3d Bosonization:
Flavor-Violated, Bordered, Quivered, and Stacked

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Abstract

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In this work we discuss several aspects of the 3d bosonization dualities, which are a conjectured equivalence between Chern-Simons matter theories in $2+1$-dimensions. We propose a new phrase diagram for the most general 3d bosonization duality when the number of matter particles is large. We discuss how the most general form the dualities can be rendered consistent on a manifold with boundaries. Novel quiver dualities, both bosonic and fermionic, are constructed which serve as a generalization of the well-known particle-vortex duality and which also have application to domain walls in $3+1$-dimensional QCD. We argue the rank-one dualities can be connected to the rank-two dualities via an orbifolding procedure. Finally, we show how such $2+1$-dimensional dualities can be related to known $3+1$-dimensional electromagnetic duality.

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DEDICATION

To Ann Nelson.

Ann was the first professor I had a chance to work with when I started my Ph.D. After deciding to switch into particle physics from electrical engineering, I was very grateful that such a brilliant and accomplished physicist was willing to guide someone so new to the field. Although my interests later took me to other lines of research, Ann and I continued to interact in the later years of my degree as members of the particle theory group. More than anything, Ann's ongoing commitment to lifting those underrepresented in physics, and in the broader community, will stick with me for the rest of my career. Her actions continually showed me how we can do better and make physics a more welcoming and diverse community.

Chapter 1

INTRODUCTION TO 3D BOSONIZATION

Behind "symmetry", "duality" is probably a high-energy theorist's favorite word. Although exact definitions of what is meant by "duality" differ, throughout this work we will take it to broadly mean *an equivalence between two theories.* As we will see, the exact nature of this equivalence can vary depending on the duality. Many of these dualities are conjectured, but they are nevertheless useful. The subject of this thesis will be investigating various aspects of a recently discovered set of dualities collectively known as "3d bosonization".

We will begin by looking at four well-known systems that exhibit dualities. Looking at all four of these systems as examples before diving into 3d bosonization may seem like overkill, but before the end of this work we will show each of them are in some way related to the 3d bosonization dualities.

1.1 *Four Systems with Dualities*

Electromagnetic Duality

Perhaps the most well-known duality is the electromagnetic duality. This duality comes from the observation that, in $3+1$-dimensions, the source-less Maxwell equations,

$$\nabla \cdot \vec{E} = 0\,, \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial \vec{B}} = 0\,, \tag{1.1a}$$

$$\nabla \cdot \vec{B} = 0\,, \qquad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial \vec{B}} = 0\,, \tag{1.1b}$$

look identical under the interchange of the electric and magnetic fields. More precisely, they are the same under the transformation

$$\vec{E} \to \vec{B} \qquad \text{and} \qquad \vec{B} \to -\vec{E}\,. \tag{1.2}$$

One can do some sort of calculation in one system, use the dictionary of (1.2), to find the equivalent calculations in the dual system. Although this transformation doesn't serve much calculational convenience, it does exhibit an underlying symmetry of Maxwell's equations that is unique to $3+1$ dimensions.

Transverse Field Ising Model

Another system that exhibits dualities is the Ising model in $1+1$ dimensions. Consider an N site lattice labeled by $j = 1, \ldots, N$ with spin degrees of freedom on each site. The on-site operators are the Pauli matrices, and we adopt the notation

$$X_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad Z_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.3}$$

We will use $|\uparrow_j\rangle$ and $|\downarrow_j\rangle$ to denote Z_j eigenstates and $|\rightarrow_j\rangle$ and $|\leftarrow_j\rangle$ for X_j eigenstates. For example, $X_j |\rightarrow_j\rangle = |\rightarrow_j\rangle$. The states are related by the usual Pauli matrix eigenstate relations, e.g. $|\rightarrow_j\rangle = \frac{1}{\sqrt{2}} (|\uparrow_j\rangle + |\downarrow_j\rangle)$. We will take the Hamiltonian of the system to be

$$H_{\text{TFIM}} = -\left(g \sum_{j=1}^{N} X_j + \sum_{j=1}^{N-1} Z_j Z_{j+1} \right) , \tag{1.4}$$

with $g > 0$. This system is invariant under the transformation $Z_j \to -Z_j$ and thus has a $\mathbb{Z}_2$ symmetry. As an example, the $\mathbb{Z}_2$ symmetry acts on the states as $|\uparrow_j\rangle \to |\downarrow_j\rangle$ and $|\rightarrow_j\rangle \to |\rightarrow_j\rangle$.

The behavior of the system in the $g = 0$ and $g \to \infty$ limits will be of interest to us. For $g \to \infty$ we can ignore the second term of H_{TFIM}, in which case the ground state is uniquely given by $|\rightarrow_1\rightarrow_2 \cdots \rightarrow_N\rangle$, which is invariant under the $\mathbb{Z}_2$ symmetry. The excitations above the ground state in this case are *single* spin flips and are created by acting Z_j on the ground state, yielding $|\rightarrow_1 \cdots \rightarrow_{j-1}\leftarrow_j\rightarrow_{j+1} \cdots \rightarrow_N\rangle$.

In the limit where $g = 0$, we can ignore the first term of H_{TFIM}, in which case there are two possible ground states: $|\uparrow_1\uparrow_2 \cdots \uparrow_N\rangle$ and $|\downarrow_1\downarrow_2 \cdots \downarrow_N\rangle$. Since neither of these ground states are invariant under the $\mathbb{Z}_2$ symmetry, the symmetry is said to be spontaneously broken. In this regime, excitations above the ground state correspond to flipping *all* spins to one side of a given site, e.g. $|\uparrow_1 \cdots \uparrow_k \downarrow_{k+1} \cdots \downarrow_N\rangle$ (it is straightforward to check flipping only a single spin yields a state of higher energy). Such a state can be created by acting on either of the ground states with the operator

$$K_k \equiv \sum_{j>k} X_j \,. \tag{1.5}$$

The operator is by no means local because it requires many spins to flip. Exicitations in this phase are solitonic and are sometimes called “kinks” (or “domain walls”). These excitations can be thought of as living *between* the two oppositely aligned spins.

Since the particles and solitons roughly serve the same purpose in their respective phases, one might be motivated to look for some relation between the two. Indeed, one can define a transformation that interchanges the behavior of the particles and solitons. Begin by defining a *dual* lattice with sites labeled by J that take on half-integer values $J = 1/2, \ldots, N + 1/2$. We can then define operators

$$X'_J \equiv Z_{J-1/2} Z_{J+1/2} \,, \qquad Z'_J = \prod_{j>J} X_j \,. \tag{1.6}$$

It is straightforward to show the Hamiltonian of (1.4) can then be rewritten

$$H'_{\text{TFIM}} = -\left(g \sum_J Z'_J + \sum_J X'_J X'_{J+1} \right) . \tag{1.7}$$

This is known as the “Kramer-Wannier” transformation [89]. Note this Hamiltonian has precisely the same form as (1.4), if we take $X_j \to Z'_J$, $Z_j \to X'_J$, $g \to 1/g$, and the rescale the whole Hamiltonian by g.[1] Since both (1.4) and (1.7) are equivalent descriptions of the same system, we say there is a duality between the two descriptions. Note under this

[1]This is overlooking some important details, since if one takes the theories described by (1.4) and (1.7) unmodified, one would come to inconsistent conclusions about the degeneracy of the ground states depending on which Hamiltonian was used. In order to reach consistent conclusions, one must be careful in treating the edges of the lattice under the transformation (1.6), in which case one arrives at boundary conditions needed to enforce equivalence. Such details are not important for our purposes, but see Ref. [93] for a detailed treatment.

transformation the particle and solitonic operators have been interchanged, which will occur for many other dualities in this work.

There exists another transformation of (1.4) that makes the system look fairly different. Define the operators

$$\chi_j \equiv Z_j \prod_{k>j} X_k \,, \qquad \tilde{\chi}_j = -iZ_j \prod_{k>j} X_k \,. \tag{1.8}$$

which obey the *anti*commutation relations

$$\{\chi_j, \chi_k\} = 2\delta_{jk} \,, \qquad \{\tilde{\chi}_j, \tilde{\chi}_k\} = 2\delta_{jk} \,, \qquad \{\chi_j, \tilde{\chi}_k\} = 0 \,. \tag{1.9}$$

As a result of these relation, we call them fermionic degrees of freedom. We can rewrite (1.4) as

$$H''_{\mathrm{TFIM}} = -\sum_j \left(i\tilde{\chi}_{j+1}\chi_j + ig\chi_j\tilde{\chi}_j\right) \,. \tag{1.10}$$

This is known as the "Jordan-Wigner" transformation [72]. Again, this is a dual description to the original system governed by H_{TFIM}. The duality changes the statistics of particles by attaching solitons to particles. More specifically, comparing our new χ_j and $\tilde{\chi}_j$ operators to (1.5), they can be thought of as attaching a particle degree of freedom to a soliton, which again will be a feature we see in other dualities.

Although it may not be obvious from this brief introduction, this transformation can be generalized to continuous $1+1$ dimensional systems. There, it is a very useful transformation because Hamiltonians which are difficult to solve in terms of the original spin degrees of freedom sometimes become easier to solve when expressed in terms of fermions or vice versa.[2]

Level-Rank Duality

The next duality we will consider is known as "level-rank duality", which is an equivalence between Chern-Simons theories in $2+1$-dimensional Chern-Simons theories (see Appendix

[2]Many subtleties of this system have only recently been understood, see Ref. [84].

A for a brief review of Chern-Simons theories) [101, 102]. Specifically, the duality states the Chern-Simons theories[3]

$$SU(N)_{-k} \qquad \text{and} \qquad U(k)_N \tag{1.11}$$

are equivalent.[4] In (1.11), the level of one Chern-Simons term is exchanged with the rank of the other, hence the name “level-rank”.

This duality can be proven explicitly using two-dimensional conformal field theory (CFT) techniques. There is an equivalence between certain CFTs based on Wess-Zumino-Witten models, which can mapped into level-rank duality using an equivalence between such CFTs and Chern-Simons theories [97].

AdS/CFT

The anti-de Sitter/conformal field theory correspondence (AdS/CFT) is the duality that has been subject to the most research within theoretical particle physics for the last twenty years. The strongest form of the duality conjectures an equivalence between $\mathcal{N} = 4$ Supersymmetric $SU(N)$ Yang-Mills theory in $3 + 1$-dimensions to type IIB superstring theory on $\text{AdS}_5 \times S^5$ [92]. Weaker forms of the duality can be reached by working in the large N and $\lambda = g^2_{\text{YM}} N$ limits, in which case we have a duality

$$\text{strongly coupled } \mathcal{N} = 4 \text{ SYM} \qquad \Leftrightarrow \qquad \text{type IIB supergravity on } \text{AdS}_5 \times S^5 \,. \tag{1.12}$$

Further limits of this duality can be taken to probe other interesting theories. For instance, compactifying one direction, choosing particular boundary conditions, and introducing a spatially varying θ-angle transforms this duality into level-rank duality, (1.11).

Unlike the three dualities above, this duality has not been proven formally. Nevertheless, there is a plethora of evidence for AdS/CFT, which we make no attempt of reviewing here. Later we will see that AdS/CFT's relation to level-rank duality will allow us to use it to make connections to 3d bosonization.

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[3]In this work we will denote a Chern-Simons term for the gauge group G and level k as G_k.

[4]This is a schematic form of the duality, the exact form is given below, see (1.24).

1.2 3d Bosonization

From the evidence above, we see dualities come in many different flavors. The primary subject of this thesis is the discussion of various aspects of a recently proposed duality that we will refer to as "3d bosonization". We begin by discussing qualitative features of 3d bonsization before discussing the quantiative details of the dualities at the level of the Lagrangian.

As its name implies, 3d bosonization is a family of dualitities where each side of the duality is a theory that lives in 2+1 dimensions that has the property of expressing fermionic degrees of freedom in terms of bosonic degrees of freedom and vice versa.

More specifically, this is a duality between Chern Simons matter theories. Aharony [2] was the first to write down the dualities in their full form, which are schematically given by

$$SU(N)_{-k+N_f/2} \text{ with } N_f\ \psi \qquad \leftrightarrow \qquad U(k)_N \text{ with } N_f\ \Phi\,, \tag{1.13a}$$

$$SU(N)_{-k} \text{ with } N_s\ \phi \qquad \leftrightarrow \qquad U(k)_{N-N_s/2} \text{ with } N_s\ \Psi\,. \tag{1.13b}$$

In the equation above and throughout this work we use the notation "$\leftrightarrow$" to mean the two theories share an infrared (IR) fixed point. On one end of each of these dualities we have scalars which are represented by ϕ or Φ. These scalars are self-interacting, the details of which will be fleshed out in what follows. On the other end of the duality we have free fermions, denoted by ψ or Ψ. The fact that one side of the duality contains bosons and the other has fermions is the origin of the name "bosonization".[5] Comparing to (1.11), 3d bosonization can also be thought of as the level-rank duality with the addition of fundamental matter on each side. The duality is conjectured to be valid for all (integer valued) N, k, and N_f so long as $N_f \leq k$, which we call a "flavor bound". Recent proposals extend the validity of this duality beyond $N_f \leq k$, the "flavor-violated" regime, and will be discussed in more detail in Chapter 2.

Bosonizaiton generalized to 2+1-dimensions is certainly not a new idea [106, 116, 48, 32],

[5]These could have equally well been called "3d fermionization dualities".

but it has only recently been established in the form of (1.13). However, since both sides of (1.13) are generally strongly coupled in the IR, it is difficult to formally prove the two sides are indeed equivalent *and thus this duality remains conjectured*. Nevertheless, there continues to be mounting evidence for the validity of 3d bosonization, which we will now review.

Evidence for 3d Bosonization

The recent resurgence of interest for 3d bosonizaiton arguably arose from evidence that certain distinct Chern-Simons matter theories in the large rank ($N \gg 1$) limit are individually dual to the same high-spin gravity theories in AdS_4 [85, 115, 123]. More recently, this motivated studies where the level (k) and rank of the Chern-Simons matter theories are taken to be large (but k/N is held fixed). In this limit certain calculations are under perturbative control [54, 6, 5, 65, 63, 96, 57] and one can confirm that many observables on both sides of the duality, such as the operator spectrum, free energy, and correlation functions, match to leading order.

In addition to matching observables at large N and k, it is also possible to perform a matching of the lowest dimension operators on either end of the duality [2]. Specifically, one can consider the lowest dimension operators that scale with N (i.e. excluding such states as mesons), and show there is a matching in classical scaling dimension, spin, and degeneracy.

As an example consider (1.13a) with $N_f = 1$. On the SU side, the obvious gauge-invariant operators that scale with N are baryons, i.e. a product of N fermions with their color indices antisymmetrized. This operator has classical dimension $\Delta \sim N$. Since each fermion has two different spin states, there are 2^N different types of such baryons. The fermions transform under $U(1)$ global symmetry which rotates their phase, and thus the baryons are also charged under said symmetry.

Meanwhile on the U side, the scalar equivalent of baryons are no longer gauge-invariant objects. What takes their place are monopole operators attached to fundamental scalars.[6] In ordinary $U(k)$ Yang-Mills monopole operators are gauge-invariant on their own, but as a result of the Chern-Simons term, flux-attachment attachment occurs and makes a lone monopole transform non-trivially. These flux-attached monopoles can be gauge-invariant by forming a bound state with the fundamental matter, i.e. the scalars in this case. The flux attachment is proportional to the level, so for a $U(k)_N$ Chern-Simons theory, the lowest dimension operator that scales with N has dimension $\Delta = \Delta_{\text{monopole}} + N\Delta_{\text{mattter}}$. Specific to (1.13a), the scalar matter in the presence of nonzero monopole flux has $\Delta_{\text{scalar}} = 1$ and so, neglecting the $\mathcal{O}(1)$ Δ_{monopole} contribution, we again have $\Delta \sim N$. Furthermore, in the presence of the monopole background, the scalar matter carries spin 1/2 [135], and thus we have 2^N possible types of such operator. Since the classical dimension and counting of these operators match on either side of the duality, this motivates the identification

$$N\text{-body baryon} \qquad \leftrightarrow \qquad \text{monopole} + N\text{-matter attachment}\,. \tag{1.14}$$

Thus we have a mapping between solitonic monopole operators and particles, again qualitatively similar to what we saw in the transverse field Ising model dualities.

A similar matching holds for (1.13b). The "baryon" operator we now form on the SU side of the duality can now be made from taking the antisymmetrized derivatives acting on scalars, which has scaling $\Delta \sim \frac{2}{3}N^{3/2}$. On the U side, we again must attach fundamental matter to the monopole, but now we have fermions in the presence of flux, which yields $\Delta \sim \frac{2}{3}N^{3/2}$. Thus we once more have an identification similar to that of (1.14).

Another piece of evidence for the dualities that we will use throughout this work is the behavior of each side of the duality under "mass deformations". Although calculations are difficult to perform at the fixed point where the duality is conjectured, one can deform each side of the duality away from said fixed point into a calculable regimes. In these calculable regimes, it is possible to show the two sides of the duality are equivalent, generally through

[6] Since we are working in $2+1$ dimensions, when we say "monopoles" we are referring to operators that are instanton-like, i.e. they are a quantized lump of magnetic flux in spacetime. See Refs. [28, 107] for more details on such operators.

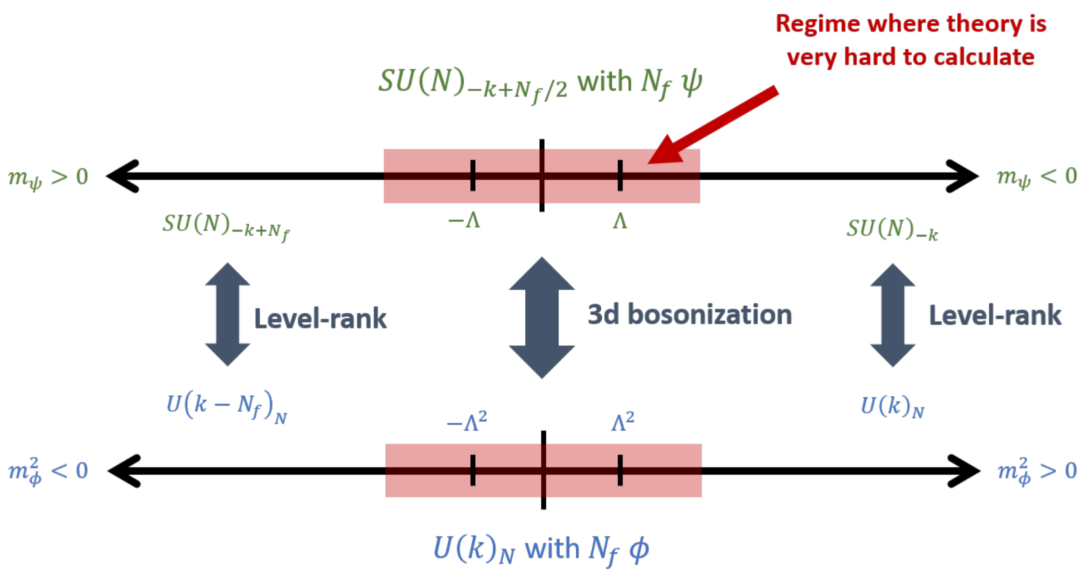

Figure 1.1: Mass deformed phases of Aharony's duality, (1.13a).

the use of (1.11). Specifically, the relevant operators we will add on either side take for the form of masses for the scalar or fermion degrees of freedom, $\mathcal{L}_{\text{relevant}} = m_\phi^2|\phi|^2$ or $m_\psi\bar{\psi}\psi$. If these mass deformations are larger than the strong coupling scale, Λ, we can integrate out the matter leaving only a topological field theory (TFT), i.e. a pure Chern-Simons theory. Since these masses can take on positive or negative values, the idea is that if we can show the dualities hold for both $m_\psi \to \pm\infty$ and $m_\phi^2 \to \pm\infty$, the dualities may continue to hold for small mass deformations and even at the conformal fixed point, see Fig. 1.1.[7]

Let us demonstrate this explicitly for (1.13a), see Fig. 1.1 for a summary. On the SU side of the duality, taking $m_\psi \to \pm\infty$ and integrating out the now massive fermion shifts the Chern-Simons term level by $\text{sgn}(m_\psi)N_f/2$. Thus we get $SU(N)_{-k}$ for large negative mass deformations and $SU(N)_{-k+N_f}$ for large positive mass deformations. Meanwhile, on the U side taking $m_\phi^2 \to \infty$ simply gaps the scalars out, leaving only $U(k)_N$. For large

[7]Specifically, starting precisely at the IR fixed point and with the mass deformations correctly identified, the claim is that the duality is (conjectured to be) exact on all scales, both above and below the scale set by the mass deformation.

negative scalar mass the scalars acquire a vacuum expectation value. If $N_f \leq k$, the scalar Higgs the gauge group, giving a Chern-Simons term of $U(k-N_f)_N$. Reassuringly, the two Chern-Simons theories we get on the U side are level-rank dual to those we get on the SU side. This leads us to identify the mass deformations across the duality as

$$m_\phi^2 \qquad \leftrightarrow \qquad -m_\psi \,. \tag{1.15}$$

Note for $N_f > k$ we instead get a non-linear sigma model on the scalar side, and this does not appear to be matched for the fermions. For this reason, the aforementioned flavor bounds were established, requiring $N_f \leq k$ for the 3d bosonization duality to be valid. We will have more to say about the duality outside of these bounds later.

Although much of the initial evidence for 3d bosonization came from large N and k studies, quite surprisingly additional evidence for these dualities arises in the exact opposite regime, where $N = k = N_f = 1$. In this case, (1.13) schematically reduce to

$$\text{Wilson-Fisher scalar} \qquad \leftrightarrow \qquad U(1)_{-1/2} + \text{fermion}, \tag{1.16a}$$

$$U(1)_1 + \text{scalar} \qquad \leftrightarrow \qquad \text{free fermion}. \tag{1.16b}$$

In this limit the dualities are known as the "Abelian dualities".

The operator mapping of 1.14 continues to hold in this regime. For example, under identical particle exchange of our "boson + flux", the phase due to Chern-Simons flux attachment when $k = \pm 1$ is the same one would pick up from interchanging two fermions without flux. If we call this exchange operation $\mathcal{O}$, we then have the schematic relation

$$\mathcal{O}\,|\text{boson} + \text{flux}\rangle = \mathcal{O}\,|\text{fermion}\rangle \tag{1.17a}$$

$$\mathcal{O}\,|\text{fermion} + \text{flux}\rangle = \mathcal{O}\,|\text{boson}\rangle\,, \tag{1.17b}$$

(see below (A.5) for more details). Thus, the statistics of bosons and fermions can be interchanged in the presence of a Chern-Simons term, and this qualitatively supports the dualities of (1.16).

Using methods familiar to supersymmetric symmetries [78], one can use the Abelian dualities to derive new dualities, also in $2+1$-dimensions. In fact, they can been used to derive an entire web of related dualities [83, 112], within which is the well-known bosonic particle-vortex duality [105, 40] and its recently discovered fermionic equivalent [119]. Both the Abelian bosonization and particle-vortex duality map the matter content of one theory to monopole operators[8] in another, again echoing the feature we saw at large N and in the transverse field Ising model. The methodology used in deriving this web of Abelian dualities has been extended to Abelian and non-Abelian linear quivers [82, 69] to generate even more novel dualities, although these are often limited in scope due to flavor bounds.

Further support for these dualities include deformations from supersymmetric cases [66, 58, 73, 74], derivation from an array of $1+1$ dimensional wires [98], a matching of global symmetries and 't Hooft anomalies [26], consistency checks of the dualities on manifolds with boundaries [9, 15], and support from Euclidean lattice constructions [30, 31, 71].

