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# Conformal Welding of Uniform Random Trees

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A dissertation  
submitted in partial fulfillment of the  
requirements for the degree of

Doctor of Philosophy

University of Washington

2014

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Program Authorized to Offer Degree:  
Mathematics



University of Washington

**Abstract**

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A conformally balanced tree is an embedding of a given planar map into the plane with constraints on the harmonic measure of its edges such that the resulting set is unique up to scale and rotation. Bishop [10] showed that there exists a conformal map from the exterior of the disc to the complement of such a tree. The preimage of the tree under the map is a conformal welding map which induces a lamination of the unit circle that corresponds exactly to the encoding of the tree as an excursion. We consider the distributional limits of the maps for the uniform measure on random walk excursions as the number of steps goes to infinity, normalized by conformal radius, and we show that subsequential limits are almost surely nontrivial. Additionally, we investigate the properties of laminations resulting from consistent distributions on ordered trees with variable edge length.

Burdzy, Pal, and Swanson [11] considered solid spheres of small radius reflecting in the unit interval with mass being added to the system from the left at constant rate, killed when reaching the right boundary. By transforming to a system with zero-width particles moving as independent Brownian motion, they derived a limiting stationary distribution for a particular initial distribution, as the width of a particle decreases to zero and the number of particles increases to infinity. This space-removing transformation has a direct analogy in the isomorphism between a new unbounded-range exclusion process and a superimposition of random walks with random boundary. We derive the hydrodynamic limit for these isomorphic processes.



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## ACKNOWLEDGMENTS

I express my deepest thanks to everyone who played any part in helping me through the graduate program to completion. Most importantly, my parents, who always told me I could do anything I wanted, as long as I took my mother on a nice vacation some day. Next, my various teachers and professors, who fostered my talents and did their best to instill a bit of work ethic whenever I let my guard down. I thank my students at St Albans, who reinvigorated my love of discovery in math, my fellow first-year students in the math department, my first advisor, Krzysztof Burdzy, and the other members of my committee, Soumik Pal and James R Lee. The final tier of thanks are awarded to my thesis advisor, Steffen Rohde, who relentlessly encouraged and challenged me, and Marie, my partner in life, who talked me through the lowest moments, and shared with me the highest.

## DEDICATION

to Marie, who inspires me

## PREFACE

The work is divided into two separate and mostly unrelated projects. The first, which bears the title of the whole work, is a study of special embeddings of random trees into the plane. The particular embedding is chosen to be balanced with respect to harmonic measure, which roughly means that a particle traveling along a random path towards the tree is equally likely to hit any edge on either side. We investigate the basic properties, the construction of such trees using elementary conformal maps (a discussion which reprints material submitted for my general exam), and finally obtain a first result about the distribution of large trees chosen uniformly at random, a topic that is new to the literature. This project was completed under the advisement and with the collaboration of Steffen Rohde.

The second project investigates the limiting distribution of the empirical measure of an interacting particle system on the integer lattice. We prove that the limit obtained is almost surely the solution to a particular free-boundary version of the heat equation called the Stefan problem. This project was completed under the advisement of Krzysztof Burdzy.



## Chapter 1

## CONFORMAL WELDING OF UNIFORM RANDOM TREES

**1.1 Introduction**

We consider an approach to embedding large random trees into the plane. As a most general description, a tree is a space with unique paths between points. Due to its general nature, it is fruitful to consider the connections between models studied in different contexts. In our case, we explore the space between the study of large random trees in probability theory (see [2], [3], [26]), and the natural embeddings of trees in the complex plane that arise in algebraic geometry and potential theory (for example, [30], [24]). Such an embedding is called a harmonically balanced tree, which we normalize by logarithmic capacity. We prove the following result:

**Theorem 1.1.1.** *Subsequential limits of the uniform measure on normalized conformally balanced trees are nontrivial.*

This theorem can be considered a waypoint towards the desired result, which seems to be well-supported by simulation.

**Conjecture 1.1.1.** *The uniform measure on normalized conformally balanced trees converges to a random conformal map  $f$  and a random compact set  $\Gamma \subset \mathbb{C}$  such that  $f$  welds the Brownian lamination almost surely, and  $\Gamma$  is a tree set obtained as the image of  $\mathbb{T}$  under  $f$ .*

Figure 1.1 is an approximation of a large conformally balanced tree generated by the author.

On the side of large random trees, our primary point of reference are the seminal works [2], [3], [4], and [5] of David Aldous, who demonstrated the convergence of a class of random trees to a random limiting object called the Brownian continuum random tree, or CRT, in the case where trees are rescaled to have a compact limit. This theory relies on the

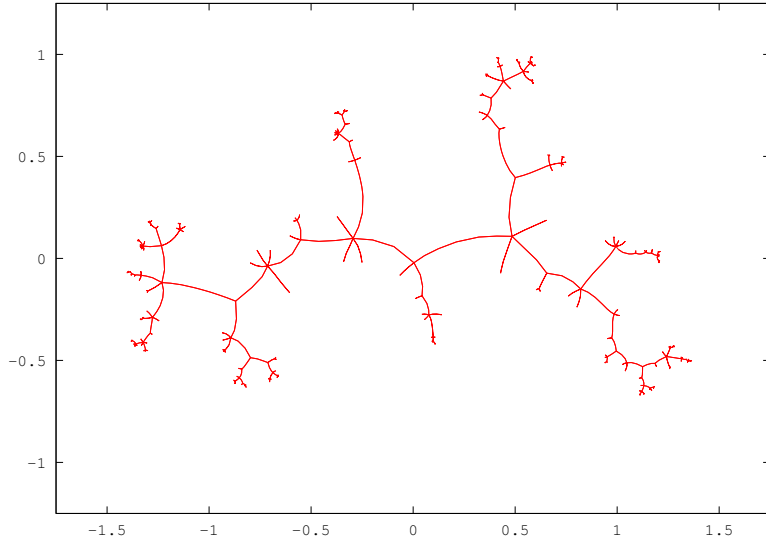


Figure 1.1: A conformally balanced tree with 500 edges, generated using the treeweld software.

connection of finite and continuum trees to continuous paths, whose values encode the distances of points on the tree from its root. Just as Brownian motion gives a universal limit for a class of random continuous functions, the normalized Brownian excursion gives a universal limit for the contour functions of certain random trees, properly rescaled.

On the other hand, we consider the problem of conformal welding. A conformal map is a complex analytic function which is a homeomorphism between two open subsets of the extended complex plane  $\mathbb{C}^*$ . Given a lamination  $L$ , which can be thought of as an equivalence relation on the unit circle  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ , a conformal welding map for  $L$  is a conformal map  $\phi$  of the disk such that  $\phi$  extends continuously to the boundary, and if  $\Gamma \in \mathbb{C}^*$  is the image of  $\mathbb{T}$  under  $\phi$ , then the equivalence classes of  $L$  consist of exactly the pre-image sets  $\phi^{-1}(z)$  for  $z \in \Gamma$ . A natural question is which laminations  $L$  admit a conformal welding map  $\phi$ . We address this question in the case of laminations corresponding to trees.

The correspondence between continuous paths and trees also gives a natural way to

construct laminations of the unit circle. In [4] and [5], Aldous considers triangulations of the circle, again encoding triangulations as continuous paths and obtaining the limit of uniform triangulations as a triangulation encoded by normalized Brownian excursion. This language can easily be translated to the language of laminations, and provides the basis for what we call the *Brownian lamination*. First, we give the relevant background on ordered trees and continuous paths (some of which is unique to this presentation), then we introduce laminations and their relationship to the former models. Along the way we prove a purely combinatorial result about certain alternating sums of real numbers. We state the uniform distributions under investigation, and discuss their relation to the Brownian processes.

Each discrete model has an approach to welding that suits it most naturally, and we consider them each separately. For the first model, which we call the *uniform arc-pairing lamination* we divide the unit circle into  $2n$  arcs between the roots of unity. Then an arc-pairing lamination is one that pairs all the arcs in non-crossing fashion, and associates the points of two paired arcs in linear decreasing fashion. Finally, nodes form equivalence classes by closure of the pairings of their neighboring intervals. The equivalence class of a node can have size (degree)  $d$  from  $d = 1$ , in the case of paired neighboring intervals, to  $d = n$ , in the case of a star-like lamination. Such laminations are in correspondence with excursions of random walk, also called Dyck paths.

The existence of a conformal welding map for an arc-pairing lamination was demonstrated by Christopher Bishop [10], and the maps yield trees embedded into the plane as the image of the boundary, which he calls conformally balanced trees. In fact, the existence of the maps in question follows from the study by George Shabat of polynomials with two critical values, called Shabat polynomials. Such polynomials correspond in one-to-one fashion (up to normalization) with the laminations described above. A reference for this topic is the book by Lando and Zvonkin [24]. The universal nature of the resulting trees and their welding maps provides ample motivation to study the particulars of this discrete model.

With Don Marshall, we gave an explicit construction of the welding map in question, allowing for an implementation in software that is capable of computing trees of up to several thousand vertices to machine accuracy, based on the zipper algorithm of Marshall [28]. A version of this software developed by the author is available at the URL

<http://github.com/oelarnes/treeweld>. This program was used to generate the images of balanced trees found in this paper. It should be stated that a numerically superior version of the same algorithm was first developed by Don Marshall. Compelling images of the distribution of the first coefficient of the Shabat polynomial (see Figure 1.7) give the motivation to investigate the complex moments of the vertices of the conformally balanced trees.

The second model, which we call the uniform proper lamination (following Aldous's definition of proper trees), satisfies a consistency condition that is well-behaved with respect to the welding operation. We present an alternative quasi-conformal welding algorithm that respects the consistency condition, and consider its asymptotics, which turn out to be insufficient for obtaining well-behaved limiting maps.

We next discuss convergence of the uniform measure on conformally balanced trees. We consider convergence in distribution in the topology of uniform convergence on compact sets, which corresponds to a topology on closed sets in the plane according to a theorem of Carathéodory. The Carathéodory topology is weaker than Hausdorff convergence of the tree set, and in fact we get the existence of subsequential limits for free from the normalization. There is no guarantee, however, that such limits will be non-trivial. We prove this result, extending it as far as allowed by the method of proof. To obtain this result, we use relatively weak properties of the coding paths and capacity of the welding sets (for example, we only require that the coding paths be uniformly  $\alpha$ -Hölder for some  $\alpha < 1/2$ ).

We conclude with a discussion of the results of simulation and some directions for future investigation, focused on the distribution of the first free coefficient of the welding map.

## **1.2 Ordered trees and paths**

The correspondence between trees and paths is well-established in the literature (see [6, 3, 26]), but we will need a full exposition before adding laminations, which are not as well explored, although the correspondence has been noted and applied, starting with [4].

We first consider ordered trees. We start with the typical notion of a mathematical graph, consisting of a set of vertices and undirected edges between them, and a rooted graph-theoretic tree, which is a connected graph  $t = (V, E)$  with no cycles and a distinguished vertex  $r$  called the root. If we think of  $t$  as a topological space where the edges are line

segments glued at their shared vertices, then by the basic properties of a tree, for each vertex  $v$  there is a unique path  $[v, r]$  from  $v$  to the root. The first vertex encountered along  $[v, r]$  is called the *parent* of  $v$ , and all of the other neighboring vertices of  $v$  are called its *children*, and compose the set  $c(v)$ . Each vertex without any children is called a *leaf*. Let  $d(v, r)$  be the graph distance between  $v$  and  $r$ , which is also the length of  $[v, r]$  if we assign unit length to each edge. Finally, for two vertices  $v$  and  $w$ , there again exists a unique path  $[v, w]$ , and we let  $b(v, w)$  be the point  $b \in [v, w]$  such that  $d(b, r)$  is minimized. The point  $b$  is clearly a unique vertex of  $t$ , and is called the branch point of  $v$  and  $w$ .

**Definition 1.2.1.** *An ordered tree with  $n$  offspring (also  $n$  edges) is a tree  $t$  together with an ordering  $o$ , which is a mapping  $o : V \setminus \{r\} \rightarrow \mathbb{N}$  that assigns to each of the children of a vertex  $v$  one of the values  $\{1, \dots, \#c(v)\}$ .*

We think of  $o$  as providing a left-to-right ordering of edges emerging from each vertex. Tracing forward from  $r$  to  $v$  along  $[v, r]$ , we can associate to each vertex a word by concatenating the orders of each vertex along the path, where the root gets the empty word. The set of words so obtained provides a complete description of the ordered tree, and a total ordering on the vertices can be obtained by the lexicographical order on words that extends the child order. By following this order and recording the edges encountered, we obtain a unique total ordering of the edges  $(e_1, e_2, \dots, e_n)$ . An ordered tree can also be called a plane tree, since the ordering suggests a way to embed the tree into the plane, unique up to orientation-preserving homeomorphism. The following classical result enumerates ordered trees:

**Proposition 1.2.1.** *The number of ordered trees with  $n$  edges is the Catalan number*

$$C_n = \frac{1}{n+1} \binom{2n}{n}. \quad (1.1)$$

*Proof.* For  $n \geq 1$ ,  $t$  decomposes into two subtrees, the tree  $t_1$ , which has the first child of  $r$  as its root, which has  $k < n$  edges, and the subtree  $t_2$  remaining after pruning away  $t_1$  and the edge leading to its root, which has  $n - k - 1$  edges. Then if  $A_n$  is the number of ordered trees with  $n$  edges, we see that it satisfies the recursive formula

$$A_n = \sum_{k=0}^{n-1} A_k A_{n-k-1},$$

which characterizes  $C_n$ , given  $A_0 = 1$ . □

We will use the symbol  $\mathcal{T}_n$  to represent the uniform probability measure on ordered trees with  $n$  edges, and the tree sampled from this distribution the *uniform ordered tree*.

Next, we briefly explain the connection between ordered trees and continuous paths. First, define a function  $g$  from the set  $\{0, \dots, 2n\}$  to the vertex set. Starting with  $g(0) = r$ , suppose  $v = g(j)$ . If  $w$  is the first (in the ordering) unexplored child of  $v$ , let  $g(j+1) = w$ , and the edge  $(v, w)$  is now considered explored. If there are no unexplored children of  $v$ ,  $g(j+1)$  is the parent of  $v$ . Since each edge will be traversed once in each direction, the process naturally ends with  $g(2n) = r$ .

Then we define the contour function  $f : [0, 2n] \rightarrow \mathbb{R}_+$  by letting  $f(i) = d(g(i), r)$  on the integers, and interpolating linearly in between. The graph of  $f$  is easily seen to be a Dyck path, that is, a path with nonnegative values and increments of  $\pm 1$  along the integers. We will usually refer to a rescaled version on the domain  $[0, 2]$  (for all  $n$ ), with increments  $\pm 1/\sqrt{n}$ . We will call these paths *excursion walks*, call the set of such paths with  $n$  increments  $E_n$ , which has cardinality  $C_n$ , as shown, and call the uniform measure on  $E_n$  by  $\mathcal{E}_n$ .

We don't use Galton-Watson trees in this paper, but since they provide important motivation for the subject of large trees, we provide a statement connecting them to the model above.

**Proposition 1.2.2.** *The Galton-Watson tree with offspring distribution*

$$P[\zeta = i] = 2^{-i}$$

*conditioned to have  $n$  total offspring has the distribution of the uniform ordered tree.*

For proof, see LeGall 2005 [26]. We now consider the uniform distribution on trees with real edge lengths, we will need to consider a subset of the ordered trees, since branching vertices of such trees will have two children almost surely. Therefore we define *proper* trees, following Aldous [3], with some modification.

**Definition 1.2.2.** *A proper ordered  $k$ -tree, for  $k \geq 1$ , is an ordered tree  $\hat{t}$  with  $2k-1$  edges, along with a set of positive real numbers  $(x_1, \dots, x_{2k-1})$  such that:*

1. The root  $r$  has exactly one child.
2. Each other vertex has either zero or two children.

Then  $\hat{t}$  is called the graph of  $t$ . Note that a proper ordered  $k$ -tree will have  $k$  leaves, since we start with one leaf, and each added branching vertex adds a leaf, giving  $k+(k-1)+1 = 2k$  vertices for  $2k - 1$  edges. We will use the notation  $t = (\hat{t}, (x_1, \dots, x_{2k-1}))$ , and the numbers  $x_j$  are considered to represent the lengths of the edge  $e_j$  in the total order given by  $o$ . A proper ordered  $k$ -tree is further called *normalized* if  $\sum_1^{2k-1} x_j = 1$ . It is called *labeled* if we further assign the labels  $\{1, 2, \dots, k\}$  to the leaves. Let  $\mathcal{P}_k^*$  be the uniform measure on labeled normalized proper ordered  $k$ -trees, and let  $\mathcal{P}_k$  be the marginal distribution without labels.

LeGall [25] described the distribution of what he called the uniform random tree in terms of the Itô measure, and connected his result to the theorem of Aldous [2] giving the finite dimensional distribution of the Brownian continuum random tree. We wish to describe the measures  $\mathcal{P}_n$  and  $\mathcal{P}_n^*$  in terms of those distributions, which will require some additional enumeration results.

Let  $T_k^\circ$  be the set of labeled proper  $k$ -tree graphs without orderings, let  $T_k^*$  be the set of the same with orderings, and let  $T_k$  be the latter without labels. That is,  $T_k$  corresponds to the first description of ordered trees, with the additional restrictions of definition 1.2.2. The following are again classical enumeration results.

**Proposition 1.2.3.**

$$\begin{aligned} \#(T_k^\circ) &= \prod_{i=1}^{k-1} (2i - 1), \\ \#(T_k^*) &= 2^{k-1} \#(T_k^\circ), \\ \#(T_k) &= C_{k-1}. \end{aligned}$$

As one might expect, there is a natural bijection between ordered binary trees with  $k$  leaves and ordered trees with  $k$  vertices (see [6]), but we will not need it here. Note that

there is no simple way to count  $T_k^0$  with labels removed, since there is no canonical labeling of the leaves like the one arising from an order.

*Proof.* An element of  $T_k^\circ$  is uniquely formed from an element of  $T_{k-1}^\circ$  by choosing one of  $2k-3$  edges, inserting a vertex and a new edge, and putting the label  $k$  on the newly created leaf, which proves the first identity.  $T_k^*$  is obtained by choosing an order for the children of each interior vertex, of which there are  $k-1$  as observed above. Finally, by identifying the  $k!$  labeled trees with the same tree graph, the third identity results from the identity

$$C_n = \frac{2^n \cdot \prod_{i=1}^n (2i-1)}{(n+1)!},$$

and the theorem is proved. □

Then we enrich the tree graphs with edge lengths  $(x_1, \dots, x_{2k-1})$ , obtaining sets  $P_k^\circ$ , etc. In the cases of labeled trees, it is not hard to think of a way to use the labels to assign a unique order to the edges, so that the values  $x_j$  can be used to determine edge lengths for the tree. In the unlabeled case, we use the lexicographical order as described above. Thus a distribution on the pair  $(\hat{t}, (x_1, \dots, x_{2k-1}))$ , where  $\hat{t}$  belongs to one of the sets above, will give a distribution on trees with edge lengths.

LeGall [25] gives an infinite uniform measure on  $P_k^\circ$ , which can be described by taking a uniform element of  $T_k^\circ$ , and independently "sampling" the  $x_k$  according to Lebesgue measure on  $[0, \infty)$ . This corresponds to the branching decomposition of an excursion path by the choice of  $k$  successive points uniformly along the supporting interval, according to Itô measure. Conditioning by a mass one distribution on the sum  $s = \sum_{j=1}^{2k-1} x_j$  obtains a probability distribution. The choice of density  $f(s) = se^{-s^2/2}$  with respect to the uniform product measure gives a probability measure on  $P_k^\circ$ , which is the distribution of uniformly chosen  $k$ -leaved subtrees from the Brownian CRT (see the reference for details).

A more concrete way to construct the random trees of  $P_k^\circ$  is the so-called "stick-breaking" construction of Aldous [2], which I will describe, since it has a parallel in a welding algorithm later in this paper. We sample a sequence of random trees  $t_k$  according to the following algorithm: Let  $\{s_j; j \geq 1\}$  be the times of a Poisson point process on  $[0, \infty)$  with intensity

$x$ , and let  $y_j = s_j - s_{j-1}$  be the increments. Let  $\{U_j; j \geq 1\}$  be a sequence of iid  $\mathcal{U}([0, 1])$  random variables. Start by letting  $t_1$  be the unique tree with one leaf with  $x_1 = s_1$ .

Given the tree  $t_k$ , the total edge length of our tree is  $s_k$ , broken up into intervals of length  $x_1, \dots, x_{2k-1}$ , and we have a tree graph  $\hat{t}_k$  with a correspondence between edges and intervals. Then to form  $t_{k+1}$ , put a point at  $U_k s_k$ , breaking some interval into two, while also adding a vertex to the corresponding edge of  $t_k$ . The two edges formed in this way are assigned lengths equal to the two newly formed intervals in order according to their distance from the root. Add a new edge at the new vertex, and assign it length  $y_{k+1}$  and label  $k$ . Now the distribution of  $\hat{t}_{k+1}$  is independent of the new sequence  $\{x_1, \dots, x_{2k-1}\}$ , and those values are exchangeable, that is, invariant in distribution under permutation.

To get measures on  $P_k^*$  and  $P_k$ , we simply renormalize the distribution on  $s$  to have total weight inverse to the respective cardinalities. One can check that the stick-breaking construction is well-behaved if one also chooses a random choice of order for each newly-added edge. Next, we condition  $s$  to be 1, to obtain the distributions  $\mathcal{P}_k^*$  and  $\mathcal{P}_k$ . For any distribution on  $s$ , we can apply a change of variables to see that the values  $x_1, \dots, x_{2k-2}$  are distributed uniformly in the  $2k - 2$ -dimensional simplex with side length  $s$ , with the value  $x_{2k-1}$  being determined by those values and the value of  $s$ .

Our purpose for normalizing  $s$  to 1 is to provide a basis for obtaining random laminations of the circle (to be defined shortly), but it also defines the random measures on the random trees described above, either by giving each vertex the delta mass  $x_j$ , or by putting rescaled Lebesgue measure on edges as line segments (it doesn't matter in the limit). The above discussion is proof of most of the following result describing the distribution of  $\mathcal{P}_k^*$  and  $\mathcal{P}_k$ .

**Proposition 1.2.4.** *The uniform normalized proper ordered  $k$ -tree  $\mathcal{P}_k$  is the distribution on  $k$ -leaved subtrees of the Brownian CRT, normalized to have total length 1. The Brownian subtree can be obtained from  $\mathcal{P}_k$  by independently sampling a length  $s$  according to the density  $f_k(s) = \frac{s^{2k-1}}{2^{k-1}(k-1)!} e^{-s^2/2}$ , and rescaling each  $x_k$  by  $s$ .*

*Proof.* The only fact that doesn't follow from the discussion above is the density  $f_k$ , which follows from the density  $f(s) = s e^{-s^2/2}$ . The calculation below shows this as well as illustrating the various uniform densities on the  $x_j$ , and showing that  $f$  is in fact a density for

$P_k^\circ$ .

$$\begin{aligned}
& \sum_{T_k^\circ} \int \cdots \int_{\{x_1 > 0, \dots, x_{2k-1} > 0\}} s e^{-s^2/2} dx_{2k-1} \cdots dx_1 \\
&= \int_0^\infty \#(T_k^\circ) s e^{-s^2/2} \int_0^s \cdots \int_0^{s-x_1-\dots-x_{2k-3}} dx_{2k-2} \cdots dx_1 ds \\
&= \int_0^\infty \frac{\prod_{i=1}^{k-1} (2i-1)}{(2k-2)!} s^{2k-1} e^{-s^2/2} ds \\
&= \int_0^\infty \frac{s^{2k-1} e^{-s^2/2}}{2^{k-1} (k-1)!} ds \\
&= \int_0^\infty \frac{s^{2k-3} e^{-s^2/2}}{2^{k-2} (k-2)!} ds = \dots = 1,
\end{aligned}$$

where the last two identities follow from successive application of integration by parts.  $\square$

Next, we consider the contour functions associated to ordered proper  $k$ -trees. While the idea is the same as that for discrete trees, it does not appear in the literature in the form we use here, which is useful for constructing and analyzing laminations. Therefore we take some time to explore some basic properties.

Given a proper ordered  $k$ -tree  $t$ , there is a natural way to make  $t$  a metric space, where the edge  $e_j$  is a line segment of length  $x_j$  glued at its endpoints. Then for  $v \in t$ ,

$$d_t(v, w) = \sum_{j: e_j \in [v, w]} x_j.$$

We explore the tree in the same way as defined above, but this time we construct the contour function directly, defining values for  $f$  for increasing  $x$  as we explore the tree. Let  $f(0) = 0$  and begin exploration at the root. Suppose  $f(x) = y$ , and our exploration process is at vertex  $v$  with some edges explored and others unexplored. If some child  $w$  is the first unexplored child of  $v$  in ordering, let  $f(x + x_i) = y + x_i$ , where  $x_i$  is the length of the edge  $e_i = (v, w)$ . If all there are no unexplored children of  $v$ , then let  $f(x + x_j) = y - x_j$ , where  $x_j$  is the length of the edge connecting  $v$  to its parent.

Thus, analogous to the Dyck path, the contour function for a proper tree is a continuous path with slope  $\pm 1$ , turning at times corresponding to branching points and leaves of  $t$ . For

a normalized proper tree,  $f$  has domain  $[0, 2]$ . We define the class of functions obtained in this way:

**Definition 1.2.3.** *A continuous function  $f$  on  $[0, 2]$  is an excursion path with  $k$  peaks if*

1.  $f(0) = f(2) = 0$ ,
2.  $f(x) > 0$  on  $(0, 2)$ ,
3.  $f'(x) = \pm\sqrt{k}$ , with  $2k - 1$  turning points  $\{s_j; 1 \leq j \leq 2k - 1\}$ .

We choose the slope  $\sqrt{k}$  to have the most natural statement of convergence.

It is natural to consider the uniform measure on such paths, which we construct by considering the distribution on turning points  $s_j$ . Then the uniform measure on excursion paths with  $k$  peaks is obtained by taking  $2k - 1$  i.i.d. uniform  $[0, 2]$  random variables  $s_j$  with  $0 < s_1 < s_2 \dots$ , and conditioning on the event that the path  $f$  is positive.

**Theorem 1.2.1.** *The distribution on paths  $f$  obtained as the contour function of the uniform proper  $k$ -tree is the uniform measure on excursion paths with  $k$  peaks.*

*Proof.* We show that for a fixed tree shape, the distribution on the  $s_j$  is according to the product measure with unit weight over the event corresponding to the particular tree shape. Note that the sequence  $s = \{s_j; 1 \leq j \leq 2k - 1\}$  uniquely determines the tree shape, which we write  $\hat{t} = t(s)$ . More precisely:

**Lemma 1.2.1.** *Suppose  $X = h(\hat{t}, x_1, \dots, x_{2k-1})$  is a random variable on the space  $P_k$ . Let  $E$  be expectation according to  $\mathcal{P}_k$  and  $E'$  according to the uniform measure on excursion paths with  $k$  peaks. Then*

$$\begin{aligned} E(X, \hat{t} = t_0) &= Z \int_0^1 \cdots \int_0^{1-x_1-\dots-x_{2k-3}} h(t_0, x_1, \dots, x_{2k-1}) dx_1 \cdots dx_{2k-2} \\ &= Z \int \cdots \int \chi_{\{t(s)=t_0\}} h(t_0, x(s_1, \dots, s_{2k-1})) ds_1 \cdots ds_{2k-2} \\ &= E'(X, t(s) = t_0), \end{aligned}$$

where  $Z$  is the normalizing factor depending only on  $k$ .

Given the lemma, summing over all possible trees, we obtain  $E(X) = E'(X)$  and the theorem is proved.

To prove lemma 1.2.1, we consider the differences  $y_j = s_j - s_{j-1}$ , and we show there is a unitriangular change-of-variables matrix (that is, upper triangular having ones on the diagonal) between the vectors  $x$  and  $y$ , for some rearrangement of the  $x_j$  and the  $y_j$ . For a given tree graph, the relationship between the variables  $x_j$  and  $y_j$  is determined by the shape of the tree:  $y_j$  is the distance between a leaf and the previous or next branch point in the exploration process.

We consider the  $x_j$  in the order they are reached in the exploration process. If the first leaf is encountered at the end of edge  $x_j$ , let

$$y_1 = \sum_{i=1}^j x_i.$$

Similarly, whenever  $x_m$  is the length of the right edge of a branching vertex, the exploration process continues consecutively upward, exploring new edges, until a leaf  $x_j$  is reached. Then let

$$y_m = \sum_{i=m}^j x_i.$$

Once a leaf is encountered, the next turning point is found the next time a left branch with length  $x_m$  is encountered, having traveled down a set of previously explored edges  $x_i$  with  $i > m$ . Then let  $I$  be the index set of the edges encountered in this way, and let

$$y_m = \sum_I x_i.$$

Since every edge  $x_m$  with  $m > 1$  is either a left or right branch of some branching vertex, there is a  $y_m$  defined as above, and the change-of-variables matrix taking  $x$  to  $y$  is unitriangular by construction. Then the  $s_j$  are obtained by a simple transformation, and the transformations determine the shape of tree, so the image of the simplex  $\{\sum_1^{2k-1} x_j = 1\}$  under  $T$  is the set  $\{t(s) = t_0\}$ , giving the desired result and proving the theorem.  $\square$

Now we have a clear picture of two models of uniform random trees, corresponding to two models of uniform random excursions. Before we move to laminations, we state a convergence result.

