SYNCHRONOUS COUPLINGS OF REFLECTED BROWNIAN MOTIONS IN SMOOTH DOMAINS

Krzysztof Burdzy, **Zhen-Qing Chen** and **Peter Jones**

Abstract. For every bounded planar domain D with a smooth boundary, we define a "Lyapunov exponent" $\Lambda(D)$ using a fairly explicit formula. We consider two reflected Brownian motions in D, driven by the same Brownian motion (i.e., a "synchronous coupling"). If $\Lambda(D) > 0$ then the distance between the two Brownian particles goes to 0 exponentially fast with rate $\Lambda(D)/(2|D|)$ as time goes to infinity. The exponent $\Lambda(D)$ is strictly positive if the domain has at most one hole. It is an open problem whether there exists a domain with $\Lambda(D) < 0$.

1. Introduction and main results.

Suppose $D \subset \mathbf{R}^2$ is an open connected bounded set with C^4-smooth boundary, not necessarily simply connected. Let $\mathbf{n}(x)$ denote the unit inward normal vector at $x \in \partial D$. Let B be standard planar Brownian motion and consider the following Skorokhod equations,

$$X_t = x_0 + B_t + \int_0^t \mathbf{n}(X_s) dL_s^X \qquad \text{for } t \geq 0, \tag{1.1}$$

$$Y_t = y_0 + B_t + \int_0^t \mathbf{n}(Y_s) dL_s^Y \qquad \text{for } t \geq 0. \tag{1.2}$$

Here L^X is the local time of X on ∂D. In other words, L^X is a non-decreasing continuous process which does not increase when X is in D, i.e., $\int_0^\infty \mathbf{1}_D(X_t) dL_t^X = 0$, a.s. Equation (1.1) has a unique pathwise solution (X, L^X) such that $X_t \in \overline{D}$ for all $t \geq 0$ (see [LS]). The reflected Brownian motion X is a strong Markov process. The same remarks apply to (1.2), so (X, Y) is also strong Markov. We will call (X, Y) a "synchronous coupling." Note that on any interval (s, t) such that $X_u \in D$ and $Y_u \in D$ for all $u \in (s, t)$, we have $X_u - Y_u = X_s - Y_s$ for all $u \in (s, t)$.

Before we state our main results, we will introduce some notation and make some technical assumptions on ∂D. We will assume that for every point $x \in \partial D$, there exists a neighborhood U of x and an orthonormal system CS_x such that $\mathbf{n}(x) = (0, 1)$ and $x = (0, 0)$ in CS_x, and $\partial D \cap U$ is a part of the graph of a function $y_2 = \psi_x(y_1)$ satisfying $\psi_x(y_1) = (1/2)\nu(x) y_1^2 + O(y_1^3)$. This defines the curvature $\nu(x)$ for ∂D at x. We will assume that there is $N_1 < \infty$ such that for every unit vector $\mathbf{m}$, there are at most N_1 points $x \in \partial D$ with $\mathbf{n}(x) = \mathbf{m}$. Recall that ∂D is assumed to be C^4-smooth. We will assume that there is only a finite number of $x \in \partial D$ with $\nu(x) = 0$ and that for every such x, we have $\psi_x(y_1) = c_x y_1^3 + O(y_1^4)$ with $c_x \neq 0$. The distance between x and y will be denoted $\mathbf{d}(x, y)$.

Research partially supported by NSF grant DMS-0303310.

Theorem 1.1. *If D satisfies the above assumptions and it has at most one hole then $\mathbf{d}(X_t, Y_t) \to 0$ as $t \to \infty$, a.s., for every pair of starting points $(x_0, y_0) \in \overline{D} \times \overline{D}$.*

The above theorem complements the results in [BC] where it has been proved that the distance between X_t and Y_t converges to 0 as $t \to \infty$ for two classes of domains: (i) polygonal domains, i.e., domains whose boundary consists of a finite number of closed polygons, and (ii) "lip domains", i.e., bounded Lipschitz domains which lie between graphs of two Lipschitz functions that have Lipschitz constants strictly less than 1. The number of holes plays no role in the case of polygonal domains but it is an open problem whether it does in the case of smooth domains (see Section 2).

Earlier research of Cranston and Le Jan ([CLJ1, CLJ2]) on synchronous couplings of reflected Brownian motions was focused on convex domains. In that case, it is clear that $t \to \mathbf{d}(X_t, Y_t)$ is non-increasing. Cranston and Le Jan proved that for a large class of convex domains, $\mathbf{d}(X_t, Y_t) > 0$ for all $t \geq 0$, a.s., if $\mathbf{d}(X_0, Y_0) > 0$. The present paper, especially Theorem 1.2 below, answers a problem posed at the end of [CLJ1] and improves on the estimate given in the Appendix of [CLJ2].

Next we will present our main technical result on the "Lyapunov exponent," which is a crucial step in the proof of Theorem 1.1. We need some more notation. Let $\sigma_t^X = \inf\{s \geq 0 : L_s^X \geq t\}$. For every bounded planar domain D we have $\lim_{t\to\infty} L_t^X = \infty$ so $\sigma_t^X < \infty$ for all $t \geq 0$, a.s. The arc length measure on ∂D will be denoted "dx", e.g., we will write $\int_{\partial D} f(x)dx$ to denote the integral of f with respect to the arc length. For any $x, y \in \partial D$, we let $\alpha(x, y)$ be the angle formed by the tangent lines to ∂D at x and y, with the convention that $\alpha(x, y) \in [0, \pi/2]$. For every point $x \in \partial D$, let $\omega_x(dy)$ be the "harmonic measure" on ∂D with the base point x, defined as follows. Let $K(x, y)$, $y \in D$, be the Martin kernel in D with the pole at x, i.e., the only (up to a multiplicative constant) positive harmonic function in D which vanishes everywhere on the boundary of D except for a pole at x. Then we let $\omega_x(dy) = a_x \frac{\partial K(x,y)}{\partial \mathbf{n}(y)} dy$ where the constant a_x is chosen so that $\lim_{y\to x} \pi \mathbf{d}(x, y)^2 \omega_x(dy)/dy = 1$. Let $|D|$ denote the area of D.

Theorem 1.2. *Let*

$$\Lambda(D) = \int_{\partial D} \nu(x)dx + \int_{\partial D}\int_{\partial D} |\log\cos\alpha(x, y)|\omega_x(dy)dx. \tag{1.3}$$

If $\Lambda(D) > 0$, then for any $x_0, y_0 \in \overline{D}$, a.s.,

$$\lim_{t\to\infty} \frac{\log \mathbf{d}(X_t, Y_t)}{t} = -\frac{\Lambda(D)}{2|D|}. \tag{1.4}$$

By the Gauss-Bonnet Theorem, the first integral in (1.3), that is, $\int_{\partial D} \nu(x)dx$, is equal to $2\pi\chi(D)$, where $\chi(D)$ is the Euler characteristic of D. In our case, $\chi(D)$ is equal to 1 minus the number of holes in D. We are not aware of a simple representation of the second (double) integral in (1.3). The integral $\int_{\partial D} \nu(x)dx$, which appeared in [CLJ2], emerges in our arguments as the limit of $(1/t)\int_0^t \nu(X_s)dL_s^X$ when $t \to \infty$. See [H] for some results involving $\int_0^t \nu(X_s)dL_s^X$.

It is elementary to check using the definition (1.3) that $\Lambda(D)$ is invariant under scaling, i.e., for any $a > 0$, $\Lambda(D) = \Lambda(aD)$, where $aD = \{x \in \mathbf{R}^2 : x = ay \text{ for some } y \in D\}$.

We will now explain the intuitive content of Theorem 1.2. The disc with center x and radius r will be denoted $\mathcal{B}(x,r)$. Suppose that at some time t, $\mathbf{d}(X_t,Y_t)$ is very small so that when one of the processes is on the boundary of the domain then ∂D looks like a very flat parabola inside the disc $\mathcal{B}(X_t, 2\mathbf{d}(X_t,Y_t))$. Suppose further that the line segment $\overline{X_t,Y_t}$ is "almost" parallel to ∂D. Then the local time components in (1.1) and (1.2) will be almost identical over a short time period $[t, t+\Delta t]$, except for a small difference between the reflection vectors due to the curvature of ∂D. This small difference translates into the first integral in (1.3). From time to time, X makes large excursions from ∂D, whose endpoints are at a distance comparable to the diameter of D. At the end of any such excursion, one and only one of the processes X or Y gets a substantial local time push, until again $\overline{X_t,Y_t}$ is almost parallel to ∂D. This results in the reduction of $\mathbf{d}(X_t,Y_t)$ by a factor very close to $\cos\alpha(x,y)$, where x and y are the endpoints of the excursion. The double integral on the right hand side of (1.3) represents the change in $\mathbf{d}(X_t,Y_t)$ due to large excursions. We find it surprising and intriguing that the magnitudes of the two phenomena affecting the distance $\mathbf{d}(X_t,Y_t)$, described above, are comparable and give rise to two "independent" terms on the right hand side of (1.3).

We will briefly sketch the idea behind the proof of Theorem 1.2. First, we prove that the distance between the particles will be small at least from time to time, so that we can apply methods appropriate for processes reflecting on very flat parabolas. The main part of the proof deals with the two phenomena described in the previous paragraph. When the line segment $\overline{X_t,Y_t}$ is "almost parallel" to ∂D and one or both processes reflect on ∂D, the change of $\mathbf{d}(X_t,Y_t)$ is "almost" deterministic in nature and so are our methods. The change in $\mathbf{d}(X_t,Y_t)$ due to "large" excursions of X_t in D is much harder to analyze and that part of the proof is very complicated. We list here several of the challenges. First, it is conceivable that even a single excursion may result in a reduction of $\mathbf{d}(X_t,Y_t)$ to 0, if the endpoints of the excursion are at $x,y\in\partial D$ with $\cos\alpha(x,y)=0$. Proving that this is not the case takes considerable effort. Second, we use excursion theory and ergodicity of X_t to prove that $\log\mathbf{d}(X_t,Y_t)$ obeys a strong law of large numbers, in the sense of (1.4). The problem here is that although X_t is recurrent and ergodic, the vector process (X_t,Y_t) is neither, and so we have to analyze the behavior of Y_t by proving that it is "close" to that of X_t. Finally, one has to find upper bounds for probabilities of various "unusual" events which clearly cannot happen, from the intuitive point of view, but which have to be accounted for in a rigorous argument.

The rest of the paper is organized as follows. Section 2 is devoted to the discussion of some open problems and examples, mostly related to Theorem 1.1. It also contains a (very short) proof of Theorem 1.1. The proof of Theorem 1.2, consisting of many lemmas, is given in Sections 3 and 4. Most arguments in Section 3 are deterministic or analytic in nature. Section 4 contains arguments based on the excursion theory.

We are grateful to Greg Lawler, Nick Makarov, Don Marshall and Bálint Virág for very useful discussions and advice.

2. Examples and open problems.

The paper was inspired by the following problem which still remains open.

Problem 2.1. *(i) Does there exist a bounded planar domain such that with positive probability,*

$$\limsup_{t\to\infty} \mathbf{d}(X_t, Y_t) > 0?$$

(ii) Does there exist a bounded domain D with $\Lambda(D) < 0$?

The two problems are related to each other via the following conjecture.

Conjecture 2.2. *If $\Lambda(D) < 0$ then with probability one,* $\limsup_{t\to\infty} \mathbf{d}(X_t, Y_t) > 0$.

We believe that the above conjecture can be proved using the same methods as in the proof of Theorem 1.2. Since we do not know whether any domains with $\Lambda(D) < 0$ exist, we have little incentive to work out the details of the proof for Conjecture 2.2.

A technical problem arises in relation to Problem 2.1 (i)—it is not obvious how to define a "synchronous coupling" of reflected Brownian motions in an arbitrary domain. It is desirable from both technical and intuitive point of view to have the strong Markov property for the process (X_t, Y_t). See [BC] for a discussion of these points. So far, the existence of synchronous couplings of reflected Brownian motions with the strong Markov property can be proved only in these domains where the stochastic Skorokhod equations (1.1)-(1.2) have a unique strong solution. A recent paper ([BBC]) shows that this is the case when D is a planar Lipschitz domain with the Lipschitz constant less than 1.

We will next present some speculative directions of research related to Problem 2.1 (ii). We start by explaining how Theorem 1.1 follows from Theorem 1.2.

Proof of Theorem 1.1. If D has at most one hole then the first integral on the right hand side of (1.3) is equal to 2π or 0, by the Gauss-Bonnet Theorem. The integrand in the double integral in (1.3) is non-negative and it is easy to see that it is strictly positive on a non-negligible set. Hence, $\Lambda(D) > 0$ and, consequently, (1.4) holds, according to Theorem 1.2. Thus, Theorem 1.1 follows from Theorem 1.2. □

The above proof suggests the following strategy for finding a domain with $\Lambda(D) < 0$. One should find a domain where the first integral on the right hand side of (1.3) is significantly less than zero. This is because the contribution from the second term is always non-negative. In other words, one has to consider domains with many holes because, as we have already mentioned in Section 1, the first term is equal to 1 minus the number of holes, multiplied by 2π. The obvious problem with this strategy is that punching holes in a domain may increase the double integral on the right hand side of (1.3), and this may offset the effect of holes on the first integral.

Here is a possible avenue of research based on the above idea. Suppose that D has a large number of small holes. Here "small" means that the holes have diameters very small in comparison with the diameter of the domain. Let us assume that distances between different holes, and distances between holes and the outside boundary of D are large in comparison with diameters of holes. Then it is not hard to see that the right hand side of (1.3) is very close to the sum of analogous formulas for each connected component of ∂D. In other words, there is little interaction between different connected components of ∂D, if the holes are small and far apart. If we can find a hole with the shape which yields $\Lambda(D) < 0$ for a single hole, then $\Lambda(D) < 0$ for a domain D with a large number of small holes of this shape.

Simple heuristic estimates show that if ∂D has "approximate" corners, like a polygonal domain (the corners have to be "approximate" because the domain has to be smooth), then the double integral on the right hand side of (1.3) is very large. Domains, or rather holes, with this property will not help us in our search for a domain D with $\Lambda(D) < 0$. The ultimate domain without corners is a disc. At the moment we are concerned with "holes" so we will find $\Lambda(D)$ for D which is the exterior of a disc.

Recall that a disc with center x and radius r is denoted $\mathcal{B}(x, r)$.

Proposition 2.3. *If* $D = \mathcal{B}((0,0),1)^c$ *then* $\Lambda(D) = 0$.

Proof. Recall that the first integral in (1.3) is equal to -2π. We will parametrize ∂D using $\theta \in [0, 2\pi)$ and writing $x = e^{i\theta}$ for $x \in \partial D$. The formula for the harmonic measure in D is well known and easy to derive using standard complex analytic methods (conformal mappings). This easily leads to the following formula for the "harmonic measure" ω_x,

$$\frac{\omega_x(dy)}{dy} = \frac{1}{4\pi \sin^2(\frac{\theta-\theta'}{2})},$$

where $x = e^{i\theta}$ and $y = e^{i\theta'}$. Thus the double integral in (1.3) is equal to

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|\log|\cos(\theta-\theta')||}{4\pi \sin^2(\frac{\theta-\theta'}{2})} d\theta d\theta' = 2\pi \int_0^{2\pi} \frac{|\log|\cos(\theta)||}{4\pi \sin^2(\theta/2)} d\theta = \int_0^{\pi} \frac{|\log|\cos(\theta)||}{\sin^2(\theta/2)} d\theta.$$

We have

$$\begin{aligned}
&\int_0^{\pi/2} \frac{|\log|\cos(\theta)||}{\sin^2(\theta/2)} d\theta = -\int_0^{\pi/2} \frac{\log\cos(\theta)}{\sin^2(\theta/2)} d\theta \\
&= [2(\theta + \cot(\theta/2)\log\cos\theta - \log(\cos(\theta/2) - \sin(\theta/2)) + \log(\cos(\theta/2) + \sin(\theta/2))]\Big|_{\theta=0}^{\theta=\pi/2} \\
&= \pi + 2\log 2,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\pi/2}^{\pi} \frac{|\log|\cos(\theta)||}{\sin^2(\theta/2)} d\theta &= -\int_{\pi/2}^{\pi} \frac{\log(-\cos(\theta))}{\sin^2(\theta/2)} d\theta \\
&= \Big[2(\theta + \cot(\theta/2)\log(-\cos\theta) - \log(\sin(\theta/2) - \cos(\theta/2)) \\
&\qquad + \log(\cos(\theta/2) + \sin(\theta/2))\Big]\Big|_{\theta=\pi/2}^{\theta=\pi} \\
&= \pi - 2\log 2,
\end{aligned}$$

so

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|\log|\cos(\theta-\theta')||}{4\pi \sin^2(\frac{\theta-\theta'}{2})} d\theta d\theta' = \int_0^{\pi} \frac{|\log|\cos(\theta)||}{\sin^2(\theta/2)} d\theta = 2\pi,$$

and $\Lambda(D) = 0$. □

We have proved that $\Lambda(D) = 0$ for the exterior of a disc by a brute force calculation. It is a natural question whether the same result follows from some elegant symmetry argument—we have not found one so far.

Since $\Lambda(D) = 0$ for the exterior of the disc, discs are not helpful as holes in the (hypothetical) construction of a domain D with $\Lambda(D) < 0$. Our next observation is that the exterior of a line segment would be a great candidate for a useful hole. This is because for any points x and y on a line segment, we have $\alpha(x, y) = 0$ and, therefore, the double integral on the right hand side of (1.3) vanishes for the exterior of a line segment. Hence, $\Lambda(D) < 0$ for the exterior of a line segment. Unfortunately, we cannot use line segments as holes because their boundaries are not smooth. Instead, we can try a domain "close" to a line segment but with a smooth boundary. A natural candidate is a very elongated ellipse. Our preliminary numerical calculations showed that $\Lambda(D) = 0$ for the exterior of any ellipse. We are grateful to Bálint Virág for the following rigorous proof of this result.

Proposition 2.4. *If D is the exterior of an ellipse then $\Lambda(D) = 0$.*

Proof. We will use complex analysis and complex notation in this proof. Recall that $\Lambda(D)$ is invariant under scaling. Hence, we can consider any ellipse with the given eccentricity. In other words, it is enough to prove that the proposition holds for any ellipse D that can be represented as $D = g(U)$, where $U = \mathcal{B}(0, 1)$, $g(z) = z + a/z$, and a is any real number in $(0, 1)$.

We start by proving the following claim. Suppose that f is an analytic function in $\overline{U}$, $f(1)$ is purely imaginary, and $f'(1)$ is real. Then

$$\frac{1}{\pi}\int_{\partial U} \frac{\Re f(x)}{|1-x|^2}|dx| = -f'(1). \tag{2.1}$$

Since $\Re f(y)$ is harmonic in U and continuous on $\overline{U}$,

$$\Re f(y) = \frac{1}{2\pi}\int_{\partial U} \Re f(x)\frac{1-|y|^2}{|y-x|^2}|dx|,$$

for $y \in U$. We have assumed that $f(1)$ is imaginary and $f'(1)$ is real, so, by dominated convergence,

$$\begin{aligned}-f'(1) &= \lim_{r\to 0, r>0} \frac{\Re f(1-r) - \Re f(1)}{r} = \lim_{r\to 0, r>0} \frac{1}{2\pi}\int_{\partial U} \Re f(x)\frac{1-|1-r|^2}{r|1-r-x|^2}|dx| \\ &= \frac{1}{\pi}\int_{\partial U}\frac{\Re f(x)}{|1-x|^2}|dx|.\end{aligned}$$

We have shown that (2.1) holds.

The first integral in (1.3) is equal to -2π. It will suffice to show that the second (double) integral is equal to 2π. The second integral in (1.3) is equal to

$$\frac{1}{\pi}\int_{\partial U}\int_{\partial U} \frac{\log \Re\left(\frac{xg'(x)}{|g'(x)|}\cdot\frac{|g'(y)|}{yg'(y)}\right)}{|x-y|^2}|dx||dy|.$$

Note that $g'(z) = 1 - a/z^2$ and for $z \in \partial U$, $\overline{z} = 1/z$. Let $\beta = 1/(2g'(y)y)$ for some $y \in \partial U$. If we write $h(x) = \Re\left(\frac{xg'(x)}{yg'(y)}\right)$ then for $x \in \partial U$, we have

$$h(x) = \Re(2\beta g'(x)x) = \beta g'(x)x + \overline{\beta}\overline{g'(x)}\overline{x} = \beta(1 - a/x^2)x + \overline{\beta}(1 - ax^2)(1/x).$$

We have

$$\log \Re\left(\frac{xg'(x)}{|g'(x)|}\cdot\frac{|g'(y)|}{yg'(y)}\right) = \log\left(\frac{|g'(y)|}{|g'(x)|}\cdot\Re\left(\frac{xg'(x)}{yg'(y)}\right)\right) = \log\left(\frac{|g'(y)|}{|g'(x)|}\right) + \log h(x)$$
$$= \Re\log\left(\frac{g'(y)}{g'(x)}\right) + \Re\log h(x) = \Re\log\left(h(x)\cdot\frac{g'(y)}{g'(x)}\right).$$

Hence,

$$\frac{1}{\pi}\int_{\partial U}\frac{\log\Re\left(\frac{xg'(x)}{|g'(x)|}\cdot\frac{|g'(y)|}{yg'(y)}\right)}{|x-y|^2}|dx| = \frac{1}{\pi}\int_{\partial U}\frac{\Re\log\left(h(x)\cdot\frac{g'(y)}{g'(x)}\right)}{|x-y|^2}|dx|.$$

Recall that $|y|=1$. Substituting $x = y/v$, we see that the last integral is equal to

$$\frac{1}{\pi}\int_{\partial U}\frac{\Re\log\left(h(y/v)\cdot\frac{g'(y)}{g'(y/v)}\right)}{|x-y|^2}\frac{|x|^2}{|y|}|dv| = \frac{1}{\pi}\int_{\partial U}\frac{\Re\log\left(h(y/v)\cdot\frac{g'(y)}{g'(y/v)}\right)}{|1-v|^2}|dv|.$$

We have

$$h(y/v)\cdot\frac{g'(y)}{g'(y/v)} = \frac{(\beta(1-a/(y/v)^2)(y/v)+\overline{\beta}(1-a(y/v)^2)(1/(y/v)))g'(y)}{1-a/(y/v)^2}$$
$$= g'(y)\frac{y}{v}\frac{v^2(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})}{y^2-av^2}.$$

Let

$$k(v) = v\left(h(y/v)\cdot\frac{g'(y)}{g'(y/v)}\right) = g'(y)y\frac{v^2(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})}{y^2-av^2},$$

and note that, since $|v|=1$,

$$\frac{1}{\pi}\int_{\partial U}\frac{\Re\log\left(h(y/v)\cdot\frac{g'(y)}{g'(y/v)}\right)}{|1-v|^2}|dv| = \frac{1}{\pi}\int_{\partial U}\frac{\Re\log k(v)}{|1-v|^2}|dv|.$$

Next we will verify that (2.1) can be applied to $f(x) = \log k(x)$. We have for $y\in\partial U$,

$$\beta = \frac{1}{2g'(y)y} = \frac{1}{2(1-a/y^2)y} = \frac{y}{2(y^2-a)},$$

$$\overline{\beta} = \frac{\overline{y}}{2(\overline{y}^2-a)} = \frac{1/y}{2((1/y)^2-a)} = \frac{y}{2(1-ay^2)},$$

$$\frac{\overline{\beta}}{\beta} = \frac{y^2-a}{1-ay^2},$$

$$\frac{\beta-a\overline{\beta}}{\overline{\beta}-a\beta} = \frac{\frac{y}{2(y^2-a)}-a\frac{y}{2(1-ay^2)}}{\frac{y}{2(1-ay^2)}-a\frac{y}{2(y^2-a)}} = \frac{1+a^2-2ay^2}{y^2-2a+a^2y^2},$$

$$k(v) = g'(y)y\frac{v^2(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})}{y^2-av^2} = \frac{1}{2\beta}\frac{v^2(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})}{y^2-av^2},$$

$$\begin{aligned}
k(1) &= \frac{1}{2\beta}\frac{(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})}{y^2-a} = \frac{(\overline{\beta}/\beta-a)+y^2(1-a\overline{\beta}/\beta)}{2(y^2-a)} \\
&= \frac{(\frac{y^2-a}{1-ay^2}-a)+y^2(1-a\frac{y^2-a}{1-ay^2})}{2(y^2-a)} = 1,
\end{aligned}$$

$$\log k(1) = 0,$$

$$k'(v) = g'(y)y\frac{2v(\overline{\beta}-a\beta)(y^2-av^2)+2av[v^2(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})]}{(y^2-av^2)^2},$$

$$k'(v)/k(v) = \frac{2v(\overline{\beta}-a\beta)(y^2-av^2)+2av[v^2(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})]}{(y^2-av^2)[v^2(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})]},$$

$$\begin{aligned}
(\log k)'(1) = k'(1)/k(1) &= \frac{2(\overline{\beta}-a\beta)(y^2-a)+2a[(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})]}{(y^2-a)[(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})]} \\
&= \frac{2(\overline{\beta}-a\beta)}{(\overline{\beta}-a\beta)+y^2(\beta-a\overline{\beta})} + \frac{2a}{(y^2-a)} \\
&= \frac{2}{1+y^2\left(\frac{\beta-a\overline{\beta}}{\overline{\beta}-a\beta}\right)} + \frac{2a}{(y^2-a)} \\
&= \frac{2}{1+y^2\left(\frac{1+a^2-2ay^2}{y^2-2a+a^2y^2}\right)} + \frac{2a}{(y^2-a)} \\
&= \frac{2(y^2-2a+a^2y^2)}{2y^2-2a+2a^2y^2-2ay^4} + \frac{2a}{(y^2-a)} \\
&= \frac{(1-a^2)y^2}{(y^2-a)(1-ay^2)} \\
&= (1-a^2)\frac{1}{(1-a/y^2)(1-ay^2)} \\
&= (1-a^2)\frac{1}{|1-ay^2|^2} \in \mathbf{R}.
\end{aligned}$$

Since $\Re\log k(1) = 0$ and $(\log k)'(1)$ is real, we can apply (2.1) to obtain

$$\frac{1}{\pi}\int_{\partial U}\frac{\Re\log k(v)}{|1-v|^2}|dv| = -(\log k)'(1),$$

and

$$\begin{aligned}
\frac{1}{\pi}\int_{\partial U}\int_{\partial U}\frac{\log\Re\left(\frac{xg'(x)}{|g'(x)|}\cdot\frac{|g'(y)|}{yg'(y)}\right)}{|x-y|^2}|dx||dy| &= \int_{\partial U}\frac{1}{\pi}\int_{\partial U}\frac{\Re\log k(v)}{|1-v|^2}|dv||dy| \\
= -\int_{\partial U}(\log k)'(1)|dy| &= -\int_{\partial U}\frac{(1-a^2)y^2}{(y^2-a)(1-ay^2)}|dy| \\
= i\int_{\partial U}\frac{(1-a^2)y^2}{(y^2-a)(1-ay^2)}\frac{1}{y}dy. \qquad &(2.2)
\end{aligned}$$

The function $\frac{(1-a^2)y}{(y^2-a)(1-ay^2)}$ has two poles inside U, at $y = \pm\sqrt{a}$, and the residue is equal to $1/2$ at each of these points. Hence, by the residue theorem, the right hand side of (2.2) is equal to 2π. This completes the proof of the proposition. □

The last result raises some questions, but before we state them as a formal conjecture, we rush to add that it is very easy to see that $\Lambda(D) > 0$ for exteriors of some convex smooth domains, for example, those that have "approximate" corners.

Conjecture 2.5 *(i) If D is the exterior of a simply connected domain then $\Lambda(D) \geq 0$.*

(ii) If D is the exterior of a simply connected domain and $\Lambda(D) = 0$ then D^c is a disc or an ellipse.

Another problem, hard to state as a formal conjecture, is to find an (easy) way to derive $\Lambda(D)$ for the exterior of an ellipse from the value of this constant for the exterior of a disc. We point out an obvious fact that $\Lambda(D)$ is not invariant under conformal mappings. It is not hard to see that $\Lambda(D)$ is not invariant under the transformation $(x_1, x_2) \mapsto (cx_1, x_2)$.

Problem 2.6. *Let D be the exterior of a disc. Is it true that $\mathbf{d}(X_t, Y_t) \to 0$ as $t \to \infty$, a.s.?*

The last problem might be hard because it deals with the "critical" case, i.e., the case when $\Lambda(D) = 0$. On the other hand, the symmetries of the disc might be the basis of a reasonably easy proof, specific to this domain.

3. Analysis of Skorokhod transforms.

Notation. The following notation will be used throughout the paper.

All constants $c_1, c_2, \dots$ will take values in $(0, \infty)$ unless stated otherwise. We will write $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Recall that the distance between $x, y \in \mathbf{R}^2$ is denoted as $\mathbf{d}(x, y)$; the same symbol will be used to denote the distance between a point and a set, etc. Our arguments will involve elements of $\mathbf{R}$ or $\mathbf{R}^2$, and one- or two-dimensional vectors. We will use $|\cdot|$ to denote the usual Euclidean norm in all such cases. For $x, y \in \mathbf{R}^2$, the meaning of $|x - y|$ is the same as that of $\mathbf{d}(x, y)$ but we will nevertheless find it convenient to use both pieces of notation. The disc with center x and radius r will be denoted $\mathcal{B}(x, r)$. Recall the definition of curvature $\nu(x)$ at a point $x \in \partial D$, from the Introduction and let $\nu^* = \sup_{x \in \partial D} |\nu(x)|$. The unit inward normal vector at $x \in \partial D$ will be denoted as $\mathbf{n}(x)$. We will indicate coordinates of points and components of vectors by writing $X_t = (X_t^1, X_t^2)$, $Y_t = (Y_t^1, Y_t^2)$, $B_t = (B_t^1, B_t^2)$, and $\mathbf{n}(x) = (\mathbf{n}_1(x), \mathbf{n}_2(x))$, but this notation may refer to a coordinate system specific to a proof and different from the usual one. The angle between vectors $\mathbf{p}$ and $\mathbf{r}$ will be denoted $\angle(\mathbf{p}, \mathbf{r})$, with the convention that it takes values in $[0, \pi]$. Recall that $\alpha(x, y) = \angle(\mathbf{n}(x), \mathbf{n}(y)) \wedge (\pi - \angle(\mathbf{n}(x), \mathbf{n}(y)))$, for $x, y \in \partial D$.

The area of D and the length of its boundary will be denoted $|D|$ and $|\partial D|$, resp.

The distribution of the solution $\{(X_t, Y_t), t \geq 0\}$ to (1.1)-(1.2) will be denoted $\mathbf{P}^{x,y}$ and the distribution of $\{X_t, t \geq 0\}$ will be denoted $\mathbf{P}^x$. We will suppress the superscripts when no confusion may arise. We will denote the usual Markov shift operator by θ_t.

In the first of our lemmas, we will prove that for an arbitrarily small $\varepsilon_0 > 0$, for any two points $x_0, y_0 \in \overline{D}$, two synchronously coupled reflecting Brownian motions X and Y starting from x_0 and y_0 respectively will come within ε-distance from each other in finite time a.s. This claim is very similar to Lemma 3.3 of [BC] but sufficiently different to make it impossible for us to use that lemma in the present paper. Regrettably, we could not find a shorter proof of this seemingly quite intuitive result.

We will write $\tau_\varepsilon^+ = \tau^+(\varepsilon) = \inf\{t > 0 : \mathbf{d}(X_t, Y_t) \geq \varepsilon\}$ and $\tau_\varepsilon^- = \tau^-(\varepsilon) = \inf\{t > 0 : \mathbf{d}(X_t, Y_t) \leq \varepsilon\}$.

We remark that the following lemma holds for smooth domains in any dimension.

Lemma 3.1. *Consider any $\varepsilon_0 > 0$, any $x_0, y_0 \in \overline{D}$, and assume that $(X_0, Y_0) = (x_0, y_0)$. Then $\tau^-(\varepsilon_0) < \infty$ a.s.*

Proof. The proof will consist of several steps. In the first two steps, we will prove some properties of the deterministic Skorokhod mapping.

Step 1. Let $\gamma = (\gamma^1, \gamma^2) : [0, \infty) \to \mathbf{R}^2$ be a continuous function with $\gamma(0) \in \overline{D}$ and finite variation on each bounded interval of $[0, \infty)$. Let $\lfloor \gamma \rfloor_{s,t}$ denote the total variation of γ on $[s, t]$. We will use analogous notation for other functions. By the results of [LS], there exists a unique pair of continuous functions $\beta : [0, \infty) \to \overline{D}$ and $\eta : [0, \infty) \to \mathbf{R}^2$ with the following properties:

(i) $\lfloor \eta \rfloor_{s,t} \leq \lfloor \gamma \rfloor_{s,t}$ for every $0 \leq s \leq t$,

(ii) $\int_0^\infty \mathbf{1}_{\{\beta_s \in D\}} d\ell_s = 0$, where $\ell_t \stackrel{\text{df}}{=} \lfloor \eta \rfloor_{0,t}$,

(iii) $\eta_t = \int_0^t \mathbf{n}(\beta_s) d\ell_s$ for every $t \geq 0$, and

(iv) $\beta_t = \gamma_t + \eta_t$, for all $t \geq 0$.

We will call (β, η) the Skorokhod transform of γ; sometimes we will call β the Skorokhod transform and denote β by $\mathcal{S}(\gamma)$.

We will show that

(1.a) $\lfloor \beta \rfloor_{s,t} \leq \lfloor \gamma \rfloor_{s,t}$ for all $t > s \geq 0$, and

(1.b) for $c_1 \in (0, 1)$, $c_2, c_3 \in (0, \infty)$, there exists $c_4 > 0$ such that if $\beta^1_{t_1} - \beta^1_0 \leq (1 - c_1)(\gamma^1_{t_1} - \gamma^1_0)$, $\gamma^1_{t_1} - \gamma^1_0 > c_2$, and $\lfloor \gamma \rfloor_{0,t_1} \leq c_3$ then $\lfloor \gamma \rfloor_{0,t_1} - \lfloor \beta \rfloor_{0,t_1} \geq c_4$.

According to (8') of [LS],

$$dl_t = -1_{\partial D}(\beta_t) \langle \mathbf{n}(\beta_t),\, d\gamma_t \rangle \tag{3.1}$$

and so

$$d\beta_t = d\gamma_t + \mathbf{n}(\beta_t) dl_t = d\gamma_t - 1_{\partial D}(\beta_t) \mathbf{n}(\beta_t) \langle \mathbf{n}(\beta_t), d\gamma_t \rangle. \tag{3.2}$$

This proves that $\lfloor \beta \rfloor_{s,t} \leq \lfloor \gamma \rfloor_{s,t}$ for any $0 \leq s < t$, i.e., this proves (1.a).

Let θ_t denote the angle between $\mathbf{n}(\beta_t)$ and $d\gamma_t$ whenever $\beta_t \in \partial D$ and $d\gamma_t$ is defined. Otherwise, define $\theta_t = \pi/2$. Note that ℓ_t is non-decreasing, by its definition in (ii), so (3.1) implies that $\theta_t \in [\frac{\pi}{2}, \pi]$. Recall that $\mathbf{n}(x) = (\mathbf{n}_1(x), \mathbf{n}_2(x))$. By (3.2),

$$\begin{aligned}\int_0^{t_1} |\cos \theta_s| |d\gamma_s| &\geq \int_0^{t_1} \mathbf{n}_1(\beta_s) \cos \theta_s d\gamma_s \\ &= (\gamma^1_{t_1} - \gamma^1_0) - (\beta^1_{t_1} - \beta^1_0) \\ &\geq c_1(\gamma^1_{t_1} - \gamma^1_0) \geq c_1 c_2.\end{aligned}$$

Since $\int_0^{t_1} |d\gamma_s| = [\gamma]_{0,t_1} \leq c_3$, for $\delta := \min\{1,\, c_1c_2/(2c_3)\}$ we have from the above

$$\int_0^{t_1} |\cos \theta_s| 1_{\{\theta_s \in [\frac{\pi}{2} + \delta,\, \pi]\}} |d\gamma_s| \geq c_1 c_2 - c_3 \sin \delta \geq c_1 c_2 / 2. \tag{3.3}$$

On the other hand, $|d\beta_t| = |d\gamma_t| \sin\theta_t$ and so

$$\begin{aligned}
\lfloor \gamma \rfloor_{0,t_1} - \lfloor \beta \rfloor_{0,t_1} &= \int_0^{t_1} (1-\sin\theta_s)|d\gamma_s| \\
&\geq \frac{1}{2}\int_0^{t_1} \cos^2\theta_s |d\gamma_s| \\
&\geq \frac{1}{2}\int_0^{t_1} 1_{\{\theta_s\in[\frac{\pi}{2}+\delta,\,\pi]\}} \cos^2\theta_s |d\gamma_s| \\
&\geq \frac{\sin\delta}{2}\int_0^{t_1} 1_{\{\theta_s\in[\frac{\pi}{2}+\delta,\,\pi]\}} |\cos\theta_s|\, |d\gamma_s| \\
&\geq \frac{c_1 c_2 \sin\delta}{4} := c_4.
\end{aligned}$$

This proves (1.b).

Step 2. Since D is bounded and has a smooth boundary, there is a constant $c_1 < \infty$ such that any two points $x, y \in \overline{D}$ can be connected by a C^∞ curve inside $\overline{D}$ of length $t_1 = t_1(x,y) < c_1$. Consider any $x, y \in \overline{D}$, and fix some C^∞ curve $\gamma : [0, t_1] \to \overline{D}$ with the natural (length) parametrization, and such that $t_1 < c_1$, $\gamma_0 = x$ and $\gamma_{t_1} = y$.

In this step, we will extend the definition of γ from $[0, t_1]$ to $[0, \infty)$. We will show that for any D and $\varepsilon > 0$, there exists a constant $c_2 \in [c_1, \infty)$ such that any curve γ defined initially on $[0, t_1]$ may be extended to $[0, \infty)$ in such a way that for some $t \leq c_2$,

$$|\gamma_t - \mathcal{S}(\gamma + y - x)_t| \leq \varepsilon. \tag{3.4}$$

Recall that $\mathcal{S}(\gamma)$ is the Skorokhod transform of γ (see Step 1).

