Growth of Reciprocal pseudo-Anosovs on Lattice Surfaces

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Abstract

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Motivated by number theory, *Reciprocal geodesics* were first introduced by Sarnak [23], who studied their asymptotic growth on the modular curve. Erlandsson-Souto [7] gave a geometric interpretation and generalization of reciprocal geodesics and a dynamical proof of asymptotic counting results in the more general setting of *hyperbolic orbifolds* $\mathbb{H}^2/\Gamma$ where Γ is a lattice. We introduce the notion of *reciprocal pseudo-Anosov maps* of translation surfaces and establish a correspondence between such maps and reciprocal geodesics. We then show how to apply the Erlandsson-Souto results to compute the asymptotic growth for particular families of highly symmetric surfaces known as *lattice surfaces* or *Veech surfaces* [26], and to in fact compute the constants for the asymptotic growth of pseudo-Anosov maps on certain families of lattices surfaces, called *Bouw-Möller* [5] and *primitive square-tiled surfaces* [24].

Contents

1 Introduction

1.1 Reciprocal Geodesics on Hyperbolic Surfaces

Reciprocal geodesics on the modular surface Sarnak [23], motivated by the theory of integral quadratic forms, considered the counting and equidistribution of conjugacy classes of what he called *reciprocal geodesics* on the modular curve $\mathbb{H}^2/\operatorname{PSL}(2,\mathbb{Z})$, which we will later think of as the *Teichmüller curve* of the torus3.4. These reciprocal geodesics can be viewed as closed geodesics passing through the projection of the point i, the unique order 2 cone point on $\mathbb{H}^2/\operatorname{PSL}(2,\mathbb{Z})$. He showed that the number of these grow exponentially (when ordered by length), and computed the exponential growth rate, precisely showing that the number of reciprocal geodesics of length at most L grows asymptotically like $\frac{3}{8}e^L$.

Theorem. [23]

$$\#\{\text{reciprocal geodesics } \rho \text{ with } \ell(\rho) < L\} \sim \frac{3}{8}e^L,$$

where $\sim$ means that the ratio tends to 1 as $L \to \infty$, and $\ell(\rho)$ is the length of the closed geodesic ρ.

Reciprocal geodesics on hyperbolic orbifolds Erlandsson and Souto [7] extended the notion of reciprocal geodesics to more general lattices $\Gamma \subset \operatorname{PSL}(2,\mathbb{R})$ containing order 2 elements, which recovered Sarnak's asymptotic for the growth of reciprocal geodesics on the modular surface, and extended the result to be able to count reciprocal geodesics on the hyperbolic orbifold $\mathbb{H}^2/\Gamma$, that is, closed geodesics passing through orbifold points of order 2. They show the existence of a constant c, involving the orbifold Euler characteristic of $\mathbb{H}^2/\Gamma$ and the normalizers of order 2 subgroups, and show that the number of reciprocal geodesics of length at most L grows asymptotically like ce^L. We will state this result more precisely in §4.

Reciprocal pseudo-Anosov maps on lattice translation surfaces In this thesis, we give a way of interpreting the Erlandsson-Souto result on counting reciprocal geodesics on hyperbolic orbifolds [7] in terms of what we call *reciprocal pseudo-Anosov maps* of *lattice translation surfaces.* Lattice translation surfaces are translation surfaces which have affine automorphism groups projecting to a lattice in $\operatorname{PSL}(2,\mathbb{R})$. *Reciprocal pseudo-Anosov maps* are pseudo-Anosov maps ρ of a translation surface $S = (X,\omega)$ whose stable and unstable foliations are interchanged by an order 2 element of the affine automorphism group $\operatorname{PSL}(X,\omega)$ of the surface (and thus, by the action of $\operatorname{SL}(2,\mathbb{R})$, can be made orthogonal). We show how *reciprocal geodesics* on the Teichmüller curves $\mathbb{H}^2/\operatorname{PSL}(X,\omega)$ of *lattice surfaces* correspond to these reciprocal pseudo-Anosov maps, count them, and for a series of important examples inspired by work of Bouw-Moller [5], McMullen [21] [21], Bainbridge [2], and Mukamel [22], we explicitly compute the precise asymptotic constants $C(\Gamma)$. Our main results (Theorem 4.3.1, Theorem 5.1.2, and Theorem 5.1.4) can be summarized as follows, with precise formulations given in §4 and §5. Let $S = (X,\omega)$ be a lattice translation surface for which the affine automorphism group contains an element σ which is of order 2 in the projection to $\operatorname{PSL}(2,\mathbb{R})$.

Definition. A *reciprocal pseudo-Anosov map* of S is a pseudo-Anosov map ρ such that the stable and unstable foliations of ρ are interchanged by σ.

We let $\lambda = \lambda(\rho) > 1$ be the expansion factor of ρ. Our main results show:

Theorem 1.1.1. With notation as above, there is a bijective correspondence between reciprocal pseudo-Anosov maps on $S = (X,\omega)$ and closed reciprocal geodesics on the hyperbolic orbifold $\mathbb{H}^2/\operatorname{PSL}(X,\omega)$. Precisely, associated to a reciprocal pseudo-Anosov ρ on S of expansion factor λ there is a reciprocal geodesic of length $2\log\lambda$ on the hyperbolic orbifold $\mathbb{H}^2/\operatorname{PSL}(X,\omega)$. Thus, counting using this interpretation, we can compute the asymptotics of the growth of these reciprocal pseudo-Anosovs (ordered by the expansion factor, or equivalently, length of the associated closed geodesic) for a family of what are known as *Bouw-Möller surfaces*, and for a family of square-tiled L-shaped surfaces.

Bouw-Möller surfaces A particular family of translation surfaces $S_q = (X_q,\omega_q)$, $q \geq 3$ we will be interested in are part of a family introduced by Bouw-Möller [5] (also studied by Hooper [12]), such that the stabilizer group $\operatorname{SL}(X_q,\omega_q)$ is the Hecke triangle group $\Delta^+(2,q,\infty)$. For these surfaces, we show that the number of reciprocal pseudo-Anosov maps on S_q of expansion factor at most $e^{L/2}$ grows asymptotically like $\frac{q}{8(q-2)}e^L$.

L-shaped surfaces The family S_q grows in genus. There is also an important family of genus 2 lattice translation surfaces admitting order 2 automorphisms, we describe work on a family of these which are *square-tiled*, that is, which are once-branched covers of the flat torus. We will state this result precisely in §5.

Further questions This work suggests many further directions of research. We are currently studying the asymptotics for non-square tiled lattice L-shaped (genus 2) surfaces. It would also be interesting to use the dynamical techniques developed by Erlandsson-Souto [7] to study the growth of reciprocal pseudo-Anosov maps of non-lattice translation surfaces admitting order 2 automorphisms, by using the mixing of the Teichmüller geodesic flow, and to compute the associated asymptotic growth constants. We plan to pursue this program in future research.

1.2 Organization

We discuss the actions of $\mathrm{SL}(2,\mathbb{R})$ on real and hyperbolic spaces in §2. We classify matrices by their actions and trace, and discuss several important linear algebra facts about hyperbolic and elliptic matrices that are important tools for our work. In §3, we define translation surfaces, their moduli spaces, the action of $\mathrm{SL}(2,\mathbb{R})$ on these moduli spaces, with a particular focus on the notion of *lattice surfaces*. We give several examples of families of lattice surfaces in §3.4. In §4, we define the notion of *reciprocal geodesics*, and state precisely the results of Erlandsson-Souto, and in §4.2, introduce the concept of reciprocal pseudo-Anosov maps and connect them to reciprocal geodesics. In Section 5, we explicitly compute asymptotic growth rates of these reciprocal pseudo-Anosov maps for some Bouw-Möller surfaces and square-tiled L-shaped tables. In 6, we discuss natural questions that arise from this work.

2 Actions of $\mathrm{SL}(2,\mathbb{R})$

Linear and fractional linear actions The groups $\mathrm{GL}^+(2,\mathbb{R})$ and $\mathrm{SL}\,(2,\mathbb{R})$ act linearly on $\mathbb{R}^2$ and via fractional linear transformations on hyperbolic space $\mathbb{H}^2$. The relationship between these actions is fundamental to the construction of the main objects of this thesis, reciprocal geodesics and reciprocal pseudo-Anosov maps. This chapter outlines general properties of these actions. A reference for these facts and further details on hyperbolic geometry is S. Katok's book [14] on Fuchsian groups.