**Proposition 1.2.5.** *The distribution on excursion paths  $f$  resulting from the measure  $\mathcal{E}_k$  converges to the distribution of the normalized Brownian excursion as  $k$  goes to infinity:*

$$f \xrightarrow{d} B_t.$$

*Proof.* This is a special case of Theorem 23 of [3], due to Proposition 1.2.2.  $\square$

### 1.2.1 A process defined by turning at random times

Because of its special distribution in terms of the Brownian CRT, a stochastic process on  $\mathbb{R}$  whose path consists of segments of fixed slope merits a digression. Let  $\{X_j; j \geq 0\}$  be a sequence of i.i.d. exponential random variables with rate  $\lambda = 1$ . Let  $Y_n = \sum_{j=1}^n X_j$  be the random times of the associated Poisson process  $N_t$ , and consider the processes

$$Z_t^+ = \sum_{n=0}^{\infty} Y_{2n+1} \wedge t - Y_{2n} \wedge t, \quad (1.2)$$

$$Z_t^- = \sum_{n=0}^{\infty} Y_{2n+2} \wedge t - Y_{2n+1} \wedge t, \quad (1.3)$$

$$Z_t = Z_t^+ - Z_t^-. \quad (1.4)$$

When  $t = Y_n$  for some  $n$ , each difference  $Y_k \wedge t - Y_{k-1} \wedge t = Y_k - Y_{k-1} = X_k$  for  $k \leq n$  and 0 for  $k \geq n$ , so  $Z_t$  is equal to the alternating sum of exponential random variables, and  $Z_t$  interpolates linearly in between turning times with slope 1.  $Z$  is not a Markov process. It would be worthwhile to prove convergence to Brownian motion for a rescaled version, since the finite dimensional distributions converge to the appropriate normal variables. We demonstrate this for the simplest case, and leave the matter open. Random subdivisions of the unit interval, which form the basis for the distribution of this process, have been studied extensively (for example, see Darling [15]), but this particular construction seems to be original.

Let

$$Z_t^n = \sqrt{n} Z_{nt}.$$

**Proposition 1.2.6.** *For  $0 \leq s < t$ ,  $Z_t^n - Z_s^n$  converges in distribution to a  $\mathcal{N}(0, t - s)$  random variable.*

*Proof.* Between  $s$  and  $t$ ,  $Z$  is determined by the distribution of the times of increase of the Poisson process  $N_t$ , which has rate  $n$  after rescaling. Conditioning on the number of points  $k = N_t - N_s$ , the times  $s = T_0, T_1, \dots, T_{k+1} = t$  are uniformly distributed on the interval  $[s, t]$ . Then if we label the interval widths  $V_j = T_j - T_{j-1}$  for  $1 \leq j \leq k+1$ , we have  $k$  random variables uniformly distributed over the  $k$ -simplex, and the distribution is invariant under any permutation. See David and Nagaraja, p. 133 [16]. For simplicity, we assume  $k$  is even and  $Z$  is increasing at  $s$ , and the other cases go through with trivial modification.

Then

$$Z_t^n - Z_s^n = \sqrt{n} \left( \sum_{j=1}^{k/2} V_{2j-1} - \sum_{j=1}^{k/2} V_{2j} \right),$$

so after applying the permutation  $\sigma$  given by  $2j-1 \mapsto k, 2j \mapsto k/2+j$ , letting  $\tilde{V}_j = V_{\sigma(j)}$ , we have

$$Z_t^n - Z_s^n = \sqrt{n}(t-s) \left( 1 - 2 \sum_1^{k/2} \frac{\tilde{V}_j}{t-s} \right).$$

By the exchangeability of the  $V$ , the sum is the  $k/2$ -th order statistic of  $k$  i.i.d. uniform random variables on  $[0, 1]$ , which has density

$$f(u)du = \frac{k}{2} \binom{k}{k/2} u^{k/2-1} (1-u)^{k/2} du.$$

Substitute  $z = \sqrt{n}(t-s)(1-2u)$ ,  $u = \frac{1}{2} - \frac{z}{2(t-s)\sqrt{n}}$ , and apply Stirling's formula to substitute

$$\binom{k}{k/2} = \frac{4^{k/2}}{\sqrt{\pi k/2}} (1 + o(k)),$$

to obtain

$$\begin{aligned} f_{Z|N=k}(z)dz &\approx \frac{4^{k/2}\sqrt{2k}}{2\sqrt{\pi}} \left( \frac{1}{\frac{1}{2} - \frac{z}{2(t-s)\sqrt{n}}} \right) \left( \frac{1}{4} - \frac{z^2}{4(t-s)^2n} \right)^{k/2} \frac{1}{2(t-s)\sqrt{n}} dz \\ &= \sqrt{\frac{k}{2\pi n}} \frac{1}{(t-s)} e^{\frac{-z^2}{2(t-s)} \cdot \frac{k}{n(t-s)}} (1 + o(k \wedge n)) dz. \end{aligned}$$

For large  $n$ , with probability within  $\epsilon$  of 1,  $|k - n(t - s)| \leq C\sqrt{n}$  for properly chosen  $C$  (Karatzas and Shreve, p. 15 [21]), and on this range the density  $f_{Z|N=k}$  converges to the density of the  $\mathcal{N}(0, t - s)$  random variable, so  $f_Z(z) - \phi(z) < \epsilon$  for large  $n$ , for all  $\epsilon$ , and  $f_Z(z)$  converges to  $\phi(z)$ , which implies convergence in distribution.  $\square$

Since we are interested in trees, we are interested in the distribution of excursions of the process  $Z$ . In particular, we look at the first excursion. Let

$$\tau_0 = \inf\{t : Z_t < 0\}.$$

We are interested in the distribution of the number of "peaks" (according to the notation of definition 1.2.3) of  $\{Z_t; 0 \leq t \leq \tau_0\}$ . Just as in the proof of Proposition 1.2.6, the distribution of turning times is uniform over the interval  $[0, \tau_0]$ . If we normalize to  $[0, 2]$  and condition on the number of peaks, we obtain the distribution of Definition 1.2.3 without any additional effort. We linger on this subject to prove an interesting combinatorial identity that gives the distribution of the number of peaks of the first excursion.

**Theorem 1.2.2.** *Given  $2k$  real numbers  $x = \{x_1, \dots, x_{2k}\}$  with the property that*

$$\sum_{i \in I} \epsilon_i x_i \neq 0$$

*for every  $I \subset \{1, \dots, 2k\}$ ,  $\epsilon_i \in \{-1, 1\}$ , there are exactly  $D_k = (2k - 1)!!(2k - 3)!!$  permutations  $\sigma$  of the  $x_i$  such that*

$$\sum_{i=1}^j (-1)^{i+1} x_{\sigma(i)} > 0 \tag{1.5}$$

*for all  $j < 2k$  and*

$$\sum_{i=1}^{2k} (-1)^{i+1} x_{\sigma(i)} < 0. \tag{1.6}$$

*We call such a permutation valid for  $x$*

Here  $(2k - 1)!! = \prod_{i=1}^k (2i - 1)$ . We call the hypothesis that no two sets of partial sums have the same value the *non-singularity* condition. Since the distribution on intervals  $V$  satisfies the non-singularity condition almost surely, this implies a probability distribution

on the number of peaks of the first excursion, which is  $k$  in the case that the first  $2k$  intervals represent a valid permutation of the values in order. Therefore

**Corollary 1.2.1.** *The probability that the first excursion of  $Z$  has  $k$  peaks is*

$$\frac{D_k}{(2k)!}.$$

It is notable that this probability is equal to

$$\frac{C_{k-1}}{2^{2k-1}},$$

the probability that simple random walk first hits  $-1$  at time  $2k - 1$ , corresponding to a tree of  $k - 1$  edges. To observe this fact we simply write the formulas and simplify.

$$\begin{aligned} \frac{D_k 2^{2k-1}}{C_{k-1}(2k)!} &= \frac{(2k-1)!!(2k-3)!!2^{2k-1}(k-1)!k!}{(2k)!(2k-2)!} \\ &= \frac{(2k-1)!!(2k-3)!!(2k-2)!!(2k)!!}{(2k)!(2k-2)!} = 1. \end{aligned}$$

*Proof.* The corollary follows from the theorem by conditioning on the unordered set of interval sizes for the first  $2k$  intervals, then observing that the probability of obtaining an excursion through permutation (which are all equally likely) is  $D_k/(2k)!$ , as desired.

To prove the theorem, we first prove the statement for a carefully chosen set of values, then show that the number of permutations remains invariant as we continuously change the values past roots. First, let  $x_j = 3^{j-1} + 1$  for  $j = 1, \dots, 2k$ , which satisfies the condition of the theorem, since each  $x_j$  is greater than the sum of all the previous elements. Because of this property, a permutation  $\sigma$  of  $1, \dots, 2k$  satisfying the theorem must have  $\sigma(2k) = 2k$  in order to satisfy (1.6), and does satisfy it in this case. In order to satisfy (1.5), it is necessary and sufficient that, progressing forward through the permutation, each time the largest index yet is encountered, that index occurs in an odd position, indicating an up step, since a down step is guaranteed to yield a negative value for the sum. Therefore, the permutations satisfying the two conditions are exactly those with  $2k$  in the last position, and otherwise for the running maximums to occur in odd positions. We give an example of a valid permutation for  $k = 4$ , written in one-line notation:

Here 3, 6, and 7 are the running maximums, in the first, third, and seventh position, respectively. Ignoring the final value, we need to find the number of permutations of  $2k - 1$  with relative maximum values only occurring in odd position, which we call  $D_k$  and we need to show  $D_k$  satisfies the given formula. For  $k = 1$ , the permutation 1 is the only option, and  $D_k = 1$ . Given  $D_k$ , we require

$$D_{k+1} = (2k + 1)(2k - 1)D_k$$

to prove the identity for our choice of  $x_i$ . Given a permutation  $\sigma$  of length  $m$  with running maximums in odd positions, we form a new one by adding a value  $j \in \{1, \dots, m + 1\}$  in position  $m + 1$ , then using  $\sigma$  to order the unused values in the first  $m$  positions. If  $m$  is odd, we cannot use  $m + 1$  for this choice, as that would create a running maximum in even position, so there are  $m$  ways to obtain a new sequence. If  $m$  is even, we can use any of the  $m + 1$  values. Applying this twice yields the formula for the induction step, and the result is proved for one choice of values. The proof of this part can be found on the Online Encyclopedia of Integer Sequences [20], posted by David Callan. Thus when  $x$  has  $x_j = 3^{j-1} + 1$  for  $1 \leq j \leq k$ , there are  $D_k$  valid permutations for  $x$ .

Now suppose  $x = \{x_1, \dots, x_{2k}\}$  satisfies the non-singularity condition, and suppose  $x'$  is a sequence differing from  $x$  in one coordinate, that is,  $x'_j \neq x_j$  for exactly one index  $j$ . As  $y$  goes from  $x$  to  $x'$ , the sequence may fail to satisfy the non-singularity condition at some finite set of values (since there are only finitely many values we need to check, which vary linearly). If the non-singularity condition continues to hold for all  $y$ , then conditions (1.5) and (1.6) also continue to either hold or not hold, and there are  $D_k$  valid permutations of  $x'$ . Next, suppose there is one value  $x^0$  between  $x$  and  $x'$  such that the non-singularity condition fails. For  $x_0$  there must be exactly one combination that sums to zero, since if there were two, we could obtain a non-trivial zero sum not including  $x_j^0$ , contradiction the hypothesis that  $x$  is non-singular and differs only at  $x_j$ . Suppose  $\sigma$  is a valid permutation for  $x$  that fails as it crosses  $x^0$ . There are 4 cases we must consider.

1.  $j$  is an up-step of  $\sigma$ , and  $x_j < x_j^0$ . Then condition (1.6) must be the condition that fails. Then when  $y_j = x'_j > x_j^0$ , since there is only one point  $x^0$  in the interval where

$\sigma$  fails the non-singularity condition, we now have

$$\sum_{i=1}^m (-1)^{i+1} x'_{\sigma(i)} > \sum_{i=1}^{2k} (-1)^{i+1} x'_{\sigma(i)} > 0,$$

for  $m < 2k$ . Let  $\tau$  be the permutation taking  $i$  to  $2k + 1 - i$ , reversing the order of the elements.  $\sigma \circ \tau$  is valid for  $x'$ , since

$$\sum_{i=1}^m (-1)^{i+1} x'_{\sigma(\tau(i))} = -1 \cdot \left( \sum_{i=1}^{2k} (-1)^{i+1} x'_{\sigma(i)} - \sum_{i=1}^{2k-m} (-1)^{i+1} x'_{\sigma(i)} \right) > 0.$$

but

$$\sum_{i=1}^{2k} (-1)^{i+1} x'_{\sigma(\tau(i))} = - \sum_{i=1}^{2k} (-1)^{i+1} x'_{\sigma(i)} < 0.$$

2.  $j$  is a down-step of  $\sigma$ , and  $x_j > x_j^0$ . This is exactly the same as the case above.  $\tau \circ \sigma$  is not valid for  $x'$ , but is valid for  $x$ .
3.  $j$  is an up-step of  $\sigma$ , and  $x_j > x_j^0$ . Now is it condition (1.5) that fails, and at some index  $m < 2k$ ,

$$\sum_{i=1}^m (-1)^{i+1} x'_{\sigma(i)} < 0.$$

Now let  $\tau$  reverse the order of the indices below  $m$ :  $i \mapsto 2m - i$  for  $i < m$ . By a similar argument to case 1, the permutation  $\sigma \circ \tau$  becomes valid when  $\sigma$  ceases being valid.

4.  $j$  is a down-step, and  $x_j < x_j^0$ . The same as case 3.

The same reasoning in the opposite direction shows that whenever a non-valid permutation becomes valid as a parameter changes, a valid permutation formed by reversing indices becomes non-valid. Therefore as each variable changes continuously, the number of valid permutations remains invariant off of singular values, proving the theorem.  $\square$

The next step would be to prove convergence in distribution of  $Z^n$  to Brownian motion and convergence of the excursions of large size to the distribution of normalized Brownian excursion, which would give the result analogous to 1.2.5 for the uniform path with  $k$  peaks. We move forward to laminations, however.

### 1.3 Laminations

A lamination, as we use it here, is an equivalence relation of points of the circle.

**Definition 1.3.1.** A lamination  $L$  is a subset of  $\mathbb{T} \times \mathbb{T}$  that is reflexive, transitive, and symmetric. It is called flat if whenever  $(\zeta_1, \zeta_2) \in L$ , and  $\xi_1, \xi_2$  are in different components of  $\mathbb{T} \setminus \{\zeta_1, \zeta_2\}$ , then  $(\xi_1, \xi_2) \in L \implies (\zeta_1, \xi_1) \in L$ .

We will sometimes specify a lamination by defining its pairs  $(\zeta, \xi)$ , and sometimes by defining an equivalence relation  $\sim$ . In either case we mean the same object.

Led by our interest in plane trees, we consider a special type of lamination which we call *arc-pairing*. Fix  $n > 0$ , and suppose we subdivide the circle into  $2n$  closed arcs  $\{A_j\}_{j=1}^{2n}$  with pairwise disjoint interiors. We pair the arcs in non-crossing fashion, recorded as  $\Pi = \{(A_{\sigma_1(k)}, A_{\sigma_2(k)}); 1 \leq k \leq n\}$ , where each arc appears in exactly one pair. Suppose for a pair  $(A, B) \in \Pi$ ,  $A$  is the arc subtending the arguments  $\alpha \leq \theta \leq \alpha + \omega$ , and  $B$  the arguments  $\beta \leq \theta \leq \beta + \omega$ , then  $L$  is the minimal lamination containing all pairs of the form

$$(e^{i\pi(\alpha+t)}, e^{i\pi(\beta+\omega-t)}),$$

for  $0 \leq t \leq \omega$ , for all such pairs  $(A, B)$ . The non-crossing property is nothing more than the constraint on pairs  $(A, B)$  necessary to make the resulting lamination flat. Note that paired arcs must subtend the same size angle, but non-paired arcs can be different sizes.

An arc-pairing lamination is called *balanced* if the endpoints of the intervals are exactly the  $2n$  roots of unity, which we will denote  $\{\zeta_{j,2n}\}$ . The set of all balanced arc-pairing laminations for a given  $n$  is  $\mathcal{L}_n$ .

Arc-pairing laminations can be obtained from excursion paths of the kind discussed above, and indeed from bridge paths more generally, although no longer in one-to-one fashion. Since analysis on random bridges is simpler, we give the more general correspondence. First we consider the special case of balanced laminations, which correspond to bridge paths with fixed step size.

Let  $T$  be the one-dimensional torus  $\mathbb{R}/(x \sim x + 2)$ .

**Definition 1.3.2.** A bridge walk with  $2n$  steps  $\beta : T \rightarrow \mathbb{R}$  is a continuous function with the following properties:

1.  $\beta(0) = \beta(2) = 0$ ,
2.  $\beta(j/n) = \beta((j-1)/n) \pm 1/\sqrt{n}$  for  $0 < j \leq 2n$ ,
3.  $\beta(s)$  linearly interpolates  $\beta(\lfloor ns \rfloor/n)$  and  $\beta(\lfloor ns \rfloor + 1/n)$ .

An excursion walk is a bridge walk with the additional constraint that  $\beta(t) \geq 0$  for all  $t \in \mathbb{R}$ .

We first define a mapping from bridge walks to arc-pairing laminations. Given a bridge walk  $\beta$ , we define a metric  $d_\beta$  on  $T$ . Given  $s, t \in T$ , let  $I_1, I_2$  be the two components of  $T \setminus \{s, t\}$ . Then let

$$d_\beta(s, t) = \beta(s) + \beta(t) - 2 \max_{i=1,2} \inf_{r \in I_i} \beta(r)$$

$d_\beta$  gives distance zero between points which face each other under the graph (if we consider the graph extended periodically to the whole real line). Then  $L$  is the equivalence relation  $s \sim t$  when  $d_\beta(s, t) = 0$ , with the obvious correspondence between  $\mathbb{T}$  and  $T$ .

$\beta$  attains its minimum value at finitely many points, which form an equivalence class  $v$  under  $d$ , which we call a vertex, since it corresponds to the vertex of the corresponding ordered tree. This procedure defines a mapping  $\Phi$  where

$$\Phi(\beta) = (L, v).$$

We show that  $\Phi$  is a bijection.

**Proposition 1.3.1.** *There is a bijection between bridge walks with  $2n$  steps and pairs  $(L, v)$ , where  $L \in \mathcal{L}_n$  and  $v$  is a equivalence class of nodes in  $L$ .*

*Proof.* We show that there is a well-defined map  $\Psi$  from pairs  $(L, v)$  to bridge walks, then we show that  $\Psi = \Phi^{-1}$ . Given a pair  $(L, v)$ , we now construct the contour function  $h : T \rightarrow \mathbb{R}$ . First we consider the case where  $0 \in v$ , which corresponds to the case where  $\beta$  is an excursion walk. The segment  $[0, 2]$  is divided into  $2n$  intervals  $I_j = [(j-1)/n, j/n]$ . For each  $j$ , there is a unique  $k$  such that  $(I_j, I_k) \in \Pi$ . Assuming  $j < k$ , let  $h(j/n) = h((j-1)/n) + 1/\sqrt{n}$  and  $h(k/n) = h((k-1)/n) - 1/\sqrt{n}$ , interpolating between nodes linearly. This process is well-defined and uniquely determines  $h$ , since each interval is accounted for exactly once in this process. We must show that  $h$  is an excursion walk.

**Lemma 1.3.1.** *The contour function  $h$  so defined is an excursion walk, and the lamination obtained as the quotient space of its tree metric  $d$  is  $L$ .*

*Proof of lemma.* This fact is a consequence of the non-crossing property.  $h$  is an excursion walk, since for each pair  $(j, k)$  there is one up-step followed by one down-step at a later time, and all intervals  $I_i$  are accounted for in this way. To check that  $L$  is obtained from  $h$ , fix a pair  $(j, k)$  with  $j < k$ . We need to show that the points of  $I_j$  and  $I_k$  are paired in linearly decreasing fashion (that is,  $d_h((j-1)/n + s, k/n - s) = 0$  for  $0 \leq s \leq 1/n$ ). Indeed,  $I_j$  is an up-step and  $I_k$  a down-step, so if  $h(t) \geq h(j/n)$  on  $[j/n, (k-1)/n]$ , the condition will be satisfied, since the points are at the same level by construction and have no lower minimum between them. By the non-crossing property, each interval  $I_i$  for  $j < i < k$  is paired to another in that same connected component, so as is the case with the whole contour function,  $h$  restricted to the interval  $[j/n, (k-1)/n]$  is an excursion walk above the level  $h(j/n)$ . Since each interior point on an interval is not a local extremum, it can only be paired with at most one point, and  $d_h$  gives the relation  $L$  on interior points.

On the other hand, given two nodes  $s < t$  with  $d_h(s, t) = 0$ , there exist a maximal set of finitely many points  $\{r_k\}_{k=0}^M$  such that  $r_k < r_{k+1}$  for each  $0 \leq k < M$ ,  $r_0 = s$ ,  $r_M = t$ , and  $d_h(r_j, r_k) = 0$  for all  $j, k \in \{0, \dots, M\}$ . The points  $r_k$  are exactly the local minima at the level  $h(s)$  between  $s$  and  $t$ . For each  $k$ ,  $r_k$  is the beginning of an up-step, and  $r_{k+1}$  the end point of a downstep, so that the intervals are paired under the lamination. Therefore  $(r_k, r_{k+1}) \in L$ , and by the transitive property, for every pair  $s, t$  with  $d_h(s, t) = 0$ ,  $(s, t) \in L$ , and the lemma is proved.  $\square$

Next, suppose  $0 \notin v$ . Fix  $t_0 \in v$ , and define  $h_0$  as above on the interval  $[t_0, t_0 + 2]$ . Recall that our domain is  $T$ , so  $h_0$  is defined on all of  $\mathbb{R}$  by periodicity. Then let  $h = h_0 - h_0(0)$ . Clearly  $h$  is a bridge walk, and the vertical shift does not change  $d_h$ , so the lamination derived from  $h$  is  $L$ . In all cases, since  $h$  is an excursion walk above  $h(t_0)$ , the members of  $v$  are the minima of  $h$ , and there can be no other minima since they would be identified by  $d_h$ .

Therefore, if we define  $\Psi(L, v) = h$  as above, then we have shown that  $\Psi = \Phi^{-1}$ , demonstrating the bijection, as desired.  $\square$

**Corollary 1.3.1.** *If  $P_n$  is the uniform measure on bridge walks, the induced distribution on laminations by the bijection  $\Phi$  is the uniform distribution.*

*Proof.* The measure is weighted by the number of equivalence classes of vertices, which is always  $n + 1$ , since these correspond to the vertices of the coded binary tree.  $\square$

Essentially the exact same construction determines the arc-pairing lamination for the excursion path with  $k$ -peaks, corresponding to the non-balanced case (which we not call proper arc-pairing laminations), and the same proof applies with only minor adjustment. The choice of 1 as the root works well in the discrete case, but in the case of proper trees leads to an unsatisfying arbitrary leaf, so we introduce an independent uniform rotational parameter  $\omega \in \mathbb{T}$  which represents the root of the tree. Then the pair  $(\omega, f)$ , where  $f$  is sampled from  $\mathcal{P}_k$ , determines the uniform measure on proper arc-pairing laminations with  $2k - 1$  arc pairs, which we denote by  $\mathcal{PL}_k$ .

We bring the discussion full circle by briefly explaining the relationship between laminations and trees, which can be deduced from the work so far. Indeed, given a lamination  $L$ , the quotient space  $\mathbb{T}/\sim$  has a tree structure. As we are interested in passing to the limit and considering the Brownian lamination, we introduce the real tree, which is a generalization of all the models so far discussed. The following is taken from [26].

**Definition 1.3.3.** *A compact metric space  $(t, d)$  is a real tree if for all  $x, y \in t$ , there is exactly one path  $[x, y] \subset t$ , and  $l([x, y]) = d(x, y)$ , where  $l$  is the length of the path according to the metric.*

It is easy to see how elements of  $\mathcal{P}_k$ , for example, can be seen as real trees. Real trees, however, can have infinite branching, and indeed generally have uncountably many leaves. Given a continuous excursion  $f : [0, T] \rightarrow [0, \infty)$ , define the tree metric as before:

$$d_f(x, y) = f(x) + f(y) - 2 \min_{x \leq t \leq y} f(t),$$

then  $[0, T]/\sim$  is a real tree, where again  $x \sim y$  when  $d_f(x, y) = 0$ . The same relation  $\sim$  defines a lamination on  $\mathbb{T}$  for an excursion function, after rescaling the domain. We now define the Brownian continuum random tree and the Brownian lamination.

**Definition 1.3.4.** *The Brownian continuum random tree and the Brownian lamination are the real tree and lamination, respectively, coded by the normalized Brownian excursion.*

We will not go into detail on the normalized Brownian excursion, except to say that it is a Brownian motion conditioned to be positive on the interval  $(0, 1)$  and return to 0 at time 1. The Brownian lamination has been studied by Aldous [4] with a slightly different definition, where he focused on obtaining it as the limit of random triangulations of regular polygons, which indeed correspond in one-to-one fashion with balanced laminations, and described self-similarity results which may have application in determining self-similarities of our desired conformal maps, to be discussed in the next section.

Before we discuss conformal maps, we extend convergence of paths to convergence of laminations.

### 1.3.1 Convergence of laminations

A sequence of laminations  $\{L_n\}_{n=1}^\infty$ ,  $L_n \in T \times T$ , will be said to converge to a lamination  $L$  if it converges in the Hausdorff metric, that is, if  $\max\{\sup_{\pi \in L} d(\pi, L_n), \sup_{\pi \in L_n} d(\pi, L)\}$  goes to zero as  $n$  goes to infinity, where  $d$  is Euclidian distance in  $\mathbb{R}^2$ . Let  $P$  be the measure on coupled sequences of uniform excursion walks (according to either model discussed earlier) converging to the normalized Brownian excursion. Specifically,  $P$  is a measure on sequences  $\{e, e_n\}_{n=1}^\infty$  such that the marginal distribution on  $e_n$  is that of the uniform measure on excursion walks,  $e$  is the normalized Brownian excursion on the interval  $[0, 2]$ , and  $e_n \rightarrow e$  almost surely. Then to each excursion associate a lamination where the mapping  $e \mapsto L$  is determined by the tree metric  $\rho_e$  according the rule  $(s, t) \in L$  if  $\rho_e(s, t) = 0$ . We obtain the following convergence result:

**Proposition 1.3.2.**  *$L_n$  converges to  $L$  almost surely under  $P$ .*

*Proof.* Fix  $\epsilon > 0$ . First suppose  $(s, t) \in L$  is a pair such that  $0 < s < t < 2$ . Suppose there does not exist  $r$  with  $0 < r < s$  or  $t < r < 2$  such that  $(r, s) \in L$  or  $(t, r) \in L$ . Then  $s$  and  $t$  are not local minima and there exist points  $s', t'$  with  $s - \epsilon < s' < s$ ,  $t < t' < t + \epsilon$  such that  $e(s') < e(s)$  and  $e(t') < e(t)$ . Let  $\eta = \min\{e(s) - e(s'), e(t) - e(t')\}$ , and consider  $n$

large enough that  $\|e - e_n\| < \eta/3$ . Then for those  $n$ , there exists a pair  $(s_n, t_n)$  in the given epsilon intervals at the level  $e(s) - \eta/2$ , since we have

$$\max\{e_n(t'), e_n(s')\} < e(s) - 2\eta/3 < e(s) - \eta/3 < \inf_{s \leq r \leq t} e(r),$$

by the intermediate value theorem.

Next, suppose instead that there exists no  $r$  with  $s < r < t$  such that  $(r, s)$  or  $(r, t) \in L$ . Now let  $\eta = \min_{s+\epsilon \leq r \leq t-\epsilon} e(r)$ , and again consider  $n$  large enough that  $\|e - e_n\| < \eta/3$ . By the same arguments as above, there exist pairs  $(s_n, t_n) \in L_n$  with  $s < s_n < s + \epsilon$ ,  $t - \epsilon < t_n < t$  at level  $e(s) - \eta/2$ .

Now consider the lamination  $L$  induced by  $e$ . Almost surely,  $L$  has no points of degree greater than three, that is, Brownian motion does not have consecutive local minima at the same level. This follows from the finite-dimensional distribution of the CRT being a proper tree (see Aldous [3]), since if there were quadruple points of Brownian excursion with positive probability, then there would be quadruple points of the subtree with positive probability. If  $(s, t) \in L$  is a double point of  $L$ , then both conditions above hold, and if  $(s, r), (r, t), (s, t) \in L$  is a triple point with  $s < r < t$ , then the pairs  $(s, r)$  and  $(r, t)$  satisfy the second condition above, and  $(s, t)$  satisfies the first, so again we can find pairs of  $L_n$  within  $\epsilon$  of each.

Since all non-trivial pairs of  $L$  are part of a double or triple point almost surely, we have  $\sup_L d(\pi, L_n) < 2\epsilon$  for  $n$  sufficiently large.

On the other hand, suppose  $\{(s_n, t_n)\}_{n=1}^\infty$  is a sequence of pairs  $(s_n, t_n) \in L_n$  such that  $d((s_n, t_n), L) > \epsilon$  infinitely often. A subsequence converges to a pair of points  $(s, t)$ , so we need to show that  $(s, t) \in L$ . Indeed,  $e(s) = e(t)$  by convergence of the functions, so  $(s, t) \notin L$  implies that there is some  $r$  with  $s < r < t$  and  $e(r) < e(s)$ . However,  $e_n(r) \rightarrow e(r)$  so eventually  $e_n(r) < e(s) \sim e_n(s)$ , contradicting  $(s_n, t_n) \in L_n$ . Therefore, for  $n$  large enough,  $\sup_{L_n} d(\pi, L) < \epsilon$ , and  $L_n \rightarrow L$  in the Hausdorff metric.  $\square$

**Corollary 1.3.2.** *The uniform balanced arc-pairing lamination converges in distribution to the Brownian lamination.*

*Proof.* This follows from the theorem and Proposition 1.2.5.  $\square$

## 1.4 Conformal Welding

We state the general welding problem for laminations of the circle. Recall that a conformal map is an injective complex analytic function. Let  $\mathbb{D}^*$  be the complement of the closed unit disk, extended to the point at infinity.

