The Bilinski dodecahedron, and assorted parallelohedra, zonohedra, monohedra, isozonohedra and otherhedra.

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Fifty years ago Stanko Bilinski showed that Fedorov's enumeration of convex polyhedra having congruent rhombi as faces is incomplete, although it had been accepted as valid for the previous 75 years. The dodecahedron he discovered will be used here to document errors by several mathematical luminaries. It also prompted an examination of the largely unexplored topic of analogous non-convex polyhedra, which led to unexpected connections and problems.

Background.

In 1885 Evgraf Stepanovich Fedorov published the results of several years of research under the title "Elements of the Theory of Figures" [9] in which he defined and studied a variety of concepts that are relevant to our story. This book-long work is considered by many to be one of the milestones of mathematical crystallography. For a long time this was, essentially, inaccessible and unknown to Western researchers except for a summary [10] in German.[1]

Several mathematically interesting concepts were introduced in [9]. We shall formulate them in terms that are customarily used today, even though Fedorov's original definitions were not exactly the same. First, a *parallelohedron* is a polyhedron in 3-space that admits a tiling of the space by translated copies of itself. Obvious examples of parallelohedra are the cube and the Archimedean six-sided prism. The analogous 2-dimensional objects are called *parallelogons*; it is not hard to show that the only polygons that are parallelogons are the centrally symmetric quadrangles and hexagons. It is clear that any prism with a parallelogonal basis is a parallelohedron, but we shall encounter many parallelohedra that are more complicated. It is clear that any non-singular affine image of a parallelohedron is itself a parallelohedron.

Another new concept in [9] is that of zonohedra. A *zonohedron* is a polyhedron such that all its faces are centrally symmetric; there are several equivalent definitions. All Archimedean prisms over even-sided bases are zonohedra, but again there are more interesting examples. A basic result about zonohedra is:

Each convex zonohedron has a center.

This result is often attributed to Aleksandrov [1] (see [5]), but in fact is contained in a more general theorem[2] of Minkowski [27, p. 118, Lehrsatz IV]. Even earlier, this was Theorem 23 of Fedorov ([9, p. 271], [10, p. 689]), although Fedorov's proof is rather convoluted and hard to follow.

We say that a polyhedron is *monohedral* (or is a *monohedron*) provided its faces are all mutually congruent. The term "isohedral" — used by Fedorov [9] and Bilinski [3] — nowadays indicates the more restricted class of polyhedra with the property that their symmetries act transitively on their faces.[3] The polyhedra of Fedorov and Bilinski are not (in general) "isohedra" by definitions that are customary today. We call a polyhedron *rhombic* if all its faces are rhombi. It is an immediate consequence of Euler's theorem on polyhedra that the only monohedral zonohedra are the rhombic ones.

One of the results of Fedorov ([9, page 267], [10, page 689]) is contained in the claim:

There are precisely four distinct types of monohedral convex zonohedra: the rhombic triacontahedron T*, the rhombic icosahedron* F*, the rhombic dodecahedron* K*, and the infinite family of rhombohedra (rhombic hexahedra)* H*.*

"Type" here is to be understood as indicating classes of polyhedra equivalent under similarities. The family of rhombohedra contains all polyhedra obtained from the cube by dilatation in any positive ratio in the direction of a body-diagonal.

These polyhedra are illustrated in Figure 1; they are sometimes called *isozonohedra*. The polyhedra T and K go back at least to Kepler [23], while F was first described by Fedorov [9]. I do not know when the family H was first found — it probably was known in antiquity.

An additional important result from Fedorov [9] is he following; notice the change to "combinatorial type" from the "affine type" that is inherent in the definition.

Every convex parallelohedron is a zonohedron of one of the five combinatorial types shown in Figure 2. Conversely, every convex zonohedron of one of the five combinatorial types in Figure 2 is a parallelohedron.[4]

Fedorov's proof is not easy to follow; a more accessible proof of Fedorov's result can be found in [2, Ch. 8].

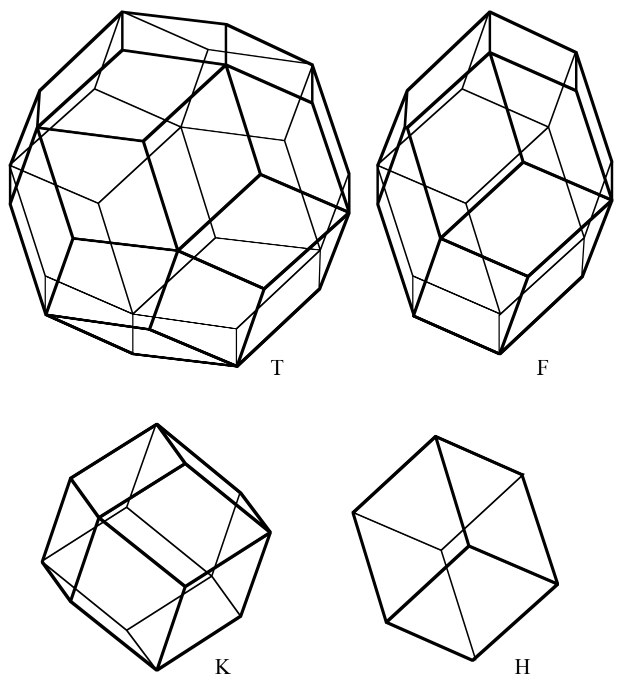


Figure 1. The four isozonohedra (convex rhombic monohedra) enumerated by Fedorov. Kepler found the triacontahedron T and the dodecahedron K, while Fedorov discovered the icosahedron F. The infinite class H of rhombic hexahedra seems to have been known much earlier.

