Tonal Composition in Multidimensional Virtual Realms

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Introduction

In two-part species counterpoint, it is common to reuse the same cantus firmus to compose several different melodies, each of which is to be sung only with the cantus firmus and not in harmony with any other. Let us imagine that all of these melodies are recorded individually at the same tempo, and each one is assigned to a separate track on a multitrack console. Then, while one person sings the underlying cantus firmus, another at the console continually switches from one melody to the next by keeping only one track unmuted at any given time. While the transitions might sound momentarily disorienting, there would be a sense of continuity throughout due to the constant presence of the cantus firmus. In time, a skilled composer could learn to compose musical pieces in which such transitions are optimally seamless, regardless of any given listener's decisions on when and which parts to switch.

Let us now imagine how a piece composed in this manner might be heard by picturing six performers forming a circle around a single listener, all facing inward, as shown in Figure 1. The listener is free to spin around in place as she pleases, but at any given time she faces only three performers: the one directly in front of her, and the two to that performer's immediate left and right. Each performer is given a different part and reads it at the correct tempo once the piece begins. At any given time, if he is one of the three performers being faced by the listener, he plays his part out loud; otherwise, he reads it in silence. Thus, throughout the piece only three parts are heard by the listener, just as in any three-part harmony.

The notation for such a piece might look like a six-part harmony rolled up to form a cylinder, as shown in Figure 2. Of course, all six parts are not meant to be played at once. Rather, we can see that each part is written to accommodate only three possibilities: to be played in harmony with the two parts immediately above it, with the two immediately below, or with both the one immediately above and the one immediately below. It can be observed from this notation that it is possible to hear a complete range of registers at all times. After all, a part can never be played together with the one directly across from it, so therefore each can be written for the same register. Of course, a composer might opt for the reverse scenario in which voices gradually shift from all basses to all sopranos as the listener makes a 180-degree turn. It seems reasonable to assume that no artistic freedoms are lost; only more are to be gained.

With some thought, it becomes obvious that the parts need not even be in the same key. For example, one part could sound a major third from the root, while its counterpart directly across sounds a minor third, as shown in Figure 3. As they are not played concurrently, they would never be in conflict; alternating between the two parts would be heard as switching between parallel keys. There are certainly many possible cases to consider, and unlike the early composers of organum, we would probably not require hundreds of years to figure them out by trial and error. Still, it would be much more practical to devise systematic tools for recognizing these harmonic possibilities, especially for the purpose of expanding into advanced chromaticism as well as increasingly complex layouts for the distribution of parts in physical space.

At this point, readers familiar with neo-Riemannian voice-leading graphs might intuit that such graphs are ideally suited for devising these systematic tools. Otherwise, I ask those who are not to simply accept this proposition as a matter of faith, for this paper will be devoted to the use of such graphs for composing a type of music that I will henceforth refer to as *amnesticism*. My decision to invent a new term was not a frivolous one; rather, the need to distinguish this method from established practices of spatial music became clear to me only after numerous lengthy attempts to explain it to others. In amnestic music, tonal harmony and physical location are interdependently linked, creating fundamentally different constraints on the composer as well as setting up completely different expectations from the listener.

I define amnesticism as a method of composition determined by strict tonal constraints previously unseen in the genre of spatial music, in which voice-leading graphs are mapped onto a physical space, either real or virtual, allowing the parts of a musical piece to intersect with each other in unique sonorities for each region of the realm such that a listener hears smooth transitions in harmony as he or she navigates from one region to the next.

I would now like to preempt any possible doubts regarding why amnesticism should be desired by offering the following arguments.

- While some find neo-Riemannian theory fascinating on its own terms, others criticize it for failing to provide the scholarly community with substantially new insights. Amnesticism suggests that the greater benefit of neo-Riemannian voice-leading graphs may lie not in their utility as analytical tools, but as compositional ones.

- Some listeners prefer conventional tonal harmony. Amnesticism will allow composers to appeal to broader audiences while remaining at the forefront of musical exploration and discovery.
- Unlike much of contemporary music relevant to academic interests, the development of amnesticism will immediately benefit from established tonal practice in counterpoint, harmony, and form.
- As interactive multimedia continue to penetrate further into our daily lives, amnesticism provides a unique opportunity for tonal music to intersect with modern technology by actively depending on it as its foundation.
- File sharing has made the violation of intellectual property rights an unfortunate reality for many musicians. Amnesticism can help stem that tide by allowing each new work to require its own unique interface.

I will end my defense with a historical observation. The Model T and the Apple II revolutionized society by bringing personal transportation and personal computing to the masses, respectively. We learn as students how these pioneering inventions demonstrated the superiority of assembly lines and efficient interfaces. However, our textbooks typically neglect one crucial detail, which is that both were actually *inferior* products for their time.

For most anyone who built or bought cars in 1908, the Model T was a loser in every single metric that mattered, and likewise with the Apple II for those who built or bought computers in 1977. Even as technology was marching steadily towards faster speed and greater power, both machines were slower and weaker than their predecessors. Their one redeeming virtue of affordability was considered irrelevant, as no market existed that valued the tradeoff. And yet, the Model T and the Apple II won their places in history precisely by creating and then dominating these new markets.[1]

I believe this is a helpful perspective to keep in mind when asking what the conceptual breakthroughs in music of the twenty-first century will be, for they may very well be along paths dismissed by some as contrary to progress. Now, I do not claim that amnesticism will sit alongside serialism and minimalism in the textbooks of decades to come; my aim is simply to promote it as a worthwhile pursuit to follow for any composer driven to explore uncharted territory. In turn, I ask the reader to put aside any doubts regarding whether an audience for this kind of music exists. One does not, and that is the challenge.

Understanding the 3D hexatonic cube

It might seem strange to introduce a new method of composition based on conventional tonality by way of the hexatonic system as a starting example, given the lack of a shared key amongst its member triads. However, the present research on voice-leading

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[1] See Christensen 1997. Christensen makes a distinction between what he calls sustaining innovations and disruptive innovations. The former are based on improvements to technology that follow predictable patterns and trajectories. The latter create new markets, oftentimes unexpectedly and not according to initial plans, because their inferior quality renders them uncompetitive in existing markets.

graphs initially developed out of a need to discuss symmetric harmonies independent of any diatonic context. I believe the advantage of being able to visualize graphs as symmetric under rotation or reflection is sufficient to warrant the extra layer of abstraction created by initially bypassing the diatonic system.

In general, the order in which concepts are explained in this paper might not seem ideal, but I believe it is the most conducive to understanding. Thus, I ask the reader to trust that every section of this paper connects to the broader picture in due course. We will begin with the hexatonic cube, which depicts voice-leading relations between all the major, minor, and augmented triads of a hexatonic system, as shown in Figure 4.[2]

In traditional music analysis, edges are treated as voice leadings. However, I wish to maintain the notion that the two ends of an edge actually represent two distinct voices, rather than pitch motion within a single voice. Thus, I will refer to edges as representative of dyads rather than voice leadings. In the hexatonic cube, these dyads are all semitones.

Each edge of the hexatonic cube meets with two other edges at each one of its ends to form a vertex. Since the end of each edge represents one pc of a semitone dyad, each vertex represents the unique triad formed by the pcs represented by the ends of three edges. There are eight vertices in all, representing the three major, three minor, and two augmented triads of a hexatonic system.[3]

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[2] My observations on the hexatonic system and its various properties are largely derived from Cohn 1996, 1998, and 2000.

[3] Douthett and Steinbach 1998 shows how voice-leading relations amongst the triads of a hexatonic system may be graphed into what they call HexaCycles, which in turn connect with each other by way of shared augmented triads to form the graph that they call Cube Dance.

It is possible to visualize the graph in a way that places emphasis on its faces. For example, parallel edges on the hexatonic cube represent the same semitone dyad. If the center of the hexatonic cube is taken as the origin, then each group of parallel edges may be merged into a single, parallel axis passing through this origin. Since there are three groups of parallel edges, three mutually perpendicular axes are formed. With edges removed from view, it is easier to recognize that each face represents a single pc, and each pair of opposite faces represents a semitone dyad, as can be seen in Figure 5. The axes, meanwhile, are simply visual aids that highlight the space between opposite faces.

Let us analogize the hexatonic cube in one final way, with our three mutually perpendicular axes seen as light switches that toggle between the two pcs of a semitone dyad. On a piano keyboard, we can cycle through the eight triads of a hexatonic system while keeping the hand stationary by having the thumb, pinky, and middle finger each alternate between the two notes of its nearest semitone dyad, much like switches independently toggling between down and up positions, as shown in Figure 6.[4]

The three axes of the hexatonic cube can be similarly viewed as switches that toggle between pcs represented by pairs of opposite faces, with the low and high pcs of each of the three semitone dyads replacing down and up positions, respectively. We should not get too comfortable with our good fortune that the number of switches perfectly matches the number of dimensions found in Euclidean space, for we will soon generalize these concepts to higher dimensions.

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[4] My understanding of toggling in the hexatonic system is derived from Cohn 2000 and Siciliano 2005.

Choosing one of two positions for each of the three switches allows for eight ($2^3 = 8$) unique combinations representing the hexatonic system's eight different triads. Naturally, the three faces representing the unique combination of any one particular triad will meet at the vertex representing that triad on the hexatonic cube.

In music analysis, the edges and vertices of a hexatonic cube are typically treated as its foundational elements, and faces as their byproduct. In this section, we have just taken the contrary view by placing emphasis on the faces, which represent pcs, and specifically on pairs of opposite faces, which represent semitone dyads. In the following section, we will see why such a visualization will aid us in mapping the hexatonic cube onto the virtual sphere.

Mapping the 3D hexatonic cube onto the 3D virtual sphere

It might seem that I contradicted myself in the previous section. I began by noting that I wished to treat graphical representations of voice leadings as two distinct voices, rather than as motion between two pcs in a single voice. However, I then used the analogy of a switch, suggesting motion back and forth between two positions after all. I shall now explain my reasoning. Each semitone dyad is represented by two voices, both of which indeed remain stationary. Rather, it is the listener who moves, not in *pitch* space but in *physical* space.

This physical movement takes place in the virtual realm, an artificial environment that is free to mimic or distort the acoustic properties of the physical world. Each listener is free to wander into any region of the virtual realm at any time, irrespective of any other listeners present. While musical works composed for these virtual realms are constructed along multiple dimensions in physical space, the listener's experience of them still takes place in only one dimension, that of time. As such, a single piece may be heard in countless different ways, namely based on each listener's chosen path of movement within the virtual realm.

Graphs map onto virtual realms and not the other way around. To prevent any possible confusion between the two, in this paper I will depict graphs as *n*-cubes (which include 4D tesseracts) and virtual realms as *n*-spheres (which include 2D circles). This lack of surface features will help reinforce the notion that the shape of a virtual realm need not be defined by any graph, and may therefore undergo continuous stretching and bending without any changes to its internal properties.[5]

The most straightforward mapping from a voice-leading graph to a virtual realm of same dimensionality is when each vertex, edge, and face of the graph corresponds to a unique sector or boundary of the virtual realm, and vice versa. This is a *bijective* mapping.[6]

In the previous section, we observed that each face of the hexatonic cube represents one of two pcs in a semitone dyad. Each vertex is then treated as the point at which the three faces representing the pcs of its respective triad meet, while individual edges by themselves possess no distinct virtue. They are simply parallel to the mutually perpendicular axes at the origin that highlight the space between opposite faces. Since edges and vertices are entirely byproducts of faces and their relative orientation, a bijective mapping of the hexatonic cube onto the virtual sphere need only take faces into account.

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[5] See Weeks 2002 for a basic understanding of topology.

[6] See Carter 2009 for a basic understanding of mappings as functions in group theory.

It is not difficult to imagine a bijective mapping of a single pair of opposite faces from the hexatonic cube onto the virtual sphere that simply involves bisecting the sphere through the origin and along the plane parallel to those two faces, creating opposite hemispheres, as shown in Figure 7. The pc represented by each face is then heard as a voice within the hemisphere onto which it was mapped. A listener physically situated in one hemisphere hears only the low pc; then, moving across the plane of the bisecting circle, she hears only the high pc. Because no listener may simultaneously occupy two opposite hemispheres, the two voices of a semitone dyad are always heard in isolation from each other.

Figure 8 shows that bisecting the sphere in the exact same manner for the other two pairs of opposite faces creates a virtual sphere bisected three ways along mutually perpendicular circles. This creates eight sectors, or *octants*, each created by the intersection of three hemispheres of different orientation, corresponding to the eight vertices of the hexatonic cube. A listener in each octant hears three simultaneous voices corresponding to the three hemispheres coincident with that octant. Not surprisingly, the pcs sounded by these three voices make up the triad represented by the vertex that corresponds with that octant in the virtual realm.

It is now easy to see that our analogy of the hexatonic cube as three mutually perpendicular switches is actually a fairly accurate description of how a virtual realm might be programmed such that a listener's path of movement triggers the particular voices that she hears at any given time. Crossing the bisecting circle from one hemisphere to the other is like toggling the switch between low and high pcs of the semitone dyad represented by a pair of opposite faces on the hexatonic cube. Had we chosen to map the graph according to its edges rather than its faces, we might consider it necessary to distinguish each of the four quadrants that result when a bisecting circle itself is bisected by each of the other two. It is now obvious that such a step would be redundant.

Understanding graph progressions

Now that we have seen how the hexatonic cube is bijectively mapped onto a virtual sphere, it might be helpful at this point to clarify exactly what all of this means in a musical context. A mapping is nothing more than the physical realization of a graph, which itself is the multidimensional equivalent of a single chord in nondimensional music. While chords specify a collection of pcs, graphs and their mappings specify the distribution of such collections within a physical space. And just as individual chords are arranged sequentially in time to form musically coherent progressions in nondimensional pieces, I postulate that progressions of graphs in their various mappings may also be used to create compositional models for pieces to be heard in a multidimensional space.[7] While I call these graph progressions, it should be noted that these progressions are determined not just by the graphs themselves, but also by how they are ordered and physically oriented as they are

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[7] See Morris 1995. Morris explains that the realization of a draft may be assisted by two preceding stages: a compositional space, which depicts specific musical properties and relations outside of time, followed by a compositional design, which takes certain elements of that compositional space and arranges them in a temporal order. Because the extra dimensions of a graph may be realized as either temporal or physical dimensions, the compositional *space* of a multidimensional piece might well resemble the compositional *design* of a nondimensional one.

mapped onto the virtual realm. In this sense, they are analogous to chord progressions that also take inversions and voice leadings into account.

When speaking of a chord, one may refer just to its quality, such as "major triad," or to its quality plus a specific pitch-class (pc) as root, such as "C-major triad." Mappings may be considered in a similar manner, except with the additional feature of physical orientation. Thus, even before specifying any pc or collection of pcs, one may speak of the general quality of a mapping, such as "bijective hexatonic," or else include how it is oriented within the virtual realm, such as "bijective hexatonic, clockwise with low augmented triad in the (+, +, +) octant."

Of course it is doubtful that such a lengthy description will ever find much practical use, but I do not wish to establish an official terminology at this point. For now, this particular example is based on the observation that there are sixteen possibilities for specifying the physical orientation of a hexatonic cube bijectively mapped onto a virtual sphere. The hexatonic cube has two forms that are mirror images of each other and thus are not superposable. The LP-cycle of one is clockwise, and the other counter-clockwise, when viewing the cube with the vertex of the high augmented triad aligned directly above the low one. After that, the low augmented triad may be used as a reference point, such that whichever of the eight octants it occupies will determine the placement of every other triad in lockstep.

Thus, by itself the bijective mapping already suggests two possible means by which one particular mapping might follow another to create a musically coherent progression. The first means is *permutation*, in which the same graph is rotated or reflected, resulting in the same chords being mapped onto different regions of the virtual realm at different points in time, as shown in Figure 9. The second means is *modulation*, in which a graph transposes to a different pc collection, possibly by pivoting around common pcs, as shown in Figure 10. It is likely that any musically coherent graph progression will involve some combination of both permutation and modulation.

Extending hexatonic cubes into supercubes

Every hexatonic cube contains two opposite vertices representing its two augmented triads, which are separated by semitonal transposition. In turn, every augmented triad belongs to two hexatonic cubes: one in which it is the higher of the two augmented triads, and one in which it is the lower. In one cube it is represented by all three switches in the high pc position, while in the cube transposed a semitone higher, it is represented by all three switches in the low pc position. Much like a pivot chord that allows modulation between related keys, the augmented triad functions like a pivot triad that allows modulation between different hexatonic cubes separated by semitonal transposition.

By joining both cubes at the vertex that represents this augmented triad and then aligning edges that share a common pc, we create a "supercube" where each combined edge now represents two semitone dyads that can be combined into one whole-tone dyad, as shown in Figure 11. As such, this supercube may be analogized as three mutually perpendicular switches, each of which now toggles amongst *three* positions: low, middle, and high pcs, as shown in Figure 12. Choosing one of three positions for each of the three switches allows for twenty-seven ($3^3 = 27$) unique combinations, each representing a different triad. Among these can be found a total of eleven different chord qualities, as shown in Figure 13.

While the hexatonic supercube allows for a broader palette of harmonic possibilities, I will mention just two of them here. First, diminished triads are now possible, completing the full range of triadic possibilities found in traditional tonal music. And second, a single system can now contain relative triads. The R operation, of course, is the third and last of the principal neo-Riemannian operations after the L and P operations. Because it involves the interval of a whole tone, which is not found in the hexatonic system, the R operation is the only one of the three missing from the hexatonic cube.

Mapping the supercube onto the virtual sphere

As the supercube is a 3D graph, it can certainly be bijectively mapped in its entirety onto a virtual sphere, with each of its twenty-seven vertices mapping onto one of twenty-seven different sectors. For our immediate purposes, however, we would like to know how it may be mapped onto a sphere divided into eight octants. The immediate solution is to reduce the three possibilities for each of its three switches down to two—between low and middle pcs, middle and high pcs, or low and high pcs—a process that I will call *simplification*. This simplifies the graph into a unit cube, as shown in Figure 14, which means that like the hexatonic cube, it may also be bijectively mapped onto a virtual sphere with eight octants.

Figure 15 shows a comprehensive list of the possible ways in which the supercube may be simplified to create a unit cube. Excluding the hexatonic cube itself, there are nine unique cube types which cannot be permuted or modulated to yield any other. They are distinguished by their orientation and placement relative to the low and high hexatonic cubes, as well as by the number of whole-tone dyads involved.

Different simplifications of the same supercube may be mapped onto the virtual realm at different points in time to create a musically coherent progression of mappings, as shown in Figure 16. It is also possible that the transitions between these mappings will sound smoother if they pivot around common edges or faces.

Understanding the 4D octatonic tesseract

Technically speaking, there should be no conceptual difficulties involved in generalizing properties of the 3D hexatonic cube to the 4D octatonic tesseract. Given that we live in a 3D universe and not a 4D one, however, trying to understand the tesseract poses a major challenge in practice. I will thus devote this section to helping us visualize the octatonic tesseract as a 4D graph.

Let us disregard the hexatonic system for a moment, and just imagine that the three pairs of opposite faces on the cube represent any three semitone dyads, regardless of any specific pc collection. If we establish that one of the three switches on this cube will always stay on one pc of its semitone dyad and never the other, then we eliminate the need to depict that switch as a pair of opposite faces. The two faces collapse into one, and our 3D cube flattens into a 2D square. It can thus be understood that *collapsing* refers to the reduction of a graph by removing one or more of its dimensions from consideration. This square now contains only two pairs of opposite edges representing two switches; the third switch is held constant and thus need not be considered by the graph.

The process can also be reversed. If we hold that there is now a third semitone dyad to consider, we duplicate the square from our previous paragraph such that all the vertices of one square occupy the space representing the low pc of this new semitone dyad, and all those of the other square occupy the space representing the high pc. Visually, each vertex on one square is connected by an edge to its counterpart on the other square. Because all these edges are parallel and represent the same semitone dyad, they may be merged into a single axis passing through the origin. In other words, we have created a third switch that toggles between the two pcs of this new semitone dyad. This additional dyad adds another dimension to the graph, solidifying a pair of opposite squares into a 3D cube. In other words, *solidification* is the opposite of collapsing, as it involves the addition of an extra dimension to a graph.