**Definition 1.4.1.** *A lamination  $L$  is called conformal if there exists a conformal map  $\phi : \mathbb{D}^* \rightarrow \mathbb{C}^*$ , continuous on  $\mathbb{T}$ , such that  $\phi(\zeta) = \phi(\xi)$  if and only if  $(\zeta, \xi) \in L$ .*

It is easy to see that a conformal map that extends continuously to the boundary yields a flat lamination in this way. It is generally a difficult question to determine which laminations are conformal. Leung [27] and Gupta [19] provide some sufficient conditions in their theses, but these results are not sufficient for our purposes.

Our efforts are therefore directed towards the following goal:

**Problem 1.4.1.** *Let  $L$  be the Brownian lamination. Find a conformal map  $f : \mathbb{D}^* \rightarrow \mathbb{C}^*$  that extends continuously to the boundary such that  $f(\zeta) = f(\xi)$  if and only if  $\zeta \sim \xi$ .*

We are not yet able to solve the stated problem. We start by solving the welding problem for arc-pairing laminations, then considering the limiting behavior of the conformal welding maps found in this way. As stated before, Bishop [10] demonstrated welding for balanced laminations, and the method is the same for the proper case as well. We give an explicit construction of the quasiconformal welding map required, and an explanation of the correcting homeomorphism, an application of the measurable Riemann mapping theorem.

### 1.4.1 Welding a curve

We consider a first example of a welding map, between the upper and lower half planes. First we fix some notation.  $\mathbb{C}$  is the complex plane, and  $\mathbb{C}^*$  is the extended plane, with the topology of the sphere.  $\mathbb{H}$  or  $\mathbb{H}_+$  will be the upper half-plane  $\{\text{Im } z > 0\}$ , and  $\mathbb{H}_-$  the lower half-plane.  $\mathbb{D}$  is the unit disc  $\{|z| < 1\}$ , and  $\mathbb{D}^*$  is the complement of its closure. For the purpose of conformal maps, we consider  $\mathbb{D}^*$  to be a simply-connected region including the point at infinity, so, by the Riemann mapping theorem, given a simply-connected region  $\Omega$

containing a neighborhood of infinity, there exists a unique conformal map  $f : \mathbb{D}^* \rightarrow \Omega$  such that  $f(\infty) = \infty$  and such that  $f$  satisfies an asymptotic expansion at infinity given by

$$f(z) = az + b + O(|z|^{-1}),$$

for  $a > 0$  and  $b$  complex. This function will be referred to as the normalized conformal map for  $\Omega$ .

Suppose

$$h : \mathbb{R} \rightarrow \mathbb{R}$$

is an increasing homeomorphism. We wish to find a Jordan curve  $J$ , dividing the plane into two regions  $\Omega^+$  and  $\Omega^-$ , and conformal maps  $f_+$  and  $f_-$  from  $\mathbb{H}^+$  and  $\mathbb{H}^-$  onto  $\Omega^+$  and  $\Omega^-$ , respectively. A conformal map from the half-plane into a region bounded by a Jordan curve extends continuously to the boundary by Carathéodory's theorem, and we require that

$$f_+(x) = f_-(y) \iff h(x) = y,$$

for all  $x, y \in \mathbb{R}$ . If  $f_+$  and  $f_-$  satisfy this condition, we let  $f$  be the piecewise map, and we say that  $f$  is the *conformal welding map* for the homeomorphism  $h$ , which we call the *welding homeomorphism*. Uniqueness depends on removability of the curve  $J$ . If  $f$  and  $g$  are two conformal welding maps for curves  $J_f$  and  $J_g$ , and if  $f \circ g^{-1}$  extends across  $J_g$ , then  $f \circ g^{-1}$  is a linear map, and we see that the welding map is unique up to a normalization. Next, we identify a class of functions  $h$  for which a conformal welding map exists.

#### 1.4.2 Quasisymmetry and quasiconformal maps

Quasiconformal maps are an important tool for constructing conformal maps. While conformal maps between given regions are uniquely determined by the images of three boundary points, quasiconformal maps may be constructed with a wide range of boundary conditions. Then composition with a solution to the Beltrami equation can be used to correct a quasiconformal map into a conformal map, obtaining the desired welding map. First, some definitions.

Given a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $f(x + iy) = u(x + iy) + iv(x + iy)$ , suppose  $f$  is absolutely continuous in  $x$  for almost every  $y$ , and vice versa, so that there exist partial derivatives

almost everywhere,

$$\begin{aligned}f_x &= u_x + iv_x \\f_y &= u_y + iv_y.\end{aligned}$$

We define the Beltrami differentials,

$$\begin{aligned}f_z &= \frac{1}{2}\left(f_x + \frac{1}{i}f_y\right) \\f_{\bar{z}} &= \frac{1}{2}\left(f_x - \frac{1}{i}f_y\right).\end{aligned}$$

We only consider orientation-preserving homeomorphisms, which are maps  $f$  such that

$$|f_{\bar{z}}| \leq |f_z|.$$

Next, suppose  $\mu_f$  is the function such that

$$f_{\bar{z}} = \mu_f f_z,$$

defined to be zero when  $f_z = 0$ , and we say that  $\mu_f$  is the *complex dilatation* of  $f$ . For a conformal map,  $f_z = f' \neq 0$ , and in fact  $\mu_f = 0$  everywhere if and only if  $f$  is conformal (Ahlfors p.33 [1]). Finally, we say that a map is *K-quasiconformal* in a region  $D$  for  $K \geq 1$  if

$$\|\mu_f\|_\infty \leq \frac{K-1}{K+1} < 1,$$

where the sup norm is taken over  $D$ . In order to construct a welding map, we need a solution to a certain boundary-value problem. Given an increasing homeomorphism  $h : \mathbb{R} \rightarrow \mathbb{R}$ , we wish to find a quasiconformal map  $g : \mathbb{H} \rightarrow \mathbb{H}$  with boundary values  $h$ . Such a map does not exist in general, so we place additional constraints on  $h$ .

For  $M > 1$ , we say that  $h$  is *M-quasisymmetric* if, for all  $x, y$  in  $\mathbb{R}$ ,  $h$  satisfies

$$\frac{1}{M} < \frac{|h(x+y) - h(x)|}{|h(x) - h(x-y)|} < M.$$

Quasisymmetry is the necessary and sufficient condition for  $h$  to determine the boundary values of a quasiconformal map.

**Theorem 1.4.1.** *Given an  $M$ -quasisymmetric function  $h : \mathbb{R} \rightarrow \mathbb{R}$ , there exists a quasiconformal map  $g : \mathbb{H} \rightarrow \mathbb{H}$ , which extends continuously to  $\mathbb{R}$ , such that  $g|_{\mathbb{R}} = h$ . Conversely, if  $g$  is a quasiconformal map from  $\mathbb{H}$  onto itself, then  $g$  extends continuously to the real line, and its restriction is a  $M$ -quasisymmetric function for some  $M$ .*

A proof can be found in [1], pp. 65 and 69. Now we can solve the welding problem for quasisymmetric welding homeomorphisms.

**Theorem 1.4.2.** *If  $h$  is a  $M$ -quasisymmetric welding homeomorphism, there exists a conformal welding map  $f$  for  $h$ , as defined above.*

*Proof.* Let  $g_+$  be the quasiconformal map with boundary values  $h$ , and  $g_-$  the identity map. Then  $g$ , defined piecewise, welds the upper- and lower- half planes according to  $h$ . Suppose there exists a quasiconformal map  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  such that  $f = \phi \circ g$  is conformal everywhere. Since  $\phi$  is continuous on the real line, the composed map  $f$  will also weld according to  $h$ , and thus be the desired map.

In order for  $f$  to be conformal, we require that  $\mu_{\phi \circ g} = 0$  at all points, which can be calculated with a chain rule on the total differential. One finds in [1] that the required complex dilatation is

$$\mu_\phi = \left[ -\frac{g_z^2}{|g_z|^2} \mu_g \right] \circ g^{-1}.$$

Any  $\phi$  satisfying the above equation will give a conformal map when composed with  $g$ . Now we have reduced the problem to finding the solution to a particular differential equation. A solution  $\phi$  to the relation

$$\frac{\phi_{\bar{z}}}{\phi_z} = \mu,$$

for given  $\mu : \mathbb{C} \rightarrow \mathbb{D}$ , is called a solution to the *Beltrami equation* with data  $\mu$ . The proof is completed by the measurable Riemann mapping theorem of Ahlfors and Bers.

**Theorem 1.4.3.** *For any measurable  $\mu$  with  $\|\mu\|_\infty < 1$ , there exists a unique normalized quasiconformal mapping  $\phi^\mu$  with complex dilatation  $\mu$  leaving  $0, 1$ , and  $\infty$  fixed.*

$\mu_\phi$  is bounded away from 1, since  $g$  is quasiconformal, so if we let  $\phi$  be the unique mapping with this dilatation, then the function  $f = \phi \circ g$  is a conformal welding map.  $\square$

Quasisymmetric boundary conditions yield image curves called quasicircles, and for such curves, the welding problem has a unique solution.

**Theorem 1.4.4.** *A welding homeomorphism from  $\mathbb{R} \rightarrow \mathbb{R}$  has a unique conformal welding map such that the image of  $\mathbb{R}$  is a quasicircle if and only if it is quasisymmetric.*

### 1.4.3 Welded Planar Trees

One can extend Theorem 1.4.1 to a welding homeomorphism between the upper and lower halves of the unit circle by applying the map  $z + 1/z$  and defining the homeomorphism to be the identity outside the interval  $[-2, 2]$ . Again, a quasisymmetric welding homeomorphism yields a conformal welding map and a curve in the plane as the image of the circle. On the other hand, we can start with the curve, and it uniquely determines a conformal welding map by the Riemann mapping theorem, and induces its welding homeomorphism. As a natural starting point for the generalization to trees, we can consider the normalized conformal map  $f$  from the complement of  $\mathbb{D}^*$  to the complement of a finite tree  $\Gamma$  embedded in  $\mathbb{C}$ . Let  $V$  be the set of vertices of the tree, and  $\Gamma \setminus V$  is a pairwise disjoint union of open arcs in  $\mathbb{C}$ . Each arc  $\gamma$  has two preimages under  $f$ , and induces a welding homeomorphism  $w_\gamma$  between them, with the property that  $f(\zeta) = f(w(\zeta))$ . The correspondence between the tree  $\Gamma$  and the welding homeomorphism  $w_\Gamma$  is our primary focus in what follows. Our main theorem gives sufficient conditions for the welding homeomorphism  $w$  to produce a conformal welding map to a planar tree  $\Gamma$ .

We have defined arc-pairing laminations earlier, but we generalize slightly here and restate things in terms of homeomorphisms. Let  $\Pi$  be a non-crossing partition of a set of arcs  $\mathcal{A}_n = \{A_j\}_{j=1}^{2n}$  as described in section 1.3.

**Definition 1.4.2.** *A function  $w : \cup_{\mathcal{A}_n} A \rightarrow \cup_{\mathcal{A}_n} A$  is a welding homeomorphism for  $\Pi$  if, restricted to the arcs of a pair  $(A, B) \in \Pi$ ,  $w$  is a homeomorphic involution that is decreasing in argument. That is, as  $\zeta$  traverses  $A$  in the clockwise direction,  $w(\zeta)$  traverses  $B$  in the counter-clockwise direction, and vice-versa.*

**Definition 1.4.3.** *Let  $L = L(\Pi, w)$  be the minimal lamination containing all pairs of the form  $(\zeta, w(\zeta))$  in  $\partial\mathbb{D} \times \partial\mathbb{D}$ .  $L$  is a lamination, and we call such laminations arc-pairing*

laminations.

It is easy to see that an embedded finite planar tree  $\Gamma$  with nice enough arcs yields an arc-pairing lamination. It will be important to have a canonical representation for a planar tree, so we define a distinguished class of arc-pairing laminations.

**Definition 1.4.4.** *Suppose  $L(\Pi, w)$  is a, arc-pairing lamination such that the arcs of  $\Pi$  are the arcs between the  $2n$ th roots of unity, and  $w$  identifies points by arc length. That is, if  $\pi = \{(\pi j/n, \pi(j+1)/n), (\pi k/n, \pi(k+1)/n)\}$ , then  $w$  is given by*

$$w(\zeta) = \frac{e^{(k+j+1)\pi i/n}}{\zeta}.$$

*Then we say that  $L$  is a balanced arc-pairing lamination.*

Henceforth we omit "arc-pairing", since we only consider this type. For each lamination, there exists a balanced lamination, unique up to a rotation, by matching the combinatorial pattern of the non-crossing partition. The tree corresponding to a balanced lamination also has special properties, and will be called a balanced tree. Each edge of a balanced tree has equal harmonic measure, and furthermore, any subset of an edge has the same harmonic measure from each side, by the symmetry of the welding homeomorphism and the symmetry of harmonic measure on the circle, and the fact that harmonic measure is invariant under conformal map.

Given a planar tree, one can easily define the corresponding balanced lamination as follows. Starting from a vertex, label the edges 1 through  $2n$ , one label on each side of each edge, following the contour of the tree in the clockwise direction. Then label the intervals between the  $2n$ th roots of unity in the same way, and pair intervals whose labels are on opposite sides of an edge on the planar tree. See Figure 1.4 for a visualization.

#### 1.4.4 Computable construction of the welding map for a balanced tree

Let  $L$  be a balanced lamination with  $n$  pairs. We label the endpoints of the arcs of the pair  $\pi_j$  as  $a_j, b_j, c_j, d_j$ , in order of increasing argument, with arcs  $(a_j, b_j)$  and  $(c_j, d_j)$ , and we choose the ordering of the indices and labels so that  $\pi_1$  is a pair of adjacent arcs with  $b_1 = c_1$ , and upon application of the  $j$ th welding map  $F_j$ , the next pair  $\pi_{j+1}$  will be taken

to an adjacent pair of arcs with  $F_j(b_{j+1}) = F_j(c_{j+1})$ . Such an ordering is possible, since every lamination contains an adjacent pair, and when an adjacent pair is welded, a new lamination is induced on the remaining arcs. Our theorem follows from Lemma 2.3 in [10] with a bit of additional work, but we present a more constructive proof.

**Theorem 1.4.5.** *Given a balanced lamination  $L$ , there exists a conformal map  $H$  from the exterior of the unit circle to the complement of a finite planar tree  $\Gamma$ . This map, extended continuously to the boundary, has the property that for a point  $z \in \Gamma$  not a vertex, the preimage of  $z$  under  $H$  is a pair of points  $\zeta, \xi$  on  $\partial\mathbb{D}$  such that  $w(\zeta) = \xi$ . Furthermore, restricted to an arc to one side of any root of unity  $\xi$ ,  $H$  has the asymptotic expansion*

$$H(\zeta) - H(\xi) = a(\zeta - \xi)^{2/d} + O((\zeta - \xi)^{2/d+\epsilon}),$$

where  $d$  is the degree of the vertex of the image of  $\zeta$ , for some  $a \in \mathbb{C}$  and  $\epsilon > 0$ .

We will refer to the final property as the *welding property*. We will prove the theorem in two steps. First, we construct a conformal map from the exterior of the circle using two classes of elementary conformal maps that satisfies the welding condition for some welding map  $\tilde{w}$  for the lamination  $\Pi$  (that is, for each  $j$ ,  $\tilde{w}_j$  is a homeomorphic involution on the pair  $\pi_j$  decreasing in angle). Then the function  $h(\zeta) = \tilde{w}(w(\zeta))$  maps arcs into themselves, and can be extended continuously to the unit circle. In the second step, we use the function  $h$  to define a quasiconformal map of the plane  $G$  with boundary values on the circle such that  $F \circ G$  satisfies the welding condition for  $w$ . Then a solution to the Beltrami equation corrects the quasiconformal modulus to produce the desired conformal map.

Let  $f_0$  be the conformal map from the exterior of the circle to the upper half plane which takes the endpoints of the adjacent arcs of  $\pi_1$  to  $-1/2, 0$ , and  $1/2$ . Define pairs of intervals  $\pi_i^0$  with endpoints  $a_i^0, b_i^0, c_i^0$ , and  $d_i^0$  as the images of the  $\pi_i$  under  $f_0$ . For  $0 < \alpha < 1$ , consider the function  $\phi_\alpha$  given by

$$\phi_\alpha(z) = (z - \alpha)^\alpha (z + 1 - \alpha)^{1-\alpha}. \tag{1.7}$$

**Proposition 1.4.1.** *The function  $\phi_\alpha$  has the following properties (see Figure 1.3).*

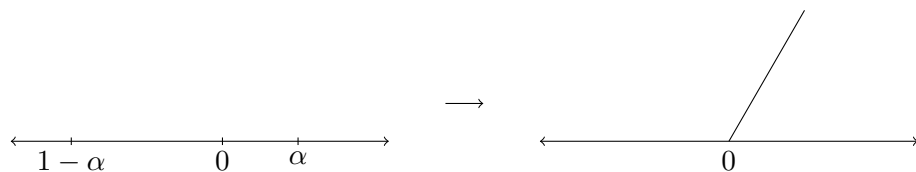


Figure 1.2: The boundary behavior of the conformal map  $\phi_\alpha$ . Here,  $\alpha = 1/3$ .

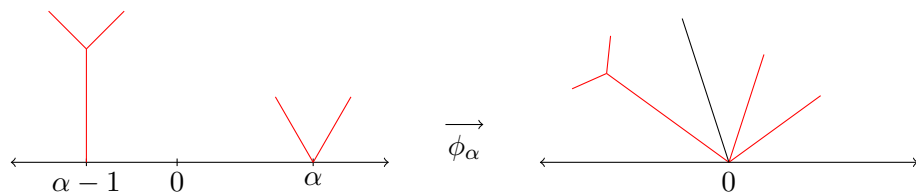


Figure 1.3:  $\alpha$  is chosen to make the angles evenly spaced in the image. Here,  $\alpha = 3/5$ .

1. It is a conformal map from the upper half plane into itself, and extends continuously to the boundary, mapping the interval  $(-\infty, \alpha - 1]$  to  $(-\infty, 0]$ ,  $[\alpha - 1, 0]$  and  $[0, \alpha]$  to the left and right sides, respectively, of a line segment from zero with argument  $\pi\alpha$  in the upper half plane, and  $[\alpha, \infty)$  to  $[0, \infty)$ . Note that 0 maps to the tip of the interval.
2.  $\phi_\alpha$  is analytic in a neighborhood of infinity, with expansion  $z + b + O(1/z)$ .
3.  $\phi_\alpha$  is analytic in a neighborhood of 0 with expansion  $f(0) + O(z^2)$ .
4.  $\phi_\alpha$  extends analytically and one-to-one across all other boundary points except  $\alpha$  and  $1 - \alpha$ .

We now construct a sequence of conformal maps  $\{f_k\}_{k=1}^{n-1}$  from the upper half plane into itself. The map  $f_k$  will be the composition of two maps: a Möbius transformation  $l_k$  sending the real line to itself, so that  $\pi_k^{k-1} = \{[1 - \alpha_k, 0], [0, \alpha_k]\}$ , for some  $\alpha_k$ , and the map  $\phi_k = \phi_{\alpha_k}$ , which will take the pair  $\pi_k^{k-1}$  into the half plane and all other intervals back to the real line. For each map  $f_k = \phi_k \circ l_k$ , we carry the correspondences forward, letting  $\pi_j^k = f_k(\pi_j^{k-1})$ ,  $a_j^k = f_k(a_j^{k-1})$ , and so on, for those segments with  $j > k$ . It remains to be

shown how to choose  $\alpha_k$ . The resulting map will not satisfy the welding property for  $w$ , but will be corrected in the following section.

#### 1.4.5 Embedding with elementary functions

We start with  $f_1 = \phi_{1/2}$ , which sends  $a_0^1$  and  $d_0^1$  to 0 and  $b_0^1$  to  $i/2$ . After applying  $f_1$ , the length of images of intervals near the zeros are asymptotic with the square root of their length. We will need to keep track of these asymptotics. We give a recursive definition of a function  $q_k$  on the endpoints of the  $\pi_j^k$  on the extended real line which corresponds to the degree of the tree under construction. Let  $E_0 = \cup_{j=1}^n \{a_j^0, b_j^0, c_j^0, d_j^0\}$  be the set of endpoints of intervals, and let  $E_k = f_k(E_{k-1})$  for  $k > 0$ . Let  $q_0(x) = 1$  for  $x \in E_0$ . Then, for each  $x \in E_k$  for  $k > 0$ ,  $x$  is the image of one or two points, which we denote  $E_{k-1}^x = \{y : f_k(y) = x\}$ . Then define

$$q_k(x) = \sum_{y \in E_{k-1}^x} q_{k-1}(y).$$

For example, the point  $x = 0$  will have  $q_1(x) = 2$ , since it is the image of  $-1/2$  and  $1/2$ , and the rest will have  $q_1(y) = 1$ . Once a point has been mapped into the upper half plane, the value of  $q$  remains constant. We choose  $\alpha$  so that, after welding, the angles on either side of the natural slit of the map will be divided equally by the branches of the tree from the point on that side. For example, if  $q_1(a_1) = 1$  and  $q_1(d_1) = 2$ , let  $\alpha_2 = 2/3$ . See Figure 1.4 for a visual reference.

**Lemma 1.4.1.** *For  $1 \leq k < n$ , let*

$$\alpha_k := \frac{q_{k-1}(d_k^{k-1})}{q_{k-1}(a_k^{k-1}) + q_{k-1}(d_k^{k-1})} = \frac{q_{k-1}(F_{k-1}(d_k))}{q_k(F_k(d_k))},$$

*and let  $F_k = f_k \circ \dots \circ f_1$  with  $f_j = \phi_j \circ l_j$  for the  $\alpha_k$  above. Then for each  $1 < k < n$ , and each point  $x \in E_0$ , in a neighborhood of  $x$ ,  $F$  is one-to-one with series expansion*

$$F_k(x \pm \delta) - F_k(x) = a_0 \delta^p + \sum_{k=1}^{\infty} a_k \delta^{p+p_k},$$

*if  $F_k(x) \in \mathbb{R}$ , or*

$$F_k(x \pm \delta) - F_k(x) = a_0 \delta^{2p} + \sum_{k=1}^{\infty} a_k \delta^{p+p_k},$$

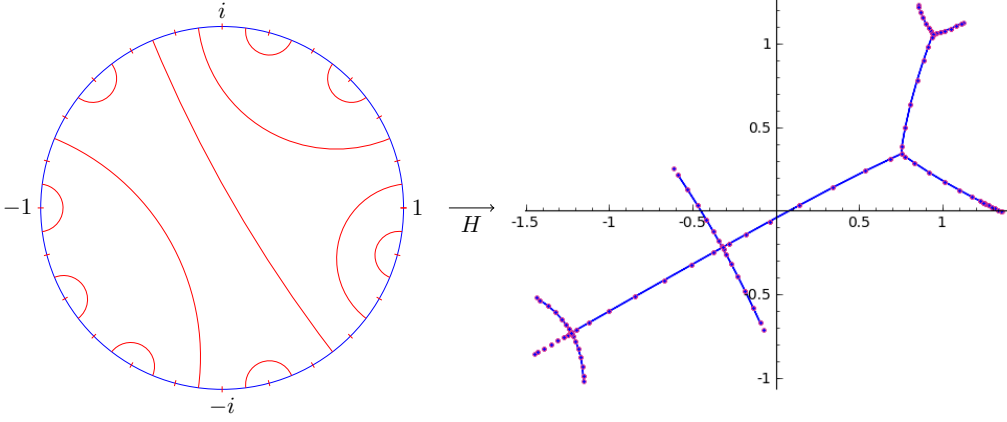


Figure 1.4: The plane tree resulting from the conformal welding map  $H$  for the balanced lamination on the left, which was sampled from the uniform distribution.

if  $F_k(x) \in \mathbb{H}$ , for some increasing positive sequence  $p_k$ ,  $a_k \in \mathbb{C}$ , and

$$p = \frac{1}{q_k(F_k(x))}.$$

The values of  $a_k$  and  $p_k$  depend on the sign of  $\pm\delta$ . The same condition applies at infinity, if necessary, after an inversion.

*Proof.* The induction hypothesis holds for  $k = 0$ . For  $j < n - 1$ , if the induction hypothesis holds for  $j$  and  $f_{j+1}$  is analytic and one-to-one near all  $e_k^j$  with  $e_k^j \neq e_{j+1}^j$  for  $e \in \{a, b, c, d\}$ , and  $q_{j+1}(F_{j+1}(e_k)) = q_j(F_j(e_k))$  so the hypothesis extends to  $j + 1$  for these points. The point  $b_{j+1}^j = c_{j+1}^j$  maps into the half plane, and since  $q_{k+1}(F_{k+1}(x)) = q_k(F_k(x))$  for this point, property (3) of the function  $\phi_\alpha$  gives the desired expansion. To the right of  $x = d_{j+1}$ , we have, by hypothesis, and conformality of  $l$ ,

$$l_{j+1}(F_j(x + \delta)) - l_{j+1}(F_j(x)) = a_0\delta^{p_0} + \sum_{k=1}^{\infty} a_k\delta^{p_0+p_k},$$

where  $p_0 = 1/q_k(F_k(x))$ , and

$$\phi_\alpha(\alpha + \epsilon) = \epsilon^\alpha + \sum_{k=1}^{\infty} b_k\epsilon^{\alpha+k},$$

so

$$\begin{aligned}
F_{j+1}(x + \delta) - F_{j+1}(x) &= \phi_{j+1}(\alpha_{j+1} + [l_{j+1}(F_j(x + \delta)) - l_{j+1}(F_j(x))]) \\
&= \left( a_0 \delta^{p_0} + \sum_{k=1}^{\infty} a_k \delta^{p_0 + p_k} \right)^{\alpha_{j+1}} + \\
&\quad \sum_{m=1}^{\infty} b_m \left( a_0 \delta^{p_0} + \sum_{k=1}^{\infty} a_k \delta^{p_0 + p_k} \right)^{\alpha_{j+1} + m} \\
&= a_0^{\alpha_{j+1}} \delta^p + \sum_{k=1}^{\infty} \tilde{a}_k \delta^{p + \tilde{p}_k},
\end{aligned}$$

where  $p = 1/q_{k+1}(F_{k+1}(x))$ ,  $\tilde{p}_1 = \min(p_1, p_0)$ , and so on. To see that the resulting series expansions converge and represent locally invertible functions, consider the series to be expansions of analytic functions in  $\delta^\eta$ , where  $\eta$  is a fractional power with lowest common denominator among the powers  $p + p_k$ . The same argument applies on the other side of  $d_{j+1}$  and on either side of  $a_{j+1}$ .  $\square$

A composition of the maps  $f_k$  up to  $n - 1$  leaves the two points  $a_n^{n-1} = d_n^{n-1}$  and  $b_n^{n-1} = c_n^{n-1}$  on the real line. By shifting one point to zero and the other to infinity, we then apply a square map, and reset infinity (that is, compose with a fractional map such that the total composed map sends infinity to infinity), so that the resulting map  $f_n$  is such that  $F_n = f_n \circ \dots \circ f_0$  is a map from the exterior of the disk to the exterior of a tree, comprised of the union of the analytic images of the segments, and normalized to be approximately the identity in a neighborhood of infinity, that is,

$$F_n(z) = z + O(|z|^{-1}).$$

Together with Lemma 1, the finished construction gives a desirable property to the elementary embedding  $F_n$ :

**Corollary 1.4.1.** *The map  $F_n$  gives another embedding  $\Gamma'$  of the same planar tree as  $\Gamma$ , and, in a neighborhood to one side of a root of unity  $\xi$ , has the same asymptotic expansion as  $H$ ,*

$$F_n(\zeta) - F_n(\xi) = a(\zeta - \xi)^{2/d} + O(|\zeta - \xi|^{2/d+\epsilon}),$$

with different constants.

To prove the corollary, simply follow the values of  $q_k$  in the proof above to see that they correspond to the degree of the planar tree as it is being constructed. In addition to providing the symmetry necessary in the following section, the choice of  $\alpha_k$  ensures equal angles around each vertex in the image, and thus produces better results in the approximation.

