HITTING A BOUNDARY POINT WITH REFLECTED BROWNIAN MOTION

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Abstract. An explicit integral test involving the reflection angle is given for the reflected Brownian motion in a half-plane to hit a fixed boundary point.

1. Introduction and main results. Let $D_* = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$, identify $\mathbf{R}^2$ with $\mathbf{C}$ and ∂D_* with $\mathbf{R}$ and suppose that $\theta : \mathbf{R} \to (-\pi/2, \pi/2)$ is a $C^{1+\varepsilon}$-function except, possibly, at 0. Then there exists a reflected Brownian motion (RBM) in D_* with the variable angle of reflection $\theta(x)$. The angle of reflection $\theta(x)$ is measured in the clockwise direction from the inward pointing normal. Here is a straightforward construction of such a process (Rogers (1991)).

Let $Y(t) = Y_1(t) + iY_2(t)$ be a standard 2-dimensional Brownian motion, $Y(0) = y_1 + iy_2, y_2 > 0$. Let

$$L_t = \max(-\inf_{s \le t} Y_2(s), 0),$$

$$X_2(t) = Y_2(t) + L_t. \tag{1.1}$$

Then the equation

$$X_1(t) = Y_1(t) + \int_0^t \tan\theta(X_1(s))dL_s \tag{1.2}$$

has a solution. The process $X(t) \stackrel{\text{df}}{=} X_1(t) + iX_2(t)$ is an RBM in D_* with the angle of reflection θ. The process X is defined only until it hits 0, i.e., it is defined on a random time interval. The same remark pertains to other related processes discussed in this paper.

Here is our main result.

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Theorem 1.1. *The reflected Brownian motion X hits 0 with positive probability if and only if*

$$\int_0^1 \frac{1}{y} \exp\left[\int_{-1}^1 \frac{\theta(x)xdx}{\pi(x^2+y^2)}\right] \cos\left[\int_{-1}^1 \frac{\theta(x)ydx}{\pi(x^2+y^2)}\right] dy < \infty. \tag{1.3}$$

Remarks 1.1. (i) If RBM X approaches 0 then it does so in a finite time because the one-dimensional RBM $\operatorname{Im} X$ cannot stay bounded forever.

(ii) Theorem 1.1 answers a problem posed by Rogers (1991). Varadhan and Williams (1985) discussed the case when θ is constant on the positive and negative part of the real axis. A partial solution in the general case is presented in Rogers (1991). See also a new article by Rogers (1990).

(iii) Note that (1.3) is equivalent to

$$\int_0^1 \frac{1}{y} \exp\left[\int_{-1}^1 \frac{\theta(x)xdx}{\pi(x^2+y^2)}\right] dy < \infty \tag{1.4}$$

in each of the following cases:

(a) when θ is an odd function, or

(b) when $|\theta|$ is bounded away from $\pi/2$, i.e., there is $c > 0$ such that $|\theta(x)| < \pi/2 - c$ for all x.

In case (a) the integral under cosine in (1.3) is zero and in case (b) the cosine of the same integral is bounded away from 0.

(iv) Our proof uses an idea of Rogers (1989, 1991). We will map D_* onto a "strip domain" D using an analytic function h so that $h(X)$ is a time-changed RBM in D with vertical vectors of reflection. The point $0 \in \partial D_*$ is mapped onto the "left endpoint" z_0 of ∂D, possibly "∞". The horizontal component of $h(X)$ is a time-changed Brownian motion and it hits z_0 with positive probability if and only if z_0 has a finite real part. This is equivalent to X hitting 0 with positive probability and may be expressed algebraically as in (1.3).

Originally Rogers (1991) mapped D_* onto a domain above the graph of a function. This approach does not result immediately in an integral test but it has some other potential. We will explore it in a forthcoming paper.

The test (1.3) may be difficult to apply as it contains complicated integrals. We will now give a few more or less concrete examples of θ's which satisfy or do not satisfy (1.3).

Corollary 1.1. *Suppose that $\alpha > 0$ and*

$$\theta(x) = \begin{cases} -\operatorname{sgn}(x)\dfrac{\alpha}{|\log|x||} & \text{if } |x| < 1/3, \\ 0 & \text{if } |x| \geq 1/3. \end{cases}$$

Then the RBM X hits 0 with positive probability if and only if $\alpha > \pi/2$.

One may consider an RBM in the strip $\{z \in \mathbf{C} : \operatorname{Im} z \in (0,\pi)\}$ rather than in D_*. This strip is conformally equivalent to D_* (use the mapping $z \to e^z$) and "$-\infty$" corresponds to $0 \in \partial D_*$. It is natural to consider periodic angles of reflection in a strip. They correspond to "geometrically periodic" θ in D_* which we discuss in the next corollary.

Corollary 1.2. *Suppose that for some $c > 1$ and all $x \in \mathbf{R}$ we have $\theta(x) = \theta(cx)$. Then the RBM X hits 0 with a positive probability if and only if*

$$\int_1^c \frac{\theta(x) - \theta(-x)}{x} dx < 0.$$

Corollary 1.3. *The event that the first hitting time T_0 of 0 is finite and there exists $\varepsilon > 0$ such that $\operatorname{Re} X(t) > 0$ for all $t \in (T_0 - \varepsilon, T_0)$ has positive probability if and only if*

$$\int_0^1 \frac{(\theta(x) + \pi/2)dx}{x} < \infty.$$

In particular, X may approach 0 from the right if for some $\alpha < -1$, we have $\theta(x) = -\pi/2 + |\log x|^\alpha$ for $x > 0$. The process will not approach 0 from the right if $\theta(x) = -\pi/2 + |\log x|^{-1}$ for $x > 0$.

