# Crouzeix's Conjecture and Beyond for Special Classes of Matrices

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A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2021

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Program Authorized to Offer Degree:
Applied Mathematics

University of Washington

## Abstract

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Let $A$ be an $n$ by $n$ matrix with numerical range $W(A) := \{q^*Aq : q \in \mathbb{C}^n,\ \|q\|_2 = 1\}$. We are interested in functions $\hat{f}$ that maximize $\|f(A)\|_2$ (the matrix norm induced by the vector 2-norm) over all functions $f$ that are analytic in $W(A)$ and satisfy $\max_{z \in W(A)} |f(z)| \leq 1$. It is known that there are functions $\hat{f}$ that achieve this maximum and that such functions are of the form $B \circ \phi$, where $\phi$ is any conformal mapping from $W(A)$ to the unit disk $\mathbb{D}$ and $B$ is a Blaschke product of degree at most $n-1$. Michel Crouzeix has conjectured that $\|\hat{f}(A)\|_2 \leq 2$ and he proved that $\|\hat{f}(A)\|_2 \leq 11.08$ [M. Crouzeix, Numerical range and functional calculus in Hilbert space, J. Funct. Anal., vol. 244, issue 2, 2007, p. 668-690]. Later Crouzeix and Palencia proved $\|\hat{f}(A)\|_2 \leq 1 + \sqrt{2}$ [M. Crouzeix and C. Palencia, The Numerical Range is a $(1+\sqrt{2})$-Spectral Set, SIMAX, vol. 38, 2017, p. 649-655]. However, the conjectured bound of 2 remains unproven in general.

Other questions about the optimal function $\hat{f}$ remain open as well. Here we consider questions about uniqueness of the optimizing function $\hat{f}$ and show that for 2 by 2 matrices, $\hat{f}$ is unique (up to multiplication by a scalar) but for certain 3 by 3 matrices this is not the case. It has been observed numerically that the optimal Blaschke product $\hat{B}$ associated with a given conformal mapping $\phi$ often has degree less than $n-1$, and we prove this for a certain class of matrices. We also evaluate $\|\hat{f}(A)\|_2$ explicitly for certain classes of matrices with elliptical numerical range. Our goal is to learn as much as possible about the optimal

function $\hat{f}$, the associated Blaschke product $\hat{B}$, and the actual value of $\|\hat{f}(A)\|_2$ for matrices with elliptical numerical range. In doing this, we make use of known and new cyclic identities involving Jacobi elliptic functions.

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# NOTATION

| | |
|---|---|
| $\mathbb{C}$ | the complex numbers |
| $\mathbb{R}$ | the real numbers |
| $\mathbb{D}$ | the unit disk |
| $\mathbb{C}^{n\times n}$ | $n \times n$ complex matrices |
| $\lvert\lvert A\rvert\rvert_2$ | the 2-norm of matrix $A$ |
| $\lvert\lvert f(z)\rvert\rvert_\Omega$ | the infinity norm of $f(z)$ on set $\Omega$ |
| $C(\Omega, n)$ | the upper bound |
| $W(A)$ | the numerical range of $A$ |
| $r(A)$ | the numerical radius of $A$ |
| $\partial W(A)$ | the boundary of the numerical range of $A$ |
| $\phi(z)$ | Conformal mapping |
| $B(z)$ | Blaschke product |
| $\mathrm{sn}(\cdot), \mathrm{cn}(\cdot), \mathrm{dn}(\cdot)$ | Jacobi elliptic functions |
| $q$ | nome |
| $k$ | elliptic modulus |
| $K$ | quarter period |
| $V$ | Vandermonde matrix |
| $N_p$ | Pick matrix |
| $\oplus$ | direct sum |

# ACKNOWLEDGMENTS

Throughout the writing of this dissertation I have received a lot of help and support.

First, I would like to thank my advisor, Professor Anne Greenbaum, who is very kind and wise. It is a great pleasure to work with her and I really benefited a lot from our weekly discussions. She always explains concepts in an insightful way, formulates fascinating and meaningful research questions and gives concise and intelligent answers to hard problems. Her expertise and patience assist me to do better research.

It is an enjoyable experience to be in the Department of Applied Math, University of Washington, Seattle. My sincere thanks to all members here. The valuable suggestions and encouragements from my committee readers, the interesting courses and talks from the faculty, the friendly help from the staff and the close cooperation with the students will be in my memory forever.

Last, I would like to thank my family. Without the support of my grandparents and parents, I can not start and finish my research here. Thanks to my wife, who takes good care of me and our daughter. Our lovely daughter makes our life happier and happier.

# DEDICATION

to my family

# Chapter 1

# INTRODUCTION

Given a square matrix $A \in \mathbb{C}^{n\times n}$, it is often necessary to estimate or bound $||f(A)||_2$, where $f$ is a given analytic function and $||\cdot||_2$ denotes the matrix norm induced by the 2-norm for vectors; i.e., the largest singular value. This problem arises, for instance, in analyzing the stability of solutions of differential equations, for example, $f(A) = \exp(tA),\ t > 0$, and in analyzing the convergence rate of iterative linear system solvers such as the GMRES algorithm.

The simplest bound is in terms of the eigenvalues and eigenvectors of $A$, or, more generally, in terms of its Jordan form; if $A = V\Lambda V^{-1}$, where $\Lambda$ is the Jordan normal form and $\kappa(V) = ||V||_2 \cdot ||V^{-1}||_2$ is the 2-norm condition number of $V$, then

$$||f(A)||_2 \leq \kappa(V)||f(\Lambda)||_2.$$

If $\kappa(V)$ is very large, however, then this may be a large overestimate of $||f(A)||_2$. Another idea is to bound $||f(A)||_2$ based on the size of $f$ on some region $\Omega \subset \mathbb{C}$ that contains the spectrum of $A$:

$$||f(A)||_2 \leq C(\Omega, n) \sup_{z\in\Omega} |f(z)|. \tag{1.1}$$

The smallest constant $C(\Omega, n)$ for which inequality (1.1) holds for all analytic functions $f$ may be independent of $n$ for some classes of matrices or some sets $\Omega$, and in this case we can write

$$C(\Omega, n) = C(\Omega).$$

M. Crouzeix [7] chose $\Omega$ to be the *numerical range* (or, field of values) of $A$:

$$W(A) = \{q^*Aq : q \in \mathbb{C}^{n\times 1} \text{ and } ||q||_2 = 1\},$$

and he proved $2 \leq C(W(A)) \leq 11.08$, independent of $n$. He conjectured that $C(W(A)) = 2$; i.e., for all analytic functions $f$:

$$||f(A)||_2 \leq 2 \sup_{z \in W(A)} |f(z)|.$$

It is easy to see that $C(W(A))$ cannot be less than 2; it was shown by M. Crabb [5] and later D. Choi [4] and M. Crouzeix [15] that the matrices

$$J_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \; J_n = \begin{bmatrix} 0 & \sqrt{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & \sqrt{2} \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \; n > 2 \qquad (1.2)$$

satisfy

$$||J_n^{n-1}||_2 = 2 \sup_{z \in W(J_n)} |z^{n-1}|, \; n \geq 2.$$

Recently, M. Crouzeix and C. Palencia [9] made a major breakthrough and showed that

$$C(W(A)) \leq 1 + \sqrt{2}.$$

The goal of my research is to study some special classes of matrices and not only show that Crouzeix's conjecture holds for these matrices (in some cases this has already been shown) but identify the smallest constant $C$ such that $||f(A)||_2 \leq C \sup_{z \in W(A)} |f(z)|$ for these classes and find a function (or functions) $\hat{f}$ for which $||\hat{f}(A)||_2 = C \sup_{z \in W(A)} |\hat{f}(z)|$. I will answer some open questions about the uniqueness of the *extremal function* $\hat{f}$. The form of $\hat{f}$ is known: $\hat{f} = \hat{B} \circ \phi$, where $\phi$ is a conformal mapping from $W(A)$ to the unit disk $\mathbb{D}$, and $\hat{B}$ is a finite Blaschke product of degree at most $n-1$. From numerical experiments, it appears that $\hat{B}$ often has degree less than $n-1$, and we give several examples that $\hat{B}$ has degree strictly less than $n-1$.

## *1.1 Some Matrices and Functions for which Crouzeix's Conjecture holds*

### *1.1.1 Example 1*

If $A$ is normal, we have $A = U^*\Lambda U$, where $U$ is unitary and $\Lambda$ is diagonal, and then

$$||f(A)||_2 = ||U^* f(\Lambda) U||_2 = ||f(\Lambda)||_2 = \sup_{z\in\sigma(A)} |f(z)| \leq \sup_{z\in W(A)} |f(z)|,$$

since the spectrum of $A$, $\sigma(A) \subset W(A)$. The upper bound is $C(W(A)) = 1$ for normal matrices, and any function $f$, whose maximum value on $W(A)$ (the convex hull of the spectrum of $A$) occurs at an eigenvalue, achieves this optimal bound.

### *1.1.2 Example 2*

If matrix $A$ is diagonalizable with a matrix of eigenvectors $V$ such that $\kappa(V) \leq 2$, we know

$$||f(A)||_2 \leq \kappa(V)||f(\Lambda)||_2 \leq 2||f(\Lambda)||_2 \leq 2 \sup_{z\in W(A)} |f(z)|,$$

since $\sigma(A) \subset W(A)$.

### *1.1.3 Example 3*

Define the numerical radius of matrix $A$ as

$$r = \max_{z\in W(A)} |z|.$$

If $W(A)$ is a disk centered at $\alpha$ with radius $r_0$, we know that the conformal mapping is linear

$$\phi(z) = \frac{1}{r_0}(z - \alpha),$$

and the numerical radius of $\phi(A)$ is

$$r(\phi(A)) = 1.$$

By Ando's theorem (See section 1.2.2), $C(W(A)) \leq 2$.

### *1.1.4 Example 4*

If $A$ is a 2-nilpotent matrix (nonnormal), $A^2 = 0$, then $W(A)$ is a disk centered at the origin [17] and

$$r(A) = \frac{1}{2}||A||_2.$$

Thus, Crouzeix's conjecture holds, the exact bound is $C(W(A)) = 2$, the conformal mapping is

$$\phi(z) = \frac{2}{||A||_2} z,$$

and the unique extremal Blaschke product is

$$\hat{B}(z) = z.$$

One interesting case is

$$A = \begin{bmatrix} 0_{m\times m} & \tilde{A} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix}_{(m+k)\times(m+k))}, \ \forall \tilde{A} \in \mathbb{C}^{m\times k}.$$

See Appendix A for a short proof. This is also a special case of the class of half-radial matrices, described in the paper [19].

### *1.1.5 Example 5*

It is also known that for any matrix the conjecture holds for special functions:

(1) $f(z) = c_0 z + c_1, \quad c_0, c_1 \in \mathbb{C}$,

(2) $f(z) = z^m, \quad m \in \mathbb{N}$,

(3) $f(z) = \sum_{k=0}^{N} a_k z^k$, if $a_k \geq 0$ and $r(A) \in W(A)$.

*Proof.* (1) For any square matrix $A$, we have [21]

$$||A||_2 \leq 2 \sup_{z\in W(A)} |z| = 2r(A),$$

and it follows that

$$||c_0 A + c_1 I||_2 \leq 2r(c_0 A + c_1 I) = 2 \sup_{z\in W(c_0 A + c_1 I)} |z| = 2 \sup_{z\in W(A)} |c_0 z + c_1|,$$

since $W(c_0A + c_1I) = c_0W(A) + c_1$.

(2)

$$||A^m||_2 \leq 2r(A^m) \leq 2r(A)^m.$$

This is the power inequality of C. Berger and C. Pearcy [30].

(3)

$$||f(A)||_2 = ||\sum_{k=0}^{N} a_k A^k||_2 \leq \sum_{k=0}^{N} a_k ||A^k||_2 \leq 2\sum_{k=0}^{N} a_k r(A)^k = 2 \sup_{z\in W(A)} |f(z)|.$$

See [21] for details. □

### *1.1.6 Example 6*

A. Greenbaum and D. Choi [4, 14] proved that the conjecture holds for perturbed Jordan blocks $J_\epsilon$, D. Choi extended this to diagonally scaled perturbed Jordan blocks, and C. Glader, M. Kurula, and M. Lindström noted that it also holds for $A = PJ_\alpha$ [11], where $P$ is any permutation matrix, and

$$J_\epsilon = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \epsilon & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n},$$

$$J_\alpha = \begin{bmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{n-1} \\ \alpha_n & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}. \tag{1.3}$$

In the same paper, C. Glader et al. proved the case of 3 × 3 matrices with elliptic numerical range centered at an eigenvalue.

### 1.1.7 Example 7

Suppose that $A \in \mathbb{C}^{n\times n}$ and $M \in \mathbb{C}^{N\times N}$ $(n < N)$. $M$ is defined as *a dilation* of $A$ if $A = PM|_{\mathbb{C}^n}$, where $P$ is the orthogonal projection from $\mathbb{C}^N$ onto $\mathbb{C}^n$. For example,

$$M = \begin{bmatrix} A & * \\ * & * \end{bmatrix}.$$

$M$ is *a power dilation* of $A$ if $M^k$ is a dilation of $A^k$ for every positive integer $k$. Then, if $M$ is similar to a normal matrix $N$ via a similarity transformation with a moderate condition number $\kappa$ ($M = S^{-1}NS$, $NN^* = N^*N$ and $\kappa = ||S||_2||S^{-1}||_2$), for any polynomial $p$ we have

$$||p(A)||_2 \leq ||p(M)||_2 \leq \kappa \sup_{z\in\sigma(M)} |p(z)|.$$

If $\kappa \leq 2$ and $\sigma(M) \in W(A)$, then Crouzeix's conjecture holds for matrix $A$. This is one method T. Caldwell, A. Greenbaum and K. Li used to study Crouzeix's conjecture in the paper [13].

## 1.2 Tools

In this section, I will discuss some useful tools in this work. First I show two inequalities, von Neumann's inequality and T. Ando's theorem, and then I explain how conformal mapping and Blaschke products are used in the analysis.

### 1.2.1 von Neumann's inequality

A matrix or operator $C$ is a *contraction* if

$$||C||_2 \leq 1.$$

J. von Neumann showed [29] that for any polynomial $p$, the 2-norm of $p(C)$ is bounded by the supremum of $|p|$ in the closed unit disk $\mathbb{D}$,

$$||p(C)||_2 \leq \sup_{z\in\mathbb{D}} |p(z)|, \text{ if } ||C||_2 \leq 1.$$

### *1.2.2 T. Ando's theorem*

Define the *numerical radius* of an operator or matrix $A$ as

$$r(A) = \sup_{z \in W(A)} |z|.$$

Given a constant $K > 0$, a set $\Omega \subset \mathbb{C}$ is said to be a $K$-spectral set of operator or matrix $A$, if the spectrum of $A$, $\sigma(A) \subset \Omega$ and

$$||f(A)||_2 \leq K \sup_{z \in \Omega} |f(z)|,$$

where $f$ is any function analytic in $\Omega$ and continuous on the boundary of $\Omega$.

**Theorem 1** (T. Ando)**.** *If $r(T) \leq 1$, then the unit disk $\mathbb{D}$ is a 2-spectral set for operator or matrix $T$.*

T. Ando [1] pointed out that the numerical radius of an operator $T$ is less than or equal to 1 if and only if $T$ can be written in the form

$$T = (I + H)^{\frac{1}{2}} C (I - H)^{\frac{1}{2}},$$

where $H$ is a self-adjoint contraction and $C$ is a contraction. Since

$$H = H^* \text{ and } ||H||_2 \leq 1,$$

we can write

$$H = QRQ^*,$$

where $Q$ is unitary and $R$ is diagonal with real elements in the interval $[-1, 1]$. Thus,

$$R = \cos(\tilde{R}),$$

where $\tilde{R}$ is diagonal with real elements in $[0, \pi]$.

$$H = Q \cos(\tilde{R}) Q^* = \cos(Q \tilde{R} Q^*) = \cos(U),$$

where $U = Q\tilde{R}Q^*$. Hence,

$$(I+H)^{\frac{1}{2}} = Q(I+\cos(\tilde{R}))Q^* = \sqrt{2}Q\cos(\tilde{R}/2)Q^* = \sqrt{2}\cos\left(\frac{U}{2}\right),$$

$$(I-H)^{\frac{1}{2}} = Q(I-\cos(\tilde{R}))Q^* = \sqrt{2}Q\sin(\tilde{R}/2)Q^* = \sqrt{2}\sin\left(\frac{U}{2}\right),$$

and

$$T = 2\cos\left(\frac{U}{2}\right)C\sin\left(\frac{U}{2}\right).$$

Taking $g(z) = \max(1, 2|\cos(z)|)$, we have

$$|\sin(z)|g(z) = \max(|\sin(z)|, |\sin(2z)|),$$

and, if we define

$$V = g\left(\frac{U}{2}\right),$$

then $V$ is nonsingular and

$$||V||_2 \le 2,\ ||V^{-1}||_2 \le 1,$$
$$||\sin\left(\frac{U}{2}\right)V||_2 \le 1,\ ||V^{-1}2\cos\left(\frac{U}{2}\right)||_2 \le 1,$$
$$||V^{-1}TV||_2 \le ||V^{-1}2\cos\left(\frac{U}{2}\right)||_2||C||_2||\sin\left(\frac{U}{2}\right)V||_2 \le 1.$$

Hence, for any analytic function $f$,

$$||f(T)||_2 \le ||V||_2||f(V^{-1}TV)||_2||V^{-1}||_2 \le 2\sup_{z\in\mathbb{D}}|f(z)|.$$

### *1.2.3 Orthogonality of singluar vectors of $\hat{f}(A)$*

**Theorem 2.** *If $\hat{f}(z)$ defined on $W(A)$ is one optimal function for $A \in \mathbb{C}^{n\times n}$, i.e.,*

$$C(W(A), n) = \frac{||\hat{f}(A)||_2}{\sup\limits_{z\in W(A)}|\hat{f}(z)|},$$

*and if $C(W(A), n) > 1$, then the left and right singular vectors corresponding to the largest singular value of $\hat{f}(A)$ are orthogonal to each other; that is, if $v_1$ is the right singular vector corresponding to the largest singular value $\sigma_1$ so that $\hat{f}(A)v_1 = u_1\sigma_1$, then $u_1^*v_1 = 0$.*

See proposition 2.2 in [8]. We use this property to argue the uniqueness of $\hat{f}(z)$ for some cases.

### *1.2.4 Conformal mapping*

From Riemann's mapping theorem, there is a bijective conformal mapping $\phi$ from the interior of $W(A)$ to the interior of the unit disk $\mathbb{D}$, and by the Caratheodory theorem, it can be extended continuously to the boundary

$$\phi : W(A) \to \mathbb{D}.$$

If we choose one point $z_0$ inside $W(A)$ such that $\phi(z_0) = 0$, then this mapping is unique up to multiplication by a scalar of modulus 1 ($e^{i\beta}$), which is the rotation of angle $\beta$ around the origin. In Crouzeix's conjecture, our goal is to find the upper bound $C(W(A))$ or

$$\sup_f \frac{||f(A)||_2}{\sup\limits_{z \in W(A)} |f(z)|} \equiv \sup_f \frac{||f(A)||_2}{||f||_{W(A)}}.$$

It is equivalent to consider

$$\sup_f \frac{||f(A)||_2}{||f||_{W(A)}} = \sup_h \frac{||h(M)||_2}{||h||_{\mathbb{D}}},$$

where $M = \phi(A)$ and $f = h \circ \phi$, since

$$||f(A)||_2 = ||h(M))||_2,$$

and

$$||f||_{W(A)} = ||h \circ \phi||_{W(A)} = ||h||_{\mathbb{D}}.$$

### *1.2.5 Blaschke products*

From the definition of functions of a matrix, $f(A)$ is uniquely determined by the values of $f$ (and perhaps some of its derivatives) at the eigenvalues of $A$. Suppose

$$\hat{f} = \arg\sup_f \frac{||f(A)||_2}{||f||_{W(A)}}.$$

Then, of all functions $f$ satisfying $f(A) = \hat{f}(A)$ (i.e., all functions $f$ that match $\hat{f}$ and appropriate derivatives at the eigenvalues of $A$), $\hat{f}$ must be the one that minimizes the denominator above, $\|f\|_{W(A)}$.

Pick-Nevanlinna interpolation shows us that of all functions that take $n$ given values (or given derivative values) at given points in the unit disk $\mathbb{D}$, the one with minimal $\infty$-norm on $\mathbb{D}$ is a scalar multiple of a Blaschke product of degree at most $n-1$, i.e.,

$$B(z) = \exp(i\gamma) \prod_{j=1}^{n-1} \frac{z - \alpha_j}{1 - \bar{\alpha_j} z}, \ |\alpha_j| \leq 1.$$

Since Blaschke products map the unit disk to itself and thus have magnitude 1 on the boundary, our goal is equivalent to find

$$\sup_f \frac{||f(A)||_2}{||f||_{W(A)}} = \sup_B ||B \circ \phi(A)||_2,$$

where $\phi$ is any conformal mapping from $W(A)$ to $\mathbb{D}$. While the optimal Blaschke product $\hat{B}$ (corresponding to a given conformal mapping $\phi$) is often uniquely determined (to within multiplication by a unit scalar), we will see in the following part that this is not always the case; thus $\hat{f} = \hat{B} \circ \phi$ is also not unique.

### *1.2.6 Jacobi elliptic functions and Jacobi theta functions*

An elliptic function is a meromorphic function that is periodic in two directions on the complex plane. In the discussion, we take the Jacobi form. One definition is the inverse of the incomplete elliptic integral. Take

$$u = \int_0^{\psi} \frac{d\theta}{\sqrt{1 - m\sin^2(\theta)}},$$

where $0 \leq m \leq 1$. The elliptic sine is defined as $\mathrm{sn}(u)$,

$$\mathrm{sn}(u) = \sin(\psi),$$

the elliptic cosine is given by $\mathrm{cn}(u)$,

$$\mathrm{cn}(u) = \cos(\psi),$$

and the delta amplitude $\mathrm{dn}(u)$ is

$$\mathrm{dn}(u) = \sqrt{1 - m\sin^2(\psi)}.$$

If we choose $\psi$ and $m$ as two parameters, Jacobi elliptic functions can be written as $\mathrm{sn}(u, m)$, $\mathrm{cn}(u, m)$, $\mathrm{dn}(u, m)$, etc. The elliptic modulus $k$ is

$$k = \sqrt{m},$$

and the complements of $k$ and $m$ are defined as

$$m' = 1 - m,$$

and

$$k' = \sqrt{m'}.$$

The quarter periods along the real axis and the imaginary axis, $K$ and $K'$, can be written in terms of $m$ or $k$. By introducing the Jacobi theta functions,

$$\vartheta_1(z, q) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+1/2)^2}\exp\left((2n+1)iz\right),$$

$$\vartheta_2(z, q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}\exp\left((2n+1)iz\right),$$

$$\vartheta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2}\exp\left(2niz\right),$$

and

$$\vartheta_4(z, q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}\exp\left(2niz\right),$$

where $q$ is the nome, we can rewrite Jacobi elliptic functions and the parameters as follows:

$$\mathrm{sn}(u, k^2) = \frac{\vartheta_3(0, q)\vartheta_1(u/2K, q)}{\vartheta_2(0, q)\vartheta_4(u/2K, q)},$$

$$k = \left(\frac{\vartheta_2(0, q)}{\vartheta_3(0, q)}\right)^2,$$

and

$$K = \frac{\pi}{2}\vartheta_3(0,q)^2.$$

Gabor Szegö [31] showed that the conformal mapping of the interior of an ellipse with foci points ±1 onto the unit disk is

$$\phi_0(z) = \sqrt{k}\,\mathrm{sn}\left(\frac{2K}{\pi}\sin^{-1} z, k^2\right),$$

if we map the origin to itself. The nome

$$q = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2,$$

where $m_1$ and $m_2$ are the lengths of the semi-major and semi-minor axes respectively. In this thesis, when $k = k(q)$, we denote

$$\mathrm{sn}(u) = \mathrm{sn}(u, k^2).$$

Otherwise, the second parameter is given explicitly, for example,

$$k_n = k(q^{n-1}),\ n = 2, 3, \cdots .$$

## Chapter 2

# SOME NEW RESULTS

It is known that Crouzeix's conjecture holds for matrices with circular numerical range. In this chapter, we show the relation between matrices with circular numerical range and matrices with elliptical numerical range, and give several examples of how we take advantage of the tools in the previous chapter to study the exact upper bound and uniqueness (up to multiplication by a scalar) of the extremal function. We point out that the extremal function is not always unique even for the circular cases.