Similarly, if we now wish to add a fourth semitone dyad to this cube, we duplicate it and connect the two cubes by parallel edges to create a pair of opposite cubes. All of these edges may be merged into a single axis that can be thought of as a switch that toggles between the low and high pcs of this fourth semitone dyad. A tesseract, then, is simply the solidification of a pair of opposite cubes into a 4D graph, much like a cube is the solidification of a pair of opposite squares into a 3D graph, as shown in Figure 17. It should be noted that despite its appearance as depicted in this 2D illustration, the tesseract is indeed regular in its symmetry, meaning that it can be superimposed on every possible rotation or reflection of itself, just like a cube. Our inability to visualize this property is simply a limitation of our 3D understanding of the universe.

Of course, this process of solidification may once again be reversed. If we establish that one of the four switches will always stay on one pc of its semitone dyad and never the other, then the switch representing that dyad on the graph becomes redundant and no longer requires representation on the graph. Our 4D tesseract "flattens" once again into a 3D cube.

We now turn to the 4D octatonic tesseract that specifically depicts voice-leading relations between all the seventh chords of an octatonic system: fully diminished, half-diminished, minor seventh and dominant seventh, as shown in Figure 18.[8]

We first visualized the tesseract by imagining a cube being duplicated; its two copies were then connected by parallel edges that all represent the same semitone dyad. Of course, there is nothing distinctive about this additional dyad. Much like the three pairs of opposite faces on the cube, the four pairs of opposite cubes that make up the octatonic tesseract are each graphically indistinguishable from the others, as demonstrated in Figure 19. By merging each group of parallel edges into a single axis that passes through the origin at the center of the tesseract, we create four mutually perpendicular axes. If it is difficult to visualize each of

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[8] Douthett and Steinbach 1998 shows how voice-leading relations amongst the seventh chords of an octatonic system may be graphed into what they call OctaTowers, which in turn connect with each other by way of shared fully diminished sevenths to form the graph that they call Power Towers.

four axes being perpendicular to the other three in 4D space, it is enough to recognize that this is a generalization of the same property observed in the hexatonic cube.

These four axes represent switches that toggle between low and high pcs of the four semitone dyads of the octatonic system, allowing for sixteen ($2^4 = 16$) unique combinations that represent the octatonic system's sixteen different seventh chords. This can be demonstrated on a piano keyboard by keeping the hand stationary while having each of four digits independently alternate between the two notes of its nearest semitone dyad, as shown in Figure 20.

Mapping the 3D hexatonic cube onto the 2D virtual circle

To facilitate our understanding of how the tesseract maps onto the virtual sphere, it will be helpful first to observe how a hexatonic cube maps onto the virtual circle that was briefly mentioned at the start of this paper. Since both involve mapping a graph onto a virtual realm of one fewer dimension, the underlying logic of the latter can be generalized to the former.

Because the hexatonic cube contains three pairs of opposite faces while a circle contains only two perpendicular axes, it will not be possible to designate corresponding sectors on the virtual circle that retain all the relationships between vertices found on the hexatonic cube. Our mapping therefore cannot be bijective.

Each of the three pairs of opposite faces maps onto the virtual circle as opposite semicircles created by a line that bisects the circle, with each line set at an equal angle from the other two. (In this case, 120 degrees.) Much like opposite hemispheres on the virtual sphere, each pair of opposite semicircles represents the two respective pcs of a semitone dyad, and each vertex maps onto a "slice" created by the intersection of three semicircles, as shown in Figure 21. A listener in each slice thus hears a different triad belonging to that particular hexatonic system. Because these bisecting lines occupy the same plane, however, there can only be six (2 × 3 = 6) slices, which means that two vertices, and thus two triads, are absent from the mapping.

The visual clarity of the virtual circle makes it relatively easy for us to recognize which two triads are missing, and why. When the hexatonic cube is bijectively mapped onto a virtual sphere, a listener in any given octant has the option to cross over any of the three bisecting circles, and the three hemispheres coincident with that octant are interchangeable in physical orientation through rotation or reflection. When the three pairs of opposite faces are mapped onto the virtual circle, however, a listener in any given slice may only cross over two bisecting lines, and the three semicircles coincident with that slice are not interchangeable; one occupies the middle, flanked by the other two at opposite ends.

Thus, it is not enough simply to observe how faces are mapped onto the virtual circle; we must now be mindful of edges as well, because only six of the twelve can be mapped. A listener moving to a neighboring slice toggles between pcs on a single switch while the other two switches remain unchanged; circumnavigating the circle in this fashion retains the same ordering of three switches toggled in repeated sequence. The virtual circle may then be seen as a mapping from an enclosed path of six edges interconnecting six vertices along the hexatonic cube, as shown in Figure 22. There are four unique paths, with each one excluding a different pair of opposite vertices.

Another way to visualize the two missing triads is to observe that there are two possible configurations of three switches in which there is no single region where a listener in the virtual circle can simultaneously occupy all three semicircles of the pcs chosen, as shown in Figure 23. Naturally, these two configurations correspond to the two triads that are absent.

Additionally, we may choose to remove an entire semitone dyad from consideration by simply establishing one of its two pcs as constant. This will result in one of several scenarios. If the pc is shared by two voices, it can be realized either into two independent parts or else as a doubled part. Otherwise, the pc is held in only one voice and realized as a single part that is heard throughout the circle. Regardless of how it is realized in composition, the graph that maps onto this arrangement is still an enclosed path along the hexatonic cube, only now it has collapsed into a 2D square consisting of four edges interconnecting four vertices, as shown in Figure 24. Collapsing the enclosed path even further yields one edge joining two vertices. The correlation between number of edges and number of vertices is still consistent, however, if we consider this to be a double edge representing two opposite directions. This maps onto a virtual circle in which two semitone dyads each sound only one pc throughout.

Mapping the 4D octatonic tesseract onto the 3D virtual sphere

We saw in the previous section that mapping a 3D cube onto a 2D circle results in six ($2^3 - 2 = 6$) slices. At the same time, it is not difficult to recognize that the bijective mapping of a 2D square onto a virtual circle results in four ($2^2 = 4$) quadrants that do not align with these six slices. It then stands to reason that the mapping of a 4D tesseract onto a 3D sphere, relative to the bijective mapping of a 3D hexatonic cube onto that same sphere, will also result in such a misalignment, this time between fourteen ($2^4 - 2 = 14$) "wedges" and eight ($2^3 = 8$) octants.

Each of the four switches of the octatonic system may be mapped onto the virtual sphere as a bisecting plane, with each plane set apart from the other three at an equal angle. This plane bisects the sphere into opposite hemispheres, representing one pair of opposite cubes on the octatonic tesseract. As with a bijective mapping from the hexatonic cube, a listener in one hemisphere hears only one pc of its respective semitone dyad; crossing the bisecting circle, she hears only the other. This time, however, every wedge in the sphere represents the intersection of four hemispheres, each representing a different seventh chord belonging to the octatonic system, as shown in Figure 25.

The absence of two seventh chords from the mapping of a tesseract onto the virtual sphere is logically equivalent to the absence of two triads from the mapping of a cube onto the virtual circle. That is, there are two possible configurations of four switches in which the four hemispheres of the pcs chosen do not overlap a single region, as shown in Figure 26. In both cases, the four pcs are those of one of the two missing seventh chords, which are always represented by opposite vertices on the octatonic tesseract.

In the same way that we eliminated an entire semitone dyad from the hexatonic cube, collapsing it from a 3D graph into a 2D one, we may choose to map the octatonic tesseract onto the sphere by establishing one pc of a semitone dyad as constant. The resulting graph would then be a unit cube and thus allow for a bijective mapping, as shown in Figure 27.

Extending octatonic tesseracts into supertesseracts

Just as two hexatonic cubes may be joined at the vertex that represents their common augmented triad, two octatonic tesseracts may be joined at the vertex that represents their common fully diminished seventh chord. Unlike with cubes, however, it is not such a simple matter to align corresponding edges on two tesseracts, nor to visualize the continuation of those edges into the 4D space that represents seventh chords outside the octatonic system of either tesseract. After all, a supercube is made up of only eight individual cubes in 3D space; a "supertesseract," on the other hand, is made up of sixteen individual tesseracts in 4D space!

For this reason, I will describe the supertesseract solely in terms of its four mutually perpendicular switches, each of which toggles amongst three positions of low, middle, and high. Figure 28 shows how this can be performed on the piano keyboard. Choosing one of three positions for each of the four switches allows for eighty-one ($3^4 = 81$) unique combinations, each representing a different seventh chord. Among these can be found eighteen different chord qualities, as shown in Figure 29.

The utility of the supertesseract lies in its capacity to be reduced before mapping onto the virtual sphere. Like the hexatonic supercube, it may be simplified into a unit tesseract by reducing the number of positions available for each switch from three to two; like the octatonic tesseract, it may be further collapsed into a supercube by establishing one pc of a dyad as constant, thus eliminating one of the four switches from consideration entirely, as shown in Figure 30.

The supertesseract enriches our harmonic palette considerably, even more so than the supercube. From the standpoint of conventional tonal music, the more useful of the new possibilities include major sevenths and minor-major sevenths. Others, such as those that alternate between low and high pcs with no intervening middle pcs, actually suggest sonorities taken from the octatonic system *not* represented by the two tesseracts being joined.

Understanding 3D diatonic cubes

The symmetry of the cube makes it ideally suited for graphing voice-leading relations within the hexatonic system, which is bookended by augmented triads that evenly divide the octave. As we have seen, however, the three switches that compose the cube need not necessarily represent semitone dyads. After all, if we continue to extend the supercube outward, each axis will eventually span twelve chromatic tones and beyond, creating what we might facetiously call an "ultracube." Our ultracube may then be simplified into a unit cube once more by pairing any two pcs into a dyad for each axis represented by one of the three switches. From a tonal perspective, the most obvious pcs to be chosen are those that make up diatonic hexachords.[9]

In the same way that we imagined cycling through the eight triads of a hexatonic system while keeping the hand stationary on a piano keyboard, we can easily recognize that given any six consecutive white keys, it is possible to play four diatonic triads made up of stacked thirds: two in root position with one lower and one higher, one in first inversion, and one in second inversion, as shown in Figure 31. There are seven different hexachords, each beginning on a different note of the C-major scale. The diminished triad may be found in four of these hexachords, and is therefore absent from the other three.

The four remaining trichords that are not diatonic triads all contain the interval of a step. Two of these trichords contain additional intervals of a third and a fifth, implying diatonic seventh chords missing a chordal third and a chordal fifth, respectively; the remaining two trichords contain additional fourths and imply suspended chords or stacked fifths. The hexachordal system that comprises these eight diatonic triads and trichords is isomorphic to the hexatonic system and thus can also be plotted as a unit cube, as shown in Figure 32.

Because we are interested in diatonic chords, it stands to reason that the transpositions of the diatonic cube to which we are interested in modulating will most likely

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[9] My exploration of diatonic cubes and their various properties is largely derived from Tymoczko 2004, 2008, and 2011.

be those that share five pcs and differ only by a single pc, namely the pc in the diatonic scale that is absent from the original hexachord. Modulation to a transposed cube will involve pivoting around a face represented by the pc either immediately above or below the missing pc on the scale, so for a given diatonic cube there are two possible neighbors towards which to modulate.

The graph on the left side of Figure 33 shows that if we intend to stay within a single diatonic collection, modulation to either of these transposed cubes will fill in the same missing pc, albeit through different switches and in opposition to different pcs. In other words, we are simply cycling through the diatonic circle of fifths and will return to our original cube after seven modulations. However, if we are concerned with preserving the set class of our chosen hexachord, then we will need to change diatonic collections by cycling through the chromatic circle of fifths, as shown on the right side of Figure 33.

Either way, modulation occurs only forward or backward, cycling through each axis one at a time and in the same order. This prevents us from constructing supercubes whose outermost vertices represent chords that interest us equally, as is the case with hexatonic supercubes (which include all the diminished triads allowed by the intersection of two hexatonic collections) and octatonic tesseracts (likewise with major sevenths). At most, we will probably only be interested in the formation of two adjacent unit cubes, or what we might call a "duplecube," as shown in Figure 34. In all cases, however, our intentions are the same: to define additional harmonic possibilities that will aid in smooth progressions from one mapping to the next in a musical context.[10]

Distorted and infinite realms

It might be obvious by now that we have been working under many arbitrary assumptions regarding the layout and logic of the virtual realm. After all, there is certainly no real imperative for the border that separates the two pcs of a semitone dyad to be straight, or a bisection, or even a single line or plane. For instance, we may choose to expand opposite semicircles across the line of bisection in the virtual circle such that each overlaps the other, creating a marginal area in between where both pcs of its respective semitone dyad are momentarily heard together, as shown in Figure 35. The transition as the listener crosses from one semicircle into the other will then sound less abrupt. This procedure, of course, can also be performed on opposite hemispheres in the virtual sphere.

I also mentioned earlier in this paper that my decision to represent virtual realms as circles and spheres is simply a matter of convenience; any topologically equivalent shape will work just as well. In a virtual setting, however, there is even more creative freedom, given that reconfiguring the layout is simply a matter of revising the source code. Wormholes may transport the listener across time and space, and the realm itself need not be finite. For

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[10] Tymoczko 2011 distinguishes between perfectly even and nearly even divisions of the scale; graphs representing the former share common vertices with their transpositions, while those representing the latter share common faces. This distinction affects how transposed cubes connect to form chains. While interesting to consider for future research, I will not explore the issue further in this paper, as I am only concerned at this time with local regions and not entire chains of graphs.

example, a listener proceeding straight along one axis in an infinite 2D realm might continuously encounter and pass through the same dividing line, toggling back and forth between the low and high pc of a single semitone dyad, as shown in Figure 36. Instead of a light switch on a wall, then, this is analogous to turning the rotary switch on a lamp.

The possibility of leaving and returning to the same sector by proceeding in a single direction along one axis allows us to retain the two chords that get excluded when we map a graph onto a virtual realm of one fewer dimension. This will be easier to demonstrate using the example of a cube mapped onto a circle. By skewing each dividing line in either direction away from the origin, we are able to create an additional sector at the center of the circle representing one of the missing triads, as shown in Figure 37. Skewing each dividing line in the other direction will result in the other missing triad.

It should now be clear why we only get six slices. The two missing triads cannot simultaneously occupy the same space, so in a perfectly balanced layout they cancel each other out completely. However, if each switch on the cube is mapped onto two dividing lines such that the inner region represents one pc of a semitone dyad and the outer regions both represent the other, then the two missing triads do not compete for space, as shown in Figure 38. In this layout, creating additional room for one triad in any given region will free up space for the other somewhere else. This procedure generalizes to the mapping of the tesseract onto the virtual sphere, which also excludes two chords given a perfectly balanced layout, as well as to infinite realms of any dimension similar to the one just described.

Constructing and realizing graph progressions

Once again, let us leave the abstract world of graphs for a moment to return to the concrete world of music heard by listeners. Works composed for virtual realms are based on progressions of graphs in their various mappings, but exactly how are these progressions constructed to create coherent musical narratives? In this section, I will summarize all the different ways in which one mapping may succeed another, in the order they were introduced in this paper.

- *Permutation:* A single graph is rotated or reflected. The pc collection remains the same.
- *Modulation:* A single graph is transposed. While its pc collection is changed, the set class remains the same.
- *Simplification:* A supercube is reduced such that the number of positions amongst which one or more of its switches may toggle is consistent with the mapping, typically from three to two.
- *Collapsing:* A graph has one or more of its switches eliminated entirely such that only one pc of the respective dyad is heard throughout the virtual realm. The reverse process is solidification. Note that while collapsing is also a type of graph reduction, its nature is different from simplification. With simplification, the dimensionality of a graph remains constant. With collapsing, the dimensionality is lowered.

- *Alternation:* A graph is followed by one of a different set class. For example, a hexatonic cube is replaced by a diatonic cube, or by an octatonic tesseract flattened to 3D.
- *Realignment:* A graph is followed by one of different dimensionality. Because the number of sectors is changed, the two mappings will not align perfectly.[11]
- *Distortion:* The layout of a mapping is changed. Dividing lines may be curved, the realm may be reshaped, or its outer boundary may be wrapped back onto itself, making the realm infinite.

In all cases, it is possible that these progressions can be made smoother by having like chords correspond to like sectors across mappings, in the same way that common tones and pivot chords are used to transition between harmonies and keys in nondimensional music.

Understanding scale graphs and key signature charts

I began this paper by using the example of two-part counterpoint exercises that share the same notated cantus firmus to demonstrate the underlying concept of multidimensional tonal music. It may therefore be assumed that the process of realizing a graph progression is not radically different from that of realizing a harmonic progression. However, while every

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[11] Callender 1998 discusses the possibility of a split/fuse transformation to address relations between sets of different cardinality. This is a possible area for future research.

part in nondimensional music can reasonably be expected to share the same key at any given time, it is possible to create multidimensional mappings in which one region of the virtual realm is in a different key than another, with some parts concurrently straddling two or more keys. For this reason, the process of realizing a progression of mappings must account for parts assigned to multiple harmonic functions.[12]

Because every diatonic key is associated with a unique collection of pcs, they behave no differently from chords and thus may be graphed in the same manner. The same graphs will result whether we treat them as scales or chords; for the sake of clarity I will refer to them as heptachords when I wish to draw upon the terminology associated with sonorities. Thus, we can treat closely related diatonic keys as heptachords that deviate by semitonal voice leading in one part, with common tones retained in all other parts. For example, the two voice-leading paths leading away from the key of C-diatonic are F to F♯, leading to G-diatonic, and B to B♭, leading to F-diatonic.[13]

In addition to diatonic keys, it is possible to include keys based on alternate scales such as the acoustic, harmonic major, and harmonic minor scales, which all deviate from the diatonic scale by semitonal voice leading in one part. For example, E to E♭ leads C-diatonic to F-acoustic; C to C♯ leads it to G-acoustic; A to A♭ leads it to C-harmonic-major; and G to G♯ leads it to A-harmonic-minor. These relationships are shown in the graph in Figure 39,

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[12] My observations on scale graphs and key signature charts in this section are largely derived from Tymoczko 2004 and 2005. The scale graph I refer to in this section is based on the scale lattice from Tymoczko 2011.

[13] I will refer to the collection of pcs associated with the key of C-major as C-diatonic; however, this does not imply that C is the root.

which has been extended to include several more diatonic keys in both directions along the circle of fifths.[14]

The cubic nature of this graph is deceptive, of course. Intuitively, a sonority composed of seven notes should result in a 7D graph, yet we are presented with a 3D one. This extreme collapse and simplification is made possible by eliminating every heptachord other than those based on the four scales mentioned, as well as by allowing each of the three axes to represent more than one part. In other words, although the graph shows G-acoustic situated across from F-acoustic along the same axis, these keys do not deviate from C-diatonic by semitonal voice leading in the same part.

In order to determine the collection of pcs permitted to be sounded by each part, it might seem to be a simple matter of identifying graphs of scales that are supersets of graphs of chords when one is overlaid above the other. Upon further consideration, however, it becomes apparent that the process is subtractive rather than additive; that is to say, it is not a question of which pcs may be included, but which are to be excluded. Let us revisit our six performers shown in Figure 3 at the beginning of this paper. If the two opposite pcs of a semitone dyad are not meant to be heard simultaneously, then it is not enough for the part assigned to one pc to avoid playing the other pc. Every other part that combines with either of these two pcs to form a triad should avoid sounding both pcs as well.[15]

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[14] Like the graphs for diatonic cubes, this graph for scales can be extended to form an enclosed chain, but at this point I am concerned only with local regions.

[15] Of course, the determination of permissible pcs for each part is merely an artistic suggestion. While it's likely that a part assigned to C-diatonic will tend to favor E over E♭, such is not necessarily the case with F and F♯, or A and A♭. As such, the intersection of scale graphs with chord graphs is undertaken as a topic in this paper mostly for musical considerations. It is possible that further research will reveal insight of greater mathematical interest.

