Whole-Number Place-Value Understanding Of Students With Learning Disabilities

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Abstract

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Place-value is an essential concept that is foundational for understanding our number system and for developing procedures for multidigit operations, but there has been limited research into the place-value understanding of students with learning disabilities. This research is a two-part mixed-methods study that links basic research to instruction. The first part of the study consists of clinical interviews investigating how fifteen students with learning disabilities understand place-value in a variety of contexts. The second part focuses on teaching base-ten numeration to four students who demonstrated difficulties with this concept in the assessment study. Findings from the assessment study found that students with learning disabilities are a heterogeneous group, with five distinct profiles of place-value understanding emerging from the fifteen students. One of the common difficulties that the students had with place-value was difficulty understanding base-ten numeration, the concept of unitizing by tens, and this impacted their ability to solve mathematical word problems correctly. Findings from the instructional study found that counting collections (Schwedtfeger & Chan, 2007) was an effective instructional activity for improving students' understanding of base-ten numeration, as measured by accuracy and problem-solving strategies. This study argues that researchers should focus more on place-value understanding for students with learning disabilities, and proposes both an assessment to measure this understanding and an instructional activity to improve students' understanding of one facet of place-value understanding.

Keywords: learning disabilities; place value; counting; clinical interviews; mathematics; special education.

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INTRODUCTION

Place-value is an essential concept that is foundational for understanding our number system and for developing procedures for multidigit operations (Fuson and Beckmann, 2012; Wearne, Hiebert, & Campbell, 1994), but it is a concept that many students have difficulty understanding (Ross, 1989), and there is some evidence that it is particularly difficult for students with learning disabilities (Hanich, Jordan, Kaplan, & Dick, 2001). However, there has been limited research into the topic of place-value understanding for students with learning disabilities (Section 1) and further research is required before anything decisive can be said about this population.

This dissertation delves deeply into the place-value understanding of students with learning disabilities, examining the topic from three different perspectives: a literature review, a basic research study, and an instructional study. Each perspective is presented in a different section of this dissertation, but the conclusions of one section inform the focus of the following section. The literature review found that the topic of place-value had been little studied for the population of students with learning disabilities, so the following section examined place-value understanding for this population. This basic research study found that many students with learning disabilities have difficulties understanding base-ten numeration and so the following section focused on instructing the students about this concept.

This dissertation is a unified whole, with the focus of each section leading logically from the conclusions of the previous section. Each section has been conceptualized as an individual article so there is some repetition of content, and as all three articles are collected into a dissertation I make references across sections.

Section 1: Mathematical Proficiency: A View Into The Learning Disabilities Literature

The first section is a systematic literature review that investigates which facets of mathematical proficiency have been addressed either in the basic research or the instructional literature on students with learning disabilities. I found that both branches of literature have favored a view of mathematics proficiency that consists of accurate whole-number calculation and word problem solving, but that students' place-value knowledge was rarely studied.

Section 2: *Whole-number place-value understanding for students with learning disabilities*

Having found that few articles had studied the place-value understanding of students with learning disabilities, the next step was to understand what these students did know about place-value. The second section is a basic research study that describes clinical interviews designed to evaluate what students in upper elementary understand about place-value and the results of these interviews for fifteen 4^{th} grade students with learning disabilities. The results of these interviews showed that students with learning disabilities are a heterogeneous group, with five different profiles of place-value understanding emerging from these fifteen students. Despite the variety within the group, many of the students had difficulties with the concept of base-ten numeration, which is the concept of unitizing by tens. Students with learning disabilities' difficulties with the concepts of place-value affected their ability to solve mental arithmetic, algorithmic calculations, and word problems correctly. Therefore when studying or teaching students with learning disabilities we need to assess their place-value understanding as well as their calculation and problem-solving skills.

Section 3: Using Counting Collections To Teach Place-Value Concepts To Students With Learning Disabilities

The third section is an instructional study that uses a single-subject design to examine what four 4th grade students with learning disabilities learned about base-ten numeration during an intervention using counting collections as an instructional activity. Over the course of the intervention all students made improvements in their accuracy and used more mature problem-solving strategies when solving place-value problems, and maintained this growth in follow-up interviews. The students who made the most progress in their accuracy were the students who used the largest variety of strategies and started off by using beginning-level strategies that emphasized unitizing. These findings imply that students with learning disabilities can benefit from engaging in the instructional activity of counting collections when they employ a variety of strategies that lead them from beginning level strategies emphasizing unitizing towards more mature strategies.

By demonstrating that it is important to consider multiple aspects of place-value understanding for students with learning disabilities, this dissertation contributes to the discourse about the mathematics learning of students with special educational needs.

SECTION 1: Mathematical Proficiency: A View Into The Learning Disabilities Literature

Students with learning disabilities often have difficulties in mathematics, but currently most of the research on this population comes from the field of special education with little integration with the field of mathematics education. This systematic literature review aims to further the conversation between these two fields by examining which facets of mathematics proficiency have been addressed in both the basic research and instructional literature on students with learning disabilities. I found that both of these branches of literature have privileged a view of mathematics proficiency that consists of accurate whole-number calculation and word problem solving. In order to gain a fuller picture of the mathematical strengths and weaknesses of students with learning disabilities we need to engage more with the mathematics education literature, which will suggest other areas of mathematics to examine with regard to this population.

Approximately 10% of elementary school children show persistently poor mathematics achievement, showing characteristics similar to that of students with mathematics learning disabilities (Mazzocco & Myers, 2003). Although this is a significant proportion of the school population, the vast majority of the research comes from the field of special education rather than the field of mathematics education (Karp, 2013). This is problematic because although special education researchers are experts at working with the population, most are not disciplinary experts and therefore there are aspects of mathematics teaching and learning that they may not fully appreciate. This systematic literature review aims to further the research on the mathematics learning of students with learning disabilities by using a definition of "mathematical proficiency" from mathematics education to analyze what this term has meant for researchers studying students with learning disabilities and to identify areas that merit further study.

The National Research Council (NRC, 2001) synthesized the relevant research on mathematics learning to produce a comprehensive description of the elements that promote successful mathematics learning, or "mathematical proficiency". They found it to be composed of five mathematical strands: procedural fluency, strategic competence, conceptual understanding, adaptive reasoning, and productive disposition.

When Rivera (1997) reviewed the mathematics learning disabilities literature she found that most of the research focused on computations and word problems, which constitute a small portion of mathematical proficiency as defined by the NRC. In this literature review I examine whether this critique of the literature still holds true fifteen years later. In the intervening years there have been several reviews of the mathematics learning disabilities literature but these have examined the effectiveness of instructional practices (Gersten et al., 2009; Miller, Butler, & Lee, 1998; Montague & Dietz, 2009) rather than the content of the research.

In this literature review, I first frame my review by examining how mathematics education researchers view mathematics proficiency, as expressed through mathematical strands. I then explore the multiple ways in which the term "learning disabilities" has been defined and my reasons for choosing certain definitions of this term while rejecting others. Finally, I conduct a systematic review of the learning disabilities literature to examine how mathematical proficiency is currently being expressed.

Theoretical Framework

In 2010 the Common Core State Standards for Mathematics (CCSSM) adopted eight standards for mathematical practice that were largely based on the five mathematical stands that the NRC (2001) considers essential to mathematical proficiency in elementary and middle schools: procedural fluency, strategic competence, conceptual understanding, adaptive reasoning, and productive disposition.

Procedural fluency is "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (National Research Council, 2001, p. 116). Procedural fluency includes accuracy and facility with rote counting, reading and writing numerals, positional place-value knowledge, basic fact knowledge, and multidigit calculation.

Strategic competence is the "ability to formulate, represent, and solve mathematical problems" (National Research Council, 2001, p. 116). While most school problems do not require formulating because they are already well defined, genuine problems that students encounter outside of school generally require the problem to be defined and the mathematics to be formalized. Representation can be done numerically, symbolically, verbally, physically, or graphically, and is often the first step in solving problems. A student with strong strategic competence can solve problems by creating and utilizing multiple representations, finding several solution strategies, and choosing the most efficient strategy.

Conceptual understanding is an integrated strand that includes comprehension of mathematical concepts, the construction of relationships between these mathematical concepts, and the ability to extend and apply these mathematical concepts to novel situations (Carpenter & Lehrer, 1999; National Research Council, 2001). Comprehension of mathematical concepts consists of many topics including place-value understanding, computation, fractions, and problem contexts. It also includes knowing and being able to use the principles of counting and arithmetic. Students with good conceptual understanding have organized their mathematical ideas into a coherent whole, which allows them to reconstruct forgotten ideas and integrate new ideas easily.

Adaptive reasoning is the "capacity to think logically about the relationships among concepts and situations" (National Research Council, 2001, p. 129). This strand includes logical thought about problems, articulating what one knows with explanations and justifications, and the ability to reflect on the experience of problem-solving.

Productive disposition is the "inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (National Research Council, 2001, p. 116). This strand includes beliefs, attitudes, and motivations regarding mathematics.

Although I have defined each of these five strands independently, they are tightly interwoven and interdependent. Even though procedural and conceptual competence are often seen as competing aspects of school mathematics, an increase in the competence in one will support the development of the other (Rittle-Johnson & Siegler, 1998). Developing conceptual understanding makes it easier to remember facts because they come in coherent mental structures that are easier to access, having many potential points of contact (Hiebert & Wearne, 1996). Equally, a degree of procedural skill supports the development of conceptual relationships between facts (Canobi, 2009). There are similar connections between all five of the strands. Therefore an increase in mathematical proficiency requires instruction that focuses on all five strands.

I based my analysis of the literature on these five mathematical strands. By analyzing each mathematical task given in the studies, I determined which mathematical strands were covered by the studies and which strands required more research. The aspects of mathematical proficiency that were privileged by the community of researchers who studied mathematics and learning disabilities emerged from the distribution of strands that they had examined.

Learning disabilities

In this literature review I focus on how researchers have studied mathematics in students with learning disabilities. However, there are difficulties with choosing which studies to include because this is an ill-defined population and there is no one definition that describes the students who can be considered to have learning disabilities in mathematics. Some researchers have selected students with a mathematics learning disability using a district diagnosis and some have used a cut-off point on a standardized mathematics achievement assessment. Even among the researchers who used the score on a standardized achievement test there was wide variety in the cut-off scores they used to define a student as having a learning disability, with some researchers selecting any student in the lowest 30th percentile (Geary, Hoard, Byrd-Craven, & DeSoto, 2004) and others selecting only students who scored in the lowest 10th percentile (Garrett, Mazzocco, & Baker, 2006; Murphy, Mazzocco, Hanich, & Early, 2007). The problem with the studies that used the 30th percentile cut-off is that this group contains two distinctive groups of children with different cognitive characteristics, growth rates, and profiles: students with mathematical learning disabilities who scored in the 10th percentile and low-achieving students who scored between the 11th and 25th percentile (Murphy, Mazzocco, Hanich, & Early, 2007). Given that students in the 10th percentile exhibit different cognitive characteristics than students in the low-achievement group, I restricted this literature review to articles on students with learning disabilities meaning that the studies either used a district diagnosis of learning disability or they used a cut-off point below the 11th percentile (see Table 2 and 3).

I also restricted this literature review to research conducted in the United States of America (U.S.A.) because of definitional issues with the term "learning disability". In the United

Kingdom (U.K.) the term "learning disability" means students with more severe disabilities who are educated in self-contained classrooms (Moscardini, 2010), and in other countries the meaning is unclear. In the U.K. and several other European countries, researchers generally use the more specific terms of "dyslexia"—a specific deficit in reading skills—or "dyscalculia"—a specific and severe deficit in the ability to process numerical information that results in a failure to develop fluent numerical computation skills (Landerl, Bevan, & Butterworth, 2004)—to express the same level of disability as is described by the general term of "learning disabilities" in the U.S.A. Given the uncertainty of the way the term "learning disabilities" is used in different countries, I restricted this literature review to studies conducted in the U.S.A.

For the purpose of this literature review I included studies conducted in the U.S.A. that used the following definitions of learning disability: a district diagnosis of a learning disability or a mathematics learning disability as defined by a cut-off point below the 11th percentile on a standardized mathematics assessment.

Present review

As Gersten et al. (2009) noted, there are two bodies of research that have emerged within the learning disabilities literature on teaching mathematics. The first one is *basic research* that examines the characteristics and growth trajectories of students with learning disabilities in mathematics. The second one consists of *instructional studies*, where the aim is teach the students a topic they find difficult. In this study I examine these branches of research separately as I analyze the similarities and differences between their views of mathematical proficiency.

Using the framework described above I examine how mathematical proficiency has been investigated and which of its aspects have been privileged within the learning disabilities field, as I address the following research questions: Which facets of mathematical proficiency have been covered in the basic research learning disabilities literature and which have not? Which facets of mathematical proficiency have been addressed in the instructional learning disabilities literature and which have not? What coherence is there between mathematical proficiency as espoused in the basic research literature and in the instructional studies?

Methodology

For this literature review, empirical studies that focused on mathematics learning for elementary students with learning disabilities and were published in peer-reviewed journals between 2000 and 2013 were reviewed. The electronic database PsycINFO was searched using the search terms *math** AND *learning disabilit**, *learning disorder** AND math*, *and learning difficult* AND math*.*

Studies that met the following criteria were included in the review:

1. Peer reviewed from 2000-2013. The cut-off date of 2000 means that the studies were written since the publishing of the current NCTM 2000 standards.
2. Empirical studies investigating aspects of mathematics learning among students with learning disabilities. These studies included experimental, quasi-experimental, single-subject, and qualitative studies.
3. In order to maintain an elementary focus, participants were between 5 and 12 years old.
4. Included a description of the mathematical tasks. This allowed evaluation of the mathematical domains and strands.
5. Took place in the United States in order to facilitate comparison between the participants.

This electronic search resulted in 572 abstracts, of which 134 studies were selected for further review, because they met the above criteria and were not duplicates.

Upon further review 106 articles were excluded. Articles were excluded if they had cutoffs between the 15th and the 40th percentile on mathematics achievement tests (n=29), were primarily about students with low mathematical achievement (n=8), did not disaggregate the results for the students with learning disabilities from the other students (n=5), were about students at risk for low mathematical achievement (n=3), or if learning disabilities was not defined (n=1). Articles were also excluded if they were primarily about students with ADHD (n=8), students with other disorders or diseases (n=16), not about student with learning disabilities (n=3), reading or writing (n=11), social or behavioral skills (n=3), assessment development/ validity (n=12), memory or cognitive processing (n=4), mainstreaming (n=2), or analyzing mathematics texts (n=1). Twenty-eight articles met the final inclusion criteria to be included in this review.

Coding procedures

Once studies had been identified, they were classified as basic research or instructional. Studies were classified as basic research if their main aim was to study the mathematical achievement of students with learning disabilities or the incidence of mathematical learning disabilities and their subtypes. Studies were classified as instructional if the intent of the study was to examine the effectiveness of a particular instructional technique or to investigate how students participated in the mathematics class.

The studies were then coded for the definition of learning disability, methodology, learning theory, number of participants, mathematical tasks addressed, mathematical strands (see Table 1), and mathematical proficiency. The mathematical tasks were used to assign the mathematical strand. As many mathematical topics can be learned either conceptually or procedurally, for example place-value or multidigit calculation, I assessed whether the students were asked to approach the tasks conceptually or procedurally. Studies were coded as procedural if students were asked to recall basic facts, to follow a specific procedure such as a standard algorithm, or to name the digits in the tens or hundreds place. Studies were coded as conceptual if students were asked to solve the problems in ways that made sense to them, if the study probed for more than a positional knowledge of place-value, or if the study explicitly mentioned student understanding.

Table 1

Tasks used to code studies into a particular mathematical strand

Mathematical strand	Tasks from the mathematical strand	Examples
Procedural fluency	• Counting	• Rote counting task from WIAT-II (Geary, Hoard, Nugent, & Byrd-Craven, 2008)
	• Reading or writing numbers	• Number discrimination task from WIAT-II (Geary, Hoard, Nugent, & Byrd-Craven, 2008)
	• Positional place-value	• What is the largest 3-digit number? (Garrett, Mazzocco & Baker, 2006)
	• Basic facts • Calculation • Speed	• Test of computational fluency: 3 minutes to solve addition and subtraction basic facts and multidigit computations (Fuchs & Fuchs, 2002)
Strategic competence	• Formulate problems	• Students wrote their own word problems (Moscardini, 2010).
	• Representations o Teacher produced representations	• Students showed where numbers should be placed on number line (Geary, Hoard, Nugent, & Byrd-Craven, 2008)
	o Student produced representations	• Students draw pictorial representation of the problem (Rodriguez, Parmar, & Signer, 2001)
	• Solve word problems	• Students solved one-step story problems of change, combination, and equalization problem types (Fuchs & Fuchs, 2002)
	• Researchers investigated strategy use	• Researchers recorded students' strategies when transforming their shapes (Grobecker & De Lisi, 2000)

Conceptual understanding	• Principles	
	○ Counting	• Students watched examiner count aloud and reported whether count was correct (Murphy, Mazzocco, Hanich, & Early, 2007)
	○ Arithmetical	• Students given 46+37=83, then asked to solve 37+46= (Jordan, Hanich, & Kaplan, 2003)
	○ Length equality	• Two unsharpened pencils, one pencil displaced relative to other and students asked if pencils same length (Grobecker & Lawrence, 2000)
	• Understanding	
	○ Understanding the problem context	• Students given points for finding relevant information and using correct operation for word problem (Fuchs & Fuchs, 2002)
	○ Place-value understanding	• How many 100s are in 1,000? (Garrett, Mazzocco & Baker, 2006)
	○ Computational understanding	• Students represented problems with base-ten blocks, relating this concrete representation to symbols (Mancl, Miller, & Kennedy, 2012)
	○ Fraction understanding	• Compared fractions in two different modes (Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013)
	• Constructing relationships between mathematical concepts	• Connected concrete, representational, and abstract strategies (Mancl, Miller, & Kennedy, 2012)
	• Extending and applying concepts to novel situations	• Students applied knowledge to contextually realistic word problem (Fuchs & Fuchs, 2002)
Adaptive reasoning	• Logical thought about situations	• For real-world problems students put together disparate parts of the problem (Fuchs & Fuchs, 2002)
	• Explanations	• Students explained how they got their answers (Baxter, Woodward, Voorhies, & Wong, 2002)
	• Justifications	• Students justified their answers (Behrend, 2003)
	• Reflections	• Students evaluated how well they had responded to place-value problems (Garrett, Mazzocco, Baker, 2006)
Productive disposition	• Beliefs about mathematics • Attitudes about mathematics • Motivations regarding mathematics	• Students completed a self-report of their mathematical competence (Montague & van Garderen, 2003)

Results

Tables 2 and 3 show that sixteen articles examined issues of basic research for mathematics learning disabilities and twelve were instructional studies, with no articles combining the two types of studies. Although this focus on one type of study could be attributed to the necessity of published articles to be concise, only 3 of the 56 authors (Fuchs, Fuchs, and Owen) have published both basic research and instructional studies suggesting that these two types of research are generally undertaken by different groups of researchers.

As different groups of researchers undertake the different types of research there may be a lack of coherence between the topics that are examined in basic research and in instructional studies. In the findings I examine the coherence between the recommendations from mathematics education as represented by CCSSM (2010) and the NRC (2001), the results of basic research studies, and the focus of the instructional studies. I investigate the differences between the two groups of studies in the methodologies used and the mathematical strands that have been studied.

Methodologies

Most of the studies were purely quantitative, but two basic research studies used mixed methods (Parmar & Signer, 2005; Rodriguez, Parmar, & Signer, 2001) and one instructional study used a qualitative case-study design (Baxter, Woodward, Voorhies, & Wong, 2002). The basic research studies tended to be large-scale longitudinal or medium-scale comparison between groups studies, while the instructional studies tended to be either medium-scale comparisons between interventions or small-scale single-subjects designs (see Table 4).

