Production planning, pricing and market coordination for news vendor problem with price dependent demand

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Abstract

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Randomness is a common and yet quite an expensive challenge that every manufacturing and marketing system faces. In this research we focus on products for which the malfunction likelihood of the production outputs are not independent of one another, such as flu vaccine and semiconductors. Production has random yield that is interpreted as the ratio of the faultless items in a production batch. We also assume that these products have a limited marketing season, suggesting the news-vendor model with randomness on the supply side is an outstanding platform to analyze the production and marketing processes of such products. Our base model studies a single manufacturer present in a single market, where the manufacturer dynamically selects the production commitment and the price at the beginning of the production season and marketing season, accordingly. Also, we assume that demand is deterministic but price dependent.

For strategically important products such as the flu vaccine, the government may find it necessary to intervene in the pricing or manufacturing process in order to coordinate the market. The need for coordinating the market may also be met by other consumer interest groups such as distribution channel owners, think-tanks or policy making agencies. We incorporate the presence of a social planner in our model by studying the centralized case and the socially optimum production commitment and pricing schema. The challenge is to find contracts that can set the conditions for the manufacturer in a way that his optimum production and pricing policy is exactly the same as the one of the social planner. Our study introduces three contracts that can coordinate the market. In the second phase of this research, we study the same problem for the cases where there are two symmetrical manufacturers with interdependent demands. By the third phase, we study the case with one manufacturer who has the option to provide the product to two different markets with different characteristics. Our study shows that production commitment is smaller if there is no randomness and the yield ratio is known. We also show that the production commitment of the centralized setting can be either less or more than the production commitment of the decentralized setting, depending on the salvage cost. Numerical analysis, managerial insights, and conclusions are presented at the end of each chapter.

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DEDICATION

To Aghajaan va Maman
who live the love.
To those who " seek decency more than excess of knowledge".
To those who are not afraid of doubt, faith, and fear.
To those who do whatever they do, for the sake of it.
To those who, also, observe.

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Chapter 1

INTRODUCTION

1.1 Research motivation

Billions of dollars are spent every year to mitigate the effects of randomness in production. Researchers have introduced different models to analyze randomness in production, and different techniques, such as keeping a safety inventory or assigning time buffers. These have been found to mitigate, more or less the production uncertainty. In some production lines, however, certain external parameters applied to the production processes can affect all of the work-in-process inventories. In other words, the items defection likelihood are not independent of one another; therefore, the scale of unpredictability is much higher than that of the case where items are I.I.D. Examples of random yield production can be found in such industries as agriculture, chemical manufacturing, pharmaceuticals, and re-manufacturing. The complexity of the problem can increase rapidly if there are other restrictions in the production such as limited marketing season, for which flu vaccines and electronic chips are two excellent examples.

The flu vaccine production serves a multi billion dollar market that has received a great deal of attention in recent years. Every year, influenza outbreaks result in 250,000 to 500,000 deaths world-wide and incur billions of dollars in costs. According to the Word Health Organization (WHO), vaccination is the best way to control flu pandemics. Thus, influenza vaccine has received a great deal of attention in news headlines, academic research, business articles, and remarks made by high-ranking officials.

Flu vaccine production has a few characteristics that make its production planning and market coordination quite challenging. The first issue is that its production follows random yield format. In other words, the production starts with a known number of production units (i.e., eggs), but only a random fraction of those units pass the quality assurance test so that their products are approved as vaccines. This is due to various uncontrollable natural parameters, e.g., minor fluctuations in temperature, which can significantly cause the final yield to deviate from the expected. A disruption can also ruin the whole production as in the case of contamination at Powerject Pharmaceuticals of Liverpool in August of 2003 that disqualified the entire production batch. At the time, Powerject was one of only two vaccine makers approved by the FDA to sell flu vaccines in the United States. [1]. In order to study such problems, it is necessary to apply probabilistic aspects of production to the model and investigate all different possible scenarios and their associated profits and expenses.

The second issue is that flu vaccine has a rather long production run. From cultivating the virus through manufacturing, testing, and packaging, the entire production takes about seven months. Such a long production cycle makes the supplier unable to respond to any unpredicted changes in supply or demand, e.g, a red alert

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[1] ”Flu, that was close” Economist, Oct 21st 2004, http://www.economist.com/node/3315387

for an upcoming flu pandemic. A number of research projects have attempted to find faster production processes, and there have been some marginal improvements. For instance, a new technology for flu vaccine production uses dog liver cells instead of eggs as the cultivating environment. This new procedure could reduce the production time by as much as one month. However, six months is still plenty of time for a widespread flu epidemic to develop, thus highlighting the importance of planning the production precisely and comprehensively.

The third issue arises from the rapid mutation of the flu virus, which limits the usability of each version of the vaccine. New strains of flu virus emerge every year bearing different characteristics. In a reversal of prior incidence figures, 87 percent of deaths in the 2009 H1N1 influenza epidemic were among people under 65 years of age[2]. Thus, new vaccines are developed annually in order to protect from strains that constantly arising. In fact, each year's flu "cocktail" vaccine contains the weakened virus of three of the most dominant strains observed at that time, thus providing the maximum standard resistance in one dose. Unfortunately, because the manufacturer cannot store the surplus, this limits the supplier's production and marketing horizon to one year, which is a shockingly short cycle in the drug development field.

Aside from these technical issues, manufacturer barriers to entry constitute a macro level complication in the flu vaccine market. The main entry barriers for new suppliers are technology limitation, high liability costs, expensive and complicated FDA approval procedures, and narrow profit margin. Such tight market conditions seem to decrease the vaccine manufacturers interest in this market in the short term.

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[2] Shrestha, Sundar S., et al. "Estimating the burden of 2009 pandemic influenza A (H1N1) in the United States (April 2009–April 2010)." Clinical Infectious Diseases 52.suppl 1 (2011): S75-S82.

Yet, a disruption in the production of one of the manufacturers can deprive the society of its products. If there are not sufficient producers, the vaccine's supply would decrease tremendously and that would put the nation's public health at risk. Such a situation makes a perfect case for government coordination of manufacturer profit and social welfare through development of a mutually beneficial contract.

Computer device manufacturers encounter similar issues in their manufacturing and marketing. Consider a specific model of a silicon-based processor that is available in different performance levels, high and low. However, all of those processors, whether high or low, are made by the same production process whereby some random proportion of the products become high performance and the rest low. In other words, the manufacturer does not know the exact percentage of his products that will be high performance. All that the manufacturer knows is the probability distribution function that indicates the proportion of high or low performance devices. Furthermore, these devices have a restricted marketing window, typically about a year, after which a new design of the product is introduced and the current model will be out of date. The manufacturer should sell most of the products in the current period before that happens. Not having the device on hand while in demand or having it on clearance shelves can incur large opportunity costs for the manufacturer. Competition with other brands is another major challenge for the computer device manufacturers and we address that in chapter 3

In this research, we study a firm with random yield production that sells its product in a market with price-dependent demand. We assume that the firm has two decision variables, production commitment and price. The firm's objective is to find the pricing strategy and production commitment that maximize its profit.

We call it the optimum strategy for the decentralized case. Into the scenario, we include a social planner who cares not only for the profit of the manufacturer but also for the welfare of the consumers. In this light, we investigate which pricing strategy and production commitment the social planner would choose, if he had the power to make the manufacturer follow him. We call it the optimum strategy for the centralized case. The ultimate objective is to find the contracts potentially capable of setting the manufacturer's parameters in a way that the decentralized optimum policy becomes equal to the centralized optimum policy. Such contracts would enable the social planner to coordinate the market and the manufacturer in a way that both parties end up achieving optimum status, also known as a win-win situation. The rest of this chapter provides the literature review and a description of the model, including the variable and assumptions.

Chapter 2 presents a monopoly setting with a single manufacturer selling products in a single market. Chapter 3 examines the model under a duopoly setting, the difference being that the firm's demand in the duopoly setting depends on its own price and on the price of its rival. Although the reasoning path and analysis objectives remain the same, the dynamic of the model and solutions fundamentally change. The duopoly model extension provides the opportunity to study the competition and make a thorough comparison between the duopoly and the monopoly settings.

In Chapter 4, we study a single manufacturer who can present his product in two different markets with different characteristics. One is larger but more sensitive to the price, which resembles the situation in developing countries or economically challenged neighborhoods, while the other is relatively small but also less price sensitive. This extension shows us how the social planner can possibly persuade the manufacturer to assign a higher proportion of its products to the poor market. It also gives us valuable insights as to the effect of market parameters on the optimum production commitment. In that chapter, we slightly relax the assumption about the linearity of the price-demand correlation by updating it to be convex and piece-wise linear. Suggestions for future research, appendices, and references are provided at Chapter 5

1.2 Literature study

This research is related to three streams of research in operations management. First, our study contributes to the academic literature of the random yield production problem. The review paper by Yano and Lee (2012) provides a well-founded classification of papers in this field. According to the paper, there have been three main approaches to modeling randomness in production in the literature. The first group of models studies production lines in which products are independent from one another. Thus, if one product is defective, it does not change the chance that other in the batch are defective. Probability distributions based on the Bernoulli distribution, such as a bimodal distribution, have been the most frequently used approach to model randomness in this subset of the literature.

The second group of models describes production processes in which products are dependent on one another or an exogenous factor. These systems are commonly studied with a proportional yield model in which the yield ratio is stochastic. The production batch size, namely production commitment, is the manufacturer's decision variable, and yield ratio is a random variable with known distribution. The third class of models describes production models in which the yield ratio is random, yet it depends on the production commitment or the number of rounds of production that have transpired in the past. Such models are commonly known as systems with positive or negative learning loops.

In this research, we study a single-stage, single-period model with stochastically proportional yield and no learning effect. There are quite a few papers in the literature similar to this one. White (1970) studied a model with linear revenue and expected salvage and shortage costs. Gerchak, Parlar, and Vickson (1986) introduced a model with linear profit and shortage and salvage costs. This was followed by another Henig and Gerchak (1990) model that extended the results for the initial inventory case. They proved the existence of an order point for initial inventory under which an order should be placed. Ehrhardt and McClelland (1987) studied a model with linear profit and shortage and salvage costs. They allowed the set-up cost to be positive. They introduced two heuristic policies for the case with no set-up cost; one is the standard newsboy solution and second is the newsboy solution adjusted for the average fraction defective. Their results show that the second heuristic is pretty accurate when the ratio of holding cost to shortage cost is greater than two.

A rich literature also exists on multiple-period production systems. Many of the papers in this literature are adjusted forms of the EOQ model. An early paper with such settings is Shih(1980), which shows that the optimum production commitment decreases as uncertainty in production increases. Lee and Rosenblatt (1985) and Xhang and Gerchak(1990) have studied similar problems, except that they allow for production inspections and adjustment before the production is completed. More recently, Cachon and Kok (2007) investigated the importance of correctly estimating salvage value.

Other related literature studies various forms of price-demand dependency in the news-vendor model. Two approaches for modeling the dependency of the price and the demand are an additive relation and a multiplicative relation; several studies compare the characteristics of these two approaches. Petruzzi and Dada (1997) studied the two demand-price model for the classic news vendor problem. Raz and Porteus (2006) studied a price-setting news vendor problem with general form of uncertainty. Salinger and Ampudia (2011) studied the Lerners relationship and ways of interpreting it for these two models by comparing them with the deterministic case.

The second stream of related operations management literature involves contract theory. Taylor and Yadav (2011) considered whether subsidizing sales would result in higher availability of the goods for the market, and they concluded that the answer depends on whether customers' valuation of the product is homogeneous or heterogeneous. They addressed positive externality and uncertainty in demand in their paper; however, their model did not cover either random yield or price dependency of the demand.

Recently, a few papers have examined a combination of contract theory and random yield. Keren (2009) combined the news vendor problem with random yield, adding a distributer between the retailer and manufacturer. A working paper by **?** Inderfurth and Clemens (2011) studies the contracts for a news-vendor problem with random yield, with and without presence of an emergency procurement source. None of these papers, however, considers the dependency between price and demand, nor the dynamic decision-time agenda that this one does.

Certain subsets of literature study specific markets. Production planning for flu vaccine is one area. Mamani et al. (2012) studied the effect of subsidizing on flu vaccine coordination. Arifoglu et al. (2010); Chik, Mamani, and Simchi-Levi (2008); and Cho (2010) studied market dynamics and the different coordinating contracts for a single period product with random yield. Wu et al. (2005) provided a review of the literature on capacity allocation in the high-tech industry. Kouvelis and Milner (2002) showed that higher supply uncertainty increases vertical integration profitability. Van Mieghem (1999) and Van Miegem and Dada (1999) studied the effect of capacity, pricing, and inventory aspects of the problem.

The literature on remanufacturing planning, recognized as a subset of industrial engineering literature, studies a similar problem. Bakal and Akcali (2009) studied a remanufacturing unit that takes loads of defective items as an input, while only a random proportion of them can be refurbished. They assume that both demand and supply are price dependent and take a dynamic decision-making approach that is similar to our study. However, they neither study the stochastic and deterministic case nor the other extensions of the model, such as market coordination or the duopoly case.

1.3 Model

We propose a single-item, single-run, single-period model that aims to maximize the manufacturer's profit and takes into account shortage, surplus, and production costs. We assume that the manufacturer uses a dynamic approach to set the production commitment and price. A list of all parameters and variables in the model is presented in Table 1.1

Assumption 1. *Demand is deterministic and a decreasing linear function of price.*

$$D_p = a - bp.$$

Assumption 2. *The final production output is proportional to the production commitment and yield ratio:*

$$q = N \times \nu.$$

Since the manufacturer sets the price, sales is the minimum of demand and final production output:

$$k = min[q, D] = min[a - bp, N\nu]. \tag{1.1}$$

Equation 1.1 indicates that there are two decision variables in this model, price and the production commitment. Also, the only random variable is the production yield ratio ν. We do not hold any specific assumption regarding the shape of the probability distribution function of the yield ratio, unless it is specifically mentioned. However, we assume that the mean and standard deviation (i.e., μ and σ) are known.

We start studying the monopoly setting with the decentralized case in which the manufacturer sets both decision variables in order to maximize his profit. A social planner is defined as an agent whose objective is to optimize the overall benefit top society. After driving the pricing strategy and the total profit function for the centralized scenario, it is possible to compare the centralized and decentralized scenarios. We intend to find contracts that can potentially align the manufacturer's optimum strategy and social interests; in other words, we intend to find contracts that can coordinate the market.

Table 1.1 Parameters and variables in the model

Parameters	a	Society's population or maximum market size
	b	Demand's sensitivity to price
	c	Marginal production cost
	c_{soc}	Maximum infection cost per person
	s	Marginal salvage cost
	h	Marginal shortage cost
	μ	Expected production yield ratio
	σ	Standard deviation of production yield ratio
Random variables	ν	Production yield ratio
	F , f	Density and cumulative probability function of ν
Decision variables	N	Production commitment
	p	Retail price
Model variables	q	Final production output
	D_p	Demand as a function of price
	k	Manufacturer's sales
	π	Manufacturer's profit function
Contracts variables	c_m	Subsidized/taxed production cost
	stx	Subsidy or tax on production
	t	Threshold for sales rebate
	r	Sales rebate value
	s^*	Adjusted salvage cost in cost sharing

In Chapter 3 we present the duopoly case, in which we assume that there are two manufacturers with interdependent demands. If one of them increases his price, his perceived demand decreases and his competitor's demand increases. The random yield ratio is assumed to be the same for both manufacturers. Initially, we study the case where the manufacturers are asymmetrical and derive optimum pricing strategy for that case. We later move on to the case in which the manufacturers are symmetrical, and derive the optimum pricing and production strategy for them. We also introduce three contracts for this setting.

The double market setting is covered in Chapter 4 using the same production model except that the manufacturer has the option to present her product in two different markets with different characteristics. Some complimentary assumptions set these markets in such a way that one of them is larger in size and more price sensitive, compared to the other one.

The model presented in this research is designed to be as robust as we can make it. It does not hold any assumptions about probability density function (pdf) so that, no matter what type of pdf a manufacturer encounters, our results stay legitimate. We also replace the assumption about the linearity of the demand-price relationship with a more general one in the double market case. Our proof shows that our suggested contracts can match manufacturer and social planner production policies, no matter which one is larger. We also discuss whether the information needed for applying the contract is available for the social planner.

Chapter 2

MONOPOLY SETTING WITH A SINGLE MARKET

This chapter explores the random yield problem with a single manufacturer who sells products in a single market. We study this problem in a dynamic setting, which means the manufacturer sets the price after production is complete and the realized production quantity is known. We begin by studying the decentralized stochastic case in which the manufacturer's goal is to maximize profit. As we evaluate this problem in a stochastic setting, we also look at the deterministic case to study the effect of randomness on the manufacturer's optimal strategy. Next, we extend the stochastic model to depict a social planner and the manufacturer taking turns to set production capacity and price. We propose three contracts that can be implemented to coordinate the market. Finally, the chapter concludes with numerical results that provide a sensitivity analysis of different parameters to the optimal policy.

2.1 Decentralized settings

In a decentralized setting, the manufacturer sets production capacity and price sequentially. Setting production capacity at the outset constitutes an effort to optimize profit. Then, in order to earn as much revenue as possible, he sets the price and synchronizes demand with the realized production quantity. A relatively high price secures high marginal revenue, but it reduces demand and potentially produces salvage costs. On the other hand, a low price increases demand with low marginal benefit and potentially incurs high shortage costs. The manufacturer's challenge is not only to find the optimum production capacity but also to set the optimum price for maximizing profit.

2.1.1 The main model

As discussed earlier, the manufacturer determines production commitment N and price p, respectively at the beginning and end of the production season. Realized production quantity q is perceived at the end of production season and the manufacturer uses this information to set the price. In order to find the optimum strategy, we solve the problem backwards. Thus, we begin by setting up the profit function at the second stage, when the manufacturer is concerned about maximizing the revenue and minimizing inventory cost. The profit function in the second stage takes the following form:

$$\begin{aligned} \pi_q(p) &= \text{Price} \times \text{Sales} - [E(\text{Salvage cost}) + E(\text{Shortage cost})] \qquad (2.1) \\ &= \begin{cases} p \times D_p - s \times (q - D_p) & \text{if } q > D_p \\ p \times q - h \times (D_p - q) & \text{if } q < D_p. \end{cases} \end{aligned}$$

Given the optimal price p^* we can write the profit function at the first stage as a function of production commitment N as follows:

$$\pi_(N) = p^* \times \text{Sales} - [E(\text{Salvage cost}) + E(\text{Shortage cost}) + \text{Production cost}]. \quad (2.2)$$

Notice that the production cost parameter appears in the profit function at the production stage depicted in Equation 2.2 but not at the pricing stage as shown in Equation 2.1. As a result of dynamic decision making, production cost is incurred at the first stage so there is no further production cost by the time that manufacturers has reached the second stage. Lemma 1 formulates the profit function for the second stage, pricing.

Lemma 1. *The manufacturer's optimum pricing strategy and profit function for a given q is:*

$$\text{if} \quad \begin{cases} q < \frac{a+bs}{2} & \Rightarrow \quad p^* = \frac{a-q}{b},\ D^* = q \ \Rightarrow \pi^* = \frac{aq-q^2}{b}, \\ q \geq \frac{a+bs}{2} & \Rightarrow \quad p^* = \frac{a-bs}{2b},\ D^* = \frac{a+bs}{2} \ \Rightarrow \pi^* = \frac{(a+bs)^2}{4b} - sq. \end{cases}$$

Proof. See Appendix A.

□

The optimum pricing strategy is a function of realized production quantity. As Lemma 1 demonstrates, if the realized production quantity is less than threshold $\theta = \frac{a+bs}{2}$, the optimum price equalizes demand and supply and clears the market, generating no surplus or shortage cost. However, if the realized production quantity goes above that threshold, it is in the manufacturer's best interest not to reduce the price below $\frac{a-bs}{2b}$, although this results in surplus charges.

Assumption 3. $\frac{a}{b} > \frac{c}{\mu}$

The maximum feasible price is $\frac{a}{b}$ because a price larger than $p = \frac{a}{b}$ would result in a negative value for demand. On the other hand, $\frac{c}{\mu}$ represents the expected unit production cost. Intuitively, the maximum feasible retail price should be higher than the expected unit production cost.

Given the optimum pricing strategy, Proposition 1 characterizes the optimal production commitment for the decentralized case.

Proposition 1. *The manufacturer's optimum production commitment in the decentralized case has a unique solution and is equal to the root of function $\mathcal{A}(N)$ as indicated in Equation 2.4.*

Proof. The manufacturer's profit function can be written as follows:

$$\begin{aligned}\pi(N) &= \frac{aq-q^2}{b}\ \Big|_{q=0}^{\frac{a+bs}{2}} + \frac{(a+bs)^2}{4b} - sq\ \Big|_{q=\frac{a+bs}{2}}^{\infty} - cN \\ &= (\frac{a}{b}+s)N\int_0^{\frac{a+bs}{2N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{N^2}{b}\int_0^{\frac{a+sb}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu + (\frac{(a+bs)^2}{4b})\bar{F}(\frac{a+sb}{2N}) - s\mu N - cN.\end{aligned} \tag{2.3}$$

By taking the derivative of equation 2.3 we have:

$$\frac{\partial}{\partial N}\pi(N^*) = \mathcal{A}(N^*) = (\frac{a}{b}+s)\int_0^{\frac{a+bs}{2N^*}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N^*}{b}\int_0^{\frac{a+bs}{2N^*}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c = 0. \tag{2.4}$$

The following properties can be proven using simple algebra:

$$\begin{aligned}\frac{\partial}{\partial N}\mathcal{A}(N) &= -\frac{2}{b}\int_0^{\frac{\theta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu < 0\,, \\ \lim_{N\to 0}\mathcal{A}_{(N)} &= \frac{a}{b}\mu - c > 0 \qquad \text{, and} \\ \lim_{N\to\infty}\mathcal{A}_{(N)} &= -s\mu - c < 0\,,\end{aligned}$$

when $\theta = \frac{a+bs}{2}$.

These results show that $\mathcal{A}_{(N)}$ is a monotonically decreasing function with a positive value at $N = 0$ and a negative limit value at the positive infinity. Thus there exists a unique N^* for which $\mathcal{A}_{(N^*)} = 0$ and $\pi_{(N^*)}$ is the maximum profit. □

Also, we can show that for any value of N, one and only one of these equations holds:

$$0 = \mathcal{A}_{N^*} \leq \mathcal{A}_N \Leftrightarrow N^* \geq N \text{ or}$$

$$0 = \mathcal{A}_{N^*} > \mathcal{A}_N \Leftrightarrow N^* < N.$$

2.1.2 Measuring the effect of uncertainty on the optimum production policy

In order to measure the effect of uncertainty, we study the decentralized model with deterministic production. Note that since there is no randomness in production, the manufacturer can perfectly match production and demand; therefore, the optimum price has a singular closed form solution. After determining the optimum strategy, we compare the optimum production commitment for the deterministic case $\dot{N}$ with that of the stochastic case N^*.

Lemma 2. *If the production yield is deterministic, the optimum production commitment is equal to* $\dot{N} = \frac{a}{2\mu} - \frac{bc}{2\mu^2}$.

Proof. If there is no randomness in production, the manufacturer can set the supply exactly equal to its goal demand and avoid surplus and shortage cost. We can show this result as:

$$\pi = pD_p - Nc.$$

Since supply is predictable and price sets the demand equal to supply, the manufacturer has only one decision variable and that is the production commitment. We can rewrite the profit function as:

$$D = q \Leftrightarrow \pi = \frac{a - D}{b} \times D - N \times c = \frac{a - \mu N}{b} * \mu N - cN = (\frac{a\mu}{b} - c)N - \frac{\mu^2}{b}N^2.$$

By using the first order condition, the optimum production commitment for the deterministic case is determined to be

$$\dot{N} = \frac{a}{2\mu} - \frac{bc}{2\mu^2} \qquad \dot{p} = \frac{1}{2}(\frac{a}{b} - \frac{c}{\mu}). \tag{2.5}$$

□

As discussed in Assumption 3, $\frac{a}{b}$ is the maximum feasible value for price and $\frac{c}{\mu}$ is the expected production cost. Therefore, the optimum price for the deterministic case $\dot{p}$ lies at the mid point of the feasible price range. Proposition 2 compares the optimum production commitment for the stochastic case with for the deterministic case.

Proposition 2. *In the decentralized setting, the optimum production commitment in the deterministic case is always less than or equal to of stochastic case.*

Proof. We define the auxiliary function $S(l)$ as

$$\mathrm{S}(l) = \int_0^{\mu(1+l)} \left((1+l)\frac{\nu}{\mu} - (\frac{\nu}{\mu})^2 \right) f(\nu)\, \mathrm{d}\nu - l.$$

In Appendix B, we show that

$$\forall l \quad \mathrm{S}(l) \geq 0.$$

In Appendix C, we can show that if $l = \frac{bs+bc/\mu}{a-bc/\mu}$,

$$\int_0^{\mu(1+l)} \left((1+l)\frac{\nu}{\mu} - (\frac{\nu}{\mu})^2 \right) f(\nu)\, \mathrm{d}\nu - l \geq 0 \quad \Longleftrightarrow \quad \mu^2 - \int_0^{\frac{\theta}{\dot{N}}} \nu^2 f(\nu)\, \mathrm{d}\nu > \frac{\theta}{\dot{N}} \left(\mu - \int_0^{\frac{\theta}{\dot{N}}} \nu f(\nu)\, \mathrm{d}\nu \right). \tag{2.6}$$

The right hand side of Equation 2.6 is the necessary and sufficient condition for $\mathcal{A}(\dot{N}) > 0$. As shown in Proposition 1, $\mathcal{A}(N)$ is a decreasing function; therefore,

$$\mathcal{A}(\dot{N}) > 0 \quad \Longleftrightarrow \quad \dot{N} \leq N^*. \tag{2.7}$$

□

Proposition 2 shows that the production commitment strictly increases as randomness increases. This result is intuitive because a higher production commitment helps the manufacturer to secure himself against high randomness. In the next section, we analyze the random yield problem when a social planner is also a decision maker.

2.2 Decision sharing scenarios with a social planner

In this section, we study the market in which there is also a social planner whose goal is to maximize the society's welfare. Following the definition used in the economics literature, we assume that social welfare is equal to the total utility gained by consumption of the product minus associated costs. In other words, the social planner's objective is to maximize the society's benefit, which includes the manufacturer's profit and the society's total consumption utility.

As Assumption 1 describes, bp represents the size of the population that does not consume the product. We assume that consumption utility U_{con} is different for different people and it follows a uniform distribution $U_{con} \subset f[0, U_{max}]$, where U_{max} is the maximum consumption utility. It can also be interpreted that for any specific price p, the proportion of society whose consumption utility is more than p is equal to $1 - \frac{p}{U_{max}}$. Accordingly, this is the proportion of society that buys the product. We can determine the proportion that buys the product by locating indifferent customers whose consumption utility is exactly equal to the retail price. For indifferent customers, we have

$$U_{con} = p, \quad \frac{U_{con}}{U_{max}} \times a = D = a - b * p \quad \Rightarrow \quad U_{max} = \frac{a}{b}.$$

While the manufacturer's goal is to optimize his own profit, the social planner's objective is to maximize the society's total utility. It is worth mentioning that the social planner's objective function does not depend on the price. From his point of view, price is a value transferred internally between different parties in the society so that total utility does not change. Using basic price theory, Lemma 3 calculates utility for each player at the supply-demand equilibrium point.

Lemma 3. *If v units of the product are produced and consumed, the utility distribution between different parties can be shown as:*

$$
\begin{aligned}
U_{v,society} &= \frac{v^2}{2b}, \\
U_{v,manufacturer} &= v\frac{a-v}{b}, \\
U_{v,lost} &= \frac{(a-v)^2}{2b}\text{, } \textit{and} \\
U_{V,total} &= \frac{v}{2} * (\frac{2a-v}{b}) = \frac{2av-v^2}{2b}.
\end{aligned}
$$

Proof. If demand is v, the equilibrium price is $p_v = \frac{a-v}{b}$. Figure 2.1 depicts the utility distribution when consumption is equal to v. The gray area indicates the society's utility gained by consumption of v units of the product and the square indicates the manufacturer's utility. □

Using the social planner's profit function, we can find the optimum production and pricing strategy. Moreover, we can ascertain which contracts can coordinate the manufacturer's optimum production commitment with that of the social planner. Two decision variables and two decision-makers combine to provide four different scenarios. We are interested in identifying the optimum pricing and production policies for each of the different scenarios. Note that all four scenarios follow the dynamic price and production setting.

In the first scenario, commonly known as the decentralized scenario, the manufacturer owns the right to set both decision variables as covered in section 2.1.1. In the second scenario, partially decentralized, the social planner decides only the

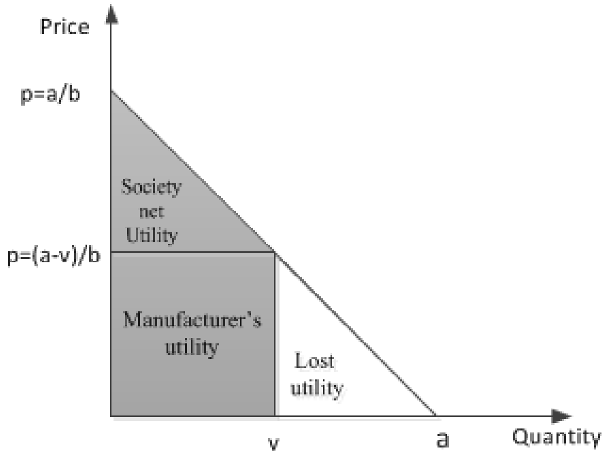


Figure 2.1 Utility gained by consumption of v units of the product

P

	Decision maker	*Manufacturer*	*Social Planner*
N	*Manufacturer*	Scenario 1, decentralized	Scenario 3, partially centralized
	Social Planner	Scenario 2, partially centralized	Scenario 4, centralized

Table 2.1 Four different decision making scenarios

production commitment. When the production season ends and realized production quantity is measured, the manufacturer sets the retail price. The third scenario, partially centralized, is the same as the second scenario except that the decision makers switch decision variables. In the fourth scenario, centralized, both decision variables are determined by the social planner. These four scenarios are summarized in Table 2.1.

2.2.1 Scenario 1, decentralized

Scenario 1, decentralized is the model we discussed in section 2.1.1. As shown in Equation 2.4, we can find the optimum production commitment for the decentralized problem by solving the $\mathcal{A}(N)$ function. Also, Lemma 1 propounds the optimum pricing strategy. Using N^*_{dec}, the optimum production commitment of this scenario, as the reference point for the decentralized case, we compare it with the optimum production commitment for the other scenarios.

2.2.2 Scenario 2, partially decentralized

In the second scenario, the social planner sets the production commitment and the manufacturer sets the retail price after the realized production quantity becomes available. Lemma 4 provides the optimal pricing and production commitment strategy for the partially decentralized case.

Lemma 4. *In the partially decentralized scenario the manufacturer has a unique optimum production commitment that is equal to the root of* $\mathrm{B}(N^*)$

$$\mathrm{B}(N^*) = (\frac{a}{b} + s) \int_0^{\frac{\theta}{N^*}} \nu f(\nu)\, \mathrm{d}\nu - \frac{N^*}{b} \int_0^{\frac{\theta}{N^*}} \nu^2 f(\nu)\, \mathrm{d}\nu - s\mu - c.$$

Proof. In the partially decentralized scenario, the optimal price is a function of the realized production quantity. As in the decentralized scenario, the manufacturer sets the price to optimize his benefit based on realized production quantity. Therefore, the optimum pricing strategy in the partially decentralized scenario is exactly the same as the optimum pricing strategy for the decentralized scenario, as shown in equation 2.3 and proved in Appendix A. The manufacturer's optimal pricing strategy in the partially decentralized scenario is as follows:

$$p^*_{2,M} = \begin{cases} \frac{a-bs}{2b} & \text{if} \quad q > \frac{a+bs}{2} \\ \frac{a-q}{b} & \text{if} \quad q < \frac{a+bs}{2} \end{cases} \Rightarrow D_{2,M} = \begin{cases} \frac{a+bs}{2} = \theta \\ q \end{cases} .$$

The society's total utility function in the second stage is:

$$TU_{2,Soc,2} = \begin{cases} \frac{2a\theta-\theta^2}{2b} - s(q-\theta) & \text{if} \quad q > \frac{a+bs}{2} \\ \frac{2aq-q^2}{2b} & \text{if} \quad q < \frac{a+bs}{2} \end{cases} .$$

Now, we can set up the society's total utility function in the first stage as:

$$TU_{2,Soc,1} = \pi_2 = (\frac{2a\theta-\theta^2}{2b}+s\theta)\bar{F}(\frac{\theta}{N})+(\frac{a}{b}+s)N\int_0^{\frac{\theta}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{N^2}{2b}\int_0^{\frac{\theta}{N}} \nu^2 f(\nu) - sN\mu - cN.$$

Using the first order condition, we have:

$$\mathrm{B}(N_2^*) = \frac{\partial}{\partial N}\pi_2 = (\frac{a}{b}+s)\int_0^{\frac{\theta}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{N}{b}\int_0^{\frac{\theta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c = 0.$$

We can search for the optimum production commitment using the first order condition, B = 0. Appendix D shows that B(N) has the three properties:

$$\frac{\partial}{\partial N}\mathrm{B} =< 0\,,$$

$$\lim_{N\to 0}\mathrm{B} => 0, \text{and}$$

$$\lim_{N\to\infty}\mathrm{B} =< 0.$$

These properties show that function B is monotonically decreasing with a unique root in N, which is the optimum production commitment for the partially decentralized case. □

2.2.3 Scenario 3, partially centralized

In the third scenario, the manufacturer chooses the production commitment and then the social planner sets the retail price. It is important to mention that the optimum pricing strategy for the partially centralized scenario sets the demand and the realized production quantity equally. That is because the social planner sees the retail price as an interior variable between manufacturer and customers and it acts to equalize demand and supply. The social planner's only interest in the price is to set it such that all of the realized production quantity is consumed and there is no shortage cost. In other words, the social planner sets the price to clear the market. Lemma 5 derives the optimum production commitment for the partially centralized scenario.