Figure 41 shows the superimposition of a very simple chord graph over a scale graph. Let us suppose that we collapse our hexatonic cube to a single dimension, with two switches held constant. In other words, we are left with two vertices representing two triads separated by a lone semitone dyad. If we overlay this 1D graph onto the scale graph by highlighting every heptachord on the scale graph that contains each of the respective triads, we are presented with two distinct sets of highlighted vertices that do not intersect. We are free to choose any vertex from one set and its counterpart from the other, thus designating the two heptachords represented by those vertices to be the keys used for our 1D graph.

There are two aspects of this approach that make it complicated from a compositional perspective. First, those parts assigned to a constant pc are constrained by both keys, not just one. Second, the inclusion of a second semitone dyad creates two more sets of highlighted vertices on the scale graph. However, because we are concerned only with their intersection into unit squares and cubes rather than their union, this means that we are adding extraneous details to the graph even as we are now interested in fewer of them, yielding diminished returns.

For this reason, it is possible that the composer may prefer to use key signature charts, either in conjunction with scale graphs or by themselves, to determine permissible pcs for each individual part. Figure 40 shows the accidentals that would be included in the key signature for each given scale. Naturally, those belonging to the diatonic scales are easiest to recognize. To see how these key signatures might be assigned to a mapping, let us return to Figure 41 and choose two unit segments at random, each representing a different intersection of one vertex from each set of highlighted vertices for two triads differentiated by a semitone dyad. The two examples at the top of Figure 42 shows how our two unit segments are mapped onto a virtual circle, with key signatures indicating the heptachord that each vertex represents.

The example at the bottom of Figure 42 shows how a unit square is mapped onto a virtual circle. Obviously, the 2D nature of this square indicates that there are now two semitone dyads being given consideration. On the circle, each key signature deviates from its immediate neighbor by one accidental. Had we attempted to arrive at this arrangement on the scale graph, we would have had to highlight four sets of vertices. Using key signature charts as our first resort, by contrast, is not only easier but perhaps more intuitive from a musical standpoint as well. Figure 43 shows all the scales to which each major, minor, augmented, and diminished triad belongs, while Figure 44 shows all those to which each fully diminished, half-diminished, minor seventh, dominant seventh, and French sixth chord belongs.

A sample composition

I have composed a short, simple piece to be heard in a virtual circle divided into six slices, that demonstrates some of the possibilities of this compositional method while by no means exhausting its potential.

Figure 45 shows the score for *Amnestic Hexagon*, and Figure 46 shows the graph progression that served as the underlying compositional design, including key signatures for different regions of the circle at any given time during the piece. Because I have not accumulated enough experience to generate meaningful graph progressions *ex novo*, this design still reflects a largely nondimensional manner of compositional thinking, which is perhaps evident in the conventional logic of its harmonies. Nevertheless, I believe it provides a useful starting point for more skilled composers to build upon.

Figure 47 shows a basic interface, programmed in Max/MSP, by which a listener may hear the piece. When the program is loaded, audio files of all six individual parts begin playing simultaneously, although three are kept silent to the listener. (This uses up unnecessary processing power, and will need to be made more efficient in the future for lengthier and more intricate pieces.) The mouse's Cartesian coordinates are converted to polar coordinates, identifying where the listener is located within the virtual circle at any given point in time. Each time the listener moves to a new slice, a signal is sent that mutes or unmutes the appropriate instrumental parts.

Just as the early composers of counterpoint had to focus on achieving consonance in local sonorities before putting thought into overall harmonic progressions and cadences, it is likely that modern composers will need to indulge in a preliminary stage of playful experimentation before a systematic method for constructing and realizing graph progressions can be formulated.

FIGURES AND TABLES

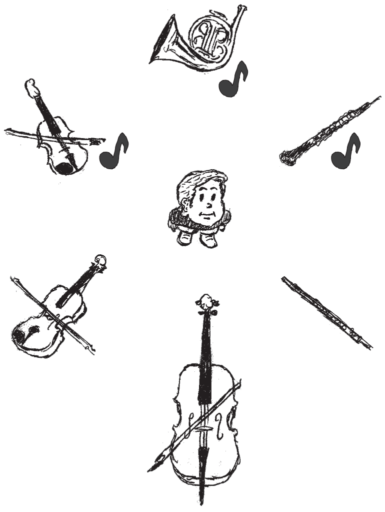


Figure 1: Six performers arranged in a circle, surrounding a listener. At any given moment, only three adjacent instruments are playing their respective parts. In the scenario shown here, those three are the violin, horn, and oboe.

Figure 2: Notated score for six performers arranged in a circle. The top figure is how the score might appear on paper. The bottom figure is a 3D representation of how the 2D piece unfolds in time. At any given moment, only three adjacent parts are heard by the listener.

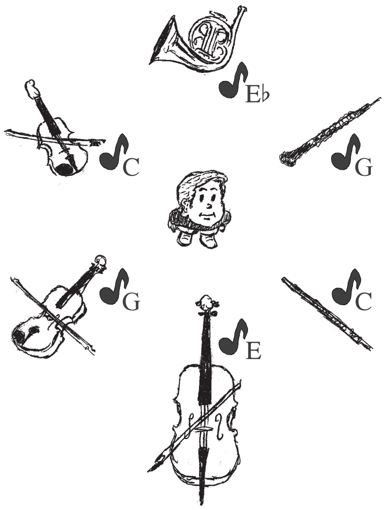


Figure 3: Six performers arranged in a circle, each assigned a single, sustained note. At some given moment, if the performer assigned to play the note E♭ is heard, then the performer assigned to play the note E stays silent. Because only three adjacent performers play their notes at any given time, the listener will never hear E♭ and E simultaneously.

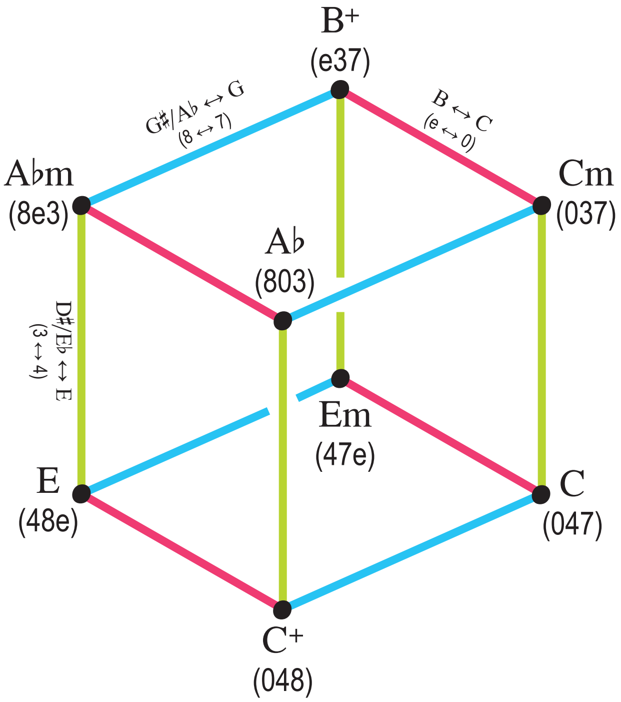


Figure 4: Hexatonic cube HEX$_{3,4}$. Vertices represent triads, and each set of parallel edges represents a single voice leading by semitone in one voice.

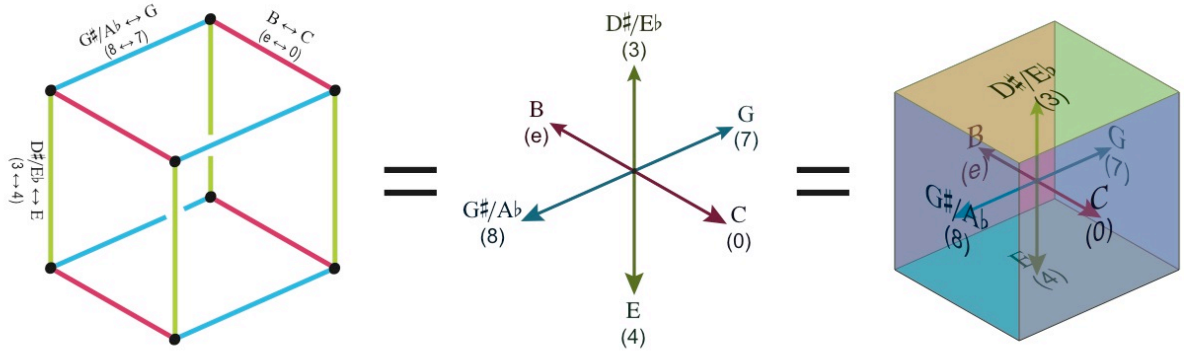


Figure 5: Hexatonic cube HEX$_{3,4}$. Faces represent pcs, and the axis between each pair of opposite faces represents a semitone dyad.

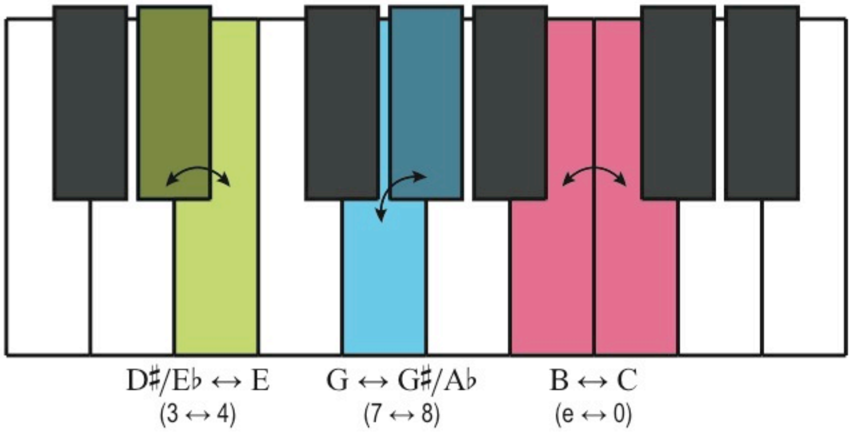


Figure 6: Keyboard layout for the HEX$_{3,4}$ cube.

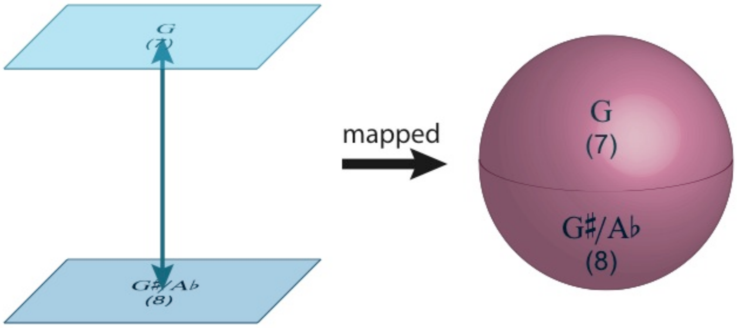


Figure 7: Mapping the "G↔G♯/A♭" switch onto the virtual sphere. A performer playing G is heard only in the top hemisphere, while one playing G♯/A♭ is heard only in the bottom hemisphere. A listener physically situated within the sphere is free to move back and forth between the two hemispheres, but can never simultaneously occupy both.

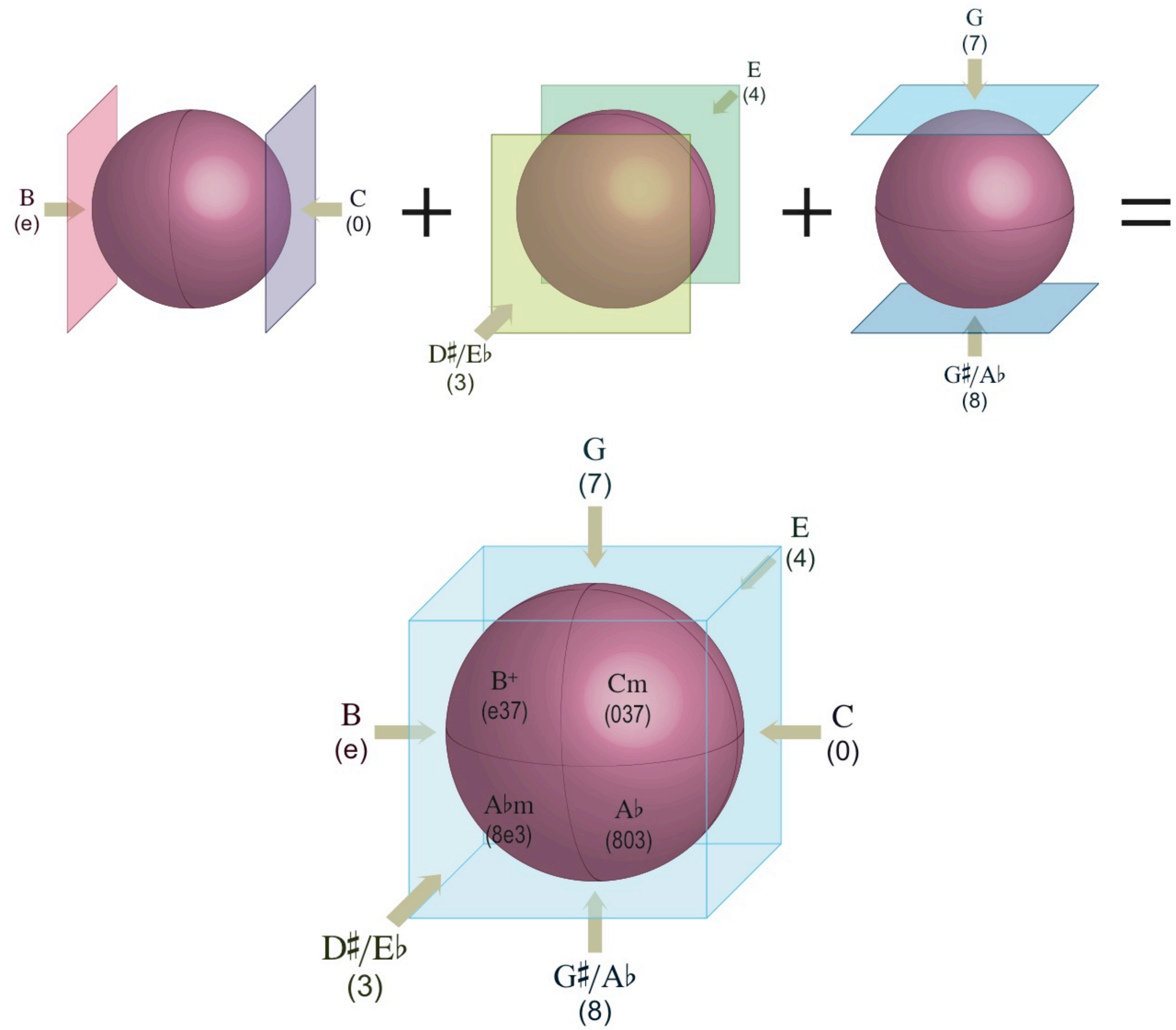


Figure 8: The three switches of the HEX$_{3,4}$ cube are mapped onto the virtual sphere as pairs of opposite hemispheres at mutually perpendicular angles, dividing the sphere into eight octants. The faces of the cube continue to be shown here for visual clarity.

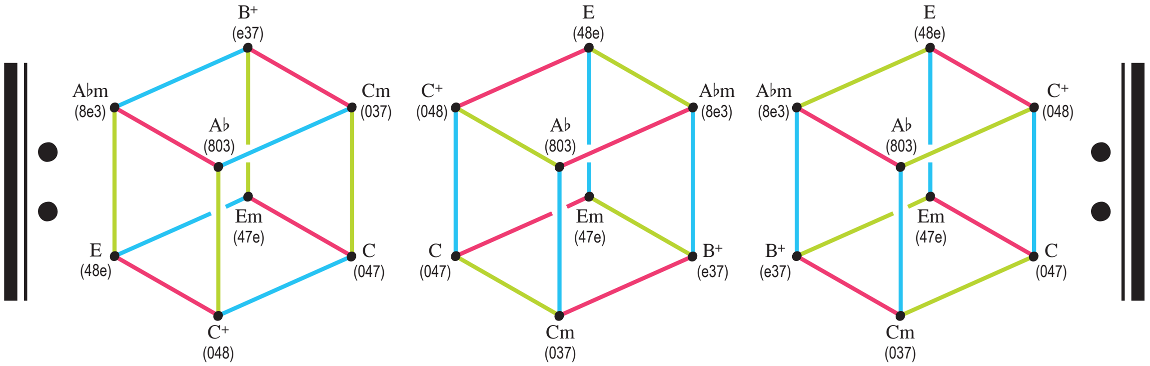


Figure 9: A graph progression involving permutations of the $\mathrm{HEX}_{3,4}$ cube. The first cube is rotated clockwise by 120 degrees around the imaginary line connecting A♭-major to E-minor to yield the second cube. This second cube is then reflected across the plane of A♭-major, E-major, E-minor, and C-minor to yield the third cube. At the repeat sign, the return to the first cube naturally involves both rotation and reflection.

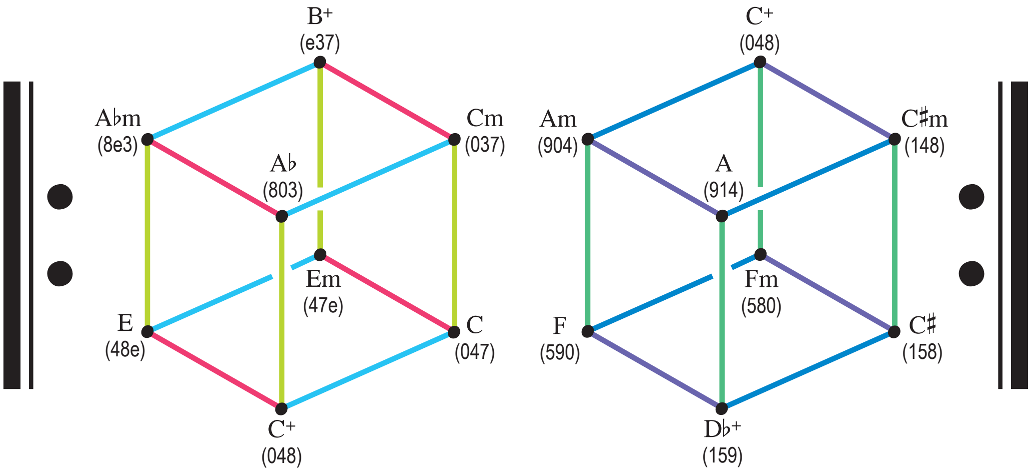


Figure 10: A graph progression involving a repeated modulation between the $\mathrm{HEX}_{3,4}$ and $\mathrm{HEX}_{0,1}$ cubes.