Table 2

Characteristics of the basic research studies

Study	Definition of mathematics learning disability	Methodology	Theory of learning	Number of participants	Mathematical Strand	Details about the Strands	Mathematical proficiency
Fuchs & Fuchs (2002).	Mathematics disability: Identified by the teacher as having mathematics goals on IEP and scored more than 1.5 standard deviations below the norm on test of computational fluency	Comparison between groups	Cognitive learning theory	40	Conceptual Procedural Strategic Adaptive	Understanding problem Extending and applying Basic facts Calculation Speed Solve problems Logical thought Explanations	Simple, multistep, and real-world word problems
Garrett, Mazzocco, & Baker (2006).	Mathematics learning disability: ≤10th percentile on TEMA for at least 2 years	Longitudinal	Information-processing theory	249	Conceptual Procedural Adaptive	Place-value understanding Positional place-value Calculation Reflection	Multidigit calculation and place-value
Geary, Hoard, Nugent, Byrd-Craven (2008).	Mathematical learning disability: ≤11th percentile in both 1st and 2nd grade	Comparison between groups	Information-processing theory	261	Procedural Strategic	Counting Reading/ writing numbers Basic facts Calculation Teacher representations	Linear use of number-line
Grobecker & De Lisi (2000).	Learning disability: IQ average but academics below grade-level expectancy in at least one subject	Comparison between groups	Piaget's cognitive theory	180	Procedural Strategic	Basic facts Calculation Teacher representations Solve problems Strategy use	Spatial geometry, e.g. transformations
Grobecker & Lawrence (2000).	Learning disability: IQ average but academics below grade-level expectancy in at least one subject	Comparison between groups	Piaget's cognitive theory	69	Conceptual Procedural Strategic	Principles of length equality Nonverbal calculation Basic facts Strategy use	Well-integrated conceptual schemas
Judge & Watson (2011).	Learning disability: district diagnosis	Longitudinal	Developmental	1,265	Conceptual Procedural	 Basic facts	Conceptual and procedural

Mazzocco & Kover (2007).	Mathematics learning disability: ≤10th percentile for at least 3 of the 6 school years	Longitudinal	Information-processing theory	178	Procedural Strategic	Counting Basic facts Calculation Teacher representations	Performance on achievement tests
Mazzocco & Myers (2003).	Mathematics learning disability: Either ≤10th percentile in both 2nd and 3rd grade, or standard scores <86, or discrepancy between IQ and achievement	Longitudinal	Developmental	209	Conceptual Procedural Strategic	Counting principles Counting Reading/ Writing numbers Basic facts Calculation Teacher representations	Performance on achievement tests
Mazzocco, Myers, Lewis, Hanich, & Murphy (2013).	Mathematical learning disability: <11th percentile on WJ-R at least twice between 3rd and 6th grade	Comparison between groups	Developmental	122	Conceptual Procedural Strategic	Fraction understanding Calculation Teacher representations	Fraction magnitude comparison
Mazzocco & Thompson (2005).	Mathematics learning disability: ≤10th percentile in both 2nd and 3rd grade	Longitudinal	Developmental	209	Conceptual Procedural Strategic	Counting principles Counting Reading/ Writing numbers Basic facts Calculation Teacher representations Solve problems	Performance on achievement tests
Montague & van Garderen (2003).	Learning disability: district diagnosis	Comparison between groups	Developmental	135	Conceptual Procedural Strategic Disposition	Computational understanding Constructing relationships Calculation Solve problems Strategy use	Problem-solving
Morgan, Farkas, & Wu (2011).	Learning disability: district diagnosis	Longitudinal	Developmental	7,400	Procedural	Reading/ writing numbers Basic facts Calculation	Performance on achievement tests

Murphy, Mazzocco, Hanich, & Early (2007).	Mathematics Learning disability: ≤10th percentile on TEMA	Longitudinal	Developmental	210	Conceptual Procedural	Counting principles Counting Reading/ Writing numbers Basic facts Positional place-value Calculation	Performance on achievement tests
Namkung & Fuchs (2012).	Computational difficulty: <7 on addition facts. Problem difficulty <7 on word problems.	Comparison between groups	Developmental	332	Conceptual Procedural Strategic	Counting principles Basic facts Calculation Speed Teacher representations Solve problems	Calculation, word problem skills, number line estimation, and counting principles
Parmar & Signer (2005).	Learning disability: district diagnosis	Comparison between groups	Cognitive learning theory	91	Conceptual Strategic Adaptive	Understanding problem Teacher representations Solve problems Logical thought	Graphs
Rodriguez, Parmar, & Signer (2001).	Learning disability: receiving supplementary mathematics instruction in a resource room. 2 grade discrepancy in mathematics	Descriptive interpretive	Cognitive learning theory	74	Conceptual Strategic Adaptive	Understanding problem Teacher representations Student representations Solve problems Strategy use Explanations	Understanding numeration and solving word problems

Table 3

Characteristics of the instructional studies

Study	Definition of mathematics learning disability	Methodology	Theory of learning	Number of participants	Mathematical Strand	Details about the Strands	Mathematical proficiency
Baxter, Woodward, Voorhies, & Wong (2002).	Mathematics learning disability: has an IEP and receives special education services in mathematics. 6th percentile on ITBS problem-solving subtest.	Case studies	Sociocultural learning theory	28	Conceptual Procedural Strategic Adaptive	Understanding problems Basic facts Solve problems Explanations	Mathematical reasoning and problem-solving
Becker, McLaughlin, Weber, Gower (2009).	Receiving special education services in mathematics and reading	Single-subject	Behaviorism	1	Procedural	Basic facts Speed	Computational fluency
Burns (2005).	Learning disability: designated by State regulations	Single-subject	Behaviorism	3	Procedural	Basic facts Speed	Computational fluency
Fuchs, Fuchs, Hamlett & Appleton (2002).	Mathematics disability: Identified by the teacher as having mathematics goals on IEP and scored more than 1.5 standard deviations below the norm on test of computational fluency	Comparison between types of instruction	Cognitive learning theory	40	Conceptual Procedural Strategic Adaptive	Understanding problem Extending and applying Basic facts Calculation Speed Solve problems Logical thought Explanations	Multistep and real-world story problems
Glover, McLaughlin, Derby, & Gower (2010).	Learning disability: district diagnosis	Single-subject	Behaviorism	2	Procedural	Basic facts Speed	Basic facts fluency
Mancl, Miller, & Kennedy (2012).	Mathematical learning disability: district diagnosis	Single-subject	Mixture of cognitive learning theory and behaviorism	5	Conceptual Procedural Strategic	Computational understanding Constructing relationships Calculation Teacher representations Solve problems Strategy use	Accuracy in calculation and word problem solving

Owen & Fuchs (2002).	Learning disability	Comparison between types of instruction	Cognitive learning theory	24	Conceptual Strategic Adaptive Disposition	Extending and applying Teacher representations Solve problems Explanations	Word problem solving
Sealander, Johnson, Lockwood, & Medina, (2012).	Specific Learning Disability: district diagnosed	Single-subject	Mostly behaviorism with some cognitive learning theory	8	Conceptual Procedural Strategic	Computational understanding Basic facts Speed Teacher representations	Basic facts fluency
Seo & Woo (2010).	Learning disability: district diagnosis	Usability study	Cognitive learning theory	17	Conceptual Strategic	Understanding problem Student representations Solve problems	Addition and subtraction word problems
Shapiro, Edwards, & Zigmond (2005).	Learning disability: district diagnosis		Behaviorism	229	Conceptual Procedural Strategic	Number concepts Counting Reading/ writing numbers Calculation Speed Solve problems	Multidigit calculation, concepts, and applications
Woodward (2006)	Learning disability: district diagnosis	Comparison between types of instruction	Information-processing theory	58	Conceptual Procedural Strategic Adaptive Disposition	Computational understanding Constructing relationships Extending & applying Basic facts Calculation Speed Solve problems Teacher representations Explanations	Basic facts, extended facts, strategies
Woodward, Monroe, & Baxter (2001).	Learning disability: district diagnosis	Comparison between types of instruction	Sociocultural learning theory	182	Conceptual Strategic Adaptive	Understanding problems Extending & applying Student representations Solve problems Explanations	Mathematical understanding

Table 4

Number of studies that used each methodology

	Basic Research, n=16	Instructional Studies, n=12
Longitudinal	7 (44%)	0 (0%)
Comparison between groups	8 (50%)	0 (0%)
Comparison between treatments	0 (0%)	5 (42%)
Case studies	0 (0%)	1 (8%)
Descriptive interpretive	1 (6%)	0 (0%)
Usability	0 (0%)	1 (8%)
Single-subject	0 (0%)	4 (33%)
Other	0 (0%)	1 (8%)

Theory of Learning

Three learning theories informed the basic research studies: cognitive learning theory, information-processing theory, and developmental perspectives (see Table 5). Half of the basic research studies took a developmental perspective, investigating the mathematical achievement growth trajectories of students with learning disabilities. A third of the basic research studies used cognitive learning theories, with investigations of how students with learning disabilities utilized schemas developed to understand numeration and mathematical problems, and the remaining studies took an information-processing perspective by investigating the impact of central executive function and metacognition skills on mathematical achievement.

Table 5

Number of studies based on each learning theory

	Basic Research, n=16	Instructional Studies, n=12
Behaviorism	0 (0%)	4 (33%)
Mixture of cognitive learning theory and behaviorism	0 (0%)	2 (17%)
Cognitive learning theory	5 (31%)	3 (25%)
Information-processing	3 (19%)	1 (8%)
Developmental	8 (50%)	0 (0%)
Sociocultural learning theory	0 (0%)	2 (17%)

The instructional studies drew inspiration from a different variety of learning theories (see Table 5). Half of the studies were inspired by behaviorism and relied on direct instruction to produce observable changes in students' calculation fluency. A third of the instructional studies were informed by cognitive learning theories, with investigations of how students with learning disabilities utilized schemas and strategies to solve word problems and computations. One instructional study justified its focus on teaching basic facts by using an information-processing perspective to explain that automaticity with basic facts allows students to think about more complex mathematical tasks (Woodward, 2006), and the remaining studies used sociocultural learning theory to examine interactions in mathematics classrooms and their impact on student learning.

Mathematical Strands

This review of the literature found that procedural fluency, conceptual understanding, and strategic reasoning have all been well represented in the mathematics learning disabilities literature, but adaptive reasoning and mathematical disposition have been understudied for this population in the U.S.A. (see Table 6). The basic research and instructional studies have had a similar distribution across the strands, except that the instructional studies were more likely to investigate adaptive reasoning and mathematical disposition than the basic research studies.

Table 6

Number of studies that examined each mathematical strand

	Basic Research, n=16	Instructional Studies, n=12
Procedural	14 (88%)	9 (75%)
Strategic	13 (81%)	9 (75%)
Conceptual	12 (75%)	9 (75%)
Adaptive	4 (25%)	5 (42%)
Disposition	1 (6%)	2 (17%)

From these results it is clear that many articles covered more than one strand, so I also analyzed which strands coexisted and which strands were examined on their own. Only procedural fluency was examined on its own, by one basic research article (Morgan, Farkas, & Wu, 2011) and three instructional studies (Becker, McLaughlin, Weber, & Gower, 2009; Burns, 2005; Glover, McLaughlin, Derby, & Gower, 2010). A quarter of the instructional studies examined only procedural fluency, indicating that these researchers prioritize procedural fluency over other mathematical competencies. The other strands were all examined in conjunction with other strands, reflecting the interwoven nature of the strands and the difficulty of investigating one strand without also investigating another.

Procedural fluency. This literature review found that basic research studies on procedural fluency were mostly focused on whether students knew the basic facts or were able to solve multidigit computational problems (see Table 7). This focus dominated the basic research literature, with 69% of the studies measuring students' skill at completing basic arithmetic facts and 81% of the studies measuring students' ability to solve multidigit computations correctly. These studies found that students with learning disabilities had difficulties with basic addition facts (Mazzocco & Thompson, 2005) and multidigit calculation (Montague & van Garderen, 2003).

Table 7

Number of studies that examined each aspect of procedural fluency

	Basic Research, n=16	Instructional Studies, n=12
Multidigit Calculation	13 (81%)	4 (33%)
Basic Facts	11 (69%)	7 (58%)
Counting Sequence	5 (31%)	1 (8%)
Reading/Writing Numbers	5 (31%)	1 (8%)
Speed	2 (13%)	7 (58%)
Positional Place-Value	2 (13%)	0 (0%)

A focus on basic facts also dominated the literature from the instructional studies, with 58% of the studies including an examination of students' basic facts knowledge and 42% of the studies designed to improve students' knowledge of basic facts. Sealander, Johnson, Lockwood, and Medina (2012) used a single-subject design to demonstrate the efficacy of the concrete-representational-abstract[1] (CRA) instructional sequence for teaching subtraction facts to students with learning disabilities. Typical of studies designed to teach multiplication facts, Glover, McLaughlin, Derby, and Gower (2010) used a single-subjects design to demonstrate the efficacy of a direct instruction flashcard system for teaching multiplication facts to students with disabilities. When Woodward (2006) used a comparison between intervention designs he found that both using timed practice drills and teaching students to use strategies for learning facts were effective ways to help students learn multiplication facts.

All of these studies measured the speed with which students recalled basic facts (see Table 7), often requiring them to recall the facts within 2 or 3 seconds. The behaviorist perspective of most of these instructional studies led to a focus on arithmetical automaticity because this is an easily observed and measured skill.

Instructional studies focused on multidigit calculations less than the basic research studies (see Table 6), with only one study designed to improve students' skills with multidigit subtraction. Mancl, Miller, and Kennedy (2012) used a single-subject design to demonstrate the efficacy of the concrete-representative-abstract instructional sequence using base-ten blocks for teaching multidigit subtraction to students with learning disabilities.

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[1] This instructional sequence starts by introducing a concept with concrete manipulatives. Once students demonstrate mastery at this concrete level they are introduced to the same concept illustrated with drawings, and only once they have mastered this representational stage are they taught to solve the same types of problems with symbols.

In the basic research studies there was some focus on the students' knowledge of the counting sequence and their ability to read and write numbers (see Table 9). Mazzocco and Thompson (2005) found that accurate reading of numerals in kindergarten was an important predictor of future mathematical achievement, while Murphy, Mazzocco, Hanich, and Early (2007) found that students with mathematics learning disabilities continued to show deficits with the counting sequence up through 2nd grade. Despite these findings, there was limited investigation of these skills in the instructional studies, and no studies designed to improve students' performance in these skills.

Only two basic research studies (Garrett, Mazzocco, & Baker, 2006; Murphy, Mazzocco, Hanich, & Early, 2007) and none of the instructional studies measured students' positional place-value knowledge. Neither of these studies reported the students' accuracy in responding to place-value questions.

Strategic reasoning. This literature review found that many basic research studies on strategic reasoning investigated students' ability to solve word problems (see Table 8). Among the articles that included problem-solving, only three analyzed the students' problem-solving skills, while the others either used the students' problem-solving scores to classify students or subsumed these scores into overall mathematics achievement. All three of these articles investigated how many problems the students answered correctly, and found that students with learning disabilities were more successful at solving easy problems than complex ones. Fuchs and Fuchs (2002) found that accuracy rates for students with mathematics disabilities dropped from 75% for one-step story problems to 14% for multistep story problems, and for students with both reading and mathematics disabilities the accuracy rate dropped from 55% to 8%. Rodriguez, Parmar, and Signer (2001) found that for the students with learning disabilities the accuracy rates dropped from 17% for the easiest comparison difference unknown question to 0% for the hardest multistep change start unknown problems, compared to 67% and 14% respectively for the students in general education. Parmar and Signer (2005) also found that students with learning disabilities were more successful solving elementary questions that involved extracting or locating data on a graph than solving advanced questions that involved extrapolating information or stating relationships between data points.

Table 8

Number of studies that examined each aspect of strategic reasoning

	Basic Research, n=16	Instructional Studies, n=12
Solve Problems	8 (50%)	8 (67%)
Teacher Representations	9 (56%)	4 (33%)
Student Representations	1 (6%)	2 (17%)
Strategy Use	4 (25%)	1 (8%)
Formulate Problems	0 (0%)	0 (0%)

Many instructional studies also analyzed students' abilities to solve word problems (see Table 8). Among the articles that included problem-solving only three analyzed the students' problem-solving skills, and the others either subsumed the word problems into a larger mathematics achievement score or analyzed another aspect of the problem-solving context. Typical of other studies that used comparisons between interventions to investigate how to improve students' problem-solving, Fuchs, Fuchs, Hamlett, and Appleton (2002) found that small-group problem-solving tutoring that explicitly taught students to make connections between novel and familiar problems increased the students' ability to solve story problems correctly.

Three of the basic research articles and one instructional study investigated students' strategy use. Although each of these studies used the phrase "strategy use" to mean that the researchers examined how the students solved problems, they each had different definitions of what constituted "strategy use". For example, Rodriguez, Parmar, and Signer (2001) stated that strategy use was computation, measurement, counting, or drawing, whereas Montague and van Garderen (2003) stated that it was counting, using a benchmark comparison, or using decomposition and recomposition. The basic research studies found that students with learning disabilities used fewer valid strategies (Montague & van Garderen, 2003; Rodriguez, Parmar, & Signer, 2001) and utilized less strategic planning (Grobecker & DeLisi, 2000) than typically achieving students. The instructional study used students' strategy use to screen participants (Mancl, Miller, & Kennedy, 2012) rather than to teach the students a broader range of strategies. Future research should examine whether it is possible to teach students with learning disabilities to use a broader range of strategies.

Another topic that many basic research studies investigated was the students' ability to use mathematical representations presented by the teacher (see Table 8). Six articles examined students' responses to a teacher representation, and the others included students' use of teacher representations as part of the overall mathematics achievement data. The effects on students' use of teacher representations varied depending on the representation used. Students with mathematics learning disabilities had some success answering questions based on the visual appearance of bar graphs (Parmar & Signer, 2005) or visual models of fractions (Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013), but had difficulties using a number line to position numbers (Geary, Hoard, Nugent, & Byrd-Craven, 2008; Namkung & Fuchs, 2012; Rodriguez, Parmar, & Signer, 2001) or pegboard paper to copy geometrical shapes (Grobecker & De Lisi, 2000).

Despite the evidence from basic research that the form of the representation made a difference as to whether students with learning disabilities were able to use a representation as a mathematical tool, the instructional studies successfully taught word problem solving with schematic diagrams (Owen & Fuchs, 2002), multidigit calculations with base-ten blocks (Mancl, Miller, & Kennedy, 2012), and basic facts with objects, arrays, and number lines (Sealander, Johnson, Lockwood, & Medina, 2012; Woodward, 2006). With the exception of the number line, the efficacy of these representations has not been studied for students with learning disabilities; therefore future research should investigate how students with learning disabilities interpret and utilize base ten blocks, arrays, and schematic diagrams.

Although there was a relatively strong focus on students' use of representations, only one basic research study (Rodriguez, Parmar, & Signer, 2001) and two instructional studies (Seo & Woo, 2010; Woodward, Monroe, & Baxter, 2001) asked students to produce their own representations, and only one of these studies analyzed the students' representations. Rodriguez, Parmar, and Signer (2001) found that when students drew their own representations they attended to the context but not the mathematics in the problem. By attending to the students' representations these researchers found an important reason that students with disabilities have difficulties with word problems; future research into students' representations may lead to other important findings.

None of the articles had the students formulate the problems by writing their own word problems (see Table 8).

Conceptual understanding. Four of the basic research studies examined students' knowledge of counting principles, one examined their knowledge of length equality principles, and none examined their knowledge of arithmetic principles (see Table 9). The studies of counting principles found that the students with the most severe learning disabilities continued to have difficulties identifying counting errors even in 2nd grade (Murphy, Mazzocco, Hanich, & Early, 2007; Namkung & Fuchs, 2012). Grobecker and Lawrence (2000) also found that students with learning differences had difficulties conserving length. Despite the evidence that students with learning disabilities have difficulties with counting principles and conservation of length, no instructional studies attempted to teach these topics to the students.

Table 9

Number of studies that examined each aspect of conceptual understanding

	Basic Research, n=16	Instructional Studies, n=12
Counting & Length Equality Principles	5 (33%)	0 (0%)
Understanding Problems	3 (19%)	4 (33%)
Numerical understanding	2 (13%)	4 (33%)
Extending & Applying Concepts	1 (6%)	4 (33%)
Constructing Relationships	1 (6%)	2 (17%)

Three basic research studies and four instructional studies mentioned the importance of student understanding when solving word problems, but only one study analyzed student understanding. Parmar and Singer (2005) found that students with learning disabilities sometimes got the problems incorrect because they misinterpreted the problems, for example they interpreted “taller than” to mean “tallest”.

Basic research studies showed that students with learning disabilities have difficulties with understanding both computational (Montague & van Garderen, 2003) and fractional concepts (Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013), although they seem to perform normally on nonverbal calculation (Grobecker & Lawrence, 2000). Several instructional studies undertook to remediate the students’ lack of computational understanding using either the concrete-representational-abstract instructional sequence (Mancl, Miller, & Kennedy, 2012; Sealander, Johnson, Lockwood, & Medina, 2012) or an approach that integrated strategy instruction with timed tests (Woodward, 2006), but none of these instructional studies analyzed whether the students gained computational understanding. None of the studies attempted to teach fractional understanding or analyzed the place-value understanding of students with learning disabilities.

Fuchs and Fuchs (2002) found that students with learning disabilities had difficulties transferring their problem solving knowledge to real-world problems. Instruction focused on both teaching strategies and transfer strategies helped students extend their skills beyond the word problems (Fuchs, Fuchs, Hamlett, & Appleton, 2002; Owen & Fuchs, 2002) and multiplication facts (Woodward, 2006) they had been taught.

Although two instructional studies encouraged students to construct relationships between concepts, neither of these articles analyzed whether the students had constructed these relationships (Mancl, Miller, & Kennedy, 2012; Woodward, 2006).

Adaptive reasoning. Although aspects of adaptive reasoning were mentioned in four basic research articles and five instructional studies (see Table 10), the analysis of the students' adaptive reasoning was limited. Two basic research articles (Fuchs & Fuchs, 2002; Rodriguez, Parmar, & Signer, 2001) and five instructional articles (Baxter, Woodward, Voorhies, Wong, 2002; Fuchs, Fuchs, Hamlett, & Appleton, 2002; Owens & Fuchs, 2002; Woodward, 2006; Woodward, Monroe, & Baxter, 2001) mentioned that they asked for students' explanations of their problem solutions, but did not analyze the quality of the students' explanations. Two basic research articles (Fuchs & Fuchs, 2002; Parmar & Signer, 2005) and one instructional study (Fuchs, Fuchs, Hamlett, & Appleton, 2002) implied that students' logical thought about problems was an important aspect to consider, but none of these studies analyzed students' logical thought. There were no studies that examined student justifications for their responses.

Only Garrett, Mazzocco, and Baker (2006) analyzed the students' adaptive reasoning skills, in particular the students' reflective skills. They found that in comparison to typically achieving students, students with mathematics disabilities were less accurate at evaluating whether they could correctly solve a problem and whether solutions to a problem were correct or incorrect.