Let $\{\beta_t, 0 \leq t \leq t_1\}$ be the Skorokhod transform of $\{\gamma_t + \gamma_{t_1} - \gamma_0, 0 \leq t \leq t_1\}$, defined as in Step 1. We will inductively define γ_t for all $t \geq 0$. Let $\gamma_t = \beta_{t-t_1}$ for $t \in [t_1, 2t_1]$, and let $\{\beta_t, t \in [t_1, 2t_1]\}$ be the Skorokhod transform of $\{\gamma_t + \gamma_{2t_1} - \gamma_{t_1}, t \in [t_1, 2t_1]\}$. We continue by induction, i.e., we let $\gamma_t = \beta_{t-kt_1}$ for $t \in [kt_1, (k+1)t_1],$, $k \geq 2$, and we let $\{\beta_t, t \in [kt_1, (k+1)t_1]\}$ be the Skorokhod transform of $\{\gamma_t + \gamma_{(k+1)t_1} - \gamma_{kt_1}, t \in [kt_1, (k+1)t_1]\}$. Note that both γ_t and β_t stay in $\overline{D}$ for all $t \geq 0$. Clearly

$$\beta = \mathcal{S}(\gamma + y - x) \qquad \text{and} \qquad \gamma_t = \beta_{t-t_1} \quad \text{for every } t \geq t_1. \tag{3.5}$$

By (1.a) in Step 1 and (3.5), we have $\lfloor \gamma \rfloor_{s,t} \leq t - s$ for all $0 \leq s \leq t < \infty$.

If $|\gamma_0 - \beta_0| \leq \varepsilon$ then we are done. Otherwise, at least one of the following inequalities holds, $|\gamma_0^1 - \beta_0^1| \geq \varepsilon/2$ or $|\gamma_0^2 - \beta_0^2| \geq \varepsilon/2$. We will assume without loss of generality that it is the first of the two inequalities that holds and we will make another harmless assumption that in fact $\gamma_0^1 - \beta_0^1 \leq -\varepsilon/2$, or, equivalently, $\gamma_{t_1}^1 - \gamma_0^1 \geq \varepsilon/2$. Let c_3 be the diameter of D. Fix some $c_4 \in (0,1)$ and integer $j > 1$ such that $\sum_{k=1}^j c_4^{k-1}\varepsilon/2 > 2c_3$. If we had

$$\gamma_{kt_1}^1 - \gamma_{(k-1)t_1}^1 \geq c_4(\gamma_{(k-1)t_1}^1 - \gamma_{(k-2)t_1}^1) \quad \text{for every } 2 \leq k \leq j, \tag{3.6}$$

then we would obtain

$$\gamma_{jt_1}^1 - \gamma_0^1 = \sum_{k=1}^j \gamma_{kt_1}^1 - \gamma_{(k-1)t_1}^1 \geq \sum_{k=1}^j c_4^{k-1}\varepsilon/2 > 2c_3,$$

and that would contradict the definition of c_3 as the diameter of D. So there must be some $k_0 \leq j$ such that

$$\gamma^1_{k_0 t_1} - \gamma^1_{(k_0-1)t_1} \leq c_4(\gamma^1_{(k_0-1)t_1} - \gamma^1_{(k_0-2)t_1}). \tag{3.7}$$

Let k_0 be the smallest integer with this property. Then

$$\gamma^1_{kt_1} - \gamma^1_{(k-1)t_1} \geq c_4(\gamma^1_{(k-1)t_1} - \gamma^1_{(k-2)t_1})$$

for all $k < k_0$ and so $\gamma^1_{(k_0-1)t_1} - \gamma^1_{(k_0-2)t_1} \geq c_4^j \varepsilon/2$. The following is equivalent to (3.7),

$$\beta^1_{(k_0-1)t_1} - \beta^1_{(k_0-2)t_1} \leq c_4(\gamma^1_{(k_0-1)t_1} - \gamma^1_{(k_0-2)t_1}).$$

By (1.b) of Step 1, for some $c_5 > 0$,

$$\lfloor \gamma \rfloor_{(k_0-2)t_1,(k_0-1)t_1} - \lfloor \beta \rfloor_{(k_0-2)t_1,(k_0-1)t_1} \geq c_5. \tag{3.8}$$

If $\lfloor \gamma \rfloor_{(k-1)t_1,kt_1} \leq \varepsilon$ for some $k \leq j+1$ then

$$|\gamma_{(k-1)t_1} - \beta_{(k-1)t_1}| = |\gamma_{(k-1)t_1} - \gamma_{kt_1}| \leq \lfloor \gamma \rfloor_{(k-1)t_1,kt_1} \leq \varepsilon,$$

and we can take $c_2 = jt_1$, i.e., there exists $t \leq jt_1$ with $|\gamma_t - \beta_t| \leq \varepsilon$.

If $\lfloor \gamma \rfloor_{(k-1)t_1,kt_1} > \varepsilon$ for all $k \leq j+1$, then by Step 1 and (3.8),

$$\begin{aligned}\lfloor \gamma \rfloor_{0,jt_1} - \lfloor \beta \rfloor_{0,jt_1} &= \sum_{k=1}^{j} \lfloor \gamma \rfloor_{(k-1)t_1,kt_1} - \lfloor \beta \rfloor_{(k-1)t_1,kt_1} \\ &\geq \lfloor \gamma \rfloor_{(k_0-2)t_1,(k_0-1)t_1} - \lfloor \beta \rfloor_{(k_0-2)t_1,(k_0-1)t_1} \geq c_5.\end{aligned}$$

Thus we have shown that either there exists $0 \leq t \leq jt_1$ with $|\gamma_t - \beta_t| \leq \varepsilon$ or $\lfloor \gamma \rfloor_{t_1,(j+1)t_1} = \lfloor \beta \rfloor_{0,jt_1} \leq \lfloor \gamma \rfloor_{0,jt_1} - c_5$. The same argument shows that either there exists $t \in [kt_1, (k+j)t_1]$ with $|\gamma_t - \beta_t| \leq \varepsilon$ or

$$\lfloor \gamma \rfloor_{(k+1)t_1,(k+1+j)t_1} = \lfloor \beta \rfloor_{kt_1,(k+j)t_1} \leq \lfloor \gamma \rfloor_{kt_1,(k+j)t_1} - c_5. \tag{3.9}$$

Recall that $\lfloor \gamma \rfloor_{0,t_1} = t_1 \leq c_1$, and $\lfloor \gamma \rfloor_{kt_1,(k+1)t_1} = \lfloor \beta \rfloor_{(k-1)t_1,kt_1} \leq \lfloor \gamma \rfloor_{(k-1)t_1,kt_1}$ for all k. Hence, $\lfloor \gamma \rfloor_{0,jt_1} \leq jc_1$, and if (3.9) holds for all $k \leq m$, then

$$0 \leq \lfloor \gamma \rfloor_{mt_1,(m+j)t_1} \leq jc_1 - mc_5.$$

This can be true only if $m \leq jc_1/c_5$. Hence, for some $k \leq jc_1/c_5 + 1$ and some $t \in [0, (k+j)t_1]$ we have $|\gamma_t - \beta_t| \leq \varepsilon$.

Step 3. First, we will present a version of the "support theorem" stronger than that given in Theorem I (6.6) in [Ba]. Recall that one calls a continuous non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ a modulus of continuity for a function $\gamma : [0,t_1] \to \mathbf{R}^2$ if for all $s,t \in [0,t_1]$ we have $|\gamma_t - \gamma_s| \leq \psi(|t-s|)$. Let $\mathcal{K}_{\psi,t_1}$ denote the family of all functions $\gamma : [0,t_1] \to \mathbf{R}^2$ with modulus of continuity ψ. Let $\mathbf{P}$ denote the Wiener measure on $C[0,t_1]^2$, i.e., the distribution of the planar Brownian motion. It follows easily from the existence of "Lévy's modulus of continuity" (see Theorem 2.9.25 in [KS]), that for every $t_1 \in (0,\infty)$ and $p_0 < 1$ there exists ψ such that

$\mathbf{P}(\mathcal{K}_{\psi,t_1}) > p_0$. This fact can be used to modify the proof of Proposition I (6.5) of [Ba] to show that there exists ψ such that for any $\varepsilon > 0$ one can find $p_1 > 0$ with

$$\mathbf{P}(\{\gamma \in \mathcal{K}_{\psi,t_1} : \sup_{0\le t\le t_1} |\gamma_t| \le \varepsilon\}) \ge p_1. \tag{3.10}$$

Let $\psi_\lambda(t) = \psi(t) + \lambda t$ and for $\phi : [0, t_1] \to \mathbf{R}^2$, let

$$\mathcal{K}_{\psi,t_1,\phi,\varepsilon} = \{\gamma \in \mathcal{K}_{\psi,t_1} : \sup_{0\le t\le t_1} |\gamma_t - \phi_t| \le \varepsilon\}.$$

If $\gamma \in \mathcal{K}_{\psi,t_1}$ and ϕ is Lipschitz with constant λ then $\gamma + \phi \in \mathcal{K}_{\psi_\lambda,t_1}$. The proof of Theorem I (6.6) in [Ba] can be easily modified to yield the following version of the support theorem. Suppose that $\varepsilon, p_1 > 0$ and ψ satisfy (3.10). Then for every $\lambda, t_1 < \infty$ and $\varepsilon' > 0$ one can find $p_2 > 0$ such that for any function $\phi : [0, t_1] \to \mathbf{R}^2$ which is Lipschitz with constant λ and satisfies $\phi(0) = 0$, we have $\mathbf{P}(\mathcal{K}_{\psi_\lambda,t_1,\phi,\varepsilon'}) \ge p_2$. The important aspect of the last assertion is that p_2 does not depend on ϕ. Fix a function ψ satisfying this statement for the rest of the proof.

Recall that $\mathcal{S}(\gamma)$ denotes the Skorokhod transform of γ (see Step 1) and let ε_0 be as in the statement of the lemma. Let c_1 be the constant defined in Step 2 and $c_2 > c_1$ be the constant in Step 2 relative to $\varepsilon_0/5$ in place of ε. By Theorem 1.1 in [LS], the Skorokhod mapping $\mathcal{S} : C([0, c_2], \mathbf{R}^2) \to C([0, c_2], \mathbf{R}^2)$ is Hölder continuous on compact sets. Let $\mathcal{K} = \{\gamma \in \mathcal{K}_{\psi_2,c_2} : \gamma_0 \in \overline{D}\}$. The set $\mathcal{K}$ is compact so one can find $\varepsilon_1 \in (0, \varepsilon_0/5)$ such that if $\gamma, \widehat{\gamma} \in \mathcal{K}$ and $|\widehat{\gamma}_t - \gamma_t| \le \varepsilon_1$ for $0 \le t \le c_2$, then $|\mathcal{S}(\widehat{\gamma})_t - \mathcal{S}(\gamma)_t| \le \varepsilon_0/5$ for $0 \le t \le c_2$.

Recall from Step 2 that for every pair of points $x, y \in \overline{D}$ there is a curve $\gamma = \gamma^{x,y} : [0,\infty) \to \overline{D}$ such that $\gamma_0 = x$, $\gamma_{t_1} = y$ for some $0 < t_1 < c_1$, and $\lfloor\gamma\rfloor_{s,t} \le t - s$, for all s and t in $[0, t_1]$. By Step 2, we can extend γ to be a curve in $\overline{D}$ satisfying (3.4) and (3.5). Note that γ is a Lipschitz curve on $[0,\infty)$ with Lipschitz constant 1.

Recall that reflected Brownian motions X_t and Y_t are defined in (1.1)-(1.2) relative to a Brownian motion B_t and assume that $X_0 = x$ and $Y_0 = y$. Find $p_2 > 0$ such that $\mathbf{P}(\mathcal{K}_{\psi_2,c_2,\phi,\varepsilon_1}) \ge p_2$ for every Lipschitz function ϕ with Lipschitz constant 1 satisfying $\phi(0) = 0$. It follows that

$$\mathbf{P}\left(\{B_t + x, 0 \le t \le c_2\} \in \mathcal{K}_{\psi_2,c_2,\gamma^{x,y},\varepsilon_1}\right) \ge p_2. \tag{3.11}$$

Consider ω such that $B.(\omega) + x \in \mathcal{K}_{\psi_2,c_2,\gamma^{x,y},\varepsilon_1} \subset \mathcal{K}_{\psi_2,c_2}$. Then

$$|\mathcal{S}(B. + x)_t - \mathcal{S}(\gamma^{x,y})_t| \le \varepsilon_0/5 \qquad \text{for every } 0 \le t \le c_2.$$

Clearly $B. + y \in \mathcal{K}_{\psi_2,c_2}$. Since $|B_t + y - (\gamma^{x,y}_t + y - x)| \le \varepsilon_1$ for $t \in [0, c_2]$, we have

$$|\mathcal{S}(B. + y)_t - \mathcal{S}(\gamma^{x,y} + y - x)_t| \le \varepsilon_0/5 \qquad \text{for } 0 \le t \le c_2.$$

Note that by Step 2 and our choice of c_2, there is some $t_0 \in [0, c_2]$ such that $|\gamma^{x,y}_{t_0} - \mathcal{S}(\gamma^{x,y} + y - x)_{t_0}| \le \varepsilon_0/5$, for some $t_0 \in [0, c_2]$. Note also that since $\gamma[0,\infty) \subset \overline{D}$, by the uniqueness of the Skorokhod problem, $S(\gamma) = \gamma$. Combining these observations, we conclude that

$$\begin{aligned}
&|X_{t_0} - Y_{t_0}| = |\mathcal{S}(B. + x)_{t_0} - \mathcal{S}(B. + y)_{t_0}| \\
&\quad = |\mathcal{S}(B. + x)_{t_0} - \gamma_{t_0}| + |\gamma_{t_0} - \mathcal{S}(\gamma + y - x)_{t_0}| + |\mathcal{S}(\gamma + y - x)_{t_0} - \mathcal{S}(B. + y)_{t_0}| \\
&\quad \le \varepsilon_0/5 + \varepsilon_0/5 + \varepsilon_0/5 < \varepsilon_0.
\end{aligned}$$

It follows from (3.11) that there exists $p_2 > 0$ such that for any $x, y \in \overline{D}$, $X_0 = x$, $Y_0 = y$, the probability that there exists $t_0 \le c_2$ with $\mathbf{d}(X_{t_0}, Y_{t_0}) \le \varepsilon_0$ is greater than p_2. By the Markov property applied at times jc_2, $j = 1, 2, \ldots$, the probability that there is no $t_0 \le kc_2$ with $\mathbf{d}(X_{t_0}, Y_{t_0}) \le \varepsilon_0$ is bounded above by $(1 - p_2)^k$. This implies easily that with probability one, there exists $t < \infty$ with $\mathbf{d}(X_t, Y_t) \le \varepsilon_0$. □

Lemma 3.2. *Let $\delta(x)$ denote the Euclidean distance between x and ∂D. Define $\tau_D = \inf\{t \ge 0 : X_t \notin D\}$ and $\tau_{\mathcal{B}(x,r)} = \inf\{t \ge 0 : X_t \notin \mathcal{B}(x, r)\}$. Then there exists $c_1 < \infty$ such that for $X_0 = x_0 \in D$,*

$$\mathbf{P}(\tau_{\mathcal{B}(x_0,r)} \le \tau_D) \le c_1 \delta(x_0)/r \qquad \text{for } r \ge \delta(x_0). \tag{3.12}$$

Proof. We are going to prove that (3.12) holds for any bounded $C^{1,1}$-smooth domain in $\mathbf{R}^n$, for any $n \ge 2$.

Since D is a bounded $C^{1,1}$-smooth domain, the "uniform" boundary Harnack principle holds for D (see [A]), that is, there exist $r_0 > 0$ and $c > 0$ such that for $z \in \partial D$, $r \in (0, r_0]$ and any non-negative harmonic functions u and v in $D \cap \mathcal{B}(z, 2r)$ that vanish continuously on $\partial D \cap \mathcal{B}(z, 2r)$, we have

$$\frac{u(x)}{v(x)} \le c\,\frac{u(y)}{v(y)} \qquad \text{for any } x, y \in D \cap \mathcal{B}(z, r).$$

Let $\{K_D(x, z);\, x \in D,\, z \in \partial D\}$ denote the Poisson kernel of the Brownian motion W killed upon leaving D; that is,

$$\mathbb{E}^x\left[\phi(W_{\tau_D})\right] = \int_{\partial D} K_D(x, z)\phi(z)\sigma(dz)$$

for every continuous function ϕ on ∂D, where σ denotes the surface area measure. Since D is bounded $C^{1,1}$-smooth, it is known (see [Z]) that there are constants $c_3 > c_2 > 0$ such that

$$\frac{c_2\delta(x)}{|x - z|^n} \le K_D(x, z) \le \frac{c_3\delta(x)}{|x - z|^n} \qquad \text{for every } x \in D \text{ and } z \in \partial D. \tag{3.13}$$

Note that $x \mapsto K_D(x, z)$ is a harmonic function in D and vanishes continuously on $\partial D \setminus \{z\}$.

The lemma clearly holds when $\delta(x_0) \ge r_0/8$. This is because, since D is bounded, there is $R > 0$ such that for every $x_0 \in D$, $D \subset \mathcal{B}(x_0, R)$ and so (3.12) holds trivially for $r > R$. Thus in the case of $\delta(x_0) \ge r_0/8$, (3.12) holds for every $r > 0$ by choosing c_1 sufficiently large.

We now assume $\delta(x_0) < r_0/8$. Without loss of generality, we may and do assume that $r > 8\delta(x_0)$ and $\mathcal{B}(x_0, 2r)^c \cap D \ne \emptyset$. We can further assume that $r \le r_0$ since D is bounded.

Define $h(x) = \mathbf{P}_x(\tau_{\mathcal{B}(x_0,r)} \le \tau_D)$. Clearly, h is a harmonic function in $D \cap \mathcal{B}(x_0, r)$ and vanishes continuously on $\partial D \cap \mathcal{B}(x_0, r)$. Let $y_0 \in \partial D$ be such that $\delta(x_0) = \mathrm{dist}(x_0, y_0)$. By the triangle inequality, $\mathcal{B}(y_0, r/2) \subset \mathcal{B}(x_0, r)$. Now take $z \in D \setminus \mathcal{B}(x_0, 2r)$. Since $x_0 \in \mathcal{B}(y_0, r/4)$, we have by the boundary Harnack inequality,

$$\frac{h(x_0)}{K_D(x_0, z)} \le c\frac{h(x)}{K_D(x, z)} \qquad \text{for every } x \in \mathcal{B}(y_0, r/4).$$

Let $x = x_0 + \frac{r}{8}(x_0 - y_0)$. Note that $h(x) \leq 1$,

$$\frac{1}{2}|z - x_0| \leq |z - x| \leq 2|z - x_0|,$$

and $\delta(x) \geq r/8$. These facts and (3.13) imply that $h(x_0) \leq c_1 \delta(x_0)/r$. This proves the lemma. $\square$

We fix parameters $a_1, a_2 > 0$ for the rest of the paper. We will impose bounds on their values later on. Let $S_0 = U_0 = 0$ and for $k \geq 1$ define

$$S_k = \inf\{t > U_{k-1} : \mathbf{d}(X_t, \partial D) \vee \mathbf{d}(Y_t, \partial D) \leq a_2 \mathbf{d}(X_t, Y_t)^2\},$$
$$U_k = \inf\{t > S_k : \mathbf{d}(X_t, X_{S_k}) \vee \mathbf{d}(Y_t, Y_{S_k}) \geq a_1 \mathbf{d}(X_{S_k}, Y_{S_k})\}.$$

We will assume that $a_1 < 1/4$. Then it is easy to see that $\mathbf{P}(U_k < \infty \mid S_k < \infty) = 1$, for every k. Finiteness of S_k's is less obvious. The next lemma contains a result that is significantly stronger than the finiteness of S_k's. This stronger result is needed in later arguments.

Lemma 3.3. *There exist $c_1, c_2, c_3, c_4 \in (0, \infty)$ and $\varepsilon_0, r_0, p_0 > 0$ with the following properties. Assume that $X_0 \in \partial D$, $\mathbf{d}(X_0, Y_0) = \varepsilon$, $\mathbf{d}(Y_0, \partial D) = r$ and let*

$$T_1 = \inf\{t \geq 0 : \mathbf{d}(X_t, X_0) \vee \mathbf{d}(Y_t, Y_0) \geq c_1 r\}.$$

(i) If $\varepsilon \leq \varepsilon_0$ and $r \leq r_0$ then $\mathbf{P}(S_1 \leq T_1, L^X_{S_1} - L^X_0 \leq c_2 r) \geq p_0$.
(ii) If $\varepsilon \leq \varepsilon_0$ and $r \leq c_3 \varepsilon$ then $\mathbf{E}(L^X_{S_1 \wedge \tau^+(\varepsilon_0)} - L^X_0) \leq c_4 r$.

Proof. (i) Recall the notation from the beginning of this section. Let CS_1 be the orthonormal coordinate system such that $X_0 = 0$ and $\mathbf{n}(X_0)$ lies on the second axis. Assume that $r_0 < \varepsilon_0 < 1/(200\nu^*)$. Let $c_5 \in (0, 1/6)$ be a small constant whose value will be chosen later. The following definitions refer to the coordinates in CS_1,

$$T_2 = \inf\{t \geq 0 : Y^2_t \geq 2r\},$$
$$T_3 = \inf\{t \geq 0 : |Y^1_t - Y^1_0| \geq c_5 r\},$$
$$T_4 = \inf\{t \geq 0 : Y_t \in \partial D\},$$
$$A_1 = \{T_4 \leq T_2 \wedge T_3\},$$
$$T_5 = \inf\{t \geq 0 : |X^1_t - X^1_0| \geq 2c_5 r\}.$$

First we will assume that $r \leq \varepsilon/2$. We will show that $T_5 \geq T_2 \wedge T_3 \wedge T_4$ if A_1 holds. We will argue by contradiction. Assume that A_1 holds and $T_5 < T_2 \wedge T_3 \wedge T_4$. Then $B^1_t - B^1_0 = Y^1_t - Y^1_0$ for $t \in [0, T_5]$ so $|B^1_t - B^1_0| \leq c_5 r$ for the same range of t's. We have

$$X^1_{T_5} - X^1_0 = B^1_{T_5} - B^1_0 + \int_0^{T_5} \mathbf{n}_1(X_t) dL^X_t,$$

so $\left| \int_0^{T_5} \mathbf{n}_1(X_t) dL^X_t \right| \geq c_5 r$. We assume that $\varepsilon_0 > 0$ is so small that for $r < \varepsilon_0$ and $x \in \mathcal{B}(0, 2c_5 r)$, we have $\mathbf{n}_2(x) \geq |\mathbf{n}_1(x)|/(2\nu^* \cdot 2c_5 r)$. It follows that

$$\int_0^{T_5} \mathbf{n}_2(X_t) dL^X_t \geq c_5 r/(2\nu^* \cdot 2c_5 r) = 1/(4\nu^*).$$

Note that $B^2_t - B^2_0 = Y^2_t - Y^2_0$ for $t \in [0, T_4]$. Since $\mathbf{d}(X_0, Y_0) = \varepsilon$, we have

$$\mathbf{d}(X_0, Y_t) \leq \varepsilon + \sqrt{(2r)^2 + (c_5 r)^2} < \varepsilon + 3r < 3\varepsilon \qquad \text{for } t \leq T_2 \wedge T_3 \wedge T_4.$$

Therefore for $t \leq T_2 \wedge T_3 \wedge T_4$, $|Y^2_t| \leq 3\varepsilon$. Since $T_5 < T_2 \wedge T_3 \wedge T_4$, it follows that

$$|B^2_t - B^2_s| = |Y^2_t - Y^2_s| \leq |Y^2_t| + |Y^2_s| \leq 6\varepsilon \qquad \text{for } s, t \in [0, T_5].$$

Thus

$$\begin{aligned} X^2_{T_5} - X^2_0 &\geq -|B^2_{T_5} - B^2_0| + \int_0^{T_5} \mathbf{n}_2(X_t) dL^X_t \\ &\geq -6\varepsilon + 1/(4\nu^*) \geq -6\varepsilon_0 + 1/(4\nu^*) \geq 44\varepsilon_0, \end{aligned}$$

and $X^2_{T_5} \geq 44\varepsilon_0 + X^2_0 = 44\varepsilon_0$. Let $T_6 = \sup\{t \leq T_5 : X_t \in \partial D\}$. Then $B^2_{T_5} - B^2_{T_6} = X^2_{T_5} - X^2_{T_6} \geq 44\varepsilon_0 - r \geq 43\varepsilon_0$, a contradiction with the fact that $|B^2_t - B^2_s| \leq 6\varepsilon \leq 6\varepsilon_0$ for $s, t \in [0, T_5]$. This proves that $T_5 \geq T_2 \wedge T_3 \wedge T_4$ if A_1 holds.

We will show that if A_1 holds then $S_1 \leq T_4$. Assume that A_1 holds and let $T_7 = \sup\{t \leq T_4 : X_t \in \partial D\}$. Note that neither X_t nor Y_t visit ∂D on the interval (T_7, T_4). Hence, $X_{T_7} - Y_{T_7} = X_{T_4} - Y_{T_4}$. If ε_0 and r_0 are sufficiently small then $|X^1_0 - Y^1_0| \geq \varepsilon/2$ because $r \leq \varepsilon/2$ and $\mathbf{d}(Y_0, \partial D) = r$. We have assumed that A_1 holds so $|Y^1_{T_4} - Y^1_0| \leq c_5 r$. We have proved that $T_5 \geq T_4$ on A_1, so $|X^1_{T_4} - X^1_0| \leq 2c_5 r$. Recall that $c_5 \leq 1/6$ and $r \leq \varepsilon/2$. It follows that

$$\begin{aligned} \mathbf{d}(X_{T_7}, Y_{T_7}) = \mathbf{d}(X_{T_4}, Y_{T_4}) &\geq |X^1_{T_4} - Y^1_{T_4}| \\ &\geq |X^1_0 - Y^1_0| - |Y^1_{T_4} - Y^1_0| - |X^1_{T_4} - X^1_0| \geq \varepsilon/2 - 3c_5 r \geq \varepsilon/4. \end{aligned}$$

On the other hand, assuming $\varepsilon_0 > 0$ is small,

$$\begin{aligned} \mathbf{d}(X_{T_7}, Y_{T_7}) &\leq \mathbf{d}(X_{T_7}, X_0) + \mathbf{d}(X_0, Y_0) + \mathbf{d}(Y_0, Y_{T_7}) \\ &\leq 2|X^1_{T_7} - X^1_0| + \varepsilon + \mathbf{d}(Y_0, Y_{T_7}) \leq 2 \cdot 2c_5 r + \varepsilon + 3r \leq 3\varepsilon. \end{aligned}$$

We have

$$|Y^1_{T_4} - Y^1_{T_7}| = |Y^1_{T_4} - Y^1_0| + |Y^1_0 - Y^1_{T_7}| \leq c_5 r + c_5 r = 2c_5 r.$$

Since $Y_{T_4} \in \partial D$, $X_{T_7} \in \partial D$, $X_{T_7} - Y_{T_7} = X_{T_4} - Y_{T_4}$, $\mathbf{d}(X_{T_7}, Y_{T_7}) \leq 3\varepsilon$, and $|Y^1_{T_4} - Y^1_{T_7}| \leq 2c_5 r$, we have $\angle(\mathbf{n}(Y_{T_4}), \mathbf{n}(X_{T_7})) \leq 2\nu^* \cdot 2(3\varepsilon + 2c_5 r) \leq 16\nu^*\varepsilon$. This and easy geometry show that $\mathbf{d}(Y_{T_7}, \partial D) \leq 2 \cdot 2c_5 r \cdot 16\nu^*\varepsilon = 64c_5 r\nu^*\varepsilon$. Hence,

$$\frac{\mathbf{d}(Y_{T_7}, \partial D)}{\mathbf{d}(X_{T_7}, Y_{T_7})} \leq \frac{64c_5 r\nu^*\varepsilon}{\varepsilon/4} = 256c_5 r\nu^* \leq 128c_5\nu^*\varepsilon \leq 32c_5\nu^*\mathbf{d}(X_{T_7}, Y_{T_7}).$$

We choose $c_5 > 0$ so small that $32c_5\nu^* \leq a_2$. Then $\mathbf{d}(Y_{T_7}, \partial D) \leq a_2\mathbf{d}(X_{T_7}, Y_{T_7})^2$. We obviously have $\mathbf{d}(X_{T_7}, \partial D) \leq a_2\mathbf{d}(X_{T_7}, Y_{T_7})^2$ because $X_{T_7} \in \partial D$. This shows that $S_1 \leq T_7$ and completes the proof that if A_1 holds then $S_1 \leq T_4$.

Assume that A_1 holds and suppose that $\int_0^{T_4} \mathbf{n}_2(X_t) dL^X_t \geq 20r$. We will show that this leads to a contradiction. Recall that $|B^2_t - B^2_s| \leq 8r$ for $s, t \in [0, T_4]$. We obtain

$$X^2_{T_4} = X^2_{T_4} - X^2_0 \geq -|B^2_{T_4} - B^2_0| + \int_0^{T_4} \mathbf{n}_2(X_t) dL^X_t \geq -8r + 20r = 12r.$$

Recall that $T_7 = \sup\{t \le T_4 : X_t \in \partial D\}$. We have $B^2_{T_4} - B^2_{T_7} = X^2_{T_4} - X^2_{T_7} \ge 12r - r = 11r$, a contradiction with the fact that $|B^2_t - B^2_s| \le 8r$ for $s,t \in [0,T_4]$. Hence, if A_1 holds then $\int_0^{T_4} \mathbf{n}_2(X_t) dL^X_t \le 20r$. Note that $\mathbf{n}_2(x) \ge 1/2$ for all $x \in \partial D \cap \mathcal{B}(0,6r)$, assuming that $\varepsilon_0 > 0$ is small and $r \le r_0 < \varepsilon_0$. We have shown that if A_1 holds then $T_5 \ge T_4$, so $\mathbf{n}_2(X_t) \ge 1/2$ for $t \in [0,T_4]$ such that $X_t \in \partial D$. This implies that,

$$(1/2)(L^X_{S_1} - L^X_0) \le (1/2)(L^X_{T_4} - L^X_0) \le \int_0^{T_4} \mathbf{n}_2(X_t) dL^X_t \le 20r.$$

We have shown that $\{S_1 \le T_1, L^X_{S_1} - L^X_0 \le 40r\} \subset A_1$. It is easy to see that $\mathbf{P}(A_1) > p_1$ for some $p_1 > 0$ which depends only on c_5. This completes the proof of part (i) in the case $r \le \varepsilon/2$, with $c_1 = 2$ and $c_2 = 40$.

Next consider the case when $r \ge \varepsilon/2$. Let

$$\begin{aligned} T_8 &= \inf\{t > 0 : \mathbf{d}(Y_t, X_0) \ge 2\varepsilon\}, \\ T_9 &= \inf\{t > 0 : X_t \in \partial D, \mathbf{d}(Y_t, \partial D) \le \mathbf{d}(X_t, Y_t)/2\}, \\ T_{10} &= \inf\{t > 0 : L^X_t - L^X_0 \ge 20\varepsilon\}, \\ A_2 &= \{T_4 \le T_8\}, \\ A_3 &= \{T_9 \le T_8 \wedge T_{10}\}. \end{aligned}$$

We will show that $A_2 \subset A_3$. Assume that A_2 holds. First, we will prove that $L^X_{T_4} - L^X_0 \le 20\varepsilon$. Suppose otherwise, i.e., $L^X_{T_4} - L^X_0 \ge 20\varepsilon$. Recall that we are using the coordinate system CS_1 with the origin at $X_0 \in \partial D$. Let $T_{11} = \inf\{t \ge 0 : |X^1_t - X^1_0| \ge 5\varepsilon\}$. We will show that $T_{11} \ge T_4$. We will argue by contradiction. Assume that $T_{11} < T_4$. We have assumed that A_2 holds, so $T_{11} < T_8$. Then $B^1_t - B^1_0 = Y^1_t - Y^1_0$ for $t \in [0, T_{11}]$ and $|B^1_t - B^1_0| \le 4\varepsilon$ for the same range of t's. We have

$$\left| \int_0^{T_{11}} \mathbf{n}_1(X_t) dL^X_t \right| = |X^1_{T_{11}} - X^1_0 - (B^1_{T_{11}} - B^1_0)| \ge |X^1_{T_{11}} - X^1_0| - |B^1_{T_{11}} - B^1_0| \ge 5\varepsilon - 4\varepsilon = \varepsilon.$$

If $\varepsilon_0 > 0$ is sufficiently small and $\varepsilon < \varepsilon_0$ then $\mathbf{n}_2(x) \ge |\mathbf{n}_1(x)|/(2\nu^* \cdot 5\varepsilon)$ for $x \in \partial D \cap \mathcal{B}(0, 5\varepsilon)$, so $\int_0^{T_{11}} \mathbf{n}_2(X_t) dL^X_t \ge \varepsilon/(2\nu^* \cdot 5\varepsilon) = 1/(10\nu^*)$. We have $B^2_t - B^2_0 = Y^2_t - Y^2_0$ for $t \in [0, T_{11}]$, because $T_{11} \le T_4$, so $|B^2_t - B^2_s| \le 4\varepsilon$ for $s,t \in [0,T_{11}]$. Recall that $\varepsilon < \varepsilon_0 < 1/(100\nu^*)$. We obtain,

$$\begin{aligned} X^2_{T_{11}} &= X^2_{T_{11}} - X^2_0 \ge -|B^2_{T_{11}} - B^2_0| + \int_0^{T_{11}} \mathbf{n}_2(X_t) dL^X_t \\ &\ge -4\varepsilon + 1/(10\nu^*) \ge 6\varepsilon. \end{aligned}$$

Let $T_{12} = \sup\{t \le T_{11} : X_t \in \partial D\}$. Then $B^2_{T_{11}} - B^2_{T_{12}} = X^2_{T_{11}} - X^2_{T_{12}} \ge 6\varepsilon - \varepsilon = 5\varepsilon$, a contradiction, because $|B^2_t - B^2_s| \le 4\varepsilon$ for $s,t \in [0,T_{11}]$. This proves that $T_{11} \ge T_4$.

Recall that we have assumed that $L^X_{T_4} - L^X_0 \ge 20\varepsilon$. We have $\mathbf{n}_2(x) \ge 1/2$ for $x \in \partial D \cap \mathcal{B}(0,10\varepsilon)$, assuming $\varepsilon_0 > 0$ is small and $\varepsilon \le \varepsilon_0$. Since $T_{11} \ge T_4$, $\mathbf{n}_2(X_t) \ge 1/2$ for $t \le T_4$ such that $X_t \in \partial D$, so

$$\begin{aligned} X^2_{T_4} &= X^2_{T_4} - X^2_0 \ge -|B^2_{T_4} - B^2_0| + \int_0^{T_4} \mathbf{n}_2(X_t) dL^X_t \ge -4\varepsilon + (1/2)(L^X_{T_4} - L^X_0) \\ &\ge -4\varepsilon + 10\varepsilon = 6\varepsilon. \end{aligned}$$

Recall that $T_7 = \sup\{t \le T_4 : X_t \in \partial D\}$. Then $B^2_{T_4} - B^2_{T_7} = X^2_{T_4} - X^2_{T_7} \ge 6\varepsilon - \varepsilon = 5\varepsilon$, a contradiction, because $|B^2_t - B^2_s| \le 4\varepsilon$ for $s,t \in [0,T_{11}]$. This proves that if A_2 holds then $L^X_{T_4} - L^X_0 \le 20\varepsilon \le 40r$.

Note that $X_{T_4} - Y_{T_4} = X_{T_7} - Y_{T_7}$, $Y_{T_4}, X_{T_7} \in \partial D$, and $T_7 \le T_4 \le T_{11}$. Assuming that $\varepsilon_0 > 0$ is small, these facts easily imply that the angle between $X_{T_7} - Y_{T_7}$ and the tangent line to ∂D at X_{T_7} is smaller than $\pi/8$, so $\mathbf{d}(Y_{T_7}, \partial D) \le \mathbf{d}(X_{T_7}, Y_{T_7})/2$. Hence, $T_9 \le T_4$ and, therefore, if A_2 occurs then $T_9 \le T_4 \le T_8 \wedge T_{10}$. This completes the proof that $A_2 \subset A_3$.

It is easy to see that $\mathbf{P}(A_2) > p_2 > 0$, where p_2 depends only on ε_0 and D. It follows that $\mathbf{P}(A_3) > p_2$.

We may now apply the strong Markov property at the stopping time T_9 and repeat the argument given in the first part of the proof, discussing the case $r \le \varepsilon/2$. It is straightforward to complete the proof of part (i), adjusting the values of $c_1, c_2, \varepsilon_0, r_0$ and p_0, if necessary.

(ii) Let c_1 and c_2 be as in part (i) of the lemma, let $T^0_5 = 0$, and for $k \ge 1$ let

$$\begin{aligned}
T^k_1 &= \inf\{t \ge T^{k-1}_5 : \mathbf{d}(X_{T^{k-1}_5}, X_t) \vee \mathbf{d}(Y_{T^{k-1}_5}, Y_t) \ge c_1 \mathbf{d}(Y_{T^{k-1}_5}, \partial D)\},\\
T^k_2 &= \inf\{t \ge T^{k-1}_5 : L^X_t - L^X_{T^{k-1}_5} \ge c_2 \mathbf{d}(Y_{T^{k-1}_5}, \partial D)\},\\
T^k_3 &= \inf\{t \ge T^{k-1}_5 : Y_t \in \partial D\},\\
T^k_4 &= T^k_1 \wedge T^k_2 \wedge T^k_3,\\
T^k_5 &= \inf\{t \ge T^k_4 : X_t \in \partial D\}.
\end{aligned}$$

Let $\varepsilon_0 > 0$ be the constant which works for part (i) of the lemma. An examination of the proof of part (i) shows that we have in fact proved a statement stronger than that in part (i) of the lemma, namely, using the notation of the first part of the proof,

$$\mathbf{P}(S_1 \le T_1 \wedge T_4, L^X_{S_1} - L^X_0 \le c_2 r) \ge p_0. \tag{3.14}$$

Next we will estimate $\mathbf{E}\mathbf{d}(Y_{T^k_5 \wedge \tau^+(\varepsilon_0)}, \partial D)$. By Lemma 3.2,

$$\begin{aligned}
\mathbf{P}(\sup_{t \in [T^k_4, T^k_5]} \mathbf{d}(X_t, X_{T^k_4}) \in [2^{-j-1}, 2^{-j}] \mid \mathcal{F}_{T^k_4}) &\le c_6 \mathbf{d}(X_{T^k_4}, \partial D)/2^{-j}\\
&\le c_7 \mathbf{d}(Y_{T^{k-1}_5}, \partial D)/2^{-j}.
\end{aligned} \tag{3.15}$$

Write $\gamma = \mathbf{d}(Y_{T^{k-1}_5}, \partial D)$, and let j_0 be the largest integer such that $2^{-j_0} \ge \mathrm{diam}(D)$. Consider j such that $\sup_{t \in [T^k_4, T^k_5 \wedge \tau^+(\varepsilon_0)]} \mathbf{d}(X_t, X_{T^k_4}) \le 2^{-j}$. It is not hard to show that if $j_0 \le j \le |\log \varepsilon_0|$ then $\mathbf{d}(Y_{T^k_5 \wedge \tau^+(\varepsilon_0)}, \partial D) \le c_7 \varepsilon_0 2^{-j}$ for some $c_7 < \infty$. If $j \ge |\log \varepsilon_0|$ then $\mathbf{d}(Y_{T^k_5 \wedge \tau^+(\varepsilon_0)}, \partial D) \le \gamma + c_8 \varepsilon_0 2^{-j}$.