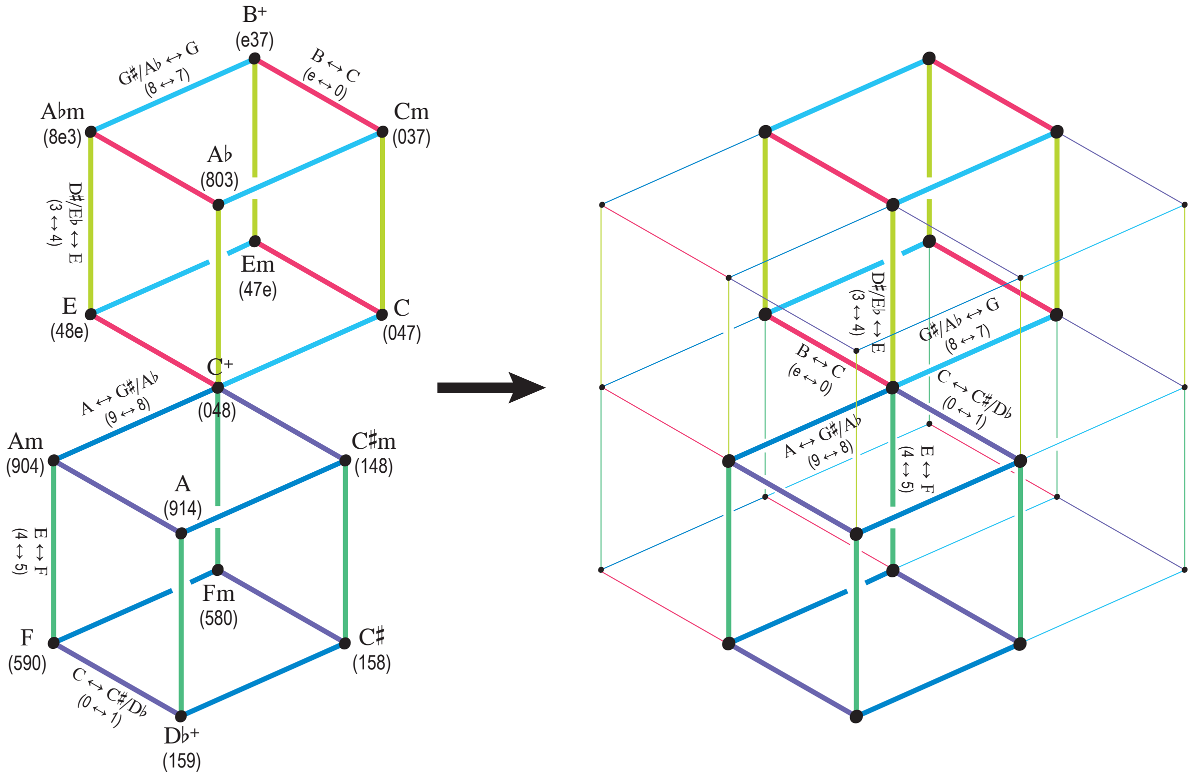


Figure 11: Supercube created by joining hexatonic cubes HEX$_{3,4}$ and HEX$_{0,1}$ and aligning like edges. This allows for a wider range of harmonic possibilities that includes diminished triads and relative chords.

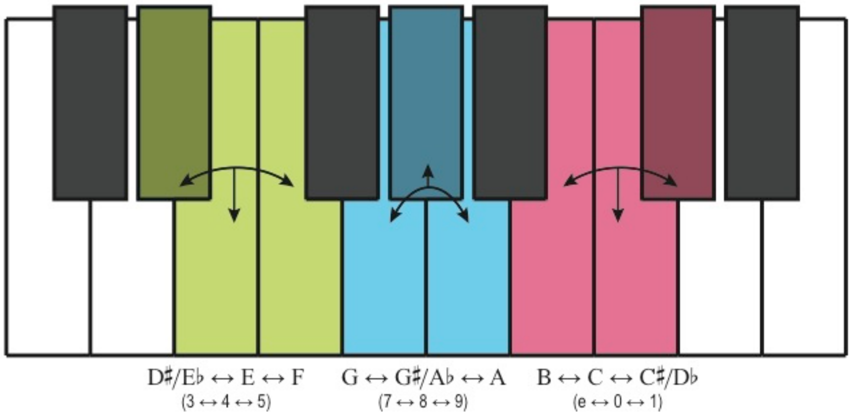


Figure 12: Keyboard layout for supercube HEX$_{3,4}$/HEX$_{0,1}$.

switch positions `e0123456789t`	no. per system	interval set class	pc set	chord name (*non-traditional in italics*)
`L--.L--.L--.`	1	[4,4,4]	(048)	augmented triad
`L--.L--.-M-.`	3	[3,4,5]	(037)	minor triad
`L--.L--.--H.`	3	[2,4,6]	(026)	dominant seventh (incomplete)
`L--.-M-.-M-.`	3	[3,5,4]	(037)	major triad
`L--.-M-.--H.`	3	[2,5,5]	(027)	*fifth, suspended fourth*
`L--.--H.-M-.`	3	[3,3,6]	(036)	diminished triad
`L--.--H.--H.`	3	[2,6,4]	(026)	*diminished fifth, minor seventh*
`-M-.-M-.-M-.`	1	[4,4,4]	(048)	augmented triad
`-M-.-M-.--H.`	3	[3,4,5]	(037)	minor triad
`-M-.--H.--H.`	3	[3,5,4]	(037)	major triad
`--H.--H.--H.`	1	[4,4,4]	(048)	augmented triad
	27 total			

Figure 13: Possible combinations of switch positions for the supercube. The pc index given is for supercube $HEX_{3,4}/HEX_{0,1}$.

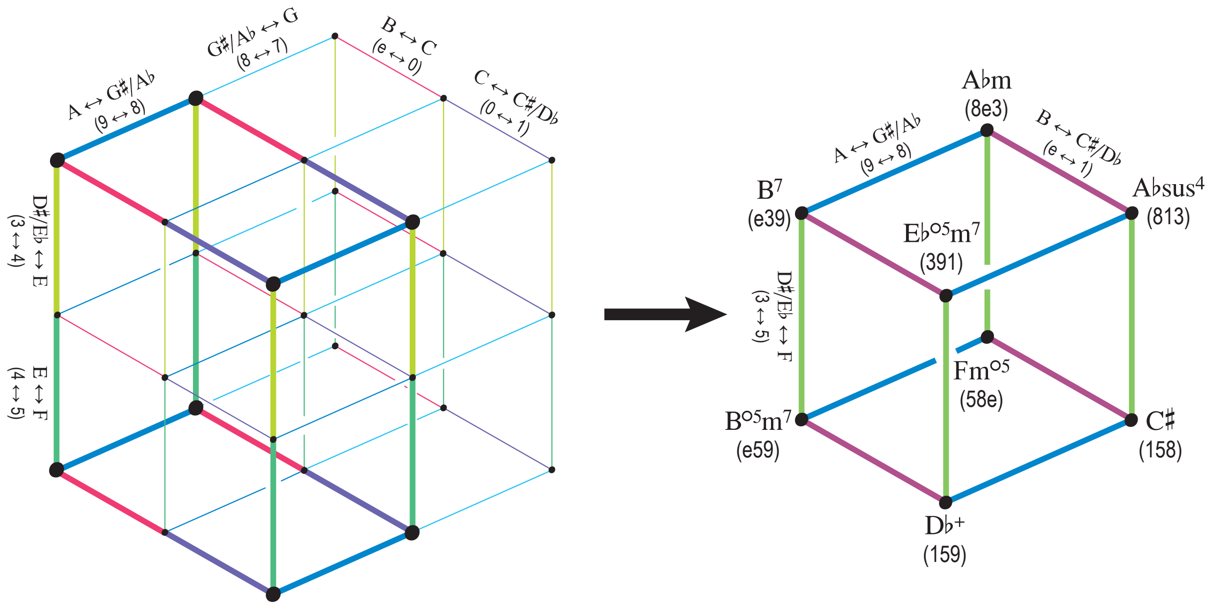
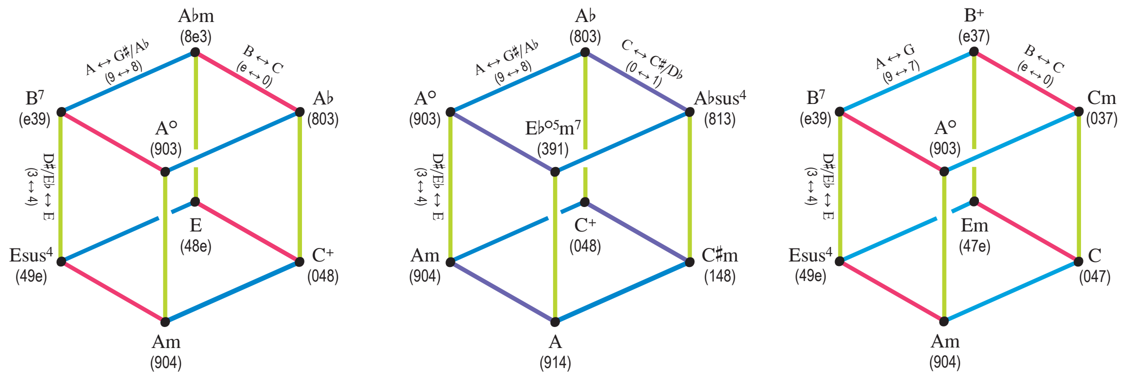


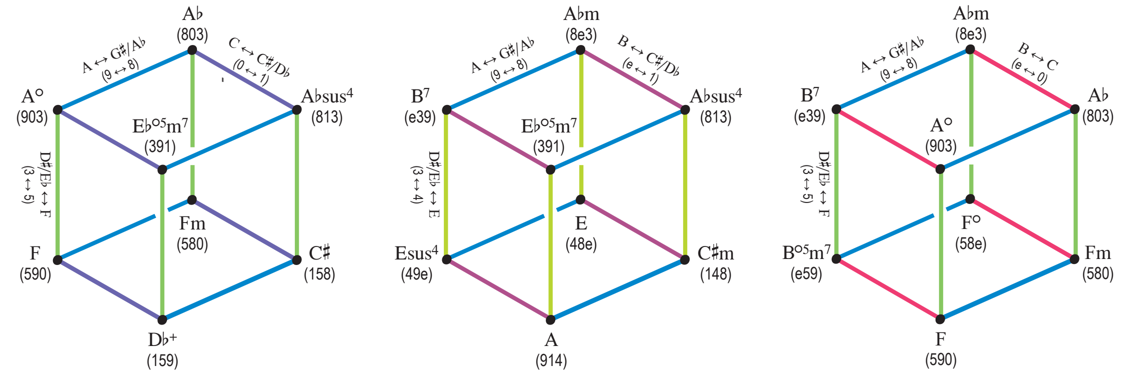
Figure 14: Simplification of supercube $HEX_{3,4}/HEX_{0,1}$ into a unit cube. For each of the three switches, either one of the two connected semitone dyads is chosen over the other, or the two are merged into a whole-tone dyad. In this instance, switches "B↔C↔C♯/D♭" and "D♯/E♭↔E↔F" are merged into "B↔C♯/D♭" and "D♯/E♭↔F," respectively, while "G♯/A♭↔A" is chosen over "G↔G♯/A♭" as the third switch.



Cube a (3x): shares face with low cube and edge with high cube

Cube b (3x): shares edge with low cube and face with high cube

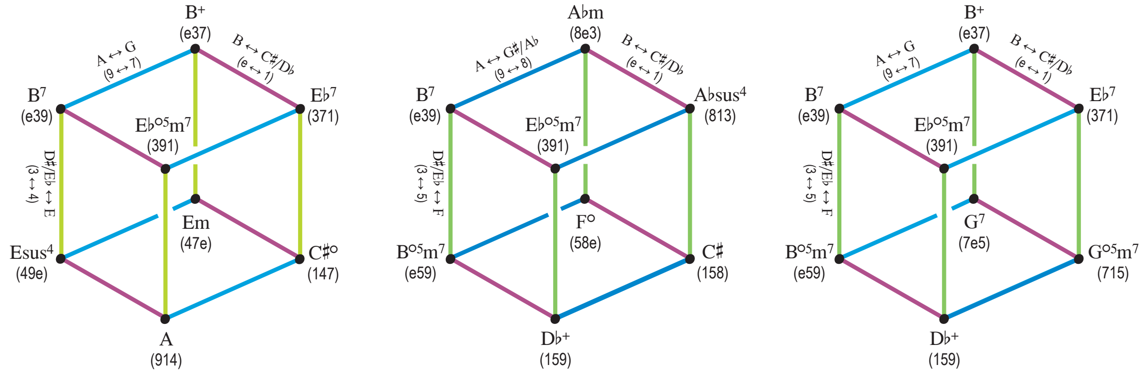
Cube c (3x): simplification of low cube plus Cube a



Cube d (3x): simplification of high cube plus Cube b

Cube e (3x): simplification of Cube a plus Cube b, polar suspended fourths

Cube f (3x): simplification of Cube a plus Cube b, polar diminished triads



Cube g (3x): simplification of low cube plus two Cube as, one Cube b

Cube h (3x): simplification of high cube plus one Cube a, two Cube bs

Cube i (1x): simplification of supercube

Figure 15: The nine possible types of unit cubes derived from simplification of the hexatonic supercube, excluding the hexatonic cube itself. For each of the first eight cube types, a single supercube will yield three possibilities that are related by permutation and modulation. Naturally, there is only one possibility of the last cube type, which results from simplification of the entire supercube.

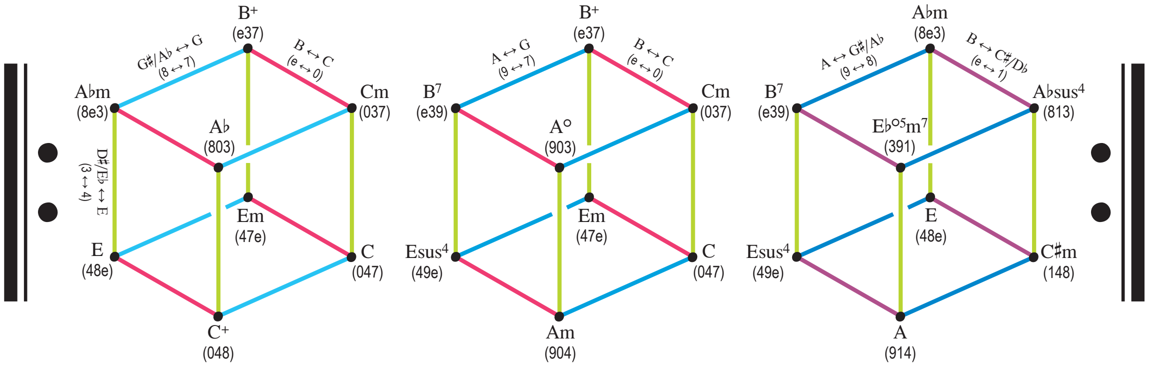


Figure 16: A graph progression involving different simplifications of the $\text{HEX}_{3,4}/\text{HEX}_{0,1}$ supercube. The first cube is $\text{HEX}_{3,4}$. The second cube remains in the same orientation as the first, but extends the "G↔G♯/A♭" switch to "G ↔ A." The third cube then reduces "G↔A" to "G♯/A♭↔A" and extends "B↔C" to "B↔C♯/D♭."

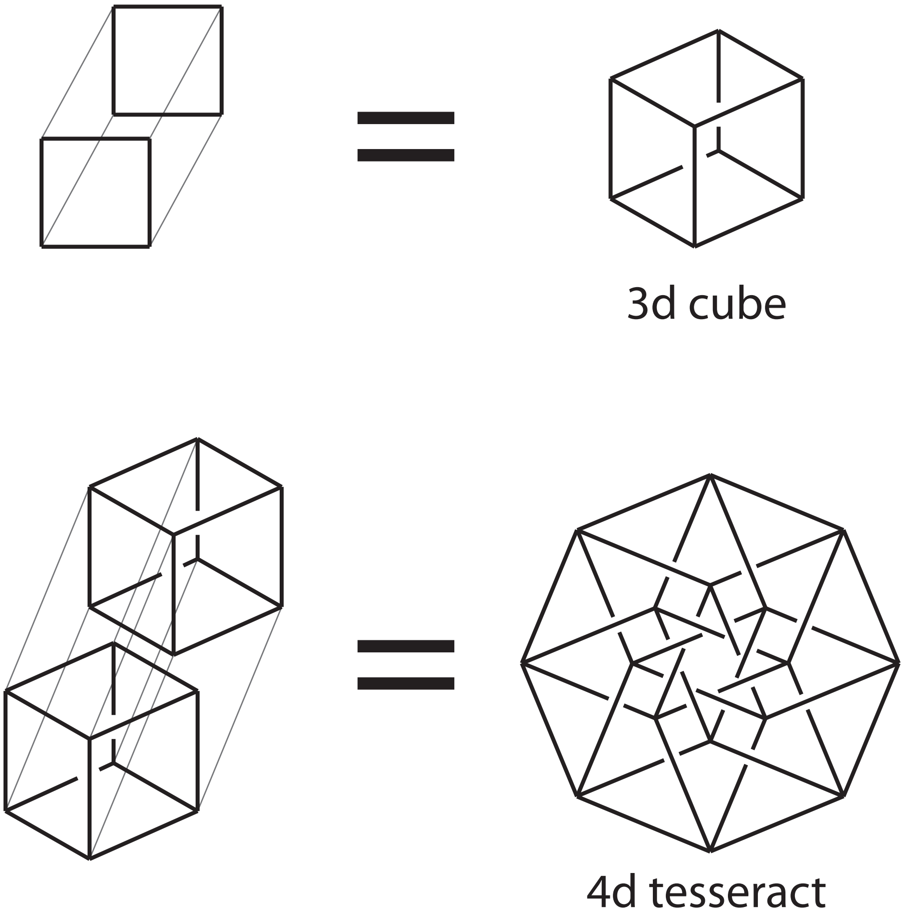


Figure 17: Just as a cube is formed by joining like vertices on opposite squares, a tesseract is formed by joining like vertices on opposite cubes.

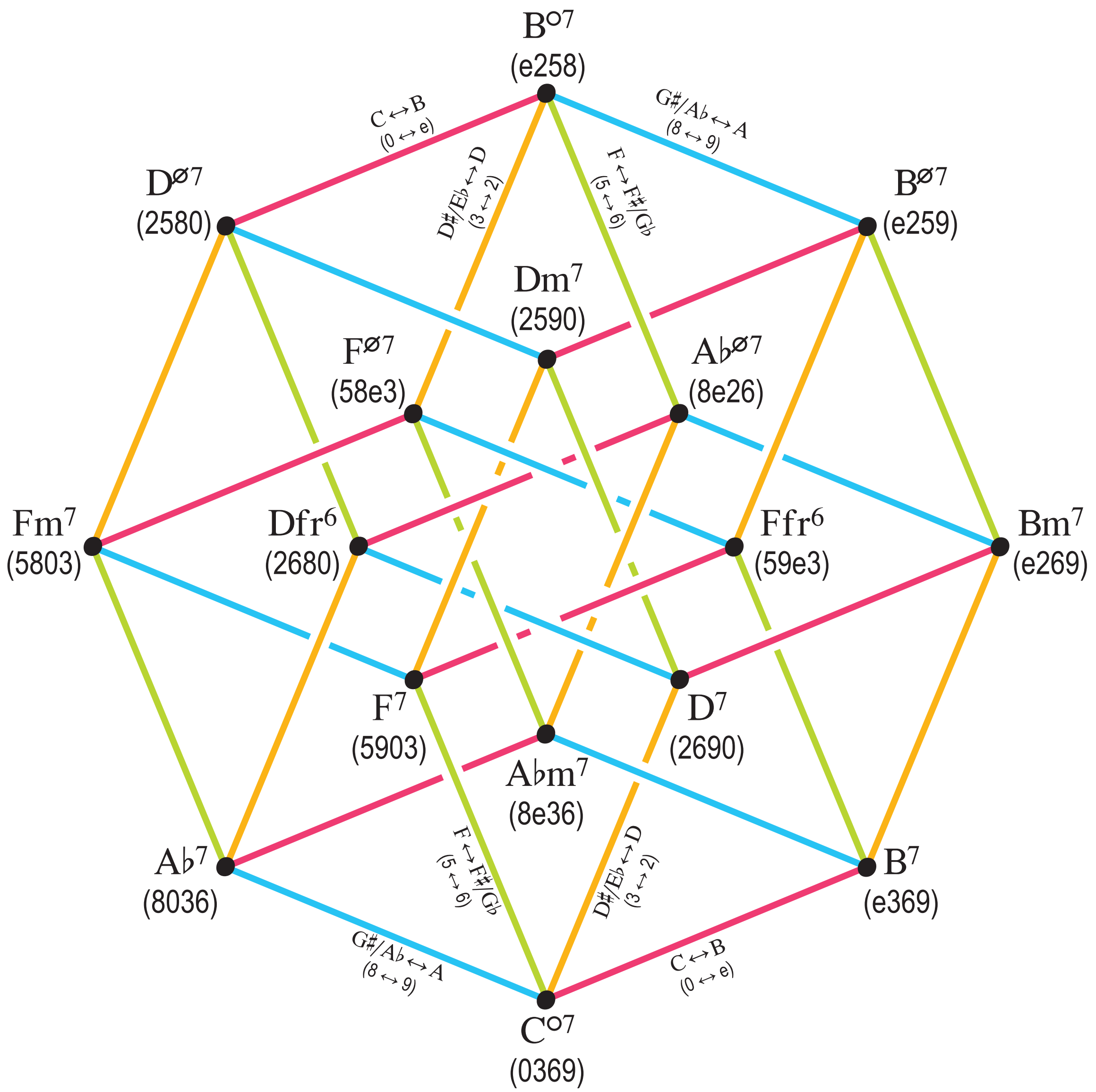


Figure 18: Octatonic tesseract OCT$_{2,3}$.