### *2.1 Theorems*

**Theorem 3.** *If $\hat{f}(z)$ defined on $W(A)$ is one optimal function for $A \in \mathbb{C}^{n\times n}$, i.e.,*

$$C(W(A), n) = \frac{||\hat{f}(A)||_2}{\sup\limits_{z\in W(A)} |\hat{f}(z)|},$$

*then $\hat{f}(z)$ is one optimal function for $A^T$, and $\overline{\hat{f}(\bar{z})}$ is one optimal function for both $\bar{A}$ and $A^*$.*

*Proof.* For any unit vector $q$, taking $p = \bar{q}$, we have

$$p^* A^T p = (p^* A^T p)^T = p^T A \bar{p} = q^* A q,$$

$$p^* \bar{A} p = \overline{\overline{p^* \bar{A} p}} = \overline{\bar{p}^* A \bar{p}} = \overline{q^* A q},$$

and

$$q^* A^* q = \overline{(q^* A^* q)^*} = \overline{q^* A q}.$$

That is,

$$W(A) = W(A^T) = \overline{W(\bar{A})} = \overline{W(A^*)}.$$

For any analytic function $g(z)$ on $W(A)$, $g(\bar{z})$ is defined on $\overline{W(A)}$, $h(z) = \overline{g(\bar{z})}$ is also analytic on $\overline{W(A)}$, and we have

$$\frac{||g(A^T)||_2}{\sup\limits_{z\in W(A^T)} |g(z)|} = \frac{||g(A)||_2}{\sup\limits_{z\in W(A)} |g(z)|} \leq \frac{||\hat{f}(A)||_2}{\sup\limits_{z\in W(A)} |\hat{f}(z)|} = C(W(A), n),$$

the equality holds when taking $g = \hat{f}$. By the same token,

$$\frac{||g(\bar{A})||_2}{\sup\limits_{z\in W(\bar{A})} |g(z)|} = \frac{||\overline{g(\bar{A})}||_2}{\sup\limits_{z\in W(A)} |\overline{g(\bar{z})}|} = \frac{||h(A)||_2}{\sup\limits_{z\in W(A)} |h(z)|} \leq \frac{||\hat{f}(A)||_2}{\sup\limits_{z\in W(A)} |\hat{f}(z)|} = C(W(A), n),$$

$$\frac{||g(A^*)||_2}{\sup\limits_{z\in W(A^*)} |g(z)|} = \frac{||g(A^*)^*||_2}{\sup\limits_{z\in W(A)} |\overline{g(\bar{z})}|} = \frac{||h(A)||_2}{\sup\limits_{z\in W(A)} |h(z)|} \leq \frac{||\hat{f}(A)||_2}{\sup\limits_{z\in W(A)} |\hat{f}(z)|} = C(W(A), n).$$

Both inequalities can hold when taking $h = \hat{f}$, i.e., $g(z) = \overline{\hat{f}(\bar{z})}$. $\square$

**Theorem 4.** *For any square matrix $A$ and complex number $\alpha \in \mathbb{C}$,*

$$W(A + \alpha A^*) = \{w + \alpha w^* | w \in W(A)\},$$

*and*

$$\partial W(A + \alpha A^*) = \{w + \alpha w^* | w \in \partial W(A)\}.$$

*Proof.* For any unit vector $q$, we have

$$q^*(A + \alpha A^*)q = q^*Aq + q^*\alpha A^*q = (q^*Aq) + \alpha(q^*Aq)^*,$$

thus,

$$W(A + \alpha A^*) = \{w + \alpha w^* | w \in W(A)\}.$$

The boundary points of $W(A)$ are the set of values $q^*Aq$ where $q$ is a normalized eigenvector corresponding to the largest eigenvalue of

$$\frac{e^{i\theta}A + e^{-i\theta}A^*}{2}, \ 0 \leq \theta < 2\pi. \tag{2.1}$$

The boundary points of $W(A+\alpha A^*)$ are the set of values $p^*(A+\alpha A^*)p = p^*Ap + \alpha(p^*Ap)^*$ where $p$ is a normalized eigenvector corresponding to the largest eigenvalue of

$$\frac{e^{i\gamma}(A+\alpha A^*) + e^{-i\gamma}(A^* + \alpha^* A)}{2} = \frac{(e^{i\gamma} + \alpha^* e^{-i\gamma})A + (\alpha e^{i\gamma} + e^{-i\gamma})A^*}{2},\ 0 \leq \gamma < 2\pi. \quad (2.2)$$

Taking $e^{i\gamma} + \alpha^* e^{-i\gamma} = re^{i\theta}$, the matrix in equation 2.2 is a positive multiple of that in equation 2.1, so eigenvectors corresponding to the largest eigenvalue of the two matrices are the same,

$$\partial W(A+\alpha A^*) = \{w + \alpha w^* | w \in \partial W(A)\}.$$

□

## 2.2 *Circular case*

### *2.2.1 Uniqueness of the extremal function for Jordan Blocks*

We know that for the $n$ by $n$ Jordan block

$$J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n},$$

$W(J)$ is a disk centered at the origin of radius $\cos(\frac{\pi}{n+1})$, and the conformal mapping $\phi$ that maps $W(J)$ to the unit disk $D$ is

$$\phi(z) = \frac{1}{\cos(\frac{\pi}{n+1})} z = cz,\ c = \frac{1}{\cos(\frac{\pi}{n+1})} > 1.$$

We can rewrite

$$\phi(J) = D^{-1}JD,$$

where

$$D = \begin{bmatrix} 1 & & & \\ & c & & \\ & & \ddots & \\ & & & c^{n-1} \end{bmatrix}_{n\times n}.$$

Since $||J||_2 = 1$, by von Neumann's inequality,

$$||B(J)||_2 \leq 1,$$

and we know for any Blaschke product $B$ that

$$B(\phi(J)) = D^{-1}B(J)D,$$

and

$$||B(\phi(J))||_2 \leq \kappa(D)||B(J)||_2 = c^{n-1}||B(J)||_2 \leq c^{n-1}.$$

One extremal Blaschke product is

$$\hat{B}(z) = z^{n-1},$$

since

$$||\hat{B}(\phi(J))||_2 = c^{n-1}.$$

To prove uniqueness, suppose the extremal $B(z)$ takes the form

$$B(z) = \prod_{j=1}^{n-1} \frac{z-\alpha_j}{1-\bar{\alpha}_j z}, \ \alpha_j \in \mathbb{C}, \ \text{and } |\alpha_j| \leq 1,$$

then

$$B(J) = p_0 I + p_1 J + \cdots + p_{n-1}J^{n-1},$$

or

$$P = B(J) = \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-2} & p_{n-1} \\ 0 & p_0 & p_1 & \cdots & p_{n-2} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & p_0 & p_1 \\ 0 & 0 & \cdots & 0 & p_0 \end{bmatrix},$$

since $J^n = 0$. Here $p_j$ is $1/j!$ times the $j^{th}$ derivative of $B(z)$ evaluated at $z = 0$ and can be expressed in terms of the $\alpha_j$'s:

$$p_0 = (-1)^{n-1} \prod_{j=1}^{n-1} \alpha_j,$$

$$p_1 = (-1)^{n-2} \sum_{j=1}^{n-1} \left( \left( \prod_{k \neq j} \alpha_k \right) (1 - |\alpha_j|^2) \right),$$

$$\vdots$$

We know that the $p_j$'s satisfy

$$||P||_2 = ||B(J)||_2 \leq 1,$$

and

$$||B(\phi(J))||_2 = ||D^{-1}PD||_2 \leq ||D^{-1}||_2 ||P||_2 ||D||_2 \leq c^{n-1}, \tag{2.3}$$

with equality in both places if $B$ is extremal. Suppose that the first right and left singular vectors of $D^{-1}PD$ are $v_1$ and $u_1$ respectively, i.e.,

$$u_1^* D^{-1} P D v_1 = c^{n-1}, \; ||v_1||_2 = ||u_1||_2 = 1.$$

Then equality holds in (2.3) only when $||P||_2 = 1$, $u_1$ is parallel to the first right singular vector of $D^{-T} = D^{-1}$, that is $e_1$ ( $e_j$'s are the standard basis), $v_1$ is parallel to the first right singular vector of $D$ (i.e., $e_n$), $D^{-1}u_1$ is parallel to the first left singular vector of $P$, and $Dv_1$ is parallel to the first right singular vector of $P$. Thus, we have

$$|p_{n-1}| = 1,$$

and

$$||[p_0 \; p_1 \cdots p_{n-1}]^T||_2 = \left( \sum_{j=0}^{n-1} |p_j|^2 \right)^{\frac{1}{2}} \leq ||P||_2 = 1$$

gives us

$$p_j = 0, \; j = 0, 1, \cdots, n-2.$$

Further, $p_0 = 0$ means $B(0) = 0$, from which we get some $\alpha_j = 0$. Without loss of generality, take $\alpha_1 = 0$. Then

$$B(z) = z \prod_{j=2}^{n-1} \frac{z - \alpha_j}{1 - \bar{\alpha_j} z}, \; \alpha_j \in \mathbb{C}, \text{ and } |\alpha_j| \leq 1.$$

Differentiating $B(z)$ we see $B'(0) = 0$ (i.e., $p_1 = 0$), which implies that one of the remaining $\alpha_j$'s is 0, i.e.,

$$B(z) = z^2 \prod_{j=3}^{n-1} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}, \ \alpha_j \in \mathbb{C}, \ \text{and } |\alpha_j| \leq 1.$$

Doing this step by step, finally we get

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0.$$

That is, $\hat{B}(z) = z^{n-1}$ is the only extremal Blaschke product.

### *2.2.2 A Diagonally Scaled 3 By 3 Jordan Block*

Here we identify one example for which the extremal function is not unique.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1-t \\ 0 & 0 & 0 \end{bmatrix}, \ t \in [0, \sqrt{3} - 1].$$

$W(A)$ is a disk centered at the origin with radius

$$r = \frac{\sqrt{(1 + (1-t)^2)}}{2} \leq 1.$$

The conformal mapping $\phi$ that maps $W(A)$ to the unit disk $\mathbb{D}$ is

$$\phi(z) = \frac{1}{r} z,$$

which means

$$\phi(A) = \frac{1}{r} A = \begin{bmatrix} 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1-t}{r} \\ 0 & 0 & 0 \end{bmatrix} = D \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} D^{-1} = D J_3 D^{-1},$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & \dfrac{r^2}{1-t} \end{bmatrix}, \ J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $0 \leq t \leq \sqrt{3}-1$, we get

$$\frac{\sqrt{3}-1}{\sqrt{2}} \leq r \leq \frac{1}{\sqrt{2}},$$

and

$$\frac{1}{2} \leq \frac{r^2}{1-t} \leq 1.$$

Then, for any Blaschke product $B$ we obtain

$$B(\phi(A)) = DB(J_3)D^{-1},$$

and

$$||B(\phi(A))||_2 \leq ||D||_2||B(J_3)||_2||D^{-1}||_2 = 1 \cdot 1 \cdot ||D^{-1}||_2 = \max\left(\frac{1-t}{r^2}, \frac{1}{r}\right).$$

Taking $B_1(z) = z$, we find

$$||B_1(\phi(A))||_2 = \frac{1}{r},$$

and taking $B_2(z) = z^2$, we find

$$||B_2(\phi(A))||_2 = \frac{1-t}{r^2}.$$

In addition, on the interval $[0, \sqrt{3}-1]$, the function $h(t) = \dfrac{1-t}{r} = \dfrac{2}{\sqrt{1+\frac{1}{(1-t)^2}}}$ decreases monotonically as $t$ increases. We find the value $t_0$ after which $||B_1(\phi(A))||_2$ starts to exceed $||B_2(\phi(A))||_2$ by solving

$$\frac{1}{r} = \frac{1-t_0}{r^2},$$

or

$$h(t_0) = 1.$$

We get

$$t_0 = 1 - \frac{1}{\sqrt{3}}.$$

When $t \leq t_0$, the extremal Blaschke product is $z^2$; when $t \geq t_0$, it is $z$; at the value $t_0$, both Blaschke products ($z$ and $z^2$) give the same extremal result

$$||B(\phi(A))||_2 = \sqrt{3}.$$

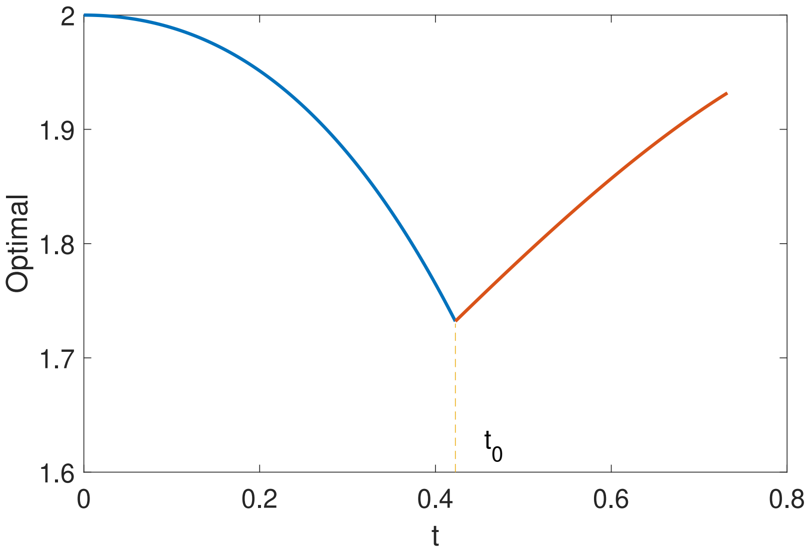


Figure 2.2.1: $||\hat{B}(\phi(J_t))||_2$ vs. $t$. Blue is $\hat{B}(z) = z^2$, and red is $\hat{B}(z) = z$.

Figure 2.2.1 shows a plot of $\max_B ||B(\phi(A))||_2 = \max\{||B_1(\phi(A))||_2, ||B_2(\phi(A))||_2\}$ as $t$ ranges between 0 and $\sqrt{3}-1$. In addition to showing nonuniqueness, the extremal Blaschke product has degree less than the maximal degree $n-1$. This has been observed in numerical experiments, and we have now proved it for this class of matrices when $1-1/\sqrt{3} < t < \sqrt{3}-1$.

At the point $t = t_0 = 1 - \frac{1}{\sqrt{3}}$, we have

$$\phi(A) = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Suppose that the first right and left singular vectors of $DB(J_3)D^{-1}$ are $v_1$ and $u_1$ respectively, i.e.,

$$u_1^* DB(J_3)D^{-1}v_1 = \sqrt{3}, \ ||v_1||_2 = ||u_1||_2 = 1.$$

We see that $u_1$ must be parallel to the first right singular vector of $D^T = D$ (i.e., $e_1$), $v_1$ should be parallel to the first right singular vector of $D^{-1}$ (i.e., $e_2$ or $e_3$). In fact, the choices of $v_1$ ($v_1 = e_2$ or $v_1 = e_3$) correspond to different extremal Blachke products $z$ and $z^2$, respectively.

### *2.3 Elliptic case*

There are several ways to extend A. Greenbaum and D. Choi's result for $J_\alpha$ in section (1.1.6). We can take $A = Q^*(\lambda I + J_\alpha)Q$, where $Q$ is a unitary matrix and $\lambda \in \mathbb{C}$; let $A = D(\lambda I + J_\alpha)D^{-1}$, where $D$ is diagonal (since $DJ_\alpha D^{-1}$ is still in the form of equation (1.1)); define

$$\tilde{J} = \begin{bmatrix} \lambda_\alpha I + J_\alpha & & & \\ & \lambda_\beta I + J_\beta & & \\ & & \ddots & \\ & & & \lambda_\omega I + J_\omega \end{bmatrix},$$

where $J_\alpha, \cdots, J_\omega$ are all in the form of equation (1.1.6) (Crouzeix's conjecture holds for each block of $\tilde{J}$, so it also holds for $\tilde{J}$. This includes the standard Jordan form); or construct $A = J_\alpha + bJ_\alpha^*,\ b \in [0,1]$.

Especially, since Crouzeix's conjecture holds for any matrix with a circular numerical range, by Theorem 4, if the numerical range of a square matrix $A$ is an ellipse with semi-major and semi-minor axes $m_1$ and $m_2$ respectively, then we can translate and rotate $A$ to the form

$$\tilde{A} = \exp(-i\beta)A + \alpha I,$$

where $\alpha$ is the center of $W(A)$ and $\beta$ is the angle between the major axis and x-axis (real). Then $W(\tilde{A})$ is an ellipse centered at the origin, the major axis is on the $x$-axis, and the minor axis is on the $y$-axis. The boundary of $W(\tilde{A})$ is

$$\partial W(\tilde{A}) = m_1\cos(\theta) + im_2\sin(\theta), \quad \theta \in [0, 2\pi).$$

If $m_1 = m_2$, this is the special case where $W(\tilde{A})$ and $W(A)$ are disks. Otherwise, taking

$$b = \frac{m_1 - m_2}{m_1 + m_2}, \quad r = \frac{m_1 + m_2}{2},$$

we know that

$$\tilde{A} = C + bC^*, \tag{2.4}$$

where

$$C = \frac{1}{1-b^2}\left(\tilde{A} - b\tilde{A}^*\right),$$

and the boundary of $W(C)$ is

$$\partial W(C) = \left\{\frac{1}{1-b^2}(z + bz^*)|z \in \partial W(\tilde{A})\right\} = r\exp(i\theta),\ \theta \in [0, 2\pi).$$

$W(C)$ is a disk centered at the origin with radius

$$r = \frac{m_1 + m_2}{2}.$$

In this way, given any matrix $A$ with elliptic numerical range, we can find $b$ and matrix $C$, such that $W(C)$ is a disk; given a matrix $C$ with circular numerical range and a parameter $b$, by equation (2.4) we can construct a matrix $A$ with elliptic numerical range. It is shown [3] that the two focal points are eigenvalues of matrix $A$. In the next part, we discuss the elliptic cases derived from some special matrices with a disk numerical range.

### *2.3.1 2 by 2 case*

For any 2 by 2 matrix, $W(A)$ is an ellipse and Crouzeix [6] proved that

$$C(W(A), 2) = 2\exp\left(-\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}\frac{2}{1+\rho^n}\right),$$

where

$$\rho = \frac{1+\sqrt{1-\epsilon^2}}{\epsilon},$$

and $\epsilon$ is the eccentricity of the ellipse $W(A)$. Here we give an equivalent expression for the bound $C(W(A), 2)$ and prove that $\hat{B}$ (or $\hat{f}$) is unique.

It can be shown that, after a proper translation and rotation, any $2 \times 2$ matrix $A \neq cI$ (trivial case) is unitarily similar to a matrix of the form

$$\begin{bmatrix} 0 & a \\ d & 0 \end{bmatrix},$$

where $a > 0, d \geq 0$. If $a < d$, it is equivalent to consider $A^T$, so without loss of generality we can assume $a \geq d \geq 0$. The matrix $A$ is normal if and only if $a = d$, and in this case $||f(A)||_2 \leq \max_{z \in W(A)} |f(z)|$; otherwise, $W(A)$ is an ellipse. From G. Szegö [31], the conformal mapping is

$$\phi_0(z) = \sqrt{k}\, \mathrm{sn}\left(\frac{2K}{\pi} \sin^{-1} z\right).$$

We can multiply $A$ by the scalar $\sqrt{k/(ad)}$ and write

$$A = \begin{bmatrix} 0 & \sqrt{\dfrac{k}{b}} \\ \sqrt{kb} & 0 \end{bmatrix},$$

where $0 \leq b = \frac{d}{a} \leq 1$ and $0 < k = k(b^2) < 1$. The conformal mapping maps $A$ to itself, since $A$ is diagonalizable and it maps the eigenvalues of $A$, $\pm\sqrt{k}$, to themselves.

$$\phi(A) = A = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\frac{b}{k}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\frac{k}{b}} \end{bmatrix} = T^{-1}CT.$$

Since $||C||_2 = 1$ and the condition number of $T$, $\kappa(T) = 1 \cdot \sqrt{\frac{k}{b}}$, for any Blaschke product $B$, by von Neumann's inequality [29],

$$||B(\phi(A))||_2 \leq \sqrt{\frac{k}{b}}.$$

The equality holds when

$$\hat{B}(z) = z,$$

see Figure 2.3.1.

Further, the extremal $B$ is unique if $A$ is nonnormal ($a > d$ or $b < 1$). Suppose an extremal $B$ is of the form

$$B(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}, \ |\alpha| \leq 1.$$

We will show that $\alpha$ must be 0.

By Theorem 2, an extremal Blaschke product $B$ satisfies

$$B(\phi(A))v_1 = ||B(\phi(A))||_2 u_1,$$

and

$$u_1^* v_1 = 0,$$

where $u_1$ and $v_1$ are the first left and right singular vectors of $B(\phi(A))$. Taking

$$V = [v_1 \; v_2] = [v_1 \; u_1],$$

we have

$$V^* V = V V^* = I,$$

and the diagonal elements of $V^* B(\phi(A))V$ are 0, i.e., the eigenvalues of $V^* B(\phi(A))V$ or $B(\phi(A))$ are of the form $\pm\lambda$; i.e.,

$$\frac{\sqrt{k} - \alpha}{1 - \bar{\alpha}\sqrt{k}} = -\frac{-\sqrt{k} - \alpha}{1 + \bar{\alpha}\sqrt{k}}, \; 0 < k \leq 1,$$

which gives

$$\alpha = \bar{\alpha} k.$$

When $0 < b < 1$, we know $0 < k < 1$ and

$$\alpha = 0;$$

when $b = 1$, we know $k = 1$ and the matrix A is normal.

### *2.3.2 3 by 3 elliptical case with one parameter*

Since $W(J_3)$ is a disk, we can construct a matrix $A$

$$A = J_3 + b J_3^*,$$

and from the previous discussion $W(A)$ is an ellipse. C. Glader, M. Kurula, and M. Lindström argued that Courzeix's conjecture holds in this case [11]. We give the exact upper bound and prove uniqueness here.

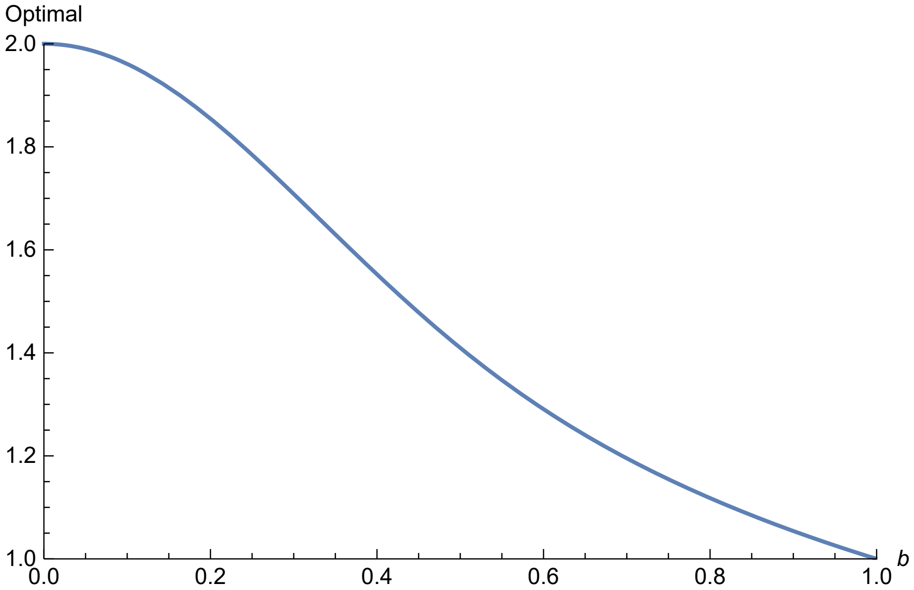


Figure 2.3.1: $||\hat{B}(\phi(A))||_2 = \sqrt{\frac{k}{b}}$ vs. $b$

Scale $A$ by the factor $\dfrac{1}{2\sqrt{b}\cos(\frac{\pi}{4})}$, thus

$$A = \frac{1}{\sqrt{2b}} \begin{bmatrix} 0 & 1 & 0 \\ b & 0 & 1 \\ 0 & b & 0 \end{bmatrix}, \; b \in (0, 1].$$

The eigenvalues of $A$ are 0 and the two foci of $W(A)$, $\pm 1$. The conformal mapping $\phi$ is

$$\phi(z) = k^{\frac{1}{2}} \operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}(z)\right),$$

that is,

$$\phi(A) = cA,$$

where $c = \phi(1) = k^{1/2}$. We can rewrite

$$\phi(A) = \begin{bmatrix} 0 & \sqrt{\dfrac{k}{2b}} & 0 \\ \sqrt{\dfrac{kb}{2}} & 0 & \sqrt{\dfrac{k}{2b}} \\ 0 & \sqrt{\dfrac{kb}{2}} & 0 \end{bmatrix} = T^{-1}CT,$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{t}} & 0 \\ 0 & 0 & \dfrac{1}{t} \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & \sqrt{\dfrac{kt}{2b}} & 0 \\ \sqrt{\dfrac{kb}{2t}} & 0 & \sqrt{\dfrac{kt}{2b}} \\ 0 & \sqrt{\dfrac{kb}{2t}} & 0 \end{bmatrix},$$

and

$$t = \frac{b(1+\sqrt{1-k^2})}{k}.$$

We can check that

$$||C||_2 = 1,$$

and

$$\frac{1}{2} < t \leq 1, \text{ if } b \in (0,1].$$

When $b \to 0$, $t \to \dfrac{1}{2}$, i.e., $b = 0$ is a removable singularity of $t$.

By von Neumann's inequality, for any Blaschke product $B$, we have

$$||B(\phi(A))||_2 = ||T^{-1}B(C)T||_2 \leq ||T^{-1}||_2 ||B(C)||_2 ||T||_2 \leq 1 \cdot 1 \cdot \frac{1}{t} = \frac{k}{b(1+\sqrt{1-k^2})}.$$

Taking

$$\hat{B}(z) = \frac{z^2 - \alpha^2}{1 - \bar{\alpha}^2 z^2},$$

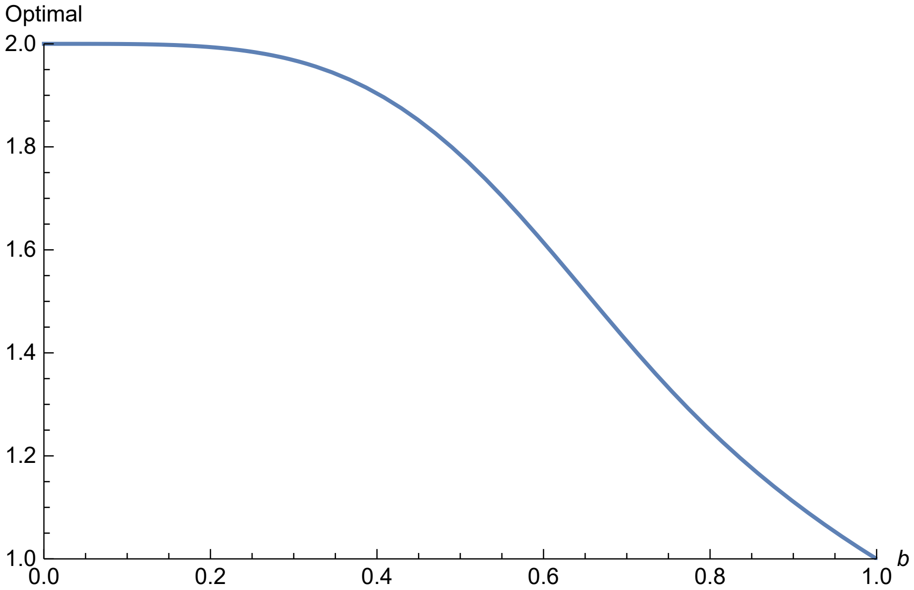


Figure 2.3.2: $||\hat{B}(\phi(A))||_2 = \frac{1}{t}$ vs. $b$

where

$$\alpha = \phi(\frac{1}{\sqrt{2}}) = \sqrt{\frac{k}{1+\sqrt{1-k^2}}},$$

we can achieve the extremal Blaschke product. That is, for this class of matrix

$$\hat{B}(C) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \dfrac{k}{1+\sqrt{1-k^2}} & 0 \\ \dfrac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} & 0 & 0 \end{bmatrix},$$

$$\hat{B}(\phi(A)) = \begin{bmatrix} 0 & 0 & \dfrac{1}{t} \\ 0 & \dfrac{k}{1+\sqrt{1-k^2}} & 0 \\ \dfrac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}t & 0 & 0 \end{bmatrix},$$

$$||\hat{B}(\phi(A))||_2 = \frac{1}{t} = \frac{k}{b(1+\sqrt{1-k^2})}.$$

To prove uniqueness, assume the extremal Blaschke product is in the form

$$B(z) = \frac{z-\alpha_1}{1-\bar{\alpha_1}z}\frac{z-\alpha_2}{1-\bar{\alpha_2}z}.$$

Since the minimum polynomial of $C$ is

$$C^3 - kC = 0,$$

we have

$$B(C) = p_0 I + p_1 C + p_2 C^2,$$

where

$$\begin{cases} p_0 = \alpha_1\alpha_2 \\ p_1 = -\alpha_1\frac{1-|\alpha_2|^2}{1-\bar{\alpha_2}^2k} - \alpha_2\frac{1-|\alpha_1|^2}{1-\bar{\alpha_1}^2k} + k(\bar{\alpha_1}+\bar{\alpha_2})\frac{1-|\alpha_1|^2}{1-\bar{\alpha_1}^2k}\frac{1-|\alpha_2|^2}{1-\bar{\alpha_2}^2k} \\ p_2 = -\alpha_1\bar{\alpha_2}\frac{1-|\alpha_2|^2}{1-\bar{\alpha_2}^2k} - \bar{\alpha_1}\alpha_2\frac{1-|\alpha_1|^2}{1-\bar{\alpha_1}^2k} + (1+k\bar{\alpha_1}\bar{\alpha_2})\frac{1-|\alpha_1|^2}{1-\bar{\alpha_1}^2k}\frac{1-|\alpha_2|^2}{1-\bar{\alpha_2}^2k} \end{cases}. \tag{2.5}$$

On the other hand,

$$||B(\phi(A))||_2 \leq ||T^{-1}||_2||B(C)||_2||T||_2 \leq \frac{1}{t},$$

with equality in both places if and only if $B$ is extremal. Then, we must have

$$||B(C)||_2 = 1,$$

and $e_1$ and $e_3$ must be parallel to the first left and right singular vectors of $B(\phi(A))$ (or $B(C)$), respectively. Thus, we have

$$B(C) = \begin{bmatrix} 0 & 0 & x_{13} \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & 0 \end{bmatrix},$$

and

$$1 = |x_{13}| = ||B(C)||_2.$$

Without loss of generality, taking $x_{13} = 1$, we can obtain

$$\begin{cases} p_0 = -\dfrac{k}{1+\sqrt{1-k^2}} \\ p_1 = 0 \\ p_2 = \dfrac{2}{1+\sqrt{1-k^2}} \end{cases},$$

and

$$B(C) = \hat{B}(C).$$

Combining with expressions (2.5) for $p_0$ and $p_1$ in terms of $\alpha_1$ and $\alpha_2$, we have

$$\alpha_1 = -\alpha_2 = \sqrt{\frac{k}{1+\sqrt{1-k^2}}} \text{ or } \alpha_1 = -\alpha_2 = -\sqrt{\frac{k}{1+\sqrt{1-k^2}}}.$$

One can double check that the expression (2.5) for $p_2$,

$$\frac{2}{1+\sqrt{1-k^2}} = p_2 = -\alpha_1\bar{\alpha_2}\frac{1-|\alpha_2|^2}{1-\bar{\alpha_2}^2 k} - \bar{\alpha_1}\alpha_2\frac{1-|\alpha_1|^2}{1-\bar{\alpha_1}^2 k} + (1+k\bar{\alpha_1}\bar{\alpha_2})\frac{1-|\alpha_1|^2}{1-\bar{\alpha_1}^2 k}\frac{1-|\alpha_2|^2}{1-\bar{\alpha_2}^2 k},$$

is also satisfied. In sum, the extremal Blaschke product is unique

$$B(z) = \hat{B}(z).$$

Further, we can find the relation of Figure 2.3.2 with Figure 2.3.1 by showing the relation between $k(q^2)$ and $k(q)$

$$\sqrt{k(q^2)} = \frac{k(q)}{1+\sqrt{1-k(q)^2}}, \ q \in [0,1].$$

*Proof.* We know that the Jacobi theta functions satisfy

$$\theta_3(q)^4 = \theta_2(q)^4 + \theta_4(q)^4, \tag{2.6}$$

and we can rewrite them in terms of the Dedekind eta function

$$\theta_2(q) = \frac{2\eta(2\tau)^2}{\eta(\tau)},$$

$$\theta_3(q) = \frac{\eta(\tau)^5}{\eta(2\tau)^2\eta\left(\frac{\tau}{2}\right)^2},$$

$$\theta_4(q) = \frac{\eta\left(\frac{\tau}{2}\right)^2}{\eta(\tau)},$$

where $q = \exp(\pi i \tau)$. Substituting all $\theta_j$ into equality (2.6), we find

$$\eta(\tau)^{24} = \eta\left(\frac{\tau}{2}\right)^8 \eta(2\tau)^8 \left(16\eta(2\tau)^8 + \eta\left(\frac{\tau}{2}\right)^8\right).$$

It follows that

$$\eta\left(\frac{\tau}{2}\right)^8 = \frac{-8\eta(2\tau)^{12} + \sqrt{64\eta(2\tau)^{24} + \eta(\tau)^{24}}}{\eta(2\tau)^4},$$

$$\eta(2\tau)^8 = \frac{-\eta\left(\frac{\tau}{2}\right)^{12} + \sqrt{64\eta(\tau)^{24} + \eta\left(\frac{\tau}{2}\right)^{24}}}{32\eta(\frac{\tau}{2})^4},$$

or

$$\eta(4\tau)^8 = \frac{-\eta(\tau)^{12} + \sqrt{64\eta(2\tau)^{24} + \eta(\tau)^{24}}}{32\eta(\tau)^4}.$$

Expressing $\eta(4\tau)$ and $\eta(\frac{\tau}{2})$ in terms of $\eta(\tau)$ and $\eta(2\tau)$ on the left hand side, we have an identity

$$\eta(\tau)^2\eta(4\tau)^4\left(\eta(\tau)^{12} + \eta(2\tau)^4\eta\left(\frac{\tau}{2}\right)^8\right) = 2\eta(2\tau)^{14}\eta\left(\frac{\tau}{2}\right)^4.$$

This can be written in terms of $\theta_j$'s,

$$\theta_2(q^2)\left(\theta_3(q)^2 + \theta_4(q)^2\right) = \theta_2(q)^2\theta_3(q^2),$$

which is equivalent to the equality

$$\sqrt{k(q^2)} = \frac{k(q)}{1 + \sqrt{1 - k(q)^2}}.$$

□

Figure 2.3.2 and Figure 2.3.1 match the functions $\sqrt{\dfrac{k(b^4)}{b^2}}$ and $\sqrt{\dfrac{k(b^2)}{b}}$ respectively.