Table 10

Number of studies that examined each aspect of adaptive reasoning

	Basic Research, n=16	Instructional Studies, n=12
Explanations	2 (13%)	5 (42%)
Logical thought about situations	2 (13%)	1 (8%)
Reflections	1 (6%)	0 (0%)
Justifications	0 (0%)	0 (0%)

Mathematical disposition. Despite evidence that mathematical anxiety can negatively impact students' performance in mathematics (Ashcraft, Krause, & Hopko, 2007), only one of the basic research articles (Montague & van Garderen, 2003) and two of the instructional studies (Owen & Fuchs, 2002; Woodward, 2006) investigated the mathematical disposition of students with learning disabilities. These studies found that students with learning disabilities generally had positive attitudes about mathematics (Woodward, 2006), their competence in mathematics (Montague & van Garderen, 2003), and the specific intervention (Owen & Fuchs, 2002). Although these studies were uniformly positive about students with learning disabilities' mathematical dispositions, these studies represent a small sample of students and further research should engage further in studying this topic.

Proficiency in mathematics

Almost two-thirds of the basic research studies classified mathematical proficiency as being able to solve word problems correctly or achieving a good score on a mathematics achievement test. Mathematical achievement tests typically consist of a mixture of computation problems, words problems, and a few questions from other topics. Other definitions of mathematical proficiency included procedural knowledge, representational understanding, and conceptual knowledge.

The instructional studies also emphasized the ability to solve word problems correctly, with half of the studies classifying this as an essential part of mathematical proficiency. The instructional studies also emphasized computational fluency, with over half of the studies classifying this as part of mathematical proficiency. Other definitions of mathematical proficiency included mathematical reasoning, conceptual understanding, and strategy use.

Discussion

This systematic review was designed to examine the current status of the literature from the U.S.A. on learning disabilities and mathematics for students in elementary school.

The theoretical frameworks that underpin much of the mathematics learning disabilities research—cognitive learning theory, developmental psychology, and behaviorism—tend to favor quantitative studies, and use experimental, quasi-experimental or single-subject designs. These designs are excellent for examining how prevalent a characteristic is within a given population or the degree to which students' responses can be changed by instruction, but qualitative studies are better suited for clarifying the causes behind problems (Flyvbjerg, 2006). Now that quantitative studies have established that students with learning disabilities have difficulties with aspects of procedural fluency, the field would benefit from qualitative research that investigates how students interpret and make sense of problem situations, the meaning they give to their actions, and their affective responses to problems. Such qualitative work could provide more information about what underlies students' difficulties.

Procedural fluency is the most frequently studied strand in the mathematics learning disabilities literature, partially because this is the mathematical strand that is most easily recognized as "mathematics" and partially because this strand can be investigated quantitatively. This strand lends itself to quantitative study because the variables can be cleanly coded with a limited number of codes, for example an addition fact is either recalled correctly within 3 seconds or it is not.

Basic number fact knowledge is the most commonly studied topic within procedural fluency, appearing both in the basic research and the instructional literature. This emphasis on basic facts is a reasonable focus because numerous studies have demonstrated that students with learning disabilities have difficulties with recalling basic facts (Geary, Hoard, Byrd-Craven, & DeSoto, 2004; Jordan & Montani, 1997; Russell & Ginsburg, 1984).

All of the instructional studies that focused on teaching basic facts to students assessed how fast they could recall these facts. Measuring how many facts are recalled in one minute is an efficient way to produce a numerical measure of basic fact knowledge but it privileges a form of performance that many students with learning disabilities have difficulties with. Jordan and Montani (1997) found that students with specific mathematics difficulties had difficulties in retrieving basic facts in timed, but not in untimed conditions. Students with learning disabilities may have difficulties processing the language in the amount of time allotted to the recall of each fact in a timed condition, but be able to accurately reconstruct the fact when given sufficient time. Thus relying on a timed measure of basic facts is likely to underestimate how much basic fact knowledge students with learning disabilities have, and may constrain them from continuing on to more complicated mathematics. Future instructional studies designed to teach basic facts should include an untimed measure of students' basic fact knowledge, for instance by measuring students' strategy use.

There is a disconnect between the basic research literature and the instructional studies, in that while the basic research literature found that multidigit calculation, counting, and reading numerals were difficult for students with learning disabilities, few studies have tried to remediate these problems. Future instructional studies should address these research gaps.

Mastery of the place-value system is fundamental to developing conceptual understanding and procedural fluency with multidigit calculations (National Research Council, 2001). Despite the importance of the place-value system, very few articles examined positional place-value knowledge, and none reported on the accuracy of students' responses to place-value questions. Future basic research studies should investigate the place-value knowledge of students with learning disabilities, and if students prove to have difficulty with this concept, there should be instructional studies to improve students' place-value knowledge.

Strategic reasoning is the second most thoroughly investigated strand in the mathematical learning disability literature. Many of the studies investigated whether students correctly solved word problems, finding that students with learning disabilities were able to solve simple word problems, but had difficulties with more complex word problems. These difficulties with more complicated problems occurred because the students with learning disabilities accessed a limited number of strategies and used limited strategic planning when solving problems. Despite the fact that there were several instructional studies designed to improve students' problem-solving skills, none of these studies focused on improving students' range of strategies or their strategic planning. Future instructional studies should focus on increasing students' range of valid strategies and their strategic planning skills.

In the basic research studies there was a focus on students' responses to mathematical representations. Despite evidence from basic research that the form of a representation made a difference as to whether students with learning disabilities were able to use a representation as a mathematical tool, the instructional studies used a variety of representations that have not yet been investigated in basic research. Therefore future research should investigate how students with learning disabilities interpret and utilize base ten blocks, arrays, and schematic diagrams.

Most of the basic research on mathematical representations was focused on representations presented by the teacher rather than student-produced representations. This emphasis is partially due to a philosophy of learning that emphasizes explicit instruction (Gersten, 2009), in which the teacher models the use of a representation and the students are expected to utilize the representation in a similar manner. With this philosophy of learning students are not expected to produce their own representations or utilize representations in a novel manner. The emphasis on teacher-produced representations may also be because these representations are easier to investigate systematically than student-produced representations. For example, when investigating where students place a number on a number line, the distance between where they place the number and where it should be can be measured to produce a single number, whereas when students produce their own representations they can produce a wide variety of representations which may be difficult to code accurately. Despite the difficulties of analyzing student-produced representations, the one study that did analyze them produced the interesting insight that students were attending to the context but not the mathematics in the problem.

Therefore including analysis of student representations would be an interesting focus for future research studies.

Various aspects of conceptual understanding have been investigated in the mathematical learning disability literature, but none of them exhaustively and few studies have demonstrated a deep interpretation of conceptual understanding, with most studies investigating only one or two aspects of conceptual understanding. This limited focus may stem from a philosophy of learning that emphasizes task analysis and the breaking of complex tasks into simpler steps that can be better accessed by the learner (Allen & Schwartz, 1996).

One complex idea that can be made more accessible to the learner by identifying its component parts is counting and its principles. There are five counting principles that can be identified and evaluated separately: the one-to-one, stable order, cardinal, abstraction, and order irrelevance principles (Gelman & Meck, 1983). Consequently several basic research studies have investigated students' knowledge of counting principles and found that even in 2^{nd} grade students with learning disabilities may have difficulties applying counting principles. Despite this evidence, no instructional studies have yet addressed this issue.

Despite the complications quantifying numerical understanding, several basic research studies investigated this concept and found that students with learning disabilities have difficulties with understanding both computational and fractional concepts, but not nonverbal calculation. Several instructional studies then undertook to remediate the students' lack of computational understanding but did not analyze whether the students made gains in this area. None of the studies attempted to analyze or teach place-value understanding of students with learning disabilities.

Extending and applying was one of the few aspects of conceptual understanding for which the topics investigated in the basic research informed the instructional studies. Basic research showed that students with learning disabilities had difficulties transferring their knowledge of how to solve simple problems to real-world problems. Follow-up studies then found that instruction that taught both teaching strategies and transfer strategies helped students extend their skills beyond the word problems that they were already familiar with.

There has been limited research analyzing the adaptive reasoning of students with learning disabilities. Although several studies mentioned the importance of students' explanations they did not analyze the quality of these explanations. Future qualitative studies should examine the quality of students with learning disabilities' explanations.

There has also been limited research analyzing the mathematical disposition of students with learning disabilities. The studies that measured mathematical disposition were uniformly positive about students with learning disabilities' mathematical dispositions, but these studies represent a small sample of students and future research should engage in studying this topic more deeply.

As has been found in previous reviews of the literature, research studies on the mathematics skills of students with learning disabilities have privileged a view of mathematics proficiency that consists of accurate whole-number calculation and word problem solving (Miller, Butler, & Lee, 1998; Rivera, 1997). The predominance of these foci mean that there have been many areas of mathematics learning that have not yet been adequately explored in the mathematics education literature and that would be good future directions for research in the field.

As special education researchers learn what mathematics education researchers mean by the term "mathematical proficiency" and vice versa, they can both broaden their research agendas to become more inclusive: the special education field can include more aspects of mathematics proficiency than it has traditionally considered important and the mathematics education field can include students with disabilities within its research agendas. The ultimate beneficiaries of these broadened research agendas will be the students, as they get exposed to richer mathematics in richer contexts.

Limitations

In order to achieve a greater consistency of language and to ensure that the studies were investigating similar populations of students, this literature review was restricted to studies from the U.S.A. However this restriction did exclude important work on mathematical learning disabilities that is occurring in Europe, Israel, Canada, and China. Future studies, having resolved the different definitions of learning disabilities used in the different countries, should broaden the scope of the literature to include literature from outside of the U.S.A.

This literature review also included only studies that had a cut-off of less than the 11^{th} percentile or a district diagnosis of learning disabilities. This restriction was placed on the literature search to ensure that the studies were investigating similar populations of students, but it did have the effect of excluding the work of certain researchers who work in the field of mathematics learning disabilities, for example Jitendra and Xin.

Connection to dissertation

In this literature review I often found a discontinuity between the basic research and the instructional studies, because it was usually different research groups who conducted the two different types of study. I also found that the place-value understanding of students with learning disabilities had hardly been studied.

In my dissertation I addressed these problems by conducting both basic research and an instructional study into the place-value understanding of students with learning disabilities. In order to accomplish this I conducted a two-part mixed methods study. In the first part of the study I interviewed fifteen students with learning disabilities as I examined their place-value knowledge and how place-value knowledge impacted their ability to solve mental arithmetic calculations and word problems accurately. In the second part of the study I took what I had learned about the students' place-value knowledge during the interview and designed an instructional intervention to improve the base-ten numeration understanding of four of the students.

I believe that all the mathematical strands are highly intertwined and that it is not possible to examine one facet of mathematical proficiency apart from other facets. Therefore, while examining students' understanding of place-value, I also examined students' counting proficiency, their understanding of the context of word problems, their ability to solve these word problems accurately, their ability to represent these problems, and their ability to explain and justify their answers. Although I did not formally ask students about their mathematical disposition, during the course of my study this topic occasionally emerged.

SECTION 2: Whole-Number Place-Value Understanding For Students With Learning Disabilities

Most research on the mathematical knowledge of students with learning disabilities has focused on their procedural fluency, with limited focus on their conceptual knowledge. This article examines what students with learning disabilities understand about place-value. Fifteen 4th grade students with learning disabilities participated in individual clinical interviews where they were asked to solve a set of tasks designed to assess their understanding of place-value. Students with learning disabilities are a heterogeneous group, with five different profiles of understanding of place-value emerging from these fifteen students. Many of the students' errors in mental arithmetic, algorithmic calculations, and word problems were due to place-value misconceptions. They have a variety of difficulties with the concepts of place-value that affect their ability to solve mathematical problems that may not be solely captured by looking at their procedural knowledge of place-value. When studying or teaching students with learning disabilities and their understanding of place-value we need to focus on all the aspects of place-value, not just procedural aspects.

The Common Core State Standards for Mathematics (CCSM) (2010) emphasize the importance of place-value as an essential concept that develops throughout the elementary school years. Place-value knowledge is foundational for understanding our number system and for developing procedures for multidigit addition and subtraction (Fuson and Beckmann, 2012; Wearne, Hiebert, & Campbell, 1994), but it takes a long time to develop because it is multifaceted (Fuson, 1990).

Students with learning disabilities are often taught to follow procedural algorithms by rote, with little focus on the conceptual basis for the algorithms (Cawley, Parmar, Lucas-Fusco, Kilian, & Foley, 2007). These students often have considerable difficulties following algorithmic procedures to solve multidigit addition and subtraction problems (Jordan, Hanich & Uberti, 2003), but it is unclear how much of these students' difficulties is due to difficulties with retrieving rote sequential procedures (Fazio, 1996) and how much it is due to a lack of a conceptual knowledge of place-value upon which to build a firm knowledge of multidigit algorithms. Few researchers have investigated the place-value knowledge of students with learning disabilities, and those that have present contradictory viewpoints about it. Donlan & Gourlay (1999) found that there was no difference between students with specific language impairments and their age-matched controls in their ability to compare double-digit numerals, whereas Hanich, Jordan, Kaplan & Dick (2001) found that both students with reading difficulties and students with mathematics difficulties did have difficulties with place-value knowledge. These contradictory viewpoints may be due to the fact that these different researchers investigated different facets of place-value. In order to try and resolve these contradictory view points, in this study I investigated a broad range of the facets of place value while addressing my research question: What do 4th grade students with learning disabilities understand about whole-number place value?

Following a brief summary of relevant research literature and the research methods, I report on the results of individual interviews designed to learn what fifteen 4th grade students with learning disabilities understood about a broad range of the aspects of place value: base-ten numeration, place-value numeration, counting, and flexibility with decomposition and composition of numbers. From these results I outline five profiles of place-value understanding for students with learning disabilities and then investigate what implications my findings have for research and the teaching of students with learning disabilities.

Place-Value

Place-value is a fundamental concept in mathematics, which underlies the structure of our number system, and yet it takes a long time to develop. In the CCSM (2010) the concept of place-value is developed gradually from kindergarten through 6^{th} grade. In the beginning, students are focused on using the regularity of the number system to help them count. Then they learn to decompose and compose numbers, developing the idea that there are ten ones in ten, ten tens in one hundred, and that the digits in a two-digit number respectively represent the number of tens and ones needed to make this number. They also learn to use this place-value understanding to help them solve operations on multidigit numbers. In 2^{nd} grade students extend this knowledge to include the idea that ten hundreds make one thousand and that the digits in a three-digit number respectively represent the number of hundreds, tens, and ones in the number. By the end of 3rd grade students are expected to fluently add and subtract within 1,000 and multiply one-digit whole numbers by multiples of ten by using algorithms based on place-value. By the end of 4^{th} grade students are expected to generalize their place-value knowledge of whole numbers up to one million, while in 5^{th} grade the students expand their knowledge of place-value into decimals, and in 6^{th} grade students expand their knowledge into negative numbers.

In the CCSM (2010) there is an emphasis on students using place-value knowledge to help them solve standard algorithms in all four operations. Fuson and Beckmann (2012) argue that the term *standard algorithm* means any algorithm that relies on decomposing numbers into base-ten units, carrying out single-digit computations within each unit but maintaining the place-value of the resulting numbers, and then composing the answer from these individual computations. If this is the definition of a standard algorithm then all the operations for whole numbers and decimals can use the same method, and only one rule needs to be learned for each operation. When students have a strong understanding of the base-ten structure of numbers the standard algorithms become powerful tools.

Place-value is a multifaceted concept. Van de Walle, Karp, and Bay-Williams (2010) enumerate four aspects of place-value: base-ten numeration, place-value numeration, counting, and the flexible composition and decomposition of numbers.

When students understand *base-ten numeration* they understand that sets of tens (and multiples of tens) can be seen as single entities that can be counted. Students with a strong understanding of base-ten numeration can flexibly count sets of tens and count by tens without confusing these two actions. Base-ten numeration is a specific example of *unitizing*, which is the idea that one can count groups of objects as well as being able to count individual objects (Fosnot & Dolk, 2001).

Students who understand *place-value numeration* know that the position of digits in written numbers determines what size group they represent. This is what is generally thought of as place-value knowledge in school mathematics, and is often known as *positional knowledge*.

Counting involves enumerating objects by following a fixed sequence of words. This skill is an element of place-value knowledge because skilled counters recognize that numbers are formed by following regular patterns, and use this recognition to help them count large numbers (Callahan & Clements, 1984).

When students are able to use *flexible composition and decomposition of numbers* they are able to compose 2 hundreds 5 tens and 6 ones into 256, and decompose this number into 1 hundred 14 tens and 16 ones, 25 tens and 6 ones, or a variety of other groupings. This flexibility with numbers is essential for understanding regrouping within the standard algorithms. (Fuson & Beckmann, 2012).

Place-value development for typically developing students

The complicated nature of place-value means that children take a long time to fully develop knowledge of these concepts and there are a number of phases that children go through in their development of place-value concepts. Fuson et al. (1997) proposed a 5-stage model, the UDSSI model, for the conceptions students develop about 2-digit whole-number place-value:

1. *Unitary single digit*—Students learn the written and spoken numbers one to nine, and how to count these quantities.
    i. *Unitary multidigit*–Students count by ones and identify the number as a whole, so 53 consists of 53 ones and they count 1, 2…53
2. *Decades*— Students can state that the digit on the right is ones and the digit on the left is tens, but they do not understand the quantities each numeral represents and they count the tens by ones, e.g. they can identify that 53 is 5 tens and 3 ones but they count 1, 2,…50, 51, 52, 53.
3. *Sequence*— Students count by tens and ones, so they count 10, 20, 30, 40, 50, 51, 52, 53.
4. *Separate*— Students understand that the digits represent tens and ones, so 53 consists of 5 tens and 3 ones.
5. *Integrated*— Students move fluidly between sequence and separate conceptualizations, so they both know that 53 is 5 tens and 3 ones and they can count by tens and ones.

Several of these conceptions of place-value may be available to children at the same time and they may use different conceptions depending on the situation.

There are also several common misconceptions that students develop about place-value. Some students develop a *concatenated single digit* conception of place-value, meaning that that they interpret the digits in 53 as independent digits consisting of a five and three (Fuson et al., 1997). This viewpoint is particularly prevalent as students operate on multidigit numbers and causes many errors with the procedures for multidigit addition and subtraction (Fuson & Briars, 1990). Some students develop a *face value* conception of place-value, meaning that they do not require that the objects represented by the tens place be equivalent to ten of the objects in the ones place (Ross, 1990). This conception becomes problematic when non-canonical partitioning is used, such as when regrouping has occurred, so that children who could easily state that 5 tens and 3 ones was 53 would have difficulty stating how many objects were there when they were grouped as 4 tens and 13 ones. This misconception is particularly likely to occur when students always use pre-grouped materials, such as base-ten blocks, because then the student do not have think about how big the groups they are working with are.

As stated above, place-value conceptions take a long time to develop. Ross (1989) found that more than half of the students in her study were still developing whole-number place-value concepts in 4^{th} grade, with one-third of her sample having a face-value conception of place-value, and one student who still retained a unitary conception of place-value.

Place-value for students with learning disabilities

There has been little research on students with learning disabilities' understanding of place-value, and most of this research has examined their procedural place-value knowledge. Most of this research has examined students' place-value numeration knowledge. Studies examining place-value numeration have examined students' ability to identify digits in the tens and hundreds place, identify the place-value of particular positions in the numbers, state the smallest and largest 1-, 2-, and 3-digit numbers, compare numbers, or pick the larger of two numbers (Cirino, Fletcher, Ewing-Cobbs, Barnes, & Fuchs, 2007; Desoete & Grégoire, 2006; Donlan, Cowan, Newton, & Lloyd, 2007; Garrett, Mazzocco, Baker, 2006; Murphy, Mazzocco, Hanich, & Early, 2007; Wise et al, 2008). A few studies have examined students' base-ten numeration by asking them to state the number of tens in one hundred or one hundreds in one thousand (Garrett, Mazzocco, Baker, 2006; Murphy, Mazzocco, Hanich, & Early, 2007). Several studies have examined students' counting by asking them to state number word sequences or object count (Cirino, Fletcher, Ewing-Cobbs, Barnes, & Fuchs, 2007; Desoete & Grégoire, 2006; Donlan, Cowan, Newton, & Lloyd, 2007; Fazio, 1996; Houssart, 2001; Murphy, Mazzocco, Hanich, & Early, 2007).

Most of the research on the counting skills of students with learning disabilities and related disorders has focused on students in the primary grades. This research has found that students with mathematics learning disabilities or specific language impairments continue to show deficits with the counting sequence up through 3rd grade (Desoete & Grégoire, 2006; Donlan, Cowan, Newton, & Lloyd, 2007; Fazio, 1996; Murphy, Mazzocco, Hanich, & Early, 2007).

An exception to the focus on students in the primary grades is Houssart's (2001) study of the counting difficulties of a girl in year 5 (9- to 10-years-old). This girl had difficulties with extended counting, which meant that the girl had difficulties counting across tens and hundreds boundaries, counting forwards and backwards in different steps and from different starting numbers. These difficulties made it hard for the girl to be successful at various mathematics problems such as counting money, making change, measuring larger distances, and subtraction of 3-digit numbers. As students with learning disabilities tend to have counting difficulties in the primary grades, future research should examine how long they continue to exhibit these difficulties and what effect counting difficulties have on their understanding of more complex mathematical concepts.