#### 1.4.6 Quasiconformal correction

Next, we turn to the quasiconformal function  $G$  which guarantees the welding property. Define  $\tilde{w}$  on the union of the open arcs to be  $\tilde{w}(\zeta) = F^{-1}(F(\zeta))$ , where the inverse is chosen so that  $F^{-1}(F(\zeta)) \neq \zeta$ . Then  $F$  satisfies

$$F(\zeta) = F(\tilde{w}(\zeta)),$$

for all  $\zeta$  except roots of unity. Let  $\mathcal{A} = \mathcal{B} \cup \mathcal{W}$ , a natural two-coloring of the arcs of the circle. On a given arc  $[a, b]$ , the function  $\zeta \mapsto \tilde{w}(w(\zeta))$  is a homeomorphism of the arc into itself, increasing in angle. Define  $h : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$  as follows.

$$h(\zeta) = \begin{cases} \zeta & \text{for } \zeta \in \mathcal{W} \\ \tilde{w}(w(\zeta)) & \text{for } \zeta \in \mathcal{B}, \end{cases}$$

extended continuously to the roots of unity. We need a quasiconformal map  $H$  such that  $H|_{\partial\mathbb{D}} = h$ . The condition for the existence of such a map is quasiasymmetry, which reduces to a bi-Lipschitz condition for the given  $h$ , because the map is linear on one side of each root of unity, the only places where it is not analytic. Thus we require that for each  $x \in \mathcal{R}$ , in a neighborhood of  $x$ , there exists  $K > 1$  such that

$$\frac{1}{K} \leq \frac{|h(\zeta_1) - h(\zeta_0)|}{|\zeta_1 - \zeta_0|} < K,$$

or, equivalently,  $K$  such that for  $z_0, z_1$  in the image of a neighborhood of  $x$ , if  $F^{-1}(z_i) = \{\zeta_i, \xi_i\}$ , for  $i = 0, 1$ , then take  $F^{-1}$  to be the inverse of  $F$  restricted to the neighborhood containing  $\xi$ . Then we need

$$\frac{1}{K} \leq \frac{|F^{-1}(F(\zeta_1)) - F^{-1}(F(\zeta_0))|}{|\zeta_1 - \zeta_0|} < K.$$

By inverting the expression for  $F$  in that neighborhood, it is easy to see that  $F^{-1}$  has an asymptotic expansion

$$F^{-1}(z) - x = a(z - F(x))^{d/2} + O(|z - F(x)|^{2/d+\epsilon}),$$

and therefore

$$F^{-1}(F(\zeta)) - x = ab(\zeta - x) + O(|\zeta - x|^\epsilon),$$

where  $\epsilon$  is chosen as needed in each case. In both cases the error term has a convergent series expansion in an appropriate neighborhood, and is thus well-behaved with respect to composition and inversion. In a neighborhood on each side of  $x$ ,  $h$  is thus approximately linear, and  $K$  can be chosen to satisfy the bi-Lipschitz condition above. Thus  $h$  is quasymmetric on the circle, and there exists a quasiconformal map  $H : \mathbb{C} \rightarrow \mathbb{C}$  with boundary values  $h$  on the unit circle. Consider  $G_0 = F \circ H$ . It is a quasiconformal map from the exterior of  $\mathbb{D}$  to the exterior of a tree. For  $z \in G_0(\partial\mathbb{D})$ ,  $F^{-1}(z)$  is a pair of points  $\{\zeta, \tilde{w}(\zeta)\}$  for  $\zeta \in \mathcal{W}$ . Then  $\zeta = h(\tilde{\zeta})$  and

$$\tilde{w}(\tilde{\zeta}) = \tilde{w}(w(w(\tilde{\zeta}))) = h(w(\tilde{\zeta})),$$

so  $G_0$  satisfies the welding condition for the desired welding map  $w$ .

Finally, there exists a quasiconformal map  $B : \mathbb{C} \rightarrow \mathbb{C}$  such that  $G = B \circ G_0$  is conformal and satisfies the normalization  $G(z) = z + O(|z|^{-1})$  at infinity. The map  $B$  is again the solution to the Beltrami equation as outlined in the previous section. This nearly concludes the proof of Theorem 1.4.5, with  $G$  being the desired map. The asymptotic expansion will follow from the existence of the Shabat polynomial, as described in the following section.

#### 1.4.7 A recursive construction of an infinite planar tree with piecewise linear maps

We now consider a welding algorithm that interacts well with the uniform proper lamination. Recall that by proposition 1.2.4 this lamination is equivalent in distribution to the tree obtained by the stick-breaking construction of Aldous. We give the equivalent construction in terms of laminations, and call it the *arc-insertion* process.

Recall that a sample from  $\mathcal{P}\mathcal{L}_k$  is a lamination with  $2k - 1$  pairs of arcs encoded by an element of  $\mathcal{P}_k$  with an independent  $\mathcal{U}(\mathbb{T})$  random variable giving the argument of the root.

Given such a lamination  $L$ , obtain a new lamination by sampling an independent random variable  $\alpha$  according to the distribution of the first order statistic of  $2k - 1$   $\mathcal{U}([0, \pi])$  random variables, that is, having density

$$f(x) = \frac{2k - 1}{\pi^{2k-1}} (\pi - x)^{2k-2}.$$

Then sample a uniform random point  $\omega$  on  $\mathbb{T}$  and insert two paired arcs of length  $\alpha$  on either side of  $\omega$ , scaling the rest of the lamination linearly away. In order to represent the resulting lamination as an arc-pairing lamination, the two paired arcs at the point  $\omega$  are each split in two, resulting in  $2k + 1$  paired arcs.

**Proposition 1.4.2.** *The mapping  $(L_k, \alpha, \omega) \mapsto L_{k+1}$  defined by the above procedure yields a lamination  $L_{k+1}$  distributed according to  $\mathcal{P}\mathcal{L}_{k+1}$ .*

*Proof.* We are translating the consistent family of Lemma 23 from [3] to the language of laminations. Since we rescale to constant total length, we have to check that if  $C_k$  are the times of a Poisson process with  $r(t) = t$ , the variable  $C_{k+1} - C_k$  has distribution according to  $f(x)$  when rescaled to unit length, which is due to the fact that edge-lengths are exchangeable. By the rotational symmetry of the process, it is clear that the independent root argument is still uniform and independent after a step of the process.  $\square$

Similarly the reverse procedure, of taking a random leaf, removing it and its neighboring paired arcs, and rescaling the remaining lamination to the whole circle, gives the distribution  $\mathcal{P}\mathcal{L}_{k-1}$ . We therefore say  $\mathcal{P}\mathcal{L}_k$  is *arc-insertion consistent*.

This construction can be imitated with quasiconformal maps, if we can find maps with the following property. We want a sequence of maps  $F_n$  from  $\mathbb{D}^*$  to the complement of a tree  $\Gamma_n$ , such that  $\Gamma_n$  a finite planar tree with  $n$  edges, and  $\Gamma_n \subset \Gamma_m$  for  $n < m$ . We can accomplish this with quasiconformal welding maps. Suppose we have a sequence of laminations  $\{L_n\}$  corresponding to welded trees  $\Gamma_n$  as the images of the circle under a yet-undefined quasiconformal map. Given the lamination  $L_n$ , after one step of the welding process, we have a map  $f_n$  (we reverse the indexing of the maps for reasons that will become clear), an induced lamination  $L_n^{n-1}$  on the circle, and a proto-tree  $P_{n-1}$ , the image of the welded arcs. Suppose  $L_n^{n-1} = L_{n-1}$ , that is, the induced lamination after one step towards

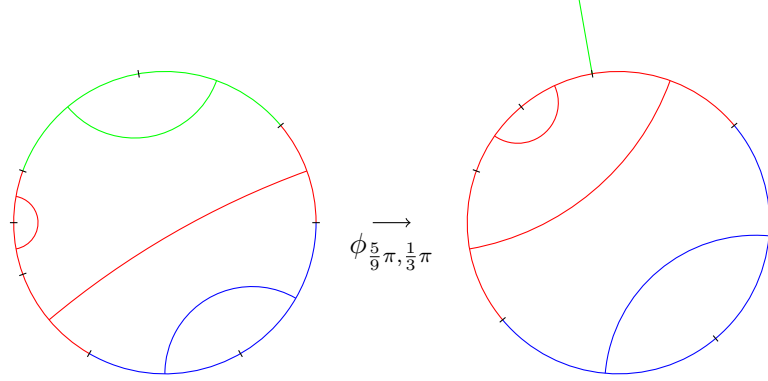


Figure 1.5: From left to right, the boundary values of the quasiconformal welding map  $\phi_{\zeta, \alpha}$  with values  $\zeta = \frac{5}{9}\pi$  and  $\alpha = \frac{1}{3}\pi$ . Also, an example of the arc-insertion sequence, showing  $L_2$  and  $L_1$ . The red intervals form a single pair in the lamination  $L_1$  on the right.

creating the tree  $\Gamma_n$  is the same as the lamination for the tree  $\Gamma_{n-1}$ . Then we find that  $\Gamma_n$  is the union of  $\Gamma_{n-1}$  and  $F_{n-1}(P_{n-1})$ .

We will accomplish this goal with two related constructions. First, a sequence of laminations  $\{L_n\}$  constructed recursively from a sequence of points  $\{\zeta_n\}$ ,  $\zeta_n \in \partial\mathbb{D}$ , and a sequence of arc-lengths  $\{\alpha_n\}$ ,  $0 < \alpha_n < \pi$ ,  $\alpha_n \rightarrow 0$ .  $L_0$  is the lamination that identifies the upper and lower halves of the circle by arc length, and  $L_n$  is formed from  $L_{n-1}$  by inserting adjacent, paired arcs of length  $\alpha_n$  at the point  $\zeta_n$  and shifting all other points, scaling linearly toward the antipodal point  $\zeta_n^* = \zeta_n + \pi$ . More specifically, let

$$T(\zeta) = \frac{\pi - \alpha_n}{\pi}(\zeta - \zeta_n^*) + \zeta_n^*,$$

where  $-\pi < \zeta - \zeta_n^* < \pi$ . Then, if  $\pi = \{A, A'\} \in \Pi_{n-1}$  and  $\zeta_n \notin A \cup A'$ ,  $\tilde{\pi} = \{T(A), T(A')\} \in \Pi_n$ , and  $w_n$  identifies points by arc length as in the balanced lamination. Typically,  $\zeta_n$  is in  $A$  for some interval of a pair  $\pi$ . Then split  $\pi$  into two pairs  $\pi_1$  and  $\pi_2$  on either side of  $\zeta_n$  and  $w_{n-1}(\zeta_n)$ , and include pairs  $\tilde{\pi}_1$  and  $\tilde{\pi}_2$  in  $\Pi_n$ . The final pair of  $\Pi_n$  is  $\{(\zeta_n - \alpha_n, \zeta_n), (\zeta_n, \zeta_n + \alpha_n)\}$ . We call this sequence of laminations the *arc-insertion* sequence of laminations for the values  $\{\zeta_n\}$  and  $\{\alpha_n\}$ .

Second, we need a two-parameter family of quasiconformal maps  $\phi_{\zeta, \alpha}$  that welds  $(\zeta - \alpha, \zeta)$  to  $(\zeta, \zeta + \alpha)$  with boundary values  $f(\zeta) = T^{-1}(\zeta)$ . Figure 1.5 shows the desired image of

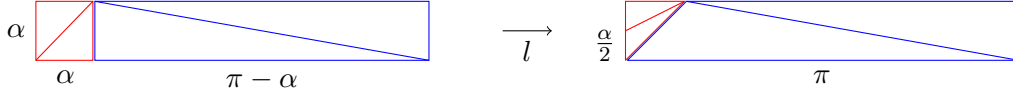


Figure 1.6: From left to right,  $T_1$  and  $T_2$  in red, and  $T_3$  and  $T_4$  in blue map to  $T'_1, T'_2, T'_3$ , and  $T'_4$  under  $\phi$ . The remainder of the rectangle and its image is the reflection of this image across the line  $x = \pi$ .

$\phi_{\zeta, \alpha}$  on the boundary. We will construct  $\phi$  as a piecewise linear map in polar coordinates. Without loss of generality, we assume  $\zeta = 1$ . Outside of the annulus  $A_\alpha = \{z : 1 < |z| < e^\alpha\}$ ,  $\phi$  is the identity. The function  $\psi(z) = i \log(z) + 2\pi$  maps  $A_\alpha \setminus (1, e^\alpha)$  to the rectangle  $R_\alpha = \{x + iy : 0 < x < 2\pi, 0 < y < \alpha\}$ . We divide the rectangle into eight triangular regions  $T_1, \dots, T_8$  as follows.

$$T_1 = \triangle(0, \alpha + i\alpha, i\alpha), \quad (1.8)$$

$$T_2 = \triangle(0, \alpha, \alpha + i\alpha), \quad (1.9)$$

$$T_3 = \triangle(\alpha, \pi, \alpha + i\alpha), \quad (1.10)$$

$$T_4 = \triangle(\pi, \pi + i\alpha, \alpha + i\alpha). \quad (1.11)$$

$T_5$  through  $T_8$  are the reflections of the others across the line  $x = \pi$ . Under  $\phi$ , each triangle  $T_i$  maps onto a corresponding triangle  $T'_i$ . See Figure 1.6 for a visual reference.

$$T'_1 = \triangle(i\alpha/2, \alpha + i\alpha, i\alpha), \quad (1.12)$$

$$T'_2 = \triangle(i\alpha/2, 0, \alpha + i\alpha), \quad (1.13)$$

$$T'_3 = \triangle(0, \pi, \alpha + i\alpha), \quad (1.14)$$

$$T'_4 = \triangle(\pi, \pi + i\alpha, \alpha + i\alpha). \quad (1.15)$$

The map for each corresponding triangle is an affine linear transformation of  $\mathbb{R}^2$ . We can write a general form for the complex dilatation of such a map. Suppose  $T = (0, x_1 + iy_1, x_2 + iy_2)$  and  $T' = (0, u_1 + iv_1, u_2 + iv_2)$ . Then if  $l$  is the linear transformation between them, we have

$$\mu_l = \frac{u_1 y_2 - u_2 y_1 - x_1 v_2 + x_2 v_1 + i(x_1 u_2 - x_2 u_1 + v_1 y_2 - v_2 y_1)}{u_1 y_2 - u_2 y_1 + x_1 v_2 - x_2 v_1 + i(x_1 u_2 - x_2 u_1 - v_1 y_2 + v_2 y_1)}.$$

For the triangles above, we have

$$\begin{aligned}\mu_1 &= \frac{1}{3}, \\ \mu_2 &= \frac{-1 + 2i}{3}, \\ \mu_3 &= \frac{\alpha + i(\pi - \alpha)}{2\pi - \alpha + i(\pi - \alpha)}, \\ \mu_4 &= 0.\end{aligned}$$

Then, if  $l$  is the piecewise linear map on  $R_\alpha$ , extended continuously across the inner boundaries, let

$$\phi = \psi^{-1} \circ l \circ \psi,$$

on  $A_\alpha$ , and the identity outside.  $\phi$  is a quasiconformal map of  $\mathbb{D}^*$  to  $\mathbb{D}^* \setminus [0, \alpha/2]$ , since it is a homeomorphism with complex dilatation a.e. bounded away from 1, since composition with a conformal map does not change the magnitude of the complex dilatation. Also,  $\phi(\zeta) = T^{-1}(\zeta)$  on  $\partial\mathbb{D} \setminus [-\alpha, \alpha]$  as desired. Conjugation with a rotation gives the general map  $\phi_{\zeta, \alpha}$ .

Given sequences  $\{\zeta_n\}$  and  $\{\alpha_n\}$ , it is now straightforward to construct a quasiconformal embedding of the tree corresponding to the lamination  $L_m$  for any  $m > 0$ . Let

$$\Phi_m = \rho \circ \phi_{\zeta_1, \alpha_1} \circ \phi_{\zeta_2, \alpha_2} \circ \cdots \circ \phi_{\zeta_m, \alpha_m},$$

where  $\rho(z) = z + z^{-1}$ , the conformal map that welds the top and bottom of the circle. It is routine to check that  $\Phi_m$  satisfies the welding condition for the arc-identifying welding homeomorphism for the lamination  $L_m$ . Then, using the same procedure as above, we can use the solution to the Beltrami equation for the composed complex dilatation to produce the conformal welding map. The advantage to the arc-insertion approach defined in this section is that we can easily produce infinite objects. The tree  $\Gamma_m$  is an increasing set, so let

$$\Gamma_\infty = \overline{\bigcup_{m=1}^\infty \Gamma_m}.$$

Of course, without additional analysis, there is no way to know whether or not  $\Gamma_\infty$  even has a tree structure. If  $\alpha_n \rightarrow 0$ , there is a limit function  $\Phi$ .

**Proposition 1.4.3.** *If  $\alpha_n \rightarrow 0$ ,  $\Phi_n$  converges uniformly on compact subsets of  $\mathbb{D}^*$  to a locally quasiconformal homeomorphism  $\Phi$ , and  $\Phi(\mathbb{D}^*) = \Gamma^c$ .*

*Proof.* Since  $\phi_{\zeta_m, \alpha_m}$  is the identity map for  $|z| > e^{\alpha_m}$ , If  $K$  is a compact neighborhood with  $d(K, \mathbb{D}) = \epsilon$ , let  $n_0$  be such that  $\alpha_n < \log(1 + \epsilon)$  for  $n > n_0$ . Then for  $m > n_0$ ,  $\Phi_m(K)$  is constant, and the limit exists. Since  $\Phi_m$  is quasiconformal on  $K$ ,  $\Phi$  is as well. To see that  $\Phi(\mathbb{D}^*) = \Gamma^c$ , consider  $z \in \mathbb{D}^*$  with  $B_\epsilon(z) \subset \mathbb{D}^*$ . Eventually,  $\Phi_m$  is constant on  $B_\epsilon(z)$ , so  $d(\Phi(z), \Gamma_m)$  is fixed for all  $m$  large enough, and  $\Phi(z) \notin \Gamma$ , so  $f(\mathbb{D}^*) \subset \Gamma^c$ . On the other hand, suppose  $z_n \rightarrow \partial\mathbb{D}$ . Each map  $\phi_{\alpha, \zeta}$  decreases the distance to the boundary, so  $\Phi(z_n) \rightarrow \Gamma$ , and  $\Gamma^c \subset f(\mathbb{D}^*)$ .  $\square$

### 1.5 Shabat polynomials

We now explain the connection to the theory of *dessins d'enfants*, or children's drawings, which is the study of maps on the Riemann sphere in the context of algebraic geometry, and Galois theory in particular. Belyi's theorem (1979) establishes that the Riemann surfaces described by maps (bipartite planar graphs) are exactly those surfaces  $X$  defined over the field of algebraic numbers  $\overline{\mathbb{Q}}$ . Each map corresponds to a meromorphic function  $h$  with critical values  $0, 1, \infty$ , and we say  $h$  is a covering of the sphere by  $X$  *ramified* over those three points. We are interested in the case of maps that are trees, corresponding to the case where  $X$  is the Riemann sphere and  $f$  is a polynomial. For a complete exposition of the algebraic point of view, see the book by Lando and Zvonkin [24].

To see the connection between our work and polynomials, we restate the basic objects and properties of interest. A *balanced lamination*  $L$  is a non-crossing pairing of the arcs of the unit circle between the roots of unity in linear decreasing fashion, and for each such lamination, there exists a conformal welding map  $f$  and a plane set  $\Gamma$  called a *conformally balanced tree*. We normalize  $f$  so that it has the expansion

$$f(z) = z + \frac{b_1}{z} + O(|z|^{-2})$$

near infinity. The following observations can be found in a paper by Biane [8], and as Lemma 2.4 in Bishop [10]. Consider the function

$$\phi_n(z) = z^n + z^{-n},$$

which composes the power function  $z^n$  with the welding map  $z + 1/z$ . The balanced lamination guarantees that arcs are pairing with opposite parity around the circle, so the lamination induced by applying  $z^n$  to the pairs of  $L$  consists of pairs  $(\zeta, \bar{\zeta})$  around the circle. Since  $z + 1/z$  welds this lamination, the function  $\phi_n$  has the property that  $\phi_n(\zeta) = \phi_n(\xi)$  for  $(\zeta, \xi) \in L$ . Let  $g$  be the inverse conformal map to  $f$ , and consider the function

$$p(z) = \phi_n \circ g,$$

a  $n$ -fold holomorphic covering of the complement of the segment  $[-2, 2]$  by the complement of the tree  $\Gamma$ , by which we mean that each point outside the segment has exactly  $n$  preimages in  $\Gamma^c$ .

**Proposition 1.5.1.** *The function  $p = \phi_n \circ g$  is a monic polynomial of degree  $n$  having at most two critical values.*

*Proof.* Because  $\phi_n$  extends continuously to  $\mathbb{T}$  and respects  $L$ ,  $p$  extends continuously across  $\Gamma$ . By Morera's theorem, since  $\Gamma$  consists of analytic arcs [10],  $p$  is entire. Since it has a degree  $n$  pole at infinity, it is a polynomial of degree  $n$ , and because of the normalization, it is monic.

For each point  $c$  in the image of a polynomial of degree  $n$ , by the fundamental theorem of algebra, there must be exactly  $n$  preimages, counting multiplicity. A multiple root of  $p - c$  of degree  $k$  is a point  $z_0$  such that the Taylor's expansion of  $p - c$  in a neighborhood of  $z_0$  has first term  $a(z - z_0)^k$ . Then the critical points of  $p$  are exactly those points which have a multiple root. Since each point off  $\{-2, 2\}$  has exactly  $n$  preimages (there are  $n$  edges providing the  $n$  preimages for points within the interval), the critical points of  $p$  can take on only the values  $\{-2, 2\}$ .

□

A polynomial with the property of having at most two critical values is called a *Shabat polynomial*, and such polynomials are special cases of the meromorphic Belyi functions described above after a translation and dilation. In fact, each such polynomial corresponds to one of our conformally balanced trees, up to rigid transformations.

Shabat polynomials are also called *generalized Chebyshev polynomials*, for a reason which we will now explain, which also provides an intuitive first example. The Chebyshev polynomial of the first kind, which has the formula

$$T_n(x) = \cos(n \arccos x)$$

in the interval  $(-1, 1)$ , has critical points at  $\cos(\pi k/n)$  for  $k = 1, \dots, n-1$ , and critical values  $\pm 1$ , which are necessarily the only critical points and values of the polynomial.

The polynomial  $T_n$  corresponds to the lamination  $L = \{(\zeta, \bar{\zeta}), \zeta \in \mathbb{T}\}$  for each  $n$ , and the polynomial  $T_n$  takes the welded "straight tree" with  $n$  subdivisions, and folds it up  $n$  times, sending it to the segment  $[-1, 1]$ . The critical points correspond to the vertices of the tree. The generalized polynomial extends this behavior to the case of trees with vertices not on the real line.

Another example is the other obvious polynomial with fewer than two critical values, which is the polynomial  $p(z) = z^n$ , which has critical points only at  $z = 0$ , with value 0. The preimage of  $[-2, 2]$  under  $p$  is the star tree with a degree  $n$  vertex at 0 and leaves at the  $2n$ -th roots of unity, with edge lengths  $\sqrt[n]{2}$ . This is the only polynomial of degree  $n$  with one critical value, corresponding to the only tree with only one non-leaf vertex (up to rigid transformations). Looking forward, we note that this tree converges to the closed unit disk, and the welding map for the associated lamination converges to the identity function uniformly on compact sets of the domain. Proving that this convergence is not generally the case for polynomials chosen uniformly at random is the subject of our main theorem.

As noted above, there has been interest in computing Shabat polynomials (see [24], [7]) and algebraic methods are only appropriate for trees up to degree eight or so. Newton's method provides a way to calculate vertices to arbitrary precision, but only converges with a good initial guess, which is hard to determine for trees with many edges and a complex shape. A description of the method and its difficulties can be found in [34]. Our algorithm with Marshall described above provides a way to initialize the data for the Newton's method algorithm, and we are able to compute vertices to arbitrarily high precision for trees up to a thousand vertices, which was not previously possible.

## 1.6 Subsequential limits

We now prove our first limit theorem, which holds that subsequential limits of uniform balanced trees are non-trivial. While not a strong result, proving non-triviality of limits for certain models such as the diffusion-limited aggregation has presented a strong challenge, and we require some interplay between the distribution of large trees and the potential theory of welding maps, which may help point the way towards stronger results. In this section we consider only the case of balanced laminations in order to reduce the number of calculations, but the same result should hold for the case of proper laminations.

For sequences of conformal maps on a fixed domain, the most convenient topology for both the sequence of maps and the image set is the Carathéodory topology. Taking from [32], we suppose  $G_n = \mathbb{C} \setminus \Gamma_n$  is the image of  $\mathbb{D}$  under a conformal map  $f_n$ . Then we say  $f_n \rightarrow f$  uniformly on compact sets if for every  $K \subset \mathbb{D}$  compact,  $\|f - f_n\|_\infty \rightarrow 0$ , and we say that  $G_n \rightarrow G$  with respect to  $w_0$  in the sense of kernel convergence if

1. either  $G = \{w_0\}$ , or  $G$  is a domain not equal to  $\mathbb{C}$  containing  $w_0$  such that some neighborhood of  $w_0$  lies in every  $G_n$  for  $n$  sufficiently large;
2. for  $w \in \partial G$ , there exist  $w_n \in \partial G_n$  such that  $w_n \rightarrow w$ .

The *Carathéodory kernel theorem* gives that if  $f_n$  is a sequence of conformal maps with  $f_n(0) = w_0$  and  $f'_n(0) > 0$ , then convergence in the two senses is equivalent, and if  $G \neq \{w_0\}$ ,  $f$  is a conformal map from  $\mathbb{D}$  with  $f(0) = w_0$  and  $f'(0) > 0$ . Furthermore, the space of domains or functions with this topology is a separable metric space.

Since we consider maps in a neighborhood of infinity, we let  $w_0 = \infty$ , and instead of  $\mathbb{D}$  we use  $\mathbb{D}^*$  as our domain, requiring  $f'(\infty) > 0$ . Convergence in  $\mathbb{C}^*$  is determined with respect to the spherical or chordal metric, which can be approximated by

$$d(z, w) = \min\{|z - w|, |1/z - 1/w|\}.$$

It is easy to see that convergence in this sense holds on the more general domain. We exhaust  $\mathbb{D}^*$  with compact domains  $D_n$ , and let

$$d_C(f, g) = \sum_1^\infty 2^{-n} \|f - g\|_{D_n},$$

for which convergence is equivalent to convergence on compact sets because  $\mathbb{C}^*$  is compact under  $d$ . To see that the space is separable, observe that on a compact neighborhood  $N$  of infinity, a conformal map  $f$  with  $f(\infty) = \infty$  has an asymptotic expansion

$$f(z) = b_{-1}z + b_0 + \frac{b_1}{z} + \dots,$$

and we can find a rational map with rational coefficients that approximates  $f$  on  $N$  to any degree of precision.

Let  $\Sigma_0$  be the set of conformal maps from  $\mathbb{D}^*$  into  $\mathbb{C}^*$  with asymptotic expansion

$$f(z) = z + \frac{b_1}{z} + O(|z|^{-2}),$$

and  $\Sigma_0$  is compact. The proof uses the bound  $|f(z)| < 2|z|$  and Montel's first theorem, which gives that a family of functions uniformly bounded on compact sets is normal. See Pommerenke, Theorem 1.7 [31].

Therefore, if  $P_n$  is the uniform measure on balanced arc-pairing laminations with  $n$  arc pairs, since  $P_n(f \in \Sigma_0) = 1$ , the sequence  $P_n$  is trivially tight, and by Prohorov's theorem [9], it is relatively compact.

**Proposition 1.6.1.** *Under the topology of Carathéodory convergence,  $P_n$  is relatively compact, and has subsequential limits in distribution. If  $P$  is such a limit, and  $f$  is a random limiting map with distribution  $P$ , then  $f(z)$  is a conformal map of  $\mathbb{D}^*$  with expansion  $f(z) = z + b_1/z + \dots$  near  $\infty$  almost surely.*

The proof follows from the above discussion. Now we state our main theorem.

**Theorem 1.6.1.** *If  $P$  is a subsequential limit of  $P_n$ ,  $P(f = \text{id}) = 0$ .*

We consider properties of  $L$  under the measure  $P_n$ . Let  $\Lambda_\alpha(E)$  be the  $\alpha$ -dimensional Hausdorff content, that is,

$$\Lambda_\alpha(E) = \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \text{diam}(U)^\alpha,$$

where the infimum is taken over all open covers of  $E$ . We seek to show that for all  $n$ , a set of large Hausdorff content is mapped far from itself in the conformal welding map

### 1.6.1 A uniform Hölder bound

We start by establishing a modulus of continuity result. Recall that  $P_n$  puts the uniform measure on bridge walks  $\beta$  with  $2n$  steps.

**Lemma 1.6.1.** *For every  $0 < \alpha < 1/2$ , for any  $p_0 < 1$ , there exists  $C > 0$  such that if  $C_\alpha$  is defined to be the smallest constant such that*

$$|\beta(t) - \beta(s)| \leq C_\alpha |t - s|^\alpha,$$

on  $0 \leq s < t \leq 2$ , then

$$P_n[C_\alpha \leq C] \geq p_0,$$

for all  $n > 0$ .

Note that although  $C_\alpha$  clearly exists almost surely for all  $n$ , for  $\alpha \geq 1/2$ ,  $C_\alpha$  is unbounded with  $n$ . The proof follows Kolmogorov's continuity estimates, which also show that the limiting process, the Brownian bridge, is Hölder- $\alpha$ .

*Proof.* Let  $S = \{S_k; k \geq 1\}$  be simple random walk, and  $X^n$  be the continuous function linearly interpolating the steps of  $S$  with interval size  $1/n$  and Brownian scaling.

We consider the expectation

$$E[|X_t^n - X_s^n|^{2p}],$$

for integers  $p > 0$ . We wish to show that there is a constant  $K_p$  independent of  $n$  such that

$$E[|X_t^n - X_s^n|^{2p}] < K_p |t - s|^p$$

for all  $0 \leq s < t \leq 2$ .