### *2.3.3 3 by 3 elliptical case with two parameters*

Let

$$A = \frac{1}{\sqrt{b(1+t^2)}}\begin{bmatrix} 0 & 1 & 0 \\ b & 0 & t \\ 0 & bt & 0 \end{bmatrix},\ b \in (0, 0.7],\ t \in [0,1].$$

We know that $A$ has eigenvalues 0 and $\pm 1$. The numerical range $W(A)$ is an ellipse with major and minor axes $\frac{1\pm b}{\sqrt{b}}$ and focal points $\pm 1$. The conformal mapping $\phi$ is an elliptic function and

$$\phi(A) = k^{\frac{1}{2}} A,$$

where $k = k(b)$ is the elliptic modulus. From the numerical tests, we seem to have different extremal Blaschke products for different choices of parameters $t$ and $b$. It is shown in Figure 2.3.3 that we can take

$$B(z) = \begin{cases} z, & (b,t) \in \text{ Region I} \\ \dfrac{z-\beta}{1-\beta z}, & (b,t) \in \text{ Region II} \\ \dfrac{z^2-\alpha^2}{1-\alpha^2 z^2}, & (b,t) \in \text{ Region III} \end{cases}, \tag{2.7}$$

where the parameter $\alpha$ depends only on $k$ (or $b$).

$$\alpha^2 = \frac{k}{1+\sqrt{1-k^2}} = \frac{1-\sqrt{1-k^2}}{k} > 0,$$

and $\beta$ can be written in terms of $b$ and $t$ as

$$\beta = \pm\sqrt{\frac{-c_3+\sqrt{\Delta}-\sqrt{(c_3-\sqrt{\Delta})^2-16c_4^2}}{4c_4}},$$

$$\Delta = c_3^2 - 4c_2c_4 + 8c_4^2.$$

See Appendix D for the details of the coefficients $c_j$'s and $e_j$'s in this section. In fact, assuming that the extremal Blaschke product is of degree 1 with parameter $\beta$, then maximizing

$$||B(\phi(A))||_2 = \left\| \frac{\phi(A)-\beta I}{I-\bar{\beta}\phi(A)} \right\|_2, \ |\beta| \le 1$$

reveals that $\beta^2$ is the minimum root of the quartic equation

$$c_4x^4 + c_3x^3 + c_2x^2 + c_3x + c_4 = 0.$$

This quartic equation has positive real roots $\beta^2 \in (0,1)$ in Region II, and extremal Blaschke products are obtained by taking $\beta$ to be $\pm$ the square root of $\beta^2$; it has no real roots in

Region I, and the boundary point $\beta = 0$ gives the extremal Blaschke product, as shown in equation (2.7).

In Region I, we have

$$B(\phi(A)) = \phi(A) = \sqrt{k}A,$$

and its norm is

$$L_I = \sqrt{k}||A||_2 = \frac{k(1+t^2b^2)}{b(1+t^2)}.$$

In Region II,

$$B(\phi(A)) = \frac{\phi(A) - \beta I}{I - \beta\phi(A)} = \begin{bmatrix} \frac{\beta\beta_1 k}{(1+t^2)} - \beta, & \beta_1\sqrt{\frac{k}{b(1+t^2)}}, & \beta\beta_1\frac{kt}{b(1+t^2)} \\ \beta_1\sqrt{\frac{kb}{(1+t^2)}}, & \beta\beta_1 k - \beta, & \beta_1 t\sqrt{\frac{k}{b(1+t^2)}} \\ \beta\beta_1\frac{kbt}{(1+t^2)}, & \beta_1 t\sqrt{\frac{kb}{(1+t^2)}}, & \beta\beta_1\frac{kt^2}{(1+t^2)} - \beta \end{bmatrix},$$

where

$$\beta_1 = \frac{1-\beta^2}{1-\beta^2 k},$$

and the norm of of $B(\phi(A))$ is

$$L_{II} = \sqrt{\frac{1}{3}\left(e_1 + \frac{-1-\sqrt{3}i}{2}\sqrt[3]{\frac{d_1 + \sqrt{d_1^2 - 4d_2^3}}{2}} + \frac{-1+\sqrt{3}i}{2}\sqrt[3]{\frac{d_1 - \sqrt{d_1^2 - 4d_2^3}}{2}}\right)},$$

where

$$d_1 = -\frac{e_1e_2}{6} + \frac{e_1^3}{27} + \frac{e_3}{2},$$
$$d_2 = \frac{e_2}{3} - \frac{e_1^2}{9}.$$

In fact, $L_{II}^2$ is the maximum root of the cubic equation

$$x^3 - e_1x^2 + e_2x - e_3 = 0.$$

In Region III,

$$B(\phi(A)) = \frac{\phi(A)^2 - \alpha^2 I}{I - \alpha^2\phi(A)^2} = \frac{k}{(1+\sqrt{1-k^2})(1+t^2)}\begin{bmatrix} (1-t^2) & 0 & \frac{2t}{b} \\ 0 & (1+t^2) & 0 \\ 2bt & 0 & (t^2-1) \end{bmatrix},$$

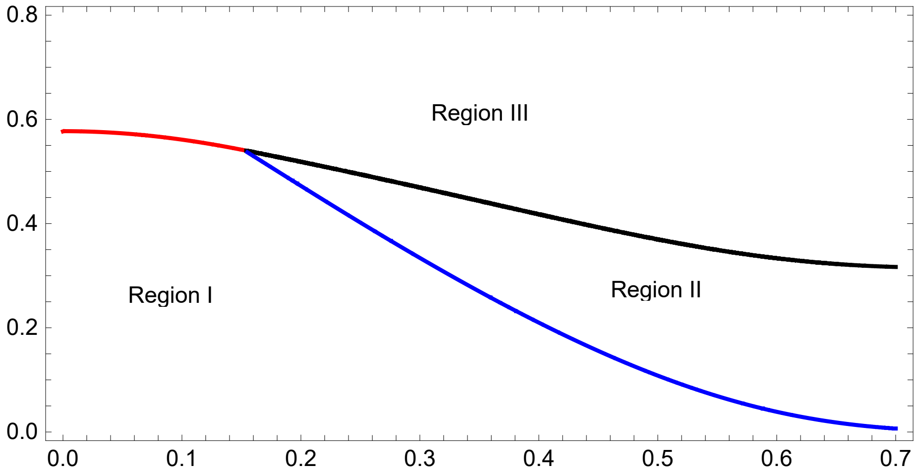


Figure 2.3.3: Turning point $t_0$ vs. $b$

and its norm is

$$L_{III} = \frac{k\left(t(1-b^2) + \sqrt{(b^2+t^2)(1+b^2t^2)}\right)}{b(1+t^2)(1+\sqrt{1-k^2})}.$$

In Figure 2.3.3, $L_I = L_{II}$ gives us the blue curve. On the left of this curve $L_I > L_{II}$, and on the right of this curve $L_I < L_{II}$. Similarly, $L_I = L_{III}$ gives us the red curve, and $L_{II} = L_{III}$ gives us the black curve. These 3 curves divide $[0, 0.7] \times [0, 1]$ into 3 regions: in Region I, $L_I > L_{II}$ and $L_I > L_{III}$; in Region II, $L_{II} > L_I$ and $L_{II} > L_{III}$; in Region III, $L_{III} > L_I$ and $L_{III} > L_{II}$. In Region I and III, we can expect a unique extremal Blaschke product up to multiplication by a scalar, while in Region II and on the curves, the extremal Blaschke product is not unique.

The figure agrees with our analysis in the previous sections. For example, when $t = 0$, we have

$$A = \frac{1}{\sqrt{b}} \begin{bmatrix} 0 & 1 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $(b, 0)$ is in Region I, we can expect a unique Blaschke product $B(z) = z$. This is the same as the 2 by 2 case in section (2.3.1). When $b \to 0$, we can consider the matrix

$$\sqrt{b(1+t^2)}A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}.$$

From section (2.2.2), we know that the turning point is $t_0 = \frac{1}{\sqrt{3}}$, which is exactly the left boundary point of the red curve. When $t = 1$, we have

$$A = \frac{1}{\sqrt{2b}} \begin{bmatrix} 0 & 1 & 0 \\ b & 0 & 1 \\ 0 & b & 0 \end{bmatrix}.$$

Since $(b, 1)$ is in Region III, we can expect a unique Blaschke product

$$B(z) = \frac{z^2 - \alpha^2}{1 - \alpha^2 z^2}.$$

This is the same as the 3 by 3 case in section (2.3.2).

# Chapter 3

## EXTENSION OF THE CHOI-CROUZEIX-CRABB MATRIX

We know that Crouzeix's conjecture holds for the Choi-Crouzeix-Crabb matrix and the upper bound 2 is achieved in this case. The numerical range of the Choi-Crouzeix-Crabb matrix is a disk, thus by equation (2.4) we can extend this matrix and analyze the exact upper bound and uniqueness of the extremal function.
The Choi-Crouzeix-Crabb matrix is defined as

$$C = \begin{bmatrix} 0 & \sqrt{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{n\times n} .$$

The numerical range $W(C)$ is a disk centered at the origin with radius 1. The conformal mapping is

$$\phi(z) = z.$$

The unique optimal function is $\hat{B}(z) = z^{n-1}$, and the upper bound is $||\hat{B}(C)||_2 = 2$, showing that the upper bound 2 can be achieved.

Take

$$A = \frac{1}{2\sqrt{b}}(C + bC^*) = \frac{1}{2\sqrt{b}} \begin{bmatrix} 0 & \sqrt{2} & 0 & \cdots & 0 & 0 & 0 \\ \sqrt{2}b & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & b & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & b & 0 & \sqrt{2} \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{2}b & 0 \end{bmatrix}_{n\times n} , b \in (0,1],$$

$$D = \begin{bmatrix} b^{\frac{0}{2}} & & & & \\ & b^{\frac{1}{2}} & & & \\ & & \ddots & & \\ & & & b^{\frac{n-2}{2}} & \\ & & & & b^{\frac{n-1}{2}} \end{bmatrix}.$$

Then

$$A = \frac{1}{2}DHD^{-1},$$

where

$$H = \begin{bmatrix} 0 & \sqrt{2} & 0 & \cdots & 0 & 0 & 0 \\ \sqrt{2} & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \sqrt{2} \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{2} & 0 \end{bmatrix}_{n\times n}.$$

Thus $H$ is real, symmetric and Hermitian. Here $W(A)$ is an ellipse centered at the origin with semi-axes $\frac{1\pm b}{2\sqrt{b}}$. In the following sections, we prove that Crouzeix's Conjecture holds for certain values of $n$, that the upper bound is $\sqrt{\frac{k(b^{2n-2})}{b^{n-1}}}$ and that the extremal function $\hat{f} = \hat{B} \circ \phi$ is unique up to multiplication by a scalar.

We use the notation:

$$q = b^2,\ k = k(b^2) = k(q),\ k_n = k(b^{2n-2}),\ K = K(k),\ K_n = K(k_n).$$

### 3.1 Find the conformal mapping of W(A) to the unit disk

We first determine the eigenvalues and eigenvectors of $H$. Solving

$$Hx = \lambda x,$$

we have

$$\begin{cases} \sqrt{2}x_2 = \lambda x_1 \\ \sqrt{2}x_1 + x_3 = \lambda x_2 \\ x_{j-1} + x_{j+1} = \lambda x_j, \quad j = 3, 4, \cdots, n-2 \\ x_{n-2} + \sqrt{2}x_n = \lambda x_{n-1} \\ \sqrt{2}x_{n-1} = \lambda x_n \end{cases} .$$

Solving the difference equations, we find

$$x_j = C_1 t_1^j + C_2 t_2^j,\ j = 2, 3, \cdots, n-1,$$

where

$$t_1 = \frac{\lambda + \sqrt{\lambda^2 - 4}}{2},\ t_2 = \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}.$$

Thus,

$$\begin{cases} x_1 = \dfrac{\sqrt{2}x_2}{\lambda} \\ x_n = \dfrac{\sqrt{2}x_{n-1}}{\lambda} \\ \dfrac{C_1}{C_2} = \dfrac{2t_1 - \lambda}{\lambda^2 t_1 - 2t_1 - \lambda} \\ \dfrac{C_1}{C_2} = \dfrac{\lambda t_1 - (\lambda^2 - 2)}{(\lambda^2 - 2)t_1^{2n-2} - \lambda t_1^{2n-3}} \end{cases} ,$$

and

$$(\lambda^2 - 1)(t_1^{2n-1} - t_1) = 0;$$

that is

$$t_1^{2n-2} = 1,$$

$$t_1 = \exp(i\frac{(j-1)\pi}{n-1}),\ j = 1, 2, \cdots, n.$$

Then,

$$t_2 = \frac{1}{t_1} = \exp(i\frac{(1-j)\pi}{n-1}),$$

and

$$\lambda_j = 2\cos(\frac{(j-1)\pi}{n-1}),\ j = 1, 2, \cdots, n.$$

The corresponding eigenvectors are

$$v_j = [v_{j1}, v_{j2}, \cdots, v_{jn-1}, v_{jn}]^T,\ j = 1, 2, \cdots n,$$

where

$$\begin{cases} v_{j1} = \dfrac{1}{\sqrt{2}} \\ v_{jk} = \cos(\dfrac{(k-1)(j-1)\pi}{n-1}),\ k = 2, 3, \cdots, n-1 \\ v_{jn} = \dfrac{(-1)^{j-1}}{\sqrt{2}} \end{cases},$$

and

$$||v_j||_2 = \begin{cases} \sqrt{n-1}, & j = 1 \text{ or } n \\ \sqrt{\dfrac{n-1}{2}}, & j = 2, 3, \cdots, n-1 \end{cases}.$$

Normalizing these eigenvectors, we obtain the real symmetric matrix

$$S = [S_1, S_2, \cdots, S_n]_{n\times n} = [S_{ij}],$$

where

$$S_{ij} = \sqrt{\frac{2^{1-\eta_1-\eta_2-\eta_3-\eta_4}}{n-1}} \cdot \cos\left(\frac{(i-1)(j-1)\pi}{n-1}\right), \tag{3.1}$$

$$\eta_1 = \begin{cases} 1, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases},\quad \eta_2 = \begin{cases} 1, & \text{if } j = 1 \\ 0, & \text{otherwise} \end{cases},$$

$$\eta_3 = \begin{cases} 1, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases}, \quad \eta_4 = \begin{cases} 1, & \text{if } j = n \\ 0, & \text{otherwise} \end{cases}.$$

We can check that

$$S^T = S = S^{-1},\ S^2 = I.$$

The eigenvalue matrix of $A$ is

$$\Lambda = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \cos\left(\frac{(j-1)\pi}{n-1}\right) & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix},$$

and

$$A = DS\Lambda S^{-1}D^{-1} = DS\Lambda SD^{-1}.$$

$W(A)$ is an ellipse with axes $\frac{2(1\pm b)}{\sqrt{b}}$ and foci $\pm 1$. The conformal mapping from the interior of the ellipse to the unit disk is

$$\phi(z) = k^{\frac{1}{2}} \operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}(z)\right),$$

where sn is one of Jacobi's elliptic functions, and $k = k(b^2)$ and $K = K(k)$ are the elliptic modulus and quarter period respectively.

We have

$$\phi(A) = DS\Psi SD^{-1}, \tag{3.2}$$

where $\Psi$ is the eigenvalue matrix of $\phi(A)$ and the diagonal elements are

$$\psi_j = k^{\frac{1}{2}} \operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}\left(\cos(\frac{(j-1)\pi}{n-1})\right)\right) = k^{\frac{1}{2}} \operatorname{sn}\left(K(1-\frac{2(j-1)}{n-1})\right), \quad j = 1, 2, \cdots, n.$$

We have

$$\psi_{n+1-j} = -\psi_j$$

$$\sqrt{k} = \psi_1 > \psi_2 > \cdots > \psi_{n-1} > \psi_n = -\sqrt{k}.$$

and $\phi(A)$ is real and persymmetric ($\phi(A)_{i,j} = \phi(A)_{n+1-j,n+1-i}$) with non-zero elements every other diagonal.

### *3.2 Construct a contraction that is similar to* $\phi(A)$

To construct a contraction from the conformal mapping of $A$, we need some Jacobi elliptic equalities involving sn(·), cn(·) and dn(·). We use a similar method from A. Khare, A. Lakshminarayan and U. Sukhatme [24–26]. We first prove the equality

$$k(q^n) = k^n \prod_{m=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mathrm{sn}^4 \left( \frac{(2m+1)K}{n} \right) = k^n \prod_{m=0}^{n-1} \mathrm{sn}^2 \left( \frac{(2m+1)K}{n} \right), \tag{3.3}$$

where $q$ is the nome, $k = k(q)$ is the elliptic modulus, $K = K(k) = K(q)$ is the quarter period and $n$ is any positive integer.

*Proof.* Since

$$\begin{aligned} 1 + x^n &= \prod_{m=0}^{n-1} (\exp(\frac{(2m+1)\pi i}{n}) + x) \\ &= \begin{cases} (1+x) \cdot \displaystyle\prod_{m=0}^{\frac{n-1}{2}-1} (1 - 2\cos(\frac{(2m+1)\pi}{n})x + x^2), & n \text{ odd}, \\ \displaystyle\prod_{m=0}^{\frac{n}{2}-1} (1 - 2\cos(\frac{(2m+1)\pi}{n})x + x^2), & n \text{ even}, \end{cases} \end{aligned}$$

and

$$\frac{\sqrt{k}\ \mathrm{sn}\,(u, k^2)}{2\sin(\dfrac{\pi u}{2K})} = q^{\frac{1}{4}} \prod_{j=1}^{\infty} \left( \frac{1 - 2\cos(\frac{\pi u}{K}) q^{2j} + q^{4j}}{1 - 2\cos(\frac{\pi u}{K}) q^{2j-1} + q^{4j-2}} \right),$$

we can interchange the order of two products (absolute convergence) to obtain

$$\begin{aligned}
k(q^n) &= 4q^{\frac{n}{2}}\prod_{j=1}^{\infty}\left(\frac{1+q^{2jn}}{1+q^{(2j-1)n}}\right)^4 \\
&= \begin{cases} 4q^{\frac{n}{2}}\displaystyle\prod_{j=1}^{\infty}\left(\left(\frac{1+q^{2j}}{1+q^{2j-1}}\right)\cdot\prod_{m=0}^{\frac{n-1}{2}-1}\left(\frac{1-2\cos(\frac{(2m+1)\pi}{n})q^{2j}+q^{4j}}{1-2\cos(\frac{(2m+1)\pi}{n})q^{2j-1}+q^{4j-2}}\right)\right)^4, & n \text{ odd} \\ 4q^{\frac{n}{2}}\displaystyle\prod_{j=1}^{\infty}\left(\prod_{m=0}^{\frac{n}{2}-1}\left(\frac{1-2\cos(\frac{(2m+1)\pi}{n})q^{2j}+q^{4j}}{1-2\cos(\frac{(2m+1)\pi}{n})q^{2j-1}+q^{4j-2}}\right)\right)^4, & n \text{ even} \end{cases} \\
&= \begin{cases} \dfrac{k^n}{4^{n-1}}\displaystyle\prod_{m=0}^{\frac{n-1}{2}-1}\left(\frac{\operatorname{sn}\left(\frac{(2m+1)K}{n}\right)}{\sin\left(\frac{(2m+1)\pi}{2n}\right)}\right)^4, & n \text{ odd} \\ \dfrac{k^n}{4^{n-1}}\displaystyle\prod_{m=0}^{\frac{n}{2}-1}\left(\frac{\operatorname{sn}\left(\frac{(2m+1)K}{n}\right)}{\sin\left(\frac{(2m+1)\pi}{2n}\right)}\right)^4, & n \text{ even} \end{cases} \\
&= \frac{k^n}{4^{n-1}}\prod_{m=0}^{\lfloor\frac{n}{2}\rfloor-1}\left(\frac{\operatorname{sn}\left(\frac{(2m+1)K}{n}\right)}{\sin\left(\frac{(2m+1)\pi}{2n}\right)}\right)^4.
\end{aligned}$$

When $q \to 0$, we have

$$k(0) = 0,\ K(0) = \frac{\pi}{2},$$

and

$$\operatorname{sn}\left(\frac{(2m+1)K}{n}, 0\right) = \sin\left(\frac{(2m+1)\pi}{2n}\right);$$

When $q \to 1$, we have

$$k(1) = 1,\ K(1) \to \infty,$$

and

$$\operatorname{sn}\left(\frac{(2m+1)K}{n}, 1\right) = 1,\ m = 0, 1, \cdots \left\lfloor\frac{n}{2}\right\rfloor - 1.$$

Thus, for any positive integer $n$

$$4^{n-1}\prod_{m=0}^{\lfloor\frac{n}{2}\rfloor-1}\sin^4\left(\frac{(2m+1)\pi}{2n}\right) = 1.$$

and

$$k(q^n) = k^n \prod_{m=0}^{\lfloor \frac{n}{2} \rfloor -1} \operatorname{sn}^4 \left( \frac{(2m+1)K}{n} \right).$$

Since $\operatorname{sn}(z) = \operatorname{sn}(2K - z)$ and $\operatorname{sn}(K) = 1$, we have

$$k(q^n) = k^n \prod_{m=0}^{\lfloor \frac{n}{2} \rfloor -1} \operatorname{sn}^4 \left( \frac{(2m+1)K}{n} \right) = k^n \prod_{m=0}^{n-1} \operatorname{sn}^2 \left( \frac{(2m+1)K}{n} \right).$$

□

Define

$$\Omega_{i,j} = \sum_{l=1}^{n} S_{i,l} \psi_l S_{l,j} = \sqrt{k} \sum_{l=1}^{n} S_{i,l} \operatorname{sn} \left( K \left( 1 - \frac{2(l-1)}{n-1} \right) \right) S_{l,j},$$

and

$$\Omega'_{i,j} = \sum_{l=1, \psi_l \neq 0}^{n} S_{i,l} / \psi_l S_{l,j} = \frac{1}{\sqrt{k}} \sum_{l=1, \psi_l \neq 0}^{n} S_{i,l} \frac{1}{\operatorname{sn} \left( K \left( 1 - \frac{2(l-1)}{n-1} \right) \right)} S_{k,j}.$$

We have that

$$\Omega_{i,j} = \Omega_{j,i} = \Omega_{n+1-i,n+1-j},$$

$$\Omega'_{i,j} = \Omega'_{j,i} = \Omega'_{n+1-i,n+1-j},$$

$$\sum_{l=1}^{n} \Omega_{i,l} \Omega'_{l,j} = \begin{cases} \delta_{i,j}, & n \text{ even} \\ \delta_{i,j} - S_{i\frac{n+1}{2}} S_{\frac{n+1}{2}j}, & n \text{ odd} \end{cases},$$

$$\Omega_{i,j} = \Omega'_{i,j} = 0, \text{ when } i, j \text{ are both odd/even.}$$

When $n$ is even, we prove that

$$k(q^{n-1}) = \frac{\Omega_{1,n}}{\Omega'_{1,n}} = k \frac{1 + \sum_{j=2}^{n-1} (-1)^{j-1} \operatorname{sn} \left( K(1 - \frac{2(j-1)}{n-1}) \right)}{1 + \sum_{j=2}^{n-1} (-1)^{j-1} / \operatorname{sn} \left( K(1 - \frac{2(j-1)}{n-1}) \right)}. \tag{3.4}$$

*Proof.* We take advantage of the properties of the Jacobi elliptic functions

$$\operatorname{sn}(z) = \operatorname{sn}(2K - z), \ \forall z \in \mathbb{C},$$

$$\operatorname{sn}(z^* - z) = -\operatorname{sn}(z^* + z),$$

where $z^* = iK' + 2miK', m \in \mathbb{Z}$ are the simple poles of sn. The left hand side is

$$k(b^{2n-2}) = k^{n-1} \prod_{m=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \operatorname{sn}^4\left(\frac{(2m+1)K}{n-1}\right) = k^{n-1} \prod_{m=0}^{n-2} \operatorname{sn}^2\left(\frac{(2m+1)K}{n-1}\right),$$

and the right hand side is

$$\begin{aligned}
&k\frac{1 + \sum_{j=2}^{n-1}(-1)^{j-1} \operatorname{sn}\left(K(1 - \frac{2(j-1)}{n-1})\right)}{1 + \sum_{j=2}^{n-1}(-1)^{j-1} / \operatorname{sn}\left(K(1 - \frac{2(j-1)}{n-1})\right)} \\
=&k\frac{\operatorname{sn}(K) + \sum_{j=2}^{n-1}(-1)^{j-1} \operatorname{sn}\left(K(1 - \frac{2(j-1)}{n-1})\right)}{\operatorname{sn}(K) + \sum_{j=2}^{n-1}(-1)^{j-1} / \operatorname{sn}\left(K(1 - \frac{2(j-1)}{n-1})\right)} \\
=&k\frac{(-1)^{\frac{n}{2}-1} \sum_{j=0}^{n-2}(-1)^{j} \operatorname{sn}\left(\frac{(2j+1)K}{n-1}\right)}{(-1)^{\frac{n}{2}-1} \sum_{j=0}^{n-2}(-1)^{j} / \operatorname{sn}\left(\frac{(2j+1)K}{n-1}\right)} \\
=&k\frac{\sum_{j=0}^{n-2}(-1)^{j} \operatorname{sn}\left(\frac{(2j+1)K}{n-1}\right)}{\sum_{j=0}^{n-2}(-1)^{j} / \operatorname{sn}\left(\frac{(2j+1)K}{n-1}\right)}.
\end{aligned}$$

Letting

$$f_1(z) = k^{n-2} \prod_{m=0}^{n-2} \operatorname{sn}^2\left(z + \frac{(2m+1)K}{n-1}\right) \sum_{j=0}^{n-2}(-1)^{j} / \operatorname{sn}\left(z + \frac{(2j+1)K}{n-1}\right),$$

$$f_2(z) = \sum_{j=0}^{n-2}(-1)^{j} \operatorname{sn}\left(z + \frac{(2j+1)K}{n-1}\right),$$

we prove $f_1(z) = f_2(z)$ and equation (2) is just $f_1(0) = f_2(0)$. We see that $f_1(z)$ and $f_2(z)$ are two doubly periodic functions, since

$$\operatorname{sn}(z^* - z) = -\operatorname{sn}(z^* + z),\ \forall z \in \mathbb{C},$$

or

$$1/\operatorname{sn}(z^* - z) = -1/\operatorname{sn}(z^* + z),\ \forall z \in \mathbb{C}.$$

The double poles of $f_1(z)$ are removable, for example,

$$\lim_{z \to iK' - K} (z + K - iK')^2 f_1(z) = 0.$$

Thus $f_1(z)$ and $f_2(z)$ only have the same simple poles as sn. Since

$$\operatorname{sn}(z + iK') = \frac{1}{k\ \operatorname{sn}(z)},$$

the residues of $f_1(z)$ and $f_2(z)$ are

$$\lim_{z \to z^*} (z - z^*) f_1(z) = \pm \frac{\displaystyle\sum_{j=0}^{n-2} (-1)^j \operatorname{cn}\left(\frac{2jK}{n-1}\right) \operatorname{dn}\left(\frac{2jK}{n-1}\right)}{\displaystyle k^{n-3} \prod_{j=1}^{n-2} \operatorname{sn}^2\left(\frac{2jK}{n-1}\right)},$$

and

$$\lim_{z \to z^*} (z - z^*) f_2(z) = \pm \frac{1}{k},$$

where $z^*$ is a simple pole of $f_1(z)$ or $f_2(z)$. Defining

$$C_1 = \frac{\displaystyle\sum_{j=0}^{n-2} (-1)^j \operatorname{cn}\left(\frac{2jK}{n-1}\right) \operatorname{dn}\left(\frac{2jK}{n-1}\right)}{\displaystyle k^{n-2} \prod_{j=1}^{n-2} \operatorname{sn}^2\left(\frac{2jK}{n-1}\right)},$$

we have that

$$f_1(z) - C_1 f_2(z) = C_2,$$

and since $\operatorname{sn}(z)$ is odd and periodic,

$$\int_{-2K}^{2K} \operatorname{sn}(z+r)dz = 0,\ \forall r \in \mathbb{R},$$

$$\int_{-2K}^{2K} f_1(z)dz = 0,$$

$$\int_{-2K}^{2K} f_2(z)dz = 0,$$

and

$$\int_{-2K}^{2K} (f_1(z) - C_1 f_2(z))dz = 0 = 4KC_2,$$

which gives $C_2 = 0$.