2.1 Linear and Projective Actions

The group $\mathrm{GL}^+(2,\mathbb{R})$ acts on $\mathbb{R}^2$ linearly by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

Projective action This linear action of $\mathrm{GL}^+(2,\mathbb{R})$ on $\mathbb{R}^2$ induces an action on $\mathbb{RP}^1$, the space of unoriented lines in $\mathbb{R}^2$ passing though the origin. This action factors through both quotients factors to an action of the group quotient group $\mathrm{PSL}\,(2,\mathbb{R}) = \mathrm{SL}\,(2,\mathbb{R})\,/\{\pm I\}$. If we consider the line $\{y = \sigma^{-1}x\}$, so $\sigma = x/y$ in the inverse slope, the induced action gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \sigma = \frac{a\sigma+b}{c\sigma+d}.$$

Note that a line is fixed by the action of a matrix g if and only if it is an eigendirection of the matrix g.

2.2 Fractional Linear Action

The fractional linear action of $\mathrm{SL}\,(2,\mathbb{R})$ on the upper-half plane $\mathbb{H}^2 = \{z \in \mathbb{C} : \Im(z) > 0\}$ is given by extending the projective linear action from $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ to $\mathbb{H}^2$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

for $z \in \mathbb{H}^2$. These mappings are isometries of $\mathbb{H}^2$ when equipped with the Riemann metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, known as the *hyperbolic metric* and are sometimes called *Möbius* transformations [14]. The fact that these maps send $\mathbb{H}^2$ to $\mathbb{H}^2$ and act as hyperbolic isometries follows from a direct computation of derivatives, as well as the fact that $\Im(g \cdot z) = 1/|cz+d|^2\Im(z)$. The action on $\mathbb{H}^2$ is transitive, and the stabilizer of the point i is the compact subgroup

$$SO(2,\mathbb{R}) = \left\{ r_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} : 0 \le \theta < 2\pi \right\}.$$

The space of lattices We record here how the linear action on $\mathbb{R}^2$ induces an action on the space of *unimodular* lattices, unit-covolume lattices in $\mathbb{R}^2$. Given $v, w \in \mathbb{R}^2$ linearly independent, let $\Lambda(v,w) = \langle v, w\rangle = \mathbb{Z}v + \mathbb{Z}w$ be the lattice in $\mathbb{R}^2$ generated by v, w. The linear action of $GL^+(2,\mathbb{R})$ on $\mathbb{R}^2$ induces an action on the space of lattices in $\mathbb{R}^2$, where a lattice is a discrete subgroup with finite covolume- all such are of the form $\Lambda(v,w)$ for some choice of v, w. This is discussed in [11]. The action on the space of lattices is in fact transitive; so is the action of the subgroup $\mathrm{SL}(2,\mathbb{R})$ the space of unimodular lattices. Given such a lattice Λ, there is a translation surface for more details structure on the quotient torus $\mathbb{R}^2/\Lambda$; details are given in §3. This action can be interpreted more generally as an action on the locus of unit-area flat tori. We note that the stabilizer of the standard integer lattice $\mathbb{Z}^2$ is the subgroup $\mathrm{SL}\,(2,\mathbb{Z})$, and by transitivity, the stabilizer of any other lattice is conjugate to $\mathrm{SL}(2,\mathbb{Z})$. We will see how to generalize this action to an action of $\mathrm{SL}(2,\mathbb{R})$ on the space of unit-area translation surfaces in §3.

2.3 Classifying Matrices

We can understand many properties of the action of a matrix $g \in \mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$ and $\mathbb{H}^2$ by looking at the trace of g, which allows us to classify elements of $\mathrm{SL}(2,\mathbb{R})$ into three categories: elliptic, parabolic, and hyperbolic. This classification inspired the Nielsen-Thurston classification of surface diffeomorphisms [8], which we recall briefly before returning to our matrices:

Theorem (Nielsen–Thurston Classification of Surface Diffeomorphisms)**.** For X a compact surface of genus $g > 1$, any $\rho \in \mathrm{Diffeo}^+(X)$ is, up to isotopy, either:

- Periodic: there exists m such that $\rho^m = \mathrm{Id}$.
- Reducible: ρ preserves a family of simple closed curves, meaning some iterate of ρ preserves a subsurface possibly with boundary.
- pseudo-Anosov: the map ρ preserves a pair of transverse measured foliations η_s and η_u, contracting along the stable foliation η_s and expanding along the unstable foliation η_u.

This classification of surfaces diffeomorphisms is not a trichotomy, because a surface diffeomorphism can be both reducible and periodic. We use the classification by absolute value of trace because it is a trichotomy.

Trace, conjugacy, and characteristic polynomials We recall that the trace $\mathrm{tr}(g) = a+d$ is a conjugacy invariant. Given $g \in \mathrm{SL}(2,\mathbb{R})$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\det(g) = ad - bc = 1.$$

The characteristic polynomial of g is thus

$$p_g(\lambda) := \det(g - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - \mathrm{tr}(g)\lambda + 1.$$

Since the determinant (and thus the characteristic polynomial) is a conjugacy invariant, so is the trace. Our classification of matrices will be via the *absolute value* of the trace. So, this descends to a classification in $\mathrm{PSL}(2,\mathbb{R})$.

Elliptic elements If $|tr(g)| < 2$, g is said to be *elliptic*, and the characteristic polynomial p_g has no real roots, but a pair of complex conjugate roots. Any elliptic element is conjugate (in $\mathrm{SL}(2,\mathbb{R})$) to a matrix to the form $r_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. Under the linear action on $\mathbb{R}^2$, r_θ acts as a rotation by angle θ around 0 in $\mathbb{R}^2$, and preserves circles. Under the fractional linear action on $\mathbb{H}^2$, r_θ acts as a rotation by angle 2θ around its unique fixed point $i \in \mathbb{H}^2$. Any elliptic matrix thus acts on $\mathbb{H}^2$ a rotation around its unique fixed point in $\mathbb{H}^2$.

Parabolic elements If $|\mathrm{tr}(g)| = 2$, g is *parabolic* and p_g has a single, repeated, real root, ± 1. Despite the name, a parabolic element g does not fix parabolas via its linear action. They are called parabolic because they are intermediate between elliptic and hyperbolic elements. Any parabolic element g is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to a matrix of the the form $h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ with $t \in \mathbb{R}$. Parabolic elements g act linearly by shears of the plane $\mathbb{R}^2$. On hyperbolic space $\mathbb{H}^2$, they preserve horocycles, curves (either circles or straight lines) which are tangent to their unique fixed point on the boundary of $\mathbb{H}^2$, corresponding to the unique eigendirection of g.

Hyperbolic elements If $|\mathrm{tr}(g)| > 2$, g is *hyperbolic*, and p_g has two distinct real eigenvalues $\lambda, 1/\lambda, |\lambda| > 1$. The linear action of hyperbolic elements g do preserve hyperbolas. Hyperbolic elements are conjugate in $\mathrm{SL}(2,\mathbb{R})$ to matrices of the form $g_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix}$, for some $t \in \mathbb{R}$. Note that the linear action of g_t preserves the hyperbolas $\{xy = c\}$. It expands $\mathbb{R}^2$ in the x-direction by $e^{\frac{t}{2}}$ and contracts the y-direction by $e^{\frac{-t}{2}}$. On the hyperbolic plane $\mathbb{H}^2$, a hyperbolic element g fixes two points on the boundary, corresponding to the (inverse) slopes of the eigendirections of g, and stabilizes the hyperbolic geodesic connecting these fixed points. In the case of g_t, this geodesic is the imaginary axis connecting 0 to ∞. Note that $g_t \cdot i = e^t i$, which is at hyperbolic distance t away from i.