This and (3.15) imply that

$$\begin{aligned}
&\mathbf{E}(\mathbf{d}(Y_{T_5^k\wedge\tau^+(\varepsilon_0)},\partial D)\mid\mathcal{F}_{T_4^k})\\
&\quad\le\sum_{j_0\le j\le|\log\varepsilon_0|}c_7\varepsilon_0 2^{-j}\mathbf{P}(\sup_{t\in[T_4^k,T_5^k]}\mathbf{d}(X_t,X_{T_4^k})\in[2^{-j-1},2^{-j}]\mid\mathcal{F}_{T_4^k})\\
&\qquad+\sum_{|\log\varepsilon_0|\le j\le|\log\gamma|}(\gamma+c_8\varepsilon_0 2^{-j})\mathbf{P}(\sup_{t\in[T_4^k,T_5^k]}\mathbf{d}(X_t,X_{T_4^k})\in[2^{-j-1},2^{-j}]\mid\mathcal{F}_{T_4^k})\\
&\qquad+\sum_{j\ge|\log\gamma|}(\gamma+c_8\varepsilon_0 2^{-j})\mathbf{P}(\sup_{t\in[T_4^k,T_5^k]}\mathbf{d}(X_t,X_{T_4^k})\in[2^{-j-1},2^{-j}]\mid\mathcal{F}_{T_4^k})\\
&\quad\le\sum_{j_0\le j\le|\log\varepsilon_0|}c_9\varepsilon_0 2^{-j}(\gamma/2^{-j})\\
&\qquad+\gamma+\sum_{|\log\varepsilon_0|\le j\le|\log\gamma|}c_{10}\varepsilon_0 2^{-j}(\gamma/2^{-j})+\sum_{j\ge|\log\gamma|}c_{10}\varepsilon_0 2^{-j}\\
&\quad\le c_{11}\varepsilon_0\gamma|\log\varepsilon_0|+\gamma+c_{12}\gamma\varepsilon_0|\log\gamma|+c_{13}\gamma\varepsilon_0\le\gamma(1+c_{13}\varepsilon_0|\log\varepsilon_0|).
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbf{E}(\mathbf{d}(Y_{T_5^k\wedge\tau^+(\varepsilon_0)},\partial D)\mid\mathcal{F}_{T_4^k})\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}}\\
&\quad\le\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}}(1+c_{13}\varepsilon_0|\log\varepsilon_0|)\mathbf{d}(Y_{T_5^{k-1}\wedge\tau^+(\varepsilon_0)},\partial D).
\end{aligned}$$

This, (3.14) and the strong Markov property yield,

$$\begin{aligned}
&\mathbf{E}(\mathbf{d}(Y_{T_5^k\wedge\tau^+(\varepsilon_0)},\partial D)\mathbf{1}_{\{S_1\ge T_4^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}})\\
&\quad=\mathbf{E}(\mathbf{1}_{\{S_1\ge T_4^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}}\mathbf{E}(\mathbf{d}(Y_{T_5^k\wedge\tau^+(\varepsilon_0)},\partial D)\mid\mathcal{F}_{T_4^k}))\\
&\quad\le(1+c_{13}\varepsilon_0|\log\varepsilon_0|)\mathbf{E}(\mathbf{1}_{\{S_1\ge T_4^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}}\mathbf{d}(Y_{T_5^{k-1}\wedge\tau^+(\varepsilon_0)},\partial D))\\
&\quad\le(1+c_{13}\varepsilon_0|\log\varepsilon_0|)\mathbf{E}(\mathbf{d}(Y_{T_5^{k-1}\wedge\tau^+(\varepsilon_0)},\partial D)\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}}\mathbf{E}(\mathbf{1}_{\{S_1\ge T_4^k\}}\mid\mathcal{F}_{T_5^{k-1}}))\\
&\quad\le(1+c_{13}\varepsilon_0|\log\varepsilon_0|)\mathbf{E}(\mathbf{d}(Y_{T_5^{k-1}\wedge\tau^+(\varepsilon_0)},\partial D)\mathbf{1}_{\{S_1\ge T_4^{k-1}\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}}\\
&\qquad\times(1-\mathbf{P}(T_5^{k-1}\le S_1\le T_4^k\mid S_1\ge T_5^{k-1}))\\
&\quad\le(1+c_{13}\varepsilon_0|\log\varepsilon_0|)(1-p_0)\mathbf{E}(\mathbf{d}(Y_{T_5^{k-1}\wedge\tau^+(\varepsilon_0)},\partial D)\mathbf{1}_{\{S_1\ge T_4^{k-1}\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}})\\
&\quad\le(1+c_{13}\varepsilon_0|\log\varepsilon_0|)(1-p_0)\mathbf{E}(\mathbf{d}(Y_{T_5^{k-1}\wedge\tau^+(\varepsilon_0)},\partial D)\mathbf{1}_{\{S_1\ge T_4^{k-1}\}}\mathbf{1}_{\{T_5^{k-2}<\tau^+(\varepsilon_0)\}}).
\end{aligned}$$

We obtain by induction,

$$\begin{aligned}
&\mathbf{E}(\mathbf{d}(Y_{T_5^k\wedge\tau^+(\varepsilon_0)},\partial D)\mathbf{1}_{\{S_1\ge T_4^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}})\\
&\quad\le(1+c_{13}\varepsilon_0|\log\varepsilon_0|)^k(1-p_0)^k\mathbf{E}\mathbf{d}(Y_{T_5^0},\partial D).
\end{aligned}$$

It follows that

$$
\begin{aligned}
\mathbf{E}(L^X_{S_1\wedge\tau^+(\varepsilon_0)} - L^X_0) &= \sum_{k=0}^\infty \mathbf{E}\left((L^X_{S_1\wedge\tau^+(\varepsilon_0)} - L^X_0)\mathbf{1}_{\{S_1\in[T_5^k,T_5^{k+1})\}}\right) \\
&= \sum_{k=0}^\infty \mathbf{E}\left(\mathbf{1}_{\{S_1\in[T_5^k,T_5^{k+1})\}} \sum_{j=0}^k \mathbf{1}_{\{T_5^j<\tau^+(\varepsilon_0)\}} (L^X_{T_5^{j+1}\wedge\tau^+(\varepsilon_0)} - L^X_{T_5^j\wedge\tau^+(\varepsilon_0)})\right) \\
&\le \sum_{k=0}^\infty \mathbf{E}\left(\mathbf{1}_{\{S_1\in[T_5^k,T_5^{k+1})\}} \sum_{j=0}^k \mathbf{1}_{\{T_5^{j-1}<\tau^+(\varepsilon_0)\}} c_2 \mathbf{d}(Y_{T_5^j\wedge\tau^+(\varepsilon_0)}, \partial D)\right) \\
&\le \sum_{k=0}^\infty c_2 \mathbf{E}(\mathbf{d}(Y_{T_5^k\wedge\tau^+(\varepsilon_0)}, \partial D)\mathbf{1}_{\{S_1\ge T_4^k\}}\mathbf{1}_{\{T_5^{k-1}<\tau^+(\varepsilon_0)\}}) \\
&\le \sum_{k=0}^\infty c_2(1+c_{13}\varepsilon_0|\log\varepsilon_0|)^k(1-p_0)^k \mathbf{E}\mathbf{d}(Y_{T_5^0}, \partial D).
\end{aligned}
$$

If we assume that $\varepsilon_0 > 0$ is sufficiently small, this is bounded by $c_{14}\mathbf{E}\mathbf{d}(Y_{T_5^0}, \partial D) = c_{14}\mathbf{d}(Y_0, \partial D)$.
□

Recall that a_1 and a_2 are parameters in the definitions of S_k's and U_k's stated at the paragraph preceding Lemma 3.3.

Corollary 3.4. *For any $a_1, a_2 > 0$, and any starting points $(X_0, Y_0) = (x_0, y_0) \in \overline{D}\times\overline{D}$, all stopping times S_k are finite a.s.*

Proof. Let ε_0 and r_0 be as in Lemma 3.3. By Lemma 3.1, there is a finite stopping time T_1 such that $\mathbf{d}(X_{T_1}, Y_{T_1}) \le \varepsilon_0\wedge r_0$. So there exist $p_1 > 0$ and $c_1 < \infty$ such that $\mathbf{P}(T_1 \le c_1) \ge p_1 > 0$. Let T_2 be the first time after T_1 when either X or Y hits ∂D, and note that $\mathbf{d}(X_{T_2}, Y_{T_2}) = \mathbf{d}(X_{T_1}, Y_{T_1}) \le \varepsilon_0 \wedge r_0$. Let

$$T_3 = \inf\{t \ge T_2 : \mathbf{d}(X_t, X_{T_2}) \vee \mathbf{d}(Y_t, Y_{T_2}) \ge c_2 r_0\}.$$

It is easy to see that $\mathbf{P}(T_2 \le T_3 < \infty \mid T_1 < \infty) = 1$ when $c_2 > 0$ is small. Select such $c_2 > 0$ and apply the strong Markov property at T_2 and Lemma 3.3 (i) to see that there exists $p_2 > 0$ such that $\mathbf{P}(S_1 \le T_3 \mid T_1 < \infty) \ge p_2$. On the other hand, for some $c_3 < \infty$, we have $\mathbf{P}(T_3 \le T_1 + c_3 \mid T_1 < \infty) \ge 1 - p_2/2$. It follows that $\mathbf{P}(S_1 \le T_3 \le c_1 + c_3) > p_1p_2/2$ and so $\mathbf{P}(S_1 > c_1 + c_3) < 1 - \frac{p_1p_2}{2}$. By the Markov property, $\mathbf{P}(S_1 > k(c_1 + c_3)) \le (1 - \frac{p_1p_2}{2})^k$ for $k \ge 1$, so $S_1 < \infty$, a.s.

Recall that $\mathbf{P}(U_k < \infty \mid S_k < \infty) = 1$ for every k, according to the remark made before the statement of Lemma 3.3. By induction and the strong Markov property applied at S_k's and U_k's, all stopping times S_k and U_k are finite a.s. □

Lemma 3.5. *For any $c_1 > 0$, one can choose $a_1, a_2 > 0$ and $\varepsilon_0 > 0$ so that for every $k \ge 1$ and all $s, t \in (S_k \wedge \tau^+(\varepsilon_0), U_k \wedge \tau^+(\varepsilon_0))$, a.s.,*

$$\angle(X_t - Y_t, X_s - Y_s) \le c_1 \mathbf{d}(X_{S_k\wedge\tau^+(\varepsilon_0)}, Y_{S_k\wedge\tau^+(\varepsilon_0)}).$$

Proof. Recall that D is assumed to be C^4-smooth. Elementary geometry shows that for any $c_1 > 0$ there exist $\varepsilon_0, a_1, a_2 > 0$ with the following properties. Suppose that $x, y \in \partial D$ and $r = \mathbf{d}(x, y) \le \varepsilon_0/2$. Let

$$\begin{aligned}
A_1 &= \mathcal{B}(x, 2a_1 r) \cap \partial D, \\
A_2 &= \mathcal{B}(y, 2a_1 r) \cap \partial D, \\
A_3 &= \{z \in \overline{D} : \mathbf{d}(x, z) \le 2a_1 r, \mathbf{d}(z, \partial D) \le 2a_2 r^2\}, \\
A_4 &= \{z \in \overline{D} : \mathbf{d}(y, z) \le 2a_1 r, \mathbf{d}(z, \partial D) \le 2a_2 r^2\}, \\
A_5 &= \{z \in \overline{D} : \exists v \in A_1, u, w \in A_2 : z = v - u + w\}, \\
A_6 &= \{z \in \overline{D} : \exists v \in A_2, u, w \in A_1 : z = v - u + w\}, \\
A_7 &= A_3 \cup A_5, \\
A_8 &= A_4 \cup A_6, \\
\beta &= \sup\{\angle(x_0 - y_0, x_1 - y_1) : x_0, x_1 \in A_7, y_0, y_1 \in A_8\}.
\end{aligned} \tag{3.16}$$

With a suitable choice of small $\varepsilon_0, a_1, a_2 > 0$, we have $\beta \le (c_1/4)r$.

We will assume that $S_k < \tau^+(\varepsilon_0)$ because otherwise $(S_k \wedge \tau^+(\varepsilon_0), U_k \wedge \tau^+(\varepsilon_0)) = \emptyset$ and there is nothing to prove. Let $x \in \partial D$ be the closest point to X_{S_k} and let $y \in \partial D$ be the closest point to Y_{S_k}. Note that if ε_0 and a_2 are small then $\mathbf{d}(x, y) = r \le 2\mathbf{d}(X_{S_k}, Y_{S_k})$. We use points x and y to define sets A_k, as in (3.16). We will argue that $\angle(X_t - Y_t, X_s - Y_s) \le 2\beta$, for all $s, t \in (S_k, U_k \wedge \tau^+(\varepsilon_0))$. Note that $X_{S_k} \in A_3$ and $Y_{S_k} \in A_4$. This and the definition of β imply that $\angle(X_{S_k} - Y_{S_k}, x - y) \le \beta$.

Suppose that there exists $t \in (S_k, U_k \wedge \tau^+(\varepsilon_0))$ with $\angle(X_t - Y_t, x - y) > \beta$ and let $T = \inf\{t \ge S_k : \angle(X_t - Y_t, x - y) > \beta\}$. By continuity, $\angle(X_T - Y_T, x - y) = \beta$. It is impossible that both X_T and Y_T are in D, because then we would have $X_t - Y_t = X_T - Y_T$ for some $t_0 > 0$ and all $t \in (T - t_0, T + t_0)$. This would imply that $\angle(X_t - Y_t, x - y) = \beta$ for $t \in (T - t_0, T + t_0)$, and, therefore, $\inf\{t \ge S_k : \angle(X_t - Y_t, x - y) > \beta\} \ge T + t_0$, a contradiction. We will show that it cannot happen that $X_T, Y_T \in \partial D$. Suppose that it is true that $X_T, Y_T \in \partial D$ and recall that we are working under assumption that $T < U_k$. The definition of U_k implies that $X_T \in A_3 \subset A_7$ and $Y_T \in A_4 \subset A_8$. Since $\angle(X_T - Y_T, x - y) = \beta$, it follows that the supremum in the definition of β is attained for points $x_0, x_1, y_0, y_1 \in \partial D$ (take $x_0 = x, y_0 = y, x_1 = X_T$ and $y_1 = Y_T$). Easy geometry shows that this cannot be the case because we can slightly move either y_0 or y_1 into the interior of D to increase the value of $\angle(x_0 - y_0, x_1 - y_1)$.

Suppose without loss of generality that $X_T \in \partial D$ and $Y_T \in D$. For some random $t_1 > 0$, the process Y will not touch the boundary within $[T, T + t_1]$, while the local time L^X will have a non-zero increment, a.s. It is easy to see that the local-time-term push that X will get over $[T, T + t_1]$ will make $\angle(X_t - Y_t, x - y)$ smaller, and hence $\angle(X_t - Y_t, x - y) \le \beta$ for $t \in [T, T + t_1]$, contradicting the definition of T. We conclude that $\angle(X_t - Y_t, x - y) \le \beta$ for $t \in (S_k, U_k \wedge \tau^+(\varepsilon_0))$. This and the fact that $\angle(X_{S_k} - Y_{S_k}, x - y) \le \beta$ imply that

$$\angle(X_t - Y_t, X_s - Y_s) \le 2\beta \le (c_1/2)r \le c_1 \mathbf{d}(X_{S_k}, Y_{S_k}),$$

for all $s, t \in (S_k, U_k \wedge \tau^+(\varepsilon_0))$. □

Recall the definition of the stopping times S_k, U_k from the paragraph preceding Lemma 3.3. For $k \geq 1$, define

$$\rho_t = \frac{\mathbf{d}(X_t, Y_t)}{\mathbf{d}(X_0, Y_0)},$$
$$\widetilde{\rho}_t = \prod_{j=1}^{\infty} \frac{\mathbf{d}(X_{U_j \wedge t}, Y_{U_j \wedge t})}{\mathbf{d}(X_{S_j \wedge t}, Y_{S_j \wedge t})},$$
$$\overline{\rho}_t = \prod_{j=0}^{\infty} \frac{\mathbf{d}(X_{S_{j+1} \wedge t}, Y_{S_{j+1} \wedge t})}{\mathbf{d}(X_{U_j \wedge t}, Y_{U_j \wedge t})},$$

with the convention $0/0 = 1$. Note that $\rho_t = \widetilde{\rho}_t \overline{\rho}_t$. Let $\mathcal{T} = \bigcup_{k \geq 1} (S_k, U_k]$ and $\mathcal{T}^c = (0, \infty) \setminus \mathcal{T}$.

Lemma 3.6. *For any $c_1 > 0$ there exist $a_0, \varepsilon_0 > 0$ such that if $a_1, a_2 \in (0, a_0)$ and $\mathbf{d}(X_0, Y_0) \leq \varepsilon_0$ then for all $t \geq 0$, a.s.,*

$$\left| \log \widetilde{\rho}_{t \wedge \tau(\varepsilon_0)} + \frac{1}{2} \int_{[0, t \wedge \tau(\varepsilon_0)] \cap \mathcal{T}} \left(\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y \right) \right| \leq c_1 \left(L^X_{t \wedge \tau(\varepsilon_0)} + L^Y_{t \wedge \tau(\varepsilon_0)} \right).$$

Proof. Since D is assumed to be C^4-smooth, for any $c_2 \in (0, 1)$ and $c_3 > 0$, we can find $\varepsilon_0 > 0$ so small that for any $x, y \in \partial D$ with $\mathbf{d}(x, y) \leq 2\varepsilon_0$,

$$\big(- (1 + c_2)(1/2)\nu(x) - c_3/2 \big) \mathbf{d}(x, y) \leq \frac{x - y}{\mathbf{d}(x, y)} \cdot \mathbf{n}(x) \leq \big(- (1 - c_2)(1/2)\nu(x) + c_3/2 \big) \mathbf{d}(x, y).$$

This, Lemma 3.5, differentiability of ν and simple geometry show that one can choose small $a_1, a_2 > 0$ and $\varepsilon_0 > 0$ so that for every $k \geq 1$ and all $t \in [S_k, U_k \wedge \tau^+(\varepsilon_0)]$ such that $X_t \in \partial D$, assuming $S_k < \tau^+(\varepsilon_0)$,

$$\begin{aligned}(-(1 + c_2)(1/2)\nu(X_t) - c_3) \mathbf{d}(X_{S_k}, Y_{S_k}) &\leq \frac{X_t - Y_t}{\mathbf{d}(X_t, Y_t)} \cdot \mathbf{n}(X_t) \\ &\leq (-(1 - c_2)(1/2)\nu(X_t) + c_3) \mathbf{d}(X_{S_k}, Y_{S_k}).\end{aligned}$$

Analogous estimates hold for $\frac{Y_t - X_t}{\mathbf{d}(Y_t, X_t)} \cdot \mathbf{n}(Y_t)$. We obtain for $t \in [S_k, U_k \wedge \tau^+(\varepsilon_0)]$,

$$\begin{aligned}\mathbf{d}(X_t, Y_t) - \mathbf{d}(X_{S_k}, Y_{S_k}) &= \int_{S_k}^t \frac{(X_s - Y_s)}{\mathbf{d}(X_s, Y_s)} \cdot \mathbf{n}(X_s) dL_s^X + \int_{S_k}^t \frac{(Y_s - X_s)}{\mathbf{d}(Y_s, X_s)} \cdot \mathbf{n}(Y_s) dL_s^Y \\ &\leq \int_{S_k}^t (-(1 - c_2)(1/2)\nu(X_s) + c_3) \mathbf{d}(X_{S_k}, Y_{S_k}) dL_s^X \\ &\quad + \int_{S_k}^t (-(1 - c_2)(1/2)\nu(Y_s) + c_3) \mathbf{d}(X_{S_k}, Y_{S_k}) dL_s^Y.\end{aligned}$$

Thus

$$\frac{\mathbf{d}(X_t, Y_t)}{\mathbf{d}(X_{S_k}, Y_{S_k})} \leq 1 - (1 - c_2)(1/2) \int_{S_k}^t (\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y) + c_3 \int_{S_k}^t (dL_s^X + dL_s^Y).$$

We obtain in a similar way,

$$\frac{\mathbf{d}(X_t, Y_t)}{\mathbf{d}(X_{S_k}, Y_{S_k})} \geq 1 - (1+c_2)(1/2) \int_{S_k}^t (\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y) - c_3 \int_{S_k}^t (dL_s^X + dL_s^Y).$$

Note that $\mathbf{d}(X_t, Y_t)/\mathbf{d}(X_{S_k}, Y_{S_k}) \in [1-2a_1, 1+2a_1]$ for $t \in [S_k, U_k \wedge \tau^+(\varepsilon_0)]$, so

$$\begin{aligned}
&\frac{\mathbf{d}(X_t, Y_t)}{\mathbf{d}(X_{S_k}, Y_{S_k})} \\
&\leq (1+2a_1) \wedge \left(1 - (1-c_2)(1/2) \int_{S_k}^t (\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y) + c_3 \int_{S_k}^t (dL_s^X + dL_s^Y)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\mathbf{d}(X_t, Y_t)}{\mathbf{d}(X_{S_k}, Y_{S_k})} \\
&\geq (1-2a_1) \vee \left(1 - (1+c_2)(1/2) \int_{S_k}^t (\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y) - c_3 \int_{S_k}^t (dL_s^X + dL_s^Y)\right).
\end{aligned}$$

We have $1 + a \leq e^a$ for all a. For any $c_4 > 0$ we can choose $a_1 > 0$ so small that $1 + a \geq e^a e^{-c_4 |a|}$ for $a \in [-2a_1, 2a_1]$. Hence, for sufficiently small a_1, and $t \in [S_k, U_k \wedge \tau^+(\varepsilon_0)]$, assuming $S_k < \tau^+(\varepsilon_0)$,

$$\begin{aligned}
\widetilde{\rho}_t &= \frac{\mathbf{d}(X_t, Y_t)}{\mathbf{d}(X_{S_k}, Y_{S_k})} \prod_{j=1}^{k-1} \frac{\mathbf{d}(X_{U_j}, Y_{U_j})}{\mathbf{d}(X_{S_j}, Y_{S_j})} \\
&\leq \left(1 - (1/2 - c_2/2) \int_{S_k}^t (\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y) + c_3 \int_{S_k}^t (dL_s^X + dL_s^Y)\right) \\
&\quad \times \prod_{j=1}^{k-1} \left(1 - (1/2 - c_2/2) \int_{S_j}^{U_j} (\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y) + c_3 \int_{S_j}^{U_j} (dL_s^X + dL_s^Y)\right) \\
&\leq \exp\Bigg(- (1/2 - c_2/2) \int_{[0, t \wedge \tau(\varepsilon_0)] \cap \mathcal{T}} (\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y) \\
&\quad + c_3 \int_{[0, t \wedge \tau(\varepsilon_0)] \cap \mathcal{T}} (dL_s^X + dL_s^Y)\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\rho}_t \geq \exp\Bigg(&- (1/2 + c_2) \int_{[0, t \wedge \tau(\varepsilon_0)] \cap \mathcal{T}} (\nu(X_s) dL_s^X + \nu(Y_s) dL_s^Y) \\
&- 2c_3 \int_{[0, t \wedge \tau(\varepsilon_0)] \cap \mathcal{T}} (dL_s^X + dL_s^Y) \\
&- c_4 \Big((1/2 + c_2) \int_{[0, t \wedge \tau(\varepsilon_0)] \cap \mathcal{T}} (|\nu(X_s)| dL_s^X + |\nu(Y_s)| dL_s^Y) \\
&\quad + 2c_3 \int_{[0, t \wedge \tau(\varepsilon_0)] \cap \mathcal{T}} (dL_s^X + dL_s^Y) \Big) \Bigg).
\end{aligned}$$

Since $|\nu|$ is bounded and c_2, c_3 and c_4 are arbitrarily small, the last two estimates yield the lemma. □

Recall that we have assumed that for every $x \in \partial D$, there are only finitely many points $y \in \partial D$ with $\alpha(x, y) = 0$.

Lemma 3.7. *Suppose that $z \in \partial D$ and let $K = \{y \in \partial D : \alpha(z, y) = 0\}$ and $M_k = \{y \in \partial D : \alpha(z, y) \in [2^{-k}, 2^{-k+1}]\}$. There exist $k_0, c_1 < \infty$ and $c_2 > 0$ not depending on z such that for $k \geq k_0$, the arc length measure of M_k is less than $c_1 2^{-k/2}$ and the distance from M_k to K is bounded below by $c_2 2^{-k}$.*

Proof. We have assumed that the boundary of D is C^4-smooth and that there exist at most a finite number of points $x_1, x_2, \dots, x_n$ such that $\nu(x_k) = 0$, $k = 1, \dots, n$. Moreover, we have assumed that the third derivative of the function representing the boundary does not vanish at any x_k. This implies that there exist $\delta_0, c_3, c_4 > 0$ such that if $x \in \partial D$ and $\mathbf{d}(x, x_k) \leq \delta_0$ for some k then $|\nu(x)| \geq c_3 \mathbf{d}(x, x_k)$; moreover, if $x \in \partial D$ and $\mathbf{d}(x, x_k) \geq \delta_0$ for every $k = 1, \dots, n$, then $|\nu(x)| \geq c_4$. We make δ_0 smaller, if necessary, so that $\mathbf{d}(x_j, x_k) \geq 4\delta_0$ for all $j \neq k$. It is elementary to see that there exists $c_5 > 0$ with the following properties (i)-(iii).

(i) For every point $x \in \partial D$ such that $\mathbf{d}(x, x_k) \geq 2\delta_0$ for every $k = 1, \dots, n$, and every $y \in \partial D$ with $\mathbf{d}(x, y) \leq \delta_0$, we have $|\alpha(x, y)| \geq c_5 \mathbf{d}(x, y)$.

(ii) If $x \in \partial D$ and $\mathbf{d}(x, x_k) < 2\delta_0$ for some k, $y \in \partial D$, $\mathbf{d}(x, y) \leq \delta_0$, and y lies on the same side of x_k as x then $|\alpha(x, y)| \geq c_5 \mathbf{d}(x, y) \mathbf{d}(x, x_k)$.

(iii) If $x = x_k$ for some k, $y \in \partial D$ and $\mathbf{d}(x, y) \leq \delta_0$ then $|\alpha(x, y)| \geq c_5 \mathbf{d}(x, y)^2$.

Make δ_0 smaller, if necessary, so that for any $x, y \in \partial D$ with $|\alpha(x, y)| \geq \pi/4$, we have $\mathbf{d}(x, y) \geq 4\delta_0$.

Consider any $z \in \partial D$ and let $z_1, z_2, \dots, z_m$ be all points in ∂D such that $\alpha(z, z_k) = 0$ or $\alpha(z, z_k) = \pi/2$. The number m of such points is bounded by a constant m_0 depending on D but not on z. The family of points $\{z_1, \dots, z_m, x_1, \dots, x_n\}$ divides ∂D into $n + m$ Jordan arcs Γ_k, $k = 1, \dots, n+m$. Let λ denote the arc length measure on ∂D, i.e., $\lambda(dx)$ is an alternative notation for dx.

Fix some Γ_k and note that the curvature $\nu(x)$ has a constant sign on this arc because there are no x_j's between the endpoints of Γ_k. Since there are no points z_j between the endpoints of Γ_k, the function $x \to \alpha(z, x)$ is monotone on this arc. For an arc Γ_k, let y_k^- and y_k^+ denote its endpoints and assume that $\alpha(z, x)$ takes the maximum on Γ_k at $x = y_k^-$. It is elementary to deduce from (i)-(iii) that for some $c_6 > 0$ depending only on D, and all j,

$$\lambda(\{x \in \Gamma_k : \alpha(x, y_k^-) \in [2^{-j}, 2^{-j-1}]\}) \leq c_6 2^{-j/2}.$$

Since the number of Γ_k's is bounded by a constant independent of z,

$$\lambda(\{x \in \partial D : \alpha(x, z) \in [2^{-j}, 2^{-j-1}]\}) \leq c_7 2^{-j/2}.$$

Conditions (i)-(iii) easily imply that $\mathbf{d}(M_j, K) \geq c_8 2^{-j}$ for some c_8 depending only on D. □

Lemma 3.8. *There exists $c_1 < \infty$ such that for any $s > 0$,*

$$\rho_s \leq \exp(c_1(L_s^X + L_s^Y)).$$

Proof. Since D is assumed to be C^4-smooth, there exists $c_2 < \infty$ such that for any $x \in \partial D$ and $y \in D$,

$$\frac{x-y}{\mathbf{d}(x,y)} \cdot \mathbf{n}(x) \leq c_2 \mathbf{d}(x,y). \tag{3.17}$$

Let $T_0 = 0$, and for $k \geq 1$,

$$\begin{aligned} T_k = &\inf\{t \geq T_{k-1} : \mathbf{d}(X_t, Y_t) \notin (\tfrac{1}{2}\mathbf{d}(X_{T_{k-1}}, Y_{T_{k-1}}), 2\mathbf{d}(X_{T_{k-1}}, Y_{T_{k-1}}))\} \\ &\wedge \inf\{t \geq T_{k-1} : L_t^X - L_{T_{k-1}}^X \geq 1\} \wedge \inf\{t \geq T_{k-1} : L_t^Y - L_{T_{k-1}}^Y \geq 1\}. \end{aligned}$$

Then, by (3.17), for any $k \geq 1$ and $t \in (T_{k-1}, T_k]$,

$$\begin{aligned} &\mathbf{d}(X_t, Y_t) - \mathbf{d}(X_{T_{k-1}}, Y_{T_{k-1}}) \\ &= \int_{T_{k-1}}^t \frac{(X_s - Y_s)}{\mathbf{d}(X_s, Y_s)} \cdot \mathbf{n}(X_s) dL_s^X + \int_{T_{k-1}}^t \frac{(Y_s - X_s)}{\mathbf{d}(Y_s, X_s)} \cdot \mathbf{n}(Y_s) dL_s^Y \\ &\leq \int_{T_{k-1}}^{T_k} c_2 \mathbf{d}(X_s, Y_s) dL_s^X + \int_{T_{k-1}}^{T_k} c_2 \mathbf{d}(X_s, Y_s) dL_s^Y \\ &\leq 2c_2 \mathbf{d}(X_{T_{k-1}}, Y_{T_{k-1}}) \int_{T_{k-1}}^{T_k} (dL_s^X + dL_s^Y). \end{aligned}$$

This implies that for any $t \in (T_{k-1}, T_k]$,

$$\begin{aligned} \frac{\mathbf{d}(X_t, Y_t)}{\mathbf{d}(X_{T_{k-1}}, Y_{T_{k-1}})} &\leq 1 + 2c_2(L_{T_k}^X - L_{T_{k-1}}^X + L_{T_k}^Y - L_{T_{k-1}}^Y) \\ &\leq \exp(2c_2(L_{T_k}^X - L_{T_{k-1}}^X + L_{T_k}^Y - L_{T_{k-1}}^Y)), \end{aligned}$$

and

$$\begin{aligned} \mathbf{d}(X_t, Y_t) &= \frac{\mathbf{d}(X_t, Y_t)}{\mathbf{d}(X_{T_{k-1}}, Y_{T_{k-1}})} \prod_{j=1}^{k-1} \frac{\mathbf{d}(X_{T_j}, Y_{T_j})}{\mathbf{d}(X_{T_{j-1}}, Y_{T_{j-1}})} \\ &\leq \prod_{j=1}^{k} \exp(2c_2(L_{T_j}^X - L_{T_{j-1}}^X + L_{T_j}^Y - L_{T_{j-1}}^Y)) \\ &\leq \exp(2c_2(L_{T_k}^X + L_{T_k}^Y)). \end{aligned}$$

This proves the lemma. □

We define a partial order for two distinct points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ by saying that $x \prec y$ if $x_1 < y_1$, or $x_1 = y_1$ and $x_2 < y_2$. Let Z_t be the closest point in ∂D to the pair $\{X_t, Y_t\}$, if there is only one such point. In the case when there are multiple points in ∂D with the minimum distance to $\{X_t, Y_t\}$, we let Z_t be the point which is the smallest one according to $\prec$; an easy argument based on compactness of ∂D shows that there exists such a point. Our choice of the tie-breaking convention is arbitrary—it plays no role in the proofs. Note that if $T_0 = \inf\{t \geq 0 : X_t \in \partial D \text{ or } Y_t \in \partial D\}$ then $Z_{T_0} = X_{T_0}$ if $X_{T_0} \in \partial D$ and $Y_{T_0} \notin \partial D$; $Z_{T_0} = Y_{T_0}$ if $Y_{T_0} \in \partial D$ and $X_{T_0} \notin \partial D$; Z_{T_0} can be either X_{T_0} or Y_{T_0} if both $X_{T_0} \in \partial D$ and $Y_{T_0} \in \partial D$.

The following piece of notation will be used in many lemmas,

$$F(s,u,x,a) = \left\{ \sup_{s \leq t \leq u} \mathbf{d}(X_t, x) \leq a \right\}.$$

The proof of the next lemma is the most complicated and delicate argument in this paper.

Lemma 3.9. *Let $T_0 = \inf\{t \geq 0 : X_t \in \partial D \text{ or } Y_t \in \partial D\}$ and $\varepsilon = \mathbf{d}(X_0, Y_0)$. There exist $\beta_0 \in (1/2, 1)$ and $c_1, c_2 < \infty$ such that the following hold. Assume that,*

$$|\pi/2 - \angle(X_0 - Y_0, \mathbf{n}(Z_0))| \leq c_1 \mathbf{d}(X_0, Y_0)^{\beta_0} \quad \text{and} \quad \mathbf{d}(X_0, \partial D) \leq c_2 \mathbf{d}(X_0, Y_0). \tag{3.18}$$

(i) There exist $c_3 < \infty$ and $\varepsilon_0 > 0$ such that whenever $\varepsilon \leq \varepsilon_0$,

$$\mathbf{E} |\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \leq c_3 \varepsilon.$$

(ii) For some $\beta_1 > 0$, $\beta_2 > 1$, $c_4 < \infty$ and $\varepsilon_0 > 0$, we have for all $\varepsilon \leq \varepsilon_0$,

$$\mathbf{E}\left(\mathbf{1}_{F^c(T_0, S_1, Z(T_0), \varepsilon^{\beta_1})} |\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \right) \leq c_4 \varepsilon^{\beta_2}.$$

(iii) Let $K = \{x \in \partial D : \tan \alpha(Z_0, x) \geq \varepsilon^{-\beta_1}\}$. For some $\beta_1 > 0$, $\beta_2 > 1$, $c_5 < \infty$ and $\varepsilon_0 > 0$, we have for all $\varepsilon \leq \varepsilon_0$,

$$\mathbf{E}\left(\mathbf{1}_{\{Z_{T_0} \in K\}} |\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \right) \leq c_5 \varepsilon^{\beta_2}.$$

Proof. (i) *Step 1.* For some $c_6 < \infty$, let

$$\begin{aligned}
T_1 &= \inf\{t > T_0 : \mathbf{d}(X_t, \partial D) \wedge \mathbf{d}(Y_t, \partial D) \leq \mathbf{d}(X_t, Y_t), \\
&\qquad |\pi/2 - \angle(\mathbf{n}(Z_t), X_t - Y_t)| \leq c_1 \mathbf{d}(X_t, Y_t)^{\beta_0}\}, \\
T_2 &= \inf\{t > T_0 : \mathbf{d}(X_t, X_{T_0}) \geq c_6 \mathbf{d}(X_{T_0}, Y_{T_0})^{\beta_0}\}, \\
T_3 &= \inf\{t > T_0 : X_t \in \partial D\} \mathbf{1}_{\{Y_{T_0} \in \partial D\}} + \inf\{t > T_0 : Y_t \in \partial D\} \mathbf{1}_{\{Y_{T_0} \notin \partial D\}}, \\
T_4 &= \inf\{t > T_2 : X_t \in \partial D \text{ or } Y_t \in \partial D\}, \\
T_5 &= \inf\{t > T_4 : \mathbf{d}(X_t, X_{T_4}) \geq c_6 \mathbf{d}(X_{T_4}, Y_{T_4})^{\beta_0}\}, \\
T_6 &= \inf\{t > T_4 : X_t \in \partial D\} \mathbf{1}_{\{Y_{T_4} \in \partial D\}} + \inf\{t > T_4 : Y_t \in \partial D\} \mathbf{1}_{\{Y_{T_4} \notin \partial D\}}.
\end{aligned}$$

It is elementary to see that for a suitable choice of c_6, $\{T_3 < T_2\} \subset \{T_1 < T_2\}$, and similarly $\{T_6 < T_5\} \subset \{T_1 < T_5\}$.

We will now estimate changes in the distance between X_t and Y_t over the interval $[T_0, T_1 \wedge T_2]$ under various scenarios, and probabilities of these scenarios.

Let $M_* = \{x \in \partial D : \alpha(x, Z_0) = \pi/2\}$. For integer k and any $x \in M_*$, let $M_k = \{y \in \partial D : \tan \alpha(y, x) \in [2^{-k}, 2^{-k+1})\}$. Let N be such that $Z_{T_0} \in M_N$. Let k_1 be the largest integer with $\mathbf{d}(M_{k_1}, M_*) \geq 4c_6 \mathbf{d}(X_0, Y_0)^{\beta_0}$. Since we are concerned with the case when $\mathbf{d}(X_0, Y_0)$ is small, we can assume that $k_1 > 0$.