We are glad to acknowledge great influence of ideas of Rogers (1989, 1991) on our research. We would also like to express our gratitude to Chris Rogers for numerous discussions of the subject.

2. Proofs. Recall that we identify $\mathbf{R}^2$ with $\mathbf{C}$ and ∂D_* with $\mathbf{R}$. Let $\mathbf{R}_+ = \{x \in \mathbf{R} : x > 0\}$ and $\mathbf{R}_- = \{x \in \mathbf{R} : x < 0\}$. The closure of a set A will be denoted $\overline{A}$. For a harmonic function φ, its conjugate function will be denoted $\widetilde{\varphi}$.

Proof of Theorem 1.1. **Step 1.** A domain D will be called a *strip domain* if whenever $x + iy_1 \in D$ and $x + iy_2 \in D$ then $x + iy \in D$ for all $y \in [y_1, y_2]$. The vector of reflection is defined by $V(x) = \tan\theta(x) + i$. We will map D_* conformally onto a "strip domain" D in such a way that V will be mapped onto a vertical vector. Moreover, $\mathbf{R}_-$ will be mapped onto the "upper boundary" of D and the image of $V(x)$ will point downwards for $x \in \mathbf{R}_-$. The positive part of the real axis will be mapped onto the "lower boundary" of D and the image of $V(x)$ will point upwards for $x \in \mathbf{R}_+$.

Let $D_1 = \{z \in \mathbf{C} : \operatorname{Im} z \in (0, \pi)\}$ and let $g(z)$ be the branch of $\log z$ which maps D_* onto D_1. For $z \in \partial D_1$, let $\varphi(z) = \theta(e^z)$. Extend φ continuously to D_1 so as to be bounded and harmonic in D_1 and let $\widetilde{\varphi}$ be a conjugate function of φ. Define an analytic function f on D_1 by

$$f'(z) = \exp(i(\varphi(z) + i\widetilde{\varphi}(z))). \tag{2.1}$$

Note that $\varphi(z) \in (-\pi/2, \pi/2)$ for $z \in D_1$. Therefore,

$$\operatorname{Re} f'(z) = e^{-\widetilde{\varphi}(z)} \cos\varphi(z) > 0. \tag{2.2}$$

Let $\gamma(t) = tz + (1-t)w$ where $z, w \in D_1$. Then $\gamma'(t) = z - w$ and

$$\begin{aligned} f(z) - f(w) &= \int_0^1 f'(\gamma(t))(z-w)dt \\ &= \left[\int_0^1 f'(\gamma(t))dt\right](z-w). \end{aligned}$$

Since the real part of the integral is strictly positive, $f(z) = f(w)$ if and only if $z = w$. In other words, the function f is univalent. Let $h = f \circ g$ on D_* and $D = h(D_*)$.

Let us establish some basic properties of h and D.

The argument of f' is always strictly between $-\pi/2$ and $\pi/2$ so $\{z \in D_1 : \operatorname{Im} z = \pi\}$ is mapped by f onto a curve Γ_1 which is the graph of a function. We obviously have $h(\mathbf{R}_-) = \Gamma_1$. By analogy, $\Gamma_2 \stackrel{\text{df}}{=} h(\mathbf{R}_+)$ is a similar curve. It follows from the argument principle that D is a "strip domain."

The derivative of h is given by

$$h'(z) = f'(g(z))g'(z) = f'(\log z)\frac{1}{z}. \tag{2.3}$$

A harmonic function composed with an analytic function is harmonic, so $\varphi(\log z) = \theta(z)$ and $\widetilde{\varphi}(\log z) = \widetilde{\theta}(z)$ for $z \in D_*$, where θ is the bounded harmonic extension of the original θ to the whole of D_* and $\widetilde{\theta}$ is a conjugate function of θ. Hence, (2.1) and (2.3) yield

$$h'(z) = \frac{1}{z}\exp[i(\theta(z) + i\widetilde{\theta}(z))]. \tag{2.4}$$

We have

$$\arg h'(x) = \theta(x) \qquad \text{for } x \in \mathbf{R}_+$$

and

$$\arg h'(x) = \theta(x) - \pi \qquad \text{for } x \in \mathbf{R}_-.$$

This implies that the horizontal component of the vector $h'(x)V(x)$ is null for $x \in \mathbf{R}$, $x \neq 0$. In other words, the vector $V(x)$ is mapped by h onto a vertical vector for $x \in \mathbf{R}\backslash\{0\}$.

Step 2. In this step, we will prove that h is C^2 on $\overline{D_*}$ (except at 0) provided $\theta \in C^{1+\varepsilon}$ away from 0. Our argument is standard but we could not find a ready reference.