Lemma 5. *Optimum production commitment for the partially centralized scenario is equal to*

$$N^*_{3,M} = \frac{a\mu - bc}{2(\sigma^2 + \mu^2)}.$$

Proof.

$$q = D = a - bp \Rightarrow p^*_{3,SP} = \frac{a-q}{b},$$

so then, we have

$$U_{3,M} = \pi_3 = p^* q - cN = \int_0^\infty \left(\frac{aN}{b}\nu - \frac{N^2}{b}\nu^2\right) f(\nu)\, \mathrm{d}\nu - cN = \frac{aN}{b}\mu - \frac{N^2}{b}E(\nu^2).$$

By taking the first order derivative, then, we have

$$N^*_{3,M} = \frac{a\mu - bc}{2(\sigma^2 + \mu^2)}. \tag{2.8}$$

□

It is worth noting that $N^*_{3,M}$ in equation 2.8 neither depends on the probability distribution function of the yield ratio nor on the salvage cost.

2.2.4 Scenario 4, centralized

In the fourth scenario, the social planner sets both decision variables. As in the partially centralized scenario, the social planner clears the market whenever he has the option to set the price.

Lemma 6. *The optimum production commitment for the centralized scenario is equal to*

$$N^*_{cen} = \frac{a\mu - bc}{E(\nu^2)} = \frac{a\mu - bc}{\mu^2 + \sigma^2}. \tag{2.9}$$

Proof. As we saw in Figure 2.1, society gains a total utility of $\frac{2aq-q^2}{2b}$ by consuming q units of the product. Therefore, we have:

$$U_{4,Soc} = \pi_4 = \int_0^\infty \frac{2aq - q^2}{2b} f(\nu)\,\mathrm{d}\nu - cN = \frac{a\mu}{b}N - \frac{E(\nu^2)N^2}{2b} - cN.$$

By taking the first order condition, we have:

$$\frac{\partial}{\partial N}\pi_4 = \frac{a\mu}{b} - \frac{E(\nu^2)}{b}N - c \Rightarrow N^*_{cen} = \frac{a\mu - bc}{E(\nu^2)} = \frac{a\mu - bc}{\mu^2 + \sigma^2}.$$

□

2.2.5 Comparison of the decision sharing scenarios

By comparing the optimum production commitments of the four different scenarios, we attempt to determine which scenario and which conditions result in the largest production commitments.

Lemma 7. *The optimum production commitment for the partially decentralized scenario is equal to or larger than that of the decentralized scenario.*

Proof. From Lemma 4 we have,

$$\mathcal{A}(N) = (\frac{a}{b} + s)\int_0^{\frac{\theta}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N}{b}\int_0^{\frac{\theta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c.$$

and from Lemma 4 we have,

$$\mathrm{B}(N) = (\frac{a}{b} + s) \int_0^{\frac{\theta}{N}} \nu f(\nu)\, \mathrm{d}\nu - \frac{N}{b} \int_0^{\frac{\theta}{N}} \nu^2 f(\nu)\, \mathrm{d}\nu - s\mu - c.$$

which implies that

$$\mathrm{B}(N) = \mathcal{A}(N) + \frac{N}{b} \int_0^{\frac{\theta}{N}} \nu^2 f(\nu)\, \mathrm{d}\nu \;\Rightarrow\; \mathrm{B}(N) \geq \mathcal{A}(N).$$

The second term on the right side of this equation is positive, suggesting that $\mathrm{B}(N)$ is always greater than $\mathcal{A}(N)$. Knowing that both $\mathcal{A}(N)$ and $\mathrm{B}(N)$ are monotonically decreasing functions and have a unique root, we can conclude that the cross point of $\mathrm{B}(N)$ with the x-axis is on the right hand side of the crosspoint of $\mathcal{A}(N)$ and the x-axis.Thus, the optimum production commitment for the partially decentralized scenario is always greater than or equal to the optimum production commitment for the decentralized scenario as follows:

$$\mathrm{B}(N) \geq \mathcal{A}(N) \Rightarrow N_2^* \geq N_{dec}^* \ .$$

□

Lemma 8. *The optimum production commitment for the centralized scenario is twice that of the optimum production commitment for the partially centralized scenario.*

Proof. From Lemma 5 and Lemma 6 we have

$$N_{cen}^* = \frac{a\mu - bc}{(\sigma^2 + \mu^2)} = 2\frac{a\mu - bc}{2(\sigma^2 + \mu^2)} = 2N_3^*.$$

□

Assumption 4. *The feasible domain of the salvage cost is*

$$s \in [-\frac{c}{\mu}, \infty).$$

Salvage cost presumably has no upper limit; however, it is limited on the negative side. In fact, the salvage value of an item cannot exceed the expected production cost. Otherwise, the manufacturer can earn profit from surplus products, which means his optimum production commitment should be positive infinity.

Comparing the decentralized and centralized scenarios yields the greatest insight in this section. We can show that, based on the model parameters, either scenario can result in a larger production commitment. Proposition 3 shows that while changing salvage cost does not affect the optimum production commitment for the centralized scenario, it changes the optimum production commitment for the decentralized scenario to be either less or more than that for the centralized scenario.

Proposition 3. *Depending on the value of the salvage cost, the optimum production commitment for the decentralized scenario can be either less or more than the optimum production commitment for the centralized scenario.*

Proof. As we discussed in Proposition 1,

$$\begin{aligned} \mathcal{A}(N) < 0 \quad &\Leftrightarrow \quad N > N^*_{dec}\text{, and} \\ \mathcal{A}(N) > 0 \quad &\Leftrightarrow \quad N < N^*_{dec}. \end{aligned} \tag{2.10}$$

While $\mathcal{A}(N)$ is the derivative of the profit function for the decentralized scenario and N^*_{dec} is the solution for the decentralized problem as shown in equation 2.4. Appendix F shows that, if we insert the optimum production commitment for the centralized scenario N^*_{cen} in $\mathcal{A}(N)$, we can tease out the following properties:

$$\lim_{s\to-\frac{c}{\mu}} \mathcal{A}_{N^*_{cen}}(s) > 0 \quad \text{and}$$

$$\lim_{s\to\infty} \mathcal{A}_{N^*_{cen}}(s) < 0 \quad \text{and}$$

$$\frac{\partial}{\partial s}\mathcal{A}_{N^*_{cen}}(s) < 0$$

According to these results, $A(N^*_{cen})$ can have either a positive or a negative value, which according to Equation 2.10 means that N^*_{cen} can either be less than or more than N^*_{dec}. □

Notice that the optimum production commitment for the centralized setting N^*_{cen} does not depend on the salvage cost as mentioned in Equation 2.9. However, N^*_{dec} changes when salvage cost varies. Section 2.3.3, covers a buy back contract and includes a detailed discussion of the effect of salvage cost on the optimum production commitment for the decentralized scenario.

2.3 Contracts and coordination policies in the monopoly setting

As shown in the previous section, the optimum production commitment from the manufacturer's perspective may be different from the production commitment that maximizes society's total utility. Therefore, the social planner may be willing to provide a package of financial incentives to the manufacturer in order to align his financial objectives with society's optimum strategy. If a specific financial incentives package, namely a contract, is capable of providing such an alignment of interests, we describe it as a coordinating contract. More specifically, a coordinating contract should be able to set the optimum production commitment of the manufacturer equal to the optimum production commitment of the social planner. The sufficient and necessary condition for this property is

$$\mathcal{A}(N^*_{cen})_{\text{contract}} = 0. \tag{2.11}$$

As Proposition 3 shows, N^*_{cen} can be either larger or smaller than N^*_{dec}. Thus, a coordinating contract should be able to either increase or decrease N^*_{dec} to match it to N^*_{cen}. We study three types of contract in this section and show how they can coordinate the optimum production commitment of the manufacturer and the social planner.

2.3.1 Subsidizing/taxing production cost

The social planner may be able to influence the manufacturer's decision-making by affecting the production cost. He can do so by subsidizing or taxing the production cost. In our model, c represents the original production cost and c_m represents the manufacturer's realized production cost. According to the contract, the social planner provides or collects the difference between c_m and c. The next proposition shows why the manufacturer's optimum production commitment can be altered by changing the manufacturer's production cost.

Proposition 4. *The subsidizing/taxing production cost contract is coordinating. We can write that in terms of formulation as:*

$$\exists\, c_m \mid \mathcal{A}(N^*_{cen})_{C_m} = (\frac{a}{b} + s) \int_0^{\frac{\theta}{N^*_{cen}}} \nu f \nu \,\mathrm{d}\nu - \frac{2N^*_{cen}}{b} \int_0^{\frac{\theta}{N^*_{cen}}} \nu^2 f(\nu) \,\mathrm{d}\nu - s\mu - c_m = 0.$$

Proof. Following Remark 2.11, we replace N^*_{cen} in the $\mathcal{A}(N)$ function and derive the formula for it so that

$$\begin{aligned}\mathcal{A}(N^*_{cen})_{C_m} =&(\frac{a}{b}+s)\int_0^{\frac{\theta}{N^*_{cen}}} \nu f \nu \,\mathrm{d}\nu - \frac{2N^*_{cen}}{b}\int_0^{\frac{\theta}{N^*_{cen}}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c_m \\ =&(\frac{a}{b}+s)\int_0^{\frac{a+bs}{2(a-\mu-bc)}\times(\mu^2+\sigma^2)} \nu f \nu \,\mathrm{d}\nu \\ &-\frac{2(a\mu-bc)}{\mu^2+\sigma^2}\int_0^{\frac{a+bs}{2(ax\mu-bc)}(\mu^2+\sigma^2)} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c_m.\end{aligned} \tag{2.12}$$

There is only one c_m in Equation 2.12, which is the only coordinating variable that the social planner has. As discussed in Proposition 1, $\mathcal{A}(N)$ is a monotonically decreasing function in N with a unique root. Equation 2.12 shows that changing c_m shifts the $\mathcal{A}(N)$ function up and down, which can change the cross point of the $\mathcal{A}(N)$ function and the x axis; i.e., N^*_{dec}. We can use this relation to set the value of N^*_{dec} equal to N^*_{cen} and have a coordinating contract. □

Lemma 9. *The optimum adjustment that the subsidizing/taxing contract applies to production cost is equal to* $stx = -\mathcal{A}(N^*_{cen})$.

Proof. Define stx as the adjustment value for the production cost that coordinates the market, such that:

$$c - stx = c_m \quad \Leftrightarrow \quad c - c_m = stx.$$

If the value is coordinating, we should have:

$$\begin{aligned}&\mathcal{A}(N^*_{cen})_{c_m} = 0 \\ &\Rightarrow \mathcal{A}(N^*_{cen})_{c_m} = (\frac{a}{b}+s)\int_0^{\frac{\theta}{N}^*_{cen}} \nu f \nu \,\mathrm{d}\nu - \frac{2N^*_{cen}}{b}\int_0^{\frac{\theta}{N}^*_{cen}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - (c - stx) = 0 \\ &= \mathcal{A}(N^*_{cen}) + stx \\ &\Rightarrow stx = -\mathcal{A}(N^*_{cen}).\end{aligned}$$

□

We can summarize the contract as follows:

If $N^*_{dec} < N^*_{cen} \iff \mathcal{A}(N^*_{cen}) < 0 \iff stx > 0 \iff$ Subsidy shifts the function up

If $N^*_{dec} > N^*_{cen} \iff \mathcal{A}(N^*_{cen}) > 0 \iff stx < 0 \iff$ Tax shifts the function down.

If the optimum production commitment for the decentralized case is less than that of the centralized case, the manufacturer's observed risk of having surplus items is relatively high. In this case, the social planner can take some of that risk off the manufacturer's shoulders by reducing production costs. This way, the manufacturer has less expected loss when production yield and sales happen to be relatively low. This incentivizes the manufacturer to increase his production commitment and equalize it to the social optimum production commitment. The taxing case is the same process in reverse, as it increases the production cost and causes the manufacturer to lower his production commitment.

The subsidizing/taxing contract produces benefits and complications. Our results show that this contract can coordinate the market in both underproduction and overproduction cases, which is a plus for this contract. Enforcing the contract also seems to be fairly straight-forward. However, acquiring the information needed to set up the contract, specifically for finding $\mathcal{A}(N^*_{cen})$, can be a challenge for the social planner since some of that information could be intellectual and technological assets of the manufacturer. We compare different contracts in terms of cost efficiency for the social planner in subsequent sections. Total cost to the social planner in this contract can be determined by:

$$stx = -\mathcal{A}(N^*_{cen}) \Rightarrow TC_{\text{social planner}} = -\mathcal{A}(N^*_{cen}) \times N^*_{cen}. \tag{2.13}$$

Note that this equation can take the form of either cost or earning for the social planner, depending on the value of $\mathcal{A}(N^*_{cen})$.

2.3.2 Sales rebate contract

In the sales rebate contract, the social planner pays r per sales unit to the manufacturer if sales exceed a previously set threshold of t. The social planner gets to set the contract parameters r and t, and he does that to match the manufacturer's optimum production commitment with his own. The profit function of the manufacturer for this contract is:

$$\pi_{reb} = p \times min(q, D) - s \times (q - min(q, D))^+ + r \times (min(q, D) - t)^+. \qquad (2.14)$$

The third phrase of Equation 2.14 represents what the social planner pays to the manufacturer according to the contract. Depending on the value of t, the contract imposes different dynamics to compensate the manufacturer. The next set of lemmas and propositions describe how the manufacturer's profit function, optimum production commitment, and pricing strategy work in different ranges of the threshold t.

Lemma 10. *If $t < \frac{a+bs}{2}$, the manufacturer's profit function and optimum pricing* *strategy are:*

$$\textit{If } q < t \qquad \Rightarrow \qquad \begin{cases} p^* = \frac{a-q}{b} \\ D^* = q \\ \pi^* = \frac{a-q}{b}q \end{cases} .$$

$$\textit{If } t < q < \frac{a+bs+br}{2} \qquad \Rightarrow \qquad \begin{cases} p^* = \frac{a-q}{b} \\ D^* = q \\ \pi^* = \frac{a-q}{b}q + (q-t)r \end{cases} .$$

$$\textit{If } \frac{a+bs+br}{2} < q \qquad \Rightarrow \qquad \begin{cases} p^* = \frac{a-bs-br}{2b} \\ D^* = \frac{a+bs+br}{2} \\ \pi^* = \frac{(a+b(s+r))^2}{4b} - \frac{rt+sq}{2} \end{cases} .$$

According to the optimum pricing strategy, the manufacturer clears the market unless realized production quantity exceeds $\frac{a+bs+br}{2}$. In other words, the manufacturer does not decrease the price any lower than $\frac{a-bs-br}{2b}$. By comparing the maximum sales to the original model, we can see that this contract shifts the maximum sale for $\frac{br}{2}$ units which can be positive or negative depending on the sign of r. One interpretation of negative values of r is if the social planner wants to decrease the manufacturer's maximum sales, he should impose a tax on any item being sold after sales reach r.

Lemma 11. *The optimum production commitment of the manufacturer in the presence of a sales rebate contract follows:*

$$\mathcal{A}_{reb}(N^*_{reb}) = \frac{a+bs+br}{b}\int_0^{\frac{a+bs+br}{2N_{reb}}} \nu f\nu \,\mathrm{d}\nu - \frac{2N_{reb}}{b}\int_0^{\frac{a+bs+br}{2N_{reb}}} \nu^2 f\nu \,\mathrm{d}\nu - r\int_0^{\frac{t}{N_{reb}}} \nu f\nu \,\mathrm{d}\nu - s\mu - c = 0.$$

Also, if the compensation rate r becomes zero, the optimum production commitment becomes equal to the optimum production commitment for the decentralized scenario.

Proof. We set up the manufacturer's profit function and check the first order condition with respect to N. If N_{reb} is the manufacturer's optimum production commitment under the sales rebate contract, we have:

$$\mathcal{A}_{reb}(N^*_{reb}) = \frac{a+bs+br}{b}\int_0^{\frac{a+bs+br}{2N_{reb}}} \nu f\nu \,\mathrm{d}\nu - \frac{2N_{reb}}{b}\int_0^{\frac{a+bs+br}{2N_{reb}}} \nu^2 f\nu \,\mathrm{d}\nu - r\int_0^{\frac{t}{N_{reb}}} \nu f\nu \,\mathrm{d}\nu - s\mu - c = 0.$$

We are interested in determining how N^*_{reb} differs from N^*_{dec}. Defining

$$\bar{\theta} = \frac{a+bs+br}{2} \qquad \text{and} \quad \theta = \frac{a+bs}{2},$$

we can rewrite $\mathcal{A}(N)$ and $\mathcal{A}_{reb}(N,r,t)$ as follows:

$$\begin{aligned}
\mathcal{A}_{reb}(N_{reb},r,t) &= \frac{2\bar{\theta}}{b}\int_0^{\frac{\theta}{N_{reb}}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N_{reb}}{b}\int_0^{\frac{\bar{\theta}}{N_{reb}}} \nu^2 f(\nu)\,\mathrm{d}\nu - r\int_0^{\frac{t}{N_{reb}}} \nu f(\nu)\,\mathrm{d}\nu - s\mu - c,\\
\mathcal{A}(N,\theta) &= \frac{2\theta}{b}\int_0^{\frac{\theta}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N}{b}\int_0^{\frac{\theta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c.
\end{aligned}$$

By applying some algebra manipulation, we can derive the following properties.

$$\begin{aligned}
&\lim_{r\to 0}\mathcal{A}_{reb}(N,r,t) = \mathcal{A}(N) \quad \Rightarrow \quad \lim_{r\to 0} N^*_{reb} = N^*_{dec}\\
&\lim_{t\to 0}\mathcal{A}_{reb}(N,\bar{\theta},t) = \mathcal{A}(N,\bar{\theta}).
\end{aligned} \tag{2.15}$$

These results show how the optimum production commitment changes as the contract parameter approaches zero. □

Proposition 5. *The sales rebate contract can coordinate the market.*

Proof. As r, and consequently θ increase, $\mathcal{A}_{reb}(N_{feb},r,t)$ shifts up. Thus,

$$\begin{aligned}
\frac{\partial}{\partial r}\mathcal{A}_{reb}(N_{reb},r,t) &= \int_0^{\frac{1}{N_{reb}}(\frac{a+bs+br}{2}-t)} \nu f(\nu)\,\mathrm{d}\nu > 0\\
\frac{\partial}{\partial \theta}\mathcal{A}_{reb}(N_{reb},r,t) &= \frac{\partial}{\partial \theta}\mathcal{A}(N) = \frac{2}{b}\int_0^{\frac{\theta}{N}} \nu f(\nu)\,\mathrm{d}\nu > 0.
\end{aligned} \tag{2.16}$$

As equation 2.15 shows, if t approaches zero, $\mathcal{A}_{reb}(N)$ merges to $\mathcal{A}(N, \bar{\theta})$. Our results also show that $\frac{\partial}{\partial r}\mathcal{A}_{reb}(N_{reb})$ is larger than zero, which means that if $t = 0$ and for a positive value of r, $\mathcal{A}_{reb}(N)$ is more than $\mathcal{A}(N)$. The same reasoning indicates that $\mathcal{A}_{reb}(N)$ is less than $\mathcal{A}(N)$, if r is less than zero. Remember that both $\mathcal{A}_{reb}(N)$ and $\mathcal{A}(N)$ are monotonically decreasing in N with a unique root in N. This means that if $\mathcal{A}_{reb}(N)$ is more than $\mathcal{A}(N)$, it hits the x-axis farther away on the positive side so that N^*_{reb} is larger than N^*_{dec}. We can summarize our results as follows:

$$\text{if} \quad t = 0 \quad \begin{array}{lll} \forall \quad r > 0 & \Rightarrow & N^*_{reb} > N^*_{dec} \\ \forall \quad r < 0 & \Rightarrow & N^*_{reb} < N^*_{dec}\,. \end{array}$$

□

Lemma 12 compares the effect of threshold t to the compensation rate r.

Lemma 12. *Decreasing the threshold t amplifies the effect of the contract.*

Proof.

$$\frac{\partial}{\partial t}\mathcal{A}_{reb}(N_{reb}, r, t) = -\frac{rt}{N_{reb}^2} f(\frac{t}{N_{reb}}) \equiv -\text{Sign}(r).$$

□

If the social planner wants to persuade the manufacturer to increase his production commitment, he sets a positive value for r. In this setting, the lower the threshold t, the higher the expected payment to the manufacturer. On the other hand, if the social planner wants the manufacturer to have a smaller production commitment, he taxes sales as they exceed t. A smaller threshold implies a larger expected tax collection. These results show that the social planner can influence the manufacturer's decision in either direction by offering this contract. The result is for any setting of the problem in which there is at least one contract setting with

$t < \frac{a+bs}{2}$ whereby, the sales rebate contract can coordinate the market.

We continue our analysis for this contract to show how setting t in different ranges can change the contract.

Lemma 13. *Assuming that r is a positive number, if*

$$\frac{a+bs}{2} < t < \frac{a+bs+br}{2},$$

the optimum pricing strategy is as follows:

$$\textit{If} \quad q < \frac{a+bs}{2} \quad \Longrightarrow \qquad \begin{cases} p^* = \frac{a-q}{b} \\ D^* = q \\ \pi^* = \frac{a-q}{b} q \end{cases},$$

$$\textit{if } \frac{a+bs}{2} < q < t \qquad \begin{cases} p^* = \frac{a-bs}{2b} \\ D^* = \frac{a+bs}{2} \\ \pi^* = \frac{(a+bs)^2}{4b} + s(a-q) \end{cases},$$

$$\textit{if } t < q < \frac{a+bs+br}{2} - b\sqrt{(\frac{r}{2})^2 + \frac{r(a+bs)}{2b} - \frac{rt+as}{b}} \qquad \begin{cases} p^* = \frac{a-bs}{2b} \\ D^* = \frac{a+bs}{2} \\ \pi^* = \frac{(a+bs)^2}{4b} + s(a-q) \end{cases}$$

$$\textit{if}\ \frac{a+bs+br}{2} - b\sqrt{(\frac{r}{2})^2 + \frac{r(a+bs)}{2b} - \frac{rt+as}{b}} < q < \frac{a+bs+br}{2} \qquad \begin{cases} p^* = \frac{a-q}{b} \\ D^* = q \\ \pi^* = \frac{(a-q)}{b}q + r(q-t) \end{cases} \quad ,$$

$$\textit{if}\ \frac{a+bs+br}{2} < q \qquad \begin{cases} p^* = \frac{a-bs-br}{2b} \\ D^* = \frac{a+bs+br}{2} \\ \pi^* = \frac{(a+b(s+br))^2}{4b} - \frac{rt+sq}{2} \end{cases} \quad .$$

As suggested above, such a contract may or may not persuade the manufacturer to increase his sales any further than $\frac{a+bs}{2}$ level, which constituted maximum sales in the absence of a contract. In other words, the manufacturer would not increase sales unless the realized production quantity is significantly larger than $\frac{a+bs}{2}$. In that case, maximum sales will be set with $\frac{a+bs+br}{2}$. This happens because the contract contribution to the manufacturer is not enough to cover the loss caused by the reduction in price when sales are only slightly larger than $\frac{a+bs}{2}$. On the other hand, if sales are significantly larger than $\frac{a+bs}{2}$, payment from the social planner can mitigate the profit loss. Such an equilibrium happens at $q = \sqrt{(\frac{r}{2})^2 + \frac{r(a+bs)}{2b} - \frac{rt+as}{b}}$.

We could not find any closed-form solution for the optimum production commitment. Yet, we can show that the sales rebate contract can increase the manufacturer's production commitment compared to the base model decentralized case. That is because the sales rebate contract creates an opportunity to earn higher profits by selling more products, which naturally leads the manufacturer to increase his production commitment. Lemma 14 studies the case where the threshold is larger than maximum sales in the contract cases discussed thus far.

Lemma 14. *If $r > 0$ and $\frac{a+bs+br}{2} < t$, the optimum pricing strategy is as follows:*

$$\text{If } q < \frac{a+bs}{2} \qquad \Rightarrow \qquad \begin{cases} p^* = \frac{a-q}{b} \\ D^* = q \\ \pi^* = \frac{(a-q)}{b} q \end{cases}$$

$$\text{if } \frac{a+bs}{2} < q < t \qquad \Rightarrow \qquad \begin{cases} p^* = \frac{a-bs}{2b} \\ D^* = \frac{a+bs}{2} \\ \pi^* = \frac{(a+bs)^2}{4b} + (a-q)s \end{cases}$$

$$\text{if } t < q \qquad \Rightarrow \qquad \begin{cases} p^* = \frac{a-bs}{2b} \\ D^* = \frac{a+bs}{2} \\ \pi^* = \frac{(a+bs)^2}{4b} + (a-q)s \end{cases} .$$

The optimum pricing and sales strategy in this case are exactly same as those in the absence of a contract, therefore the contract fails to affect the manufacturer's behavior. There is only one mismatch in profit function between this case and the original one and that is the $(a-t)r$ term in the profit function, if $t < q$. This expression does not depend on q; therefore, we can conclude that the derivative of the profit function for this case is exactly the same as that for the decentralized case. Thus, the production commitments of these two cases are identical. In other words, the expected payment from the social planner does not surpass the lost profit caused by the reduction in the price so that the manufacturer does not change his original production policy.

To conclude the sales rebate contract, we find that if the threshold is less than $\frac{a+bs}{2}$, the contract is coordinating; otherwise, it may or may not be able to coordinate the market. Specifically, if the threshold is larger than $\frac{a+bs+br}{2}$, the contract cannot coordinate the market.

2.3.3 Buy-back contract

In this contract, the social planner offers a subsidy or tax to salvage the unsold products. This way, the manufacturer encounters a different expected salvage cost and changes his production commitment accordingly. Lemma 15 gives us the analytical tools to set up the contract.

Lemma 15. *The following properties hold for the $\mathcal{A}(N)$ function with regard to the salvage cost:*

$$\lim_{s \to -\frac{c}{\mu}} \mathcal{A}_N(s) > 0, \textit{ and}$$

$$\lim_{s \to \infty} \mathcal{A}_N(s) < 0 \qquad , \textit{if } \frac{1}{2}\frac{\mu - \frac{bc}{a}}{\mu^2 + \sigma^2} < \frac{N}{a}.$$

Proof. See Appendix E. □

As described in Lemma 15, there is a condition for the case where s approaches infinity, which we need in order to find out if the contract can be coordinating. Therefore the only N that needs to meet this condition is the optimum production commitment for the centralized case N_{cen}. By replacing it in that inequality, we have

$$\frac{1}{2}\frac{\mu - \frac{bc}{a}}{\mu^2 + \sigma^2} < \frac{N_{cen}}{a} \Leftrightarrow \frac{N_{cen}}{a} = \frac{\frac{a\mu - bc}{\mu^2 + \sigma^2}}{a} = \frac{\mu - \frac{bc}{a}}{\mu^2 + \sigma^2} > \frac{1}{2}\frac{\mu - \frac{bc}{a}}{\mu^2 + \sigma^2},$$

and so the condition holds.

Lemma 16. *$A(N, s)$ is a monotonically decreasing function in s because*

$$\frac{\partial}{\partial s}\mathcal{A}(N, s) = \int_0^{\frac{a+bs}{2N}} \nu f(\nu)\,\mathrm{d}\nu - \mu < 0.$$

This implies that, by decreasing the salvage cost, we can shift the $\mathcal{A}(N)$ function up. That is an appropriate strategy when the decentralized optimum production commitment is less than the optimum production commitment for the centralized scenario, as when the social planner wants the manufacturer to make a larger production commitment.

Proposition 6. *The buy-back contract can coordinate the market.*

If the social planner compensates a proportion of the potential salvage cost, $\mathcal{A}(N)$ shifts up and the optimum production commitment of the manufacturer increases. The same reasoning applies to the case in which the decentralized optimum production commitment is higher than that for the centralized scenario. In that case, the social planner can decrease his production commitment by putting some pressure on the manufacturer; e.g., assigning a tax to salvage the unsold products.

As discussed in section 2.2.5, the necessary and sufficient condition for coordinating the market is that

$$\mathcal{A}(N^*_{cen}, s^*) = 0\,.$$

By rewriting the equation we have

$$\mathcal{A}(N_{cen}, s^*_{N_{cen}}) = (\frac{a}{b}+s^*)\int_0^{\frac{a+bs^*}{2(a\mu-bc)}(\mu^2+\sigma^2)} \nu f(\nu)\,\mathrm{d}\nu - \frac{2(a\mu-bc)}{b(\mu^2+\sigma^2)}\int_0^{\frac{a+bs^*}{2(a\mu-bc)}(\mu^2+\sigma^2)} \nu^2 f(\nu)\,\mathrm{d}\nu - s^*\mu - c = 0,$$

where s^* is the optimum salvage cost that equalizes the optimum production commitment for the centralized and decentralized scenarios.

Lemma 17. *The buy-back contract has a unique coordinating salvage cost and the cost of applying this contract is*

$$TC_{salvage} = (s-s^*)[q - \frac{a+bs}{2}]^+ = (s-s^*)\left((N^*_{cen} \times \int_{\frac{a+bs^*}{2N^*_{cen}}}^{\infty} \nu f(\nu)\,\mathrm{d}\nu) - \frac{a+bs^*}{2}\right).$$

Parameters	a	b	c	s	μ	σ
Values	500	1	10	10	1	.25

Table 2.2 Values of the simulation parameters

Proof. See Appendix F. □

The results in Appendix F show that $\mathcal{A}(N^*_{cen}, s)$ is a monotonically decreasing function in s with a unique answer for s^*. Although we cannot find the closed form solution for s, we can show that it has a unique root, and we can also find it for a specific probability distribution function by conducting numerical analysis.

2.4 Numerical studies of the monopoly setting

In this section, we provide numerical results to study the sensitivity of our models in regards to cost, market, and production parameters. Table 2.2 shows the original values of the model parameters that we study in this section. The yield ratio is assumed to follow a uniform distribution throughout this study; however, we study some properties of the model when the yield ratio follows anormal distribution.

Before starting the sensitivity analysis, the following lemma explains why securing the probability distribution function can result in having a better understanding of the relation between the decentralized and centralized cases.

Lemma 18. *If the yield ratio follows a uniform probability distribution, there exists a minimum salvage cost that guarantees the optimum production commitment for the centralized scenario will be higher than the optimum production commitment for the* *decentralized scenario. Thus,*

$$\frac{a * .14619}{b} - \frac{c}{\mu} = s_{min} \leq s \Longleftrightarrow N^*_{dec} \leq N^*_{cen}.$$

Proof.

$$A(N^*_{cen}) = -(a\mu - bc) + e^{-\frac{(a+bs)\mu}{a\mu-bc}}[(a+bs)\mu + 2(a\mu - bc)] = 0\,.$$

If we set this equation equal to zero, we have

$$2 + \frac{(a+bs)\mu}{a\mu - bc} = e^{-\frac{(a+bs)\mu}{a\mu-bc}} \equiv 2 + x = e^x.$$

By simple algebraic operations, we have

$$s_{min} = \frac{a * .14619}{b} - \frac{c}{\mu}.$$

□

This is the largest salvage cost required to keep the production commitment of the centralized scenario higher than the production commitment for the decentralized scenario. We can multiply that number by -1 to yield the smallest salvage value that would be needed.

2.4.1 Sensitivity analysis for salvage and production cost

Figures 2.2 and 2.3 show the profit function for different values of salvage cost and production cost, accordingly. The blue curve in both graphs represents the case with lowest cost. As you can see in both figures, the optimum profit as well as the optimum production commitment decrease as the cost increases. That relationship is what production subsidizing and buy-back contracts utilize to adjust the production commitment according to the social planner's objective.