Suppose that $-k_1 \leq k \leq 0$. Then, by Lemmas 3.2 and 3.7, $\mathbf{P}(N = k) \leq c_7 \mathbf{d}(X_0, Y_0) 2^{-k}$. If $N = k$ then $\mathbf{d}(X_{T_0}, \partial D) \vee \mathbf{d}(Y_{T_0}, \partial D) \leq c_8 \mathbf{d}(X_0, Y_0) 2^k$. This and Lemma 3.2 imply that

$$\mathbf{P}(T_1 \geq T_2 \mid N = k) \leq \mathbf{P}(T_3 \geq T_2 \mid N = k) \leq c_9 \mathbf{d}(X_{T_0}, Y_{T_0})^{1-\beta_0} 2^k,$$

and, therefore,

$$\mathbf{P}(N = k, T_1 \geq T_2) \leq c_{10} \mathbf{d}(X_{T_0}, Y_{T_0})^{2-\beta_0}.$$

Elementary geometry shows that the distance between X_t and Y_t is reduced by at most a factor of $1 - c_{11} 2^{2k}$ over the interval $[T_0, T_1 \wedge T_2]$, so we have $\mathbf{d}(X_{T_1 \wedge T_2}, Y_{T_1 \wedge T_2}) \geq (1 - c_{11} 2^{2k}) \mathbf{d}(X_0, Y_0)$.

Next assume that $0 < k \leq k_1$. It follows from Lemmas 3.2 and 3.7 that $\mathbf{P}(N = k) \leq c_{12} \mathbf{d}(X_0, Y_0) 2^{-k/2}$. We obviously have $\mathbf{d}(X_{T_0}, \partial D) \vee \mathbf{d}(Y_{T_0}, \partial D) \leq \mathbf{d}(X_0, Y_0)$. This and Lemma 3.2 imply that

$$\mathbf{P}(T_1 \geq T_2 \mid N = k) \leq \mathbf{P}(T_3 \geq T_2 \mid N = k) \leq c_{13} \mathbf{d}(X_{T_0}, Y_{T_0})^{1-\beta_0},$$

and, therefore,

$$\mathbf{P}(N = k, T_1 \geq T_2) \leq c_{14} \mathbf{d}(X_{T_0}, Y_{T_0})^{2-\beta_0} 2^{-k/2}.$$

We have $\angle(\mathbf{n}(Z_t), X_t - Y_t) \in (c_{15} 2^{-k}, \pi - c_{15} 2^{-k})$ for $t \in [T_0, T_2]$. It follows that the distance between X_t and Y_t is reduced by at most a factor of $c_{16} 2^k$ over the interval $[T_0, T_1 \wedge T_2]$, so $\mathbf{d}(X_{T_1 \wedge T_2}, Y_{T_1 \wedge T_2}) \geq c_{16} 2^{-k} \mathbf{d}(X_0, Y_0)$.

The next case is $N \leq -k_1$. We trivially have $\mathbf{P}(N \leq -k_1) \leq 1$. If $N \leq -k_1$ then $\mathbf{d}(X_{T_0}, \partial D) \vee \mathbf{d}(Y_{T_0}, \partial D) \leq c_{17} \mathbf{d}(X_0, Y_0)^{1+\beta_0}$. Lemma 3.2 implies that

$$\begin{aligned} \mathbf{P}(N \leq -k_1, T_1 \geq T_2) &\leq \mathbf{P}(T_1 \geq T_2 \mid N \leq -k_1) \\ &\leq \mathbf{P}(T_3 \geq T_2 \mid N \leq -k_1) \leq c_{18} \mathbf{d}(X_{T_0}, Y_{T_0}). \end{aligned} \tag{3.19}$$

The distance between X_t and Y_t is reduced by at most a factor of $1 - c_{19} \mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0}$ over the interval $[T_0, T_1 \wedge T_2]$, so $\mathbf{d}(X_{T_1 \wedge T_2}, Y_{T_1 \wedge T_2}) \geq (1 - c_{19} \mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0}) \mathbf{d}(X_0, Y_0)$.

Let N' be such that $X_{T_4} \in M_{N'}$ if $X_{T_4} \in \partial D$ and $Y_{T_4} \in M_{N'}$ if $Y_{T_4} \in \partial D$. We will analyze the change to $\mathbf{d}(X_t, Y_t)$ over the interval $[T_0, T_1 \wedge T_5]$ for different values of N', assuming that $T_1 \geq T_2$ and $N \leq -k_1$.

Suppose that $-k_1 \leq k \leq 0$. Since $\mathbf{d}(X_{T_2}, \partial D) \leq c_{20} \mathbf{d}(X_0, Y_0)^{\beta_0}$, Lemmas 3.2 and 3.7 imply that $\mathbf{P}(N' = k \mid N \leq -k_1, T_1 \geq T_2) \leq c_{21} \mathbf{d}(X_0, Y_0)^{\beta_0} 2^{-k}$. If $N' = k$ then $\mathbf{d}(X_{T_4}, \partial D) \vee \mathbf{d}(Y_{T_4}, \partial D) \leq c_{22} \mathbf{d}(X_0, Y_0) 2^k$. This and Lemma 3.2 imply that

$$\begin{aligned} \mathbf{P}(T_1 \geq T_5 \mid N \leq -k_1, T_1 \geq T_2, N' = k) &\leq \mathbf{P}(T_6 \geq T_5 \mid N \leq -k_1, T_1 \geq T_2, N' = k) \\ &\leq c_{23} \mathbf{d}(X_{T_0}, Y_{T_0})^{1-\beta_0} 2^k. \end{aligned}$$

We combine estimates of probabilities in this paragraph with (3.19) to obtain,

$$\mathbf{P}(N \leq -k_1, N' = k, T_1 \geq T_5) \leq c_{24} \mathbf{d}(X_{T_0}, Y_{T_0})^2.$$

The distance between X_t and Y_t is reduced by at most a factor of $1 - c_{25}2^{2k}$ over the interval $[T_0, T_1 \wedge T_5]$, so $\mathbf{d}(X_{T_1\wedge T_5}, Y_{T_1\wedge T_5}) \geq (1 - c_{25}2^{2k})\mathbf{d}(X_0, Y_0)$.

Next consider the case $0 < k \leq k_1$. By Lemmas 3.2 and 3.7,

$$\mathbf{P}(N' = k \mid N \leq -k_1, T_1 \geq T_2) \leq c_{26}\mathbf{d}(X_0, Y_0)^{\beta_0}2^{-k/2}.$$

If the event $\{N \leq -k_1, T_1 \geq T_2\}$ holds then $\mathbf{d}(X_{T_4}, \partial D) \vee \mathbf{d}(Y_{T_4}, \partial D) \leq \mathbf{d}(X_0, Y_0)$. This and Lemma 3.2 imply that

$$\begin{aligned}\mathbf{P}(T_1 \geq T_5 \mid N \leq -k_1, T_1 \geq T_2, N' = k) &\leq \mathbf{P}(T_6 \geq T_5 \mid N \leq -k_1, T_1 \geq T_2, N' = k)\\ &\leq c_{27}\mathbf{d}(X_{T_0}, Y_{T_0})^{1-\beta_0},\end{aligned}$$

and, using (3.19),

$$\mathbf{P}(N \leq -k_1, N' = k, T_1 \geq T_5) \leq c_{28}\mathbf{d}(X_{T_0}, Y_{T_0})^2 2^{-k/2}.$$

We have $\angle(\mathbf{n}(Z_t), X_t - Y_t) \in (c_{29}2^{-k}, \pi - c_{29}2^{-k})$ for $t \in [T_0, T_5]$. It follows that the distance between X_t and Y_t is reduced by at most a factor of $c_{30}2^k$ over the interval $[T_0, T_1 \wedge T_5]$, so $\mathbf{d}(X_{T_1\wedge T_5}, Y_{T_1\wedge T_5}) \geq c_{30}2^{-k}\mathbf{d}(X_0, Y_0)$.

Consider the case $N' \leq -k_1$. We trivially have $\mathbf{P}(N' \leq -k_1 \mid N \leq -k_1, T_1 \geq T_2) \leq 1$. If $N' \leq -k_1$ then $\mathbf{d}(X_{T_4}, \partial D) \vee \mathbf{d}(Y_{T_4}, \partial D) \leq c_{31}\mathbf{d}(X_0, Y_0)^{1+\beta_0}$. This and Lemma 3.2 imply that

$$\begin{aligned}\mathbf{P}(T_1 \geq T_5 \mid N \leq -k_1, T_1 \geq T_2, N' \leq -k_1) &\leq \mathbf{P}(T_6 \geq T_5 \mid N \leq -k_1, T_1 \geq T_2, N' \leq -k_1)\\ &\leq c_{32}\mathbf{d}(X_{T_0}, Y_{T_0}),\end{aligned}$$

and, using (3.19),

$$\mathbf{P}(N \leq -k_1, N' \leq -k_1, T_1 \geq T_5) \leq c_{33}\mathbf{d}(X_{T_0}, Y_{T_0})^2.$$

If $N' \leq -k_1$, the distance between X_t and Y_t is reduced by at most a factor of $1 - c_{34}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0}$ over the interval $[T_0, T_1 \wedge T_5]$, so we have $\mathbf{d}(X_{T_1\wedge T_5}, Y_{T_1\wedge T_5}) \geq (1 - c_{34}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0)$.

An argument similar to those given above yields

$$\mathbf{P}(N \leq -k_1, T_1 \geq T_2, N' \geq k_1) \leq c_{35}\mathbf{d}(X_0, Y_0)^{1+3\beta_0/2}.$$

If $\{N \leq -k_1, T_1 \geq T_2, N' \geq k_1\}$ holds then $\mathbf{d}(X_{T_4}, Y_{T_4}) \geq (1 - c_{36}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0)$.

Finally, by Lemmas 3.2 and 3.7, we have $\mathbf{P}(N \geq k_1) \leq c_{37}\mathbf{d}(X_0, Y_0)^{1+\beta_0/2}$.

Step 2. Recall that θ denotes the usual Markov shift operator and let

$$\begin{aligned}A &= \big(\{|N| \leq k_1\} \cap \{T_1 \leq T_2\}\big) \cup \big(\{N < -k_1\} \cap \{N' \leq k_1\} \cap \{T_1 \leq T_5\}\big),\\ T_7 &= T_1\mathbf{1}_A + T_0 \circ \theta_{T_2}\mathbf{1}_{\{|N|\leq k_1\}\cap\{T_1>T_2\}} + T_0\mathbf{1}_{\{N>k_1\}}\\ &\quad + T_0 \circ \theta_{T_5}\mathbf{1}_{\{N<-k_1\}\cap\{N'\leq k_1\}\cap\{T_1>T_5\}} + T_4\mathbf{1}_{\{N<-k_1\}\cap\{N'>k_1\}}.\end{aligned}$$

Note that $\mathbf{d}(X_{T_7}, Y_{T_7}) \leq \mathbf{d}(X_0, Y_0)$.

Let $D(a) = \{x \in D : \mathbf{d}(x, \partial D) \leq a\}$. We will define a number of stopping times and events involving a parameter $\beta_3 > 1$ whose value will be chosen later. Recall that Z_t is the closest point on ∂D to the pair $\{X_t, Y_t\}$. Let $\delta = \mathbf{d}(X_{T_7}, Y_{T_7})$ and for $k \geq 0$,

$$\begin{aligned}V_k &= \inf\{t \geq T_7 : X_t, Y_t \in D(\delta^{\beta_3^k})\},\\ G_k &= \{|\pi/2 - \angle(\mathbf{n}(Z_{V_k}), X_{V_k} - Y_{V_k})| > c_1\mathbf{d}(X_{V_k}, Y_{V_k})^{\beta_0}\}.\end{aligned}$$

We will define some stopping times V_k^j and related events A_k^j assuming that $V_k > T_7$ (otherwise V_k^j's and A_k^j's can be defined in an arbitrary way). If G_k^c holds, we let $V_k^j = V_k$ for all j. We will state the definitions in the case when G_k holds and $X_{V_k} \in \partial D(\delta^{\beta_3^k}) \setminus \partial D$. In the case when G_k holds and $Y_{V_k} \in \partial D(\delta^{\beta_3^k}) \setminus \partial D$, the roles of X_t and Y_t should be interchanged in the definitions of V_k^j's and A_k^j's. Let CS_k be the orthonormal coordinate system with the origin at the point in ∂D that is closest to X_{V_k}, whose first axis is tangent to ∂D. We will write $X_t = (X_t^1, X_t^2)$ in this coordinate system. Note that $X_{V_k}^1 = 0$ in CS_k. Let

$$
\begin{aligned}
V_k^1 &= \inf\{t \geq V_k : X_t \in D(\delta^{\beta_3^k}/2)\}, \\
A_k^1 &= \left\{ V_k^1 \leq \inf\{t \geq V_k : X_t \in D(2\delta^{\beta_3^k})^c \text{ or } |X_t^1| = \delta^{\beta_3^k}\} \right\}, \\
V_k^2 &= \inf\{t \geq V_k^1 : X_t \in D(\delta^{\beta_3^k})^c\}, \\
A_k^2 &= \left\{ V_k^2 \leq \inf\{t \geq V_k^1 : X_t \in D(\delta^{\beta_3^k}/4) \text{ or } |X_t^1| = 2\delta^{\beta_3^k}\} \right\}.
\end{aligned}
$$

If δ is small then $\partial D \cap \mathcal{B}(X_{V_k}, 2\mathbf{d}(X_{V_k}, Y_{V_k}) \vee 2\delta^{\beta_3^k})$ is almost flat. If events A_k^1 and A_k^2 occur, the process X_t moves towards the boundary of D and then away from the boundary, without moving too much in the horizontal direction in CS_k. The result is that the distance from $Y_{V_k^2}$ to ∂D is greater than $\delta^{\beta_3^k}/8$.

It follows from Lemma 3.7 and its proof that there exists $\delta_0, c_{38} > 0$ depending only on D, such that if $\delta \leq \delta_0$ then either (i) $\angle(\mathbf{n}(x), \mathbf{n}(y)) \geq c_{38}\delta^{2\beta_3^k}$ for all $x = (x^1, x^2) \in \partial D$ and $y = (y^1, y^2) \in \partial D$ with $3\delta^{\beta_3^k} \leq -x^1 \leq 9\delta^{\beta_3^k}$ and $3\delta^{\beta_3^k} \leq y^1 \leq 9\delta^{\beta_3^k}$, or (ii) $\angle(\mathbf{n}(x), \mathbf{n}(y)) \geq c_{38}\delta^{2\beta_3^k}$ for all $x, y \in \partial D$ with $10\delta^{\beta_3^k} \leq -x^1 \leq 16\delta^{\beta_3^k}$ and $10\delta^{\beta_3^k} \leq y^1 \leq 16\delta^{\beta_3^k}$, in CS_k. We have to consider cases (i) and (ii) because there might be a (single) $z \in \partial D$ with $-16\delta^{\beta_3^k} \leq z^1 \leq 16\delta^{\beta_3^k}$ and $\nu(z) = 0$. Depending on the sign of $X_{V_k^2}^1 - Y_{V_k^2}^1$, one of the following events holds,

$$\angle(X_{V_k^2} - Y_{V_k^2}, \mathbf{n}(x)) \geq c_{38}\delta^{2\beta_3^k}, \qquad \text{for } x \in \partial D,\ 3\delta^{\beta_3^k} \leq x^1 \leq 9\delta^{\beta_3^k}, \tag{3.20}$$

$$\angle(X_{V_k^2} - Y_{V_k^2}, \mathbf{n}(x)) \geq c_{38}\delta^{2\beta_3^k}, \qquad \text{for } x \in \partial D,\ 3\delta^{\beta_3^k} \leq -x^1 \leq 9\delta^{\beta_3^k}, \tag{3.21}$$

$$\angle(X_{V_k^2} - Y_{V_k^2}, \mathbf{n}(x)) \geq c_{38}\delta^{2\beta_3^k}, \qquad \text{for } x \in \partial D,\ 10\delta^{\beta_3^k} \leq x^1 \leq 16\delta^{\beta_3^k}. \tag{3.22}$$

$$\angle(X_{V_k^2} - Y_{V_k^2}, \mathbf{n}(x)) \geq c_{38}\delta^{2\beta_3^k}, \qquad \text{for } x \in \partial D,\ 10\delta^{\beta_3^k} \leq -x^1 \leq 16\delta^{\beta_3^k}. \tag{3.23}$$

We will discuss only cases (3.20) and (3.22). The other cases are symmetric—we leave them to the reader. In case (3.20) we let

$$
\begin{aligned}
V_k^3 &= \inf\{t \geq V_k^2 : X_t^1 = 6\delta^{\beta_3^k}\}, \\
A_k^3 &= \left\{ V_k^3 \leq \inf\{t \geq V_k^2 : X_t^1 = -3\delta^{\beta_3^k} \text{ or } X_t \in D(2\delta^{\beta_3^k})^c \cup D(\delta^{\beta_3^k}/2)\} \right\}, \\
V_k^4 &= \inf\{t \geq V_k^3 : X_t \in D(\delta^{\beta_3^{k+1}})\}, \\
A_k^4 &= \left\{ V_k^4 \leq \inf\{t \geq V_k^3 : X_t \in D(3\delta^{\beta_3^k})^c \text{ or } |X_t^1 - 6\delta^{\beta_3^k}| = \delta^{\beta_3^k}\} \right\}.
\end{aligned}
$$

In case (3.22), we let

$$\begin{aligned} V_k^3 &= \inf\{t \geq V_k^2 : X_t^1 = 13\delta^{\beta_3^k}\}, \\ A_k^3 &= \left\{ V_k^3 \leq \inf\{t \geq V_k^2 : X_t^1 = -3\delta^{\beta_3^k} \text{ or } X_t \in D(2\delta^{\beta_3^k})^c \cup D(\delta^{\beta_3^k}/2)\} \right\}, \\ V_k^4 &= \inf\{t \geq V_k^3 : X_t \in D(\delta^{\beta_3^{k+1}})\}, \\ A_k^4 &= \left\{ V_k^4 \leq \inf\{t \geq V_k^3 : X_t \in D(3\delta^{\beta_3^k})^c \text{ or } |X_t^1 - 13\delta^{\beta_3^k}| = \delta^{\beta_3^k}\} \right\}. \end{aligned}$$

In either case, let $A_k = A_k^1 \cap A_k^2 \cap A_k^3 \cap A_k^4$, and similarly in cases (3.21) and (3.23).

We will assume that $\delta_0 > 0$ is so small that $\delta^{\beta_3^{k+1}} < \delta^{\beta_3^k}/4$ for $\delta \leq \delta_0$. We will later impose an upper bound on β_3 which, in turn, will impose an upper bound on δ_0. Note that given this assumption about δ_0, if $\mathbf{d}(X_0, Y_0) \leq \delta_0$ and A_k holds then $V_k^4 \leq V_{k+1}$, so we can estimate the probability of the intersection of consecutive A_k's using the strong Markov property at times V_k. Let

$$\begin{aligned} V_k^5 &= \inf\{t > V_k : X_t \in \partial D\}, \\ C_k &= \left\{ \sup_{t \in [V_k, V_k^5]} \mathbf{d}(X_t, X_{V_k}) \leq \delta^{\beta_3^{k-1}}/2 \right\}. \end{aligned}$$

By Lemma 3.2,

$$\mathbf{P}(C_k^c \mid X_{V_k} \in \partial D(\delta^{\beta_3^k}) \setminus \partial D) \leq c_{39}\delta^{\beta_3^k}/\delta^{\beta_3^{k-1}} = c_{39}\delta^{\beta_3^{k-1}(\beta_3 - 1)}. \tag{3.24}$$

We will find a lower bound for $\sup_{t \in [V_k, V_k^5]} \mathbf{d}(X_t, Y_t)$ assuming that $\mathbf{d}(X_{T_7}, Y_{T_7}) \geq \delta^{\beta_3^k}$ and $G_k \cap A_k \cap C_{k+1}$ occurred. First consider the case when $|X_{V_k}^1 - Y_{V_k}^1| > \delta^{\beta_3^k}/8$. Then it is easy to see that the distance between X_t and Y_t is reduced between times V_k and V_k^5 by at most a constant factor c_{40}, so $\mathbf{d}(X_t, Y_t) \geq c_{41}\delta^{\beta_3^k}$ for $t \in [V_k, V_k^5]$. Next suppose that $|X_{V_k}^1 - Y_{V_k}^1| \leq \delta^{\beta_3^k}/8$. Then $|X_t^1 - Y_t^1| \leq \delta^{\beta_3^k}/4$ for $t \in [V_k, V_k^5]$, because the boundary of D is "flat" in the neighborhood under consideration. After time V_k^2, processes X_t and Y_t move along ∂D without touching it, to the place where the angle between the line passing through both particles and the normal to the boundary of the domain is bounded below by $c_{38}\delta^{2\beta_3^k}$. It follows that for $t \in [V_k, V_k^5]$, the process Y_t is reflecting on the part of the boundary where the angle between the line passing through both particles and the normal to the boundary of the domain is bounded below by $c_{38}\delta^{2\beta_3^k}$. Hence, the distance between X_t and Y_t is reduced between times V_k and V_k^5 by at most a factor of $c_{42}\delta^{2\beta_3^k}$. This implies that if $G_k \cap A_k \cap C_{k+1}$ holds then $\mathbf{d}(X_t, Y_t) \geq c_{43}\delta^{3\beta_3^k}$ for $t \in [V_k, V_k^5]$.

Let

$$F_k = \bigcup_{k \leq m < 2k} \left(G_m^c \cup (G_m \cap A_m \cap C_{m+1})\right),$$

and note that

$$\bigcup_{k \leq m < 2k} A_m \cap \bigcap_{k \leq m < 2k} C_{m+1} \subset \bigcup_{k \leq m < 2k} (G_m^c \cup (G_m \cap A_m)) \cap \bigcap_{k \leq m < 2k} C_{m+1} \subset F_k.$$

It is elementary to see that the probability of A_k is bounded below by $p_1 > 0$, not depending on k or β_3, assuming δ is small. By the strong Markov property applied at stopping times V_m, we have $\mathbf{P}(\bigcap_{k \leq m < 2k} A_m^c) \leq (1 - p_1)^k$, so

$$\mathbf{P}\left(\bigcup_{k \leq m < 2k} A_m \right) \geq 1 - (1 - p_1)^k. \tag{3.25}$$

By (3.24),

$$\mathbf{P}\left(\bigcap_{k\le m<2k} C_m\right) \ge 1 - \sum_{k\le m<2k} c_{39}\delta^{\beta_3^{m-1}(\beta_3-1)}.$$

The quantity on the right hand side is bounded below by $1-(1-p_1)^k$, for small δ. This and (3.25) imply that $\mathbf{P}(F_k)\ge 1-2(1-p_1)^k$, for small δ. Let

$$\begin{aligned} T_8 = \inf\{t > T_7 : \mathbf{d}(X_t,\partial D)\wedge \mathbf{d}(Y_t,\partial D) \le \mathbf{d}(X_t,Y_t), \\ |\pi/2 - \angle(\mathbf{n}(Z_t), X_t - Y_t)| \le c_1 \mathbf{d}(X_t,Y_t)^{\beta_0}\}. \end{aligned}$$

Note that if $G_k^c \cup (G_k\cap A_k\cap C_{k+1})$ occurs then $T_8 \le V_k^5$ and $\mathbf{d}(X_{T_8},Y_{T_8}) \ge c_{43}\delta^{3\beta_3^k}$. In view of the estimate for the probability of F_k, we see that for $k\ge 1$,

$$\mathbf{P}(\mathbf{d}(X_{T_8},Y_{T_8}) \le c_{43}\delta^{3\beta_3^{2k}}) \le 2(1-p_1)^k.$$

We choose $\beta_3>1$ so that $\beta_3^2(1-p_1)<1$.

Step 3. Let c_{44} be the same as c_2 in the statement of Lemma 3.3, and let c_{45} be the same as c_1 in the statement of that lemma. Let $T_9^1 = T_8$ and for $k\ge 1$,

$$\begin{aligned}
T_{10}^k &= \inf\{t\ge T_9^k : \mathbf{d}(X_t, X_{T_9^k})\vee \mathbf{d}(Y_t,Y_{T_9^k}) \ge 2(\mathbf{d}(X_{T_9^k},\partial D)\vee \mathbf{d}(Y_{T_9^k},\partial D))\},\\
T_{11}^k &= \inf\{t\ge T_9^k : X_t\in\partial D\},\\
T_{12}^k &= \inf\{t\ge T_9^k : Y_t\in\partial D\},\\
T_{13}^k &= \inf\{t\ge T_{11}^k : L_t^X - L_{T_{11}^k}^X \ge c_{44}\mathbf{d}(Y_{T_{11}^k},\partial D)\}\mathbf{1}_{\{T_{11}^k\le T_{12}^k\}},\\
T_{14}^k &= \inf\{t\ge T_{12}^k : L_t^Y - L_{T_{11}^k}^Y \ge c_{44}\mathbf{d}(X_{T_{11}^k},\partial D)\}\mathbf{1}_{\{T_{11}^k> T_{12}^k\}},\\
T_{15}^k &= \inf\{t\ge T_{11}^k : Y_t\in\partial D\}\mathbf{1}_{\{T_{11}^k\le T_{12}^k\}},\\
T_{16}^k &= \inf\{t\ge T_{11}^k : X_t\in\partial D\}\mathbf{1}_{\{T_{11}^k> T_{12}^k\}},\\
T_9^{k+1} &= \inf\{t\ge T_9^k : \mathbf{d}(X_t, X_{T_9^k})\vee \mathbf{d}(Y_t,Y_{T_9^k}) \ge 2c_{45}(\mathbf{d}(X_{T_9^k},\partial D)\vee \mathbf{d}(Y_{T_9^k},\partial D))\}\\
&\qquad \wedge (T_{13}^k + T_{14}^k)\wedge (T_{15}^k+T_{16}^k).
\end{aligned}$$

Note that $\mathbf{d}(X_{T_9^1},Y_{T_9^1}) = \mathbf{d}(X_{T_8},Y_{T_8}) \le \mathbf{d}(X_0,Y_0)$. It is easy to see that if $\varepsilon_0>0$ is small and $\mathbf{d}(X_0,Y_0)\le\varepsilon_0$, we have $\mathbf{P}(T_{11}^k < T_{10}^k)\ge p_2>0$, where p_2 depends only on D. By the strong Markov property applied at $T_{11}^k\wedge T_{12}^k$ and Lemma 3.3 (i), for some $p_3>0$,

$$\mathbf{P}(S_1\le T_9^{k+1} \mid \mathcal{F}_{T_9^k}, \{S_1 > T_9^k\}) \ge p_3.$$

Let k_2 be such that $(1-p_3)^{k_2-1}\le 1/2$. Then

$$\mathbf{P}(S_1\ge T_9^{k_2}) \le (1-p_3)^{k_2-1}\le 1/2.$$

Step 4. Let $T_{17} = T_9^{k_2}$ and note that

$$\mathbf{d}(X_{T_9^1}, X_{T_{17}})\vee \mathbf{d}(Y_{T_9^1},Y_{T_{17}}) \le c_{46}(\mathbf{d}(X_{T_9^1},\partial D)\vee \mathbf{d}(Y_{T_9^1},\partial D)),$$

where $c_{46} = (2c_{45})^{k_2}$. For t between T_9^1 and T_{17}, the angle between the vector of reflection for any of the processes and $X_t - Y_t$ is bounded below by a quantity depending on N; we will next discuss this dependence and its consequences. We will examine various cases in the same order as in Step 1 and we will also recall some estimates from Step 1. There will be many cases to consider—we will label them for future reference.

We start with a general remark that applies to many of the cases discussed below. If $T_1 \le T_2$ then the lower bounds for $\mathbf{d}(X_{T_1\wedge T_2}, Y_{T_1\wedge T_2})$ obtained in Step 1 apply also to $\mathbf{d}(X_{T_{17}}, Y_{T_{17}})$, for the same reasons, but with constants that may be different. The same is true when $T_1 \le T_5$.

(a) Consider the case when $N = k$ and $-k_1 \le k \le 0$. Then we have $\mathbf{P}(N = k) \le c_{47}\mathbf{d}(X_0, Y_0)2^{-k}$,

$$\mathbf{P}(N = k, T_1 \ge T_2) \le c_{48}\mathbf{d}(X_{T_0}, Y_{T_0})^{2-\beta_0},$$

and $(1 - c_{49}2^{2k})\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_1\wedge T_2}, Y_{T_1\wedge T_2}) \le \mathbf{d}(X_0, Y_0)$. If $T_1 \le T_2$ then $\mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \ge (1 - c_{50}2^{2k})\mathbf{d}(X_0, Y_0)$. Note that the distance between X_t and Y_t does not increase before time $T_8 = T_9^1$. The increase of the local time L_t^X between times T_9^1 and T_{17} is bounded by $c_{51}\mathbf{d}(X_0, Y_0)$, and a similar bound holds for the increment of L_t^Y, so, according to Lemma 3.8, $\mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le \mathbf{d}(X_0, Y_0)(1 + c_{52}\mathbf{d}(X_0, Y_0))$, assuming that $\mathbf{d}(X_0, Y_0)$ is small. Combining the two estimates, we obtain

$$(1 - c_{50}2^{2k})\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le \mathbf{d}(X_0, Y_0)(1 + c_{52}\mathbf{d}(X_0, Y_0)).$$

(b) Next consider the case when $N = k$, $-k_1 \le k \le 0$, and $T_2 < T_1$. Then, $\mathbf{d}(X_7, Y_7) \le \mathbf{d}(X_0, Y_0)$, $\mathbf{d}(X_{17}, Y_{17}) \le c_{53}\mathbf{d}(X_8, Y_8)$, and using Step 2, for $n \ge 1$,

$$\mathbf{P}(N = k, T_1 \ge T_2, \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le c_{54}\mathbf{d}(X_0, Y_0)^{3\beta_3^{2n}}) \le c_{55}\mathbf{d}(X_{T_0}, Y_{T_0})^{2-\beta_0}(1 - p_1)^{n+1}.$$

(c) Assume that $0 < k \le k_1$. Then $\mathbf{P}(N = k) \le c_{56}\mathbf{d}(X_0, Y_0)2^{-k/2}$,

$$\mathbf{P}(N = k, T_1 \ge T_2) \le c_{57}\mathbf{d}(X_{T_0}, Y_{T_0})^{2-\beta_0}2^{-k/2},$$

and $c_{58}2^{-k}\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_1\wedge T_2}, Y_{T_1\wedge T_2}) \le \mathbf{d}(X_0, Y_0)$. If $T_1 \le T_2$ then we have $\mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \ge c_{59}2^{-k}\mathbf{d}(X_0, Y_0)$. The bound $\mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le \mathbf{d}(X_0, Y_0)(1 + c_{60}\mathbf{d}(X_0, Y_0))$ holds for the same reason as in case (a). Combining the two estimates, we see that

$$c_{58}2^{-k}\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le \mathbf{d}(X_0, Y_0)(1 + c_{60}\mathbf{d}(X_0, Y_0)).$$

(d) Next consider the case when $N = k$, $0 < k \le k_1$, and $T_2 < T_1$. Then, $\mathbf{d}(X_7, Y_7) \le \mathbf{d}(X_0, Y_0)$, $\mathbf{d}(X_{17}, Y_{17}) \le c_{61}\mathbf{d}(X_8, Y_8)$, and using Step 2, for $n \ge 1$,

$$\begin{aligned}&\mathbf{P}(N = k, T_1 \ge T_2, \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le c_{62}(c_{58}2^{-k}\mathbf{d}(X_0, Y_0))^{3\beta_3^{2n}}) \\ &\quad\le c_{63}\mathbf{d}(X_{T_0}, Y_{T_0})^{2-\beta_0}2^{-k/2}(1 - p_1)^{n+1}.\end{aligned}$$

(e) The next case is when $N \le -k_1$. We will use the trivial estimate $\mathbf{P}(N \le -k_1) \le 1$. We have $\mathbf{d}(X_{T_1\wedge T_2}, Y_{T_1\wedge T_2}) \ge (1 - c_{64}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0)$. If $T_1 \le T_2$ then

$$(1 - c_{65}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le \mathbf{d}(X_0, Y_0)(1 + c_{66}\mathbf{d}(X_0, Y_0)).$$

(f) Recall that N' is defined by the following conditions, $X_{T_4} \in M_{N'}$ if $X_{T_4} \in \partial D$, and $Y_{T_4} \in M_{N'}$ if $Y_{T_4} \in \partial D$. Suppose that $-k_1 \le k \le 0$. Then

$$\mathbf{P}(N \le -k_1, N' = k, T_1 \ge T_5) \le c_{67}\mathbf{d}(X_{T_0}, Y_{T_0})^2,$$

and $(1 - c_{68}2^{2k})\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_1\wedge T_5}, Y_{T_1\wedge T_5}) \le \mathbf{d}(X_0, Y_0)$. If $N \le -k_1$, $N' = k$, $-k_1 \le k \le 0$, and $T_1 \le T_5$ then we have $\mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \ge (1 - c_{69}2^{2k})\mathbf{d}(X_0, Y_0)$ and

$$(1 - c_{69}2^{2k})\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le \mathbf{d}(X_0, Y_0)(1 + c_{70}\mathbf{d}(X_0, Y_0)).$$

(g) If $N \le -k_1$, $N' = k$, $-k_1 \le k \le 0$, and $T_5 < T_1$ then, $\mathbf{d}(X_7, Y_7) \le \mathbf{d}(X_0, Y_0)$, $\mathbf{d}(X_{17}, Y_{17}) \le c_{71}\mathbf{d}(X_8, Y_8)$, and using Step 2, for $n \ge 1$,

$$\begin{aligned}&\mathbf{P}(N \le -k_1, N' = k, T_1 \ge T_5, \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le c_{72}\mathbf{d}(X_0, Y_0)^{3\beta_3^{2n}})\\ &\quad \le c_{73}\mathbf{d}(X_{T_0}, Y_{T_0})^2(1 - p_1)^{n+1}.\end{aligned}$$

(h) If $0 < k \le k_1$ then

$$\mathbf{P}(N \le -k_1, N' = k, T_1 \ge T_5) \le c_{74}\mathbf{d}(X_{T_0}, Y_{T_0})^2 2^{-k/2},$$

and $c_{75}2^{-k}\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_1\wedge T_5}, Y_{T_1\wedge T_5}) \le \mathbf{d}(X_0, Y_0)$. If $N \le -k_1$, $N' = k$, $0 < k \le k_1$, and $T_1 \le T_5$ then $\mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \ge c_{76}2^{-k}\mathbf{d}(X_0, Y_0)$ and

$$c_{76}2^{-k}\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le \mathbf{d}(X_0, Y_0)(1 + c_{77}\mathbf{d}(X_0, Y_0)).$$

(i) If $N \le -k_1$, $N' = k$, $0 < k \le k_1$, and $T_5 < T_1$ then, $\mathbf{d}(X_7, Y_7) \le \mathbf{d}(X_0, Y_0)$, $\mathbf{d}(X_{17}, Y_{17}) \le c_{78}\mathbf{d}(X_8, Y_8)$, and using Step 2, for $n \ge 1$,

$$\begin{aligned}&\mathbf{P}(N \le -k_1, N' = k, T_1 \ge T_5, \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le c_{79}(c_{75}2^{-k}\mathbf{d}(X_0, Y_0))^{3\beta_3^{2n}})\\ &\quad \le c_{80}\mathbf{d}(X_{T_0}, Y_{T_0})^2 2^{-k/2}(1 - p_1)^{n+1}.\end{aligned}$$

(j) The next case to be considered is when $N \le -k_1$ and $N' \le -k_1$. We have $\mathbf{P}(N' \le -k_1 \mid N \le -k_1, T_1 \ge T_2) \le 1$,

$$\mathbf{P}(N \le -k_1, N' \le -k_1, T_1 \ge T_5) \le c_{81}\mathbf{d}(X_{T_0}, Y_{T_0})^2,$$

and $\mathbf{d}(X_{T_1\wedge T_5}, Y_{T_1\wedge T_5}) \ge (1 - c_{82}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0)$. If $N \le -k_1$, $N' \le -k_1$, and $T_1 \le T_5$ then $\mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \ge (1 - c_{83}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0)$ and

$$(1 - c_{83}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0) \le \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le \mathbf{d}(X_0, Y_0)(1 + c_{84}\mathbf{d}(X_0, Y_0)).$$

(k) If $N \le -k_1$, $N' \le -k_1$, and $T_5 < T_1$ then, $\mathbf{d}(X_7, Y_7) \le \mathbf{d}(X_0, Y_0)$, $\mathbf{d}(X_{17}, Y_{17}) \le c_{85}\mathbf{d}(X_8, Y_8)$, and using Step 2, for $n \ge 1$,

$$\begin{aligned}&\mathbf{P}(N \le -k_1, N' \le -k_1, T_1 \ge T_5, \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \le c_{86}\mathbf{d}(X_0, Y_0)^{3\beta_3^{2n}})\\ &\quad \le c_{87}\mathbf{d}(X_{T_0}, Y_{T_0})^2(1 - p_1)^{n+1}.\end{aligned}$$

(l) Consider the case when $N' \geq k_1$ and note that $\mathbf{P}(N \leq -k_1, T_1 \geq T_2, N' \geq k_1) \leq c_{88}\mathbf{d}(X_0, Y_0)^{1+3\beta_0/2}$. If $N \leq -k_1$, $N' \geq k_1$, and $T_1 \geq T_2$ then we have $\mathbf{d}(X_{T_4}, Y_{T_4}) \geq (1 - c_{89}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0)$. If, in addition, $T_1 \leq T_5$ then

$$\mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \geq (1 - c_{90}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0)$$

and

$$(1 - c_{90}\mathbf{d}(X_{T_0}, Y_{T_0})^{2\beta_0})\mathbf{d}(X_0, Y_0) \leq \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \leq \mathbf{d}(X_0, Y_0)(1 + c_{91}\mathbf{d}(X_0, Y_0)).$$

(m) If $N \leq -k_1$, $N' \geq k_1$, and $T_5 < T_1$ then, $\mathbf{d}(X_7, Y_7) \leq \mathbf{d}(X_0, Y_0)$, $\mathbf{d}(X_{17}, Y_{17}) \leq c_{92}\mathbf{d}(X_8, Y_8)$, and using Step 2, for $n \geq 1$,

$$\begin{aligned}
&\mathbf{P}(N \leq -k_1, N' \leq -k_1, T_1 \geq T_5, \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \leq c_{93}\mathbf{d}(X_0, Y_0)^{3\beta_3^{2n}}) \\
&\quad \leq c_{94}\mathbf{d}(X_{T_0}, Y_{T_0})^2(1 - p_1)^{n+1}.
\end{aligned}$$