Suppose  $t_0$  and  $t_1 = t_0 + 1/n$  are lattice points and  $t_0 - s \geq 1/n$ . Then as  $t = t_0 + h$  goes from  $t_0$  to  $t_1$ ,

$$E[|X_t^n - X_s^n|^{2p}] = E[|X_{t_0}^n - X_s^n + hY|^{2p}],$$

and letting  $X = X_{t_0}^n - X_s^n$ ,  $Y$  is a Bernoulli random variable independent of  $X$ , we have

$$E[|X + hY|^{2p}],$$

which is increasing in  $h$ , since  $X$  and  $Y$  are centered, symmetric, independent random variables. Indeed,

$$\begin{aligned} E[|X + hY|^{2p}] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x + hy|^{2p} f_X(x) f_Y(y) dy dx \\ &= \int_0^{\infty} \int_{-\infty}^{\infty} (x + hy)^{2p} + (x - hy)^{2p} f_X(x) f_Y(y) dy dx, \end{aligned}$$

so

$$\frac{d}{dh} E[|X + hY|^{2p}] = \int_0^{\infty} \int_{-\infty}^{\infty} 2py((x + hy)^{2p-1} - (x - hy)^{2p-1}) f_X(x) f_Y(y) dy dx,$$

and  $(x + hy)^{2p-1} - (x - hy)^{2p-1}$  has the same sign as  $y$  for  $x > 0$ , which shows that the moment increases in  $h$ . Suppose there exists  $K$  that works when  $t$  is a lattice point, then

$$E[|X_t^n - X_s^n|^{2p}] < E[|X_{t_1}^n - X_s^n|^{2p}] < K|t_1 - s|^p < 2^p K|t - s|^p,$$

since  $t - s \geq 1/n$ . When  $t - s < 2/n$ ,

$$E[|X_t^n - X_s^n|^{2p}] < (\sqrt{n}(t - s))^{2p} < 2^p (t - s)^p,$$

so if we can find a  $K$  that works whenever  $t$  is a lattice point, we can find a  $K'$  that works for all  $t$ , independent of  $n$ . We can apply the same argument for  $s$ , so we need only check lattice points. The next step is to observe that the moments of simple random walk increase to those of Brownian motion. Let  $X_t$  be standard Brownian motion.

**Lemma 1.6.2.** *For an integer  $p > 0$  and  $t = k/n$ ,  $E(|X_t^n|^{2p}) < E(|X_t|^{2p})$ .*

The moment generating function for a sum of Bernoulli random variables is

$$g(\lambda) = E[\exp \lambda S_k] = \cosh^k(\lambda),$$

and we find the  $2p$ -th moment by taking derivatives. Let

$$m_{k,2p} = g^{(2p)}(0) = E[S_k^{2p}].$$

The derivatives of  $g$  are polynomials in  $x = \cosh(\lambda)$  and  $y = \sinh(\lambda)$ .  $x' = y$  and  $y' = x$ ,

so

$$\begin{aligned} g^{(q)} &= \sum_{j=0}^k a_j^q x^{k-j} y^j \\ &= \sum_{j=0}^k a_j^{q-1} [(k-j)x^{k-j-1} y^{j+1} + jx^{k-j+1} y^{j-1}], \end{aligned}$$

and  $g^{(q)}$  satisfies the following recursive formula:

$$\begin{aligned} a_0^0 &= 1 \\ a_j^0 &= 0 \text{ for } j > 0 \\ a_0^q &= a_1^{q-1} \\ a_j^q &= (k - j + 1)a_{j-1}^{q-1} + (j + 1)a_{j+1}^{q-1} \text{ for } j > 0. \end{aligned}$$

A value  $a_i^q$  can be found by taking a path from  $(0, 0)$  to  $(q, i)$  along the lattice with diagonal steps, multiplying by  $j$  or  $k - j$  at each step depending on whether the step is an up-step or a down-step, then summing over all possible paths.

Then

$$m_{k,2p} = a_0^{2p},$$

which consists of the sum of terms of the form

$$\prod_{q=1}^p (k - j_q) j_q,$$

over all possible excursion paths, since every path from  $a_0^0$  to  $a_0^{2p}$  contains  $p$  up-steps and  $p$  down-steps, so an up-step at height  $j_q$ , gaining a factor of  $k - j_q$ , is followed by a later downstep at the same height, gaining a factor of  $j_q$ . Now suppose  $X_t^n = S_{tn}/\sqrt{n}$  at a lattice point, then

$$E[|X_t^n|^{2p}] = E\left[\left|\frac{S_{nt}}{\sqrt{n}}\right|^{2p}\right] = n^{-p} m_{nt,2p} = \sum_{e \in \mathcal{E}_{2p}} \prod_{q=1}^p \left(t - \frac{j_q}{n}\right) j_q.$$

Now let  $M_{t,2p}$  be the  $2p$ -th moment of normalized Brownian motion at  $t$ . The centered normal variable with variance  $t$  has moment generating function

$$\begin{aligned} g(\lambda) &= e^{t\lambda^2/2} \\ g'(\lambda) &= t\lambda g(\lambda), \end{aligned}$$

and we follow the same argument, writing  $g^{(q)}(\lambda)/g(\lambda)$  as a polynomial in  $\lambda$  with coefficients  $b_j^q$ , obtaining the recurrence relation

$$\begin{aligned} b_0^0 &= 1 \\ b_j^0 &= 0 \text{ for } j > 0 \\ b_0^q &= b_1^{q-1} \\ b_j^q &= tb_{j-1}^{q-1} + (j+1)b_{j+1}^{q-1} \text{ for } j > 0. \end{aligned}$$

Again we obtain  $b_j^q$  by following lattice paths from  $(0, 0)$  to  $(q, j)$ , accumulating factors of  $j$  as we take down-steps, and we get

$$M_{t,2p} = g^{(2p)}(0) = b_0^{2p} = \sum_{e \in \mathcal{E}_{2p}} t^p \prod_{q=1}^p j_q,$$

over the same lattice paths, and therefore the same combinations  $j_q$ . Thus we have

$$E[|X_t^n - X_s^n|^{2p}] < K_p |t - s|^p$$

for all  $n$ , where  $K_p$  is the  $2p$ -th moment of a standard normal random variable.

We obtain the same result for the bridge by observing that the distribution is stochastically dominated by  $X_n$  away from zero. Indeed, the Radon-Nikodym derivative  $\lambda$  for the measure of  $|\beta|$  with respect to that of  $|X^n|$  is determined by counting the number of paths from a given height  $x$  to zero, which is decreasing in  $|x|$ . Therefore  $P_n[\{|\beta(t)| \geq x\}] \leq P[\{|X^n(t)| \geq x\}]$  for all  $x > 0$  and  $t > 0$ , and

$$E_n[|\beta(k/n)|^{2p}] < E[|X_{k/n}^n|^{2p}],$$

for all  $p > 0$ . To prove stochastic domination, for  $j > 0$ , let  $p_{n,j} = P[|X_{k/n}^n| = j/\sqrt{n}]$ , and let

$$\lambda_n(j) = \frac{P_n[|\beta(k/n)| = j/\sqrt{n}]}{p_{n,j}},$$

the discrete Radon-Nikodym derivative. Then

$$P_n[|\beta(k/n)| \geq j/\sqrt{n}] - P[|X_{k/n}^n| \geq j/\sqrt{n}] = \sum_{i=j}^{\infty} (\lambda_n(i) - 1)p_{n,i}.$$

If the sum is positive for any  $j$ , it must be for a  $j$  such that  $\lambda_n(j) > 1$ . But then all terms before that are positive, and the sum  $\sum_{i=j}^{\infty} (\lambda_n(i) - 1)p_{n,i}$  is positive, which is impossible since both are probability measures. Therefore  $|\beta|$  is stochastically dominated, as above.

Then, since  $\beta((j+k)/n) - \beta(j/n)$  has the same distribution as  $\beta(k/n)$ , we have

$$E_n[|\beta((j+k)/n) - \beta(j/n)|^{2p}] \leq K \left(\frac{k}{n}\right)^p,$$

for all  $n > 0$  and for all  $0 \leq k < j+k \leq 2n$ .

Next, by Chebyshev's inequality, we have

$$P_n[\{|\beta(t) - \beta(s)| > \epsilon^\alpha\}] \leq \frac{E_n[|\beta(t) - \beta(s)|^{2p}]}{\epsilon^{2p\alpha}}.$$

With  $\epsilon = 1/2^N$ ,  $s = j/(2^N)$  and  $t = (j+1)/(2^N)$ , consider

$$P_n[\cup_{j=0}^{2^N} \{|\beta((j+1)/2^N) - \beta(j/2^N)| > (1/2^N)^\alpha\}] \leq K_p 2^{N(1+p(2\alpha-1))},$$

which is summable over  $N$  for large enough  $p$ . Thus for  $M$  large enough, for any  $q$  close to one, we can get

$$P_n[\cup_{N=M}^{\infty} \cup_{j=0}^{2^N} \{|\beta((j+1)/2^N) - \beta(j/2^N)| > (1/2^N)^\alpha\}] < 1 - q.$$

From here follow the proof of Kolmogorov's continuity estimate in Karatzas and Shreve [21] to show that on a set of probability  $p_0$  close to one, each process is Hölder continuous with the same constant  $C$ .  $\square$

We now apply our result to conformal mapping. We consider the question of the size of the sets glued together across distant parts of the circle in terms of Hausdorff content.

Consider the following random times:

$$s_1 = \sup\{t : t \leq 1, \beta(t) \leq 1\}$$

$$s_2 = \sup\{t : t \leq 1, \beta(t) \leq 2\}$$

$$t_2 = \inf\{t : t \geq 1, \beta(t) \leq 2\}$$

$$t_1 = \inf\{t : t \geq 1, \beta(t) \leq 1\}$$

Let  $E$  be the event that we have

$$\frac{1}{4} \leq s_1 \leq s_2 \leq \frac{3}{4} \leq \frac{5}{4} \leq t_2 \leq t_1 \leq \frac{7}{4}.$$

More generally, given  $x \in [0, 2]$  and  $h > 0$ , let  $E_{x,h}$  be the event that  $\beta$  up-crosses  $(\beta(x) + \sqrt{h}, \beta(x) + 2\sqrt{h})$  in the interval  $(x, x+h)$ , stays above  $\beta(x) + 2\sqrt{h}$  on  $(x+h, x+2h)$ , and downcrosses the same interval on  $(x+2h, x+3h)$ .

Let  $F(C)$  be the event that  $\beta$  is Hölder- $\alpha$  with constant  $C$ . As the  $P_n \rightarrow P$ , the distribution of Brownian bridge, there exists some positive probability  $q$  such that  $P_n(E) > q$ , since the Brownian path will cross any threshold or stay above a given value with some positive probability (see Karatzas and Shreve, p. 95 [21]), so for  $C$  large enough,

$$P_n[E \cap F(C)] \geq p + q - 1 = \lambda > 0. \quad (1.16)$$

Let  $G = E \cap F(C)$  for  $C$  so chosen, and we consider paths in the event  $G$  in what follows. Given a lamination  $L = L_\beta$  for  $\beta \in G$ , for each  $1 \leq x \leq 2$ , there is a pair  $(\zeta_x, \xi_x) \in L$  with  $|\zeta_x - \xi_x| > 1$ , since by construction we have

$$\frac{1}{4} \leq s_1 \leq s_x \leq s_2 \leq \frac{3}{4} \leq \frac{5}{4} \leq t_2 \leq t_x \leq t_1 \leq \frac{7}{4},$$

and if  $\phi$  is the obvious map from  $[0, 2]$  to  $\mathbb{T}$ , we let  $\zeta_x = \phi(s_x)$  and  $\xi_x = \phi(t_x)$ , and we see they are on opposite sides of the circle. It is easy to see that  $d_\beta(s_x, t_x) = 0$ , so  $(\zeta_x, \xi_x) \in L$ , and if  $f$  is the conformal welding map for  $L$ , for each  $1 \leq x \leq 2$ ,  $f(\zeta_x) = f(\xi_x)$  and either  $|f(\zeta_x) - \zeta_x| > 1/2$  or  $|f(\xi_x) - \xi_x| > 1/2$ . Let  $B$  be the points

$$\left\{s : \frac{1}{4} \leq s \leq \frac{3}{4}, |f(\phi(s)) - \phi(s)| > \frac{1}{2}\right\},$$

and let  $B'$  be similarly defined for the second interval.

**Lemma 1.6.3.** *There exists a constant  $D > 0$  depending on  $C$  such that on  $G$ ,  $\Lambda_\alpha(A) > D$  for either  $A = B$  or  $A = B'$ .*

**Corollary 1.6.1.** *For all  $n$  large enough, on an event of some positive probability  $p$ , there exists a set  $A \subset \mathbb{T}$  of Hausdorff- $\alpha$  content at least  $D$  such that  $|f(z) - z| > 1/2$  for  $z \in A$ .*

*Proof.*  $\beta(B \cup B')$  covers the interval  $[a, b]$ , so if  $\mathcal{U}$  is a cover of  $B \cup B'$  by open intervals  $U$ , there is a cover  $\mathcal{U}' = \{[s_i, t_i]\}_{i=1}^m$  by sub-intervals such that  $[1, 2] \in \cup_i [\beta(s_i), \beta(t_i)]$ . Since we have  $|\beta(t) - \beta(s)| \leq C|t - s|^\alpha$  in  $G$ , then

$$\sum_{\mathcal{U}} \text{diam}(U)^\alpha \geq \sum_i |t_i - s_i|^\alpha \geq \frac{\sum |\beta(t_i) - \beta(s_i)|}{C} \geq \frac{b - a}{C},$$

and the lemma is proved. The corollary then follows from the construction of  $B$  and  $B'$ .  $\square$

**Corollary 1.6.2.** *For each  $0 < p < 1$ , there exist  $\eta > 0$ ,  $D > 0$  such that for all  $n > 0$ , on an event of probability greater than  $p$ , there exists a set  $A \subset \mathbb{T}$  of Hausdorff- $\alpha$  content at least  $D$  such that  $|f(z) - z| > \eta$  for  $z \in A$ .*

*Proof.* From the proof of Lemma 1.6.1, we can choose  $C$  large enough so that the sample paths are Hölder- $\alpha$  continuous with constant  $C$  for all  $n$  with probability arbitrarily large.

For small  $h$ , we have  $P_n[\cup_x E_{x,h}]$  arbitrarily close to one for large  $n$ . Indeed, for Brownian motion,  $P[E_{x,h}]$  is nonzero and does not depend on  $x$  or  $h$  by the invariance of Brownian motion under scaling and shifting, and since the distributions on successive intervals are independent,  $P[\cup_{x \in [0, \delta]} E_{x,h}]$  goes to 1 as  $h$  goes to zero for all  $\delta > 0$  by the law of large numbers, by checking successive intervals of width  $h$ . Then the distribution of Brownian bridge is arbitrary close to that of Brownian motion as  $\delta$  goes to zero, so by first fixing  $\delta$ , then  $h$ , then  $n$ , we obtain the desired result.

By the same arguments as in Lemma 1.6.3, the corollary holds for the chosen probability and constants.

### 1.6.2 Pfluger's Theorem

We next require the following version of Pfluger's Theorem:

**Lemma 1.6.4.** *Let  $U = \{z : 0 < \text{Re } z < 1, 0 < \text{Im } z < 1\}$  with left side  $L$  and right side  $R$ . Let  $A \subset R$  be a finite union of closed arcs, and let  $\Gamma$  be the family of curves joining  $L$  to  $A$  in  $U$ . Then are  $c > 0$ ,  $b \in \mathbb{R}$  such that*

$$\log \text{cap } A \leq -\frac{c}{\text{mod } \Gamma} + b \tag{1.17}$$

*Proof.* Our proof uses a reflection principle from [1]. It requires little additional work to provide a complete proof, so the reference is not required for what follows. First we consider the region

$$\Omega = \{z : 1 < |z| < 2, \text{Im } z > 0\},$$

where  $L$  is the lower half-circle of the boundary,  $R$  is the upper,  $A \subset R$  is a finite union of closed arcs, and  $\Gamma$  is the family of curves joining  $L$  to  $A$  in  $\Omega$ . We now show (1.17) for this new region. This is accomplished with the following consideration:

**Proposition 1.6.2.** *Let  $T(z) = \bar{z}$ . Let  $\hat{\Omega} = \{1 < |z| < 2\}$ , and let  $\hat{A} = A \cup T(A)$  be the union of  $A$  with its reflection about  $\mathbb{R}$ . Then let  $\hat{\Gamma}$  be the family of curves joining  $\partial\mathbb{D}$  to  $\hat{A}$ . Then*

$$\text{mod } \hat{\Gamma} = 2 \text{ mod } \Gamma \tag{1.18}$$

*Proof.* First let  $\Gamma'$  be  $\Gamma$  extended to allow curves  $\gamma \in \bar{\Omega}$  that intersect  $\mathbb{R}$ . If  $\rho : \bar{\Omega} \rightarrow \mathbb{R}$  is admissible for  $\Gamma'$ , then define  $\hat{\rho}$  to be  $\rho$  on  $\bar{\Omega}$  and  $\rho \circ T$  on the reflected region. Then, given a curve  $\hat{\gamma} \in \hat{\Gamma}$ , let  $\gamma$  be the curve  $T^{-1}(\hat{\gamma}) \subset \bar{\Omega}$ .  $\gamma \in \Gamma'$ , so

$$\int_{\hat{\gamma}} \hat{\rho} ds = \int_{\gamma} \rho ds \geq 1,$$

and  $\hat{\rho}$  is admissible for  $\hat{\Gamma}$ . Then, since

$$\int_{\hat{\Omega}} \hat{\rho}^2 dA = 2 \int_{\Omega} \rho^2 dA,$$

by taking the infimum over metrics for  $\Gamma'$ , we see that

$$\text{mod } \Gamma' \geq \frac{1}{2} \text{ mod } \hat{\Gamma}.$$

We must show that the modulus of the extended family  $\Gamma'$  is equal to that of  $\Gamma$ . Since  $A$  and  $L$  are unions of closed intervals on the boundary of  $\Omega$ , a Jordan domain, there exists a conformal map to a slit rectangle  $U$  such that  $L$  maps to the left side,  $A$  to the right side, and  $\partial\Omega \setminus L \cup A$  to a union of horizontal slits. This construction can be found in the proof of Theorem IV.4.1 in [17]. Then for the images of both  $\Gamma$  and  $\Gamma'$  under the conformal map, the modulus is equal to the ratio of side lengths by the standard calculation for the rectangle. By conformal invariance of the modulus, we have  $\text{mod } \Gamma = \text{mod } \Gamma'$ , and

$$\text{mod } \Gamma \geq \frac{1}{2} \text{mod } \hat{\Gamma}.$$

On the other hand, if  $\rho$  is admissible for  $\hat{\Gamma}$ ,  $(\rho + \tilde{\rho})/2$  is admissible on  $\Gamma$ , and

$$\begin{aligned} \text{mod } \Gamma &\leq \int_{\Gamma} \frac{1}{4} (\rho + \tilde{\rho})^2 dA = \frac{1}{4} \int_{\Omega} \rho^2 + \tilde{\rho}^2 dA + \frac{1}{4} \int_{\Omega} \rho \tilde{\rho} + \tilde{\rho} \rho dA \\ &= \frac{1}{4} \int_{\hat{\Omega}} \rho^2 dA + \frac{1}{4} \int_{\hat{\Omega}} \rho \tilde{\rho}^2 dA \\ &\leq \frac{1}{2} \int_{\hat{\Omega}} \rho^2 dA, \end{aligned}$$

and

$$\text{mod } \Gamma \leq \frac{1}{2} \text{mod } \hat{\Gamma}.$$

□

Now Pfluger's theorem [32], gives us

$$\log\left(\frac{\sqrt{2}}{3} \text{cap } \hat{A}\right) \leq -\frac{\pi}{\text{mod } \hat{\Gamma}} = -\frac{\pi}{2 \text{mod } \Gamma}, \quad (1.19)$$

and  $\text{cap } A < \text{cap } \hat{A}$  by monotonicity, giving us the desired result for the region  $\Omega$ . Now let  $U = l(\log(\Omega))$ , where  $l$  is an affine linear map rescaling the rectangle  $\log(\Omega)$  to the unit square given in the statement of the lemma. The map scales distances between points of  $A$  by at most a bounded factor, so the capacity is scaled by at most the same linear factor, which can be seen directly from the definition of capacity. The modulus of  $\Gamma$  is unchanged by the conformal map, and the affine linear map also has a predictable effect on the modulus: if  $l$  transforms a rectangle of width  $w$  and height  $h$  to  $U$ , and  $\Gamma$  is a curve family in  $R$ , then

$$\frac{w}{h} \text{mod } \Gamma \leq \text{mod } l(\Gamma) \leq \frac{h}{w} \text{mod } \Gamma,$$

assuming that  $w \geq h$ . To see that this is true, consider a curve  $\gamma$  in  $U$ , and let  $\rho$  be a metric on  $U$ . Integrating a metric  $\rho$  over the curve  $l(\gamma)$  gives a change-of-variables factor  $h < \frac{d\gamma}{dl(\gamma)} < w$ , so the admissible metrics vary from those on  $R$  by at most these factors. Integrating  $\rho^2$  over  $U$  with the change of variables  $dA = wh \cdot dl(A)$  gives the desired result. Thus the inequality (1.19) applies as well, up to a fixed constant factor, to the unit rectangle  $U$ , and the result is proved. □

We now prove our theorem. First we prove a slightly simpler lemma to get the idea of the proof:

**Lemma 1.6.5.** *If  $P$  is a subsequential limit of  $P_n$ , then  $P \neq \delta_{\text{id}}$ .*

*Proof.* Suppose  $P_n \rightarrow \delta_{\text{id}}$ . Then for  $m > 0, \epsilon > 0$ , let  $E_{m,\epsilon}$  be the event

$$\sup_{\{|z| \geq 1 + \frac{1}{m}\}} |f(z) - z| < \epsilon, \quad (1.20)$$

and  $P_n[E_{m,\epsilon}] \rightarrow_n 1$ . Next let  $A = \{z \in \mathbb{T} : |f(z) - z| > \frac{1}{2}\}$ , and let  $F_D$  be the event that  $\Lambda_\alpha(A) > D$ . By Lemma 1.6.3, we can choose  $D$  so that there exists  $p_0 > 0$  such that for all  $m > 0, \epsilon > 0$ , and  $n$  large enough,

$$P_n[E_{m,\epsilon} \cap F_D] > p_0.$$

Then let  $U = \{z : 1 < |z| < 1 + 1/m\}$ , and divide  $U$  into  $m$  boxes  $U_k$ , each covering an arc of length  $2\pi/m$  on  $\partial\mathbb{D}$  and having radial segments as sides. Let  $A_k = A \cap \overline{U_k}$  for  $k = 1, \dots, m$ . Hausdorff content is sub-additive, so  $\sum_k \Lambda_\alpha(A_k) > D$ . On the other hand,  $\Lambda_\alpha(A_k) \leq (2\pi/m)^\alpha$ , so if  $d$  is the number of  $A_k$  such that  $\Lambda_\alpha(A_k) > D/2m$ , we have

$$\begin{aligned} D &\leq (m-d) \frac{D}{2m} + d \left(\frac{2\pi}{m}\right)^\alpha \\ \frac{D}{2} &\leq d \left(\left(\frac{2\pi}{m}\right)^\alpha - \frac{D}{2m}\right), \end{aligned}$$

and  $d \geq cm^\alpha$  for some constant  $c > 0$ , for  $m$  large enough.

Let  $\Gamma_k$  be the curve family joining  $\{|z| = 1 + 1/m\} \cap \overline{U_k}$  to  $A_k$  inside  $\overline{U_k}$ , and let  $A'_k = mA_k$ ,  $\Gamma' = m\Gamma_k$ , that is, rescale everything by  $m$  to obtain a box of constant size. Then by Lemma 1.6.4,

$$\log \text{cap } A'_k \leq -\frac{c}{\text{mod } \Gamma'_k} + b. \quad (1.21)$$

Then  $\text{cap } A'_k = m \text{cap } A_k$ ,  $\text{mod } \Gamma_k = \text{mod } \Gamma'_k$ , and by Pommerenke, p.225 [32], we have

$$\Lambda_\alpha(A_k)^{\frac{1}{\alpha}} \leq M \text{cap } A_k,$$

so combining inequalities, we obtain

$$\text{mod } \Gamma_k \geq \frac{-1/c}{\log m - b + \log(1/M) + \log \Lambda_\alpha(A_k)/\alpha}. \quad (1.22)$$

Sum over the  $A_k$  such that  $\Lambda_\alpha(A_k) > D/2m$  to obtain

$$\sum_k \text{mod } \Gamma_k \geq \frac{cm^\alpha}{\log m}, \quad (1.23)$$

for large  $m$ .

On the other hand, for a curve  $\gamma \in \Gamma_k$  joining  $\zeta \in A_k$  to  $\xi$  with  $|\xi| = 1 + 1/m$ ,

$$\int_\gamma |f'(z)| |dz| \geq |f(\zeta) - f(\xi)| \geq |f(\zeta) - \zeta| - |\zeta - \xi| - |\xi - f(\xi)| > \frac{1}{2} - \frac{2\pi + 1}{m} - \epsilon,$$

so  $3|f'(z)|$  is admissible for  $\Gamma_k$ . Then

$$\sum_k \text{mod } \Gamma_k \leq \sum_k \int_{U_k} 9|f'(z)|^2 dA \leq 9\pi(1 + 1/m + \epsilon)^2, \quad (1.24)$$

since  $f$  is a conformal map of  $D_m \setminus \bar{D}$  into  $\{|z| < 1 + \frac{1}{m} + \epsilon\}$ , and  $\text{Area } f(U) = \int_U |f'(z)|^2 dA$ . Combining (1.23) and (1.24), for properly chosen  $m$  and  $\epsilon$ , we obtain a contradiction, and the lemma is proved.  $\square$

To prove the theorem, we suppose  $P(f = \text{id}) = q > 0$ , in which case we let  $p > 1 - q$ , and by Corollary 1.6.2 there exist  $\eta, D$  such that on a set  $F_{\eta, D}$  with probability at least  $p$ , for all  $n$ , there exists  $A \in \mathbb{T}$  with  $\Lambda_\alpha(A) > D$  and  $|f(\zeta) - \zeta| > \eta$  on  $A$ . Then for each  $\epsilon > 0$ , if  $E_{m, \epsilon}$  is as above,  $P_n[E_{m, \epsilon}] \geq q$  for  $n$  large enough. Now we proceed as above, on the set  $E_{m, \epsilon} \cap F_{\eta, D}$ , which has positive probability for large  $n$ , substituting  $\eta$  for the fixed distance  $1/2$ , using the metric  $K|f'|$  where  $K$  is a constant greater than  $1/\eta$ . We obtain the same contradiction, proving the theorem.  $\square$

### 1.7 The first coefficient

We conclude by examining the distribution of the first free coefficient  $b_1$  in the asymptotic expansion of the conformal map  $f$ . The main purpose for doing so is that it might be possible to derive an invariance principle on welded CRTs that gives a relationship on  $b_1$  through self-similarity, which seems to be the most promising way to obtain usable distributions for the map. Furthermore,  $b_1$  has a strong geometric interpretation. If  $f$  is the conformal map

$$f(z) = z + \frac{a^2}{4z}, \quad (1.25)$$

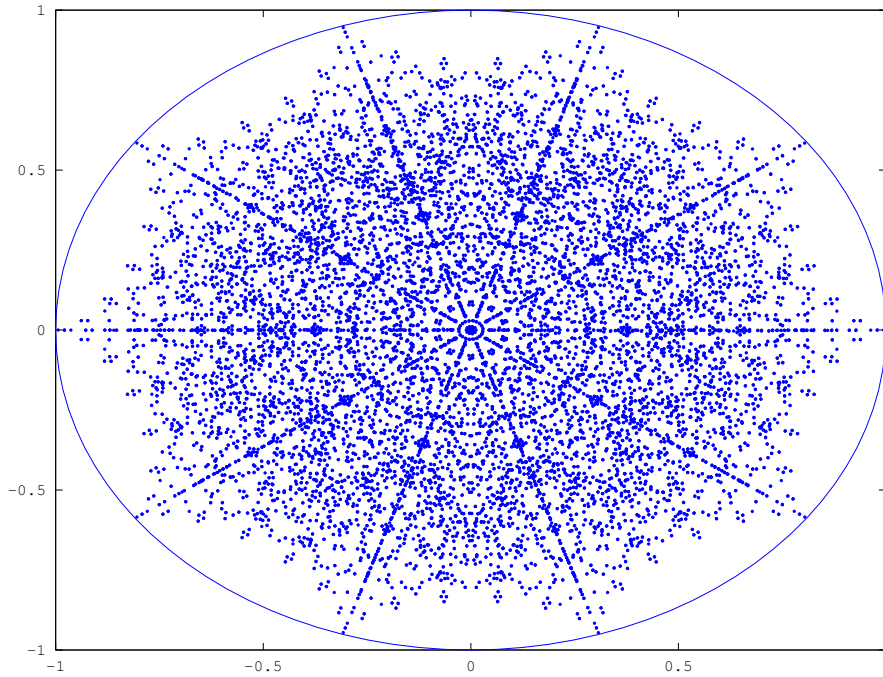


Figure 1.7: A plot of all the values of  $b_1$  for ordered trees with 10 edges.

then  $a$  and  $-a$  are the focal points of an ellipse that is the complement of the image domain. In this example, and in general for conformal maps, we have the constraint  $|b_1| \leq 1$ . The case  $b_1 = 0$  corresponds to the trivial map, and  $b_1 = 1$  corresponds to the conformal map to the line interval  $[-2, 2]$ . In general, we can think of  $b_1$  as a sort of second moment of the plane set  $\Gamma$ . By calculating numerical approximations to all trees of a given number of vertices, we can visualize the distribution of  $b_1$  for small values of  $n$ . Figure 1.7 shows all values of  $b_1$  for  $n = 10$ , and is a striking and motivating image. We do not yet have a conjectured limiting distribution, although it seems to be very nearly uniform near the center of the disk, decaying rapidly between  $r = .5$  and  $r = .6$ . A useful direction of effort would be to try to determine the theoretical distribution of  $b_1$  from self-similarity properties of the CRT.