Let

$$\alpha(x) = \begin{cases} \theta(x) & \text{for } x \in \mathbf{R}_+, \\ \theta(x) - \pi & \text{for } x \in \mathbf{R}_-. \end{cases}$$

Extend α boundedly and harmonically to D_* and let $\widetilde{\alpha}$ be the conjugate function. Observe that $h'(z) = \exp(i(\alpha(z) + i\widetilde{\alpha}(z)))$.

First we will localize our argument. Let I be an open interval in $\mathbf{R}_+$ or $\mathbf{R}_-$ and let J be an open subinterval of I with $\overline{J} \subset I$. Let $\psi \in C^\infty(\mathbf{R})$ with $\operatorname{supp}(\psi) \subset I$ and $\psi \equiv 1$ on J. Then $\psi\alpha \in C^{1+\varepsilon}$. Moreover $(\alpha + i\widetilde{\alpha}) - (\psi\alpha + i\widetilde{\psi\alpha})$ extends analytically across J, by the Schwartz reflection principle, since $\alpha - \psi\alpha = 0$ on J. Hence, $h \in C^2(D_* \cup J)$ provided the analogous function corresponding to $\psi\alpha$ has the same property. We will assume without loss of generality that $\alpha \in C^{1+\varepsilon}(\mathbf{R})$ and has compact support which lies in $\mathbf{R}_+$ or $\mathbf{R}_-$.

Let $\beta(x) = \alpha'(x)$ for $x \in \mathbf{R}$ and

$$\beta(z) \stackrel{\text{df}}{=} \int_{-\infty}^{\infty} \frac{y}{v^2 + y^2}\beta(x+v)\frac{dv}{\pi}, \qquad z = x + iy \in D_*,$$

be the harmonic extension of β to D_*. We have

$$\alpha(z) = \int_{-\infty}^{\infty} \frac{y}{v^2+y^2}\alpha(x+v)\frac{dv}{\pi}, \qquad z = x+iy \in D_*.$$

By interchanging integration and differentiation we see that $\beta(z) = \frac{\partial}{\partial x}\alpha(z)$ for $z \in D_*$. Since β is continuous on $\mathbf{R}$, its harmonic extension to D_* is continuous on $\overline{D_*}$ and equal to $\beta = \alpha'$ on $\mathbf{R}$. In other words, $\frac{\partial}{\partial x}\alpha$ is continuous on $\overline{D_*}$.

By Theorem 6.8 of Zygmund (1979, vol. I, p. 54) transported to D_*, $\widetilde{\beta}$ extends to be continuous on $\overline{D_*}$. Likewise, $\widetilde{\alpha}$ is continuous on $\overline{D_*}$.

Since the analytic functions $\frac{\partial}{\partial x}[\alpha(z)+i\widetilde{\alpha}(z)]$ and $\beta(z)+i\widetilde{\beta}(z)$ have the same real part, we have $\frac{\partial}{\partial x}\widetilde{\alpha}(z) = \widetilde{\beta}(z) + ic$ where c is a real constant. Thus $\frac{\partial}{\partial x}\widetilde{\alpha}$ extends to be continuous on $\overline{D_*}$. Moreover, on $\mathbf{R}$ this extension equals $\frac{\partial}{\partial x}\widetilde{\alpha}(x)$ since

$$\begin{aligned}\widetilde{\alpha}(x_1+iy) - \widetilde{\alpha}(x_2+iy) &= \int_{x_2}^{x_1} \frac{\partial}{\partial x}\widetilde{\alpha}(v+iy)dv \\ &= \int_{x_2}^{x_1} [\widetilde{\beta}(v+iy)+ic]dv \\ &\underset{y\to 0}{\longrightarrow} \int_{x_2}^{x_1} [\widetilde{\beta}(v)+ic]dv.\end{aligned}$$

Divide $\widetilde{\alpha}(x_1) - \widetilde{\alpha}(x_2)$ by $x_1 - x_2$ and let $x_1 - x_2 \to 0$. Thus $\frac{\partial}{\partial x}\widetilde{\alpha}(x) = \widetilde{\beta}(x) + ic$ on $\mathbf{R}$.

Let $x \in \mathbf{R}$. By the mean value theorem

$$\frac{\alpha(x+is)-\alpha(x)}{s} = \frac{\partial \alpha}{\partial y}(x+iy)|_{y=t}$$

for some $t \in [0, s]$. Since $\frac{\partial}{\partial y}\alpha = -\frac{\partial}{\partial x}\widetilde{\alpha}$, $\frac{\partial}{\partial y}\alpha$ extends to be continuous on $\overline{D_*}$ and hence, for $x \in \mathbf{R}$,

$$\lim_{t\downarrow 0}\frac{\partial \alpha}{\partial y}(x+iy)|_{y=t} = \lim_{s\downarrow 0}\frac{\alpha(x+is)-\alpha(x)}{s} = \frac{\partial \alpha}{\partial y}(x+iy)|_{y=0}.$$

A similar statement applies to $\frac{\partial}{\partial y}\widetilde{\alpha} = \frac{\partial}{\partial x}\alpha$.

Thus we have shown that $\alpha + i\widetilde{\alpha}$ is a C^1 function on $\overline{D_*}$.