(n) Finally, we consider the case $N \geq k_1$. We have $\mathbf{P}(N \geq k_1) \leq c_{95}\mathbf{d}(X_0, Y_0)^{1+\beta_0/2}$. Then, $\mathbf{d}(X_7, Y_7) \leq \mathbf{d}(X_0, Y_0)$, $\mathbf{d}(X_{17}, Y_{17}) \leq c_{96}\mathbf{d}(X_8, Y_8)$, and using Step 2, for $n \geq 1$,

$$\begin{aligned}
&\mathbf{P}(N \leq -k_1, N' \leq -k_1, T_1 \geq T_5, \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) \leq c_{97}\mathbf{d}(X_0, Y_0)^{3\beta_3^{2n}}) \\
&\quad \leq c_{98}\mathbf{d}(X_{T_0}, Y_{T_0})^{1+\beta_0/2}(1 - p_1)^{n+1}.
\end{aligned}$$

The estimates for values of $\mathbf{d}(X_{T_{17}}, Y_{T_{17}})$ and the corresponding probabilities listed above as (a)-(n) yield the following inequality. Its lines are labelled according to the case they represent.

$$\begin{aligned}
&\mathbf{E}|\log \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) - \log \mathbf{d}(X_0, Y_0)| && (3.26)\\
&\leq \sum_{-k_1 \leq k \leq 0} c_{99}\mathbf{d}(X_0, Y_0)2^{-k}(c_{100}2^{2k} + c_{101}\mathbf{d}(X_0, Y_0)) && (a)\\
&+ \sum_{-k_1 \leq k \leq 0} \sum_{n \geq 1} c_{102}\mathbf{d}(X_0, Y_0)^{2-\beta_0}(1 - p_1)^{n+1}(c_{103} + (3\beta_3^{2n} - 1)|\log \mathbf{d}(X_0, Y_0)|) && (b)\\
&+ \sum_{0 \leq k \leq k_1} c_{104}\mathbf{d}(X_0, Y_0)2^{-k/2}(c_{105}k + c_{106}\mathbf{d}(X_0, Y_0)) && (c)\\
&+ \sum_{0 \leq k \leq k_1} \sum_{n \geq 1} c_{107}\mathbf{d}(X_0, Y_0)^{2-\beta_0}2^{-k/2}(1 - p_1)^{n+1} \\
&\qquad \times (c_{108} + c_{109}\beta_3^{2n} + c_{110}k\beta_3^{2n} + (3\beta_3^{2n} - 1)|\log \mathbf{d}(X_0, Y_0)|) && (d)\\
&+ c_{111}\mathbf{d}(X_0, Y_0)^{2\beta_0} + c_{112}\mathbf{d}(X_0, Y_0) && (e)\\
&+ \sum_{-k_1 \leq k \leq 0} c_{113}\mathbf{d}(X_0, Y_0)^2(c_{114}2^{2k} + c_{115}\mathbf{d}(X_0, Y_0)) && (f)\\
&+ \sum_{-k_1 \leq k \leq 0} \sum_{n \geq 1} c_{116}\mathbf{d}(X_0, Y_0)^2(1 - p_1)^{n+1}(c_{117} + (3\beta_3^{2n} - 1)|\log \mathbf{d}(X_0, Y_0)|) && (g)\\
&+ \sum_{0 \leq k \leq k_1} c_{118}\mathbf{d}(X_0, Y_0)^2 2^{-k/2}(c_{119}k + c_{120}\mathbf{d}(X_0, Y_0)) && (h)\\
&+ \sum_{0 \leq k \leq k_1} \sum_{n \geq 1} c_{121}\mathbf{d}(X_0, Y_0)^2 2^{-k/2}(1 - p_1)^{n+1} \\
&\qquad \times (c_{122} + c_{123}\beta_3^{2n} + c_{124}k\beta_3^{2n} + (3\beta_3^{2n} - 1)|\log \mathbf{d}(X_0, Y_0)|) && (i)
\end{aligned}$$

$$+ c_{125}\mathbf{d}(X_0,Y_0)^{2\beta_0} + c_{126}\mathbf{d}(X_0,Y_0) \tag{j}$$
$$+ \sum_{n\ge1} c_{127}\mathbf{d}(X_0,Y_0)^2(1-p_1)^{n+1}(c_{128} + (3\beta_3^{2n}-1)|\log\mathbf{d}(X_0,Y_0)|) \tag{k}$$
$$+ c_{129}\mathbf{d}(X_0,Y_0)^{1+3\beta_0/2}(c_{130}\mathbf{d}(X_0,Y_0)^{2\beta_0} + c_{131}\mathbf{d}(X_0,Y_0)) \tag{l}$$
$$+ \sum_{n\ge1} c_{132}\mathbf{d}(X_0,Y_0)^2(1-p_1)^{n+1}(c_{133} + (3\beta_3^{2n}-1)|\log\mathbf{d}(X_0,Y_0)|) \tag{m}$$
$$+ \sum_{n\ge1} c_{134}\mathbf{d}(X_0,Y_0)^{1+\beta_0/2}(1-p_1)^{n+1}(c_{135} + (3\beta_3^{2n}-1)|\log\mathbf{d}(X_0,Y_0)|). \tag{n}$$

Recall that $\beta_0 \in (1/2,1)$ and $\beta_3^2(1-p_1) < 1$. Given these constraints on the values of the parameters, it is straightforward to check that (3.26) implies that

$$\mathbf{E}|\log\mathbf{d}(X_{T_{17}},Y_{T_{17}}) - \log\mathbf{d}(X_0,Y_0)| \le c_{136}\mathbf{d}(X_0,Y_0).$$

Step 5. It is easy to check that our estimates on the size of $\mathbf{d}(X_t,Y_t)$ apply not only at T_{17} but on the whole interval $[0,T_{17}]$. Hence,

$$\mathbf{E}\sup_{t\in[0,T_{17}]}|\log\mathbf{d}(X_t,Y_t) - \log\mathbf{d}(X_0,Y_0)| \le c_{136}\mathbf{d}(X_0,Y_0).$$

At several places in our argument we have assumed that $\mathbf{d}(X_0,Y_0)$ is small. Let $\varepsilon_1 > 0$ be such that the last inequality holds if $\mathbf{d}(X_0,Y_0) \le \varepsilon_1$. Let $Q_0 = 0$, $Q_1 = T_{17} \wedge \tau^+(\varepsilon_1)$ and $Q_k = Q_1 \circ \theta_{Q_{k-1}}$ for $k \ge 2$. Note that if $\mathbf{d}(X_{Q_k},Y_{Q_k}) = \varepsilon_1$ then $\mathbf{d}(X_{Q_n},Y_{Q_n}) = \varepsilon_1$ for all $n \ge k$. If $Q_k = T_{17}\circ\theta_{Q_{k-1}} < \tau^+(\varepsilon_1)\circ\theta_{Q_{k-1}}$ then we can apply the argument given in Steps 1-4 to the post-Q_k process, by the strong Markov property, because condition (3.18) is satisfied for $t = Q_k$ in place of $t = 0$. It follows that if $\mathbf{d}(X_0,Y_0) \le \varepsilon_1$ then

$$\mathbf{E}\sup_{t\in[0,Q_1]}|\log\mathbf{d}(X_t,Y_t) - \log\mathbf{d}(X_0,Y_0)| \le c_{136}\mathbf{d}(X_0,Y_0),$$

and

$$\mathbf{E}\Big(\sup_{t\in[Q_{k-1},Q_1]}|\log\mathbf{d}(X_t,Y_t) - \log\mathbf{d}(X_{Q_{k-1}},Y_{Q_{k-1}})| \mid \mathcal{F}_{Q_{k-1}}\Big)$$
$$\le c_{137}\mathbf{d}(X_{Q_{k-1}},Y_{Q_{k-1}}).$$

The argument given in part (a) of Step 4 shows that, a.s.,

$$\mathbf{d}(X_{Q_k},Y_{Q_k}) \le \mathbf{d}(X_{Q_{k-1}},Y_{Q_{k-1}}) + c_{138}\mathbf{d}(X_{Q_{k-1}},Y_{Q_{k-1}})^2.$$

If $\mathbf{d}(X_{Q_{k-1}},Y_{Q_{k-1}}) \le 2\mathbf{d}(X_0,Y_0)$ then

$$c_{138}\mathbf{d}(X_{Q_{k-1}},Y_{Q_{k-1}})^2 \le 4c_{138}\mathbf{d}(X_{Q_0},Y_{Q_0})^2$$

and, therefore, $\mathbf{d}(X_{Q_k},Y_{Q_k}) \le 2\mathbf{d}(X_0,Y_0)$ for

$$k \le \mathbf{d}(X_{Q_0},Y_{Q_0})/(4c_{138}\mathbf{d}(X_{Q_0},Y_{Q_0})^2) = 1/(4c_{138}\mathbf{d}(X_0,Y_0)).$$

This implies that for $k \le 1/(4c_{138}\mathbf{d}(X_0, Y_0))$,

$$\mathbf{E}(\sup_{t\in[Q_{k-1},Q_k]} |\log \mathbf{d}(X_t, Y_t) - \log \mathbf{d}(X_{Q_{k-1}}, Y_{Q_{k-1}})| \mid \mathcal{F}_{Q_{k-1}}) \le c_{139}\mathbf{d}(X_0, Y_0),$$

and

$$\mathbf{E}(\sup_{t\in[Q_{k-1},Q_k]} |\log \mathbf{d}(X_t, Y_t) - \log \mathbf{d}(X_0, Y_0)| \mid \mathcal{F}_{Q_{k-1}}) \le c_{140}k\mathbf{d}(X_0, Y_0).$$

For $k \ge 1/(4c_{138}\mathbf{d}(X_0, Y_0))$, we use the bound $\mathbf{d}(X_{Q_{k-1}}, Y_{Q_{k-1}}) \le \varepsilon_1$ to conclude that

$$\mathbf{E}(\sup_{t\in[Q_{k-1},Q_k]} |\log \mathbf{d}(X_t, Y_t) - \log \mathbf{d}(X_{Q_{k-1}}, Y_{Q_{k-1}})| \mid \mathcal{F}_{Q_{k-1}}) \le c_{141},$$

and

$$\mathbf{E}(\sup_{t\in[Q_{k-1},Q_k]} |\log \mathbf{d}(X_t, Y_t) - \log \mathbf{d}(X_0, Y_0)| \mid \mathcal{F}_{Q_{k-1}}) \le c_{142}k.$$

By Step 3, if $\mathbf{d}(X_0, Y_0) \le \varepsilon_1$ then $\mathbf{P}(S_1 \ge Q_k) \le 2^{-k}$. Hence

$$\begin{aligned}
&\mathbf{E}\left(|\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \cdot \mathbf{1}\left(\bigcup_{k\ge 0} \{S_1 \in [Q_k, Q_{k+1}]\} \right)\right) \qquad (3.27)\\
&\le \sum_{k\ge 0} \mathbf{E}(\mathbf{1}_{\{S_1\in[Q_k,Q_{k+1}]\}} \sup_{t\in[Q_k,Q_{k+1}]} |\log \mathbf{d}(X_t, Y_t) - \log \mathbf{d}(X_0, Y_0)|)\\
&\le \sum_{k\ge 0} \mathbf{E}(\mathbf{1}_{\{S_1\ge Q_k\}} \sup_{t\in[Q_k,Q_{k+1}]} |\log \mathbf{d}(X_t, Y_t) - \log \mathbf{d}(X_0, Y_0)|)\\
&\le \sum_{k\le 1/(4c_{138}\mathbf{d}(X_0,Y_0))} 2^{-k}c_{140}k\mathbf{d}(X_0, Y_0) + \sum_{k> 1/(4c_{138}\mathbf{d}(X_0,Y_0))} 2^{-k}c_{142}k\\
&\le c_{143}\mathbf{d}(X_0, Y_0) + c_{144}\mathbf{d}(X_0, Y_0)^{-1} \exp(-c_{145}\mathbf{d}(X_0, Y_0)^{-1})\\
&\le c_{146}\mathbf{d}(X_0, Y_0).
\end{aligned}$$

Suppose that $\mathbf{d}(X_0, Y_0) \le \varepsilon_1^2$ and let

$$\begin{aligned}
W' &= \min\{Q_k : \mathbf{d}(X_{Q_k}, Y_{Q_k}) = \varepsilon_1\},\\
W'' &= \inf\{t \ge 0 : X_t \in \partial D, \mathbf{n}(X_t) \cdot (Y_t - X_t) \le 0\}\\
&\qquad \wedge \inf\{t \ge 0 : Y_t \in \partial D, \mathbf{n}(Y_t) \cdot (X_t - Y_t) \le 0\},
\end{aligned}$$

with the convention that $\inf \emptyset = \infty$. Note that $\sup_{t\in[0,W'')} \mathbf{d}(X_t, Y_t) \le \mathbf{d}(X_0, Y_0)$. This implies that $W'' \le W'$, and if $W'' < \infty$ then,

$$\mathbf{d}(X_{W''}, \partial D) \vee \mathbf{d}(Y_{W''}, \partial D) \le c_{147}\mathbf{d}(X_{W''}, Y_{W''})^2 \le c_{147}\mathbf{d}(X_0, Y_0)^2.$$

By Lemma 3.8,

$$\begin{aligned}
(L^X_{W'} - L^X_{W''}) + (L^Y_{W'} - L^Y_{W''}) &\ge c_{148}|\log \mathbf{d}(X_{W''}, Y_{W''}) - \log \mathbf{d}(X_{W'}, Y_{W'})| \qquad (3.28)\\
&\ge c_{148}|\log \mathbf{d}(X_0, Y_0) - \log \mathbf{d}(X_{W'}, Y_{W'})| \ge c_{149}.
\end{aligned}$$

Suppose ε_1 is less than ε_0 in Lemma 3.3. Then, by Lemma 3.3 (ii) and the strong Markov property applied at W'',

$$\mathbf{E}(L^X_{S_1\wedge W'} - L^X_{W''}) + \mathbf{E}(L^Y_{S_1\wedge W'} - L^Y_{W''}) \le c_{150}\mathbf{d}(X_0,Y_0)^2.$$

This and (3.28) imply that,

$$\mathbf{P}(W' \le S_1) \le c_{151}\mathbf{d}(X_0,Y_0)^2. \tag{3.29}$$

Let $W''' = \inf\{t \ge W' : \mathbf{d}(X_t,Y_t) = \mathbf{d}(X_0,Y_0)\}$ and $W_1 = T_1 \circ \theta_{W'''}$. By the last formula in Step 2 and the strong Markov property,

$$\mathbf{E}|\log \mathbf{d}(X_{W_1},Y_{W_1}) - \log \mathbf{d}(X_{W'''},Y_{W'''})| \le c_{152}. \tag{3.30}$$

Let $Q^{W_1}_k = Q_k \circ \theta_{W_1}$. Then, by the strong Markov property at W_1, (3.27), (3.29) and (3.30),

$$\mathbf{E}\left(|\log \mathbf{d}(X_{S_1},Y_{S_1}) - \log \mathbf{d}(X_0,Y_0)| \cdot \mathbf{1}\left(\bigcup_{k\ge 0}\{S_1 \in [Q^{W_1}_k, Q^{W_1}_{k+1}]\}\right)\mathbf{1}_{\{W'\le S_1\}}\right)$$
$$\le c_{153}\mathbf{d}(X_0,Y_0)^2.$$

Let $W_k = W_1 \circ \theta_{W_{k-1}}$ and $Q^{W_j}_k = Q_k \circ \theta_{W_j}$. By induction we have

$$\mathbf{P}(W_k \le S_1) \le c_{154}\mathbf{d}(X_0,Y_0)^{2k},$$

and for $j \ge 1$,

$$\mathbf{E}\left(|\log \mathbf{d}(X_{S_1},Y_{S_1}) - \log \mathbf{d}(X_0,Y_0)| \cdot \mathbf{1}\left(\bigcup_{k\ge 0}\{S_1 \in [Q^{W_j}_k, Q^{W_j}_{k+1}]\}\right)\mathbf{1}_{\{W_j\le S_1\}}\right)$$
$$\le c_{155}\mathbf{d}(X_0,Y_0)^{2j}. \tag{3.31}$$

Note that $\mathbf{d}(X_0,Y_0)$ is bounded by the diameter of the domain so

$$\log \mathbf{d}(X_{S_1},Y_{S_1}) - \log \mathbf{d}(X_0,Y_0) \le c_{156} + c_{157}|\log \mathbf{d}(X_0,Y_0)|.$$

This, (3.29) and (3.30) imply that

$$\mathbf{E}(|\log \mathbf{d}(X_{S_1},Y_{S_1}) - \log \mathbf{d}(X_0,Y_0)| \cdot \mathbf{1}_{\{W'\le S_1\le W_1\}}) \le c_{158}\mathbf{d}(X_0,Y_0)^2|\log \mathbf{d}(X_0,Y_0)|. \tag{3.32}$$

Let $W'_k = W' \circ \theta_{W_k}$. Then, for $k \ge 1$,

$$\mathbf{E}(|\log \mathbf{d}(X_{S_1},Y_{S_1}) - \log \mathbf{d}(X_0,Y_0)| \cdot \mathbf{1}_{\{W'_k\le S_1\le W_{k+1}\}}) \le c_{159}\mathbf{d}(X_0,Y_0)^{2k+2}|\log \mathbf{d}(X_0,Y_0)|. \tag{3.33}$$

It is straightforward to check that

$$\bigcup_k [Q_k, Q_{k+1}] \cup \bigcup_j \bigcup_k [Q^{W_j}_k, Q^{W_j}_{k+1}] \cup [W', W_1] \cup \bigcup_k [W'_k, W_{k+1}] = [0,\infty).$$

Hence, part (i) of the lemma follows from (3.27), (3.31), (3.32) and (3.33).

(ii) Let $A^* = F^c(T_0, S_1, Z(T_0), \mathbf{d}(X_0,Y_0)^{\beta_1})$ and assume that $\beta_1 \in (0,1)$. Recall the estimates for the probabilities that N takes values in $[-k_1, 0]$ or $[0, k_1]$ from Step 1 of part (i) of the proof. Analogous estimates, the strong Markov property applied at time T_0, and an argument similar to that given in Step 3 but with k_2 replaced by $k_3 = \min\{k : 2^{-k} \leq \mathbf{d}(X_0,Y_0)^{\beta_1}\}$, yield for some $\beta_4 > 0$,

$$\begin{aligned} \mathbf{P}(A^*) &\leq c_{160}\mathbf{d}(X_0,Y_0)^{\beta_4}, \\ \mathbf{P}(\{-k_3 \leq N \leq 0\} \cap A^*) &\leq c_{161}\mathbf{d}(X_0,Y_0)^{1+\beta_4}2^{k_3}, \\ \mathbf{P}(\{0 < N \leq k_3\} \cap A^*) &\leq c_{162}\mathbf{d}(X_0,Y_0)^{1+\beta_4}. \end{aligned} \tag{3.34}$$

We will estimate

$$\mathbf{E}(|\log \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) - \log \mathbf{d}(X_0,Y_0)|\mathbf{1}_{A^*})$$

by splitting the integral into the sum of integrals over various events, as in (3.26). An upper bound for the above expectation can be obtained by using the same estimates as in (3.26), lines (b), (d), and (f)-(n), and replacing estimates in lines (a), (c) and (e) by the following estimates. By estimates similar to those in Step 2 (a), for some $\beta_5 > 0$,

$$\begin{aligned} &\mathbf{E}(|\log \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) - \log \mathbf{d}(X_0,Y_0)|\mathbf{1}_{\{-k_1 \leq N \leq 0, T_1 \leq T_2\} \cap A^*}) \\ &\quad \leq \sum_{-k_1 \leq k \leq -k_3} \mathbf{P}(N = k, T_1 \leq T_2)(c_{163}2^{2k} + c_{164}\mathbf{d}(X_0,Y_0)) \\ &\qquad + \mathbf{P}(\{-k_3 < N \leq 0\} \cap A^*)(c_{165}2^{2k_3} + c_{166}\mathbf{d}(X_0,Y_0)) \\ &\quad \leq \sum_{-k_1 \leq k \leq -k_3} c_{167}\mathbf{d}(X_0,Y_0)2^{-k}(c_{168}2^{2k} + c_{169}\mathbf{d}(X_0,Y_0)) \\ &\qquad + c_{170}\mathbf{d}(X_0,Y_0)^{1+\beta_4}2^{k_3}(c_{171}2^{-2k_3} + c_{172}\mathbf{d}(X_0,Y_0)) \\ &\quad \leq c_{173}\mathbf{d}(X_0,Y_0)^{1+\beta_1} + c_{174}\mathbf{d}(X_0,Y_0)^{2-\beta_0} \\ &\qquad + c_{175}\mathbf{d}(X_0,Y_0)^{1+\beta_4+\beta_1} + c_{176}\mathbf{d}(X_0,Y_0)^{2+\beta_4-\beta_1} \\ &\quad \leq c_{177}\mathbf{d}(X_0,Y_0)^{1+\beta_5}. \end{aligned} \tag{3.35}$$

Similarly, by estimates similar to Step 2 (c), for some $\beta_6 > 0$,

$$\begin{aligned} &\mathbf{E}(|\log \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) - \log \mathbf{d}(X_0,Y_0)|\mathbf{1}_{\{0 < N \leq k_1, T_1 \leq T_2\} \cap A^*}) \\ &\quad \leq \sum_{k_3 \leq k \leq k_1} \mathbf{P}(N = k, T_1 \leq T_2)(c_{178}k + c_{179}\mathbf{d}(X_0,Y_0)) \\ &\qquad + \mathbf{P}(\{0 < N < k_3\} \cap A^*)(c_{180}k_3 + c_{181}\mathbf{d}(X_0,Y_0)) \\ &\quad \leq \sum_{k_3 \leq k \leq k_1} c_{182}\mathbf{d}(X_0,Y_0)2^{-k/2}(c_{183}k + c_{184}\mathbf{d}(X_0,Y_0)) \\ &\qquad + c_{185}\mathbf{d}(X_0,Y_0)^{1+\beta_4}(c_{186}k_3 + c_{187}\mathbf{d}(X_0,Y_0)) \\ &\quad \leq c_{188}\mathbf{d}(X_0,Y_0)^{1+\beta_4/2}|\log \mathbf{d}(X_0,Y_0)| + c_{189}\mathbf{d}(X_0,Y_0)^{2+\beta_4/2} \\ &\qquad + c_{190}\mathbf{d}(X_0,Y_0)^{1+\beta_4}|\log \mathbf{d}(X_0,Y_0)| + c_{191}\mathbf{d}(X_0,Y_0)^{2+\beta_4} \\ &\quad \leq c_{192}\mathbf{d}(X_0,Y_0)^{1+\beta_6}. \end{aligned} \tag{3.36}$$

Recall that $\beta_0 \in (1/2,1)$. We apply estimates similar to those in Step 2 (e) to see that for some $\beta_7 > 0$,

$$\begin{aligned}
&\mathbf{E}(|\log \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) - \log \mathbf{d}(X_0,Y_0)| \mathbf{1}_{\{N \le -k_1, T_1 \le T_2\} \cap A^*}) \\
&\quad \le \mathbf{P}(A^*)(c_{193}\mathbf{d}(X_0,Y_0)^{2\beta_0} + c_{194}\mathbf{d}(X_0,Y_0)) \\
&\quad \le c_{195}\mathbf{d}(X_0,Y_0)^{\beta_4}(c_{196}\mathbf{d}(X_0,Y_0)^{2\beta_0} + c_{197}\mathbf{d}(X_0,Y_0)) \\
&\quad \le c_{198}\mathbf{d}(X_0,Y_0)^{1+\beta_7}.
\end{aligned} \tag{3.37}$$

Note that the sum of lines (b), (d), and (f)-(n) in (3.26) is bounded by $c_{199}\mathbf{d}(X_0,Y_0)^{\beta_8}$ for some c_{199} and $\beta_8 > 1$. This and (3.35)-(3.37) imply that for some c_{200} and $\beta_9 > 1$,

$$\mathbf{E}(|\log \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) - \log \mathbf{d}(X_0,Y_0)| \mathbf{1}_{A^*}) \le c_{200}\mathbf{d}(X_0,Y_0)^{\beta_9}.$$

As in part (i), we note that in fact we have proved that

$$\mathbf{E}(\sup_{t \in [0,T_{17}]} |\log \mathbf{d}(X_t,Y_t) - \log \mathbf{d}(X_0,Y_0)| \mathbf{1}_{A^*}) \le c_{200}\mathbf{d}(X_0,Y_0)^{\beta_9}.$$

Recall from part (i) that for $k \le 1/(4c_{138}\mathbf{d}(X_0,Y_0))$ we have $\mathbf{d}(X_{Q_k}, Y_{Q_k}) \le 2\mathbf{d}(X_0,Y_0)$. Let

$$A_k^* = F^c(T_0 \circ \theta_{Q_{k-1}}, S_1, Z(T_0 \circ \theta_{Q_{k-1}}), \mathbf{d}(X_{Q_{k-1}}, Y_{Q_{k-1}})^{\beta_1}).$$

Then for $k \le 1/(4c_{138}\mathbf{d}(X_0,Y_0))$,

$$\begin{aligned}
&\mathbf{E}(\sup_{t \in [Q_{k-1},Q_k]} |\log \mathbf{d}(X_t,Y_t) - \log \mathbf{d}(X_{Q_{k-1}}, Y_{Q_{k-1}})| \mathbf{1}_{A^*}) \\
&\quad \le \mathbf{E}(\sup_{t \in [Q_{k-1},Q_k]} |\log \mathbf{d}(X_t,Y_t) - \log \mathbf{d}(X_{Q_{k-1}}, Y_{Q_{k-1}})| \mathbf{1}_{A_k^*}) \\
&\quad \le c_{201}\mathbf{d}(X_0,Y_0)^{\beta_9}.
\end{aligned}$$

We have the following estimate analogous to (3.27),

$$\begin{aligned}
&\mathbf{E}\left(|\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0,Y_0)| \cdot \mathbf{1}\left(\bigcup_{k \ge 0} \{S_1 \in [Q_k, Q_{k+1}]\}\right) \mathbf{1}_{A^*}\right) \\
&\le \sum_{k \ge 0} \mathbf{E}(\mathbf{1}_{\{S_1 \in [Q_k,Q_{k+1}]\}} \sup_{t \in [Q_k,Q_{k+1}]} |\log \mathbf{d}(X_t,Y_t) - \log \mathbf{d}(X_0,Y_0)| \mathbf{1}_{A^*}) \\
&\le \sum_{k \ge 0} \mathbf{E}(\mathbf{1}_{\{S_1 \in [Q_k,Q_{k+1}]\}} \sup_{t \in [Q_k,Q_{k+1}]} |\log \mathbf{d}(X_t,Y_t) - \log \mathbf{d}(X_0,Y_0)| \mathbf{1}_{A_{k+1}^*}) \\
&\le \sum_{k \ge 0} \mathbf{E}(\mathbf{1}_{\{S_1 \ge Q_k\}} \sup_{t \in [Q_k,Q_{k+1}]} |\log \mathbf{d}(X_t,Y_t) - \log \mathbf{d}(X_0,Y_0)| \mathbf{1}_{A_{k+1}^*}) \\
&\le \sum_{k \le 1/(4c_{138}\mathbf{d}(X_0,Y_0))} 2^{-k} c_{201} k \mathbf{d}(X_0,Y_0)^{\beta_9} + \sum_{k > 1/(4c_{138}\mathbf{d}(X_0,Y_0))} 2^{-k} c_{142} k \\
&\le c_{202}\mathbf{d}(X_0,Y_0)^{\beta_9} + c_{203}\mathbf{d}(X_0,Y_0)^{-1} \exp(-c_{204}\mathbf{d}(X_0,Y_0)^{-1}) \\
&\le c_{205}\mathbf{d}(X_0,Y_0)^{\beta_9}.
\end{aligned} \tag{3.38}$$

The expectation

$$\mathbf{E}\left(|\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \mathbf{1}_{F^c(T_0, S_1, Z(T_0), \varepsilon^{\beta_1})}\right)$$

is bounded by the estimates on the right hand sides of (3.31), (3.32), (3.33) and (3.38). This easily implies part (ii) of the lemma.

(iii) Let $C = \{Z_{T_0} \in K\}$. Note that C signifies the event discussed in part (n) of Step 4 in part (i) of the proof. Recall from that step that $\mathbf{P}(C) \leq c_{206}\mathbf{d}(X_0, Y_0)^{\beta_{10}}$ for some $\beta_{10} > 1$. If we add the factor $\mathbf{1}_C$ to the left hand side of (3.26), the right hand side of (3.26) is reduced to line (n), and we obtain

$$\begin{aligned}
&\mathbf{E}\left(|\log \mathbf{d}(X_{T_{17}}, Y_{T_{17}}) - \log \mathbf{d}(X_0, Y_0)| \mathbf{1}_C\right) \\
&\quad \leq \sum_{n \geq 1} c_{207}\mathbf{d}(X_0, Y_0)^{1+\beta_0/2}(1-p_1)^{n+1}(c_{208} + (3\beta_3^{2n} - 1)|\log \mathbf{d}(X_0, Y_0)|) \\
&\quad \leq c_{209}\mathbf{d}(X_0, Y_0)^{\beta_{11}},
\end{aligned}$$

for some $\beta_{11} > 1$. We have the following formula similar to (3.38),

$$\begin{aligned}
&\mathbf{E}\left(|\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \cdot \mathbf{1}\left(\bigcup_{k \geq 0}\{S_1 \in [Q_k, Q_{k+1}]\}\right) \mathbf{1}_C\right) \\
&\leq \mathbf{E}(\mathbf{1}_{\{S_1 \leq Q_1\}}|\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \mathbf{1}_C) \\
&\quad + \sum_{k \geq 1} \mathbf{E}(\mathbf{1}_{\{S_1 \in [Q_k, Q_{k+1}]\}}|\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \mathbf{1}_C) \\
&\leq \mathbf{E}(\mathbf{1}_{\{S_1 \leq Q_1\}}|\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \mathbf{1}_C) \\
&\quad + \sum_{k \geq 1} \mathbf{E}(\mathbf{1}_{\{S_1 \geq Q_k\}} \sup_{t \in [Q_k, Q_{k+1}]} |\log \mathbf{d}(X_t, Y_t) - \log \mathbf{d}(X_0, Y_0)| \mathbf{1}_C) \\
&\leq c_{209}\mathbf{d}(X_0, Y_0)^{\beta_{11}} + \sum_{k \leq 1/(4c_{210}\mathbf{d}(X_0, Y_0))} 2^{-k} c_{211} k \mathbf{d}(X_0, Y_0)^{\beta_{10}+1} \\
&\quad + \sum_{k > 1/(4c_{210}\mathbf{d}(X_0, Y_0))} 2^{-k} c_{212} k \mathbf{d}(X_0, Y_0)^{\beta_{10}+1} \\
&\leq c_{209}\mathbf{d}(X_0, Y_0)^{\beta_{11}} + c_{213}\mathbf{d}(X_0, Y_0)^{\beta_{10}} + c_{214}\mathbf{d}(X_0, Y_0)^{-1+\beta_{10}} \exp(-c_{215}\mathbf{d}(X_0, Y_0)^{-1}) \\
&\leq c_{216}\mathbf{d}(X_0, Y_0)^{\beta_{12}},
\end{aligned} \tag{3.39}$$

for some $\beta_{12} > 1$. We can bound

$$\mathbf{E}\left(|\log \mathbf{d}(X_{S_1}, Y_{S_1}) - \log \mathbf{d}(X_0, Y_0)| \mathbf{1}_{\{Z_{T_0} \in K\}}\right)$$

by the sum of the right hand sides of (3.31), (3.32), (3.33) and (3.39). Part (iii) of the lemma follows. □

4. Arguments based on excursion theory.

We start this section with a review of the excursion theory. See, e.g., [M] for the foundations of the theory in the abstract setting and [B] for the special case of excursions of Brownian motion.

Although [B] does not discuss reflected Brownian motion, all results we need from that book readily apply in the present context. We will use two different but closely related "exit systems." The first one, presented below, is a simple exit system representing excursions of a single reflected Brownian motion from ∂D. The second exit system, presented after Lemma 4.2, is more complex as it encodes the information about two reflected Brownian motions X and Y, and stopping times S_k and U_k.

An "exit system" for excursions of the reflected Brownian motion X from ∂D is a pair (L^*_t, H^x) consisting of a positive continuous additive functional L^*_t and a family of "excursion laws" $\{H^x\}_{x\in\partial D}$. We will soon show that $L^*_t = L^X_t$. Let Δ denote the "cemetery" point outside $\mathbf{R}^2$ and let $\mathcal{C}$ be the space of all functions $f : [0,\infty) \to \mathbf{R}^2 \cup \{\Delta\}$ which are continuous and take values in $\mathbf{R}^2$ on some interval $[0,\zeta)$, and are equal to Δ on $[\zeta,\infty)$. For $x \in \partial D$, the excursion law H^x is a σ-finite (positive) measure on $\mathcal{C}$, such that the canonical process is strong Markov on (t_0,∞), for every $t_0 > 0$, with the transition probabilities of Brownian motion killed upon hitting ∂D. Moreover, H^x gives zero mass to paths which do not start from x. We will be concerned only with the "standard" excursion laws; see Definition 3.2 of [B]. For every $x \in \partial D$ there exists a unique standard excursion law H^x in D, up to a multiplicative constant.

Excursions of X from ∂D will be denoted e or e_s, i.e., if $s < u$, $X_s, X_u \in \partial D$, and $X_t \notin \partial D$ for $t \in (s,u)$ then $e_s = \{e_s(t) = X_{t+s},\ t \in [0, u-s)\}$ and $\zeta(e_s) = u - s$. By convention, $e_s(t) = \Delta$ for $t \geq \zeta$, so $e_t \equiv \Delta$ if $\inf\{s > t : X_s \in \partial D\} = t$. Let $\mathcal{E}_u = \{e_s : s \leq u\}$.

Let $\sigma_t = \inf\{s \geq 0 : L^*_s \geq t\}$ and let I be the set of left endpoints of all connected components of $(0,\infty) \setminus \{t \geq 0 : X_t \in \partial D\}$. The following is a special case of the exit system formula of [M],

$$\mathbf{E}\left[\sum_{t\in I} V_t \cdot f(e_t)\right] = \mathbf{E}\int_0^\infty V_{\sigma_s} H^{X(\sigma_s)}(f)ds = \mathbf{E}\int_0^\infty V_t H^{X_t}(f)dL^*_t, \tag{4.1}$$

where V_t is a predictable process and $f : \mathcal{C} \to [0,\infty)$ is a universally measurable function which vanishes on excursions e_t identically equal to Δ. Here and elsewhere $H^x(f) = \int_{\mathcal{C}} f dH^x$.

The normalization of the exit system is somewhat arbitrary, for example, if (L^*_t, H^x) is an exit system and $c \in (0,\infty)$ is a constant then $(cL^*_t, (1/c)H^x)$ is also an exit system. One can even make c dependent on $x \in \partial D$. Let $\mathbf{P}^y_D$ denote the distribution of Brownian motion starting from y and killed upon exiting D. Theorem 7.2 of [B] shows how to choose a "canonical" exit system; that theorem is stated for the usual planar Brownian motion but it is easy to check that both the statement and the proof apply to the reflected Brownian motion. According to that result, we can take L^*_t to be the continuous additive functional whose Revuz measure is a constant multiple of the arc length measure on ∂D and H^x's to be standard excursion laws normalized so that

$$H^x(A) = \lim_{\delta\downarrow 0} \frac{1}{\delta} \mathbf{P}^{x+\delta\mathbf{n}(x)}_D(A), \tag{4.2}$$

for any event A in a σ-field generated by the process on an interval $[t_0,\infty)$, for any $t_0 > 0$. The normalization of the local time is linked to the normalization of ω_x, given before the statement of Theorem 1.2.

The Revuz measure of L^X is the measure $dx/(2|D|)$ on ∂D, i.e., if the initial distribution of X is the uniform probability measure μ in D then $\mathbf{E}^\mu \int_0^1 \mathbf{1}_A(X_s)dL^X_s = \int_A dx/(2|D|)$ for any Borel set $A \subset \partial D$, see Example 5.2.2 of [FOT].

We will show that $L^*_t = L^X_t$. It is sufficient to verify that the normalization $L^*_t = L^X_t$ works for the half-space $D_* = \{(x_1,x_2) : x_2 > 0\}$. Let $K_a = \{(x_1,x_2) \in \mathbf{R}^2 : x_2 = a\}$ and $T_a = \inf\{t >$

$0 : B_t \in K_a\}$. Note that $\mathbf{P}^{(0,0)+\delta\mathbf{n}((0,0))}(T_a < T_{\partial D_*}) = \delta/a$ for $\delta < a$, so $H^{(0,0)}(T_a < T_{\partial D_*}) = 1/a$, assuming that $H^{(0,0)}$ is normalized as in (4.2). The reflected Brownian motion X_t in the half-plane D_* with $X_0 = (0,0)$, and its local time L_t^X on ∂D_* may be constructed from the planar Brownian motion $B_t = (B_t^1, B_t^2)$ starting from $(0,0)$ by the following formula

$$(X_t, L_t^X) = \left(\left(B_t^1,\, B_t^2 - \min_{0\le s\le t} B_s^2 \right),\, - \min_{0\le s\le t} B_s^2 \right).$$

Note that the y-coordinate of an excursion of the reflecting Brownian motion X from ∂D^* is just an excursion of 1-dimensional Brownian motion away from 0. It is well-known that such 1-dimensional excursions form a Poisson point process. The event that $X_t - (0, L_t^X)$ does not hit K_1 before time σ_1^X is the same as the event that there is no excursion of X from ∂D_* such that it starts at a time t with $L_t^X = a$, $a \in [0,1]$, and the height of the y-coordinate of the excursion exceeds $1 + a$. If we assume that $L_t^* = L_t^X$, then according to the exit system formula (4.1), the probability of this event is equal to the probability that a Poisson random variable with parameter $\int_0^1 \frac{1}{1+a} da$ takes value 0, i.e., this probability is equal to

$$\exp\left(- \int_0^1 \frac{1}{1+a} da \right) = 1/2. \tag{4.3}$$

The event "$X_t - (0, L_t^X)$ does not hit K_1 before time σ_1^X" is the same as the event "B_t does not hit K_1 before hitting K_{-1}," and, obviously, the last event has probability 1/2. This agrees with (4.3) so the assumption that $L_t^* = L_t^X$ is correct. In other words, the normalization of the local time L_t^X contained implicitly in (1.1) and the normalization of excursion laws H^x given in (4.2) match so that (dL_t^X, H^x) is an exit system for X_t from ∂D.