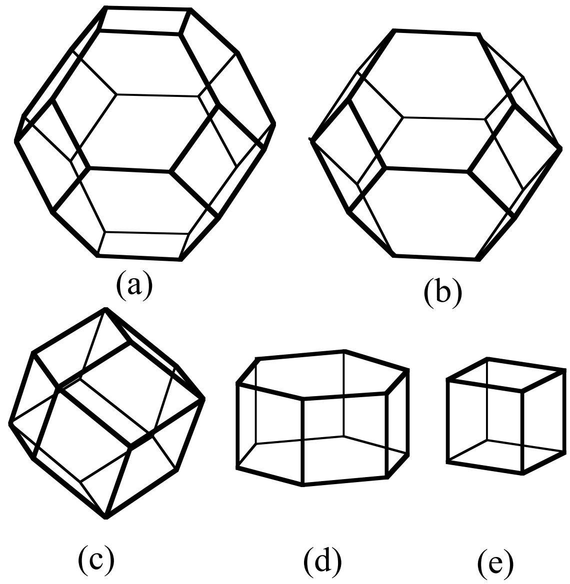


Figure 2. Representatives of the five combinatorial types of convex parallelohedra, as determined by Fedorov [9]. (a) is the truncated octahedron (an Archimedean polyhedron); (b) is an elongated dodecahedron (with regular faces, but not Archimedean); (c) is Kepler's rhombic dodecahedron K (a Catalan polyhedron); (d) is the Archimedean 6-sided prism; and (e) is the cube.

Bilinski's rhombic dodecahedron.

Fedorov's enumeration of monohedral rhombic isohedra (called *isozonohedra* by Fedorov and Bilinski, and by Coxeter [7]) mentioned above claimed that there are precisely four distinct types (counting all rhombohedra as one type). Considering the elementary character of such an enumeration, it is rather surprising that it took three-quarters of a century to find this to be mistaken.[5] Bilinski [3] found that there is an additional isozonohedron and proved:

Up to similarity, here are precisely five distinct convex isozonohedra.

The rhombic monohedral dodecahedron found by Bilinski shall be denoted B; it is not affinely equivalent to Kepler's dodecahedron (denoted K) although it is of the same combinatorial type. Bilinski also proved that there are no other isozonohedra. To ease the comparison of B and K, both are shown in Figure 3.

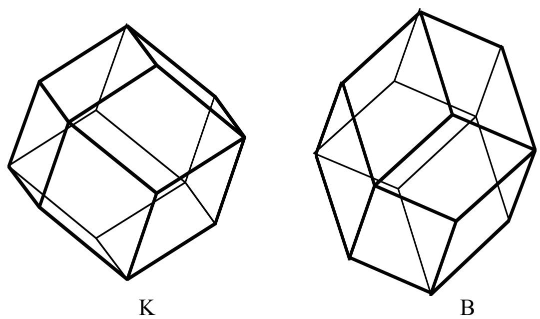


Figure 3. The two convex rhombic monohedra (isozonohedra): Kepler's K and Bilinski's B.

Bilinski's proof of the existence of the dodecahedron B is essentially trivial, and this makes it even more mysterious how could Fedorov have missed it.[6] The proof is based on two observation:

(i) All faces of every convex zonohedron are arranged in *zones*, that is families of faces in which all members share parallel edges of the same length; and

(ii) All edges of such a zone may be lengthened or shortened by the same factor while keeping the polyhedron zonohedral.

In particular, all such edges on one zone can be deleted (shrunk to 0). Performing such a *zone deletion* — a process mentioned by Fedorov — starting with Kepler's rhombic triacontahedron T yields (successively) Fedorov's icosahedron F, Bilinski's dodecahedron B, and two rhombohedra, the obtuse H_o and the acute H_a. This family of isozonohedra that are descendants of the triacontahedron is shown in Figure 4. The proof that there are no other isozonohedra is slightly more complicated and is not of particular interest here.

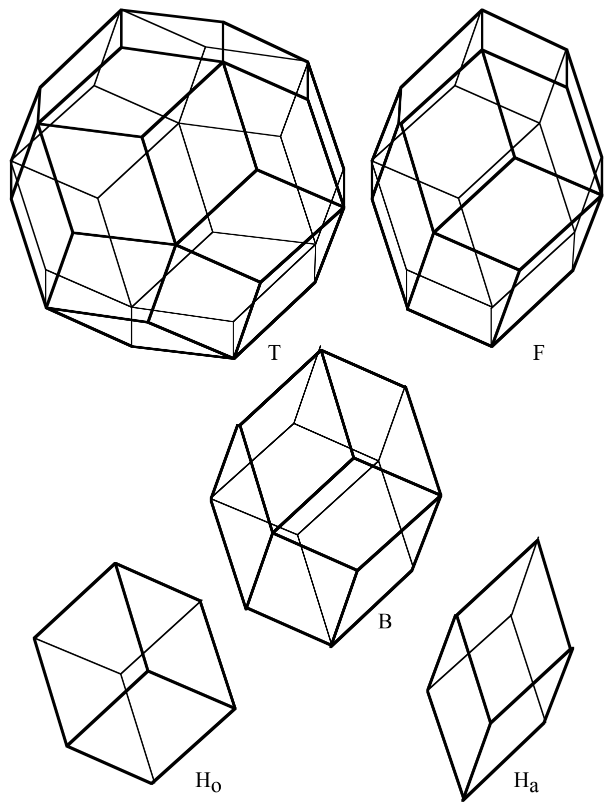


Figure 4. The triacontahedron and its descendants: Kepler's triacontahedron T, Fedorov's icosahedron F, Bilinski's dodecahedron B, and the two hexahedra, the obtuse H_o and the acute H_a. The first three are shown by *.wrl* illustrations in [25], and other web pages,

The family of "direct" descendants of Kepler's rhombic dodecahedron K is smaller; it contains only one rhombohedron H^*_o, see Figure 5. However, one may wish to include in the family a "cousin" H^*_a — consisting of the same rhombi as H^*_o, but in an *acute* conformation.