Recall that h is analytic in D_* with $h'(z) = \exp(i(\alpha(z)+i\widetilde{\alpha}(z)))$ for $z \in D_*$. By the above remarks, $h' \in C^1(\overline{D_*})$. Since h is the integral of the derivative (which is bounded), h is continuous on $\overline{D_*}$. By the reasoning above $\frac{\partial}{\partial x}h$ and $\frac{\partial}{\partial y}h$ are continuous on $\overline{D_*}$ with $\frac{\partial}{\partial x}h = h'$ and $\frac{\partial}{\partial y}h = ih'$ for $z \in \overline{D_*}$. Again, using the result above, h, $\frac{\partial}{\partial x}h$, $\frac{\partial}{\partial y}h$, $\frac{\partial^2}{\partial x^2}h$, $\frac{\partial^2}{\partial y^2}h$, $\frac{\partial^2}{\partial x\partial y}h$ and $\frac{\partial^2}{\partial y\partial x}h$ are all continuous on $\overline{D_*}$. In other words, h is C^2 on $\overline{D_*}$ (except at 0, since we used a localization argument).

Step 3. Let

$$\begin{aligned} a = a_D &= \inf\{\operatorname{Re} z : z \in D\}, \\ b = b_D &= \inf\{\operatorname{Re} h(z) : \operatorname{Re} z = 0, z \in D_*\}.\end{aligned}$$

Clearly $a \leq b$, though there are domains for which $a \neq b$. We will prove that $a = -\infty$ if and only if $b = -\infty$.

It follows from (2.2) that $\operatorname{Re} f$ is increasing on horizontal lines. This implies that $\operatorname{Re} h(z)$ is an increasing function of $|z|$ along the half lines in D_* ending at 0 and for $z \in U \stackrel{\text{df}}{=} \{z \in D_* : |z| < 1\}$ we have

$$\operatorname{Re} h(z) = \operatorname{Re} f(\log z) \leq \sup_{\substack{v \in D_1 \\ \operatorname{Re} v = 0}} \operatorname{Re} f(v) \stackrel{\text{df}}{=} M < \infty.$$

Thus $M - \operatorname{Re} h$ is a positive harmonic function on U and is continuous on $\overline{U} \setminus \{0\}$. Therefore, it has the following representation

$$M - \operatorname{Re} h(z) = PI(M - \operatorname{Re} h)(z) + \frac{cy}{x^2 + y^2}, \qquad z = x + iy,$$

where "PI" is the analog of the Poisson integral and c is a non-negative constant. The above representation is well known for the disc and can be transported to U by a conformal mapping.

Suppose that $a = -\infty$. If $c \neq 0$ in the above formula then clearly $M - \operatorname{Re} h(iy) \to \infty$ as $y \to 0$. If $c = 0$ then we also have $M - \operatorname{Re} h(iy) \to \infty$. This follows easily from the maximum principle and the fact that $M - \operatorname{Re} h(z)$ increases as $|z|$ decreases, $z \in \partial D_*$. In both cases we have $b = -\infty$.

Step 4. Equations (1.1) and (1.2) may be rewritten as

$$X(t) = Y(t) + \int_0^t V(X_s) dL_s$$

where V is the vector of reflection introduced in Step 1. The mapping h is of class C^2 in $\overline{D_*} \setminus \{0\}$ and analytic in D_* so the Itô formula is applicable to $h(X)$ and we obtain

$$h(X(t)) = h(X(0)) + \int_0^t h'(X(s)) dY(s) + \int_0^t h'(X(s)) V(X(s)) dL_s.$$

By the abuse of notation, h' denotes in the above formula the Jacobian matrix of $h(x, y)$. The process X spends zero time on ∂D_* and h is analytic in D_* so $\int_0^t h'(X(s)) dY_s$ is a time-change of Brownian motion. The local time L does not increase unless X is at the boundary of D_* and $\operatorname{Re} h'(x) V(x) = 0$ for $x \in \partial D_*$, $x \neq 0$, so $\int_0^t h'(X(s)) V(X(s)) dL_s$ has null real component. It follows that $\operatorname{Re} h(X(t))$ is a time-changed one-dimensional Brownian motion run for a random amount of time.

Note that $h'(z) \in \mathbf{C} \setminus \{0\}$ for $z \neq 0$. If we time-change $\operatorname{Re} h(X(t))$ so that it becomes a Brownian motion, it cannot stop or converge unless X reaches 0 or ∞.

Whether X hits 0 with positive probability, does not depend on the values of $\theta(x)$ for $|x| > 1$. Thus we may assume without loss of generality that $\theta(x) = 0$ for $|x| > 1$. Then $\Gamma_1 = h(\mathbf{R}_-)$ and $\Gamma_2 = h(\mathbf{R}_+)$ cannot intersect at a finite right extreme point of D and $\sup\{\operatorname{Re} z : z \in D\} = \infty$. Since $\operatorname{Re} h(X(t))$ is a time-change of Brownian motion, it cannot converge to $+\infty$ and it follows that $\operatorname{Re} h(X(t))$ cannot stop or converge unless X hits 0.