### 1.7.1 Complex moments

The coefficient  $b_1$  has a deeper relationship to the geometry of the tree, suggesting a deep combinatorial relationship between all the coefficients of  $f$  and  $p$  that we explore here.

Complex moments of the form  $M_k = \int_{\Gamma} z^k \omega(dz) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta)^k |d\zeta|$  can be used to investigate the geometry of the plane trees. In this section we investigate the connection between the values  $M_k$  and the coefficients of the conformal maps  $f, g$  and the Shabat polynomial  $p$ . In fact, we find that  $M_k$  takes exactly the value of the Riemann Sum approximation  $m_k = \frac{1}{2^n} \sum_i^{2^n} d_i v_i^p$  for at least the first four values of  $k$ . We conjecture that  $m_k = M_k$  for all  $k > 0$ . In fact, this seems to be a property of polynomials in general, except for the interpretation as an integral over a tree. There are well-known identities for symmetric polynomials generalizing the substitutions used here, so further investigation should start there.

Let  $f(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$  in a neighborhood of infinity be the conformal welding map for a degree  $n$  Shabat polynomial  $p$ . Let  $g(z) = f^{-1}(z) = z + \sum_{n=1}^{\infty} c_n z^{-n}$  in a neighborhood of infinity. For this discussion, we require the first three coefficients,  $c_1, c_2, c_3$ , and we observe that by expanding the power series of the relation  $f(g(z)) = z$ , we obtain our first set of relations,

$$\begin{aligned} c_1 &= -b_1, \\ c_2 &= -b_2, \\ c_3 &= -b_3 - b_1^2. \end{aligned}$$

Let  $\{v_i\}_{i=1}^{n+1} = \{f(\zeta_{i,2n})\}_{i=1}^{2n}$  be the vertices of  $\Gamma$ . We write  $f(\zeta_{i,2n}) = f_i$  in what follows.

If  $p(z)$  is the Shabat polynomial, let  $P(z) = p^2(z) - 4 = \prod_{i=1}^{2n} (z - f_i)$ . If  $P(z) = \sum_{j=0}^{2n} a_j z^{2n-j}$ , we have a relation between the values  $f_i$  and  $a_j$ , which we express as follows:

$$a_j = (-1)^j \sum_{|S|=j} \prod_{i \in S} f_i$$

We also obtain  $P$  by the composition  $P(z) = g(z)^{2n} + g(z)^{-2n}$ . We consider the expansion at infinity to obtain the coefficients  $a_j$  in terms of the coefficients of  $g$ . Expanding, we get

$$g(z)^{2n} = z^{2n} + 2nc_1z^{2n-2} + 2nc_2z^{2n-3} + (n(2n-1)c_1^2 + 2nc_3)z^{2n-4} + \dots,$$

and therefore,

$$\begin{aligned} a_1 &= 0, \\ a_2 &= 2nc_1, \\ a_3 &= 2nc_2, \\ a_4 &= n(2n-1)c_1^2 + 2nc_3. \end{aligned}$$

Exploiting  $a_1 = \sum f_i = 0$ , we can write the  $M_k$  in terms of the  $a_i$ , as shown:

$$\begin{aligned} 0 &= \left(\sum_i f_i\right)^2 = \sum_i f_i^2 + \sum_{i \neq j} f_i f_j \\ &= 2nm_2 + 2a_2, \end{aligned}$$

so  $m_2 = -a_2/n$ , then

$$\begin{aligned} 0 &= \left(\sum_i f_i\right)^3 = \sum_i f_i^3 + 3 \sum_{i \neq j} f_i^2 f_j + 6a_3 \\ &= 2nm_3 + 3 \sum_i f_i^2 \left(\sum_j f_j - f_i\right) - 6a_3 \\ &= 2nm_3 - 6nm_3 - 6a_3, \end{aligned}$$

so  $m_3 = -3a_3/2n$ . Finally,

$$\begin{aligned}
0 &= \left(\sum_i f_i\right)^4 = \sum_i f_i^4 + 4 \sum_{i \neq j} f_i^3 f_j + 3 \sum_{i \neq j} f_i^2 f_j^2 + 6 \sum_{i \neq j \neq k} f_i^2 f_j f_k + 24a_4 \\
&= 2nm_4 + 4 \sum_i f_i^3 \left(\sum_j f_j - f_i\right) + 3 \sum_i f_i^2 \left(\sum_j f_j^2 - f_i^2\right) + \\
&6 \sum_i f_i^2 \left(2a_2 - 2f_i \sum_j f_j + 2f_i^2\right) + 24a_4 \\
&= 2nm_4 - 8nm_4 + 12n^2 m_2^2 - 6nm_4 + 24nm_2 a_2 + 24nm_4 + 24a_4 \\
&= 12nm_4 - 12a_2^2 + 24a_4,
\end{aligned}$$

so  $m_4 = (a_2^2 - 2a_4)/n$ . Collecting these identities, we have

$$\begin{aligned}
m_1 &= 0, \\
m_2 &= -\frac{a_2}{n}, \\
m_3 &= -\frac{3a_3}{2n}, \\
m_4 &= -\frac{2a_4 - a_2^2}{n}.
\end{aligned}$$

Finally, we can evaluate the integral  $M_k$  in terms of the coefficients of  $f$ , since

$$\frac{1}{2\pi} \int_{\mathbb{T}} f(\zeta)^k |d\zeta| = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)^k}{\zeta} d\zeta,$$

so we are looking for the constant terms of the expansion of  $(z + b_1/z + \dots)^k$ , since all other terms vanish.

For  $k = 2, 3, 4$ , we get the following:

$$\begin{aligned}
M_2 &= 2b_1 \\
M_3 &= 3b_2 \\
M_4 &= 4b_3 + 6b_1^2.
\end{aligned}$$

It is immediately clear that  $m_2 = M_2$  and  $m_3 = M_3$ . We perform the substitutions for the last identity:  $m_4 = -\frac{2a_4 - a_2^2}{n} = -\frac{2n(2n-1)c_1^2 - 4nc_3 - 4n^2c_1^2}{n} = -4c_3 + 2c_1^2 = 4b_3 + 6b_1^2 = M_4$ .

## Chapter 2

**HYDRODYNAMIC LIMIT OF A BOUNDARY-DRIVEN  
REFLECTING EXCLUSION PROCESS AND A STEFAN PROBLEM**

**2.1 Preliminaries**

This paper is primarily concerned with the hydrodynamic limit of a exclusion process  $Z_t$  on a bounded one-dimensional lattice of grid size  $1/N$  with the following dynamics: a particle  $p$  moves into an adjacent unoccupied site at rate proportional to the size of the block of occupied particles of which  $p$  is a member. We think of each particle in the block as having internal energy transferred to the outermost particle elastically. In addition, the process is boundary driven: at constant rate, the leftmost block of particles (possibly empty) is shifted to the right one position, and a new particle is added to the vacant first position. Finally, particles are killed when they move to the rightmost site.

We call the model boundary-driven reflecting Brownian exclusion. Recall that simple exclusion process can also be considered a model for Brownian motion, but in this case particles in adjacent sites are considered to swap places when diffusion should occur, whereas in the current model particles undergo elastic reflection, transferring their energy to the outermost particle. We are interested in the limiting shape of the empirical distribution for all times as the grid size scales to zero and the dynamics scale appropriately, and in theorem 5.1 we prove that this hydrodynamic limit satisfies the differential equation

$$\partial_t z(x, t) = \partial_x \left( \frac{1}{(1 - z(x, t))^2} \partial_x z(x, t) \right), \quad (2.1)$$

with appropriate boundary conditions. This particle system was chosen to approximate the system of one-dimensional crowded Brownian spheres defined in [11], and the connection is of interest because the hydrodynamic limit of the exclusion process  $Z_t$  matches the conjectured hydrodynamic limit of the Brownian process in that paper. Because of the connection, and because the method of proof in that paper also describes the key isomorphism that we use to derive the limiting equation, we briefly describe that process and the key transformation.

H. Rost [33] also considered reflecting Brownian intervals and derived the hydrodynamic limit above in the case of the entire real line.

Consider intervals  $I_t^k = (B_t^k, B_t^k + 1/N)$ , such that when  $B_t^k \geq 0$  and  $|B_t^k - B_t^j| > 1/N$  for all  $j$  such that  $j \neq k$ ,  $B_t^k$  moves as independent Brownian motion. Intervals  $I_t^k$  reflect instantaneously and symmetrically, and are killed when  $B_t^k + 1/N = 1$ . Finally, for  $k$  greater than some  $k_0$ ,  $B_0^k = -(k - k_0)/N$  and  $B_t^k = B_0^k + at$  until  $B_t^k = 0$ , so that particles continuously enter the interval at rate  $aN$ . In order to derive the limiting stationary distribution in the case  $k_0 = 0$ , the authors of [11] consider the transformations  $T_t : (-\infty, 1] \rightarrow [0, S_t]$  with

$$T_t(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x - \int_0^x \mathbb{1}_{\cup_k I_t^k}(z) dz & \text{for } 0 < x \leq 1. \end{cases}$$

$T_t$  maps  $I_t^k$  to a point  $C_t^k$ , and simply translates unoccupied space, so the  $C_t^k$  move as independent, symmetrically reflecting Brownian motions with drift  $-adt$ , due to the continually inserted intervals. Furthermore,  $T_t(1) = S_t$  is a random boundary that changes proportionally to the number of particles entering or leaving the system. Since the distribution of symmetrically reflecting Brownian motions is identical to that of independent particles, we are reduced to the case of independent  $dA_t^k = dW_t^k - adt$ , reflecting at 0 and killed at  $S_t$ . This leads us to conjecture the following hydrodynamic limit, which is a form of the well-known Stefan melting-freezing problem:

**Definition 2.1.1.** *Given  $s_0 \geq 0$ ,  $v_0 \in C^1([0, s_0])$  with  $s_0 = 1 - \int_0^{s_0} v_0(x) dx$ , and  $a > 0$ , a pair  $(v, s)$  such that  $s \in C^1([0, T])$ ,  $s(0) = s_0$ ,  $s > 0$ , and  $v \in C^2(D_T) \cap C^1(\overline{D_T})$ , where  $D_T = \{(x, t) : 0 < x < s(t), 0 < t \leq T\}$ , satisfying*

$$\partial_t v(x, t) = \partial_{xx} v(x, t) + a \partial_x v(x, t) \quad 0 < x < s(t), t > 0, \quad (2.2)$$

$$v(x, 0) = v_0 \quad 0 \leq x \leq s(0), \quad (2.3)$$

$$(\partial_x v(x, t) + av(x, t))|_{x=0} = -a \quad t > 0, \quad (2.4)$$

$$v(s(t), t) = 0 \quad t > 0, \quad (2.5)$$

$$s(t) = 1 - \int_0^{s(t)} v(x, t) dx \quad t \geq 0, \quad (2.6)$$

is called a solution to the Stefan problem 2.2-2.6.

Condition (2.6) is more familiar in the differential form  $s'(t) = -a - \partial_x u(x, t)|_{s(t)}$ . The Stefan problem has been well studied in many forms, though perhaps not this exact form. The book [29] by Meirmanov is an excellent reference. In particular, with  $a = 0$ , this equation is the classical one-dimensional, one-phase melting problem, where in the region  $0 \leq x < s(t)$ ,  $v$  represents the temperature of water above freezing, and  $x \geq s(t)$  represents a region of ice with temperature 0. Strong existence and uniqueness of this elementary case is covered in Cannon's book [13]. We should also note that the condition  $v_0 \in C^1([0, 1])$  may not be necessary, but serves only to make the definitions simpler. Indeed, the main theorem below holds whenever the initial condition is in  $L^2$  and the solution satisfies the integral form of the solution derived in section 3. The Stefan problem has been studied in a probabilistic context as well, as a hydrodynamic limit by Chayes and Swindle [14], Gravner and Quastel [18], and Landim and Valle [23]. In [14] and [23], the particle model is simple exclusion, with different particle types representing the liquid and solid regions. Our model is close to that of Gravner and Quastel, who use the Stefan hydrodynamic limit to prove shape theorems for internal diffusion-limited aggregation, but our proofs are not similar and the application is different.

We now describe the second discrete process,  $Y_t^N$ , which corresponds to the distribution of the transformed Brownian motion process. For notational simplicity we will omit  $N$  from the process, but it will always be used in the corresponding probability measure  $P^N$ . Let

$$A_N = \left\{ \frac{1}{2N}, \dots, \frac{2j+1}{2N}, \dots, \frac{2N-1}{2N} \right\}.$$

Our state space for  $Y_t$  is the subset  $\mathcal{E}_N$  of  $\mathbb{N}^{A_N}$  such that for  $\eta \in \mathcal{E}_N$ ,  $\eta_x$  counts the number of particles at site  $x$  for a distribution of particles on  $A_N$  with the following restriction: there must be  $M$  particles, with  $M < N$ , and the particles may only occupy sites  $(2j+1)/(2N)$  with  $j < N-M$ . Let  $M_t$  be the number of particles at time  $t$ , and define a random boundary  $S_t$  with

$$S_t = 1 - \frac{M_t}{N} + \frac{1}{2N}.$$

At exponential random times with rate  $N^2$ , particles move as independent random walks, reflecting at zero. If a particle hits  $S_t$  at time  $t$ , it is killed (removed from the system), and  $S_t = S_{t-} + 1/N$ . In addition, there is a drift effect, occurring at rate  $aN$ , for  $a \geq 0$  constant, where every particle except those at zero shift one site towards the origin, an additional particle is added at  $1/(2N)$ , and  $S_t$  shifts one site left. The only state  $\eta \in \mathcal{E}_N$  for which this does not happen is  $\eta_{1/2N} = N - 1$ , in which case there is no change (think of the generated particle being immediately killed).

The sum of delta masses of weight  $Y_t/N$  at each site gives a measure  $\mu_{Y_t}$  or just  $\mu_t$ , depending on context, of mass less than one. The object  $\mu.$  is an element of the Skorohod space of right-continuous paths on the metric space  $\mathcal{M}$  of positive measures on  $[0, 1]$ ,  $D([0, 1], \mathcal{M})$ . Let  $P^N$  be the probability measure on right-continuous paths in  $\mathcal{E}_N$  that determines the process  $Y_t$ . Let the corresponding probability measure on  $D([0, 1], \mathcal{M})$  be  $Q^N$ . The hydrodynamic limit of the  $Q^N$  is the subject of our first theorem:

**Theorem 2.1.1.** *If for each  $P^N$ ,  $Y_0^N$  is a random variable with values a.s. in  $\mathcal{E}_N$  such that  $\sup_N E^N[1/N \sum_{x \in A_N} Y_0^N(x)^2] < \infty$ , with empirical measures  $\mu_0^N$  such that  $\{\mu_0^N(dx)\}_N$  converges weakly to the delta measure on a fixed absolutely continuous measure  $v_0(x)dx$ , then a limiting measure  $Q^\infty$  of the corresponding measures  $Q^N$  on  $D([0, T], \mathcal{M})$  exists and is the delta measure on the unique solution of a weak version of the Stefan problem (2.2)-(2.6) with initial data  $v_0$ , the density of which is a solution to that problem when such a solution exists.*

The weak version of the problem is defined in section 3. The methods used to derive the hydrodynamic limit are largely based on those in the book [22] by Kipnis and Landim, so we identify our contribution in two areas. First, although the Stefan problem has been well studied as a hydrodynamic limit, the exclusion process we describe and the application of the free boundary problem to such a process is new. Second, one would normally eliminate a drift term through transformation rather than build it into the process, but in our case the asymmetric process was necessary as the transformed version of the exclusion process. The unusual process and the simple setting in the unit interval allows for an interesting application of elementary harmonic analysis for the  $H_{-1}$  bound and the uniqueness proof.

Our general approach, following [18], is to make the free boundary go away by building it into a zero-range effect of a process  $X_t^N$  as described below. From the other direction, the differential equation transforms into a nonlinear integral equation which mirrors the form of the process. The proof of Theorem 2.1.1 is in four steps. In Lemma 2.2.2, we prove that the Markov process describes a relatively compact sequence of probability measures. Lemma 2.3.1 guarantees that any limit points lie in  $L^2([0, T] \times [0, 1])$  almost surely. Lemma 2.4.2 shows that such limit points must satisfy a weak version of equation (2.2)-(2.6). Finally, Lemma 2.5.1 proves that the solution to such an equation is unique. Combining these results, we see that the process converges to a measure which is the delta measure on the solution to an integral form of the problem which coincides with the solution to that problem when it exists.

In section 5, we show that the two discrete processes described in this section are in fact isomorphic, and use the isomorphism to prove the hydrodynamic limit (2.1) in Theorem 2.6.1.

## 2.2 Construction and Relative Compactness of the Markov Process

We construct a Markov process  $X_t^N$ , henceforth  $X_t$ , by defining its infinitesimal generator. For  $N > 0$  let  $A_N = \{1/(2N), 3/(2N), \dots, (2N-1)/(2N)\}$  and  $\mathcal{M}_N = \mathbb{N}^{A_N}$ . Let  $\mathcal{M}$  be the set of finite measures on  $[0, 1]$ , and we associate  $\eta \in \mathcal{M}_N$  with its empirical measure in  $\mathcal{M}$ ,  $\mu_\eta = \sum_{x \in A_N} \frac{\eta_x}{N} \delta_x$ . Consider the following generator on functions  $f : \mathcal{M}_N \rightarrow \mathbb{R}$ :

$$\begin{aligned} \mathcal{L}_N f(\eta) &= \mathcal{L}_N^1 f(\eta) + \mathcal{L}_N^2 f(\eta), \\ \mathcal{L}_N^1 f(\eta) &= N^2 \sum_{x \in A_N} \lambda(\eta_x) \left[ f(\eta^{x, x-1/N}) - f(\eta) + f(\eta^{x, x+1/N}) - f(\eta) \right], \\ \mathcal{L}_N^2 f(\eta) &= aN (f(\sigma(\eta)) - f(\eta)), \end{aligned}$$

where  $\Phi(x) = (x-1) \vee 0$ ,

$$\eta_y^{x, x+i/N} = \begin{cases} \eta_y - 1 & \text{for } y = x \text{ and } x + i/N \in A_N, \\ \eta_y + 1 & \text{for } y = x + i/N, \\ \eta_y & \text{otherwise,} \end{cases}$$

and

$$\sigma(\eta)_y = \begin{cases} \eta_y + \eta_{y+1/N} & \text{for } y = 1/(2N), \\ 0 & \text{for } y = (2N - 1)/(2N), \\ \eta_{y+1/N} & \text{otherwise.} \end{cases}$$

The appendix of [22] describes the construction of a Markov process  $X_t$  on  $\mathcal{M}_N$  from such a generator (such that  $d/dt E[f(X_t) | X_s = \eta] |_{t=s} = \mathcal{L}f(\eta)$ ), the idea being that states are changed at the minimum of exponential random times with rates  $N^2\Phi(\eta_x)$  and  $aN$ , to the corresponding state, with the minimum itself being an exponential random time, well defined almost surely.

Let  $\mathcal{F}_N$  be the set of states  $\eta$  such that  $\sum_{x \in A_N} \eta_x = N$ ,  $\eta_x > 0$  for  $x$  less than some  $b$  and  $\eta_x = 0$  for  $x \geq b$ . When  $X_0 \in \mathcal{F}_N$  a.s., the process  $\Phi(X_t)$  has the same dynamics as the process  $Y_t^N$  described in the introduction. The reflecting random walk effect is due to the fact that a pile at site  $x$  loses particles at a rate proportional to the height  $\Phi(X_t(x))$ , as if each particle is moving independently at rate  $N^2$ . We now consider the drift and random boundary. Let  $S_t = \min\{x \in A_N : X_t(x) = 0\}$ . When a particle moves from  $S_t - 1/N$  to  $S_t$ , which can only happen if  $X_t(S_t - 1/N) \geq 2$ , it is killed, in the sense that  $\Phi(X_t)$  no longer counts it, and the boundary  $S_t$  is incremented by  $1/N$ , as desired. With one exception, when the process moves from  $\eta$  to  $\sigma(\eta)$ ,  $\Phi(\sigma(\eta_x)) = \Phi(\eta_{x+1})$  except at  $x = 0$ , where  $\Phi(\sigma(\eta)_0) = \Phi(\eta_{1/N}) + \Phi(\eta_0) + 1$ , representing the generated particle. The only exception is the state  $X_t(1/2N) = N$ ,  $S_t = 3/2N$ , in which case  $\sigma(\cdot)$  has no effect, again as desired. Thus  $\Phi(X_t)$  with boundary  $S_t$  is identical to  $Y_t$  in its dynamics, and therefore in distribution, given corresponding initial distributions. Finally, note that  $\mathcal{F}_N$  is closed under the process, and can function as the state space, corresponding to the state space  $\mathcal{E}_N$  of  $Y_t$ .

Next, we calculate the generator applied to a linear functional. First, we have

$$\mathcal{L}_N \eta_x = \Delta_N^* \Phi(\eta_x) - aD_N^* \eta_x,$$

where  $\Delta_N$  and  $D_N$  are operators with  $N \times N$  matrices:

$$\Delta_N = N^2 \begin{pmatrix} -1 & 1 & 0 & 0 & \dots \\ 1 & -2 & 1 & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & 0 & 1 & -2 & 1 \\ \dots & 0 & 0 & 1 & -1 \end{pmatrix}$$

and

$$D_N = N \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Here  $A^*$  denotes the transpose of  $A$ . Note that  $\Delta_N$  and  $D_N$  represent the second symmetric and first left difference quotients, respectively, for functions  $f \in C^2([0, 1])$  with  $f'(0) = f'(1) = 0$ , in that for these functions,  $D_N f(x)$  converges to  $f''(x)$  as  $N$  goes to infinity, uniformly in  $A_N$ .

We briefly digress to discuss the choice of our class of functions. Throughout the paper, the functions  $f \in C^2([0, 1])$  with  $f'(0) = f'(1) = 0$  will be used for test functions, as they serve several purposes. First, as noted, it is these functions for which the operators above converge uniformly to  $f''$  and  $f'$ , respectively. Second, they are as a dense class of functions in  $C([0, 1])$  and can be used to define a metric on  $\mathcal{M}$ . Third, it is against these test functions that the weak form of the Stefan problem holds. Finally, the subfamily  $\{\sqrt{2} \cos(\pi kx)\}$  is an orthonormal basis for the discrete and continuous domains, and are used to prove essential  $L^2$  bounds later in the paper.

Returning to our calculation, for  $f : [0, 1] \rightarrow \mathbb{R}$ , let

$$\langle f, \eta \rangle_N = \frac{1}{N} \sum_{x \in A_N} f(x) \eta_x,$$

and by linearity of  $\mathcal{L}_N$ ,

$$\mathcal{L}_N \langle f, \eta \rangle_N = \frac{1}{N} \sum_{x \in A_N} f(x) (\Delta_N^* \Phi(\eta_x) - a D_N^* \eta_x) = \langle \Delta_N f, \Phi(\eta) \rangle_N - a \langle D_N f, \eta \rangle_N.$$

Next we prove relative compactness of  $X$  in  $D([0, T], \mathcal{M})$ . Let

$$\langle f, \mu \rangle = \int f(x) d\mu(x).$$

We define the metric on  $\mathcal{M}$ , the space of positive measures on  $[0, 1]$ , letting

$$d(\nu, \mu) = \sum_{j=0}^{\infty} \frac{|\langle f_j, \nu \rangle - \langle f_j, \mu \rangle| \wedge 1}{2^j},$$

where  $f_j$  are in  $C^2([0, 1])$  with  $f_j'(0) = f_j'(1) = 0$ , a set which is dense in  $C([0, 1])$ . When  $\mu(dx) = u(x)dx$ , we will use  $\langle f, u \rangle$  and  $\langle f, \mu \rangle$  interchangeably. Since each  $f_j$  is bounded,  $A \in \mathcal{M}$  is precompact if and only if  $\langle 1, \mu \rangle$  is bounded over  $A$  (each  $\langle f_j, \mu \rangle$  is bounded and converges along a subsequence, which implies subsequential convergence in the metric, and  $\mathcal{M}$  is complete with respect to  $d$ ). Note that the supports of all  $Q_N$  are contained in  $\{\mu : \langle 1, \mu \rangle = 1\}$ , a compact set. Convergence is weak convergence (convergence of expectations of continuous functions) in the space of probability measure on the Skohorod space  $D([0, T], \mathcal{M})$  of right-continuous functions on  $\mathcal{M}$ . For our purposes, convergence in this space can be convergence in  $\mathcal{M}$ , uniformly in  $t$ , since our limit points are continuous. Thus if  $f(\mu)$  is continuous on  $\mathcal{M}$ , then  $\int_0^T f(\mu_t) dt$  and  $\sup_{0 \leq t \leq T} f(\mu_t)$  are continuous on  $D([0, T], \mathcal{M})$ . By  $\{X_t\}_{t=0}^T$ , we will mean the Markov process on  $A_N$  with probability measures  $P^N$ . By  $\mu_t$  we will mean the corresponding coordinate process on  $\mathcal{M}_1$ , with probability measure  $Q^N$ , so that, for example, for  $A$  Borel,

$$P^N [\langle f, X_t \rangle_N \in A] = Q^N [\langle f, \mu_t \rangle \in A].$$

Relative compactness in this space follows from the following conditions, found in Chapter 2 of [22]. Let  $\mathcal{T}_T$  be the space of stopping times of the usual filtration, bounded by  $T$ .

**Lemma 2.2.1.** *Let  $Q^N$  be a sequence of probability measures on  $D([0, T], \mathcal{M})$ . The sequence is relatively compact (in the sense of weak convergence) if:*

1. *For every  $t$  in  $[0, T]$  and every  $\epsilon > 0$ , there is a compact  $K(t, \epsilon) \subset \mathcal{M}$  such that  $\sup_N Q^N [\mu_t \notin K(t, \epsilon)] \leq \epsilon$ .*

2.

$$\lim_{\gamma \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\tau \in \mathcal{T}_T, \theta \leq \gamma} P^N [\rho(\mu_\tau, \mu_{(\tau+\theta) \wedge T}) \geq \epsilon] = 0$$

We prove the following lemma by checking these conditions.

**Lemma 2.2.2.** *The sequence  $\{Q^N\}$  is relatively compact in  $D([0, T], \mathcal{M})$ .*

*Proof.* Note that condition (1) is automatically satisfied since, for all  $N$ ,  $P^N[\mu_t \notin \mathcal{M}_1] = 0$ . To check (2), we determine the square variation process for the  $P^N$ -martingale  $M_t = \langle f, X_t \rangle_N - \int_0^t \mathcal{L}_N \langle f, X_s \rangle_N ds$  for  $f \in C^2([0, 1])$  with  $f''(0) = f''(1) = 0$ . Exactly as in the proof of Theorem 3.2 of [12], we can show that  $M_t^2 - \int_0^t B_s ds$  is a martingale, where, if  $X_t = \eta$ ,

$$\begin{aligned} B_t &= \lim_{s \rightarrow 0^+} (1/s) E^N [(\langle f, X_{t+s} \rangle_N - \langle f, X_t \rangle_N)^2 | X_t] \\ &= N^2 \sum_{x \in A_N} \Phi(\eta_x) (\langle f, \eta^{x, x+1} \rangle_N - \langle f, \eta \rangle_N)^2 + (\langle f, \eta^{x, x-1} \rangle_N - \langle f, \eta \rangle_N)^2 \\ &\quad + aN (\langle f, \sigma(\eta) \rangle_N - \langle f, \eta \rangle_N)^2 \\ &= N^2 \sum_{x \in A_N} \Phi(\eta_x) \frac{1}{N^4} (D_N f(x+1)^2 + D_N f(x)^2) + a \frac{1}{N} \langle D_N f, \eta \rangle_N^2. \end{aligned}$$

Since  $|D_N f(x)| \leq \|f'\|_\infty$ , and  $1/N \sum_{x \in A_N} \Phi(\eta_x) \leq 1/N \sum_{x \in A_N} \eta_x = 1$ ,

$$|B_t| \leq \frac{1}{N} \|f'\|^2 + \frac{a}{N} \|f'\|^2$$

Fix  $\tau \in \mathcal{T}_T$ , and by  $\tau + \theta$  we will mean  $(\tau + \theta) \wedge T$ , and

$$E^N \left[ M_{\tau+\theta}^2 - \int_0^{\tau+\theta} B_s ds \mid \mathcal{F}_\tau \right] = M_\tau^2 - \int_0^\tau B_s ds,$$

so

$$E^N [M_{\tau+\theta}^2 - M_\tau^2] = E^N \left[ \int_\tau^{\tau+\theta} B_s ds \right] \leq \frac{C\theta}{N}.$$

Now

$$\begin{aligned} |\langle f, X_{\tau+\theta} \rangle - \langle f, X_\tau \rangle| &\leq |M_{\tau+\theta} - M_\tau| + \left| \int_\tau^{\tau+\theta} \mathcal{L}_N \langle f, X_s \rangle ds \right|, \\ P[|M_{\tau+\theta} - M_\tau| \geq \epsilon] &\leq \frac{E[(M_{\tau+\theta} - M_\tau)^2]}{\epsilon^2} \\ &= \frac{E[M_{\tau+\theta}^2 - M_\tau^2]}{\epsilon^2} \\ &\leq \frac{C\theta}{N\epsilon^2}, \end{aligned}$$

and  $|\int_{\tau}^{\tau+\theta} \mathcal{L}_N \langle f, X_s \rangle ds| \leq C\theta$ , since, again, the generator is an inner product of a derivative of  $f$  with a mass 1 measure. So

$$P^N [|\langle f, X_{\tau+\theta} \rangle - \langle f, X_{\tau} \rangle| \geq \epsilon] \leq C_{\epsilon}\theta.$$

To bound the metric by a given  $\epsilon$ , we only need consider finitely many  $f_k$  and choose  $\epsilon_k$  appropriately for each of these. The bound is independent of  $\tau$ , so

$$\sup_{\tau \in \mathcal{T}_T, \theta \leq \gamma} P^N [\rho(\mu_{\tau}, \mu_{\tau+\theta}) \geq \epsilon] \leq C_{\epsilon}\gamma$$

and (2) is satisfied. Thus  $Q^N$  is relatively compact and has subsequential limits.  $\square$

### 2.3 Limit Measures Are $L^2$ Almost Surely

In this section we prove two lemmas which together characterize the subsequential limits of the measures  $Q^N$ .