3d bosonization dualities have found application toward the half-filled fractional quantum Hall effect as well as surface states of topological insulators [119, 125, 95, 112]. Apart from these curious dualities, Chern-Simons matter theories at IR fixed points are models of critical phenomena in low-dimensional condensed matter systems, capture phenomena such as the fractional quantum Hall effect, and play a role in topological quantum computing [103]. Further, there have been lessons taken from these dualities and imported back into theories closer to ones recognizable to high energy physicists [86, 52].

A quick note on vocabulary moving forward: soon we will see there is a far more general form of the duality, and we will reserve the name "3d bosonization" to refer to such dualities in general. Since Aharony was the first to propose the most general form of the single-species dualities, sometimes we will refer to (1.13) as "Aharony's dualities".

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[8]This may seem contradictory with the name "particle-vortex", but one can think of monopole operators as vortex-creation operators. This is easiest to see in the Higgs phase, where all monopole flux lines are combined into the time-like direction. See Ref. [28] for an explanation.

1.2.1 Details

Now let us discuss 3d bosonization in a more quantitative fashion. Explicitly, while starting from different UV theories, (1.13) is the conjecture that there exists a duality between the Euclidean-space partition functions[9]

$$\int \mathcal{D}(\cdots)\, e^{-\int d^3x\, \mathcal{L}_{SU}} \qquad \leftrightarrow \qquad \int \mathcal{D}(\cdots)\, e^{-\int d^3x\, \mathcal{L}_{U}}, \tag{1.18}$$

again with "$\leftrightarrow$" representing a duality holding in the IR limit.[10]

Each side of the duality has both gauge symmetries and global symmetries. The global symmetries on either side of the duality match, and for (1.13) they are $SU(N_f)\times U(1)$ and $SU(N_s)\times U(1)$, respectively.[11] Keeping careful track of the matching of global symmetries and their associated 't Hooft anomalies can be achieved by coupling background gauge fields to their associated conserved currents and making sure their associated Chern-Simons levels match across the duality. To this end, in what follows we will make the background fields explicit.

For (1.13b), the Lagrangians are,

$$\mathcal{L}_{SU} = \left|D_{b'+B+\tilde{A}_1}\phi\right|^2 + \alpha\left|\phi^4\right| - i\left[-\frac{k}{4\pi}\mathrm{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right) - \frac{Nk}{4\pi}\tilde{A}_1 d\tilde{A}_1\right], \tag{1.19a}$$

$$\mathcal{L}_{U} = i\bar{\Psi}\not{D}_{c+B}\Psi - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right) - \frac{N}{2\pi}\mathrm{Tr}_k(c)d\tilde{A}_1 + 2Nk\mathrm{CS}_{\mathrm{grav}}\right], \tag{1.19b}$$

where ϕ represents the self-interacting scalars, Ψ represents the fermions, and the various Chern-Simons and BF terms are collected in the $[\cdots]$. A summary of the various gauge fields is given in Table 1.1. In (1.19) and the following, we use uppercase letters to denote background gauge fields and lowercase for a dynamical gauge fields. In what will be a

[9]In Euclidean space, what was the a level-k Minkowski space Chern-Simons terms, i.e. (A.1), picks up an additionally coefficient of $-i$ in the Lagrangian. As such, for the body of this text we will write a level-k Chern-Simons terms as $\mathcal{L} \supset -i\left(\frac{k}{4\pi}AdA\right)$.

[10]More precisely, there exists an equivalence when the theories defining the partition functions are tuned to describe exact CFTs. In this limit, it is conjectures the two theories flow to the same IR fixed point.

[11]More precisely, the global $SU(N)$ symmetry is $\left(SU(N)\times U(1)\right)/\mathbb{Z}_N$. We ignore mod factors for much of this work when discussing global symmetries for brevity. Their omission will not affect our analysis except where such factors are made explicit. See Ref. [26] for a detailed account of global symmetries.

	Dynamical Gauge Fields		Background Fields	
Symmetry	$SU(N)$	$U(k)$	$SU(N_f)$	$U(1)_{m,b}$
Field	b'_μ	c_μ	B_μ	$\tilde{A}_{1\mu}$

Table 1.1: Various gauge fields used Aharony's duality, (1.23b).

necessary distinction for later application, we will denote dynamical (background) spin$_c$ valued connections with a (A), while ordinary connections will be denoted with b (B), c (C), and so on. $U(1)$ gauge fields will generally be denoted with a tilde, e.g. $\tilde{A}_1$.

The absence of a Maxwell term for, say b', i.e. $\frac{1}{4e^2}(db')^2$, can easily be seen because the IR limit requires $e^2 \to \infty$. Moreover, the action for the Wilson-Fisher scalar is obtained by tuning the scalar mass $m_\varphi^2 \to 0$ and quartic coupling $\alpha \to \infty$.[12] The only effect of the quartic coupling in what follows will be to stabilize the potential in the presence of negative scalar mass deformations. We will often drop the $\alpha|\phi|^4$ term when we write Lagrangians in what follows.

Our notation for covariant derivatives is

$$\left(D_{b'+B+\tilde{A}_1}\right)_\mu \phi = \left[\partial_\mu - i\left(b'_\mu \mathbb{1}_{N_s} + B_\mu \mathbb{1}_N + \tilde{A}_{1\mu}\mathbb{1}_{NN_s}\right)\right]\phi, \tag{1.20a}$$

$$\left(D_{c+B}\right)_\mu \Psi = \left[\partial_\mu - i\left(c_\mu \mathbb{1}_{N_s} + B_\mu \mathbb{1}_k\right)\right]\Psi, \tag{1.20b}$$

with $\mathbb{1}_N$ the $N \times N$ identity matrix. As expected, both types of matter couple to the respective dynamical gauge fields on their respective end of the duality. Additionally, both couple to the same background gauge field associated with the $SU(N_s)$ flavor symmetry, B_μ. On the SU side of the duality, the matter also couples directly to the $U(1)$ background gauge field, $\tilde{A}_1$.

We denote the gravitational Chern-Simons term by $\mathrm{CS}_{\mathrm{grav}}$, and it defined by

$$\int_{\mathcal{M}=\partial X} \mathrm{CS}_{\mathrm{grav}} \equiv \frac{1}{192\pi}\int_X \mathrm{Tr} R \wedge R, \tag{1.21}$$

[12]These limits must be taken in a carefully correlated fashion, as these limits are very non-uniform.

where X is a $d = 4$ manifold, $\mathcal{M}$ is its $d = 3$ boundary, and R is the Riemann tensor of X.[13]

Fermions in 2 + 1-dimenions require regularization and we will choose to use Pauli-Villars regulation. The heavy Pauli-Villars regulators give rise to η-invariant phases which we will denote by half-level Chern-Simons terms [133, 134]. We will use the convention for η-invariants where a *positive* mass deformation for the fermion will not change the level of the Chern-Simons term. This is the notation established in [112]. Being very explicit, for N_f, N-component fermions, we have absorbed into the kinetic term what is often written as a half-integer Chern-Simons term that results from integrating out heavy regulator fermions, i.e.

$$i\bar{\psi}\not{D}_A\psi - \frac{1}{2}\times i\left[-\frac{N_f}{4\pi}\mathrm{Tr}_N\left(AdA - i\frac{2}{3}A^3\right) - 2NN_f\mathrm{CS}_{\mathrm{grav}}\right] \qquad \Rightarrow \qquad i\bar{\psi}\not{D}_A\psi\,. \tag{1.22}$$

Again, note that this convention is chosen such that when integrating out positive mass dynamical fermions the hidden η-invariant term is canceled, which leaves the Chern-Simons levels unchanged. However, when a negative mass fermion is integrated out, the overall effect is to shift the associated Chern-Simons levels by N_f. This will be the convention we use for fermions throughout this work.[14] We will continue to denote fermion half-levels when specifying the level of a Chern-Simons theory outside of a Lagrangian.

Returning to the other duality in (1.13), the Lagrangians of (1.13a) are,

$$\begin{aligned}\mathcal{L}_{SU} &= i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1}\psi - i\left[\frac{N_f-k}{4\pi}\mathrm{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right) + \frac{N}{4\pi}\mathrm{Tr}_{N_f}\left(CdC - i\frac{2}{3}C^3\right)\right]\\ &\quad - i\left[\frac{N\left(N_f-k\right)}{4\pi}\tilde{A}_1 d\tilde{A}_1 + 2N\left(N_f-k\right)\mathrm{CS}_{\mathrm{grav}}\right], \end{aligned}\tag{1.23a}$$

$$\mathcal{L}_U = \left|D_{c+C}\phi\right|^2 - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right) - \frac{N}{2\pi}\mathrm{Tr}_k\left(c\right)d\tilde{A}_1\right]. \tag{1.23b}$$

[13] The details of the gravitational Chern-Simons term will not be important for much of this work, so many times in what follows we will drop its dependence. When it is important, it will simply serve as additional consistency check between topological field theories. More details behind such a term can be found in Refs. [112] and [60].

[14] This will slightly complicate things when we time-reverse the duality, because this transformation should also flip the η-invariant term. However, we will keep the same convention whether or not we are talking about the original or time-reversed duality. The net effect of this will mean time-reversal comes with a shift in Chern-Simons terms as well.

The field identifications are identical to that of (1.23b), but now the $SU(N_f)$ global symmetry has the associated background field C_μ. Similar to (1.23b), we see the background $U(1)$ gauge field couples directly to the matter on the SU side of the duality.

It is possible to derive (1.19) from (1.23) and vice versa. To do so, one can promote the $U(1)$ global symmetry to be dynamical, which we denote by $\tilde{A}_1 \to \tilde{a}_1$. The U side of the duality now has two dynamical $U(1)$ gauge symmetries, one of which can be integrated out. After a sort calculation, one recovers (1.23) up to relabeling and background Chern-Simons terms (which can always be added to both sides).

As mentioned above, the core of these recently discovered non-supersymmetric $2+1$ dimensional dualities is the well-known level-rank duality,

$$SU(N)_{-k} \leftrightarrow U(k)_{N,N} \tag{1.24}$$

where we are using the notation

$$U(N)_{P,Q} = \frac{SU(N)_P \times U(1)_{NQ}}{\mathbb{Z}_N} \tag{1.25}$$

to denote Chern-Simons terms where the Abelian and non-Abelian parts have different levels [60]. Earlier we neglected the additional mod $\mathbb{Z}_N$ factor for simplicity. In what follows, we use the shorthand $U(k)_{N,N} \equiv U(k)_N$ when the levels are equal, up to rank dependence. This duality has been rigorously proven to hold at all scales, and is part of a larger class of level-rank dualities, which also includes [60]

$$U(N)_{k,k\pm N} \qquad \Leftrightarrow \qquad U(k)_{-N,-N\mp k}. \tag{1.26}$$

1.3 The Master Bosonization Duality

Recently, a generalization of Aharony's dualities has been discovered [25, 68] where each side of the duality has fermions *and* scalars. Since Aharony's dualities are a special case of this more general duality, we will refer to it as the "master duality" in this work. Schematically, this duality conjectures

$$SU(N)_{-k+\frac{N_f}{2}} \text{ with } N_s\ \phi \text{ and } N_f\ \psi \qquad \leftrightarrow \qquad U(k)_{N-\frac{N_s}{2}} \text{ with } N_f\ \Phi \text{ and } N_s\ \Psi. \tag{1.27}$$

Note that this duality reduces to Aharony's dualities, (1.13b) and (1.13a), when $N_f = 0$ and $N_s = 0$, respectively. Novel to the master duality is the fact the scalar and fermionic matter on each side of the duality interact with one another through a quartic term and each type of matter is subject to its own flavor bound.

1.3.1 Details

The Lagrangians of the master bosonization duality [25, 68], are given by[15]

$$\begin{aligned}\mathcal{L}_{SU} = &\left|D_{b'+B+\tilde{A}_1+\tilde{A}_2}\phi\right|^2 + i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1}\psi + \mathcal{L}_{\text{int}} - i\left[\frac{N_f-k}{4\pi}\text{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right)\right]\\ &- i\left[\frac{N}{4\pi}\text{Tr}_{N_f}\left(CdC - i\frac{2}{3}C^3\right) + \frac{N(N_f-k)}{4\pi}\tilde{A}_1 d\tilde{A}_1 + 2NN_f\text{CS}_{\text{grav}}\right],\end{aligned} \tag{1.28a}$$

$$\begin{aligned}\mathcal{L}_{U} = &\left|D_{c+C}\Phi\right|^2 + i\bar{\Psi}\not{D}_{c+B+\tilde{A}_2}\Psi + \mathcal{L}'_{\text{int}}\\ &- i\left[\frac{N}{4\pi}\text{Tr}_k\left(cdc - i\frac{2}{3}c^3\right) - \frac{N}{2\pi}\text{Tr}_k(c)d\tilde{A}_1 + 2Nk\text{CS}_{\text{grav}}\right]\end{aligned} \tag{1.28b}$$

with the mass identifications $m_\psi \leftrightarrow -m_\Phi^2$ and $m_\phi^2 \leftrightarrow m_\Psi$. Our definitions of fields are shown in Table 1.2. Again, we will use uppercase letters for background gauge fields, lowercase for dynamical gauge fields, and Abelian fields carry a tilde. This duality is subject to the flavor bound $(N_f, N_s) \leq (k, N)$, but excludes the case $(N_f, N_s) = (k, N)$. Our notation for covariant derivatives is

$$\left(D_{b'+B+\tilde{A}_1+\tilde{A}_2}\right)_\mu\phi = \left[\partial_\mu - i\left(b'_\mu\mathbb{1}_{N_s} + B_\mu\mathbb{1}_N + \tilde{A}_{1\mu}\mathbb{1}_{NN_s} + \tilde{A}_{2\mu}\mathbb{1}_{NN_s}\right)\right]\phi, \tag{1.29a}$$

$$\left(D_{b'+C+\tilde{A}_1}\right)_\mu\psi = \left[\partial_\mu - i\left(b'_\mu\mathbb{1}_{N_f} + C_\mu\mathbb{1}_N + \tilde{A}_{1\mu}\mathbb{1}_{NN_f}\right)\right]\psi, \tag{1.29b}$$

$$\left(D_{c+C}\right)_\mu\Phi = \left[\partial_\mu - i\left(c_\mu\mathbb{1}_{N_f} + C_\mu\mathbb{1}_k\right)\right]\Phi, \tag{1.29c}$$

$$\left(D_{c+B+\tilde{A}_2}\right)_\mu\Psi = \left[\partial_\mu - i\left(c_\mu\mathbb{1}_{N_s} + B_\mu\mathbb{1}_k + \tilde{A}_{2\mu}\mathbb{1}_{kN_s}\right)\right]\Psi. \tag{1.29d}$$

---

[15]These Lagrangians are based on those in [68] with $\tilde{A}_1 \to N\tilde{A}_1$ for simplicity. This amounts to saying quarks have charge 1 under $U(1)_m$ rather than baryons. Additionally, we follow the conventions outlined in Ref. [15]. We have dropped all gravitational Chern-Simons terms since they are not relevant for our purposes. Note there is a slight difference in convention in the sign of the BF term and the $\tilde{A}_2$ coupling on the U side of the duality. However since the difference always amounts to an even number of sign changes the TFTs still match under mass deformations. Additionally, the flux attachment procedure picks up two minus signs from this effect as well, meaning the quantum numbers of the baryon and monopole operators still match.

The interaction terms are

$$\mathcal{L}_{\text{int}} = \alpha \left(\phi^{\dagger a_c a_s}\phi_{a_c a_s}\right)^2 - C\left(\bar{\psi}^{a_c a_f}\phi_{a_c a_s}\right)\left(\phi^{\dagger b_c a_s}\psi_{b_c a_f}\right) \tag{1.30a}$$

$$\mathcal{L}'_{\text{int}} = \alpha \left(\Phi^{\dagger a_c a_f}\Phi_{a_c a_f}\right)^2 + C'\left(\bar{\Psi}^{a_c a_s}\Phi_{a_c a_f}\right)\left(\Phi^{\dagger b_c a_f}\Psi_{b_c a_s}\right) \tag{1.30b}$$

where a_c, b_c are indices associated with the color symmetries; a_f, b_f with the $SU(N_f)$ symmetry; and a_s, b_s with the $SU(N_s)$ symmetry. C and C' are coefficients of the associated interactions which we will later fix.

The interactions terms represent all possible relevant and marginal operators consistent with symmetries.[16] The quartic scalar terms will henceforth be implied anytime a scalar is present, but we will make note of the scalar/fermion interaction terms when they exist. In what follows we will often drop the explicit indices and denote the interaction terms by, e.g., $|\phi|^4$ and $\bar{\Psi}\Psi|\Phi|^2$.

The $|\phi|^4$ and $|\Phi|^4$ are the usual interactions that are present at the Wilson-Fischer fixed point. The effect of the scalar-fermion mixing term is to give a subset of the N_f (or N_s) fermions a mass when the scalars in the theory acquire a nonzero vacuum expectation value. The additional effect of this mass from the mixing term is necessary to get complete agreement between the two sides of the duality, and the relative sign between the mixing terms in $\mathcal{L}_{\text{int}}$ and $\mathcal{L}'_{\text{int}}$ is important to match the phases. The $\tilde{A}_2$ field is associated with a new symmetry that arises due to the presence of both scalars and fermions on each side of the duality.

As mentioned in the introduction, Aharony's dualities (1.13) can be found by taking the $N_s = 0$ and $N_f = 0$ limits of (1.28).

Because the duality exactly at the IR fixed point is between what are in general strongly coupled theories, the best evidence for validity of 3d bosonization dualities comes from gapped phases where the identification can be directly verified. The dictionary for mass

[16]Although the mixed scalar and fermion interactions are marginal in the IR at leading order in the large N limit, the sign of the subleading corrections are currently unknown. As in [68, 25] we will assume such operators are at least marginal since they are vital for the consistency of the master duality.

	Dynamical Gauge Fields		Background Fields			
Symmetry	$SU(N)$	$U(k)$	$SU(N_s)$	$SU(N_f)$	$U(1)_{m,b}$	$U(1)_{F,S}$
Field	b'_μ	c_μ	B_μ	C_μ	$\tilde{A}_{1\mu}$	$\tilde{A}_{2\mu}$

Table 1.2: Various gauge fields used in the master duality. Dynamical fields are denoted by lowercase letters while background fields by uppercase. $\tilde{A}_{1\mu}$ is associated with the monopole/baryon number $U(1)$ symmetry also present in Aharony's dualities. $\tilde{A}_{2\mu}$ is associated to the $U(1)$ symmetry which couples to the additional fermion/scalar matter in the master duality.

terms across the master duality is given by [68][17]

$$m_\psi \leftrightarrow -m_\Phi^2\,, \qquad m_\phi^2 \leftrightarrow m_\Psi\,. \tag{1.31}$$

Since we have two types of matter on each side of the duality, naively one would expect there to be four different mass-deformed phases. However, it has been shown that the interactions of (1.30a) and (1.30a) separate one of these four phases into two separate phases, giving us five phases total [25, 68]. Specifically, when the scalar acquires a vacuum expectation value the interactions give the so-called "singlet fermions", which are neutral under the unbroken gauge group, a mass shift.

[17]Note the opposite convention appears in [25] since it is the time-reversed version of the duality considered in [68].

The five massive phases are shown in Fig. 1.2. On the SU side we expect to find

$$
\begin{aligned}
\text{(I)} &: [SU(N)_{-k+N_f} \times U\left(NN_f\right)_{-1}] \times SU\left(N_f\right)_N \times SU\left(N_s\right)_0 \times J_I \,, && \text{(1.32a)} \\
\text{(II)} &: [SU(N)_{-k} \times U\left(0\right)_{-1}] \times SU\left(N_f\right)_0 \times SU\left(N_s\right)_0 \times J_{II} \,, && \text{(1.32b)} \\
\text{(III)} &: [SU\left(N - N_s\right)_{-k} \times U\left(0\right)_{-1}] \times SU\left(N_f\right)_0 \times SU\left(N_s\right)_{-k} \times J_{III} \,, && \text{(1.32c)} \\
\text{(IVa)} &: [SU\left(N - N_s\right)_{-k+N_f} \times U\left(N_f\left(N - N_s\right)\right)_{-1}] \\
&\quad \times SU\left(N_f\right)_{N-N_s} \times SU\left(N_s\right)_{-k} \times J_{IVa} \,, && \text{(1.32d)} \\
\text{(IVb)} &: [SU\left(N - N_s\right)_{-k+N_f} \times U\left(NN_f\right)_{-1}] \times SU\left(N_f\right)_N \times SU\left(N_s\right)_{-k+N_f} \times J_{IVb} \,. && \text{(1.32e)}
\end{aligned}
$$

Meanwhile, on the U side,

$$
\begin{aligned}
\text{(I)} &: [U\left(k - N_f\right)_N \times U\left(kN\right)_{-1}] \times SU\left(N_f\right)_N \times SU\left(N_s\right)_0 \times J_{I'} \,, && \text{(1.33a)} \\
\text{(II)} &: [U(k)_N \times U\left(kN\right)_{-1}] \times SU\left(N_f\right)_0 \times SU\left(N_s\right)_0 \times J_{II'} \,, && \text{(1.33b)} \\
\text{(III)} &: [U(k)_{N-N_s} \times U\left(k\left(N - N_s\right)\right)_{-1}] \times SU\left(N_f\right)_0 \times SU\left(N_s\right)_{-k} \times J_{III'} \,, && \text{(1.33c)} \\
\text{(IVa)} &: [U\left(k - N_f\right)_N \times U\left(k\left(N - N_s\right)\right)_{-1}] \times SU\left(N_f\right)_{N-N_s} \times SU\left(N_s\right)_{-k} \times J_{IVa'} \,, && \text{(1.33d)} \\
\text{(IVb)} &: [U\left(k - N_f\right)_{N-N_s} \times U\left(kN + (N_f - k)N_s\right)_{-1}] \\
&\quad \times SU\left(N_f\right)_N \times SU\left(N_s\right)_{-k+N_f} \times J_{IVb'} \,. && \text{(1.33e)}
\end{aligned}
$$

The bracketed are level-rank dual by (1.24), while the rest of the terms are global symmetries and should be the same on both sides. The Abelian factors unique to each phase are given by

$$
J_i \equiv J_i^{ab} \frac{1}{4\pi} \tilde{A}_a d\tilde{A}_b \tag{1.34}
$$

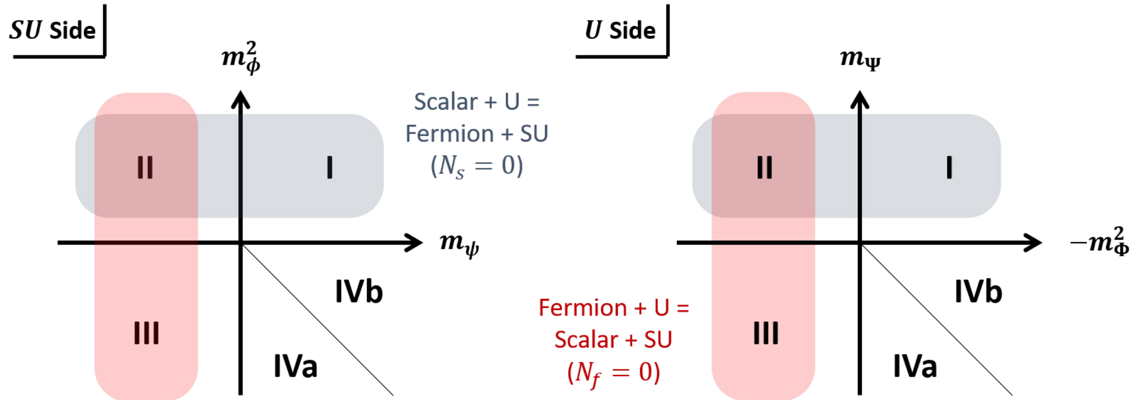

Figure 1.2: Various phases of the master duality and the non-Abelian reductions. The shaded red and blue correspond to the single-matter non-Abelian dualities.

with $a, b = 1, 2$, i indexing the phase $\{I, \ldots, IVb\}$, and

$$J_I^{ab} = \begin{pmatrix} -N\left(k - N_f\right) & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.35a}$$

$$J_{II}^{ab} = \begin{pmatrix} -Nk & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.35b}$$

$$J_{III}^{ab} = -\frac{Nk}{N - N_s} \begin{pmatrix} N & N_s \\ N_s & N_s \end{pmatrix}, \tag{1.35c}$$

$$J_{IVa}^{ab} = -\frac{N\left(k - N_f\right)}{N - N_s} \begin{pmatrix} N & N_s \\ N_s & N_s \end{pmatrix} - N_f N_s \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.35d}$$

$$J_{IVb}^{ab} = -\frac{N\left(k - N_f\right)}{N - N_s} \begin{pmatrix} N & N_s \\ N_s & N_s \end{pmatrix}. \tag{1.35e}$$

Since the massive phases are dual to one another, this is taken as good evidence that the master duality remains true at the conformal fixed point. A similar matching can be performed on the five critical lines that separate the five phases [68, 25].

