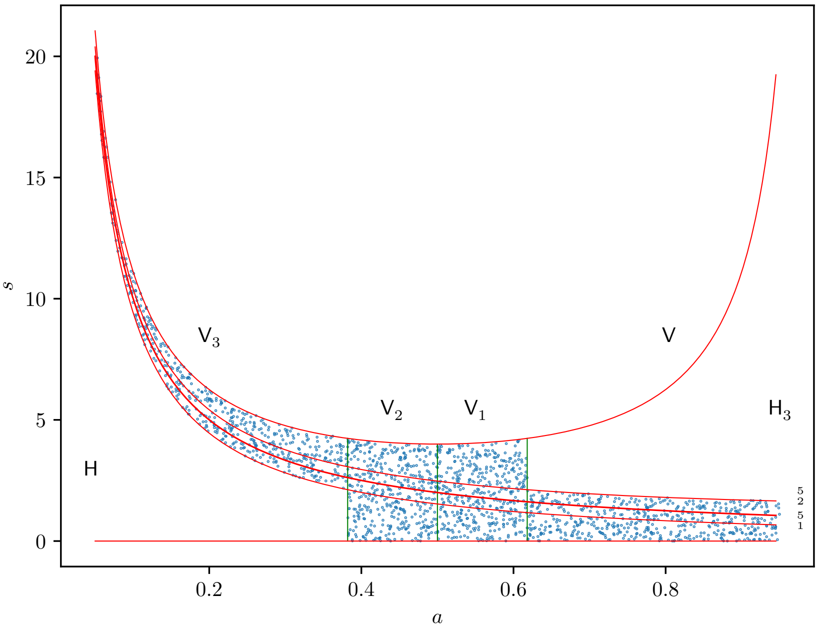

Figure 6: An orbit of the natural extension $\widetilde{\mathcal{G}}_5$ with the regions $\{\mathsf{V}_i^3\}_{i=0}^3$ and $\{\mathsf{H}_i^3\}_{i=0}^3$ from proposition 3.1 indicated.

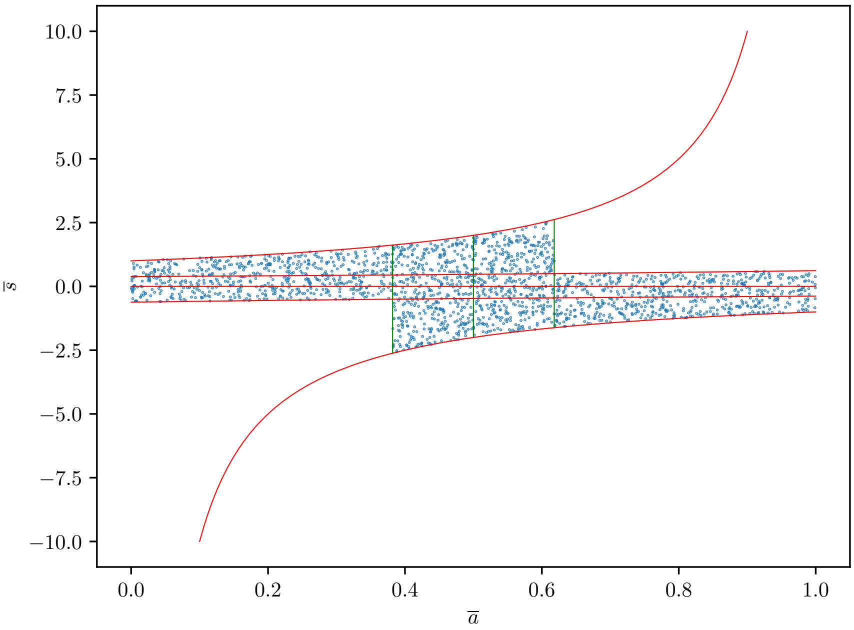

Figure 7: An orbit of the natural extension $\widetilde{\mathcal{G}}_5$ conjugated by $(a, s) \mapsto (\overline{a}, \overline{s}) := (a, s - 1/a)$.

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