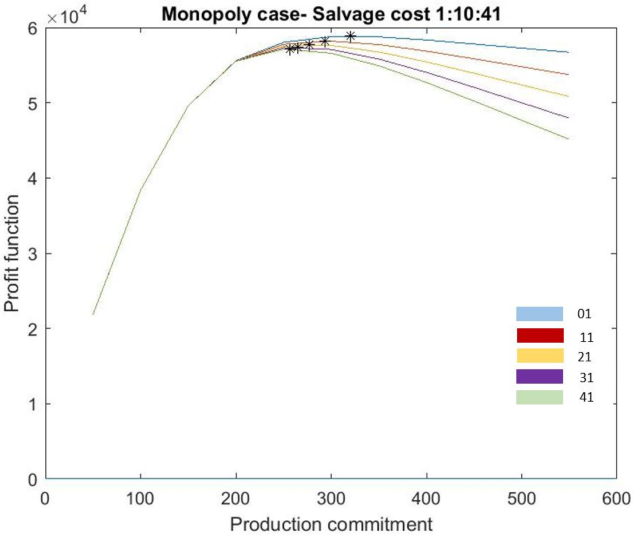


Figure 2.2 Profit function for salvage costs 1:10:41

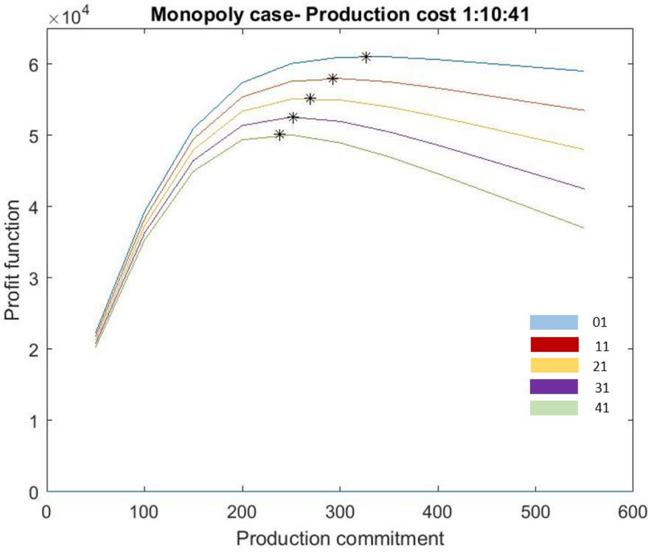


Figure 2.3 Profit function for production costs 1:10:41

The profit function, however, does not respond equally to these two types of cost parameter. As Figure 2.2 shows, profit function for smaller values of production commitment stays the same as salvage cost varies. On the other hand, the profit function for different values of salvage cost diversifies as the production commitment increases. In other words, variation in production cost affects the whole curve to shift up and down, but salvage cost variation seems to influence only the right side of the graph. That is because small production commitment reduces the chance of having surplus items, and so the expected salvage cost is negligible. Naturally, the expected salvage cost increases as the production commitment rises. Thus, the salvage cost affects the function values associated with larger production commitment values.

2.4.2 Sensitivity analysis for market parameters

The manufacturer's optimum strategy depends on the market size and price sensitivity index. In this section, we study how these parameters affect the optimum production commitment. We present simulation results for five different values of a, from 200 to 1000 in increments of 200. As shown in Figure 2.4, the optimum profit and optimum production commitment shift up as the market size increases. Not surprisingly, having a larger market motivates the manufacturer to increase his production commitment, and, since there are expanded opportunities for selling, the manufacturer's profit also increases.

We repeat the same experiment by changing the price sensitivity b. Figure 2.5 illustrates the derivative of the profit function changing for different values of b from 0.5 to 2.0 As b increases, the graph shifts down and optimum production slightly decreases. This can be interpreted as the manufacturer's optimal pricing becoming more conservative and increasing the overall risk of production as price sensitivity

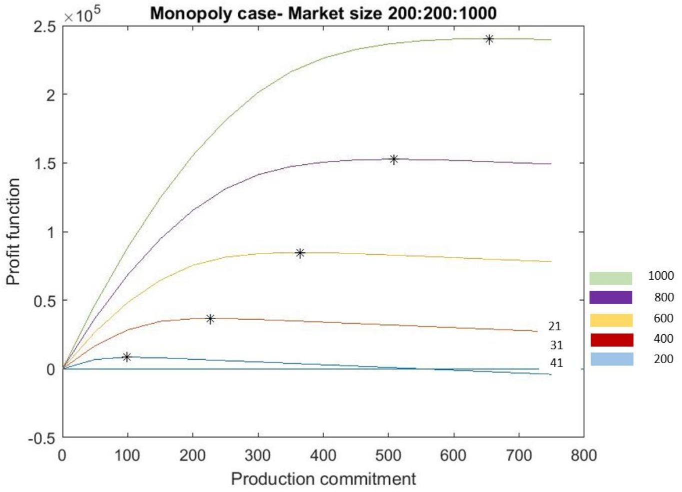


Figure 2.4 Profit function for market size 500:500:2000

increases. A higher risk then naturally leads to a down-scaling of the production commitment.

2.4.3 Sensitivity analysis for probability distribution parameters

In the real world, the probability distribution function of the yield ratio is the intellectual property of the manufacturer. Therefore, we do not make any assumptions about its type in the analytical part of this research. This also makes our model more suitable for different applications. In the numerical studies, however, we assume that yield ratio follows uniform distribution. Under that assumption, we can examine the effects of having different values of mean and standard deviation.

Figure 2.6 shows how changing the mean of yield ratio μ affects manufacturer's profit function and optimum production commitment. It also shows that the manufacturer's optimum profit increases with the mean of the yield ratio. One explanation

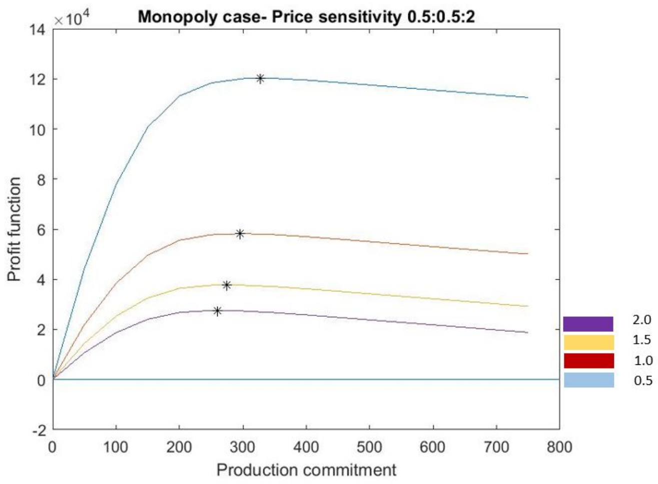


Figure 2.5 Profit function for price sensitivity 0.5: 0.5: 2

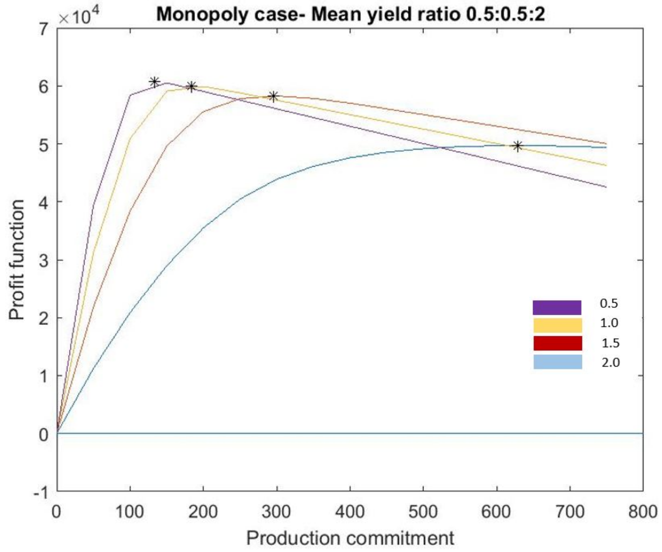


Figure 2.6 Profit function for yield ratio mean .5:.5:2

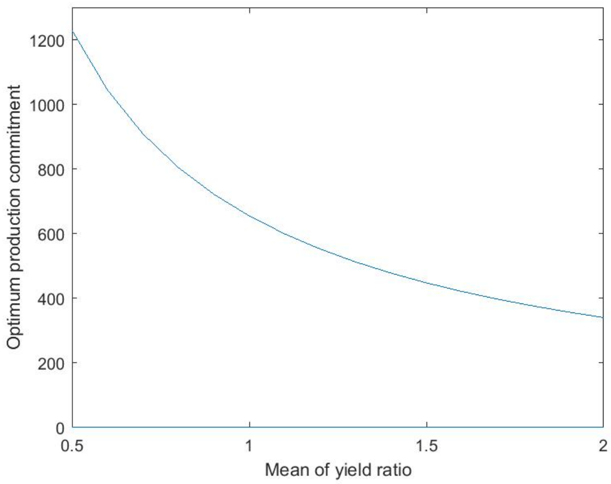


Figure 2.7 Optimum production commitment for different values of yield ratio mean

is that production of certain amounts of product, vaccine for example, costs less if the mean of the yield ratio is higher. Increasing the mean of the yield ratio reduces the manufacturer's optimum production commitment. Doing so does not necessarily counter the social planner's interest since it does not affect the realized production quantity. In fact, as Figure 2.7 depicts, the optimum production quantity presents a convex shape in regards to the mean of the yield ratio. That means that as the mean of the yield ratio increases linearly, production commitment decrease more slowly than linear, therefore the expected value for realized production quantity expands if mean of yield ratio increases.

Studying the effect of the magnitude of randomness, we fix the mean of the yield ratio μ at 1 and expand the standard deviation σ from .1 to .25. Having a standard deviation of .1 means that the yield ratio in within [0.83 , 1.17]. Similarly, if standard deviation is 0.25, the yield ratio falls in interval of [0.13, 1.86]. Figure 2.8

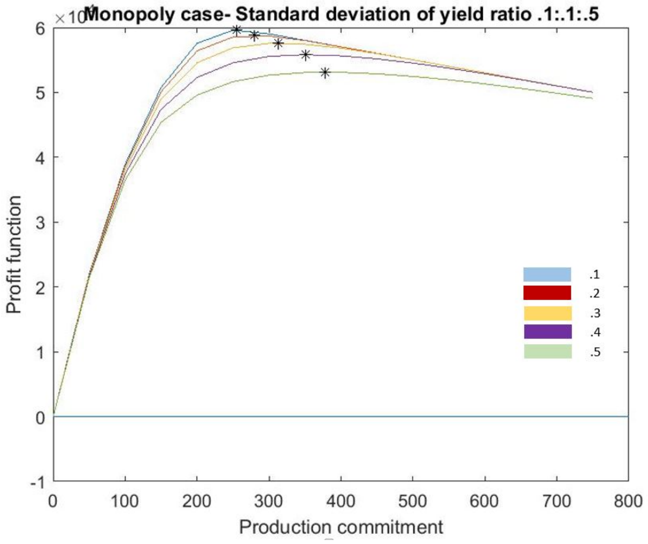


Figure 2.8 Profit function for yield ratio standard deviation .1:.05:.25

shows that optimum production commitment increases as the production uncertainty increases. That makes sense since having a higher production commitment can partially mitigate the potential disruption, although it also results in a higher expected salvage cost. One might also notice that the profit function becomes less convex as the variation in the yield ratio increases. That, indeed, supports Proposition 3 in which we showed that the optimum production commitment is less in the absence of randomness.

Chapter 3

STUDYING COMPETITION OF TWO MANUFACTURERS AND SINGLE MARKET

In a single market with two competing manufacturers producing perfectly substitutable products, the demand for each firm's products depends on its own price and that of its rival. As in the monopoly setting, manufacturers deal with randomness in their yield ratio and use a dynamic decision-making process to set their price and production commitment. These manufacturers have interdependent demands; i.e., demand for each firm's products linearly increases with the retail price for products from the other firm. The model assumes complete information sharing, which means both firms have equal and perfect information about one another's parameters and realized production quantity.

At the beginning of this chapter, we study asymmetrical firms which have different parameter settings. Next, we study symmetrical firms and derive the optimum pricing strategy and production commitment. After that we examine the effect of uncertainty and introduce various coordinating contracts. The chapter ends with numerical analysis results.

3.1 Competition between two asymmetrical firms

In this section, we study a case concerning two asymmetrical manufacturers. We derive the optimum pricing strategy and profit function for firm 1, based on the known values of q_1, q_2, and p_2. Since retail prices are interdependent, we can find the optimum pricing strategy based only on known values of q_1 and q_2. The demand function of the firms can be formulated as:

$$\begin{aligned} D_1 =& a_1 - b_{1,1}p_1 + b_{1,2}p_2, \\ D_2 =& a_2 + b_{2,1}p_1 - b_{2,2}p_2. \end{aligned}$$

By adjusting the retail price, each firm can set its demand either above or below its realized production quantity and fall into either a surplus or shortage situation. In Lemma 19, we study these two situations separately. *Sh* and *Su* represent shortage and surplus situations, respectively. This lemma provides the optimum pricing strategy for the case of asymmetrical firms.

Lemma 19. *Given the realized production quantities and the competitor's retail price, the optimum pricing strategy and profit function for the first firm are as follows:*

$$p_1^* = \begin{cases} : \text{(1Su)} \quad \frac{a_1+b_{1,2}p_2-sb_{1,1}}{2b_{1,1}}, & \text{if } q_1 \geq \frac{a_1+sb_{1,1}+b_{1,2}p_2}{2} \quad \text{, and} \\ : \text{(1Sh)} \quad \frac{a_1-q_1+b_{1,2}p_2}{b_{1,1}} & \text{if } q_1 \leq \frac{a_1+sb_{1,1}+b_{1,2}p_2}{2}. \end{cases}$$

$$\pi^*_{1|q_1,q_2,p_2} = \begin{cases} : \text{(1Su)}\frac{(a_1+b_{1,2}p_2-sb_{1,1})^2}{2b_{1,1}} + s(a_1+b_{1,2}p_2-q_1), & \text{if } q_1 \geq \frac{a_1+sb_{1,1}+b_{1,2}p_2}{2} \quad \text{, and} \\ : \text{(1Sh)}\frac{a_1-q_1+b_{1,2}p_2}{b_{1,1}}q_1, & \text{if } q_1 \leq \frac{a_1+sb_{1,1}+b_{1,2}p_2}{2}. \end{cases}$$

Proof. For the surplus quantity case we have

$$q_1 > D_1 \Leftrightarrow q_1 > a_1 - b_{1,1}p_1 + b_{1,2}p_2 \Leftrightarrow p_1 > \frac{a_1 - q_1 + b_{1,2}p_2}{b_{1,1}}. \tag{3.1}$$

The condition above guarantees that there are surplus items left unsold. Assuming that q_1, q_2, and p_2 are known and the market is in a surplus situation, we can write the profit function for firm 1 as

$$\begin{aligned}\pi_{1|q_1,q_2,p_2} &= p_1 D_1 - s(q_1 - D_1)\\ &= p_1(a_1 - b_{1,1}p_1 + b_{1,2}p_2) - s(q_1 - a_1 + b_{1,1}p_1 - b_{1,2}p_2)\\ &= -b_{1,1}p_1^2 + (a_1 + b_{1,2}p_2 - sb_{1,1})p_1 - s(q_1 - a_1 - b_{1,2}p_2).\end{aligned}$$

By applying the first order condition we have

$$p_1^* = \frac{a_1 + b_{1,2}p_2 - sb_{1,1}}{2b_{1,1}} \Rightarrow \pi^*_{1|q_1,q_2,p_2} = \frac{(a_1 + b_{1,2}p_2 - sb_{1,1})^2}{2b_{1,1}} - s(q_1 - a_1 - b_{1,2}p_2).$$

However, if the inequality in Equation 3.1 does not hold, demand exceeds realized production quantity and the market falls into the shortage stage. We show that the optimal pricing strategy under the shortage assumption makes the demand equal to the realized production quantity; i.e., the manufacturer clears its stock and shortage cost becomes zero. Based on these results we can form the profit function for firm 1 as follows:

$$\pi^*_{1|q_1,q_2,p_2} = \begin{cases} \text{(1Su): } \frac{(a_1+b_{1,2}p_2-sb_{1,1})^2}{2b_{1,1}} + s(a_1 + b_{1,2}p_2 - q_1) & \text{if } p_1 \geq \frac{a_1-q_1+b_{1,2}p_2}{b_{1,1}}, \text{ and} \\ \text{(1Sh): } \frac{a_1-q_1+b_{1,2}p_2}{b_{1,1}} q_1 & \text{if } p_1 \leq \frac{a_1+b_{1,2}p_2-q_1}{b_{1,1}}. \end{cases}$$

□

It is worth mentioning that number assignment to the firms is arbitrary. In other words, our model, either in the symmetrical or asymmetrical setting, is a symmetric game. While we focus on the formulations for firm 1, firm 2 formulas can be derived according to the symmetry of the game, and we provide results for firm 2 whenever necessary. Similar to the conditions in Lemma 19, the optimum price strategy and profit function for the second firm are:

$$p_2^* = \begin{cases} \text{(2Su): } \frac{a_2+b_{2,1}p_1-sb_{2,2}}{2b_{2,2}}, & \text{if } q_2 \geq \frac{a_2+sb_{2,2}+b_{2,1}p_1}{2}, \text{ and} \\ \text{(2Sc): } \frac{a_2-q_2+b_{2,1}p_1}{b_{2,2}}, & \text{if } q_2 \leq \frac{a_2+sb_{2,2}+b_{2,1}p_1}{2}. \end{cases} \tag{3.2}$$

$$\pi^*_{2|q_1,q_2,p_1} = \begin{cases} \text{(2Su): } \frac{(a_2+b_{2,1}p_1-sb_{2,2})^2}{2b_{2,2}} + s(a_2+b_{2,1}p_1-q_2), & \text{if } q_2 \geq \frac{a_2+sb_{2,2}+b_{2,1}p_1}{2}, \text{ and} \\ \text{(2Sh): } \frac{a_2-q_2+b_{2,1}p_1}{b_{2,2}}q_2, & \text{if } q_2 \leq \frac{a_2+sb_{2,2}+b_{2,1}p_1}{2}. \end{cases}$$

We introduce these auxiliary variables to simplify the formulations:

$$\omega_i = a_i - sb_{i,i} \quad , \gamma_i = a_i - q_i \quad , \Delta = b_{1,1}b_{2,2} - b_{1,2}b_{2,1}.$$

As indicated in Lemma 19, there is a threshold for the realized production quantity, before and after which the optimum pricing strategy behaves differently. In the next three lemmas we study the optimum pricing strategy under three different situations; first, the case where both firms have enough realized production quantity to pass the threshold; second, when none of the firms passes the threshold, and third; when only one firm has enough realized production quantity to pass the threshold.

Lemma 20. *If both firms have adequate realized production quantity to pass these conditions:*

$$\begin{cases} \textit{(1)} : q_1 \geq \frac{\theta_1}{2} + \frac{b_{1,2}}{2}\frac{b_{2,1}\omega_1+2b_{11}\omega_2}{3b_{1,1}b_{2,2}+\Delta}, \textit{ and} \\ \textit{(3)} : q_2 \geq \frac{\theta_2}{2} + \frac{b_{2,1}}{2}\frac{2b_{2,2}\omega_1+b_{1,2}\omega_2}{3b_{1,1}b_{2,2}+\Delta}. \end{cases}$$

then, the optimum pricing strategy is

$$(1Su)\&(2Su):\begin{cases} p_1^* = \frac{a_1+b_{1,2}p_2-sb_{1,1}}{2b_{1,1}} & ,and \\ p_2^* = \frac{a_2+b_{2,1}p_1-sb_{2,2}}{2b_{2,2}}. \end{cases} \Rightarrow \begin{cases} p_1^* = \frac{2b_{2,2}}{3b_{1,1}b_{2,2}+\Delta}\omega_1 + \frac{b_{1,2}}{3b_{1,1}b_{2,2}+\Delta}\omega_2, \\ p_2^* = \frac{2b_{1,1}}{3b_{1,1}b_{2,2}+\Delta}\omega_2 + \frac{b_{2,1}}{3b_{1,1}b_{2,2}+\Delta}\omega_1. \end{cases}$$

$$\Rightarrow \begin{cases} \pi^*_{1|q_1,q_2} = 2b_{1,1}(\frac{2b_{2,2}\omega_1+b_{1,2}\omega_2}{3b_{1,1}b_{2,2}+\Delta})^2 + s(\gamma_1 + \frac{2b_{1,1}b_{1,2}}{3b_{1,1}b_{2,2}+\Delta}\omega_2 + \frac{b_{1,2}b_{2,1}}{3b_{1,1}b_{2,2}+\Delta}\omega_1),\ and \\ \pi^*_{2|q_1,q_2} = 2b_{2,2}(\frac{2b_{1,1}\omega_2+b_{2,1}\omega_1}{3b_{1,1}b_{2,2}+\Delta})^2 + s(\gamma_2 + \frac{2b_{2,1}b_{2,2}}{3b_{1,1}b_{2,2}+\Delta}\omega_1 + \frac{b_{1,2}b_{2,1}}{3b_{1,1}b_{2,2}+\Delta}\omega_2). \end{cases}$$

This lemma shows that if both firms have relatively high realized production quantity on hand, they both set a minimum price and do not lower their price below that threshold. In other words, the firms do not find it in their best interest to clear the market if their realized production quantity exceeds a certain level. That is aligned with what we found in the monopoly model in Chapter 2, although the formulations are much more complicated here.

Another interesting result is that the optimal profit and optimal pricing strategy for each firm is independent of the realized production quantity of the other firm. That is because, if manufacturers have ample supply of realized production quantity, only the market and cost parameters affect the optimum price and maximum sales. Using the results provided in this lemma we can rewrite conditions (*1Su*) and (*2Su*) as follows:

$$\begin{cases} (1) : q_1 \geq \frac{\theta_1}{2} + \frac{b_{1,2}}{2}\frac{b_{2,1}\omega_1+2b_{11}\omega_2}{3b_{1,1}b_{2,2}+\Delta}, \text{ and} \\ (3) : q_2 \geq \frac{\theta_2}{2} + \frac{b_{2,1}}{2}\frac{2b_{2,2}\omega_1+b_{1,2}\omega_2}{3b_{1,1}b_{2,2}+\Delta}. \end{cases} \tag{3.3}$$

In this new formulation, the pricing strategy of each firm depends only on the realized production quantity and market parameters, not the competitor's price. That makes sense simply because competitor's price also depends on realized production quantities and market parameters; therefore, it is just a dependent variable in this setting that can be disclosed.

Lemma 21. *If neither of the firms has adequate realized production quantity, meaning conditions 1Sh and 2Sh in equation 3.4 hold, the optimum pricing strategy is*

$$(1Sh)\&(2Sh):\begin{cases} p_1^* = \frac{a_1-q_1+b_{1,2}p_2}{b_{1,1}} & , \\ p_2^* = \frac{a_2-q_2+b_{2,1}p_1}{b_{2,2}}. \end{cases} \Rightarrow \begin{cases} p_1^* = \frac{b_{2,2}\gamma_1+b_{1,2}\gamma_2}{\Delta}, \\ p_2^* = \frac{b_{1,1}\gamma_2+b_{2,1}\gamma_1}{\Delta}. \end{cases}$$

$$\Rightarrow \begin{cases} \pi^*_{1|q_1,q_2} = \frac{q_1}{\Delta}(\gamma_1 b_{2,2} + b_{1,2}\gamma_2), \\ \pi^*_{2|q_1,q_2} = \frac{q_2}{\Delta}(\gamma_2 b_{1,1} + b_{2,1}\gamma_1). \end{cases}$$

This lemma shows that if neither of the firms has high realized production quantity, it is in both of their best interests to clear the market. Unlike the case in Lemma 20, the firms' optimal price as well as their optimal profit depend on one another's realized production quantity. Now, by rewriting the conditions, we have

$$\begin{cases} \textit{(1Sh)}: (1+\frac{b_{1,2}b_{2,1}}{2\Delta})q_1 + \frac{b_{1,2}b_{1,1}}{2\Delta}q_2 \leq \frac{\theta_1}{2} + \frac{b_{1,2}b_{2,1}}{2\Delta}a_1 + \frac{b_{1,2}b1,1}{2\Delta}a_2 \\ \textit{(2Sh)}: (1+\frac{b_{1,2}b_{2,1}}{2\Delta})q_2 + \frac{b_{2,1}b_{2,2}}{2\Delta}q_1 \leq \frac{\theta_2}{2} + \frac{b_{1,2}b_{2,1}}{2\Delta}a_2 + \frac{b_{,2,1}b2,2}{2\Delta}a_1. \end{cases} \tag{3.4}$$

These equations represent a linear combination of q_1 and q_2 as demonstrated in Figure 3.1. Similar to what we see in equation 3.3, we do not have the competitor's price in the formulas after we rewrite them based on results in Lemma 21.

Lemma 22. *If firm 1 has adequate realized production quantity to be in a surplus stage but firm 2 is in a shortage stage, as shown in equation 3.5, the optimum pricing* *strategy is*

$$(1Su)\,\&\,(2Sh):\begin{cases} p_1^* = \frac{a_1+b_{1,2}p_2-sb_{1,1}}{2b_{1,1}}, \\ p_2^* = \frac{a_2-q_2+b_{2,1}p_1}{b_{2,2}}. \end{cases} \Rightarrow \begin{cases} p_1^* = \frac{1}{b_{1,1}b_{2,2}+\Delta}(b_{2,2}\omega_1 + b_{1,2}\gamma_2), \\ p_2^* = \frac{1}{b_{1,1}b_{2,2}+\Delta}(2b_{1,1}\gamma_2 + b_{2,1}\omega_1). \end{cases}$$

$$\Rightarrow \begin{cases} \pi^*_{1|q_1,q_2} &= 2b_{1,1}(\frac{(b_{2,2}\omega_1+b_{1,2}\gamma_2)}{b_{1,1}b_{2,2}+\Delta})^2 + s(\gamma_1 + \frac{b_{1,2}b_{2,1}\omega_1+2b_{1,1}b_{1,2}\gamma_2}{b_{1,1}b_{2,2}+\Delta}), \\ \pi^*_{2|q_1,q_2} &= \frac{b_{2,1}\omega_1+2b_{1,1}\gamma_2}{b_{1,1}b_{2,2}+\Delta}q_2. \end{cases}$$

We rewrite the conditions in order to eliminate the competitor's price from the formula:

$$\begin{cases} \textit{(1Su)} : 2q_1 + 2\frac{b_{11}b1,2}{b_{1,1}b_{2,2}+\Delta}q_2 & \geq \theta_1 + \frac{b_{1,2}b_{2,1}}{b_{1,1}b_{2,2}+\Delta}\omega_1 + 2\frac{b_{1,1}b_{1,2}}{b_{1,1}b_{2,2}+\Delta}a_2\,, \\ \textit{(Sh2)} : (2 + \frac{b_{1,2}b_{2,1}}{b_{1,1}b_{2,2}+\Delta})q_2 & \leq \theta_2 + \frac{b_{1,2}b_{2,1}}{b_{1,1}b_{2,2}+\Delta}a_2 + \frac{b_{2,2}b_{2,1}}{b_{1,1}b_{2,2}+\Delta}\omega_1. \end{cases}$$

If firm 1 is in a shortage stage and firm 2 is in a surplus stage, the optimum pricing strategy and profit function are

$$(1Sh)\&(2Su):\begin{cases} p_1^* = \frac{a_1-q_1+b_{1,2}p_2}{b_{1,1}}, \\ p_2^* = \frac{a_2+b_{2,1}p_1-sb_{2,2}}{2b_{2,2}}. \end{cases} \Rightarrow \begin{cases} p_1^* = \frac{1}{b_{1,1}b_{2,2}+\Delta}(2b_{2,2}\gamma_1 + b_{1,2}\omega_2), \\ p_2^* = \frac{1}{b_{1,1}b_{2,2}+\Delta}(b_{2,1}\gamma_1 + b_{1,1}\omega_2). \end{cases}$$

$$\Rightarrow \begin{cases} \pi^*_{1|q_1,q_2} &= \frac{b_{1,2}\omega_2+2b_{2,2}\gamma_1}{b_{1,1}b_{2,2}+\Delta}q_1\,, \\ \pi^*_{2|q_1,q_2} &= 2b_{2,2}(\frac{b_{1,1}\omega_2+b_{2,1}\gamma_1}{b_{1,1}b_{2,2}+\Delta})^2 + s(\gamma_2 + \frac{b_{2,1}b_{1,2}\omega_2+2b_{2,1}b_{2,2}\gamma_1}{b_{1,1}b_{2,2}+\Delta}). \end{cases}$$

We can also rewrite the (*1Sh*) and (*2Su*) conditions in this way:

$$\begin{cases} \text{(1Sh)} : (2 + \frac{b_{1,2}b_{2,1}}{b_{1,1}b_{2,2}+\Delta})q_1 & \leq \theta_1 + \frac{b_{1,2}b_{2,1}}{b_{1,1}b_{2,2}+\Delta}a_1 + \frac{b_{1,1}b_{1,2}}{b_{1,1}b_{2,2}+\Delta}\omega_2\,, \\ \text{(2Su)} : 2q_2 + 2\frac{b_{2,2}b_{2,1}}{b_{1,1}b_{2,2}+\Delta}q_1 & \geq \theta_2 + \frac{b_{2,1}b_{1,2}}{b_{1,1}b_{2,2}+\Delta}\omega_2 + 2\frac{b_{2,2}b_{2,1}}{b_{1,1}b_{2,2}+\Delta}a_1. \end{cases} \tag{3.5}$$

As Lemma 22 shows, the optimal price for both firms has a negative correlation with the value of the realized production quantity of the firm that is in a shortage stage. In other words, while the optimum price for the high yield firm does not rely on his own realized production quantity, it has a linearly negative relation with the other firm's realized production quantity. That basically shows the mechanism through which competing firms can cover one another's shortage, which can reduce the risk of an overall shortage for society.

Deriving the optimal production commitment for the case of asymmetrical firms turned out to be too computationally challenging. Adding the property of symmetry allows us to find the optimum production policies as well as to find contracts that can coordinate the market.

3.2 Competition between two symmetrical firms

In this section, we study a model with a single market and two symmetrical firms with random yet equal yield ratios. Although the firms' demands are interdependent, we frame the market size as it is divided between the firms. As depicted in Assumption 5, each firm's maximum market size is equal to $\bar{a}$, which is half of the total market size a.

Assumption 5.

$$\bar{a} = \frac{a}{2}.$$

Following Assumption 5, the demand function is:

$$\begin{aligned} D_1 =& \bar{a} - bp_1 + b'p_2, \text{ and} \\ D_2 =& \bar{a} - bp_2 + b'p_1. \end{aligned}$$

Lemma 23. *In the monopoly case, if we scale the market parameters a and b, up or down, the optimum production commitment also changes proportionally.*

Proof. From Proposition 1 we have

$$\mathcal{A}(N) = (\frac{a}{b} + s) \int_0^{\frac{a+bs}{2N}} \nu f(\nu) \, \mathrm{d}\nu - \frac{2N}{b} \int_0^{\frac{a+bs}{2N}} \nu^2 f(\nu) \, \mathrm{d}\nu - s\mu - c = 0.$$

If we multiply a, b, and N by X, the same equation remains valid, as X is crossed off in the nominators and denominators. Since we know that $\mathcal{A}(N)$ has a unique root, XN must be the only optimum production commitment for the new setting. □

To illustrate this lemma in practice, imagining two scenarios. In the first, we have two identical and isolated markets with two independent manufacturers. In the second scenario, we have one manufacturer and one market for which the size and demand slope are twice as large as of the markets in the first scenario. Using Lemma 23, we can conclude that the sum of the optimum production commitments in the duopoly setting is the same optimum production commitment in the monopoly setting.

Before we start deriving the optimum pricing strategy, we need to clarify the relationship between the competition parameter b' and the price sensitivity parameter b. Assumption 6 indicates that the competitor's price cannot affect a firm's demand more than the firm's own price.

Assumption 6.

$$0 \leq b' \leq b.$$

The next series of lemmas derive the optimum pricing strategy for firm 1, in different potential regions of market sample space.