1.3.2 USp and SO Generalizations

There exists analogs of the 3d bosonization dualities for the SO and USp groups. The master dualities for these groups are given by [25, 68][18]

$$SO(N)_{-k+\frac{N_f}{2}} \text{ with } N_s\ \phi \text{ and } N_f\ \psi \quad \leftrightarrow \quad SO(k)_{N-\frac{N_s}{2}} \text{ with } N_f\ \Phi \text{ and } N_s\ \Psi\,, \tag{1.36}$$

$$USp(2N)_{-k+\frac{N_f}{2}} \text{ with } N_s\ \phi \text{ and } N_f\ \psi \quad \leftrightarrow \quad USp(2k)_{N-\frac{N_s}{2}} \text{ with } N_f\ \Phi \text{ and } N_s\ \Psi\,. \tag{1.37}$$

Here, the matter is still in the fundamental representation of the respective gauge groups. The difference now is that the scalars are real, and the fermions are Majorana. There are five massive phases following the same pattern as those considered for the U/SU master duality. Note that the mass deformed phases match under the level-rank dualities generalized to the SO and USp cases [3],

$$SO(N)_{-k} \leftrightarrow SO(k)_N \times SO(kN)_{-1}\,, \tag{1.38}$$

$$USp(2N)_{-k} \leftrightarrow USp(2k)_N \times SO(4kN)_{-1}\,. \tag{1.39}$$

For the most part, throughout this work we will focus on the SU and U 3d bosonization dualities. Occasionally we will generalize to the SO and USp groups.

1.4 Outline

The outline of this thesis is as follows. Chapter 2 discusses extension of the 3d bosonization dualities into the flavor-violated regime. In Chapter 3 we discuss how the 3d bosonization dualities behave on a manifold with boundaries, which is important for self-consistency of the dualities as well as potentially connecting with experiment. We begin our discussion of quiver gauge theories in Chapter 4 with bosonic quivers. Such quivers have application to the description of domain walls in 3+1-dimensional QCD and also provide a natural generalization

[18] Here we follow the notation of [3], where $USp(2N) = Sp(N)$ and the levels of SO groups are normalized to give Chern-Simons terms $\frac{k}{8\pi}\mathrm{Tr}\left(AdA - i\frac{2}{3}A^3\right)$. Also note that Majorana fermions come with an regularizing phase of $\exp(-i\pi\eta/4)$ instead of $\exp(-i\pi\eta/2)$, with η the η-invariant.

particle-vortex duality. Chapter 5 continues our discussion of quivers by considering fermion-fermion dualities. Although this is a somewhat trivial extension of the bosonic analog, the main focus of this chapter is procedure for connecting the aforementioned 3d bosonization dualities to a class of similar dualities with higher representation matter. Lastly, Chapter 6 contains work elucidating the connection between known 3d and 4d dualities. We consider a supersymmetric analog of 3d bosonization, and then argue the results seem to generalize to the nonsupersymmetric version.

The appendices contain many details of the work in the body of this text. Appendix A contains a (brief) review of Chern-Simons theories. In Appendix B, we provide some additional details about the 3d bosonization dualities, including details about spin considerations. Appendix C contains additional details regarding the flavor-violated master duality. Finally, Appendices D and E contain supplementary details about the bosonic and fermionic quiver constructions, respectively.

Disclaimer: Much of the body of this work contains significant text overlap from various published works written, in part, by the author of this thesis. Chapter 2 contains text from Ref. [7], of which the author contributed a significant portion of the text with the exception of Section 3.2. Chapter 3 contains text from Refs. [9, 15], in which the author contributed significant portions of the text throughout both works. Chapter 4 contains text from Refs. [8, 7]. In the former, the author contributed significant portions of the text throughout with the exception of Section 4.2. Chapter 5 contains text from Ref. [13], where the author contributed significant portions of the text throughout. Chapter 6 contains text from [14], where the author primarily contributed to Section 5, with minor contributions throughout.

Chapter 2

FLAVOR-VIOLATED DUALITIES

In Chapter 1 we discussed the 3d bosonization dualities in the form they were originally proposed by Aharony [2], (1.13), which contain several integer parameters one could tune – N, k, and N_f (or N_s). These parameters were subject to what we will call the "flavor-bound" which requires $k \leq N_f$ for (1.13a) and $N \leq N_s$ for (1.13b). The reason for this bound can be seen by considering the behavior of these dualities under mass deformations. Consider (1.13a) in the "flavor-violated" regime where $N_f > k$ and we deform the fermions to positive mass and, per the mass mapping (1.15), expect this to be equivalent to tuning the scalars to negative mass. On the fermion side, nothing changes from the flavor-bounded regime and we find the TFT $SU(N)_{-k+N_f}$. However, on the scalar side the excess flavor symmetry of the scalars now not only completely breaks the gauge symmetry but also breaks the flavor symmetry, yielding a non-linear sigma model with target space

$$\mathcal{M}(N_f, k) \equiv \frac{U(N_f)}{U(k) \times U(N_f - k)}. \tag{2.1}$$

Clearly, such a model does not match the TFT $SU(N)_{-k+N_f}$, hence the reason for the flavor bounds.

More recently, a proposal has arisen for an extension of (1.13a) to the "flavor-violated" regime where $k < N_f < N_*(N, k)$ [86] where N_* is some undetermined upper bound.[1] It is conjectured that the fermionic side of (1.13a) also has a non-linear sigma model phase. For light fermion mass, $m_\psi < |m_*|$, the fermions form a condensate and spontaneously break their associated flavor symmetry.[2] This means both sides of the duality grow a quantum

[1] Although not considered in [86], an extension of (1.13b) to the regime $N < N_s < N'_*(N, k)$ should also be possible.

[2] In general, the phase transition points do not necessarily need to exist at the same mass magnitude m_*.

region described by a complex Grassmannian, see Fig. 2.1. We will review the details of this proposal in Sec. 2.1.

A natural next step is to combine the aforementioned flavor-extension of the 3d bosonization dualities to the master duality discussed in Sec. 1.3. As such, in this chapter we work to extend the master duality to the flavor-violated regime. Specifically, we will extend it into the regime where $k < N_f < N_*(N,k)$ and $N_s < N$. Naively one may think this extension is fairly trivial and the master duality grows a single new quantum region when fermion mass is light. However, with certain strong coupling assumptions, we find the phase diagrams grows several new quantum regions and exhibits behavior not yet seen in the context of 3d bosonization. We will also propose a form of the master duality which holds in the "double-saturated" case where $k = N_f$ and $N_s = N$.

We will begin by reviewing the construction of [86], which corresponds to the $N_s = 0$ limit of the mater duality. Like the ordinary master duality, the phase diagrams for the $N_s < N$ and $N_s = N$ cases look fairly different due to particular cancellations which occur in the latter case. We will begin with the phase diagram for the more general $N_s < N$ since it is slightly simpler to analyze. The $N = N_s$ case will be considered second and it will be applicable to the quivers we are constructing to model the domain wall behavior of QCD_4 in Chapter 4. Finally, we will discuss extending the master duality to the double saturated case in Sec. 2.3.

2.1 Review of QCD_3 Symmetry-Breaking

As was mentioned in the the introduction, the flavor bounds of (1.13a) are imposed because when $N_f > k$, the Higgs phase of the U side of the theory becomes a non-linear sigma model. This does not appear to be matched on the SU side, because the corresponding fermion mass deformation simply shifts the Chern-Simons level. Hence the bound $N_f \leq k$ was imposed to avoid the mismatch from the non-linear sigma model phase.

However, one can tune the bare mass parameter to make two transition points symmetrical, and we will assume that has been done throughout this work.

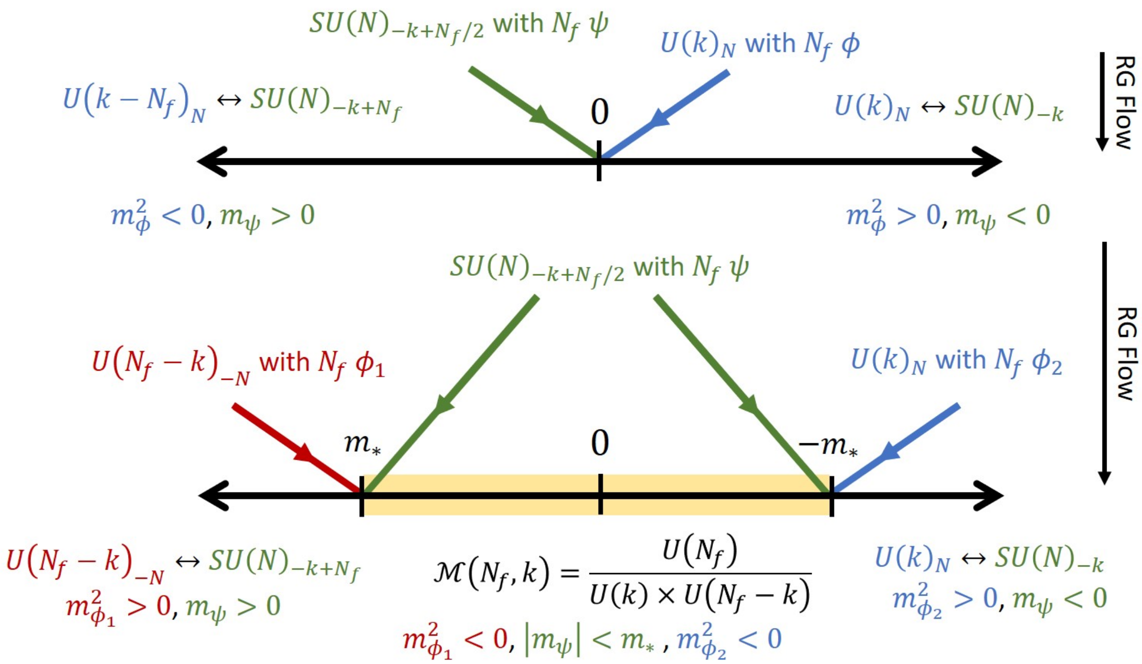

Figure 2.1: *Top:* Mass deformation diagrams of the flavor-bounded 3d bosonization dualities of (1.13a) where $N_f \leq k$. Here the blue and green lines are the RG flows of the respectively theories, and the black line is the mass deformation axis. *Bottom:* The flavor-violated extension of the usual 3d bosonization dualities proposed by [86]. Here we are working in the regime where $k < N_f < N_*$ and $N > 2$. The portion of the axis which is shaded yellow is the Grassmannian phase. The SU levels are *without* the additional $-N_f/2$ shift Ref. [86] (and many others) use to make things look more symmetric. We review the details of this diagram in Sec. 2.1.

The proposal of Ref. [86] is that when $k < N_f < N_*(N,k)$, for small fermion masses, i.e. $|m_\psi| < m_*$, the SU Chern-Simons theory causes the fermions to condense, yielding a non-linear sigma model whose target space is the complex Grassmannian manifold of (2.1).[3] Said another way, the SU side exhibits "chiral" symmetry breaking, yielding as its low energy "pion" Lagrangian the target space of (2.1). Importantly, it is the number of fermions with $|m_\psi| < m_*$ which determines whether or not the associated gauge group confines, resulting in the Grassmannian (2.1). Specifically, the assumption is that if the number of light fermions exceeds the bare level of the gauge group (the level without the η-invariant offset), the fermions still condense. When we construct quivers below we will further split the N_f symmetry into smaller subgroups and individually tune their masses. This can lead to a rich structure with many different Grassmannians, but in this chapter we will generally be concerned with phases outside of this region. More details on such subtleties of the phase diagram can be found in Refs. [18, 24].

A potential issue with this assumption is that the Grassmannian of the SU side only exists for small mass deformations, see Fig. 2.1. This is not matched on the U side if we use the phase diagram of the flavor-bounded duality (1.13a), which has a Grassmannian phase for unbounded negative scalar mass deformations. To rectify this, the authors of ref. [86] propose that there are *two* scalar theories dual to the SU end that are patched together to describe the full phase diagram. More explicitly, the duality proposed is

$$SU(N)_{-k+N_f/2} \text{ with } N_f\,\psi \qquad \leftrightarrow \qquad \begin{cases} U(k)_N \text{ with } N_f\,\Phi_1 & m_\psi = -m_* \qquad \text{(blue)} \\ U(N_f-k)_{-N} \text{ with } N_f\,\Phi_2 & m_\psi = m_* \qquad \text{(red)} \end{cases}, \tag{2.2}$$

where the colors in parenthesis correspond to those in Fig. 2.1 and figures which follow. We will sometimes label the bosonized duals by whether they are dual to the "$m_\psi > 0$" or "$m_\psi < 0$" region of the fermionic phase diagram (with appropriate mass deformations).

[3]Note, here we are using a slightly different convention than [86] for the levels of the Chern-Simons terms. The SU levels we use do not have the additional $-N_f/2$ shift used in [86] to make the phases look more symmetric.

Note that in both scalars theories, when the mass becomes negative we obtain the same aforementioned Grassmannian manifold of (2.1). Accompanying the Grassmannian is a Wess-Zumino-Witten term with a coefficient that is determined by the level of the Chern-Simons theories on the U side. This is important for matching of the operators across the duality, but will not play an essential role in our story. For more details on this term see [86]. This model is summarized in Fig. 2.1.

One feature of this construction that we will see occurs also in the master duality case is that the SU side of the duality does not yield a good description of the quantum phase or the theories which exist at the phase transition. That is, the true IR description on the SU side is hidden by strong dynamics and so we must pass to the U side to obtain a full description of the phase diagram.

2.2 *Flavor-Violated Master Duality*

We now generalize the analysis of [86] to the master duality. In this Chapter, we will use a slightly different labeling convention of the phases of the mater duality, shown in Fig. 2.2.

In particular, we are interested in mapping out the phase diagram of

$$SU(N)_{-k+N_f/2} \text{ with } N_f\ \psi \text{ and } N_s\ \phi \tag{2.3}$$

when $N_f > k$ and $N_s \leq N$ as a function of the scalar and fermion masses. We will begin our analysis in the asymptotic regimes where the mass deformations are large compared to the strong scale and we don't expect to see quantum phases. When one or both mass deformations are small, we again expect to find phases that may be described by Grassmannian manifolds.

When we give an asymptotically large positive mass to the scalars, we can integrate them out and they have no effect on the gauge group. The theory we are left with is

$$SU(N)_{-k+N_f/2} \text{ with } N_f\ \psi. \tag{2.4}$$

This is nothing more than the left-hand side of (2.2) and so this portion of the phase diagram is identical to what was found for QCD$_3$. We can do the same for a large neg-

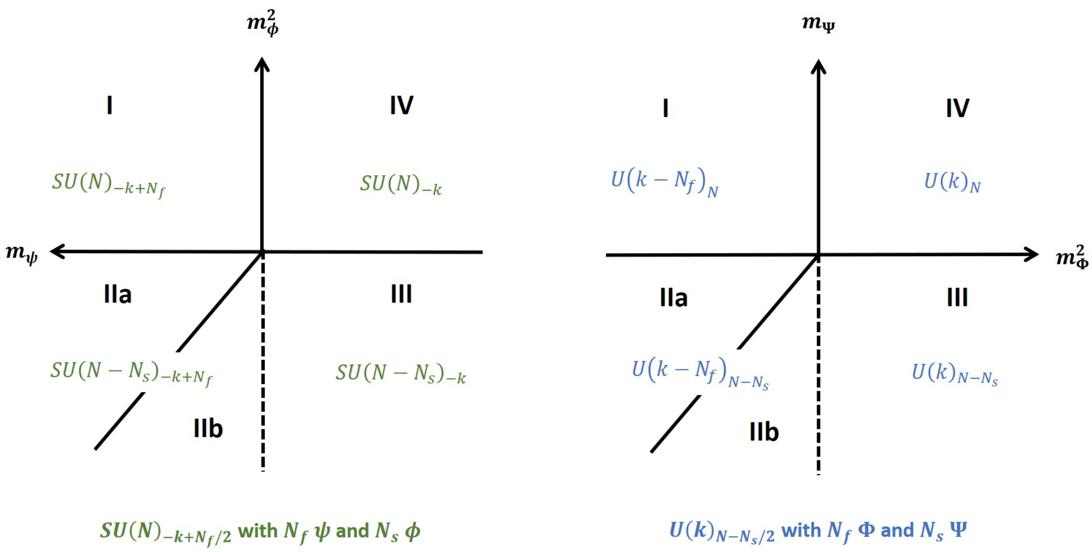

Figure 2.2: Phase diagram of the flavor-bounded master duality with the labelling convention used in Chapter 2 and Appendix C. Critical theories separating phases are the black lines. The dotted black lines represent a critical theory that would be present for $N > N_s$, but is not present for $N = N_s$.

ative mass deformation. Assuming the Higgsing is maximal,the gauge group breaks as $SU(N) \to SU(N-N_s)$. The resulting theory is

$$SU(N-N_s)_{-k+\frac{N_f}{2}} \text{ with } N_f\ \psi \tag{2.5}$$

which again appears to be described by (2.2) so long as $N - N_s > 1$. For now, we are ignoring effects due to the interaction term, but we will return to such considerations when we analyze the Grassmannian regime. We will see such terms make the Grassmannian portion of (2.5) different than that of (2.2) with $N \to N - N_s$.

Since we can match onto (2.2) for the asymptotically large mass phases of the SU side of the master duality, we follow the same reasoning for the U side. We thus assume once more we cannot use a single dual theory to describe all the phases of the U side. This motivates the "flavor-violated master duality" shown in Fig. 2.3:

$$\begin{aligned} & SU(N)_{-k+N_f/2} \text{ with } N_f\ \psi \text{ and } N_s\ \phi \\ & \qquad \leftrightarrow \quad \begin{cases} U(k)_{N-N_s/2} \text{ with } N_f\ \Phi_1 \text{ and } N_s\ \Psi_1 & m_\psi = -m_* \text{ (blue)} \\ U(N_f-k)_{-N+N_s/2} \text{ with } N_f\ \Phi_2 \text{ and } N_s\ \Psi_2 & m_\psi = m_* \text{ (red).} \end{cases} \end{aligned} \tag{2.6}$$

As a first consistency check, we make sure all four asymptotic phases match across the duality

$$\text{(I)}: \qquad SU(N)_{-k+N_f} \quad \leftrightarrow \quad U(N_f-k)_{-N}, \tag{2.7a}$$

$$\text{(II)}: \qquad SU(N-N_s)_{-k+N_f} \quad \leftrightarrow \quad U(N_f-k)_{-N+N_s}, \tag{2.7b}$$

$$\text{(III)}: \qquad SU(N-N_s)_{-k} \quad \leftrightarrow \quad U(k)_{N-N_s}, \tag{2.7c}$$

$$\text{(IV)}: \qquad SU(N)_{-k} \quad \leftrightarrow \quad U(k)_{N}. \tag{2.7d}$$

The levels for both the dynamical and background fields can be fixed by making sure the mass deformations of the two U theories match that of the SU theories for the asymptotically large deformations we have thus far discussed. The details of this analysis as well as explicit Lagrangians are given in Appendix C. Phases I and II follow from the first dual description in (2.6), while phases III and IV correspond to second dual description in (2.6). Note that for $N_f \leq k$, the flavor-bounded master duality instead yields the phases

$$\text{(I')}: \qquad SU(N)_{-k+N_f} \quad \leftrightarrow \quad U(k-N_f)_N, \tag{2.8a}$$

$$\text{(II')}: \qquad SU(N-N_s)_{-k+N_f} \quad \leftrightarrow \quad U(k-N_f)_{N-N_s}, \tag{2.8b}$$

but phases III and IV are the same as the $N_f > k$ case. We will use this fact to save us some work later on.

The mass mappings between the two sides are slightly complicated by the two scalar descriptions of the U side. For $m_\psi < 0$ we have

$$m_\psi + m_* \leftrightarrow -m_{\Phi_1}^2 \qquad m_\phi^2 \leftrightarrow m_{\Psi_1} \tag{2.9}$$

while for $m_\psi > 0$

$$m_\psi - m_* \leftrightarrow m_{\Phi_2}^2 \qquad m_\phi^2 \leftrightarrow -m_{\Psi_2}. \tag{2.10}$$

Once more we see the scalar mass vanishes when the magnitude of the fermion mass is equal to m_*.

Now let us discuss what we expect to occur as we move to smaller mass deformations. Once more, we can use (2.2) as a guiding principle for particular phases. Since phases I and IV precisely correspond to the large mass deformations of (2.2), we expect to find the very same Grassmannian in between them, which we have called phase V. This is indeed consistent with the two U theories of (2.6). The story gets slightly more complicated for other regions of the phase diagram. Let us now analyze the $N_s < N$ and $N_s = N$ cases in turn.

Flavor-Violation for $N > N_s$

Summarizing our results first, the phase diagram of (2.3) and the claimed duals for $N > N_s$ are given in Fig. 2.3. As we have learned, many aspects of the phase diagram are better described on the U side, so much of what is drawn on the SU is learned by considering how

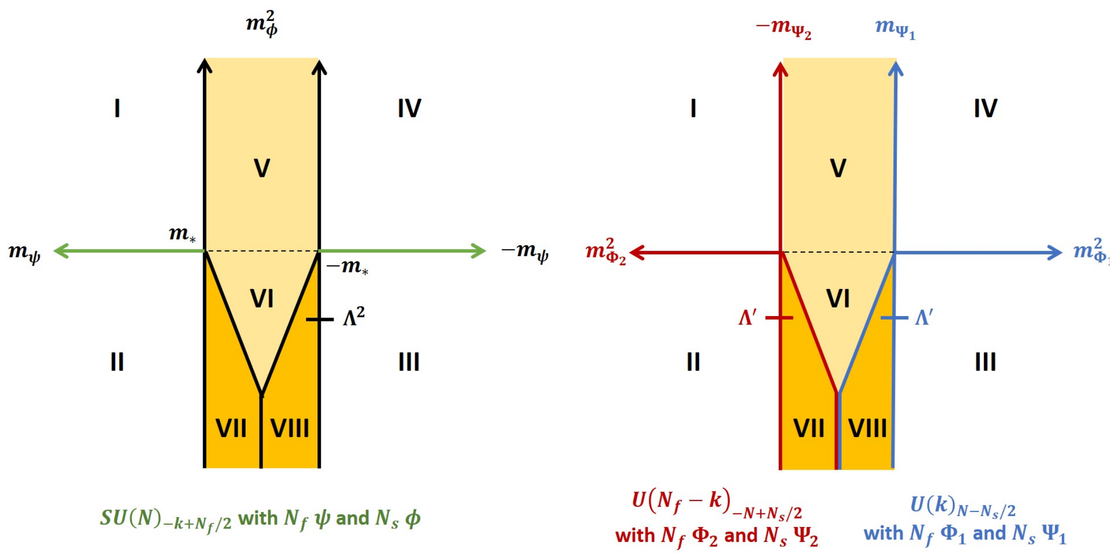

Figure 2.3: SU and U sides of the flavor-violated master duality for $N > N_s$. On the SU side, the critical lines in green are well-described by the corresponding SU theory. The critical lines in black are best described by the U duals.

the two dual U theories can be consistent in the presence of a finite interaction term. Phases I, V, and IV correspond to the three phases of (2.2) conjectured by Ref. [86].