2.4 Eigenvectors for Hyperbolic Elements

We now record some important information about eigenvalues and eigenvectors for hyperbolic matrices. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a hyperbolic matrix in $\mathrm{SL}(2, \mathbb{R})$, so $ad - bc = 1$ and $|\operatorname{tr}(g)| = |a + d| > 2$. By the quadratic formula applied to the characteristic polynomial p_g, we see that the roots of p_g (that is, the eigenvalues of g) are real and distinct, and have the form

$$\lambda^{\pm} = \frac{\operatorname{tr}(g) \pm \sqrt{\operatorname{tr}(g)^2 - 4}}{2}.$$

Note that $\lambda^+ \lambda^- = 1$. To find the eigenvectors, we must solve the system of equations

$$\begin{aligned} ax + by &= \lambda^{\pm} x \\ cx + dy &= \lambda^{\pm} y. \end{aligned}$$

Thinking in terms of (inverse) slopes $\sigma^{\pm} = x^{\pm}/y^{\pm}$ of the eigendirections, we have to find the fixed points of the projection action of g, that is, to solve

$$\frac{a\sigma + b}{c\sigma + d} = \sigma.$$

Again, we can use the quadratic equation, as we need to solve

$$c\sigma^2 + (d - a)\sigma - b = 0.$$

This gives us

$$\sigma^{\pm} = \frac{(a - d) \pm \sqrt{(d - a)^2 + 4bc}}{2c} = \frac{(a - d) \pm \sqrt{\operatorname{tr}(g)^2 - 4}}{2c}.$$

Lengths and traces A hyperbolic element g with $|\operatorname{tr}(g)| = \tau$ is conjugate to a matrix

$$g_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix}, 2\cosh(t/2) = \tau.$$

If we consider the invariant geodesic γ of g, and the quotient $\gamma/\langle g \rangle$, this yields a *closed* geodesic of hyperbolic length $\ell(g) = t$, so the relationship between length of the geodesic $\ell(g)$ and the trace τ is given by

$$\ell(g) = 2\cosh^{-1}(\tau/2).$$

Orthogonality We will be interested in hyperbolic elements for which the eigendirections are *orthogonal*, so $\sigma^+\sigma^- = -1$. That is,

$$\begin{aligned} \sigma^+\sigma^- &= \frac{((a - d) + \sqrt{(\operatorname{tr}(g)^2 - 4)}}{2c} \frac{((a - d) - \sqrt{(\operatorname{tr}(g)^2 - 4)}}{2c} \\ &= \frac{(a - d)^2 - ((a + d)^2 - 4)}{4c^2} \\ &= \frac{-4ad - 4(bc - ad)}{4c^2} \\ &= -\frac{bc}{c^2}, \end{aligned}$$

that is, we need $b = c$, so the matrix g must be symmetric.

Hyperbolic geodesics As we discussed above, every hyperbolic element g preserves the hyperbolic geodesic connecting its eigendirections; for example, the matrices g_t preserve the imaginary axis. We now restrict to the case of symmetric hyperbolic elements, those for which the eigendirections are orthogonal. Suppose the eigendirections have (inverse) slopes σ and $-\frac{1}{\sigma}$. The hyperbolic geodesic connecting these points is the Euclidean semi-circle intersecting the x-axis at these points, and has equation

$$\left(x - \frac{(\sigma - \frac{1}{\sigma})}{2}\right)^2 + y^2 = \left(\frac{(\sigma + \frac{1}{\sigma})}{2}\right)^2, y > 0.$$

We note that the point $i \in \mathbb{H}^2$ lies on this geodesic. To check this, we need to check that $x = 0, y = 1$ satisfies this equation, i.e.,

$$\begin{aligned}\left(-\frac{(\sigma - \frac{1}{\sigma})}{2}\right)^2 + 1^2 &= \frac{1}{4}(\sigma^2 + \sigma^{-2} - 2) + 1\\ &= \frac{1}{4}(\sigma^2 + \sigma^{-2} - 2 + 4)\\ &= \frac{1}{4}(\sigma^2 + \sigma^{-2} + 2)\\ &= \frac{1}{4}(\sigma^2 + \sigma^{-2} + 2\sigma\sigma^{-1})\\ &= \left(\frac{(\sigma + \frac{1}{\sigma})}{2}\right)^2.\end{aligned}$$

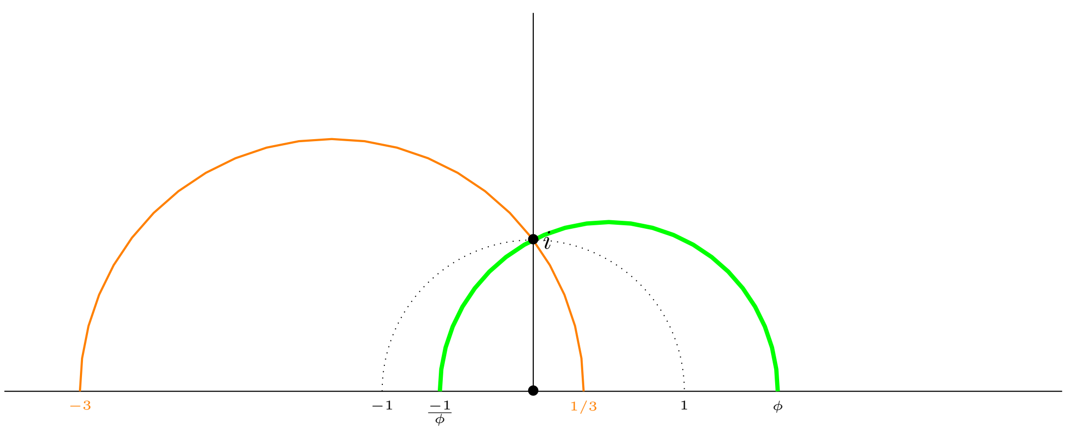


Figure 1: Geodesics passing through i which have boundary points dotted: $\{1, -1\}$, green: $\{\phi, -1/\phi\}$ with $\phi = \frac{1+\sqrt{5}}{2}$, and orange: $\{-3, 1/3\}$ respectively.

2.5 Order 2 Matrices

We now turn to discussing a special class of elliptic elements. We first note that if an element $[g] \in \mathrm{PSL}\,(2, \mathbb{R})$ is of order 2, it must be elliptic, since any representative $g \in \mathrm{SL}(2, \mathbb{R})$ must satisfy $g^2 = \pm I_2$. Thus, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$g^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + ca \end{pmatrix} = \pm I_2,$$

so we must have either $a + d = \mathrm{tr}(g) = 0$, or $b = c = 0$, in which case $a = \pm 1, d = \mp 1$

2.6 Lattices

We conclude this section by recording some facts about lattices in $\mathrm{PSL}(2,\mathbb{R})$. A lattice $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a discrete subgroup of finite covolume. We note that the discreteness requirement implies that any elliptic element $\gamma \in \Gamma$ must be of finite order: since any elliptic element of infinite order is conjugate to a matrix r_θ with θ an irrational multiple of 2π, then its powers accumulate on the identity. A lattice Γ is cocompact if and only if it does not contain parabolic elements. The quotient $\mathbb{H}^2/\Gamma$ is a *hyperbolic orbifold*. Orbifold points (also called *cone points*) on $\mathbb{H}^2/\Gamma$ correspond to conjugacy classes of elliptic elements conjugate in Γ, not $\mathrm{PSL}(2,\mathbb{R})$. Closed geodesics on $\mathbb{H}^2/\Gamma$ correspond to conjugacy classes of hyperbolic elements of Γ, and in the setting where Γ is not cocompact, cusps of $\mathbb{H}^2/\Gamma$ correspond to closed horocycles, that is, conjugacy classes of parabolic elements.

3 Translation Surfaces

In this chapter, we describe the definition of translation surfaces and their moduli spaces, and the $\mathrm{GL}^+(2,\mathbb{R})$-action on these moduli spaces, which subsequently induces an action of $\mathrm{SL}(2,\mathbb{R})$ on the locus of unit-area translation surfaces. References for this background material include Athreya-Masur [1], Massart [17], Wright's survey [28], and Zorich's survey [29]. We will then turn to a particular class of surfaces with large affine symmetry group, known as *lattice surfaces* (or *Veech surfaces*), with a particular focus on some families of examples which we study in greater depth in this thesis.

3.1 Three Definitions

A translation surface has three *equivalent* definitions, coming from complex analysis, (singular) Riemannian geometry, and flat geometry respectively. We will use aspects of all three, but will only formally state the complex analytic definition. The equivalence of these definitions can be found, for example, in in Chapter 2 of [1], [28].

Definition 3.1.1. A *compact, genus g translation surface* $S = (X, \omega)$ is a pair consisting of a compact Riemann genus surface X and ω, a holomorphic 1-form on X which locally is dz.