Suppose that $a_D > -\infty$. If $\operatorname{Re} h(X(t))$ stops at a finite time or converges then X hits 0 and we are done. Otherwise $\operatorname{Re} h(X(t))$ will hit a_D with probability 1. Let $T_0 = \inf\{t > 0 : \operatorname{Re} h(X(t)) = a_D\}$. Then $\{X(t), 0 < t < T_0\}$ is a curve in $\overline{D_*}$ which must converge to 0 as $t \to T_0$. We have already pointed out in Remark 1.1(i) that $T_0 < \infty$ a.s.

Now consider the case $a_D = -\infty$. If $\operatorname{Re} h(X(t))$ stops at a finite time or converges then X converges to 0 and $\operatorname{Re} h(X(t))$ converges to $-\infty$. This is impossible for a time-changed Brownian motion and therefore $\operatorname{Re} h(X(t))$ will take arbitrarily large values in every interval (t_0, ∞). According to Step 3, $M - \operatorname{Re} h$ is positive in a neighborhood of 0 so $X(t)$ will never approach 0.

We have just shown that X hits 0 with positive probability if and only if $a_D > -\infty$ and this is equivalent to $b_D > -\infty$ by Step 3. Recall that for $y > 0$

$$\begin{aligned} \frac{\partial}{\partial y} \operatorname{Re} h(iy) &= -\operatorname{Im} h'(iy) \\ &= -\operatorname{Im}\left[\frac{1}{iy} \exp(i(\theta(iy) + i\widetilde{\theta}(iy)))\right] \\ &= \frac{1}{y} \exp(-\widetilde{\theta}(iy)) \cos\theta(iy) > 0. \end{aligned} \tag{2.5}$$

Thus $b_D > -\infty$ if and only if

$$\int_0^1 \frac{\partial}{\partial y} \operatorname{Re} h(iy) dy < \infty. \tag{2.6}$$

We may use the following formula to express the harmonic extension of θ and its conjugate, $\widetilde{\theta}$, which vanishes at i.

$$\begin{aligned} (\theta(iy) + i\widetilde{\theta}(iy)) &= \int_{-\infty}^{\infty} \left(\frac{1}{x - iy} - \frac{x}{1 + x^2} \right) \frac{\theta(x)dx}{\pi} \\ &= \int_{-\infty}^{\infty} \left(\frac{x}{x^2 + y^2} - \frac{x}{1 + x^2} \right) \frac{\theta(x)dx}{\pi} + i \int_{-\infty}^{\infty} \frac{y}{x^2 + y^2} \frac{\theta(x)dx}{\pi}. \end{aligned} \tag{2.7}$$

In view of (2.5) and (2.7), condition (2.6) becomes

$$\int_0^1 \frac{1}{y} \exp\left[\int_{-\infty}^{\infty} \left(\frac{x}{x^2 + y^2} - \frac{x}{1 + x^2} \right) \frac{\theta(x)dx}{\pi} \right] \cos\left[\int_{-\infty}^{\infty} \frac{y}{x^2 + y^2} \frac{\theta(x)dx}{\pi} \right] dy < \infty. \tag{2.8}$$

As before, we may assume that $\theta(x) = 0$ for $|x| > 1$ and rewrite (2.8) as

$$\int_0^1 \frac{1}{y} \exp\left[\int_{-1}^{1} \left(\frac{x}{x^2 + y^2} - \frac{x}{1 + x^2} \right) \frac{\theta(x)dx}{\pi} \right] \cos\left[\int_{-1}^{1} \frac{y}{x^2 + y^2} \frac{\theta(x)dx}{\pi} \right] dy < \infty. \tag{2.9}$$

Note that

$$\left| \int_{-1}^{1} \frac{x}{1 + x^2} \frac{\theta(x)dx}{\pi} \right| < (\log 2)/2$$

since $|\theta(x)| < \pi/2$. We can drop the corresponding integral from (2.9) and obtain an equivalent inequality

$$\int_0^1 \frac{1}{y} \exp\left[\int_{-1}^1 \frac{x}{x^2+y^2} \frac{\theta(x)dx}{\pi}\right] \cos\left[\int_{-1}^1 \frac{y}{x^2+y^2} \frac{\theta(x)dx}{\pi}\right] dy < \infty.$$

This completes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.1. Since θ is an odd function, (1.3) reduces to (1.4).

Suppose that $\alpha \le \pi/2$. We have for $y < 1/3$

$$\begin{aligned}\int_0^{1/3} \frac{x\,dx}{(x^2+y^2)|\log x|} &= \left(\int_0^y + \int_y^{1/3}\right) \frac{x\,dx}{(x^2+y^2)|\log x|} \\ &\le \int_0^y \frac{xdx}{y^2} + \int_y^{1/3} \frac{xdx}{x^2|\log x|} \\ &= c_1 + \log|\log y|.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^1 \frac{1}{y} \exp\left[\int_{-1}^1 \frac{x}{x^2+y^2} \frac{\theta(x)dx}{\pi}\right] dy &= \int_0^1 \frac{1}{y} \exp\left[-2\int_0^{1/3} \frac{x}{x^2+y^2} \frac{\alpha dx}{\pi|\log x|}\right] dy \\ &\ge \int_0^{1/3} \frac{1}{y} \exp\left[-\frac{2\alpha}{\pi}(c_1 + \log|\log y|)\right] dy \\ &\ge c_2 \int_0^{1/3} \frac{dy}{y|\log y|^{2\alpha/\pi}} = \infty.\end{aligned}$$

Hence, (1.3) is not satisfied when $\alpha \le \pi/2$.