Let $T_{\partial D_*} = \inf\{t > 0 : B_t \in \partial D_*\}$. Then by Theorem II.1.16 of [Ba],

$$\mathbf{P}^{(0,0)+\delta\mathbf{n}((0,0))}(B^1(T_{\partial D_*}) \in dy) = \frac{1}{\delta\pi(1 + (y/\delta)^2)} dy.$$

Hence,

$$\begin{aligned} H^{(0,0)}(e(\zeta-) \in dy) &= \lim_{\delta\downarrow 0} \frac{1}{\delta} \mathbf{P}^{(0,0)+\delta\mathbf{n}((0,0))}(B^1(T_{\partial D_*}) \in dy) \\ &= \lim_{\delta\downarrow 0} \frac{1}{\delta} \cdot \frac{1}{\delta\pi(1 + (y/\delta)^2)} dy = \frac{1}{\pi y^2}. \end{aligned}$$

This means that $\omega_x(dy) = H^x(e(\zeta-) \in dy)$, and it is easy to see that this result extends to all C^2-smooth domains D.

Lemma 4.1. *For some $c_1, c_2, c_3 \in (0,\infty)$ and any $x, y \in \overline{D}$ and $a > 0$,*

(i) $\mathbf{P}^{x,y}(L_1^X > a) \le 2e^{-c_1 a}$, and

(ii) $\mathbf{P}^{x,y}(L_{\sigma^X(1)}^Y > a) \le c_2 e^{-c_3 a}$.

Proof. (i) Since D is a bounded C^2-smooth domain in $\mathbf{R}^2$, it is known (cf. [BH]) that the transition density function $p(t,x,y)$ of the reflecting Brownian motion in D satisfies the estimate

$$p(t,x,y) \le c_4 t^{-1} e^{-c_5|x-y|^2/t} \qquad \text{for } t \in (0,1] \text{ and } x, y \in \overline{D}.$$

Therefore,

$$\sup_{x\in\overline{D}} \mathbf{E}^x[L_1^X] = c_6 \sup_{x\in\overline{D}} \int_{\partial D}\int_0^1 p(s,x,y)dsdy < \infty.$$

Take $c_7 > 0$ so that $\sup_{x\in\overline{D}} \mathbf{E}^x[L_1^X] < 1/(2c_7)$. It follows from Khasminskii's inequality that

$$\sup_{x\in\overline{D}} \mathbf{E}^x\left[e^{c_7 L_{t_0}^X}\right] < 2.$$

This implies that for any $a > 0$,

$$\mathbf{P}^{x,y}(L_1^X > a) < 2e^{-c_7 a}.$$

(ii) We have proved in part (i) that $\sup_{x\in\overline{D}} \mathbf{E}^x e^{c_7 L_{t_0}^X} < 2$. This, the additivity of L^X and the Markov property of (X,Y), imply that $\mathbf{E}\exp(c_7(L_j^X - L_{j-1}^X)) < 2$. A routine application of the Markov property at times $t = 1,2,\dots$ shows that

$$\mathbf{E}^{x,y}\exp(c_7 L_k^X) = \mathbf{E}^{x,y}\exp\left(c_7 \sum_{1\le j\le k}(L_j^X - L_{j-1}^X)\right) \le 2^k.$$

It follows that if $c_8 > 0$ is sufficiently small, then for integer k of the form $k = c_8 a$,

$$\begin{aligned}\mathbf{P}^{x,y}(L_k^X \ge a) &= \mathbf{P}^{x,y}\left(\exp(c_7 L_k^X) \ge \exp(c_7 a)\right)\\ &\le 2^k \exp(-c_7 a) = 2^{c_8 a}\exp(-c_7 a) \le \exp(-c_7 a/2)\end{aligned}$$

Since $y \mapsto \mathbf{P}^y(L_1^Y > 1) = \int_D p(\frac{1}{2},y,z)\mathbf{P}^z(L_{1/2}^Y > 1)dz$ is continuous on $\overline{D}$, there is $p_1 > 0$ such that $\inf_{x\in\overline{D}} \mathbf{P}^y(L_1^Y > 1) \ge p_1$. Hence

$$\mathbf{P}^{x,y}(L_{c_8 a}^Y \le 1) = \mathbf{P}^{x,y}(L_k^Y \le 1) \le (1-p_1)^k = (1-p_1)^{c_8 a} = e^{-c_9 a}.$$

For a of the form $a = k/c_8$, where $k \ge 1$ is an integer, we obtain,

$$\begin{aligned}\mathbf{P}^{x,y}(L_{\sigma^Y(1)}^X > a) &\le \mathbf{P}^{x,y}(L_{c_8 a}^X > a) + \mathbf{P}^{x,y}(\sigma^Y(1) \ge c_8 a)\\ &\le \mathbf{P}^{x,y}(L_{c_8 a}^X > a) + \mathbf{P}^{x,y}(L_{c_8 a}^Y \le 1)\\ &\le e^{-c_7 a/2} + e^{-c_9 a} \le c_{10} e^{-c_{11} a}.\end{aligned}$$

It is elementary to see that by adjusting the values of the constants c_{10} and c_{11} we can make the formula valid for all $a \ge 0$. By interchanging the roles of X_t and Y_t, we obtain part (ii) of the lemma. □

Recall that $|D|$ and $|\partial D|$ denote the area of D and the length of its boundary.

Lemma 4.2. *Let $\varphi \in C(\partial D)$. For any $\delta, p > 0$ there exists $t_0 < \infty$ such that for every $x \in \overline{D}$, we have*
(i)

$$\mathbf{P}^x\left(\sup_{t\ge t_0}\left|\frac{1}{t}\int_0^t \varphi(X_s)dL_s^X - \frac{1}{2|D|}\int_{\partial D}\varphi(y)dy\right| \ge \delta\right) < p.$$

(ii) In particular,

$$\mathbf{P}^x \left(\sup_{t \geq t_0} \left| \frac{L_t^X}{t} - \frac{|\partial D|}{2|D|} \right| \geq \delta \right) < p.$$

Proof. Let μ be the uniform probability distribution on D. By Lemma 4.1 (i), $\sup_{x\in\overline{D}} \mathbf{E}^x L_1^X < \infty$. Since $\varphi(x)$ is bounded, it follows that $\sup_{x\in\overline{D}} \mathbf{E}^x |\int_0^1 \varphi(X_s) dL_s^X| < \infty$ and $\mathbf{E}^\mu |\int_0^1 \varphi(X_s) dL_s^X| < \infty$. Since μ is the stationary distribution for X, the ergodic theorem shows that $\mathbf{P}^\mu$-a.s.,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi(X_s) dL_s^X = \lim_{k\to\infty} (1/k) \sum_{1\leq n\leq k} \int_{n-1}^n \varphi(X_s) dL_s^X = \mathbf{E}^\mu \int_0^1 \varphi(X_s) dL_s^X.$$

The Revuz measure of L^X is $dx/(2|D|)$ on ∂D and $\varphi(x)$ is a continuous function, so it is easy to see that

$$\mathbf{E}^\mu \int_0^1 \varphi(X_s) dL_s^X = \frac{1}{2|D|} \int_{\partial D} \varphi(y) dy.$$

Therefore, $\mathbf{P}^\mu$-a.s.,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi(X_s) dL_s^X = \frac{1}{2|D|} \int_{\partial D} \varphi(y) dy,$$

and

$$\mathbf{P}^\mu \left(\sup_{t\geq t_0} \left| \frac{1}{t} \int_0^t \varphi(X_s) dL_s^X - \frac{1}{2|D|} \int_{\partial D} \varphi(y) dy \right| \geq \delta/6 \right) < p/6, \tag{4.4}$$

for any given $\delta, p > 0$ and some $t_0 < \infty$.

The arguments on p. 6 of [BB] or Theorem 2.4 in [BH] show that there are constants $c_1, c_2 > 0$ such that

$$\sup_{x,y\in D} \left| p_t(x,y) - \frac{1}{|D|} \right| \leq c_1 e^{-c_2 t} \qquad \text{for } t \geq 1. \tag{4.5}$$

Let $t_1 < \infty$ be so large that the right hand side of (4.5) is $p/6$. Then the Markov property applied at t_1, (4.4) and (4.5) imply that for $x \in \overline{D}$,

$$\begin{aligned} &\mathbf{P}^x \left(\sup_{t\geq t_0} \left| \frac{1}{t} \int_{t_1}^{t_1+t} \varphi(X_s) dL_s^X - \frac{1}{2|D|} \int_{\partial D} \varphi(y) dy \right| \geq \delta/6 \right) \\ &\leq \mathbf{P}^\mu \left(\sup_{t\geq t_0} \left| \frac{1}{t} \int_{t_1}^{t_1+t} \varphi(X_s) dL_s^X - \frac{1}{2|D|} \int_{\partial D} \varphi(y) dy \right| \geq \delta/6 \right) + \frac{p}{6} < \frac{p}{3}. \end{aligned} \tag{4.6}$$

We can increase t_0, if necessary, so that for all $x \in \overline{D}$,

$$\mathbf{P}^x \left(\sup_{t\geq t_0} \left| \frac{1}{t} \int_{t_1}^{t_1+t} \varphi(X_s) dL_s^X - \frac{1}{t_1+t} \int_{t_1}^{t_1+t} \varphi(X_s) dL_s^X \right| \geq \delta/3 \right) \leq p/3, \tag{4.7}$$

and

$$\mathbf{P}^x \left(\sup_{t\geq t_0} \left| \frac{1}{t_1+t} \int_0^{t_1} \varphi(X_s) dL_s^X \right| \geq \delta/3 \right) < p/3. \tag{4.8}$$

Part (i) of the lemma follows from (4.6)-(4.8) and the triangle inequality. Part (ii) follows from (i) by taking $\varphi = 1$. $\square$

We will have to analyze excursions of X from ∂D containing intervals (S_k, U_k). The exit system (L_t^X, H^x) is inadequate for this purpose so we will now introduce a "richer" version of this exit system, capable of keeping track of some extra information.

Let $\ell(t) = \max\{S_k : k \geq 0, S_k \in [0,t]\}$. Consider the strong Markov process $(X_t, Y_t, X_{\ell(t)}, Y_{\ell(t)})$ and let $e_s(t) = (X_{s+t}, Y_{s+t}, X_{\ell(s+t)}, Y_{\ell(s+t)})$ for $t \geq 0$ and s such that $X_s \in \partial D$ and $\zeta(e_s) \stackrel{\text{df}}{=} \inf\{t > s : X_t \in \partial D\} - s > 0$. For all other s, we let $e_s \equiv \Delta$ (a cemetery state added to $\mathbf{R}^8$). Note a technical difference with the previous version of the exit system—here, the excursions are not killed at $\zeta(e_s)$ but are continued after that time; this version of the exit system is discussed in Maisonneuve [M].

We will describe an exit system $(L_t^X, H^{(x_1,y_1,x_2,y_2)})$ for the process $(X_t, Y_t, X_{\ell(t)}, Y_{\ell(t)})$ from the set $\partial D \times \overline{D}^3$. For any $(x_1, y_1, x_2, y_2) \in \partial D \times \overline{D}^3$, $H^{(x_1,y_1,x_2,y_2)}$ is a σ-finite measure defined as follows. The first component of $(X_t, Y_t, X_{\ell(t)}, Y_{\ell(t)})$ under $H^{(x_1,y_1,x_2,y_2)}$ has the same distribution on $[0,\zeta)$ as an excursion under H^{x_1}, defined previously. Under $H^{(x_1,y_1,x_2,y_2)}$, the process X_t continues after ζ as a reflected Brownian motion in D, starting from $X_{\zeta-}$, but otherwise independent of $\{X_t, t \in [0,\zeta)\}$. The other components of $(X_t, Y_t, X_{\ell(t)}, Y_{\ell(t)})$ are determined by the first component as follows. First, we find B, the Brownian motion driving X, using the uniqueness of the solution to (1.1). Then we use B and (1.2) to define a reflected Brownian motion Y in D staring from y_1. We set $S_1^e = U_0^e = 0$ and for $k \geq 1$,

$$\begin{aligned}
U_1^e &= \inf\{t > 0 : \mathbf{d}(X_t, x_2) \vee \mathbf{d}(Y_t, y_2) \geq a_1 \mathbf{d}(x_2, y_2)\}, \\
S_k^e &= \inf\{t > U_{k-1}^e : \mathbf{d}(X_t, \partial D) \vee \mathbf{d}(Y_t, \partial D) \leq a_2 \mathbf{d}(X_t, Y_t)^2\}, \\
U_k^e &= \inf\{t > S_k^e : \mathbf{d}(X_t, X_{S_k^e}) \vee \mathbf{d}(Y_t, Y_{S_k^e}) \geq a_1 \mathbf{d}(X_{S_k^e}, Y_{S_k^e})\}.
\end{aligned}$$

Let $\ell^e(t) = \max\{S_k^e : k \geq 1, S_k^e \in [0,t]\}$. The last two components of the process under $H^{(x_1,y_1,x_2,y_2)}$ are defined to be $X_{\ell^e(t)}$ and $Y_{\ell^e(t)}$ if $\ell^e(t) > 0$, and x_2 and y_2, if $\ell^e(t) = 0$.

Note that the exit system for $(X_t, Y_t, X_{\ell(t)}, Y_{\ell(t)})$ from $\partial D \times \overline{D}^3$ is equivalent, in a sense, to the exit system of X_t from ∂D because $\overline{D}^3$ is the state space for $(Y_t, X_{\ell(t)}, Y_{\ell(t)})$. Moreover, since X and Y are strong solutions to the stochastic Skorokhod equations (1.1)-(1.2) driven by the same Brownian motion, it follows that $(Y_t, X_{\ell(t)}, Y_{\ell(t)})$ is a deterministic function of $\{X_s, 0 \leq s \leq t\}$. We included $(Y_t, X_{\ell(t)}, Y_{\ell(t)})$ in the process so that we can keep track of the stopping times S_k and T_k inside the excursions of X_t away from ∂D.

In the present context, the exit system formula of [M] changes its form from that in (4.1) to

$$\begin{aligned}
\mathbf{E} \sum_{t \geq 0} V_t \cdot f(e_t) &= \mathbf{E} \int_0^\infty V_{\sigma_s^X} H^{(X(\sigma_s^X), Y(\sigma_s^X), X(\ell(\sigma_s^X)), Y(\ell(\sigma_s^X)))}(f) ds \qquad (4.9) \\
&= \mathbf{E} \int_0^\infty V_t H^{(X(t), Y(t), X(\ell(t)), Y(\ell(t)))}(f) dL_t^X,
\end{aligned}$$

where V_t is a predictable process and f is a non-negative universally measurable function which vanishes on excursions identically equal to Δ, and those with $\zeta(e) = 0$.

Let

$$T^e = \inf\{t \geq U_1^e : X_t \in \partial D \text{ or } Y_t \in \partial D\},$$

and recall that $Z_{T^e} = X_{T^e}$ if $X_{T^e} \in \partial D$ and $Z_{T^e} = Y_{T^e}$ otherwise. Recall also that $F(s, u, x, a)$ denotes $\{\sup_{s \leq t \leq u} \mathbf{d}(X_t, x) \leq a\}$. Let $T_X(A) = \inf\{t \geq 0 : X_t \in A\}$.

We assume that the constant a_1 in the next lemma satisfies Lemma 3.6.

Lemma 4.3. *(i) For any $c_0 > 0$ there exist $c_1 < \infty$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \geq c_0 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1} \mathbf{d}(x_1, y_1)$, then*

$$H^{(x_1,y_1,x_2,y_2)} \big(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mid U_1^e \leq \zeta\big) \leq c_1 \varepsilon.$$

(ii) For any $c_0, \beta_1 > 0$, there exist $c_1 < \infty$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \geq c_0 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \leq \varepsilon \leq \varepsilon_0$, $k \geq 1$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1} \mathbf{d}(x_1, y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)} \big(|\log \rho_{U_k^e} - \log \rho_{S_{k+1}^e}| \mathbf{1}_{\{U_k^e \leq \tau^+(\varepsilon^{\beta_1}) \wedge \zeta\}} \mid U_1^e \leq \zeta\big) \leq c_1 \varepsilon^{1+(k-1)\beta_1}.$$

(iii) For any $c_0 > 0$ there exist $\beta_1 > 0$, $\beta_2 > 1$, and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \geq c_0 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1} \mathbf{d}(x_1, y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)} \Big(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{T^e \leq S_2^e\}} \mathbf{1}_{F^c(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\Big) \leq \varepsilon^{\beta_2}.$$

(iv) For any $c_0 > 0$ there exist $\beta_1 > 0$, $\beta_2 > 1$, and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \geq c_0 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1} \mathbf{d}(x_1, y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)} \Big(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{S_2^e \leq T^e\}} \mathbf{1}_{F^c(S_2^e, T^e, Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\Big) \leq \varepsilon^{\beta_2}.$$

(v) For any $c_0, \beta_0 > 0$ there exist $\beta_1 > 0$, $\beta_2 > 1$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \geq c_0 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1} \mathbf{d}(x_1, y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)} \Big(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \mathbf{1}_{F^c(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\Big) \leq \varepsilon^{\beta_2}.$$

(vi) Let $K = K(x_1, \varepsilon, \beta_1) = \{x \in \partial D : \tan \alpha(x_1, x) \geq \varepsilon^{-\beta_1}\}$. For any $c_0 > 0$ there exist $c_1 < \infty$, $\beta_1 > 0$, $\beta_2 > 1$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \geq c_0 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1} \mathbf{d}(x_1, y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)} \big(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{Z_{T^e} \in K\}} \mid U_1^e \leq \zeta\big) \leq c_1 \varepsilon^{\beta_2}.$$

(vii) Let K be defined as in (vi). For any $c_0 > 0$ there exist $\beta_0, \beta_1 > 0$, $\beta_2 > 1$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \geq c_0 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1} \mathbf{d}(x_1, y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)} \Big(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \mathbf{1}_{\{X(T_X(\partial D)) \in K\}} \mid U_1^e \leq \zeta\Big) \leq \varepsilon^{\beta_2}.$$

Proof. Parts (i), (iii) and (vi) of the lemma follow from the strong Markov property applied at U_1^e and Lemma 3.9 (i), (ii) and (iii). Part (vii) follows from (v) (proved below) and (vi). It remains to prove (ii), (iv) and (v).

(ii) For $k \geq 2$, if $\mathbf{d}(X_{S^e_{k-1}}, Y_{S^e_{k-1}}) \leq \varepsilon^{\beta_1}$ then both processes are within distance $c_2\varepsilon^{2\beta_1}$ of ∂D at time S^e_{k-1}. Brownian motion starting at a point z at most $c_2\varepsilon^{2\beta_1}$ units away from ∂D will hit ∂D before hitting $\partial\mathcal{B}(z, \varepsilon^{\beta_1})$ with probability no less than $1 - c_3\varepsilon^{\beta_1}$. This and the strong Markov property applied at S^e_{j-1}, $j = 2, 3, \ldots, k$, imply that

$$H^{(x_1,y_1,x_2,y_2)}\left(U^e_k \leq \tau^+(\varepsilon^{\beta_1}) \wedge \zeta \mid U^e_1 \leq \zeta\right) \leq c_4\varepsilon^{(k-1)\beta_1}.$$

We combine this with part (i), using the strong Markov property at time U^e_k, to see that (ii) holds.

(iv) If $S^e_2 \leq T^e$ then $\mathbf{d}(X_{S^e_2}, Y_{S^e_2}) \leq \varepsilon$ and $\mathbf{d}(X_{S^2_e}, \partial D) \leq c_1\varepsilon^2$. Suppose that $\beta_1 \in (0, 2)$. Brownian motion starting at a point z at most $c_1\varepsilon^2$ units away from ∂D will hit ∂D before hitting $\partial\mathcal{B}(z, \varepsilon^{\beta_1})$ with probability not less than $1 - c_2\varepsilon^{2-\beta_1}$. By the strong Markov property applied at S^e_2,

$$H^{(x_1,y_1,x_2,y_2)}\left(F^c(S^e_2, T_X(\partial D), X_{S^e_2}, \varepsilon^{\beta_1}) \mid U^e_1 \leq \zeta, S^e_2 \leq T^e, \mathcal{F}_{S^e_2}\right) \leq c_3\varepsilon^{2-\beta_1}.$$

This, the strong Markov property applied at S^e_2 and part (i) of the lemma imply part (iv), for a suitable choice of β_1 and β_2.

(v) We have for some $\beta_2 > 1$,

$$\begin{aligned}
&H^{(x_1,y_1,x_2,y_2)}\left(|\log\rho_{U^e_1} - \log\rho_{S^e_2}|\mathbf{1}_{\{\tau^+(\varepsilon^{\beta_0})\geq S^e_2\}}\mathbf{1}_{F^c(T^e,T_X(\partial D),Z_{T^e},\varepsilon^{\beta_1})} \mid U^e_1 \leq \zeta\right)\\
&\leq H^{(x_1,y_1,x_2,y_2)}\left(|\log\rho_{U^e_1} - \log\rho_{S^e_2}|\mathbf{1}_{F^c(T^e,S^e_2,Z_{T^e},\varepsilon^{\beta_1}/2)} \mid U^e_1 \leq \zeta\right)\\
&\quad + H^{(x_1,y_1,x_2,y_2)}\Big(|\log\rho_{U^e_1} - \log\rho_{S^e_2}|\mathbf{1}_{\{S^e_2\leq\tau^+(\varepsilon^{\beta_0})\wedge T_X(\partial D)\}}\mathbf{1}_{F(T^e,S^e_2,Z_{T^e},\varepsilon^{\beta_1}/2)}\\
&\qquad \times \mathbf{1}_{F^c(S^e_2,T_X(\partial D),X(S^e_2),\varepsilon^{\beta_1}/2)} \mid U^e_1 \leq \zeta\Big)\\
&\leq \varepsilon^{\beta_2} + H^{(x_1,y_1,x_2,y_2)}\Big(|\log\rho_{U^e_1} - \log\rho_{S^e_2}|\mathbf{1}_{\{S^e_2\leq\tau^+(\varepsilon^{\beta_0})\wedge T_X(\partial D)\}}\\
&\qquad \times \mathbf{1}_{F^c(S^e_2,T_X(\partial D),X(S^e_2),\varepsilon^{\beta_1}/2)} \mid U^e_1 \leq \zeta\Big),
\end{aligned} \tag{4.10}$$

where the last inequality follows from (iii). It will suffice to bound the second term on the right hand side. If $\mathbf{d}(X_{S^e_2}, Y_{S^e_2}) \leq \varepsilon^{\beta_0}$, then $\mathbf{d}(X_{S^2_e}, \partial D) \leq c_1\varepsilon^{2\beta_0}$. Choose $\beta_1 > 0$ so that $2\beta_0 - \beta_1 > 0$. Brownian motion starting at a point z at most $c_1\varepsilon^{2\beta_0}$ units away from ∂D will hit ∂D before hitting $\partial\mathcal{B}(z, \varepsilon^{\beta_1}/2)$ with probability not less than $1 - c_2\varepsilon^{2\beta_0-\beta_1}$. By the strong Markov property applied at S^e_2,

$$H^{(x_1,y_1,x_2,y_2)}\left(F^c(S^e_2, T_X(\partial D), X_{S^e_2}, \varepsilon^{\beta_1}/2) \mid U^e_1 \leq \zeta, S^e_2 \leq \tau^+(\varepsilon^{\beta_0}), \mathcal{F}_{S^e_2}\right) \leq c_3\varepsilon^{2\beta_0-\beta_1}.$$

This, the strong Markov property applied at S^e_2 and part (i) of the lemma imply that the second term on the right hand side of (4.10) is bounded by $c_4\varepsilon^{1+2\beta_0-\beta_1}$. This completes the proof of the lemma. □

Lemma 4.4. *(i) There exists $c_1 < \infty$ such that for $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$,*

$$H^{(x_1,y_1,x_2,y_2)}|\log\cos\alpha(x_1, X_{\zeta-})| \leq c_1.$$

(ii) Let $K = K(x_1,\varepsilon,\beta_1) = \{x \in \partial D : \tan\alpha(x_1,x) \geq \varepsilon^{-\beta_1}\}$. For any $c_0 > 0$ there exist $\beta_1 > 0$, $\beta_2 > 1$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1,x_2) \vee \mathbf{d}(y_1,y_2) < a_1\mathbf{d}(x_2,y_2)$, $\mathbf{d}(x_1,y_1) \geq c_0\mathbf{d}(x_2,y_2)$, $\mathbf{d}(x_1,y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1}\mathbf{d}(x_1,y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{\{X(T_X(\partial D))\in K\}} \mid U_1^e \leq \zeta\right) \leq \varepsilon^{\beta_2}.$$

(iii) For any $c_0 > 0$ there exist $\beta_1 > 0$, $\beta_2 > 1$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1,x_2) \vee \mathbf{d}(y_1,y_2) < a_1\mathbf{d}(x_2,y_2)$, $\mathbf{d}(x_1,y_1) \geq c_0\mathbf{d}(x_2,y_2)$, $\mathbf{d}(x_1,y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1}\mathbf{d}(x_1,y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{\{T^e\leq S_2^e\}}\mathbf{1}_{F^c(T^e,S_2^e,Z_{T^e},\varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right) \leq \varepsilon^{\beta_2}.$$

(iv) For any $c_0 > 0$ there exist $\beta_1 > 0$, $\beta_2 > 1$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1,x_2) \vee \mathbf{d}(y_1,y_2) < a_1\mathbf{d}(x_2,y_2)$, $\mathbf{d}(x_1,y_1) \geq c_0\mathbf{d}(x_2,y_2)$, $\mathbf{d}(x_1,y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1}\mathbf{d}(x_1,y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{\{S_2^e\leq T^e\}}\mathbf{1}_{F^c(S_2^e,T^e,Z_{T^e},\varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right) \leq \varepsilon^{\beta_2}.$$

(v) For any $c_0 > 0$ there exist $\beta_1 > 0$, $\beta_2 > 1$ and $\varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1,x_2) \vee \mathbf{d}(y_1,y_2) < a_1\mathbf{d}(x_2,y_2)$, $\mathbf{d}(x_1,y_1) \geq c_0\mathbf{d}(x_2,y_2)$, $\mathbf{d}(x_1,y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1}\mathbf{d}(x_1,y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{F^c(T^e,T_X(\partial D),Z_{T^e},\varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right) \leq \varepsilon^{\beta_2}.$$

Proof. (i) Recall from Section 1 that we have assumed that the boundary of D is C^4-smooth and that there exist at most a finite number of points $x_1, x_2, \ldots, x_n$ such that $\nu(x_k) = 0$, $k = 1, \ldots, n$, and the third derivative of the function representing the boundary does not vanish at any x_k. This implies that there exist $\delta_0, c_2, c_3 > 0$ such that if $x \in \partial D$ and $\mathbf{d}(x, x_k) \leq \delta_0$ then $|\nu(x)| \geq c_2\mathbf{d}(x, x_k)$; moreover, if $x \in \partial D$ and $\mathbf{d}(x, x_k) \geq \delta_0$ for every $k = 1, \ldots, n$, then $|\nu(x)| \geq c_3$. We make δ_0 smaller, if necessary, so that we can assume that $\mathbf{d}(x_j, x_k) \geq 4\delta_0$ for all $j \neq k$. It is elementary to see that there exists $c_4 > 0$ with the following properties (a)-(c).

(a) For every $x \in \partial D$ such that $\mathbf{d}(x, x_k) \geq 2\delta_0$ for $k = 1, \ldots, n$, and every $y \in \partial D$ with $\mathbf{d}(x, y) \leq \delta_0$, we have $|\alpha(x, y)| \geq c_4\mathbf{d}(x, y)$.

(b) If $x \in \partial D$ and $\mathbf{d}(x, x_k) < 2\delta_0$ for some k, $y \in \partial D$, $\mathbf{d}(x, y) \leq \delta_0$, and y lies on the same side of x_k as x then $|\alpha(x, y)| \geq c_4\mathbf{d}(x, y)\mathbf{d}(x, x_k)$.

(c) Suppose that $x = x_k$ for some k. If $y \in \partial D$ and $\mathbf{d}(x, y) \leq \delta_0$ then $|\alpha(x, y)| \geq c_4\mathbf{d}(x, y)^2$.

Make δ_0 smaller, if necessary, so that for any $x, y \in \partial D$ with $|\alpha(x, y)| \geq \pi/4$, we have $\mathbf{d}(x, y) \geq 4\delta_0$.

It is standard to prove, using the same methods as in the proof of Lemma 3.2, that for some $c_5, c_6 < \infty$ and all $x \in \partial D$, we have $\omega_x(dy) \leq c_5\mathbf{d}(x, y)^{-2}dy$ if $\mathbf{d}(x, y) \leq \delta_0$, and $\omega_x(dy) \leq c_6 dy$ if $\mathbf{d}(x, y) \geq \delta_0$.

Consider any $z \in \partial D$ and let $z_1, z_2, \ldots, z_m$ be all points in ∂D such that $\alpha(z, z_k) = \pi/2$. The number m of such points is bounded by a constant m_0 depending on D but not on z. Let Γ_1 be the set of points on the same connected component of ∂D as z, within the distance δ_0 from z. The family of points $\{z_1, \dots, z_m, x_1, \dots, x_n\}$ divides $\partial D \setminus \Gamma_1$ into Jordan arcs Γ_k, $2 \le k \le n_0$, with $n_0 \le n+m$. For an arc Γ_k, let y_k^- and y_k^+ denote its endpoints, and let $\Gamma_k^- = \{x \in \Gamma_k : \mathbf{d}(y_k^-, x) \le \delta_0\}$, $\Gamma_k^+ = \{x \in \Gamma_k : \mathbf{d}(y_k^+, x) \le \delta_0\}$, and $\Gamma_k^0 = \Gamma_k \setminus (\Gamma_k^- \cup \Gamma_k^+)$.

Since

$$H^{(z,y_1,x_2,y_2)} |\log \cos \alpha(z, X_{\zeta-})| = H^z |\log \cos \alpha(z, X_{\zeta-})| = \int_{\partial D} |\log \cos \alpha(z, y)| \omega_z(dy),$$

we will estimate the integral on the right hand side. We will split the integral into the sum of integrals over Γ_k's. Let λ denote the arc length measure on ∂D, i.e., $\lambda(dx)$ is an alternative and equivalent notation for dx.

Since $\nu(x)$ is bounded over ∂D, for $x \in \Gamma_1$ we have $\alpha(x, z) \le c_7 \mathbf{d}(x, z)$ and so $|\log \cos \alpha(x, z)| \le c_8 \mathbf{d}(x, z)^2$. Recall that $\omega_x(dy) \le c_5 \mathbf{d}(x, y)^{-2} dy$ if $\mathbf{d}(x, y) \le \delta_0$. This implies that

$$\int_{\Gamma_1} |\log \cos \alpha(z, y)| \omega_z(dy) \le 2 \int_0^{\delta_0} c_8 u^2 c_5 u^{-2} du \le c_9 \delta_0. \tag{4.11}$$

Consider an arc Γ_k with $k \ge 2$. First assume that the distance between y_k^- and y_k^+ is greater than δ_0. Since the curvature $\nu(x)$ has the constant sign on Γ_k, the function $x \to \alpha(z, x)$ takes its maximum at one or both endpoints of Γ_k. Recall that, by convention, $\alpha(z, x) \le \pi/2$. This and (a)-(c) show that

$$\lambda(\{x \in \Gamma_k^- : \pi/2 - \alpha(x, y_k^-) \in [2^{-j}, 2^{-j-1}]\}) \le c_{10} 2^{-j/2}.$$

We have $\omega_z(dx) \le c_6 dx$ for $x \in \Gamma_k$ so

$$\int_{\Gamma_k^-} |\log \cos \alpha(z, y)| \omega_z(dy) \le c_{11} \sum_{j \ge 1} j 2^{-j/2} = c_{12}, \tag{4.12}$$

and similarly

$$\int_{\Gamma_k^+} |\log \cos \alpha(z, y)| \omega_z(dy) \le c_{12}.$$

By (a), for $x \in \Gamma_k^0$, $\pi/2 - \alpha(x, z) \ge c_{13} > 0$, where c_{13} does not depend on z or k. Hence,

$$\int_{\Gamma_k^0} |\log \cos \alpha(z, y)| \omega_z(dy) \le c_{14} |\partial D| = c_{15}.$$

Next suppose that the distance between y_k^- and y_k^+ is less than δ_0. Then one of the endpoints of Γ_k, say y_k^-, is a point x_{j_1}, and y_k^+ is a point z_{j_2}. By Lemma 3.7,

$$\lambda(\{x \in \Gamma_k : \pi/2 - \alpha(x, y_k^-) \in [2^{-j}, 2^{-j-1}]\}) \le c_{16} 2^{-j/2},$$

for $2^{-j} \le c_{17} \mathbf{d}(y_k^-, y_k^+)$. Hence,

$$\int_{\Gamma_k} |\log \cos \alpha(z, y)| \omega_z(dy) \le c_{18} \sum_{2^{-j} \le c_{17} \mathbf{d}(y_k^-, y_k^+)} j 2^{-j/2} \le c_{19}. \tag{4.13}$$

Since the number of arcs n_0 is bounded by a constant independent of z, we obtain

$$H^{(z,y_1,x_2,y_2)}|\log\cos\alpha(z,X_{\zeta-})| \le \sum_k \int_{\Gamma_k} |\log\cos\alpha(z,y)|\omega_z(dy) \le c_{20}.$$

(ii) It is not hard to prove, using (4.2), that $H^{(x_1,y_1,x_2,y_2)}(U_1^e \le \zeta) \ge c_2\varepsilon^{-1}$, so part (i) implies that

$$H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1,X_{\zeta-})| \mid U_1^e \le \zeta\right) \le c_3\varepsilon. \tag{4.14}$$

The proof of part (ii) can be finished by applying the same ideas as in part (i). The only modification that is needed is to restrict the range of j in (4.12) to $2^{-j} \le \varepsilon^{\beta_1}$, and similarly for (4.13).

(iii) We have

$$\begin{aligned}
&H^{(x_1,y_1,x_2,y_2)}\Big(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{\{T^e\le S_2^e\}}\mathbf{1}_{F^c(T^e,S_2^e,Z_{T^e},\varepsilon^{\beta_1})} \mid U_1^e \le \zeta\Big) \qquad (4.15)\\
&\le H^{(x_1,y_1,x_2,y_2)}\Big(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{F^c(T^e,S_2^e,X_{T^e},\varepsilon^{\beta_1})}\mathbf{1}_{\{T^e\le S_2^e,X_{T^e}=Z_{T^e}\}} \mid U_1^e \le \zeta\Big)\\
&\quad + H^{(x_1,y_1,x_2,y_2)}\Big(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{F^c(T_X(\partial D),S_2^e,Z_{T^e},\varepsilon^{\beta_1}/2)}\\
&\qquad \times \mathbf{1}_{\{Y_{T^e}=Z_{T^e},T_X(\partial D)\le S_2^e\}} \mid U_1^e \le \zeta\Big)\\
&\quad + H^{(x_1,y_1,x_2,y_2)}\Big(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{F^c(T^e,T_X(\partial D),Y_{T^e},\varepsilon^{\beta_1}/2)}\\
&\qquad \times \mathbf{1}_{\{Y_{T^e}=Z_{T^e},T^e\le T_X(\partial D)\}} \mid U_1^e \le \zeta\Big).
\end{aligned}$$

The first term on the right hand side is bounded by $c_1\varepsilon^{\beta_3}$ for some $\beta_3 > 1$ by (4.14), the strong Markov property applied at $T_e(\partial D)$, and (3.34). A similar bound holds for the second term, by the strong Markov property applied at $T_X(\partial D)$. To estimate the last term, we fix a $\beta_4 \in (1, 1+\beta_1)$ and note that,

$$\begin{aligned}
&H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{F^c(T^e,T_X(\partial D),Y_{T^e},\varepsilon^{\beta_1}/2)}\mathbf{1}_{\{Y_{T^e}=Z_{T^e},T^e\le T_X(\partial D)\}} \mid U_1^e \le \zeta\right)\\
&\le H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{\{\mathbf{d}(T_X(\partial D),x_1)\le 2\varepsilon^{\beta_1}\}} \mid U_1^e \le \zeta\right)\\
&\quad + H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{\{\mathbf{d}(X_{U_1^e},\partial D)\le\varepsilon^{\beta_4}\}} \mid U_1^e \le \zeta\right)\\
&\quad + H^{(x_1,y_1,x_2,y_2)}\Big(|\log\cos\alpha(x_1,X_{\zeta-})|\mathbf{1}_{\{\mathbf{d}(T_X(\partial D),x_1)>2\varepsilon^{\beta_1},\mathbf{d}(X_{U_1^e},\partial D)>\varepsilon^{\beta_4}\}}\\
&\qquad \times \mathbf{1}_{F^c(T^e,T_X(\partial D),Y_{T^e},\varepsilon^{\beta_1}/2)}\mathbf{1}_{\{Y_{T^e}=Z_{T^e},T^e\le T_X(\partial D)\}} \mid U_1^e \le \zeta\Big). \qquad (4.16)
\end{aligned}$$

To bound the first term on the right hand side we use the same idea that underlies (4.11). The conditioning on $\{U_1^e \le \zeta\}$ transforms the excursion law H into a probability distribution. The event in question concerns Brownian motion starting at $X(U_1^e)$ and killed upon hitting the boundary. The starting point, $X(U_1^e)$, is at most $c_2\varepsilon$ units away from x_1. For points $y \in \partial D$ with $\mathbf{d}(x_1,y) \in (2^{-k}, 2^{-k+1}]$, we have $|\log\cos\alpha(x_1,y)| \le c_3 2^{-2k}$, and the probability of hitting the set of such points is bounded by $c_4\varepsilon 2^k$, by Lemma 3.2. Let k_1 be the smallest integer such that $2^{-k_1} \le \varepsilon$, and let k_2 be the smallest integer such that $2^{-k_2} \le 2\varepsilon^{\beta_1}$. Then the first term on the right hand side of (4.16) is bounded by

$$\sum_{k=k_1}^{k_2} c_3 2^{-2k} c_4 \varepsilon 2^k \le c_5\varepsilon^2.$$

We turn to the second term on the right hand side of (4.16). It is rather easy to show, using (4.14) and the same ideas as in part (i) of the proof, that for any $r > 0$,

$$H^{(x_1,y_1,x_2,y_2)}(|\log\cos\alpha(x_1, X_{\zeta-})| \mid U_1^e \le \zeta, \mathbf{d}(X_{U_1^e}, \partial D) = r) \le c_6\varepsilon.$$

It is straightforward to see that

$$H^{(x_1,y_1,x_2,y_2)}(\mathbf{d}(X_{U_1^e}, \partial D) \le r \mid U_1^e \le \zeta) \le c_7 r\varepsilon^{-1}.$$

One can use these estimates to find a bound of the form $c_8\varepsilon^{\beta_5}$ with $\beta_5 > 1$, for the second term on the right hand side of (4.16).