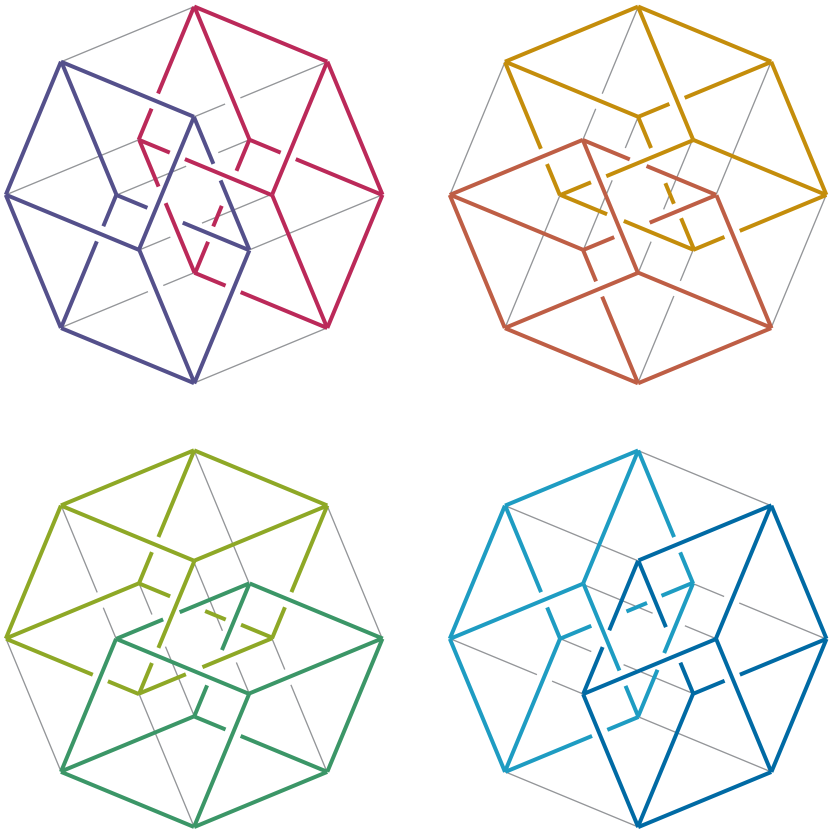


Figure 19: The four pairs of opposite cubes on a tesseract.

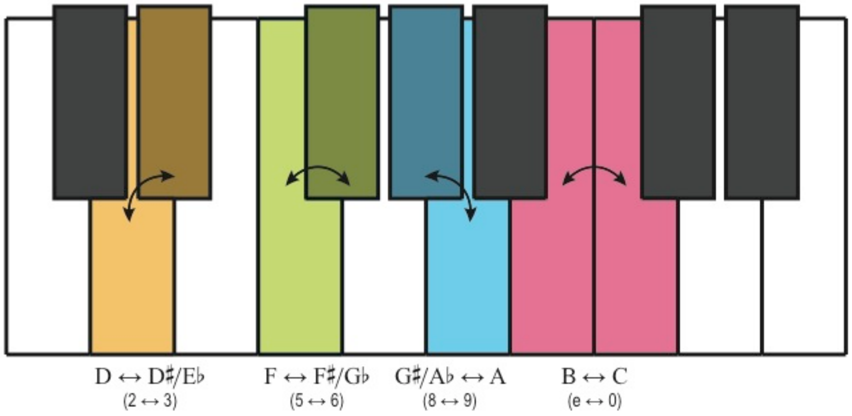


Figure 20: Keyboard layout for the $\mathrm{OCT}_{2,3}$ tesseract.

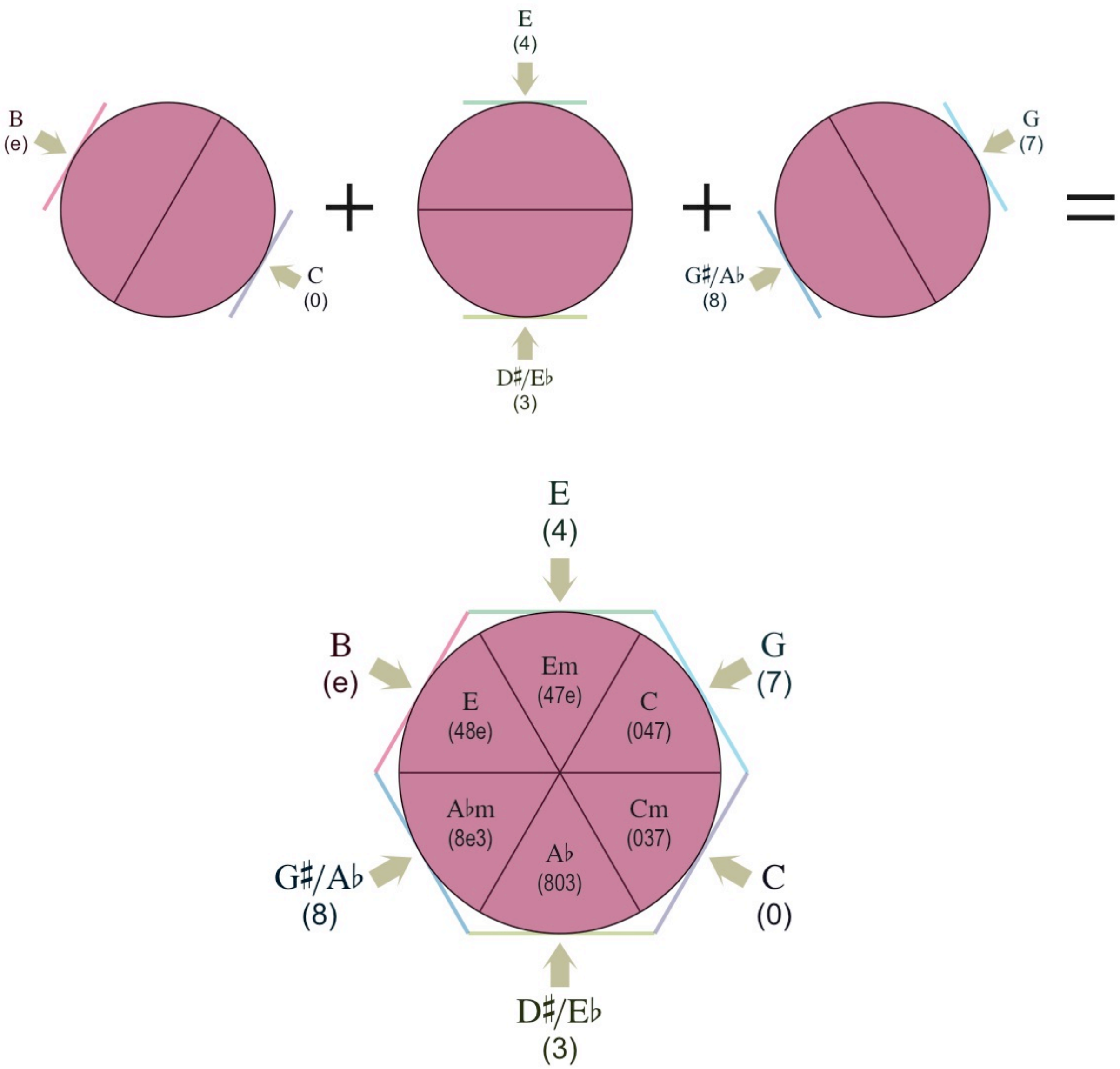


Figure 21: The three switches of the $\text{HEX}_{3,4}$ cube are mapped onto the virtual circle as pairs of opposite semicircles at equal angles, dividing the circle into six slices.

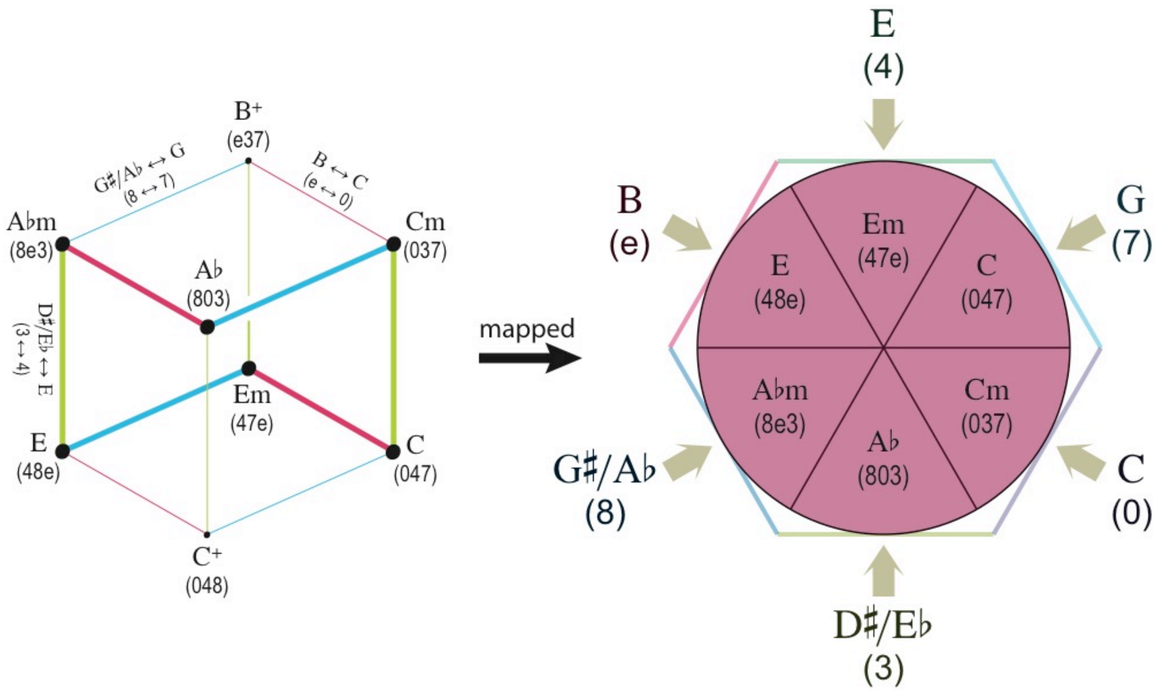


Figure 22: Hexatonic cube HEX$_{3,4}$ cannot be mapped onto the virtual circle without excluding two triads. In this case, the two augmented triads are missing. The actual graph being mapped may thus be depicted as an enclosed path of six edges interconnecting the cube's three major and three minor triads.

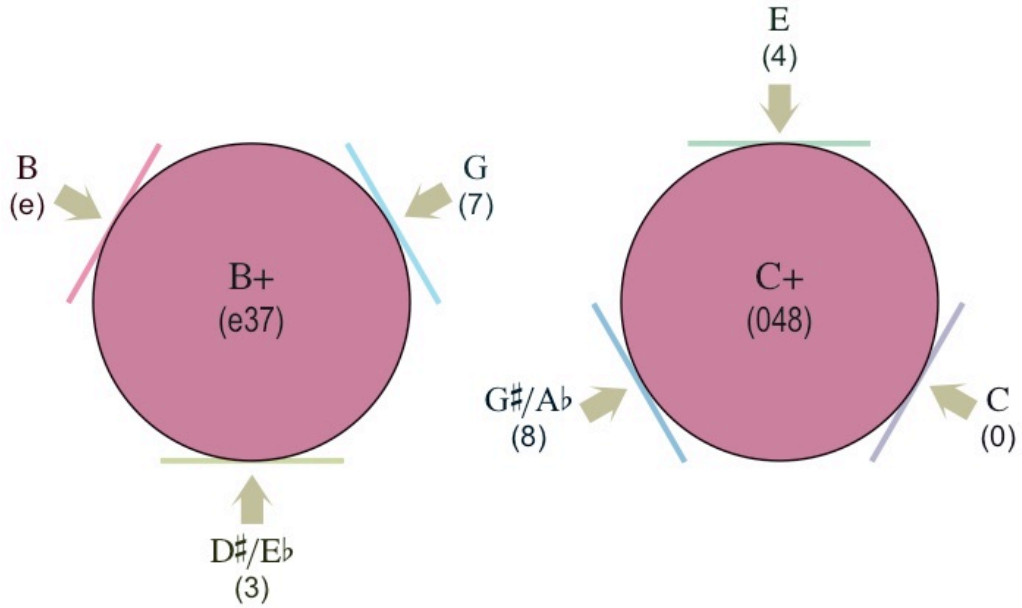


Figure 23: There is no region within the virtual circle in which the three pcs of each missing triad can be heard concurrently.

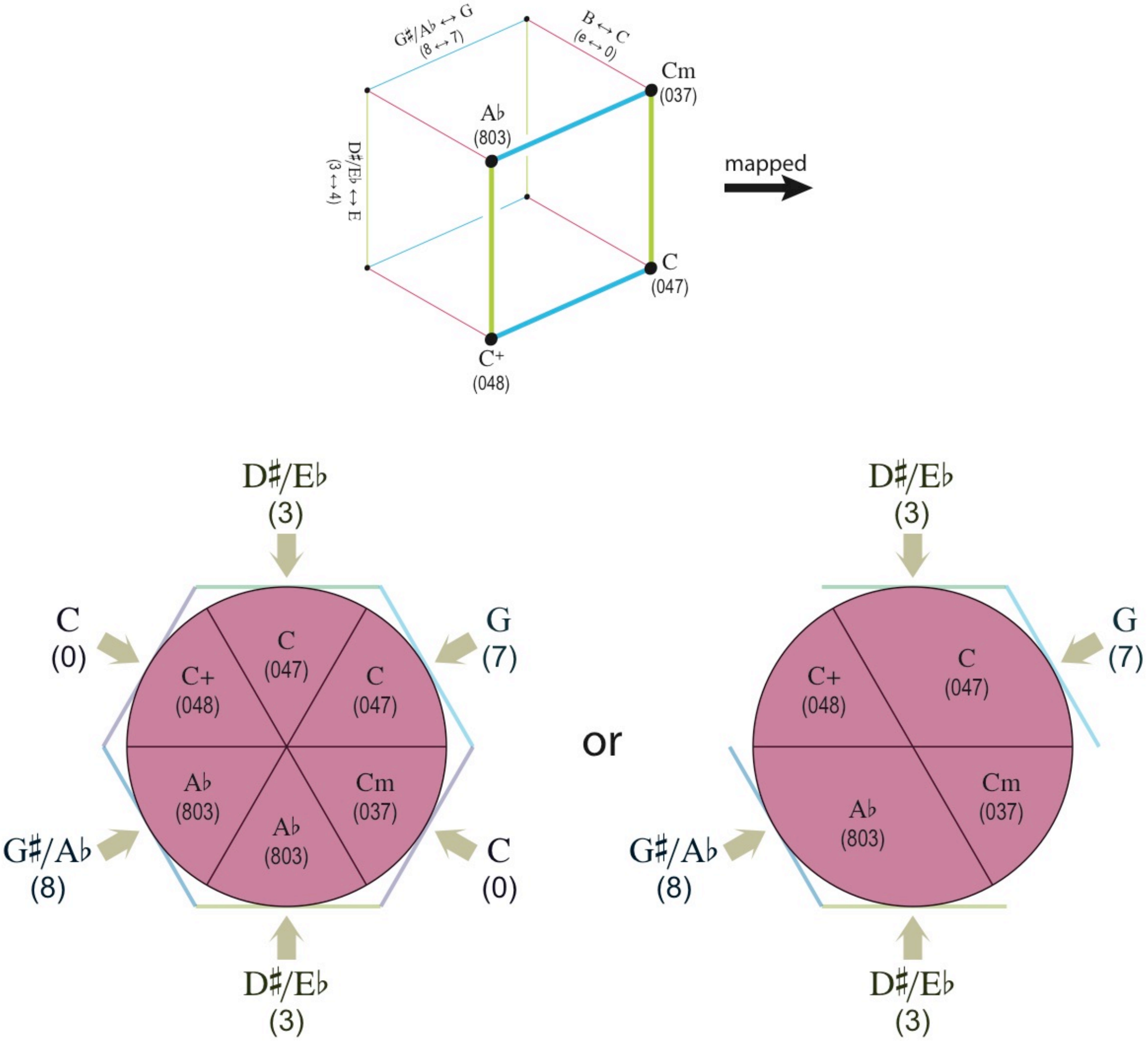


Figure 24: The “B↔C” switch of the HEX$_{3,4}$ cube has been removed from consideration by holding C as constant when it is mapped onto the virtual circle. The left circle shows two separate parts playing C. The right side shows a single part playing C throughout the circle. The actual graph being mapped may be depicted as a cube flattened into a square.

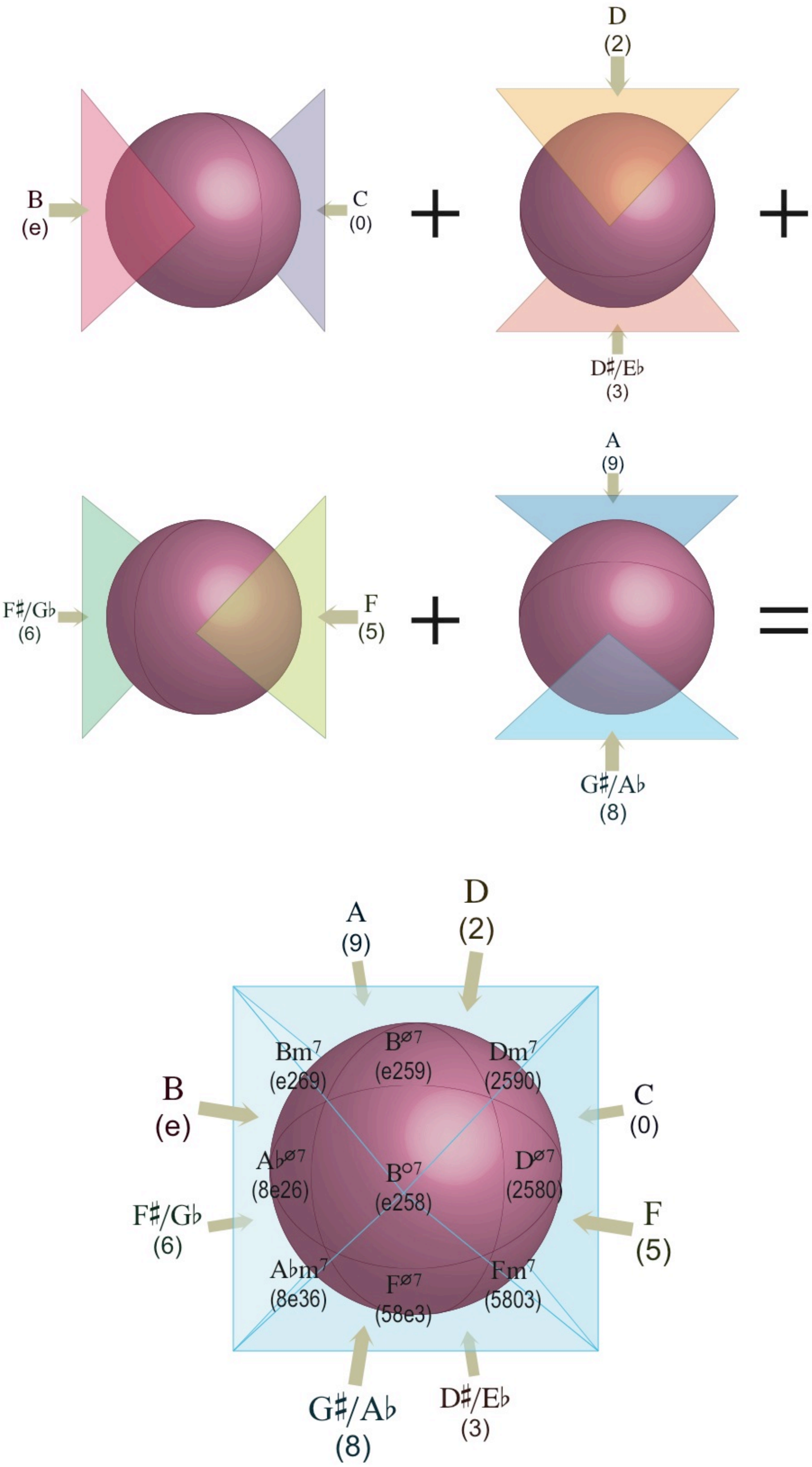


Figure 25: The four switches of the OCT$_{2,3}$ cube are mapped onto the virtual sphere as pairs of opposite hemispheres at equal angles, dividing the sphere into fourteen wedges.