Now assume that $\alpha > \pi/2$ and choose $\varepsilon > 0$ and $a < \infty$ such that

$$\frac{a^2}{a^2+1} \frac{2\alpha}{\pi} > 1 + \varepsilon.$$

Then, for $y < 1/(3a)$,

$$\begin{aligned}\int_0^{1/3} \frac{x\,dx}{(x^2+y^2)|\log x|} &\ge \int_{ay}^{1/3} \frac{x\,dx}{(x^2+y^2)|\log x|} \\ &\ge \int_{ay}^{1/3} \frac{x\,dx}{(x^2+x^2/a^2)|\log x|} \\ &= \frac{a^2}{a^2+1} \int_{ay}^{1/3} \frac{dx}{x|\log x|} \\ &= c_3 + \frac{a^2}{a^2+1} \log|\log ay|.\end{aligned}$$

We obtain

$$\begin{aligned}
&\int_0^{1/3a} \frac{1}{y} \exp\left[\int_{-1}^{1} \frac{x}{x^2+y^2} \frac{\theta(x)dx}{\pi}\right] dy \\
&\quad = \int_0^{1/3a} \frac{1}{y} \exp\left[-2\int_0^{1/3} \frac{x}{x^2+y^2} \frac{\alpha dx}{\pi|\log x|}\right] dy \\
&\quad \le \int_0^{1/3a} \frac{1}{y} \exp\left[-\frac{2\alpha}{\pi}\left(c_3 + \frac{a^2}{a^2+1}\log|\log ay|\right)\right] dy \\
&\quad \le c_4 \int_0^{1/3a} \frac{dy}{y|\log ay|^{1+\varepsilon}} < \infty.
\end{aligned}$$

This implies (1.4). □

Proof of Corollary 1.2. First we will derive a formula for

$$\frac{\partial}{\partial y} \operatorname{Re} h(i(cy)) / \frac{\partial}{\partial y} \operatorname{Re} h(iy), \quad y > 0,$$

where h is the function defined in the proof of Theorem 1.1.

The analytic functions $\theta(z) + i\widetilde{\theta}(z)$ and $\theta(cz) + i\widetilde{\theta}(cz)$ have the same real part and hence differ by a purely imaginary constant. When we evaluate the difference at i and take into account that $\widetilde{\theta}(i) = 0$ we see that the constant is equal to $-i\widetilde{\theta}(ic)$. This fact and (2.4) yield

$$h'(cz)/h'(z) = e^{-\widetilde{\theta}(ic)}/c > 0.$$

Since $\frac{\partial}{\partial y} \operatorname{Re} h(iy) = \operatorname{Re} ih'(iy)$,

$$\frac{\partial}{\partial y} \operatorname{Re} h(i(cy)) / \frac{\partial}{\partial y} \operatorname{Re} h(iy) = e^{-\widetilde{\theta}(ic)}/c.$$

Now

$$\widetilde{\theta}(ic) = \int_{-\infty}^{\infty} \left(\frac{x}{1+x^2} - \frac{x}{x^2+c^2}\right) \frac{\theta(x)dx}{\pi}.$$

Since $\theta(cx) = \theta(x)$, we have

$$\begin{aligned}
&\int_0^{\infty} \left(\frac{x}{1+x^2} - \frac{x}{x^2+c^2}\right) \frac{\theta(x)dx}{\pi} \\
&\quad = \lim_{n\to\infty} \sum_{k=-n}^{n} \int_{c^k}^{c^{k+1}} \left(\frac{x}{1+x^2} - \frac{x}{x^2+c^2}\right) \frac{\theta(x)dx}{\pi} \\
&\quad = \lim_{n\to\infty} \sum_{k=-n}^{n} \int_1^{c} \left(\frac{c^k v}{1+(c^k v)^2} - \frac{c^k v}{(c^k v)^2+c^2}\right) \frac{\theta(c^k v)c^k dv}{\pi} \\
&\quad = \lim_{n\to\infty} \sum_{k=-n}^{n} \int_1^{c} \left(\frac{c^{2k} v}{1+c^{2k} v^2} - \frac{c^{2(k-1)} v}{c^{2(k-1)} v^2+1}\right) \frac{\theta(v)dv}{\pi} \\
&\quad = \lim_{n\to\infty} \int_1^{c} \left(\frac{c^{2n} v}{1+c^{2n} v^2} - \frac{c^{2(-n-1)} v}{1+c^{2(-n-1)} v^2}\right) \frac{\theta(v)dv}{\pi} \\
&\quad = \int_1^{c} \frac{\theta(v)dv}{\pi v}.
\end{aligned}$$