One of the errors in the literature dealing with Bilinski's dodecahedron is the assertion by Coxeter [7, p. 148] that the two rhombic dodecahedra — Kepler's and Bilinski's — are affinely equivalent. To see the affine non-equivalence of the two dodecahedra (easily deduced even from the drawings

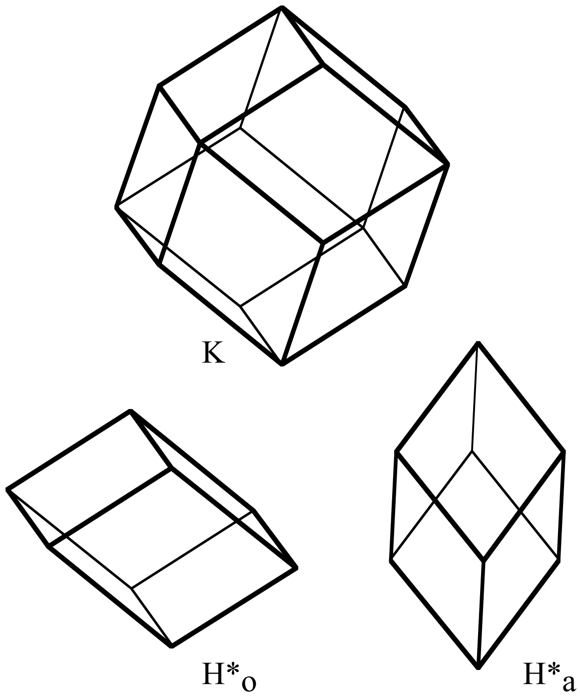


Figure 5. Kepler's rhombic dodecahedron K and its descendant, rhombohedron H^*_o. The rhombohedron H^*_a is "related" to them since its faces are congruent to those of the other two isozonohedra shown; however, it is not obtainable from K by zone elimination.

in Figure 3), consider the long (vertical) body-diagonal of Bilinski's dodecahedron (Figure 3(b)). It is parallel to four of the faces, and in each face to one of the diagonals. In two faces this is the short diagonal, in the other two the long one. But in the Kepler dodecahedron the corresponding diagonals are all of the same length. Since ratios of lengths of parallel segments are preserved under affinities, this establishes the non-equivalence.

If one has a model of Bilinski's dodecahedron in hand, one can look at one of the other diagonals connecting opposite 4-valent vertices, and see that no face diagonal is parallel to it. This is in contrast to the situation with Kepler's dodecahedron.

By the theorems of Fedorov mentioned above, since Bilinski's dodecahedron B is a zonohedron combinatorially equivalent to Kepler's, it is a parallelohedron. This can be easily established directly, most simply by manipulating three or four models of B. It is strange that Bilinski does not mention the fact that B is a parallelohedron.

In this context we have to mention a serious error committed by A. Schoenflies [30, pages 467 and 470] and very clearly formulated by E. Steinitz. It is more subtle than Coxeter's, who may have been misguided by the following statement of Steinitz [34, page 130]:

> The aim [formulated previously in a different form] is to determine the various *partitions of the space into congruent polyhedra in parallel positions*. Since an affine image of such a partition is a partition of the same kind, affinely related partitions are not to be considered as different. Then there are only five convex partitions of this kind. [My translation and comments in brackets]

How did excellent mathematicians come to commit such errors? The confusion illustrates the delicate interactions among the concepts involved, considered by Fedorov, Dirichlet, Voronoi, and others. A correct version of Steinitz's statement would be (see Delone [8]):

Every convex parallelohedron P is affinely equivalent to a parallelohedron P' such that a tiling by translates of P' coincides with the tiling by the Dirichlet-Voronoi regions of the points of a lattice L'. The lattice L' is affinely related to the lattice L associated with one of the five Fedorov parallelohedra P''. But P' need not be the image of P'' under that affinity. *Affine transformations do not commute with the formation of Dirichlet-Voronoi regions.*

In particular, isozonohedra other than rhombohedra are not mapped onto isozonohedra under affine transformations that are not similarities.

As an illustration of this situation, it is easy to see that Bilinski's dodecahedron B is affinely equivalent to a polyhedron B' that has an insphere (a sphere that touches all its faces). The centers of a tiling by translates of B' form a lattice L' such that this tiling is formed by Dirichlet-Voronoi regions of the points of L'. The *lattice* L' has an affine image L such that the tiling by Dirichlet-Voronoi regions of the points of L is a tiling by copies of the Kepler dodecahedron K. However, since the Dirichlet domain of a lattice is not affinely associated with the lattice, there is no implication that either B or B' is affinely equivalent to K.

A simple illustration of the analogous situation in the plane is possible with hexagonal parallelogons (as mentioned earlier, a *parallelogon* is a polygon that admits a tiling of the plane by translated copies). As shown in Figure 6, the tiling is by the Dirichlet regions of a lattice of points. This lattice is affinely equivalent to the lattice associated with regular hexagons, but the tiling is obviously not affinely equivalent to the tiling by regular hexagons.

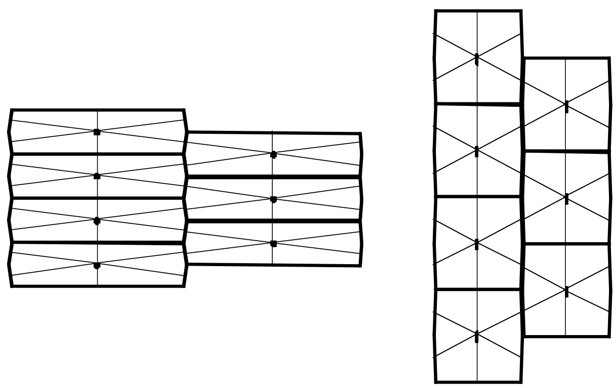


Figure 6. An affine transform of the lattice of centers at left leads to the lattice of the tiling by regular hexagons. The Dirichlet domains of the points of the lattice are transformed into the hexagons at right, which clearly are not affinely equivalent to regular hexagons.

It is appropriate to mention here that for simple parallelohedra (those in which all vertices have valence 3) that tile face-to-face Voronoi proved [38] that each is the affine image of a Dirichlet-Voronoi region. For various strengthenings of this result see [26].

Non-convex parallelohedra.

Bilinski's completion of the enumeration of isozonohedra needs no correction. However, it may be of interest to examine the situation if *non-convex* rhombic monohedra are admitted; we shall modify the original definition and call them *isozonohedra* as well. Moreover, there are various reasons why one should investigate — more generally — non-convex parallelohedra.