The SU phase diagram contains the following phases

$$\text{(I)}: \quad SU(N)_{-k+N_f} \times [SU(N_f)_N \times SU(N_s)_0 \times J_{\text{I}}] \tag{2.11a}$$

$$\text{(II)}: \quad SU(N-N_s)_{-k+N_f} \times \left[SU(N_f)_N \times SU(N_s)_{-k+N_f} \times J_{\text{II}}\right] \tag{2.11b}$$

$$\text{(III)}: \quad SU(N-N_s)_{-k} \times [SU(N_f)_0 \times SU(N_s)_{-k} \times J_{\text{III}}] \tag{2.11c}$$

$$\text{(IV)}: \quad SU(N)_{-k} \times [SU(N_f)_0 \times SU(N_s)_0 \times J_{\text{IV}}] \tag{2.11d}$$

$$\text{(V) to (VIII)}: \quad \text{Better described by } U \text{ side} \tag{2.11e}$$

where the Chern-Simons theories in $[\cdots]$ belong to background gauge fields. Meanwhile, the U side is given by

$$\begin{aligned}
(\mathrm{I}):&\quad U(N_f-k)_{-N}\times[SU(N_f)_N\times SU(N_s)_0\times J_{\mathrm{I}}] &(2.12\mathrm{a})\\
(\mathrm{II}):&\quad U(N_f-k)_{-N+N_s}\times\left[SU(N_f)_N\times SU(N_s)_{-k+N_f}\times J_{\mathrm{II}}\right] &(2.12\mathrm{b})\\
(\mathrm{III}):&\quad U(k)_{N-N_s}\times[SU(N_f)_0\times SU(N_s)_{-k}\times J_{\mathrm{III}}] &(2.12\mathrm{c})\\
(\mathrm{IV}):&\quad U(k)_N\times[SU(N_f)_0\times SU(N_s)_0\times J_{\mathrm{IV}}] &(2.12\mathrm{d})\\
(\mathrm{V}),(\mathrm{VI}):&\quad \mathcal{M}(N_f,k)\times[SU(k)_N\times SU(N_f-k)_0\times SU(N_s)_0\times J_{\mathrm{V,VI}}] &(2.12\mathrm{e})\\
(\mathrm{VII}):&\quad \mathcal{M}(N_f,k)\times\left[SU(k)_N\times SU(N_f-k)_{N_s}\times SU(N_s)_{N_f-k}\times J_{\mathrm{VII}}\right] &(2.12\mathrm{f})\\
(\mathrm{VIII}):&\quad \mathcal{M}(N_f,k)\times[SU(k)_{N-N_s}\times SU(N_f-k)_0\times SU(N_s)_{-k}\times J_{\mathrm{VIII}}]\,. &(2.12\mathrm{g})
\end{aligned}$$

We label the critical theories by the two phases which they separate. They are

$$\begin{aligned}
(\mathrm{I\text{-}II}):&\quad U(N_f-k)_{-N+N_s/2}\text{ with }N_s\ \Psi_2 \quad\leftrightarrow\quad SU(N)_{-k+N_f}\text{ with }N_s\ \phi &(2.13\mathrm{a})\\
(\mathrm{III\text{-}IV}):&\quad U(k)_{N-N_s/2}\text{ with }N_s\ \Psi_1 \quad\leftrightarrow\quad SU(N)_{-k}\text{ with }N_s\ \phi &(2.13\mathrm{b})\\
(\mathrm{I\text{-}V}):&\quad U(N_f-k)_{-N}\text{ with }N_f\ \Phi_2 &(2.13\mathrm{c})\\
(\mathrm{IV\text{-}V}):&\quad U(k)_N\text{ with }N_f\ \Phi_1 &(2.13\mathrm{d})\\
(\mathrm{II\text{-}VII}):&\quad U(N_f-k)_{-N+N_s}\text{ with }N_f\ \Phi_2 &(2.13\mathrm{e})\\
(\mathrm{III\text{-}VIII}):&\quad U(k)_{N-N_s}\text{ with }N_f\ \Phi_1 &(2.13\mathrm{f})\\
(\mathrm{VI\text{-}VII}):&\quad \mathcal{M}(N_f,k)\text{ with }(N_f-k)N_s\ \psi_s &(2.13\mathrm{g})\\
(\mathrm{VI\text{-}VIII}):&\quad \mathcal{M}(N_f,k)\text{ with }kN_s\ \psi'_s &(2.13\mathrm{h})\\
(\mathrm{VII\text{-}VIII}):&\quad \mathcal{M}(N_f,k)\text{ with }(N_f-k)N_s\ \psi_s\text{ and }kN_s\ \psi'_s. &(2.13\mathrm{i})
\end{aligned}$$

Most of the critical theories are described by U side alone except (I-II) and (III-IV) which have an SU description related by (1.13). As a reminder, ψ_s and ψ'_s denote neutral fermions in the phases where dynamical gauge group is completely broken. By construction, each of the critical theories correctly describes the transition between the two phases which it separates, consistent with anomaly constraints.

The J_i for $i =$ I, II, III, IV are the Abelian Chern-Simons levels of the two $U(1)$ background gauge fields $\tilde{A}_1$ and $\tilde{A}_2$ in the i-th phase. When one enters the Grassmannian phases, the breaking of the $SU(N_f)$ global symmetry yields an additional background $U(1)$ field, which we call $\tilde{A}_3$. We write these $U(1)$ levels as

$$J_i \equiv J_i^{ab} \frac{1}{4\pi} \tilde{A}_a d\tilde{A}_b \tag{2.14}$$

where the J_i^{ab} are[4]

$$J_{\mathrm{I}}^{ab} = \begin{pmatrix} N(N_f - k) & 0 \\ 0 & 0 \end{pmatrix} \tag{2.15a}$$

$$J_{\mathrm{II}}^{ab} = \frac{N(N_f - k)}{N - N_s} \begin{pmatrix} N & N_s \\ N_s & N_s \end{pmatrix} \tag{2.15b}$$

$$J_{\mathrm{III}}^{ab} = \frac{-Nk}{N - N_s} \begin{pmatrix} N & N_s \\ N_s & N_s \end{pmatrix} \tag{2.15c}$$

$$J_{\mathrm{IV}}^{ab} = \begin{pmatrix} -Nk & 0 \\ 0 & 0 \end{pmatrix} \tag{2.15d}$$

$$J_{\mathrm{V,VI}}^{ab} = \begin{pmatrix} 0 & 0 & -Nk \\ 0 & 0 & 0 \\ -Nk & 0 & Nk \end{pmatrix} \tag{2.15e}$$

$$J_{\mathrm{VII}}^{ab} = \begin{pmatrix} 0 & 0 & -Nk \\ 0 & N_s(N_f - k) & -N_s k \\ -Nk & -N_s k & \frac{N_s k^2}{N_f - k} + Nk \end{pmatrix} \tag{2.15f}$$

$$J_{\mathrm{VIII}}^{ab} = \begin{pmatrix} 0 & 0 & -Nk \\ 0 & -N_s k & -N_s k \\ -Nk & -N_s k & (N - N_s)k \end{pmatrix}. \tag{2.15g}$$

See appendix C for more detailed discussion about the Lagrangian and background fields. We now explain in more detail how we determined these phases and critical lines.

[4]We suppress the third column/row for phases outside of the Grassmannian regime for brevity. For more details on how these terms are calculated, see [68, 25, 15].

Construction of Flavor-Violated Master Duality As mentioned above, we follow the guiding principle of Ref. [86] to conjecture the phase diagram on the U side of the duality. For Aharony's duality, the natural way of constructing the U side of the flavor-violated phase diagram was to overlay the phase diagram of two scalar theories, which was consistent since both scalar theories exhibited the same non-linear sigma model phase. It follows that as one traverses the quantum phase of the U side one must switch over from one scalar description to another. This is the same principle by which we have constructed the phase diagram for the master duality.

What complicates the phase diagram description for the flavor-violated master duality is the presence of the interaction term, which leads to an additional splitting of phases. It is straightforward to see the additional critical line from the interaction term is present in what will become the overlapping Grassmannian phases of the U theories. Since phases I, V, and IV should correspond to (2.2), the new critical line should have no effect deep into said regions. One can also verify that the overlapping scalar theories of (2.3) are not consistent outside of phase V without an interaction term. Per these constraints, we conjecture the interaction term's coefficients are chosen such that it splits the Grassmannian region below phase V. Furthermore, since such critical lines cannot simply terminate, at the crossover between the two scalar theories we conjecture the two critical lines merge. Rather remarkably, the merged critical line VII-VIII is precisely the theory needed to be consistent under anomaly constraints given the two phases it separates.

More explicitly, to reach a consistent intermediate phase we require the interaction for the blue theory to have a positive coefficient for its interaction term so as to give the fermions a positive mass when Φ_1 is Higgsed. The interaction takes the form $\mathcal{L}^1_{\text{int}} \supset c'\left(\Phi_1^{\dagger\alpha M}\Psi^1_{\alpha N}\right)\left(\bar{\Psi}_1^{\beta N}\Phi^1_{\beta M}\right)$ with α, β indices for the $U(k)$ gauge field, M an index for the $SU(N_f)$ global symmetry, N index for the $SU(N_s)$ global symmetry, and $c' > 0$. We then must give the fermions a negative mass to make the singlets light. This gives kN_s light fermions living on the critical theory between phases VI and VIII. Since the axis of the fermion mass deformation of the red theory is flipped relative to the blue one, it requires a negative coefficient for the interaction term, and so $\mathcal{L}^2_{\text{int}} \supset -c' \left(\Phi_2^{\dagger \alpha M} \Psi^2_{\alpha N}\right) \left(\bar{\Psi}_2^{\beta N} \Phi^2_{\beta M}\right)$ with α, β now $U(N_f - k)$ indices. This gives $(N_f - k)N_s$ light fermions on the critical line between phases VII and VI. At the crossover between (VI-VII) and (VI-VIII), the two critical theories unite into $N_f N_s$ light fermions, which is consistent with the transition from VII to VIII. This separates the Grassmannian phase into the three different regimes shown in Fig. 2.3. These phases all share the same Grassmannian $\mathcal{M}(N_f, k)$, but have distinct non-Abelian background Chern-Simons terms. [5]

For the rest of the phase diagram, calculating the critical theories is straightforward. As conjectured in [86], the critical lines corresponding to I-V and IV-V, as well as II-VII and III-VIII occur at some finite fermion mass deformation $\pm m_*$. These are obtained by application of (2.2). As such these lines are better described by the U side of the theory. This is different from the I-II and III-IV critical lines, which follow from a straight forward application of (1.13b).

Although much of this phase diagram lives in the strong coupling regime, it offers a plausible scenario for the matching of the dynamical and background gauge terms, which we discuss explicitly in Appendix C. The gravitational Chern-Simons terms can also be calculated and are consistent with expectations from level-rank duality.

Flavor-Violation for $N_s = N$

The conjectured phase diagram for $N = N_s$ is shown in Fig. 2.4, with the corresponding phases given in (2.16) and (2.17).

To see why $N = N_s$ needs to be considered separately, we begin by looking at the SU side. In the phase where $m_\phi^2 < 0$, the gauge group is $SU(N - N_s)$ and is thus completely broken when $N = N_s$. Hence for large enough scalar mass, we assume the strong dynamics of the gauge groups which was responsible for the condensation of the fermions is eliminated.

[5]Also note that phase VII and VIII have a different coefficient of the Wess-Zumino-Witten term from phases V and VI. This comes from the Chern-Simons level of the U gauge theories bordering the Grassmannian phases.

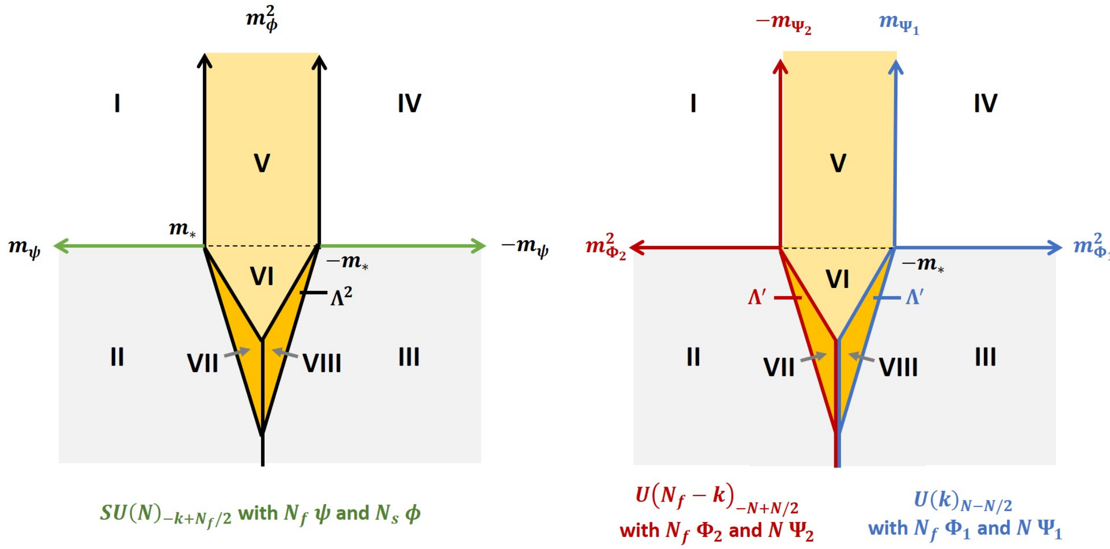

Figure 2.4: SU and U sides of the flavor-violated master duality for $N = N_s$ and $|k| < N_f < N_*$.

If this occurs, the Grassmannian phase cannot persist as we approach $m_\phi^2 \to -\infty$. We will take the termination to occur around $m_\phi^2 \sim \Lambda^2$. When one goes beyond this scale, we assume the gauge bosons of $SU(N)$ pick up too large of a mass (relative to the mass from the Chern-Simons terms) to cause the fermions to condense, meaning the Grassmannian phase terminates as one goes $\left|m_\phi^2\right| \gg \Lambda^2$. Note that this is very different from the $m_\phi^2 > 0$ phase, where the scalars are simply gapped out and the fermion condensate is unaffected.

This should be mirrored on the U side of the duality for $N = N_s$, where now we have the special case where the level of the U gauge group is zero and thus the gauge bosons are truly massless in the classical theory. Since we are in the strongly coupled regime we can only make conjectures, but we offer a plausible mechanism to match the SU side. Like the SU side, we will make the assumption that the termination comes when the matter, this time the fermion, has a large enough mass (in magnitude) to be integrated out. We conjecture that for large enough fermion mass, the level being $N - N_s = 0$ means the U gauge group is confining and it induces a mass gap for the scalars. This gap is large, but can be cancelled off by an appropriate mass deformation, similar to the singlet fermion picture described above. This allows us to still have a critical theory of light scalars along the II-VII and VIII-III transition that can drive us into the Grassmannian and give a mass to the singlet fermions. Once these theories become degenerate, i.e. when the two lines of light scalar meet in Fig. 2.4 the description breaks down and the Grassmannian terminates, leaving just the singlet fermions on the II-III transition.

The phases of the SU side are[6]

$$\begin{aligned}
&(\mathrm{I}): && SU(N)_{-k+N_f} \times [SU(N_f)_N \times SU(N_s)_0 \times J_{\mathrm{I}}] && (2.16a)\\
&(\mathrm{II}): && U(1)_0 \times \left[SU(N_f)_N \times SU(N_s)_{-k+N_f} \times U(1)_{N(N_f-k)}\right] && (2.16b)\\
&(\mathrm{III}): && U(1)_0 \times [SU(N_f)_0 \times SU(N_s)_{-k} \times U(1)_{-Nk}] && (2.16c)\\
&(\mathrm{IV}): && SU(N)_{-k} \times [SU(N_f)_0 \times SU(N_s)_0 \times J_{\mathrm{IV}}] && (2.16d)\\
&(\mathrm{V}) \text{ to } (\mathrm{VIII}): && \text{Better described by } U \text{ side.} && (2.16e)
\end{aligned}$$

As we saw for Aharony's duality, the U side is a better description for certain phases. Phases

[6]Here we use the fact that both the theories $SU(0)$ and $U(k)_0$ lead to a decoupled $U(1)_0$ theory. To see this on the SU side, note the gauge group is completely broken down by the scalar vacuum expectation value, but this leaves a single light degree of freedom which is the Goldstone boson associated with the spontaneous breaking of the $U(1)$ global symmetry. On the U side, the Chern-Simons term disappears and thus we are left with $U(k)$ Yang-Mills in the IR. The $SU(k)$ part of this confines, leaving the Abelian part of the gauge group which has a light photon. The photon is also associated with the spontaneously broken $U(1)$ symmetry corresponding to conserved particle flux (this is easiest to see in the dual photon language). Hence both of these light degrees of freedom can be associated with a spontaneous symmetry breaking of the $U(1)$ global symmetry (i.e. the one associated with $\tilde{B} = \tilde{A}_1 + \tilde{A}_2$).

of the U side is given by

$$\text{(I)}: \quad U(N_f-k)_{-N} \times [SU(N_f)_N \times SU(N_s)_0 \times J_{\text{I}}] \tag{2.17a}$$

$$\text{(II)}: \quad U(1)_0 \times \left[SU(N_f)_N \times SU(N_s)_{-k+N_f} \times U(1)_{N(N_f-k)}\right] \tag{2.17b}$$

$$\text{(III)}: \quad U(1)_0 \times [SU(N_f)_0 \times SU(N_s)_{-k} \times U(1)_{-Nk}] \tag{2.17c}$$

$$\text{(IV)}: \quad U(k)_N \times [SU(N_f)_0 \times SU(N_s)_0 \times J_{\text{IV}}] \tag{2.17d}$$

$$\text{(V), (VI)}: \quad \mathcal{M}(N_f,k) \times [SU(k)_N \times SU(N_f-k)_0 \times SU(N_s)_0 \times J_{\text{V}}] \tag{2.17e}$$

$$\text{(VII)}: \quad \mathcal{M}(N_f,k) \times \left[SU(k)_N \times SU(N_f-k)_{N_s} \times SU(N_s)_{N_f-k} \times J_{\text{VII}}\big|_{N=N_s}\right] \tag{2.17f}$$

$$\text{(VIII)}: \quad \mathcal{M}(N_f,k) \times \left[SU(k)_0 \times SU(N_f-k)_0 \times SU(N_s)_{-k} \times J_{\text{VIII}}\big|_{N=N_s}\right]. \tag{2.17g}$$

We now have a single $U(1)$ background gauge field for phases II and III, which is $\tilde{A} \equiv \tilde{A}_1 = -\tilde{A}_2$. This is because the global $U(1)$ baryon/monopole symmetry is spontaneously broken in the phases II and III for $N = N_s$ (see Appendix C for more details).

Finally, critical theories are given by

$$\text{(I-II)}: \quad U(N_f-k)_{-N/2} \text{ with } N\ \Psi_2 \quad \leftrightarrow \quad SU(N)_{-k+N_f} \text{ with } N\ \phi \tag{2.18a}$$

$$\text{(III-IV)}: \quad U(k)_{N/2} \text{ with } N\ \Psi_1 \quad \leftrightarrow \quad SU(N)_{-k} \text{ with } N\ \phi \tag{2.18b}$$

$$\text{(I-V)}: \quad U(N_f-k)_{-N} \text{ with } N_f\ \Phi_2 \tag{2.18c}$$

$$\text{(IV-V)}: \quad U(k)_N \text{ with } N_f\ \Phi_1 \tag{2.18d}$$

$$\text{(II-VII)}: \quad U(N_f-k)_0 \text{ with } N_f\ \Phi_2 \tag{2.18e}$$

$$\text{(III-VIII)}: \quad U(k)_0 \text{ with } N_f\ \Phi_1 \tag{2.18f}$$

$$\text{(VI-VII)}: \quad \mathcal{M}(N_f,k) \text{ with } (N_f-k)N\ \psi_s \tag{2.18g}$$

$$\text{(VI-VIII)}: \quad \mathcal{M}(N_f,k) \text{ with } kN\ \psi'_s \tag{2.18h}$$

$$\text{(VII-VIII)}: \quad \mathcal{M}(N_f,k) \text{ with } (N_f-k)N\ \psi_s \text{ and } kN\ \psi'_s \tag{2.18i}$$

$$\text{(II-III)}: \quad U(1)_0 \text{ with decoupled } NN_f\ \tilde{\psi}_s. \tag{2.18j}$$

In the above, ψ_s, ψ'_s and $\tilde{\psi}_s$ denote neutral fermions in the phases where dynamical gauge group is completely broken. Note that in contrast to $N > N_s$ case, here we have critical line (II-III) which can only be described by SU side using semiclassical analysis.

2.3 Double-Saturated Flavor Bound

We now investigate the validity of the doubly saturated flavor bound, which will be applicable to the flavored quiver we construct in Chapter 4. That is, can the master duality be extended to also hold for the case $(N_s, N_f) = (N, k)$?

Let us review in more detail why the master duality was invalid in such a limit. In [68] it is argued that the phases (shown in Fig. 2.2) for $N_s = N$ are

$$\text{(I)}: \qquad SU(N)_0 \quad \leftrightarrow \quad U(0)_N \tag{2.19a}$$

$$\text{(IIa)}: \qquad U(1)_0 \quad \leftrightarrow \quad U(0)_0 \tag{2.19b}$$

$$\text{(IIb)}: \qquad SU(0)_0 \quad \leftrightarrow \quad U(0)_0 \tag{2.19c}$$

$$\text{(III)}: \qquad U(1)_0 \quad \leftrightarrow \quad U(1)_0 \tag{2.19d}$$

$$\text{(IV)}: \qquad SU(N)_{-k} \quad \leftrightarrow \quad U(k)_N \tag{2.19e}$$

with mass mapping $m_\phi^2 \leftrightarrow m_\Psi$ and $m_\psi \leftrightarrow -m_\Phi^2$. Phase I, III, and IV should all match, since these phases reduce to Aharony's dualities, (1.13), and we know the flavor saturated cases pass all tests in those dualities. There appears to be a conflict already in phase I, where one side of the theory is a completely broken $U(0)$ theory, while the other end of the theory is simply $SU(N)$ Yang-Mills with no Chern-Simons term. By assumption, the latter of these confines, so the low energy limit is empty. Meanwhile, in the completely broken theory, the lightest excitations are finite mass vortices. Thus there are no light degrees of freedom in either case, which is consistent. The matching in phase III occurs because each side is described by a decoupled $U(1)_0$, as we saw earlier in the $N = N_s$ flavor violated master.

The conflict in the double saturated master duality then arises in phase IIa and IIb. Let us compare phase IIa to what we had in phase III. On the SU side, we have gone from $SU(0)_{-k}$ to $SU(0)_0$. The breaking and subsequent Goldstone boson which occurred in phase

III was irrespective of the Chern-Simons level, so there is no difference in the resulting light degrees of freedom in this phase. Meanwhile on the U side we have gone from $U(k)_0$ to $U(0)_0$. When the rank of the gauge group is reduced to zero, there is no light photon left in the theory. Thus we have a mismatch in the degrees of freedom on either side, leading Refs. [25, 68] to postulate that the master duality no longer holds for such a case.