**Lemma 2.3.1.** *If for each  $P^N$ ,  $X_0$  is a random variable with values a.s. in  $\mathcal{F}_N$  such that  $\sup_N E^N [1/N \sum_{x \in A_N} X_0(x)^2] < \infty$ , a subsequential limit  $Q^{\infty}$  of the corresponding  $Q^N$  on  $D([0, T], \mathcal{M})$  has the property that  $\mu$  is absolutely continuous with density  $u \in L^2([0, T] \times [0, 1])$   $Q^{\infty}$ -a.s.*

We prove Lemma 2.3.1 by looking at the evolution of a variant of the  $H_{-1}$  norm. Let  $\psi_0 = 1$  and  $\psi_k = \sqrt{2} \cos(k\pi x)$ , and let  $\phi_0 = 0$  and  $\phi_k = \sqrt{2} \sin(k\pi x)$ . Recall that  $A_N = \{1/(2N), \dots, (2j+1)/(2N), \dots, (2N-1)/(2N)\}$  and let  $\psi_{\mathbf{k}}^{\mathbf{N}}, \phi_{\mathbf{k}}^{\mathbf{N}}$  represent the vectors of the respective functions evaluated on  $A_N$  and  $A_N - 1/(2N)$ , respectively. Then  $\{\psi_k\}_{k=0}^{\infty}$  has the following properties:

1.  $\Delta_N \psi_{\mathbf{k}}^{\mathbf{N}} = -4N^2 \sin^2(\pi k/(2N)) \psi_{\mathbf{k}}^{\mathbf{N}}$ .
2.  $D_N \psi_{\mathbf{k}}^{\mathbf{N}} = -2N \sin(\pi k/(2N)) \phi_{\mathbf{k}}^{\mathbf{N}}$ .
3.  $\{\psi_{\mathbf{k}}^{\mathbf{N}}\}_{k=0}^{N-1}$  is an orthonormal basis for  $\mathbb{R}^N$ .
4.  $\langle \eta_1, \eta_2 \rangle_N = \sum_0^{N-1} \langle \psi_k, \eta_1 \rangle_N \langle \psi_k, \eta_2 \rangle_N$  for  $\eta_1, \eta_2 \in \mathbb{N}^{A_N}$ .

$$5. \eta = \sum_0^{N-1} \langle \psi_k, \eta \rangle_N \psi_k^{\mathbf{N}} \text{ for } \eta \in \mathbb{N}^{A_N}.$$

The first three can easily be checked by calculations, noting that  $\psi_k(1/(2N)) - \psi_k(-1/(2N)) = \psi_k((2N+1)/(2N)) - \psi_k((2N-1)/(2N)) = 0$ , and the last two are consequences of (3). By  $\langle \phi_k, \eta \rangle_N$  we will always mean the offset sum  $1/N \sum_{x \in A_N} \phi_k(x - 1/(2N)) \eta_x$ . Let  $\lambda_{k,N} = 2N \sin(\pi k/(2N))$ . Next we consider, for  $\eta \in \mathbb{N}^{A_N}$ ,

$$h_N(\eta) = \sum_{k=1}^{N-1} \frac{\langle \psi_k, \eta \rangle_N^2}{\lambda_{k,N}^2}.$$

We restrict to  $\eta \in \mathcal{F}_N$  so that  $\langle \psi_0, \eta \rangle_N = 1$ . For  $1 \leq k < N$ ,  $\lambda_{k,n} > C > 0$ , independent of  $k$  and  $N$ , so

$$h_N(\eta) \leq \sum_{k=0}^{N-1} \langle \psi_k, \eta \rangle^2 - 1 \leq \frac{1}{N} \sum_{A_N} \eta_x^2 - 1.$$

Apply the generator

$$\begin{aligned} \mathcal{L}_N^1 \langle \psi_k, \eta \rangle^2 &= N^2 \sum_{x \in A_N} \Phi(\eta_x) (\langle \psi_k, \eta^{x,x+1} \rangle_N^2 - \langle \psi_k, \eta \rangle_N^2 + \langle \psi_k, \eta^{x,x-1} \rangle_N^2 - \langle \psi_k, \eta \rangle_N^2) \\ &= N^2 \sum_{x \in A_N} \Phi(\eta_x) [(2\langle \psi_k, \eta \rangle_N - N^{-1}[\psi_k(x+1/N) - \psi_k(x)])N^{-1}(\psi_k(x+1/N) - \psi_k(x)) \\ &\quad + (2\langle \psi_k, \eta \rangle_N - N^{-1}[\psi_k(x-1/N) - \psi_k(x)])N^{-1}(\psi_k(x-1/N) - \psi_k(x))] \\ &\leq -2\lambda_{k,N}^2 \langle \psi_k, \Phi(\eta) \rangle_N \langle \psi_k, \eta \rangle_N + \lambda_{k,N}^2/N, \end{aligned}$$

where the last bound follows from the inequalities  $N(\psi_k(x-1/N) - \psi_k(x)) \leq \lambda_{k,N}$  and  $\langle 1, \Phi(\eta) \rangle_N \leq 1$ . Then,

$$\begin{aligned} \mathcal{L}_N^2 \langle \psi_k, \eta \rangle_N^2 &= aN(\langle \psi_k, \sigma(\eta) \rangle_N^2 - \langle \psi_k, \eta \rangle_N^2) \\ &= a(2\langle \psi_k, \eta \rangle_N + N^{-1}\lambda_{k,N}\langle \phi_k, \eta \rangle_N)(-\lambda_{k,N}\langle \phi_k, \eta \rangle_N) \\ &\leq -2a\lambda_{k,N}\langle \psi_k, \eta \rangle_N \langle \phi_k, \eta \rangle_N + \lambda_{k,N}^2/N, \end{aligned}$$

and therefore we obtain

$$\begin{aligned}
\mathcal{L}_N h_N(\eta) &\leq -2 \sum_{k=1}^{N-1} \langle \psi_k, \Phi(\eta) \rangle_N \langle \psi_k, \eta \rangle_N \\
&\quad - 2a \sum_{k=1}^{N-1} \frac{\langle \psi_k, \eta \rangle_N \langle \phi_k, \eta \rangle_N}{\lambda_{k,N}} + C \\
&\leq -2 \langle \Phi(\eta), \eta \rangle^2 - C_1 \sum_{k=1}^{N-1} \frac{\langle \psi_k, \eta \rangle_N \langle \phi_k, \eta \rangle_N}{\lambda_{k,N}} + C_2.
\end{aligned}$$

Next, let  $b_k = \langle \psi_k, \eta \rangle_N$ , and consider

$$\begin{aligned}
-\sum_{k=1}^{N-1} \frac{\langle \psi_k, \eta \rangle_N \langle \phi_k, \eta \rangle_N}{\lambda_{k,N}} &= -\sum_{k=1}^{N-1} \frac{b_k \langle \phi_k, \sum_{j=0}^{N-1} b_j \psi_j \rangle_N}{\lambda_{k,N}} \\
&= -\sum_{k=1}^{N-1} \sum_{j=0}^{N-1} \frac{b_k b_j \langle \phi_k, \psi_j \rangle_N}{\lambda_{k,N}}.
\end{aligned}$$

We claim that

$$\begin{aligned}
\sum_{i=0}^{M-1} \sin\left(\frac{\pi k i}{N}\right) \cos\left(\frac{\pi j(2i+1)}{2N}\right) &= \\
\frac{1}{4} \csc\left(\frac{\pi(j+k)}{2N}\right) \left( \cos\left(\frac{2\pi j + \pi k}{2N}\right) - \cos\left(\frac{2\pi j M + 2\pi k M - \pi k}{2N}\right) \right) \\
+ \frac{1}{4} \csc\left(\frac{\pi(j-k)}{2N}\right) \left( \cos\left(\frac{2\pi j M - 2\pi k M + \pi k}{2N}\right) - \cos\left(\frac{2\pi j - \pi k}{2N}\right) \right), \quad (2.7)
\end{aligned}$$

and we prove by induction in  $M$ . Since the constant terms of the right hand side are the variable terms evaluated at  $M = 1$ , we see that we can check the difference  $S(M+1) - S(M)$  to obtain a telescoping sum on the right hand side. In other words, we require

$$\begin{aligned}
&\sin\left(\frac{\pi k M}{N}\right) \cos\left(\frac{\pi j(2M+1)}{2N}\right) = \frac{1}{4} \csc\left(\frac{\pi(j+k)}{2N}\right) \\
&\times \left( \cos\left(\frac{2\pi j M + 2\pi k M - \pi k}{2N}\right) - \cos\left(\frac{2\pi j(M+1) + 2\pi k(M+1) - \pi k}{2N}\right) \right) \\
&+ \frac{1}{4} \csc\left(\frac{\pi(j-k)}{2N}\right) \\
&\times \left( \cos\left(\frac{2\pi j(M+1) - 2\pi k(M+1) + \pi k}{2N}\right) - \cos\left(\frac{2\pi j M - 2\pi k M + \pi k}{2N}\right) \right).
\end{aligned}$$

Recalling that  $\cos(A+B) - \cos(A) = -2 \sin(B/2) \sin(A+B/2)$ , first with  $A = (2\pi j M + 2\pi k M - \pi k)/(2N)$  and  $B = \pi(j+k)/N$ , line one becomes

$$\frac{1}{2} \sin\left(\frac{2\pi j M + 2\pi k M + \pi j}{2N}\right),$$

and with  $A = (2\pi jM - 2\pi kM + \pi k)/(2N)$  and  $B = \pi(j - k)/N$ , line two becomes

$$-\frac{1}{2} \sin \left( \frac{2\pi jM - 2\pi kM + \pi j}{2N} \right).$$

Finally, the difference formula for sin is  $\sin(A + B) - \sin(A) = 2 \sin(B/2) \cos(A + B/2)$ , and substituting  $A = (2\pi jM - 2\pi kM + \pi j)/(2N)$  and  $B = 2\pi k/N$ , we get exactly the desired formula.

Therefore, for  $j - k$  even, we get

$$\begin{aligned} \langle \phi_k, \psi_j \rangle_N &= \frac{1}{4N} \csc \left( \frac{\pi(j+k)}{2N} \right) \left( \cos \left( \frac{2\pi j + \pi k}{2N} \right) - \cos \left( \frac{-\pi k}{2N} \right) \right) \\ &\quad + \frac{1}{4N} \csc \left( \frac{\pi(j-k)}{2N} \right) \left( \cos \left( \frac{\pi k}{2N} \right) - \cos \left( \frac{2\pi j - \pi k}{2N} \right) \right) \\ &= \frac{1}{4N} \csc \left( \frac{\pi(j+k)}{2N} \right) \left( \cos \left( \frac{\pi k}{2N} + \frac{\pi j}{N} \right) - \cos \left( \frac{\pi k}{2N} \right) \right) \\ &\quad + \frac{1}{4N} \csc \left( \frac{\pi(j-k)}{2N} \right) \left( \cos \left( \frac{-\pi k}{2N} \right) - \cos \left( \frac{-\pi k}{2N} + \frac{\pi j}{N} \right) \right), \end{aligned}$$

and using the identity  $\cos(A + B) - \cos(A) = -2 \sin(A + B/2) \sin(B/2)$ , we get

$$\langle \phi_k, \psi_j \rangle_N = \frac{-1}{2N} \left( \sin \left( \frac{\pi j}{2N} \right) + \sin \left( \frac{-\pi j}{2N} \right) \right) = 0.$$

For  $j - k$  odd, we get

$$\begin{aligned} \langle \phi_k, \psi_j \rangle_N &= \frac{1}{4N} \csc \left( \frac{\pi(j+k)}{2N} \right) \left( \cos \left( \frac{2\pi j + \pi k}{2N} \right) + \cos \left( \frac{-\pi k}{2N} \right) \right) \\ &\quad + \frac{1}{4N} \csc \left( \frac{\pi(j-k)}{2N} \right) \left( -\cos \left( \frac{\pi k}{2N} \right) - \cos \left( \frac{2\pi j - \pi k}{2N} \right) \right), \end{aligned}$$

this time with  $\cos(A + B) + \cos(A) = 2 \cot(A + B/2) \cos(B/2)$ ,

$$\langle \phi_k, \psi_j \rangle_N = \frac{1}{2N} \cos \left( \frac{\pi j}{2N} \right) \left( \cot \left( \frac{\pi(j+k)}{2N} \right) - \cot \left( \frac{\pi(j-k)}{2N} \right) \right)$$

$\cot(A + B) - \cot(A) = -\csc(A) \sin(B) \csc(A + B)$ , giving

$$\frac{\langle \phi_k, \psi_j \rangle_N}{2N \sin \left( \frac{\pi k}{2N} \right)} = -\frac{\cos \left( \frac{\pi j}{2N} \right) \cos \left( \frac{\pi k}{2N} \right)}{2N^2 \sin \left( \frac{\pi(j-k)}{2N} \right) \sin \left( \frac{\pi(j+k)}{2N} \right)}.$$

We can conclude that

$$\frac{b_k b_j \langle \phi_k, \psi_j \rangle_N}{2N \sin \left( \frac{\pi k}{2N} \right)} = -\frac{b_j b_k \langle \phi_j, \psi_k \rangle_N}{2N \sin \left( \frac{\pi j}{2N} \right)},$$

and, after canceling pairs for  $j > 0$ , and recalling that  $b_0 = 1$ , the double sum reduces to

$$\begin{aligned} -\sum_{k=1}^{N-1} \sum_{j=0}^{N-1} \frac{b_k b_j \langle \phi_k, \psi_j \rangle_N}{\lambda_{k,N}} &= -\sum_{k=1}^{N-1} \frac{b_k \langle \phi_k, \psi_0 \rangle}{\lambda_{k,N}} \\ &= \sum_{k=1}^{N-1} \frac{b_k \sigma(k) \cos\left(\frac{\pi k}{2N}\right)}{2N^2 \sin^2\left(\frac{\pi k}{2N}\right)}, \end{aligned}$$

where  $\sigma(k) = k \bmod 2$ . Next we obtain an upper bound for this expression.  $b_k \leq \sqrt{2}b_0$ ,  $\cos(x) \leq 1$  and  $2N^2 \sin^2\left(\frac{\pi k}{2N}\right) \geq Ck^2$  for  $1 \leq k \leq N-1$ , so

$$-\sum_{k=1}^{N-1} \sum_{j=0}^{N-1} \frac{b_k b_j \langle \phi_k, \psi_j \rangle_N}{\lambda_{k,N}} \leq C \sum_{k=1}^{\infty} \frac{1}{k^2} \leq C.$$

Finally, noting that  $\langle \eta, \eta \rangle_N \leq \langle \Phi(\eta), \eta \rangle_N + 1$  we obtain an estimate for the generator of  $h_N$ ,

$$\mathcal{L}_N h_N(\eta) + 2\langle \eta, \eta \rangle_N \leq C.$$

$h_N(X_t) - \int_0^t \mathcal{L}_N h_N(X_s) ds$  is a martingale, so

$$E_N \left( h_N(X_T) + 2 \int_0^T \langle X_t, X_t \rangle_N dt \right) \leq E_N(h_N(X_0)) + CT \quad (2.8)$$

and since  $h_N \geq 0$ ,  $E_N(\int_0^T \langle X_t, X_t \rangle_N dt) \leq E_N(h_N(X_0)) + CT$ , where  $C$  does not depend on  $N$ . We can conclude as in Corollary 5.6.3 in [22] (proof provided in the appendix) that when

$$\sup_N E_N(h_N(X_0)) \leq \sup_N E_N \left[ \frac{1}{N} \sum_{A_N} X_0(x)^2 \right] < \infty,$$

limit points  $Q^*$  of the sequence  $\{Q^N, N \geq 1\}$  are concentrated on paths  $\pi(t, dx) = u(t, x) dx$  such that

$$\int_0^T \int_0^1 u(t, x)^2 dx dt < \infty$$

and  $u \in L^2([0, 1] \times [0, T])$  a.s.

## 2.4 The Hydrodynamic Equation

In this section we prove that the points of the limiting measure satisfy a weak version of the hydrodynamic equation 2.2. First we identify the equation.

**Proposition 2.4.1.** *Let  $v$  be a solution to equations 2.2-2.6. Then the function  $u : [0, 1] \times [0, T] \rightarrow \mathbb{R}$  defined by*

$$u(x, t) = \mathbb{1}_{\{(x,t):x < s(t)\}}(v(x, t) + 1)$$

*satisfies the integral equation*

$$\begin{aligned} \int_0^1 f(x)u(x, t)dx - \int_0^1 f(x)u(x, 0)dx \\ = \int_0^t \int_0^1 f''(x)\Phi(u(x, \sigma))dx d\sigma - a \int_0^t \int_0^1 f'(x)u(x, \sigma)dx d\sigma \end{aligned} \quad (2.9)$$

*for each  $t \leq T$ .*

*Proof.* This is a calculation by integration by parts. Recall conditions 2.2-2.6 and that  $f'(0) = 0$ . Define  $v = 0$  for  $x > s(t)$ .

$$\begin{aligned} \int_0^{s(t)} f(x)v(x, t)dx - \int_0^{s(0)} f(x)v(x, 0)dx \\ = \int_0^t \partial_r \int_0^{s(r)} f(x)v(x, r)dx dr \\ = \int_0^t s'(r)f(s(r))v(s(r), r) + \int_0^t \int_0^{s(r)} f(x)\partial_r v(x, r)dx dr \\ = \int_0^t \int_0^{s(r)} f(x)(v_{xx}(x, r) + av_x(x, r))dx dr \\ = \int_0^t f(x)(v_x + av)|_{x=0}^{s(r)} - a \int_0^{s(r)} f'(x)v(x, r)dx dr \\ - \int_0^t f'(x)v(x, r)|_{x=0}^{s(r)} + \int_0^{s(r)} f''(x)v(x, r)dx dr \\ = \int_0^t f(s(r))(-a - s'(r)) + af(0) \\ + \int_0^{s(t)} (f''(x) - af'(x))v(x, r)dx dt \\ = \int_0^t \int_0^{s(t)} -af'(x)dx - d_r \int_0^s (r)f(x)dx \\ + \int_0^{s(t)} (f''(x) - af'(x))v(x, r)dx dt \\ = - \int_0^{s(t)} f(x)dx + \int_0^{s(0)} f(x)dx \\ + \int_0^t \int_0^1 f''(x)v(x, t)dx - \int_0^1 f'(x)(v(x, t) + 1)\mathbb{1}_{x < s(t)}dx dt. \end{aligned}$$

If it can be shown that  $v \geq 0$  in  $D_T$  for  $v_0 \geq 0$ , following our intuition for the heat equation, then the proposition is proven, since  $\Phi(u) = v$ . Indeed, this is true, and is the subject of the next lemma.

**Lemma 2.4.1.** *Suppose  $u \in C^2(D_T) \cap C^1(\overline{D_T})$  and satisfies conditions 2.2-2.3 and boundary conditions*

$$\begin{aligned} u_x + au &\leq 0 \text{ on } \{0\} \times (0, T] \\ u &\geq 0 \text{ on } (0, s_0] \times \{0\} \cup \{(s(t), t) : 0 \leq t \leq T\}. \end{aligned}$$

Then  $u \geq 0$  on  $\overline{D_T}$ .

*Proof.* Suppose  $u < 0$  somewhere on  $\overline{D_T}$ . Let  $\Phi((x, t)) = (x+at, t)$ , then define  $v : \Phi(D_T) \rightarrow \mathbb{R}^+$  by  $v = u \circ \Phi^{-1}$ . Then  $v$  satisfies  $v_{xx} = v_t$  in  $\Phi(D_T)$ , so by Theorem 1.6.1 in [13], the minimum of  $v$  occurs on the boundary  $\Phi(B_T)$ , where  $B_T = \overline{D_T} \setminus D_T$ . Let  $l > \sup_{0 \leq t \leq T} s(t)$ , and let  $\tilde{u} = u + \epsilon e^{-ax} - \epsilon e^{-al}$ . Then

$$\begin{aligned} \tilde{u}_{xx} &= u_{xx} + a^2 \epsilon e^{-ax} \\ a\tilde{u}_x &= au_x - a^2 \epsilon e^{-ax} \\ \tilde{u}_t &= \tilde{u}_{xx} + a\tilde{u}_x \end{aligned}$$

so  $\tilde{v}$  as before satisfies the heat equation, and the minimum principle. Since  $e^{-ax} > e^{-al}$  in  $D_T \cup B_T$ , we still have  $\tilde{u} \geq 0$  on  $[0, s_0] \times \{0\} \cup \{(s(t), t) : 0 \leq t \leq T\}$ . So for some  $\epsilon > 0$ ,  $\tilde{u} \leq -\epsilon e^{-al}/2$  somewhere, and the minimum occurs on the left boundary. Suppose this is the case, and let  $t_0 = \inf\{T, t : \tilde{u}(0, t) \leq -\epsilon e^{-al}/2\}$ . Now wlog,  $T = t_0$  and  $T$  is a minimum for  $\tilde{u}$  and  $\tilde{u}(0, T) = -\epsilon e^{-al}/2$ .

$$\tilde{u}_x(0, T) + a\tilde{u}(0, T) \leq -a\epsilon e^{-al}, \text{ so } \tilde{u}_x(0, T) \leq -a\epsilon e^{-al}/2$$

This is a contradiction, since  $(0, T)$  is a minimum for  $\tilde{u}$ , and the lemma is proved. □

□

Next we turn to convergence. Our goal is the following lemma, which together with uniqueness for such results and existence of the strong solution, proves Theorem (0.1):

**Lemma 2.4.2.** *If, for each  $P^N$ ,  $X_0 = \eta_N$  a.s. with empirical measures  $\mu_N$  such that  $\sup_N \{\langle \mu_N, \mu_N \rangle\} < \infty$  and  $\mu_N(dx) \rightarrow u_0(x)dx$  in  $\mathcal{M}$ , then under  $Q^\infty$ , for  $0 < t \leq T$ ,  $\mu_t(dx) = u(x, t)dx$  a.s. and*

$$\langle f, u(\cdot, t) \rangle - \langle f, u(\cdot, 0) \rangle = \int_0^t \langle f'', \Phi(u(\cdot, s)) \rangle ds - a \int_0^t \langle f', u(\cdot, s) \rangle ds$$

a.s.

*Proof.* In previous sections we have already shown that for  $f \in C([0, 1])$ , the  $P^N$ -martingale  $M_t = \langle f, X_t \rangle_N - \int_0^t \mathcal{L}_N \langle f, X_s \rangle_N ds$  satisfies

$$P^N [ |M_t - M_0| > \epsilon ] < \frac{Ct}{N\epsilon^2},$$

and

$$\mathcal{L}_N \langle f, X_s \rangle_N = \langle \Delta_N f, \Phi(X_s) \rangle_N - a \langle D_N f, X_s \rangle_N.$$

For  $f \in C^2([0, 1])$  such that  $f'(0) = f'(1) = 0$ ,  $\Delta_N f \rightarrow f''$  and  $D_N f \rightarrow f'$  uniformly, so

$$|(\langle \Delta_N f, \Phi(X_s) \rangle_N - a \langle D_N f, X_s \rangle_N) - (\langle f'', \Phi(X_s) \rangle_N - a \langle f', X_s \rangle_N)| \rightarrow 0$$

uniformly in  $\mathcal{M}_1$ . Now  $-a \langle f', X_s \rangle_N = -a \langle f', \mu_s \rangle$  in distribution (under  $Q^N$ ), which is a continuous functional on  $\mathcal{M}_1$ , but the term with  $\lambda$  is not, so we must make some additional estimates. For  $\mu \in \mathcal{M}$ , let

$$K_\epsilon \mu(x) = \frac{1}{2\epsilon} \int \mathbb{1}_{|x-y| \leq \epsilon} d\mu(y).$$

For the Markov process  $(X, A_N, P^N)$ , let  $S_t = \max\{x \in A_N : X_t > 0\}$ . Recall that under  $P^N$ ,  $X_t \in \mathcal{E}_N$  for all  $t$  a.s., so for  $x \leq S_t$ ,  $X_t(x) \geq 1$ . We claim that for  $0 < t \leq T$ , given  $\delta > 0$ , for  $\epsilon$  small enough, as  $N$  goes to infinity,

$$Q_N \left[ \left| \langle f, \mu_t \rangle - \langle f, \mu_0 \rangle - \left( \int_0^t \langle f'', \Phi(K_\epsilon \mu_s) \rangle ds - a \int_0^t \langle f', \mu_s \rangle ds \right) \right| > \delta \right] \rightarrow 0.$$

By our previous comments, it will be enough to show that, for some small  $\epsilon$ , for  $N$  large enough,

$$|\langle f'', \Phi(K_\epsilon \mu_\eta) \rangle - \langle f'', \mu_{\Phi(\eta)} \rangle| \leq \frac{\delta}{2t}$$

for all  $\eta \in \mathcal{E}_N$ . Define

$$h(x) = K_\epsilon \mu_\eta(x) - \Phi(K_\epsilon \mu_\eta(x)),$$

and note that  $0 \leq h(x) \leq 1$  and  $h(x) = 0$  for  $x > S_t + \epsilon$ . For  $x \in [0, \max\{0, S_t - \epsilon\}]$ ,  $\Phi(\eta_y) = \eta_y - 1$  for  $y \in A_N$  such that  $|y - x| \leq \epsilon$ . So

$$\begin{aligned} K_\epsilon \mu_\eta &= \frac{1}{2\epsilon} \int \mathbb{1}_{|x-y| \leq \epsilon} d\mu_\eta(y) \\ &= \frac{1}{2\epsilon N} \sum_{y \in A_N \cap \{x: |x-y| \leq \epsilon\}} \eta_y \\ &\geq \frac{|A_N \cap \{x: |x-y| \leq \epsilon\}|}{2\epsilon N} \\ &\geq \frac{2\epsilon N - 1}{2\epsilon N} \end{aligned}$$

and on this interval,  $1 - h(x) = e(x) \leq 1/(2\epsilon N)$ . Now we will use the decompositions

$$\int_0^1 f''(x) \Phi(K_\epsilon \mu_\eta(x)) dx = \int_0^1 f''(x) K_\epsilon \mu_\eta(x) dx + \int_0^{S_t + \epsilon} f''(x) h(x) dx$$

and

$$\int_0^1 f''(x) d\mu_{\Phi(\eta)}(x) = \int_0^1 f''(x) d\mu_\eta(x) - \frac{1}{N} \sum_{x \in A_N, x \leq S_t} f''(x).$$

Let

$$U = \{(x, y) : |x - y| \leq \epsilon, 0 \leq x \leq 1\}$$

$$U_\epsilon = \{(x, y) \in U : y < \epsilon \text{ or } y > 1 - \epsilon\}$$

$$V = U \setminus U_\epsilon,$$

let

$$I_\epsilon = \frac{1}{2\epsilon} \int \int \mathbb{1}_{U_\epsilon} f''(x) d\mu_\eta(y) dx,$$

and

$$\begin{aligned} \int_0^1 f''(x) K_\epsilon \mu_\eta(x) dx &= I_\epsilon + \frac{1}{2\epsilon} \int \int \mathbb{1}_V f''(x) dx d\mu_\eta(y) \\ &= I_\epsilon + \int_\epsilon^{1-\epsilon} \left( \frac{1}{2\epsilon} \int_{y-\epsilon}^{y+\epsilon} f''(x) dx \right) d\mu_\eta(y). \end{aligned}$$

$f''_\epsilon(y) = 1/2\epsilon \int_{y-\epsilon}^{y+\epsilon} f''(x) dx$  converges uniformly to  $f''(y)$  on  $[\epsilon, 1 - \epsilon]$ , so

$$\left| \int_\epsilon^{1-\epsilon} f''_\epsilon(x) - f''(x) d\mu_\eta(y) \right| = C_\epsilon$$

converges to zero with  $\epsilon$  independently of  $N$ . Next, look at

$$\int_0^1 f''(x)h(x) = \int_0^{S_t-\epsilon} f''(x)dx - \int_0^{S_t-\epsilon} f''(x)e(x)dx + \int_{S_t-\epsilon}^{S_t+\epsilon} f''(x)h(x)dx.$$

The second and third terms are bounded by  $M/(2\epsilon N)$  and  $M\epsilon$ , respectively, and

$$\left| \int_0^{S_t-\epsilon} f''(x)dx - \frac{1}{N} \sum_{x \in A_N, x \leq S_t-\epsilon} f''(x) \right| \leq C_N$$

vanishes as  $N$  goes to infinity, as the Riemann sum approximation of  $f''$ . Additionally

$$\frac{1}{N} \sum_{x \in A_N, S_t-\epsilon < x \leq S_t} f''(x)$$

is bounded by  $M(\epsilon N + 1)/N$ . We now decompose 2.4:

$$\begin{aligned} |\langle f'', \Phi(K_\epsilon \mu_\eta) \rangle - \langle f'', \mu_{\Phi(\eta)} \rangle| &\leq \left| I_\epsilon + \int_0^\epsilon f''(x)d\mu_\eta(x) + \int_{1-\epsilon}^1 f''(x)d\mu_\eta(x) \right| \\ &\quad + \frac{M}{2\epsilon N} + M\epsilon + C_\epsilon + C_N + \frac{M(\epsilon N + 1)}{N}. \end{aligned}$$