Transition maps By integrating the one-form ω, we obtain an atlas of charts to the complex plane $\mathbb{C}$ away from the zeros of ω. This allows us to view S as a 2-dimensional smooth manifold with an atlas of charts to $\mathbb{C}$ whose transition maps are Euclidean translations, possibly outside of a finite set, and this interpretation motivates the name translation surface. We emphasize that we *do not* include rotations in our possible transition maps. The zero set Σ of ω is the possible finite set. Away from these points, in these local coordinates $\omega = dz$. At any point where ω has a zero of order k, there is a local coordinate for which $\omega = z^k dz = \frac{1}{k+1} d(z^{k+1})$. All but these many points are regular, that is, the total angle at each point is 2π, but at a zero of order k, we have a total angle of $2\pi(k+1)$, coming from the fact that $z \mapsto z^{k+1}$ is a degree $(k+1)$ map from $\mathbb{C}^*$ to $\mathbb{C}^*$.

Polygons By fixing a basis for the homology of the surface S relative to the singularity set Σ, we can cut the surface up into topological disks. Integrating ω along this basis allows us to represent the surface as a collection of Euclidean polygons with parallel sides identified by Euclidean translations. Indeed, it is possible to define translation surfaces in terms of a collection of polygons. Using a collection of Euclidean polygons with parallel sides of the same length identified by translation, we will in fact present many of our examples of translation surfaces in this way. Given such a presentation the Riemann surface structure X comes from the fact that translations are holomorphic, and the one-form dz descends from $\mathbb{C}$ to the surface since $d(z + c) = dz$. Vertices in a polygonal representation could glue to have an angle which is possibly in excess of 2π. These are referred to as singular points, and all angles are an integer multiple of 2π [13].

Equivalences Returning to the complex analytic definition, we that say two translation surfaces $S_1 = (X_1, \omega_1), S_2 = (X_2, \omega_2)$ are *equivalent* if there is there is a biholomorphism $f : X_1 \to X_2$ where $f_*(\omega_2) = \omega_1$. For the polygonal definition, there is a natural notion of equivalence called *scissors congruence*, and that the equivalences are also equivalent is non-trivial to show; see [1] Chapter 2 and [28].

3.2 Strata of Translation Surfaces

We now define a *moduli space* Ω_g of compact genus g translation surfaces up to the aforementioned surface equivalence. This space Ω_g is stratified according to the number and multiplicity of the zeros of ω. By the Gauss-Bonnet theorem or the Riemann-Roch theorem [1], the orders of the zeros of ω must add to $2g - 2$. If we fix $\alpha = (\alpha_1, \dots, \alpha_k)$ to be an integer partition of $2g - 2$, we can consider the space $\Omega_g(\alpha)$ where ω has k distinct cone points of order $(\alpha_1, \dots, \alpha_k)$. As examples, the regular decagon with parallel sides identified gives a genus 2 surface with two cone points each of order 4π, i.e., a surface in $\Omega_2(1, 1)$. The regular octagon with opposite sides identified gives a translation surface of genus two with one singularity of order 6π, so a surface in $\Omega_2(2)$. Each stratum looks locally like $\mathbb{C}^N$ where $N = 2g + k - 1$ where k denotes the number of distinct zeros of ω. Kontsevich-Zorich [15] showed that each stratum has up to three connected components which are described by exactly two invariants: hyper-ellipticity and spin parity.

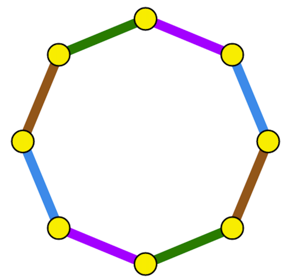


Figure 2: A regular octagon with parallel sides identified by Euclidean translations. The yellow vertex glues to a single cone point of 6π. This figure is used with permission from A. Artiles.

Actions on strata The linear action of $\mathrm{GL}^+(2,\mathbb{R})$ on $\mathbb{R}^2$ induces an action on the moduli space Ω_g which is most easily seen in the polygonal definition as the linear action on polygons in the plane. We note that parallel sides of equal length remain parallel and of equal length, so given a polygonal presentation and a matrix $h \in \mathrm{GL}^+(2,\mathbb{R})$, we obtain a new translation surface. This action preserves strata $\Omega_g(\alpha)$. The area of a translation surface $S = (X, \omega)$ is the area of the corresponding polygons (in complex analytic notation, it is given by $\mathrm{Area}\,(X,\omega) = \frac{i}{2}\int_X \omega \wedge \overline{\omega}$). The action of the subgroup $\mathrm{SL}\,(2,\mathbb{R})$ preserves the locus $\mathcal{H}(\alpha)$ of surfaces with area 1, that is

$$\mathcal{H}(\alpha) = \{(X,\omega) \in \Omega_g(\alpha) : \mathrm{Area}(X,\omega) = 1\}.$$

The action of the rotational subgroup $SO(2,\mathbb{R})$ corresponds to multiplying ω by a nonzero complex number with norm one.

Measures There are natural local coordinates called period coordinates [28] to $\mathbb{C}^{2g+k-1}$ on strata $\Omega_g(\alpha)$ which correspond to viewing a one-form as an element of relative cohomology of the surface, relative to the zeros. The pull-back of the Lebesgue measure from $\mathbb{C}^{2g+k-1}$ to strata gives a natural measure, which induces (via a cone construction) a $\mathrm{SL}(2,\mathbb{R})$-invariant measure on the area 1 locus $\mathcal{H}(\alpha)$. Independent results of Masur [18] and Veech [26] [27] show that the measure on each stratum $\mathcal{H}(\alpha)$ is finite, and in fact the $\mathrm{SL}(2,\mathbb{R})$-action is ergodic. Loosely, ergodic actions are ones where the only non-zero measure invariant sets under the action are either $\emptyset$ or the whole measure space.

Lattice surfaces The ergodicity of the $\mathrm{SL}(2,\mathbb{R})$-action implies that almost every (with respect to Masur-Veech measure) translation surface $S = (X,\omega)$ has *dense* $\mathrm{SL}(2,\mathbb{R})$-orbit in $\mathcal{H}(\alpha)$ and trivial stabilizer under the $\mathrm{SL}(2,\mathbb{R})$-action. We will be interested in a class of surfaces at the other extreme: for which the orbit is *closed* and the stabilizer, in some sense, is as large as possible. Let

$$\mathrm{SL}\,(X,\omega) = \{h \in \mathrm{SL}(2,\mathbb{R}) : h \cdot (X,\omega) = (X,\omega)\}$$

be the stabilizer subgroup of (X,ω). This is the group of derivatives of affine automorphisms of the translation surface.

Definition. A translation surface is called a *lattice surface* if the stabilizer $\mathrm{SL}(X,\omega)$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$.

By a theorem of Smillie [19], the $\mathrm{SL}(2,\mathbb{R})$ orbit of (X,ω) is closed if and only if (X,ω) is a lattice surface, and in this case can be identified with the quotient $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SL}(X,\omega)$.
The projection of this closed orbit from *Teichmüller space* $\mathcal{T}_g$ to the moduli space $\mathcal{M}_g$ of compact genus g Riemann surfaces can be identified with $\mathbb{H}^2/\,\mathrm{PSL}(X,\omega)$, which is an isometrically embedded image of

a hyperbolic orbifold into moduli space. The isometry is with respect to the *Teichmüller metric*, which measures the distance between two Riemann surfaces in terms of the difference in their complex structures measure by quasiconformal dilatation. Several examples which we have mentioned (the flat torus, the regular octagon and decagon are lattice surfaces). We will discuss these surfaces in §3.4.

3.3 Teichmüller Geodesic Flow and pseudo-Anosov Maps

The positive diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$ is the set of matrices

$$\left\{ g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{\frac{-t}{2}} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

The orbits of the diagonal subgroup on $\mathcal{T}_g$ project to geodesics in Ω_g. Thus, this action is known as the *Teichmüller geodesic flow*. We note that if (X,ω) has a *closed* orbit of this flow, say $g_{t_0}(X,\omega) = (X,\omega)$, the element $g_{t_0} \in \mathrm{SL}(X,\omega)$, and the induced affine automorphism of the surface is what is known as a *pseudo-Anosov map*, stretching the horizontal direction on (X,ω) by $e^{t_0/2}$ and contracting the vertical by the same amount.