Lemma 24. *Given that the competitor's retail price is known, the optimum pricing strategy for firm 1 is as follows:*

$$p_1^* = \begin{cases} \frac{\bar{a}+b'p_2-sb}{2b}, & \textit{if } q_1 \geq \frac{\bar{a}+sb+b'p_2}{2}, \textit{ and} \\ \frac{\bar{a}-q_1+b'p_2}{b}, & \textit{if } q_1 \leq \frac{\bar{a}+sb+b'p_2}{2}. \end{cases} \tag{3.6}$$

Proof. For the surplus quantity case we have

$$q_1 > D_1 \Leftrightarrow q_1 > \bar{a} - bp_1 + b'p_2 \Leftrightarrow p_1 > \frac{\bar{a} - q_1 + b'p_2}{b}.$$

$$\begin{aligned}\pi_{1,Su|q_1,p_2} &= p_1 D_1 - s(q_1 - D_1) = p_1(\bar{a} - bp_1 + b'p_2) - s(q_1 - \bar{a} + bp_1 - b'p_2)\\ &= -bp_1^2 + (\bar{a} + b'p_2 - sb)p_1 + s(\bar{a} - q_1 + b'p_2).\end{aligned}$$

By applying the first order condition we have

$$\begin{aligned}p_1^* &= \frac{\bar{a} + b'p_2 - bs}{2b}, \text{ and}\\ D_1^* &= \frac{\bar{a} + b'p_2 + bs}{2}, \text{ and}\\ \pi^*_{1,Su|q_1,p_2} &= \frac{(\bar{a} + b'p_2 - bs)^2}{4b} - s(q_1 - \bar{a} - b'p_2) = \frac{(\bar{a} + b'p_2 + bs)^2}{4b} - sq_1.\end{aligned}$$

Under the shortage scenario, it can easily be shown that the optimal price makes the demand equal to available production quantity; i.e., the firm clears the market so there is no shortage cost. Thus, for the shortage case we have

$$p_1^* = \frac{\bar{a} - q_1 + b'p_2}{b}, \quad \pi^*_{1,Sh|q_1,p_2} = \frac{\bar{a} - q_1 + b'p_2}{b} * q_1.$$

Defining $u = \bar{a} + b'p_2 + bs$, we can show that the optimum price under the surplus case results in either more or equal profit than it would be the shortage case, as follows:

$(u - 2q_1)^2 \geq 0 \Longrightarrow u^2 \geq 4(u - q_1)q_1.$

$$\Rightarrow \quad \frac{(\bar{a} + b'p_2 + bs)^2}{4} \geq (\bar{a} + b'p_2 + bs - q_1)q_1 \Longrightarrow \frac{(\bar{a} + b'p_2 + bs)^2}{4b} - sq_1 \geq \frac{\bar{a} - q_1 + b'p_2}{b} q_1.$$

$$\pi^*_{1,Su|q_1,p_2} \geq \pi^*_{1,Sh|q_1,p_2}.$$

We can summarize the optimum pricing strategy as:

$$p_1^* = \begin{cases} \frac{\bar{a} + b'p_2 - sb}{2b}, & \text{if } q_1 \geq \frac{\bar{a} + b'p_2 + sb}{2}, \text{ Surplus Scenario, and} \\ \frac{\bar{a} + b'p_2 - q_1}{b}, & \text{if } q_1 \leq \frac{\bar{a} + b'p_2 + sb}{2}, \text{ Shortage Scenario.} \end{cases} \tag{3.7}$$

□

Similar to the decentralized monopoly problem, the optimum price for the surplus case guarantees the manufacturer's greatest profit, given that the realized production quantity is large enough to meet its associated demand. Otherwise, it is in the firm's best interest to clear the market.

We showed that there are two possible cases for each firm, the low realized production quantity and high realized production quantity. That gives us a sample space with four possible cases. We study these cases in Lemmas 25 to 27.

Lemma 25. *If the realized production quantity of both firms is high enough to meet the conditions in equation 3.8, the optimum price and demand is*

$$p_1^* = p_2^* = \frac{\bar{a} - bs}{2b - b'}, \textit{ and,}$$

$$D_1^* = D_2^* = \frac{b}{2b - b'}\bar{a} + \frac{b - b'}{2b - b'}bs.$$

Proof.

$$p_1^* = \frac{\bar{a} + b'\frac{\bar{a} + b'p_1 - bs}{2b} - bs}{2b} \quad \Rightarrow \quad p_1^*(1 - \frac{b'^2}{4b^2}) = \frac{\bar{a}(1 + \frac{b'}{2b}) - bs(1 + \frac{b'}{2b})}{2b}$$

$$\Rightarrow p_1^* = \frac{a - bs}{2b - b'}.$$

By replacing the optimum price formula in the demand function, one can easily find the optimum demand. These results are the same for both firms, since the firms operate under the same conditions and formulas are independent of the realized production quantities. We can also rewrite the conditions as follows:

$$\text{if} \begin{cases} q_1 \geq D_1^* = \frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs, \\ q_2 \geq D_2^* = \frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs \end{cases} \quad \Rightarrow \quad p_1^* = p_2^* = \frac{\bar{a} - bs}{2b - b'}. \tag{3.8}$$

□

Lemma 26. *If the realized production quantity of both firms is low enough to meet equation 3.9, the optimum price and demand are*

$$\begin{aligned}
p_1^* =& \frac{\bar{a}}{b-b'} - \frac{bq_1 + b'q_2}{b^2 - b'^2}, \text{ and} \\
p_2^* =& \frac{\bar{a}}{b-b'} - \frac{bq_2 + b'q_1}{b^2 - b'^2}.
\end{aligned}$$

$$D_1^* = q_1 \quad D_2^* = q_2.$$

Proof.

$$\begin{aligned}
p_1 =& \frac{\bar{a} + b'p_2 - q_1}{b} = \frac{a + b'\frac{\bar{a}+b'p_1-q_2}{b} - q_1}{b} \\
&\Rightarrow p_1\left(1 - \frac{b'^2}{b^2}\right) = \bar{a}\frac{b+b'}{b^2} - \frac{b'q_2 + bq_1}{b^2} \\
&\Rightarrow p_1^* = \frac{\bar{a}}{b-b'} - \frac{bq_1 + b'q_2}{b^2 - b'^2}.
\end{aligned}$$

The formula for p_2^* can be derived by the same approach. Inserting the new price formulas in equation 3.7, we can reconstruct the conditions as follows:

$$\begin{cases}
q_1 + \frac{bb'}{2b^2-b'^2}q_2 < \frac{b(b+b')}{2b^2-b'^2}a + \frac{b(b^2-b'^2)}{2b^2-b'^2}s & \text{, and} \\
\\
q_2 + \frac{bb'}{2b^2-b'^2}q_1 < \frac{b(b+b')}{2b^2-b'^2}a + \frac{b(b^2-b'^2)}{2b^2-b'^2}s.
\end{cases} \tag{3.9}$$

□

As in the monopoly case, the manufacturers clear the market when their realized production quantity drops below a certain level. Since manufacturers' demand functions are interdependent, these equations are written in terms of both q_1 and

q_2. However, we can show that if the competition parameter b' approaches zero, this conditions also approach those of the monopoly case in the previous chapter.

Lemma 27. *If the realized production quantity of firm 1 is fairly high but that of firm 2 is low, as described in equation 3.11, the optimum price and demand are*

$$\begin{aligned}
p_1^* =& \frac{b+b'}{2b^2-b'^2}\bar{a} - \frac{b'}{2b^2-b'^2}q_2 - \frac{b^2}{2b^2-b'^2}s,\\
p_2^* =& \frac{2b+b'}{2b^2-b'^2}\bar{a} - \frac{2b}{2b^2-b'^2}q_2 - \frac{bb'}{2b^2-b'^2}s,
\end{aligned}$$

and

$$\begin{aligned}
D_1 =& \frac{b^2+b'b}{2b^2-b'^2}\bar{a} + \frac{b(b^2-b'^2)}{2b^2-b'^2}s - \frac{bb'}{2b^2-b'^2}q_2,\\
D_2 =& q_2.
\end{aligned} \tag{3.10}$$

Proof.

$$\begin{aligned}
p_1^* &= \frac{\bar{a} + b'p_2 - bs}{2b} = \frac{\bar{a} + b'\frac{\bar{a}+b'p_1-q_2}{b} - bs}{2b},\\
p_1^*\left(\frac{2b^2-b'^2}{2b^2}\right) &= \bar{a}\frac{b+b'}{2b^2} - q_2\frac{b'}{2b^2} - \frac{s}{2}.\\
\Rightarrow p_1^* &= \frac{b+b'}{2b^2-b'^2}\bar{a} - \frac{b'}{2b^2-b'^2}q_2 - \frac{b^2}{2b^2-b'^2}s.
\end{aligned}$$

$$\begin{aligned}
p_2^* &= \frac{\bar{a} + b'p_1 - q_2}{b} = \frac{\bar{a} + b'\frac{\bar{a}+b'p_2-bs}{2b} - q_2}{b}.\\
p_2^*\left(\frac{2b^2-b'^2}{2b^2}\right) &= \bar{a}\frac{2b+b'}{2b^2} - \frac{b'}{2b}s - \frac{q_2}{b},\\
\Rightarrow p_2^* &= \frac{2b+b'}{2b^2-b'^2}\bar{a} - \frac{2b}{2b^2-b'^2}q_2 - \frac{bb'}{2b^2-b'^2}s.
\end{aligned}$$

We can also rewrite the conditions as follows

$$\begin{cases}
q_1 + \frac{bb'}{2b^2-b'^2}q_2 > \frac{b^2+bb'}{2b^2-b'^2}\bar{a} + \frac{b(b^2-b'^2)}{2b^2-b'^2}s \text{, and}\\
q_2 < \frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}s.
\end{cases} \tag{3.11}$$

$\square$

By summarizing the previous lemmas, we derive the optimum pricing strategy as a function of realized production quantity:

$$\text{If } \begin{cases} q_1 \geq \frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs, \\ q_2 \geq \frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs \end{cases} \Rightarrow p_1 = p_2 = \frac{\bar{a}-sb}{2b-b'} \Rightarrow D_1 = D_2 = \frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs.$$

$$\text{If } \begin{cases} q_1 \leq \frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs, \\ \frac{bb'}{2b^2-b'^2}q_1 + q_2 \geq \frac{b(b+b')}{2b^2-b'^2}a + \frac{b(b^2-b'^2)}{2b^2-b'^2}s \end{cases} \begin{cases} p_1^* = & \frac{2b+b'}{2b^2-b'^2}\bar{a} - \frac{2b}{2b^2-b'^2}q_1 - \frac{bb'}{2b^2-b'^2}s, \\ p_2^* = & \frac{b+b'}{2b^2-b'^2}\bar{a} - \frac{b'}{2b^2-b'^2}q_1 - \frac{b^2}{2b^2-b'^2}s. \\ & \\ D_1 = & q_1, \\ D_2 = & \frac{b^2+b'b}{2b^2-b'^2}\bar{a} + \frac{b(b^2-b'^2)}{2b^2-b'^2}s - \frac{bb'}{2b^2-b'^2}q_1. \end{cases}$$

$$\text{If } \begin{cases} q_1 + \frac{bb'}{2b^2-b'^2}q_2 \geq \frac{b(b+b')}{2b^2-b'^2}a + \frac{b(b^2-b'^2)}{2b^2-b'^2}s, \\ q_2 \leq \frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs \end{cases} \begin{cases} p_1^* = & \frac{b+b'}{2b^2-b'^2}\bar{a} - \frac{b'}{2b^2-b'^2}q_2 - \frac{b^2}{2b^2-b'^2}s, \\ p_2^* = & \frac{2b+b'}{2b^2-b'^2}\bar{a} - \frac{2b}{2b^2-b'^2}q_2 - \frac{bb'}{2b^2-b'^2}s. \\ & \\ D_1 = & \frac{b^2+b'b}{2b^2-b'^2}\bar{a} + \frac{b(b^2-b'^2)}{2b^2-b'^2}s - \frac{bb'}{2b^2-b'^2}q_2, \\ D_2 = & q_2. \end{cases}$$

$$\text{If } \begin{cases} q_1 + \frac{bb'}{2b^2-b'^2}q_2 < \frac{b(b+b')}{2b^2-b'^2}a + \frac{b(b^2-b'^2)}{2b^2-b'^2}s, \\ q_2 + \frac{bb'}{2b^2-b'^2}q_1 < \frac{b(b+b')}{2b^2-b'^2}a + \frac{b(b^2-b'^2)}{2b^2-b'^2}s. \end{cases} \Rightarrow \begin{cases} p_1^* = & \frac{\bar{a}}{b-b'} - \frac{bq_1+b'q_2}{b^2-b'^2}, \\ p_2^* = & \frac{\bar{a}}{b-b'} - \frac{bq_2+b'q_1}{b^2-b'^2} \end{cases} \Rightarrow D_1^* = q_1 \quad D_2^* = q_2.$$

Three terms repeatedly appear in the optimum pricing strategy. Thus, in order to simplify the formulas, we define auxiliary variables, Z, I, and $\bar{\theta}$ to replace them:

$$\begin{aligned}
Z &= \frac{bb'}{2b^2 - b'^2},\\
I &= \frac{b(b+b')}{2b^2 - b'^2}a + \frac{b(b^2 - b'^2)}{2b^2 - b'^2}s \;=\; b * (b+b') * \frac{a + (b-b')s}{2b^2 - b'^2},\\
\bar{\theta} &= \frac{b}{2b - b'}\bar{a} + \frac{b - b'}{2b - b'}s \;=\; \frac{\bar{a} + (b-b')s}{2b - b'}b.
\end{aligned}$$

Remark 1.

$$\frac{I}{1+Z} = \bar{\theta}.$$

By rewriting the optimum pricing strategy using the new auxiliary variable, we have:

$$\text{If } \begin{cases} q_1 \geq \bar{\theta}, \\ q_2 \geq \bar{\theta} \end{cases} \Rightarrow \qquad p_1 = p_2 = \frac{\bar{a} - bs}{2b - b'} \Rightarrow D_1^* = D_2^* = \bar{\theta}.$$

$$\text{If } \begin{cases} q_1 \leq \bar{\theta}, \\ Zq_1 + q_2 \geq I \end{cases} \Rightarrow \qquad \begin{cases} p_1^* = \frac{2b+b'}{2b^2-b'^2}\bar{a} - \frac{2b}{2b^2-b'^2}q_1 - \frac{bb'}{2b^2-b'^2}s, \\ p_2^* = \frac{b+b'}{2b^2-b'^2}\bar{a} - \frac{b'}{2b^2-b'^2}q_1 - \frac{b^2}{2b^2-b'^2}s. \\ \\ D_1^* = q_1, \\ D_2^* = I - Zq_1. \end{cases}$$

$$\text{If } \begin{cases} q_1 + Zq_2 \geq I, \\ q_2 \leq \bar{\theta} \end{cases} \Rightarrow \qquad \begin{cases} p_1^* = \frac{b+b'}{2b^2-b'^2}\bar{a} - \frac{b'}{2b^2-b'^2}q_2 - \frac{b^2}{2b^2-b'^2}s, \\ p_2^* = \frac{2b+b'}{2b^2-b'^2}\bar{a} - \frac{2b}{2b^2-b'^2}q_2 - \frac{bb'}{2b^2-b'^2}s. \\ \\ D_1^* = I - Zq_2, \\ D_2^* = q_2. \end{cases}$$

$$\text{If}\ \begin{cases} q_1 + Zq_2 < I, \\ q_2 + Zq_1 < I \end{cases} \qquad \Rightarrow \qquad \begin{cases} p_1^* = \frac{\bar{a}}{b-b'} - \frac{bq_1+b'q_2}{b^2-b'^2}, \\ p_2^* = \frac{\bar{a}}{b-b'} - \frac{bq_2+b'q_1}{b^2-b'^2}. \\ \\ D_1^* = q_1, \\ D_2^* = q_2. \end{cases}$$

Four different inequality sets in the optimum pricing strategy separate the sample space of q_1-q_2 into four different regions. The optimum pricing strategy offers a unique set of formulas for each of these four regions, all of which meet at point A:

$$A \begin{cases} q_1 = \bar{\theta} \quad \text{, and} \\ q_2 = \bar{\theta}. \end{cases}$$

In order to plot the sample space we define lines e_1 and e_2 as follows

$$e_1 : \quad q_1 = -Zq_2 + I,$$
$$e_2 : \quad q_1 = -\frac{q_2}{Z} + \frac{I}{Z}.$$

Using algebra, it is straightforward to show that e_1 and e_2 cross one another at point A. The sample space for the optimum pricing strategy is depicted in Figure 3.1. This result shows that the aforementioned conditions cover the sample space of all possible values of q_1 and q_2, with no overlaps.

Lemma 28. *The absolute value of the slope of e_1 is always less than 1, while the absolute value of the slope of e_2 is always greater than 1. Also, both lines are downward sloping.*

$$0 \leq Z \leq 1.$$

Proof. Define $l = \frac{b'}{b}$. Using Assumption 6, we have $0 \leq l \leq 1$. Given the definition of l, we can reformulate Z as follows:

$$Z = \frac{bb'}{2b^2 - b'^2} = \frac{\frac{bb'}{b^2}}{\frac{2b^2-b'^2}{b^2}} = \frac{l}{2 - l^2}$$

$$\begin{cases} \lim_{l \to 0} Z = \frac{l}{2-l^2} = 0 & \text{, and} \\ \lim_{l \to 1} Z = \frac{l}{2-l^2} = 1 & \text{, and} \\ \frac{\partial}{\partial l} Z = \frac{2-l^2+2l^2}{(2-l^2)^2} = \frac{2+l^2}{(2-l^2)^2} > 0 \end{cases} \Rightarrow \quad 0 \leq Z \leq 1.$$

□

In order to build the profit function of firm 1, we assume that the production commitment of firm 2 is known. Lemma 29 presents the profit function of firm 1 for the case that N_1 is less than N_2 and Lemma 30 presents the profit function for firm 1 for the reverse. The blue lines in Figure 3.1 show the difference between these two cases and show how each case crosses different regions of the production sample space. Also, it is easy to show that the profit function formulas described in Lemmas 29 and 30 converge if N_1 and N_2 approach one another.

Lemma 29. *If $N_1 \leq N_2$, the profit function of firm 1 for a known N_2 is as follows:*

$$\begin{aligned} \pi_1(N_1|N_2) = (&\frac{\bar{a}}{b - b'} + s)N_1 \int_0^{\frac{I}{(N_1 Z + N_2)}} \nu f(\nu)\, \mathrm{d}\nu - \frac{bN_1^2 + b'N_1N_2}{b^2 - b'^2} \int_0^{\frac{I}{(N_1 Z + N_2)}} \nu^2 f(\nu)\, \mathrm{d}\nu \\ &+ \frac{I}{b} N_1 \int_{\frac{I}{(N_1 Z + N_2)}}^{\frac{\bar{\theta}}{N_1}} \nu f(\nu)\, \mathrm{d}\nu - \frac{2b}{2b^2 - b'^2} N_1^2 \int_{\frac{I}{(N_1 Z + N_2)}}^{\frac{\bar{\theta}}{N_1}} \nu^2 f(\nu)\, \mathrm{d}\nu \\ &+ \frac{\bar{\theta}^2}{b} \bar{F}(\frac{\bar{\theta}}{N_1}) - sN_1\mu - cN_1. \end{aligned}$$

Proof. See Appendix N. □

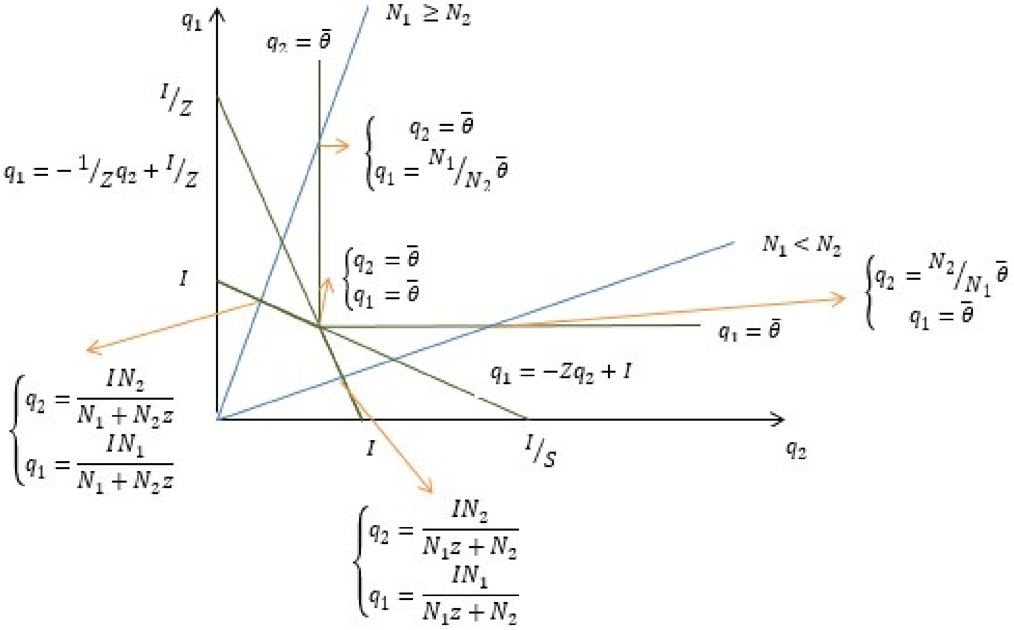


Figure 3.1 The pricing strategy sample space for q_1 and q_2

Lemma 30. *If $N_1 \geq N_2$, the profit function of firm 1 is as follows:*

$$\begin{aligned}\pi_1(N_1|N_2) = (&\frac{\bar{a}}{b-b'} + s)N_1 \int_0^{\frac{I}{N_1+N_2Z}} \nu f(\nu)\,\mathrm{d}\nu - \frac{bN_1^2 + b'N_1N_2}{b^2-b'^2} \int_0^{\frac{I}{N_1+N_2Z}} \nu^2 f(\nu)\,\mathrm{d}\nu \\ &+ \frac{I^2}{b} \int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}} f(\nu)\,\mathrm{d}\nu - \frac{2ZI}{b}N_2 \int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}} \nu f(\nu)\,\mathrm{d}\nu + \frac{Z^2}{b}N_2^2 \int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu \\ &+ \frac{\bar{\theta}^2}{b}\bar{F}(\frac{\bar{\theta}}{N_2}) - s\mu N_1 - cN_1.\end{aligned}$$

Proof. See Appendix O. □

Remark 2.

$$\lim_{N_1 \to \infty} \pi_1(N_1|N_2) = -\infty.$$

Proof. See Appendix P. □

Lemma 31. *If $N_1 \leq N_2$, the derivative of the profit function with respect to N_1 is*

$$\begin{aligned}\frac{\partial}{\partial N_1}\pi_1(N_1|N_2) = (&\frac{\bar{a}}{b-b'} + s) \int_0^{\frac{I}{N_1Z+N_2}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2bN_1 + b'N_2}{b^2-b'^2} \int_0^{\frac{I}{N_1Z+N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu \\ &+ \frac{I}{b} \int_{\frac{I}{N_1Z+N_2}}^{\frac{\bar{\theta}}{N_1}} \nu f(\nu)\,\mathrm{d}\nu - \frac{4b}{2b^2-b'^2}N_1 \int_{\frac{I}{N_1Z+N_2}}^{\frac{\bar{\theta}}{N_1}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c.\end{aligned}$$

Proof. See Appendix K. □

Lemma 32. *If $N_1 \geq N_2$, the derivative of the profit function with respect to N_1 is*

$$\frac{\partial}{\partial N_1}\pi_1(N_1|N_2) = (\frac{\bar{a}}{b-b'} + s) \int_0^{\frac{I}{N_1+ZN_2}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2bN_1 + b'N_2}{b^2-b'^2} \int_0^{\frac{I}{N_1+ZN_2}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c.$$

Proof. Proof is similar to the case of $N_1 < N_2$ as indicated in Appendix K. □

Proposition 7. *Firms in the duopoly setting have an equal and unique optimum production commitment, which is equal to the root of function $\mathcal{A}_{\mathcal{D}}$:*

$$\mathcal{A}_{\mathcal{D}}(N) = (\frac{\bar{a}}{b-b'} + s) \int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N \int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c = 0. \quad (3.12)$$

Proof. Since the firms are symmetrical, the production commitment of both firms is equal at the equilibrium point. That is the point where the marginal profit of both firms is equal to zero, although that might not be the only equilibrium point. In order to find the optimum production commitment, we set the profit function derivative for firm 1 equal to zero. Also, since the production commitments are equal, we replace N_1 and N_2 with general term N, so that

$$\begin{aligned}
\frac{\partial}{\partial N_1}\pi_1(N_1|N_2) &= (\frac{\bar{a}}{b-b'}+s)N\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2br+b'}{b^2-b'^2}N^2\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - sN\mu - cN \\
&= N\left((\frac{\bar{a}}{b-b'}+s)\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N_2\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c\right) \\
&= N_2 * \mathcal{A}_{\mathcal{D}}(N) = 0.
\end{aligned}$$

The following properties show that $\mathcal{A}_{\mathcal{D}}(N)$ has a unique root which is the optimum production commitment for each of the two firms:

$$\begin{aligned}
&\lim_{N\to 0}\mathcal{A}_{\mathcal{D}}(N) = (\frac{\bar{a}}{b-b'}+s)\mu - s\mu - c = (\frac{\bar{a}}{b-b'}-\frac{c}{\mu})\mu > 0, \\
&\lim_{N\to\infty}\mathcal{A}_{\mathcal{D}}(N) = -\frac{2b+b'}{b^2-b'^2}\frac{\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu}{\frac{1}{N}} - s\mu - c = -\frac{2b+b'}{b^2-b'^2}\frac{\frac{\bar{\theta}^3}{N^4}f(\frac{\bar{\theta}}{N})}{-\frac{1}{N^2}} - s\mu - c = -s\mu - c < 0, \\
&\frac{\partial}{\partial N}\mathcal{A}_{\mathcal{D}}(N) = \frac{\bar{\theta}^3}{N^3(b-b')} - \frac{2b+b'}{b^2-b'^2}\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu < 0 \qquad \exists K| \quad \forall N \quad N > k.
\end{aligned}$$

□

Notice that in order to obtain a negative value for $\frac{\partial}{\partial N}\mathcal{A}_{\mathcal{D}}(N)$, the production commitment should be larger than the lower bound of k. The exact value of k depends on the probability distribution function of the yield ratio. Nevertheless, it exists. It is also worth mentioning that the $\mathcal{A}_{\mathcal{D}}(N)$ function may not be monotonically decreasing for production commitment values less than k. That also depends on the type of probability distribution function of the yield ratio.

In order to conduct numerical analyses, we need to find the production capacity range in which the formulas are feasible. We start by proving that the upper limits of the profit function integrals are larger than the lower limits. Then, we find a feasible range for capacity, in which potential yield ratios remain between the upper limit and lower limit of its affiliated probability distribution function. The following remark and lemma provide the inequalities that are required for running the numerical results.

Remark 3.

$$\textit{If } N_1 < N_2 \Rightarrow \frac{I}{N_1 Z + N_2} < \frac{\bar{\theta}}{N_1},$$
$$\textit{if } N_1 > N_2 \Rightarrow \frac{I}{N_1 + N_2 Z} < \frac{\bar{\theta}}{N_2}.$$

Proof. If $N_1 < N_2 \Rightarrow ZN_1 + N_1 < ZN_1 + N_2 \Rightarrow \frac{I}{ZN_1+N_2} < \frac{I}{ZN_1+N_1} = \frac{\bar{\theta}}{N_1}$. The proof for the second expression follows the same steps. □

Lemma 33. *The manufacturers' capacity lower and upper bounds are as follows:*

$$\frac{1}{UB} * \frac{\bar{b}}{2\bar{b} - b'} < \frac{N}{a} < \infty.$$

Proof.

$$\frac{\bar{\theta}}{N} < UB \Rightarrow \frac{\bar{\theta}}{\bar{a}UB} < \frac{N}{a} \Rightarrow \frac{1}{UB}\left[\frac{\bar{b}}{2\bar{b} - b'} + \frac{\bar{b} - b'}{2\bar{b} - b'}\frac{bs}{\bar{a}}\right] < \frac{N}{\bar{a}}.$$

Since $\frac{bs}{\bar{a}}$ is negligible,

$$\Rightarrow \frac{1}{UB}\frac{\bar{b}}{2\bar{b} - b'} < \frac{N}{\bar{a}}.$$

Also we have

$$LB < \frac{I}{N_1 Z + N_2} \Rightarrow \frac{N_1}{\bar{a}} < \frac{I}{\bar{a}(Z + \frac{N_2}{N_1})} * \frac{1}{LB}.$$

Assuming that $\frac{N_1}{N_2}$ is not too large, as LB reaches zero, the upper limit of $\frac{N}{\bar{a}}$ approaches infinity. □

3.3 Measuring the impact of uncertainty in the symmetrical setting

In this section, we study the symmetrical duopoly model under the deterministic setting. We use the results of this section as a benchmark for the stochastic model. Since production is deterministic, manufacturers can match the realized production quantity with the optimum demand; therefore, there is no surplus or shortage. The next lemma provides the optimum production commitment and pricing strategy.

Lemma 34. *If the production process is deterministic, the optimum production commitment for each of the firms is equal to*

$$N^* = \frac{b}{2(b-b')\mu}a.$$

Proof. Since there is no randomness in production, the firms can exactly set their production to the demand they plan to meet, which means

$$\begin{aligned}
&q = \mu \times N, \\
&D_{i(p_i)} = q_i, \qquad\qquad \Rightarrow \quad p_1 = \frac{\bar{a} + b'p_2 - q_1}{b} \qquad \pi_1 = p_1 \times q_1 = -\frac{q_1^2}{b} + \frac{\bar{a}+b'p_2}{b}q. \\
&D1 = \bar{a} - b * p_1 + b' * p_2
\end{aligned}$$

By applying the first order condition we have

$$q_1^* = \frac{\bar{a}+b'p_2}{2}, \quad q_2^* = \frac{\bar{a}+b'p_1}{2}.$$

We would like to find a formula for q_1^* that is independent of p_2. To do so, we derive the formula for p_1 and replace p_2 in that formula as follows:

$$\begin{aligned}
p_1 &= \frac{\bar{a}+b'p_2-q_1}{b}, \\
&\qquad\qquad\qquad \Rightarrow p_1 = \frac{\bar{a} + b'\frac{\bar{a}+b'p_1-q_2}{b} - q_1}{b}. \\
p_2 &= \frac{\bar{a}+b'p_1-q_2}{b}
\end{aligned}$$

With some simple algebra we get

$$p_1 = \frac{\bar{a}(b+b')}{(b^2-b'^2)} - \frac{b'q_2+bq_1}{b^2-b'^2}, \quad p_2 = \frac{\bar{a}(b+b')}{(b^2-b'^2)} - \frac{b'q_1+bq_2}{b^2-b'^2}.$$

The profit function for the manufacturer is as follows:

$$\pi_1 = p_1 \times q_1 - cN_1 = p_1 \mu N_1 - cN_1 = (\frac{\bar{a}(b+b')}{(b^2-b'^2)} - \frac{b'\mu N_2 + b\mu N_1}{b^2-b'^2}) * \mu N_1 - cN_1.$$

Assuming that $\dot{N}_1 = \dot{N}_2$ and by applying the first order condition, we have

$$\dot{N}_1 = \dot{N}_2 = \frac{\bar{a}}{2\mu} - \frac{(b-b')c}{2\mu^2}.$$

□

Remark 4. *Competition increases the production commitment for the deterministic case.*

In the next lemma we show that randomness causes the production commitment to increase. Later in the numerical analysis chapter, we will see that the optimum production commitment increases if the standard deviation grows.

Lemma 35. *Uncertainty increases maximum sales for both firms*

Proof. The maximum sales for the stochastic case is equal to $\frac{\bar{a}b}{2b-b'} + \frac{b(b-b')s}{2b-b'}$, while the sales for the deterministic case is equal to $\frac{\bar{a}b}{2b-b'}$. The firms in the deterministic case do not end up with any surplus items so the salvage cost is zero. That can justify the difference between the maximum sales in the deterministic case and the stochastic case.

□

This is compatible with what we found for the monopoly case and for what one intuitively expects. In the next section, we study how changes in different parameters affect optimal production commitment and manufacturer's optimal profit. This result introduces the examination of contracts that can coordinate the market, which is presented in section 3.5.

$\bar{a}$	$\bar{b}$	b'	s	c	μ	σ
1000	10	2.5	10	10	1	.25

Table 3.1 Default parameter values for the numerical analysis in the duopoly case

3.4 Sensitivity analysis of the duopoly symmetrical setting

In this section, we provide the simulation results of the firm 1 profit function to get an idea of how it changes under different circumstances. In each subsection, we have the different measurements of the profit function as well as graphs of the extreme parameter values. We repeat that experiment for different values for the market parameters, cost parameters, and production parameters.

In each experiment, we change N_1 and N_2 from 100 to 1000, using steps of 100 units, while we alter one parameter at a time. The base parameter setting of the simulation is reported in Table 3.1, and Figure 3.2 shows the values of the profit function in that setting. The straight blue line represents $N1 = N2$ on the $\pi = 0$ plane. The blue circles also show the best response function for firm 1.

There are a few points worth mentioning about this graph:

- As discussed in Lemma 29, the profit function of firm 1 goes to minus infinity if we increase its production commitment toward infinity. Firm 1's profit function is also decreasing in N_2, but it does not necessarily go to minus infinity for high values of N_2. In other words, the profit function of firm 1 is more sensitive to N_1, than to N_2.

- The peak of the profit function for firm 1 occurs when N_2 is fairly small. That

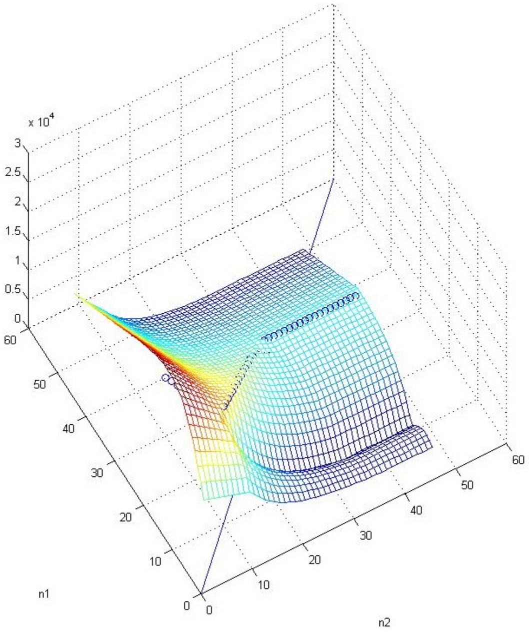


Figure 3.2 Profit function for N_1 and N_2 for parameter values in Table 3.1

is what we call the **semi-monopoly** area, largely because the competitor is in a weak position so that firm 1 would have its highest feasible profit. The optimal response of N_1 depends greatly on N_2

- As N_2 increases, the competition enters the **equal response** area, where the best response N_1 is equal to N_2, and the profit function decreases as N_1 and N_2 increase. This area is the most important part to the sample space since the equilibrium production commitment falls into this interval. We cover it in more detail in the next section.