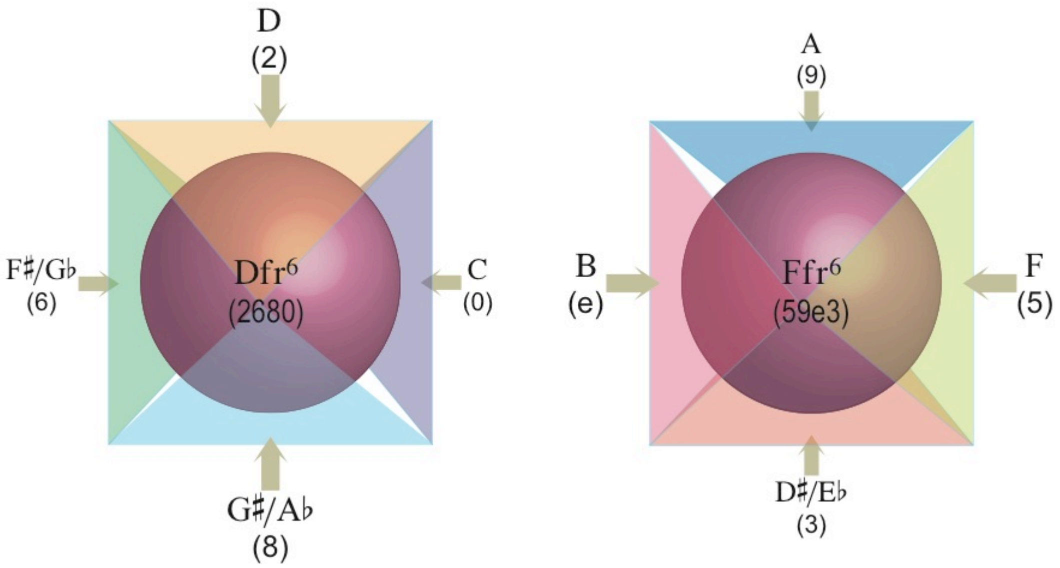


Figure 26: There is no region within the virtual sphere in which the four pcs of each missing seventh chord can be heard concurrently.

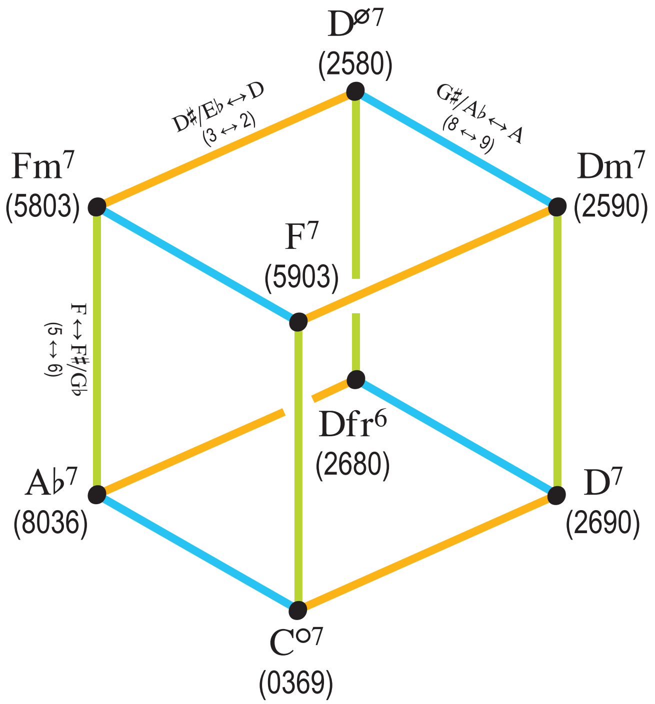


Figure 27: Octatonic tesseract $\text{OCT}_{2,3}$ collapsed into a cube. The "B↔C" switch has been removed from consideration by holding C as constant.

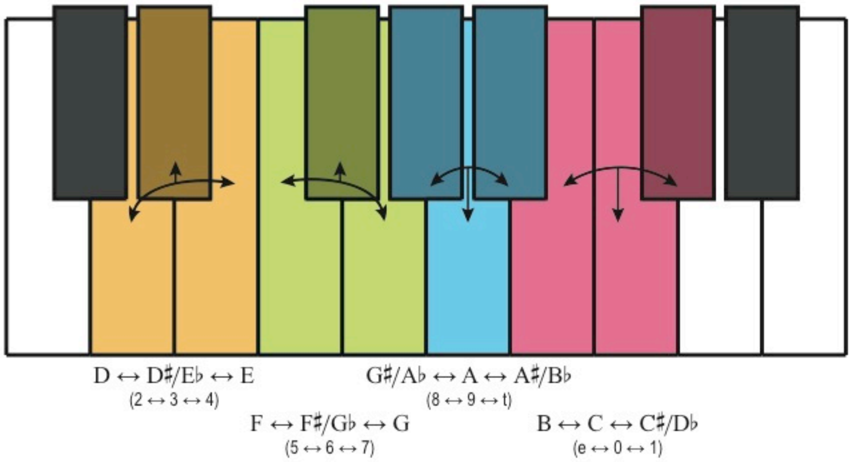


Figure 28: Keyboard layout for supertesseract $OCT_{2,3}/OCT_{0,1}$.

switch positions e0123456789t	no. per system	interval set class	pc set	chord name *(non-traditional in italics)*
L--L--L--L--	1	[3,3,3,3]	(0369)	fully diminished seventh
L--L--L---M-	4	[2,3,3,4]	(0258)	half-diminished seventh
L--L--L----H	4	[1,3,3,5]	(0147)	*diminished fifth, major seventh*
L--L---M--M-	4	[2,3,4,3]	(0358)	minor seventh
L--L---M---H	4	[1,3,4,4]	(0148)	minor major seventh
L--L----H-M-	4	[2,2,3,5]	(0247)	*major triad, suspended second*
L--L----H--H	4	[1,3,5,3]	(0347)	*minor-major triad*
L---M-L---M-	2	[2,4,2,4]	(0268)	French sixth
L---M-L----H	4	[1,4,2,5]	(0157)	*half-diminished seventh, suspended fourth*
L---M--M--M-	4	[2,4,3,3]	(0258)	dominant seventh
L---M--M---H	4	[1,4,3,4]	(0158)	major seventh
L---M---H-M-	4	[2,2,4,4]	(0248)	*dominant seventh, raised fifth*
L---M---H--H	4	[1,4,4,3]	(0148)	*augmented fifth, major seventh*
L----HL----H	2	[1,5,1,5]	(0167)	*“Petrushka fifths”*
L----H-M--M-	4	[2,3,2,5]	(0257)	*minor seventh, suspended fourth*
L----H-M---H	4	[1,5,2,4]	(0157)	*major seventh, suspended fourth*
L----H--H-M-	4	[2,2,5,3]	(0247)	*minor seventh, suspended second*
L----H--H--H	4	[1,5,3,3]	(0147)	*minor triad, diminished-perfect fifth*
-M--M--M--M-	1	[3,3,3,3]	(0369)	fully diminished seventh
-M--M--M---H	4	[2,3,3,4]	(0258)	half-diminished seventh
-M--M---H--H	4	[2,3,4,3]	(0358)	minor seventh
-M---H-M---H	2	[2,4,2,4]	(0268)	French sixth
-M---H--H--H	4	[2,4,3,3]	(0258)	dominant seventh
--H--H--H--H	1	[3,3,3,3]	(0369)	fully diminished seventh
	81 total			

Figure 29: Possible combinations of switch positions for the supertesseract. The pc index given is for supertesseract $OCT_{2,3}/OCT_{0,1}$.

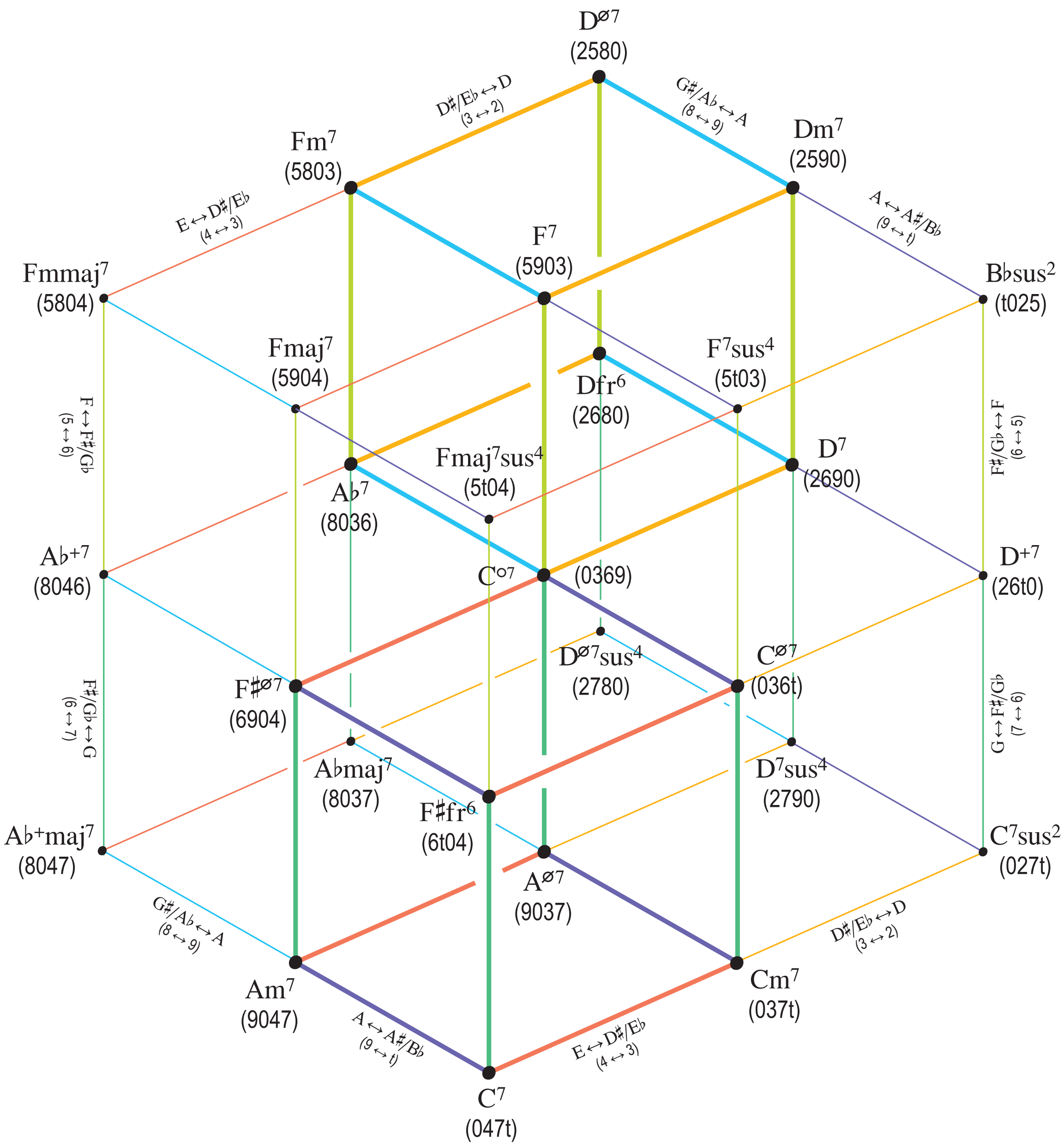


Figure 30: Octatonic supertesseract $\mathrm{OCT}_{2,3}/\mathrm{OCT}_{0,1}$ collapsed into a supercube. The "B↔C" and "C↔C♯/D♭" switches have been removed from consideration by holding the middle position C as constant. The asymmetry of this supercube due to the removal of one axis means that no unit cube derived from simplification can be permuted or modulated to form any other. Every possible unit cube thus constitutes its own unique cube type. Note that those combinations of switch positions that do not include a single middle position are necessarily absent from this graph.

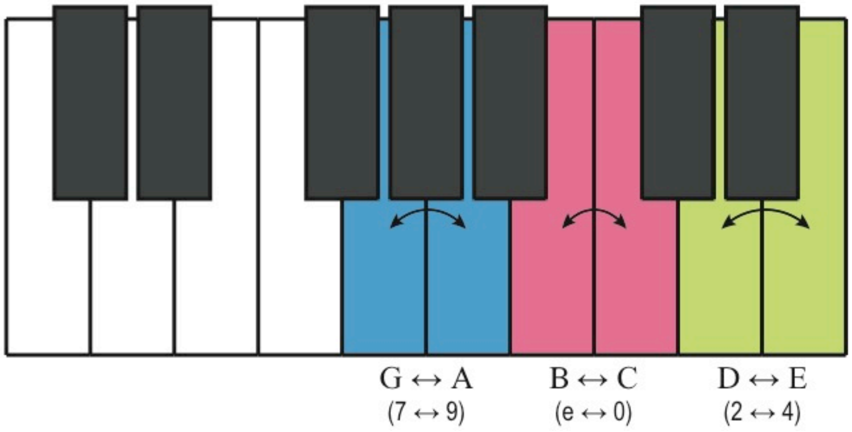


Figure 31: Keyboard layout for the diatonic cube of hexachord [7,9,e,0,2,4].

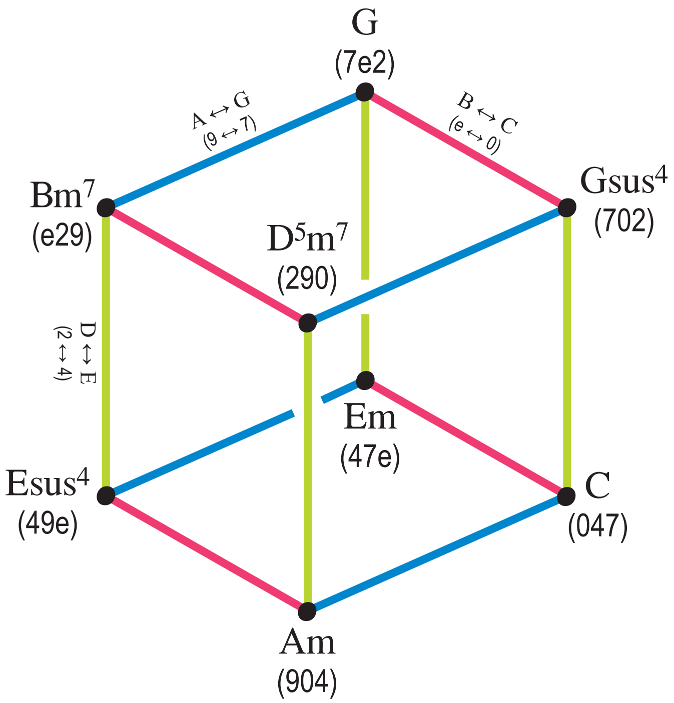


Figure 32: Diatonic cube of hexachord [7,9,e,0,2,4].

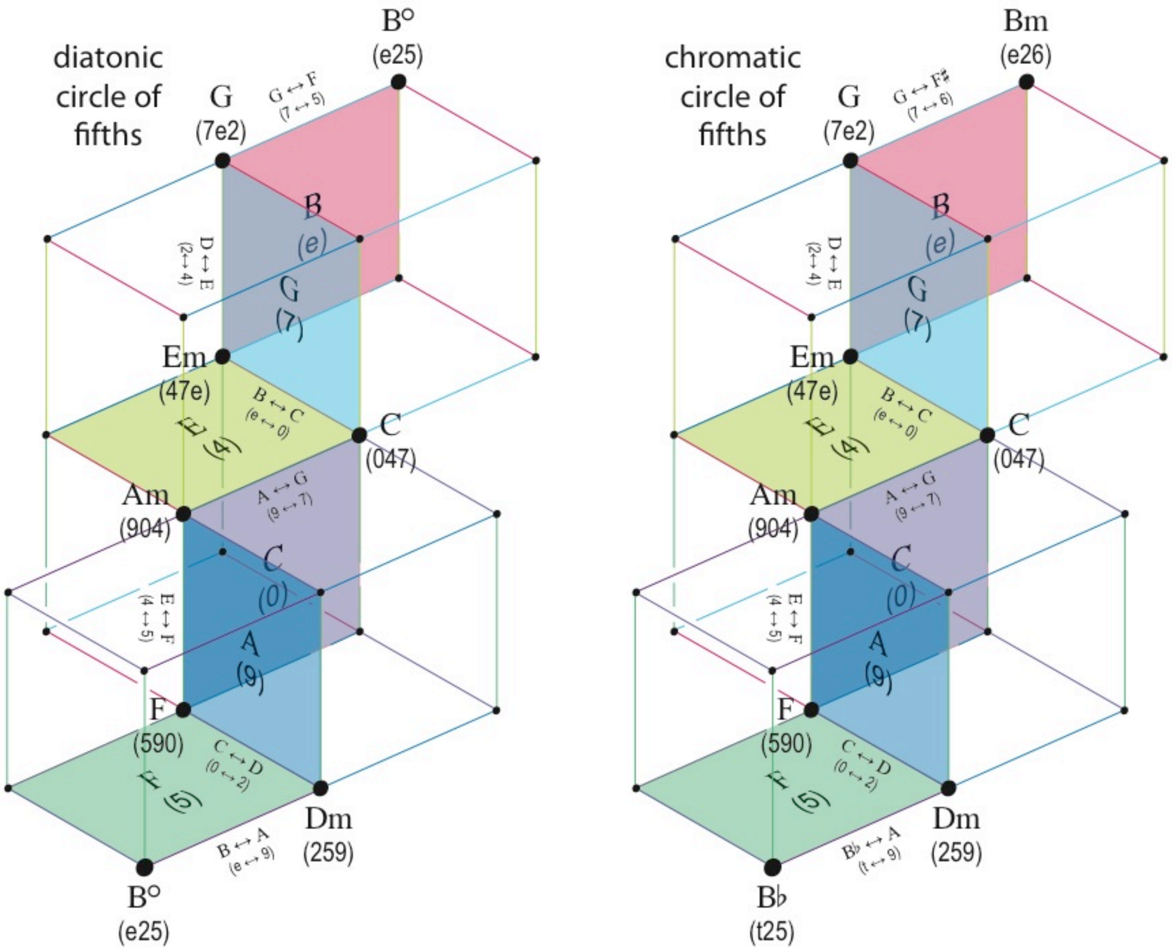


Figure 33: Chains of diatonic cubes joined by common faces. The left chain stays within the C-diatonic system and thus returns to the starting cube after seven modulations, while the right chain moves through the entire chromatic system and returns to the starting cube after twelve modulations.