The research on students with learning disabilities' understanding of procedural place-value is mixed. Donlan, Cowan, Newton, and Lloyd, (2007) found that 8-year-old students with specific language impairments had severe deficits in place-value knowledge when they compared numbers up to ten thousand but not when dealing with 2-digit numbers. Cirino, Fletcher, Ewing-Cobbs, Barnes, and Fuchs, (2007) found that 3rd and 4th grade students with mathematical difficulties had difficulties with identifying numbers and identifying digits in the tens or the hundreds place. This seems to contradict the finding that 8-year-old students with specific language impairments can compare 2-digit numbers (Donlan, Cowan, Newton, & Lloyd, 2007). This discrepancy may be due to a different population (students with different disabilities and living in different countries) or due to slightly different tasks (comparing numbers versus identifying numeral in the tens place). Desoete and Grégoire (2006) found that most 3rd grade students with mathematical learning disabilities had more difficulties with knowledge of oral numbers versus written numerals when judging which number is larger. So far the research suggests that students with learning disabilities have difficulties with procedural aspects of place-value, but it is not clear what aspects of place-value are most problematic.

Only a few articles have examined conceptual measures of place-value for students with disabilities (Cawley, Parmar, Lucas-Fusco, Kilian, & Foley, 2007; Desoete & Grégoire, 2006; Jordan & Hanich, 2000; Jordan, Hanich, & Uberti, 2003; Russell & Ginsburg, 1984). All of these researchers agree that students with disabilities have difficulties with some aspects of place-value, but due to reporting conventions it is hard to tell which aspects of place-value the students find particularly difficult.

Cawley, Parmar, Lucas-Fusco, Kilian, and Foley (2007) examined the conceptual knowledge of place-value in students with mild disabilities in primary, intermediate, and junior high self-contained classes. They asked the students to complete tasks that examined counting, base-ten numeration, place-value numeration, and the flexible composition of numbers in 2-digit numbers. They found that the students had difficulties with conceptual aspects of place-value even though they could answer procedural questions. This study only looked at students' place-value knowledge of 2-digit numbers, which led to ceiling effects with the intermediate and junior high students.

Russell and Ginsburg (1984) asked 4th grade students with mathematical difficulties to solve questions that focused on counting, base-ten numeration, and place-value numeration. These students seemed to have elementary concepts of base-ten notation but experienced difficulty in related enumeration skills, especially if large numbers were involved.

Jordan, Hanich, and Uberti (2003) asked several groups of 8-11-year-old students to solve problems that examined place value skills in base-ten numeration, place-value numeration, and flexible composition and decomposition of numbers. They found that students' difficulties with place-value depended on their diagnosis. They found that students with mathematical difficulties but no reading difficulties understood place-value but had difficulty with addition and subtraction of multidigit numbers, whereas students with reading disabilities but no mathematical difficulties did not understand place-value concepts but were better at following procedures for solving multidigit addition and subtraction problems.

In the report of these studies all the conceptual place-value tasks were reported together. It would be interesting to know which tasks the students found easy and which they found difficult.

Previous research into place-value knowledge has mostly focused on procedural place-value knowledge and has had mixed results. Those studies that did examine conceptual place-value aggregated the results of the place-value tasks and the scores of the students, which makes it difficult to determine whether there were group differences in which aspects of place-value the students found difficult. In this study I examined four aspects of the conceptual place-value knowledge of individual students with learning disabilities, including how they used their place-value knowledge to solve problems in all four operations, which is the major use of place-value knowledge in elementary school (Fuson, 2012). The study described in this section is basic research that will inform the design of the instructional study described in the following section.

Learning and Assessment

Before we teach mathematics to students we need to know what they already understand about the subject (Allsopp, Kyger, & Lovin, 2007), and this requires us to assess their knowledge. Most standardized assessments assess what the students already know independently, or their zone of actual development. However, Vygotsky (1978) posited that in order to assess for instructional purposes we should not assess what ideas have already developed but assess which ideas are in the process of development. He proposed the idea of the zone of proximal development as the "distance between actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance" (p. 86). Determining the zone of proximal development allows us to find out which ideas are in the process of maturing and are therefore in the ideal space for learning, whereas tests that measure the zone of actual development measures what has already been taught.

Ginsburg (1997) proposed clinical interviews as a powerful tool for assessing students and determining their zone of proximal development because, unlike in standardized tests or other written tests, it is possible to scaffold the children's understanding during a clinical interview. The main purpose of a clinical interview is to understand the thinking underlying a student's response to a task. In order to support the interviewer's understanding of the child's thinking the interviewer can check whether the student understands the question, ask questions that prompt the student to rethink their answer, and/or adjust the difficulty of the questions to match the level of the child's understanding. As an experimental method the clinical interview has a good history of replication, in that any competent interviewer can obtain similar results (Ginsburg, 1997).

The clinical interview is particularly appropriate for students with disabilities because these students may understand problems and numbers differently than typically developing students, and the clinical interview allows the interviewer to probe for these different understandings (Allsopp, Kyger, & Lovin, 2007). In this study I use clinical interviews in order to understand how students with learning disabilities understand and use place-value concepts.

Methods

Subjects

Fifteen fourth grade students attending an independent school for students with dyslexia and related language difficulties participated in this study during the 2012-2013 school year. All students at the school have received or are in the process of receiving a diagnosis of dyslexia or a specific language disability by a diagnostician who specializes in evaluating students with learning and/or language disabilities. The advantage of conducting this study at this independent school was that all the students in the school had language-related disabilities, whether in reading, writing, or oral language, so their disabilities had a commonality that I might not have found in a public school where students with disabilities in reading and students with disabilities only in mathematics are both placed under the broad category of specific learning disabilities. This school was in the Pacific Northwest of the United States of America.

All the participants spoke English as their first language. Six of the participants were female and nine were male.

Recruitment. The participant in this study were 4th grade students, most of whom have been in the school for at least a year. Therefore at the end of the 2011-2012 school year an email was sent to all the parents of the 3rd grade students introducing me as a researcher interested in learning disabilities, explaining the purposes of the study, and alerting the parents that the study would occur in the new school year.

At the beginning of the 2012-2013 school year a recruitment letter and the consent form were emailed to all the parents of the 4th grade students, with a request that this form be returned to the students' teacher. This letter was followed by a presentation at the school's Open House at the beginning of the school year, where the Vice-Principal introduced me and the purposes of my study.

These recruitment efforts resulted in the parents of 16 of the 28 fourth grade students signing the consent form. All but one of these students subsequently signed the assent form once they understood the purpose, procedures, risks, and benefits involved in the study. All fifteen of the remaining students were selected to participate in the study.

Procedures

The study reported in this section was the first phase of a larger study in which I first examined what students with learning disabilities understood about whole-number place-value and then designed an intervention based on the results of this interview. In this part of the study I investigated what the students knew about counting, base ten numeration, composing and decomposing numbers, and how they used this knowledge to solve computation and word problems.

In order to explore the students' understanding of place-value I conducted an individual clinical interview with each of the participants. These interviews took place in the school's tutoring center. The tutoring center consisted of eight cubicles, each consisting of two tables and several chairs, and some posters about phonics or other reading skills on the walls. The tutoring center was next to the music room, so occasionally we could hear other students playing music.

The entire interview consisted of 17 questions and took between 38 and 98 minutes, with most of the interviews taking between 50 and 70 minutes. If it seemed as if the interview was going to take more than an hour, I curtailed the interview between 43 and 61 minutes and then recommenced it several days later. The length of the first interview was more constrained by the demands of the school day such as lunch or specialists than by the students' ability to focus on mathematics. None of the students showed evidence of fatigue or inattention towards the end of their interview sessions.

The full interview had 17 questions, 11 of which I analyze in this section (see Appendix A). The first set of questions examined students' ability to count. First I asked the students to count 120 tiles and then represent their count (Schwedtfeger & Chan, 2007). Then students counted by tens as I placed completed tens frames on the table. There were thirteen completed tens frames so the students counted by tens to 130. Then students again counted by tens as I placed tens frames on the table but this time they started from 14 with one completed tens frame and one tens frame with only four dots. Again there were thirteen completed tens frames so they counted by tens from 14 to 134 (Wright, Martland, & Stafford, 2006). I included these questions because Desoete and Grégoire, (2006) found that many 3rd grade students with mathematical learning disabilities still have difficulties with the counting sequence, and I wanted to know whether these difficulties with counting by ones and by tens continued into the fourth grade. Students' responses to these questions gave an indication of their fluency and accuracy with verbal counting (see Table 11).

The tens frames tasks also gave information about which students were having difficulties with base-ten numeration. Some students switched units part way through the task suggesting they were not fluent with distinguishing counting by tens and counting by sets of tens.

Table 11

Summary of place-value aspects explored in each question

	Base-ten numeration	Place-value numeration	Counting	Decomposition and composition of numbers
Count 120 tiles			X	
Count Tens Frame	X		X	
Multiples of ten	X			X
Division by ten	X			X
Mental arithmetic		X		X
Word problems		X		X

The next set of questions consisted of word problems designed to expose the students' knowledge of the base-ten system, involving either multiples of ten or division by ten. The first question was “There are 10 candies in a pack. How many candies are in X packs?” There were five variants on this problem: (a) How many candies are in 4 packs? (b) How many candies are in 10 packs? (c) How many candies are in 13 packs? (d) How many candies in 20 packs? (e) How many candies if there are 15 packs and 4 single pieces? (Hiebert & Wearne, 1996). The second question was “Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?” These questions gave information about which students were fluent at composing and decomposing numbers or using base-ten numeration, and which students used recall or other strategies that broke down above one hundred.

The next set of questions I analyzed were the mental arithmetic problems. I asked students to solve the following problems mentally and explain how they solved the problem: (a) 38+10= (b) 38+24=, (c) 30–16=, (d) 33–16=. The last set of questions I analyzed were five word problems: a Join Result Unknown problem (JRU), a Separate Result Unknown problem (SRU), a Join Change Unknown problem (JCU), a Comparison problem, and a Grouping problem (Carpenter et al., 1999). These problems gave information about which students were comfortable with using their place-value knowledge to flexibly compose and decompose numbers in the context of problems, and which students preferred to use an algorithm. If the students used an algorithm these questions also gave information about their place-value numeration skills.

For all of the word problems, the students had a selection of manipulatives available to use if they wanted to. These manipulatives included connecting cubes stacked in towers of ten, a hundreds number chart, base ten blocks, tens frames, and colored tiles. I read the students the word problems, asked them to retell the problem, and then asked them how to solve it. The problem was printed so that the students could refer back to the problem as they solved it.

After they responded to each question, I asked follow-up questions to elicit more detailed descriptions and explanations of their strategies. I asked questions such as, "How did you figure that out?", "Can you do it out loud?", "Can you show me how you did it?", "Is there another way to solve this problem?", or "How do you know?" (Ginsburg, 1997).

Data Collection & Analysis

I analyzed the students' responses to the questions for accuracy, strategy use, and errors. This analysis helped me understand how much the students knew about counting, base-ten numeration, place-value numeration, composing and decomposing numbers, and how they used this knowledge to solve computation and word problems.

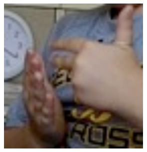
I video-recorded the clinical interviews. As the students were working I documented whether the students got the problem correct, where the students made errors, the students' strategies, the tools used by the students, and other relevant comments on an interview form. After each interview I summarized the student's explanations and made a content log of the videos for each student with the time when each question occurred in the video and notable incidents that occurred within each question.

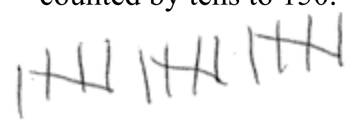
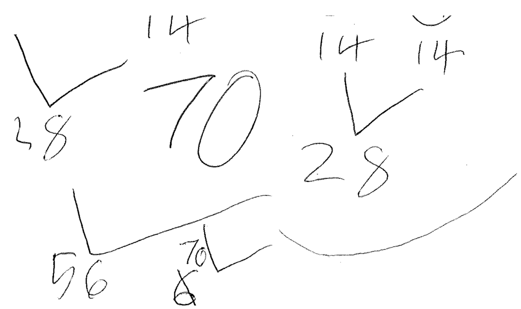
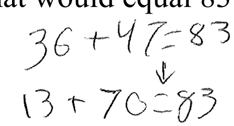
I coded the strategies of the word problems using the classification system of Cognitively Guided Instruction (Carpenter et al., 1999). I used the following codes: direct modeling by ones, counting by ones, skip counting, repeated addition, direct modeling by tens, counting by tens, combining tens and ones, incrementing, compensating, partial products, multiplying up, direct place value, rule, standard algorithm, recall, none, invalid, incorrect invented algorithm, or other (see Table 12 for examples). Once I had assigned strategies to each problem, I documented whether the students used place value knowledge to solve the problems and whether their errors were due to incompletely understood place value concepts.

From the interview videos I transcribed 92 responses to questions where students made errors and 29 responses where they got the problem correct. I transcribed all incorrect answers, but only transcribed correct answers when the students said something particularly interesting or when their responses to a question contrasted with their answers on a similar problem. As I transcribed I adapted conventions commonly used in interaction analysis (Schegloff, 1997) to fit the mathematical context of the questions (see Table 13). I used these conventions because they focus attention on both the words and actions of the interviewees and their surroundings. The words, hesitations, actions, manipulatives used, and mathematical representations used were important clues to how the students understood the mathematics and how confident they were about their answers. I made some adaptations to the standard notations of interaction analysis. I used "," to separate the numbers in the counting sequence. I used "…" to mean that the counting sequence was correct between the stated numbers, instead of its normal meaning of pause. I used this unconventional use of "…" in order to focus the readers' eyes on the errors rather than taking up space and energy by writing out all the correct numbers. I used numerals when the students said the numbers in the standard manner, but wrote them out in words if the student said something other than the standard number sequence. Again, I used this notation to help focus the readers' attention on the errors.

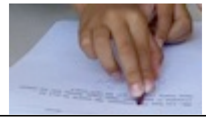
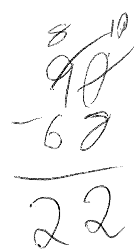
Table 12

Codes used to assign strategies to mental arithmetic and word problems

Codes	Specific Codes	Description	Examples of strategies	Problem
Beginning-Level Strategies	Direct modeling by ones	Student modeled the problem with blocks or pictures representing every numeral with a block or a picture. Student then counted every block or picture by ones	Student made 5 groups of 14 tiles. He counted by ones from 14 to 70. 	Ben has 5 packs of markers. There are 14 markers in each pack. How many markers does he have altogether?
	Counting by ones	Student started counting by ones from the second set, or student counted back by ones.	"So 38 and then it got to 81." Student put a block on 38 and a block on 81. She counted on the hundreds chart from 38 to 81, "1...43[2]. I think his Mom gave him 43". 	Jayden had 38 marbles. His Mom gave him some more marbles for his birthday. Now Jayden has 81 marbles. How many marbles did his Mom give him?
	Skip counting	Students skip counted by a number other than ten. They used fingers to keep track of how many times they counted.	"14 times 5 or 5 times 14. 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70. [Kept track on his fingers]" 	Ben has 5 packs of markers. There are 14 markers in each pack. How many markers does he have altogether?

Beginning-Level Unitizing Strategies	Direct modeling by tens	Student modeled the problem with blocks, representing groups of ten with a stick of tens. Student counted by tens.		Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?
	Counting by tens	Student counted by tens. They usually used tallies, blocks, or their fingers to keep track of how many times they counted.	Student made tally marks as she counted by tens to 150. 	Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?
More Mature Strategies	Repeated addition	Student used repeated addition to solve the grouping problem.		Ben has 5 packs of markers. There are 14 markers in each pack. How many markers does he have altogether?
	Combining tens and ones	Student added the tens and the ones separately, and then added the results of these two problems together.	"So 36 plus 47 equals so 3+4 is 7 6+7 is that would be thirteen, so thirteen plus that would equal 83." 	Sarah had 36 pennies. Her Mom gave her 47 more pennies. How many pennies does she have in all?

Incrementing	Student started with one of the numbers and then added or subtracted from that amount in progressive jumps.	38+20=58 58+4=62	38+24
Compensating	Student started with one of the numbers and made an easy jump, which they then adjusted to get to the number they were supposed to be working with.	"22 books so basically I knew that ninety minus thirty would be sixty so I took 8 of those away and it came up with 22." {90-30=60, but needed 68, so need 8 less than 30 which is 22}	Jim read 68 books. Brian read 90 books. How many more books did Brian read than Jim?
Partial products	Student solved the problem by breaking number into parts, and multiplying each section separately.	"So that's 5 multiply 14. So then I'd pretend it's ten and ten five times is fifty so then it's 50 and then 4 times would be 4, 8, 16 8 plus 8 is 16, so then 16 plus 4 is 20, so then I'd do plus 20 equals 70. So he had 70 markers." 5 x 14 50+20 =70 10	Ben has 5 packs of markers. There are 14 markers in each pack. How many markers does he have altogether?
Multiplying up	Student solved the division problem by multiplying by the divisor until as close as possible to the dividend. Student needed to keep track of how many times they multiplied by the divisor in order to find the quotient.	"He'll need fifteen boxes and he'll have eight left over. It's really easy because you just go ten ten ten because there's a hundred and then you got to go to the fifties and since eight is not actually a ten, that makes it eight left over" {100 crayons in 10 boxes, 50 crayons in 5 boxes. 10+5 is 15 boxes. There are 8 crayons left over.}	Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?

	Direct place value	Student looked at the number and immediately knew how many groups of tens were in the number.	Covered up the 8 with his pen and said, "Covered up the eight and I got fifteen so there's fifteen boxes and there's eight left." 	Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?
Procedural Strategies	Rule	Student used a rule for multiplying by ten. 10*n=n0	"Write down four and then you'd take the zero from ten cos it's ten times four and you'd take the zero from ten and you'd put it next to the four and then it would say forty."	There are 10 candies in a pack. How many candies are in 4 packs?
	Standard algorithm	Student used a standard algorithm to solve the problem.	"I'm going to do 90 at the top minus 68. So zero minus eight you can't do that so if there's more on the floor go next door. That's ten. Take away one from the nine that'll be eight and then cross out the zero and make it a ten and now since I borrowed now I'm going to do eight minus ten. So ten minus one, two, three, four, five, six, seven, eight [Counted back on his fingers] that's two. And then eight minus six one, two, three, four, five, six [Counted back on his fingers] so that's two again. So I think Brian read 22 more books than Jim." 	Jim read 68 books. Brian read 90 books. How many more books did Brian read than Jim?

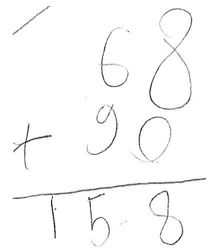
	Recall	Student heard the number and immediately knew how many pieces of candy there were.	Student immediately responded “Forty pieces of candy”.	There are 10 candies in a pack. How many candies are in 4 packs?
Incorrect Strategies	None	Student gave no answer to the problem.	“I don't I would I I can't really remember what the equation would would be to make that thi— to make the answer so I would need to do it on paper.”	33-16
	Invalid	Student used a strategy that cannot result in the correct answer.	For the comparison problem the student added the numbers in the problem. 	Jim read 68 books. Brian read 90 books. How many more books did Brian read than Jim?
	Incorrect invented algorithm	Student used an invented algorithm to solve the problem.	“You would need a hundred and nine boxes but um six are left over.” {hto/10=h0t+o}	Mr. Lin has 196 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?
	Other	Student used some other strategy.	“Two hundred. Well I thought that since thirteen was thirty then I thought that twenty was twenty but then I’m like no because twenty is a higher number than thirteen.”	There are 10 candies in a pack. How many candies are in 20 packs?

Table 13

Summary of Transcript Conventions. Adapted from Schegloff (1997).

Symbol	Interpretation
,	Denotes tagging or that a new tens frame was placed down
…	Means that the counting sequence was correct in between the numbers.
[ ]	Descriptions of actions
.	Downward intonation
::	Lengthening of the previous sound. The more colons, the greater the lengthening.
?	Upward intonation
!	Sudden increase in volume
—	Cut-off of the previous sound.
( )	Problematic or uncertain hearings
{ }	Teacher's understanding of unclear statement
[speaker's initials: words]	Overlapping speech

After transcribing the counts by ones and tens, I coded for student errors in sequencing, base-ten numeration, writing numerals, one-to-one correspondence, and errors due to attention (see Table 14). Having coded for these errors I further coded for whether the sequencing errors occurred at the places where children frequently make errors such as at the decades boundaries or above one hundred, or because the student started counting at the wrong number. I also coded for whether the one-to-one correspondence errors seemed related to the student's uncertainty with the number system or seemed to be more related to attentional issues.

After transcribing the mental arithmetic and word problems I coded for the types of errors students made when they solved the problems incorrectly (see Table 15). I coded these errors as counting error, base-ten numeration error, place-value numeration error, composition and decomposition of numbers error, tracking error, or invalid. The counting errors were further coded as sequencing or one-to-one correspondence errors. The base-ten numeration errors were further coded as sole units or units errors. The place-value numeration errors were further coded as independent digits, independent units, algorithm, or value of digits errors. One problem could have more than one type of error.

Table 14

Codes used during the analysis of the counts by ones and tens

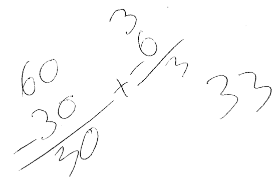
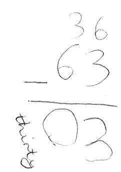
Code	Description	Example
Place-value Errors		
Sequencing Error	Student stated a number that does not come next in the counting sequence.	"87,[3] 86, 88, 87"
Decades Error	After a number ending in 9 the student stated a number that does not come next in the counting sequence. The error occurred below one hundred.	"78, 79, 30"
Hundreds Error	Student stated a number that does not come next in the counting sequence. The error occurred at one hundred or above.	"14, 24…104, 204"
Starting Error	Student started with an assumption that there were more then ten dots within each tens frame.	HT: So how many is that? [Indicating a tens frame] A: Twelve
Base-ten Numeration Error	Student started counting by tens but then switched to counting the tens as ones.	"14, 24, 25, 26, 27"
Writing Numerals Error	Student wrote the numeral incorrectly	Said 120, but wrote 112
Other Errors		
One-to-one Correspondence Error	Student took two or more tiles for one count, or counted one tile more than once.	"16, [Dropped 2 tiles] 17"
Attention Errors	Errors that seemed unrelated to student's knowledge of the count sequence, and seemed more related to distractions within the environment.	"69, 70, 70 [Dropped tile on floor, picked it up and continued counting] 70, 71"

[3] "," denotes tagging.
"…" denotes that the counting sequence is correct between the numbers that are written.