To summarize the issue: changing the *level* on the SU side has no effect on the theory, but changing the *rank* does affect the light degrees of freedom on the U side of the theory, but only for the case of $N_f = k$ where the rank of the U group is reduced to zero. As we mentioned above, the light degrees of freedom in this phase should be associated with a spontaneous symmetry breaking of the $U(1)$ global symmetry associated to $\tilde{A}_1$.

To extend the validity of the master duality to the double saturated case is simple – we choose to *explicitly* break the $U(1)$ global symmetry on both sides of the duality. This will eliminate the Goldstone bosons associated with the spontaneous breaking of said symmetry, and thus we will again have a matching in phases IIa and IIb. Note this will also modify the light degrees of freedom in said phases.

Perhaps not by coincidence, we will see in Chapter 4 we will be motivated to explicitly break the same $U(1)$ global symmetry in the construction of the $SU(N)$ Yang-Mills n-node quivers [8]. Specifically, we will introduce a $\det(Y_{i,i+1})$ terms and appropriate monopole term on the U side to eliminate the $U(1)^{n-1}$ symmetry. Hence, by eliminating these $U(1)$ global symmetries on both sides of the duality, we have also extended the master duality to validity in the $(N_f, N_s) = (k, N)$ case. This allows the master duality to be valid for the $N_F = 1$ quivers, which we construct Chapter 4.

2.4 Disucssion and Future Work

In this chapter we extended the master duality presented in [68, 25] to the flavor violated regime where $N_f > k$. It is important to keep in mind the general philosophy used in deriving the phase diagrams of Figs. 2.3 and 2.4: the dynamics in the quantum phase is hidden by strong dynamics in the SU theory. It is only when we pass to the U theory that we can see interesting structure emerge, similar to approached used in [86] as well as in [56, 37, 33, 36]. Like these prior works, our dualities pass all the required consistency checks short of explicit large N calculations. And although it is difficult to prove the existence of these new quantum phases exactly, 3d bosonization gives good evidence that they indeed exist.

Obvious generalizations of this approach include the flavor extension of the master duality to other regimes. Since one can effectively interchange the SU and U sides of the duality by promoting the background global symmetry associated to $\tilde{A}_1$ to be dynamical, it should be straightforward to extend our work here to the $N_s > N$ and $N_f \leq k$ regime. The double flavor violated case, where $(N_f, N_s) > (k, N)$ would also be interesting to consider.[7]

[7] Recently, the authors of [21] analyzed the phase diagram of QCD_3 in the large N limit under minimal assumption and found there should be sequence of first-order phase transitions coming from a multitude of metastable vacua. This phase structure is realized in the dual description with the help of potential term beyond quartic scalar term. It would also be interesting to see how these large N features modify the analysis of the main text in more detail.

Chapter 3

3D BOSONIZATION WITH BOUNDARIES

One aspect of 3d bosonization that has received little attention is the role of boundaries in the duality. If there is any testable prediction to come from the duality, it is necessary to understand how to describe the behavior of systems with dual bulk theories in the presence of a boundary in order to make contact with quantities measurable on a finite sample. After all, the physics of edge modes is the most easily accessible physical manifestation in quantum Hall samples. Furthermore, including boundaries gives us one more check of the dualities which remain conjectural. If there is any intent on exporting the lessons from 2+1 dimensional bosonization to inform any aspect of experimental protocol for real world samples, then it is important to understand how the dualities can be made consistent with the introduction of a boundary.

To understand both these points, in the first part of this chapter we will study how Abelian theories on flat, half space $\mathbb{R}^{2,1}_{+}$ are related by bosonization. In particular, we will restrict our investigation to flux attachment between IR descriptions of (1.16), i.e. free fermions and scalars with quartic self-coupling near or at the conformal fixed point [9]. Despite our restriction to $\mathbb{R}^{2,1}_{+}$, we believe our results generalize to curved manifolds with arbitrary boundaries so long as they are topologically trivial.

In the Abelian case, we will show that a prescriptive method exists to assign boundary conditions for Abelian bosonization and particle-vortex dualities. The key realization in arriving at the correct accounting for necessary boundary conditions and edge modes in Abelian dualities was that the Chern-Simons terms were best thought of in terms motivated by their UV origins. That is, the Chern-Simons terms are replaced by a theory of heavy, well-regulated “fiducial fermions”.

In the second part of this chapter we generalize this construction to the non-Abelian Aharony dualities, (1.13), and then the master dualitiy, (1.27). At first glance, one might assume that the application of the fiducial fermion analysis in Abelian dualities in the presence of a boundary would be a trivial generalization to the non-Abelian dualities. However because both sides of the single species non-Abelian and master bosonization dualities have dynamical gauge fields, the introduction of a boundary is more subtle. Choosing certain boundary conditions on bulk dynamical gauge fields can alter the counting of global symmetries present on the boundary. The interplay between boundary conditions and global symmetries presents an interesting, non-trivial extension of the study in [9] and will require a careful analysis in order to find duality-consistent boundary conditions.

An ambitious program considering boundary conditions in 2+1 dimensional dualities has been outlined by Gaiotto in a talk [50]. In that talk, he conjectures dual pairs of boundary conditions based on constructing interfaces between a theory and its dual. Assuming that at low energies the theories decouple across the interface, an interesting web of Abelian and non-Abelian dualities emerges with subtle, non-trivial interplay of boundary conditions imposed on scalars and gauge fields. In this chapter, we construct a duality that agrees with one of the examples in [50] and gives evidence that these conjectures – which are based on the decoupling assumption – are possibly true more broadly. While not attempted in the following chapter, it would be very interesting to flesh out the details of Gaiotto's program.

To the end of finding duality-consistent boundary conditions for the master bosonization duality, this chapter is organized as follows. In Sec.3.1, we will briefly review the master bosonization duality and establish some new notation. We begin our discussion of the Abelian bosonization with boundaries in Sec. 3.2. Specifically, in 3.2.1 we construct the appropriate theories participating in bosonization on $\mathbb{R}^{2,1}_{+}$ including possible boundary conditions and requirements for the theory to be non-anomalous. Sec. 3.2.2 is where we formulate the role of boundary conditions and self-consistency in describing dual theories on $\mathbb{R}^{2,1}_{+}$. Further, in 3.2.3, we will give evidence for the continuum duality by writing down the microscopic theory on a Euclidean three-dimensional cubic lattice.

Sec. 3.3 obtains the consistent boundary conditions and necessary edge modes for non-Abelian dualities with a single species of matter, schematically following what was done for the Abelian case. Using the method developed to address the single species non-Abelian dualities, we will show how the new prescription works in all massive phases of the master bosonization duality in Sec. 3.4. Finally, we briefly comment on generalization to the SO and USp versions of the master duality in Sec. 3.4.1.

3.1 Notation and Preliminaries

In this chapter it will be useful to introduce shorthand for Chern-Simons and BF terms to keep equations brief. To that end, we define the notation for rank N gauge groups

$$\mathrm{CS}_N[b] \equiv \frac{1}{4\pi}\mathrm{Tr}_N\left(bdb - i\frac{2}{3}b^3\right), \tag{3.1}$$

$$\mathrm{BF}[f;\mathrm{Tr}_N b] \equiv \frac{1}{2\pi} f d\mathrm{Tr}_N b. \tag{3.2}$$

The master duality Lagrangians of (1.28) become

$$\begin{aligned}\mathcal{L}_{SU} &= \left|D_{b'+B+\tilde{A}_1+\tilde{A}_2}\phi\right|^2 + i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1}\psi + \mathcal{L}_{\text{int}} - i\left((N_f-k)\mathrm{CS}_N[b'] + N\mathrm{CS}_{N_f}[C]\right)\\ &\quad - i\left(N(N_f-k)\mathrm{CS}_1[\tilde{A}_1] + 2NN_f\mathrm{CS}_{\text{grav}}\right),\end{aligned} \tag{3.3a}$$

$$\mathcal{L}_U = |D_{c+C}\Phi|^2 + i\bar{\Psi}\not{D}_{c+B+\tilde{A}_2}\Psi + \mathcal{L}'_{\text{int}} - iN\left(\mathrm{CS}_k[c] + \mathrm{BF}[\mathrm{Tr}_k(c);\tilde{A}_1] + 2k\mathrm{CS}_{\text{grav}}\right). \tag{3.3b}$$

Below we will need to keep careful track of $U(1)$ gauge field levels and it will be helpful to introduce a Lagrange multiplier field f on the SU side of the Lagrangian,

$$\begin{aligned}\mathcal{L}_{SU} &= |D_{b+B}\phi|^2 + i\bar{\psi}\not{D}_{b+C-\tilde{A}_2}\psi - i\left((N_f-k)\mathrm{CS}_N[b] + \mathrm{BF}\left[f;\mathrm{Tr}_N\left(b - \mathbb{1}_N\left(\tilde{A}_1+\tilde{A}_2\right)\right)\right]\right)\\ &\quad - iN\left(\mathrm{CS}_{N_f}[C] + (k-N_f)\left(\mathrm{BF}[\tilde{A}_1;\tilde{A}_2] + \mathrm{CS}_1[\tilde{A}_2]\right) + 2N_f\mathrm{CS}_{\text{grav}}\right) + \mathcal{L}_{\text{int}},\end{aligned} \tag{3.4}$$

This effectively transforms the dynamical gauge field to $SU(N) \to U(N)\times U(1)$. Analyzing the symmetry breaking pattern for $U(N)\times U(1)$ is easier than for $SU(N)$ [25, 68, 60]. Although the presence of the Lagrange multiplier f makes coupling slightly obscure, on the U side $\tilde{A}_1$ only appears through a BF coupling to the monopole current $\star j_m = \frac{1}{2\pi}d\mathrm{Tr}_k c$, while on the SU side it still couples directly to the particle number current.

3.2 *Abelian Bosonization with Boundaries*

More explicitly, the Abelian duality of (1.16b) have the Lagrangians

$$\mathcal{L}_{\mathrm{WF+flux}}[A] \equiv \mathcal{L}_{\mathrm{WF}}[\Phi,\, c] - i\left(\mathrm{CS}_1[c] + \mathrm{BF}[c;A]\right) \qquad \leftrightarrow \qquad \mathcal{L}_f[\psi, \lambda,\, A], \tag{3.5}$$

where we have taken the $N = k = N_f = 1$ and $N_s = 0$ limit of (3.3), canceled the $\mathrm{CS}_{\mathrm{grav}}$ terms on each side, and taken $\tilde{A}_1 \to \tilde{A}$. We will drop all tildes from the Abelian fields for this section. The duality (1.16a) is

$$\mathcal{L}_{\mathrm{WF}}[\phi,\, B] \qquad \leftrightarrow \qquad \mathcal{L}_f[\Psi, \lambda,\, a] - i\left(-\mathrm{BF}[a;B] - \mathrm{CS}_1[B]\right) \equiv \mathcal{L}_{\mathrm{f+flux}}[B], \tag{3.6}$$

which is the *time reversed* version of the $N_s = N = k = 1$ and $N_f = 0$ limit of the master duality and we have relabeled $\tilde{c} - \tilde{A}_2 \to \tilde{a}$ and $\tilde{B} = \tilde{A}_1 + \tilde{A}_2$, with $\tilde{c}$ the Abelian part of the gauge field c.[1] The Lagrangians for the various matter fields participating in the above dualities are given by

$$\mathcal{L}_{\mathrm{WF}}[\phi,\, B] = |(\partial_\mu - iB_\mu)\phi|^2 - \alpha|\phi|^4, \tag{3.7a}$$

$$\mathcal{L}_f[\psi, \lambda,\, A] = \lim_{m_\lambda \to -\infty} \; i\bar{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi + i\bar{\lambda}\gamma^\mu(\partial_\mu - iA_\mu)\lambda - m_\lambda\bar{\lambda}\lambda\,. \tag{3.7b}$$

A useful map between dualities will be between global symmetry currents. Since we identify the global $U(1)$ symmetries on either side of the duality, it is natural to also identify the conserved currents associated with said symmetries. For example, the duality (3.5) implies the identification of

$$j^\mu_{\mathrm{WF+flux}}(x) \equiv \frac{\delta S_{\mathrm{WF+flux}}[A]}{\delta A_\mu(x)} \qquad \leftrightarrow \qquad j^\mu_f(x) \equiv \frac{\delta S_f[A]}{\delta A_\mu(x)} \tag{3.8}$$

where we have defined $S_{\mathrm{WF+flux}} = \int d^3x\; \mathcal{L}_{\mathrm{WF+flux}}$ and similarly for other actions. For the side of the duality with a dynamical $U(1)$ gauge field, the global $U(1)$ is associated with a flux current. Meanwhile, the side with just matter has a global $U(1)$ that is associated with particle number.

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[1] Due to our η-invariant convention, a time-reversal must be accompanied by an extra $\mathrm{CS}_1[a] + 2\mathrm{CS}_{\mathrm{grav}}$ Chern-Simons terms on the fermion end. The reason for the time reversal is to better match onto the form of the dualities used in Refs. [83, 112].

3.2.1 Theories on half-space

Let us begin by addressing the subtleties associated with the theories on the half-space, $\mathbb{R}^{2,1}_{+}$. We will explore the space of boundary conditions consistent with (3.7b) or (3.7a) defined on $\mathbb{R}^{2,1}_{+}$.

To do so, we must remind ourselves of how to be honest about boundary conditions in field theories. Consider a theory with action S defined on the manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. By taking the variation δS we will find two classes of terms

$$\delta S = \delta S_{\text{bulk}} + \delta S_{\text{bdry}} = \int_{\mathcal{M}} \delta\mathcal{L}_{\text{bulk}} + \int_{\partial\mathcal{M}} \delta\mathcal{L}_{\text{bdry}}. \tag{3.9}$$

The bulk part of the action is still extremized by the classical equations of motion, and consistency of the variation amounts to choosing conditions on the field configurations such that δS_{bdry} vanishes as well. In the classical limit of the theory, the field configuration that satisfies the equations of motion should also satisfy boundary conditions. In the full quantum theory this is not necessarily the case. One way to proceed is by manually restricting the space of allowed field configurations by inserting delta funtions in the path integral which impose the desired boundary conditions. This method excludes fluctuations where $\delta S_{\text{bdry}} \neq 0$. Alternatively, we could do the path integral over all boundary field configurations. In that case the boundary conditions would only be obeyed by the dominant field configurations in the path integral–those which extremize the action. Below we will see that for all fields we consider there will be multiple boundary conditions which satisfy $\delta S_{\text{bdry}} = 0$. The boundary conditions will be chosen such that the theory remains non-anomalous and we keep the global symmetries on either side of the duality consistent.

In addition to the field conventions listed above, we will take coordinates on $\mathbb{R}^{2,1}_{+}$ to be $\{t, x, y\}$ where $t, x \in (-\infty, \infty)$ and $y \geq 0$. The boundary of $\mathbb{R}^{2,1}_{+}$ is the surface at $y = 0$. Indices i, j will be used to denote coordinates on the boundary and μ, ν in the bulk.

Boundary conditions

Applying the above approach to (3.7a), we take the theory defined on $\mathbb{R}^{2,1}_+$ by fiat and vary such that

$$\delta S_{WF}[\phi, B] = \ldots + \int_{\partial\mathcal{M}} d^2x \left(\delta\phi^\dagger D_y\phi + \delta\phi D_y\varphi^\dagger\right) \tag{3.10}$$

where "…" contains bulk terms which vanish on-shell. This implies that both Dirichlet $\delta\phi|_\partial = 0$ and Neumann $D_y\phi|_\partial = 0$ are valid boundary conditions, where "$|_\partial$" denotes an expression which holds at the boundary.

Now consider the boundary conditions for a Dirac fermion. We write the Dirac fermion (3.7b) evaluated on $\mathbb{R}^{2,1}_+$ in terms of left and right handed components:

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \qquad \text{i.e.} \quad \psi_\pm = P_\pm\psi \qquad \text{with} \qquad P_\pm = \frac{\mathbb{1} \pm \gamma^y}{2}, \tag{3.11}$$

and γ^y is the gamma matrix in the direction perpendicular to the boundary. $\gamma^y = i\gamma^t\gamma^x$ is the 'γ_5' in the boundary theory. Now, on $\mathbb{R}^{2,1}_+$ the terms in (3.7b) that depend on ψ in this language read

$$\begin{aligned} S_f[\psi, \lambda, A] = \int_{\mathbb{R}^{2,1}_+} d^3x \, \Big(& i\bar{\psi}_+ \not{D}_{A_i}\psi_+ + i\bar{\psi}_- \not{D}_{A_i}\psi_- + \bar{\psi}_- A_y\psi_+ - \bar{\psi}_+ A_y\psi_- \\ & + \frac{i}{2}(\bar{\psi}_-\partial_y\psi_+ - \partial_y\bar{\psi}_-\psi_+ + \partial_y\bar{\psi}_+\psi_- - \bar{\psi}_+\partial_y\psi_-)\Big) + \ldots \end{aligned} \tag{3.12}$$

where the ellipses denote the terms that only depend on the Pauli-Villars regulator field and $\not{D}_{A_i} \equiv \gamma^i(\partial_i - iA_i)$.

The boundary terms generated by the variation of (3.12) are

$$\delta S_f[\psi, A] = \ldots + \int_{\partial\mathcal{M}} d^2x \, \left(\psi_-\delta\bar{\psi}_+ - \psi_+\delta\bar{\psi}_- + \bar{\psi}_-\delta\psi_+ - \bar{\psi}_+\delta\psi_-\right). \tag{3.13}$$

We can consistently impose Dirichlet boundary conditions on *either* of the chiral components,[2]

$$\psi_+|_\partial = 0 \qquad \text{or} \qquad \psi_-|_\partial = 0. \tag{3.14}$$

[2]We will always take $P_\pm\psi|_\partial$ to imply the corresponding relation on the conjugate field, namely $\bar{\psi}P_\mp|_\partial = 0$.

However, choosing *both* $\psi_+|_\partial = 0$ and $\psi_-|_\partial = 0$ over-constrains the equations of motion at the boundary [38]. Either choice in (3.14) leaves behind a chiral edge mode as seen in the current running parallel to the boundary, $j^i_\psi = \bar{\psi}\gamma^i\psi$. In Sec. 3.2.2, we will explore how requiring a non-anomalous theory forces us to choose one boundary condition over the other.

Since the action for the Pauli-Villars regulator fields is identical to that for the Dirac fermions, the analysis above applies in kind. In particular, we apply chiral boundary conditions on the Pauli-Villars regulator as well, i.e.

$$\lambda_+|_\partial = 0 \qquad \text{or} \qquad \lambda_-|_\partial = 0. \tag{3.15}$$

In what follows, we will show boundary condition of the Dirac fermion and Pauli-Villars field are related. In order to keep track of which boundary condition we are imposing on the two fields, we introduce a superscript $\mathcal{L}^{\pm}_f[\psi, \lambda, A]$ to indicate imposing the boundary conditions $\psi_\mp|_\partial = \lambda_\pm|_\partial = 0$.

We will use similar notation for the time reversed fermion actions $\bar{\mathcal{L}}^{\mp}_f[\psi, \lambda, A]$ with the superscript indicating the type of boundary conditions imposed. Note that the time reversed version of $\mathcal{L}^+_f$ is $\bar{\mathcal{L}}^-_f$ and vice versa. In addition, we should keep in mind that $\mathcal{L}^\pm_f$ itself was defined with a large negative Pauli-Villars mass, and since fermion mass terms are time-reversal odd, $\bar{\mathcal{L}}^\pm_f$ is defined with a large positive Pauli-Villars mass. This means that $\bar{\mathcal{L}}_f$ can be thought of as coming with a $k = +\frac{1}{2}$ Chern-Simons term rather than $k = -\frac{1}{2}$.

Next, consider the possible boundary conditions for our dynamical gauge fields. To constrain such fields, we will consider the action at the level of the microscopic description in which the Maxwell term is still dominant. Upon variation, we find

$$\delta S_{\text{Maxwell}}[b] = \ldots - \frac{1}{e^2}\int_\partial d^2x\ F^{yi}\delta b_i, \tag{3.16}$$

with $F_{yi} = \partial_y b_i - \partial_i b_y$. Once more, we see we can impose either Dirichlet or Neumann boundary conditions. The former requires the variation along the boundary to vanish, i.e. $b_i = 0$. Neumann boundary conditions require the field strength adjacent and oriented perpendicular to the boundary be flat, $F_{iy} = 0$.

Lastly, we will consider the boundary conditions for a level-k Chern-Simons term. Such terms will only come up in the IR limit of the dualities. Varying a Chern-Simons term gives

$$\delta\left(\int d^3x\ k\mathrm{CS}_1[b]\right) = \ldots + \frac{k}{4\pi}\int_{\partial\mathcal{M}} d^2x\ \epsilon^{ij} b_i \delta b_j. \tag{3.17}$$

While we could impose $b_t = 0$ or $b_x = 0$ at the boundary, requiring the general, sufficient condition that

$$(b_t - v_b b_x)|_\partial = 0, \tag{3.18}$$

makes the boundary physics clear in the context of (3.5) and (3.6). That is, we maintain a chiral edge mode with velocity v_b and chirality set by $\mathrm{sgn}(v_b)$. In order for the boundary kinetic term to be positive definite, the velocity must be chosen such that $v_b k > 0$ [47]. In what follows we will be mostly interested in relativistic theories fixing the magnitude $|v_b| = 1$. Since a gauge transformation of a Chern-Simons term also produces a boundary term, any gauge choice that we make must be consistent with (3.18). The simplest solution is to promote the boundary condition to a gauge fixing condition, i.e. we let $(b_t - v_b b_x) = 0$ in the bulk as well. As we will see in the next section, the freedom to choose v_b is actually tied to the choice of fermionic boundary conditions. The consistency requirement on the sign of v_b will then pick a preferred fermionic boundary condition, which we will hardwire into the path integral.

In this section, we have seen that there are multiple choices of boundary conditions for all of the fields in our theories. However, the choices will be constrained by requiring the theory to be non-anomalous and that the global symmetries on either side of the duality match.

Boundary modes and anomalies

The discussion of the previous subsection will prove sufficient to study the duality between the conformal field theories related by bosonization. However, to check the consistency of our dualities under deformations, we will also be interested in adding mass gaps to the theories on $\mathbb{R}^{2,1}_{+}$. Before formulating dualities like (3.5) and (3.6) with boundary conditions, we will highlight additional subtleties in gapped phases in the presence of a boundary.

Our main concern in this section is the possible existence of domain wall fermions (DWFs) and their interplay with anomalies.[3] DWFs are typically discussed in the context of Dirac fermions defined on $\mathbb{R}^{2,1}$ with a spatially varying mass term – specifically, a mass term that changes sign across an interface. But the same basic construction also allows us to look for the existence of massless boundary modes on $\mathbb{R}^{2,1}_{+}$. A massless chiral mode localized on the boundary will exist when the mass profile leaves

$$\xi\left(y\right)=e^{\pm\int_0^y dy'\,m(y')} \tag{3.19}$$

finite for all $y\in\mathbb{R}^{2,1}_{+}$ [76]. Unlike the DWF descending from the construction on $\mathbb{R}^{2,1}$, any constant, non-zero mass profile ($m(y)=m$) in (3.19) yields a normalizable zero mode for a fermionic theory on $\mathbb{R}^{2,1}_{+}$. The chirality of the DWF is set by the sign of the mass: $\mathrm{sgn}(m)=+1$ gives a left-mover and $\mathrm{sgn}(m)=-1$ a right-mover. In either case, the chiral current is not conserved, and so the boundary theory on its own is anomalous. While this is not necessarily an inconsistency in the case when the fermion number is not gauged, we are only interested in theories in which our global symmetry currents are in fact conserved and so can be consistently coupled to background fields.[4]

It has long been known that a level-k Chern-Simons in the bulk can precisely account for the anomalous chiral modes living on the defect so long as they satisfy the relation

$$k=n_+-n_-, \tag{3.20}$$

where (n_+) n_- are the number of (right-) left-moving modes. More precisely, the nonzero anomaly of the bulk Chern-Simons term under gauge transformations of its associated gauge

[3]Strictly speaking, our "domain wall" is really the boundary of our material, but we will continue to use this slight abuse of vocabulary.