It is of some interest to note that the characterization of plane parallelogons (convex or not) is completely trivial. A version is formulated as Exercise 1.2.3(i) of [16, page 24]: A closed topological disk M is a parallelogon if and only if it is possible to partition the boundary of M into four or six arcs, with opposite arcs translates of each other. Two examples of such partitions are shown in Figure 7.

Another reason for considering non-convex parallelohedra is that there is no intrinsic justification for their exclusion, while — as we shall see — many interesting forms become possible, and some tantalizing problems arise. The *crosses*, *semicrosses* and other *clusters* studied by Stein [32] and others provide examples of such questions and results[7]. It also seems reasonable that the use of parallelohedra in applications need not be limited to convex ones.

It is worth noting that by Fedorov's Definition 24 (page 285 of [9], page 691 of [10]) and earlier ones, a *parallelohedron* need not be convex, nor do its faces need to be centrally symmetric.

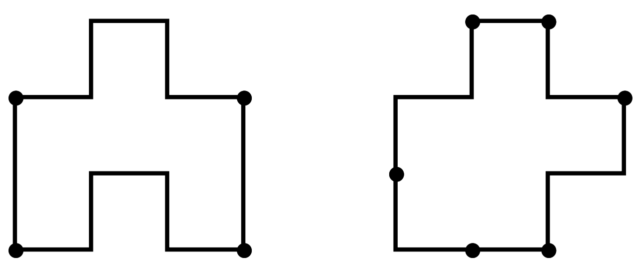


Figure 7. Planigons without center have boundary partitioned into 4 or 6 arcs, such that the opposite arcs are translates of each other.

Two non-convex rhombic monohedra (in fact, isohedra) have been described in the nineteenth century; see Coxeter [7, pages 102 – 103, 115 – 116]. Both are triacontahedra, and are selfintersecting. This illustrates the need for a precise description of the kinds of polyhedra we wish to consider here.

Convex polyhedra discussed so far need little explanation, even though certain variants in the definition are possible. However, now we are concerned with wider classes of polyhedra regarding which there is no generally accepted definition.[8] Unless the contrary is explicitly noted, in the present note we consider only polyhedra *with surface homeomorphic to a sphere and adjacent faces not coplanar*. We say they are of *spherical type*. There are infinitely many combinatorially different *rhombic monohedra* of this type — to obtain new ones it is enough to "appropriately paste together" along common faces two or more smaller polyhedra. This will interest us a little bit later.

The two triacontahedra mentioned above are not accepted in our discussion. However, a remarkable non-convex rhombic hexecontahedron of the spherical type was found by Unkelbach [37]; it is shown in Figure 8. Its rhombi are the same as those in Kepler's triacontahedron T. It is one of almost a score of rhombic hexecontahedra described in the draft of [15]; however, all except U are not of the spherical type.

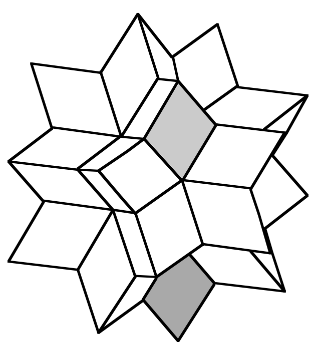


Figure 8. Unkelbach's hexecontahedron. It has pairs of disjoint, coplanar but not adjacent faces, which are parts of the faces of the great stellated triacontahedron. All its vertices are distinct, and all edges are in planes of mirror symmetry.

For a more detailed investigation of *non-convex* isozonohedra, we first *restrict attention to rhombic dodecahedra*. We start with the two *convex* ones – Kepler's K and Bilinski's B – and apply a modification we call *indentation.* An indentation is carried out at a 3-valent vertex of a isozonohedron. It consists of the removal of the three incident faces and their replacement by the three "inverted" faces – that is the triplet of faces that has the same outer boundary as the original triplet, but fits on the other side of that boundary. This is illustrated in Figure 9, where we start from Kepler's dodecahedron K shown in (a), and indent the nearest 3-valent vertex (b). It is clear that this

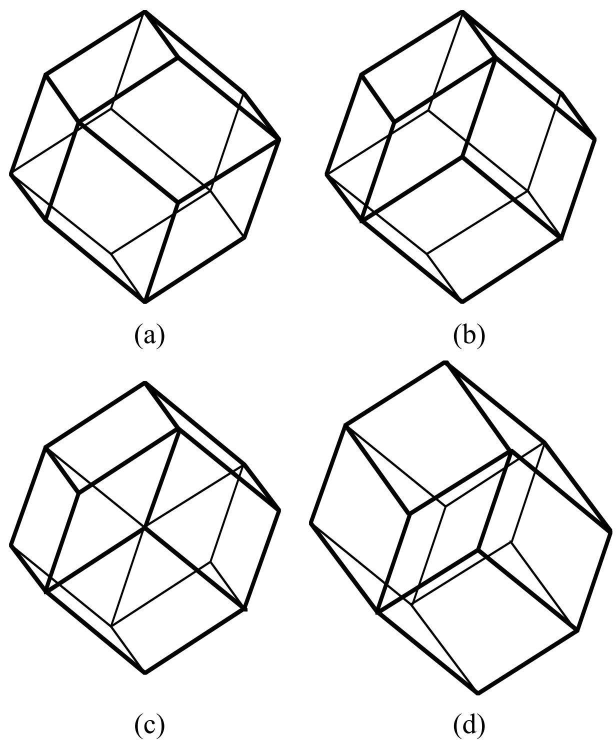
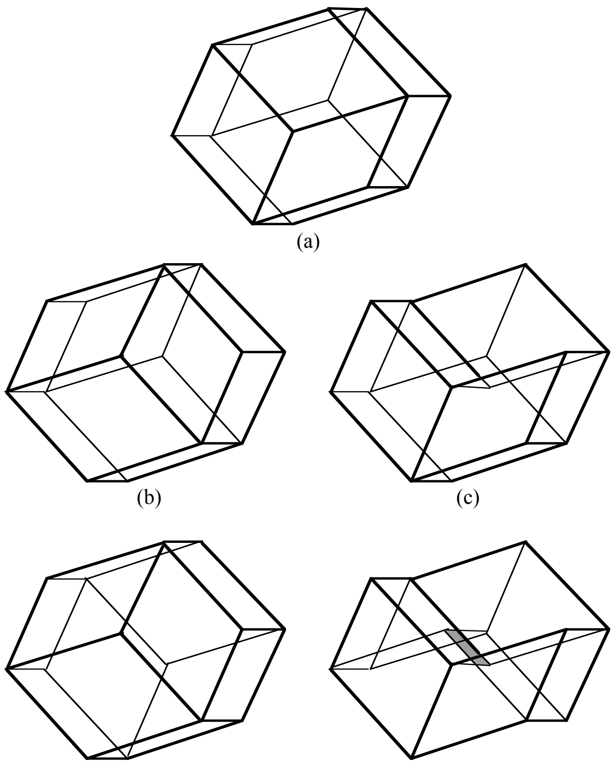


