Algebraic Skeleton Transform: A symbolic computation challenge

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Abstract

We introduce an algebraic formulation for the shape skeleton transform (also known as the medial axis transform) and present a concrete low-degree example. While we can now construct an algorithmic approach and provide a proof-of-concept example, small increases in algebraic degree or geometric complexity cause the algebraic computations to become intractable, and we propose the algebraic skeleton transform as a challenge problem for algebraic computation.

Introduction

The *shape skeleton* is a reduced-dimension geometry descriptor consisting of the locus of centers of maximal balls; i.e. balls that are contained within an object but not within any other ball in the object. The concept was introduced by Blum [1] as the *medial axis* (to identify objects in images) and has been extended under various names including *symmetry sets*, *central sets*, *cut loci*, *geometric skeletons*, and *shape skeletons* (that we use here). The skeletal concept has turned out to be important enough to garner the attention of numerous researchers and has produced enough results that entire books exist on the subject under generic names like *medial representations* [2]. There are papers dedicated to applications of medial/skeletal methods [3] including applications to robotic collision detection [4] and vision systems [5]; and 3D skeletons remain an area of interest in the graphics community [6]. Here we focus on introducing a new algebraic perspective.

Object representation

Since the shape skeleton concept involves maximal balls contained in a geometric object, it is inherently tied to *solid modeling* [7], the study of shape representation. Classifications for solid modeling techniques include boundary-representations (b-reps) [8], and cell decomposition methods (including voxels [9] and octrees [10]) but here we focus on representing shapes directly in terms of mathematical functions now known as function-based representations (f-reps) [11]. Various approaches to f-reps may employ a general class of computable functions or restricted classes such as the semi-algebraic functions typical of R-functions [12], but here we restrict further to algebraic functions; specifically polynomials with rational coefficients. The idea of designing geometry using polynomials has been introduced previously [13] with the intent of exploiting the capabilities of what, at the time, were recently developed computer algebra systems. The available computing power supported basic and blended boolean operations and some property evaluation (e.g. object extent via silhouettes obtained by variable elimination using resultants or Gröbner bases). Since then 25 years have passed, the capabilities of computers and computer algebra systems have advanced significantly, and we return to consider applications not previously considered feasible.

Skeletal data and algebraic sweeps

The skeleton is defined as the locus of maximal sphere centers, but that is insufficient for recapturing the original shape. The additional information needed for reconstruction is specification of the maximal sphere radii. The skeleton together with the radius specification, which we refer to as the *skeletal data* [14], constitutes an alternative, reduced-dimension representation of the geometry [15]. Here we present an algebraic approach (which can extend to $\mathbb{R}^3$) for computing the skeletal data of a 2D body $\Omega = \{(x, y) : f(x, y) \leq 0\}, f \in \mathbb{Q}[x, y]$. The critical element in the definition of the skeleton is the maximal circle. We use the standard quadratic description $g(x,y;x_0,y_0,r)=(x-x_0)^2+(y-y_0)^2-r^2$ so $g=0$ represents a circle of radius r centered at $(x,y)=(x_0,y_0)$. For the circle to be maximal, it must have multiple points of first order contact with the boundary $\partial\Omega$. Labeling the contact points (x_1,y_1) and (x_2,y_2), multiple distinct 1^{st}-order contact imposes the conditions (with the final condition ensuring that the distance between the contact points has a real inverse k):

$$\begin{aligned} f_1,f_2: &\quad f(x_1,y_1)=0,\ f(x_2,y_2)=0 \\ g_1,g_2: &\quad g(x_1,y_1;x_0,y_0,r)=0,\ g(x_2,y_2;x_0,y_0,r)=0 \\ h_1,h_2: &\quad df_1\wedge dg_1=0,\ df_2\wedge dg_2=0 \\ d: &\quad k((x_2-x_1)^2+(x_2-x_1)^2)-1=0 \end{aligned}$$