[4]Dualities between theories which have non-vanishing boundary anomalies for global symmetries can also be formulated, as long as the anomalies on both sides of the duality agree. We do not consider such dualities in this work, but they have been outlined in [50] along with the theories we consider.

field can be exactly compensated by the axial anomaly of chiral edge movers on the boundary. This is know as the Callan-Harvey mechanism [29].

In addition to the chiral anomaly, there is also a framing anomaly of such edge theories which arises under diffeomorphism transformations. There is a condition analogous to the Callan-Harvey mechanism which accounts for anomalies associated with diffeomorphism transformations of the gravitational Cherm-Simons terms we will consider. In particular, a manifold $\mathcal{M}$ with a boundary will not be diffeomorphism invariant unless the theory satisfies

$$k_\Omega = \frac{1}{2}\left(\tilde{n}_+ - \tilde{n}_-\right) \tag{3.21}$$

where k_Ω is the coefficient of gravitational Chern-Simons term, $-i\left(2\mathrm{CS}_{\mathrm{grav}}\right)$ of (1.21), and $\tilde{n}_\pm$ the number of right- and left-moving Majorana-Weyl fermions, respectively. Fortuantely, a single chiral Dirac fermion is equivalent to two Majorana-Weyl fermions, i.e. $n_\pm = 2\tilde{n}_\pm$ [112]. Hence, so long as $k = k_\Omega = \pm 1$, a single chiral fermion can render the theory non-anomalous for both the chiral and framing anomalies. In what follows, our calculations will be organized such that keeping track of (3.20) is completely equivalent to (3.21).

We will see that requiring our theories to be non-anomalous – such that (3.20) is satisfied – arranges for us the pieces laid out above into a working conjecture for Abelian dualities with a boundary. Furthermore, this counting will naturally appear as an organizational tool in the lattice construction in later sections.

Let us now take account of the possible edge modes that can appear in the context of bosonization dualities. To start, we will only consider matter fields. The scalars will never give rise to a chiral edge mode. For gapped fermions, we naturally get DWFs subject to the boundary conditions of (3.14), which select any possible surviving edge mode. As hinted by (3.20) these DWFs are intimately connected to Chern-Simons terms.

By the same reasoning that our gapped fermions give rise to DWFs, so too do the Pauli-Villars fields. We will always take the boundary conditions on the Pauli-Villars regulators to kill off the would-be DWF. If we do not kill off the Pauli-Villars DWF, this would give us massless ghosts localized to the boundary. This would be orthogonal to the Pauli-Villar's field original purpose, which was to regulate high energy degrees of freedom giving rise to the parity anomaly.

Consider a spin$_c$ valued connection, A, coupled to a heavy Dirac fermion, χ, and a heavy Pauli-Villars regulator, λ, with negative masses. Here we will take A to be a background field, but analogous results hold for dynamical spin$_c$ valued connections up to potential boundary conditions which we will discuss later. The Lagrangian terms generated by integrating out a heavy Dirac fermion is $-i\left(-\mathrm{CS}_1[A] - 2\mathrm{CS}_{\mathrm{grav}}\right)$. Furthermore from (3.19), χ gives rise to a DWF of negative chirality, and so we can satisfy (3.20) by imposing $\chi_+|_\partial = 0$ to leave the DWF unaffected. The same DWF is precisely the edge mode we also need to account for the framing anomaly. To remove the DWF associated with the Pauli-Villars regulator, we impose $\lambda_-|_\partial = 0$. Analogous results follow choosing positive mass Dirac fermions and Pauli-Villars regulators with a flipped Chern-Simons level and the opposite boundary conditions. Choosing the signs of the fermion and Pauli-Villars masses to be anti-aligned, the Chern-Simons terms cancel. Furthermore, both the fermion and Pauli-Villars boundary conditions prevent any DWFs from arising. As promised, for $k = \pm 1$ only one of the two possible fermionic boundary conditions yields a theory consistent with (3.20).

Returning to the IR boundary conditions on the gauge fields, we saw the Chern-Simons term gave us a chiral edge mode whose handedness was set by the sign by the velocity in (3.18) and hence by k. If this Chern-Simons term is generated by integrating out a massive fermion, the bosonic chiral edge mode from the gauge field can be understood as a $1+1$-dimensional bosonized DWF. Thus, the IR physics still retains some memory of the microscopic picture due to the gapless chiral edge mode furnished by the underlying DWF, which appropriately accounts for the anomalies. Together the massive fermions, Maxwell term for the gauge field, and chiral edge mode give a complete microscopic picture of the theory. This implies that (3.18) emerges from the boundary conditions imposed on the microscopic fermions.

In fact, we would like to promote this to an operating principle for how to deal with Chern-Simons terms when analyzing theories in the presence of boundaries. We want to view *all* spin$_c$ and gravitational Chern-Simons terms as being generated by integrating out massive fermions. This is the easiest way to get a consistent microscopic picture accounting for all the resulting boundary modes and anomaly inflows. In particular, this means we will have the following view of the Chern-Simons terms appearing in Abelian bosonization:

$-i\left(\mathbf{CS}_1[A]+2\mathbf{CS}_{\mathbf{grav}}\right)$ **Chern-Simons terms:** Pauli-Villars regulator and a free fermion with m_λ, $m_\chi > 0$ and the $\chi_-|_\partial = 0$ and $\lambda_+|_\partial = 0$ boundary conditions.

$-i\left(-\mathbf{CS}_1[A]-2\mathbf{CS}_{\mathbf{grav}}\right)$ **Chern-Simons terms:** Pauli-Villars regulator and a free fermion with m_λ, $m_\chi < 0$ and the $\chi_+|_\partial = 0$ and $\lambda_-|_\partial = 0$ boundary conditions.

The signs of the masses of the fermion and Pauli-Villars fields and their appropriate boundary conditions are completely determined by the sign of the Chern-Simons level. We will use this microscopic description both for Chern-Simons terms for dynamical spin$_c$ fields and for Chern-Simons terms associated with background spin$_c$ valued connections. For clarity, we will denote the fermions that appear in eqs. (3.5) and (3.6) as ψ and refer to them as "dynamical", while "fiducial" fermions χ refer to the microscopic description of the Chern-Simons term. More explicitly, we will view every Chern-Simon terms as arising from

$$e^{\pm\int d^3x\; -i(\mathrm{CS}_1[A]+2\mathrm{CS}_{\mathrm{grav}})} = \int \mathcal{D}\chi\,\mathcal{D}\lambda\; e^{-\int d^3x\;\mathcal{L}^{\pm}_{ff}[\chi,\lambda,A]}, \tag{3.22}$$

where

$$\mathcal{L}^{\pm}_{ff}[\chi,\lambda,A] = \lim_{|m_\chi|,|m_\lambda|\to\infty}\Big(i\bar{\chi}\not{D}_A\chi \pm |m_\chi|\bar{\chi}\chi + i\bar{\lambda}\not{D}_A\lambda \pm |m_\lambda|\bar{\lambda}\lambda\Big). \tag{3.23}$$

The superscript on fiducial fermion action denotes the sign of the fermion and Pauli-Villars masses[5] as well as the corresponding boundary conditions, $\chi_\mp|_\partial = \lambda_\pm|_\partial = 0$. As usual, we have chosen the convention that the fermionic mass term appears generically as $V(\psi) = +m_\psi\bar{\psi}\psi$.

[5]This is to be contrasted with our definition of $S^{\pm}_f[\psi,\lambda,A]$ for the dynamical fermions. The latter were massless to begin with and we always took the Pauli-Villars mass to be negative and large. The superscript in that case only referred to the boundary conditions.

The only difference between dynamical and background spin$_c$ valued connections is the possibility of imposing boundary conditions on the former. Since Dirichlet boundary conditions set the gauge field at the boundary to zero, imposing them will eliminate anomalous current flow onto the boundary from a dynamical Chern-Simons term. Hence, we do not need to put any additional chiral boundary modes to compensate for such currents. However, employing Dirichlet boundary conditions changes the boundary gauge symmetry to a global symmetry; thus, introducing a second global $U(1)$ symmetry into the theory. On the dual side, new boundary localized matter has to be added to account for this enhanced global symmetry. In this section, we will only consider Neumann boundary conditions on the dynamical gauge fields so that (3.20) needs to be satisfied for all types of gauge fields. Additional dualities with Dirichlet boundary conditions on gauge fields have been outlined in [50] and will be considered when we move to the non-Abelian case.

We will see that the boundary modes associated with the fiducial fermions will be crucial in developing a consistent picture of boundary modes. This is particularly interesting when the Chern-Simons terms involved describe only background fields. In this case the fiducial fermion still can contribute massless boundary modes, even though the Chern-Simons term does not involve any fluctuating fields. From the point of view of the low energy theory it appears that these fermionic boundary modes have to be added "by hand" in order for the duality to hold.

3.2.2 Dualities with Boundaries

We now turn to establishing three-dimensional bosonization and particle-vortex duality in the presence of a boundary. Our starting point is the conjecture that dualities (1.16) are valid on $\mathbb{R}^{2,1}_{+}$ provided the boundary conditions are correctly applied to dynamical and fiducial fermions. From this conjecture, we will also be able to establish a web of Abelian dualities – i.e. scalar-vortex and fermion-QED$_3$ – in the presence of boundaries. The derivation will give us a setting to establish checks between chiral degrees of freedom on the boundary and Chern-Simons levels such that (3.20) is satisfied at every step of the way. All partition functions in this and subsequent sections are understood to be defined on the half-space and distinct from their full-space equivalents.

Taking inspiration from the microscopic description of bosonization [30], the coupling of the dynamical field b to the background field A can alternatively be written

$$\mathrm{CS}_1[c] + \mathrm{BF}[c; A] = \mathrm{CS}_1[c + A] - \mathrm{CS}_1[A]. \tag{3.24}$$

We will see in later sections that rewriting (3.5) and (3.6) with only Chern-Simons terms will be useful in understanding edge modes.

Scalar+Flux = Fermion

Our conjecture for the form of the seed duality with a boundary starts with rewriting the flux attachment to Wilson-Fisher scalars using (3.24),

$$i\mathcal{L}_{\mathrm{WF+flux}} = i\mathcal{L}_{WF}[\Phi, c] + \mathrm{CS}_1[c + A] - \mathrm{CS}_1[A]. \tag{3.25}$$

In this form, the coupling of the statistical gauge field c to the background A can be understood entirely in terms of the microscopic fiducial description via heavy fermions:

$$e^{-\int d^3x\, -i(\mathrm{CS}_1[c+A]+2\mathrm{CS}_{\mathrm{grav}})} = \int \mathcal{D}\chi_1\, \mathcal{D}\lambda_1\; e^{-\int d^3x\;\mathcal{L}^{+}_{ff}[\chi_1, \lambda_1, c+A]}, \tag{3.26a}$$

$$e^{-\int d^3x\, -i(-\mathrm{CS}_1[A]-2\mathrm{CS}_{\mathrm{grav}})} = \int \mathcal{D}\chi_2\, \mathcal{D}\lambda_2\; e^{-\int d^3x\;\mathcal{L}^{-}_{ff}[\chi_2, \lambda_2, A]}, \tag{3.26b}$$

where once again the superscripts are chosen such that they generate the corresponding Chern-Simons terms appearing in (3.25). Implicit in the above expression is the fact the gravitational Chern-Simons terms coming from each of the fiducial fermions cancel,

$$(\mathrm{CS}_1[c + A] + 2\mathrm{CS}_{\mathrm{grav}}) + (-\mathrm{CS}_1[A] - 2\mathrm{CS}_{\mathrm{grav}}) = \mathrm{CS}_1[c + A] - \mathrm{CS}_1[A]. \tag{3.27}$$

This particular combination of Chern-Simons terms will be used many times in what follows.

We should reemphasize that this rewriting has actual content in the case of a theory with boundary: Even though A is a non-dynamical background gauge field, $\mathcal{L}^{-}_{ff}[\chi_2, \lambda_2, A]$

	Scalar + flux	**Fermion**
Boundary conditions	$\Phi = 0$	$\psi_+ = 0$
	$\partial_y c_i - \partial_i c_y = 0$	
Additional edge modes	Left-mover coupled to A	None
	Right-mover coupled to $c + A$	

Table 3.1: Summary of boundary conditions and additional edge movers for (3.28).

will give rise to massless chiral boundary modes associated with the fiducial fermion χ_2 despite working in the $|m_{\chi_2}| \to \infty$ limit. As noted above, from the perspective of the coarse-grained, Chern-Simons formulation of the theory in (3.25) these gapless edge modes appear to be added by hand.

The fermionic side of the duality (3.5) does not need any additional work: It is already in a form that makes the chiral edge modes obvious. We can simply apply the chiral boundary conditions on dynamical fermions ($\psi_+|_\partial = 0$) and Pauli-Villars regulator ($\lambda_-|_\partial = 0$). Our conjecture is then that

$$\begin{aligned} &\int \mathcal{D}\Phi\ \mathcal{D}c\ \prod_{j=1,2} \mathcal{D}\chi_j\, \mathcal{D}\lambda_j\ e^{-\int d^3x\ \mathcal{L}_{WF}[b] + \mathcal{L}^+_{ff}[\chi_1,\lambda_1, c+A] + \mathcal{L}^-_{ff}[\chi_2,\lambda_2,A]} \\ &\qquad\qquad \leftrightarrow \int \mathcal{D}\psi\ \mathcal{D}\lambda\, e^{-\int d^3x\ \mathcal{L}^-_f[\psi,\lambda,A]} \end{aligned} \tag{3.28}$$

holds as an equivalence at the conformal point. Additionally, we choose the dynamical gauge field to obey Neumann boundary conditions, $(\partial_y c_i - \partial_i c_y)|_\partial = 0$, and the scalar to obey the Dirichlet condition, $\Phi|_\partial = 0$. These results are summarized in Table 3.1.

In order to establish some guiding principle for the conjectured duality of CFTs, we can gap both theories and track whether our putative equivalence holds for positive and negative mass deformations. We will see the boundary conditions in our conjecture naturally arise by requiring the theory to be non-anomalous and have consistent global symmetries. We should find consistent dualities between theories in the bulk and on the boundary for positive and negative mass deformations away from the CFT.

Let us start with the free fermion side of (3.5). Making the mass deformation explicit, the action is given by the replacement

$$\mathcal{L}_f^+[\psi, \lambda, A] \to \mathcal{L}_f^+[\psi, \lambda, A] - m_\psi \bar{\psi}\psi \tag{3.29}$$

where $\psi_+|_\partial = 0$. In the IR limit of the theory, integrating out the massive degrees of freedom of the fermion yields

$$i\mathcal{L}_f = -\frac{1}{2}\left(1 - \text{sgn}(m_\psi)\right)\left(\text{CS}_1 + 2\text{CS}_{\text{grav}}\right) \qquad \text{(IR Limit)}. \tag{3.30}$$

When the Pauli-Villars field and the fermion have the same sign of mass, corresponding to a $-\text{CS}_1[A] - 2\text{CS}_{\text{grav}}$ Chern-Simons term, we need a single left-moving chiral edge mode to account for the anomalous term in order for this to be consistent with (3.20). Since $m_\psi < 0$, the DWF which arises from our analysis of Sec. 3.2.1 is exactly the anomaly cancelling edge mode we need. If instead we had imposed the condition $\psi_-|_\partial = 0$, then this would have suppressed the DWF. Hence, if we demand a non-anomalous theory, we are forced into choosing $\psi_+|_\partial = 0$.

We should now check to make sure everything is consistent for $m_\psi > 0$. In this case we get no ordinary or gravitational Chern-Simons terms and ψ's mass profile naturally gives rise to a right-moving DWF. It seems like we are in trouble. Fortunately, applying $\psi_+|_\partial = 0$ prevents any right-movers on the boundary. We are thus left with no chiral edge modes and (3.20) is satisfied for both signs of m_ψ.

For the Wilson-Fisher scalar with flux, introducing a mass deformation $m_\Phi^2 > 0$ corresponds to a gapped scalar. Flowing to the IR, the only term with c dependence is $\text{CS}_1[c+A]$. As reviewed in above and in appendix B of [112], this theory is completely determined by its framing anomaly and thus equal to $-2\text{CS}_{\text{grav}}$. This results in an overall $-\text{CS}_1[A] - 2\text{CS}_{\text{grav}}$ Chern-Simons term, consistent with the fermionic side when $m_\psi < 0$. We should also check that the anomaly inflow condition (3.20) is still satisfied on this side of the duality. It is here where our microscopic description of the Chern-Simons term in (3.26) will be important. Integrating out c caused the first Chern-Simons term to vanish leaving behind $-\mathrm{CS}_1[A] - 2\mathrm{CS}_{\mathrm{grav}}$. From the micrscopic perspective, this can be viewed as the condition

$$\int \mathcal{D}\chi\ \mathcal{D}\lambda\ \mathcal{D}c\ e^{-\int d^3x\ \mathcal{L}^{\pm}_{ff}[\chi,\lambda,c+A]} = 1. \tag{3.31}$$

That is, the fiducial fermions provide no ordinary or gravitational Chern-Simons terms as well as no corresponding edge movers. Per our prescription, the remaining fiducial fermion associated with $-\mathrm{CS}_1[A]-2\mathrm{CS}_{\mathrm{grav}}$ has the correct mass profile and boundary condition such that it contains a left-moving DWF. Thus, (3.20) is satisfied.

To complete our discussion of massive phases we need to check that everything is consistent when $m_\Phi^2 < 0$. This gives a negative mass squared term in $V(\Phi)$, spontaneously breaking the emergent $U(1)$ in the scalar theory. This kills off the Chern-Simons term for c, and so integrating out Φ and c leaves behind no Chern-Simons terms. As expected, this means that the IR theory in the Higgs phase is identical to the 'vacuum' region. When $c = 0$, the edge modes of the fiducial fermions associated with $\mathrm{CS}_1[c + A]$ and $-\mathrm{CS}_1[A]$ have the same gauge coupling but opposite chiralities, and hence cancel one another. Since no Chern-Simons terms or fermions are left behind, there are no possible chiral modes that can arise and make this theory anomalous. Hence, we have found a consistent story for the duality on either side of the mass deformation.

That last step is to see if the scalar boundary conditions is constrained. To do so, we rely on our identification of global symmetry currents on either side of the duality, (3.8). For this purpose, it becomes useful to reinterpret the cancellation of the anomaly from (3.20) in a slightly different, but equivalent, language. The Chern-Simons term of the bulk is anomalous on its own under the global $U(1)$ topological symmetry because the corresponding current has a nonzero divergence at the boundary. This seems to imply that the symmetry is broken at the boundary. However, the Chern-Simons anomaly is compensated via the axial $U(1)$ symmetry of the DWFs, and hence the theory is non-anomalous under a simultaneous topological $U(1)$ transformation in the bulk and the axial $U(1)$ transformation on the DWFs. If the two symmetries are identified, the global topological $U(1)$ symmetry is restored on the boundary by the transformation of the DWFs and is unbroken everywhere. This is in agreement with the fermion side of the duality where the global $U(1)$ symmetry of particle number is unbroken in the bulk and on the boundary.

Returning to the constraints on the boundary condition of the scalar, recall that the equations of motion for the scalar and Chern-Simons term tie the matter current to the topological current,

$$j^{\mu}_{\text{flux}} \equiv \frac{k}{2\pi}\epsilon^{\mu\nu\rho}\partial_{\nu}c_{\rho} = -j^{\mu}_{\text{scalar}}. \tag{3.32}$$

Here, j^{μ}_{scalar} is the usual scalar matter current and we have temporarily set the background fields to zero. However, as we have argued above, on the boundary it is not the flux which accounts for the topological $U(1)$ symmetry, but the DWFs. Hence, we should have $j^{i}_{\text{flux}}|_{\partial} = 0$ and by (3.32) should also take $j^{i}_{\text{scalar}}|_{\partial} = 0$. Such a condition on the scalar current can only be achieved by Dirichlet boundary conditions, $\Phi|_{\partial} = 0$. Dirichlet boundary conditions are usually referred to as the "ordinary transition" boundary conditions of the $O(2)$ Wilson-Fisher fixed point. See [91] for a recent discussion.

The above constructions leads us to conjecture what happens to the DWFs at the conformal fixed point: As the mass deformation becomes smaller, according to (3.19) the DWF becomes less and less localized to the boundary. In the massless limit, the DWF recombines with a DWF of opposite chirality living on – in the case of a finite interval $y \in [0, L]$ – the other boundary. Note that on the semi-infinite interval that we have used for $\mathbb{R}^{2,1}_{+}$, the oppositely chiral fermion is not explicitly seen as the boundary condition at $y = L$ is replaced by a condition on the asymptotic behavior of the matter fields. At the conformal fixed point, we then have an ordinary Dirac fermion which lives in the bulk.

	Fermion + flux	Scalar
Boundary conditions	$\Psi_- = 0$	$\phi = 0$
	$\partial_y a_i - \partial_i a_y = 0$	
Additional edge modes	Left-mover coupled to $a + B$	None
	Right-mover coupled to a	

Table 3.2: Summary of boundary conditions and additional edge movers for (3.33).

Fermion+Flux = Scalar

Having established a set of conventions in the first seed duality in the presence of a boundary, we can carry the above notation through into the second seed duality. Our conjecture is that

$$\begin{aligned} &\int \mathcal{D}\Psi\ \mathcal{D}a\ \mathcal{D}\lambda_0 \prod_{j=1,2} \mathcal{D}\chi_j\, \mathcal{D}\lambda_j\ e^{-\int d^3x\ \mathcal{L}_f^+[\Psi,\lambda_0,a]+\mathcal{L}_{ff}^-[\chi_1,\lambda_1,a+B]+\mathcal{L}_{ff}^+[\chi_2,\lambda_2,a]} \\ &\qquad\qquad \leftrightarrow \int \mathcal{D}\phi\ e^{-\int d^3x\ \mathcal{L}_{\mathrm{WF}}[\phi,B]} \end{aligned} \tag{3.33}$$

holds as an equivalence at the conformal point. Once more, we have imposed Neumann boundary conditions on the dynamical gauge field a and Dirichlet boundary conditions on the scalar. These results are summarized in Table 3.2. We should recall the procedure that maps from (3.5) to (3.6) and make sure that it is consistent with our boundary picture.

In the bulk, this duality can be derived from the first seed duality by promoting the background spin$_c$ valued connection A to a dynamical field, a, introducing an ordinary background $U(1)$ field B, and adding $-i\,(-\mathrm{BF}[a;B] - \mathrm{CS}_1[B])$ to the Lagrangian. Looking first at the scalar side of this procedure and starting with (3.25), it becomes useful to define a new recipe for moving from the first seed duality to the second in the presence of a boundary by rewriting the BF term:

> **New Promotion:** Promote A to a dynamical field, a, introduce a new background field B, and add $-i\,(\mathrm{CS}_1[a] - \mathrm{CS}_1[a+B])$ to the Lagrangian.

The Chern-Simons terms should be understood throughout the process in their microscopic descriptions with appropriate boundary conditions such that they give rise to chiral modes on the boundary to satisfy (3.20). Once more, we have introduced the combination whose gravitational Chern-Simons terms cancel one another. Note the old and new promotions are completely equivalent in the bulk where there are no surface terms from integration by parts or chiral modes to consider on the boundary.

Applying this procedure to (3.25) gives

$$i\mathcal{L} = i\mathcal{L}_{WF}[\Phi, c] + \mathrm{CS}_1[c+a] - \mathrm{CS}_1[a+B]. \tag{3.34}$$

For brevity, we will leave the process of rewriting Chern-Simons terms as fermion and Pauli-Villars fields as implied moving forward. When integrating out the dynamical fields, we find in the absence of holonomies, an assumption we will always make from now on, $0 = c + a$, $0 = a + B$, and thus $c = -a = B$.