3.4 Examples of Lattice Surfaces

The torus and the modular curve The seminal example of a translation surface is the square flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. As we discussed in §2, the stabilizer of this surface under the natural linear action is the same as the stabilizer of the lattice $\mathbb{Z}^2$, which is the modular group $\mathrm{SL}(2,\mathbb{Z})$. In general, given a unit covolume lattice $\Lambda = h\mathbb{Z}^2 \subset \mathbb{R}^2$, $h \in SL(2,\mathbb{R})$, we can form the quotient torus $\mathbb{R}^2/\Lambda$, which is flat and has unit area (we can think of it as $(\mathbb{C}/\Lambda, dz)$, or as a parallelogram fundamental domain for Λ with opposite sides identified). Then, we can think of $\mathrm{SL}(2,\mathbb{R}) / \mathrm{SL}(2,\mathbb{Z})$ as the space of unit-area flat tori $\mathcal{H}(\emptyset)$. If we further identify lattices (or tori) which differ by a rotation, we obtain the quotient space $\mathrm{SO}(2)\backslash \mathrm{SL}(2,\mathbb{R}) / \mathrm{SL}(2,\mathbb{Z}) = \mathbb{H}^2 / \mathrm{SL}(2,\mathbb{Z})$; recall the identification $\mathrm{SO}(2)\backslash \mathrm{SL}(2,\mathbb{R}) \equiv \mathbb{H}^2$. The square torus has no cone points under edge identifications; the associated Teichmúller curve $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$ is called the *modular curve*, shown in 3.4. It has two cone points of order 2 and 3; these come exactly from the conjugacy classes of torsion elements in the stabilizer group $\mathrm{PSL}(2,\mathbb{Z})$, and correspond to the square and hexagonal torus respectively, these are illustrated in green and red below in Figure 3.4.

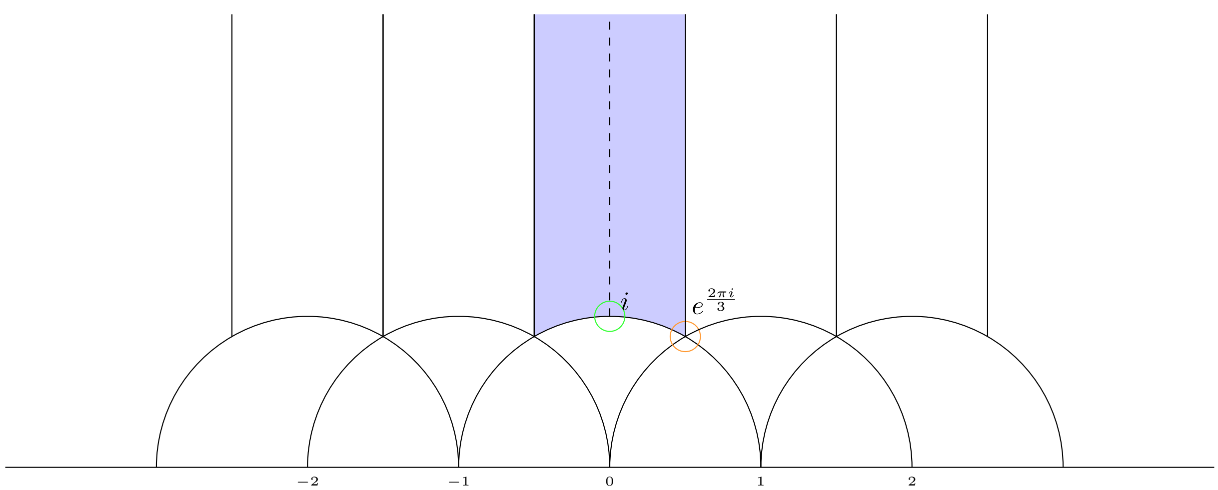


Figure 3: An illustration of the fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$ acting on $\mathbb{H}^2$, with cone points of order 2 at i and order 3 at $e^{\frac{2\pi i}{3}}$, indicated in green and orange respectively.

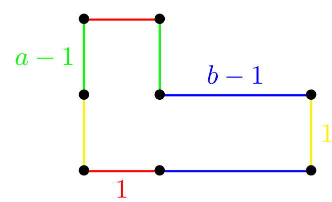
Regular Polygons The torus, regular octagon, and regular decagon examples generalize to regular polygons with $2k$ edges and parallel sides glued by Euclidean translations; regular polygons with parallel edges of the same length are translation surfaces. If $k = 2g$, we obtain a genus g surface with one cone point of angle $2\pi(2g-1)$, and if $k = 2g+1$, we obtain a genus g surface with two cone points of angle $2\pi g$ respectively. For regular polygons with $2k+1$ sides, we take a *double* of the surface and then identify pairs of parallel edges. For $k = 1$ and $k = 2$, we get the triangle and square polygons $\mathbb{R}^2$. In either case , the quotient by this as a fundamental domain is a flat torus. Figure 3.2 demonstrates a regular octagon with sides identified.

L-shaped surfaces We will be particularly interested in a family of genus 2 surfaces $L(a, b)$ in $\Omega_2(2)$ given by L-shaped translation surfaces determined by two positive real parameters a, b, shown in 3.4. These surfaces are related to the work of McMullen [20], Calta [6], and Bainbridge [2], which we discuss in Section 5.3. They provide us with several examples of Teichmüller curves in genus 2.

Lattice L-shaped surfaces McMullen [21] showed that the L-shaped surface $L(a, b)$ is a lattice surface if and only if a and b are both rational or $a = x + z\sqrt{D}$ and $b = y + z\sqrt{D}$ for some $x, y, z \in \mathbb{Q}$ with $x + y = 1$ and $0 \leq D \in \mathbb{Z}$.

The Golden L The translation surface $L\left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ is called the *Golden L*. The stabilizer group of this surface can be directly computed to be the Hecke triangle group $\Delta^+(2, 5, \infty)$, which we will describe in more detail in §5 below.

Figure 4: An L-Shaped surface with parameters a, b, with side identifications indicated by color. This results in a genus 2 surface with one singular point of order 2.



Square-tiled surfaces These surfaces, also known as *origamis*, are finite covers of the square flat torus branched over zero; that is, they can be tiled by unit squares. The number of squares in the tiling determines the degree of the covering map between a square-tiled surface and the square torus. Their stabilizer groups are closely related to $\mathrm{SL}(2, \mathbb{Z})$§5, and an algorithm for computing them in terms of covering data is given by Schmithüsen [25]. Special cases of these are so-called *X-origamis* and *staircase* origamis.

Definition. A square-tiled surface is called *primitive* if there are no intermediate coverings between the surface and the torus.

In the case of a primitive square-tiled surface, we are guaranteed by Schmithüsen [24] that the stabilizer group is a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$.These will be discussed further in 5. To understand the relationship between stabilizers of square-tiled surfaces and $\mathrm{PSL}(2, \mathbb{Z})$, we need to define the notions of commensurability and arithemticity.

Definition. Two subgroups $\Gamma_1, \Gamma_2 \subset \mathrm{SL}(2, \mathbb{R})$ are *commensurable* if there are finite index subgroups $G_1 \subset \Gamma_1$ and $G_2 \subset \Gamma_2$ which are conjugate in $\mathrm{SL}(2, \mathbb{R})$. A subgroup of $\mathrm{SL}(2, \mathbb{R})$ is *arithmetic* if it is commensurable to $\mathrm{SL}(2, \mathbb{Z})$.

The Gutkin-Judge theorem Gutkin-Judge showed the following characterization of square-tiled surfaces in terms of the stabilizer group:

Theorem. [10] Let (X, ω) be a translation surface. The following are equivalent.

- (X, ω) is a branched cover of a torus.
- $\mathrm{SL}(X, \omega)$ is an arithmetic lattice.

4 Reciprocal Geodesics and Reciprocal pseudo-Anosov Maps

In this chapter, we state the Sarnak [23] and Erlandsson-Souto [7] results, and then show how to interpret these results in terms of reciprocal pseudo-Anosov maps in the setting when the hyperbolic orbifolds we consider arise as the Teichmüller curve of a lattice surface.

4.1 Reciprocal Geodesics on the Modular Curve

Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{R})$ containing an order 2 element. A matrix in Γ is called *primitive* if it is not a proper power of another element of Γ. We now define the notion of a reciprocal geodesic for a general lattice Γ containing non-trivial order two elements.

Definition. A closed (primitive) geodesic ρ on $\mathbb{H}^2/\Gamma$ is called *reciprocal* when there is a nontrivial element $\gamma \in \Gamma$ where $\gamma^2 = \mathrm{Id}$ and $\gamma(\rho) = -\rho$ acts as an involution of a geodesic ρ.