- The third area is **constant response**, where the profit function of firm 1 decreases in N_1 but remains almost constant in N_2. That happens when the market is saturated and the firm finds it advantageous not to increase its sales i.e., not to decrease its price any further than a certain level, no matter how many realized production units retained on hand. In this case, having a higher level of stock does not generate any further profit; instead, it incurs higher salvage cost. On the other hand, the competitor is also in saturated market status, a level that does not affect the sales of firm 1. Therefore, the profit function in this area looks to be linearly decreasing in N_1 and relatively insensitive to N_2.

- One can divide the optimum profit of firm 1 into the *competitive leverage* and the *base production* profit. For smaller values of N_2, the firm 1 optimum profit is remarkably high, and that is due to the competitive leverage. As firm 2 increases its production commitment, firm 1 starts to lose some of its benefit, yet it holds on to its base production profit. The base profit and its affiliated production commitment do not change much if the competitor increases its

b	$(N1, N2)_{max}$	$\pi(N1, N2)_{max}$	$\pi(100, 100)$	$\pi(1000, 1000)$	$N_{1,Const}$
1	(500, 100)	314500	156500	204400	590
5	(400,100)	59000	30500	26700	602
10	(300, 100)	27300	14700	4500	617
50	(200, 100)	3200	2200	-13000	100
100	(200, 100)	580	570	-15000	100

Table 3.2 The optimal production commitment for different values of b

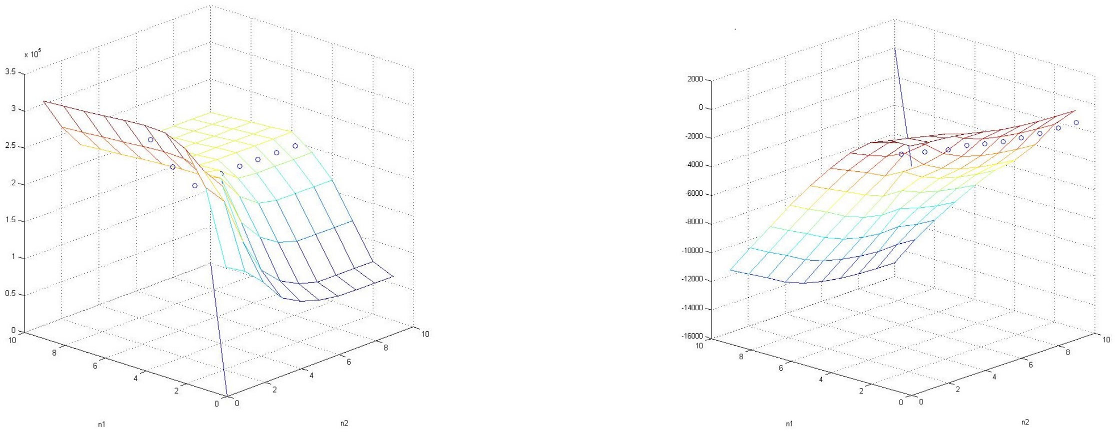
production commitment even more.

We study the sensitivity of the profit function and optimum production commitment to the market parameters, cost parameters, and production yield parameters in the next sub-sections.

3.4.1 Market and competition parameters

By running two sets of simulations, we can study the effect of market and competition parameters. In the first one, we fix market size and the ratio of competition parameter to the price sensitivity parameter $\bar{a}$ & $b'/b = .2$, and assign different values to the price sensitivity parameter, b. Table 3.2 and Figure 3.3 demonstrate the maximum and equilibrium profit function and affiliated production commitments.

The graphs in Figure 3.3 show the profit function for $b = 1$ and $b = 100$, which are the lowest and highest value of price sensitivity parameter in our study. As these graphs indicate, increasing the price sensitivity decreases the semi-monopoly effect. This makes sense because a highly price-sensitive market does not leave much room



a: $b = 1$

b: $b = 100$

Figure 3.3 Profit function for different values of the price sensitivity index

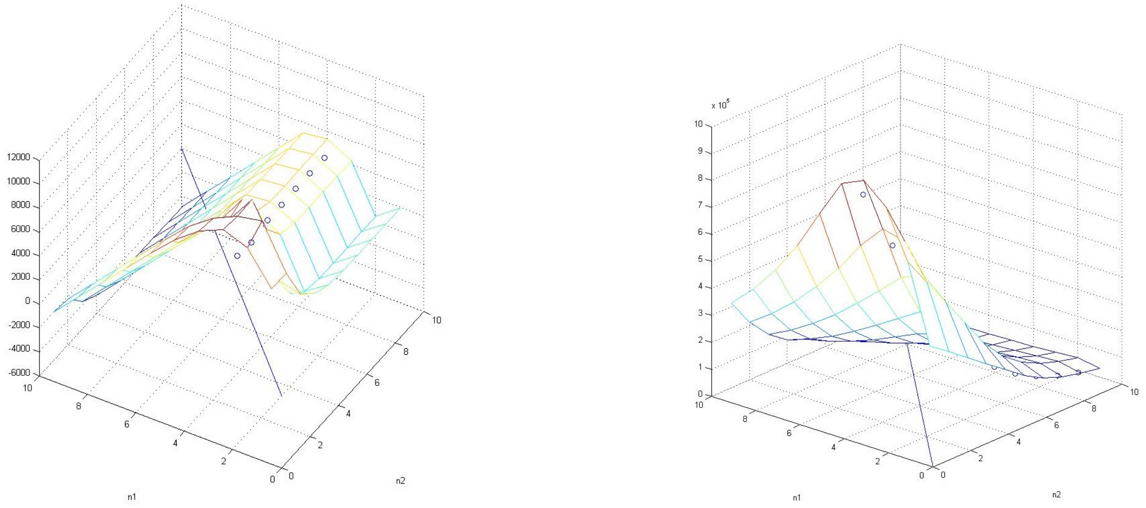
to generate revenue, so survival becomes more challenging and competition becomes less critical. Having a high value of b also makes the firm 1 profit function highly dependent on the value of N_1. Thus, throughout the graph, the second graph looks pretty much like a monotonically descending function in N_1 with not much sensitivity to N_2.

In the second set of numbers here, we fix the market size and price sensitivity parameter, $\bar{a}$ and b, and assign different numbers to the competition index b'. As we discussed in Assumption 6, the proportion of b' to b should be less than 0.5 Therefore, we report b' as a proportion of b in Table 3.3.

Figure 3.4 presents a profit function graph for a highly competitive case versus a no-competition case. The graph on the right is for a highly competitive setting

$\frac{b'}{b}$	$(N1, N2)_{max}$	$\pi(N1, N2)_{max}$	$\pi(100, 100)$	$\pi(1000, 1000)$	$N_{1,Const}$
0	(200, 200)	10440	6870	-4870	453
0.01	(300,100)	10600	7000	-4600	457
0.1	(300, 100)	14400	8800	-2000	511
0.25	(300, 100)	27300	14700	4400	617
0.49	(400, 100)	927000	393000	28000	210

Table 3.3 Numerical results for different values of competition index



a: $b' = 0$

b: $b' = .49$

Figure 3.4 Profit function for different values of the competition index in which $b' = .49$, so that the competitor's price is almost as influential on each firm's demand as its own price is. Naturally, the profit function seems to be fairly symmetrical with respect to N_1 and N_2. Another interesting observation for a highly competitive setting is that the maximum possible profit and the minimum values are significantly higher in this setting. In other words, the firm's profit is higher if the competition is tougher.

Note that the constant response area seems not to exist in a highly competitive setting, or at least it exists only if the production commitment is fairly high. One way to justify this result is that, if the competitor's price affects a firm's demand just as much as its own price, it does not make much sense to set a minimum price hoping that demand will not fall below a certain level. We can also look at the formulas for the saturation setting. If $b' = b$ then the saturation level $\bar{\theta}$ is equal to $\bar{a}$. In other words, the firm will not fix its price unless its sales reach maximum possible coverage $\bar{a}$.

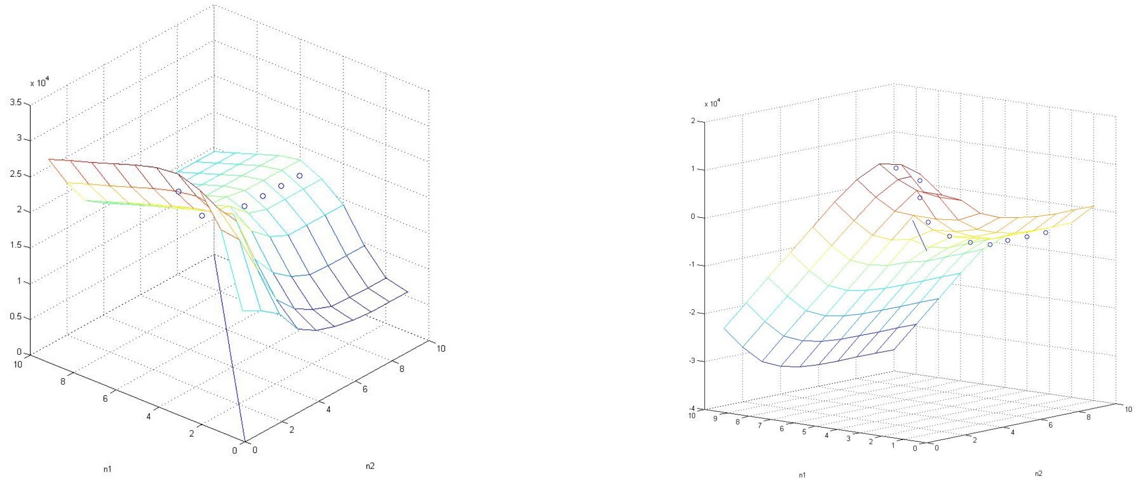
3.4.2 Cost parameters

This section studies the the effect of salvage and production cost on optimum profit function and production commitment. This is particularly important because it leads us to determine whether contracts such as production subsidizing or salvage rebate can possibly coordinate the market. Table 3.4 describes our simulation results for different values of the production cost c.

The first graph on the left side of Figure 3.5 represents the low production cost $c = 0.01$, while the right one demonstrates the high production cost $c = 50$. As

c	$(N1, N2)_{max}$	$\pi(N1, N2)_{max}$	$\pi(100, 100)$	$\pi(1000, 1000)$	$N_{1,Const}$
0.01	(400, 100)	31200	15700	14400	453
1	(400,100)	31000	15600	13500	617
10	(300, 100)	27300	14700	4500	617
25	(200, 100)	20800	13200	-10500	438
50	(200, 100)	15800	10700	-35500	330

Table 3.4 Numerical results for different values of c



a: $c = 1$

b: $c = 50$

Figure 3.5 Profit function for different values of production cost

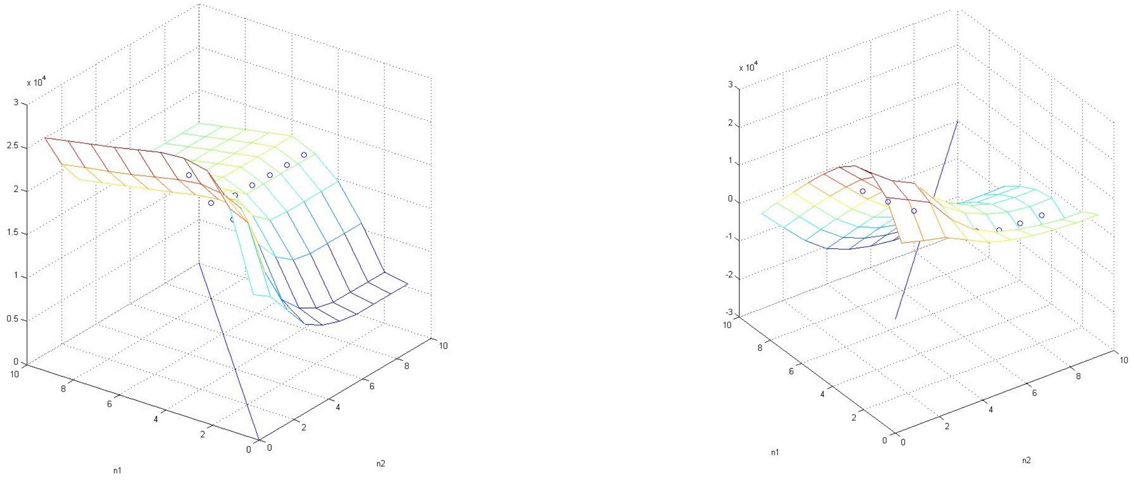
s	$(N1, N2)_{max}$	$\pi(N1, N2)_{max}$	$\pi(100, 100)$	$\pi(1000, 1000)$	$N_{1,Const}$
-5	(400, 100)	28000	14700	16000	573
0	(400, 100)	27700	14700	12200	587
10	(400,100)	27300	14700	4500	617
25	(300, 100)	27200	14700	-6800	484
50	(300, 100)	27200	14700	-25000	421

Table 3.5 Numerical results for different values of salvage cost

Lemma 29 and 30 demonstrated, production cost appears in the profit function once and as the term $-cN_1$. Comparing the profit function with high or low production costs, one can add a downward-in-N_1 plane to the graph on the left and get the graph on the right. Notice the significant decrease in the scale of the graphs, the graph on the left is entirely in the positive space while the right one is mostly in the negative space.

Next we have the sensitivity analysis results about salvage cost s. The salvage cost can have a negative value; i.e., salvage cost can act as salvage value. However, the absolute value of salvage should be less than the expected production cost, $-\frac{c}{\mu}$. Table 3.5 demonstrates the simulation results on the salvage cost, s.

As Table 3.5 presents, changing salvage cost does not significantly affect the profit function for the smaller values of the production commitment. Yet, it significantly affects the larger values, such as $\pi(1000, 1000)$. Intuitively, since over-supply is more likely to result when the production commitment is larger, the profit function for larger production commitments is more sensitive to the salvage cost. In other words,



a: $s = -5$ b: $s = 50$

Figure 3.6 Profit function for different values of salvage cost

increasing salvage cost does not significantly affect the $\{N_{1,small}, N_{2,small}\}$ corner, but as N_1 increases, it substantially influences the profit function. The graphs in Figure 3.6 compare the profit function for minimum and maximum salvage cost in our study.

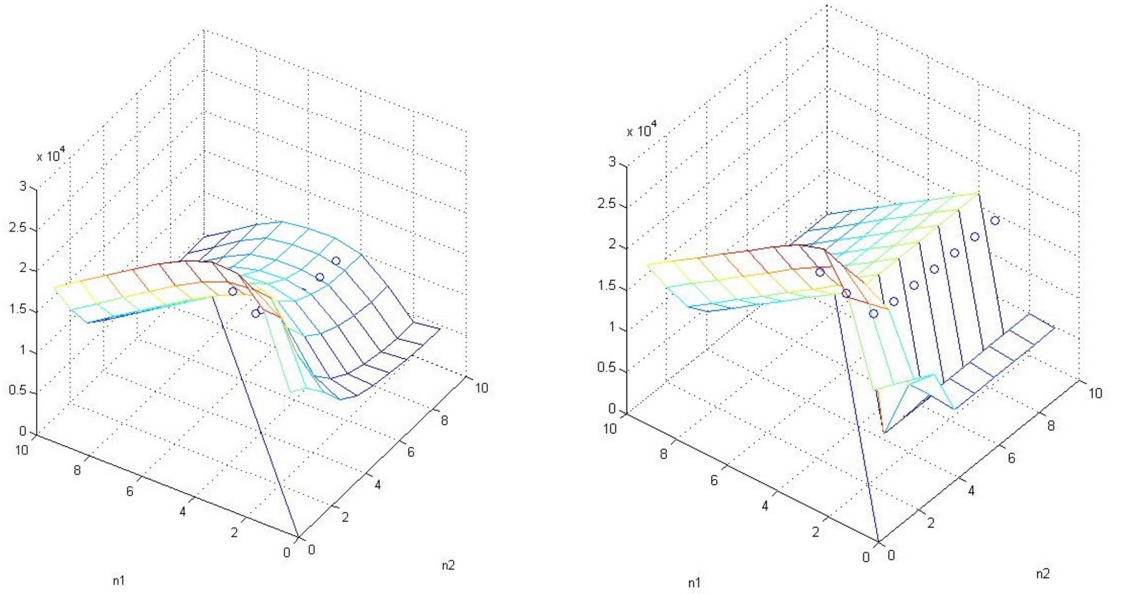
3.4.3 Production parameters, μ and σ

In this section, we study the behavior of the profit function for different values of mean and standard deviation. We use a normal probability distribution function for this experiment with the mean equal to 1. Standard deviation σ, should not exceed .33 to ensure a negligible chance of having a negative random yield.

Table 3.6 shows that the maximum feasible profit for firm 1 decreases as the standard deviation increases. That is justifiable because the expected salvage cost is negligible for the less variable scenario. For the highly variable scenario, however,

σ	$(N1,N2)_{max}$	$\pi(N1,N2)_{max}$	$\pi(100,100)$	$\pi(1000,1000)$	$N_{1,Const}$
0.01	(300, 100)	29000	15000	4500	356
0.1	(300, 100)	28700	15000	4500	423
.25	(300,100)	27300	14750	4500	617
0.3	(400, 100)	26700	14600	4500	531

Table 3.6 Numerical results for different values of σ



a: $\sigma = 0.3$ b: $\sigma = 0.01$

Figure 3.7 Profit function for different values of standard deviation of the yield ratio the expected salvage cost is significant and decreases the maximum profit of the function.

Figure 3.7 compares a highly variable case to an almost-deterministic case. The graph on the left represents the $\sigma = .3$ setting, while the right one shows the profit function for $\sigma = .01$. The sections of the profit function are clearly segregated in the deterministic setting, but the graph is fairly smooth in the highly variable setting.

3.5 Contracting and coordination policies in symmetrical duopoly case

In this section, we introduce two contracts that can coordinate the firms optimum strategy with the social planner's optimum strategy for the symmetrical duopoly case. It is fair to assume that if a social planner offers the contract to both firms and if one firm shows that that contract can coordinate the market, that contract is capable of coordinating both firms. Before we discuss the contracts, Lemma 36 formulates the optimum production commitment for the centralized case, which is the social planner's optimum strategy.

Lemma 36. *The optimum centralized production commitment for the duopoly case is as follows:*

$$N^*_{4,Du} = \frac{2\bar{a}\mu - 2bc}{\mu^2 + \sigma^2}.$$

Proof. Similar to Lemma 6, we have

$$U_{4,Total} = \int_0^\infty \frac{2aq - q^2}{2b} f(\nu)\,\mathrm{d}\nu - 2cN = \frac{a\mu}{b}N - \frac{E(\nu^2)N^2}{2b} - cN.$$

By taking the F.O.C, we have:

$$\frac{\partial}{\partial N}U_{4,Du} = \frac{1}{2}(\frac{a\mu}{b} - \frac{E(\nu^2)}{b}N - c) \Rightarrow N^*_{4,Du} = \frac{1}{2}\frac{a\mu - bc}{\mu^2 + \sigma^2} = \frac{\bar{a}\mu - bc}{\mu^2 + \sigma^2}.$$

□

As Lemma 36 shows, the competition does not affect the optimum production quantity for the centralized case. That is because the social planner cares only about the firms' costs and society's total consumption. Therefore, the pricing mechanism and the interdependency of demands do not affect his optimum production commitment. Next, we discuss the first coordinating contract for the duopoly setting.

3.5.1 Production subsidizing or taxing contract

In this contract the social planner either decreases the production cost for the firms by subsidizing the production process or increases the production cost for the firms by imposing a tax over the production process. We show that, this way, the social planner can increase or decrease the firms' optimum production commitment. By doing so, the social planner can set the firms' optimum production commitment exactly equal to the centralized optimum production commitment.

Looking back at Proposition 7, we can see that the derivative of the profit function $\mathcal{A}_{\mathcal{D}}(N)$ has a linear relation with the production cost parameter c. By increasing or decreasing c, one can shift $\mathcal{A}_{\mathcal{D}}(N)$ up or down and accordingly increase or decrease the cross point of the curve and the x axis; i.e., the optimum production capacity. Lemma 37 illustrates the dynamic of the contract.

Lemma 37. *If the social planner intends to increase the production commitment, he should subsidize the production cost and if he intends to decrease the overall production cost, he should tax the production process.*

What this lemma indicates is that if the social planner subsidizes the production commitment, the manufacturer gets a smaller production cost. Since $\frac{\partial \mathcal{A}_{\mathcal{D}}(N)}{\partial c}$ is negative, a smaller c works as if the $\mathcal{A}_{\mathcal{D}}(N)$ is shifted up. As the properties of $\mathcal{A}_{\mathcal{D}}(N)$ function in Proposition 7 indicates, $\mathcal{A}_{\mathcal{D}}(N)$ is a monotonically decreasing function in

N with a unique root so if it shifts up, its cross point with the X-axis moves to the right. In other words, a lower production cost results in a higher optimum production commitment. The amount of subsidy, stx, depends on the optimum production commitment that the social planner wants the firms to target. It, however, is limited to $s\mu + c$, since that is the value for which it is profitable for the firms to limitlessly expand their capacity.

On the other hand, if the social planner wants the overall production commitment to be less than the decentralized setting, he can impose a tax on the production commitment. Such a tax shifts the $\mathcal{A}_{\mathcal{D}}(N)$ function down so that its cross point with the x axis shifts to the left and the optimum production commitment decreases. Similar to what we had in lemma 9 in the monopoly case, we have

$$stx = -\mathcal{A}(\frac{N_4^*}{2}).$$

That is because the total production commitment is divided between the manufacturers. It is also important to notice that this contract needs to be applied to both manufacturers as our analysis is based on the assumption that firms observe symmetrical markets. Otherwise, our results do not necessarily hold.

3.5.2 Buy-back contract

In this contract, the social planner pledges to make a partial payment to the firms for surplus items left at the end of the marketing season. This contract mitigates the surplus cost and reduces the financial risk of having surplus items.

Proposition 8 explains why the buy-back contract is coordinating. Our analysis is based on the result that the derivative of the profit function $\mathcal{A}_{\mathcal{D}}(N)$ is a monotonically decreasing function in surplus cost, s. In this way, the social planner can change the intersection of the $\mathcal{A}_{\mathcal{D}}(N)$ function and the x- axis by changing the value of s, thus, taxing or subsidizing the surplus items. This proposition provides the necessary properties based on which we analyze the possibility of the buy-back contract to coordinate the market.

Proposition 8. *The buy-back contract can coordinate the market.*

Proof. First we study the $\mathcal{A}_{\mathcal{D}}(N)$ function's in extreme values of the salvage cost. As presented in Appendix M, $\mathcal{A}_{\mathcal{D}}(N)$ has the following properties:

$$\begin{aligned}
\lim_{s\to -\frac{c}{\mu}} \mathcal{A}_{\mathcal{D}}(N,s) &= (\frac{\bar{a}}{b-b'}+s)\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c\\
&= (\frac{\bar{a}}{b-b'}-\frac{c}{\mu})\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - (-\frac{c}{\mu})\mu - c\\
&= (\frac{\bar{a}}{b-b'}-\frac{c}{\mu})\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - (-\frac{c}{\mu})\mu - c
\end{aligned}$$

while $\lim_{s\to -\frac{c}{\mu}} \frac{\bar{\theta}}{N} = \frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})$

$$\begin{aligned}
\lim_{s\to -\frac{c}{\mu}} \mathcal{A}_{\mathcal{D}}(N,s) &= (\frac{\bar{a}}{b-b'}-\frac{c}{\mu})\int_0^{\frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})} \nu f(\nu)\,\mathrm{d}\nu\\
&\quad - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})} \nu^2 f(\nu)\,\mathrm{d}\nu\\
&= (\frac{b+b'}{N(2b+b')})(\bar{a}-(b-b')\frac{c}{\mu})\int_0^{\frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})} \nu f(\nu)\,\mathrm{d}\nu\\
&\quad - \int_0^{\frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})} \nu^2 f(\nu)\,\mathrm{d}\nu\\
&= \int_0^{\frac{b}{(2b-b')}\frac{(\bar{a}-(b-b')\frac{c}{\mu})}{N}} (\frac{b+b'}{(2b+b')}\frac{(\bar{a}-(b-b')\frac{c}{\mu}}{N}-\nu)f(\nu)\,\mathrm{d}\nu > 0
\end{aligned}$$

We can numerically show that this inequality is correct for the reasonable values of the parameters and for three common probability distribution functions.

$$\lim_{s\to -\infty} \mathcal{A}_{\mathcal{D}}(N,s) = (\frac{\bar{a}}{b-b'}+s)\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c$$

$$= (\frac{\bar{a}}{b-b'} + s)\mu - \frac{2b+b'}{b^2-b'^2}N(\mu^2+\sigma^2) - s\mu - c$$
$$= (\frac{\bar{a}}{b-b'})\mu - \frac{2b+b'}{b^2-b'^2}N(\mu^2+\sigma^2) - c.$$

If we assume that $\frac{\bar{a}}{2} < N$ and $1 < \mu$, then the equations hold and $\mathcal{A}_{\mathcal{D}}(N)$ is a decreasing function of s with a positive value at the left side of its domain and a negative value on the positive infinity extreme. The following formulas present the same results in terms of formulas:

$$\lim_{s \to -\frac{c}{\mu}} \mathcal{A}_{\mathcal{D}}(N,s) > 0$$
$$\lim_{s \to \infty} \mathcal{A}_{\mathcal{D}}(N,s) < 0$$
$$\frac{\partial}{\partial s}\mathcal{A}_{\mathcal{D}}(N,s) < 0$$

Based on these properties, we know that the $\mathcal{A}_{\mathcal{D}}(N)$ function is a monotonically decreasing function in s. That means that for any value of production commitment, we can find a salvage cost for which that production commitment is the firms' optimum. Now we can apply that to the centralized optimum production commitment. □

3.6 Comparison of the the monopoly case and the duopoly case

In this section, we compare the symmetrical duopoly case with the monopoly case. We can show how competition affects the firm's total sales and pricing strategy. These results are helpful for the policy makers in order to decide if letting different manufacturers in the same market aligns with their policy.

Lemma 38. *The maximum total sales are higher in the duopoly case than the monopoly case.*

Proof. The maximum total sales in the duopoly of symmetrical firms are equal to $2 * \bar{\theta}$, and the maximum sales in duopoly case are equal to θ. If we study maximum sales $\bar{\theta}$ as a function of b', we get the following properties:

$$\begin{aligned}
&\lim_{b' \to 0} \bar{\theta} = \frac{b}{2b - b'}\bar{a} + \frac{b - b'}{2b - b'}bs = \frac{\bar{a} + bs}{2} = \frac{\theta}{2} \\
&\lim_{b' \to b} \bar{\theta} = \bar{a} = \frac{a}{2} \\
&\frac{\partial}{\partial b'}\bar{\theta} > 0
\end{aligned}$$

Remember that the maximum sales in the monopoly case are equal to θ. Considering that total sales in the duopoly case are simply twice that of an individual manufacturer's sales, we can conclude that the maximum total sales are more when demand of the manufacturers is dependent. □

Lemma 39. *The minimum price is higher in the symmetrical duopoly case than in the monopoly case.*

Proof.

$$p_{\text{duo}} = \frac{\bar{a} - bs}{2b - b'} = \frac{a - bs}{2b - 2b'} \quad p_{\text{mono}} = \frac{a - bs}{2b} \quad \Rightarrow \quad p_{\text{duo}} > p_{\text{mono}}$$

□

This lemma shows that the interdependency between the firms' demand increases the reserved price of the firms. One justification for this result is that the higher the competition is between two firms, the more risk averse they become in terms of their pricing; therefore, their reserved price becomes higher.

Chapter 4

THE DOUBLE MARKET SETTING

In this section, we study the case of a single manufacturer and a double market. The markets are assumed to be different in size and price sensitivity. We start by deriving the optimum pricing strategy and production commitment. The analysis follows by finding the first order condition of the profit function that provides the optimum production commitment. The chapter continues with introducing three coordinating contracts and ends with numerical analysis.

Following the linear price-demand correlation assumption, the demand function in market 1 and 2 are as follows:

$$D_1 = a_1 - b_1 p_1 \qquad D_2 = a_2 - b_2 p_2$$

while p_i is the price in market i. We characterize the difference between the two markets by introducing two indexes. The first index, scale parameter, shows how much larger or smaller market 2 is, compared to market 1; i.e.,

$$ms = \frac{a_2}{a_1}$$

The second index, price capacity, shows the ratio of the maximum price of firm 1 to the maximum price of firm 2. Remember that maximum price in the market i is $\frac{a_i}{b_i}$. Naturally if price capacity is larger than 1, it means the people in firm 1 are willing to pay a higher price for the first unit of product. The manufacturer is more interested in a market if that market is richer and has a higher maximum price. The next assumption sets up our model in a way that each of these markets has one of these two qualities so we get to study the competition between the markets to get a higher proportion of the products. The next assumption sets

Assumption 7. $a_1 < a_2 \qquad b_1 < b_2 \qquad \frac{a_2}{b_2} < \frac{a_1}{b_1}$

This assumption implies that market 1 represents a small yet rich market while market 2 is a large but poor one. One can think of developed and developing countries as a good example for such markets. That is because the developing countries are typically larger in size, yet their maximum price is significantly less than the developed country market. Although our model assumes that markets are segregated, this model can describe different sub markets in one country when the distribution locations are geographically far away. In other words, our model does not support the case in which third parties buy the product in one market and re-sell it in the other one.

We show the market scale parameter with ms and the price capacity parameter with pc. Using these parameters, we can rewrite the second market parameters as a function of the first market's parameter as follows:

$$\begin{array}{l} ms = \dfrac{a_2}{a_1} \\ pc = \dfrac{{}^{a_1}\!/_{b_1}}{{}^{a_2}\!/_{b_2}} \end{array} \quad \Rightarrow \quad \begin{array}{l} a_2 = a_1 * ms \\ b_2 = ms * pc * b_1 \end{array} \quad \text{while } \; ms \geq 1 \text{ and } \; pc \geq 1$$

We can also model the price capacity parameter by $|a_1 b_2 - a_2 b_1|$. The larger this index is, the more different those markets are in terms of their price capacity. If two markets have the same price capacity parameter, we call them proportional market as described below:

$$|a_1b_2 - a_2b_1| = 0 \quad \Leftrightarrow \quad \frac{a_2}{b_2} = \frac{a_1}{b_1} \quad \Leftrightarrow \quad \text{Market 1 and 2 are proportional}$$

4.1 The main model

Assuming that realized product quantity q is known, we examine the profit function in either surplus or shortage stage and find the optimum pricing strategy in each case. As discussed earlier, the optimum pricing strategy sets the demand to be equal or less than the realized production quantity so the manufacturer never ends up in the shortage stage. That is because price has to be lower than equilibrium price, which not only reduces the partial profit of the manufacturer but also results in shortage cost. Therefore, we set the manufacturer's profit function as in a surplus stage and check for the conditions that guarantees that the surplus status holds and the answers are feasible. The profit function of the double market case is as follows:

$$\begin{aligned}
\pi &= p_1 * D_1 + p_2 * D_2 - s(q - D_1 - D_2) \\
&= p_1 * (a_1 - b_1p_1) + p_2 * (a_2 - b_2p_2) - s(q - a_1 + b_1p_1 - a_2 + b_2p_2) \\
\pi &= -b_1p_1^2 - b_2p_2^2 + (a_1 - b_1s)p_1 + (a_2 - b_2s)p_2 + s(a_1 + a_2 - q)
\end{aligned}$$

$$s.t:$$

$$\begin{aligned}
&D_1 + D_2 \leq q \\
&p_1 \leq \frac{a_1}{b_1} \\
&p_2 \leq \frac{a_2}{b_2}
\end{aligned}$$

Lemma 40. *Optimum pricing strategy for the double market problem is as follows:*

$$if\ q < \frac{a_1b_2 - a_2b_1}{2b_2} \quad \Rightarrow \begin{cases} p_1^* = \frac{a_1-q}{b_1} \\ D_1^* = q \\ \pi_1 = \frac{a_1}{b_1}q - \frac{q^2}{b_1} \\ \qquad\qquad\qquad\qquad D_1^* + D_2^* = q \\ P_2^* = \frac{a_2}{b_2} \\ D_2^* = 0 \\ \pi_2 = 0 \end{cases}$$

$$if\ \frac{a_1b_2 - a_2b_1}{2b_2} < q < \frac{a_1 + b_1s}{2} + \frac{a_2 + b_2s}{2} \quad \Rightarrow \begin{cases} p_1 = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_1}{2b_1} - \frac{q}{b_1+b_2} \\ D_1^* = \frac{a_1b_2-a_2b_1}{2(b_1+b_2)} + \frac{b_1q}{b_1+b_2} \\ \pi_1 = -\frac{b_1}{(b_1+b_2)^2}q^2 + \frac{(a_1+a_2)b_1}{(b_1+b_2)^2}q + \frac{(a_1b_2-a_2b_1)(a_2b_1+a_1b_2+2a_1b_1)}{4(b_1+b_2)^2b_1} \\ D_1^* + D_2^* = q \quad \pi = -\frac{q^2}{b_1+b_2} + \frac{a_1+a_2}{b_1+b_2}q + \frac{(a_1b_2-a_2b_1)^2}{4(b_1+b_2)b_1b_2} \\ p_2 = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_2}{2b_2} - \frac{q}{b_1+b_2} \\ D_2^* = -\frac{a_1b_2-a_2b_1}{2(b_1+b_2)} + \frac{b_2q}{b_1+b_2} \\ \pi_2 = -\frac{b_2}{(b_1+b_2)^2}q^2 + \frac{(a_1+a_2)b_2}{(b_1+b_2)^2}q - \frac{(a_1b_2-a_2b_1)(a_2b_1+a_1b_2+2a_2b_2)}{4(b_1+b_2)^2b_2} \end{cases}$$

$$\textit{if } \frac{a_1+b_1s}{2b_1}+\frac{a_2+b_2s}{2b_2}<q \qquad \Rightarrow \begin{cases} p_1^*=\frac{a_1-b_1s}{2b_1} \\ D_1^*=\frac{a_1+b_1s}{2} \\ \pi_1=\frac{\theta_1^2}{b_1} \\ \\ p_2^*=\frac{a_2-b_2s}{2b_2} \\ D_2^*=\frac{a_2+b_2s}{2} \\ \pi_2=\frac{\theta_2^2}{b_2} \end{cases} \qquad D_1^*+D_2^*<q \qquad \pi=\frac{\theta_1^2}{b_1}+\frac{\theta_2^2}{b_2}-sq$$

Proof. See Appendix L. □

Assumption 8.

$$\frac{a_1}{2b_1}-s<\frac{a_2}{b_2} \quad \Rightarrow \quad 1<ms<2$$

This assumption guarantees the competition between the markets. In other words, it shows that the minimum price in the rich market is smaller than the maximum price in the poor market so somewhere on the price spectrum, the markets compete on product allocation. Without this assumption, we have two separate markets and the manufacturer does not distribute any product in the poor market, unless the rich market is saturated. We relax this assumption in the first contract and study the possibility that the social planner gets to affect the product allocation by offering sales subsidizing the product in the poor market.