Further, from $f_1(0) - C_1 f_2(0) = 0$ and $f_1(iK') - C_1 f_2(iK') = 0$, we get

$$\begin{cases} C_1 = \dfrac{f_1(0)}{f_2(0)} = \dfrac{k^{n-2} \displaystyle\prod_{m=0}^{n-2} \operatorname{sn}^2\left(\frac{(2m+1)K}{n-1}\right) \sum_{j=0}^{n-2} (-1)^j / \operatorname{sn}\left(\frac{(2j+1)K}{n-1}\right)}{\displaystyle\sum_{j=0}^{n-2} (-1)^j \operatorname{sn}\left(\frac{(2j+1)K}{n-1}\right)} \\ C_1 = \dfrac{f_1(iK')}{f_2(iK')} = \dfrac{\displaystyle\sum_{j=0}^{n-2} (-1)^j \operatorname{sn}\left(\frac{(2j+1)K}{n-1}\right)}{k^{n-2} \displaystyle\prod_{m=0}^{n-2} \operatorname{sn}^2\left(\frac{(2m+1)K}{n-1}\right) \sum_{j=0}^{n-2} (-1)^j / \operatorname{sn}\left(\frac{(2j+1)K}{n-1}\right)} = \dfrac{1}{C_1} \end{cases},$$

thus, $C_1 = 1$. In sum, $f_1(z) = f_2(z)$.

We not only prove equation (3.4), but also get two equalities

$$f_1(z) = f_2(z),$$

and

$$\sum_{j=0}^{n-2} (-1)^j \operatorname{cn}\left(\frac{2jK}{n-1}\right) \operatorname{dn}\left(\frac{2jK}{n-1}\right) = k^{n-2} \prod_{j=1}^{n-2} \operatorname{sn}^2\left(\frac{2jK}{n-1}\right).$$

□

When $n$ is even, we prove

$$\Omega_{1,j+1}\Omega_{1,n+1-k}\Omega'_{1,n}\Omega'_{k,j+1} = \Omega'_{1,j+1}\Omega'_{1,n+1-k}\Omega_{1,n}\Omega_{k,j+1}. \tag{3.5}$$

Similarly, when $n$ is odd, we prove

$$h = (-1)^{\frac{n-1}{2}} \frac{(\Omega^2_{1,i+1} + \Omega^2_{n,i+1}) - ((-1)^{\frac{n-1}{2}}\Omega'_{1,i+1} - \Omega'_{n,i+1})^2}{\Omega_{1,i+1}\Omega_{n,i+1}}, \; i = 1, 3, \cdots, n-2, \tag{3.6}$$

where $h$ does not depend on $i$.

When $n$ is odd, we prove $k(q^{n-1})$ is the smaller root of the quadratic equation

$$x^2 - hx + 1 = 0. \tag{3.7}$$

We proved these equations for the first few integers $n = 2, 3, 4, 5, 6$, and we numerically tested a lot of cases (different $n$ and $q = b^2$) in Mathematica.

**Lemma 5.** *Given a matrix $A \in \mathbb{C}^{m\times n}$, if $\tilde{A}$ is a submatrix of $A$ with selected row indices $\{r_1, r_2, \cdots, r_p\}$ and selected column indices $\{c_1, c_2, \cdots, c_q\}$, and the entries of rows $\{r_1, r_2, \cdots, r_p\}$ and columns $\{c_1, c_2, \cdots, c_q\}$ are 0's except for those in $\tilde{A}$, then the singular values of $\tilde{A}$ are a subset of the singular values of $A$.*

*Proof.* Taking the standard basis $\{e_j\}$, we have

$$\tilde{A} = P_1 A P_2, \; P_1 = \begin{bmatrix} e^T_{r_1} \\ e^T_{r_2} \\ \vdots \\ e^T_{r_p} \end{bmatrix}, \; P_2 = \begin{bmatrix} e_{c_1}, e_{c_2}, \cdots, e^T_{c_q} \end{bmatrix}.$$

We can add the missing indices to extend $P_1$ (adding rows) and $P_2$ (adding columns) to be two permutation matrices $\tilde{P}_1$ and $\tilde{P}_2$. Thus,

$$\tilde{P}_1 A \tilde{P}_2 = \tilde{P}_1 U_A \Sigma_A V_A^* \tilde{P}_2 = \begin{bmatrix} \tilde{A} & 0_{p\times(n-q)} \\ 0_{(m-p)\times q} & X \end{bmatrix} = \begin{bmatrix} U_{\tilde{A}} & 0 \\ 0 & U_X \end{bmatrix} \begin{bmatrix} \Sigma_{\tilde{A}} & 0 \\ 0 & \Sigma_X \end{bmatrix} \begin{bmatrix} V^*_{\tilde{A}} & 0 \\ 0 & V^*_X \end{bmatrix}.$$

□

Now we have the conformal mapping of $A$, and we can choose a diagonal matrix $T$ such that $T\phi(A)T^{-1}$ is a contraction,

$$T = \begin{bmatrix} 1 & & & \\ & T_1 & & \\ & & \ddots & \\ & & & T_{n-1} \end{bmatrix}.$$

Letting $t_j = T_j b^{\frac{j-1}{2}}$, when $n$ is even, we take

$$\begin{cases} t_1^2 = \dfrac{\Omega_{1,2}}{\Omega'_{1,2}} \\ t_2^2 = t_{n-1}^2 / t_{n-3}^2 \\ t_3^2 = \dfrac{\Omega_{1,4}}{\Omega'_{1,4}} \\ \quad \vdots \\ t_{n-3}^2 = \dfrac{\Omega_{1,n-2}}{\Omega'_{1,n-2}} \\ t_{n-2}^2 = t_{n-1}^2 / t_1^2 \\ t_{n-1}^2 = \dfrac{\Omega_{1,n}}{\Omega'_{1,n}} = k \dfrac{1 + \sum_{j=2}^{n-1} (-1)^{j-1} \operatorname{sn}\left( K(1 - \dfrac{2(j-1)}{n-1}) \right)}{1 + \sum_{j=2}^{n-1} (-1)^{j-1} / \operatorname{sn}\left( K(1 - \dfrac{2(j-1)}{n-1}) \right)} = k(b^{2n-2}) \end{cases}, \tag{3.8}$$

we get that

$$T\phi(A)T^{-1} = \begin{bmatrix} 1 & & & \\ & t_1 & & \\ & & \ddots & \\ & & & t_{n-1} \end{bmatrix} M \begin{bmatrix} 1 & & & \\ & t_1 & & \\ & & \ddots & \\ & & & t_{n-1} \end{bmatrix}^{-1},$$

where

$$M = [\Omega_{ij}] = \sum_{l=1}^{n} S_{il} \psi_l S_{lj}.$$

$T\phi(A)T^{-1}$ is symmetric along the secondary diagonal, i.e., $N = T\phi(A)T^{-1}P$ is symmetric, where

$$P = \begin{bmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

From equation (3.4) and (3.5) we can check

$$(N+I)(N-I)(N+\prod_{j=1}^{n}\psi_j I) = 0, \tag{3.9}$$

that is, the singular values of $N$ and $T\phi(A)T^{-1}$ are $n-1$ 1's and $\prod_{j=1}^{n}\psi_j < 1$,

$$||T\phi(A)T^{-1}||_2 = 1.$$

We can also use lemma (5) to solve a simpler version of equation (3.9). Since $\phi(A)$ has all 0's along its main diagonal and every other diagonal, we can divide $N$ into 2 submatrices. For example, when $n = 4$, we have

$$N = \begin{bmatrix} 0 & M_{12}\frac{1}{t_1} & 0 & M_{14}\frac{1}{t_3} \\ M_{21}\frac{t_1}{1} & 0 & M_{23}\frac{t_1}{t_2} & 0 \\ 0 & M_{32}\frac{t_2}{t_1} & 0 & M_{34}\frac{t_2}{t_3} \\ M_{41}\frac{t_3}{1} & 0 & M_{43}\frac{t_3}{t_2} & 0 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix},$$

and the two submatrices $N_1$ and $N_2$ are

$$N_1 = \begin{bmatrix} M_{14}\frac{1}{t_3} & M_{12}\frac{1}{t_1} \\ M_{34}\frac{t_2}{t_3} & M_{32}\frac{t_2}{t_1} \end{bmatrix},$$

$$N_2 = \begin{bmatrix} M_{23}\frac{t_1}{t_2} & M_{21}\frac{t_1}{1} \\ M_{43}\frac{t_3}{t_2} & M_{41}\frac{t_3}{1} \end{bmatrix}.$$

By checking

$$N_1^T N_1 = I,$$

and

$$(N_2+I)(N_2-I)(N_2-\prod_{j=1}^{n}\psi_j I)=0,$$

we also get

$$||T\phi(A)T^{-1}||_2=1.$$

Further,

$$1\le ||T||_2||T^{-1}||_2=\sqrt{\frac{k(b^{2n-2})}{b^{n-1}}}\le 2,\ 0\le b\le 1.$$

When $n$ is odd, take

$$\begin{cases} t_1^2=t_{n-1}^2/t_{n-2}^2 \\ t_2^2=\dfrac{t_{n-1}^2\Omega_{n,2}-(-1)^{\frac{n-1}{2}}\Omega_{1,2}}{\Omega'_{n,2}-(-1)^{\frac{n-1}{2}}\Omega'_{1,2}} \\ t_3^2=t_{n-1}^2/t_{n-4}^2 \\ \vdots \\ t_{n-3}^2=t_{n-1}^2/t_2^2 \\ t_{n-2}^2=\dfrac{t_{n-1}^2\Omega_{n,n-1}-(-1)^{\frac{n-1}{2}}\Omega_{1,n-1}}{\Omega'_{n,n-1}-(-1)^{\frac{n-1}{2}}\Omega'_{1,n-1}} \\ t_{n-1}^2=k(b^{2n-2}) \end{cases}, \tag{3.10}$$

and we can check that equation (3.9) is also satisfied. Since $\psi_{\frac{n+1}{2}}=0$, we have

$$(N+I)(N-I)N=0.$$

In sum, when $n$ is a positive integer

$$||T\phi(A)T^{-1}||_2=1,$$

and

$$1\le ||T||_2||T^{-1}||_2=\sqrt{\frac{k(b^{2n-2})}{b^{n-1}}}\le 2,\ 0<b\le 1.$$

### *3.3 Find one extremal mapping*

Find one mapping $\hat{B}$ (which will be proved to be the unique extremal Blaschke product in the next section) such that

$$||\hat{B}(\phi(A))||_2 = ||T||_2||T^{-1}||_2 = \sqrt{\frac{k_n}{b^{n-1}}},$$

so the upper bound should be exactly $\sqrt{\frac{k_n}{b^{n-1}}}$. It might be hard to solve nonlinear equations to obtain the roots of a Blaschke product as we did in the previous section. Instead, by the Nevanlinna–Pick theorem, we check the Pick matrix to prove the existence and uniqueness of $\hat{B}$. Suppose (without assuming that $\hat{B}$ is a Blaschke product) that $\hat{B}$ has the form

$$\hat{B}(\phi(A)) = \sum_{j=0}^{n-1} p_j \phi(A)^j.$$

We have

$$||T\hat{B}\left(\phi(A)\right)T^{-1}||_2 = 1,$$

and $e_1$ and $e_n$ must be the first left and right singular vectors of $\hat{B}(\phi(A))$ (or $T\hat{B}(\phi(A))T^{-1}$) respectively. Hence, $\hat{B}(\phi(A))$ takes the form

$$\hat{B}(\phi(A)) = \begin{bmatrix} 0 & \cdots & 0 & \sqrt{\frac{k_n}{b^{n-1}}} \\ * & * & * & 0 \\ * & * & * & \vdots \\ * & * & * & 0 \end{bmatrix},$$

and by taking the first row or last column we can solve a linear system to get the $p_j$'s. Let

$$Z = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{bmatrix} = \sum_{j=0}^{n-1} p_j \Psi^j.$$

We know

$$\hat{B}(\phi(A)) = DSZSD^{-1},$$

or

$$SZS = \begin{bmatrix} 0 & \cdots & 0 & \sqrt{k_n} \\ * & * & * & 0 \\ * & * & * & \vdots \\ * & * & * & 0 \end{bmatrix}.$$

Taking the first row or last column on both sides, we get the linear system

$$S \begin{bmatrix} S_{11} & & & & & \\ & S_{12} & & & & \\ & & \ddots & & & \\ & & & S_{1j} & & \\ & & & & \ddots & \\ & & & & & S_{1n} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_j \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ \sqrt{k_n} \end{bmatrix},$$

$$\begin{bmatrix} S_{11} & & & & & \\ & S_{12} & & & & \\ & & \ddots & & & \\ & & & S_{1j} & & \\ & & & & \ddots & \\ & & & & & S_{1n} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_j \\ \vdots \\ z_n \end{bmatrix} = S \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ \sqrt{k_n} \end{bmatrix} = \sqrt{k_n} \begin{bmatrix} S_{1n} \\ S_{2n} \\ \vdots \\ S_{jn} \\ \vdots \\ S_{nn} \end{bmatrix},$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_j \\ \vdots \\ z_n \end{bmatrix} = \sqrt{k_n} \begin{bmatrix} 1 \\ -1 \\ \vdots \\ (-1)^{j-1} \\ \vdots \\ (-1)^{n-1} \end{bmatrix}.$$

Thus,

$$\hat{B}(\phi(A)) = DSZSD^{-1} = \sqrt{\frac{k_n}{b^{n-1}}}\begin{bmatrix} & & & & & & 1 \\ & & & & & b & \\ & & & & b^2 & & \\ & & & \iddots & & & \\ & & b^{n-3} & & & & \\ & b^{n-2} & & & & & \\ b^{n-1} & & & & & & \end{bmatrix}_{n\times n}, b \in (0,1]. \quad (3.11)$$

In addition, we can find $p_j$'s by solving (there is no need to solve for the $p_j$'s here)

$$Z = \begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_n \end{bmatrix} = \sum_{j=0}^{n-1} p_j \Psi^j,$$

$$V\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix} = \sqrt{k_n}\begin{bmatrix} 1 \\ -1 \\ \vdots \\ (-1)^{n-1} \end{bmatrix}, \quad V = \begin{bmatrix} 1 & \psi_1 & \cdots & \psi_1^{n-1} \\ 1 & \psi_2 & \cdots & \psi_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \psi_n & \cdots & \psi_n^{n-1} \end{bmatrix}. \quad (3.12)$$

Since all $\psi_j$'s are distinct, the Vandermonde matrix $V$ is nonsingular and we expect a unique solution of $p_j$'s.

### *3.4 Uniqueness of the extremal Blaschke product*

From part 3, we know that the mapping $\hat{B}$ maps the eigenvalues of $\phi(A)$ to the eigenvalues of $\hat{B}(\phi(A))$, i.e., $\pm\sqrt{k_n}$. By the Nevanlinna–Pick interpolation theorem, we check the Pick matrix $N_P$(real and symmetric)

$$N_P = \left[\frac{1-\overline{z_j}z_i}{1-\overline{\psi_j}\psi_i}\right]_{i,j=1}^{n} = \left[\frac{1-z_j z_i}{1-\psi_j\psi_i}\right]_{i,j=1}^{n}. \quad (3.13)$$

If the matrix $N_P$ is semi-positive definite with rank $n-1$, then $\hat{B}$ is the unique extremal Blaschke product of degree $n-1$. We will check this matrix for $n = 2, 3, 4, 5, 6$ in the next chapter by analytically determining its eigenvalues or doing Cholesky decomposition. We also performed numerical tests for many integers $n$ in Mathematica.

In addition, we can get this Blaschke product by the method in paper [12]. The matrix $C_n$ is real,

$$C_n = \sqrt{k_n} V^{-1} \begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & \ddots & & & \\ & & & (-1)^{j-1} & & \\ & & & & \ddots & \\ & & & & & (-1)^{n-1} \end{bmatrix} VP, \tag{3.14}$$

and the matrix $M_{2n}$ is also real

$$M_{2n} = \begin{bmatrix} C_n & 0 \\ 0 & -C_n \end{bmatrix},$$

thus we can get the real coefficients of the extremal Blachke product $\{\beta_0, \beta_1, \cdots, \beta_{n-1}\}$, which is the eigenvector of $C_n$ corresponding to the eigenvalue 1,

$$C_n \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix} = 1 \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix},$$

$$\hat{B}(z) = \frac{\beta_0 + \beta_1 z + \cdots + \beta_{n-1} z^{n-1}}{\beta_0 z^{n-1} + \beta_1 z^{n-2} + \cdots + \beta_{n-1}}.$$

### 3.5 *Another look at the result matrix*

Given an $n \times n$ per-diagonal matrix

$$M = \begin{bmatrix} & & & & & 1 \\ & & & & b & \\ & & & b^2 & & \\ & & \iddots & & & \\ & & b^{n-3} & & & \\ & b^{n-2} & & & & \\ b^{n-1} & & & & & \end{bmatrix}_{n\times n}, b \in (0,1].$$

We have

$$W(M) = \{(\bar{x}_1 x_n + b^{n-1} x_1 \bar{x}_n) + (b\bar{x}_2 x_{n-1} + b^{n-2} x_2 \bar{x}_{n-1}) + \cdots | \quad ||x||_2 = 1\},$$

$$W\left(\begin{bmatrix} 0 & 1 \\ b^{n-1} & 0 \end{bmatrix}\right) = \{(\bar{x}_1 x_2 + b^{n-1} x_1 \bar{x}_2)| \quad ||x||_2 = 1\}.$$

On the one hand, we know

$$W\left(\begin{bmatrix} 0 & 1 \\ b^{n-1} & 0 \end{bmatrix}\right) \subseteq W(M),$$

by taking $x_2 = x_3 = \cdots = x_{n-1} = 0$ in $W(M)$.

On the other hand, since $0 \le b \le 1$,

$$\cdots \subset W\left(\begin{bmatrix} 0 & b^2 \\ b^{n-3} & 0 \end{bmatrix}\right) \subset W\left(\begin{bmatrix} 0 & b \\ b^{n-2} & 0 \end{bmatrix}\right) \subset W\left(\begin{bmatrix} 0 & 1 \\ b^{n-1} & 0 \end{bmatrix}\right),$$

we have

$$W(M) = \{(\bar{x}_1 x_n + b^{n-1} x_1 \bar{x}_n)| \quad ||x||_2 = 1\}.$$

Thus,

$$W\left(\begin{bmatrix} 0 & 1 \\ b^{n-1} & 0 \end{bmatrix}\right) = W(M).$$

$W(M)$ is an ellipse with two axes $1 \pm b^{n-1}$. The eigenvalues of $M$ are $\pm b^{\frac{n-1}{2}}$. The conformal mapping for $M$ is written as

$$\phi(z) = \sqrt{k_n}\ \operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}(b^{-\frac{n-1}{2}}z)\right),$$

$$\phi(M) = \sqrt{k_n}b^{-\frac{n-1}{2}}M,$$

where $k_n = k(b^{2n-2})$. Here the parameter is $b^{n-1}$ instead of $b$. Taking

$$T = \begin{bmatrix} t_0 & & & & & \\ & t_1 & & & & \\ & & t_2 & & & \\ & & & \ddots & & \\ & & & & t_{n-2} & \\ & & & & & t_{n-1} \end{bmatrix},$$

$$\phi(M) = T^{-1}\left[T\phi(M)T^{-1}\right]T,$$

we choose $t_j$ as

$$t_{n-1-j} = \max\left(1, k^{\frac{1}{2}}b^{-\frac{n-1}{2}}b^j\right),$$

$$t_j = 1,$$

where $j = 0, 1, \cdots, \left\lfloor \frac{n-1}{2} \right\rfloor$. Hence,

$$1 = t_0 \le t_1 \le \cdots \le t_{n-2} < t_{n-1} = k_n^{\frac{1}{2}}b^{-\frac{n-1}{2}},$$

and

$$||T\phi(M)T^{-1}||_2 = 1.$$

For any Blaschke product,

$$B(\phi(M)) = T^{-1}B(T\phi(M)T^{-1})T,$$

$$||B(\phi(M))||_2 \le ||T^{-1}||_2||B(T\phi(M)T^{-1})||_2||T||_2 \le \kappa(T) = k_n^{\frac{1}{2}}b^{-\frac{n-1}{2}} = \sqrt{\frac{k_n}{b^{n-1}}}.$$

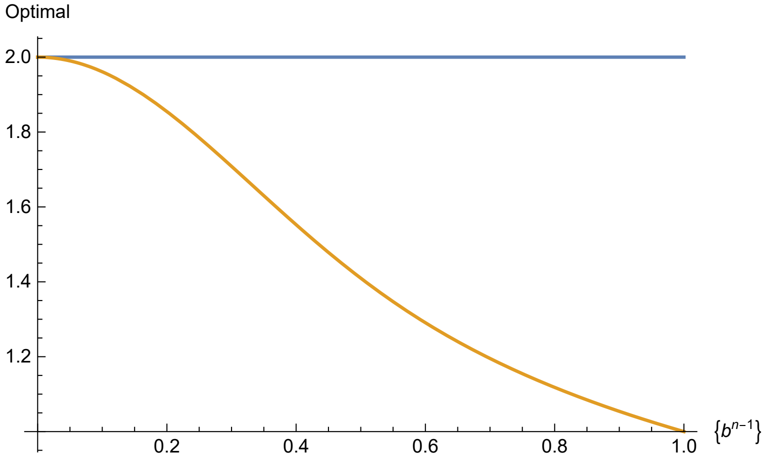


Figure 3.5.1: $||\hat{B} \circ \phi(M)||_2 = \sqrt{\frac{k_n}{b^{n-1}}}$ vs $b^{n-1}$

By taking

$$\hat{B}(z) = z,$$

we ensure the equality holds, i.e.,

$$||\hat{B}(\phi(M))||_2 = \sqrt{\frac{k_n}{b^{n-1}}} \leq 2.$$

See figure 3.5.1, which illustrates that the Crouzeix's Conjecture holds, and we get 2 when $b = 0$, which is the result using the Crabb-Choi-Crouzeix matrix. We see the upper bound decreases as $b$ increases, and if $b \neq 0$, the upper bound increases as $n$ increases. Further, the Blaschke product is unique by checking the Pick matrix (all ones)

$$P = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n \times n} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

It is a rank 1 positive semi-definite matrix with eigenvalues/singular values $n$ and $n-1$ 0's, so we can expect a unique Blaschke product of degree 1, as above $\hat{B}(z) = z$.

# Chapter 4

# EXAMPLES AND NUMERICAL RESULTS

We will discuss several examples for specific integers $n$ mentioned in the previous chapter and compare the results with numerical tests. The numerical methods include several steps: 1. Use Chebfun [10] to find the boundary of the numerical range of matrix $A$; 2. Find the conformal mapping $\phi$ from $W(A)$ to the unit disk $\mathbb{D}$ using the Szegö kernel or Kerzmann-Stein integral equation [23]; 3. Calculate a Cauchy integral to find $\phi(A)$; 4. Optimize the roots $\alpha_j$'s of a Blaschke product to maximize $||B(\phi(A))||_2$.