In this context, we call the involution a *reciprocal element.* Sarnak [23] gave a parameterization of reciprocal geodesics on the modular surface $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$ using quadratic forms, and proved the following asymptotic counting result. The trace of the hyperbolic element ρ is directly related to the length $\ell(\rho)$ of the reciprocal geodesic it determines (we are abusing notation slightly by using ρ as both a geodesic and a hyperbolic element)

$$\mathrm{tr}(\rho) = 2\cosh\left(\frac{1}{2}\ell(\rho)\right).$$

For large lengths, $2\cosh(\ell(\rho)/2) \sim e^{\ell(\rho)/2}$, that is, the ratio goes to 1 as $\ell(\rho) \to \infty$.

Theorem 4.1.1. [23] For $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, we have

$$\#\{\text{reciprocal geodesics } \rho \text{ on } \mathbb{H}^2/\Gamma \text{ with } \ell(\rho) < L\} \sim \frac{3}{8}e^L,$$

where $\sim$ means that the ratio tends to 1 as $L \to \infty$.

This means that in the modular surface there are asymptotically $\frac{3}{8}e^L$ reciprocal geodesics with length at most L.

4.2 Generalization to Hyperbolic Orbifolds

Erlandsson-Souto [7] generalize the work of Sarnak [23] and Bourgain-Kontorovich [4] to the general setting of lattices $\Gamma \subset PSL(2,\mathbb{R})$ containing order 2-elements. In addition to their counting result, which we state below, they also show that reciprocal geodesics become *equidistributed* on the unit tangent bundle $T^1(\mathbb{H}^2/\Gamma)$.

Theorem 4.2.1. [7] Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a lattice containing order 2 elements. Then

$$\#\{\text{ reciprocal geodesics } \rho \text{ on } \mathbb{H}^2/\Gamma \text{ with } \ell(\rho) \leq L\} \sim \frac{C(\Gamma)}{|\chi^{orb}(\mathbb{H}^2/\Gamma)|} \cdot e^L,$$

where $\chi^{orb}(\mathbb{H}^2/\Gamma)$ is the Euler characteristic of the orbifold $\mathbb{H}^2/\Gamma$ and

$$C(\Gamma) = \frac{1}{4}\left(\sum_{\sigma \in \Gamma \backslash I_\Gamma} \frac{1}{|N_\Gamma(\sigma)|}\right)^2.$$

Here $I_\Gamma = \{\gamma \in \Gamma \setminus \mathrm{Id} : \gamma^2 = \mathrm{Id}\}$ is the set of order two elements, and $\Gamma \backslash I_\Gamma$ is the set of their conjugacy classes. For each conjugacy class $\sigma \in \Gamma\backslash I_\Gamma$, $|N_\Gamma(\sigma)|$ is the cardinality of the normalizer of any element in the conjugacy class σ. We emphasize that the conjugacy is inside Γ.

Application to lattice surfaces We will apply these results to the geometric context when the lattice Γ arises as the stabilizer group $\mathrm{PSL}(X,\omega)$ of a translation surface (X,ω). When we explicitly compute these constants for several examples in 5, we are computing the growth of reciprocal geodesics passing through a cone point of order two on the Teichmüller curve $\mathbb{H}^2/\Gamma$. We now show how to connect these geodesics to reciprocal pseudo-Anosov maps.

4.3 Reciprocal pseudo-Anosov Maps

Recall from §1 that *reciprocal pseudo-Anosov maps* are pseudo-Anosov maps ρ of a translation surface $S = (X,\omega)$ whose stable and unstable foliations are interchanged by an order 2 element γ of the affine automorphism group $\mathrm{PSL}\,(X,\omega)$ of the surface. We now specialize to the setting of lattice surfaces, that is, $\Gamma = \mathrm{PSL}(X,\omega)$ is a lattice. We have seen in §3.2 that pseudo-Anosov maps correspond to closed geodesics on strata. The interchanging of the foliations by γ corresponds to the geodesic passing through the cone point on the Teichmüller curve $\mathbb{H}^2/\Gamma$ corresponding to the conjugacy class of γ. Thus, our main theorem is a reinterpretation of the Erlandsson-Souto result in this setting:

Theorem 4.3.1. Let (X,ω) be a lattice translation surface such that the stabilizer group $\Gamma = \mathrm{PSL}(X,\omega)$ contains a order 2 element. There is a bijective correspondence between reciprocal pseudo-Anosov maps on $S = (X,\omega)$ and closed reciprocal geodesics on the hyperbolic orbifold $\mathbb{H}^2/\,\mathrm{PSL}\,(X,\omega)$. Precisely, associated to a reciprocal pseudo-Anosov ρ on S of expansion factor λ there is a reciprocal geodesic of length $2\log\lambda$ on the hyperbolic orbifold $\mathbb{H}^2/\,\mathrm{PSL}\,(X,\omega)$. Thus

$$\#\{\text{ reciprocal pseudo-Anosov maps } \rho \text{ of } (X,\omega) \text{ with expansion factor } \leq e^{L/2}\} \sim \frac{C\,(\Gamma)}{|\chi^{orb}(\mathbb{H}^2/\Gamma)|}\cdot e^{L},$$

where $C\,(\Gamma)$ and $\chi^{orb}(\mathbb{H}^2/\Gamma)$ are as in Theorem 4.2.1.

Proof. The order 2 element γ interchanging the stable and unstable foliations of ρ means that γ preserves the axis of ρ when viewed acting on $\mathbb{H}^2$, equivalently, that the axis of ρ passes through the fixed point of γ. The length ℓ of the geodesic and the expansion factor λ of the pseudo-Anosov are related by $\ell = 2\log\lambda$; thus, we have our correspondence between reciprocal pseudo-Anosovs of (X,ω) and reciprocal geodesics on $\mathbb{H}^2/\,\mathrm{PSL}(X,\omega)$. ☐

Applications to specific families In §5 below, we will compute the constants to give precise asymptotics of the growth of these reciprocal pseudo-Anosovs for a family of what are known as *Bouw-Möller surfaces*, and for a family of Square-tiled L-shaped surfaces. We count the reciprocal pseudo-Anosov maps ordered by the expansion factor, or equivalently, length of the associated closed geodesic.

5 Asymptotic Growth Rates of Reciprocal pseudo-Anosov Maps

In this section, we state and prove our main counting theorems. To do this, we combine the results of Erlandsson-Souto [7] and our correspondence to calculate the precise constants for the asymptotic growth of reciprocal pseudo-Anosov maps on two families of surfaces: a subfamily of Bouw-Möller surfaces whose stabilizer groups are Hecke triangle groups $\Delta^+(2,q,\infty)$ and a family of genus 2 L-shaped surfaces $\{L(a,a) : a \in \mathbb{N}\}$ which are primitive and square-tiled.

5.1 Reciprocal Pseudo-Anosov Maps on Bouw-Möller Surfaces

Hecke triangle groups We introduce a family of surfaces known as Bouw-Möller surfaces whose stabilizer groups are Hecke triangle groups, which we define below. We will be particularly interested in those with order 2 elements in the stabilizer group.

Definition. Let $p, q \in \mathbb{Z}_+$ such that $1/p + 1/q < 1$. The *Hecke triangle group* $\Delta(p,q,\infty)$ is the discrete group of reflections of a hyperbolic triangle with interior angles $\left(\frac{\pi}{p}, \frac{\pi}{q}, 0\right)$, where $\frac{\pi}{p} + \frac{\pi}{q} < \pi$. We will consider the subgroup $\Delta^+(p,q,\infty) \subset \Delta(p,q,\infty)$ of orientation preserving isometries of $\mathbb{H}^2$, and we will focus on the case $p = 2$ (right hyperbolic triangles). Since these groups are discrete, $\Delta^+(2,q,\infty)$ acts properly discontinuously on $\mathbb{H}^2$, and in fact is a lattice. The quotient $\mathbb{H}^2/\Delta^+(2,q,\infty)$ is a genus 0 hyperbolic orbifold with cone points of order 2 and q. A useful generating set for $\Delta^+(2,q,\infty)$ is given in [16]:

$$\Delta^+(2,q,\infty) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} \right\rangle$$

where $\lambda_q = 2\cos\left(\frac{\pi}{q}\right)$.