Thus

$$\widetilde{\theta}(ic) = \int_1^c \frac{\theta(v) - \theta(-v)}{\pi v} dv.$$

Recall that (1.3) is equivalent to (2.6). We have

$$\begin{aligned}
\int_0^1 \frac{\partial}{\partial y} \operatorname{Re} h(iy) dy &= \sum_{k=0}^{\infty} \int_{c^{-k-1}}^{c^{-k}} \frac{\partial}{\partial y} \operatorname{Re} h(iy) dy \\
&= \sum_{k=0}^{\infty} \int_{1/c}^{1} \frac{\partial}{\partial y} \operatorname{Re} h(i(c^{-k} y)) c^{-k} dy \\
&= \sum_{k=0}^{\infty} \int_{1/c}^{1} \frac{\partial}{\partial y} \operatorname{Re} h(iy) (\frac{1}{c} e^{-\widetilde{\theta}(ic)})^{-k} c^{-k} dy \\
&= \int_{1/c}^{1} \frac{\partial}{\partial y} \operatorname{Re} h(iy) dy \sum_{k=0}^{\infty} (e^{-\widetilde{\theta}(ic)})^{-k}.
\end{aligned}$$

The last expression is finite if and only if $e^{-\widetilde{\theta}(ic)} > 1$. Hence, (1.3) is equivalent to $\widetilde{\theta}(ic) < 0$, i.e.,

$$\int_1^c \frac{\theta(v) - \theta(-v)}{\pi v} dv < 0. \quad \square$$

Proof of Corollary 1.3. If the RBM in D_* may approach 0 from one side only, the values of θ on the other side are irrelevant and we may assume without loss of generality that θ is an odd function. If $\int_0^1 (\theta(x) + \pi/2) x^{-1} dx < \infty$ then $\theta(x) \to -\pi/2$ as $x \downarrow 0$ and a computation analogous to the one in the proof of Corollary 1.1 shows that (1.3) holds. Hence it will suffice to discuss the case when $T_0 < \infty$ a.s.

Step 1. First we will show that with positive probability there is a random interval $(T_0 - \varepsilon, T_0)$ such that $\operatorname{Re} X(t) > 0$ for all $t \in (T_0 - \varepsilon, T_0)$ if and only if with positive probability there is a random interval $(T_0 - \varepsilon, T_0)$ such that $\operatorname{Re} X(t) > 0$ for all $t \in (T_0 - \varepsilon, T_0)$ such that $X(t) \in \partial D_*$.

Let

$$\begin{aligned}
T_1 &= \inf\{t \in (0, T_0] : \operatorname{Re} X(t) = 0\}, \\
U_1 &= \inf\{t \in (T_1, T_0] : X(t) \in \mathbf{R}\}, \\
T_k &= \inf\{t \in (U_{k-1}, T_0] : \operatorname{Re} X(t) = 0\}, \quad k \geq 2, \\
U_k &= \inf\{t \in (T_k, T_0] : X(t) \in \mathbf{R}\}, \quad k \geq 2.
\end{aligned}$$

There are two possible cases. First, suppose that, with positive probability, $T_k = T_0$ for some k and, consequently, $T_m = \infty$ for $m > k$. Then our claim follows with $\varepsilon = T_0 - T_{k-1}$ (if $k = 1$ we let $\varepsilon = T_0$). Note that although $X((T_{k-1}, T_0))$ may lie in the left half plane, it may also lie in the right half plane with positive probability, by the symmetry of θ.

Now suppose that $T_k < T_0$ for all k a.s. The events $\{\operatorname{Re} X(U_k) > 0\}$ are independent by the strong Markov property and each one has probability 1/2, by symmetry. It follows that infinitely many events $\{\operatorname{Re} X(U_k) > 0\}$ happen a.s. and the same is true for $\{\operatorname{Re} X(U_k) < 0\}$. In this case, with probability 1, for every

$\varepsilon > 0$ there are $t_1, t_2 \in (T_0 - \varepsilon, T_0)$ such that $X(t_1) \in \mathbf{R}_+$ and $X(t_2) \in \mathbf{R}_-$ and our claim holds.

Step 2. We will sketch an idea which allows us to look at RBM in D in a new way.

Suppose that D_2 is a domain with the property that if $x+iy \in D_2$ then $x+iy_1 \in D_2$ for all $y_1 > y$. Let Y be a 2-dimensional Brownian motion and let $N(t)$ be the supremum of non-positive numbers such that $D_2 + iN(t)$ contains $Y([0,t])$. Then $Y(t) - iN(t)$ is an RBM in D_2 with the vertical vector of reflection (pointing upwards) on ∂D_2.

The idea goes back to Lévy in the 1-dimensional case (see (1.1)). It was first used by El Bachir (1983) and Le Gall (1987) in the 2-dimensional case. See also Burdzy (1989).

Let Z be the time-change of $h(X)$ so that its martingale part is a Brownian motion. Then Z admits a similar representation $Z(t) = Y(t) + iM(t)$, where Y is a 2-dimensional Brownian motion and M is a suitable real process with locally bounded variation. The process $M(t)$ may be decomposed as $M(t) = M_1(t) - M_2(t)$, where $M_1(t)$ increases only when $Z(t) \in \Gamma_1$ and $M_2(t)$ increases only when $Z(t) \in \Gamma_2$.

We will discuss this idea in greater detail in a forthcoming paper.

Step 3. Recall that we assume that θ is an odd function. Then Γ_1 and Γ_2 are symmetric and have a common endpoint $z_0 \in \mathbf{C}$.