With the methods we used in the first seed duality, it is straightforward to establish a duality between non-anomalous theories in the second. After integrating out the dynamical fields, there are no ordinary or gravitational Chern-Simons terms left over for either mass deformation. This is easiest to understand on the scalar side. There are no Chern-Simons terms present regardless of the mass deformation, and hence, there are no edge movers required for the theory to be non-anomalous. Since the scalar fields give rise to no chiral edge modes, we are consistent with (3.20).

Following our process for promotion for the free fermion gives

$$i\mathcal{L}_{\text{fermion+flux}} = i\mathcal{L}_f[\Psi,\, a] + \mathrm{CS}_1[a] - \mathrm{CS}_1[a+B]. \tag{3.35}$$

In the IR limit, integrating out the fermion gives

$$i\mathcal{L}_{\text{fermion+flux}} = -\frac{1}{2}\left(1-\mathrm{sgn}(m_\Psi)\right)\left(\mathrm{CS}_1[a] + 2\mathrm{CS}_{\text{grav}}\right) - \mathrm{CS}_1[a+B] + \mathrm{CS}_1[a]. \tag{3.36}$$

For $m_\Psi > 0$, integrating out the fermion gives no Chern-Simons terms, and the equations of motion for a imply $B = 0$; leaving behind no Chern-Simons terms and the edge modes of the fiducial fermions exactly cancel. For $m_\Psi < 0$, the first and last Chern-Simons terms and DWFs cancel and we are left with a $-\mathrm{CS}_1[a+B] - 2\mathrm{CS}_{\mathrm{grav}}$. Here we can again find a theory completely determined by its framing anomaly and hence it can be replaced by $+2\mathrm{CS}_{\mathrm{grav}}$.[6] Microscopically, this amounts to

$$\int \mathcal{D}\chi\ \mathcal{D}\lambda\ \mathcal{D}a\ e^{-\int d^3x\ \mathcal{L}^{\pm}_{ff}[\chi,\lambda,a+B]} = 1. \tag{3.37}$$

This leaves behind no ordinary or gravitational Chern-Simons terms and hence no edge modes are left behind. Thus, we find that after integrating out the dynamical degrees of freedom requiring the absence of anomalies for each of the Chern-Simons terms individually gives us a consistent theory.

Note that the fiducial fermion picture may not seem strictly necessary in this duality since there are no nonzero Chern-Simons terms from mass deformations and hence no edge movers are necessary to make the theory non-anomalous. However, the fiducial fermions *do* play an integral role in the above analysis since they cancel the would-be dynamical DWF, which cannot be eliminated without additional edge movers.

As with the first duality, imposing boundary conditions on the scalar requires a closer look at the global symmetry currents. Choosing Neumann boundary conditions on the dynamical gauge field a implies a constraint to field configurations which obey $(\partial_y a_i - \partial_i a_y)|_\partial = 0$. This also means the topological current parallel to the boundary vanishes, since $j^i_{\mathrm{flux}} \propto \partial_y a_i - \partial_i a_y$. Since this topological current should be identified with the particle number current on the scalar side of the duality, consistency requires $j^i_{\mathrm{scalar}}|_\partial = 0$. Again, this can only be achieved by imposing Dirichlet boundary conditions on the scalar.

Lastly, one can easily check consistency of the above prescriptions by applying the promotions again to get back to the first seed duality The only subtlety is the sign of all the Chern-Simons terms in the promotion need to be flipped. This means that our prescription is to promote B to a dynamical field in (3.33), introduce a new background field A,

[6]This follows in an analogous manner to (3.31). To see this, rewrite the dynamical spin$_c$ valued connection as the sum of a background spin$_c$ valued connection and a dynamical $U(1)$ connection $a = c + A$. Then, we can simply shift away the extra B to recover the usual expression.

add $-i\left(\mathrm{CS}_1[b+A]-\mathrm{CS}_1[A]\right)$ to the Lagrangian, and integrate out the dynamical fields. Following this through, we are left with the appropriate chiral modes for the remaining Chern-Simons terms to satisfy (3.20).

Time-reversed dualities

The time-reversed version of the seed dualities follow in a completely analogous manner. Since the Chern-Simons terms are time-reversal odd, in order to satisfy (3.20) we also need to swap the chiralities of the fermionic boundary terms. Additionally, due to our η-invariant convention we need to add $-i\left(\mathrm{CS}_1[A]+2\mathrm{CS}_{\mathrm{grav}}\right)$, with A the field which the fermion is coupled to. Other than the minor consistency check required by the fermionic and Pauli-Villars boundary conditions, the time-reversed analogs of (3.5) and (3.6) are

$$\begin{aligned}\bar{\mathcal{L}}_{\mathrm{WF+flux}} \equiv \mathcal{L}_{WF}[\Phi,\, c] - i\left(-\mathrm{CS}_1[c+A]+\mathrm{CS}_1[A]\right)&\\ \leftrightarrow \qquad \bar{\mathcal{L}}_f^{+}[\psi,\lambda,\, A] - i\left(\mathrm{CS}_1[A]+2\mathrm{CS}_{\mathrm{grav}}\right),&\end{aligned} \tag{3.38}$$

and

$$\bar{\mathcal{L}}_{\mathrm{f+flux}} \equiv \bar{\mathcal{L}}_f^{-}[\Psi,\lambda,\, a] - i\left(\mathrm{CS}_1[a+B]+2\mathrm{CS}_{\mathrm{grav}}\right) \qquad \leftrightarrow \qquad \mathcal{L}_{WF}[\phi,\, B]. \tag{3.39}$$

As in the previous versions of the dualities, we can simply identify the correct number of boundary modes needed to ensure the absence of anomalies by looking at the sign and level of the Chern-Simons term directly.

Scalar-Vortex duality

Moving deeper into the web of dualities in [83, 112], we will start with finding the influence of a boundary on

$$\overline{\mathcal{L}}_{\mathrm{WF}}[C] \leftrightarrow \mathcal{L}_{\text{scalar-QED}}[C]. \tag{3.40}$$

Beginning with (3.33), this duality is derived by promoting B to be dynamical, introducing a new background field C, and adding $-\mathrm{CS}_1[b]+\mathrm{CS}_1[b+C]-\mathrm{CS}_1[C]$ Chern-Simons terms to both sides of the duality. Note these terms are equivalent to $\text{BF}[b;C]$ in the absence of boundaries. However, there would appear to be an issue of applying our fiducial fermion prescription to this duality. That is we have Chern-Simons terms of ordinary $U(1)$ – rather than spin$_c$ valued – connections.[7] The coupling of the fiducial fermions to such fields violates the relation forced by (B.4) discussed in Appendix B.1. However, we can work around that by rewriting the BF term including a spin$_c$ valued connection as [60]

$$\text{BF}[b;C] = \text{CS}_1[b+C+A] - \text{CS}_1[b+A] - \text{CS}_1[C+A] + \text{CS}_1[A]. \tag{3.41}$$

Note that all of the gravitational Chern-Simons terms that would have accompanied each CS_1 on the right hand side of (3.41) cancel and have thus been ignored. Now, the promotion of the ordinary background connection, $B \to b$, and the subsequent coupling to another ordinary background connection C can be realized as a system of four fiducial fermions in the usual way.

Proceeding with the prescription, the scalar side of the duality becomes

$$i\mathcal{L}_{\text{scalar-QED}}[C] = i\mathcal{L}_{WF}[\phi, b] + \text{CS}_1[b+C+A] - \text{CS}_1[b+A] - \text{CS}_1[C+A] + \text{CS}_1[A]. \tag{3.42}$$

The analysis of Chern-Simons terms and edge modes follows in a similar fashion to the WF + flux case. In the phase where the scalar is massive, the equations of motion for b imply $C = 0$, which causes the four Chern-Simons terms and associated edge modes cancel. In the Higgsed phase, $b = 0$, and once more all Chern-Simons terms cancel and there are no edge modes. The modified fermionic theory is

$$\begin{aligned} i\mathcal{L}_{f'}[C] =& i\mathcal{L}_f^+[\Psi, \lambda, a] + \text{CS}_1[a] - \text{CS}_1[a+b] + \text{CS}_1[b+C+A] \\ &- \text{CS}_1[b+A] - \text{CS}_1[C+A] + \text{CS}_1[A]. \end{aligned} \tag{3.43}$$

Integrating out b implies $b = C - a$ and plugging this back into the above expression yields

$$\mathcal{L}_f^+[\Psi, \lambda, a] - i\left(\text{CS}_1[a-C] + 2\text{CS}_{\text{grav}}\right). \tag{3.44}$$

[7] Recall, a $U(1)$ Chern-Simons term is well defined modulo $\pi\mathbb{Z}$ in general. It is only picking a spin structure that makes it well defined modulo $2\pi\mathbb{Z}$.

Up to the sign of the mass terms, the two terms in the Lagrangian of (3.44) are exactly the time-reversed alternate seed duality, (3.39), with $B \to -C$, so that

$$\mathcal{L}_{\mathrm{f}'}[C] = \overline{\mathcal{L}}_{\mathrm{f+flux}}[-C] \leftrightarrow \overline{\mathcal{L}}_{\mathrm{WF}}[C]. \tag{3.45}$$

This confirms the desired relation in (3.40). This is consistent with the scalar-QED side of the duality.

There is one caveat to the use of the time reversed duality connected to our use of $\overline{\mathcal{L}}_{\mathrm{WF}}$ rather than $\mathcal{L}_{\mathrm{WF}}$. The time reversal operation changes the sign on the fermion mass term. This has the effect of flipping the relationship between the way mass deformations in the two scalar theories are mapped to one another: positive mass deformations in $\overline{\mathcal{L}}_{\mathrm{WF}}$ correspond to *negative* mass deformations in $\mathcal{L}_{\text{scalar-QED}}$.However, at the conformal fixed point $\overline{\mathcal{L}}_{\mathrm{WF}}$ is completely equivalent to $\mathcal{L}_{\mathrm{WF}}$. This is a nice check, since it reproduces the equivalence $m_\phi^2 \leftrightarrow -m_\Phi^2$ on the two sides of the bosonic particle-vortex duality.

Fermion-Vortex duality

The last duality we will consider in the presence of a boundary is the fermionic particle-vortex duality, which has some additional nuances. This duality,

$$\overline{\mathcal{L}}_{\mathrm{f}}[A] - i\left(\frac{1}{2}\mathrm{CS}_1[A]\right) \qquad \leftrightarrow \qquad \mathcal{L}_{\mathrm{QED}_3}[A], \tag{3.46}$$

was originally formulated with theories which are $\mathcal{T}$-invariant on both sides, similar to the bosonic case [119].

Recall that with our definition of $\mathcal{L}_f$ in (3.7b) this partition function contains the contribution of the negative mass, heavy Pauli-Villars field λ. Often the regulator is treated as producing a level -1/2 Chern-Simons term when integrated out. More precisely, we get the η-invariant of A. This factor means that $\mathcal{L}_{\mathrm{f}}$ is not time reversal invariant: $m_\lambda \to -m_\lambda$. The purpose of the the additional level 1/2 in (3.46) is to cancel the η-invariant and produce a time-reversal invariant fermionic partition function. However, from our normalization of the Chern-Simons term we require that $k \in \mathbb{Z}$ for the Chern-Simons term to be gauge-invariant.

Thus, multiplying with half-integer Chern-Simons terms is not a consistent procedure in a purely $2+1$ dimensional theory. To avoid this issue, this term can be viewed as arising as a boundary insertion in a theory on a $3+1$ dimensional bulk manifold, X [95, 126, 112]. More precisely, one promotes A to a spin$_c$ valued connection on X and adds

$$\frac{1}{8\pi}\int_X dA \wedge dA \tag{3.47}$$

to the Lagrangian. This promotion of A to a spin$_c$ valued connection is possible for any (orientable) choice of bulk X as all such $3+1$ dimensional manifolds admit a spin$_c$ structure. This cancels the contribution of the regulator; rendering the fermionic partition function real and both sides of the duality time-reversal invariant. All of this is perfectly valid in the $2+1$ dimensional bulk, but in the present context – where $\mathbb{R}^{2,1}_+$ would need to be realized as a boundary surface – this prescription fails. Indeed, had we proceeded through with adding $-i\left(\frac{1}{2}\text{CS}_1[A]\right)$ to $\mathcal{L}_\text{f}$ as in [83], we would have found the Chern-Simons levels of $\pm\frac{1}{2}$ on either side of the mass deformation. This is a clear contradiction with the assertion that the boundary is non-anomalous: We cannot generate "half" a DWF to satisfy (3.20).

Thus we find that in order to have a purely $2+1$ dimensional description of fermionic particle-vortex duality, we must either abandon time-reversal invariance at the conformal fixed point or find some other means of canceling the η-invariant of A.

Let us first explore what happens when we give up time reversal invariance. It is no longer necessary to transfer the $k=\frac{1}{2}$ Chern-Simons term from one side of the duality to the other. In this case, it will be convenient to begin our derivation with (3.33). We then promote the background field to be dynamical, $B \to b$, and couple to a new background spin$_c$ valued connection A via $-i\left(\text{CS}_1[b+A]+\text{CS}_1[A]\right)$, the fermion+flux side is

$$i\mathcal{L}_{\text{QED}_3'}[A] = i\mathcal{L}_f^+[\Psi,\lambda,a] - \text{CS}_1[a+b] + \text{CS}_1[a] - \text{CS}_1[b+A] + \text{CS}_1[A]. \tag{3.48}$$

where the prime is being used to distinguish this from $\mathcal{T}$-invariant QED_3. We proceed as usual in the IR limit and integrate out the dynamical fields a and b.[8] For $m_\Psi > 0$ we find

[8]More precisely, we must integrate out a before b to avoid imposing conditions which violate the spin-

no Chern-Simons terms, while for $m_\Psi < 0$ we find $\mathrm{CS}_1[A] + 2\mathrm{CS}_{\mathrm{grav}}$. The fiducial fermion associated with $\mathrm{CS}_1[A]$ provides the necessary right-mover.

Meanwhile, the scalar side yields

$$i\mathcal{L}_{\mathrm{scalar}'}[A] = i\mathcal{L}_{\mathrm{WF}}[\phi,\, b] - \mathrm{CS}_1[b + A] + \mathrm{CS}_1[A]. \tag{3.49}$$

However, we recognize this as the time-reversed first seed duality, (3.38). This ultimately gives

$$\mathcal{L}_{\mathrm{QED}_3'}[A] \leftrightarrow \overline{\mathcal{L}}_{\mathrm{f}}[A] - i\left(\mathrm{CS}_1[A] + 2\mathrm{CS}_{\mathrm{grav}}\right). \tag{3.50}$$

Again, we end up with level-0 and 1 ordinary and gravitational Chern-Simons terms on either side of the mass deformation. This time, the dynamical fermion can provide consistent chiral edge modes satisfying (3.20).

The other way to proceed is to insist on time-reversal invariance at the fixed point and doubly quantize the fields to avoid issues associated with half-integer Chern-Simons terms. With this redefinition of our fields, cancelling the $\mathcal{T}$-violating η-invariant term can be achieved with a term which meets the quantization requirements of Chern-Simons terms. However, taking $A = 2A'$ for some new spin$_c$ valued connection A' is in violation of the spin-charge relation, which would mean such an effective theory is not relevant to usual condensed matter systems [112, 114].

Following similar steps to that above, we find

$$\begin{aligned} \mathcal{L}_{\mathrm{QED}_3''}[A] \equiv \mathcal{L}_f^+[\Psi, \lambda, 2a] - i\left(-2\mathrm{CS}_1[a] + 2\mathrm{CS}_1[a + A] - 2\mathrm{CS}_1[A]\right) \\ \leftrightarrow \qquad \bar{\mathcal{L}}_f^+[\Psi, \lambda,\, 2A] = \bar{\mathcal{L}}_{\mathrm{f}}[2A]. \end{aligned} \tag{3.51}$$

It is straightforward to show edge movers are consistent with (3.20) with an ordinary $U(1)$ connection fiducial fermion prescription, analogous to (3.22),

$$e^{\pm \int d^3x\, -i(\mathrm{CS}_1[B])} = \int \mathcal{D}\chi\, \mathcal{D}\lambda\; e^{-\int d^3x\, \mathcal{L}_{ff}^{\pm}[\chi, \lambda, B]}. \tag{3.52}$$

charge relation of our connections, i.e. imposing $2b = -a - A$ [112]. The same condition prevents us from simplifying (3.48) by integrating out b.

One needs to keep in mind the double gauge field coupling causes the edge modes to contribute double the anomalous current, but this is still compensated by the Chern-Simons current inflow.

3.2.3 Lattice construction

In this section, we will build on recent work that realized the Abelian dualities in [83, 112] using exact techniques. We will consider the complex XY model on a Euclidean cubic lattice in $d = 3$ as in [30]. We will introduce a boundary to this formalism in order to find the microscopic description of one of the dualities described in Sec. 3.2.2, the claim that scalars with flux are equivalent to a theory of fermions.

Our conventions for the lattice will be that the matter living at lattice sites are denoted by a subscript n and the link variables are labeled by $n\mu$ designated to mean pointing from site n in the direction $\hat{\mu}$. A boundary will be implemented by simply truncating the lattice in the y-direction, rendering it semi-infinite. We use the index β for sites on the boundary. Link variables transverse and parallel to the boundary will be denoted by βy and $\beta i \in \{\beta t, \beta x\}$, respectively.

To realize the scalar + flux theory, we start with the XY model for a complex scalar living at lattice site n, $\Phi_n \sim e^{i\theta_n}$ given in terms of a set of phase variables $\theta_n \in [0, 2\pi)$ and background $U(1)$ gauge fields living on links $A_{n\mu}$ by

$$\mathcal{Z}_{\mathrm{XY}}[A] = \Big(\prod_n \int_{-\pi}^{\pi} \frac{d\theta_n}{2\pi}\Big) \exp\Big\{\frac{1}{T}\sum_{n\mu} \cos(\theta_{n+\hat{\mu}} - \theta_n - A_{n\mu})\Big\} \equiv \int \mathcal{D}\theta \; e^{-\frac{1}{T}H_{XY}[A]}. \tag{3.53}$$

To generate the necessary Chern-Simons term, we will employ the trick of coupling (3.53) to two-component Grassmann fields χ_n and $\bar{\chi}_n$. This is equivalent to our fiducial fermion prescription in the continuum case. The fermionic sector of the theory is given by

$$\mathcal{Z}_{\mathrm{W}}[A] = \prod_n \int d^2\bar{\chi}_n d^2\chi_n \; e^{-H_W[A](M) - H_{\mathrm{int}}(U)}, \tag{3.54}$$

where the Wilson action H_W and hopping-hopping interaction H_{int} are

$$-H_W[A](M) = \sum_{n\mu} \left(D_{n\mu} e^{-iA_{n\mu}} + D^*_{n\mu} e^{iA_{n\mu}} \right) + \sum_n (M - R) \bar{\chi}_n \chi_n, \tag{3.55a}$$

$$-H_{\text{int}}(U) = U \sum_{n\mu} D_{n\mu} D^*_{n\mu}. \tag{3.55b}$$

with $D_{n\mu}$ and $D^*_{n\mu}$ the fermionic forward and backward hopping terms, respectively

$$D_{n\mu} \equiv \left(\bar{\chi}_n \frac{\sigma^\mu + R}{2} \chi_{n+\hat{\mu}} \right), \qquad D^*_{n\mu} \equiv \left(\bar{\chi}_{n+\hat{\mu}} \frac{-\sigma^\mu + R}{2} \chi_n \right). \tag{3.56}$$

This particular form of H_{int} is chosen in [30] to reproduce the known continuum results. Similar to the continuum theory, integrating out these Wilson fermions will produce the Chern-Simons term. However, as a consequence of fermion doublers, the level of the resulting Chern-Simons theory is dependent on the relative magnitudes of M and the Wilson term, R, as well as the sign of R. Compiling the above components of the theory and including the analog of the dynamical $U(1)$ gauge field present in the continuum theory, the scalar coupled to flux is

$$\mathcal{Z}[A] = \int \mathcal{D}a \; \mathcal{Z}_{\text{XY}}[a] \, \mathcal{Z}_{\text{W}}[A - a], \qquad \int \mathcal{D}a \equiv \prod_{n\mu} \int_{-\pi}^{\pi} \frac{da_{n\mu}}{2\pi}. \tag{3.57}$$

For the remainder of this section, we will assume $|R| = 1$, which is motivated by reflection positivity. Additionally, we assume we have chosen T, $U \lesssim 0$, and $M \lesssim 6$ in order to hit the IR critical point, as explained in [30].[9] That is, these values are tuned such the theory (3.57) flows in the IR to

$$\mathcal{Z}_{\text{W}}[A] = \prod_n \int d^2\bar{\chi}_n d^2\chi_n \; e^{-H_W[A](M') - H_{\text{int}}(U')}, \tag{3.58}$$

with $M' = 6$ and $U' = 0$.

[9]We have chosen to define (3.55a) and (3.55b) such that it matches [55, 67, 117, 76] and thus differs slightly from that of [30]. To translate back, take $(M - 3R) \to M$ and then $R \to -R$.

Boundary conditions

To study the effect of the presence of a boundary on (3.57), we need to understand how boundary conditions come about on the site and link variables. We will start with the scalar fields, Φ_β. Ideally, we would have a direct analogy to the continuum case where either Neumann or Dirichlet boundary conditions are possible. The former can be implemented by requiring the scalar hopping terms perpendicular to the boundary vanish. However, due to our construction of scalar fields as having magnitude one, $\Phi_n \sim e^{i\theta_n}$, it is not actually possible to enforce Dirichlet boundary conditions, i.e. $\Phi_\beta = 0$. Instead, we will enforce Dirichlet boundary conditions by requiring the scalar current along the boundary to be zero.

The fermionic boundary conditions are such that either

$$P_+\chi_\beta = \bar{\chi}_\beta P_- = 0, \qquad \text{or} \qquad P_-\chi_\beta = \bar{\chi}_\beta P_+ = 0, \tag{3.59}$$

extremize the boundary variation term [118]. We will use as our convention $\sigma^{\hat{y}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ such that the chiral projectors in (3.59) are $P_\pm = \frac{1}{2}(1 \pm \sigma^{\hat{y}})$. From the assumption that $|R| = 1$ and up to a sign, the chiral projectors are equivalent to the matrices $\frac{1}{2}(\pm\sigma^{\hat{y}} - R)$ appearing in the fermionic hopping terms perpendicular to the boundary in (3.55a) and (3.55b). Either of the conditions in (3.59) will remove one chiral mode worth of degrees of freedom, while the other chiral mode is left unconstrained. These conditions can be compared to those in (3.14) and be seen to agree – albeit by construction [118].

Lastly, we need to consider the link variables. We again draw inspiration for the appropriate boundary conditions from the continuum case. That is, Neumann boundary conditions correspond to the condition that plaquettes perpendicular and adjacent to the boundary must vanish. On the lattice, this will correspond to the constraint

$$a_{\beta i} + a_{(\beta+\hat{\imath})y} - a_{(\beta+\hat{y})i} - a_{\beta y} = 0. \tag{3.60}$$

Alternatively, we could choose Dirichlet boundary conditions which simply require $a_{\beta i} = 0$. We would like to reproduce the results of the continuum duality (3.28) and this will guide us in choosing the corresponding boundary conditions on the lattice.

Implementation

The main results of ref. [30] – following the choice of hopping-hopping interaction H_{int} – are contained in the identification of a suitable UV map for the conserved currents built out of θ_n and $a_{n\mu}$ into a theory of free fermions. Those theories are then flowed to the IR where one can then compare to continuum results. Following these general principles, we identify the effects of truncating the lattice at some arbitrary boundary site. We will show the derivation of [30] holds in the presence of a truncated boundary and is non-anomalous for $M > 0$ so long as $R = 1$ and the $P_{-}\chi_\beta = \bar{\chi}_\beta P_{+} = 0$ boundary condition is chosen. We will also verify mass deformations away from the conformal fixed point yield equivalent results to the continuum case.