Triangle groups as stabilizers Work of Bouw-Möller [5] and Hooper [12], using semi-regular polygons and grid graphs respectively, together show how to construct explicit examples of translation surfaces whose stabilizers are Hecke triangle groups.

Theorem 5.1.1. [12, 5] For any positive integers p, q with $1/p + 1/q < 1$, there is a translation surface $S_{p,q} = (X_{p,q}, \omega_{p,q})$ such that the (projective) stabilizer group $\mathrm{PSL}(X_{p,q}, \omega_{p,q}) = \Delta^+(p,q,\infty)$.

We will focus on the setting $p = 2$, and write $S_q = S_{2,q}$ for the particular Bouw-Möller surface with stabilizer $\Delta^+(2,q,\infty)$. Our main result regarding these surfaces is:

Theorem 5.1.2. Let $S_q = (X_q, \omega_q)$ be a Bouw-Möller surface such that its stabilizer group is the Hecke triangle group $\Delta^+(2,q,\infty)$. Then

$$\#\{\text{ reciprocal pseudo-Anosov maps on } S_q \text{ with } tr\,(\rho) \leq L\} \sim \frac{q}{8\,(q-2)} e^L \tag{5.1.3}$$

Below, we discuss important low-genus examples corresponding to $q = 3, 5$.

The $q = 3$ case The triangle group $\Delta^+(2,3,\infty)$ is isomorphic to $\mathrm{PSL}\,(2,\mathbb{Z})$. We can see this by noting that $\lambda_3 = 2\cos(\pi/3) = 1$, so

$$\Delta^+(2,3,\infty) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle = \mathrm{PSL}\,(2,\mathbb{Z})\,.$$

Since the stabilizer group of the torus $\mathbb{R}^2/\mathbb{Z}^2$ is $\mathrm{SL}\,(2,\mathbb{Z})$, we see that the $q = 3$ case does occur as a stabilizer group of a lattice surface. A direct computation shows that every order 2 element of $PSL(2,\mathbb{Z})$ is conjugate (in $PSL(2,\mathbb{Z})$) to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, that is, there is one conjugacy class of order 2 elements. We will show below how to compute the size of the normalizer, and thus the growth rate of reciprocal geodesics using the Erlandsson-Souto Theorem 4.2.1, to recover Sarnak's result Theorem 1.1.

Non-arithmetic triangle groups The groups $\Delta^+(2,q,\infty)$ with $q > 3$ are *non-arithmetic*, that is, not commensurable to $PSL(2,\mathbb{Z})$, so in particular they are not stabilizers of square-tiled surfaces. For all q, there is a unique order 2 cone point on $\mathbb{H}^2/\Delta^+(2,q,\infty)$, corresponding to the conjugacy class of the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, by our main result Theorem 4.3.1, to count reciprocal pseudo-Anosovs on S_q, we will count reciprocal geodesics on $\mathbb{H}^2/\Delta^+(2,q,\infty)$, that is, closed geodesics passing through this unique order 2 cone point. To compute the constants, we will need to compute the order of the normalizer of this element, and the orbifold Euler characteristic of $\mathbb{H}^2/\Delta^+(2,q,\infty)$, which we will do in §5.2 below.

The $q = 5$ case: the golden L For $q = 5$, the group $\Delta^+(2,5,\infty)$ is the stabilizer group of the *Golden L*, a well-studied genus 2 surface, corresponding to $L(\phi,\phi)$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Square-tiled L-shaped surfaces In fact L-shaped surfaces provide an abundant source of lattice surfaces. We will consider the family $L(a,b)$ of surfaces we introduced in §3.4 shown in Figure 3.4. We follow Bainbridge [2, page 1890] in our notation. We fill focus on the setting $L(a,a)$, with a integer. McMullen showed that the surface associated to $L(a,b)$ is a lattice surface if and only if there are $x, y, z \in \mathbb{Q}$ with $x + y = 1$, and $d \in \mathbb{Z}$ such that $a = x + z\sqrt{D}$, $b = y + z\sqrt{D}$. Since we are setting $a = b$, and assuming $a \in \mathbb{N}$, we will take $x = y = z = 1/2$, so $a = \frac{1+\sqrt{D}}{2}$, and D to be an odd square. For example, $D = 9$ corresponds to the surface $L(2,2)$.

Theorem 5.1.4. For every positive integer a, there is a constant e_a such that the growth of reciprocal pseudo-Anosov maps on the genus 2 primitive square-tiled surface $L(a,a)$ is given by

$$\#\{\text{ reciprocal pseudo-Anosov maps on } L(a,a) \text{ with } \operatorname{tr}(\rho) \le L\} \sim e_a \cdot e^L \tag{5.1.5}$$

5.2 Growth Rates of Reciprocal pseudo-Anosov Maps on Bouw-Möller Surfaces

We now prove Theorem 5.1.2. The family of Teichmüller curves $\mathbb{H}^2/\Gamma$ arising from $\Gamma = \Delta^+(2,q,\infty)$ have one orbifold point of order 2, or equivalently, there is one conjugacy class of order 2 elements in Γ. Therefore, our counting of reciprocal pseudo-Anosov maps correspond to counting closed geodesics which pass through this unique order 2 point on $\mathbb{H}^2/\Delta^+(2,q,\infty)$.

Orbifold Euler Characteristic Next, we note that the orbifold characteristic of the quotient orbifold $\chi^{orb}(\mathbb{H}^2/\Gamma)$ depends only on q; we have

$$\chi^{orb}(\mathbb{H}^2/\Gamma) = 2 - \left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{q}\right) - \left(1 - \frac{1}{\infty}\right) = -\frac{1}{2} + \frac{1}{q}$$

As $q \to \infty$, $\chi^{orb}(\mathbb{H}^2/\Gamma) \sim -\frac{1}{2}$. The genus of $\mathbb{H}^2/\Delta^+(2,q,\infty)$ is 0. Like the torus, we have one conjugacy class of order 2 elements, and the normalizer consists of the element itself and the identity, so $C(\Gamma) = \frac{1}{16}$. Thus, we can express the exponential asymptotic growth rate of reciprocal pseudo-Anosov maps for the translation surfaces S_q with stabilizer group $\Delta^+(2,q,\infty)$ as

$$\frac{\frac{1}{16}}{|-\frac{1}{2}+\frac{1}{q}|} = \frac{q}{8(q-2)}$$

by [7, Corollary 1.2]. This recovers Sarnak's $\frac{3}{8}e^L$ asymptotic for the modular surface for $q = 3$.

5.3 Growth Rates of Reciprocal pseudo-Anosov Maps on $L(a,a)$

We now turn to proving Theorem 5.1.4. We recall the relationship between the orbifold Euler characteristic and volume of $\mathbb{H}^2/\Gamma$: the orbifold Euler characteristic of $\mathbb{H}^2/\Gamma$ is proportional to the volume of $\mathbb{H}^2/\Gamma$. For example, Belolipetsky [3] gives:

$$\operatorname{vol}(\mathbb{H}^2/\Gamma) = -\frac{\operatorname{vol}(S^2)}{2}\chi^{orb}(\mathbb{H}^2/\Gamma)$$

D	Index	$\#\{\lvert\sigma\rvert\}$	Generators of Γ
9	3	1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$
25	9	1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 7 & -9 \\ 4 & -5 \end{pmatrix}$
49	54	1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 17 & -20 \\ 6 & -7 \end{pmatrix}, \begin{pmatrix} 31 & -44 \\ 12 & -17 \end{pmatrix}, \begin{pmatrix} 132 & -205 \\ 85 & -132 \end{pmatrix}, \begin{pmatrix} 46 & -81 \\ 24 & -44 \end{pmatrix}$ $\begin{pmatrix} 188 & -293 \\ 77 & -120 \end{pmatrix}, \begin{pmatrix} 169 & -269 \\ 70 & -111 \end{pmatrix}, \begin{pmatrix} 191 & -308 \\ 80 & -129 \end{pmatrix}, \begin{pmatrix} 113 & -186 \\ 48 & -79 \end{pmatrix}, \begin{pmatrix} 34 & -59 \\ 15 & -26 \end{pmatrix}$
81	81	1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 21 & -25 \\ 16 & -19 \end{pmatrix}, \begin{pmatrix} 29 & -40 \\ 8 & -11 \end{pmatrix}, \begin{pmatrix} 78 & -121 \\ 49 & -76 \end{pmatrix}, \begin{pmatrix} 71 & -115 \\ 21 & -34 \end{pmatrix}$ $\begin{pmatrix} 139 & -295 \\ 42 & -71 \end{pmatrix}, \begin{pmatrix} 344 & -593 \\ 105 & -181 \end{pmatrix}, \begin{pmatrix} 515 & -891 \\ 226 & -391 \end{pmatrix}, \begin{pmatrix} 235 & -409 \\ 104 & -181 \end{pmatrix}, \begin{pmatrix} 73 & -130 \\ 41 & -731 \end{pmatrix}$ $\begin{pmatrix} 139 & -295 \\ 42 & -71 \end{pmatrix}, \begin{pmatrix} 31 & -75 \\ 12 & -29 \end{pmatrix}, \begin{pmatrix} 114 & -314 \\ 41 & -114 \end{pmatrix}, \begin{pmatrix} 19 & -36 \\ 9 & -17 \end{pmatrix} \begin{pmatrix} 28 & -81 \\ 9 & -26 \end{pmatrix}, \begin{pmatrix} 89 & -205 \\ 33 & -76 \end{pmatrix}$