The next two lemmas discuss the manufacturer's profit function and optimum production commitment.

Lemma 41. *The manufacturer's profit function in double market case is as follows:*

$$\pi = \frac{a_1}{b_1}N\int_0^{\frac{\Delta}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{N^2}{b_1}\int_0^{\frac{\Delta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu$$

$$
\begin{aligned}
&-\frac{1}{b_1+b_2}N^2\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu+\frac{a_1+a_2}{b_1+b_2}N\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu f(\nu)\,\mathrm{d}\nu+\frac{(a_1b_2-a_2b_1)^2}{4(b_1+b_2)b_1b_2}\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} f(\nu)\,\mathrm{d}\nu\\
&+(\frac{\theta_1^2}{b_1}+\frac{\theta_2^2}{b_2})\bar{F}(\frac{\theta_1+\theta_2}{N})-sN\int_{\frac{\theta_1+\theta_2}{N}}^{\infty} \nu f(\nu)\,\mathrm{d}\nu-cN
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\pi=&\int_0^{\frac{a_1b_2-a_2b_1}{2b_2N}} \frac{q(a_1-q)}{b_1}f(\nu)\,\mathrm{d}\nu+\int_{\frac{a_1b_{[}2-a_2b_1}{2b_2}}^{\frac{a_1+b_1s}{2N}+\frac{a_2+b_2s}{2N}} (\pi_1+\pi_2)f(\nu)\,\mathrm{d}\nu\\
&+\int_{\frac{a_1+b_1s}{2N}+\frac{a_2+b_2s}{2N}}^{\infty} (\pi_1+\pi_2-sq)f(\nu)\,\mathrm{d}\nu-cN
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{a_1N}{b_1}\int_0^{\frac{\Delta}{N}} \nu f(\nu)\,\mathrm{d}\nu-\frac{N^2}{b_1}\int_0^{\frac{\Delta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu\\
&-\frac{1}{b_1+b_2}N^2\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu+\frac{a_1+a_2}{b_1+b_2}N\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu f(\nu)\,\mathrm{d}\nu+\frac{(a_1b_2-a_2b_1)^2}{4(b_1+b_2)b_1b_2}\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} f(\nu)\,\mathrm{d}\nu\\
&+(\frac{\theta_1^2}{b_1}+\frac{\theta_2^2}{b_2})\bar{F}(\frac{\theta_1+\theta_2}{N})-sN\int_{\frac{\theta_1+\theta_2}{N}}^{\infty} \nu f(\nu)\,\mathrm{d}\nu-cN
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{a_1}{b_1}N\int_0^{\frac{\Delta}{N}} \nu f(\nu)\,\mathrm{d}\nu-\frac{N^2}{b_1}\int_0^{\frac{\Delta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu\\
&-\frac{1}{b_1+b_2}N^2\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu+\frac{a_1+a_2}{b_1+b_2}N\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu f(\nu)\,\mathrm{d}\nu+\frac{(a_1b_2-a_2b_1)^2}{4(b_1+b_2)b_1b_2}\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} f(\nu)\,\mathrm{d}\nu\\
&+(\frac{\theta_1^2}{b_1}+\frac{\theta_2^2}{b_2})\bar{F}(\frac{\theta_1+\theta_2}{N})-sN\int_{\frac{\theta_1+\theta_2}{N}}^{\infty} \nu f(\nu)\,\mathrm{d}\nu-cN
\end{aligned}
$$

□

Lemma 42. *The derivative of the manufacturer's profit function in respect to the* *production commitment in double market case is as follows:*

$$\frac{\partial}{\partial N}\pi = A_{DM}(N) = \frac{a_1}{b_1}\int_0^{\frac{\Delta}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N}{b_1}\int_0^{\frac{\Delta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - \frac{2}{b_1+b_2}N\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu$$

$$+ \frac{a_1+a_2}{b_1+b_2}\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu f(\nu)\,\mathrm{d}\nu - s\int_{\frac{\theta_1+\theta_2}{N}}^{\infty} \nu f(\nu)\,\mathrm{d}\nu - c$$

Lemma 43. *The first derivative of the profit function has one and only one root. In other words, the profit function has a singular maximum point.*

Proof. We can show that the first derivative function owns the following properties:

$$\lim_{n\to 0}\frac{\partial}{\partial N}\pi = \frac{a_1}{b_1} - c > 0$$

$$\lim_{n\to\infty}\frac{\partial}{\partial N}\pi = -s\mu - c < 0$$

$$\frac{\partial^2}{\partial N^2}\pi = -\frac{2}{b_1}\int_0^{\frac{\Delta}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - \frac{2}{b_1+b_2}N\int_{\frac{\Delta}{N}}^{\frac{\theta_1+\theta_2}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu < 0$$

The set of these properties concludes that the profit function is monotonically decreasing with a unique root in N. □

4.2 Comparison of the double market case and single market case

The manufacturer will be concerned to know how expanding the markets affects its pricing strategy and production commitment as well as the maximum sales in each market. The social planner, also, needs to know how he can motivate the manufacturer to expand its market to the less rich areas. This section provides the results to answer those questions and help the manufacturer and social planner to set their optimum strategies.

Lemma 44. *Assuming that there is no entry cost and competitor, entering a new market is the right decision for the manufacturer.*

Proof. Comparing the optimum pricing strategy for the double market case in Lemma 40 to the optimum pricing strategy for single market case presented in Lemma 1, the first question has a straight forward answer. Given any certain realized production quantity, the optimum price for the double market case is greater than or equal to of the single market case. That also suggests that the optimum production commitment is greater for the single market case. This observation is fairly easy to justify; having a larger audience, even if it is a rather poor sub market, does not hurt the manufacturer's sales and pricing. At best, it results in more competition among customers that leads to higher price for a specific demand or more demand for a specific price. □

Lemma 45. *If both markets have the same maximum price ,i.e., $PC = 1$, the pricing strategy of the double market case is identical to the single market case.*

Proof. Imagine that we have a market T that is the result of these two markets combined. For market T we have

$$a_T = a_1 + a_2 \qquad b_T = b_1 + b_2. \tag{4.1}$$

If $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ then $a_1b_2 - a_2b_1 = 0$ and the first threshold in the range for q is equal to zero, so practically, it does not exist. With some simple calculations we can show that

$$\frac{a_2}{b_2} = \frac{a_1}{b_1} = \frac{a_1+a_2}{b_1+b_2} \Rightarrow \begin{cases} p_1 = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_1}{2b_1} - \frac{q}{b_1+b_2} = \frac{a_1+a_2}{b_1+b_2} - \frac{q}{b_1+b_2} = \frac{a_T}{b_T} - \frac{q}{b_T} \\ p_2 = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_2}{2b_2} - \frac{q}{b_1+b_2} = \frac{a_1+a_2}{b_1+b_2} - \frac{q}{b_1+b_2} = \frac{a_T}{b_T} - \frac{q}{b_T} \\ \\ \pi = -\frac{q^2}{b_1+b_2} + \frac{a_1+a_2}{b_1+b_2}q + \frac{(a_1b_2-a_2b_1)^2}{4(b_1+b_2)b_1b_2} = -\frac{q^2}{b_1+b_2} + \frac{a_1+a_2}{b_1+b_2}q = -\frac{q^2}{b_T} + \frac{a_T}{b_T}q \end{cases}$$

The right hand equations show the price and profit for the market T while it has not reached its saturation point ,i.e

$$\theta_T = \frac{a_T + b_T s}{2} = \frac{a_1 + a_2 + b_1 s + b_2 s}{2} = \theta_1 + \theta_2$$

We can also show that the same results holds for the case that the production quantity is larger than the saturation point.

$$\frac{a_2}{b_2} = \frac{a_1}{b_1} = \frac{a_1 + a_2}{b_1 + b_2} \Rightarrow \begin{cases} p_1 = \frac{a_1 - b_1 s}{2b_1} = \frac{a_2 - b_2 s}{2b_2} = p_2 = \frac{a_T - b_T s}{2b_T} = p_T \\ D = D_1 + D_2 = \theta_1 + \theta_2 = \theta_T = D_T \\ \\ \pi = \frac{\theta^2}{b_1} + \frac{\theta^2}{b_2} - sq = \frac{\theta_T^2}{b_T} - sq = \pi_T \end{cases}$$

□

Remember that the comparison between market T and the dual market is meaningful only if the maximum price of the markets are equal. Otherwise, the relation between price and demand in the double market case does not have the linearity property unless both markets have the same maximum price. Figure 4.1 and 4.2 demonstrate how having the same maximum price results in linearity of the price-demand relation in the double market case.

Lemma 46. *If both markets have the same maximum price i,e,* $a_1 b_2 = a_2 b_1$*, the profit function of the double market case in comparison to the single market case is as follows:*

$$\pi_{dm}(n) = k\pi_{sm}(n/k) \ , \qquad \textit{while} \quad \frac{a_1 + a_2}{b_1 + b_2} = k\frac{a_1}{b_1}$$

while $\pi_{dm}(n)$ *is the profit function for the double market case and* $\pi_{sm}(n/k)$ *is the profit function for the single market case.*

Figure 4.1 Schematic demonstration of a proportional market

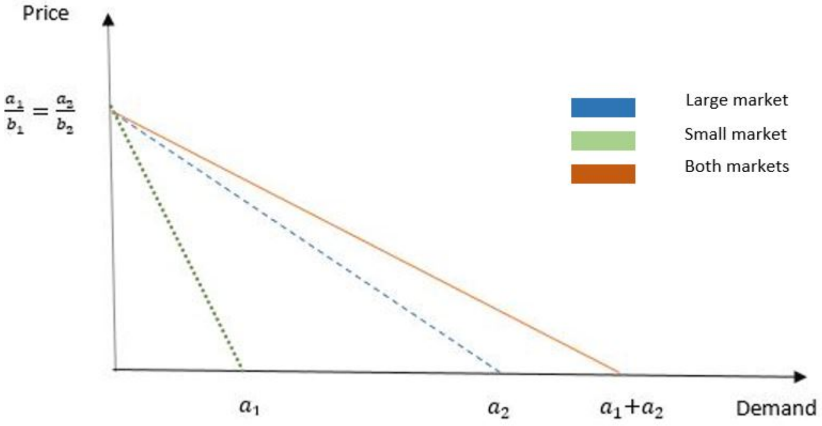
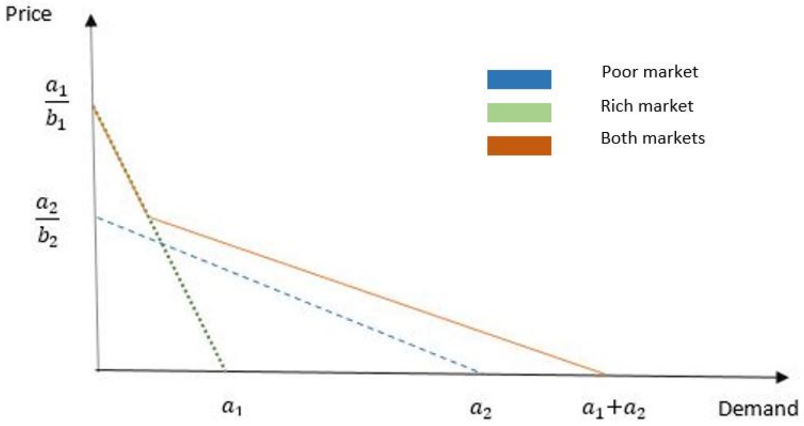


Figure 4.2 Schematic demonstration of an disproportionate market



4.3 *Contracts and coordination policies in dual market settings*

This chapter presents two contracts that can coordinate the market and change the manufacturer's optimum production commitment to become equal to the social optimum production capacity i.e., N^*_{cen}. We show that by changing the coordinating parameters of the contract, we can shift the derivative of the profit function up or down so the following property is met:

$$\frac{\partial}{\partial N}\pi(n = N^*_{cen}) = A_{DM}(n = N^*_{cen}) = 0$$

As shown in Lemma 6, the optimum production commitment for market i is as follows:

$$N^*_{c,i} = \frac{a_i\mu - b_i c}{\mu^2 + \sigma^2}$$

Also as Proposition 3 demonstrated, the optimum production commitment in the centralized scenario, N^*_{cen}, is not necessarily more or less than the optimum production commitment in the decentralized scenario, N^*_{dec}. That is because N^*_{dec} is a function of the salvage cost while N^*_{cen} does not depend on the salvage cost. The same result holds for the double market case since it is just an extension of the single market case and the same reasoning carries on in this case. We study the contracts that can coordinate the market by either increasing or decreasing the manufacturer's optimum production capacity, depending on the salvage cost.

The first contract that we study provides the option for the social planner to affect only one of the markets while the second one can affect both of the markets.

4.3.1 Asymmetrical subsidizing contract

In this contract, the social planner subsidizes the price in the poor market therefore the price that the market experiences is t units less than the price that the manufacturer charges. We can formulate this contract in the following way:

$$\begin{aligned} D_1 =& a_1 - b_1 p_1 \\ D_2 =& a_2 - b_2(p_2 - t) = a_2 - b_2 p_2 + b_2 t \end{aligned}$$

therefore, we can rewrite the profit function as follows:

$$\begin{aligned} \pi &= p_1 D_1 + p_2 D_2 - s(q - D_1 - D_2) \\ &= -b_1 p_1^2 - b_2 p_2^2 + (a_1 - b_1 s)p_1 + (a_2 - b_2 s + b_2 t)p_2 + s(a_1 + a_2 + b_2 t - q) \end{aligned}$$

$$\begin{aligned} &s.t. \\ &\quad p_1 \leq \frac{a_1}{b_1} \\ &\quad p_2 \leq \frac{a_2}{b_2} + t \\ &\quad a_1 - b_1 p_1 + a_2 - b_2 p_2 + b_2 t \leq q \end{aligned}$$

$$\begin{cases} p_2 = p_1 + \frac{a_2}{2b_2} - \frac{a_1}{2b_1} + \frac{t}{2} \\ p_2 = \frac{a_1+a_2-q}{b_2} - \frac{b_1 p_1}{b_2} + t \end{cases} \Rightarrow \begin{cases} p_1 = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_1}{2b_1} - \frac{q}{b_1+b_2} + \frac{b_2 t}{2(b_1+b_2)} \\ p_2 = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_2}{2b_2} - \frac{q}{b_1+b_2} + \frac{(b_1+2b_2)t}{2(b_1+b_2)} \end{cases}$$

Lemma 47. *The firm's optimum pricing strategy for the asymmetrical subsidizing contract is as follows:*

$$if\, q < \frac{a_1b_2 - a_2b_1}{2b_2} - \frac{b_1}{2}t \Rightarrow \begin{cases} p_1^* = \frac{a_1-q}{b_1} \\ D_1^* = q \\ \pi_1 = \frac{a_1}{b_1}q - \frac{q^2}{b_1} \\ & D_1^* + D_2^* = q \\ P_2^* = \frac{a_2}{b_2} \\ D_2^* = 0 \\ \pi_2 = 0 \end{cases}$$

$$if\ \frac{a_1b_2-a_2b_1}{2b_2}-\frac{b_1}{2}t<q<\frac{a_1+b_1s}{2}+\frac{a_2+b_2s+b_2t}{2}\ \Rightarrow\ \begin{cases} p_1=\frac{a_1+a_2}{2(b_1+b_2)}+\frac{a_1}{2b_1}-\frac{q}{b_1+b_2}+\frac{b_2t}{2(b_1+b_2)} \\ D_1^*=\frac{a_1b_2-a_2b_1}{2(b_1+b_2)}+\frac{b_1q}{b_1+b_2}-\frac{b_1b_2}{2(b_1+b_2)}t \\ \pi_1=-\frac{b_1}{(b_1+b_2)^2}q^2+\frac{(a_1+a_2)b_1}{(b_1+b_2)^2}q \\ +\frac{(a_1b_2-a_2b_1)(a_2b_1+a_1b_2+2a_1b_1)}{4(b_1+b_2)^2b_1} \\ -\frac{b_1b_2(a_1+a_2)}{2(b_1+b_2)^2}t+\frac{b_1b_2}{(b_1+b_2)^2}qt-\frac{b_1b_2^2}{4(b_1+b_2)^2}t^2 \\ \\ \\ D_1^*+D_2^*=q \\ \pi=-\frac{q^2}{b_1+b_2}+\frac{a_1+a_2}{b_1+b_2}q+\frac{(a_1b_2-a_2b_1)^2}{4(b_1+b_2)b_1b_2} \\ -\frac{a_1b_2-a_2b_1}{2(b_1+b_2)}+\frac{b_2q}{b_1+b_2}t+\frac{b_1b_2}{4(b_1+b_2)^2}t^2 \\ \\ \\ p_2=\frac{a_1+a_2}{2(b_1+b_2)}+\frac{a_2}{2b_2}-\frac{q}{b_1+b_2}+\frac{(b_1+2b_2)t}{2(b_1+b_2)} \\ D_2^*=-\frac{a_1b_2-a_2b_1}{2(b_1+b_2)}+\frac{b_2q}{b_1+b_2}+\frac{b_1b_2}{2(b_1+b_2)}t \\ \pi_2=-\frac{b_2}{(b_1+b_2)^2}q^2+\frac{(a_1+a_2)b_2}{(b_1+b_2)^2}q \\ -\frac{(a_1b_2-a_2b_1)(a_2b_1+a_1b_2+2a_2b_2)}{4(b_1+b_2)^2b_2} \\ +\frac{b_2^2}{(b_1+b_2)^2}qt+(\frac{a_2b_1}{2(b_1+b_2)}-\frac{b_2(a_1b_2-a_2b_1)}{2(b_1+b_2)^2})t \\ +\frac{b_1b_2(b_1+2b_2)}{4(b_1+b_2)^2}t^2 \end{cases}$$

$$if\ \frac{a_1+b_1s}{2}+\frac{a_2+b_2s+b_2t}{2}<q\ \Rightarrow \qquad \begin{cases} p_1^* = \frac{a_1-b_1s}{2b_1} \\ D_1^* = \theta_1 = \frac{a_1+b_1s}{2} \\ \pi_1 = \frac{\theta_1^2}{b_1} \\ \\ \quad D_1^* + D_2^* < q \qquad \pi = \frac{\theta_1^2}{b_1} + \frac{\theta_{t,2}^2}{b_2} - sq \\ \\ p_2^* = \frac{a_2-b_2s+b_2t}{2b_2} \\ D_2^* = \theta_{t,2} = \frac{a_2+b_2s+b_2t}{2} \\ \pi_2 = \frac{\theta_{t,2}^2}{b_2} \end{cases}$$

As shown in the optimum pricing strategy, the asymmetrical subsidizing contract changes the product allocation in favor of the developing market; i.e., market 2, in two ways. First, it decreases the product quantity threshold for introducing the product to market 2. If there is no contract, the firm does not allocate any product to market 2 unless the realized production quantity exceeds $\frac{a_1b_2-a_2b_1}{2b_2}$, while in the presence of an asymmetrical contract this threshold is decreased by $\frac{b_1t}{2}$. It also keeps that leverage in the product allocation in favor of market 2, while the markets are not saturated. In comparison to the original case, the optimum pricing strategy for this contract shifts $\frac{b_1b_2}{2(b_1+b_2)}t$ number of units from market 1 to market 2.

Second, it increases the saturation point in market 2. By increasing the residual profit, the contract increases manufacturer's maximum sales in market 2. That is because the contract keeps the price high while the demand keeps expanding from market two's original saturation point, θ_2.

4.3.2 Buy back contract

Lemma 48. *The derivative of the profit function is monotonically decreasing in salvage cost and it has a unique root.*

$$\lim_{s \to -\infty} A_{DM} = +\infty > 0$$
$$\lim_{s \to +\infty} A_{DM} = -\infty < 0$$
$$\frac{\partial}{\partial s} A_{DM} = -\int_{\frac{\theta_1+\theta_2}{N}}^{\infty} \nu f(\nu)\,\mathrm{d}\nu < 0$$

We show that for any value of N, the derivative function has a positive limit on the left side and a negative limit on the right infinity of the spectrum for s. It can be interpreted in this way that for every value of the production commitment, there exists s for which the derivative function equals zero and the manufacturer's profit function is in its maximum. One might point out that according to Assumption 3, salvage cost should be bounded by $-\frac{c}{\mu}$ and that is to guarantee that the model does not allow the production of surplus items to bring any profit for the manufacturer. However, we can relax that assumption here because we are assuming that the social planner wants to change the optimum production commitment of the manufacturer purposefully. Indeed, the social planner has to monitor the production commitment to not pass the social optimum level if it let the s_{con} offered by the contract to be less than $-\frac{c}{\mu}$.

Using the A_{DM} function from Lemma 42, the application of this result can be interpreted as follows:

$$\forall \;\; N^*_{cen} \;\; \exists \;\; s_{con} \;\;\mid\;\; A_{DM}(s = s_{con}, n = N^*_{cen}) = 0$$

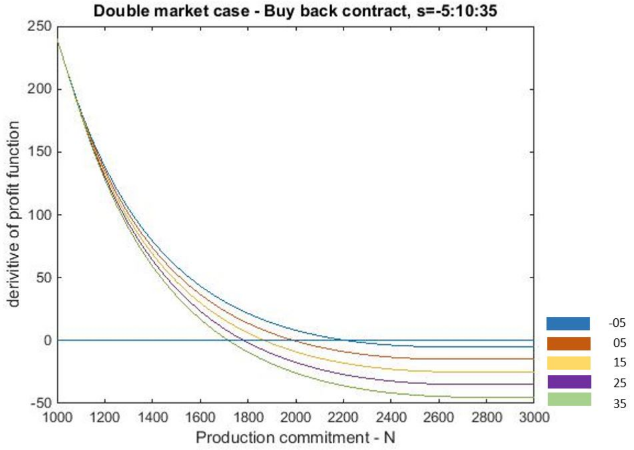


Figure 4.3 Derivative of the profit function for different values of s

This equation can be interpreted in this way that the social planner wants the manufacturer to set its production commitment on N^*_{cen}, it can change the cost of surplus recognized by the manufacturer to s_{con} and sets the manufacturer's optimum production commitment to the social optimum commitment. The expected implementing cost of this contract for the social planner is as follows:

$$Exp(\text{Cost}) = (s - s_{con}) \int_{\frac{\theta_1+\theta_2}{N^*_{cen}}}^{\infty} \nu f(\nu)\, \mathrm{d}\nu$$

Indeed, if the social planner wants to decrease the production commitment, he can increase the salvage cost by taxing it so the expected cost turns to be a negative number. Figure 4.3 shows the derivative of the profit function when the social planner changes the salvage cost at different levels.

4.3.3 Production cost sharing contract

Production cost sharing contract is the only contract that we have shown to be coordinating in all different cases of this research. That is because the production cost appears to be effective in the same phase as the production commitment is determined plus the fact that it is linearly related to the production commitment. As we can see in Lemma 42, the term c is a single term in the equation and we can shift the equation up or down by changing that term. We define c_{con} as the coordinating partial production cost for which the manufacturer's optimum production commitment is equal to the social production commitment; i.e., N^*_{cen}. In order to find the coordinating production cost we have

$$c_{con} = c - A_{DM}(N^*_{cen}) \;\Rightarrow\; A_{DM}(c = c_{con}, n = N^*_{cen}) = 0$$

while the expected implementing cost of this contract for the social planner is

$$Exp(\text{Cost}) = -A_{DM}(N^*_{cen}) * N^*_{cen}$$

Just like the previous contracts, if the social planner intends to decrease the production commitment of the manufacturer, he can impose a tax on production cost, in which case the contract implementation cost is negative, or better to say, it turns out to be contract income for the social planner. Figure 4.4 shows the derivative of the profit function when the social planner explores different values of the production cost.

4.4 Numerical analysis

In this section we study how changing the scale index and price capacity index changes the optimum production capacity and profit function. Table 4.1 presents the default parameter values for the simulations in this section.

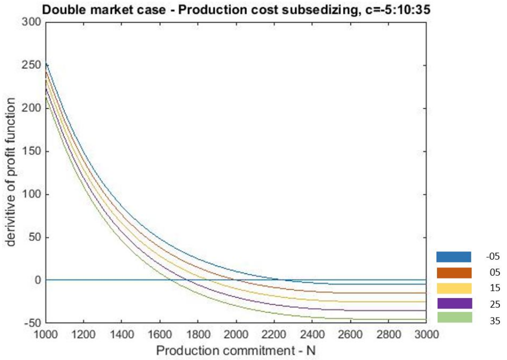


Figure 4.4 Derivative of the profit function for different values of c

a_1	b_1	pc	ms	s	c	μ	σ
1000	1	2	1	10	10	1	.25

Table 4.1 Default parameter values for double market case

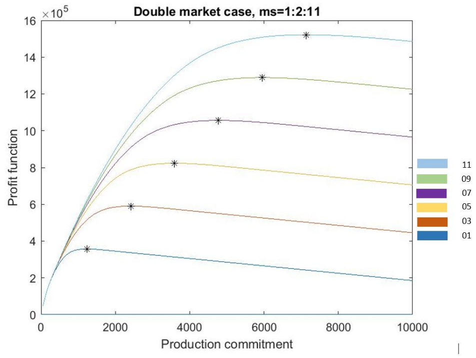


Figure 4.5 Profit function for different market size index values

4.4.1 Changing the market size index

Figure 4.5 show the profit function for different values of market size. The blue line represents the lowest value for the market size $ms = 1$. It is clear that when the market size increases and maximum price index is fixed, maximum profit and production commitment also increase.

4.4.2 Changing the price capacity index

Next, we study the effect of changing the price capacity on the manufacturer's profit function. We keep the maximum price of market 1 untouched and decrease the maximum price capacity of market 2 in order to increase the price capacity index. The graphs in Figure 4.6 depict the profit function for different scale parameters. The highest graph, the blue one, represents the $pc = 1$ case where both markets are equally rich. However, as pc increases and the market 2 gets poorer, the profit function moves down to the point that the maximum profit for the $pc = 10$ case is almost equal to half of the $pc = 1$ case.

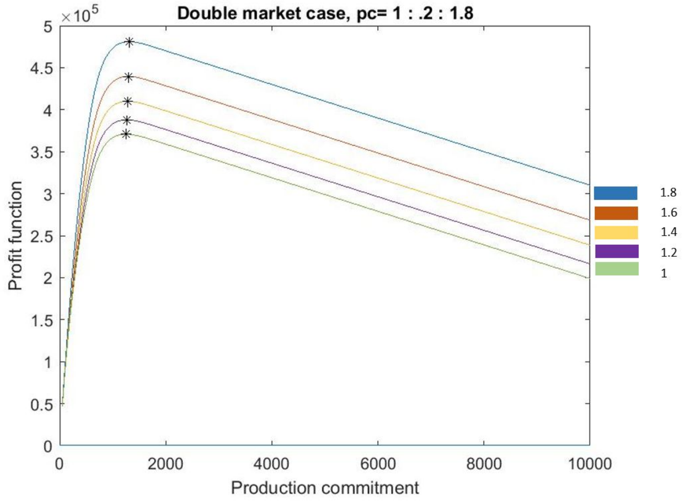


Figure 4.6 Profit function for different price capacity index values

4.4.3 Studying the effect of dis proportionality of the market

Assuming that total utility available is constant, we can compare the dual market setting and the single market setting. This way, we study whether a combination of two disproportionate markets or one market is more beneficial for the manufacturer. In other words, the question is if wealth diversity increases the company's optimal profit and production commitment or not. We represent the single market case with two proportional markets and run a numerical experiment to compare them with a disproportionate markets case. As one can see in Figure 2.1 in Chapter 2, the total utility in one market is equal to $\frac{a^2}{2b}$. We split this total utility into two markets with different maximum prices. As Assumption 8 indicates, the price capacity index should not exceed 2; therefore, we set the price capacity at its most. As a result, we have two scenarios to compare; a single market scenario, and a double market with

	Single market	Proportional markets		Disproportionate markets	
		Market 1	Market 2	Market 1	Market 2
Market size	1000	500	500	1414	707
Price sensitivity	1	.5	.5	.25	1

Table 4.2 Parameter values to study dis-proportionality

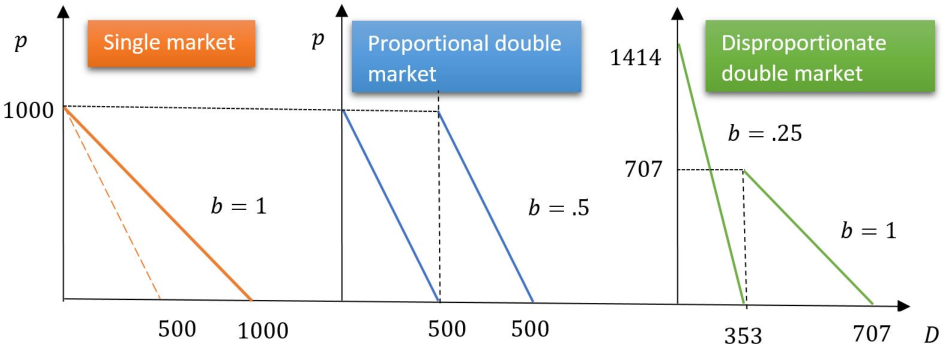


Figure 4.7 Demand function of the proportional Vs. disproportionate markets

$pc = 2$ scenario, while the total utility in both scenarios are the same. Table 4.2 depicts the setting of the scenarios and Figure 4.7 provides a schematic form for the demand function of the single market, the equivalent proportional double markets, and disproportional double markets

The blue line in Figure 4.8 shows the profit function for a single market case and the red line represents the double market case. This figure shows that it is in the manufacturer's benefit if the markets are disproportionate.