By choosing first  $\epsilon$  then letting  $N$  go to infinity, we make all terms arbitrarily small except the integrals near the boundary. For these we seem to require the  $L^2$  bound obtained in the last section. Combining all of our estimates, it now suffices to show that for  $\delta > 0$ ,  $\epsilon$  sufficiently small,

$$Q^N \left[ \int_0^t \left| I_\epsilon + \int_0^\epsilon f''(x)d\mu_s(x) + \int_{1-\epsilon}^1 f''(x)d\mu_s(x) \right| ds > \delta \right] \rightarrow 0$$

as  $N \rightarrow \infty$ . Let  $g < 2M$  be a bounded function supported on an interval of width  $2\epsilon$ . Then under  $P^N$ , for  $N$  large enough, by Schwarz's Lemma,

$$\begin{aligned} \left( \int_0^t \int g(x)d\mu_s(x)ds \right)^2 &\leq t \int_0^t \left( \frac{1}{N} \sum_{x \in A_N} g(x)X_s(x) \right)^2 ds \\ &\leq 8t\epsilon M^2 \int_0^t \frac{1}{N} \sum_{x \in A_N} X_s(x)^2 ds. \end{aligned}$$

After checking that  $\phi$  such that  $I_\epsilon = \int \phi(x)d\mu_s(x)$  is of this form, we apply (2.8) to see that

$$E^N \left[ \left( \int_0^t \int \phi(x) + \mathbb{1}_{[0,\epsilon] \cup [1-\epsilon,1]} |f''(x)| d\mu_s(x) ds \right)^2 \right] \leq 8t\epsilon M (Ct + \sup_N E^N [\langle \mu_0, \mu_0 \rangle]),$$

and by Markov's inequality we have proven the claim. Next we need to show that  $\Phi(K_\epsilon\mu_\eta)$  is a continuous operator on  $\mathcal{M}$ , so that the expectation of

$$\left| \langle f, \mu_t \rangle - \langle f, \mu_0 \rangle - \left( \int_0^t \langle f'', \Phi(K_\epsilon\mu_s) \rangle ds - a \int_0^t \langle f', \mu_s \rangle ds \right) \right|$$

converges. But  $K_\epsilon\mu_\eta(x)$  is a continuous functional at a.e.  $x$ , wherever  $x - \epsilon$  and  $x + \epsilon$  are continuous points of the distribution function for  $\mu$ . Since it is also bounded by  $1/(2\epsilon)$ , and  $\lambda$  is continuous, and we apply the dominated convergence theorem. Now, since the integral expression is bounded, it is uniformly continuous, and for all  $\delta > 0$ , for  $\epsilon$  small enough,

$$E^* \left[ \left| \langle f, \mu_t \rangle - \langle f, \mu_0 \rangle - \left( \int_0^t \langle f'', \Phi(K_\epsilon\mu_s) \rangle ds - a \int_0^t \langle f', \mu_s \rangle ds \right) \right| \right] \leq \delta.$$

$\mu \in L^2([0, T] \times [0, 1])$   $Q^*$ -a.e., and for such  $\mu$ ,  $K_\epsilon\mu \rightarrow \mu$  in  $L^2$ , so

$$\left| \int_0^t \langle f'', \Phi(K_\epsilon\mu_s) \rangle ds - \int_0^t \langle f'', \Phi(\mu_s) \rangle ds \right| \rightarrow 0$$

a.e. Again, all integral expressions are bounded and uniformly integrable in  $Q^*$  as  $\epsilon \rightarrow 0$ , so the expectation of

$$\left| \langle f, \mu_t \rangle - \langle f, \mu_0 \rangle - \left( \int_0^t \langle f'', \Phi(\mu_s) \rangle ds - a \int_0^t \langle f', \mu_s \rangle ds \right) \right|$$

is zero, and that quantity is zero almost everywhere, and the lemma is proved.  $\square$

## 2.5 Uniqueness of Weak Solutions

In the previous section, we found that the probability measure of a sub-sequential limit is concentrated on solutions to the following integral equation:

$$\langle f, u(\cdot, t) \rangle - \langle f, u(\cdot, 0) \rangle = \int_0^t \langle f'', \Phi(u(\cdot, s)) \rangle - a \langle f', u(\cdot, s) \rangle ds, \quad (2.10)$$

where angle brackets represent  $L^2([0, 1])$  inner product,  $u(t, \cdot) \in L^2([0, 1])$ ,  $f \in C^2([0, 1])$ ,  $f'(0) = f'(1) = 0$ ,  $\Phi(x) = (x - 1) \vee 0$ , and  $u(0, x) = u_0(x)$ ,  $u_0 \in L^2([0, 1])$ . We need a uniqueness theorem for such  $u$ .

**Lemma 2.5.1.** *Given functions  $u_0 \in L^2([0, 1])$ ,  $a \in C([0, T])$ , a function  $u \in L^2([0, 1] \times [0, T])$  that satisfies the integral equation (2.10) for all  $f \in C^2([0, 1])$  with  $f'(0) = f'(1) = 0$  is unique.*

*Proof.* The proof is based on the method in A2.4 of [22]. Suppose  $u$  and  $u'$  are two solutions for a given  $(u_0, a)$ , and let  $\bar{u}_t(x) = u(x, t) - u'(x, t)$  and  $\bar{\lambda}_t(x) = \Phi(u(x, t)) - \Phi(u'(x, t))$ . Then

$$\partial_t \langle f, \bar{u}_t \rangle = \langle f'', \bar{\lambda}_t \rangle - a(t) \langle f', \bar{u}_t \rangle \quad (2.11)$$

Let  $\psi_k(x) = \cos(\pi kx)$ ,  $\phi_k(x) = \sin(\pi kx)$ , and  $\{\psi_k\}_{k=0}^\infty$  is an orthonormal basis for  $L^2([0, 1])$  (the eigenfunctions for the Neumann problem). Note that  $\langle \psi_0, \bar{u}_t \rangle = 0$  for all  $t$ , and let  $b_k(t) = \langle \psi_k, \bar{u}_t \rangle$ . For positive integer  $N$ , define

$$R_N(t) = \sum_{k=1}^N \frac{b_k^2(t)}{k^2},$$

a positive, differentiable function with

$$\partial_t R_N(t) = -2\pi^2 \sum_{k=1}^N b_k(t) \langle \psi_k, \bar{\lambda}_t \rangle - 2\pi a(t) \sum_{k=1}^N \frac{b_k \langle \phi_k, \bar{u}_t \rangle}{k}$$

in the second term, we expand  $\bar{u}_t$ , and since  $\phi_k$  is a continuous linear functional, we get

$$\sum_{k=1}^N \sum_{j=1}^\infty \frac{b_k b_j \langle \phi_k, \psi_j \rangle}{k}.$$

Defining  $\sigma(k, j)$  to be 1 when  $k - j$  is odd and 0 otherwise,

$$\langle \phi_k, \psi_j \rangle = \frac{2k\sigma(k, j)}{\pi(k^2 - j^2)},$$

defined to be 0 for  $k = j$ . Now we have

$$R_N(t) = \int_0^t -2\pi^2 \sum_{k=1}^N b_k(s) \langle \psi_k, \bar{\lambda}_s \rangle - 4a(s) \sum_{k=1}^N \sum_{j=1}^\infty \frac{b_k(s) b_j(s) \sigma(k, j)}{k^2 - j^2} ds.$$

Absolute convergence of these sums to an  $L^1([0, T])$  function will allow us to use dominated convergence. The first term is bounded by  $\|\bar{u}_t\|_{L^2([0,1])} \|\bar{\lambda}_t\|_{L^2([0,1])}$  by Schwarz's inequality.

$|\bar{\lambda}_t| \leq |\bar{u}_t|$ , so that is bounded by  $\|\bar{u}_t\|_{L^2([0,1])}^2$ , which is in  $L^1([0, T])$  by hypothesis. Then,

$$\begin{aligned}
\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| \frac{b_k(s)b_j(s)\sigma(k, j)}{k^2 - j^2} \right| &= \sum_{k=1}^{\infty} |b_k| \sum_{j=0}^{\infty} \left| \frac{b_j\sigma(k, j)}{k^2 - j^2} \right| \\
&\leq \sum_{k=1}^{\infty} |b_k| \left( \sum_{j=0}^{\infty} b_j^2 \right)^{1/2} \left( \sum_{j=0}^{\infty} \frac{\sigma(k, j)}{(k^2 - j^2)^2} \right)^{1/2} \\
&\leq \sum_{k=1}^{\infty} |b_k| \left( \sum_{j=0}^{\infty} b_j^2 \right)^{1/2} \left( \frac{1}{k^2} \sum_{m=1}^{\infty} \frac{2}{m^2} \right)^{1/2} \\
&= C \|\bar{u}_t\|_{L^2([0,1])} \sum_{k=1}^{\infty} \frac{b_k}{k} \\
&\leq C \|\bar{u}_t\|_{L^2([0,1])}^2
\end{aligned}$$

Observe that  $(k^2 - j^2)^2 \geq k^2(k - j)^2$ , and for all  $k$ ,

$$\sum_{j=1}^{\infty} \frac{\sigma(k, j)}{(j - k)^2} \leq \sum_{m=1}^{\infty} \frac{2}{m^2}.$$

By hypothesis,  $\|\bar{u}_t\|_{L^2([0,1])}^2 \in L^1([0, T])$ , so we apply dominated convergence to conclude

$$\begin{aligned}
R(t) &= \int_0^t -2\pi^2 \sum_{k=1}^{\infty} b_k(s) \langle \psi_k, \bar{\lambda}_s \rangle - 4a(s) \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{b_k(s)b_j(s)\sigma(k, j)}{k^2 - j^2} ds \\
&= \int_0^t -2\pi^2 \sum_{k=1}^{\infty} b_k(s) \langle \psi_k, \bar{\lambda}_s \rangle ds. \quad (2.12)
\end{aligned}$$

Now,

$$\sum_{k=1}^{\infty} b_k(s) \langle \psi_k, \bar{\lambda}_s \rangle = \langle \bar{u}_t, \bar{\lambda}_t \rangle \geq 0,$$

since  $\lambda$  is increasing. Finally, since  $R(0) = 0$  by hypothesis,  $R(t) \leq 0$  so  $R(t) = 0$  for all  $t \geq 0$ , each  $b_k(t) = 0$ , and  $\bar{u}_t = 0$ , as desired.  $\square$

We now see that the empirical measures of the process  $X_t$  converge to the measure with density  $u(x, t) = v(x, t) + \mathbb{1}_{v>0}$ , where  $v$  is the solution to the Stefan problem 2.2-2.6, when it exists, since this solution satisfies the same integral equation that the limiting measure of  $X_t$  satisfies, and solutions to such equations are unique. Now Theorem 0.1 refers to the process  $\Phi(X_t)$ , and  $v(x, t) = \Phi(u(x, t))$ , so we need  $\mu_{\Phi(X_t)} \rightarrow \Phi(u(x, t))dx$ . Indeed, this fact is proved in the proof of Lemma 3.2, and the theorem is proved.

## 2.6 Reflecting Exclusion

Finally, we return to the regime of reflecting intervals by describing the hydrodynamic limit of the exclusion process  $Z_t$ . First we describe the dynamics of  $Z_t$  more precisely. Each site is occupied by zero or one particles, except for  $1 - 1/(2N)$ , which is always empty. Given  $Z_t = \theta$ , a state  $\zeta$  is accessible in one of three cases: first, if for some  $1 \leq k < N - 1$ ,  $\theta_{(2k-1)/2N} = 1, \theta_{(2k+1)/2N} = 0, \zeta_{(2k-1)/2N} = 0, \zeta_{(2k+1)/2N} = 1$ , and otherwise the states are equal. Second, if for  $k = N - 1$ , we instead have  $\zeta_{(2k+1)/2N} = 0$ , so that the particle is "killed" here. In either of these case,  $\theta$  moves to  $\zeta$  with rate  $N^2$  times the length of the block of occupied sites to the left of  $(2k + 1)/2N$  in  $\theta$ . Third, if  $x$  is the first unoccupied site of  $\theta$ , the state  $\zeta$  such that  $\zeta_x = 1$  and  $\zeta_y = \theta_y$  elsewhere is reached at rate  $aN$ , and we think of the entire block being pushed over to make room for a new particle. For each  $N$ , define, for  $x \in A_N, x < S_t$

$$U_t(x) = x + \frac{1}{N} \sum_{z \leq x} \Phi(X_t(z))$$

Note that  $U_t(S_t - 1/N) = 1 - 1/2N$ . Define

$$\Psi(X_t)(y) = \begin{cases} 0 & \text{if } y \in U_t(A_N \cap [0, S_t)) \\ 1 & \text{else} \end{cases}$$

so that between unoccupied sites  $y$  and  $y'$  of  $\Psi(X_t)$  there are  $\Phi(X_t(x))$  occupied sites for  $x < S_t$  such that  $U_t(x) = y'$ . One need only take a moment to check that the dynamics of  $\Psi(X_t)$  are exactly the described dynamics of  $Z_t$ . We can write the inverse map explicitly:

For  $y \in A_N$  such that  $Z_t(y) = 0$ ,

$$T_t(y) \equiv U_t^{-1}(y) = y - \frac{1}{N} \sum_{z < y} Z_t(z)$$

and, given  $Z_t$ , we extend the domain of  $T_t$  to all of  $A_N$  with the same formula. We also define equivalent transformations for the solution  $v$  of 2.2-2.6. Let

$$v_t(x) = x + \int_0^x v(x, t) dx$$

and, since  $v \geq 0$  and is differentiable on  $[0, s(t))$ , we can define a differentiable inverse  $\tau_t = v_t^{-1}$ , and let

$$z(y, t) = v(\tau_t(y), t) \tau_t'(y)$$

so that

$$\tau_t(y) = y - \int_0^y z(y, t) dy.$$

and

$$v(x, t) = z(v_t(x), t)v_t'(x)$$

Given  $z : [0, 1] \rightarrow [0, 1)$ , we can similarly define  $\tau$  and  $v$  with the last two equations, where  $v$  is defined to be the inverse to  $\tau$ . Next we identify the hydrodynamic equation that functions  $z$  should satisfy:

**Lemma 2.6.1.** *Given functions  $z$  and  $v$ , defined on the appropriate regions, satisfying the relations above for all  $t$ ,  $z$  is a solution to the differential equation:*

$$\partial_t z(y, t) = \partial_x (K(z(y, t)) \partial_y z(y, t)) \quad 0 < y < 1, t > 0, \quad (2.13)$$

$$z(x, 0) = z_0(x) \quad 0 \leq x \leq 1 \quad (2.14)$$

$$K(z(y, t)) \partial_y z(y, t)|_{y=0} = a \quad t > 0, \quad (2.15)$$

$$z(0, t) = 0 \quad t > 0, \quad (2.16)$$

with  $K(z) = 1/(1 - z)^2$ , if and only if  $v$  is a solution to 2.2-2.6

*Proof.* We will expand the equation

$$\partial_{xx} v(x, t) + a \partial_x v(x, t) = \partial_t v(x, t)$$

using

$$\partial_x v_t(x) = 1 + v(x, t) = \frac{1}{1 - z(v_t(x), t)}.$$

Now let  $z = z(v_t(x), t)$ , let  $z_t$  and  $z_y$  refer to the partial derivatives of  $z$  with respect to its variables evaluated at  $(v_t(x), t)$ . We first check the equivalence at the left boundary.  $v$  is continuous up to the boundary so  $v$  has a derivative at 0 and the above equation holds.

This gives us equivalence of

$$\begin{aligned} v_x + a(v + 1) &= 0 \\ \frac{-z_y}{(1 - z)^3} + \frac{a}{1 - z} &= 0 \\ \frac{-z_y}{(1 - z)^2} &= -a \end{aligned}$$

at  $x = v_t(x) = 0$ . Next, for  $0 < x < s(t)$ ,  $0 < v < 1$ ,

$$\begin{aligned} v_x(x, t) &= \frac{-z_y}{(1-z)^3} \\ v_{xx}(x, t) &= \partial_x \left( \frac{z_y}{(1-z)^2} \cdot \frac{-1}{1-z} \right) \\ &= \frac{-1}{1-z} \partial_x \frac{z_y}{(1-z)^2} + \frac{z_y^2}{(1-z)^5} \end{aligned}$$

so

$$v_{xx} + av_x = \frac{-1}{(1-z)^2} \partial_{v_t(x)} \frac{z_y}{(1-z)^2} + \frac{z_y^2}{(1-z)^5} - \frac{az_y}{(1-z)^3}.$$

Then

$$\begin{aligned} \partial_t v_t(x) &= - \int_0^x \partial_t v(r, t) dr \\ &= v_x(x, t) + av(x, t) - (v_x(0, t) + av(0, t)). \end{aligned}$$

Under either set of conditions, the last term is equal to  $-a$ , giving

$$\partial_t v_t(x) = \frac{-z_x}{(1-z)^3} + \frac{a}{1-z},$$

so

$$\begin{aligned} \partial_t v(x, t) &= \frac{-\partial_t v_t(x) z_y - z_t}{(1-z)^2} \\ &= \frac{-z_t}{(1-z)^2} + \frac{z_y^2}{(1-z)^5} - \frac{az_y}{(1-z)^3}, \end{aligned}$$

and  $v_{xx} + av_x = v_t$  is equivalent, for  $y = v_t(x)$ , to

$$\partial_t(z(y, t)) = \partial_y \frac{\partial_y z(y, t)}{(1-z(y, t))^2},$$

as desired. □

**Theorem 2.6.1.** *The process  $Z_t = \Psi(X_t)$ ,  $X_t$  with initial data  $u_0$  as in Theorem 0.1, converges weakly in  $D([0, T], \mathcal{M})$  to the unique solution to the differential equation 2.13-2.16 with initial data  $z_0 = \psi(u_0)$*

*Proof.* We will prove that for  $\delta > 0$ ,  $f \in C([0, 1])$ ,

$$P^N \left[ \sup_{0 \leq t \leq T} \left| \int_0^1 f(y) \mu_{Z_t(y)} dy - \int_0^1 f(y) z(y, t) dy \right| > \delta \right] \rightarrow 0,$$

Which suffices to prove the theorem. First, since  $v$  is sufficiently smooth, if we extend  $f(v_t(x))$  to be 0 where it is not defined, then our previous theorem gives

$$P^N \left[ \sup_{0 \leq t \leq T} \left| \int_0^1 f(v_t(x)) \mu_{\Phi(X)_t(x)} dx - \int_0^1 f(v_t(x)) v(x, t) dx \right| > \delta \right] \rightarrow 0$$

and since the last terms in the two differences are equivalent by a change of variables, we need to look at the difference of

$$\int_0^1 f(y) \mu_{Z_t(y)} = \frac{1}{N} \sum_{y \in A_N} f(y) Z_t(y)$$

and

$$\int_0^1 f(v_t(x)) \mu_{\Phi(X)_t(x)} dx = \frac{1}{N} \sum_{x \in A_N} f(v_t(x)) \Phi(X_t(x)) = \frac{1}{N} \sum_{y \in A_N} f(v_t(T_t(y))) Z_t(y).$$

Since  $Z \leq 1$  and  $f$  is continuous on a compact interval, the problem is reduced to the difference of  $y$  and  $v_t(T_t(y))$ . Theorem 0.1 and the  $L^2$  bound give us

$$P^N \left[ \sup_{0 \leq t \leq T} \sup_{0 \leq x \leq 1} \left| \int_0^x v(z, t) dz - \mu_{\Phi(X)_t}(dz) \right| > \delta \right] = P^N [E] \rightarrow 0,$$

so for fixed  $t$ , suppose  $y = v_t(x) = x + \int_0^x v(z, t) dz$ , and we claim that for large enough  $N$ , in the set  $E$ ,

$$|T_t(y) - x| < \delta.$$

Indeed, suppose  $T_t(y) < x - \delta$ . Then

$$\begin{aligned} y - \frac{1}{N} \sum_{z < y} Z_t(z) &< x - \delta \\ \frac{1}{N} \sum_{z \leq T_t(y)} \Phi(X_t) &> \int_0^x v(z, t) dz + \delta \\ \frac{1}{N} \sum_{z \leq T_t(y)} \Phi(X_t) &> \frac{1}{N} \sum_{z \leq x} \Phi(X_t)(z), \end{aligned}$$

and  $T_t(y) > x$ , a contradiction. A contradiction results from the opposite inequality in the same way. Then,  $v_t$  being continuous (and nonrandom), we obtain a bound on the difference

$$|v_t(T_t(y)) - v_t(x)| = |v_t(T_t(y)) - y|,$$

which gives the result we need and completes the proof of the theorem.  $\square$

## 2.7 Supplementary Proofs

Next we prove the following theorem, needed to complete the proof of Lemma 2.1. A similar result is stated without proof in [22] so it should not be new.

**Theorem 2.7.1.** *Let  $Q^N \rightarrow Q^*$  be a weakly convergent sequence of probability measures on  $D([0, T], \mathcal{M})$  representing the empirical measures of Markov processes  $(X_t, P^N)$  on  $\mathbb{N}^{A_N}$ . Suppose that*

$$\sup_N E^N \left[ \int_0^T \frac{1}{N} \sum_{x \in A_N} X_t(x)^2 dt \right] < \infty.$$

*Then  $\mu(dx, t) = u(x, t)dx$  with  $u \in L^2([0, T] \times [0, 1])$   $Q^*$ -a.s.*

We prove this statement by considering the mollification  $K_\epsilon \mu$  defined by

$$K_\epsilon \mu(x, t) = \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \mu^*(dx, t),$$

where  $\mu^*$  is the projection of  $\mu$  onto the torus  $\mathbb{T}$  created by identifying 0 and 1. We will prove that  $\mu^*$  is  $L^2$ , which implies that  $\mu$  is. Henceforth we use the notation  $\mu$  for simplicity. For given  $N$ , let  $\epsilon = \epsilon_0 + m/N$ , where  $0 \leq \epsilon_0 < 1/N$ . Then let

$$\tilde{\mu}_\epsilon(x) = \frac{1}{2\epsilon} \int \chi_{(-m/N+\epsilon_0, m/N+\epsilon_0]} d\mu(y).$$

If  $\mu$  is the empirical measure for  $\eta \in \mathbb{N}_N^A$ , then we can calculate  $\int_0^1 \tilde{\mu}_\epsilon(x)^2 dx$ .

$$\tilde{\mu}_\epsilon(x) = \frac{1}{2\epsilon N} \sum_{x \in A_N} \chi_{[x-\epsilon_0, x-\epsilon_0+\frac{1}{N})} \sum_{k=1}^{2m} \eta_{x+\frac{k-m}{N}}.$$

So we calculate

$$\begin{aligned}
\int_0^1 \tilde{\mu}_\epsilon(x)^2 dx &= \frac{1}{4\epsilon^2 N^3} \sum_{x \in A_N} \left( \sum_{k=1}^{2m} \eta_{x+\frac{k-m}{N}} \right)^2 \\
&= \frac{1}{4\epsilon^2 N} \left[ 2m \sum_{x \in A_N} \eta_x^2 + \sum_{k=1}^{2m-1} 2(2m-k) \sum_{x \in A_N} \eta_x \eta_{x+\frac{k}{N}} \right] \\
&\leq \frac{1}{4\epsilon^2 N^3} \left( 2m + 2 \sum_{k=1}^{2m-1} k \right) \sum_{x \in A_N} \eta_x^2 \\
&= \frac{m^2}{\epsilon^2 N^2} \left( \frac{1}{N} \sum_{x \in A_N} \eta_x^2 \right),
\end{aligned}$$

which gives us

$$\int_0^1 K_\epsilon \mu(x)^2 dx \leq \int_0^1 \tilde{\mu}_\epsilon(x)^2 dx \leq (1 + o(1)) \frac{1}{N} \sum_{x \in A_N} \eta_x^2.$$

If we can prove that the function  $\mu(dx, t) \mapsto \int_0^T \int_0^1 K_\epsilon \mu(x, t)^2 dx dt$  is continuous on  $D([0, T], \mathcal{M})$ , then  $E^N(\int_0^1 K_\epsilon \mu(x)^2 dx)$  converges to  $E^*(\int_0^1 K_\epsilon \mu(x)^2 dx)$  for each  $N$ , and by the above inequality, this quantity is bounded by  $\sup_N E^N \left[ \int_0^T \frac{1}{N} \sum_{x \in A_N} X_t(x)^2 dt \right]$ , which is finite by hypothesis. Therefore to prove our theorem, we need two lemmas.

**Lemma 2.7.1.** *If  $\limsup_{\epsilon \rightarrow 0} E^*(\int_0^T \int_0^1 K_\epsilon \mu(x)^2 dx dt) < \infty$ , then with probability one,  $\mu$  is absolutely continuous, with  $\mu(dx, t) = u(x, t) dx$  and  $E^*(\int_0^T \int_0^1 u^2(x) dx dt) < \infty$ .*

*Proof.* Let  $f_k(x) = e^{2\pi i k x}$ , and  $\{f_k\}_{k \in \mathbb{Z}}$  is an orthonormal basis for  $L^2([0, 1])$ , with each  $f_k$  continuous on  $\mathbb{T}$ . For each measurable function  $g : [0, 1] \rightarrow \mathbb{R}$ ,  $\int_0^T \int_0^1 g(x)^2 dx dt = \int_0^T \sum_{\mathbb{Z}} |\langle f_k, g \rangle|^2 dt$ . Let

$$N = \{ \mu \in \mathcal{M}_1 : \liminf_{\epsilon > 0} \int_0^T \sum_{k=-\infty}^{\infty} \langle f_k, K_\epsilon \mu \rangle^2 dt = \infty \}.$$

$\langle \psi_0, K_\epsilon \mu \rangle = 1$  for all  $\mu \in \mathcal{M}_1$ , and we can calculate for  $k \neq 0$ ,

$$\begin{aligned} \langle f_k, K_\epsilon \mu \rangle &= \int_0^1 e^{2\pi i k x} \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} \mu(dy) dx \\ &= \frac{1}{2\epsilon} \int_0^1 \int_{y-\epsilon}^{y+\epsilon} e^{2\pi i k x} dx \mu(dy) \\ &= \frac{1}{4\pi k i \epsilon} \int_0^1 (e^{2\pi k i \epsilon} - e^{-2\pi k i \epsilon}) e^{2\pi k i y} \mu(dy) \\ &= \frac{\sin(2\pi k \epsilon)}{2\pi k \epsilon} \langle f_k, \mu \rangle. \end{aligned}$$

We conclude that as  $\epsilon$  goes to zero,  $|\langle f_k, K_\epsilon \mu \rangle|$  increases to  $|\langle f_k, \mu \rangle|$ . Thus by the monotone convergence theorem,

$$\lim_{\epsilon \rightarrow 0} \int_0^T \sum_{k=-\infty}^{\infty} |\langle f_k, K_\epsilon \mu(t) \rangle|^2 dt = \int_0^T \sum_{k=-\infty}^{\infty} |\langle f_k, \mu(t) \rangle|^2 dt$$

for all  $\mu \in D([0, T], \mathcal{M})$ . Let  $N = \{\mu \in D([0, T], \mathcal{M}_1) : \int_0^T \sum_{k=-\infty}^{\infty} |\langle f_k, \mu \rangle|^2 dt = \infty\}$ . Clearly  $P^*(N) = 0$  by elementary measure-theoretical considerations. Thus,  $\int_0^T \sum_{k=-\infty}^{\infty} |\langle f_k, \mu(t) \rangle|^2 dt < \infty$   $P^*$ -a.e. For such  $\mu$ , the sequence  $\langle f_k, \mu(t) \rangle$  is in  $l^2$  for almost all  $t$ , so there is a function  $u(x, t) = \sum_{k=-\infty}^{\infty} \langle f_k, \mu(t) \rangle f_k$  such that  $\int_0^T \|u\|_2^2 dt < \infty$ . To see that  $\mu(dx, t) = u(x, t) dx$  as a measure for each  $t$  such that  $u$  is finite, note that the collection  $\{f_k\}$  forms an algebra which is dense in  $C(\mathbb{T})$ . For each function  $g$  in the algebra,  $\langle g, \mu(t) \rangle = \langle g, u(t) \rangle$  and the measures must be the same. Now  $\mu(dx, t) = u(x, t) dx$  for almost all  $t$  and therefore they are equal as functions in  $L^2([0, T] \times [0, 1])$ , as desired.

□

**Lemma 2.7.2.** *The function  $\mu(dx, t) \mapsto \int_0^T \int_0^1 K_\epsilon \mu(x, t)^2 dx dt$  is continuous on  $D([0, T], \mathcal{M})$ .*

*Proof.* In the proof of the previous lemma, we see that  $\int_0^1 K_\epsilon \mu(x, t)^2 dx = \sum_{\mathbb{Z}} |\langle f_k, g \rangle|^2$ , and the calculation shows that  $\langle f_k, K_\epsilon \mu \rangle$  is a continuous function of  $\mu$ . Since  $K_\epsilon \mu(x)^2$  is bounded by  $\epsilon^{-2}$  for  $\mu \in \mathcal{M}_1$ , we can apply dominated convergence to see that as  $\mu \rightarrow \nu$  in  $\mathcal{M}_1$ ,  $\int_0^1 K_\epsilon \mu(x, t)^2 dx \rightarrow \int_0^1 \nu_\epsilon(x, t)^2 dx$ . Convergence in  $D([0, T], \mathcal{M})$  then follows from comments in section 1.

□

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