Given a polynomial f defining a region Ω, the conditions correspond to a set of 7 polynomials $\{f_1,f_2,g_1,g_2,h_1,h_2,d\}$ in the variables $\{x_0,y_0,r,x_1,y_1,x_2,y_2,k\}$. Computing the skeletal data is now a problem in variable elimination. Applying a Gröbner basis method with an elimination ordering, the first element will be a polynomial in $\{x_0,y_0\}$ that provides an implicit definition of the variety containing the skeleton, and the succeeding element(s) will be polynomial(s) in $\{x_0,y_0,r\}$ that complete the skeletal data by specifying the radius of maximal spheres. To be concrete, let's consider the axis-aligned ellipse with major axis 2 and minor axis 1 defined by $f(x,y)=x^2+4y^2-1$. The polynomials defining the skeleton are:

$$\left\{\begin{matrix} f_1,f_2, \\ g_1,g_2, \\ h_1,h_2, \\ d \end{matrix}\right\} = \left\{\begin{matrix} x_1^2+4y_1^2-1,x_2^2+4y_2^2-1, \\ -r^2+(x_1-x_0)^2+(y_1-y_0)^2,-r^2+(x_2-x_0)^2+(y_2-y_0)^2, \\ -4x_1y_0+16x_0y_1-12x_1y_1,-4x_2y_0+16x_0y_2-12x_2y_2, \\ k\left((x_2-x_1)^2+(y_2-y_1)^2\right)-1 \end{matrix}\right\}$$

The lexicographic Gröbner basis B includes 16 polynomials:

$$\begin{aligned} B=\{&x_0y_0,12r^2x_0+4x_0^3-3x_0,3r^2y_0-4y_0^3-3y_0,36r^4-45r^2-4x_0^4-9x_0^2-64y_0^4-36y_0^2+9, \\ &3x_0x_1-4x_0^2,9r^2x_1-12x_1y_0^2+4x_0^3+9x_0-9x_1,-12r^2-20x_0^2+9x_1^2+20y_0^2+3,y_0^2+3y_1y_0, \\ &36r^2y_1+12x_0^2y_1+16y_0^3+9y_0-9y_1,x_1y_0-4x_0y_1+3x_1y_1, \\ &3r^2+5x_0^2-5y_0^2+9y_1^2-3,-8x_0+3x_1+3x_2,2y_0+3y_1+3y_2, \\ &16kx_0^3-9kx_0+9x_0,64kx_0^2+16ky_0^2-36k-36r^2-12x_0^2+48y_0^2+45, \\ &12kr^2+84kx_0^2-48k-48r^2-16x_0^2+64y_0^2+57,16kx_0^2y_1-3ky_0-9ky_1+3y_0+9y_1\} \end{aligned}$$

The first line of B contains the critical skeletal data. $B_1=x_0y_0$ defines the skeletal variety, and the skeletal sphere radii are described by $(B\cap\mathbb{Q}[x_0,y_0,r])\setminus(B\cap\mathbb{Q}[x_0,y_0])=\{B_2,B_3,B_4\}$. The factored versions of these elements are:

$$B_2=x_0(12r^2+4x_0^2-3),B_3=y_0(3r^2-4y_0^2-3),B_4=(3r^2-x_0^2-3)(12r^2+4x_0^2-3)$$

B_1 has factors x_0 and y_0, so the skeleton could have components on the x-axis and y-axis. Examining the factored skeletal data, we exclude as trivial any factors that also divide B_1 (since they vanish on the skeleton). For compact bodies, we also exclude factors that admit real values of r for arbitrarily large coordinate values. Thus, the factors $x_0,y_0,(3r^2-4y_0^2-3),\text{ and }(3r^2-x_0^2-3)$ are excluded, and $R:=12r^2+4x_0^2-3$

remains to govern the sphere radius. We note that, since B_1 vanishes when $x_0 = 0$, the x_0 factor of B_1 can play only a trivial role and the skeleton lies on the x-axis corresponding to the variety of $Q := y_0$. The skeleton does not occupy the full variety, but terminates where the contact points become indistinct due to higher order contact as shown in Fig.1a. Terminal points of the skeleton are centers of curvature and therefore must also lie on the evolute. Terminal skeletal points can be found by computing skeleton-evolute intersections, but such computations are generally unnecessary.