Figure 9. Indentations of the Kepler rhombic dodecahedron K, shown in (a). In (b) is presented the indentation at the vertex nearest to the observer; this is the only indentation arising from (a). A double indentation of the dodecahedron in (a), which is a single indentation of (b), is shown in (c); it fails to be a polyhedron of the spherical type, since two distinct vertices coincide at the center; hence it is not admitted. By stretching one of the zones, as in (d), an admissible polyhedron is obtained — but it is not a rhombic monohedron.

results in a non-convex polyhedron. Since all 3-valent vertices of Kepler's dodecahedron are equivalent, there is only kind of indentation possible. On the other hand, Bilinski's dodecahedron B in Figure 10(a) has two distinct kinds of 3-valent vertices, so the indentation construction leads to two distinct polyhedra; see parts (b) and (c) of Figure 10.



(a)

(b) (c)

(d) (e)

Figure 10. Indentations of the Bilinski dodecahedron shown in (a). The two different indentations are illustrated in (b) and (c), the former at an "obtuse" 3-valent vertex, the latter at an "acute" vertex. The double indentation of (a), resulting from a single indentation of (b), is presented in (d); (e) shows an additional indentation of (c) which, however, is not a polyhedron in the sense adopted here, since two faces overlap in the gray rhombus.

Returning to Figure 9, we may try to indent one of the 3-valent vertices in (b). However, none of the indentation produces a polyhedron of spherical type. The minimal departure from this type occurs on indenting the vertex opposite to the one indented first; in this case the two indented triplets of faces meet at the center of the original dodecahedron (see Figure 9(c)). We may eliminate this coincidence by stretching the polyhedron along the zone determined by the family of parallel edges that do not intrude into the two indented triplets. This yields a parallelogram-faced dodecahedron that is of spherical type (but not a rhombic monohedron); see Figure 9(d). A related polyhedron is shown in a different perspective as Figure 121 in Fedorov's book [9].

It is of significant interest that all the isozonohedra in Figures 9 and 10, — even the ones we do not quite accept, shown in Figures 9(c) and 10(e) — are parallelohedra. This can most easily be established by manipulating a few models; however, graphical or other computational verification is also readily possible.

To summarize the situation concerning dodecahedral rhombic monohedra, we have the following polyhedra of spherical type:

Two *convex* dodecahedra (Kepler's and Bilinski's);
Three *simply indented* dodecahedra (one from Kepler's polyhedron, two from Bilinski's)
One *doubly indented* dodecahedron (from Bilinski's polyhedron).

We turn now to the two larger isozonohedra, Fedorov icosahedron F and Kepler's triacontahedron T. Since each has 3-valent vertices, it is possible to indent them, and since the 3-valent vertices of each are all equivalent under symmetries, a unique indented polyhedron results in each case (Figure 11).

The icosahedron F admits several non-equivalent double indentations, see Figure 12; two are of special interest, and we shall denote them by D_1 and D_2. There are many other multiple — up to sixfold — indentations; their precise number has not been determined. An eightfold indentations of the triacontahedron T is shown in [39, page 196]; it admits several additional indentations.

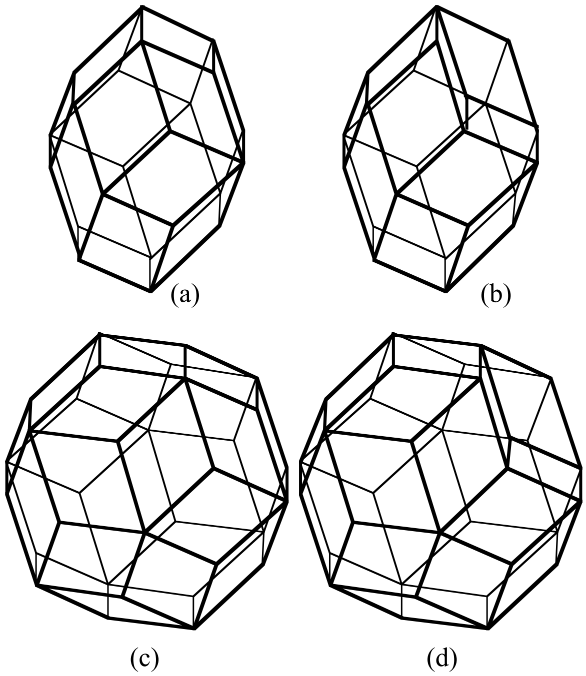
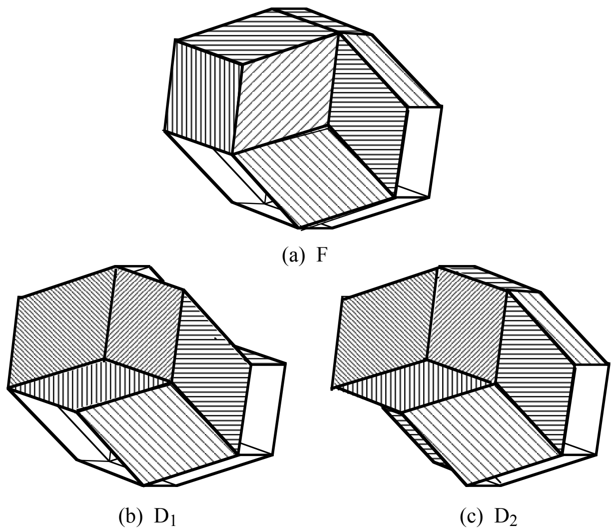


Figure 11. (a) Icosahedron F and (b) its indentation; (c) Triacontahedron T and (d) its indentation.