Figure 5: Table of the indices and generators of the stabilizer group $\Gamma_a \subset \mathrm{SL}\,(2,\mathbb{Z})$ for $L_{a,a}$, $a = \frac{1+\sqrt{D}}{2}$.

Using $\mathrm{vol}(S^2) = 4\pi$, we re-express the equality as

$$\chi(\mathbb{H}^2/\Gamma) = \frac{-\mathrm{vol}(\mathbb{H}^2/\Gamma)}{2\pi}.$$

The surfaces $L(a,a)$ are primitive square-tiled surfaces, so their stabilizer groups Γ_a are finite index subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ with index which we denote by $[\mathrm{PSL}\,(2,\mathbb{Z}) : \Gamma] = N_a$. Thus $\mathbb{H}^2/\Gamma_a$ is a degree N_a cover of $\mathbb{H}^2/\,\mathrm{PSL}\,(2,\mathbb{Z})$, so $\mathrm{vol}(\mathbb{H}^2/\Gamma) = N_a\,\mathrm{vol}(\mathbb{H}^2/\,\mathrm{PSL}\,(2,\mathbb{Z}))$. By direct computation of the area of a fundamental domain, $\mathrm{vol}(\mathbb{H}^2/\,\mathrm{PSL}\,(2,\mathbb{Z})) = \pi/3$, so $\mathrm{vol}(\mathbb{H}^2/\Gamma) = \frac{N_a \pi}{3}$. Using our reformulation, we recover that $\chi^{orb}(\mathbb{H}^2/\,\mathrm{PSL}\,(2,\mathbb{Z})) = -\frac{1}{6}$, and so $\chi^{orb}(\mathbb{H}^2/\Gamma_a)) = -\frac{N_a}{6}$.

Computation of Stabilizer Groups Recall from 3.4 that the stabilizer group of a square-tiled lattice surface is commensurable to $\mathrm{SL}(2,\mathbb{Z})$, and when the square-tiled surface is primitive, the stabilizer group is a finite index *subgroup* of $\mathrm{SL}\,(2,\mathbb{Z})$. In particular, Schmithüsen [24] gives an algorithm to compute the stabilizer group of a square-tiled surface, we use an implementation of such an algorithm, in the python package `FlatSurf` to compute the index of the stabilizer group Γ_a, $a = \frac{1+\sqrt{D}}{2}$ given the parameter D; the initial output is given in 5.

Computation of $|N_\Gamma(\sigma)|$ To apply the counting theorem of Erlandsson-Souto, we need to compute the cardinality of the normalizer of a conjugacy class of order 2 elements in our group Γ_a, an index N_a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. To compute the normalizer of an order 2 element, we consider the element $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The normalizer is given by $N_\Gamma(\sigma) = \{\gamma \in \Gamma \mid \gamma\sigma\gamma^{-1} = \sigma\}$. We can compute this set directly. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (we are abusing notation slightly by using a in different ways, but the context should be clear). Then we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This gives us

$$\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

Thus we have $b = -c$ and $a = d$. The possible matrices in the normalizer would be of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Since our stabilizer group Γ_a is a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, the matrix entries are integers. The set of possible matrices reduces to the identity and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus, $|N_\Gamma(\sigma)| = 2$ for any order 2 element σ. Now we can apply 4.3.1:

$$\begin{aligned} C(\Gamma_a) &= \frac{1}{4}\left(\sum_{[\sigma]} \frac{1}{|N_{\Gamma_a}(\sigma)|}\right)^2 \\ &= \frac{1}{4} n_a \left(\frac{1}{2}\right)^2 \\ &= \frac{n_a}{16}, \end{aligned}$$

where the sum is taken over the collection of conjugacy classes $[\sigma]$ of order 2 elements, and n_a is the number of such classes in Γ_a. Putting

$$e_a = \frac{\left(\frac{n_a}{16}\right)}{\left(\frac{N_a}{6}\right)} = \frac{3}{8}\frac{n_a}{N_a},$$

we have proved Theorem 5.1.4.

Computing n_a and N_a As we have shown in Figure 5, the index N_a can be computed directly using `FlatSurf`. Computing the number of conjugacy classes N_a of order 2 elements in Γ_a is more subtle: we already see from Figure 5 that the elliptic elements $\begin{pmatrix} 132 & -205 \\ 85 & -132 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & -0 \end{pmatrix}$ are distinct order 2 generators in the index 54 group Γ_4 (corresponding to $D = 49$), and thus are non-conjugate in Γ_4. We note that the program [9] used to generate the data in Figure 5 returns generating sets which are minimal, so these conjugacy classes are indeed distinct. We note that is not sufficient to know the index N_a, since there are finite index subgroups of $\mathrm{PSL}(2, \mathbb{Z})$ which contain no order 2 elements, such as the principal congruence subgroup of level 2, consisting of the kernel of the reduction $\pmod 2$.

6 Further Work

The research in this thesis opens up several new questions for exploration in computing the asymptotic growth rates of reciprocal pseudo-Anosov maps on translation surfaces beyond the situations we have explored. In this section, we outline some of these questions.

Computing the constants n_a and N_a As we discussed just above, while the constants N_a can be easily computed using `FlatSurf`, it would be useful to create an algorithm to compute the number n_a of conjugacy classes of order 2 elements in Γ_a, to give a more explicit formulation of e_a in terms of a.

Lattice L-shaped surfaces The surfaces $L(a,b)$ give a large family of lattice surfaces, as we described in §3.4. Letting $\Gamma_{a,b}$ denote their stabilizer group, the orbifold Euler characteristics of the associated Teichmüller curves $\mathbb{H}^2/\Gamma_{a,b}$ has been computed by Bainbridge [2] and Mukamel [22], building on earlier work of McMullen [20]. Mukamel [22] has also computed which of these Teichmüller curves have order 2 cone points; what remains is to compute the size of the normalizer of these cone points in order to compute the precise asymptotics for these lattice surfaces.

Primitive square-tiled surfaces Returning to square-tiled surfaces, we note that our proof of Theorem 5.1.4 in fact proves a more general result for primitive square tiled surfaces. If (X,ω) is a primitive square-tiled surface, and $\Gamma = \mathrm{PSL}(X,\omega) \subset \mathrm{PSL}(2,\mathbb{Z})$ its projective stabilizer group, let $N = [\mathrm{PSL}(2,\mathbb{Z}) : \Gamma]$ be the index of Γ. Let n denote the number of distinct conjugacy classes of order 2 elements in Γ (possibly $n = 0$, as we discussed above). The constant for the exponential asymptotic growth rate of reciprocal pseudo-Anosov maps on (X,ω) is then given by $\frac{3}{8}\frac{n}{N}$. It would be interesting to write an algorithm to compute n in general.

Non-lattice surfaces We can also consider reciprocal pseudo-Anosov maps of non-lattice translation surfaces (X,ω) which have an order 2 element in their (non-lattice) projective stabilizer group $\mathrm{PSL}(X,\omega)$. It is not clear that these grow exponentially in any particular case, but it would be interesting to investigate this using the dynamical ideas in Erlandsson-Souto [7] extended beyond the setting of hyperbolic surfaces. In particular, we think that mixing of the Teichmüller geodesic flow could be useful in this situation.

Higher-order cone points It would be interesting to give a translation surface interpretation of closed geodesics passing through order 3 or higher cone points on Teichmüller curves of lattice surfaces.

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