## *4.1 Numerical Algorithm*

### *4.1.1 Chebfun*

We know the numerical range of $A$ is a closed convex set in the complex plane $\mathbb{C}$ [16], and we use the method in [22] and Chebfun package [10] to get the boundary of the numerical range of $A$ up to the limits of machine precision in Matlab. This boundary $\sigma(t)$ consists of values $v_t^* A v_t$, where $v_t$ is a normalized eigenvector of $(e^{it}A + e^{-it}A^*)/2$ corresponding to the largest eigenvalue of this Hermitian matrix. If no eigenvalue of $A$ lies on $\partial W(A)$, then the boundary of $W(A)$ is smooth, but if $A$ has a *normal* eigenvalue (one whose eigenspace is orthogonal to the eigenspaces of all eigenvectors corresponding to different eigenvalues), then that eigenvalue lies on $\partial W(A)$ and creates a corner point. Since Chebfun attempts to represent this boundary by a linear combination of smooth Chebyshev polynomials, such a corner point can cause problems. Consider, for example, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

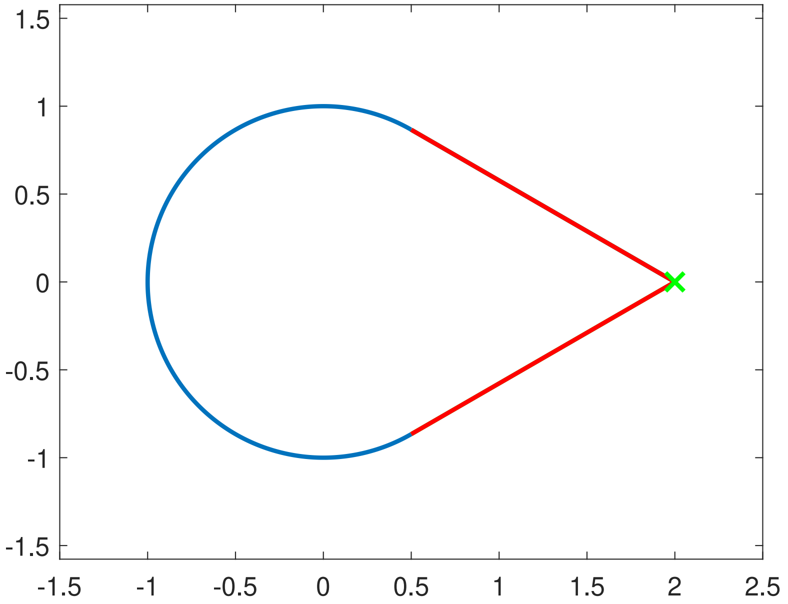


Figure 4.1.1: Numerical range of $A$

whose numerical range is plotted in Figure (4.1.1). For such matrices, one can find the eigenvector(s) $v$ corresponding to this normal eigenvalue $\lambda \in \partial W(A)$ (Chapter 1 1.6.5 [21]) and construct a unitary matrix

$$U = [v, q_1, \cdots, q_{n-1}],$$

such that

$$U^* A U = \begin{bmatrix} \lambda & 0 \\ 0 & A_1 \end{bmatrix}.$$

Since $\|f(A)\|_2 = \max\{\|f(A_1)\|_2, |f(\lambda)|\}$, if one works instead with the submatrix $A_1$ and shows that $\|f(A_1)\|_2 \leq C \sup_{z \in W(A_1)} |f(z)|$, for some $C \geq 1$, then it follows that

$$\|f(A)\|_2 \leq C \sup_{z \in W(A_1) \cup \lambda} |f(z)| \leq C \sup_{z \in W(A)} |f(z)|.$$

Thus, $C$ is an upper bound on the Crouzeix constant for $A$. We can repeat this step again and again until the matrix we consider has no normal eigenvalues, and then the boundary of the numerical range is smooth for the numerical test.

### *4.1.2 Szegö kernel*

Suppose $\sigma(t) = \partial W(A)$, where now we parameterize by arc length $t$ (counter clockwise). Define

$$\dot{\sigma}(t) = \frac{d\sigma(t)}{dt},$$
$$\dot{\gamma}(\sigma) = \frac{\dot{\sigma}(t)}{|\dot{\sigma}(t)|},$$
$$\phi'(z) = \frac{d\phi(z)}{dz},\ z \in W(A).$$

Then $\dot{\gamma}(\sigma)$ is the unit tangent vector of $\partial W(A)$ at point $\sigma(t)$. From [23], we know the conformal mapping $\phi(z)$ satisfies the integral equation

$$\sqrt{\phi'(z)} + \int_0^L k(z, w(s))\sqrt{\phi'(w)}ds = \frac{1}{2\pi i}\overline{\left(\frac{\dot{\gamma}(z)}{a-z}\right)}, \quad z, w \in \sigma(t),\ 0 \leq s \leq L, \tag{4.1}$$

where $L$ is the length of $\partial W(A)$, $a$ is the point mapped to the origin (i.e., $\phi(a) = 0$ and $\phi'(a) > 0$), and the kernel is

$$k(z,w) = \begin{cases} 0, & \text{if w=z,} \\ -\frac{1}{2\pi i}\left(\overline{\left(\frac{\dot{\gamma}(z)}{z-w}\right)} + \frac{\dot{\gamma}(w)}{w-z}\right), & \text{if } w \neq z. \end{cases}$$

We solve the integral equation (4.1) for $\phi'(z)$ and then we get $\phi(z)$ from

$$\phi(z) = -\frac{i\dot{\gamma}(z)\phi'(z)}{|\phi'(z)|}.$$

We follow the method in [23] to solve this integral equation. We choose $N$ equidistant collocation points

$$s_j = \frac{j}{N}L,\ j = 0, 1, \cdots, N,$$

and use the trapezoidal rule to discretize the integral equation (4.1) to get an $N$ by $N$ system,

$$(I + B)x = y,$$

where $I$ is the identity matrix,

$$B_{jk} = k\left(\sigma(s_j), \sigma(s_k)\right)\Delta s = \frac{L}{N} k\left(\sigma(s_j), \sigma(s_k)\right),$$

and

$$y_j = \frac{1}{2\pi i} \overline{\left(\frac{\gamma(\sigma(s_j))}{a - \sigma(s_j)}\right)}.$$

Since

$$||I + B||_2 \leq ||I||_2 + ||B||_2 \leq 1 + ||B||_F,$$

and $||B||_F$ is small in most cases, this system is well-conditioned and we use Gaussian elimination to get $x$ (i.e., $\sqrt{\phi'(z)}$).

Finally, one choice of $a$ is the average of $\partial W(A)$,

$$a = \frac{\int_0^L \sigma(s) ds}{L}.$$

For the class of matrices in the previous part, since $W(A)$ is an ellipse centered at the origin, this results in $a = 0$.

### *4.1.3 Cauchy Integral*

Now that we have the numerical conformal mapping of $W(A)$, i.e., $\phi(z)$, there are several ways to get the conformal mapping of the matrix $\phi(A)$. One way is to find the Jordan normal form of matrix $A$,

$$A = P_A J P_A^{-1},$$

then we can get $\phi(J)$ from

$$\phi^{(m)}(\lambda_j) = \frac{m!}{2\pi i} \oint_{W(A)} \frac{\phi(z)}{(z - \lambda_j)^{m+1}} dz,$$

and thus

$$\phi(A) = P_A \phi(J) P_A^{-1}.$$

Unfortunately, one cannot reliably find the Jordan form numerically when there are nontrivial Jordan blocks or when the condition number of $P_A$ is huge.

Another way is to calculate $\phi(A)$ directly from the Cauchy integral,

$$\phi(A) = \frac{1}{2\pi i} \oint_{W(A)} \frac{\phi(z)}{zI - A} dz.$$

However, when one eigenvalue of the matrix $A$ is on the boundary of the numerical range of $A$, $zI - A$ is singular. We work instead with the matrix $A_1$ discussed in section 4.1.1 to avoid this difficulty.

### *4.1.4 Optimal Blaschke Product*

To get the optimal Blaschke product

$$B(z) = \exp(i\gamma) \prod_{j=1}^{n-1} \frac{z - \alpha_j}{1 - \bar{\alpha_j} z}, \; |\alpha_j| \leq 1,$$

that maximizes $||B(\phi(A))||_2$, we can rewrite the parameters $\alpha_j$, $j = 1, \ldots, n-1$, as

$$\alpha_j = r_j \exp(i\theta_j),$$

where

$$0 \leq r_j \leq 1, \; 0 \leq \theta_j < 2\pi, \; j = 1, 2 \cdots, n-1.$$

These $2(n-1)$ parameters $r_j$'s and $\theta_j$'s are used in the Matlab function *fmincon* with lower bound 0 and upper bound 1 and $2\pi$ respectively. Note that $||B(\phi(A))||_2$ is not a convex function of these parameters; thus 100 random inital values are chosen to search for the global maximum point, but in certain cases it remains difficult to get it, as shown below.

Following Chapter 3, we obtain matrix $S$ from equation (3.1), $\phi(A)$ from equation (3.2), $T$ from equations (3.8) or (3.10) ($n$ even/odd), $N_p$ from equation (3.13), $V$ from equation (3.12) and $C_n$ from equation (3.14), and then we check that $||C||_2 = ||T\phi(A)T^{-1}||_2 = 1$ is a contraction and we calculate the coefficients $\beta$ and roots $\alpha_j$'s of the extremal Blaschke product.

### 4.2 Example n=2

When $n = 2$, we get that

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix},$$

$$\phi(A) = \begin{bmatrix} 0 & \sqrt{\frac{k}{b}} \\ \sqrt{kb} & 0 \end{bmatrix}.$$

The eigenvalues of $\phi(A)$ are $\pm\sqrt{k}$. Also

$$T = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\frac{k}{b}} \end{bmatrix},$$

$$C = T\phi(A)T^{-1} = \begin{bmatrix} 0 & 1 \\ \sqrt{k} & 0 \end{bmatrix},$$

the singular values of $C$ are 1 and $\sqrt{k}$. Thus $C$ is a contraction and

$$\hat{B}\left(\phi(A)\right) = \begin{bmatrix} 0 & \sqrt{\frac{k}{b}} \\ \sqrt{kb} & 0 \end{bmatrix} = \phi(A).$$

The Pick matrix is

$$N_P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

$N_p$ is positive semidefinite with eigenvalues 2 and 0. Therefore the optimal Blaschke product of degree 1 is unique. Let

$$V = \begin{bmatrix} 1 & \sqrt{k} \\ 1 & -\sqrt{k} \end{bmatrix},$$

$$C_2 = \sqrt{k}V^{-1} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} VP = \begin{bmatrix} k & \\ & 1 \end{bmatrix},$$

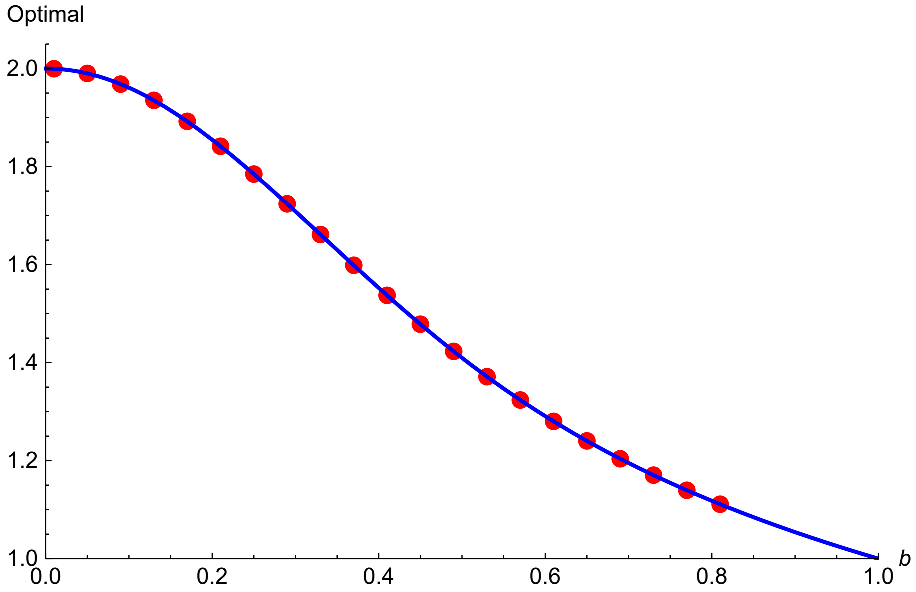


Figure 4.2.1: The solid blue line is the analytic result and red dots represent the numerical result

the eigenvector of $C_2$ corresponding to eigenvalue $\lambda = 1$ is

$$\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus,

$$\hat{B}(z) = z.$$

All the results are the same as in section (2.3.1). In addition, we can compare our analytic results with the numerical tests in figure (4.2.1) and table (4.2).

Table 4.2.1: Analytic and numerical $\alpha_1$

| b | $\alpha_1 = 0$ | Numerical $\alpha_1$ |
|---|---|---|
| 0.01 | 0 | -9.067e-05 + 5.775e-06i |
| 0.05 | 0 | -4.380e-05 - 1.587e-06i |
| 0.09 | 0 | -6.721e-05 + 1.214e-05i |
| 0.13 | 0 | -4.630e-05 - 4.931e-05i |
| 0.17 | 0 | -8.857e-05 - 1.591e-06i |
| 0.21 | 0 | -7.422e-05 + 1.537e-07i |
| 0.25 | 0 | -1.256e-04 - 1.188e-05i |
| 0.29 | 0 | -4.099e-06 - 3.197e-06i |
| 0.33 | 0 | -3.423e-06 - 3.284e-06i |
| 0.37 | 0 | 7.513e-06 - 2.014e-06i |
| 0.41 | 0 | -1.254e-07 + 2.484e-06i |
| 0.45 | 0 | 1.351e-05 - 1.462e-06i |
| 0.49 | 0 | 3.677e-06 + 1.011e-06i |
| 0.53 | 0 | 1.626e-04 - 5.533e-06i |
| 0.57 | 0 | 2.888e-05 - 5.881e-07i |
| 0.61 | 0 | 2.435e-04 + 0.0001097i |
| 0.65 | 0 | 8.711e-05 + 3.137e-07i |
| 0.69 | 0 | 2.777e-04 - 3.495e-07i |
| 0.73 | 0 | 3.965e-04 - 6.768e-07i |
| 0.77 | 0 | 3.723e-04 - 1.065e-06i |
| 0.81 | 0 | 4.303e-04 - 1.217e-06i |

### *4.3 Example n=3*

When $n = 3$, we get that

$$S = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix},$$

$$\phi(A) = \begin{bmatrix} 0 & \sqrt{\frac{k}{2b}} & 0 \\ \sqrt{\frac{kb}{2}} & 0 & \sqrt{\frac{k}{2b}} \\ 0 & \sqrt{\frac{kb}{2}} & 0 \end{bmatrix},$$

the eigenvalues of $\phi(A)$ are 0 and $\pm\sqrt{k}$.

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{\sqrt{k_3}}{b}} & 0 \\ 0 & 0 & \frac{\sqrt{k_3}}{b} \end{bmatrix},$$

$$C = T\phi(A)T^{-1} = \begin{bmatrix} 0 & \sqrt{\frac{k}{2\sqrt{k_3}}} & 0 \\ \sqrt{\frac{k\sqrt{k_3}}{2}} & 0 & \sqrt{\frac{k}{2\sqrt{k_3}}} \\ 0 & \sqrt{\frac{k\sqrt{k_3}}{2}} & 0 \end{bmatrix},$$

where

$$k_3 = \frac{k^2}{(1+\sqrt{1-k^2})^2}.$$

The singular values of the matrix $C$ are two 1's and one 0, thus $C$ is a contraction.

$$\hat{B}\left(\phi(A)\right) = \begin{bmatrix} 0 & 0 & \dfrac{k}{(1+\sqrt{1-k^2})b} \\ 0 & \dfrac{k}{1+\sqrt{1-k^2}} & 0 \\ \dfrac{kb}{(1+\sqrt{1-k^2})} & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & \sqrt{\dfrac{k_3}{b^2}} \\ 0 & \sqrt{k_3} & 0 \\ \sqrt{k_3 b^2} & 0 & 0 \end{bmatrix}.$$

the eigenvalues of $\hat{B}\left(\phi(A)\right)$ are

$$\pm\sqrt{k_3} = \pm\frac{k}{(1+\sqrt{1-k^2})}.$$

Hence, the Pick matrix is

$$N_P = \frac{2}{1+\sqrt{1-k^2}} \begin{bmatrix} \sqrt{1-k^2} & 1 & 1 \\ 1 & \dfrac{\sqrt{1-k^2}}{1-k} & \dfrac{\sqrt{1-k^2}}{1+k} \\ 1 & \dfrac{\sqrt{1-k^2}}{1+k} & \dfrac{\sqrt{1-k^2}}{1-k} \end{bmatrix},$$

and $N_p$ is positive semidefinite with eigenvalues

$$\begin{cases} \sigma_1 = \lambda_1 = & \dfrac{4}{\sqrt{1-k^2}} - \dfrac{6}{1+\sqrt{1-k^2}} + 2 \\ \sigma_2 = \lambda_2 = & \dfrac{4k}{\sqrt{1-k^2}} - \dfrac{4k}{1+\sqrt{1-k^2}} \\ \sigma_3 = \lambda_3 = & 0 \end{cases}.$$

Thus, the optimal Blaschke product of degree 2 is unique. Let

$$V = \begin{bmatrix} 1 & \sqrt{k} & k \\ 1 & 0 & 0 \\ 1 & -\sqrt{k} & k \end{bmatrix},$$

$$C_3 = \sqrt{k_3} V^{-1} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} VP = \frac{k}{1+\sqrt{1-k^2}} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & \dfrac{2}{k} \end{bmatrix}.$$

The eigenvector of $C_3$ corresponding to eigenvalue $\lambda = 1$ is

$$\beta = \begin{bmatrix} -\dfrac{k}{1+\sqrt{1-k^2}} \\ 0 \\ 1 \end{bmatrix},$$

thus,

$$\hat{B}(z) = \frac{z^2 - \dfrac{k}{1+\sqrt{1-k^2}}}{1 - \dfrac{k}{1+\sqrt{1-k^2}} z^2}.$$

The roots of this Blaschke product are

$$\alpha_{1,2} = \pm \sqrt[4]{k_3} = \pm \sqrt{\frac{k}{1+\sqrt{1-k^2}}}.$$

All the results are the same as in section (2.3.2). We can compare our analytic results with the numerical tests in figure (4.3.1) and table (4.3).

## 4.4 Example n=4

When $n = 4$, we get that

$$S = \begin{bmatrix} \dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{6}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\sqrt{6}} & -\dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & -\dfrac{1}{\sqrt{6}} \end{bmatrix},$$

Table 4.3.1: Analytic and numerical $\alpha_1$ and $\alpha_2$

| b | $\alpha_1, \alpha_2 = \pm\sqrt[4]{k_3}$ | Numerical $\alpha_1$ and $\alpha_2$ | |
|---|---|---|---|
| 0.01 | ±0.1414 | -0.14142+1.9078e-04i | 0.14142-2.0249e-04i |
| 0.05 | ±0.3162 | -0.3162-7.061e-05i | 0.3162+9.412e-05i |
| 0.09 | ±0.4242 | -0.4242+6.089e-05i | 0.4242-1.009e-04i |
| 0.13 | ±0.5098 | -0.5098+2.861e-05i | 0.5098-5.856e-05i |
| 0.17 | ±0.5826 | -0.5826-1.956e-05i | 0.5826+4.943e-05i |
| 0.21 | ±0.6468 | -0.6468+4.77e-07i | 0.6468-1.437e-06i |
| 0.25 | ±0.7044 | -0.7044-9.232e-06i | 0.7044+3.552e-05i |
| 0.29 | ±0.7563 | -0.7563-1.758e-07i | 0.7563+9.503e-07i |
| 0.33 | ±0.8030 | -0.8030+8.241e-07i | 0.8030-8.818e-06i |
| 0.37 | ±0.8447 | -0.8447+2.423e-06i | 0.8447-1.779e-05i |
| 0.41 | ±0.8813 | -0.8813-9.579e-07i | 0.8813+9.109e-06i |
| 0.45 | ±0.9128 | -0.9128-8.895e-07i | 0.9128+1.076e-05i |
| 0.49 | ±0.9389 | -0.9389-1.497e-06i | 0.9389+2.295e-05i |
| 0.53 | ±0.9598 | -0.9598-8.054e-07i | 0.9598+1.612e-05i |
| 0.57 | ±0.9755 | -0.9755-3.966e-07i | 0.9755+1.06e-05i |
| 0.61 | ±0.9865 | -0.9865-1.692e-07i | 0.9865+6.379e-06i |
| 0.65 | ±0.9935 | -0.9935-5.408e-08i | 0.9935+3.395e-06i |
| 0.69 | ±0.9974 | -0.9974-5.115e-09i | 0.9974+1.695e-06i |
| 0.73 | ±0.9992 | -0.9992+1.986e-08i | 0.9992+2.917e-07i |
| 0.77 | ±0.9998 | -0.9998+2.543e-09i | 0.9998+1.322e-07i |
| 0.81 | ±0.99998 | -0.99998+1.154e-08i | 0.99998+1.3183e-08i |

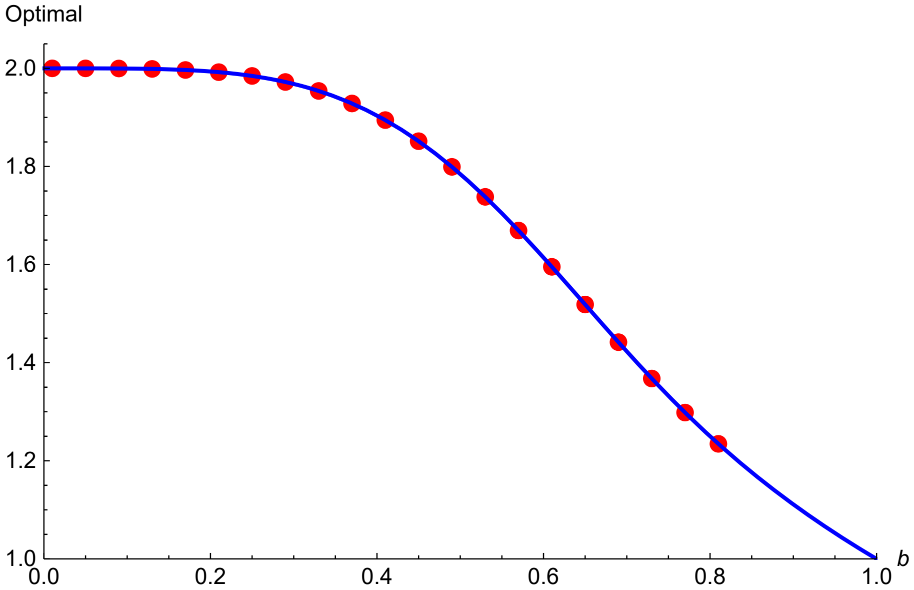


Figure 4.3.1: The solid blue line is the analytic result and red dots represent the numerical result

$$\phi(A)=\sqrt{k}\begin{bmatrix} 0 & \frac{\sqrt{2}}{3}\left(1+\varphi_1\right)b^{-\frac{1}{2}} & 0 & \frac{1-2\varphi_1}{3}b^{-\frac{3}{2}} \\ \frac{\sqrt{2}}{3}\left(1+\varphi_1\right)b^{\frac{1}{2}} & 0 & \frac{2-\varphi_1}{3}b^{-\frac{1}{2}} & 0 \\ 0 & \frac{2-\varphi_1}{3}b^{\frac{1}{2}} & 0 & \frac{\sqrt{2}}{3}\left(1+\varphi_1\right)b^{-\frac{1}{2}} \\ \frac{1-2\varphi_1}{3}b^{\frac{3}{2}} & 0 & \frac{\sqrt{2}}{3}\left(1+\varphi_1\right)b^{\frac{1}{2}} & 0 \end{bmatrix},$$

where

$$\varphi_1=\operatorname{sn}\left(\frac{K}{3}\right).$$

The eigenvalues of $\phi(A)$ are $\pm\sqrt{k}$ and $\pm\sqrt{k}\varphi_1$. We can also write $\varphi_1$ in terms of $k$ and use it to simplify our calculation below. By the addition formulas

$$\begin{cases}\operatorname{sn}(x+y)=\dfrac{\operatorname{sn}(x)\operatorname{cn}(y)\operatorname{dn}(y)+\operatorname{sn}(y)\operatorname{cn}(x)\operatorname{dn}(x)}{1-k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)}, \\ \operatorname{cn}(x+y)=\dfrac{\operatorname{cn}(x)\operatorname{cn}(y)-\operatorname{sn}(x)\operatorname{sn}(y)\operatorname{dn}(x)\operatorname{dn}(y)}{1-k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)},\end{cases}$$

and the equality

$$\begin{cases} \operatorname{sn}^2(u) + \operatorname{cn}^2(u) = 1 \\ \operatorname{dn}^2(u) + k^2 \operatorname{sn}^2(u) = 1 \end{cases},$$

we have

$$0 = \operatorname{cn}(K) = \operatorname{cn}\left(\frac{K}{3}\right)\operatorname{cn}\left(\frac{2K}{3}\right) - \operatorname{sn}\left(\frac{K}{3}\right)\operatorname{sn}\left(\frac{2K}{3}\right)\operatorname{dn}\left(\frac{K}{3}\right)\operatorname{dn}\left(\frac{2K}{3}\right),$$

and

$$\operatorname{sn}\left(\frac{2K}{3}\right) = 2\frac{\operatorname{sn}\left(\frac{K}{3}\right)\operatorname{cn}\left(\frac{K}{3}\right)\operatorname{dn}\left(\frac{K}{3}\right)}{1 - k^2\operatorname{sn}^4\left(\frac{K}{3}\right)},$$

that is

$$k^2\operatorname{sn}^4\left(\frac{K}{3}\right) - 2k^2\operatorname{sn}^3\left(\frac{K}{3}\right) + 2\operatorname{sn}\left(\frac{K}{3}\right) - 1 = 0, \tag{4.2}$$

or

$$\varphi_1 = \frac{1}{2}\left(1 - \sqrt{1 + \left(\frac{4(1-k^2)}{k^4}\right)^{1/3}} + \sqrt{2 - \left(\frac{4(1-k^2)}{k^4}\right)^{1/3} - \frac{2 - \frac{4}{k^2}}{\sqrt{1 + \left(\frac{4(1-k^2)}{k^4}\right)^{1/3}}}}\right).$$

Taking

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{k\varphi_1}{b}} & 0 & 0 \\ 0 & 0 & k\varphi_1\sqrt{\frac{\varphi_1}{b^2}} & 0 \\ 0 & 0 & 0 & k\varphi_1^2\sqrt{\frac{k}{b^3}} \end{bmatrix},$$

then

$$\begin{aligned} C &= T\phi(A)T^{-1} \\ &= \begin{bmatrix} 0 & \frac{\sqrt{2}}{3\sqrt{\varphi_1}}(1+\varphi_1) & 0 & \frac{1-2\varphi_1}{3k\varphi_1^2} \\ \frac{k\sqrt{2\varphi_1}}{3}(1+\varphi_1) & 0 & \frac{2-\varphi_1}{3\varphi_1} & 0 \\ 0 & \frac{k\varphi_1(2-\varphi_1)}{3} & 0 & \frac{\sqrt{2}}{3\sqrt{\varphi_1}}(1+\varphi_1) \\ \frac{k^2\varphi_1^2(1-2\varphi_1)}{3} & 0 & \frac{k\sqrt{2\varphi_1}}{3}(1+\varphi_1) & 0 \end{bmatrix}. \end{aligned}$$

Taking

$$N_1 = \begin{bmatrix} \frac{\sqrt{2}}{3\sqrt{\varphi_1}}(1+\varphi_1) & \frac{1-2\varphi_1}{3k\varphi_1^2} \\ \frac{k\varphi_1(2-\varphi_1)}{3} & \frac{\sqrt{2}}{3\sqrt{\varphi_1}}(1+\varphi_1) \end{bmatrix},$$

$$N_2 = \begin{bmatrix} \frac{k\sqrt{2\varphi_1}}{3}(1+\varphi_1) & \frac{2-\varphi_1}{3\varphi_1} \\ \frac{k^2\varphi_1^2(1-2\varphi_1)}{3} & \frac{k\sqrt{2\varphi_1}}{3}(1+\varphi_1) \end{bmatrix},$$

the singular values of matrix $N_1$ are two 1's and the singular values of matrix $N_2$ are 1 and $k^2\varphi_1^2$. Thus, by lemma (5) the singular values of matrix $C$ are three 1's and $k^2\varphi_1^2$, i.e., $C$ is a contraction.