The double indentations D_1 and D_2 of F shown in Figure 12 are quite surprising and deserves special mention: They are parallelohedra! Again, the simplest way to verify this is by using a few models, and investigating how they fit. This contrasts with the singly indented icosahedron, which is not a parallelohedron. None of the other isozonohedra obtainable by indentation of F or T seem to be parallelohedra.

A different construction of isozonohedra is through the union of two or more given ones along whole faces, but without coplanar adjacent faces; clearly this means that all those participating in the union must belong to the same family of rhombic monohedra – either the family of the triacontahedron, or of Kepler's dodecahedron, or of rhombohedra (with equal rhombi) not in either of these families. Besides a brief notice of this possibility by Fedorov, the only other reference is to the union of two rhombohedra mentioned by Kappraff [22, page 381].[9]



(a) (b)

Figure 12. (a) The Fedorov rhombic icosahedron F; (b) A double indentation of the F yields a non-convex rhombic icosahedron D_1 of the spherical type that is a parallelohedron; (c) A different double indentation D_2 is also a parallelohedron.

For an example of this last construction, by attaching two rhombohedra in allowable ways one can obtain three distinct decahedra, one of which is shown in Figure 13. Another is chiral, that is, comes in two mirror-image forms. This construction can be extended to arbitrarily long chains of rhombohedra; from n rhombohedra there results a parallelohedron with 4n + 2 faces, see Figure 13 for n = 3. For another example, from three acute and one obtuse rhombohedra of the triacontahedron family, that share an edge, one can form a decahexahedron E. It is chiral, but it has an axis of 2-fold rotational symmetry. By suitable unions of one of these decahexahedron with a chain of n rhombohedra ($n \geq 2$), on can obtain isozonohedra with 4n + 16 faces. All isozonohedra mentioned in this paragraph happen to be parallelohedra as well. Hence there are rhombic monohedral parallelohedra for all even $k \geq 6$ except for k = 8.

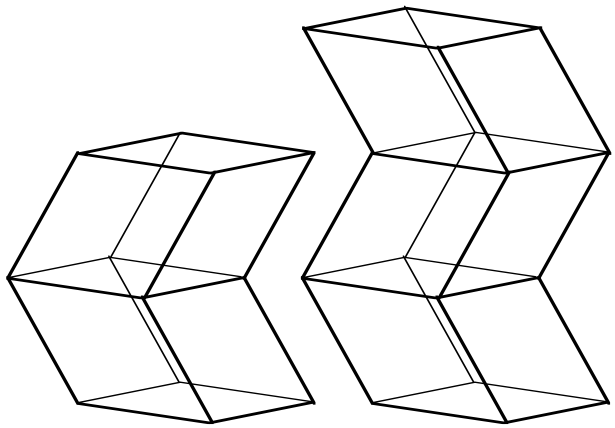


Figure 13. Isozonohedra with 10 and 14 faces.

The isozonohedra just described show that there exist rhombic monohedral parallelohedra with arbitrarily long zones. However, there is a related open problem:

Given an even integer k ≥ 4, is there a rhombic monohedral parallelohedron such that every zone has exactly k faces?

The cube has k = 4, the rhombic dodecahedra K and B have k = 6, and the doubly indented icosahedra D_1 and D_2 are examples with k = 8. No information is available for any k ≥ 10.

While the number of examples non-convex isozonohedra and parallelohedra could be increased indefinitely, in the next section we shall propose a possible explanation of which isozonohedra are parallelohedra.[10]

Remarks.

(i) The parallelohedra discussed above lack a center of symmetry, which was traditionally taken as present in parallelohedra and more generally – in zonohedra. *Convex* zonohedra have been studied extensively; they have many interesting properties, among them central symmetry[11]. However, the assumption of central symmetry (of the faces, and hence of the polyhedra) amounts to putting the cart before the horse if one wishes to study *parallelohedra* — that is, polyhedra that *tile space by translated copies*.

In fact, the one and only *immediate* consequence of the assumed property of polyhedra that allow tilings by translated copies is that their faces come in pairs that are *translationally* equivalent. For example, the octagonal prism in Figure 14 is not centrally symmetric, and its bases have no center of symmetry either. But even so, it clearly is a parallelohedron. The dodecahedra in Figures 9(b) and 10(b),(c) have no center of symmetry although their faces are rhombi and have a center of symmetry each. On the other hand, the doubly indented polyhedron is Figure 10(d) has a center. As mentioned before, each of these is a parallelohedron.

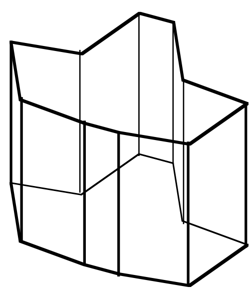


Figure 14. A non-convex parallelohedron without a center of symmetry.

We wish to claim that central symmetry is a red herring as far as parallelohedra are concerned. The reason that the requirement of central symmetry may appear to be natural is that studies of parallelohedra have practically without exception been restricted to convex ones. Now, for convex polyhedra the pairing of parallel faces by translation implies that they have equal area, whence by a theorem of Minkowski (see endnote 2) the polyhedron has a center, which implies that the paired faces coincide with their image by reflection in a point – that is, are necessarily centrally symmetric, and therefore are zonohedra. But this argument is not valid for non-convex parallelohedra, hence such polyhedra need not have a center of symmetry.

In his first short description of non-convex parallelohedra Fedorov writes (§83 in [9, p. 306]):

> The preceding deduction of simple [that is, centrally symmetric polyhedra with pairwise parallel and equal faces] convex parallelohedra is equally applicable to simple concave [that is, non-convex] ones, and hence we bring here only illustrations. We do not show the concave tetraparallelohedron [the hexagonal prism] since this is simply a prism with a concave par-hexagon as basis. Fig. 121 presents the ordinary, and Fig. 122 the elongated concave hexaparallelohedron [the rhombic dodecahedron and the elongated dodecahedron]; Fig. 123 shows the concave heptaparallelohedron [the truncated octahedron]. Obviously, there exists no concave triparallelohedron [cube]. (My translation and bracketed remarks)

Fedorov's parallelohedron in Figure 121 of [9] is isomorphic to the polyhedron shown in our Figure 9(d). A monohedral rhombic dodecahedron combinatorially equivalent to it is shown in our Figure 10(d) and derived from the Bilinski dodecahedron.