Let $D_3 = \{z \in \mathbf{C} : \operatorname{Re} z > \operatorname{Re} z_0\}$. We will show that Z may approach z_0 by hitting only one of the curves Γ_1 or Γ_2 if and only if D is a minimal fine neighborhood of z_0 in D_3. See Burdzy (1987) and Doob (1984) for the discussion of the minimal fine topology and its relationship with Brownian paths.

Suppose first that D is a minimal fine neighborhood of z_0 in D_3. Let T be the first hitting time of ∂D_3 by Y and let $D_4 = D + (Y(T) - z_0)$. By the probabilistic interpretation of the minimal fine topology, w.p.1 there is $\varepsilon > 0$ such that $Y((T - \varepsilon, T)) \subset D_4$. Then, with positive probability $Y([0,T]) \subset D_4$. If this event happens and $\operatorname{Im} z_0 > \operatorname{Im} Y(T)$ then Z hits only the lower part of the boundary of D before hitting z_0 because all that is needed to move the path of Y into D is an occasional push upwards. Since $Y([0,T]) \subset D_4$, the resulting path will not hit the upper boundary of D. Hence, X hits only the positive part of the real line prior to hitting 0, with positive probability.

Conversely, suppose that D is not a minimal fine neighborhood of z_0 in D_3. Then for each $\varepsilon > 0$ w.p.1 there is $t \in (T - \varepsilon, T)$ such that $Y(t) \notin D_4$. In this case Z must hit both Γ_1 and Γ_2 before approaching z_0 a.s. This is equivalent to saying that X hits $\mathbf{R}_+$ and $\mathbf{R}_-$ before hitting 0 a.s.

Step 4. We have proved that X may approach 0 from one side with positive probability if and only if D is a minimal fine neighborhood of z_0 in D_3. According to Theorem 9.2 of Burdzy (1987), D has this property if and only if

$$\lim_{a \downarrow 0} \frac{1}{a} G_D(z_0 + a, z_1) > 0, \tag{2.10}$$

where G_D is the Green function of D and z_1 is a fixed point in D. By the conformal invariance of the Green function, (2.10) is equivalent to

$$\lim_{a \downarrow 0} \frac{1}{a} G_{D_*}(h^{-1}(z_0 + a), h^{-1}(z_1)) > 0. \tag{2.11}$$

Note that $\operatorname{Re} h^{-1}(z_0+a)=0$ and

$$\operatorname{Im} h^{-1}(z_0+a)/G_{D_*}(h^{-1}(z_0+a),h^{-1}(z_1)) \underset{a\to 0}{\longrightarrow} c\in(0,\infty).$$

Thus, (2.11) holds if and only if

$$\lim_{a\downarrow 0}\frac{1}{a}\operatorname{Im} h^{-1}(z_0+a)>0. \tag{2.12}$$

Let $a=\operatorname{Re}(h(ib)-z_0)$ for $b>0$. Then (2.12) may be rewritten as

$$\lim_{b\downarrow 0}\frac{1}{b}\operatorname{Re}(h(ib)-z_0)<\infty. \tag{2.13}$$

According to the proof of Theorem 1.1,

$$\operatorname{Re}(h(ib)-z_0)=\int_0^b\frac{1}{y}\exp\left[\int_{-1}^1\frac{x}{x^2+y^2}\frac{\theta(x)dx}{\pi}\right]dy. \tag{2.14}$$

We have

$$\int_0^1\frac{x}{x^2+y^2}\frac{(\pi/2)dx}{\pi}=\frac{1}{2}\log\sqrt{1+1/y^2}$$

and, therefore,

$$\begin{aligned}&\exp\left[\int_{-1}^1\frac{x}{x^2+y^2}\frac{\theta(x)dx}{\pi}\right]\\ =&\frac{1}{\sqrt{1+1/y^2}}\exp\int_0^1\frac{x}{x^2+y^2}\frac{2(\pi/2+\theta(x))dx}{\pi}\\ \le&\frac{1}{\sqrt{1+1/y^2}}\exp\int_0^1\frac{2(\pi/2+\theta(x))dx}{\pi x}.\end{aligned} \tag{2.15}$$

Assume that

$$\int_0^1\frac{\pi/2+\theta(x)}{x}dx<c<\infty.$$

Combine (2.13), (2.14) and (2.15) to see that

$$\lim_{b\downarrow 0}\frac{1}{b}\operatorname{Re}(h(ib)-z_0)\le\lim_{b\downarrow 0}\frac{1}{b}\int_0^b\frac{1}{y}\frac{dy}{\sqrt{1+1/y^2}}e^{2c/\pi}=e^{2c/\pi}<\infty.$$

If

$$\int_0^1\frac{\pi/2+\theta(x)}{x}dx=\infty$$

then

$$\exp\left[\int_0^1\frac{x}{x^2+y^2}\frac{2(\pi/2+\theta(x))dx}{\pi}\right]$$

increases monotonically to ∞ as $y\to 0$. It follows that

$$\lim_{b\downarrow 0}\frac{1}{b}\operatorname{Re}(h(ib)-z_0)=\lim_{b\downarrow 0}\frac{1}{b}\int_0^b\frac{1}{y}\frac{1}{\sqrt{1+1/y^2}}\exp\left[\int_0^1\frac{x}{x^2+y^2}\frac{2(\pi/2+\theta(x))dx}{\pi}\right]dy=\infty.\quad\square$$

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