$$\hat{B}\left(\phi(A)\right) = \begin{bmatrix} 0 & 0 & 0 & \sqrt{\frac{k_4}{b^3}} \\ 0 & 0 & \sqrt{\frac{k_4}{b}} & 0 \\ 0 & \sqrt{k_4 b} & 0 & 0 \\ \sqrt{k_4 b^3} & 0 & 0 & 0 \end{bmatrix},$$

where

$$k_4 = k^3\varphi_1^4.$$

the eigenvalues of $\hat{B}\left(\phi(A)\right)$ are

$$\pm\sqrt{k_4} = \pm k\sqrt{k}\varphi_1^2.$$

Hence, the Pick matrix is

$$N_P = \begin{bmatrix} \dfrac{1-k^3\varphi_1^4}{1-k} & \dfrac{1+k^3\varphi_1^4}{1-k\varphi_1} & \dfrac{1-k^3\varphi_1^4}{1+k\varphi_1} & \dfrac{1+k^3\varphi_1^4}{1+k} \\ \dfrac{1+k^3\varphi_1^4}{1-k\varphi_1} & \dfrac{1-k^3\varphi_1^4}{1-k\varphi_1^2} & \dfrac{1+k^3\varphi_1^4}{1+k\varphi_1^2} & \dfrac{1-k^3\varphi_1^4}{1+k\varphi_1} \\ \dfrac{1-k^3\varphi_1^4}{1+k\varphi_1} & \dfrac{1+k^3\varphi_1^4}{1+k\varphi_1^2} & \dfrac{1-k^3\varphi_1^4}{1-k\varphi_1^2} & \dfrac{1+k^3\varphi_1^4}{1-k\varphi_1} \\ \dfrac{1+k^3\varphi_1^4}{1+k} & \dfrac{1-k^3\varphi_1^4}{1+k\varphi_1} & \dfrac{1+k^3\varphi_1^4}{1-k\varphi_1} & \dfrac{1-k^3\varphi_1^4}{1-k} \end{bmatrix},$$

$N_p$ is positive semidefinite with eigenvalues

$$\begin{cases} \sigma_1 = \lambda_1 = & \dfrac{3-8\varphi_1+11\varphi_1^2-2\varphi_1^3+\sqrt{\Delta}}{2(1-\varphi_1)^2(2-\varphi_1)} \\ \sigma_2 = \lambda_2 = & \dfrac{k\varphi_1(1-2\varphi_1+5\varphi_1^2)}{(1-\varphi_1)^2} \\ \sigma_3 = \lambda_3 = & \dfrac{3-8\varphi_1+11\varphi_1^2-2\varphi_1^3-\sqrt{\Delta}}{2(1-\varphi_1)^2(2-\varphi_1)} \\ \sigma_4 = \lambda_4 = & 0 \end{cases},$$

where

$$\Delta = 25 - 88\varphi_1 + 114\varphi_1^2 - 108\varphi_1^3 + 137\varphi_1^4 - 84\varphi_1^5 + 20\varphi_1^6.$$

Thus, the optimal Blaschke product of degree 3 is unique. Let

$$V = \begin{bmatrix} 1 & \sqrt{k} & k & k\sqrt{k} \\ 1 & \sqrt{k}\varphi_1 & k\varphi_1^2 & k\sqrt{k}\varphi_1^3 \\ 1 & -\sqrt{k}\varphi_1 & k\varphi_1^2 & -k\sqrt{k}\varphi_1^3 \\ 1 & -\sqrt{k} & k & -k\sqrt{k} \end{bmatrix},$$

$$C_4 = \sqrt{k_4}V^{-1}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} VP$$

$$= \frac{\varphi_1}{1-\varphi_1}\begin{bmatrix} -k^3\varphi_1^3 & 0 & -k^2\varphi_1^2 & 0 \\ 0 & -k^2\varphi_1^2 & 0 & -k(1-\varphi_1+\varphi_1^2) \\ k^2\varphi_1(1-\varphi_1+\varphi_1^2) & 0 & k\varphi_1\dfrac{2}{k} & 0 \\ 0 & k\varphi_1 & 0 & 1 \end{bmatrix},$$

the eigenvector of $C_4$ corresponding to eigenvalue $\lambda = 1$ is

$$\beta = \begin{bmatrix} 0 \\ -k\varphi_1(2-\varphi_1) \\ 0 \\ 1 \end{bmatrix},$$

thus,

$$\hat{B}(z) = z\frac{z^2 - k\varphi_1(2-\varphi_1)}{1 - k\varphi_1(2-\varphi_1)z^2}.$$

The roots of this Blaschke product are 0 and $\pm\sqrt{k\varphi_1(2-\varphi_1)}$.

We compare our analytic results with the numerical tests in Figure (4.1(a)) and Table (4.4). We see in the figure that the numerical methods do not give an accurate result when the ellipse is slim or some eigenvalues of the matrix are close to the boundary of the numerical range, since in this case we may have difficulty in evaluating the Szegö kernel or in optimizing the Blaschke product as shown in the table. This happens when the upper bound $\sqrt{\frac{k_{n-1}}{b^{n-1}}}$ drops dramatically with respect to the parameter $b$, i.e., when the magnitude of the derivative of $\sqrt{\frac{k_{n-1}}{b^{n-1}}}$ with respect to $b$ is large, and some of the optimal parameters $\alpha_j$'s are also sensitive to $b$. Thus, when $n \geq 4$, we expect this happens and the dropping region moves to 1 as the integer $n$ increases. For example, when $n = 4$ and $b > 0.6$, the eccentricity is

$$\epsilon = \frac{2\sqrt{b}}{1+b} > 0.9682,$$

meaning that the ratio of major to minor semiaxis is about 4. Although somewhat long and narrow, we have found that the conformal mapping routine can accurately map ellipses whose ratio of major to minor semiaxis is as large as about 9, so we do not expect that this is the source of the error. In fact, the relative error of the conformal mapping of the matrix A is

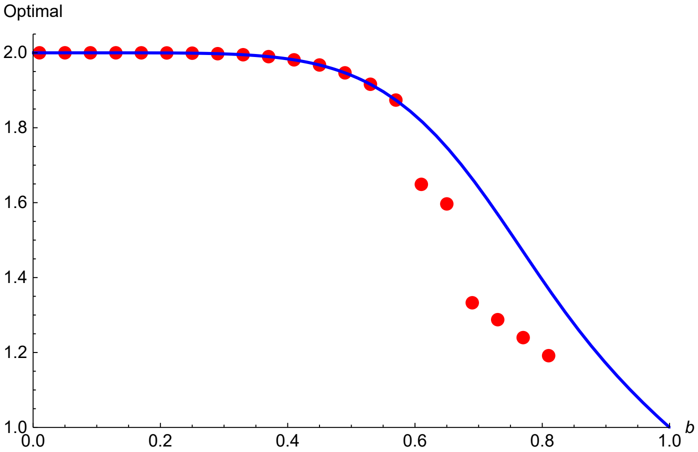
$$e_{\phi(A)} = \frac{||\phi(A)_e - \phi(A)_n||_2}{||\phi(A)_e||_2} \sim \mathcal{O}(10^{-10}),\ \ b \in [0, 0.81],$$

where $\phi(A)_e$ is the exact result and $\phi(A)_n$ represent the numerical result from the Cauchy integral. The numerical conformal mapping is accurate while we can see in Figure (4.4.2) that the numerical values of $||\hat{B}(\phi(A))||_2$ are below the accurate upper bound since the numerical roots of the Blaschke product are quite different from the analytic ones in Table (4.4). One way to improve the optimization results is to force the roots $\alpha$'s to be real as in our analysis. See details in Figure (4.1(b)) and (4.5.1).

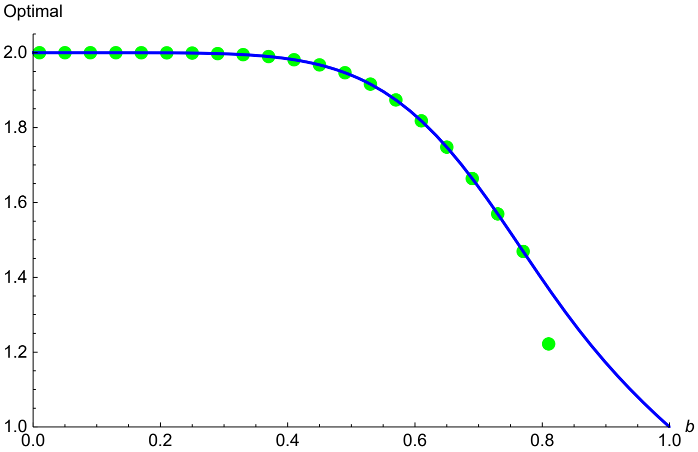
## *4.5 Example n=5*

When $n = 5$, we get that

$$S = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & 1 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & -\sqrt{2} & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ \frac{1}{\sqrt{2}} & -1 & 1 & -1 & \frac{1}{\sqrt{2}} \end{bmatrix},$$



(a) Complex $\alpha$'s



(b) Real $\alpha$'s

Figure 4.4.1: The solid blue line is the analytic result and the red and green dots represent the numerical results

Table 4.4.1: Analytic and numerical $\alpha_1$, $\alpha_2$ and $\alpha_3$

| b | $\alpha_1 = 0,\ \alpha_{2,3} = \pm\sqrt{k\varphi_1(2-\varphi_1)}$ | Numerical $\alpha_1$, $\alpha_2$ and $\alpha_3$ | | |
|---|---|---|---|---|
| 0.01 | 0, ±0.1731 | -0.003552+0.005717i | -0.1715-0.002851i | 0.175-0.002861i |
| 0.05 | 0, ±0.3855 | -4.012e-04+0.001288i | -0.3861-5.542e-04i | 0.3865-6.988e-04i |
| 0.09 | 0, ±0.5138 | -2.618e-04+7.03e-04i | -0.5153-2.558e-04i | 0.5156-3.88e-04i |
| 0.13 | 0, ±0.6119 | -8.323e-05-1.615e-04i | -0.6141+4.683e-05i | 0.6142+8.838e-05i |
| 0.17 | 0, ±0.6916 | -2.054e-05+1.589e-04i | -0.6941-3.811e-05i | 0.6941-7.579e-05i |
| 0.21 | 0, ±0.7577 | -8.874e-05+1.485e-04i | -0.7602-2.485e-05i | 0.7602-6.801e-05i |
| 0.25 | 0, ±0.8128 | -1.656e-04+2.555e-04i | -0.8151-3.042e-05i | 0.8151-1.004e-04i |
| 0.29 | 0, ±0.8585 | -1.973e-04+1.999e-04i | -0.8604-1.429e-05i | 0.8604-7.112e-05i |
| 0.33 | 0, ±0.8960 | -2.073e-04-1.679e-04i | -0.8974+6.818e-06i | 0.8974+5.042e-05i |
| 0.37 | 0, ±0.9260 | -2.361e-04-1.261e-04i | -0.927+1.758e-06i | 0.927+3.421e-05i |
| 0.41 | 0, ±0.9495 | -2.563e-04-9.851e-05i | -0.9501-6.257e-08i | 0.9501+2.274e-05i |
| 0.45 | 0, ±0.9672 | -2.779e-04-8.05e-05i | -0.9675-5.212e-07i | 0.9675+1.458e-05i |
| 0.49 | 0, ±0.9800 | -6.847e-04-1.493e-04i | -0.9801-1.144e-06i | 0.9801+1.982e-05i |
| 0.53 | 0, ±0.9887 | -8.568e-04-1.597e-04i | -0.9888-7.84e-07i | 0.9888+1.266e-05i |
| 0.57 | 0, ±0.9942 | -5.064e-04+5.569e-05i | -0.9943+3.062e-07i | 0.9943-3.305e-06i |
| 0.61 | 0, ±0.9974 | -0.172-6.084e-06i | -0.2985+0.9544i | 0.9973+1.317e-06i |
| 0.65 | 0, ±0.9990 | -0.1808+4.992e-06i | -0.2853+0.9584i | 0.999-2.833e-07i |
| 0.69 | 0, ±0.9997 | -5.455e-04-5.7e-06i | -0.2749-0.9615i | -0.3951+0.9186i |
| 0.73 | 0, ±0.9999 | -0.003328-8.579e-04i | -0.2613-0.963i | 0.9999-4.318e-07i |
| 0.77 | 0, ±1.000 | -0.00145-2.554e-07i | -0.1429-0.9897i | -0.116+0.9932i |
| 0.81 | 0, ±1.000 | -2.473e-05+9.98e-05i | -0.3548-0.9349i | -0.1307+0.9914i |

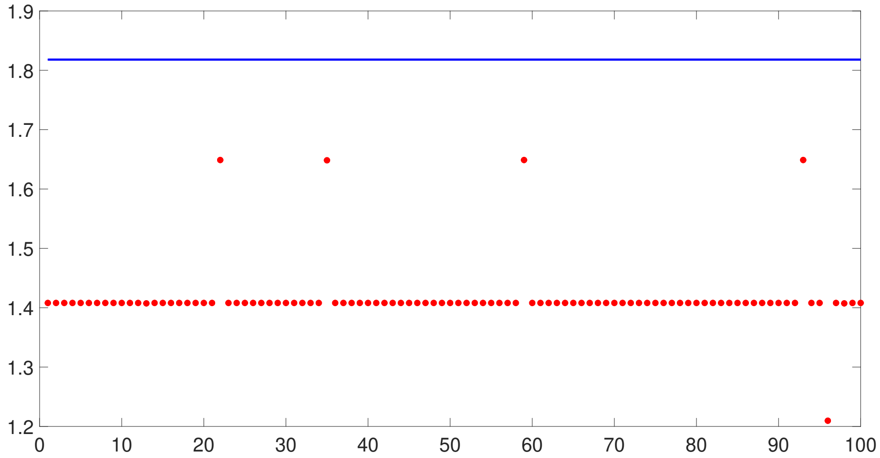


Figure 4.4.2: The red dots are the optimal results from 100 random inital guesses at $n = 4,\ b = 0.61$ and the blue line is the analytic value

$$\phi(A) = D \begin{bmatrix} 0 & \frac{\sqrt{2k}}{4} + \frac{\sqrt[4]{k_3}}{2} & 0 & \frac{\sqrt{2k}}{4} - \frac{\sqrt[4]{k_3}}{2} & 0 \\ \frac{\sqrt{2k}}{4} + \frac{\sqrt[4]{k_3}}{2} & 0 & \frac{1}{2}\sqrt{k} & 0 & \frac{\sqrt{2k}}{4} - \frac{\sqrt[4]{k_3}}{2} \\ 0 & \frac{1}{2}\sqrt{k} & 0 & \frac{1}{2}\sqrt{k} & 0 \\ \frac{\sqrt{2k}}{4} - \frac{\sqrt[4]{k_3}}{2} & 0 & \frac{1}{2}\sqrt{k} & 0 & \frac{\sqrt{2k}}{4} + \frac{\sqrt[4]{k_3}}{2} \\ 0 & \frac{\sqrt{2k}}{4} - \frac{\sqrt[4]{k_3}}{2} & 0 & \frac{\sqrt{2k}}{4} + \frac{\sqrt[4]{k_3}}{2} & 0 \end{bmatrix} D^{-1}$$

$$= \sqrt{k} \begin{bmatrix} 0 & \frac{\sqrt{2} + 2\,\mathrm{sn}\left(\frac{K}{2}\right)}{4\sqrt{b}} & 0 & \frac{\sqrt{2} - 2\,\mathrm{sn}\left(\frac{K}{2}\right)}{4\sqrt{b^3}} & 0 \\ \frac{(\sqrt{2} + 2\,\mathrm{sn}\left(\frac{K}{2}\right))\sqrt{b}}{4} & 0 & \frac{1}{2\sqrt{b}} & 0 & \frac{\sqrt{2} - 2\,\mathrm{sn}\left(\frac{K}{2}\right)}{4\sqrt{b^3}} \\ 0 & \frac{\sqrt{b}}{2} & 0 & \frac{1}{2\sqrt{b}} & 0 \\ \frac{(\sqrt{2} - 2\,\mathrm{sn}\left(\frac{K}{2}\right))\sqrt{b^3}}{4} & 0 & \frac{\sqrt{b}}{2} & 0 & \frac{\sqrt{2} + 2\,\mathrm{sn}\left(\frac{K}{2}\right)}{4\sqrt{b}} \\ 0 & \frac{(\sqrt{2} - 2\,\mathrm{sn}\left(\frac{K}{2}\right))\sqrt{b^3}}{4} & 0 & \frac{(\sqrt{2} + 2\,\mathrm{sn}\left(\frac{K}{2}\right))\sqrt{b}}{4} & 0 \end{bmatrix}.$$

The eigenvalues of $\phi(A)$ are 0, $\pm\sqrt{k}$ and $\pm\sqrt{k}\,\mathrm{sn}(\frac{K}{2}) = \pm\sqrt[4]{k_3}$. We take

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & T_1 & 0 & 0 & 0 \\ 0 & 0 & T_2 & 0 & 0 \\ 0 & 0 & 0 & T_3 & 0 \\ 0 & 0 & 0 & 0 & T_4 \end{bmatrix},$$

where

$$\begin{cases} T_1 = \sqrt[4]{\dfrac{\sqrt{1+k_5}+1-\sqrt{k_5}}{\sqrt{1+k_5}-1+\sqrt{k_5}}\dfrac{k_5}{b^2}} = \sqrt{\dfrac{1+\sqrt{1-k_3}}{1+\sqrt{1-k_3^2}}}\sqrt[4]{\dfrac{k_3}{b^2}}, \\ T_2 = \sqrt{T_4}, \\ T_3 = \dfrac{T_4}{T_1}, \\ T_4 = \sqrt{\dfrac{k_5}{b^4}} = \dfrac{k^2}{(1+\sqrt{1-k^2})(1+\sqrt[4]{1-k^2})^2 b^2}, \end{cases}$$

and

$$k_5 = k^4\,\mathrm{sn}^4\left(\frac{K}{4}\right)\mathrm{sn}^4\left(\frac{3K}{4}\right).$$

In the following simplification, we take advantage of the relations

$$\begin{cases} \mathrm{sn}\left(\dfrac{K}{2}\right) = \sqrt{\dfrac{1}{1+\sqrt{1-k^2}}} \\ \mathrm{cn}\left(\dfrac{K}{2}\right) = \sqrt{\dfrac{\sqrt{1-k^2}}{1+\sqrt{1-k^2}}} \\ \mathrm{sn}\left(\dfrac{K}{4}\right) = \sqrt{\dfrac{1}{(1+\sqrt{1-k^2})(1+\sqrt[4]{1-k^2})(1+\mathrm{cn}(\frac{K}{2}))}} \\ \mathrm{sn}\left(\dfrac{3K}{4}\right) = \sqrt{\dfrac{1+\mathrm{cn}(\frac{K}{2})}{1+\sqrt[4]{1-k^2}}} \end{cases}.$$

The matrix $C$ is

$$C = T\phi(A)T^{-1}$$

$$= \sqrt{k}\begin{bmatrix} 0 & \dfrac{\sqrt{2}+2\operatorname{sn}\left(\frac{K}{2}\right)}{4T_1} & 0 & \dfrac{\sqrt{2}-2\operatorname{sn}\left(\frac{K}{2}\right)}{4T_3} & 0 \\ \dfrac{(\sqrt{2}+2\operatorname{sn}\left(\frac{K}{2}\right))T_1}{4} & 0 & \dfrac{T_1}{2T_2} & 0 & \dfrac{(\sqrt{2}-2\operatorname{sn}\left(\frac{K}{2}\right))T_1}{4T_4} \\ 0 & \dfrac{T_2}{2T_1} & 0 & \dfrac{T_2}{2T_3} & 0 \\ \dfrac{(\sqrt{2}-2\operatorname{sn}\left(\frac{K}{2}\right))T_3}{4} & 0 & \dfrac{T_3}{2T_2} & 0 & \dfrac{(\sqrt{2}+2\operatorname{sn}\left(\frac{K}{2}\right))T_3}{4T_4} \\ 0 & \dfrac{(\sqrt{2}-2\operatorname{sn}\left(\frac{K}{2}\right))T_4}{4T_1} & 0 & \dfrac{(\sqrt{2}+2\operatorname{sn}\left(\frac{K}{2}\right))T_4}{4T_3} & 0 \end{bmatrix}.$$

Taking

$$N_1 = \sqrt{k}\begin{bmatrix} \dfrac{\sqrt{2}+2\operatorname{sn}\left(\frac{K}{2}\right)}{4T_1} & \dfrac{\sqrt{2}-2\operatorname{sn}\left(\frac{K}{2}\right)}{4T_3} \\ \dfrac{T_2}{2T_1} & \dfrac{T_2}{2T_3} \\ \dfrac{(\sqrt{2}-2\operatorname{sn}\left(\frac{K}{2}\right))T_4}{4T_1} & \dfrac{(\sqrt{2}+2\operatorname{sn}\left(\frac{K}{2}\right))T_4}{4T_3} \end{bmatrix},$$

and

$$N_2 = \sqrt{k}\begin{bmatrix} \dfrac{(\sqrt{2}+2\operatorname{sn}\left(\frac{K}{2}\right))T_1}{4} & \dfrac{T_1}{2T_2} & \dfrac{(\sqrt{2}-2\operatorname{sn}\left(\frac{K}{2}\right))T_1}{4T_4} \\ \dfrac{(\sqrt{2}-2\operatorname{sn}\left(\frac{K}{2}\right))T_3}{4} & \dfrac{T_3}{2T_2} & \dfrac{(\sqrt{2}+2\operatorname{sn}\left(\frac{K}{2}\right))T_3}{4T_4} \end{bmatrix},$$

we have

$$N_1^T N_1 = I,$$

and

$$N_2 N_2^T = I,$$

where $I$ is the 2 by 2 identity matrix. The singular values of matrix $N_1$ and $N_2$ are 1's. Thus, by lemma (5) the singular values of matrix $C$ are four 1's and one 0, i.e., $C$ is a contraction.

The extremal Blaschke product of $\phi(A)$ is

$$\hat{B}\left(\phi(A)\right) = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{\dfrac{k_5}{b^4}} \\ 0 & 0 & 0 & \sqrt{\dfrac{k_5}{b^2}} & 0 \\ 0 & 0 & \sqrt{k_5} & 0 & 0 \\ 0 & \sqrt{k_5 b^2} & 0 & 0 & 0 \\ \sqrt{k_5 b^4} & & & & \end{bmatrix}.$$

The eigenvalues of $\hat{B}\left(\phi(A)\right)$ are

$$\pm\sqrt{k_5} = \pm k^2 \operatorname{sn}^2\left(\frac{K}{4}\right)\operatorname{sn}^2\left(\frac{3K}{4}\right).$$

Hence, the Pick matrix is

$$N_P = \begin{bmatrix} \dfrac{1-k_5}{1-k} & \dfrac{1+k_5}{1-\sqrt{k\sqrt{k_3}}} & 1-k_5 & \dfrac{1+k_5}{1+\sqrt{k\sqrt{k_3}}} & \dfrac{1-k_5}{1+k} \\ \dfrac{1+k_5}{1-\sqrt{k\sqrt{k_3}}} & \dfrac{1-k_5}{1-\sqrt{k_3}} & 1+k_5 & \dfrac{1-k_5}{1+\sqrt{k_3}} & \dfrac{1+k_5}{1+\sqrt{k\sqrt{k_3}}} \\ 1-k_5 & 1+k_5 & 1-k_5 & 1+k_5 & 1-k_5 \\ \dfrac{1+k_5}{1+\sqrt{k\sqrt{k_3}}} & \dfrac{1-k_5}{1+\sqrt{k_3}} & 1+k_5 & \dfrac{1-k_5}{1-\sqrt{k_3}} & \dfrac{1+k_5}{1-\sqrt{k\sqrt{k_3}}} \\ \dfrac{1-k_5}{1+k} & \dfrac{1+k_5}{1+\sqrt{k\sqrt{k_3}}} & 1-k_5 & \dfrac{1+k_5}{1-\sqrt{k\sqrt{k_3}}} & \dfrac{1-k_5}{1-k} \end{bmatrix},$$

$N_p$ is semi-positive definite with eigenvalues/singularvalues

$$\begin{cases} \sigma_1 = \lambda_1 = & \dfrac{2(4-2k^2+\sqrt{1-k^2})}{(1-k^2)^{\frac{3}{4}}(1+\sqrt[4]{1-k^2})^2} + \dfrac{2\sqrt{\Delta_1}}{(1-k^2)^{\frac{3}{4}}(1+\sqrt[4]{1-k^2})^2(1+\sqrt{1-k^2})^{\frac{5}{2}}} \\ \sigma_2 = \lambda_2 = & 2k\dfrac{(2+\sqrt{1-k^2})+\sqrt{9-5k^2}}{(1-k^2)^{\frac{3}{4}}(1+\sqrt[4]{1-k^2})^2} \\ \sigma_3 = \lambda_3 = & \dfrac{2(4-2k^2+\sqrt{1-k^2})}{(1-k^2)^{\frac{3}{4}}(1+\sqrt[4]{1-k^2})^2} - \dfrac{2\sqrt{\Delta_1}}{(1-k^2)^{\frac{3}{4}}(1+\sqrt[4]{1-k^2})^2(1+\sqrt{1-k^2})^{\frac{5}{2}}}, \\ \sigma_4 = \lambda_4 = & 2k\dfrac{(2+\sqrt{1-k^2})-\sqrt{9-5k^2}}{(1-k^2)^{\frac{3}{4}}(1+\sqrt[4]{1-k^2})^2} \\ \sigma_5 = \lambda_5 = & 0 \end{cases}$$

where

$$\begin{aligned}\Delta_1 =&400 - 1124k^2 + 1193k^4 - 559k^6 + 96k^8 - 2k^{10} + 400\sqrt{1-k^2}\\ &- 924k^2\sqrt{1-k^2} + 781k^4\sqrt{1-k^2} - 259k^6\sqrt{1-k^2} + 22k^8\sqrt{1-k^2}.\end{aligned}$$

Thus, we find the unique optimal Blaschke product of degree 4

$$V = \begin{bmatrix} 1 & \sqrt{k} & k & k\sqrt{k} & k^2 \\ 1 & \sqrt[4]{k_3} & \sqrt{k_3} & \sqrt{k_3\sqrt{k_3}} & k_3 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -\sqrt[4]{k_3} & \sqrt{k_3} & -\sqrt{k_3\sqrt{k_3}} & k_3 \\ 1 & -\sqrt{k} & k & -k\sqrt{k} & k^2 \end{bmatrix},$$

$$\begin{aligned} C_5 &= \sqrt{k_5}V^{-1}\begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 1 \end{bmatrix} VP \\ &= \sqrt{\frac{k_5}{1-k^2}}\begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{1-k^2} \\ 0 & -2k & 0 & -2-\sqrt{1-k^2} & 0 \\ -2k & 0 & -2-\sqrt{1-k^2} & 0 & -\frac{2(1+\sqrt{1-k^2})^2}{k} \\ 0 & 2+\sqrt{1-k^2} & 0 & \frac{2(1+\sqrt{1-k^2})}{k} & 0 \\ 2+\sqrt{1-k^2} & 0 & \frac{2(1+\sqrt{1-k^2})}{k} & 0 & \frac{2(1+\sqrt{1-k^2})^2}{k^2} \end{bmatrix}, \end{aligned}$$

the eigenvector of $C_5$ corresponding to eigenvalue $\lambda = 1$ is

$$\beta = \begin{bmatrix} \sqrt{k_5} \\ 0 \\ -(1+\sqrt{k_5})k \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} k^2\operatorname{sn}^2(\frac{K}{4})\operatorname{sn}^2(\frac{3K}{4}) \\ 0 \\ -\left(1 + k^2\operatorname{sn}^2(\frac{K}{4})\operatorname{sn}^2(\frac{3K}{4})\right)k \\ 0 \\ 1 \end{bmatrix},$$

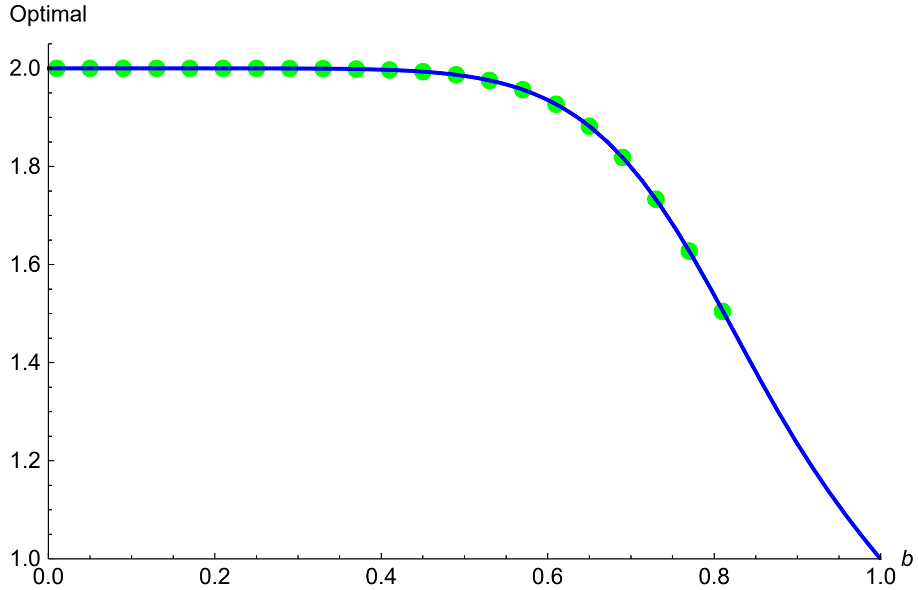


Figure 4.5.1: The solid blue line is the analytic result and the green dots represent the numerical result

thus,

$$\hat{B}(z) = \frac{z^4 - k(1+\sqrt{k_5})z^2 + \sqrt{k_5}}{1 - k(1+\sqrt{k_5})z^2 + \sqrt{k_5}z^4}.$$

The roots of this Blaschke product are

$$\alpha_{1,2,3,4} = \pm\sqrt{\frac{(1+\sqrt{k_5})k \pm \sqrt{(1+\sqrt{k_5})^2k^2 - 4\sqrt{k_5}}}{2}}.$$

We compare our analytic results with the numerical tests in Figure (4.5.1) and Table (4.5). We choose our $\alpha$'s to be real in the numerical tests.