However, Fedorov does not provide any proof for his assertion, and in fact it is not valid in general. For example, his Figure 123 does not show a polyhedron of spherical type, since one of the edges is common to four faces. This can be remedied by lengthening the short horizontal edges, but shows the need for care in carrying out the construction.

(ii) The study of non-convex parallelohedra necessitates the revision of various well-established facts concerning convex parallelohedra. For example, one of the crucial insights in the enumeration of parallelohedra (and parallelotopes in higher dimensions) is the property that every zone has either four or six faces. This is not true for non-convex parallelohedra. For example, the double indentation D_1 of Fedorov's F shown in Figure 12(b) is a parallelohedron — even though all zones of D_1 have 8 faces.

For another example, in some cases changing of the lengths of edges of a zone has limitations if the spherical type is to be preserved.

At present, there seems to be no clear understanding of the requirements on a polyhedron of spherical type to be a parallelohedron. As mentioned earlier, the three indented polyhedra in Figures 9(b) and 10(b),(c) are parallelohedra: They can be stacked like six-sided prisms. In fact, with a grain of salt added, starting with suitably chosen six-sided prisms, they may be considered as examples of Fedorov's second construction of non-convex polyhedra [9, p. 306]:

> If we replace one or several faces of a parallelohedron, or parts of these, by some arbitrary surfaces supported on these same broken lines, in such a way that a closed surface is obtained, and observing that precisely the same [translated] replacement is made in parallel position on the faces that correspond to the first ones or their parts, then, obviously the new figure will be a parallelohedron, though without a center … .

It seems clear that Fedorov did not consider this construction important or interesting, since he did not provide even a single illustration. But it does lead to parallelohedra with some or all faces triangular, in contrast to the convex case; an example is shown in Figure 15. A more elaborate example of a non-convex parallelohedron with some triangular faces, that does not admit a lattice tiling, is described by Szabo [35].

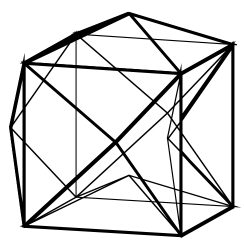


Figure 15. A monohedral parallelohedron with triangles as faces.

Another difference between convex and non-convex parallelohedra is that the convex ones can be decomposed into rhombohedra; this is of interest in various contexts – see, for example, Hart [18], Ogawa [28]. In general, such decomposition is not possible for non-convex parallelohedra. For example, the doubly indented dodecahedron in Figure 10(d) is not a union of rhombohedra.

(iii) Examination of the various isozonohedra that are — or are not — parallelohedra, together with the observation that questions of central symmetry appear irrelevant in this context, lead to the following conjecture:

Conjecture. *Let P be a sphere-like polyhedron, with no pairs of coplanar faces. If the boundary of P can be partitioned into pairs of non-overlapping "patches"* $\{S_1, T_1\}; \{S_2, T_2\}; \ldots ; \{S_r, T_r\}$, *each patch a union of contiguous faces, such that the members in each pair* $\{S_i, T_i\}$ *are translates of each other, and the complex of "patches" is topologically equivalent as a cell complex to one of the parallelohedra in Figure 2, then P is a parallelohedron. Conversely, if no such partition is possible then P is not a parallelohedron.*

As illustrations of the conjecture we can list the following examples:

(a) The three singly indented dodecahedra in Figures 9 and 10 satisfy the conditions, with the patches S_1, T_1 formed by the triplet of indented faces and their opposites, and the other pairs formed by pairs of opposite faces. Then this cell complex is topologically equivalent to the cell complex of the faces of the six-sided prism (Figure 2(d)). As we know, these dodecahedra are parallelohedra. Note that the fact that they are combinatorially equivalent to the convex dodecahedra K and B is irrelevant, since the complex of pairs of faces of the indented polyhedra is not isomorphic to that of the un-indented ones: Some pairs $\{S_i, T_i\}$ of parallel faces are separated by only a single other face while in K and B they are separated by two other faces.

(b) The doubly indented dodecahedron in Figure 10(d) complies with the requirements of the conjecture in a different way: Each pair $\{S_i, T_i\}$ consists of just a pair of parallel faces; the complex so generated is isomorphic to the one arising from Kepler's K.

(c) The doubly indented icosahedron D_1 of Fedorov's F, shown in Figure 12(b), provides additional support for the conjecture. Two of the pairs — say $\{S_1, T_1\}$and$\{S_2, T_2\}$ — are formed by the indented triplets and their opposites. The other pairs $\{S_i, T_i\}$ are the remaining four pairs of parallel faces. The complex they form is isomorphic to the face complex of the elongated dodecahedron shown in Figure 2(b). The same situation prevails with the doubly indented icosahedron D_2 of Figure 12(c). Other double indentations of the icosahedron F, as well as the single indentation of F, fail to satisfy the assumptions of the conjecture and are not parallelohedra.

(d) No indentation of the rhombic triacontahedron satisfies the assumptions of the conjecture, and in fact none is a parallelohedron.

(e) The decahexahedron E mentioned above has a decomposition into pairs $\{S_i, T_i\}$ that is isomorphic to the complex of the faces of the cube. The same situation prevails with regard to the chains of rhombohedra mentioned above.

(iv) The present paper leaves open all questions regarding parallelohedra that are not rhombic monohedra. In particular, it would be of considerable interest to generalize the above conjecture to these parallelohedra. Such an extension would also have to cover the results on "clusters" of cubes such as the crosses and semicrosses investigated by S. K. Stein and others [32], [33], [14]. One can also raise the question what are analogues for suitably defined "clusters" of rhombohedra, or other parallelohedra.