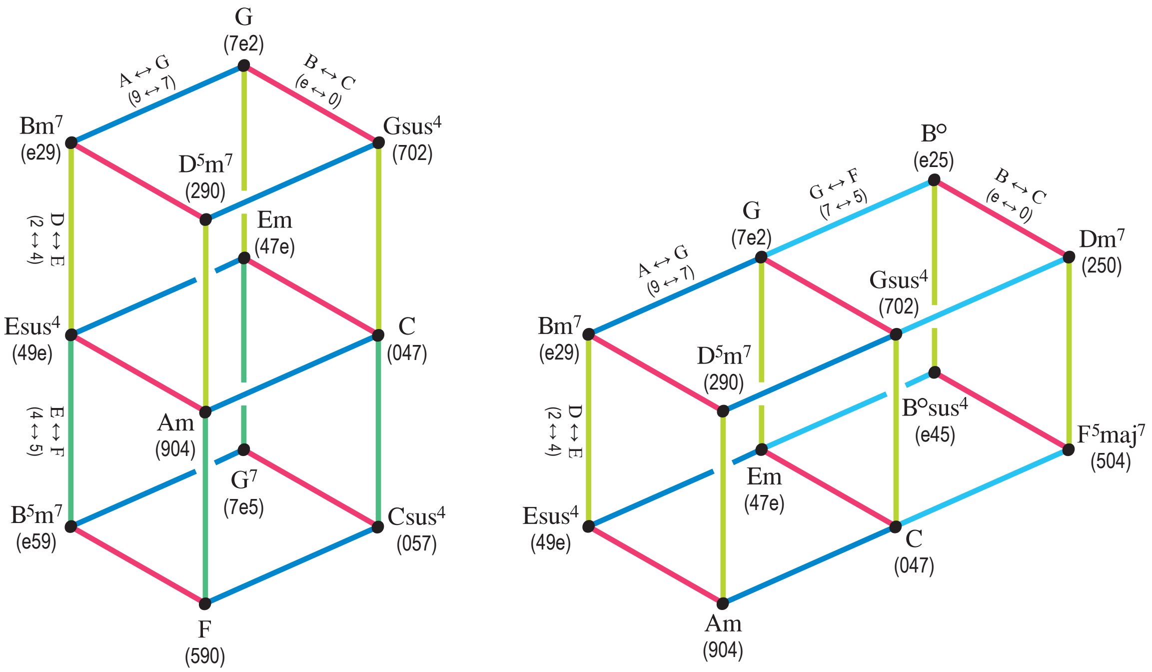


Figure 34: Duplecubes created by joining the diatonic cube of hexachord [7,9,e,0,2,4] with that of [5,7,9,e,0,2] in the left graph, and with that of [9,e,0,2,4,5] in the right graph, respectively.

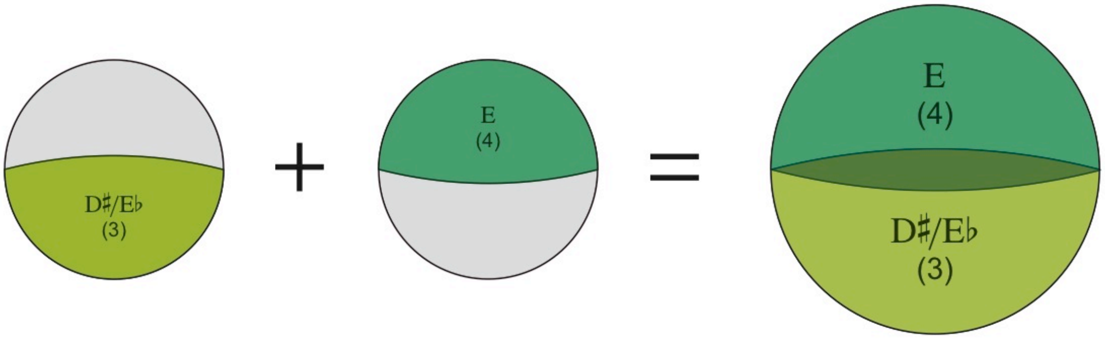


Figure 35: Opposite semicircles overlap, creating a transitional region between them in which a listener crossing from one into the other will hear both pcs of their respective semitone dyad.

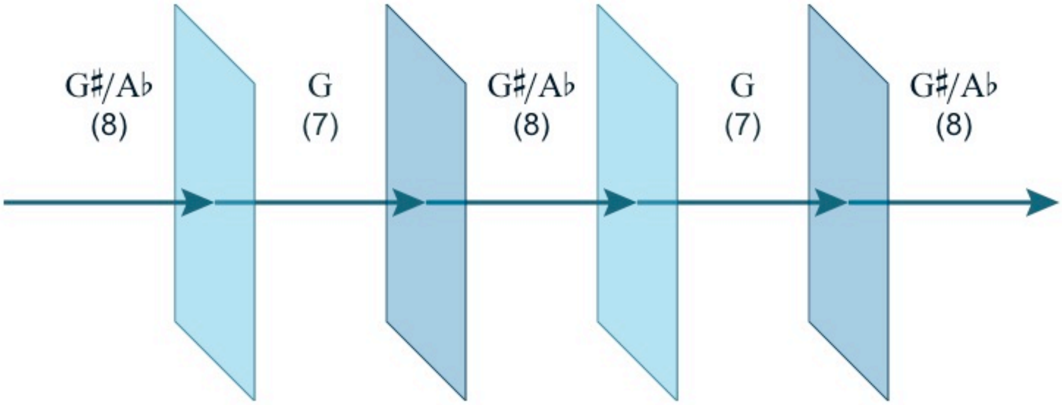


Figure 36: In this infinite realm, a listener moving parallel to the axis of the "G↔G♯/A♭" switch will continuously toggle between G and G♯/A♭.

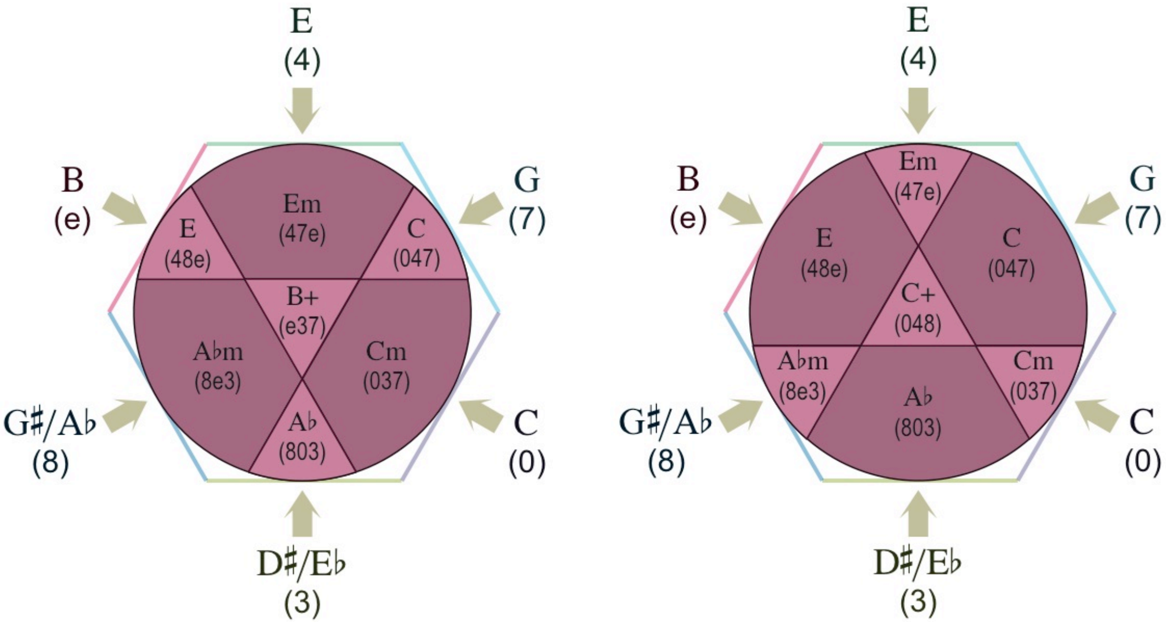


Figure 37: In the virtual circle, skewing each dividing line away from the origin results in a sector at the center that uncovers one of the missing triads. Skewing each line in the opposite direction uncovers the other.

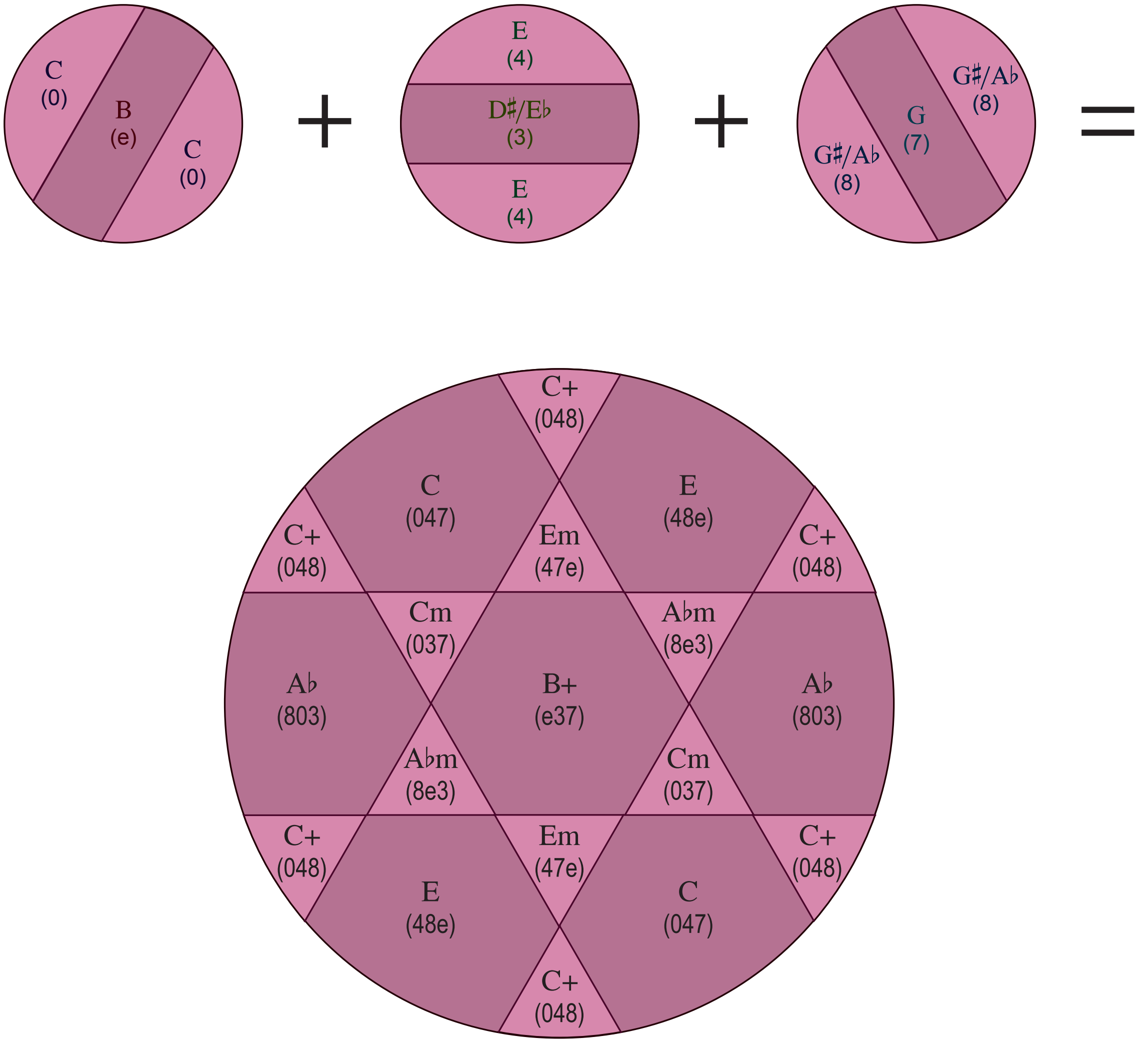


Figure 38: The three switches of hexatonic cube HEX$_{3,4}$ are each mapped as two parallel lines that divide the virtual circle into three regions, with the inner region hearing one pc of the respective semitone dyad and the two outer regions hearing the other. This uncovers both missing triads and allows each of the cube's eight triads to be heard in at least one unique sector of the virtual circle.

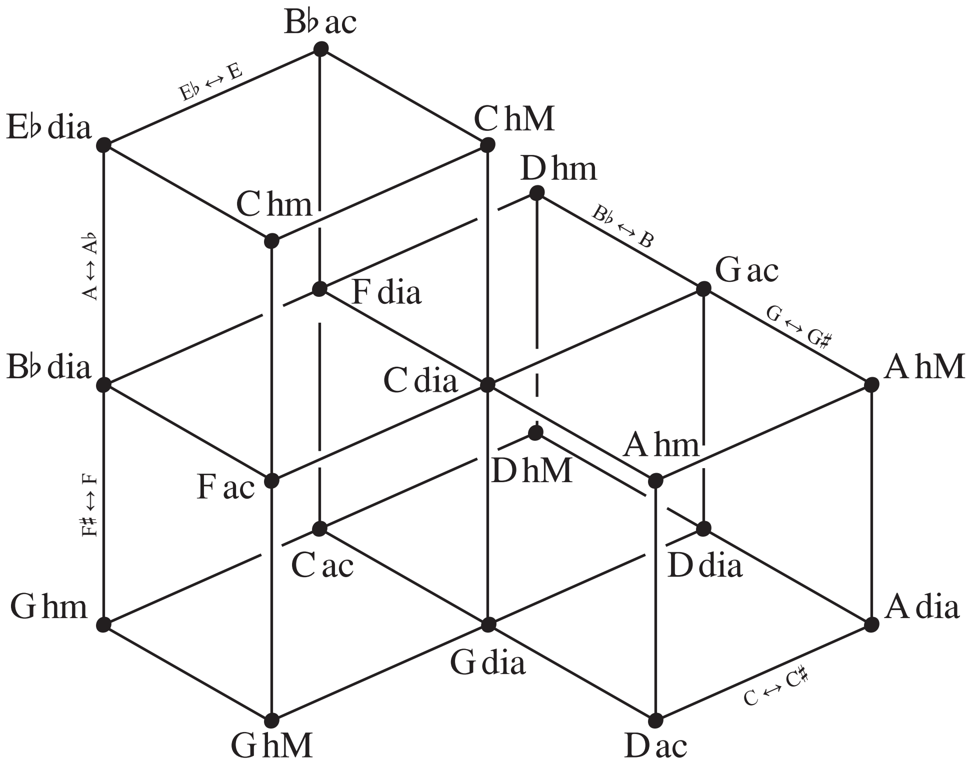


Figure 39: Scale graph showing semitonal voice-leading relations between diatonic, acoustic, harmonic major, and harmonic minor scales.

Diatonic, both major and minor		
G♭	E♭m	C♭ G♭ D♭ A♭ E♭ B♭
D♭	B♭m	G♭ D♭ A♭ E♭ B♭
A♭	Fm	D♭ A♭ E♭ B♭
E♭	Cm	A♭ E♭ B♭
B♭	Gm	E♭ B♭
F	Dm	B♭
C	Am	---
G	Em	F♯
D	Bm	F♯ C♯
A	F♯m	F♯ C♯ G♯
E	C♯m	F♯ C♯ G♯ D♯
B	G♯m	F♯ C♯ G♯ D♯ A♯
F♯	D♯m	F♯ C♯ G♯ D♯ A♯ E♯

Acoustic (major with raised 4th and lowered 7th)	
G♭	F♭ --- G♭ D♭ A♭ E♭ B♭
D♭	C♭ --- D♭ A♭ E♭ B♭
A♭	G♭ --- A♭ E♭ B♭
E♭	D♭ --- E♭ B♭
B♭	A♭ --- B♭
F	E♭ ---
C	B♭ --- F♯
G	--- C♯
D	F♯ --- G♯
A	F♯ C♯ --- D♯
E	F♯ C♯ G♯ --- A♯
B	F♯ C♯ G♯ D♯ --- E♯
F♯	F♯ C♯ G♯ D♯ A♯ --- B♯

Harmonic major (major with lowered 6th)	
G♭	E♭♭ --- --- C♭ G♭ D♭ A♭ E♭ B♭
D♭	B♭♭ --- --- G♭ D♭ A♭ E♭ B♭
A♭	F♭ --- --- D♭ A♭ E♭ B♭
E♭	C♭ --- --- A♭ E♭ B♭
B♭	G♭ --- --- E♭ B♭
F	D♭ --- --- B♭
C	A♭ --- ---
G	E♭ --- --- F♯
D	B♭ --- F♯ C♯
A	--- C♯ G♯
E	F♯ --- G♯ D♯
B	F♯ C♯ --- D♯ A♯
F♯	F♯ C♯ G♯ --- A♯ E♯

Harmonic minor (minor with raised 7th)	
E♭m	C♭ G♭ --- A♭ E♭ B♭
B♭m	G♭ D♭ --- E♭ B♭
Fm	D♭ A♭ --- B♭
Cm	A♭ E♭ ---
Gm	E♭ B♭ --- F♯
Dm	B♭ --- --- C♯
Am	--- --- G♯
Em	F♯ --- --- D♯
Bm	F♯ C♯ --- --- A♯
F♯m	F♯ C♯ G♯ --- --- E♯
C♯m	F♯ C♯ G♯ D♯ --- --- B♯
G♯m	F♯ C♯ G♯ D♯ A♯ --- --- F𝄪
D♯m	F♯ C♯ G♯ D♯ A♯ E♯ --- --- C𝄪

Figure 40: Accidentals indicated in the key signatures for all the diatonic, acoustic, harmonic major, and harmonic minor scales.

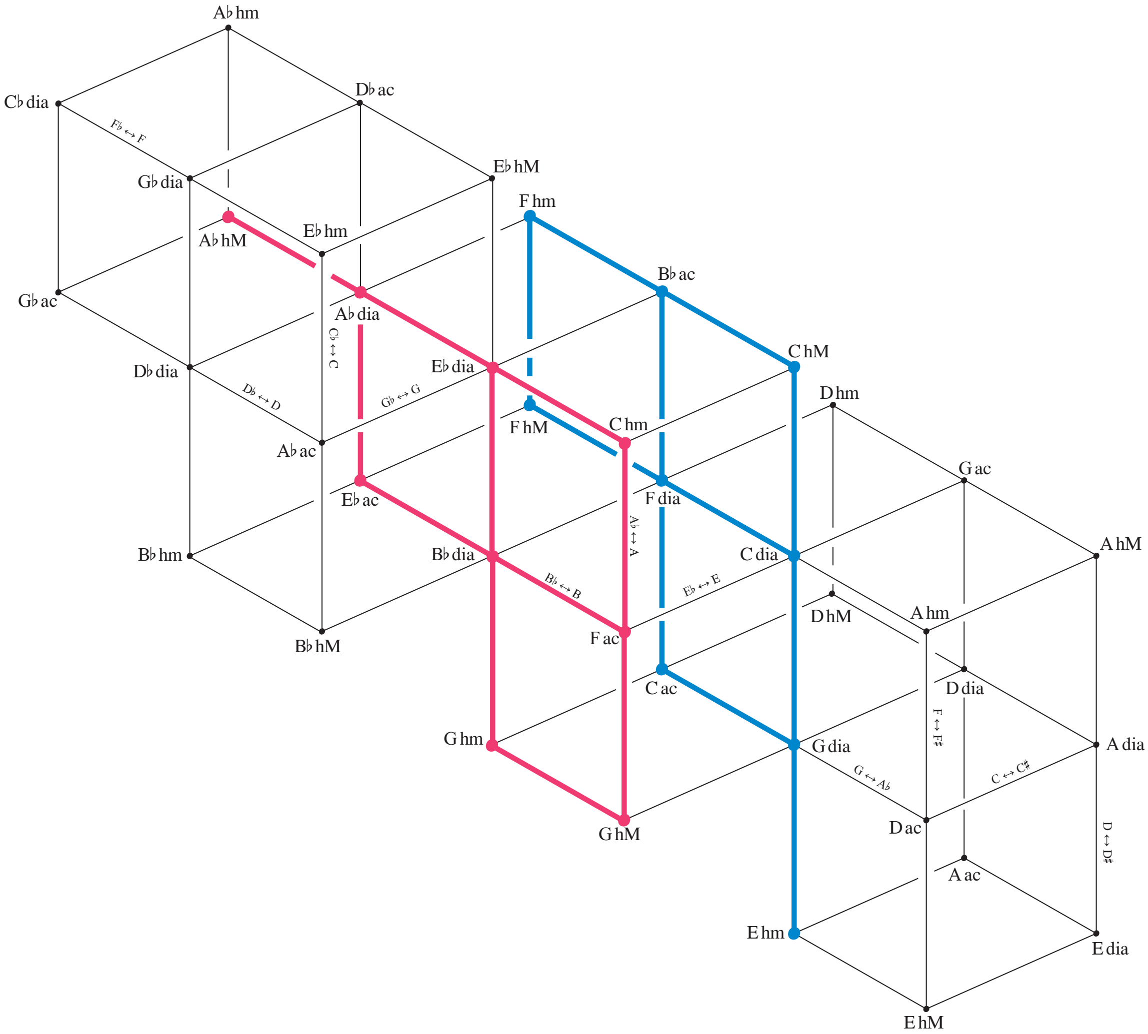


Figure 41: The set of blue vertices represents all the heptachords that contain the C-major triad. The set of red vertices represents all the heptachords that contain the C-minor triad. Every unit segment consisting of one vertex from each set separated by a single semitone dyad may be mapped onto a virtual circle that contains only C-major and C-minor triads.