Table 15

Codes used during the analysis of the incorrectly solved mental arithmetic and word problems

Type of Error	Description	Example	Problem
Place-value Errors			
Counting			
Sequencing Error	Valid strategy but error in counting sequence.	5, 10, 15, 20, 25, 30, 35, 40 45, 50 55 60 75 80 [Skipped 65 and 70]	Ben has 5 packs of markers. There are 14 markers in each pack. How many markers does he have altogether?
One-to-one Correspondence Error	Valid strategy but skipped a cube when counting.	Built 5 towers of 14 cubes. Counted, “1…17 [Skips a cube], 18…21”	Ben has 5 packs of markers. There are 14 markers in each pack. How many markers does he have altogether?
Base-ten Numeration			
Sole Units Error	Valid strategy but student focused on only one of the units.	“This is five that makes fifty. But I still have to add the fours four five times. Okay so ei:ght plus ei:ght okay so it would be sixteen. So sixteen plus four twenty is the answer. Twenty is my main answer.”	Ben has 5 packs of markers. There are 14 markers in each pack. How many markers does he have altogether?
Units Error	Valid strategy but student switched units mid-way through the problem.	||||||||||||||||||| 150 boxes 8 crayons left over. Student counted to 150 by tens making a	Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?

Place-value Numeration			
Independent Digits	Student treated the digits in the numbers as if they were independent, meaning they added each digit separately. The digits had no positional value.	"Ninety-two. I started with eighty-one. Eighty-three more." {81+8+3=92}	Jayden had 38 marbles. His Mom gave him some more marbles for his birthday. Now Jayden has 81 marbles. How many marbles did his Mom give him?
Independent Units	Valid strategy but student acted on the tens independently from the ones.		I had 63 video games. I gave 36 video games to Jordan. How many video games did I have left?
Algorithm Error	Made a mistake when using standard algorithm that shows not using positional value of digits.		I had 63 video games. I gave 36 video games to Jordan. How many video games did I have left?
Value of Digits Error	Valid strategy but counted the tens as if they were ones.	 6+7=13, 3+4=7	Sarah had 36 pennies. Her Mom gave her 47 more pennies. How many pennies does she have in all?
Composition and Decomposition of Number	Student used incorrect invented algorithm.	"One hundred and thirteen" [10*n=100+n]	There are 10 candies in a pack. How many candies are in 13 packs?

Other Errors Tracking Error	Valid strategy but student had difficulties keeping track of the numbers in the problem.	 "[Under breath counts by tens using fingers to keep track] 10, 20, 30…170, hundred and eight—, hundred and nine—"	Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?
Invalid	Strategy cannot result in correct answer. Error involved not understanding the problem context and often involved choosing the wrong operation.	"So is it ten plus thirteen?"	There are 10 candies in a pack. How many candies are in 13 packs?

Results

In this section I present what the 4th grade students with learning disabilities who participated in this study understood about whole-number place-value. First I examine the results across the whole group, and then I look at the responses of five individual students who illustrate the different types of place-value understanding that emerged in the course of the study.

Whole group

Most of the students in this study were able to solve the **multiples** of tens problems less than or equal to one hundred, and many students could solve both the mental addition and addition word problems correctly. However, many students found the counting tasks, multiples of ten above one hundred, mental subtraction, and subtraction, multiplication, and division word problems difficult (see Table 16). Particularly in the counting tasks and the word problems many of the errors that the students made were due to place-value errors.

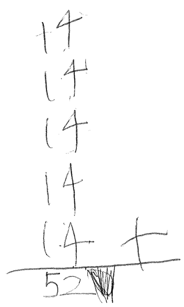
Table 16

Accuracy rates on each problem

Problem Type	Accuracy %	% incorrect answers due to place-value errors
Count tiles	40	89
Count Tens Frames	57	82
Multiples of ten, $n \leq 10$	93	0
Multiples of ten, $n > 10$	62	41
Division by ten	27	82
Mental addition	77	67
Mental subtraction	47	44
Addition word problems	67	60
Subtraction word problems	51	68
Multiplication word problems	33	50

Most students automatically knew the answers to the multiples of tens problems that were one hundred or less, but only 9 students could solve the problem with 13 packs of candy, 6 students could solve the problem with 15 packs of candy and 4 singles, and 4 students could solve the division by ten problems correctly. Although most of the students knew that it is possible to count by tens and that the number of sets of ten informs one of the total number of objects, the extent of their base-ten knowledge was limited, particularly with numbers larger than one hundred.

As place-value numeration was not directly measured it only became apparent when students used written algorithms. Only 8 students used written algorithms during the interview and 7 of these students made positional place-value errors as they completed the algorithms. For example, Rosie[4]'s solution for five 14s has several examples of positional place-value errors. First she wrote five fourteens vertically, and added the five ones to get five, which she wrote in the tens place. She then added the five fours to get twenty, which she wrote to the right of the 5, for an answer of 520.



This answer seemed too large to her so she crossed out the 0 in 520 saying, "It's fifty-two. Because zero means nothing so the zero technically isn't there." In this problem Rosie made two positional place-value errors: she did not combine the tens and ones and she did not appreciate the value of zero as a placeholder in our number system. Most of the students who used written algorithms made positional place-value errors as they completed the algorithms.

[4] All names are pseudonyms.

Eleven of the fifteen students made number sequencing mistakes either when counting by tens or by ones. Nine students counted the 120 tiles inaccurately, with eight of them making number sequencing errors. For example, Paul made a number sequencing error at the decade boundary when he counted, “79, 30, 31”. In the tens frames tasks, three students made errors when counting multiples of tens and six students made errors when counting on by tens from fourteen. For example, Abigail made a number sequencing error when she counted by tens counting, “14, 24, 34, 54”. Most of these fourth grade students with learning disabilities had difficulties counting accurately.

The students’ accuracy on the problems measuring their flexibility with numbers varied depending on the context of the problem. Most students were able to compose numbers when the process involved multiples of ten up to one hundred. Few students could decompose a 3-digit number into tens and ones.

Profiles of place-value understanding

A lot of variability was apparent when I examined the results of the individual students. Students with learning disabilities appeared to be a heterogeneous group with regard to their place value understanding, with five profiles of place value understanding emerging from the analysis (see Table 17): (1) Good place value understanding; (2) Reliance on algorithms; (3) Counting errors; (4) Beginning place value understanding; and, (5) Concatenation of digits.

Individual students also showed variability across the different aspects of place-value. For example, several students showed strengths in the flexible composition and decomposition of numbers, but showed difficulties with the concept of base-ten numeration (see Table 17). The characteristics of these place-value profiles are summarized in Table 18 and then explained in more detail in the following section.

Table 17

Students' performance in each of the place-value aspects

Student	**Base-ten numeration**	**Place-value numeration**	**Counting**	**Flexible composition and decomposition of numbers**	**Profile group**
Lewis	[5] Green	Green	Green	Green	Good place-value understanding
Sydney	Green	Green	Green	Green	Good place-value understanding
Luke	Green	White	Yellow	Green	Good place-value understanding
Hunter	Yellow	White	Green	Green	Reliance on algorithms
Robbie	Yellow	White	Green	Green	Reliance on algorithms
Kyle	Yellow	White	Red	Green	Counting errors
Abigail	Yellow	White	Red	Yellow	Counting errors
Sam	Yellow	White	Red	Yellow	Counting errors
Paul	Yellow	White	Red	Yellow	Counting errors
Kennedy	Yellow	Red	Green	Yellow	Beginning place-value understanding
Lily	Yellow	Red	Red	Yellow	Beginning place-value understanding
Robin	Yellow	Red	Red	Yellow	Beginning place-value understanding
Rosie	Yellow	Red	Red	Red	Beginning place-value understanding
Kevin	Green	Red	Yellow	Green	Concatenation of digits
Francesca	Red	Red	Yellow	White	Concatenation of digits

[5] Green represents good understanding of this aspect of place-value, yellow represents mixed understanding, and red represents poor understanding. White means that the student didn't access this aspect of place-value while answering the questions.

Table 18

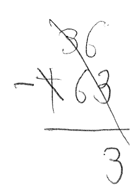
Characteristics of the place-value understanding profiles

Profile	Characteristics of place-value profile
Good place-value understanding	• Demonstrates base-ten numeration understanding • Demonstrates ability to compose and decompose numbers flexibly • Errors due to language or attention difficulties
Reliance on algorithms	• Solves problems using algorithms • Counts fluently and accurately • Demonstrates flexible composition and decomposition of numbers • Forgets steps in algorithms • Overgeneralizes rules • Confuses counting by tens and counting sets of tens
Counting errors	• Demonstrates some knowledge of base-ten numeration • Demonstrates some decomposition and composition of numbers • Reliance on beginning-level strategies • Frequent miscounts
Beginning place-value understanding	• Demonstrates some knowledge of base-ten numeration • Demonstrates some decomposition and composition of numbers • Errors due to positional place-value errors when using standard algorithms • Errors due to limited base-ten numeration understanding • Errors due to limited understanding of decomposition and composition of numbers
Concatenation of digits	• Acts upon the digits in multidigit numbers as if they were independent and valueless • Inconsistent conceptions of multidigit numbers that depend on the context of the problem

Good place-value understanding. Even students who demonstrated good place-value understanding across all problems did not respond to all of the questions correctly. However, their errors tended to be related to language or tracking difficulties rather than errors due to a misunderstanding of place-value.

Reliance on algorithms. Robbie is an example of a student who showed strengths in the flexible composition and decomposition of numbers, but had a tendency to rely on imperfectly remembered algorithms or overgeneralized rules when solving addition and subtraction problems.

When Robbie solved the mental subtraction problems he showed flexibility to decompose numbers. For example, when solving 30-16 he used an incremental strategy in which he broke the 16 up into 10 and 6, calculating " Thirty minus ten is twenty and twenty minus six one, two, three, four, five, six [Counting on his fingers] fourteen."

While Robbie was able to correctly compute mental subtraction problems using decomposition of numbers, for many of the other problems he used standard algorithms in a procedural manner that did not use his flexibility with numbers. This was not an effective strategy for Robbie, because it led him to make errors. For example, Robbie relied on the standard subtraction algorithm for completing the Separate Result Unknown (63 video games, gave 36 away) problem but inverted the numbers and wrote . This made it difficult for him to complete the problem because he was not sure how to solve 3-6 in the tens place. In this case, Robbie had forgotten that the first step of the algorithm is to place the bigger number above the smaller number.

Robbie made errors with the mental addition problems because he overgeneralized a rule that he had learned for adding ten to a single digit. When Robbie solved the problem 8+10 he used the following rule, “if you take away the zero and replace it with the eight it's eighteen,” which can be represented as 8+1~~0~~=18. In the problem 38+10, Robbie started to use the same rule when he replaced the zero with thirty-eight to get one hundred thirty-eight, or 38+1~~0~~=138. As Robbie completed this problem he realized that 138 did not make sense so he revised his answer using an incrementing strategy, “thirty plus ten is forty so that's forty-eight.” However in the 38+24 problem, Robbie’s reliance on the rule trumped his knowledge of number. He correctly worked out that 8+4=12, and 20+30=50 but then had difficulties combining these two numbers. He tried to add 12+3~~0~~=312, although he knew that this could not be the correct answer because 312 was too large a number. Robbie used his knowledge of number to reject the answer he got from using the rule, but then failed to use this knowledge to complete the problem. With this problem, Robbie’s overgeneralization of a rule overcame his flexibility with numbers.

Robbie’s reliance on algorithms may have partially been due to an incomplete grasp of base-ten numeration skills. Although Robbie used counting by tens as a strategy to solve several problems, he showed an incomplete grasp of base-ten numeration skills when solving the division by ten problem. In this problem he switched units mid-way through the problem; he started out by figuring out how many sets of ten he needed by writing a tally as he counted by tens to 150, but when he answered the question he gave the total of 150 as the answer. Robbie confused counting sets of ten and counting by tens. This confusion may have made Robbie reluctant to rely on his place-value knowledge and more likely to rely instead on algorithms.

The students in this group tended to use procedural algorithms and rules to solve problems, rather than using their flexibility at decomposing and composing numbers. Mathematics lessons in schools often emphasize the use of algorithms and so these students are “doing school” when they use algorithms. However, as students with learning disabilities often have difficulties remembering all of the steps in an algorithm (Allsopp, Kyger, & Lovin, 2007), they frequently made errors when following procedural algorithms. The students in this group also had difficulties tracking whether they were counting sets of tens or counting by tens.

Counting errors. Sam is an example of a student who made multiple counting errors during the counting tasks and when solving problems. He made six sequencing errors as he was counting the 120 tiles; these errors seemed to be related to attentional issues because most of them occurred mid-decade when he was searching for a particular tile or when another teacher entered the room.

Although in the counting task Sam’s counting errors seemed to be related to attention, elsewhere in the interview he made counting errors that suggested that his knowledge of the counting sequence was insecure. As Sam solved the division problem by counting out 158 tiles in groups of ten he counted, “1…69, seven—eighty, 81…158”, skipping the seventies decade. Unlike during the counting task, Sam did not seem to be distracted by anything other than the task of keeping track of how many tiles there were in each group at the same time as keeping track of how far he had counted.

When solving the word problems Sam resorted to beginning-level strategies, using counting by ones for all the subtraction word problems and direct modeling by ones for the grouping and the division problems, and never used based-ten numeration skills to solve any of the problems. Sam’s reliance on these beginning-level strategies compounded his difficulties with counting, because he had to keep track of many more numbers when counting or direct modeling by ones than if he had used unitizing skills.

Sam also relied on procedural algorithms that were not grounded in an understanding of place-value. For example, Sam stated that 13 packs of candy would have 113 pieces of candy in it. Although he would not have expressed his procedure in arriving at this answer in an algebraic expression, the procedure he used can be expressed as $10*n=100+n$ where $n>10$. This procedure is more than an aberration as it was also used by three other students—Abigail, Rosie and Kennedy—and was used by Sam several times during the second phase of the study.

Despite Sam's reliance on beginning-level strategies to solve word problems Sam showed flexible decomposition and composition of numbers in the mental arithmetic problems by using the strategy of combining tens and ones. For example, when solving 38+24, he first decomposed the numbers into tens and ones, by adding the ones 8+4=12 and the tens 30+20=50, and then composed these two answers into the final answer 12+50=62.

The students in this group all made miscounts during both the counting tasks and the word problems. When solving word problems their difficulties with number sequences were emphasized by their choice of beginning-level strategies that required extensive counting. These students chose to use beginning-level strategies even though they had demonstrated some capacity to use base-ten numeration and flexible decomposition and composition of numbers earlier in the interview.

Beginning place-value understanding. Robin is an example of a student who had a beginning level of place-value understanding. She had some knowledge of the decomposition and composition of numbers, base-ten numeration, and using patterns in the number system, but with harder problems the weaknesses in her understanding became apparent.

Robin showed flexibility with decomposition and composition of numbers when she used partial products to solve the 20 packs of candy problem, saying, "at first you had ten packs so that would be a hundred and there's twenty packs so that'd be a hundred more. One hundred more and that's two hundred". She used her knowledge that ten packs of candy would be one hundred pieces of candy and that there are two tens in twenty to decide that there were two hundred pieces of candy. She showed flexible decomposition and composition of numbers as she decomposed twenty into two tens and then recomposed the numbers into two one hundreds.

However with harder problems that involved both packs and singles Robin added the numbers in the candy problems rather than multiplying them. She added the numbers in the problem for the 15 packs and 4 singles problem saying, "fifteen plus four that'd be nineteen," instead of (15x10)+4. Robin showed that she had an incomplete grasp on place-value; when the problems were easy she had valid strategies to solve the multiples of tens problems, but when the problems got harder she did not use a valid strategy and resorted to a simple adding the numbers strategy.

When solving the word problems, Robin tended to use beginning-level strategies that used a beginning knowledge of base-ten numeration. She often used direct modeling by ten to solve the word problems, which showed that she understood that she could group and count objects by ten.

However, her knowledge of base-ten numeration was insecure. When counting by tens from 4 Robin showed that she had some difficulties with base-ten numeration. She counted, "14, 24, 25, 26, 27", making a switch of units from counting by tens to counting by ones. She had difficulties distinguishing between counting by tens and counting sets of tens.

Switching units was a mistake that Robin repeated when solving the division problem. She correctly direct modeled by tens up to a hundred, putting ten sticks of ten in a box and saying, "that's ninety, a hundred", but she had difficulties with direct modeling this problem beyond one hundred. First, she counted out five single cubes but realized that this made "a hundred and five", so she added five more single cubes to the first lot and declared this to be "fifty". She switched units midway through the count as she went from counting tens sticks by ten to counting single cubes by ten, although it is not clear how she then decided that 11 tens sticks became 150.

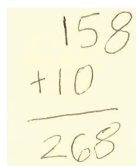
When solving 33-16 Robin made a mistake with the standard subtraction algorithm, she subtracted the smaller digit from the larger digit. She explained it as "six minus three is three and one minus three is two" giving 23 as the answer. This error reflected an inaccurate procedural knowledge of the standard subtraction algorithm that may have indicated a lack of understanding of place-value numeration (Allsopp, Kyger, & Lovin, 2007).

Robin also had difficulties counting by tens beyond one hundred. When counting by tens from 4, Robin counted, "74, 84, 94, 104, 204". She was trying to use a pattern to help her count by tens from a non-decade number, but was not sure whether the patterns affected the tens or the hundreds place.

The students in this group had a beginning level of place-value knowledge; they used direct modeling by tens, counting by tens, or composition and decomposition of numbers when the problems were easy, but reverted to invalid strategies when the problems became more complex. These students all showed inaccurate procedural knowledge of the standard algorithms, often using incorrect procedures that disregarded the value of the digits. Except for Kennedy, all of these students made miscounts during the interview.

Concatenation of digits. Francesca is an example of a student who regarded multidigit numbers as concatenated. When asked to solve 38+24, she said, "This equals seven [Pointed to 24] if you add these together and these two equal ten [Pointed to 38] so it'd be on— seventeen." She added all the digits within the numbers to get 17, as if the digits were independent and had no value. Francesca demonstrated this form of concatenation of multidigit numbers several times during the interview, producing answers that made sense only if multidigit numbers were not viewed as a unified whole but as consisting of independent digits.

Francesca was inconsistent in her understanding of multidigit numbers. In the Join Result Unknown problem she represented the 36 pennies by using 36 blocks, which suggests that she often had a unified multidigit conception of multidigit numbers when given the context of word problems.

Francesca combined a unified multidigit and a concatenated view of multidigit numbers when she used standard algorithms within the context of word problems. When she wrote an addition algorithm for the division problem, 

she placed the "1" of the ten in the hundreds column. As she did not add the digits within the numbers it seems that she was regarding the 158 and 10 as unitary multidigits. However by placing the "1" of the ten in the hundreds column she showed that she was not considering the value of the digits within the numbers.

Although Francesca's conception of multidigit numbers as consisting of valueless digits that could be added within the number was unusual, acting on the digits as if they were valueless within an algorithm was common across the students who fit into this profile. Additionally Kevin demonstrated a different form of concatenation of numbers in that he often acted on the tens independently from the ones. Both of the students in this category had inconsistent conceptions of multidigit numbers that depended on the context of the problem.

Discussion

These students with learning disabilities made many errors as they completed the mathematical tasks presented in this paper. Despite prior research suggesting that students with learning disabilities have difficulties learning basic facts (Mazzocco & Thompson, 2005) and comprehending word problems (Fuchs & Fuchs, 2002), these were not the sources of most of these students' errors. Instead, many of these students' errors in mental arithmetic, written calculations, and word problems were due to errors in applying place-value concepts.

Place-value is a complex construct consisting of four main parts: 1) base-ten numeration, 2) place-value numeration, 3) counting, and 4) flexible composition and decomposition of numbers. As place-value is complex, students may understand and apply some aspects of place-value correctly while misunderstanding other aspects of place-value. The students in this study did show varied competencies across the aspects of place-value, and even within one specific aspect. This is not unexpected because students with learning difficulties in mathematics may be particularly prone to variabilities in their performance across different aspects of number and from task to task (Houssart, 2007).

Base-ten numeration

Base-ten numeration was the strongest aspect of place-value knowledge for this group of students. All but one student showed some knowledge of base-ten numeration, demonstrating that they knew that they could count the tens frames as units of ten. Only Francesca seemed to have little understanding that there was a difference between counting by tens and counting sets of ten.

Six students had an insecure knowledge of base-ten numeration. When direct modeling by ten, Robin and Rosie were both secure in their base-ten numeration up to one hundred but made errors above one hundred. Kyle, Luke, Hunter, and Robbie all showed base-ten numeration knowledge when solving the division by ten problem but switched units when stating the answer to the problem.