(v) There just possibly may be a prehistory to the Bilinski dodecahedron. As was noted by George Hart [17] [18], a net for a rhombic dodecahedron was published by John Lodge Cowley [6] in the mid-eighteenth century, see Figure 16. The rhombi in this net appear more similar to those of the Bilinski dodecahedron than to the rhombi of Kepler's. However, these rhombi do not have the correct shape and cannot be folded to form any polyhedron with planar faces. (Since the angles of the rhombi are, as close as can be measured, 60° and 120°, the obtuse angles of the shaded rhombus would be incident with two other 120° angles – which is impossible.) An internet discussion about the net mentioned the possibility that the engraver misunderstood the author's instructions; however, it is not clear what the author actually had in mind, since no text describes the polyhedron. The later edition of [6] mentioned by Hart [17] was not available to me.

* * * * *

The author appreciates the helpful comments of a referee.

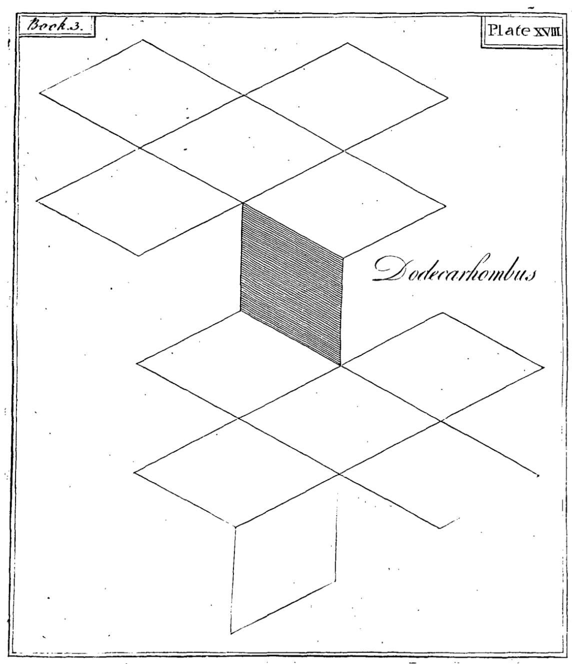


Figure 16. Cowley's net for a rhombic dodecahedron.

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Endnotes

[1] The only somewhat detailed description of Fedorov's work available in English (and in French) is in [31]. Fedorov's book [9] was never translated to any Western language, and its results have been rather inadequately described in the Western literature. The lack of a translation is probably at least in part to blame for ignorance of its results, and an additional reason may be the fact that it is very difficult to read [31, page 6].

[2] Minkowski's theorem establishes that a convex polyhedron with pairwise parallel faces of the same area has a center; the congruence of the faces in each pair follows, regardless of the existence of centers of faces (which is assumed for zonohedra).

[3] The term "gleichflächig" (= with equal surfaces) was quite established at the time of Fedorov's writing, but what it meant seems to have been more than the word implies. As explained in Edmund Hess's second note [21] excoriating Fedorov [10] and [11], the interpretation as "congruent faces" (that is, *monohedral*) is mistaken. Indeed, by "gleichflächig" Hess means something much more restrictive. Hess formulates it in [21] very clumsily, but it amounts to symmetries acting transitively on the faces, that is, to *isohedral*. It is remarkable that even the definition given by Brückner (in his well-known book [4, page 121], repeating the definition by Hess in [19] and several other places) *states* that "gleichflächig" is the same as "monohedral" but Brückner (like Hess) *takes it to mean* "isohedral". Fedorov was aware of the various papers that use "gleichflächig", and it is not clear why he used "isohedral" for "monohedral" polyhedra. In any case, this led Fedorov to claim that his results disprove the assertion of Hess [19] that every "gleichflächig" polyhedron admits an insphere. Fedorov's claim is unjustified, but with the rather natural misunderstanding of "gleichflächig" he was justified to think that his rhombic icosahedron is a counterexample. This, and disputed priority claims, led to protests by Hess (in [20] and [21]), repeated by Brückner [4, page 162], and a rejoinder by Fedorov [11]. Neither side pointed out that the misunderstanding arises from inadequately explained terminology; from a perspective of well over a century later, it seems that both Fedorov and Hess were very thin-skinned, inflexible and stubborn.

[4] In different publications Fedorov uses different notions of "type". In several (for example, [10], [12]) he has only four "types" of parallelohedra, since the rhombic dodecahedron and the elongated dodecahedron ((c) and

(b) in Figure 2) are of the same type in these classifications. Since we are interested in combinatorial types, we accept Fedorov's original enumeration illustrated in Figure 2.

5 This is a nice illustration of the claim that errors in mathematics do get discovered and corrected in due course. I can only hope that if there are any errors in the present work they will be discovered in my lifetime.

6 A possible explanation is in a tendency that can be observed in other enumerations as well: After some necessary criteria for enumeration of objects of a certain kind have been established, the enumeration is deemed complete by providing an example for each of the sets of criteria — without investigating whether there are more than one object per set of criteria. This failure of observing the possibility of a second rhombic dodecahedron (besides Kepler's) is akin to the failure of so many people that were enumerating the Archimedean solids (polyhedra with regular faces and congruent vertices, that is, congruent vertex stars) but missed the pseudorhombicuboctahedron (sometimes called "Miller's mistake"); see the detailed account of this "enduring error" in [13].

7 Recent results on crosses and semicrosses can be found in [14].

8 Many different classes of non-convex polyhedra have been defined in the literature. It would seem that the appropriate definition depends on the topic considered, and that a universally accepted definition is not to be expected.

9 In carrying out this construction we need to remember that adjacent faces may not be coplanar. This excludes the "semicrosses" of Stein [32] and other authors, although it admits the (1,3) cross. For more information see [33].

10 Crystallographers are interested in parallelohedra far more general than the ones considered here: The objects they study in most cases are not polyhedra in the sense understood here, but object combinatorially like polyhedra but with "faces" that need not be planar. The interested reader should consult [29] and [24] for more precise explanations and details.

11 It is worth mentioning that Fedorov did not require any central symmetry in the definition of zonohedra ([9, page 256], [10, page 688]). However, he switched without explanation to considering only zonohedra with

centrally symmetric faces. As pointed out by Taylor [36], this has become the accepted definition.