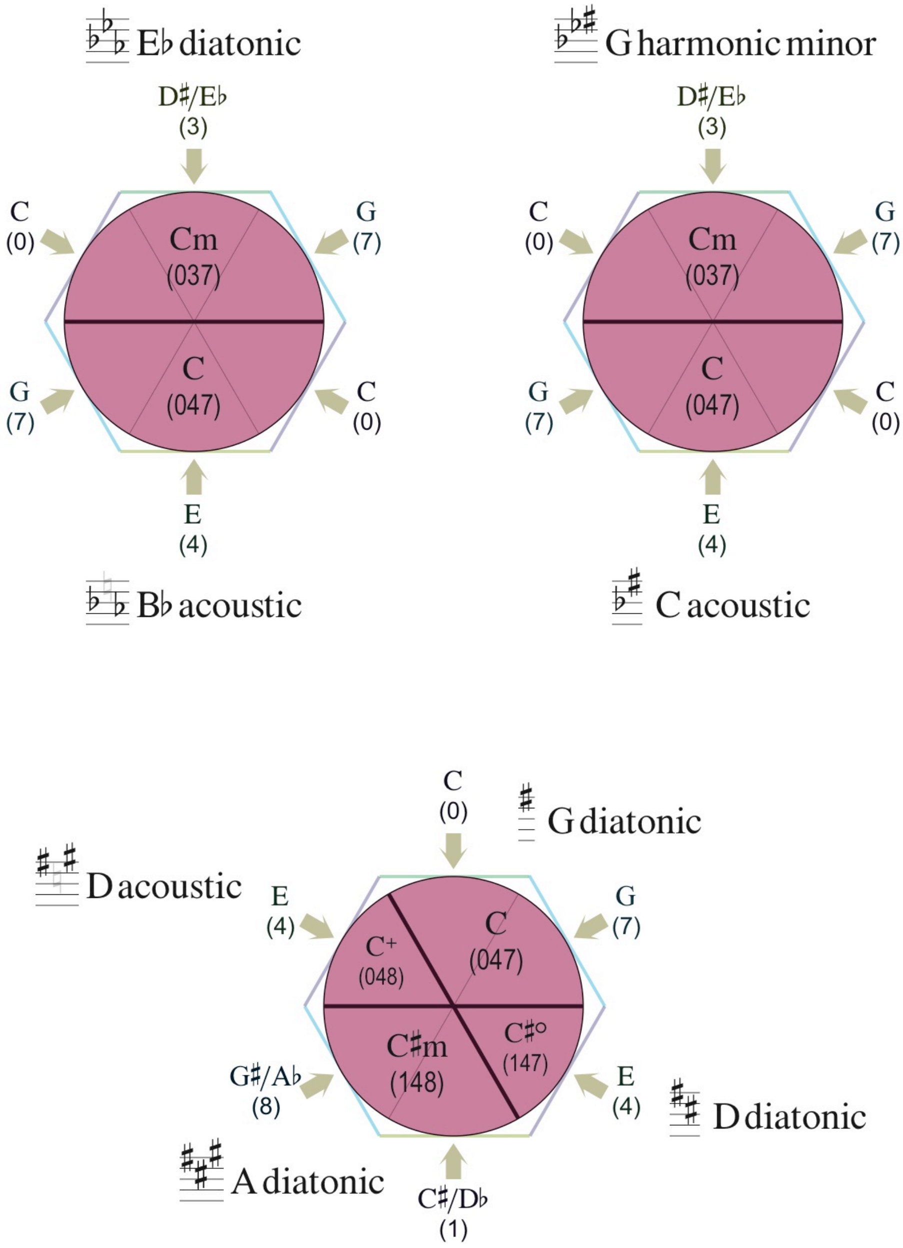


Figure 42: The top two examples show unit segments from the scale graph mapped onto the virtual circle. The bottom example shows a unit square mapped in the same manner. Key signatures show which key is heard by a listener located in any particular slice of the circle. For example, in the top left circle, a listener in the upper left slice will hear the key of E♭-diatonic, while in the lower left slice she hears the key of B♭-acoustic. Note, however, that the parts assigned to pcs C and G in the upper left and lower left slices, respectively, straddle both keys. Thus, in realizing this graph, a composer would be mindful that neither of the melodic lines assigned to each part contradicts either one of these two keys.

Maj	dia	ac	hM	hm
G♭	G♭ C♭ D♭	F♭ G♭	G♭ C♭	B♭m C♭m
D♭	D♭ G♭ A♭	C♭ D♭	D♭ G♭	Fm G♭m
A♭	A♭ D♭ E♭	G♭ A♭	A♭ D♭	Cm D♭m
E♭	E♭ A♭ B♭	D♭ E♭	E♭ A♭	Gm A♭m
B♭	B♭ E♭ F	A♭ B♭	B♭ E♭	Dm E♭m
F	F B♭ C	E♭ F	F B♭	Am B♭m
C	C F G	B♭ C	C F	Em Fm
G	G C D	F G	G C	Bm Cm
D	D G A	C D	D G	F♯m Gm
A	A D E	G A	A D	C♯m Dm
E	E A B	D E	E A	G♯m Am
B	B E F♯	A B	B E	D♯m Em
F♯	F♯ B C♯	E F♯	F♯ B	A♯m Bm

min	dia	ac	hM	hm
E♭m	G♭ C♭ D♭	G♭ A♭	B♭ C♭	B♭m E♭m
B♭m	D♭ G♭ A♭	D♭ E♭	F G♭	Fm B♭m
Fm	A♭ D♭ E♭	A♭ B♭	C D♭	Cm Fm
Cm	E♭ A♭ B♭	E♭ F	G A♭	Gm Cm
Gm	B♭ E♭ F	B♭ C	D E♭	Dm Gm
Dm	F B♭ C	F G	A B♭	Am Dm
Am	C F G	C D	E F	Em Am
Em	G C D	G A	B C	Bm Em
Bm	D G A	D E	F♯ G	F♯m Bm
F♯m	A D E	A B	C♯ D	C♯m F♯m
C♯m	E A B	E F♯	G♯ A	G♯m C♯m
G♯m	B E F♯	B C♯	D♯ E	D♯m G♯m
D♯m	F♯ B C♯	F♯ G♯	A♯ B	A♯m D♯m

Aug	ac	HM	hm
G♭+ B♭+ D+ F♯+	F♭ A♭ C E	G♭ B♭ D F♯	E♭m Gm Bm D♯m
D♭+ F+ A+ C♯+	C♭ E♭ G B	D♭ F A C♯	B♭m Dm F♯m A♯m
A♭+ C+ E+ G♯+	G♭ B♭ D F♯	A♭ C E G♯	Fm Am C♯m E♯m
E♭+ G+ B+ D♯+	D♭ F A C♯	E♭ G B D♯	Cm Em G♯m B♯m

dim	dia	ac	HM	hm
F°	G♭	C♭ D♭	E♭ G♭	E♭m G♭m
C°	D♭	G♭ A♭	B♭ D♭	B♭m D♭m
G°	A♭	D♭ E♭	F A♭	Fm A♭m
D°	E♭	A♭ B♭	C E♭	Cm E♭m
A°	B♭	E♭ F	G B♭	Gm B♭m
E°	F	B♭ C	D F	Dm Fm
B°	C	F G	A C	Am Cm
F♯°	G	C D	E G	Em Gm
C♯°	D	G A	B D	Bm Dm
G♯°	A	D E	F♯ A	F♯m Am
D♯°	E	A B	C♯ E	C♯m Em
A♯°	B	E F♯	G♯ B	G♯m Bm
E♯°	F♯	B C♯	D♯ F♯	D♯m F♯m

Figure 43: Scale membership for all the major, minor, augmented, and diminished triads.

Dom7	dia	ac	hM	hm
G♭^{7}	C♭	F♭	C♭	C♭m
D♭^{7}	G♭	C♭	G♭	G♭m
A♭^{7}	D♭	G♭	D♭	D♭m
E♭^{7}	A♭	D♭	A♭	A♭m
B♭^{7}	E♭	A♭	E♭	E♭m
F^7	B♭	E♭	B♭	B♭m
C^7	F	B♭	F	Fm
G^7	C	F	C	Cm
D^7	G	C	G	Gm
A^7	D	G	D	Dm
E^7	A	D	A	Am
B^7	E	A	E	Em
F♯^{7}	B	E	B	Bm

min7	dia	ac	hM	hm
E♭m^7	G♭ C♭ D♭	G♭	C♭	B♭m
B♭m^7	D♭ G♭ A♭	D♭	G♭	Fm
Fm^7	A♭ D♭ E♭	A♭	D♭	Cm
Cm^7	E♭ A♭ B♭	E♭	A♭	Gm
Gm^7	B♭ E♭ F	B♭	E♭	Dm
Dm^7	F B♭ C	F	B♭	Am
Am^7	C F G	C	F	Em
Em^7	G C D	G	C	Bm
Bm^7	D G A	D	G	F♯m
F♯m^7	A D E	A	D	C♯m
C♯m^7	E A B	E	A	G♯m
G♯m^7	B E F♯	B	E	D♯m
D♯m^7	F♯ B C♯	F♯	B	A♯m

Maj7	dia	hM	hm
G♭maj^7	G♭	G♭	E𝄫m
D♭maj^7	D♭	D♭	B𝄫m
A♭maj^7	A♭	A♭	F♭m
E♭maj^7	E♭	E♭	C♭m
B♭maj^7	B♭	B♭	G♭m
Fmaj^7	F	F	D♭m
Cmaj^7	C	C	A♭m
Gmaj^7	G	G	E♭m
Dmaj^7	D	D	B♭m
Amaj^7	A	A	Fm
Emaj^7	E	E	Cm
Bmaj^7	B	B	Gm
F♯maj^7	F♯	F♯	Dm

dim7	HM	hm
B^{o7} D^{o7} F^{o7} A♭^{o7}	F♯ A C E♭	F♯m Am Cm E♭m
F♯^{o7} A^{o7} C^{o7} E♭^{o7}	C♯ E G B♭	C♯m Em Gm B♭m
C♯^{o7} E^{o7} G^{o7} B♭^{o7}	B D F A♭	Bm Dm Fm A♭m

m7dim5	dia	ac	HM	hm
F^{ø7}	G♭	C♭ D♭	E♭	E♭m
C^{ø7}	D♭	G♭ A♭	B♭	B♭m
G^{ø7}	A♭	D♭ E♭	F	Fm
D^{ø7}	E♭	A♭ B♭	C	Cm
A^{ø7}	B♭	E♭ F	G	Gm
E^{ø7}	F	B♭ C	D	Dm
B^{ø7}	C	F G	A	Am
F♯^{ø7}	G	C D	E	Em
C♯^{ø7}	D	G A	B	Bm
G♯^{ø7}	A	D E	F♯	F♯m
D♯^{ø7}	E	A B	C♯	C♯m
A♯^{ø7}	B	E F♯	G♯	G♯m
E♯^{o}	F♯	B C♯	D♯	D♯m

French 6th	ac
E♭fr^6 Afr^6	E♭ A
B♭fr^6 Efr^6	B♭ E
Ffr^6 Bfr^6	F B
Cfr^6 F♯fr^6	C F♯
Gfr^6 C♯fr^6	G C♯
Dfr^6 G♯fr^6	D G♯

Figure 44: Scale membership for all the dominant, minor, major, fully diminished, and half-diminished seventh chords, and French sixths.

Amnestic Hexagon

Bennett Samuel Lin

Figure 45: Music score for *Amnestic Hexagon*.

Figure 45: Music score for *Amnestic Hexagon*. (cont'd.)

Figure 45: Music score for *Amnestic Hexagon*. (cont'd.)

Figure 45: Music score for *Amnestic Hexagon*. (cont'd.)

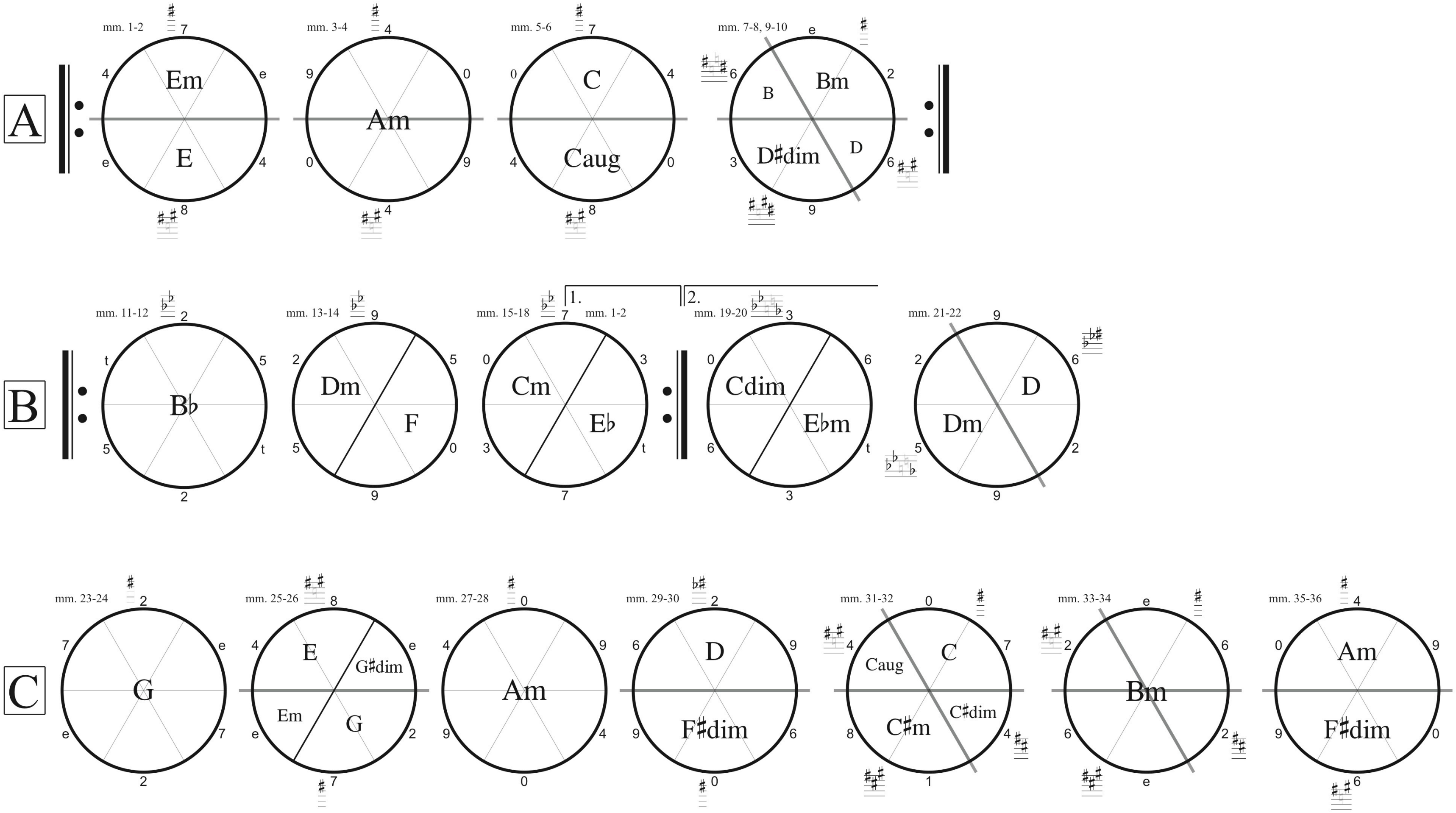


Figure 46: Compositional design for *Amnestic Hexagon*.

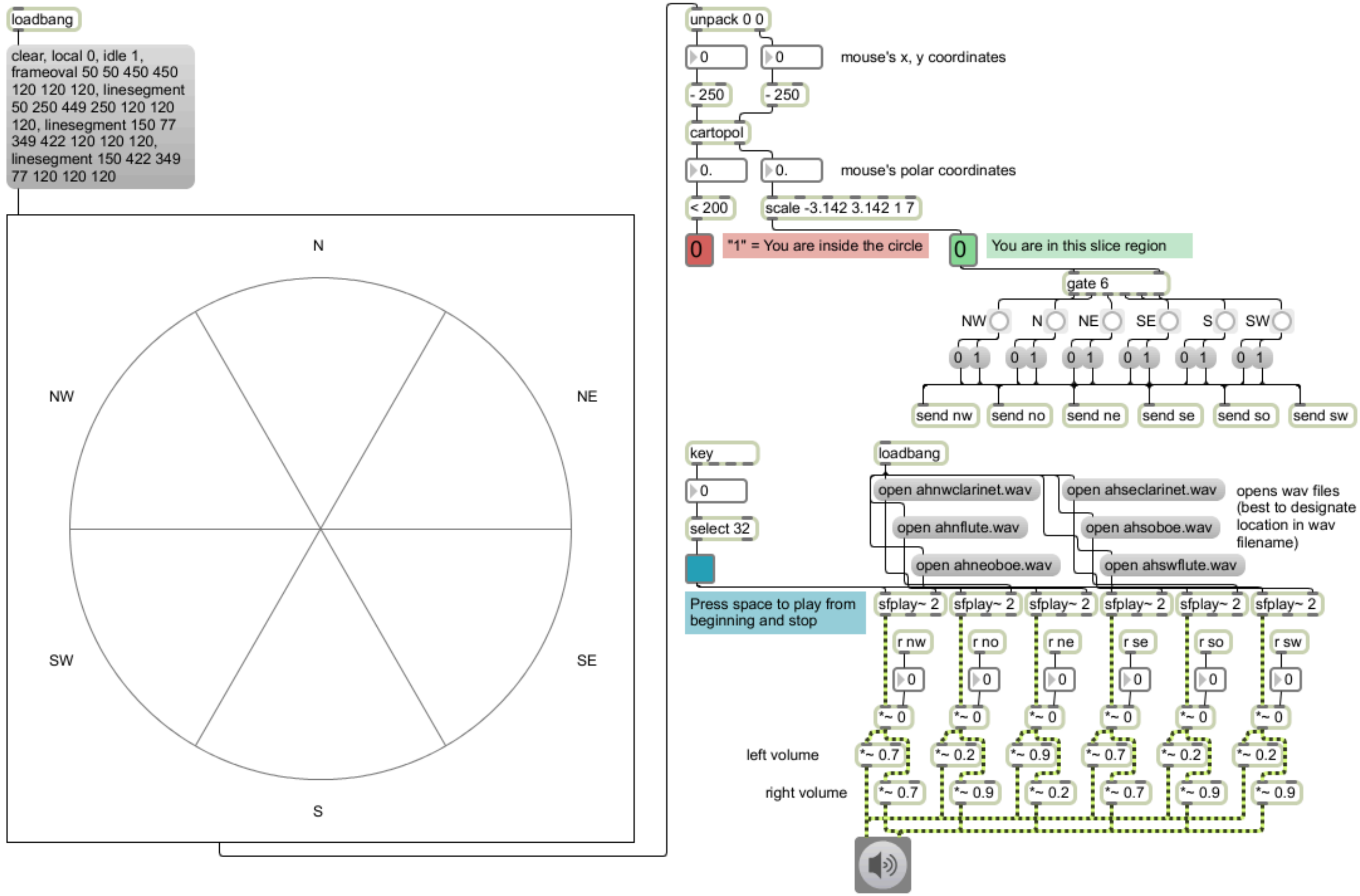


Figure 47: Max/MSP patch for cube mapped onto virtual circle.

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