## *4.6 A Short Summary*

In this thesis, we proved that Crouzeix's conjecture holds for several special classes of matrices with elliptical numerical range and found the extremal function(s) and the exact upper bound. The Jacobi elliptic function $sn(\cdot)$ maps the interior of an ellipse onto the unit disk,

Table 4.5.1: Analytic and numerical $\alpha_1$, $\alpha_2$ and $\alpha_3$

| b | Analytic $\alpha$'s | Numerical $\alpha$'s | | | |
|---|---|---|---|---|---|
| 0.01 | ±0.07655,±0.1847 | -0.07694 | 0.07693 | -0.1846 | 0.1846 |
| 0.05 | ±0.1717,±0.4117 | -0.1717 | 0.1717 | -0.4117 | 0.4117 |
| 0.09 | ±0.2322,±0.5481 | -0.2322 | 0.2322 | -0.5481 | 0.5481 |
| 0.13 | ±0.2825,±0.6507 | -0.2825 | 0.2825 | -0.6507 | 0.6507 |
| 0.17 | ±0.3284,±0.7322 | -0.3284 | 0.3284 | -0.7322 | 0.7322 |
| 0.21 | ±0.3723,±0.7977 | -0.3723 | 0.3723 | -0.7977 | 0.7977 |
| 0.25 | ±0.4158,±0.8502 | -0.4158 | 0.4158 | -0.8502 | 0.8502 |
| 0.29 | ±0.4598,±0.8919 | -0.4598 | 0.4598 | -0.8919 | 0.8919 |
| 0.33 | ±0.5048,±0.9243 | -0.5048 | 0.5048 | -0.9243 | 0.9243 |
| 0.37 | ±0.5512,±0.9489 | -0.5512 | 0.5512 | -0.9489 | 0.9489 |
| 0.41 | ±0.5991,±0.9671 | -0.5991 | 0.5991 | -0.9671 | 0.9671 |
| 0.45 | ±0.6484,±0.9799 | -0.6484 | 0.6484 | -0.9799 | 0.9799 |
| 0.49 | ±0.6987,±0.9886 | -0.6987 | 0.6987 | -0.9886 | 0.9886 |
| 0.53 | ±0.7494,±0.9940 | -0.7494 | 0.7494 | -0.9940 | 0.9940 |
| 0.57 | ±0.7996,±0.9972 | -0.7996 | 0.7996 | -0.9972 | 0.9972 |
| 0.61 | ±0.8477,±0.9989 | -0.8477 | 0.8477 | -0.9989 | 0.9989 |
| 0.65 | ±0.8921,±0.9996 | -0.8921 | 0.8921 | -0.9996 | 0.9996 |
| 0.69 | ±0.9305,±0.9999 | -0.9305 | 0.9305 | -0.9999 | 0.9999 |
| 0.73 | ±0.9611,±1.000 | -0.9611 | 0.9611 | -1.000 | 1.000 |
| 0.77 | ±0.9823,±1.000 | -0.9823 | 0.9823 | -1.000 | 1.000 |
| 0.81 | ±0.9943,±1.000 | -0.9942 | 0.9946 | -1.000 | 1.000 |

and thus finding the extremal function and proving its uniqueness is equivalent to optimizing the finite Blaschke product and arguing its uniqueness. The exact upper bound depends on the nome (and the dimension of the matrix). We gave several examples that the extremal function is unique up to multiplication of a scalar and several cases that are not. It is interesting to see the transition from unique extremal function to nonunique ones as the elements of the matrix change.

Our analysis of elliptical cases introduces new equalities involving Jacobi elliptic functions and trigonometric functions. It is the extension of cyclic identities involving Jacobi elliptic functions, and we should develop a general method to simplify (i.e., lower their degree) these equalities.

It remains an open problem if Crouzeix's conjecture holds for the elliptical numerical range case, and we need to explore more about the structures and properties of the matrix $A$ and $f(A)$.

# BIBLIOGRAPHY

[1] T. Ando. Structure of operators with numerical radius one. *Acta. Sci. Mat. (Szeged)*, 34:11–15, 1973.

[2] T. Caldwell, A. Greenbaum, and K. Li. Some extensions of the Crouzeix–Palencia result. *SIAM J. Matrix Anal. Appl.*, 39(2):769–780, May 2018.

[3] W. Cheung and C. Li. Elementary proofs for some results on the circular symmetry of the numerical range. *Linear and Multilinear Algebra*, 61(5):596–602, 2013.

[4] D. Choi. A proof of Crouzeix's conjecture for a class of matrices. *Linear Algebra and Its Applications*, 438(8):3247–3257, 2013.

[5] M. J. Crabb. The powers of an operator of numerical radius one. *Michigan Math. J.*, 18(3):253–256, 07 1971.

[6] M. Crouzeix. Bounds for analytical functions of matrices. *Integral Equations and Operator Theory*, 48(4):461–477, Apr 2004.

[7] M. Crouzeix. Numerical range and functional calculus in Hilbert space. *Journal of Functional Analysis*, 244(2):668 – 690, 2007.

[8] M. Crouzeix, F. Gilfeather, and J. Holbrook. Polynomial bounds for small matrices. *Linear and Multilinear Algebra*, 62(5):614–625, 2014.

[9] M. Crouzeix and C. Palencia. The numerical range is a $(1+\sqrt{2})$-spectral set. *SIAM Journal on Matrix Analysis and Applications*, 38(2):649–655, 2017.

[10] T. A Driscoll, N. Hale, and L. N Trefethen. Chebfun guide, 2014.

[11] C. Glader, M. Kurula, and M. Lindström. Crouzeix's conjecture holds for tridiagonal 3 x 3 matrices with elliptic numerical range centered at an eigenvalue. *SIAM Journal on Matrix Analysis and Applications*, 39(1):346–364, 2018.

[12] C. Glader and M. Lindström. Finite Blaschke product interpolation on the closed unit disc. *Journal of Mathematical Analysis and Applications*, 273(2):417–427, 2002.

[13] A. Greenbaum, T. Caldwell, and K. Li. Near normal dilations of nonnormal matrices and linear operators. *SIAM Journal on Matrix Analysis and Applications*, 37(4):1365–1381, 2016.

[14] A. Greenbaum and D. Choi. Crouzeix's conjecture and perturbed Jordan blocks. *Linear Algebra and its Applications*, 436(7):2342 – 2352, 2012.

[15] A. Greenbaum and M. L. Overton. Numerical investigation of Crouzeix's conjecture. *Linear Algebra and its Applications*, 542:225 – 245, 2018. Proceedings of the 20th ILAS Conference, Leuven, Belgium 2016.

[16] K. E. Gustafson, D. K. M. Rao, and P. R. Halmos. *Numerical Range: The Field of Values of Linear Operators and Matrices.* Universitext. Springer New York, New York, NY, 1996.

[17] U. Haagerup and P. De La Harpe. The numerical radius of a nilpotent operator on a Hilbert space. *Proceedings of the American Mathematical Society*, 115(2):371–379, 1992. ID: unige:12076.

[18] P. Henrici. *Applied and Computational Complex Analysis. Vol. 3.* John Wiley & Sons, Inc., USA, 1986.

[19] I. Hnětynková and P. Tichý. Characterization of half-radial matrices. *Linear Algebra and its Applications*, 559:227 – 243, 2018.

[20] R. Horn and V. Sergeichuk. Canonical forms for unitary congruence and *congruence. *Linear and Multilinear Algebra*, 57, 10 2007.

[21] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, 1991.

[22] C. R. Johnson. Numerical determination of the field of values of a general complex matrix. *SIAM Journal on Numerical Analysis*, 15(3):595–602, 1978.

[23] N. Kerzman and M. R. Trummer. Numerical conformal mapping via the Szegö kernel. *Journal of Computational and Applied Mathematics*, 14(1):111–123, 1986.

[24] A. Khare, A. Lakshminarayan, and U. Sukhatme. Cyclic identities for jacobi elliptic and related functions. ii. *Journal of Mathematical Physics*, 44(4):1822–1841, Apr 2003.

[25] A. Khare, A. Lakshminarayan, and U. Sukhatme. Local identities involving Jacobi elliptic fucntions. *Pramana*, 62(6):1201, June 2004.

[26] A. Khare and U. Sukhatme. Cyclic identities involving jacobi elliptic functions. *Journal of Mathematical Physics*, 43:3798–3806, 2002.

[27] K. Li. On the uniqueness of functions that maximize the Crouzeix ratio. *Linear Algebra and its Applications*, 599:105–120, 2020.

[28] B.S. Nagy, C. Foias, H. Bercovici, and L. Kérchy. *Harmonic Analysis of Operators on Hilbert Space*. Universitext. Springer New York, 2010.

[29] J. von Neumann. Eine spektraltheorie für allgemeine operatoren eines unitären raumes. erhard schmidt zum 75. geburtstag in verehrung gewidmet. *Mathematische Nachrichten*, 4(1-6):258–281, 1950.

[30] C. Pearcy. An elementary proof of the power inequality for the numerical radius. *Michigan Math. J.*, 13(3):289–291, 1966.

[31] G. Szego. Conformal mapping of the interior of an ellipse onto a circle. *The American Mathematical Monthly*, 57(7):474–478, 1950.

# Appendix A

# 2-NILPOTENT MATRIX

**Theorem 6.** *For any matrix $A \in \mathbb{C}^{n\times n}$ with $A^2 = \mathbf{0}$ (2-nilpotent matrix) and $A \neq \mathbf{0}$ (trivial), $W(A)$ is a disk centered at the origin with radius $r(A) = \frac{1}{2}||A||_2$. Hence Crouzeix's conjecture holds and the upper bound 2 is achieved by the extremal function $\hat{f}(z) = z$. This extremal function is unique up to multiplication by a scalar.*

*Proof.* Since $A^2 = \mathbf{0}$, all eigenvalues of $A$ are 0, and $A$ has the Jordan normal form

$$A = V\Lambda V^{-1},$$

where $\Lambda$ has $k = \text{rank}(A)$ $J_2$ blocks and $(n-2k)$ $J_1$ blocks along its diagonal

$$\Lambda = \begin{bmatrix} J_2 & & & & & \\ & \ddots & & & & \\ & & J_2 & & & \\ & & & J_1 & & \\ & & & & \ddots & \\ & & & & & J_1 \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 0 \end{bmatrix},$$

$$V = [v_1, v_2, \cdots, v_n],$$

$$V^{-1} = \begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_n^* \end{bmatrix},$$

and all $v_i, u_j \in \mathbb{C}^n$ for $i, j = 1, 2, \cdots, n$.

Thus,

$$A = \sum_{j=1}^{k} v_{2j-1} u_{2j}^*, \tag{A.1}$$

and $V^{-1}V = I$ gives us

$$u_i^* v_j = \delta_{ij}. \tag{A.2}$$

Assuming

$$\tilde{V} = \mathrm{span}\{v_1, v_3, \cdots, v_{2k-1}\},$$

and

$$\tilde{U} = \mathrm{span}\{u_2, u_4, \cdots, u_{2k}\},$$

we know that for any $v \in \tilde{V}$ and $u \in \tilde{U}$

$$u^* v = v^* u = 0.$$

Taking $\tilde{W}$ to be the subspace of $\mathbb{C}^n$, such that

$$\tilde{V} \oplus \tilde{U} \oplus \tilde{W} = \mathbb{C}^n,$$

we have

$$v^* u = v^* w = u^* w = 0, \ \forall v \in \tilde{V}, \ \forall u \in \tilde{U}, \ \forall w \in \tilde{W}.$$

We can decompose any vector $x \in \mathbb{C}^n$,

$$x = \alpha v_x + \beta u_x + \gamma w_x, \ v_x \in \tilde{V}, \ u_x \in \tilde{U}, \ w_x \in \tilde{W}.$$

By the definition of $||A||_2$,

$$||A||_2 = \max_{||x_1||_2 = ||x_2||_2 = 1} x_1^* A x_2.$$

Taking

$$x_1 = \alpha_1 v_{x_1} + \beta_1 u_{x_1} + \gamma_1 w_{x_1},$$

$$x_2 = \alpha_2 v_{x_2} + \beta_2 u_{x_2} + \gamma_2 w_{x_2},$$

where $v_{x_1}, v_{x_2} \in V$,$u_{x_1}, u_{x_2} \in U$, and $w_{x_1}, w_{x_2} \in W$ are all unit vectors, we have

$$|\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 = 1,$$

$$|\alpha_2|^2 + |\beta_2|^2 + |\gamma_2|^2 = 1,$$

and from equations (A.1) and (A.2),

$$||A||_2 = \max_{||x_1||_2=||x_2||_2=1} x_1^* A x_2 = \max_{||x_1||_2=||x_2||_2=1} \alpha_1^* \beta_2 v_{x_1}^* A u_{x_2}.$$

Thus, we need

$$|\alpha_1| = |\beta_2| = 1, \ \alpha_2 = \beta_1 = \gamma_1 = \gamma_2 = 0,$$

and $v_{x_1} \in \tilde{V}$ and $u_{x_2} \in \tilde{U}$ are the corresponding first left and first right singular vectors of $A$ up to some scalar $\alpha_1 \beta_2^*$.

For any unit vector $q$, we can write

$$q = \alpha v_q + \beta u_q + \gamma w_q,$$

where $v_q \in \tilde{V}, \ u_q \in \tilde{U}, \ w_q \in \tilde{W}$ are all unit vectors. Again, we have

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1,$$

and

$$|q^* A q| = |\alpha^* \beta v_q^* A u_q| \le \frac{|\alpha|^2 + |\beta|^2}{2} ||A||_2 \le \frac{1}{2} ||A||_2.$$

Hence,

$$r(A) = \max_{||q||_2=1} |q^* A q| \le \frac{1}{2} ||A||_2.$$

On the other side, for any matrix $A$

$$||A||_2 \le 2r(A),$$

and we obtain

$$r(A) = \frac{1}{2} ||A||_2.$$

The equality holds if we take

$$\alpha = \beta = \frac{1}{\sqrt{2}},\ \gamma = 0,$$

and $v_q$ and $u_q$ to be the corresponding first left and first right singular vectors of $A$ respectively, that is

$$q = \frac{1}{\sqrt{2}}\left(v_q + u_q\right).$$

In fact, taking

$$q(\theta) = \frac{1}{\sqrt{2}}\left(v_q + e^{i\theta}u_q\right),$$

we can get the boundary of the numerical range of $A$,

$$\partial W(A) = q(\theta)^* A q(\theta) = \frac{1}{2}e^{i\theta}||A||_2.$$

$W(A)$ is a disk centered at the origin with radius $\frac{1}{2}||A||_2$. By T. Ando's theorem, Crouzeix's conjecture holds for this case. The upper bound 2 can be achieved by taking $\hat{f}(z) = z$. To prove the uniqueness of the extremal function, we know the conformal mapping from the numerical range of $A$ to the unit disk is

$$\phi(z) = \frac{1}{r(A)}z = \frac{2z}{||A||_2},$$

thus,

$$\phi(A) = \frac{2}{||A||_2}A.$$

Suppose the extremal Blaschke product is

$$B(z) = \prod_{j=1}^{n-1} \frac{z-\alpha_j}{1-\overline{\alpha_j}z},\ |\alpha_j| \leq 1.$$

We have

$$B(\phi(A)) = p_0 I + p_1 \phi(A),$$

since $\phi(A)^2 = \mathbf{0}$. We have the expressions of $p_0$ and $p_1$ in terms of the $\alpha_j$'s,

$$p_0 = (-1)^{n-1}\prod_{j=1}^{n-1}\alpha_j,$$

and

$$p_1 = (-1)^{n-2} \sum_{l=1}^{n-1} \prod_{j \neq l} \alpha_j (1 - |\alpha_l|^2).$$

Since the first left and right singular vectors of the extremal Blaschke product $B(\phi(A))$, $u_1, v_1$, are orthogonal, we have

$$u_1^* B(\phi(A)) v_1 = p_0 u_1^* v_1 + p_1 u_1^* \phi(A) v_1 = 2,$$

that is

$$\frac{p_1}{||A||_2} u_1^* A v_1 = 1.$$

Since $p_1$ depends only on the $\alpha_j$'s, we must have (the choice of $v_1$ and $u_1$ may not be unique, since the first two singular values of $A$ may be the same)

$$|u_1^* A v_1| = ||A||_2,$$

and

$$|p_1| = 1.$$

On the other hand, taking the unitary matrix $Q$,

$$Q = [u_1, q_2, q_3, \cdots, q_{n-1}, v_1],$$

we have

$$2 = ||B(\phi(A))||_2 = ||Q^* B(\phi(A)) Q||_2 = \left\| \begin{bmatrix} p_0 + \frac{2p_1 u_1^* A u_1}{||A||_2} & 0 & \cdots & 0 & 2 \\ & & & & 0 \\ & & \ddots & & \vdots \\ & & & & 0 \\ & & & & p_0 \end{bmatrix} \right\|_2,$$

hence,

$$p_0 = 0,$$

which requires that at least one root of the Blaschke product is 0. Without loss of generality, taking $\alpha_1 = 0$, we have

$$p_1 = (-1)^{n-2} \prod_{j=2}^{n-1} \alpha_j,$$

thus,

$$|p_1| \leq 1.$$

Since

$$|p_1| = 1,$$

we have

$$|\alpha_2| = |\alpha_3| = \cdots = |\alpha_{n-1}| = 1,$$

and $B(z) = z$ is the only extremal Blaschke product up to multiplication by a scalar. $\square$

Alternatively, we can take a look at the unitary similarity transformation of the 2-nilpotent matrix $A$,

$$\tilde{A} = Q^* A Q,$$

where $Q$ is a unitary matrix, since $\hat{f}(z)$ is one extremal function of $A$ if and only if it is one extremal function of $\tilde{A}$. We can choose $Q$ as

$$Q = [q_1, q_2, \cdots, q_n],$$

where $q_1, \cdots, q_k$ is an orthonormal basis of $\tilde{V}$, $q_{k+1}, \cdots, q_{n-k}$ is an orthonormal basis of $\tilde{W}$, and $q_{n-k+1}, \cdots, q_n$ is an orthonormal basis of $\tilde{U}$. That is, the 2-nilpotent matrix $A$ is unitarily similar to a block matrix

$$\tilde{A} = Q^* A Q = \begin{bmatrix} \mathbf{0}_{k,n-k} & M_{k,k} \\ \mathbf{0}_{n-k,n-k} & \mathbf{0}_{n-k,k} \end{bmatrix} = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}, \ 2 \leq 2k \leq n.$$

Then, we can argue for $\tilde{A}$ in a similar way.

If $A^2 = \mathbf{0}$ and $A$ is nonnormal, taking

$$A_b = A + bA^*, \ b \in (0, 1],$$

then $W(A_b)$ is an ellipse centered at the origin with semiaxes $\frac{||A||_2(1\pm b)}{2}$. Crouzeix's conjecture holds for $A_b$ with the upper bound $\sqrt{\dfrac{k(b^2)}{b}}$.

# Appendix B

# INVOLUTORY MATRIX

**Theorem 7.** *For any involutory matrix* $A \in \mathbb{C}^{n\times n}$*, i.e.,* $A^2 = I$*, and* $A$ *nonnormal (trivial),* $W(A)$ *is an ellipse centered at the origin with focal points* $\pm 1$ *and two semiaxes* $\frac{1\pm b}{2\sqrt{b}}$*, where* $b = \frac{1}{||A||_2^2}$*. Crouzeix's conjecture holds and the upper bound* $\sqrt{\frac{k}{b}}$ *is achieved by the extremal function*

$$\hat{f}(z) = \operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}(z)\right),$$

*where* $k = k(b^2)$ *is the elliptic modulus and* $K$ *is the quarter period. This extremal function is unique up to multiplication by a scalar.*

*Proof.* Suppose the singular decomposition of $A$ is

$$A = \sum_{j=1}^{n} \sigma_j u_j v_j^*, \ ||A||_2 = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0,$$

since $A = A^{-1}$ is invertible.

If $A$ is normal, the singular values of $A$ are the absolute values of the eigenvalues of $A$, that is

$$\sigma_1 = \cdots = \sigma_n = 1,$$

and

$$A = \sum_{j=1}^{n} u_j v_j^*,$$

thus, $A$ is unitary.

If $A$ is nonnormal, then we have $\sigma_1 > 1$, otherwise

$$1 \geq \sigma_1 \geq \cdots \geq \sigma_n,$$

and

$$1 = |\det(A)| = \prod_{j=1}^{n} \sigma_j \leq 1.$$

The only case where the equality holds is

$$\sigma_1 = \cdots = \sigma_n = 1,$$

but then

$$A = \sum_{j=1}^{n} u_j v_j^*,$$

or $A$ is normal. This is a contradiction!

We can get the singular decomposition of $A^{-1} = A$:

$$A = A^{-1} = \sum_{j=n}^{1} \frac{1}{\sigma_j} v_j u_j^*, \ ||A||_2 = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0,$$

thus, the singular values of $A$ come in pairs:

$$\sigma_n = \frac{1}{\sigma_1}, \ \sigma_{n-1} = \frac{1}{\sigma_2}, \cdots .$$

Hence, if $n$ is odd, we know $A$ has at least one singular value of 1.

In addition, since $A^2 = I$, we have

$$\sigma_j \sigma_k v_j^* u_k = u_j^* v_k = \overline{v_k^* u_j},$$

when $\sigma_j \neq 1$,

$$\sigma_j^2 v_j^* u_j = \overline{v_j^* u_j},$$

and we obtain

$$v_j^* u_j = 0, \ v_j \perp u_j.$$

Especially, since $\sigma_1 > 1$ and $\sigma_n = \frac{1}{\sigma_1} < 1$, we can get

$$v_1 \perp u_1,$$

$$A v_1 = \sigma_1 u_1,$$

and

$$Au_1 = \frac{1}{\sigma_1} v_1.$$

Taking the unitary matrix

$$Q_1 = [u_1, v_1, q_3, \cdots, q_n],$$

we can obtain

$$Q_1^* A Q_1 = \begin{bmatrix} u_1^* A u_1 & u_1^* A v_1 & u_1^* A q_3 & \cdots & u_1^* A q_n \\ v_1^* A u_1 & v_1^* A v_1 & v_1^* A q_3 & \cdots & v_1^* A q_n \\ q_3^* A u_1 & q_3^* A v_1 & q_3^* A q_3 & \cdots & q_3^* A q_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_n^* A u_1 & q_n^* A v_1 & q_n^* A q_3, & \cdots, & q_n^* A q_n \end{bmatrix} = \begin{bmatrix} 0 & \sigma_1 & \mathbf{0} \\ \frac{1}{\sigma_1} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_1 \end{bmatrix}.$$

The numerical range of the left upper submatrix

$$T_1 = \begin{bmatrix} 0 & \sigma_1 \\ \frac{1}{\sigma_1} & 0 \end{bmatrix},$$

$W(T_1)$ is an ellipse centered at the origin with focal points ±1 and two semiaxes $\frac{1}{2}\left(\sigma_1 \pm \frac{1}{\sigma_1}\right)$ for $\sigma_1 > 1$.

Since

$$||Q_1^* A Q_1||_2 = ||A||_2 = \sigma_1,$$

and

$$(Q_1^* A Q_1)^2 = I,$$

we have

$$||A_1||_2 \leq \sigma_1,$$

and

$$A_1^2 = I_{n-2},$$

where $I_{n-2}$ is the identity matrix of dimension $n-2$, i.e., $A_1$ is also an involutory matrix. If $A_1$ is normal, then $W(A_1)$ is the interval $[-1, 1]$ or just one point $\pm 1$. All three cases satisfy

$$W(A_1) \subset W\left(\begin{bmatrix} 0 & \sigma_1 \\ \dfrac{1}{\sigma_1} & 0 \end{bmatrix}\right),$$

and $W(A) = W(T_1)$.

If $A_1$ is nonnormal, then we can repeat this process. Choosing another unitary matrix

$$Q_2 = \begin{bmatrix} I_2 & \\ & \tilde{Q}_2 \end{bmatrix},$$

such that

$$\tilde{Q}_2^* A_1 \tilde{Q}_2 = \begin{bmatrix} 0 & ||A_1||_2 & \mathbf{0} \\ \dfrac{1}{||A_1||_2} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_2 \end{bmatrix},$$

$$||A_2||_2 \le ||A_1||_2,$$

and

$$A_2^2 = I_{n-4}.$$

The numerical range of the submatrix

$$T_2 = \begin{bmatrix} 0 & ||A_1||_2 \\ \dfrac{1}{||A_1||_2} & 0 \end{bmatrix},$$

$W(T_2)$ is an ellipse centered at the origin with focal points ±1 and two semiaxes $\frac{1}{2}\left(||A_1||_2 \pm \frac{1}{||A_1||_2}\right)$. Also, $W(T_2) \subset W(T_1)$ since $\sigma_1 \ge ||A_1||_2$.

By induction, if $A_l$ is normal for some integer $l$, then we can get

$$W(A_l) \subset W(T_l) \subset \cdots \subset W(T_1),$$

and then $W(A) = W(T_1)$. Otherwise, taking

$$Q = Q_1 Q_1 \cdots Q_{\lfloor \frac{n}{2} \rfloor},$$

the matrix $A$ is unitarily similar to the matrix

$$Q^*AQ = \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_p \end{bmatrix},$$

where

$$p = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ even} \\ \dfrac{n+1}{2}, & \text{if } n \text{ odd} \end{cases},$$

$$T_j = \begin{bmatrix} 0 & \sigma_j \\ \dfrac{1}{\sigma_j} & 0 \end{bmatrix}, \ j = 1, 2, \cdots, p-1,$$

and

$$T_p = \begin{cases} \begin{bmatrix} 0 & \sigma_p \\ \dfrac{1}{\sigma_p} & 0 \end{bmatrix}, & \text{if } n \text{ even} \\ 1 \text{ or } -1, & \text{if } n \text{ odd} \end{cases}.$$

Again,

$$W(T_p) \subset W(T_{p-1}) \subset \cdots \subset W(T_2) \subset W(T_1),$$

and we have that $W(A) = W(T_1)$.