Inverting the transform: refleshing the skeleton

Having computed the skeletal data, the next task is to formulate the inverse transform that starts with the key factors of the skeletal data (e.g. the polynomials $Q = y_0 \text{ and } R = 12r^2 + 4x_0^2 - 3$) and reconstructs a polynomial defining Ω. Since Q describes the locus of centers and R describes the radii, the geometric operation corresponds to computing the envelope of a family of balls with varying radius or, in geometric modeling terminology, constructing a variable radius swept solid. In the skeletal context, this is referred to as "refleshing" the skeleton. Parametric formulations of sweep operations are common [16,17,18,19], sometimes referred to as envelopes and sometimes as *discriminants* because they involve simultaneous roots of a function (the expression describing the family of shapes) and its derivative (with respect to the parameter). For a single-parameter family, the variable elimination can be performed as a resultant. Here the formulation is not only implicit, and the implicitly defined envelope problem is less frequently encountered [20,21], but also algebraic. Our goal is to present a computational algebraic approach.

Refleshing involves computing the envelope of a sphere with implicitly defined radius swept along an implicit skeleton. For algebraic refleshing in $\mathbb{R}^2$, the skeleton lies on a polynomial variety that defines one or more curves. For variable radius sweep along a curve, it is convenient to embed the problem in $\mathbb{R}^3$ so that the curve is defined by a pair of polynomials $G \text{ and } H$. (For curves lying in the x, y-plane, simply set $H = z$.) A family of spheres of radius r with centers at $\{x_0, y_0, z_0\}$ are defined by $F := (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - r^2 = 0$, the sphere centers are constrained to lie on a curve by $G(x_0, y_0, z_0) = 0$ and $H(x_0, y_0, z_0) = 0$, and the sphere radii are defined implicitly by $R(x_0, y_0, r) = 0$. The envelope criterion (equivalent to Boltyanski's determinantal condition [19]) is $E := dF \wedge dG \wedge dH \wedge dR = 0$.

The algebraic envelope formulation for sweeping variable radius spheres along a curve involves eliminating the variables $\{x_0, y_0, z_0, r\}$ from the polynomials $P = \{E, F, G, H, R\}$ depending on $\{x, y, z, x_0, y_0, z_0, r\}$. For the inverse transform in $\mathbb{R}^2$, z and z_0 can be trivially removed so the computation involves eliminating $\{x_0, y_0, r\}$ from the polynomials $\overline{P} = \{E, F, G, R\}|_{z=z_0=0}$ depending on $\{x, y, x_0, y_0, r\}$. For the ellipse,

$$\overline{P} = \{r(3x - 4x_0), (x - x_0)^2 + (y - y_0)^2 - r^2, y_0, r^2 + 4x_0^2 - 3\}.$$

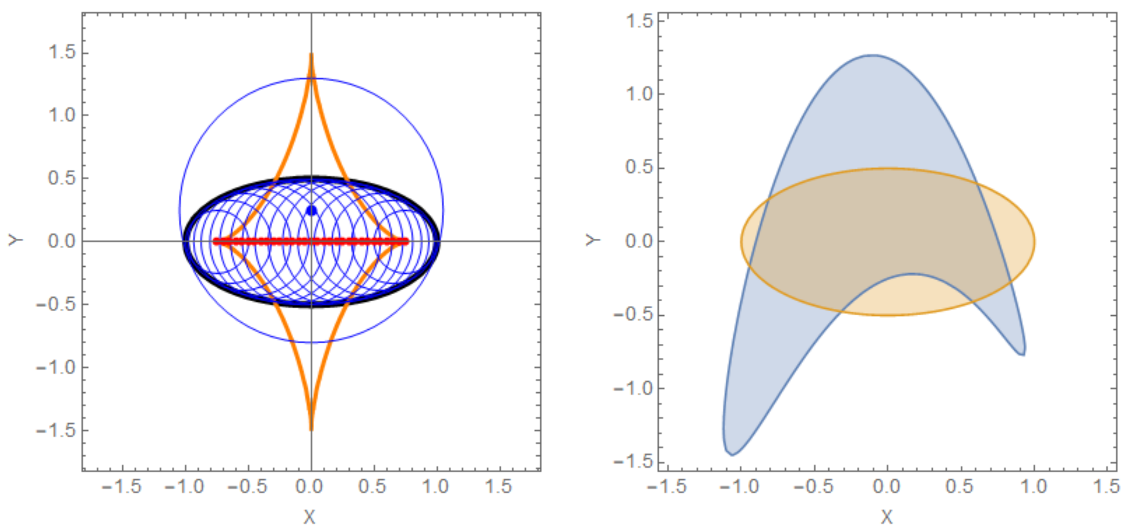
The Gröbner basis $\overline{B}$ computed from $\overline{P}$ contains a single polynomial in $\mathbb{Q}[x, y]$ which factors into:

$$\overline{B} \cap \mathbb{Q}[x, y] = \left(x^2 + 4y^2 - 1\right)\left(16x^4 + 32x^2y^2 - 24x^2 + 16y^4 + 24y^2 + 9\right)$$

The first factor recovers the ellipse; the second factor is positive semi-definite with zeros corresponding to roots of $R|_{r=0}$. This extraneous factor can be avoided by excluding the factor r from $\overline{P}_1$ to produce the revised polynomial set $\overline{P} = \{3x - 4x_0, (x - x_0)^2 + (y - y_0)^2 - r^2, y_0, r^2 + 4x_0^2 - 3\}$ with Gröbner basis $\overline{B} = \{x^2 + 4y^2 - 1, 16r^2 + 3x^2 - 4, 4x_0 - 3x, y_0\}$ so that $\overline{B} \cap \mathbb{Q}[x, y] = \{x^2 + 4y^2 - 1\}$ precisely recovers the ellipse.

Skeletal editing

The skeletal transform supports a variety of skeleton-based geometric editing techniques [14]. From an initial polynomial, we can compute and edit the skeletal data and then apply the inverse transform to obtain a polynomial defining the edited shape. For example, to create a polynomial inspired by the starfleet insignia, scale the sphere radii non-uniformly by replacing R with the numerator of $R|_{r \to r/(x_0 - 3/2)}$ and bend the skeleton along the parabola $Q = 2y_0 + 5x_0^2 - 1$. The inverse transform produces $f(x, y) = 2133x^4 - 432x^3 + 2880x^2y - 1088x^2 + 576x + 1024y^2 - 1024y - 320$. The original ellipse and the edited shape are shown in Fig. 1b.



a) b)

Fig.1 - *a) Ellipse with maximal circles centered on skeleton terminating at evolute. Point on y-axis is not skeletal due to exterior contact circle. b) Ellipse and inverse transform of edited skeletal data.*

The challenge of even slightly higher degrees

With a simple degree 2 example, we have demonstrated a foundation for using computer algebra to compute skeletal data, perform skeletal editing, and reflesh. The factor limiting usefulness of the skeleton transform is that the computational demands grow rapidly with problem complexity. In some sense, the next example in terms of complexity is the degree 4 superellipse with defining function $f_4(x, y) = x^4 + y^4 - 1$. While both the skeletal transform and inverse skeletal transforms for the ellipse are readily computed with general purpose hardware and software, attempts to run the same code with a defining function of degree 4 instead of degree 2 fail spectacularly. The computer algebra system runs continuously until system memory is depleted and the computer becomes unresponsive. Note that the problem of computing the skeltal data for f_4 has a known answer. Based on symmetry, the skeleton is associated with the variety of $(x - y)(x + y)$ and a function governing the sphere radii can be constructed by hand, but what is really needed is a way to use a computer algebra system that can overcome the swelling of intermediate results and complete this class of computation without specialzed knowledge or significant human intervention. The problem becomes more challenging still in $\mathbb{R}^3$ where multiple contact and distinctness conditions arise. The hope is that the ISSAC community will be able offer tools and/or insights to help make progress toward that goal.

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