Place-value numeration

I have the least information about the students' place-value numeration skills, with only eight students using strategies that demonstrated their knowledge of place-value numeration. Of these eight students, seven used written algorithms in such a way that it was clear that they were not using the positional value of the digits in their calculations. As these seven students solved multidigit calculations they misaligned numbers, acted on the tens and ones separately, and acted on the tens as if they were ones. Disregarding the place-value of the digits when aligning numbers in standard algorithms is a common error seen among students with learning disabilities (Behrend, 2001).

Two students showed that they viewed the digits within a number as concatenated. Francesca often did not regard the positions within the number as having any value beyond the digit, whereas Kevin usually attributed value to the different positions but regarded the units as independent. Fuson and Briars (1990) commented that a vision of multidigit numbers as consisting of independent single digits is not uncommon in the United States because children are commonly "taught multidigit addition and subtraction as sequential procedures of adding and subtracting single-digit numbers and writing digits in certain locations" (p. 181). What may be unusual about Kevin is that he only applied this independent digits conception of multidigit numbers to addition and subtraction but with division he used a more integrated vision of the multidigit numbers.

In future studies it would be helpful to include questions that specifically examine place-value numeration, so that there is data on this aspect of place-value from all students. However, most tasks that examine place-value numeration are low-level procedural tasks and many students with disabilities can respond correctly to procedural questions about place-value while demonstrating limited conceptual place-value knowledge (Cawley, Parmar, Lucas-Fusco, Kilian, & Foley, 2007). This means that it will be important to include the problems from this study that examined place-value numeration in context.

Counting

Although the expectations are that children should have mastered object counting by the end of 2nd grade (CCSSM, 2010), this study shows that many students with learning disabilities still have significant difficulties with object counting in 4th grade. Twelve of the fifteen students in this study had difficulties with at least one of the counting tasks, and eleven of these students had difficulties with number sequences at some point during the interview. This study confirms and expands previous research that found that students with language-related disorders have difficulties with counting in lower elementary school (Donlan, Cowan, Newton, & Lloyd, 2007; Fazio, 1996; Murphy, Mazzocco, Hanich, & Early, 2007). These difficulties with counting suggest that students with learning disabilities may have difficulty with more advanced arithmetic because facility with the order of numbers allows attentional resources to be devoted to more complex arithmetic problem-solving (Desoete, Stock, Schepens, Baeyens, & Roeyers, 2009).

Composing and decomposing numbers

Composition and decomposition of numbers was one of the stronger aspects of this group of students' knowledge of place-value. Just under half of the students could flexibly compose and decompose numbers and use this skill in the context of problems.

Several students could compose and decompose numbers when the problems were easy but then resorted to other strategies when the problems became harder. Several students had difficulties composing numbers because they only focused on one of the units. For example, when Kennedy was solving the problem 15 packs of candy and 4 singles, he responded with 150 because he only focused on the tens. In future studies I would also ask the question, "I have 7 packs of candy and 3 singles. How much candy do I have?" This question will help to distinguish between the students who have problems with tens and ones and the students who have difficulties with multiples of ten when the number is greater than ten.

Profiles of place-value understanding

Although my sample of students was small I found a great diversity in their understanding of place-value, with five different profiles emerging for what they understood and did not understand about place-value: (1) Good place-value understanding; (2) Reliance on algorithms; (3) Counting errors; (4) Beginning place-value understanding; and, (5) Concatenation of digits. The heterogeneity that I found in my sample reflects the findings of other researchers that students with learning disabilities are a heterogeneous group with regards to mathematics (Pieters, Desoete, Roeyers, Vanderswalmen, & Van Waelvelde, 2012; Kaufmann, Handl, & Thöny, 2003).

Students with good place-value knowledge. For the students who did display good place value knowledge in this study, it would have been interesting to test how far their knowledge of place value extended. Could they still solve the problems if the word problems had numbers with 3-, 4-, or 5-digits in them? According to the CCSM (2010) students in fourth grade are supposed to "generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place", so extensions to these problems could have gone up to 4- or 5- digits.

Reliance on standard algorithms. Of the eight students who used a written algorithm to solve a problem, seven had difficulties following the procedures correctly. The only student who did not make a mistake when using the standard algorithm was Lewis, one of the students who had good understanding of place-value.

Students with some strengths in their place-value knowledge had a tendency to resort to the standard algorithms for subtraction, despite finding it difficult to follow the correct procedures in the standard subtraction algorithm and having other strategies that they used successfully when computing mentally. Robbie and Hunter had shown good flexibility with the composition and decomposition of numbers but when presented with subtraction problems, a topic they had been working on for several years, they reverted to following a half-remembered procedure. Kamii and Dominick (1998) state that when children learn algorithms without understanding they focus on remembering the steps rather than logically solving the problem and "lose conceptual knowledge when they learn these rules" (p. 131). Focusing on the steps in an algorithm may be particularly problematic for students with learning disabilities because they often have memory difficulties that make correctly following a sequence of steps difficult (Allsopp, Kyger, & Lovin, 2007). Both of these observations appeared to be true for Robbie and Hunter.

Students with beginning place-value understanding also had difficulties remembering all of the steps in the standard algorithms, particularly if the steps did not make sense to them (Allsopp, Kyger, & Lovin, 2007), and yet they often used a standard algorithm as their preferred strategy.

These students also tended to disregard the value of the digits in the standard algorithms by aligning digits incorrectly, placing answers to the single-digit operations in the wrong columns, disregarding the importance of zero as a place-holder, or failing to fully cross-multiply. These students did not understand the importance of place-value when operating on numbers. Therefore, it is important to develop students with learning disabilities' conceptual knowledge of place-value along with their procedural knowledge of algorithms because then the procedures make more sense and are not just memorized.

Counting Errors. The students who made multiple counting errors during the counting tasks also made counting errors as they completed the word problems. These students tended to solve word problems using beginning-level strategies that required extensive counting, thus increasing the chances that they would solve the word problems incorrectly due to counting errors.

Incorrect invented algorithms

One third of the students developed their own incorrect invented algorithms based on partially understood rules. Four students used the same incorrect invented algorithm to solve the problems with more than ten packs of candy. This algorithm can be expressed as $10*n=100+n$ where $n>10$. This incorrect invented algorithm showed a misunderstanding of place-value because the students composed the numbers incorrectly. It was unclear why four students had the same incorrect error; were they misapplying a rule their teacher had told them or did they all come up with this error spontaneously?

Robbie made errors with the mental addition problems because he tried to overgeneralize a rule that he had learned for adding ten to a single digit. Students in Robbie's class had been taught to solve a single digit plus ten by taking away the zero in the ten and replacing it with the single digit. This can be represented as $n+10=1n$, if $n<10$. Robbie took this rule and over-generalized it to apply any time that he was adding a number to a multiple of ten.

Mercer, Jordan and Miller (1994) recommend the use of mnemonics as "instructional components that help students with memory problems acquire, remember, and apply specific math strategies" (p. 300). However, the findings from this study suggest that teachers of students with learning disabilities should be very careful about which mnemonics they teach the students. Teaching rules that only work for a very small subset of numbers is likely to confuse the students more than it helps them, because in their effort to reduce their memory load they may overgeneralize the rules to apply to a large set of numbers.

Implications

When researchers are studying the mathematical knowledge of students with learning disabilities they need to focus on these students' place-value knowledge, with a broad conception of place-value that encompasses all four aspects. Up until this point the few researchers who have studied students with learning disabilities and their knowledge of place-value have mostly examined students' procedural knowledge of place-value, with only a few researchers examining conceptual knowledge of place-value. Although the prior research on the conceptual knowledge of place-value has examined all aspects of place-value, the results have been reported in an aggregated form so that it was not possible to know which aspects of place-value were particularly difficult for students with learning disabilities. This study found that students with learning disabilities found some aspects of place-value more difficult than others, so future studies of place-value should investigate each aspect of place-value separately.

Teachers of students with learning disabilities need to assess and teach all aspects of place-value. The findings from this study suggest that teachers of students with learning disabilities should help students construct a broad understanding of place-value at the same time as they are developing an understanding of standard algorithms. These findings additionally suggest that teachers should be careful about which mnemonics they introduce to students, being careful not to introduce any "rules" unless they are broadly generalizable.

Limitations

There are several limitations to this study. One is the small sample size of the study. A larger sample size would have made it easier to distinguish dominant profiles of understanding of place-value.

Another limitation is that it is unclear whether the students' errors are due to their disability or due to their instruction. All the students in the study had spent at least one year in the school so they had all received the same instruction for at least one year. This may be why so many students relied on the standard subtraction algorithm despite not remembering the steps of the procedure. In my experience as a special education teacher this last concern is not specific to this school; in special education classrooms throughout the U.S.A. there is an overwhelming emphasis on learning the procedures of the standard algorithms with little focus on conceptual knowledge. This emphasis is also reflected in the research literature from the U.S.A. on teaching mathematics to students with learning disabilities, where a third of the instructional articles published in the field examined students' procedural fluency with multidigit calculations (Section 1).

Due to human subjects restrictions I was unable to look at the students' diagnoses, so I am unsure what exact types of learning disabilities they had. However, given the school's admission policies I can be fairly certain that the students had language-based learning disabilities, whether dyslexia or a language-processing disorders. It seems likely that the results from this study are specific to the types of errors with place-value that students with language-based learning disabilities show rather than the type of errors that students with mathematics learning disabilities show.

Future Studies

Future studies should continue to investigate the different aspects of place-value understanding for students with learning disabilities. These future studies should have a larger sample size and examine how the students' diagnoses correspond with the different profiles of understanding of place-value.

Another future study would be longitudinal investigation of the development of place-value concepts for students with learning disabilities and to compare this with the learning trajectory in place-value understanding of typically developing students.

In section 3 I describe a follow-up study, in which I took what I had learned about these students' understanding of place value and designed and implemented an instructional sequence for four of the students. In this instructional study I used the activity of counting collections to improve students' understanding of base-ten numeration.

SECTION 3: Using Counting Collections To Teach Place-Value Concepts To Students With Learning Disabilities

Most interventions designed to improve the mathematical knowledge of students with learning disabilities have focused on their procedural fluency, with limited focus on their conceptual understanding. This section uses a single-subject design to examine what four 4th grade students with learning disabilities learned about base-ten numeration during an intervention using counting collections as an instructional activity. Over the course of the intervention all the students made improvements in their accuracy and used more mature problem-solving strategies when solving place-value problems and maintained this growth in follow-up interviews. The students who made the most progress in their accuracy were the students who used the largest variety of strategies and started off by using beginning-level strategies that emphasized unitizing. These findings imply that students with learning disabilities can benefit from engaging in the instructional activity of counting collections when they employ a variety of strategies that lead them from beginning level strategies emphasizing unitizing towards more mature strategies.

The importance of conceptual understanding for all students is emphasized in the Common Core State Standards for Mathematics (CCSSM, 2010) because students who lack understanding of a mathematical topic tend to rely on procedures without having the flexibility to generalize these procedures to similar problems or engage in the mathematical practices. Unfortunately many students with learning disabilities (and other students receiving special education services in mathematics) are primarily taught mathematics procedurally (Gersten et al, 2009) and are given few opportunities to develop conceptual understanding. In this study I argue that students with learning disabilities can, and indeed should, be taught in a way to enhance their conceptual understanding.

This study is a follow-up to a previous study that examined what aspects of place-value were difficult for students with learning disabilities (Section 2), where it was found that nine of the fifteen students had difficulties with the concept of base-ten numeration, the concept that it is possible to count groups of tens. Therefore this exploratory study examines the efficacy of an instructional program that uses the instructional activity of counting collections (Schwedtfeger & Chan, 2007), in which students count a collection of objects efficiently and accurately, to develop the base-ten numeration understanding of students with learning disabilities. I use a single-subjects design to examine the following research questions: Is counting collections an effective activity to increase the base-ten numeration understanding of students with learning disabilities? How do students' strategies for solving the place-value problems change during the intervention?

Theoretical Approach

In this study I draw upon two disciplines: special education and mathematics education. The focus on individual student learning and the single-subjects research design comes from the field of special education. The focus on student strategy use and the development of conceptual understanding comes from the field of mathematics education.

All students need to learn mathematical topics both conceptually and procedurally because conceptual knowledge supports the development of procedural knowledge and vice versa (Canobi, 2009). *Procedural knowledge* is the knowledge of the rules, strategies, and algorithms required to complete a task (Baroody, Feil, & Johnson, 2007), while *conceptual understanding* is constructed from facts, mathematical principles, generalizations, and connections between pieces of knowledge (Baroody, Feil, & Johnson, 2007; Rittle-Johnson & Siegler, 1998).

The fluent and extensive relationships present within conceptual knowledge create coherent mental structures with multiple entry points. These multiple entry points make knowledge learned with understanding easy to remember; even if students forget individual facts or procedures, they should be able to easily reconstruct the facts or procedures from the remaining conceptual structure. The flexible connections present within the conceptual framework should also make the framework malleable enough to expand to incorporate new knowledge. On the other hand, students who learn isolated facts and procedures are likely to make errors in the execution of these skills or forget these skills altogether because they only have one entrance into the problem and if this entrance becomes blocked or distorted the fact is lost (Hiebert & Wearne, 1996). Additionally, students who learn procedures as rules to be followed are likely to find that they need to learn a whole set of new rules as they learn more about the complexity of numbers, for example as they learn about decimals (Wearne, Hiebert, & Campbell, 1994).

Explicit instruction is a major feature of instruction and instructional research for students with learning disabilities (Gersten et al, 2009). In this context, explicit instruction means that the teacher models a step-by-step plan, which the students are expected to utilize to solve similar types of problems. This type of instruction tends to present mathematics as a series of isolated procedures to be learned and repeated, and makes little attempt to help students form connections across topics. This emphasis on solely procedural knowledge is inequitable as it is an instructional strategy generally only used to teach certain groups of students, such as students with learning disabilities, and it makes it difficult for these students to develop conceptual understanding of the topic and thus makes it unlikely for them to develop more than a partial understanding of mathematics. Therefore all students, including students with learning disabilities, should be taught in a manner that helps them develop conceptual understanding.

Literature Review

In the following review of the literature I start by discussing what place-value is and what are the typical phases of development that children go through as they develop this concept. I then discuss what is known about the place-value knowledge of students with disabilities and the research that has been done on place-value instruction for both typically developing students and low-achieving students, with and without diagnosed learning disabilities. Finally I name elements that should be in future instructional studies designed to teach place-value to students with learning disabilities.

Place-value

Place-value is a fundamental concept that underlies the structure of our number system. Despite being a basic concept of mathematics it is complex and multifaceted, and contains the following four elements: base-ten numeration, place-value numeration, counting, and the flexible composition and decomposition of numbers (Van de Walle, Karp, & Bay-Williams, 2010). *Base-ten numeration* is the understanding that it is possible to count sets of tens (and multiples of tens), count by tens, and to count the individual objects. This concept is fundamental to our symbolic numerical system, in that the number 536 represents counting both 5 hundreds 3 tens and 6 ones, and 53 tens and 6 ones, as well as 536 individual objects. Base-ten numeration is related to *unitizing*, in that unitizing is the concept that it is possible to count sets of objects as well as counting the individual objects (Fosnot & Dolk, 2001). *Place-value numeration* is the knowledge that the position of the digits in written numbers determines what size group they represent. This is often known as *positional place-value* knowledge and is the focus of most place-value instruction in school mathematics. *Counting* involves enumerating objects by following a fixed sequence of words. This skill is an element of place-value knowledge because skilled counters recognize that numbers are formed by following regular patterns, and use this recognition to help them count large numbers (Callahan & Clements, 1984). *Flexible composition and decomposition of numbers* is a facility with constructing and deconstructing numbers into a variety of component place-value groupings. For example, 2 hundred 5 tens and 6 ones can be composed into the number 256, and this number can be decomposed into 1 hundred 15 tens and 6 ones, 25 tens and 6 ones, or into a variety of other groupings. This flexibility with numbers is essential for understanding regrouping within standard algorithms (Fuson & Beckmann, 2012).

The complicated nature of place-value means that children take a long time to fully develop knowledge of these concepts and there are a number of phases that children go through in their development of place-value concepts. Fuson et al. (1997) proposed a 5-stage model, the UDSSI model, for the conceptions students develop about 2-digit whole-number place-value:

1. *Unitary single digit*—Students learn the written and spoken numbers one to nine, and how to count these quantities.
    i. *Unitary multidigit*–Students count by ones and identify the number as a whole, so 53 consists of 53 ones and they count 1, 2…53
2. *Decades*— Students can state that the digit on the right is ones and the digit on the left is tens, but they do not understand the quantities each numeral represents and they count the tens by ones, e.g. they can identify that 53 is 5 tens and 3 ones but they count 1, 2,…50, 51, 52, 53.
3. *Sequence*— Students count by tens and ones, so they count 10, 20, 30, 40, 50, 51, 52, 53.
4. *Separate*— Students understand that the digits represent tens and ones, so 53 consists of 5 tens and 3 ones.

5. *Integrated*— Students move fluidly between sequence and separate conceptualizations, so they both know that 53 is 5 tens and 3 ones and they can count by tens and ones.

Several of these conceptions of place-value may be available to children at the same time and they may use different conceptions depending on the situation.

There are also a couple of common misconceptions that students develop about place-value. Some students develop a *concatenated single digit* conception of place-value, meaning that that they interpret the digits in 53 as independent digits consisting of a five and three (Fuson et al., 1997). This viewpoint is particularly prevalent as students operate on multidigit numbers and causes many errors with the procedures for multidigit addition and subtraction (Fuson & Briars, 1990). Some students develop a f*ace value* conception of place-value, meaning that they do not require that the objects represented by the tens place be equivalent to ten of the objects in the ones place (Ross, 1990). This conception becomes problematic when non-canonical partitioning is used, such as when regrouping has occurred, so that children who can easily state that 5 tens and 3 ones is 53 would have difficulty stating how many objects there are when they are grouped as 4 tens and 13 ones. This misconception is particularly likely to occur when students always use pre-grouped materials, such as base-ten blocks, because then the student do not have think about the “tenness” of the groups and what this means.

As stated above, place-value conceptions take a long time to develop. Ross (1989) found that more than half of the students in her study were still developing whole-number place-value concepts in 4th grade, with one-third of her sample having a face-value conception of place-value, and one student who still retained a unitary conception of place-value.

Place-value knowledge for students with learning disabilities

There has been little research on students with learning disabilities’ understanding of place-value, and most of this research has examined their procedural place-value knowledge (Section

1), using tasks such as identifying the digit in the tens place, comparing numbers, or identifying the largest 3-digit number. This research has had mixed results, with some researchers finding that students with specific language impairments could compare 2-digit but not 5-digit numbers (Donlan, Cowan, Newton, & Lloyd, 2007), while others found that similarly aged students with mathematical difficulties had difficulties identifying numbers and digits even in the tens and the hundreds place (Cirino, Fletcher, Ewing-Cobbs, Barnes, & Fuchs, 2007). These contradictory results may be due to different populations (students with different disabilities and living in different countries) or tasks (comparing numbers versus identifying digits in a particular place). Desoete and Grégoire (2006) found that students with mathematical learning disabilities had more difficulties comparing oral numbers than written numerals, so it is possible that the format in which the tasks were presented could also have impacted the results. So far the research suggests that students with learning disabilities have difficulties with procedural aspects of place-value, but it is not yet clear which aspects are the most problematic, or how the format of presentation or the group membership of the students affects the results.

Only a few articles have examined conceptual measures of place-value for students with disabilities, using tasks such as, “How many Xs [where X is a multiple of ten] are in Y?”, “How many Lifesavers are in Z packs of ten?”, or digit correspondence tasks where the student is asked to match the digits in a number with its meaning in physical objects. Russell and Ginsburg (1984) found that students with mathematical difficulties had elementary concepts of base-ten notation but experienced difficulty enumerating large numbers. In this study all of the results for the base-ten concept tasks were consolidated so it is not possible to know which elements of base-ten notation the students understood and which elements they had difficulties with. Cawley, Parmar, Lucas-Fusco, Kilian, and Foley (2007) examined the place-value knowledge of students with mild disabilities in more depth and found that they had difficulties with conceptual aspects of place-value even though they could answer procedural questions. Section 2 further investigated the conceptual place-value knowledge of students with learning disabilities and found that they had a variety of difficulties with the four aspects of place-value and that these difficulties impacted their ability to solve mathematical problems correctly. Jordan, Hanich, and Uberti (2003) found that students' difficulties with place-value depended on their diagnosis, which may explain the variation in place-value knowledge found in Section 2. All of these researchers agreed that students with disabilities have difficulties with place-value, and Cawley et al. (2007) and Section 2 suggested that these difficulties may be particularly related to conceptual aspects of place-value.

Place-value instruction for typically developing students

There have been several studies of place-value instruction for typically developing students. Most of these articles were primarily focused on the importance of place-value as a foundation for multidigit addition and subtraction (Fuson & Briars, 1990; Hiebert & Wearne, 1996; Wearne, Hiebert, & Campbell, 1994; Nagel & Swingen, 1998). Fuson and Briars found that students who were taught in a way that linked concrete, verbal, and symbolic representations of the quantities in the problems developed high levels of understanding of place-value and multidigit arithmetic. Further Wearne, Hiebert, and Campbell found that this level of conceptual understanding of whole numbers generalized to the students' understanding of decimals.

A few instructional studies focused on the topic of place-value independently of multidigit arithmetic. Ross (1990) investigated the 2-digit place-value knowledge of 2^{nd} -5^{th} grade students in two groups; one group had been taught to represent place-value with base-ten blocks and the other had followed a traditional curriculum. The effect of the place-value instruction was mixed;

the students in the instructional group were better at using the base-ten blocks canonically and the 2nd grade students in the instructional group were better at the positional task than the control group, but for most tasks there were few differences between the two groups. In fact the students in the instructional group were more likely to have face-value knowledge of place-value than the other group, which could be a result of using pregrouped manipulatives, such as base-ten blocks, because these students did not have to think about the significance of the "tenness" of the groupings as the blocks were already grouped in tens.