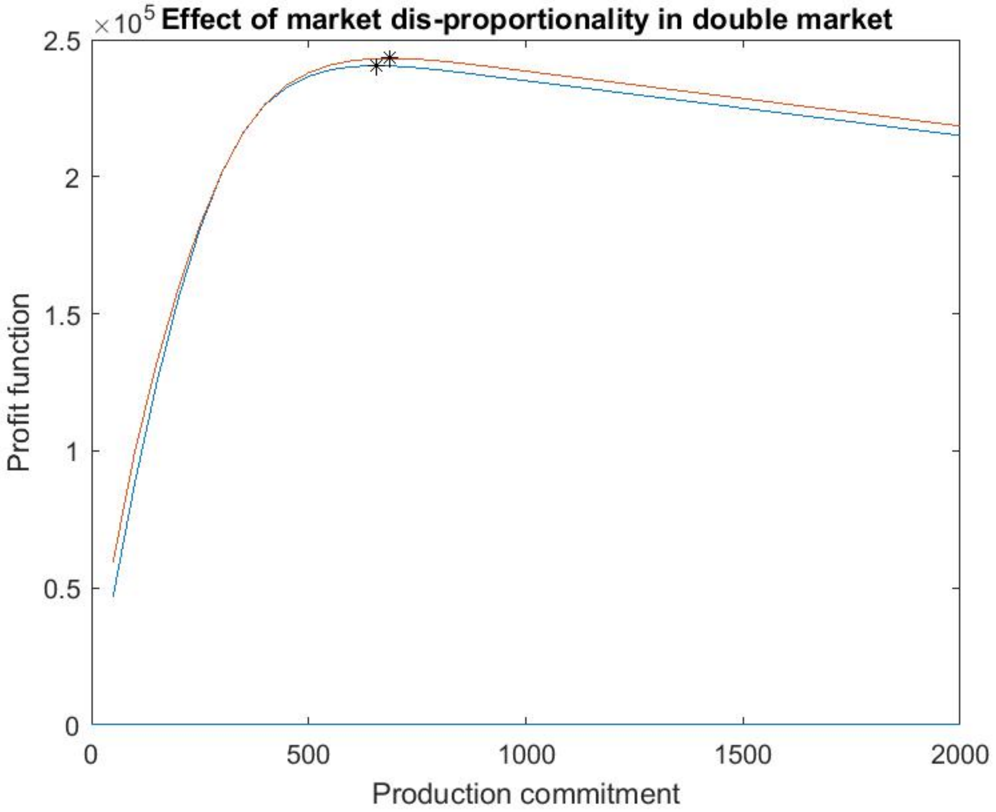


Figure 4.8 Profit function for proportional Vs. disproportionate markets

Chapter 5

CONTRIBUTIONS AND INSIGHTS

5.1 Contributions

In this research we studied a news-vendor problem with random yield and price-dependent demand, while production commitment and price are the decision variables. We incorporate a dynamic model in which, the manufacturer sets the production commitment first. He sets the price later, when the production yield is realized. In this way, he gets to adjust the price and demand according to the realized production quantity.

We show that it is in the best interest of the manufacturer to clear the market unless the realized production quantity exceeds a certain level θ. In that case, it is in the best interest of the manufacturer to not set its price any lower than a certain lower limit. Our results also show that randomness always increases the optimum production commitment. Adding a social planner to the model, we provided three contracts that can coordinate the market. Our model suggests that the optimum production commitment for the decentralized setting is not necessarily larger or smaller than that of the centralized setting. That is why our suggested contracts can cause the optimum production commitment to move either up or down.

In Chapter 3 we discussed the optimum pricing and production commitment as well as two coordinating contracts for the case with two competing symmetrical firms. We studied the effect of competition and randomness on the firms' optimum strategy as well as on the market and show that greater competition increases the optimum production commitment and can also result in higher profit for the firms.

Chapter 4 studied a scenario when a manufacturer can present its products in two markets, a rich yet small market and an a poor but larger market. We derived the optimum pricing strategy and production commitment for the manufacturer and introduced two contracts that can coordinate the market. One of the contract, the asymmetrical subsidizing contract, enables the social planner to change the production commitment and also affect the production distribution in these markets.

Our results show that randomness always increases the production commitment. We also discuss how changes in the market or cost parameters change the optimum policy. An outstanding feature of this research is that our results do not depend on the format of the probability distribution function of the yield ratio. This is particularly important because it allows for a wider range of applications. Another highlight of our research is that although we start our analysis assuming that price and demand are linearly dependent, we partially relax that assumption. We show that a market with linear and convex demand curve is equivalent of the case when we have two markets. That is of course if the markets are separable in practice. Our results assist the manufacturers policy makers in public health and technology fields make better decisions in terms of profitability and market coordination.

5.2 Managerial insights

Random yield production is common in many industries such as semiconductor manufacturing, influenza vaccine production, chemical productions and so on. One rather obvious result is that randomness increases the optimum production commitment. Our results provide the optimum production commitment for a specific case. We also show that salvage cost has a significant impact on the manufacturer and the social planner can manipulate the manufacturer's optimum strategy by providing a buy-back contract to the manufacturer. We also find that production subsidizing and taxing can also be an effective, highly robust and easy to implement contract that can coordinate all the variations that we studied.

In the duopoly case, the model suggests that having a more interdependent market does not necessarily lower the firms' profit. It will make both of the firms lower their prices and get a greater customer range. This is, to some extent, consistent with the changes in the semiconductor and computer industry.

In the dual market case, our results suggest that while entry cost is negligible, it is always beneficial to be present in a variety of markets. This result is more valid if the variable cost is relatively less than the price range. In this way, the manufacturer has once paid the fixed cost to make the products and later can just expand its production and send the products to other markets, including potentially less financially prosperous markets. That may be something that global health planner organization may be interested to promote and our suggested contracts are capable of providing such a mechanism.

5.3 Research extensions and ideas

This research is adjacent to a number of different research areas; therefore, there are multiple expansions of the models available in each of these research fields. One immediate expansion is to consider the oligopoly case and multiple market case. The next suggested expansion, in terms of complexity, is double market and duopoly case. Studying the same problem with a convex demand function also seems to be a highly useful and analytically tractable problem.

Appendix A

OPTIMUM PRICING STRATEGY IN DECENTRALIZED SETTING

For any given realized production quantity, we can split the spectrum of the retail price into two intervals, the lower interval that is associated with the shortage case and the higher interval that is associated with the surplus case. We find the optimum pricing strategy by finding the best price function within each of these intervals and compare their corresponding maximum profit in order to find the optimum price. The surplus case and its associated price interval can be calculated as follows:

$$D < q \Rightarrow a - bp < q \Rightarrow p > \frac{a-q}{b}$$

The profit function for surplus case is

$$\pi_{su} = p * D - \text{salvage cost} = p * D - s(q - D) = -bp^2 + (a - 2b)p + s(a - q)$$

By taking first order condition by p, we have

$$p^* = \frac{a - 2bs}{2b} \quad D^* = \frac{a + bs}{2}, \pi^*_{su} = \frac{a^2}{4b} + s(\frac{a}{2} + \frac{bs}{4} - q)$$

For the shortage case we have:

$$D \geq q \Rightarrow p \leq \frac{a-q}{b}$$

The profit function for the shortage case is

$$\pi_{sh} = pq - B(D - q) = B(q - a) + (b + q)p$$

By taking first order condition, we have

$$\frac{\partial}{\partial p}\pi = q + Bb > 0$$

which means that under the shortage scenario, p^* is the largest possible price, i.e,

$$p^* = \frac{a-q}{b} \Rightarrow D^* = q \Rightarrow \pi^*_{sh} = p^*q - B(D^* - q) = \frac{aq - q^2}{b}$$

Now we can compare π^*_{su} and $\pi^*_{sh}$to find out the optimal price for any q. As shown above, the optimal demand for the surplus case is $D^* = \frac{a+bs}{2}$. Since the condition for the surplus case is $q \geq D$, the condition for feasibility of the optimal surplus case is $q \geq \frac{a+bs}{2}$, which can be :

$$(q - \frac{a+bs}{2})^2 \geq 0 \Rightarrow q^2 - (a+bs)q + \frac{a^2}{4b} + \frac{asb}{2} + \frac{b^2a^2}{4} \geq 0 \Rightarrow \frac{a^2}{4} + \frac{asb}{2} + \frac{b^2a^2}{4} - qs \geq \frac{aq - q^2}{b}$$

so we can conclude that

$$q \geq \frac{a+bs}{2} \Rightarrow \pi^*_{su} \geq \pi^*_{sh}$$

This means that the optimum profit under the surplus scenario is always greater than or equal to the optimum profit of the shortage scenario, as far as the conditions for feasibility of optimum profit under shortage holds.

Otherwise, it can be shown that $\frac{\partial}{\partial p}\pi_{su}$ is positive in the range of $[0, \frac{a+bs}{2}]$ so D^*_{su} in this range is equal to q. In such a case, $\pi^*_{su} = \frac{aq-q^2}{b}$, which is exactly the same as π^*_{sh}

Appendix B

UNI-MODULARITY AND SIGN OF FUNCTION $S(L)$

We investigate the function's behavior in extremes of its domain, $l \in [0, \infty)$

$$\lim_{l \to 0} \mathrm{S}(l) = \int_0^{\mu(1+l)} ((1+l)\frac{\nu}{\mu} - (\frac{\nu}{\mu})^2) f(\nu)\,\mathrm{d}\nu \ - l = \int_0^{\mu} \left(\frac{\nu}{\mu} - (\frac{\nu}{\mu})^2 \right) f(\nu)\,\mathrm{d}\nu \geq 0$$

$$\lim_{l \to \infty} \mathrm{S}(l) = \infty - \infty \overset{\mathrm{l}}{\equiv} \frac{\int_0^{\mu(1+l)} \left((1+l)\frac{\nu}{\mu} - (\frac{\nu}{\mu})^2 \right) f(\nu)\,\mathrm{d}\nu}{l} - \frac{l}{l} = \frac{\int_0^{\infty} \left((\infty)\frac{\nu}{\mu} - (\frac{\nu}{\mu})^2 \right) f(\nu)\,\mathrm{d}\nu}{\infty} = \frac{\infty}{\infty} - 1$$

$$\overset{\text{Hopital}}{=} \frac{\mu\left((1+l)^2 - (1+l)^2\right) + \int_0^{\mu(1+l)} \frac{\nu}{\mu} f(\nu)\,\mathrm{d}\nu}{1} = \frac{1}{1} - 1 = 0$$

$$\frac{\partial}{\partial l}\mathrm{S}(l) = \mu \left((1+l)^2 - (1+l)^2] f(1+l) + \int_0^{\mu(1+l)} \frac{\nu}{\mu} f(\nu)\,\mathrm{d}\nu \right) - 1 = \frac{1}{\mu} \int_0^{\mu(1+l)} \nu f(\nu)\,\mathrm{d}\nu - 1 < 0$$

These inequalities show that $\mathrm{S}(l)$ is a monotonically decreasing function, with a positive value at $l = 0$ and zero value as l goes to infinity. This means that $\mathrm{S}(l)$ always has a positive value.

$$\forall l \quad \mathrm{S}(l) \geq 0$$

Appendix C

DETERMINISTIC CASE VS. DYNAMIC CASE

we define variable l as

$$l = \frac{\frac{bc}{\mu} + bs}{a - \frac{bc}{\mu}}.$$

By replacing l in the equation, we have

$$\frac{\theta}{\dot{N}} = \frac{a + bs}{a - \frac{bc}{\mu}}\mu = (1 + l)\mu.$$

It can be shown that

$$\dot{N} < N^* \Longleftrightarrow \mu^2 - \int_0^{\frac{\theta}{\dot{N}}} \nu^2 f(\nu)\,\mathrm{d}\nu > \frac{\theta}{\dot{N}}(\mu - \int_0^{\frac{\theta}{\dot{N}}} \nu f(\nu)\,\mathrm{d}\nu).$$

We can rewrite the right side inequality as follows:

$$\begin{aligned}
\mu^2 - \int_0^{(1+l)\mu} \nu^2 f(\nu)\,\mathrm{d}\nu &> (1+l)\mu(\mu - \int_0^{(1+l)\mu} \nu f(\nu)\,\mathrm{d}\nu) \\
\mu^2 - \int_0^{(1+l)\mu} \nu^2 f(\nu)\,\mathrm{d}\nu &> (\mu + l\mu)(\mu - \int_0^{(1+l)\mu} \nu f(\nu)\,\mathrm{d}\nu) \\
-\int_0^{(1+l)\mu} \nu^2 f(\nu)\,\mathrm{d}\nu &> l\mu^2 - (\mu + l\mu)\int_0^{(1+l)\mu} \nu f(\nu)\,\mathrm{d}\nu) \\
-\int_0^{(1+l)\mu} (\frac{\nu}{\mu})^2 f(\nu)\,\mathrm{d}\nu &> l - (1+l)\int_0^{(1+l)\mu} \frac{\nu}{\mu} f(\nu)\,\mathrm{d}\nu) \\
\int_0^{\mu(1+l)} ((1+l)\frac{\nu}{\mu} - (\frac{\nu}{\mu})^2) f(\nu)\,\mathrm{d}\nu \ &- l \geq 0
\end{aligned}$$

Appendix D

PARTIALLY DECENTRALIZED SCENARIO

The total utility function in the first stage is as follows:

$$
\begin{aligned}
TU_{2,Soc,1} &= \int_{\frac{\theta}{N}}^{\infty} (\frac{2a\theta - \theta^2}{2b} - s(q-\theta)) f(\nu)\,\mathrm{d}\nu + \int_0^{\frac{\theta}{N}} \frac{2aq-q^2}{2b} f(\nu)\,\mathrm{d}\nu - cN \\
&= (\frac{2a\theta-\theta^2}{2b} + s\theta)\bar{F}(\frac{\theta}{N}) - sN\int_{\frac{\theta}{N}}^{\infty}\nu f(\nu) + \frac{a}{b}N\int_0^{\frac{\theta}{N}}\nu f(\nu)\,\mathrm{d}\nu - \frac{N^2}{2b}\int_0^{\frac{\theta}{N}}\nu^2 f(\nu) - cN \\
&= (\frac{2a\theta-\theta^2}{2b}+s\theta)\bar{F}(\frac{\theta}{N}) + (\frac{a}{b}+s)N\int_0^{\frac{\theta}{N}}\nu f(\nu)\,\mathrm{d}\nu - \frac{N^2}{2b}\int_0^{\frac{\theta}{N}}\nu^2 f(\nu) - sN\mu - cN
\end{aligned}
$$

First order condition with respect to N:

$$
\begin{aligned}
\frac{\partial}{\partial N}TU_{2,Soc,1} =&(\frac{2a\theta-\theta^2}{2b}+s\theta)\frac{-\theta}{N^2}(-f(\frac{\theta}{N})) - s\mu + (\frac{a}{b}+s)G(\frac{\theta}{N}) + (\frac{a}{b}+s)N(-\frac{\theta}{N^2})\frac{\theta}{N}f(\frac{\theta}{N}) \\
&- \frac{N}{b}\int_0^{\frac{\theta}{N}}\nu^2 f(\nu)\,\mathrm{d}\nu - \frac{N^2}{2b}(\frac{-\theta}{N^2})(\frac{\theta}{N})^2 f(\frac{\theta}{N}) - c \\
=& \frac{\theta}{N^2}f(\frac{\theta}{N})(\frac{2a\theta-\theta^2}{2b}+s\theta-(\frac{a}{b}+s\theta)+\frac{\theta^2}{2b}) - s\mu + (\frac{a}{b}+s)G(\frac{\theta}{N}) - \frac{N}{b}\int_0^{\frac{\theta}{N}}\nu^2 f(\nu)\,\mathrm{d}\nu - c \\
=& (\frac{a}{b}+s)G(\frac{\theta}{N}) - \frac{N}{b}\int_0^{\frac{\theta}{N}}\nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c.
\end{aligned}
$$

We find function B as the polynomial whose root is the optimum production commitment for scenario 2.

$$
\frac{\partial}{\partial N}TU_{2,Soc,1} = B(N_2^*) = (\frac{a}{b}+s)\int_0^{\frac{\theta}{N^*}}\nu f(\nu)\,\mathrm{d}\nu - \frac{N_2^*}{b}\int_0^{\frac{\theta}{N_2^*}}\nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c = 0.
$$

We study the values of this function at the extremes of its domain as well as its derivative function in order to determine if it has any roots.

$$B(N_2 = 0) = (\frac{a}{b} + s)G(\infty) - \frac{0}{b}\int_0^{\infty} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c = \frac{\mu}{b} - c > 0$$

$$\begin{aligned}
\lim_{\to\infty} \mathrm{B}(N_2) &= (\frac{a}{b} + s)G(0) - \frac{\infty}{b}\int_0^{0} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c = \frac{a\mu}{b} - c > 0 \\
&= \frac{\int_0^{\frac{\theta}{N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu}{\frac{b}{N}2} - s\mu - c \\
&= \frac{\frac{-\theta}{N_2^2}(\frac{\theta^2}{N_2^2})f(\frac{\theta}{N_2})}{\frac{-b}{N_2^2}} - s\mu - c \\
&= \frac{\theta^3}{N_2^2 b} f(\frac{\theta}{N_2}) - s\mu - c \\
\Rightarrow \lim_{N\to\infty} \mathrm{B}(N_2) &= -s\mu - c < 0 \\
\frac{\partial}{\partial N}\mathrm{B}(N_2) &= (\frac{a}{b} + s)(\frac{-\theta}{N_2^2})(\frac{\theta}{N_2})f(\frac{\theta}{N_2}) - \frac{1}{b}\int_0^{\frac{\theta}{N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu - (\frac{N_2}{b})(\frac{-\theta}{N_2^2})(\frac{\theta}{N_2})^2 f(\frac{\theta}{N_2}) \\
&= [\frac{\theta^2}{N_2^3} f(\frac{\theta}{N_2})](-\frac{a}{b} - s + \frac{\theta}{b}) - \frac{1}{b}\int_0^{\frac{\theta}{N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu = \\
&= \frac{\theta^3}{2bN_2^3} f(\frac{\theta}{N_2}) - \frac{1}{b}\int_0^{\frac{\theta}{N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu \leq 0
\end{aligned}$$

These results show that B is a monotonically decreasing function. It has a positive value at the beginning of its domain set and a negative limit for the value of the function on the positive infinite side. Thus, the function B crosses the horizontal axes one time and only one time.

Appendix E

CHECKING THE DERIVITIVE FUNCTION BEHAVIOR IN IN EXTREME VALUES OF THE SALVAGE COST

$$\mathcal{A}(N,s) = (\frac{a}{b}+s)\int_0^{\frac{a+bs}{2N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N}{b}\int_0^{\frac{a+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c$$

$$\begin{aligned}
\lim_{s\to -\frac{c}{\mu}} \mathcal{A}(N,s) &= (\frac{a}{b}(-\frac{c}{\mu}))\int_0^{\frac{a+b(-\frac{c}{\mu})}{2N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N}{b}\int_0^{\frac{a+b(-\frac{c}{\mu})}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu - (-\frac{c}{\mu})\mu - c \\
&= (\frac{a}{b}-\frac{c}{\mu})\int_0^{\frac{a-\frac{bc}{\mu}}{2N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N}{b}\int_0^{\frac{a-\frac{bc}{\mu}}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu \\
&= \frac{2N}{b}(\frac{a-\frac{bc}{\mu}}{2N}\int_0^{\frac{a-\frac{bc}{\mu}}{2N}} \nu f(\nu)\,\mathrm{d}\nu - \int_0^{\frac{a-\frac{bc}{\mu}}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu) \\
&= \frac{2N}{b}(\int_0^{\frac{a-\frac{bc}{\mu}}{2N}} (\frac{a-\frac{bc}{\mu}}{2N}-\nu)\nu f(\nu)\,\mathrm{d}\nu) > 0
\end{aligned}$$

$$\begin{aligned}
\lim_{s\to\infty} \mathcal{A}(N,s) = \mathcal{A}(N,s) &= (\frac{a}{b}+s)\int_0^{\frac{a+bs}{2N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N}{b}\int_0^{\frac{a+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c \\
&= (\frac{a}{b}+s)\int_0^{\infty} \nu f(\nu)\,\mathrm{d}\nu - \frac{2N}{b}\int_0^{\infty} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c \\
&= (\frac{a}{b}+s)(\mu) - \frac{2N}{b}(\mu^2+\sigma^2) - s\mu - c \\
&= \frac{a\mu}{b} - \frac{2N}{b}(\mu^2+\sigma^2) - c < 0 \quad \Leftrightarrow \quad \frac{1}{2}\frac{\mu-\frac{bc}{a}}{\mu^2+\sigma^2} < \frac{N}{a}
\end{aligned}$$

Appendix F

COMPARISON OF PRODUCTION COMMITMENT IN CENTRALIZED AND DECENTRALIZED SCENARIOS

$$\begin{aligned}
\lim_{s\to -\frac{c}{\mu}} \mathcal{A}(N_4^*) &= 2\frac{a\mu - bc}{b}[\frac{\int_0^{\frac{\mu^2+\sigma^2}{2\mu}} \nu f\nu \,\mathrm{d}\nu}{2\mu} - \frac{\int_0^{\frac{\mu^2+\sigma^2}{2\mu}} \nu^2 f\nu \,\mathrm{d}\nu}{\mu^2+\sigma^2}] - s\mu - c \\
&= \frac{a\mu - bc}{b\mu}[\int_0^{\frac{\mu^2+\sigma^2}{2\mu}} \nu f\nu \,\mathrm{d}\nu - \frac{\int_0^{\frac{\mu^2+\sigma^2}{2\mu}} \nu^2 f\nu \,\mathrm{d}\nu}{\frac{\mu^2+\sigma^2}{2\mu}}] \\
\text{If } \frac{\mu^2+\sigma^2}{2\mu} &= l : \\
&= \text{Const} * [\int_0^{l} \nu f\nu \,\mathrm{d}\nu - \frac{\int_0^{l} \nu^2 f\nu \,\mathrm{d}\nu}{l}]
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \lim_{l\to 0} \int_0^{l} \nu f\nu \,\mathrm{d}\nu - \frac{\int_0^{l} \nu^2 f\nu \,\mathrm{d}\nu}{l} = 0 - lim[\frac{\int_0^{l} \nu^2 f\nu \,\mathrm{d}\nu}{l}] = \frac{l^2 f(l)}{1} = 0 \\
&\lim_{l\to \infty} \int_0^{l} \nu f\nu \,\mathrm{d}\nu - \frac{\int_0^{l} \nu^2 f\nu \,\mathrm{d}\nu}{l} = \mu - \frac{\sigma^2+\mu^2}{\infty}] = \mu \\
&\frac{\partial}{\partial l}[\int_0^{l} \nu f\nu \,\mathrm{d}\nu - \frac{\int_0^{l} \nu^2 f\nu \,\mathrm{d}\nu}{l}] = lf(l) - \frac{l^3 f(l) - \int_0^{l} \nu^2 f\nu \,\mathrm{d}\nu}{l^2} = \frac{\int_0^{l} \nu^2 f\nu \,\mathrm{d}\nu}{l^2} > 0
\end{aligned}$$

$$\Rightarrow \lim_{s\to -\frac{c}{\mu}} \mathcal{A}(N_4^*) > 0$$

$$\lim_{s\to \infty} \mathcal{A}(N_4^*) = (\frac{a}{b} + s) \int_0^{\frac{a+bs}{2(a\mu - bc)}(\sigma^2+\mu^2)} \nu f(\nu) \,\mathrm{d}\nu - 2\frac{a\mu - bc}{b(\sigma^2+\mu^2)} \int_0^{\frac{a+bs}{2(a\mu - bc)}(\sigma^2+\mu^2)} \nu^2 f(\nu) \,\mathrm{d}\nu - s\mu - c$$

$$= (\frac{a}{b} + s) \int_0^\infty \nu f(\nu)\,\mathrm{d}\nu - 2\frac{a\mu - bc}{b(\sigma^2 + \mu^2)} \int_0^\infty \nu^2 f(\nu)\,\mathrm{d}\nu\,\mathrm{d}\nu - s\mu - c$$

$$= (\frac{a}{b} + s)\mu - 2\frac{a\mu - bc}{b(\sigma^2 + \mu^2)}(\mu^2 + \sigma^2) - s\mu - c$$

$$\Rightarrow \lim_{s\to\infty} \mathcal{A}(N_4^*) = -\mu(\frac{a}{b} - \frac{c}{\mu}) < 0$$

Note that $\frac{a}{b}$ is the price for which demand is zero, so it is the maximum price feasible. The term $\frac{c}{\mu}$ is the expected production cost of one unit of the product. In order to have positive value of sales $\frac{a}{b}$ should be greater than $\frac{c}{\mu}$.

$$\frac{\partial}{\partial s} A(N_4^*) = \int_0^{\frac{a+bs}{2(a\mu - bc)}(\sigma^2+\mu^2)} \nu f(\nu)\,\mathrm{d}\nu - \mu < 0$$

Appendix G

FINDING THE MINIMUM SALVAGE COST FOR NORMAL DISTRIBUTION CASE

If $\nu \sim N(\mu, \sigma)$, we have these results for the normal distribution:

$$\begin{cases} \phi(x) \sim N(0,1), \Phi(r) = \int_r^\infty \phi(x)dx \\ f(\nu) \sim N(\mu,\sigma), \Phi(r) = \int_r^\infty f(\nu)\,\mathrm{d}\nu \end{cases} \quad \Rightarrow f(\nu) = \phi(\tfrac{\nu-\mu}{\sigma}), x = \tfrac{\nu-\mu}{\sigma}, \nu = \sigma x + \mu$$

$$\begin{aligned} \int_r^\infty \nu^2 f(\nu)\,\mathrm{d}\nu &= \int_r^\infty \nu^2 \phi(\frac{\nu-\mu}{\sigma})\,\mathrm{d}\nu \\ &= \sigma^2 \int_r^\infty (\frac{\nu^2+\mu^2-2\nu\mu-\mu^2+2\nu\mu}{\sigma^2})\phi(\frac{\nu-\mu}{\sigma})\,\mathrm{d}\nu \\ &= \sigma^2 \int_r^\infty (\frac{\nu-\mu}{\sigma})^2\phi(\frac{\nu-\mu}{\sigma})\,\mathrm{d}\nu - \mu^2 \int_r^\infty \phi(\frac{\nu-\mu}{\sigma})\,\mathrm{d}\nu + 2\mu \int_r^\infty (\sigma x+\mu)\phi(\frac{\nu-\mu}{\sigma})\,\mathrm{d}\nu \\ &= \sigma^2 \int_{\frac{r-\mu}{\sigma}}^\infty x^2\phi(x)\,\mathrm{d}\nu - \mu^2 \int_r^\infty \phi(x)\,\mathrm{d}\nu + 2\mu \int_r^\infty (\sigma x+\mu)\phi(x)\,\mathrm{d}\nu \\ &= \sigma^2 \int_{\frac{r-\mu}{\sigma}}^\infty x^2\phi(x)\,\mathrm{d}\nu + \mu^2 \int_{\frac{r-\mu}{\sigma}}^\infty \phi(x)dx + 2\mu\sigma \int_r^\infty x\phi(x)dx \end{aligned}$$

$$\begin{aligned} \int_r^\infty \nu f(\nu)\,\mathrm{d}\nu &= \int_r^\infty \nu\phi(\frac{\nu-\mu}{\sigma})\,\mathrm{d}\nu = \sigma \int_r^\infty (\frac{\nu-\mu}{\sigma})\phi(\frac{\nu-\mu}{\sigma})\,\mathrm{d}\nu + \mu \int_r^\infty \phi(\frac{\nu-\mu}{\sigma})d\nu \\ &= \sigma \int_{\frac{r-\mu}{\sigma}}^\infty x\phi(x)dx + \mu \int_{\frac{r-\mu}{\sigma}}^\infty \phi(x)dx \end{aligned}$$

Given that $\int_r^\infty x\phi(x)dx = \phi(r)$ and$\int_r^\infty x^2\phi(x)dx = \Phi(r) + r\phi(r)$ we have:

$$\int_r^\infty \nu^2 f(\nu)\,\mathrm{d}\nu = \sigma^2(\Phi(\frac{r-\mu}{\sigma}) + (\frac{r-\mu}{\sigma})\phi(\frac{r-\mu}{\sigma})) + \mu^2\Phi(\frac{r-\mu}{\sigma}) + 2\mu\sigma\phi(\frac{r-\mu}{\sigma})$$

$$= (\sigma^2 + \mu^2)\Phi(\frac{r-\mu}{\sigma}) + \sigma(r+\mu)\phi(\frac{r-\mu}{\sigma})$$

$$\int_r^\infty \nu f(\nu)\,\mathrm{d}\nu = \sigma\phi(\frac{r-\mu}{\sigma}) + \mu\Phi(\frac{r-\mu}{\sigma})$$

Also if $r = \frac{\theta}{N_4^*}$ then we have $\frac{r-\mu}{\sigma} = \frac{1}{b}[\frac{\sigma}{\mu}(\frac{bs+bc\mu}{a-bc\mu} + 1) + \frac{\mu}{\sigma}(\frac{bs+bc\mu}{a-bc\mu} - 1)]$. We show this equation with γ

Now by replacing these equations in $A(N_4^*)$ we have:

$$\begin{aligned}
A(N_4^*) &= \frac{a+bs}{b}\int_0^{\frac{a+bs}{2(a\mu-bc)}\mu^2+\sigma^2} \nu f\nu \mathrm{d}\nu - 2\frac{N_4^*}{b}\int^{\frac{a+bs}{2(a\mu-bc)}\mu^2+\sigma^2} {}_0\nu^2 f\nu\,\mathrm{d}\nu - s\mu - c \\
&= \frac{a+bs}{b}[\mu - \sigma\phi(\gamma) - \mu\Phi(\gamma) - \frac{2(a\mu-bc)}{b(\mu^2+sigma^2)}[(\mu^2+\sigma^2) - \Phi(\gamma) - \sigma(\mu + \frac{a+bs}{2(a\mu-bc)}(\mu^2+\sigma^2)]\phi() \\
&- s\mu - c \\
&= \frac{a+bs}{b}\mu - \frac{2}{b}(a\mu - bc) - s\mu - c \\
&+ \phi(\gamma)[-\frac{a+bs}{b}\sigma + \frac{a+bs}{b}\sigma + \frac{2(a\mu-bc)}{b(\sigma^2+\mu^2)}\sigma\mu] \\
&+ \Phi(\gamma)[-\frac{(a+bs)\mu}{b} + \frac{2(a\mu-bc)}{b}] \\
&= \frac{a\mu + bs\mu - 2a\mu + 2bc - bs\mu - bc}{b} + \phi(\gamma)[\frac{2(a\mu-bc)}{b(\sigma^2+\mu^2)}\sigma\mu] + \Phi(\gamma)\frac{-a\mu - bs\mu + 2a\mu - 2bc}{b} \\
&= \frac{-a\mu + bc}{b} + \phi(\gamma) \times \frac{2}{b}\frac{a\mu - bc}{\sigma^2+\mu^2}\sigma\mu + \Phi(\gamma) \times [\frac{a\mu - bc}{b} - s\mu - c]
\end{aligned}$$

Appendix H

OPTIMUM PRODUCTION COMMITMENT FOR SUBSIDIZING

$$
\begin{aligned}
\frac{\partial}{\partial N}\pi^* =&(\frac{a}{b}+k+s)\int_0^{\frac{a-bk+bs}{2N}} \nu f \nu - (\frac{a}{b}+k+s)N(\frac{a-bk+bs}{2N^2})\frac{a-bk+bs}{2N}f(\frac{a-bk+bs}{2N})\\
&-\frac{2N}{b}\int_0^{\frac{a-bk+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu + \frac{N^2}{b}\frac{a-bk+bs}{2N^2}(\frac{a-bk+bs}{2N})^2 f(\frac{a-bk+bs}{2N})\\
&+\frac{(a+bk+bs)^2}{4b}\frac{a-bk+bs}{2N^2}f(\frac{a-bk+bs}{2N})\\
&-s\mu-c\\
=&(\frac{a}{b}+k+s)\int_0^{\frac{a-bk+bs}{2N}} \nu f \nu - \frac{2N}{b}\int_0^{\frac{a-bk+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c\\
&+f(\frac{a-bk+bs}{2N})[-(a+bk+bs)(\frac{(a-bk+bs)^2}{4bN^2})+(\frac{(a-bk+bs)^3}{8bN^2})+(a+bk+bs)^2\frac{a-bk+bs}{8bN^2}]\\
=&(\frac{a}{b}+k+s)\int_0^{\frac{a-bk+bs}{2N}} \nu f \nu - \frac{2N}{b}\int_0^{\frac{a-bk+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c\\
&+f(\frac{a-bk+bs}{2N})[\frac{a-bk+bs}{8bN^2}][-2(a+bk+bs)(a-bk+bs)+(a-bk+bs)^2+(a+bk+bs)^2]\\
=&(\frac{a}{b}+k+s)\int_0^{\frac{a-bk+bs}{2N}} \nu f \nu - \frac{2N}{b}\int_0^{\frac{a-bk+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c\\
&+f(\frac{a-bk+bs}{2N})[\frac{a-bk+bs}{8bN^2}]\\
&[-2(a+bs)^2+2b^2k^2+(a+bs)^2+b^2k^2-2bk(a+bs)+(a+bs)^2+b^2k^2+2bk(a+bs)]\\
=&(\frac{a}{b}+k+s)\int_0^{\frac{a-bk+bs}{2N}} \nu f \nu - \frac{2N}{b}\int_0^{\frac{a-bk+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu + f(\frac{a-bk+bs}{2N})[\frac{(a-bk+bs)bk^2}{2N^2}] - s\mu - c
\end{aligned}
$$