Since Crouzeix's conjecture holds for all the block matrices $\{T_1,\ T_2, \cdots, T_p\}$ or $\{T_1,\ T_2, \cdots, T_l,\ A_l\}$, it also holds for the involutory matrix $A$. From the discussion for the 2 by 2 matrix, we know the upper bound for $A$ is

$$\max_j \left\{ \sigma_j \sqrt{k\left(\frac{1}{\sigma_j^4}\right)} \right\} = ||A||_2 \sqrt{k\left(\frac{1}{||A||_2^4}\right)} = \sqrt{\frac{k}{b}}, \tag{B.1}$$

where $k$ is the elliptic modulus $k = k(b^2)$ and

$$b = \frac{1}{||A||_2^2}.$$

The conformal mapping from $W(A)$ to the unit disk is

$$\phi(z) = \sqrt{k}\ \operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}(z)\right),$$

where $K$ is the quarter period $K = K(k) = K(b)$. Thus,

$$\phi(A) = \sqrt{k}A,$$

and the eigenvalues of $\phi(A)$ are $\pm\sqrt{k}$.

Assuming the extremal Blaschke product is

$$B = \exp(i\beta)\prod_{j=1}^{n-1}\frac{z-\alpha_j}{1-\overline{\alpha_j}z},\ |\alpha_j| \leq 1.$$

Here $\beta$ is chosen so that

$$B(\sqrt{k}) \geq 0.$$

Since $\sqrt{k}$ is in the interior of $W(\phi(A))$,

$$B(\sqrt{k}) < 1.$$

We know

$$B(\phi(A)) = p_0 I + p_1 A,$$

since $\phi(A)^2 = kI$. We can find the epxressions of $p_0$ and $p_1$ in terms of $\alpha_j$'s. Suppose the expansion of $B(z)$ about $z = 0$ is

$$B(z) = \sum_{j=0}^{\infty} a_j z^j,$$

then we have

$$\begin{aligned} B(\phi(A)) &= \sum_{j=0}^{\infty} a_j \phi(A)^j \\ &= \left(\sum_{j=0}^{\infty} a_{2j}k^j\right) I + \left(\sum_{j=0}^{\infty} a_{2j+1}k^j\right)\phi(A) \\ &= \frac{B(\sqrt{k}) + B(-\sqrt{k})}{2} I + \frac{B(\sqrt{k}) - B(-\sqrt{k})}{2} A. \end{aligned}$$

That is,

$$p_0 = \frac{B(\sqrt{k}) + B(-\sqrt{k})}{2},$$
$$p_1 = \frac{B(\sqrt{k}) - B(-\sqrt{k})}{2},$$

and

$$|p_0| \le \frac{|B(\sqrt{k})| + |B(-\sqrt{k})|}{2} \le 1,$$
$$|p_1| \le \frac{|B(\sqrt{k})| + |-B(-\sqrt{k})|}{2} \le 1.$$

Since the first left and right singular vectors of the extremal Blaschke product $B(\phi(A))$, $\tilde{u}_1, \tilde{v}_1$, are orthogonal, we have

$$||B(\phi(A))||_2 = \tilde{u}_1^* B(\phi(A)) \tilde{v}_1 = p_0 \tilde{u}_1^* \tilde{v}_1 + p_1 \tilde{u}_1^* A \tilde{v}_1 = p_1 \tilde{u}_1^* A \tilde{v}_1.$$

To maximize $||B(\phi(A))||_2$, $\tilde{u}_1$ and $\tilde{v}_1$ should be parallel to the first left and right singular vectors of $A$ (the choice of $\tilde{u}_1$ and $\tilde{v}_1$ may not be unique, since the first two singular values of $A$ may be the same), thus,

$$||B(\phi(A))||_2 = |p_1| \cdot ||A||_2. \tag{B.2}$$

From equation (B.1), we obtain

$$|p_1| = \sqrt{k}. \tag{B.3}$$

To find the extremal Blaschke product, taking the unitary matrix $\tilde{Q}$

$$\tilde{Q} = [\tilde{u}_1, q_2, q_3, \cdots, q_{n-1}, \tilde{v}_1],$$

we have

$$||B(\phi(A))||_2 = ||\tilde{Q}^* B(\phi(A)) \tilde{Q}||_2 = \left\| \begin{bmatrix} p_0 & 0 & \cdots & 0 & |p_1| \cdot ||A||_2 \\ & & & & 0 \\ & & \ddots & & \vdots \\ & & & & 0 \\ & & & & p_0 \end{bmatrix} \right\|_2,$$

hence,

$$p_0 = 0,$$

which gives

$$B(\sqrt{k}) = -B(-\sqrt{k}),$$

and

$$p_1 = B(\sqrt{k}) = \sqrt{k},$$

since $B(\sqrt{k}) \geq 0$ and $|p_1| = \sqrt{k}$. This extremal Blaschke product maps the eigenvalues of $\phi(A)$, $\pm\sqrt{k}$, to $\pm B(\sqrt{k}) = \pm\sqrt{k}$, respectively. Thus, the Pick matrix is

$$N_P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

$N_P$ is positive semi-definite with

$$\operatorname{rank}(N_p) = 1.$$

By the Nevanlinna–Pick theorem, we can expect a unique extremal Blaschke product of degree 1, and $p_1 = B(\sqrt{k}) = \sqrt{k}$ gives

$$B(z) = z.$$

The extremal function

$$\hat{f}(z) = B(\phi(z)) = \sqrt{k}\ \operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}(z)\right),$$

is unique up to multiplication by a scalar. □

Take

$$C = \frac{||A||_2^4}{||A||_2^4 - 1}\left(A - \frac{A^*}{||A||_2^2}\right).$$

If $A^2 = I$ and $A$ is nonnormal, then by Theorem (4) the numerical range of matrix $C$ is a disk centered at the origin with radius $\frac{||A||_2}{2}$.

# Appendix C

## MATRIX WITH DEGREE 2 MINIMAL POLYNOMIAL

A. Horn and V. Sergeichuk [20] proved that if $A^2$ or $\bar{A}A$ is normal then $A$ is unitarily similar to a direct sum of $1 \times 1$ and $2 \times 2$ blocks, so it follows that Crouzeix's conjecture holds and we can get the upper bound from the $2 \times 2$ blocks (similar to equation (B.1)). We can obtain the extremal function for some special cases, and one example follows.

**Corollary 8.** *Suppose the degree of the minimal polynomial of the matrix $A$ is at most 2, then Crouzeix's conjecture holds for $A$. If $A$ is nonnormal, the extremal function $\hat{f}(z)$ is unique up to multiplication by a scalar. When $W(A)$ is elliptic, the upper bound can be achieved by taking $\hat{f}(z)$ to be Jacobi elliptic sn($\cdot$); when $W(A)$ is circular, the upper bound 2 can be achieved by $\hat{f}(z) = z$.*

*Proof.* Suppose the minimal polynomial of the matrix $A$ is

$$a_2 A^2 + a_1 A + a_0 I = \mathbf{0}.$$

If $a_2 = 0$, $A$ is normal. This is a trivial case.
If $a_2 \neq 0$, we have

$$\left(\sqrt{a_2}A + \frac{a_1}{2\sqrt{a_2}} I\right)^2 = \frac{a_1^2 - 4a_0a_2}{4a_2} I.$$

Taking

$$\Delta = a_1^2 - 4a_0a_2,$$

when $\Delta = 0$, we can consider the matrix

$$\tilde{A} = \sqrt{a_2}A + \frac{a_1}{2\sqrt{a_2}} I.$$

Then $\tilde{A}^2 = \mathbf{0}$, and this is discussed in Appendix A.

When $\Delta \neq 0$, we can consider the matrix

$$\tilde{A} = \frac{1}{\sqrt{\Delta}}(2a_2 A + a_1 I).$$

Then $\tilde{A}^2 = I$, and this is discussed in Appendix B. $\square$

# Appendix D

# COEFFICIENTS TABLE

These coefficients are printed from Mathematica.

$$\begin{aligned}
c_4 =&4b^5k^4t^{12} - 8b^5k^3t^{12} + 4b^5k^2t^{12} + 6b^7k^4t^{10} + 4b^6k^4t^{10} + 4b^5k^4t^{10} + 4b^4k^4t^{10} + 6b^3k^4t^{10}\\
&+ 2b^8k^3t^{10} - 12b^7k^3t^{10} - 10b^6k^3t^{10} - 8b^5k^3t^{10} - 10b^4k^3t^{10} - 12b^3k^3t^{10} + 2b^2k^3t^{10}\\
&- 2b^8k^2t^{10} + 6b^7k^2t^{10} + 6b^6k^2t^{10} + 4b^5k^2t^{10} + 6b^4k^2t^{10} + 6b^3k^2t^{10} - 2b^2k^2t^{10} + 2b^9k^4t^8\\
&+ 5b^8k^4t^8 + 6b^7k^4t^8 + 11b^6k^4t^8 + 12b^5k^4t^8 + 11b^4k^4t^8 + 6b^3k^4t^8 + 5b^2k^4t^8 + 2bk^4t^8\\
&- b^9k^3t^8 - 6b^8k^3t^8 - 16b^7k^3t^8 - 26b^6k^3t^8 - 22b^5k^3t^8 - 26b^4k^3t^8 - 16b^3k^3t^8 - 6b^2k^3t^8\\
&- bk^3t^8 + b^8k^2t^8 + 6b^7k^2t^8 + 15b^6k^2t^8 + 16b^5k^2t^8 + 15b^4k^2t^8 + 6b^3k^2t^8 + b^2k^2t^8 + b^{10}k^4t^6\\
&+ 2b^9k^4t^6 + 7b^8k^4t^6 + 8b^7k^4t^6 + 16b^6k^4t^6 + 12b^5k^4t^6 + 16b^4k^4t^6 + 8b^3k^4t^6 + 7b^2k^4t^6\\
&+ 2bk^4t^6 + k^4t^6 - 2b^9k^3t^6 - 16b^8k^3t^6 - 8b^7k^3t^6 - 32b^6k^3t^6 - 44b^5k^3t^6 - 32b^4k^3t^6\\
&- 8b^3k^3t^6 - 16b^2k^3t^6 - 2bk^3t^6 + 6b^8k^2t^6 + 18b^6k^2t^6 + 32b^5k^2t^6 + 18b^4k^2t^6 + 6b^2k^2t^6\\
&+ 2b^9k^4t^4 + 5b^8k^4t^4 + 6b^7k^4t^4 + 11b^6k^4t^4 + 12b^5k^4t^4 + 11b^4k^4t^4 + 6b^3k^4t^4 + 5b^2k^4t^4\\
&+ 2bk^4t^4 - b^9k^3t^4 - 6b^8k^3t^4 - 16b^7k^3t^4 - 26b^6k^3t^4 - 22b^5k^3t^4 - 26b^4k^3t^4 - 16b^3k^3t^4\\
&- 6b^2k^3t^4 - bk^3t^4 + b^8k^2t^4 + 6b^7k^2t^4 + 15b^6k^2t^4 + 16b^5k^2t^4 + 15b^4k^2t^4 + 6b^3k^2t^4 + b^2k^2t^4\\
&+ 6b^7k^4t^2 + 4b^6k^4t^2 + 4b^5k^4t^2 + 4b^4k^4t^2 + 6b^3k^4t^2 + 2b^8k^3t^2 - 12b^7k^3t^2 - 10b^6k^3t^2\\
&- 8b^5k^3t^2 - 10b^4k^3t^2 - 12b^3k^3t^2 + 2b^2k^3t^2 - 2b^8k^2t^2 + 6b^7k^2t^2 + 6b^6k^2t^2 + 4b^5k^2t^2\\
&+ 6b^4k^2t^2 + 6b^3k^2t^2 - 2b^2k^2t^2 + 4b^5k^4 - 8b^5k^3 + 4b^5k^2
\end{aligned}$$

$$\begin{aligned}
c_3 ={}&16b^5k^4t^{12} - 8b^6k^3t^{12} - 48b^5k^3t^{12} - 8b^4k^3t^{12} + 16b^6k^2t^{12} + 48b^5k^2t^{12} + 16b^4k^2t^{12} - 8b^6kt^{12} \\
&- 16b^5kt^{12} - 8b^4kt^{12} + 20b^7k^4t^{10} + 16b^6k^4t^{10} + 24b^5k^4t^{10} + 16b^4k^4t^{10} + 20b^3k^4t^{10} \\
&+ 2b^8k^3t^{10} - 72b^7k^3t^{10} - 82b^6k^3t^{10} - 80b^5k^3t^{10} - 82b^4k^3t^{10} - 72b^3k^3t^{10} + 2b^2k^3t^{10} \\
&- 8b^8k^2t^{10} + 76b^7k^2t^{10} + 120b^6k^2t^{10} + 104b^5k^2t^{10} + 120b^4k^2t^{10} + 76b^3k^2t^{10} - 8b^2k^2t^{10} \\
&+ 6b^8kt^{10} - 24b^7kt^{10} - 54b^6kt^{10} - 48b^5kt^{10} - 54b^4kt^{10} - 24b^3kt^{10} + 6b^2kt^{10} + b^8k^5t^8 \\
&- 2b^7k^5t^8 - b^6k^5t^8 + 4b^5k^5t^8 - b^4k^5t^8 - 2b^3k^5t^8 + b^2k^5t^8 + 3b^9k^4t^8 + 16b^8k^4t^8 + 36b^7k^4t^8 \\
&+ 48b^6k^4t^8 + 34b^5k^4t^8 + 48b^4k^4t^8 + 36b^3k^4t^8 + 16b^2k^4t^8 + 3bk^4t^8 - 10b^9k^3t^8 - 54b^8k^3t^8 \\
&- 124b^7k^3t^8 - 194b^6k^3t^8 - 196b^5k^3t^8 - 194b^4k^3t^8 - 124b^3k^3t^8 - 54b^2k^3t^8 - 10bk^3t^8 \\
&+ 3b^9k^2t^8 + 40b^8k^2t^8 + 148b^7k^2t^8 + 264b^6k^2t^8 + 290b^5k^2t^8 + 264b^4k^2t^8 + 148b^3k^2t^8 \\
&+ 40b^2k^2t^8 + 3bk^2t^8 - 3b^8kt^8 - 42b^7kt^8 - 117b^6kt^8 - 156b^5kt^8 - 117b^4kt^8 - 42b^3kt^8 \\
&- 3b^2kt^8 + b^{10}k^5t^6 - 2b^9k^5t^6 - b^8k^5t^6 + 4b^7k^5t^6 - 4b^5k^5t^6 + 4b^3k^5t^6 - b^2k^5t^6 - 2bk^5t^6 \\
&+ k^5t^6 - 2b^{10}k^4t^6 + 10b^9k^4t^6 + 38b^8k^4t^6 + 16b^7k^4t^6 + 60b^6k^4t^6 + 76b^5k^4t^6 + 60b^4k^4t^6 \\
&+ 16b^3k^4t^6 + 38b^2k^4t^6 + 10bk^4t^6 - 2k^4t^6 - 3b^{10}k^3t^6 - 14b^9k^3t^6 - 103b^8k^3t^6 - 128b^7k^3t^6 \\
&- 246b^6k^3t^6 - 292b^5k^3t^6 - 246b^4k^3t^6 - 128b^3k^3t^6 - 103b^2k^3t^6 - 14bk^3t^6 - 3k^3t^6 + 6b^9k^2t^6 \\
&+ 96b^8k^2t^6 + 144b^7k^2t^6 + 320b^6k^2t^6 + 468b^5k^2t^6 + 320b^4k^2t^6 + 144b^3k^2t^6 + 96b^2k^2t^6 \\
&+ 6bk^2t^6 - 18b^8kt^6 - 36b^7kt^6 - 142b^6kt^6 - 248b^5kt^6 - 142b^4kt^6 - 36b^3kt^6 - 18b^2kt^6 \\
&+ b^8k^5t^4 - 2b^7k^5t^4 - b^6k^5t^4 + 4b^5k^5t^4 - b^4k^5t^4 - 2b^3k^5t^4 + b^2k^5t^4 + 3b^9k^4t^4 + 16b^8k^4t^4 \\
&+ 36b^7k^4t^4 + 48b^6k^4t^4 + 34b^5k^4t^4 + 48b^4k^4t^4 + 36b^3k^4t^4 + 16b^2k^4t^4 + 3bk^4t^4 - 10b^9k^3t^4 \\
&- 54b^8k^3t^4 - 124b^7k^3t^4 - 194b^6k^3t^4 - 196b^5k^3t^4 - 194b^4k^3t^4 - 124b^3k^3t^4 - 54b^2k^3t^4 \\
&- 10bk^3t^4 + 3b^9k^2t^4 + 40b^8k^2t^4 + 148b^7k^2t^4 + 264b^6k^2t^4 + 290b^5k^2t^4 + 264b^4k^2t^4 \\
&+ 148b^3k^2t^4 + 40b^2k^2t^4 + 3bk^2t^4 - 3b^8kt^4 - 42b^7kt^4 - 117b^6kt^4 - 156b^5kt^4 - 117b^4kt^4 \\
&- 42b^3kt^4 - 3b^2kt^4 + 20b^7k^4t^2 + 16b^6k^4t^2 + 24b^5k^4t^2 + 16b^4k^4t^2 + 20b^3k^4t^2 + 2b^8k^3t^2 \\
&- 72b^7k^3t^2 - 82b^6k^3t^2 - 80b^5k^3t^2 - 82b^4k^3t^2 - 72b^3k^3t^2 + 2b^2k^3t^2 - 8b^8k^2t^2 + 76b^7k^2t^2 \\
&+ 120b^6k^2t^2 + 104b^5k^2t^2 + 120b^4k^2t^2 + 76b^3k^2t^2 - 8b^2k^2t^2 + 6b^8kt^2 - 24b^7kt^2 - 54b^6kt^2 \\
&- 48b^5kt^2 - 54b^4kt^2 - 24b^3kt^2 + 6b^2kt^2 + 16b^5k^4 - 8b^6k^3 - 48b^5k^3 - 8b^4k^3 + 16b^6k^2 \\
&+ 48b^5k^2 + 16b^4k^2 - 8b^6k - 16b^5k - 8b^4k
\end{aligned}$$

$$
\begin{aligned}
c_2 = & 4b^7t^{12} + 16b^6t^{12} + 24b^5t^{12} + 16b^4t^{12} + 24b^5k^4t^{12} + 4b^3t^{12} - 16b^6k^3t^{12} - 80b^5k^3t^{12} \\
& - 16b^4k^3t^{12} + 4b^7k^2t^{12} + 48b^6k^2t^{12} + 112b^5k^2t^{12} + 48b^4k^2t^{12} + 4b^3k^2t^{12} - 8b^7kt^{12} \\
& - 48b^6kt^{12} - 80b^5kt^{12} - 48b^4kt^{12} - 8b^3kt^{12} + 24b^7t^{10} + 96b^6t^{10} + 144b^5t^{10} + 96b^4t^{10} \\
& + 28b^7k^4t^{10} + 24b^6k^4t^{10} + 40b^5k^4t^{10} + 24b^4k^4t^{10} + 28b^3k^4t^{10} + 24b^3t^{10} + 4b^8k^3t^{10} \\
& - 112b^7k^3t^{10} - 148b^6k^3t^{10} - 160b^5k^3t^{10} - 148b^4k^3t^{10} - 112b^3k^3t^{10} + 4b^2k^3t^{10} - 4b^8k^2t^{10} \\
& + 180b^7k^2t^{10} + 316b^6k^2t^{10} + 312b^5k^2t^{10} + 316b^4k^2t^{10} + 180b^3k^2t^{10} - 4b^2k^2t^{10} - 120b^7kt^{10} \\
& - 288b^6kt^{10} - 336b^5kt^{10} - 288b^4kt^{10} - 120b^3kt^{10} + 60b^7t^8 + 240b^6t^8 + 360b^5t^8 + 2b^8k^5t^8 \\
& - 4b^7k^5t^8 - 2b^6k^5t^8 + 8b^5k^5t^8 - 2b^4k^5t^8 - 4b^3k^5t^8 + 2b^2k^5t^8 + 240b^4t^8 + 2b^9k^4t^8 \\
& + 22b^8k^4t^8 + 60b^7k^4t^8 + 74b^6k^4t^8 + 44b^5k^4t^8 + 74b^4k^4t^8 + 60b^3k^4t^8 + 22b^2k^4t^8 + 2bk^4t^8 \\
& + 60b^3t^8 - 2b^9k^3t^8 - 80b^8k^3t^8 - 248b^7k^3t^8 - 352b^6k^3t^8 - 316b^5k^3t^8 - 352b^4k^3t^8 \\
& - 248b^3k^3t^8 - 80b^2k^3t^8 - 2bk^3t^8 + 6b^9k^2t^8 + 110b^8k^2t^8 + 408b^7k^2t^8 + 706b^6k^2t^8 \\
& + 780b^5k^2t^8 + 706b^4k^2t^8 + 408b^3k^2t^8 + 110b^2k^2t^8 + 6bk^2t^8 - 54b^8kt^8 - 300b^7kt^8 \\
& - 666b^6kt^8 - 840b^5kt^8 - 666b^4kt^8 - 300b^3kt^8 - 54b^2kt^8 + 80b^7t^6 + 320b^6t^6 + 480b^5t^6 \\
& + 2b^{10}k^5t^6 - 4b^9k^5t^6 - 2b^8k^5t^6 + 8b^7k^5t^6 - 8b^5k^5t^6 + 8b^3k^5t^6 - 2b^2k^5t^6 - 4bk^5t^6 \\
& + 2k^5t^6 + 320b^4t^6 - 6b^{10}k^4t^6 + 16b^9k^4t^6 + 62b^8k^4t^6 + 16b^7k^4t^6 + 88b^6k^4t^6 + 128b^5k^4t^6 \\
& + 88b^4k^4t^6 + 16b^3k^4t^6 + 62b^2k^4t^6 + 16bk^4t^6 - 6k^4t^6 + 80b^3t^6 + 10b^{10}k^3t^6 - 24b^9k^3t^6 \\
& - 198b^8k^3t^6 - 192b^7k^3t^6 - 420b^6k^3t^6 - 592b^5k^3t^6 - 420b^4k^3t^6 - 192b^3k^3t^6 - 198b^2k^3t^6 \\
& - 24bk^3t^6 + 10k^3t^6 + 12b^9k^2t^6 + 228b^8k^2t^6 + 464b^7k^2t^6 + 876b^6k^2t^6 + 1160b^5k^2t^6 \\
& + 876b^4k^2t^6 + 464b^3k^2t^6 + 228b^2k^2t^6 + 12bk^2t^6 - 108b^8kt^6 - 376b^7kt^6 - 852b^6kt^6 \\
& - 1168b^5kt^6 - 852b^4kt^6 - 376b^3kt^6 - 108b^2kt^6 + 60b^7t^4 + 240b^6t^4 + 360b^5t^4 + 2b^8k^5t^4 \\
& - 4b^7k^5t^4 - 2b^6k^5t^4 + 8b^5k^5t^4 - 2b^4k^5t^4 - 4b^3k^5t^4 + 2b^2k^5t^4 + 240b^4t^4 + 2b^9k^4t^4 \\
& + 22b^8k^4t^4 + 60b^7k^4t^4 + 74b^6k^4t^4 + 44b^5k^4t^4 + 74b^4k^4t^4 + 60b^3k^4t^4 + 22b^2k^4t^4 + 2bk^4t^4 \\
& + 60b^3t^4 - 2b^9k^3t^4 - 80b^8k^3t^4 - 248b^7k^3t^4 - 352b^6k^3t^4 - 316b^5k^3t^4 - 352b^4k^3t^4 \\
& - 248b^3k^3t^4 - 80b^2k^3t^4 - 2bk^3t^4 + 6b^9k^2t^4 + 110b^8k^2t^4 + 408b^7k^2t^4 + 706b^6k^2t^4 \\
& + 780b^5k^2t^4 + 706b^4k^2t^4 + 408b^3k^2t^4 + 110b^2k^2t^4 + 6bk^2t^4 - 54b^8kt^4 - 300b^7kt^4 \\
& - 666b^6kt^4 - 840b^5kt^4 - 666b^4kt^4 - 300b^3kt^4 - 54b^2kt^4 + 24b^7t^2 + 96b^6t^2 + 144b^5t^2
\end{aligned}
$$

$$
\begin{aligned}
&+96b^4t^2+28b^7k^4t^2+24b^6k^4t^2+40b^5k^4t^2+24b^4k^4t^2+28b^3k^4t^2+24b^3t^2+4b^8k^3t^2\\
&-112b^7k^3t^2-148b^6k^3t^2-160b^5k^3t^2-148b^4k^3t^2-112b^3k^3t^2+4b^2k^3t^2-4b^8k^2t^2\\
&+180b^7k^2t^2+316b^6k^2t^2+312b^5k^2t^2+316b^4k^2t^2+180b^3k^2t^2-4b^2k^2t^2-120b^7kt^2\\
&-288b^6kt^2-336b^5kt^2-288b^4kt^2-120b^3kt^2+4b^7+16b^6+24b^5+16b^4+24b^5k^4\\
&+4b^3-16b^6k^3-80b^5k^3-16b^4k^3+4b^7k^2+48b^6k^2+112b^5k^2+48b^4k^2+4b^3k^2-8b^7k\\
&-48b^6k-80b^5k-48b^4k-8b^3k
\end{aligned}
$$

$$
\begin{aligned}
e_1 =&\frac{1}{b^2\left(1+t^2\right)^2\left(1-\beta^2k\right)^2}(3b^2\beta^2+b^2\beta^6k^2+2b^2\beta^2k^2+b^2\beta^6k^2t^4+b^4\beta^6k^2t^2-2b^4\beta^4k^2t^2\\
&+4b^2\beta^4k^2t^2+2b^2\beta^2k^2t^4+b^4\beta^2k^2t^2+2b^2\beta^2k^2t^2+b^3\beta^4k-2b^2\beta^4k-2b^3\beta^2k-4b^2\beta^2k\\
&+b^3\beta^4kt^4-2b^2\beta^4kt^4+2b^3\beta^4kt^2-4b^2\beta^4kt^2-2b^3\beta^2kt^4-4b^2\beta^2kt^4-4b^3\beta^2kt^2\\
&-8b^2\beta^2kt^2+b^3kt^4+2b^3kt^2+b^3k+3b^2\beta^2t^4+6b^2\beta^2t^2+b\beta^4k-2b\beta^2k+b\beta^4kt^4\\
&+2b\beta^4kt^2-2b\beta^2kt^4-4b\beta^2kt^2+bkt^4+2bkt^2+bk+\beta^6k^2t^2-2\beta^4k^2t^2+\beta^2k^2t^2)\\
e_2 =&\frac{1}{b^2\left(1+t^2\right)^2\left(1-\beta^2k\right)^2}(3b^2\beta^4+2b^2\beta^4k^2+2b^2\beta^4k^2t^4+b^4\beta^4k^2t^2+2b^2\beta^4k^2t^2+b\beta^2k\\
&+4b^2\beta^2k^2t^2+b^2k^2t^4+b^4k^2t^2+b^2k^2+b^3\beta^6k-2b^3\beta^4k-4b^2\beta^4k+b^3\beta^2k-2b^2\beta^2k\\
&+b^3\beta^6kt^4+2b^3\beta^6kt^2-2b^3\beta^4kt^4-4b^2\beta^4kt^4-4b^3\beta^4kt^2-8b^2\beta^4kt^2+b^3\beta^2kt^4+k^2t^2\\
&-2b^2\beta^2kt^4+2b^3\beta^2kt^2-4b^2\beta^2kt^2+3b^2\beta^4t^4+6b^2\beta^4t^2+b\beta^6k-2b\beta^4k-2b^4\beta^2k^2t^2\\
&+b\beta^6kt^4+2b\beta^6kt^2-2b\beta^4kt^4-4b\beta^4kt^2+b\beta^2kt^4+2b\beta^2kt^2+\beta^4k^2t^2-2\beta^2k^2t^2)\\
e_3 =&\frac{\beta^2(k-2\beta^2)^2}{\left(1-\beta^2k\right)^2}
\end{aligned}
$$