To bound the third term on the right hand side of (4.16), we condition the process $\{X_t, t \in (U_1^e, \zeta)\}$ under $H^{(x_1,y_1,x_2,y_2)}$ on its endpoints $X_{U_1^e}$ and $X_{T_X(\partial D)}$. The result is an h-process R, in the sense of Doob, starting at $X_{U_1^e}$ and killed at $X_{T_X(\partial D)}$. Consider random sets

$$K_1 = \{x \in D : \mathbf{d}(x, \partial D) \le c_9\varepsilon^{1+\beta_1}, \mathbf{d}(x, x_1) \le \varepsilon^{\beta_1}/2, \mathbf{d}(x, X_{T_X(\partial D)}) \ge \varepsilon^{\beta_1}/2\},$$
$$K_2 = \{x \in D : \mathbf{d}(x, \partial D) \le \varepsilon, \mathbf{d}(x, x_1) \ge \varepsilon^{\beta_1}/2, \mathbf{d}(x, X_{T_X(\partial D)}) \ge \varepsilon^{\beta_1}/2\},$$

for some c_9. Suppose that the events in the indicator functions in the last term on the right hand side of (4.16) hold. If $\mathbf{d}(x_1, Z_{T^e}) \le \varepsilon^{\beta_1}/2$ then the process R must hit K_1, otherwise it must hit K_2. The following two estimates follow from standard properties of harmonic measure. If $\mathbf{d}(X_{U_1^e}, \partial D) > \varepsilon^{\beta_4}$ then the probability that the h-process R hits K_1 is bounded by $c_{10}\varepsilon^{1+\beta_1-\beta_4}$, and the probability that it hits K_2 is bounded by $c_{11}\varepsilon^{1-\beta_1}$. The conditioning on $\{U_1^e \le \zeta\}$ contributes a factor of $c_{12}\varepsilon$ (see (4.14)), so we obtain a bound $c_{13}(\varepsilon^{2+\beta_1-\beta_4} + \varepsilon^{2-\beta_1}) \le c_{14}\varepsilon^{\beta_5}$ for the third term on the right hand side of (4.16), for some $\beta_5 > 1$. All terms on the right hand side of (4.16) have bounds of this form so this finishes the proof of part (ii) of the lemma.

(iv) We have

$$\begin{aligned} &H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1, X_{\zeta-})|\mathbf{1}_{\{S_2^e \le T^e\}}\mathbf{1}_{F^c(S_2^e, T^e, Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \le \zeta\right) \\ &\le H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1, X_{\zeta-})|\mathbf{1}_{\{S_2^e \le T_X(\partial D)\}}\mathbf{1}_{F^c(S_2^e, T_X(\partial D), X_{T_X(\partial D)}, \varepsilon^{\beta_1})} \mid U_1^e \le \zeta\right). \end{aligned}$$

This can be estimated just like the last term on the right hand side of (4.15), i.e., (4.16). The crucial point is that the distance from X to ∂D must be less than $c_1\varepsilon^2$ at time S_2^e. The estimates of the third term on the right hand side of (4.16) are based on the fact that the distance from X to ∂D at time T^e is bounded by $c_2\varepsilon^{1+\beta_1}$ (see the definition of K_1).

(v) This estimate has been already proved in part (iii) because the relevant expression appears as the last term on the right hand side in (4.15). □

Lemma 4.5. *(i) There exist $\beta_0, \beta_1 > 0$, $\beta_2 > 1$ and $c_0, \varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1\mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \ge c_0\mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \le \varepsilon \le \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \le c_0^{-1}\mathbf{d}(x_1, y_1)$, then*

$$\begin{aligned} H^{(x_1,y_1,x_2,y_2)}\Big(\left|\log\rho_{U_1^e} - \log\rho_{S_2^e} - |\log\cos\alpha(x_1, X_{\zeta-})|\right|\mathbf{1}_{\{S_2^e \le \tau^+(\varepsilon^{\beta_0})\}} \\ \times \mathbf{1}_{F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})}\mathbf{1}_{F(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \le \zeta\Big) \le \varepsilon^{\beta_2}. \end{aligned}$$

(ii) There exist $\beta_0 > 0$, $\beta_1 > 1$ and $c_0, \varepsilon_0 > 0$ such that if $x_1 \in \partial D$, $y_1, x_2, y_2 \in \overline{D}$, $\mathbf{d}(x_1, x_2) \vee \mathbf{d}(y_1, y_2) < a_1 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \geq c_0 \mathbf{d}(x_2, y_2)$, $\mathbf{d}(x_1, y_1) \leq \varepsilon \leq \varepsilon_0$, and $|\pi/2 - \angle(y_1 - x_1, \mathbf{n}(x_1))| \leq c_0^{-1} \mathbf{d}(x_1, y_1)$, then

$$H^{(x_1,y_1,x_2,y_2)} \left(\left| \log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log \cos \alpha(x_1, X_{\zeta-})| \right| \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \mid U_1^e \leq \zeta \right) \leq \varepsilon^{\beta_1}.$$

Proof. (i) Let $z_1, z_2, \dots, z_{m_0} \in \partial D$ be all points with $\alpha(x_1, z_k) = \pi/2$. For integer k, let $M_k \subset \partial D$ be the set of all points y such that $\tan \alpha(y, x_1) \in [2^{-k}, 2^{-k+1}]$. Fix some β_3 such that $0 < \beta_3 < \beta_1 < 1$. Let k_0 be the smallest integer with $2^{-k_0} \leq \varepsilon^{-\beta_3}$ and $K = \bigcup_{k \leq k_0} M_k$. By Lemmas 4.3 (vii) and 4.4 (ii), we have for some $\beta_0 > 0$, $\beta_2 > 1$,

$$\begin{aligned}
&H^{(x_1,y_1,x_2,y_2)} \Big(\left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log \cos \alpha(x_1, X_{\zeta-})| \right| \\
&\qquad \times \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \mathbf{1}_{\{X(T_X(\partial D)) \in K\}} \mid U_1^e \leq \zeta \Big) \\
&\leq H^{(x_1,y_1,x_2,y_2)} \Big(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \mathbf{1}_{\{X(T_X(\partial D)) \in K\}} \mid U_1^e \leq \zeta \Big) \\
&\quad + H^{(x_1,y_1,x_2,y_2)} \Big(|\log \cos \alpha(x_1, X_{\zeta-})| \mathbf{1}_{\{X(T_X(\partial D)) \in K\}} \mid U_1^e \leq \zeta \Big) \\
&\leq \varepsilon^{\beta_2}.
\end{aligned}$$

It follows that it will suffice to show that for some $\beta_4 > 1$,

$$\begin{aligned}
&H^{(x_1,y_1,x_2,y_2)} \Big(\left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log \cos \alpha(x_1, X_{\zeta-})| \right| \\
&\quad \times \mathbf{1}_{\{X(T_X(\partial D)) \notin K\}} \mathbf{1}_{F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mathbf{1}_{F(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta \Big) \leq \varepsilon^{\beta_4}.
\end{aligned}$$

Let

$$\begin{aligned}
T_1 = S_2^e &\wedge \inf\{t \geq U_1^e : Y_t \in \partial D \text{ and } |\angle(\mathbf{n}(Y_t), X_t - Y_t) - \pi/2| \leq c_1 \mathbf{d}(x_1, y_2)^{\beta_1}\} \\
&\wedge \inf\{t \geq U_1^e : X_t \in \partial D \text{ and } |\angle(\mathbf{n}(X_t), X_t - Y_t) - \pi/2| \leq c_1 \mathbf{d}(x_1, y_2)^{\beta_1}\}.
\end{aligned}$$

for some c_1. We will assume that events $F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})$ and $F(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})$ hold and we will estimate $\left|\log \rho_{U_1^e} - \log \rho_{T_1} - |\log \cos \alpha(x_1, X_{\zeta-})|\right|$ using this assumption. Let k_1 be the smallest integer with $\mathbf{d}(x_1, M_{k_1}) \leq \varepsilon^{\beta_1}$. If $X(T_X(\partial D)) \in \bigcup_{k \geq k_1} M_k$ then $T_1 = T_X(\partial D) \wedge T_Y(\partial D)$. Hence, $\mathbf{d}(X_{U_1^e}, Y_{U_1^e}) = \mathbf{d}(X_{T_1}, Y_{T_1})$ and $|\log \cos \alpha(x_1, X_{\zeta-})| \leq c_2 \varepsilon^{2\beta_1}$. Thus, in this case,

$$\left|\log \rho_{U_1^e} - \log \rho_{T_1} - |\log \cos \alpha(x_1, X_{\zeta-})|\right| \leq c_2 \varepsilon^{2\beta_1}.$$

Now we make an extra assumption that $\beta_1 > 1/2$ and we conclude that

$$\begin{aligned}
&H^{(x_1,y_1,x_2,y_2)} \Big(\left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log \cos \alpha(x_1, X_{\zeta-})| \right| \mathbf{1}_{\{X(T_X(\partial D)) \in \bigcup_{k \geq k_1} M_k\}} \\
&\quad \times \mathbf{1}_{F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mathbf{1}_{F(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta \Big) \leq c_2 \varepsilon^{2\beta_1} = c_2 \varepsilon^{\beta_5},
\end{aligned} \tag{4.17}$$

for some $\beta_5 > 1$.

Suppose that $X(T_X(\partial D)) \in M_k$ with $k_1 \le k \le k_0$. Note that, by assumption, $|\alpha(x_1, Z_t) - \alpha(x_1, X_{\zeta-})| \le c_3 \varepsilon^{\beta_1}$ for all $t \in [U_1^e, T_1]$ such that one of the processes X or Y is on the boundary at time t. We have $(d/d\gamma)|\log\cos\gamma| = \tan\gamma$ for $\gamma \in (0, \pi/2)$. This and easy geometry imply that the change in ρ_t on the interval $[U_1^e, T_1]$ is equal to $\log\cos\alpha(x_1, X_{\zeta-})$ up to an additive constant bounded by $c_4 2^{-k}\varepsilon^{\beta_1}$, i.e.,

$$\left|\log\rho_{U_1^e} - \log\rho_{T_1} - |\log\cos\alpha(x_1, X_{\zeta-})|\right| \le c_4 2^{-k}\varepsilon^{\beta_1}.$$

By the strong Markov property applied at U_1^e and Lemma 3.2,

$$H^{(x_1,y_1,x_2,y_2)}(X(T_X(\partial D)) \in M_k \mid U_1^e \le \zeta) \le c_5 \varepsilon 2^k,$$

so for $k_1 \le k \le k_0$,

$$\begin{aligned} H^{(x_1,y_1,x_2,y_2)}\Big(&\left|\log\rho_{U_1^e} - \log\rho_{S_2^e} - |\log\cos\alpha(x_1, X_{\zeta-})|\right| \mathbf{1}_{\{X(T_X(\partial D))\in M_k\}} \\ &\times \mathbf{1}_{F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mathbf{1}_{F(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \le \zeta\Big) \le c_6 \varepsilon^{1+\beta_1}, \end{aligned}$$

and

$$\begin{aligned} H^{(x_1,y_1,x_2,y_2)}\Big(&\left|\log\rho_{U_1^e} - \log\rho_{S_2^e} - |\log\cos\alpha(x_1, X_{\zeta-})|\right| \mathbf{1}_{\{X(T_X(\partial D))\in \bigcup_{k_1\le k\le k_0} M_k\}} \\ &\times \mathbf{1}_{F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mathbf{1}_{F(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \le \zeta\Big) \le c_7 \varepsilon^{1+\beta_1}|\log\varepsilon| \le c_8 \varepsilon^{\beta_6}, \end{aligned}$$

for some $\beta_6 > 1$. This and (4.17) imply that for some $\beta_7 > 1$,

$$\begin{aligned} H^{(x_1,y_1,x_2,y_2)}\Big(&\left|\log\rho_{U_1^e} - \log\rho_{T_1} - |\log\cos\alpha(x_1, X_{\zeta-})|\right| \\ &\times \mathbf{1}_{\{X(T_X(\partial D))\notin K\}} \mathbf{1}_{F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mathbf{1}_{F(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \le \zeta\Big) \le c_9 \varepsilon^{\beta_7}. \end{aligned} \tag{4.18}$$

The following definitions assume that $X_{T_1^k} \in \partial D$. If $Y_{T_1^k} \in \partial D$ then the roles of X and Y should be interchanged in the definitions of $T_2^k, T_3^k, T_4^k, T_5^k$ and T_1^{k+1}. Let $T_1^1 = T_1$ and for $k \ge 1$,

$$\begin{aligned} T_2^k &= \inf\{t \ge T_1^k : \mathbf{d}(Y_t, Y_{T_1^k}) \ge 2\mathbf{d}(Y_{T_1^k}, \partial D)\}, \\ T_3^k &= \inf\{t \ge T_1^k : Y_t \in \partial D\}, \\ T_4^k &= \inf\{t \ge T_1^k : L_t^X - L_{T_1^k}^X \ge c_{10}\mathbf{d}(Y_{T_1^k}, \partial D)\}, \\ T_5^k &= T_2^k \wedge T_3^k \wedge T_4^k, \\ T_1^{k+1} &= \inf\{t \ge T_5^k : X_t \in \partial D \text{ or } Y_t \in \partial D\}. \end{aligned}$$

If $T_1^{k+1} \le \tau^+(\varepsilon^{\beta_0})$ and $\mathbf{d}(X_{T_1^k}, X_{T^e}) \le \varepsilon^{\beta_1}$ then $|\pi/2 - \angle(\mathbf{n}(Z_t), X_t - Y_t)| \le c_{11}\varepsilon^{\beta_1}$ for $t \in [T_1^k, T_1^{k+1}]$ and $L_{T_1^{k+1}}^X - L_{T_1^k}^X \le c_{12}\mathbf{d}(X_{T_1^k}, Y_{T_1^k})\varepsilon^{\beta_1}$. The change of $\mathbf{d}(X_t, Y_t)$ on the interval $[T_1^k, T_1^{k+1}]$ is bounded by the product of these numbers, that is $c_{13}\mathbf{d}(X_{T_1^k}, Y_{T_1^k})\varepsilon^{2\beta_1}$. This implies that the increment of $|\log\rho_t|$ on the interval $[T_1^k, T_1^{k+1}]$ is bounded by $c_{14}\varepsilon^{2\beta_1}$. By Lemma 3.3 (i), the probability that $S_2^e \geq T_5^k$ is bounded by p_1^k, for some $p_1 < 1$. We obtain, by the strong Markov property applies at T_1,

$$H^{(x_1,y_1,x_2,y_2)}\Big(\big|\log \rho_{T_1} - \log \rho_{S_2^e}\big| \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \mathbf{1}_{F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\Big) \\ \leq \sum_{k\geq 1} p_1^k c_{14} \varepsilon^{2\beta_1} \leq c_{15}\varepsilon^{2\beta_1} = c_{15}\varepsilon^{\beta_8}.$$

The exponent β_8 is greater than 1 provided $\beta_1 > 1/2$. Part (i) of the lemma follows from the last estimate and (4.18).

(ii) We have

$$\begin{aligned}
&H\left(\left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log\cos\alpha(x_1, X_{\zeta-})|\right| \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \mid U_1^e \leq \zeta\right) \\
&\quad \leq H^{(x_1,y_1,x_2,y_2)}\Big(\left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log\cos\alpha(x_1, X_{\zeta-})|\right| \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \\
&\qquad \times \mathbf{1}_{F(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mathbf{1}_{F(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\Big) \\
&\qquad + H^{(x_1,y_1,x_2,y_2)}\left(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{T^e \leq S_2^e\}} \mathbf{1}_{F^c(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right) \\
&\qquad + H^{(x_1,y_1,x_2,y_2)}\left(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{S_2^e \leq T^e\}} \mathbf{1}_{F^c(S_2^e, T^e, Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right) \\
&\qquad + H^{(x_1,y_1,x_2,y_2)}\left(|\log \rho_{U_1^e} - \log \rho_{S_2^e}| \mathbf{1}_{\{S_2^e \leq \tau^+(\varepsilon^{\beta_0})\}} \mathbf{1}_{F^c(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right) \\
&\qquad + H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1, X_{\zeta-})| \mathbf{1}_{\{T^e \leq S_2^e\}} \mathbf{1}_{F^c(T^e, S_2^e, Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right) \\
&\qquad + H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1, X_{\zeta-})| \mathbf{1}_{\{S_2^e \leq T^e\}} \mathbf{1}_{F^c(S_2^e, T^e, Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right) \\
&\qquad + H^{(x_1,y_1,x_2,y_2)}\left(|\log\cos\alpha(x_1, X_{\zeta-})| \mathbf{1}_{F^c(T^e, T_X(\partial D), Z_{T^e}, \varepsilon^{\beta_1})} \mid U_1^e \leq \zeta\right).
\end{aligned}$$

Part (ii) of the lemma follows from the above formula, part (i) of this lemma, and estimates in Lemma 4.3 (iii)-(v) and Lemma 4.4 (iii)-(v). □

Lemma 4.6. *For any $\beta_1 \in (1,2)$ there exist $\beta_2 > 0$, and $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $\mathbf{d}(X_0, Y_0) \leq \varepsilon$ then*

$$\mathbf{P}(\mathbf{d}(Y_{\sigma_1^X}, \partial D) \geq \varepsilon^{\beta_1}) \leq \varepsilon^{\beta_2}.$$

Proof. By Lemma 4.1 (ii), $\mathbf{P}(L^Y_{\sigma_1^X} \geq a) \leq c_1 e^{-c_2 a}$. Hence, for any $\beta_3 > 0$ and some $\beta_4 > 0$,

$$\mathbf{P}(L^Y_{\sigma_1^X} \geq \beta_3 |\log\varepsilon|) \leq c_1 \exp(-c_2\beta_3 |\log \varepsilon|) = \varepsilon^{\beta_4}.$$

If the event $A_1 \stackrel{\text{df}}{=} \{L^Y_{\sigma_1^X} \leq \beta_3 |\log\varepsilon|\}$ holds then, by Lemma 3.8,

$$\sup_{t\in[0,\sigma_1^X]} \mathbf{d}(X_t, Y_t) \leq \mathbf{d}(X_0, Y_0)\exp(c_4(1+\beta_3|\log\varepsilon|)) \leq c_5 \varepsilon^{1-c_4\beta_3} = c_5\varepsilon^{1-\beta_5}.$$

Choose $\beta_3 > 0$ so small that we can find β_6 and β_7 such that $\beta_5 < \beta_6 < \beta_7 < 1 - \beta_5$ and $\beta_1 = 1 - \beta_5 + \beta_6$.

Let $T_1 = \inf\{t \geq 0 : X_t \in \partial D\}$ and $\{V_t, 0 \leq t \leq \sigma_1^X - T_1\} \stackrel{\text{df}}{=} \{X_{\sigma_1^X - t}, 0 \leq t \leq \sigma_1^X - T_1\}$. If we condition on the values of X_{T_1} and $X_{\sigma_1^X}$, the process V is a reflected Brownian motion in D starting from $X_{\sigma_1^X}$ and conditioned to approach X_{T_1} at its lifetime. It is easy to see that $\mathbf{P}(\mathbf{d}(X_{T_1}, X_{\sigma_1^X}) \leq \varepsilon^{\beta_6}) \leq c_6 \varepsilon^{\beta_6}$.

Suppose that the event $A_2 = \{\mathbf{d}(X_{T_1}, X_{\sigma_1^X}) \geq \varepsilon^{\beta_6}\}$ holds. Conditional on this event, the probability that V does not spend ε^{β_7} units of local time on the boundary of ∂D before leaving the disc $\mathcal{B}(V_0, \varepsilon^{\beta_6})$ is bounded by $\varepsilon^{\beta_7 - \beta_6}$. Let A_3 be the event that V spends ε^{β_7} or more units of local time on the boundary of ∂D before leaving the disc $\mathcal{B}(V_0, \varepsilon^{\beta_6})$. If A_3 holds then it is easy to see that $\mathbf{d}(Y_{\sigma_1^X}, \partial D) \leq \varepsilon^{1-\beta_5+\beta_6} = \varepsilon^{\beta_1}$.We have shown that $\mathbf{d}(Y_{\sigma_1^X}, \partial D) \leq \varepsilon^{\beta_1}$ if $A_1 \cap A_2 \cap A_3$ holds. Since $\mathbf{P}((A_1 \cap A_2 \cap A_3)^c) \leq \varepsilon^{\beta_4} + c_6 \varepsilon^{\beta_6} + \varepsilon^{\beta_7 - \beta_6}$, the lemma follows. □

Recall that e_s denotes an excursion of X from the boundary of D starting at time s and let $\alpha(e_s) = \alpha(e_s(0), e_s(\zeta-))$.

Lemma 4.7. *For any $\delta, p > 0$ there exist $t_0, c_0 < \infty$ such that for every $x \in \overline{D}$, we have*

(i)

$$\mathbf{P}^x \left(\sum_{e_s \in \mathcal{E}_{t_0}} |\log \cos \alpha(e_s)| \geq c_0 \right) < p,$$

(ii)

$$\mathbf{P}^x \left(\sup_{u \geq t_0} \left| \frac{1}{u} \sum_{e_s \in \mathcal{E}_u} |\log \cos \alpha(e_s)| - \frac{1}{2|D|} \int_{\partial D} \int_{\partial D} |\log \cos \alpha(z,y)| \omega_z(dy) dz \right| \geq \delta \right) < p.$$

Proof. (i) It suffices to show that $\sum_{e_s \in \mathcal{E}_{t_0}} |\log \cos \alpha(e_s)|$ has a finite expectation, bounded by a constant independent of x. This follows from the exit system formula (4.1), Lemma 4.1 (i) and Lemma 4.4 (i).

(ii) Suppose that X_0 has the uniform distribution in D. Then, by the exit system formula, and since the Revuz measure of L_t^X is $dx/(2|D|)$,

$$\begin{aligned} \mathbf{E} \sum_{e_s \in \mathcal{E}_1} |\log \cos \alpha(e_s)| &= \mathbf{E} \int_0^1 H^{X(s)}(|\log \cos \alpha(e_s)|) dL_s^X \\ &= \frac{1}{2|D|} \int_{\partial D} \int_{\partial D} |\log \cos \alpha(z,y)| \omega_z(dy) dz. \end{aligned}$$

By Lemma 4.4 (i) and its proof, the last integral is finite.

Let $V_k = \sum_{e_s \in \mathcal{E}_k \setminus \mathcal{E}_{k-1}} |\log \cos \alpha(e_s)|$. By the ergodic theorem,

$$\begin{aligned} \lim_{u \to \infty} \frac{1}{u} \sum_{e_s \in \mathcal{E}_u} |\log \cos \alpha(e_s)| &= \lim_{k \to \infty} (1/k) \sum_{1 \leq n \leq k} V_n \\ &= \frac{1}{2|D|} \int_{\partial D} \int_{\partial D} |\log \cos \alpha(z,y)| \omega_z(dy) dz. \end{aligned}$$

Recall from (4.5) that the transition density $p_t(x,y)$ of reflected Brownian motion converges to $1/|D|$ exponentially fast as $t \to \infty$, uniformly in $(x,y) \in \overline{D}^2$. This can be used to finish the proof of part (ii) of the present lemma, using the same argument as in the proof of Lemma 4.2 (i). $\square$

Recall that $\mathcal{T}^c = \bigcup_k (U_k, S_{k+1}]$.

Lemma 4.8. *There exist $c_1, \varepsilon_0 > 0$ such that if $\mathbf{d}(X_0, Y_0) \le \varepsilon \le \varepsilon_0$ then for any $t \ge 1$,*

$$\mathbf{E} \int_{\mathcal{T}^c \cap [0, \sigma_t^X \wedge \tau_\varepsilon^+]} dL_s^X \le c_1 t \varepsilon |\log \varepsilon|.$$

Proof. It is easy to deduce from Lemma 3.3 that

$$\mathbf{E}(L^X_{S_1 \wedge \tau_\varepsilon^+} - L^X_{U_0}) \le c_2 \varepsilon. \tag{4.19}$$

Next we will estimate $(L^X_{S_{k+1}} - L^X_{U_k}) \mathbf{1}_{\{U_k < \tau_\varepsilon^+\}}$. Fix some $k \ge 1$ and assume that $U_k < \tau_\varepsilon^+$. Note that $\mathbf{d}(X_{U_k}, \partial D) \vee \mathbf{d}(Y_{U_k}, \partial D) \le c_3 \mathbf{d}(X_{U_k}, Y_{U_k})$. Let T_1 be the first time after U_k when either X or Y is in ∂D. Let n_0 be the greatest integer such that 2^{-n_0} is greater than the diameter of D and let n_1 be the least integer greater than $|\log \mathbf{d}(X_{U_k}, Y_{U_k})| / \log 2$. By Lemma 3.2,

$$\mathbf{P}(\mathbf{d}(X_{U_k}, X_{T_1}) \ge 2^{-n}) \le c_3 \mathbf{d}(X_{U_k}, Y_{U_k}) 2^n,$$

for $n_0 \le n \le n_1$. This obviously implies that

$$\mathbf{P}(\mathbf{d}(X_{U_k}, X_{T_1}) \in [2^{-n}, 2^{-n+1}]) \le c_3 \mathbf{d}(X_{U_k}, Y_{U_k}) 2^n,$$

for $n_0 \le n \le n_1$. Simple geometry shows that if $\mathbf{d}(X_{U_k}, X_{T_1}) \in [2^{-n}, 2^{-n+1}]$ and $Z_{T_1} = X_{T_1}$ then $\mathbf{d}(Y_{T_1}, \partial D) \le c_4 \mathbf{d}(X_{U_k}, Y_{U_k}) 2^{-n}$, and if $Z_{T_1} = Y_{T_1}$ then $\mathbf{d}(X_{T_1}, \partial D) \le c_4 \mathbf{d}(X_{U_k}, Y_{U_k}) 2^{-n}$. Hence,

$$\begin{aligned} &\mathbf{P}(\mathbf{d}(X_{T_1}, \partial D) \vee \mathbf{d}(Y_{T_1}, \partial D) \in [c_4 \mathbf{d}(X_{U_k}, Y_{U_k}) 2^{-n-1}, c_4 \mathbf{d}(X_{U_k}, Y_{U_k}) 2^{-n}]) \\ &\quad \le c_3 \mathbf{d}(X_{U_k}, Y_{U_k}) 2^n, \end{aligned}$$

for $n_0 \le n \le n_1$, and

$$\mathbf{E}(\mathbf{d}(X_{T_1}, \partial D) \vee \mathbf{d}(Y_{T_1}, \partial D)) \le c_5 \mathbf{d}(X_{U_k}, Y_{U_k})^2 |\log \mathbf{d}(X_{U_k}, Y_{U_k})|.$$

By Lemma 3.3 (ii), assuming that ε_0 is small,

$$\mathbf{E}\left(L^X_{S_{k+1}} - L^X_{U_k} \mid U_k < \tau_\varepsilon^+\right) \le c_6 \mathbf{d}(X_{U_k}, Y_{U_k})^2 |\log \mathbf{d}(X_{U_k}, Y_{U_k})|.$$

It is elementary to check that

$$\mathbf{E}\left(L^X_{U_k} - L^X_{S_k} \mid S_k < \tau_\varepsilon^+\right) \ge c_7 \mathbf{d}(X_{S_k}, Y_{S_k}),$$

and the conditional distribution of $L^X_{U_k} - L^X_{S_k}$ given $\{S_k < \tau^+_\varepsilon\}$ is stochastically bounded by an exponential random variable with mean $c_8 \mathbf{d}(X_{S_k}, Y_{S_k})$. Note that $\mathbf{d}(X_{U_k}, Y_{U_k}) \le c_9 \mathbf{d}(X_{S_k}, Y_{S_k})$. Hence,

$$N_m \stackrel{\text{df}}{=} \sum_{k=1}^{m} c_{10}\varepsilon|\log\varepsilon|(L^X_{U_k} - L^X_{S_k})\mathbf{1}_{\{S_k<\tau^+_\varepsilon\}} - (L^X_{S_{k+1}} - L^X_{U_k})\mathbf{1}_{\{U_k<\tau^+_\varepsilon\}}$$

is a submartingale with respect to the filtration $\mathcal{F}^*_m = \mathcal{F}^{X,Y}_{S_{m+1}}$. If

$$M = \inf\{m : \sum_{k=1}^{m}(L^X_{U_k} - L^X_{S_k}) \ge t\}$$

and $M_j = M \wedge j$ then

$$\mathbf{E}\sum_{k=1}^{M_j}\Big(c_{10}\varepsilon|\log\varepsilon|(L^X_{U_k} - L^X_{S_k})\mathbf{1}_{\{S_k<\tau^+_\varepsilon\}} - (L^X_{S_{k+1}} - L^X_{U_k})\mathbf{1}_{\{U_k<\tau^+_\varepsilon\}}\Big) \ge 0,$$

and

$$\mathbf{E}\sum_{k=1}^{M_j}(L^X_{S_{k+1}} - L^X_{U_k})\mathbf{1}_{\{U_k<\tau^+_\varepsilon\}} \le \mathbf{E}\sum_{k=1}^{M_j} c_{10}\varepsilon|\log\varepsilon|(L^X_{U_k} - L^X_{S_k})\mathbf{1}_{\{S_k<\tau^+_\varepsilon\}}.$$

We let $j \to \infty$ and obtain by the monotone convergence

$$\mathbf{E}\sum_{k=1}^{M}(L^X_{S_{k+1}} - L^X_{U_k})\mathbf{1}_{\{U_k<\tau^+_\varepsilon\}} \le \mathbf{E}\sum_{k=1}^{M} c_{10}\varepsilon|\log\varepsilon|(L^X_{U_k} - L^X_{S_k})\mathbf{1}_{\{S_k<\tau^+_\varepsilon\}} \le c_{11}t\varepsilon|\log\varepsilon|.$$

Hence,

$$\mathbf{E}\int_{\mathcal{T}^c\cap[U_1,\sigma^X_t\wedge\tau^+(\varepsilon)]} dL^X_s \le \mathbf{E}\sum_{k=1}^{M}(L^X_{S_{k+1}} - L^X_{U_k})\mathbf{1}_{\{U_k<\tau^+_\varepsilon\}} \le c_{11}t\varepsilon|\log\varepsilon|.$$

This and (4.19) imply the lemma. □

Recall that e_s denotes an excursion of X from ∂D starting at time s, $\alpha(e_s) = \alpha(e_s(0), e_s(\zeta-))$, and $\mathcal{E}_t$ is the family of excursions e_s with $s \le t$. See the beginning of Section 3 for the definition of $\overline{\rho}_t$.

Lemma 4.9. *Let $\mathcal{E}^*(t)$ be the restriction of $\mathcal{E}_t$ to those excursions e_u that satisfy the condition $\sup_{s\in[u,u+\zeta(e_u))} \mathbf{d}(X_s, Y_s) \le \varepsilon^{\beta_0}$. For any $\beta_0 \in (0,1)$ there exist $\beta_1 \in (3/2, 2)$, $\varepsilon_0, \beta_2 > 0$ and $c_1 < \infty$ such that if $X_0 \in \partial D$, $\varepsilon < \varepsilon_0$, $\mathbf{d}(X_0, Y_0) \le \varepsilon$ and $\mathbf{d}(Y_0, \partial D) \le \varepsilon^{\beta_1}$ then,*

$$\mathbf{E}\left|\log\overline{\rho}_{\sigma^X_1} - \sum_{e_s\in\mathcal{E}^*(\sigma^X_1)} |\log\cos\alpha(e_s)|\right| \le c_1\varepsilon^{\beta_2}.$$

Proof. Recall the "rich" version of the exit system introduced before Lemma 4.3, and the accompanying notation, i.e., stopping times S^e_k and U^e_k. Let k_1 be the smallest (random) integer such that $S_{k_1} \geq \sigma^X_1$. We will show that the triangle inequality yields

$$\begin{aligned}
\left|\log \overline{\rho}_{\sigma_1^X} - \sum_{e_s\in\mathcal{E}^*(\sigma_1^X)} |\log\cos\alpha(e_s)|\right| &\leq \sum_{e_s\in\mathcal{E}^*(\sigma_1^X), s\in\mathcal{T}} \mathbf{1}_{\{U_1^e\geq\zeta(e_s)\}}|\log\cos\alpha(e_s)| \\
&+ \sum_{e_s\in\mathcal{E}^*(\sigma_1^X), s\in\mathcal{T}} \mathbf{1}_{\{U_1^e\leq\zeta(e_s)\}}\left|\log\rho_{U_1^e} - \log\rho_{S_2^e} - |\log\cos\alpha(e_s)|\right| \\
&+ \sum_{e_s\in\mathcal{E}^*(\sigma_1^X)}\sum_{k\geq 2} \mathbf{1}_{\{U_k^e\leq\zeta(e_s)\}}\left|\log\rho_{U_k^e} - \log\rho_{S_{k+1}^e}\right| \\
&+ \sum_{e_s\in\mathcal{E}^*(\sigma_1^X), s\notin\mathcal{T}} |\log\cos\alpha(e_s)| \\
&+ |\log\rho_{S_1} - \log\rho_0| + |\log\rho_{S_{k_1}} - \log\rho_{\sigma_1^X}|. \qquad (4.20)
\end{aligned}$$

We will argue that the right hand side properly accounts for all the terms on the left hand side of the last formula. All the terms of the sum $\sum_{e_s\in\mathcal{E}^*(\sigma_1^X)} |\log\cos\alpha(e_s)|$, appearing on the left hand side, are accounted for on the first, second and fourth lines on the right hand side. The quantity $\log\overline{\rho}_{\sigma_1^X}$ can be represented as the sum of $\log\rho_{U_k} - \log\rho_{S_{k+1}}$, for all k such that $0 \leq U_k \leq S_{k+1} \leq \sigma_1^X$, except that there are two extra terms corresponding to subintervals at the very beginning and at the end of $[0, \sigma_1^X]$. The two extra subintervals are accounted for on the last line of (4.20). The intervals $[U_k, S_{k+1}] \subset [0, \sigma_1^X]$ are matched with excursions e_s in the following way. Consider a U_k and find an excursion $e_s = \{e_s(t), t \in [s, s+\zeta(e_s))\}$ such that $U_k \in [s, s+\zeta(e_s)]$. Then U_k is one of the times U^e_k for this excursion. Note that if $U^e_1 > \zeta(e_s)$ for an excursion e_s then there are no k such that $U_k \in [s, s+\zeta(e_s)]$, so we restrict the sums on the second and third lines appropriately. We split the sums according to whether $k = 1$ or $k \geq 2$, and whether $s \in \mathcal{T}$ or not. The sums on the second and third lines do not contain terms corresponding to U^e_1 with $s \notin \mathcal{T}$. This is because if $s \notin \mathcal{T}$ then $[s, S^e_2]$ is a subinterval of $[U_k, S_{k+1}]$ with $S_{k+1} = S^e_2$. Then $U_k = U^{e^*}_j$ for some j and some excursion e^*_u with $u < s$ but note that we cannot have $j = 1$ and $u \notin \mathcal{T}$. Hence, there is already a term accounting for the interval $[U_k, S_{k+1}]$.