Even in 5th grade only half of the students in each group reached an integrated understanding of the digits in 2-digit numbers, which Ross interpreted as evidence that students should not be exposed to larger numbers until they had mastered 2-digit numbers. However, in English the verbal 2-digit numbers are irregular in their formation and the regularities within the base-ten system do not become obvious until children are working with 3- and 4- digit numbers (Franke, 2003), so it may be better to introduce children to these larger and more regular numbers early in their experiences with place-value.

Burris (2013) studied the utility of concrete representations of place-value by comparing what 3rd grade students learned about 3- and 4-digit place-value when using virtual or concrete base-ten blocks. He found that both groups successfully learned base-ten numeration and place-value numeration with the manipulatives, but the group with the virtual manipulatives generated more nonstandard representations. The virtual manipulatives supported the students to go beyond a face-value conception of place-value because the virtual manipulatives were easier to break apart and put together in nonstandard ways than the concrete manipulatives.

These instructional studies show that students can be taught place-value by linking the concrete, verbal, and symbolic representations of numbers, but that it is important to consider what types of understanding the different representations and the sizes of the numbers afford.

Place-value instruction for students with learning disabilities

There are have been limited instructional studies focused on teaching place-value to students with learning disabilities. Two notable exceptions both successfully used the Concrete-Representational-Abstract (CRA) instructional sequence to teach aspects of 2-digit place-value knowledge to students with learning disabilities. In Peterson (1988) the teachers taught the students how to identify tens and ones in a double-digit number using the following instructional sequence: three lessons using concrete manipulatives, then three lessons using drawings of these manipulatives, and finally three lessons using just numbers. While the students did learn the specified tasks, they showed little evidence of generalization to 3- or 4- digit numbers or to situations other than that which was specifically taught (Flores, 2009; Peterson, 1988). The lack of generalization in these studies may have been due to the restricted affordances provided by the limited concrete base-ten models used and/or the focus on 2-digit numbers. The success of the CRA method for teaching specific skills but the lack of generalization means that the current CRA instructional strategies are too time-consuming to have general utility, but there may be promise in the principle of teaching students to link different representations of number.

Several instructional studies have taught place-value concepts to low-achieving students with mixed results. Bryant et al. (2008) were unsuccessful in teaching place-value concepts as a part of a larger number sense program for low-achievers. Despite a program that included a focus on making and counting tens and ones, using base-ten language, translating base-ten models to verbal and written numbers, and positional place-value, there was no significant effect on the students' place-value knowledge and no explanation for why there was no effect. Although Bryant et al. did not discuss the reasons for their lack of success in teaching place-value, it seems that their focus on 2-digit numbers is likely to be the reason for their lack of success, because the patterns in the base-ten number system do not become apparent until 3-digit numbers (Fuson & Briars, 1990).

On the other hand, Ellemor-Collins & Wright (2009) found that a low-achieving student learned a lot about conceptual place-value when she focused on establishing consistent links between number words, numerals, and materials in 2-, 3-, and 4-digit problems. She learned base-ten numeration skills as she learned to increment and decrement by 1s, 10s, and 100s, and flexibility with numbers as she learned to add 2-digit numbers using these incrementing skills.

Although there has been limited research on the topic of teaching place-value to low-attaining students and students with learning disabilities, there appears to be consensus that it is necessary to form links between the different representations of the numbers. There also seems to be a pattern that successful interventions do not restrict themselves to teaching 2-digit place-value, but expand the concept of place-value into 3- and 4-digits. Further research should be conducted that teaches students with learning disabilities about place-value by linking different representations of numbers, using a variety of concrete models including groupable and pregrouped base-ten models, while expanding instruction into 3- and 4-digits.

Methods

Research design

In this study I used a multiple probe across groups design to investigate whether counting collections was an effective instructional activity to increase the base-ten numeration knowledge of students with learning disabilities. I used a multiple probe approach to establish a relationship between the instructional approach and the students' performance and strategic changes.

Participants

The participants in this study were four 4th grade students who showed in an initial interview that they had difficulties with base-ten numeration knowledge. There were three girls and one boy assigned to two groups based on their classroom teacher.

All the students had been diagnosed with dyslexia or an associated language delay before they were admitted to their current school.

Setting

The participants in this study came from an independent school for 1st-8th grade students with dyslexia and related language difficulties in the Pacific Northwest. All lessons and tests took place in a cubicle in the study center.

Experimenter

The author was the instructor for this intervention. I have 10 years experience of teaching in elementary schools, including six years as a special education teacher. I conducted the pre-interview assessment and all of the instruction during the intervention.

Procedures

Initial interview. The students were selected after completing an initial interview, during which there were multiple tasks, which are described fully in Section 2. There were three tasks relevant to this article:

1. I have 13 packs of candy. There are 10 pieces of candy in each pack. How many pieces of candy do I have?
2. I have 15 packs of candy. There are 10 pieces of candy in each pack. I also have 4 single pieces of candy. How many pieces of candy do I have?
3. Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?

These are three of the five questions used during baseline (see question 1, 3, and 5 in Table 19).

The students that were selected for this study solved all three of these problems incorrectly and showed issues with their base-ten numeration understanding that are described in Section 2.

Baseline. During baseline the students were videotaped as they completed six tasks, one of which was a counting task, which was not analyzed for this article. The main tasks were to respond to five questions about place-value (see Table 19); each day during baseline these questions were presented in a different order and the numbers in the questions changed (for the actual numbers used see Table 24, Appendix B). Once students had responded to the questions, they were asked to explain what they had done to solve the problems. As the students responded to the questions, the correctness of their answers, their actual answers, and their strategies were recorded. During baseline the students were unable to complete all the questions during the half an hour set aside for the tests, therefore they completed as many questions as they could during the allotted time period. I established a stable baseline for all students of at least three consistent data points for the place-value questions. Once I had established a stable baseline I started instruction with the first pair of students.

Table 19

Place-Value Questions

1. I have [11-99] packs of candy. There are 10 pieces of candy in each pack. How many pieces of candy do I have?

2. I have [1-9] boxes of candy. There are 100 pieces of candy in a box. How many pieces of candy do I have?

3. I have [11-99] packs of candy. There are 10 pieces of candy in each pack. I also have [1-9] single pieces of candy. How many pieces of candy do I have?

4. I have [1-9] boxes of candy. There are 100 pieces of candy in a box. I also have [1-9] packs of candy. There are 10 pieces of candy in each pack. How many pieces of candy do I have?

5. I have [111-999] crayons. I want to put 10 crayons in each box. How many boxes do I need? How many crayons will be left over?

Activity. The instruction in this study was based on counting collections (Schwedtfeger & Chan, 2007), an instructional activity in which students worked in pairs to count a collection of objects, grouping the objects to increase efficiency and accuracy. The students worked in pairs so that they could exchange ideas and build on each other's knowledge. Once the students finished counting they recorded their count as a labeled drawing, typically labeled with both the size of the groups they made and how they counted, which linked the concrete, drawing, and symbolic representations of the problem together.

In this version of counting collections the students used the collections to solve a problem that was grounded in the counting collections (see Table 20). The problems were carefully chosen and sequenced to build up concepts of the base-ten system. During the first week of instruction the collections had between 200 and 400 objects and the students were required to count the objects and represent their count accurately. During the second week of instruction the collections contained objects that come in groups of ten and had between 10 and 40 groups of ten. The emphasis this week was thinking about the relationship between the number of groups of tens and ones and the 3-digit number. During the third and fourth week of instruction the collections were of objects that come in groups of hundreds and contained between 100 and 1,200 objects. The emphasis these weeks was thinking about the relationship between the number of groups of hundreds, tens, and ones and the 3-digit number. During the final two weeks of instruction counting collections were used as a basis for thinking about both Join Change Unknown and division by ten problems.

All of the instructional counting collections in this study had more than one hundred objects, and some of them had had more than a thousand objects pregrouped in hundreds. This was because it is important for the students to work with 3- and 4- digit numbers in order to appreciate the regularity in our place-value system.

Students worked on counting collections for half an hour a day, four days a week for six weeks.

Table 20

Counting collection problem types

Week	Problem type	Example
1	Counting	Count the Skittles. There are ______ Skittles altogether. There are ____ groups of ten. There are ___ left over. Show how you counted:
2	Groups of ten	I have 14 packs of pens. There are ten pens in each pack. How many pens do I have? There are ___ pens altogether. Show how you solved this problem:
2	Groups of tens and ones	I have 93 dimes and 13 pennies. How much money do I have? I have ______. Show how you solved this problem:
3	Groups of hundred	I have 3 boxes of paper clips. There are 100 paper clips in a box. How many paper clips do I have? I have ____ paper clips. Show how you solved this problem:
3/4	Groups of hundreds and tens	I have 2 boxes of cubes and 1 tower of cubes. There are 100 cubes in a box and 10 cubes in a tower. How many cubes do I have? I have ____cubes. Show 2 different ways that the cubes could have been grouped.
3/4	Groups of hundreds and ones	I have 221 pennies. I want to swap the pennies for dollars. How many dollars will I get? How many pennies will I be left with? I will get ______ dollars. I will have _______ pennies left over. Show how you solved this problem:
5/6	Join Change Unknown	Robin has 29 strings of beads. Lily has 50 strings of beads. How many more strings of beads does Lily have than Robin? __________________ Show two ways to solve the problem.
5/6	Division by ten	I have 718 beads. There are ten beads in each string. There are __ strings of beads. There are __beads left over. Show how you solved this problem:

Dependent Measures. Place-value knowledge was the dependent measure, which was measured with five place-value questions similar to those posed during baseline, except with different numbers, presented in a different order, and often with a different context. The students individually answered the place-value questions once a week. They were video-recorded as they completed these tasks and notes were taken about their answers, strategies, number sequence, and the time they took to complete the problem.

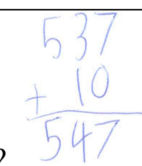
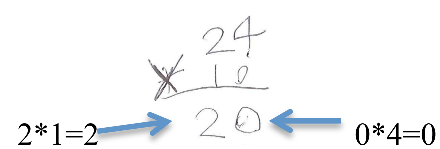
Criterion was set as 4 out of 5 problems correct. When the students in group 1 both met criterion they went into a maintenance phase where they were given no place-value instruction for four weeks and then were tested on the place-value measures again. Sam never met criterion, but because Rosie had already met criterion it was decided to stop group 2's instruction after 6 weeks. Due to the positioning of the school holidays the maintenance test for group 2 was six weeks after the end of instruction.

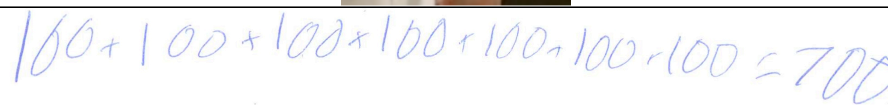
Analytic methods

This study analyzed results from the place-value questions. I used quantitative methods, recording how many questions the students got right, what strategies they used, and how long it took them to produce an answer. I coded the students' strategies using the codes: ran out of time, invalid, beginning-level, and more mature (see Table 21). Beginning-level strategies involved concrete representations, whereas mature strategies were more abstract and efficient. Both of these codes consisted of valid strategies, meaning that the strategy would produce a correct answer if correctly applied, and were further coded based on the classification system of Cognitively Guided Instruction (Carpenter et al., 1999), while the invalid strategies were identified with open coding. I used the following codes for the invalid strategies: invalid operation, invalid algorithm, units error, and incorrect algorithm. The beginning-level strategies were direct modeling and counting, while the more mature strategies were repeated addition, partial products, recall/rule, correct standard algorithm, and direct place-value. I classified the students' strategies based on their explanations of their strategy and upon my observations of what they actually did.

Table 21

Codes used to assign strategies to the place-value problems

Codes	Specific codes	Description	Examples of strategies
Ran out of time		Student did not attempt the problem because they ran out time or ran out of time before they gave an answer.	
Invalid strategies	Invalid operation	Student used an operation that could not result in the correct answer.	537 crayons, ten in a box. How many boxes will I need? 
	Invalid algorithm	Student used following algorithm: n*10=100+n, if n>10	15 packs of candy, ten pieces in a pack. How many pieces? 115 “Ten would be 100 so 15 would just be 15 more”
	Units error	Student confused units	For 5 boxes and 5 packs of candy, with 100 pieces in a box and 10 pieces in a pack. How many pieces altogether? “there's fifty [Points to 5 boxes]”
	Incorrect algorithm	Student used standard multiplication algorithm incorrectly.	

Beginning-level strategies	Direct modeling by tens (or hundreds)	Student modeled the problem with blocks, representing each group of ten with a stick of tens. Student counted by tens.	
	Counting by tens (or hundreds)	Student counted by tens. They used blocks, their fingers, or a number chart to keep track of how many times they counted.	
More mature strategies	Repeated addition	Student used repeated addition to solve the grouping problem	
	Partial products	Student solved the problem by breaking number into parts, and multiplying each section separately.	
	Recall/ Rule	Student either immediately knew the answer or used a rule for multiplying by ten or hundred, e.g. 10*n=n0	"anything times ten is just that number with zero on end if that number is a two-digit number"
	Standard Algorithm	Student used a standard algorithm to solve the problem.	3x100=300
	Direct place-value	Student looked at the number and immediately knew how many groups of tens were in the number.	 40 boxes, 5 extra

Results

Accuracy

Student performance on the place-value probes is represented in Figure 1. Results of this intervention study were examined through visual inspection of the data and the effectiveness of the intervention was examined using the percentage of non-overlapping data.

Visual inspection of the data revealed that all the students made progress in their understanding of place-value, but only Robin[6] and Lily reached the criterion of 4/5 problems correct and maintained this knowledge. Rosie reached criterion but did not maintain this level of progress. While Sam made some progress in his understanding of place-value he made less progress than the other students. There was not an immediate change between baseline and the intervention data for any of the students; the intervention took a couple of weeks to show changes in the accuracy data.

The percentage of non-overlapping data (PND) is a statistical measure used to measure the efficacy of an intervention. The PND was evaluated by converting into a percentage the number of data points from the intervention and maintenance phases of the study that were higher than the highest data points from the baseline (Scruggs, Mastropieri, & Casto, 1987). The PND for Robin was 57%, Lily 71%, Rosie 71%, and Sam 86%, with a median PND of 71%, which indicates that the intervention was fairly effective (Wendt, 2009).

Three of the four students showed a “zero baseline”, meaning that all the data at baseline were equal to zero, so the effectiveness of the intervention should be treated with

[6] All names are pseudonyms.

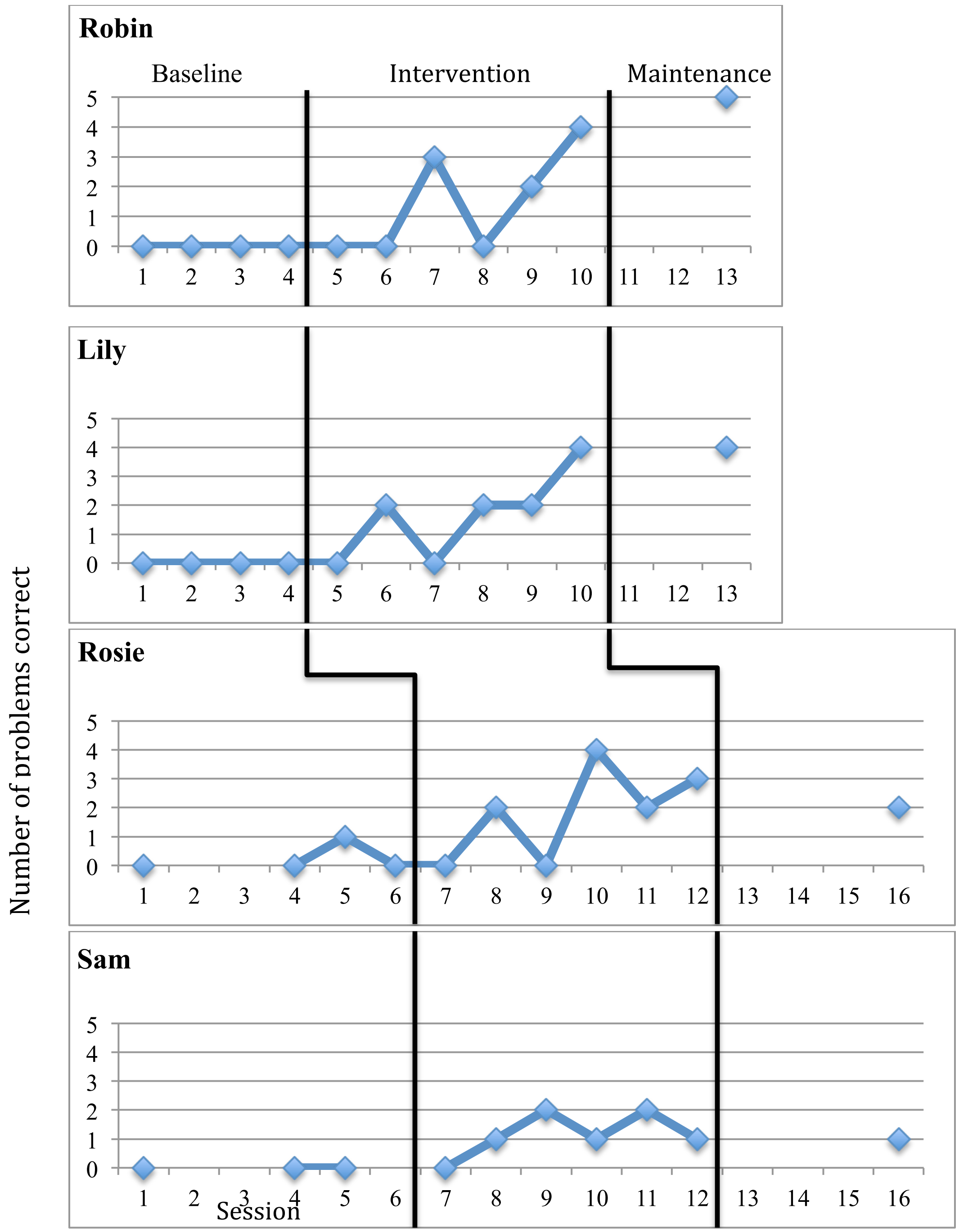


Figure 1 Place-value accuracy[7]

[7] Sam was absent for session 6 and so this data is missing.

caution because there is a floor effect at baseline (Scruggs, Mastropieri, & Casto, 1987). However, I argue that the gains that the students did make on solving these problems during the course of this intervention suggest that these problems were difficult but attainable, which leads to “higher effort and achievement than do easier goals” (Mercer, Jordan, & Miller, 1994, p. 303).

Strategy use

Although investigating the accuracy data gives some information about the effectiveness of the intervention, this does not give the full story nor explain why Lily, Robin, and Rosie made visibly more progress than Sam. Therefore the next section of the paper will investigate the students’ strategy use.

Development of valid strategies over time. Figure 2 shows the distribution of mature strategies, beginning strategies, invalid strategies, and ran out of time for each student during each session. During baseline and at the beginning of the intervention the students were unable to complete all the questions during the half an hour set aside for the tests. They completed as many questions as they could during the allotted time period, but there were several questions that they did not have time to attempt[8].

Figure 2 shows that as the intervention progressed Robin and Lily used more valid strategies to solve these place-value problems. Robin progressed from using only one valid strategy after the first week of instruction to using valid strategies for all of the questions during the follow-up interview. Lily also progressed from using only one valid strategy after the first week of instruction to using valid strategies for all of the questions after the last week of instruction.

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8 In week 6 and week 7 there are no data for Robin and Rosie respectively on the place-value questions because they took so long to complete the counting task, which was also tested weekly but is not analyzed in this study, that they did not get the opportunity to attempt any of the place-value questions.