In order to find the F.O.C of A_{sub} we have:

$$\begin{aligned}\frac{\partial}{\partial N}A_{sub}(N) =& -(\frac{a+bs+bk}{b})(\frac{a+bs-bk}{2N^2})(\frac{a+bs-bk}{2N})f(\frac{a+bs-bk}{2N})\\ &- \frac{2}{b}\int_0^{\frac{a-bk+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu + \frac{2N}{b}(\frac{a-bk+bs}{2N^2})(\frac{a-bk+bs}{2N})^2 f(\frac{a-bk+bs}{2N})\\ &- \frac{a-bk+bs}{2N^2}\acute{f}(\frac{a-bk+bs}{2N})\frac{(a-bk+bs)bk^2}{2N^2} - f(\frac{a-bk+bs}{2N})\frac{(a-bk+bs)bk^2}{N^3}\end{aligned}$$

$$\begin{aligned}=& -\frac{2}{b}\int_0^{\frac{a-bk+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu - \frac{a-bk+bs}{2N^2}\acute{f}(\frac{a-bk+bs}{2N})\frac{(a-bk+bs)bk^2}{2N^2}\\ &- f(\frac{a-bk+bs}{2N})\frac{(a-bk+bs)}{bN^3}[b^2k^2 + \frac{(a+bs)^2-(bk)^2}{4} - \frac{(a+bs-bk)^2}{4}]\end{aligned}$$

$$\begin{aligned}=& -\frac{2}{b}\int_0^{\frac{a-bk+bs}{2N}} \nu^2 f(\nu)\,\mathrm{d}\nu - \acute{f}(\frac{a-bk+bs}{2N})\frac{(a-bk+bs)^2b^2k^2}{4bN^4}\\ &- f(\frac{a-bk+bs}{2N})\frac{(a-bk+bs)}{bN^3}[\frac{b^2k^2}{2} + \frac{(a+bs)(bk)}{2}]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial k}A_{sub}(N) =& \int_0^{\frac{a-bk+bs}{2N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{b}{2N}(\frac{a+bs+bk}{b})(\frac{a+bs-bk}{2N})f(\frac{a+bs-bk}{2N})\\ &+ (\frac{a+bs-bk}{2N})^2 f(\frac{a+bs-bk}{2N})\\ &- \frac{b}{2N}\acute{f}(\frac{a-bk+bs}{2N})\frac{(a-bk+bs)^2b^2k^2}{2bN^2} + f(\frac{a-bk+bs}{2N})\frac{+2bk(a+bs-1.5bk)}{2N^2}\end{aligned}$$

$$\begin{aligned}=& \int_0^{\frac{a-bk+bs}{2N}} \nu f(\nu)\,\mathrm{d}\nu - (\frac{(a+bs)^2-b^2k^2}{4N^2})f(\frac{a+bs-bk}{2N})\\ &+ (\frac{a+bs-bk}{2N})^2 f(\frac{a+bs-bk}{2N})\\ &- \acute{f}(\frac{a-bk+bs}{2N})\frac{(a-bk+bs)^2b^2k^2}{4N^3} + f(\frac{a-bk+bs}{2N})\frac{bk(a+bs-1.5bk)}{N^2}\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{a-bk+bs}{2N}} \nu f(\nu)\,\mathrm{d}\nu \\
&\quad + \Big(\frac{(a+bs-bk)^2-(a+bs)^2+b^2k^2+4bk(a+bs)-6b^2k^2}{4N^2}\Big) f\Big(\frac{a+bs-bk}{2N}\Big) \\
&\quad - \acute{f}\Big(\frac{a-bk+bs}{2N}\Big)\frac{(a-bk+bs)^2b^2k^2}{4N^3}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\frac{a-bk+bs}{2N}} \nu f(\nu)\,\mathrm{d}\nu \\
&\quad + \Big(\frac{2bk(a+bs)-4b^2k^2}{4N^2}\Big) f\Big(\frac{a+bs-bk}{2N}\Big) \\
&\quad - \acute{f}\Big(\frac{a-bk+bs}{2N}\Big)\frac{(a-bk+bs)^2b^2k^2}{4N^3}
\end{aligned}
$$

Appendix I

SALES REBATE CONTRACT

$$\frac{\partial}{\partial r}A_{reb}(N) = \int_0^{\frac{a+bs+br}{2N_{reb}}} \nu f\nu - \,\mathrm{d}\nu \int_0^{\frac{t}{N_{reb}}} \nu f\nu \,\mathrm{d}\nu = \int_0^{\frac{a+bs+br}{2N_{reb}} - \frac{t}{N_{reb}}} \nu f\nu \,\mathrm{d}\nu > 0 \quad \text{(I.0)}$$

Also, if we define $\bar{\theta} = \frac{a+bs+br}{2}$, we can rewrite A_{reb} as a function of θ:

$$\begin{aligned} A_{reb}(\theta) &= \frac{2\bar{\theta}}{b} \int_0^{\frac{\theta}{N_{reb}}} \nu f\nu \,\mathrm{d}\nu - \frac{2N_{reb}}{b} \int_0^{\frac{\theta}{N_{reb}}} \nu^2 f\nu \,\mathrm{d}\nu - r \int_0^{\frac{t}{N_{reb}}} \nu f\nu \,\mathrm{d}\nu - s\mu - c \\ A(\theta) &= \frac{2\theta}{b} \int_0^{\frac{\theta}{N}} \nu f\nu \,\mathrm{d}\nu - \frac{2N}{b} \int_0^{\frac{\theta}{N}} \nu^2 f\nu \,\mathrm{d}\nu - s\mu - c \\ \frac{\partial}{\partial\theta}A_{reb}(N) &= \frac{\partial}{\partial\theta}A(N) = \frac{2}{b} \int_0^{\frac{\theta}{N}} \nu f\nu \,\mathrm{d}\nu > 0 \end{aligned}$$

Let's start with the assumption that $t = 0$. By this assumption, we have:

$\frac{\partial}{\partial r}A_{reb}(N)$ is larger than zero, which means that for a positive value of r, $A_{reb}(N)$ is above $A(N)$. Since both of the functions are decreasing in N, a positive value at $N = 0$, and a negative limit at $N \to \infty$, we can conclude that

$$\text{if } t = 0 \to N^*_{reb} > N^*_1$$

Appendix J

SALVAGE COST SHARING

$$\begin{aligned}
\mathcal{A}(N) =& (\frac{a}{b} + s) \int_0^{\frac{a+bs}{2N}} \nu f(\nu) \,\mathrm{d}\nu - \frac{2N}{b} \int_0^{\frac{a+bs}{2N}} \nu^2 f(\nu) \,\mathrm{d}\nu - s\mu - c = 0 \\
\frac{\partial}{\partial s}\mathcal{A}(N) =& \int_0^{\frac{a+bs}{2N}} \nu f(\nu) \,\mathrm{d}\nu + (\frac{a}{b} + s) \times \frac{b}{2N} \times \frac{a+bs}{2N} f(\frac{a+bs}{2N}) - \frac{2N}{b} \times \frac{b}{2N} \times (\frac{a+bs}{2N})^2 f(\frac{a+bs}{2N}) - \mu \\
=& \int_0^{\frac{a+bs}{2N}} \nu f(\nu) \,\mathrm{d}\nu + (\frac{a+bs}{2N})^2 f(\frac{a+bs}{2N}) - (\frac{a+bs}{2N})^2 f(\frac{a+bs}{2N}) - \mu \\
=& \int_0^{\frac{a+bs}{2N}} \nu f(\nu) \,\mathrm{d}\nu - \mu
\end{aligned}$$

Appendix K

OPTIMUM N FOR SYMMETRICAL MANUFACTURERS

$$
\begin{aligned}
\frac{\partial}{\partial N_1}\pi_1(N_1|N_2) =& (\frac{\bar{a}}{b-b'}+s)\int_0^{\frac{I}{N_1 Z+N_2}} \nu f(\nu)\,\mathrm{d}\nu \\
&-(\frac{\bar{a}}{b-b'}+s)N_1 * \frac{IZ}{(N_1Z+N_2)^2} * \frac{I}{N_1Z+N_2} f(\frac{I}{N_1Z+N_2}) \\
&-\frac{2bN_1+b'N_2}{b^2-b'^2}\int_0^{\frac{I}{N_1 Z+N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu \\
&+(\frac{bN_1^2+b'N_1N_2}{b^2-b'^2})\frac{IZ}{(N_1Z+N_2)^2}(\frac{I}{N_1Z+N_2})^2 f(\frac{I}{N_1Z+N_2}) \\
&+\frac{(2b+b')\bar{a}-bb's}{2b^2-b'^2}N_2\int_{\frac{I}{(rZ+1)N_2}}^{\frac{\bar{\theta}}{rN_2}} \nu f(\nu)\,\mathrm{d}\nu \\
&+\frac{(2b+b')\bar{a}-bb's}{2b^2-b'^2}rN_2\left(-\frac{\bar{\theta}}{r^2N_2} * \frac{\bar{\theta}}{rN_2}f(\frac{\bar{\theta}}{rN_2}) + \frac{IZ}{(rZ+1)^2N_2}\frac{I}{(rZ+1)N_2}f(\frac{I}{(rZ+1)N_2})\right) \\
&-2\frac{2b}{2b^2-b'^2}rN_2^2\int_{\frac{I}{(rZ+1)N_2}}^{\frac{\bar{\theta}}{rN_2}} \nu^2 f(\nu)\,\mathrm{d}\nu \\
&+\frac{2b}{2b^2-b'^2}r^2N_2^2\left(\frac{\bar{\theta}}{r^2N_2} * (\frac{\bar{\theta}}{rN_2})^2 f(\frac{\bar{\theta}}{rN_2}) - \frac{IZ}{(rZ+1)^2N_2}(\frac{I}{(rZ+1)N_2})^2 f(\frac{I}{(rZ+1)N_2})\right) \\
&+b(\frac{\bar{a}+(b-b')s}{2b-b'})^2\frac{\bar{\theta}}{r^2N_2}f(\frac{\bar{\theta}}{rN_2}) \\
&-sN_2\int_{\frac{\bar{\theta}}{rN_2}}^{\infty} \nu f(\nu)\,\mathrm{d}\nu \\
&-srN_2\frac{\bar{\theta}}{r^2N_2} * \frac{\bar{\theta}}{rN_2}f(\frac{\bar{\theta}}{rN_2})
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{rI^2Z}{(rZ+1)^3N_2}f\Big(\frac{I}{rZ+1}\Big)*\\
&\qquad\left(-\frac{a}{b-b'}+\frac{br+b'}{b^2-b'^2}*\frac{I}{rZ+1}+\frac{(2b+b')\bar{a}-bb's}{2b^2-b'^2}-\frac{2b}{2b^2-b'^2}*\frac{rI}{rZ+1}\right)\\
&+\frac{\bar{\theta}^2}{r^2N}*f\Big(\frac{\bar{\theta}}{rN_2}\Big)\\
&\qquad\left(-\frac{(2b+b')\bar{a}-bb's}{2b^2-b'^2}+\frac{2b}{2b^2-b'^2}*\bar{\theta}+\frac{\bar{\theta}}{b}-s\right)\\
&+\frac{\bar{a}}{b-b'}N_2\int_0^{\frac{I}{(rZ+1)N_2}}\nu f(\nu)\,\mathrm{d}\nu-\frac{2br+b'}{b^2-b'^2}{N_2}^2\int_0^{\frac{I}{(rZ+1)N_2}}\nu^2 f(\nu)\,\mathrm{d}\nu\\
&+\frac{(2b+b')\bar{a}-bb's}{2b^2-b'^2}N_2\int_{\frac{I}{(rZ+1)N_2}}^{\frac{\bar{\theta}}{rN_2}}\nu f(\nu)\,\mathrm{d}\nu-\frac{4b}{2b^2-b'^2}rN_2^2\int_{\frac{I}{(rZ+1)N_2}}^{\frac{\bar{\theta}}{rN_2}}\nu^2 f(\nu)\,\mathrm{d}\nu\\
&-sN_2\int_{\frac{\bar{\theta}}{rN_2}}^{\infty}\nu f(\nu)\,\mathrm{d}\nu
\end{aligned}
$$

Appendix L

PRICING STRATEGY FOR THE CASE WITH DOUBLE MARKET

Using K.K.T. to solve the problem we have:

$$\begin{aligned}\pi =& -b_1p_1^2 - b_2p_2^2 + (a_1 - b_1 s)p_1 + (a_2 - b_2 s)p_2 + s(a_1 + a_2 - q)\\ & - \alpha(a_1 + a_2 - b_1p_1 - b_2p_2 - q) - \beta_1(\frac{a_1}{b_1} - p_1) - \beta_2(\frac{a_2}{b_2} - p_2)\end{aligned}$$

while according to F.O.C the following conditions should be met:

$$\begin{aligned}&\frac{\partial\pi}{\partial p_1} = -2b_1p_1 + a_1 - b_1 s + b_1\alpha - \beta_1 = 0\\ &\frac{\partial\pi}{\partial p_2} = -2b_2p_2 + a_2 - b_2 s + b_2\alpha - \beta_2 = 0\\ &\frac{\partial\pi}{\partial\alpha} = a_1 + a_2 - b_1p_1 - b_2p_2 - q = 0\\ &\frac{\partial\pi}{\partial\beta_1} : p_1 \leq \frac{a_1}{b_1} \;\text{ or }\; 0 \leq D_1\\ &\frac{\partial\pi}{\partial\beta_2} : p_2 \leq \frac{a_2}{b_2} \;\text{ or }\; 0 \leq D_2\end{aligned}$$

Using the first two equations, we have

$$\frac{a_1}{b_1} - 2p_1 - s = \frac{a_2}{b_2} - 2p_2 - s \Rightarrow \quad \frac{a_1}{b_1} - 2p_1 = \frac{a_2}{b_2} - 2p_2$$

Now by comparing this result with the third equation we get the following results:

$$\begin{cases} p_2 = p_1 + \frac{a_2}{2b_2} - \frac{a_1}{2b_1} \\ p_2 = \frac{a_1+a_2-q}{b_2} - \frac{b_1p_1}{b_2} \end{cases} \Rightarrow \begin{cases} p_1^* = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_1}{2b_1} - \frac{q}{b_1+b_2} \\ p_2^* = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_2}{2b_2} - \frac{q}{b_1+b_2} \end{cases}$$

By incorporating the fourth and fifth conditions, we have:

$$\begin{cases} p_1^* = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_1}{2b_1} - \frac{q}{b_1+b_2} \leq \frac{a_1}{b_1} \\ p_2^* = \frac{a_1+a_2}{2(b_1+b_2)} + \frac{a_2}{2b_2} - \frac{q}{b_1+b_2} \leq \frac{a_2}{b_2} \end{cases} \Rightarrow q \geq \frac{a_1b_2 - a_2b_1}{2b_2}$$

This condition guarantees that affiliated conditions to the equations L, L, L are satisfied so $\alpha = \beta_1 = \beta_2 = 0$.Therefore, we can rewrite the profit function as follows:

$$\pi = -b_1p_1^2 - b_2p_2^2 + (a_1 - b_1s)p_1 + (a_2 - b_2s)p_2 + s(a_1 + a_2 - q)$$

By taking the first order condition with respect to the price variables we have

$$\frac{\partial \pi}{\partial p_1} = -2b_1p_1 + a_1 - b_1s = 0 \Rightarrow \quad p_1^* = \frac{a_1 - b_1s}{2b_1}$$
$$\frac{\partial \pi}{\partial p_2} = -2b_2p_2 + a_2 - b_2s = 0 \Rightarrow \quad p_2^* = \frac{a_2 - b_2s}{2b_2}$$

In appendix L, we show that both markets reach their saturation point simultaneously, i.e, the optimum pricing strategy sets the demands in both markets in a way that when product quantity reaches $\theta_1 + \theta_2$, both markets reach their maximum capacity.

In this appendix, we provide the supporting formula for the pricing strategy of the case with double market. Our proof demonstrates that if $q > \delta$, the production assignment to both markets happen in a way that both markets reach their saturation point, θ_i at the same time. If

$$\begin{cases} \Delta = \frac{a_1b_2 - a_2b_1}{2b_2} \\ \theta_i = \frac{a_i + b_is}{2} \\ q : \text{total production quantity} \\ q_i : \text{the production quantity assigned to market } i \text{ given that } q > \Delta \\ q_{sat,i} : \text{the total product quantity for market } i \text{ is saturated} \end{cases}$$

then we have

$$\begin{cases} q_1 = & \Delta + \frac{b_1}{b_1+b_2}(q-\Delta) \\ q_2 = & \frac{b_2}{b_1+b_2}(q-\Delta) \end{cases} \Rightarrow \begin{cases} \theta_1 = & \Delta + \frac{b_1}{b_1+b_2}q_{sat,1} - \Delta \\ \theta_2 = & \frac{b_2}{b_1+b_2}(q_{sat,2}-\Delta) \end{cases} \Rightarrow \begin{cases} \theta_1 = & \Delta + \frac{b_1}{b_1+b_2}(q_{sat,1}-\Delta) \\ \theta_2 = & \frac{b_2}{b_1+b_2}(q_{sat,2}-\Delta) \end{cases}$$

$$\Rightarrow \begin{cases} q_{sat,1} = & \frac{b_1+b_2}{b_1}\theta_1 - \frac{b_2}{b_1}\Delta \\ q_{sat,2} = & \frac{b_1+b_2}{b_2}\theta_2 + \Delta \end{cases}$$

$$\Rightarrow \quad q_{sat,1} = q_{sat,2} = \theta_1 + \theta_2$$

Appendix M

PROPERTIES OF $\mathcal{A}_{\mathcal{D}}(N)$ AS A FUNCTION OF SALVAGE COST

$$\begin{aligned}
\lim_{s\to -\frac{c}{\mu}} \mathcal{A}_{\mathcal{D}}(N,s) &= (\frac{\bar{a}}{b-b'}+s)\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c\\
&= (\frac{\bar{a}}{b-b'}-\frac{c}{\mu})\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - (-\frac{c}{\mu})\mu - c\\
&= (\frac{\bar{a}}{b-b'}-\frac{c}{\mu})\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - (-\frac{c}{\mu})\mu - c
\end{aligned}$$

while $\lim_{s\to -\frac{c}{\mu}} \frac{\bar{\theta}}{N} = \frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})$

$$\begin{aligned}
\lim_{s\to -\frac{c}{\mu}} \mathcal{A}_{\mathcal{D}}(N,s) &= (\frac{\bar{a}}{b-b'}-\frac{c}{\mu})\int_0^{\frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})} \nu f(\nu)\,\mathrm{d}\nu\\
&\quad - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})} \nu^2 f(\nu)\,\mathrm{d}\nu\\
&= (\frac{b+b'}{N(2b+b')})(\bar{a}-(b-b')\frac{c}{\mu})\int_0^{\frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})} \nu f(\nu)\,\mathrm{d}\nu\\
&\quad - \int_0^{\frac{b}{N(2b-b')}(\bar{a}-(b-b')\frac{c}{\mu})} \nu^2 f(\nu)\,\mathrm{d}\nu\\
&= \int_0^{\frac{b}{(2b-b')}\frac{(\bar{a}-(b-b')\frac{c}{\mu})}{N}} (\frac{b+b'}{(2b+b')}\frac{(\bar{a}-(b-b')\frac{c}{\mu}}{N}-\nu)f(\nu)\,\mathrm{d}\nu > 0
\end{aligned}$$

We can numerically show that this inequality is correct for the reasonable values of the parameters and for three common probability distribution functions.

$$\lim_{s\to -\infty} \mathcal{A}_{\mathcal{D}}(N,s) = (\frac{\bar{a}}{b-b'}+s)\int_0^{\frac{\bar{\theta}}{N}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b+b'}{b^2-b'^2}N\int_0^{\frac{\bar{\theta}}{N}} \nu^2 f(\nu)\,\mathrm{d}\nu - s\mu - c$$

$$= (\frac{\bar{a}}{b-b'} + s)\mu - \frac{2b+b'}{b^2-b'^2}N(\mu^2+\sigma^2) - s\mu - c$$

$$= (\frac{\bar{a}}{b-b'})\mu - \frac{2b+b'}{b^2-b'^2}N(\mu^2+\sigma^2) - c$$

If we assume that $\frac{\bar{a}}{2} < N$ and $1 < \mu$ then the equations hold and $\mathcal{A}_{\mathcal{D}}(N)$ is a decreasing function of s with positive value at the left side of its domain and negative value on the positive infinity extreme. The following formulas present the same results in terms of formulas:

Appendix N

π_1, DUOPOLY CASE, $N_1 < N_2$

$$
\begin{aligned}
N\pi_1(N_1|N_2) &= \int_0^{\frac{N_1}{(N_1 z+N_2)}I} p_1^* q_1^* \,\mathrm{d}q + \int_{\frac{N_1}{(N_1 z+N_2)}I}^{\bar{\theta}} p_1^* q_1^* \,\mathrm{d}q + \int_{\bar{\theta}}^{\infty} ((p_1^*+s)D_1^* - sq_1)\,\mathrm{d}q - cN_1 \\
&= \int_0^{\frac{N_1}{N_1 Z+N_2}I} (\frac{\bar{a}}{b-b'} - \frac{bq_1+b'q_2}{b^2-b'^2})q_1^*\,\mathrm{d}q + \int_{\frac{N_1}{N_1 Z+N_2}I}^{\bar{\theta}} (\frac{2b+b'}{2b^2-b'^2}\bar{a} - \frac{2b}{2b^2-b'^2}q_1 - \frac{bb'}{2b^2-b'^2}s)q_1^*\,\mathrm{d}q \\
&+ \int_{\bar{\theta}}^{\infty} ((\frac{\bar{a}-sb}{2b-b'}+s)(\frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs) - sq_1)\,\mathrm{d}q - cN_1
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\frac{I}{N_1 Z+N_2}} (\frac{\bar{a}}{b-b'} - \frac{b\nu N_1 + b'\nu N_2}{b^2-b'^2})\nu N_1 f(\nu)\,\mathrm{d}\nu \\
&+ \int_{\frac{I}{N_1 Z+N_2}}^{\frac{\bar{\theta}}{N_1}} (\frac{(2b+b')\bar{a} - bb's}{2b^2-b'^2} - \frac{2b}{2b^2-b'^2}\nu N_1)\nu N_1 f(\nu)\,\mathrm{d}\nu \\
&+ \int_{\frac{\bar{\theta}}{N_1}}^{\infty} ((\frac{\bar{a}+(b-b')s}{2b-b'})b(\frac{\bar{a}+(b-b')s}{2b-b'}) - s\nu N_1)f(\nu)\,\mathrm{d}\nu - cN_1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\bar{a}}{b-b'}N_1 \int_0^{\frac{I}{N_1 Z+N_2}} \nu f(\nu)\,\mathrm{d}\nu - \frac{bN_1^2 + b'N_1N_2}{b^2-b'^2} \int_0^{\frac{I}{N_1 Z+N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu \\
&+ \frac{(2b+b')\bar{a} - bb's}{2b^2-b'^2}N_1 \int_{\frac{I}{N_1 Z+N_2}}^{\frac{\bar{\theta}}{N_1}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b}{2b^2-b'^2}N_1^2 \int_{\frac{I}{N_1 Z+N_2}}^{\frac{\bar{\theta}}{N_1}} \nu^2 f(\nu)\,\mathrm{d}\nu \\
&+ b(\frac{\bar{a}+(b-b')s}{2b-b'})^2 \bar{F}(\frac{\bar{\theta}}{N_1}) - sN_1 \int_{\frac{\bar{\theta}}{N_1}}^{\infty} \nu f(\nu)\,\mathrm{d}\nu - cN_1
\end{aligned}
$$

$$
\begin{aligned}
&= (\frac{\bar{a}}{b-b'} + s)N_1 \int_0^{\frac{I}{N_1 Z + N_2}} \nu f(\nu)\,\mathrm{d}\nu - \frac{bN_1^2 + b'N_1N_2}{b^2 - b'^2} \int_0^{\frac{I}{N_1 Z + N_2}} \nu^2 f(\nu)\,\mathrm{d}\nu \\
&+ (\frac{(2b+b')\bar{a} - bb's}{2b^2 - b'^2} + s)N_1 \int_{\frac{I}{N_1 Z + N_2}}^{\frac{\bar{\theta}}{N_1}} \nu f(\nu)\,\mathrm{d}\nu - \frac{2b}{2b^2 - b'^2} N_1^2 \int_{\frac{I}{(N_1 Z + N_2}}^{\frac{\bar{\theta}}{N_1}} \nu^2 f(\nu)\,\mathrm{d}\nu \\
&+ b(\frac{\bar{a} + (b-b')s}{2b-b'})^2 \bar{F}(\frac{\bar{\theta}}{N_1}) - sN_1 \int_0^{\infty} \nu f(\nu)\,\mathrm{d}\nu - cN_1
\end{aligned}
$$

Appendix O

π_1, DUOPOLY CASE, $N_1 > N_2$

$$\begin{aligned}
\pi_1(N_1|N_2) &= \int_0^{\frac{N_1}{N_1+N_2 z}I} p_1^* q_1^* \,\mathrm{d}q + \int_{\frac{N_1}{N_1+N_2 z}I}^{\frac{N_1}{N_2}\bar{\theta}} (p_1^*+s)D_1^* - sq_1 \,\mathrm{d}q + \int_{\frac{N_1}{N_2}\bar{\theta}}^{\infty} (p_1^*+s)D_1^* - sq_1 - cN_1 \,\mathrm{d}q \\
&= \int_0^{\frac{N_1}{N_1+N_2 z}} (\frac{\bar{a}}{b-b'} - \frac{bq_1+b'q_2}{b^2-b'^2})q_1^* \,\mathrm{d}q \\
&+ \int_{\frac{IN_1}{N_1+N_2 z}}^{\frac{N_1}{N_2}\bar{\theta}} (\frac{b+b'}{2b^2-b'^2}\bar{a} - \frac{b'q_2}{2b^2-b'^2} + \frac{b^2-b'^2}{2b^2-b'^2}s)b(\frac{b+'b}{2b^2-b'^2}\bar{a} - \frac{b'q_2}{2b^2-b'^2} + \frac{b^2-b'^2}{2b^2-b'^2}s) - sq_1 \,\mathrm{d}q \\
&+ \int_{\frac{N_1}{N_2}\bar{\theta}}^{\infty} (\frac{\bar{a}-sb}{2b-b'} + s)(\frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs) - sq_1 \,\mathrm{d}q - cN_1
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{I}{N_1+N_2 z}} (\frac{\bar{a}}{b-b'} - \frac{b\nu N_1 + b'\nu N_2}{b^2-b'^2})\nu N_1 f(\nu) \,\mathrm{d}\nu \\
&+ \int_{\frac{I}{N_1+N_2 Z}}^{\frac{\bar{\theta}}{N_2}} b\left(\frac{b+'b}{2b^2-b'^2}\bar{a} - \frac{b'}{2b^2-b'^2}\nu N_2 + \frac{b^2-b'^2}{2b^2-b'^2}s\right)^2 - s\nu N_1 f(\nu) \,\mathrm{d}\nu \\
&+ \int_{\frac{\bar{\theta}}{N_2}}^{\infty} (\frac{\bar{a}+(b-b')s}{2b-b'})(\frac{b}{2b-b'}\bar{a} + \frac{b-b'}{2b-b'}bs) - s\nu N_1 f(\nu) \,\mathrm{d}\nu
\end{aligned}$$

$$\begin{aligned}
&= \frac{\bar{a}}{b-b'}N_1 \int_0^{\frac{I}{N_1+N_2 Z}} \nu f(\nu) \,\mathrm{d}\nu - (\frac{bN_1+b'N_2}{b^2-b'^2})N_2 \int_0^{\frac{I}{N_1+N_2 Z}} \nu^2 f(\nu) \,\mathrm{d}\nu \\
&+ b\left((b+b')(\frac{\bar{a}+(b-b')s}{2b^2-b'^2}\right)^2 \int_{\frac{I}{N_1+N_2 Z}}^{\frac{\bar{\theta}}{N_2}} f(\nu) \,\mathrm{d}\nu
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{2bb'}{2b^2-b'^2}(b+b')\frac{\bar{a}+(b-b')s}{2b^2-b'^2}N_2+sN_1\right)\int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}}\nu f(\nu)\,\mathrm{d}\nu \\
&+b(\frac{b'N_2}{2b^2-b'^2})^2\int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}}\nu^2 f(\nu)\,\mathrm{d}\nu \\
&+b(\frac{\bar{a}+(b-b')s}{2b-b'})^2\bar{F}(\frac{\bar{\theta}}{N_2})-sN_1\int_{\frac{\bar{\theta}}{N_2}}^{\infty}\nu f(\nu)\,\mathrm{d}\nu-cN_1
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\bar{a}}{b-b'}N_1\int_0^{\frac{I}{N_1+N_2Z}}\nu f(\nu)\,\mathrm{d}\nu-\frac{bN_1^2+b'N_1N_2}{b^2-b'^2}\int_0^{\frac{I}{N_1+N_2z}}\nu^2 f(\nu)\,\mathrm{d}\nu \\
&+b\left(\frac{(b+b')\bar{a}+(b^2-b'^2)s}{2b^2-b'^2}\right)^2\int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}}f(\nu)\,\mathrm{d}\nu \\
&-\left(2bb'(b+b')\frac{\bar{a}+(b-b')s}{(2b^2-b'^2)^2}N_2+sN_1\right)\int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}}\nu f(\nu)\,\mathrm{d}\nu \\
&+b(\frac{b'N_2}{2b^2-b'^2})^2\int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}}\nu^2 f(\nu)\,\mathrm{d}\nu+b(\frac{\bar{a}+(b-b')s}{2b-b'})^2\bar{F}(\frac{\bar{\theta}}{N_2})-sN_1\int_{\frac{\bar{\theta}}{N_2}}^{\infty}\nu f(\nu)\,\mathrm{d}\nu-cN_1
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{\bar{a}}{b-b'}N_1\int_0^{\frac{I}{N_1+N_2Z}}\nu f(\nu)\,\mathrm{d}\nu-(\frac{bN_1^2+b'N_1N_2}{b^2-b'^2})\int_0^{\frac{I}{N_1+N_2Z}}\nu^2 f(\nu)\,\mathrm{d}\nu \\
&+b\left(\frac{(b+b')\bar{a}+(b^2-b'^2)s}{2b^2-b'^2}\right)^2\int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}}f(\nu)\,\mathrm{d}\nu \\
&-\left(2bb'(b+b')\frac{\bar{a}+(b-b')s}{(2b^2-b'^2)^2}N_2+sN_1\right)\int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}}\nu f(\nu)\,\mathrm{d}\nu \\
&+b(\frac{b'N_2}{2b^2-b'^2})^2\int_{\frac{I}{N_1+N_2Z}}^{\frac{\bar{\theta}}{N_2}}\nu^2 f(\nu)\,\mathrm{d}\nu+b(\frac{\bar{a}+(b-b')s}{2b-b'})^2\bar{F}(\frac{\bar{\theta}}{N_2})-sN_1\int_{\frac{\bar{\theta}}{N_2}}^{\infty}\nu f(\nu)\,\mathrm{d}\nu-cN_1
\end{aligned}
$$

Appendix P

PROOF FOR REMARK 3

$$
\begin{aligned}
\lim_{N_1\to\infty}\pi_1(N_1|N_2) &= (\frac{\bar{a}}{b-b'}+s)\frac{\int_0^{\frac{I}{N_1+N_2Z}}\nu f(\nu)\,\mathrm{d}\nu}{\frac{1}{N_1}} - \frac{\int_0^{\frac{I}{N_1+N_2Z}}\nu^2 f(\nu)\,\mathrm{d}\nu}{\frac{b^2-b'^2}{bN_1^2+b'N_1N_2}} \\
&+ \frac{I^2}{b}\int_0^{\frac{\bar{\theta}}{N_2}} f(\nu)\,\mathrm{d}\nu - \frac{2ZI}{b}N_2\int_0^{\frac{\bar{\theta}}{N_2}}\nu f(\nu)\,\mathrm{d}\nu + \frac{Z^2}{b}N_2^2\int_0^{\frac{\bar{\theta}}{N_2}}\nu^2 f(\nu)\,\mathrm{d}\nu \\
&+ \frac{\bar{\theta}^2}{b}\bar{F}(\frac{\bar{\theta}}{N_2}) - s\mu N_1 - cN_1
\end{aligned}
$$

$$
\begin{aligned}
&= (\frac{\bar{a}}{b-b'}+s)\frac{-\frac{I^2}{(N_1+N_2Z)^3}f(\frac{I}{N_1+N_2Z}}{\frac{-1}{N_1^2}} - \frac{-\frac{I^3}{(N_1+N_2Z)^4}f(\frac{I}{N_1+N_2Z})}{\frac{2bN_1(b^2-b'^2)}{(bN_1^2+b'N_1N_2})^2} \\
&+ \frac{I^2}{b}\int_0^{\frac{\bar{\theta}}{N_2}} f(\nu)\,\mathrm{d}\nu - \frac{2ZI}{b}N_2\int_0^{\frac{\bar{\theta}}{N_2}}\nu f(\nu)\,\mathrm{d}\nu + \frac{Z^2}{b}N_2^2\int_0^{\frac{\bar{\theta}}{N_2}}\nu^2 f(\nu)\,\mathrm{d}\nu \\
&+ \frac{\bar{\theta}^2}{b}\bar{F}(\frac{\bar{\theta}}{N_2}) - s\mu N_1 - cN_1 \\
&= -\infty
\end{aligned}
$$

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