The following estimate is based on the same ideas as the proof of Lemma 4.4 (i). If $\mathbf{d}(x_1, y_1) \leq \varepsilon$ then

$$\begin{aligned}
H^{(x_1,y_1,x_2,y_2)}(\mathbf{1}_{\{U_1^e\geq\zeta(e_s)\}}|\log\cos\alpha(x_1, X_{\zeta-})|) &\leq \int_0^\varepsilon c_2 r^{-2}|\log\cos(c_3 r)|dr \\
&\leq \int_0^\varepsilon c_2 r^{-2} c_4 (c_3 r)^2 dr = c_5\varepsilon.
\end{aligned}$$

Hence, by the exit system formula (4.9),

$$\mathbf{E}\sum_{e_s\in\mathcal{E}^*(\sigma_1^X), s\in\mathcal{T}} \mathbf{1}_{\{U_1^e\geq\zeta(e_s)\}}|\log\cos\alpha(e_s)| \leq c_5\varepsilon. \qquad (4.21)$$

We have $H^{(x_1,y_1,x_2,y_2)}|\log\cos\alpha(x_1, X_{\zeta-})| \leq c_6$ for all (x_1, y_1, x_2, y_2) by Lemma 4.4 (i) so, by the exit system formula (4.9) and Lemma 4.8,

$$\mathbf{E}\sum_{e_s\in\mathcal{E}^*(\sigma_1^X), s\notin\mathcal{T}} |\log\cos\alpha(e_s)| \leq c_6\mathbf{E}\int_{\mathcal{T}^c\cap[0,\sigma_1^X\wedge\tau^+(\varepsilon^{\beta_0})]} dL_s^X \leq c_7\varepsilon^{\beta_0}|\log\varepsilon|. \qquad (4.22)$$

We have by Lemma 4.5 (ii), for some $\beta_3 > 1$,

$$H^{(x_1,y_1,x_2,y_2)}\left(\left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log\cos\alpha(x_1, X_{\zeta-})|\right| \mathbf{1}_{\{S_2^e \le \tau^+(\varepsilon^{\beta_0})\}} \mid U_1^e \le \zeta\right)$$
$$\le c_7 \mathbf{d}(x_1, y_1)^{\beta_3},$$

so, by the strong Markov property applied at U_1^e,

$$H^{(x_1,y_1,x_2,y_2)}\left(\left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log\cos\alpha(x_1, X_{\zeta-})|\right| \mathbf{1}_{\{S_2^e \le \tau^+(\varepsilon^{\beta_0})\}} \mathbf{1}_{\{U_1^e \le \zeta\}}\right)$$
$$\le c_7 \mathbf{d}(x_1, y_1)^{\beta_3} H^{(x_1,y_1,x_2,y_2)}\left(U_1^e \le \zeta\right).$$

This and the exit system formula (4.9) yield,

$$\begin{aligned}
&\mathbf{E} \sum_{e_s \in \mathcal{E}^*(\sigma_1^X), s\in\mathcal{T}} \mathbf{1}_{\{U_1^e \le \zeta(e_s)\}} \left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log\cos\alpha(e_s)|\right| \mathbf{1}_{\{S_2^e \le \tau^+(\varepsilon^{\beta_0})\}} \\
&\le \mathbf{E} \sum_{e_s \in \mathcal{E}^*(\sigma_1^X), s\in\mathcal{T}} c_7 \mathbf{d}(X_s, Y_s)^{\beta_3} \mathbf{1}_{\{U_1^e \le \zeta(e_s)\}}. \qquad (4.23)
\end{aligned}$$

We will now estimate the right hand side of (4.23). Let

$$M_j = \left(\varepsilon^{\beta_3 - 1}(L^X_{U_j} - L^X_{S_j}) - c_8 \mathbf{d}(X_{U_j}, Y_{U_j})^{\beta_3}\right) \mathbf{1}_{\{S_j \le \tau^+(\varepsilon^{\beta_0})\}}.$$

It is not hard to check that $\mathbf{E}(L^X_{U_j} - L^X_{S_j} \mid \mathcal{F}_{S_j}) \ge c_9 \mathbf{d}(X_{S_j}, Y_{S_j})$ and $\mathbf{d}(X_{U_j}, Y_{U_j}) \le c_{10}\mathbf{d}(X_{S_j}, Y_{S_j})$. Hence for an appropriate $c_8 > 0$, we have $\mathbf{E}M_j > 0$ and the process $N_k = \sum_{j\le k} M_j$ is a submartingale. Let $K = \inf\{k : \sum_{j\le k}(L^X_{U_j} - L^X_{S_j}) \ge 1\}$. It is easy to check that $L^X_{U_j} - L^X_{S_j}$ is stochastically majorized by an exponential random variable with mean $c_{11}\mathbf{d}(X_{S_j}, Y_{S_j}) \le c_{11}\varepsilon^{\beta_0}$. By the strong Markov theorem applied at time σ_1^X, we have $\mathbf{E}\sum_{j\le K}(L^X_{U_j} - L^X_{S_j}) \le 1 + c_{11}\varepsilon^{\beta_0}$. By the optional stopping theorem we have $\mathbf{E}N_{K\wedge n} \ge 0$ for any fixed n, so

$$\begin{aligned}
c_8 \mathbf{E} \sum_{j \le K\wedge n} \mathbf{d}(X_{U_j}, Y_{U_j})^{\beta_3} \mathbf{1}_{\{S_j \le \tau^+(\varepsilon^{\beta_0})\}} &\le \varepsilon^{\beta_3 - 1} \mathbf{E} \sum_{j \le K\wedge n} (L^X_{U_j} - L^X_{S_j}) \mathbf{1}_{\{S_j \le \tau^+(\varepsilon^{\beta_0})\}} \\
&\le \varepsilon^{\beta_3 - 1}(1 + c_{11}\varepsilon^{\beta_0}).
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\mathbf{E} \sum_{j \le K} \mathbf{d}(X_{U_j}, Y_{U_j})^{\beta_3} \mathbf{1}_{\{S_j \le \tau^+(\varepsilon^{\beta_0})\}} \le c_{12}\varepsilon^{\beta_3 - 1}. \qquad (4.24)$$

Note that

$$\mathbf{E} \sum_{e_s \in \mathcal{E}^*(\sigma_1^X), s\in\mathcal{T}} c_7 \mathbf{d}(X_s, Y_s)^{\beta_3} \mathbf{1}_{\{U_1^e \le \zeta(e_s)\}} \le \mathbf{E} \sum_{j \le K} \mathbf{d}(X_{U_j}, Y_{U_j})^{\beta_3} \mathbf{1}_{\{S_j \le \tau^+(\varepsilon^{\beta_0})\}},$$

so this, (4.23) and (4.24) imply that

$$\begin{aligned}
&\mathbf{E} \sum_{e_s \in \mathcal{E}^*(\sigma_1^X), s\in\mathcal{T}} \mathbf{1}_{\{U_1^e \le \zeta(e_s)\}} \left|\log \rho_{U_1^e} - \log \rho_{S_2^e} - |\log\cos\alpha(e_s)|\right| \mathbf{1}_{\{S_2^e \le \tau^+(\varepsilon^{\beta_0})\}} \\
&\le c_{13}\varepsilon^{\beta_3 - 1}. \qquad (4.25)
\end{aligned}$$

By Lemma 4.3 (ii) and the strong Markov property applied at U_1^e, if $\mathbf{d}(x_1,y_1) \le \varepsilon^{\beta_0}$ and $k \ge 2$ then

$$H^{(x_1,y_1,x_2,y_2)}\big(|\log \rho_{U_k^e} - \log \rho_{S_{k+1}^e}| \mathbf{1}_{\{U_k^e \le \tau^+(\varepsilon^{\beta_0/2})\wedge\zeta\}} \mathbf{1}_{\{U_1^e\le\zeta\}}\big) \le c_{14}\varepsilon^{\beta_0(k-1)/2},$$

so the exit system formula (4.9) implies

$$\mathbf{E} \sum_{e_s\in\mathcal{E}^*(\sigma_1^X)} \sum_{k\ge 2} \mathbf{1}_{\{U_1^e\le\zeta(e_s)\}} \left|\log \rho_{U_k^e} - \log \rho_{S_{k+1}^e}\right| \le \sum_{k\ge 2} c_{14}\varepsilon^{\beta_0(k-1)/2} \le c_{15}\varepsilon^{\beta_4}, \tag{4.26}$$

for some $\beta_4 > 0$.

By Lemma 3.9 (i),

$$\mathbf{E}|\log \rho_{S_1} - \log \rho_0| \le c_{16}\varepsilon. \tag{4.27}$$

By Lemma 4.6, for some $\beta_5 > 0$,

$$\mathbf{P}(\mathbf{d}(Y_{\sigma_1^X}, \partial D) \ge \varepsilon^{\beta_1}) \le \varepsilon^{\beta_2}. \tag{4.28}$$

By Lemma 3.9 (i),

$$\mathbf{E}\left(|\log \rho_{S_{k_1}} - \log \rho_{\sigma_1^X}| \mathbf{1}_{\{\mathbf{d}(Y_{\sigma_1^X},\partial D)\le\varepsilon^{\beta_1}\}}\right) \le c_{17}\varepsilon^{\beta_0}. \tag{4.29}$$

The lemma follows from (4.20), (4.21), (4.22), (4.25), (4.26), (4.27), (4.28) and (4.29). □

Proof of Theorem 1.2. The proof will consist of three steps. First we are going to define some events. Then we will estimate their probabilities and choose the values of the parameters so that the probabilities of the events defined in Step 1 are large. Finally, we will prove that if all of the events defined in Step 1 hold then $\mathbf{d}(X_t, Y_t)$ has the asymptotic behavior asserted in the theorem.

Step 1. Suppose that $\Lambda(D) > 0$ and fix arbitrarily small $\delta, p > 0$. Assume that $\delta < \Lambda(D)/8$.

We will define a number of events and stopping times, depending on parameters $\varepsilon, k_0, c_1, \dots, c_6$, whose values will be specified later on. We will assume that all these parameters are reals in $(0,\infty)$, except that $k_0 > 0$ is a (large) integer. The constant ε will represent the initial distance between the two Brownian particles, i.e., $\varepsilon = \mathbf{d}(X_0, Y_0)$. Let

$$A_1 = \left\{1-\delta \le (s/\sigma^X(s))(2|D|/|\partial D|) \le 1+\delta, \quad \forall s \ge k_0\right\},$$

$$A_2 = \left\{\sup_{t\ge k_0} \left(L^Y_{\sigma^X(t)} - 2t\right) \le 0\right\} \cap \left\{\sup_{t\le\sigma^X(k_0)} \mathbf{d}(X_t,Y_t) \le c_1\right\},$$

$$A_3 = \left\{\sup_{t\ge\sigma^X(k_0)} \left|\frac{1}{t}\int_0^t \nu(X_s)dL_s^X - \frac{1}{2|D|}\int_{\partial D}\nu(y)dy\right| < \delta\right\},$$

$$\widetilde{A}_3 = \left\{\sup_{t\ge\sigma^X(k_0)} \left|\frac{1}{t}\int_0^t \nu(Y_s)dL_s^Y - \frac{1}{2|D|}\int_{\partial D}\nu(y)dy\right| < \delta\right\},$$

$$A_4 = \left\{\sup_{u\ge\sigma^X(k_0)} \left|\frac{1}{u}\sum_{e_s\in\mathcal{E}_u} |\log\cos\alpha(e_s)| - \frac{1}{2|D|}\int_{\partial D}\int_{\partial D} |\log\cos\alpha(z,y)|\omega_z(dy)dz\right| < c_2\delta\right\},$$

$$A_5 = \left\{ \left| \log \overline{\rho}_{\sigma^X(k_0)} - \sum_{e_s \in \mathcal{E}_{\sigma^X(k_0)}} |\log \cos \alpha(e_s)| \right| \le \sigma^X(k_0)\delta \right\},$$

$$A_6 = \left\{ \left| \int_{\mathcal{T}^c \cap [0, \sigma^X(k_0)]} \nu(X_s) dL^X_s \right| \le \sigma^X(k_0)\delta \right\},$$

$$\widetilde{A}_6 = \left\{ \left| \int_{\mathcal{T}^c \cap [0, \sigma^X(k_0)]} \nu(Y_s) dL^Y_s \right| \le \sigma^X(k_0)\delta \right\}.$$

For each event A_j on the above list, let A'_j denote the event defined in the same way except that k_0 is replaced by $k_0 - 1$. The same remark applies to $\widetilde{A}_j$ and $\widetilde{A}'_j$.

For integer $k \ge 0$, let

$$A^k_7 = \left\{ L^Y_{\sigma^X(k_0+k+1)} - L^Y_{\sigma^X(k_0+k)} \le (c_3 \log(k+2))^2 \right\},$$

$$T_k = \sigma^X(k_0+k+1) \wedge \tau^+\Big(\varepsilon \exp(-c_2(k_0+k)(\Lambda(D) - 7\delta) + c_4(1 + (c_3\log(k+2))^2))\Big),$$

$$T^*_k = \inf\{t > \sigma^X(k_0+k+1) : L^Y_t - L^Y_{\sigma^X(k_0+k)} \ge (c_3 \log(k+2))^2\}$$
$$\wedge \tau^+\Big(\varepsilon \exp(-c_2(k_0+k)(\Lambda(D) - 7\delta) + c_4(1 + (c_3\log(k+2))^2))\Big),$$

$$A^k_8 = \Bigg\{ \mathbf{1}_{\{\mathbf{d}(X_{\sigma^X(k_0+k)}, Y_{\sigma^X(k_0+k)}) \le \varepsilon \exp(-c_2(k_0+k)(\Lambda(D)-7\delta))\}}$$
$$\times \left| \int_{\mathcal{T}^c \cap [\sigma^X(k_0+k), T_k]} \nu(X_s) dL^X_s \right| \le c_2\delta \Bigg\},$$

$$\widetilde{A}^k_8 = \Bigg\{ \mathbf{1}_{\{\mathbf{d}(X_{\sigma^X(k_0+k)}, Y_{\sigma^X(k_0+k)}) \le \varepsilon \exp(-c_2(k_0+k)(\Lambda(D)-7\delta))\}}$$
$$\times \left| \int_{\mathcal{T}^c \cap [\sigma^X(k_0+k), T^*_k]} \nu(Y_s) dL^Y_s \right| \le c_2\delta \Bigg\},$$

$$A^k_9 = \{\mathbf{d}(X_{\sigma^X(k_0+k)}, Y_{\sigma^X(k_0+k)}) > \varepsilon \exp(-c_2(k_0+k)(\Lambda(D) - 7\delta))\}$$
$$\cup \{\mathbf{d}(Y_{\sigma^X(k_0+k+1)}, \partial D) \le \varepsilon^{c_5} \exp(-c_5 c_2(k_0+k)(\Lambda(D) - 7\delta))\}.$$

Let $\mathcal{E}^{k,\varepsilon}(s,t)$ be the subset of $\mathcal{E}_t \setminus \mathcal{E}_s$ consisting of these excursions e_u that satisfy

$$\sup_{v \in [u, u+\zeta(e_u))} \mathbf{d}(X_v, Y_v) \le \varepsilon^{c_6} \exp(-c_6 c_2(k_0+k)(\Lambda(D) - 7\delta)),$$

and let

$$A^k_{10} = \Bigg\{ \prod_{0 \le j \le k} \mathbf{1}_{\{\mathbf{d}(X_{\sigma^X(k_0+j)}, Y_{\sigma^X(k_0+j)}) \le \varepsilon \exp(-c_2(k_0+j)(\Lambda(D)-7\delta))\}}$$
$$\times \left| \log \frac{\overline{\rho}_{\sigma^X(k_0+k+1)}}{\overline{\rho}_{\sigma^X(k_0+k)}} - \sum_{e_s \in \mathcal{E}^{k,\varepsilon}(\sigma^X(k_0+k), \sigma^X(k_0+k+1))} |\log \cos \alpha(e_s)| \right| \le c_2\delta \Bigg\}.$$

Step 2. In this step, we will choose the parameters so that all events defined in the previous step have large probabilities. All the bounds on probabilities will hold uniformly for all starting points $(X_0, Y_0) \in \overline{D}^2$ with $\mathbf{d}(X_0, Y_0) = \varepsilon$. The starting points will not be reflected in the notation. We can assume that $\varepsilon > 0$ is arbitrarily small in view of Lemma 3.1.

First we use Lemma 4.2 (ii) to choose k_0 so large that $\mathbf{P}(A_1) > 1 - p$. Let $c_2 \in (0, 1)$ be such that if A_1 holds then for all $k \geq 0$,

$$c_2(k_0 + k) \leq \sigma^X(k_0 + k). \tag{4.30}$$

We choose c_1 to be the constant ε_0 of Lemma 3.6, assuming that the constant c_1 in that lemma takes the value $c_2\delta/3$.

Lemma 4.2 is stated for the process X_t but it applies equally to Y_t. It is an easy consequence of part (ii) of that lemma applied to both X_t and Y_t that if we enlarge k_0, if necessary, then $\mathbf{P}\left(\sup_{t \geq k_0}(L^Y_{\sigma^X(t)} - 2t) \leq 0\right) > 1 - p$. It follows from Lemma 3.8 that if $L^Y_{\sigma^X(k_0)} \leq 2k_0$ and ε is sufficiently small then $\sup_{t \leq \sigma^X(k_0)} \mathbf{d}(X_t, Y_t) \leq c_1$. Hence, for sufficiently small ε, $\mathbf{P}(A_2) > 1 - p$.

Using Lemma 4.2 (i) we can find t_1 so large that

$$\mathbf{P}\left(\sup_{t \geq t_1}\left|\frac{1}{t}\int_0^t \nu(X_s)dL^X_s - \frac{1}{2|D|}\int_{\partial D}\nu(y)dy\right| < \delta\right) > 1 - p/2. \tag{4.31}$$

By part (ii) of the same lemma, we can enlarge k_0, if necessary, so that $\mathbf{P}(\sigma^X(k_0) > t_1) > 1 - p/2$. Hence, for this value of k_0 we have $\mathbf{P}(A_3) > 1 - p$. Since (4.31) holds with X replaced by Y, our argument shows that $\mathbf{P}(\widetilde{A}_3) > 1 - p$ for the same value of k_0.

Enlarge k_0, if necessary, so that $\mathbf{P}(\sigma^X(k_0) > t_0) > 1 - p/2$, where t_0 is the constant in the statement of Lemma 4.7 (ii), assuming that in that lemma p is replaced with $p/2$ and δ is replaced with $c_2\delta$. Then it is easy to check, using Lemma 4.7 (ii), that $\mathbf{P}(A_4) > 1 - p$ holds with this choice of k_0.

We will next show that with an appropriate choice of the parameters, $\mathbf{P}(A_5) > 1 - p$. Suppose that $\beta_0 > 0$ and $\beta_1 \in (3/2, 2)$ so that we can apply Lemma 4.9 with these parameters. Recall the notation $\mathcal{E}^*(t)$ from that lemma. Let

$$\begin{aligned}
F_k &= \{\mathbf{d}(X_{\sigma^X_k}, Y_{\sigma^X_k}) \leq \varepsilon_1\} \cap \{\mathbf{d}(Y_{\sigma^X_k}, \partial D) \leq \varepsilon_1^{\beta_1}\}, \\
G^*_k &= \left\{\left|\log(\overline{\rho}_{\sigma^X_{k+1}}/\overline{\rho}_{\sigma^X_k}) - \sum_{e_s \in \mathcal{E}^*(\sigma^X_{k+1}) \setminus \mathcal{E}^*(\sigma^X_k)} |\log\cos\alpha(e_s)|\right| \leq \sigma^X(k_0)\delta/(2k_0)\right\}, \\
G_k &= \left\{\left|\log(\overline{\rho}_{\sigma^X_{k+1}}/\overline{\rho}_{\sigma^X_k}) - \sum_{e_s \in \mathcal{E}(\sigma^X_{k+1}) \setminus \mathcal{E}(\sigma^X_k)} |\log\cos\alpha(e_s)|\right| \leq \sigma^X(k_0)\delta/(2k_0)\right\}.
\end{aligned}$$

Choose $\varepsilon_1 > 0$ so small that for $k = 0, 1, \dots, k_0 - 1$, using Lemma 4.9 and the strong Markov property at σ^X_k, $\mathbf{P}(G^*_k \mid F_k) \geq 1 - p/(4k_0)$. This implies that $\mathbf{P}(G^*_k) \geq 1 - p/(4k_0) - \mathbf{P}(F^c_k)$.

By Lemma 4.6, we can find $\varepsilon_2 > 0$ so small that for $k = 1, \dots, k_0 - 1$, the conditional probability of $\{\mathbf{d}(Y_{\sigma^X_k}, \partial D) \leq \varepsilon_1^{\beta_1}\}$ given $\{\mathbf{d}(X_{\sigma^X_{k-1}}, Y_{\sigma^X_{k-1}}) \leq \varepsilon_2\}$ is greater than $1 - p/(8k_0)$. By Lemmas 3.8 and 4.1 (ii), we can make ε so small that $\mathbf{d}(X_{\sigma^X_{k-1}}, Y_{\sigma^X_{k-1}}) \leq \varepsilon_1 \wedge \varepsilon_2$ for $k = 1, \dots, k_0 - 1$ with probability greater than $1-p/(8k_0)$. With this choice of ε we have $\mathbf{P}(G_k^*) \geq 1-p/(2k_0)$ for $k=0,\dots,k_0-1$, so $\mathbf{P}(\bigcup_{0\leq k\leq k_0-1} G_k^*) \geq 1-p/2$.

It follows from Lemmas 3.8 and 4.1 (ii) that if ε is sufficiently small then $\mathbf{P}(\tau^+(k_0^{\beta_1}) \geq k_0) \geq 1-p/2$. If $\tau^+(k_0^{\beta_1}) \geq k_0$ then $\mathcal{E}^*(k_0) = \mathcal{E}(k_0)$, so we obtain $\mathbf{P}(\bigcup_{0\leq k\leq k_0-1} G_k) \geq 1-p$. It is easy to check that $\bigcup_{0\leq k\leq k_0-1} G_k \subset A_5$, so with our choice of parameters ε and k_0, $\mathbf{P}(A_5) > 1-p$.

Recall that $\nu^* = \sup_{x\in\partial D} |\nu(x)| < \infty$. Lemma 4.8 implies that for some $\varepsilon_1 > 0$, $C_1 < \infty$, and $\varepsilon \leq \varepsilon_0 \leq \varepsilon_1$,

$$\mathbf{E}\int_{\mathcal{T}^c\cap[0,\sigma^X(k_0)\wedge\tau^+(\varepsilon_0)]} |\nu(X_s)| dL_s^X \leq \nu^* C_1 k_0 \varepsilon_0 |\log\varepsilon_0|.$$

It follows that,

$$\mathbf{P}\left(\int_{\mathcal{T}^c\cap[0,\sigma^X(k_0)\wedge\tau^+(\varepsilon_0)]} |\nu(X_s)| dL_s^X \geq c_2 k_0\delta\right) \leq \nu^* C_1 k_0\varepsilon_0|\log\varepsilon_0|/(c_2k_0\delta). \tag{4.32}$$

According to (4.30), if A_1 holds then $c_2k_0\delta \leq \sigma^X(k_0)\delta$. Suppose that ε_0 is so small that we have $\nu^*C_1k_0\varepsilon_0|\log\varepsilon_0|/(c_2k_0\delta) < p/2$. We have shown that $\mathbf{P}(\sup_{t\leq\sigma^X(k_0)} \mathbf{d}(X_t,Y_t) \leq c_1) \geq \mathbf{P}(A_2) \geq 1-p$ if ε is small. The same argument applied with ε_0 in place of c_1 shows that we can choose ε so small that $\mathbf{P}(\tau^+(\varepsilon_0) > \sigma^X(k_0)) > 1-p/2$. This and (4.32) imply that $\mathbf{P}(A_1\cap A_6) > 1-2p$.

We make k_0 larger, if necessary, so that by Lemma 4.2 (ii) we have $\mathbf{P}(\sigma^Y(2k_0) \leq \sigma^X(k_0)) \leq p/6$. By the same lemma, we can choose C_2 so that $\mathbf{P}(C_2k_0\delta \leq \sigma^X(k_0)\delta) \geq 1-p/6$. Then we can make ε so small that the same argument that leads to (4.32) gives

$$\mathbf{P}\left(\int_{\mathcal{T}^c\cap[0,\sigma^Y(2k_0)\wedge\tau^+(\varepsilon_0)]} |\nu(Y_s)| dL_s^Y \geq C_2k_0\delta\right) \leq p/6.$$

Recall that $\mathbf{P}(\tau^+(\varepsilon_0) > \sigma^X(k_0)) > 1-p/2$. Combining all these estimates, we obtain $\mathbf{P}(A_1\cap A_6\cap \widetilde{A}_6) > 1-3p$.

Recall the definition of events A_j' and $\widetilde{A}_j'$. By enlarging k_0, if necessary, and making ε smaller, we obtain the same estimates for events A_j' and $\widetilde{A}_j'$ as for A_j and $\widetilde{A}_j$, for example, $\mathbf{P}(A_1'\cap A_6'\cap\widetilde{A}_6') > 1-3p$.

By Lemma 4.1 (ii), for some $C_3, C_4 \in (0,\infty)$,

$$\mathbf{P}(A_7^k) \leq C_3\exp(-C_4(c_3\log(k+2))^2).$$

We choose c_3 so large that $\sum_{k=0}^\infty \mathbf{P}((A_7^k)^c) \leq p$.

Let c_4 be the constant called c_1 in Lemma 3.8.

We make ε smaller, if necessary, so that

$$\gamma \stackrel{\text{df}}{=} \varepsilon\exp(-c_2(k_0+k)(\Lambda(D)-7\delta) + c_4(1+(c_3\log(k+2))^2))$$

is smaller than the constant ε_0 in Lemma 4.8, for $k\geq 0$. Let C_5 be the constant c_1 of Lemma 4.8. Using the strong Markov property at $\sigma^X(k_0+k)$, and applying Lemma 4.8, we see that for $k\geq 0$,

$$\begin{aligned}
\mathbf{P}((A_8^k)^c) &\leq \frac{1}{c_2\delta}\mathbf{E}\left|\int_{\mathcal{T}^c\cap[\sigma^X(k_0+k),T_k]} \nu(X_s)dL_s^X\right| \\
&\leq \frac{1}{c_2\delta}\nu^*\mathbf{E}\int_{\mathcal{T}^c\cap[\sigma^X(k_0+k),T_k]} dL_s^X \\
&\leq \frac{1}{c_2\delta}\nu^*C_5\gamma|\log\gamma|.
\end{aligned}$$

We make ε smaller, if necessary, so that $\sum_{k=0}^{\infty} \mathbf{P}((A_8^k)^c) \le p$.

We apply Lemma 4.8 to Y_t in place of X_t to see that for $k \ge 0$,

$$\begin{aligned}\mathbf{P}((\widetilde{A}_8^k)^c) &\le \frac{1}{c_2\delta}\mathbf{E}\left|\int_{\mathcal{T}^c\cap[\sigma^X(k_0+k),T_k^*]} \nu(Y_s)dL_s^Y\right| \\ &\le \frac{1}{c_2\delta}\nu^*\mathbf{E}\left|\int_{\mathcal{T}^c\cap[\sigma^X(k_0+k),T_k^*]} dL_s^Y\right| \\ &\le \frac{1}{c_2\delta}\nu^* C_5(c_3\log(k+2))^2\gamma|\log\gamma|.\end{aligned}$$

We make ε smaller, if necessary, so that $\sum_{k=0}^{\infty} \mathbf{P}((\widetilde{A}_8^k)^c) \le p$.

We choose $c_5 \in (3/2, 2)$ and C_6 such that, by Lemma 4.6,

$$\mathbf{P}((A_9^k)^c) \le \varepsilon^{C_6}\exp(-C_6c_2(k_0+k)(\Lambda(D)-7\delta)).$$

By making ε smaller, if necessary, we obtain $\sum_{k=-1}^{\infty} \mathbf{P}((A_9^k)^c) \le p/2$. Note that the summation index starts from $k=-1$, not $k=0$ (obviously, we can assume that $k_0 > 1$).

Choose a $c_6 \in (0,1)$. By Lemma 4.9, for some C_7, C_8, and $k \ge 0$,

$$\mathbf{P}((A_{10}^k)^c \mid A_9^{k-1}) \le \frac{1}{c_2\delta}C_7\varepsilon^{C_8}\exp(-C_8c_2(k_0+k)(\Lambda(D)-7\delta)).$$

We make ε smaller, if necessary, so that $\sum_{k=0}^{\infty} \mathbf{P}((A_9^k)^c \cup (A_{10}^k)^c) \le p$.

We decrease the value of ε once again so that the argument of τ^+ in the definition of T_k is less than the constant ε_0 in the statement of Lemma 3.6 for $k \ge k_0$, assuming that the constant c_1 in Lemma 3.6 takes the value $c_2\delta/3$.

By our choice of the parameters we arrive at the following bound,

$$\begin{aligned}\mathbf{P}\Big(&A_1\cap A_2\cap A_3\cap\widetilde{A}_3\cap A_4\cap A_5\cap A_6\cap\widetilde{A}_6 \\ &\cap A_1'\cap A_2'\cap A_3'\cap\widetilde{A}_3'\cap A_4'\cap A_5'\cap A_6'\cap\widetilde{A}_6' \\ &\cap\bigcap_{k=0}^{\infty}(A_7^k\cap A_8^k\cap\widetilde{A}_8^k\cap A_9^k\cap A_{10}^k)\Big) > 1-21p. \end{aligned} \tag{4.33}$$

Step 3. We will assume that all parameters have the values chosen in the previous step and that all events that appear in (4.33) hold. Given this assumption, we will prove that $(\log\rho_t)/t \in [-\Lambda(D)-8\delta, -\Lambda(D)+8\delta]$ for all $t \ge \sigma^X(k_0)$.

Recall that $\mathbf{d}(X_0,Y_0)=\varepsilon>0$. First we will deal with the case $t=\sigma^X(k_0)$. Since $A_3, \widetilde{A}_3, A_6$ and $\widetilde{A}_6$ hold,

$$\left|\int_{\mathcal{T}\cap[0,\sigma^X(k_0)]}\nu(X_s)dL_s^X - \sigma^X(k_0)\frac{1}{2|D|}\int_{\partial D}\nu(y)dy\right| \le 2\sigma^X(k_0)\delta, \tag{4.34}$$

and

$$\left|\int_{\mathcal{T}\cap[0,\sigma^X(k_0)]}\nu(Y_s)dL_s^Y - \sigma^X(k_0)\frac{1}{2|D|}\int_{\partial D}\nu(y)dy\right| \le 2\sigma^X(k_0)\delta. \tag{4.35}$$

Since A_1 and A_2 hold, we can use Lemma 3.6 and (4.30) to conclude that

$$\begin{aligned}\left|\log \widetilde{\rho}_{\sigma^X(k_0)} - (1/2)\int_{\mathcal{T}\cap[0,\sigma^X(k_0)]} (\nu(X_s)dL_s^X + \nu(Y_s)dL_s^Y)\right| &\le (c_2\delta/3)(k_0 + L^Y_{\sigma^X(k_0)}) \\ &\le \sigma^X(k_0)\delta.\end{aligned}$$

This and (4.34)-(4.35) yield,

$$\left|\log \widetilde{\rho}_{\sigma^X(k_0)} - \sigma^X(k_0)\frac{1}{2|D|}\int_{\partial D}\nu(y)dy\right| \le 5\sigma^X(k_0)\delta. \tag{4.36}$$

Since A_4 and A_5 are assumed to hold,

$$\left|\log \overline{\rho}_{\sigma^X(k_0)} - \sigma^X(k_0)\frac{1}{2|D|}\int_{\partial D}\int_{\partial D} |\log\cos\alpha(z,y)|\omega_z(dy)dz\right| \le 2\sigma^X(k_0)\delta.$$

This and (4.36) imply

$$\begin{aligned}&\left|\log \rho_{\sigma^X(k_0)} - \sigma^X(k_0)\frac{1}{2|D|}\int_{\partial D}\nu(y)dy - \sigma^X(k_0)\frac{1}{2|D|}\int_{\partial D}\int_{\partial D} |\log\cos\alpha(z,y)|\omega_z(dy)dz\right| \\ &\le 7\sigma^X(k_0)\delta.\end{aligned}$$

Hence we have $\log\rho_{\sigma^X(k_0)}/\sigma^X(k_0) \in [-\Lambda(D)-7\delta, -\Lambda(D)+7\delta]$. Using events A'_j and $\widetilde{A}'_j$ in place of A_j and $\widetilde{A}_j$, we can also prove that $\log\rho_{\sigma^X(k_0-1)}/\sigma^X(k_0-1) \in [-\Lambda(D)-7\delta, -\Lambda(D)+7\delta]$.

Suppose that

$$\log\rho_{\sigma^X(k_0+j)}/\sigma^X(k_0+j) \in [-\Lambda(D)-7\delta, -\Lambda(D)+7\delta]$$

for some $k \ge 0$ and all $j \le k$. We will show that the same holds for $j = k+1$.

The event A_7^k holds so

$$L^X_{\sigma^X(k_0+k+1)} - L^X_{\sigma^X(k_0+k)} + L^Y_{\sigma^X(k_0+k+1)} - L^Y_{\sigma^X(k_0+k)} \le 1 + (c_3\log(k+2))^2. \tag{4.37}$$

By the induction assumption,

$$\mathbf{d}(X_{\sigma^X(k_0+k)}, Y_{\sigma^X(k_0+k)}) \le \varepsilon\exp(-\sigma^X(k_0+k)(\Lambda(D)-7\delta)). \tag{4.38}$$

This, (4.37) and Lemma 3.8 imply that

$$\sup_{t\in[\sigma^X(k_0+k),\sigma^X(k_0+k+1)]} \mathbf{d}(X_t,Y_t) \le \varepsilon\exp(-\sigma^X(k_0+k)(\Lambda(D)-7\delta) + c_4(1+(c_3\log(k+2))^2).$$

This and the assumption that A_7^k holds show that $T_k = \sigma^X(k_0+k+1) \le T_k^*$.

By (4.38) and (4.30),

$$\mathbf{d}(X_{\sigma^X(k_0+k)}, Y_{\sigma^X(k_0+k)}) \le \varepsilon\exp(-c_2(k_0+k)(\Lambda(D)-7\delta)),$$

so the indicator functions in the definitions of events A_8^k and $\widetilde{A}_8^k$ take values 1. Since these events are assumed to hold, we obtain

$$\left| \int_{\mathcal{T}^c \cap [\sigma^X(k_0+k), \sigma^X(k_0+k+1)]} \nu(X_s) dL_s^X \right| \leq c_2 \delta,$$

and

$$\left| \int_{\mathcal{T}^c \cap [\sigma^X(k_0+k), \sigma^X(k_0+k+1)]} \nu(Y_s) dL_s^Y \right| \leq c_2 \delta.$$

By the induction assumption, these estimates hold for all $n = 0, 1, \ldots, k$ in place of k, so

$$\left| \int_{\mathcal{T}^c \cap [0, \sigma^X(k_0+k+1)]} \nu(X_s) dL_s^X \right| \leq c_2 (k+1) \delta,$$

and

$$\left| \int_{\mathcal{T}^c \cap [0, \sigma^X(k_0+k+1)]} \nu(Y_s) dL_s^Y \right| \leq c_2 (k+1) \delta.$$

This, (4.30), and the inequalities in A_3 and $\widetilde{A}_3$ imply that

$$\left| \int_{\mathcal{T} \cap [0, \sigma^X(k_0+k+1)]} \nu(X_s) dL_s^X - \sigma^X(k_0+k+1) \frac{1}{2|D|} \int_{\partial D} \nu(y) dy \right| \leq 2\sigma^X(k_0+k+1)\delta, \tag{4.39}$$

and

$$\left| \int_{\mathcal{T} \cap [0, \sigma^X(k_0+k+1)]} \nu(Y_s) dL_s^Y - \sigma^X(k_0+k+1) \frac{1}{2|D|} \int_{\partial D} \nu(y) dy \right| \leq 2\sigma^X(k_0+k+1)\delta. \tag{4.40}$$

Since $T_k = \sigma^X(k_0+k+1) \leq T_k^*$ and A_2 holds, Lemma 3.6 and (4.39)-(4.40) imply that,

$$\begin{aligned} &\left| \log \widetilde{\rho}_{\sigma^X(k_0+k+1)} - \sigma^X(k_0+k+1) \frac{1}{2|D|} \int_{\partial D} \nu(y) dy \right| \\ &\quad \leq (c_2\delta/3)(k_0 + L^Y_{\sigma^X(k_0)}) + 4\sigma^X(k_0+k+1)\delta \\ &\quad \leq 5\sigma^X(k_0+k+1)\delta. \end{aligned} \tag{4.41}$$

In view of the assumption that A_4, A_5 and A_{10}^j hold for all $j = 0, \ldots, k$, and using (4.30), we have

$$\begin{aligned} &\left| \log \overline{\rho}_{\sigma^X(k_0+k+1)} - \sigma^X(k_0+k+1) \frac{1}{2|D|} \int_{\partial D} \int_{\partial D} |\log \cos \alpha(z,y)| \omega_z(dy) dz \right| \\ &\leq 2\sigma^X(k_0+k+1)\delta. \end{aligned}$$

This combined with (4.41) shows that

$$\begin{aligned} &\left| \log \rho_{\sigma^X(k_0+k+1)} - \sigma^X(k_0+k+1) \frac{1}{2|D|} \int_{\partial D} \nu(y) dy \right. \\ &\quad \left. - \sigma^X(k_0+k+1) \frac{1}{2|D|} \int_{\partial D} \int_{\partial D} |\log \cos \alpha(z,y)| \omega_z(dy) dz \right| \leq 7\sigma^X(k_0+k+1)\delta. \end{aligned}$$

In other words, $\log \rho_{\sigma^X(k_0+k+1)}/\sigma^X(k_0+k+1) \in [-\Lambda(D)-7\delta, -\Lambda(D)+7\delta]$. This completes the induction step.

We have proved that if the events in (4.33) hold then $\log \rho_{\sigma^X(k_0+k)}/\sigma^X(k_0+k) \in [-\Lambda(D)-7\delta, -\Lambda(D)+7\delta]$ for all integer $k \geq 0$. We will extend this claim to all real t greater than some $t_1 < \infty$. By Lemma 4.2 (ii), $\lim_{k\to\infty} \sigma^X(k_0+k+1)/\sigma^X(k_0+k) = 1$, a.s. Lemma 3.8 and (4.37) imply that for $t \in [\sigma^X(k_0+k), \sigma^X(k_0+k+1)]$ we have

$$\log \rho_t - \log \rho_{\sigma^X(k_0+k)} \leq C_9(1+(c_3\log(k+2))^2),$$
$$\log \rho_{\sigma^X(k_0+k+1)} - \log \rho_t \leq C_9(1+(c_3\log(k+2))^2).$$

These observations easily imply that for some $t_1 < \infty$ and all real $t \geq t_1$ we have $(\log \rho_t)/t \in [-\Lambda(D)-8\delta, -\Lambda(D)+8\delta]$. Recall from Step 2 that this holds with probability greater than $1-21p$. Since p and δ are arbitrarily small, the proof is complete. □

REFERENCES

[A] H. Aikawa, Potential-theoretic characterizations of nonsmooth domains. *Bull. London Math. Soc.* **36** (2004), 469–482.

[BB] R. Bañuelos and K. Burdzy, On the "hot spots" conjecture of J. Rauch *J. Func. Anal.* **164**, (1999) 1–33.

[Ba] R.F. Bass, *Probabilistic Techniques in Analysis*, Springer, New York, 1995.

[BBC] R. Bass, K. Burdzy and Z.-Q. Chen (2002) Uniqueness for reflecting Brownian motion in lip domains. *Ann. Inst. H. Poincaré Probab. Statist.* (to appear)

[BH] R.F. Bass and P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains *Ann. Probab.* **19** (1991) 486–508.

[B] K. Burdzy, *Multidimensional Brownian excursions and potential theory*, Longman Sci. Tech., Harlow, 1987.

[BC] K. Burdzy and Z.-Q. Chen, Coalescence of synchronous couplings, *Probab. Theory Related Fields* **123** (2002), no. 4, 553–578.

[CLJ1] M. Cranston and Y. Le Jan, On the noncoalescence of a two point Brownian motion reflecting on a circle, *Ann. Inst. H. Poincaré Probab. Statist.* **25** (1989), no. 2, 99–107.

[CLJ2] M. Cranston and Y. Le Jan, Noncoalescence for the Skorohod equation in a convex domain of $\mathbf{R}^2$, *Probab. Theory Related Fields* **87** (1990), 241–252.

[FOT] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes.* Walter de Gruyter, Berlin, 1994

[H] E. Hsu (2002) Multiplicative functional for the heat equation on manifolds with boundary, *Michigan Math. J.* **50** (2002), 351–367.

[KS] I. Karatzas and S.E. Shreve *Brownian Motion and Stochastic Calculus*, Second edition, Springer, New York, 1991.

[LS] P. L. Lions and A. S. Sznitman, Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37** (1984), 511-537.

[M] B. Maisonneuve, Exit systems, *Ann. Probability* **3** (1975), no. 3, 399–411.
[T] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.
[Z] Z. Zhao, Uniform boundedness of conditional gauge and Schrödinger equations *Commun. Math. Phys.* **93** (1984), 19–31.

K.B. and Z.C. Address: Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98115-4350
(burdzy@math.washington.edu, zchen@math.washington.edu)

P.J. Address: Department of Mathematics, Yale University, PO Box 208283, New Haven, CT 06520-8283
(jones@math.yale.edu)