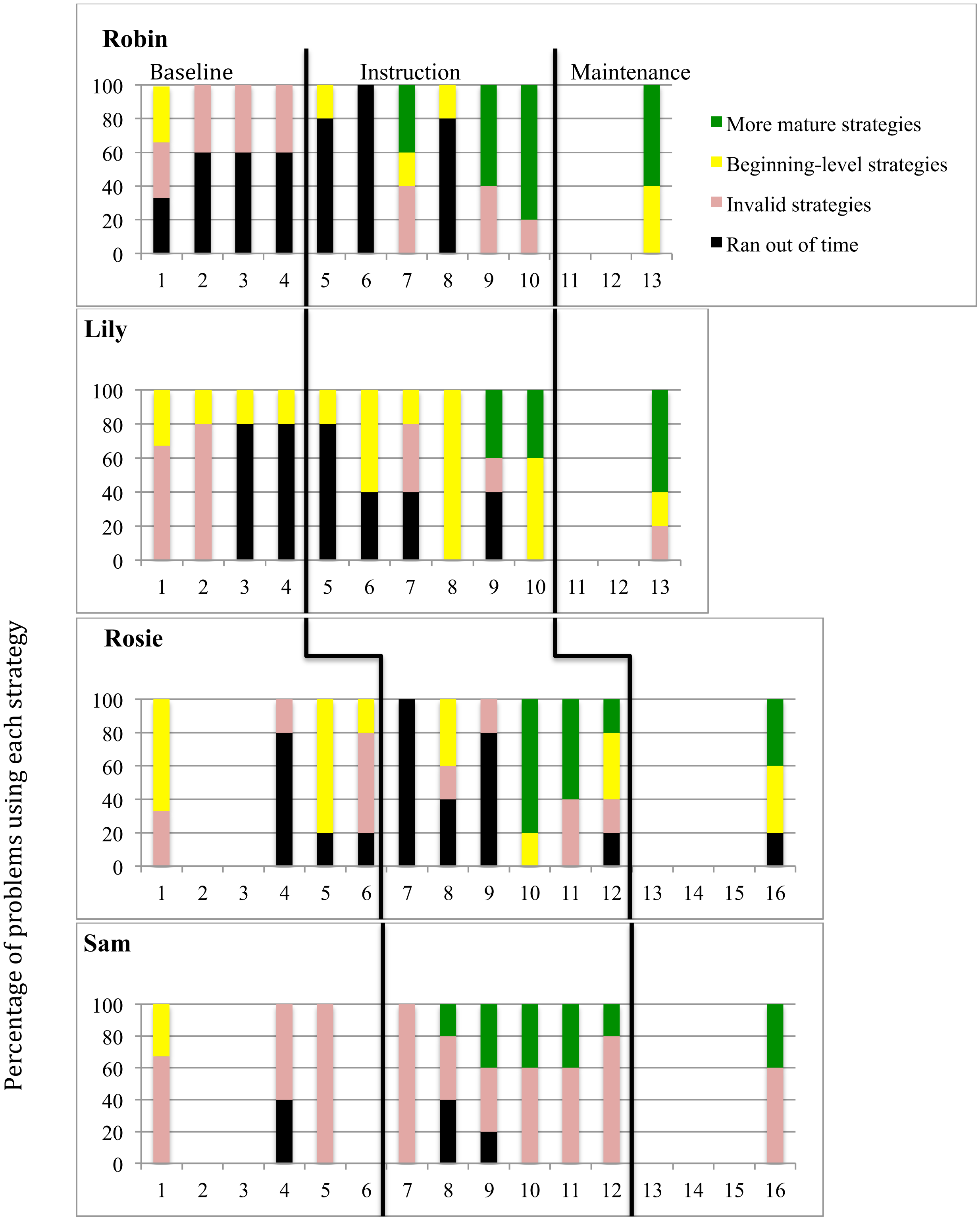


Figure 2 Percentage of valid and invalid strategies used over time

Rosie's use of valid strategies was more variable than the other students (see Figure 2). On one of the baseline days she used no valid strategies and on another she used 4 valid strategies. Despite this variability, Rosie showed progress in her use of valid strategies. During the first half of the intervention she used a mean of 1 valid strategy per week, whereas during the second half of the intervention and during the follow-up interview she used a mean of 4 valid strategies per week.

Sam also started to use more valid strategies as the instruction progressed, but unlike the other students he never used a valid strategy for all of the problems (see Figure 2). During the follow-up interview he used the same valid strategies for the two problems for which he used valid strategies during the third week of instruction.

Development of more mature strategies. In addition to using more valid strategies, all the students started using more mature strategies (see Figure 2). At first Lily, Robin, and Rosie's valid strategies were mostly the beginning-level strategies of direct modeling by tens and counting by tens. They used these strategies to solve both the multiple of tens and hundreds problems and the division by ten problems. These beginning-level strategies were prone to errors and took a long time to execute (see Table 22), with neither of these strategies having more than 69% accuracy rate or taking less than an average of 86 seconds. Lily and Rosie both made counting errors when using direct modeling and counting strategies.

The length of time that the beginning-level strategies took explains why these students so often ran out of time before completing all of the problems. Robin and Rosie both ran out of time in the middle of solving a problem when utilizing each type of beginning-level strategy. Once they started to use more mature strategies they were less likely to run out of time before completing all of the problems.

More mature strategies, such as repeated addition, partial products, recall, using a correct algorithm, and using knowledge of place-value, were more accurate and were much more time efficient (see Table 22) than the beginning-level strategies. The least accurate of the more mature strategies was used correctly 73% of the time, while the other more mature strategies were always used correctly. The slowest of the more mature strategies took an average of 85 seconds, but the mean time it took for the students to use these more mature strategies was 33 seconds. Robin started to use these more mature strategies during the third week, Rosie during the fourth week, and Lily during the fifth week of instruction.

Table 22

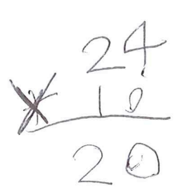
Percentage use, percentage correct, and median problem-solving time for each valid strategy during the intervention

Strategy	Percentage use (n=140)	Percentage correct	Median problem-solving time (s)
Direct Modeling	9	46	286
Counting	9	69	86
Repeated addition	1	100	39
Partial products	11	73	31
Recall	14	100	15
Correct algorithm	1	100	22
Place-value	1	100	85

Sam also started to use more mature valid strategies as the instruction progressed but unlike the other students he only once used a beginning-level strategy—direct modeling by ones—for the problems, and never used a strategy that showed unitizing, such as direct modeling by tens or counting by tens. He progressed directly to the more mature strategies of recall and partial products during the second and third week of instruction.

Variety of strategies. The students who made the most progress across the course of the intervention are the students who tried the most variety of strategies during the probes (see Table

23). These were strategies that they had learned from the instruction during the week. Both Lily and Robin used a variety of valid strategies across the course of the intervention, with Robin using six valid strategies and Lily using five, each of them trying all but one strategy multiple times. When these students first used a new strategy they often solved the problem incorrectly or ran out of time before they could use the strategy to solve the problem, but once they had used the same strategy several times their accuracy and speed improved so that they no longer made errors with that strategy. For example, the first time that Robin used the partial products strategy she said, “Ten times sixty-one then it's … six hundred and one but if there's five singles left over then I would actually have six hundred and six.” The following week she corrected her error when multiplying by ten by saying, “Since it's ten times the other number if it's a two-digit it's just the same with the zero at the end plus the one single then that would mean it would equal a hundred and eleven”, and then correctly used partial products twice more. These students benefited from trying multiple strategies and from using these strategies multiple times so that they could refine their technique.

Rosie used five valid strategies, and she used each of them multiple times. Unlike Lily and Robin, her errors were not towards the beginning of her use of a certain strategy, but the third or fourth time she used a strategy. The exception to this is her use of the standard multiplication algorithm, which she used incorrectly both times. In solving how much candy is in 24 packs of candy she wrote 

because she solved 2*1=2 and 4*0=0 and put these answers together to make 20. Except for her use of the standard multiplication algorithm, Rosie’s errors when using valid strategies seemed to be related to her level of focus on the day, rather than due to her figuring out how the strategy worked.

Sam only used two valid strategies: recall and partial products. From week 2 of the intervention Sam was either using a rule or recall to correctly solve multiples of hundreds, as he said “because like I said anything times ten … just add the like two or one two or three zeroes to it to make it comes the right number. That's how I know it. I learned it last year.” In the maintenance phase of the testing he could generalize this strategy to solve 16 boxes of 100 and say that this contained 1,600 pieces of candy.

He had difficulties using partial products consistently. He consistently knew how to multiply by a hundred but had difficulties multiplying by ten. For example, when he was solving 4 boxes of 100 and 4 packs of 10 he knew that 4 boxes contained 400 pieces of candy but he thought that 4 packs contained 14 pieces of candy because 10+4=14, and so the total was 414 pieces of candy. Throughout the intervention, Sam had difficulties multiplying by ten, as will be discussed in the next section.

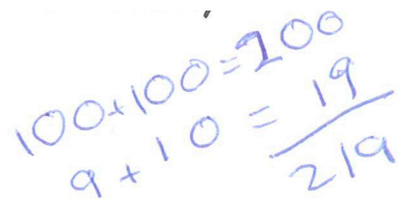
Invalid algorithm. Despite the ease with which both Lily and Sam solved multiples of hundreds problems, they continued to have difficulties with multiples of ten. Both Lily and Sam used an incorrect invented algorithm to solve problems where the numbers of packs were greater than ten. This algorithm is the same one that emerged during their interviews in Section 2: $10*n=100+n$ when $n>10$. Their use of this algorithm persisted throughout the intervention, recurring approximately every other testing time for Sam and every third testing period for Lily. These students’ previously formulated algorithm for multiplying numbers by ten was resistant to change.

Table 23

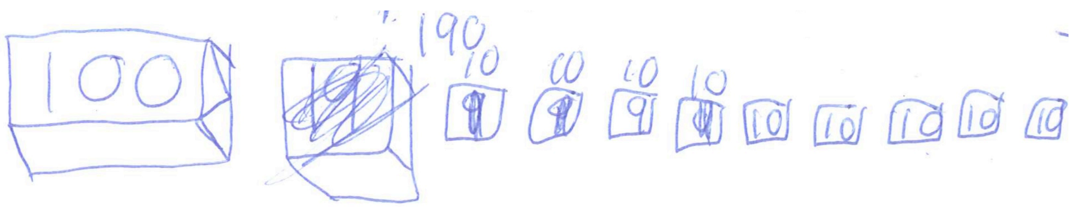
Percentage of problems (n=35) to which each child applied each strategy during the intervention

Student	Ran out of time	Invalid operation	Invalid algorithm	Units error	Incorrect algorithm	Direct modeling	Counting	Repeated addition	Partial Products	Recall	Correct algorithm	Place-value
Lily	29	3	9	0	0	26	14	3	9	9	0	0
Robin	37	9	0	6	0	3	11	0	11	11	6	6
Rosie	37	9	0	0	6	9	11	0	11	17	0	0
Sam	9	43	11	9	0	0	0	0	11	17	0	0

Invalid operation. Sam frequently used the invalid strategy of simply adding the numbers in the problem. He was very proficient and efficient at addition, rarely making an error with this operation. Sam used this invalid strategy fifteen times as his initial strategy, which suggests that he was not independently engaging with the meaning of the word problems. When I helped him think through the meaning of the problems, he was able to illustrate and solve the problems correctly. For example, for the problem 1 box of 100 and 9 packs of 10 candies at first he added all the numbers in the problem



(he knew 1 box was 100 pieces of candy), but once I challenged him to draw a picture of the problem he drew



and realized that there were 190 pieces of candy. These problems were within Sam's zone of proximal development but his reliance on the use of algorithms inhibited his readiness to independently engage with the meaning of the numbers.

The other students made operational errors between once and three times during the intervention. This level of error suggests that these other students were not having great difficulties understanding the context of the word problems.

Discussion

Summary of results

The results of this study showed that the intervention was fairly effective. All the students were more accurate at solving place-value problems at the end of the intervention than at the beginning. All the students used a larger number of valid and more mature strategies at the end of the intervention than at the beginning.

Variety of strategies

The students who made the most progress during this intervention used the widest variety of strategies. They started off using beginning-level strategies that were error prone and inefficient but were an essential stepping-stone to the more mature strategies.

When the students used direct-modeling by tens or counting tens strategies to solve the place-value problems, they were using base-ten numeration knowledge. By using these strategies for both the division and the multiples of tens problems, they demonstrated that they realized that it was possible to both count sets of tens and count by tens. The three students who made the most progress in their understanding of place-value started by using beginning-level strategies to solve both the multiplication and division problems. They solidified their understanding of base-ten numeration by using their unitizing knowledge before they progressed onto more mature strategies. Sam, who made the least progress in his understanding of place-value, never used the beginning-level strategies of direct modeling by ten or counting by ten. It seems that it was necessary for the students to utilize a variety of strategies and go through a phase of solidifying their understanding of unitizing by using the beginning-level strategies before they could take on the more mature strategies. This progression from concrete representations to more abstract and efficient strategies is a typical learning progression for students as they learn to calculate with multidigit numbers (National Research Council, 2001).

The finding that the students in this study made more progress when they used a variety of strategies contradicts the advice that is commonly given to special education teachers that they should only teach one strategy for a given skill to students with learning disabilities because otherwise the students will get confused by the multitude of possible pathways and will end up mastering no strategies (Stein, Silbert, & Carnine, 1997). This finding more closely matched Siegler and Crowley's (1991) finding that typically developing students try a variety of strategies before they settle on one (the one that fulfills the logical requirements for the problem and is also efficient), and that the students who are not producing many different ideas will not be able to make the connections across ideas that allow for growth in their thinking. Therefore, teachers need to allow all students to experiment with a number of different ideas and strategies, not just typically developing students.

Sam used a limited number of strategies during the weekly probe, being focused on addition and other procedural strategies, so he did not make much growth in his thinking throughout the intervention. What could have helped Sam produce a variety of strategies? Sam seemed to value completing the problems quickly, as demonstrated by his tendency to use strategies that could be utilized quickly—recall and partial products—so he needed to slow down and understand the meaning of the problems. If I asked him to read through each problem again, retell and model the problem, tying the concrete representation to a pictorial representation of the problem, he was able to solve the problems correctly. The problems were within his zone of proximal development but he did not solve the problems involving multiples of ten independently. He might have made better progress had he been given strategy-based instruction using the following steps: 1) Read the problem; 2) Retell the problem in your own words; 3) Model the problem with objects; 4) Draw a picture of your model; 5) Label your picture to tell how many objects are in each group; 6) Label your picture to show how you counted; 7) Show another way you could have solved this problem. Sam's lack of direct modeling by tens or counting by tens during baseline was an indication that he was unlikely to use these strategies during the instructional phase of intervention. As these strategies seem to be vital stepping-stones to more sophisticated strategies, in the future any student who does not use one of these beginning-level strategies during baseline could be taught to approach the problems using the above strategy-based instruction. Other students who use these strategies independently can be allowed to use the strategies that make the most sense to them.

Implications

Counting collections based on problems can be an effective way to help students learn about base-ten numeration. Students should be encouraged to use a variety of strategies, starting with beginning-level unitizing strategies, but gradually moving towards more mature and efficient strategies. When students do not independently engage with unitizing concepts it may be necessary to teach them to do so by using a form of strategy-based instruction.

Limitations and future studies

There were several limitations during this study. First, during the instructional portion of the intervention both groups spent a lot of time using concrete objects to investigate unitizing and investigating how the answers were related when counting by sets of ten and counting by tens, but only 3 of the 4 students took this practice on. To fully investigate this question it would be necessary to analyze the students' interactions with the mathematics and with each other during the instruction, which is beyond the scope of this article. Future analysis of the instructional portion of this study is planned.

Second, this study took place in an independent school which is a different environment from public schools. However there were many commonalities between the conditions found in this school and in special education classes in public schools, so the results may be extrapolated to these cases. Third, in this study I did not know the specific diagnoses of the students so I am unable to specify whether this intervention is most appropriate for students with mathematical learning disabilities or reading learning disabilities. Fourth, I focused on the students' knowledge of base-ten numeration, so I do not know how the intervention impacted the students' knowledge of the other aspects of place-value. Fifth, while the first group reached criterion, only one member of the second group reached criterion although the direction of their progress was upwards, so in an ideal world the second group's intervention would have been longer. Sixth, only two groups were included in the research, whereas in single-subject research it is better to include at least 3 groups. Last, the researcher was the instructor during this intervention. In the future it is would be better to have the teachers teach the intervention to ensure that it is possible to make the same progress when taught by a teacher independent from the research.

Now that this preliminary investigation has established that this intervention was fairly effective with two groups of students, a future study should be undertaken that addresses the limitations of this current study. The utility of the instruction should be evaluated when delivered by other instructors, which will involve partnering with special education teachers to refine the instruction. This will include strategy-based instruction for students who do not independently engage in unitizing during baseline, and a longer intervention for students who do not reach criterion on the tests but are making progress towards criterion. As part of the evaluation of the intervention, tests of all four aspects of place-value knowledge will be included and at least three groups of students will participate in the instruction.

CONCLUSIONS

The goal of this dissertation was to further the research into the mathematics knowledge and learning of students with learning disabilities. In particular this dissertation focused on the place-value understanding of students with learning disabilities by linking the results of an assessment study to a subsequent instructional study.

This research helps to build bridges between the mathematics education and special education fields by integrating the best aspects from both fields. This study brings an emphasis on understanding and building on students' thinking from mathematics education and a focus on the strengths and weaknesses of all individual students from special education.

I found a great diversity in the students' understanding of place-value, with five different profiles emerging for what they understood and did not understand about place-value: (1) Good place-value understanding; (2) Reliance on algorithms; (3) Counting errors; (4) Beginning place-value understanding; and, (5) Concatenation of digits. These profiles reflected a range of understandings across the four aspects of place-value: base-ten numeration, place-value numeration, counting, and flexibility with decomposition and composition of numbers. Misconceptions and lack of skill with these place-value aspects led to many of the students' errors in mental arithmetic, algorithmic calculations, and word problems.

There are several implications for future researchers from this study. The first implication is that researchers need to focus more on the place-value understanding of students with learning disabilities because difficulties with place-value understanding can impact other area of mathematics, such as solving multidigit calculations and word problems. The second implication is that when researchers do examine place-value knowledge they need to examine more than positional place-value knowledge, they need to examine all four of the aspects of place-value.

There are also implications for teachers of students with learning disabilities. The findings from this study suggest that these teachers should help students construct a broad understanding of place-value at the same time as they are developing an understanding of standard algorithms. One way to help students develop an understanding of base-ten numeration is to use the instructional activity of counting collections based on problems, but it may be necessary to teach them to engage with the unitizing concepts by using a form of strategy-based instruction. The findings additionally suggest that teachers should be careful about which mnemonics they introduce to students and avoid introducing any "rules" unless they are broadly generalizable.

This study suggests several questions that could be answered by future basic research studies on the topic of place-value understanding for students with learning disabilities:

- What are the group differences in place-value profiles between students with various types of learning disabilities?
- What are the growth trajectories for place-value understanding for students with various types of learning disabilities?

The instructional data collected for this study suggests another question that could still be analyzed:

- How do the students' interactions during instruction impact their understanding of place-value?

and suggests several other questions that could be addressed in future research:

- What is the effectiveness of counting collections as an instructional activity to teach base-ten numeration instruction when taught by the students' teachers?
- What is the effectiveness of implementing strategy-based instruction for students who do not independently engage in unitizing during baseline?

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Appendix A

Problems used in the interview

1. Give the student 120 blocks in a bag (Adapted from Schwedtfeger & Chan, 2007).
    a. *Count the blocks in this bag.*
    b. *How many blocks were in the bag?*
    c. *Show how many you counted.*
2. Put down a completed Ten Frame (Adapted from Wright, Martland, & Stafford, 2006).
    a. *How many do we have? How many dots do we have here?*
    b. Put down Ten Frames one at a time, up to 13 Ten Frames. Each time ask, *How many altogether?*
3. Put down a Ten Frame with only 4 dots (Adapted from Wright, Martland, & Stafford, 2006).
    a. *How many dots are there?*
    b. Place a complete Ten Frame to the right of the Frame with 4 dots. *How many dots are there altogether?*
    c. Continue to place complete Ten Frames to the right of the Frame with 4 dots, up to 13 Ten Frames. Each time ask, *How many dots are there altogether?*
4. *There are 10 candies in a pack.* (Hiebert & Wearne, 1996).
    a. *How many candies are in 4 packs?*
    b. *How many candies are in 10 packs?*
    c. *How many candies are in 13 packs?*
    d. *How many candies in 20 packs?*
    e. *How many candies if there are 15 packs and 4 single pieces?*
5. *Solve the following problems mentally and explain your thinking as you solve the problems.* Write problems on the paper horizontally as well as reading them orally (Adapted from Wright, Martland, & Stafford, 2006).
    a. *38+10*
    b. *38+24*
    c. *30-16*
    d. *33-16*

For the following problems, have the problems written down. Read the problems to the students as they follow along.

6. Sarah had 36 pennies. Her Mom gave her 47 more pennies. How many pennies does she have in all?
7. I had 63 video games. I gave 36 video games to Jordan. How many video games did I have left?
8. Jayden had 38 marbles. His Mom gave him some more marbles for his birthday. Now Jayden has 81 marbles. How many marbles did his Mom give him?
9. Jim read 68 books. Brian read 90 books. How many more books did Brian read than Jim?
10. Ben has 5 packs of markers. There are 14 markers in each pack. How many markers does he have altogether?
11. Mr. Lin has 158 crayons. He wants to put 10 crayons in each box. How many boxes will he need? How many crayons will be left over?

Appendix B

Table 24

Numbers used in probes[9]

Questions	Initial Interview	Baseline	Baseline	Baseline	Intervention	Intervention	Intervention	Intervention	Intervention	Maintenance
11-99 packs of candy	13	68	81	48	24	15	45	30	84	13
1-9 boxes of candy	N/A	9	5	9	6	6	8	3	4	7
11-99 packs of candy & 1-9 singles	15,4	45, 8	25, 6	77, 7	72, 8	82, 5	27, 7	61, 5	11, 1	15, 4
1-9 boxes and 1-9 packs of candy	N/A	2, 3	4, 2	6, 3	7, 1	5, 5	4, 1	4, 4	9, 3	1, 9
111-999 crayons	158	370	614	173	537	405	496	922	858	158

VITA

Helen Thouless first became interested in how to teach students receiving special education services when she volunteered at a school for students with special educational needs while studying mathematics and science for A'Levels in Cambridge, England. After high school she moved back to the U.S.A. where she received a B.A. in Psychology from Reed College, Oregon. On returning from two years serving in the Peace Corps in the Republic of Guinea, Helen received a dual certification in elementary education and special education from the Masters in Teaching program at the University of Washington—Seattle. After five years teaching in elementary schools in both Tanzania and the Seattle region, Helen returned to the University of Washington—Seattle to study how to provide high-quality mathematics teaching for students with special educational needs. While pursuing her graduate degree, Helen taught as a mathematics specialist and a special education teacher at a local elementary school, taught graduate mathematics education courses, and supported a mathematics school improvement project. Helen received her PhD in Learning Sciences with a focus on mathematics education from the University of Washington in March 2014.