Recall the existence of a DWF at the boundary was of particular importance in our continuum picture for self-consistency checks away from the conformal point. A truncated lattice also gives rise to massless chiral modes localized to the boundary [117]. In particular, there are fermionic modes obeying

$$\Psi_{\pm}(x,y,t) = \xi(y)(1 \pm \sigma^{\hat{y}})\psi_{\pm}, \qquad \xi(y) \equiv [1 - F^2(k_\mu)]^{\frac{1}{2}}[F(k_\mu)]^y \tag{3.61}$$

with $\psi_{\pm}$ a right/left helicity eigenstate and

$$F(k_i) = R - M + R\sum_{i=t,x}(1 - \cos k_i). \tag{3.62}$$

For a given k_i, this solution can be normalized only if $|F(k_i)| < 1$ [117]. At the limit $|F(k_i)| = 1$ the DWF becomes a continuum eigenstate.

Now let's turn to the derivation of the duality. We will follow the derivation of ref. [30] and point out where subtleties of the boundary come into play. To begin, rewrite the bosonic hopping term to make the bosonic currents explicit

$$e^{\frac{1}{T}\cos(\theta_{\beta+\hat{\mu}} - \theta_\beta - a_{\beta\mu})} = \sum_{j_{\beta\mu}=-\infty}^{\infty} I_{\beta_\mu}(T^{-1})\Phi_\beta\Phi^*_{\beta+\hat{\mu}}e^{-ia_{\beta\mu}j_{\beta\mu}}, \tag{3.63}$$

where I_j is the j^{th} modified Bessel function. As mentioned in the previous section, we will enforce Dirichlet boundary conditions on the scalar by requiring the scalar current in the boundary to vanish, i.e. $j_{\beta i} = 0$. The bosonic degrees of freedom can be integrated out explicitly and this simply enforces Gauss's law for the scalar currents at the boundary sites. By current conservation, this implies current onto the boundary also vanishes, $j_{\beta y} = 0$.

The implementation of boundary conditions for the Grassmann variable and their effect on eqs. (3.55a) and (3.55b) is more subtle. In the continuum case, one of these boundary conditions will kill off the DWF on the boundary, while the other will leave it untouched. This had important implications relating to the anomalous nature of the theory. Is this feature also realized in the lattice? To see this is still consistent with the Callan-Harvey mechanism on either side of the mass deformation, we need to take a closer look at the interplay between Chern-Simons terms and DWFs on the semi-infinite lattice.

On the lattice, the Chern-Simons term is determined by the masses of the $2^3 = 8$ chiral Dirac fermion modes in the continuum. These correspond to the eight extrema of the Brillouin zone at $k_{t,x,y} = \{0, \pi\}$. The effective masses of these eights modes are determined by [55, 30]

$$m_{\text{eff}}(k_\mu) = M - R \sum_\mu (1 - \cos k_\mu). \tag{3.64}$$

Since the value of R is important in eqs. (3.62) and (3.64), we should see if we can first fix its sign. Recall that it is the current of the Chern-Simons term flowing onto the boundary which renders the theory non-anomalous. This current is nonzero only when the R and M in (3.55b) have the same sign [55]. Hence, given our choice of $M > 0$ we must take $R = 1$ to allow for anomaly inflow.

From (3.56), the choice of $R = 1$ has the effect of projecting onto the right-moving chiral mode for hopping terms perpendicular to the boundary. For reasons that will become clear shortly, the correct fermionic boundary condition to choose in this case is $P_- \chi_\beta = \bar{\chi}_\beta P_+ = 0$. Together with the choice of R, this implies $D^*_{\beta y} \neq 0$ and $D_{\beta y} \neq 0$ in general. Had we chosen the opposite boundary conditions or $R = -1$ we would have found no current flow onto the boundary.

With R fixed, the value of M – or equivalently, M' of (3.54) – determines both the

Brillouin Zone	**Chirality**	**Mass Parameter**		
Extremum, k_μ		$M = 6 - \epsilon$	$M = 6$	$M = 6 + \epsilon$
$(0, 0, 0)$	R	$+$	$+$	$+$
$(0, 0, \pi)$	L	$-$	$-$	$-$
$(0, \pi, 0)$	L	$-$	$-$	$-$
$(\pi, 0, 0)$	L	$-$	$-$	$-$
$(0, \pi, \pi)$	R	$+$	$+$	$+$
$(\pi, \pi, 0)$	R	$+^*$	$+$	$+$
$(\pi, 0, \pi)$	R	$+$	$+$	$+$
(π, π, π)	L	$+^*$	0	$-$
Total CS Level		1	$\frac{1}{2}$	0
Total DWF		one R, one L	none	none

Table 3.3: Chirality, mass, and existence of a DWF for the eight modes at the extremum of the Brillouin zone, $k_\mu = (k_t, k_x, k_y)$, as calculated using eqs. (3.62) and (3.64). Positive and negative masses are denoted by a $+$ and $-$, respectively and an astrix denotes a mode which meets the condition to be a DWF.

Chern-Simons level and the existence of DWFs for each of the k_μ. For our present purposes, we will only be concerned with the behavior of the theory in the vicinity of the critical mass, $M = 6$, and so we will check the behavior of the k_μ extrema for these values.

Our results are summarized in Table 3.3. For $M = 6$, corresponding to the IR fixed point, the Chern-Simons term is level-$\frac{1}{2}$ and there are no DWFs. More precisely, the would-be DWF is at the limit where $|F(k_i)| = 1$ and has become a continuum eigenstate. This is consistent with the proposed continuum behavior at the conformal fixed point. For $M' = 6 + \epsilon$, the Chern-Simons level is zero and we have no DWFs since (3.62) is not satisfied for any k_i. Again, this is in agreement with (3.20).

$M' = 6 - \epsilon$ is slightly more subtle. This value corresponds to the UV sector of the theory where we need to level-1 Chern-Simons to generate the $e^{i\mathrm{CS}_1[A-a]}$ term as well as negative mass deformations at the IR fixed point. For this case we find a Chern-Simons level of 1 and two DWFs, since both $k_\mu = (\pi, \pi, 0)$ and $k_\mu = (\pi, \pi, \pi)$ satisfy (3.62). However, this is where our fermionic boundary conditions we enforced earlier come back into play. Since we have a Chern-Simons level of 1, we have chosen our boundary condition to kill off the left-mover, namely $P_-\chi_\beta = \bar{\chi}_\beta P_+ = 0$. This gives the correct chiral modes on the boundary to satisfy (3.20). Interestingly, since they supply a level-$\frac{1}{2}$ Chern-Simons term with no DWF, it is the fermion doublers that play the role of the Pauli-Villars regulator on the lattice. Thus, we are self-consistent with the Callan-Harvey mechanism all the way through. This analysis follows similarly for $M < 0$ in which case we would need to choose $R = -1$ and kill off right-movers with the fermionic boundary condition.

Note that by imposing the fermionic boundary conditions, we have fixed two of the Grassmann variables we would normally integrate over on the boundary sites. The fermionic current conservation imposed by Grassmann integration will still hold for such links, but now each site has only two Grassmann degrees of freedom instead of four. The contribution of the double hopping/interaction term with any boundary site is very limited in such cases, since it already contains both Grassmann degrees of freedom. To have a non-vanishing contribution it must be isolated from any other links.

Finally, we need to understand the effect of the Neumann boundary conditions on the dynamical gauge field, i.e. (3.60). The bulk integration over the link variables tied the bosonic and fermionic currents together. From the above construction, the boundary scalar current vanishes, which would seem to imply the boundary fermionic current does as well. This would present a problem for satisfying (3.20) if not for the gauge field boundary conditions. Enforcing (3.60) on, e.g., the βi link kills off the link integration along the boundary and transforms the fermionic current terms as

$$D_{\beta i} e^{-i(A_{\beta i} - a_{\beta i})} \qquad \rightarrow \qquad D_{\beta i} e^{-i(A_{\beta i} + a_{(\beta+\hat{\imath})y} - a_{(\beta+\hat{y})i} - a_{\beta y})}. \tag{3.65}$$

Hence, there is no tying of the fermionic current to the vanishing scalar current, but we should still verify that is possible to get a non-vanishing fermionic current on the boundary so our DWFs are still allowed solutions.

First, consider $e^{ia_{\beta y}}$, which naïvely would be problematic for the survival of terms like (3.65) upon integration over the corresponding link variable unless it is canceled by $e^{-ia_{\beta y}}$ from somewhere else in the path integral. With no scalar current flowing onto the boundary, we could use fermionic current term such as $D^*_{\beta y} e^{i(A_{\beta y}-a_{\beta y})}$ to cancel $e^{ia_{\beta y}}$. However, such a term means the fermionic current flows off the boundary. Since the number of Grassmann variables at the site β is saturated by the two fermionic currents due to our fermion boundary conditions, a double-hopping term to return the fermionic current to the same site is forbidden. Relying on such a cancellation would mean the boundary fermionic current is only supported for a single link.

Fortunately, there are additional contributions that work to cancel $e^{ia_{\beta y}}$. Consider the form of (3.65) for neighboring boundary links. The $(\beta-\hat{i})i$ link contains an exponential of the form $e^{-ia_{\beta y}}$ which can cancel $e^{ia_{\beta y}}$. This has the interpretation of a fermionic current flowing from the $(\beta-\hat{i})i$ link to the βi link. The cancellation generalizes over a chain of adjacent boundary links with nonzero fermionic current and causes all exponentials with dynamical gauge links perpendicular to the boundary to vanish.

The only remaining term the needs to be cancelled in (3.65) is $e^{-ia_{(\beta+\hat{y})i}}$. This can easily be achieved by either the fermionic or bosonic currents living on the $(\beta+\hat{y})i$ link. Combining this with the cancellation of $e^{ia_{\beta y}}$ and $e^{-ia_{(\beta+\hat{i})y}}$, it is possible to have an uninterrupted fermionic current flowing along the boundary in spite of having chosen scalar boundary conditions which set bosonic currents on the boundary to zero.[10] Furthermore, the chirality of this boundary current is set by our choice of fermionic boundary conditions. This is completely analogous to the continuum case.

[10] It is also possible to have a nonzero fermionic current on the boundary with Dirichlet boundary conditions on the gauge field. This is still consistent with the continuum case, but would require killing off edge movers in order to get a non-anomalous theory.

3.2.4 Discussion

In this section, we presented a generalization of Abelian bosonization that remains valid in the presence of a boundary. Our main finding is that, for the duality to be valid in the presence of boundaries, one carefully needs to account for edge modes that are associated with Chern-Simons terms. Most importantly, we require edge modes even for Chern-Simons terms in the action that only involve non-dynamical fields. We implemented this consistently by replacing all Chern-Simons terms with heavy "fiducial" fermions.

Given the fact that even the $\mathrm{BF}[b;C]$ term of (3.41) can be rewritten using our fiducial fermion prescription to yield a consistent theory, a natural question one might ask if this is always the case. In other words, can we ever run into some combination of Chern-Simons terms which is consistent with the spin-charge relation of a spin$_c$ but cannot be rewritten in terms of our fiducial building blocks? Reassuringly, the answer appears to be no. In [60] it was shown that any consistently quantized Chern-Simons term which can be put on a spin$_c$ manifold can be rewritten as

$$\mathrm{BF}[B;C] = \mathrm{CS}_1[B+C+A] - \mathrm{CS}_1[B+A] - \mathrm{CS}_1[C+A] + \mathrm{CS}_1[A], \tag{3.66a}$$

$$\mathrm{CS}_1[B] + \mathrm{BF}[B;A] = \mathrm{CS}_1[A+B] - \mathrm{CS}_1[A], \tag{3.66b}$$

$$16\mathrm{CS}_{\mathrm{g}} = 9\mathrm{CS}_1[A] - \mathrm{CS}_1[3A]. \tag{3.66c}$$

All such terms lend themselves to a description in terms of fiducial fermions.

From a condensed matter perspective, the fermionic particle/vortex was originally proposed as a $\mathcal{T}$-symmetric UV completion of the half-filled lowest Landau level. However, the need to view it as the surface of a $3+1$ dimensional topological insulator lead the authors of [126, 127] to conclude that there is no strictly $2+1$ dimensional UV completion for this system. Our analysis suggests such a completion does exist so long as one is willing to lose the spin-charge relation of a spin$_c$ valued connection or $\mathcal{T}$-invariance. One can ask whether the projection onto the lowest Landau level is somehow inconsistent with formulating the theory on a spin$_c$ manifold. If such inconsistencies arise, then the doubly quantized theory would provide a purely $2+1$ dimensional UV completion that is manifestly $\mathcal{T}$ invariant.

This would require a rigorous study of lowest Landau level projectors on spin$_c$ manifolds – a problem we leave to future work.

Since there have been other microscopic descriptions of the bulk Abelian dualities, e.g. [100, 99], one could wonder how those models realize the boundary physics as presented above. In [100], a discrete 2+1 dimensional lamination of 1-dimensional quantum wires was used to derive the Abelian bosonization and particle-vortex duality. Each wire supporting a 1+1-dimensional continuum theory suggests a natural microscopic realization of the above results; the study of which is also left for future work.

Lastly, left unexplored in this analysis among the transitions enumerated in [91] are the "extraordinary" type where the boundary scalar gets a vev and drives a surface transition in addition to gapping the bulk. That the extraordinary transition is believed to admit no relevant boundary deformations sets it apart from the boundary conditions studied in this work and warrants further study in the context of the $2+1$ dimensional dualities studied here. A rich network of dualities making along these lines has been laid out in [50] based on conjectures about the infrared behavior of "duality walls". It would be very interesting to generalize our work to these other options as well.

3.3 Single Species non-Abelian Bosonization with Boundaries

In this section we generalize the analysis of the previous Abelian cases to Aharony's non-Abelian single-species dualities, (1.13). This corresponds to the master duality where one sets either $N_f = 0$ or $N_s = 0$, so we will take the point of view of simply working from the master duality in such limits. Additionally, we make connections with the Abelian limit where we take $N_s = k = N = 1$ or $N_f = k = N = 1$ and find results consistent with our previous analysis in Sec. 3.2. We will also discuss additional subtleties involving the connections coupled to the fermion.

Setting either N_s or $N_f = 0$ eliminates one type of matter from each side of the master duality. This has the effect of making the additional $U(1)_{S,F}$ symmetry redundant. Specifically, $U(1)_{S,F}$ becomes a linear combination of the global $U(1)_{m,b}$ symmetry and the dynamical gauge group. Since $U(1)_{S,F}$ does not appear in [2], the redundancy should be expected. Importantly, $U(1)_{S,F}$ becoming redundant does not amount to just setting $\tilde{A}_2 = 0$ in the master duality. We will see keeping careful track of the $\tilde{A}_2$ dependence in Sec. 3.3.3 allows us to correctly distinguish ordinary and spin$_c$ connections.

3.3.1 Adding Boundaries for the Non-Abelian Case

For simplicity, we will continute to consider the theory on the half-space $\mathbb{R}^{2,1}_+$ with coordinates $\{t, x, y\}$ with $t, x \in (-\infty, \infty)$ and $y \geq 0$. As in the Abelian duality, the results should be largely independent of the choice of $\mathbb{R}^{2,1}_+$ as our background [9]. We will again use $i, j = \{x, t\}$ to refer to indices parallel to the boundary.

In this section we will start by briefly summarizing our conventions for boundary conditions for single-component fields as prescribed Sec. 3.2, which generalize fairly trivially to non-Abelian theories. Further, we will review the impact of the choice of boundary conditions on the presence of edge modes and anomalies in the boundary theory. We will then generalize the method for properly accounting for edge modes by introducing "fiducial fermions".

Boundary conditions

From the perspective of the action, boundary conditions arise from partial integration and demanding a well-defined variational principle. The most basic conditions one encounters require either the variation of a dynamical variable ("Dirichlet") or its coefficient ("Neumann") to vanish at the boundary. Consistent boundary conditions for a scalar with non-derivative couplings can be either Neumann or Dirichlet,

$$(D_b)_y \phi_{\alpha M}\big|_\partial = 0 \qquad \text{or} \qquad \delta\phi_{\alpha M}\big|_\partial = 0 \tag{3.67}$$

where, once again, we are using "$|_\partial$" to denote an expression which holds at the boundary. Equivalent boundary conditions hold for $\Phi_{\rho I}$. In order to derive the boundary conditions for a given Dirac fermion ψ, it is convenient to decompose ψ into its left- and right-handed components, $\psi^{\pm}$,

$$\psi = \begin{pmatrix} \psi^{+} \\ \psi^{-} \end{pmatrix}, \qquad \text{i.e.} \quad \psi^{\pm} = P_{\pm}\psi. \tag{3.68}$$

The projector $P_{\pm} = (\mathbb{1} \pm \gamma^{y})/2$ where γ^{y} is the gamma matrix which is perpendicular to the boundary. The boundary conditions are then

$$\psi^{+}_{\alpha I}\big|_{\partial} = 0 \qquad \text{or} \qquad \psi^{-}_{\alpha I}\big|_{\partial} = 0. \tag{3.69}$$

Equivalent boundary conditions hold for $\Psi_{\rho M}$. The boundary conditions on the Pauli-Villars fields follow in an analogous manner. As in the Abelian case, we chose boundary conditions for the Pauli-Villars that will never give rise to edge modes.

The boundary conditions for gauge fields also fall into the category of Neumann and Dirichlet boundary conditions,

$$F_{iy}\big|_{\partial} = \left(\partial_y b_i - \partial_i b_y + [b_y, b_i]\right)\big|_{\partial} = 0, \qquad b_i\big|_{\partial} = 0, \tag{3.70}$$

respectively.[11] In the Abelian dualities, there are only dynamical gauge fields on one side of the duality, and thus only one boundary condition is necessary [9], which obviates the complications in choosing consistent boundary conditions in both theories. In this section, we will need to be more careful in choosing boundary conditions for all of the dynamical gauge fields.

For Neumann boundary conditions on dynamical gauge fields, we will need to worry about anomaly inflow. Note that since we do not assign boundary conditions for background fields, their corresponding Chern-Simons terms can produce anomalies. The cancellation of anomalies will be achieved by introducing “fiducial fermions”, which will give rise to edge modes and will be discussed in the next section.

If we choose Dirichlet boundary conditions for the dynamical gauge fields, there is no chiral current flow off the boundary and, hence, no anomalies. This follows from the fact

[11]Neumann boundary conditions can be modified by coupling boundary matter to the bulk gauge sector by $\epsilon^{ij}F_{jy}|_{\partial} = j^{i}_{\text{bdry}}$ where j^{i}_{bdry} is the boundary matter current [45]. Since we do not add any additional charged boundary matter, we will always set $j^{i}_{\text{bdry}} = 0$.

$j^y_{\text{flux}}|_\partial \sim F_{ij}|_\partial = 0$. Since Dirichlet boundary conditions break the gauge symmetry to the group that leaves the boundary condition invariant, an additional global symmetry emerges at the boundary [45].

We will show that the only way the global symmetries are consistent with the duality is to choose Dirichlet boundary conditions on one side and Neumann boundary conditions on the other. These results align with those discussed in [50].

Lastly, we should mention that choosing the same boundary condition on all flavors is necessary in order to maintain the full $SU(N_s)$ and $SU(N_f)$ global symmetries as well as the respective gauge symmetries. For future work, it may be interesting to consider a set of boundary conditions that breaks the flavor symmetries or mixing Neumann and Dirichlet boundary conditions for subsets of the gauge fields in a given theory.

Edge modes and anomalies

In studying Chern-Simons matter theories in the presence of a boundary, we must reconcile the theories against possible edge modes allowed by the boundary conditions and any anomaly inflow. In particular, introducing gapped fermions to a manifold with a boundary can create gapless, chiral fermionic modes localized to the boundary, i.e. domain wall fermions (DWFs). If we allow the mass of the bulk fermions to vary in the direction normal to the boundary ($m(y)$), then by the standard construction [29, 77] DWFs will exist when the profile of the spatially varying mass leaves the function

$$\xi^{\pm}(y) = e^{\pm \int_0^y dy' m(y')} \tag{3.71}$$

finite for all $y \in \mathbb{R}^{2,1}_+$. In fact in $\mathbb{R}^{2,1}_+$, *any* constant, non-zero mass will give a normalizable zero mode with chirality determined by the sign of the mass. That is, we have left-moving DWFs for $\text{sgn}(m) = +1$ and right-moving DWFs for $\text{sgn}(m) = -1$.

In addition to the possible anomalies associated with non-vanishing chiral currents on our boundary, we also need to take care of potential anomaly inflow from the gauge sector. Chern-Simons theories in the presence of a boundary are not *a priori* gauge invariant everywhere.

However, the non-trivial anomaly associated with a bulk $SU(N)$ Chern-Simons term of level k can be compensated for by the chiral anomaly through the Callan-Harvey mechanism provided [29]

$$k = n_+ - n_-, \tag{3.72}$$

where k is the level of the bulk Chern-Simons theory and $n_\pm$ are the number of (right-) left-movers in the fundamental representation of $SU(N)$ living on the boundary. This of course generalizes to the Abelian case as well. Similarly the gravitational Chern-Simons term with coefficient k_Ω has an anomaly associated with diffeomorphisms, which can be compensated for by having excess right- or left-moving ($\tilde{n}_\pm$ resp.) Majorana-Weyl fermions satisfying

$$k_\Omega = \frac{1}{2}(\tilde{n}_+ - \tilde{n}_-). \tag{3.73}$$

Equivalently, we could use a single right- or left-moving Weyl fermion for every two corresponding Majorana-Weyl fermions to accomplish the same compensation.

Fiducial fermions

Informed by lattice realization of Abelian dualities, the accounting for edge modes above led to the prescriptive replacement of Chern-Simons terms by heavy fermions [9]. These "fiducial fermions" act to display the UV physics captured in the IR by the Chern-Simons terms while more directly enumerating the gauge sector edge modes. The non-trivial IR theory left behind after integrating out heavy Dirac fermions coupled to a background spin$_c$ connection A is $\mathrm{CS}_1[A] + 2\mathrm{CS}_{\mathrm{grav}}$. More importantly, the fiducial fermions give rise to DWFs which automatically render their associated Chern-Simons terms non-anomalous. Thus, the fiducial fermion prescription reads

$$e^{\pm i\int d^3x\,(\mathrm{CS}_1[A]+2\mathrm{CS}_{\mathrm{grav}})} \to \int \mathcal{D}\chi\mathcal{D}\lambda\, e^{i\int d^3x\,\mathcal{L}^{\pm}_{ff}[\chi,\lambda,A]}, \tag{3.74}$$

where

$$\mathcal{L}^{\pm}_{ff}[\chi,\lambda,A] \equiv \lim_{|m_\chi|,|m_\lambda|\to\infty} \left(i\bar{\chi}\not{D}_A\chi \mp |m_\chi|\,\bar{\chi}\chi + i\bar{\lambda}\not{D}_A\lambda \mp |m_\lambda|\,\bar{\lambda}\lambda\right). \tag{3.75}$$

Here χ is the fiducial fermion, λ is the Pauli-Villars regulator field, and the their respective masses $|m_\chi|$, $|m_\lambda|$ are taken to be parametrically heavy.

This procedure generalizes to the case where B is a non-Abelian background gauge field of $SU(N)$ and we have

$$e^{\pm i\int d^3x\,(k\mathrm{CS}_N[B]+2Nk\mathrm{CS}_{\mathrm{grav}})} = \int \prod_{M=1}^{k} \mathcal{D}\chi_M \mathcal{D}\lambda_M \, e^{i\int d^3x\, \mathcal{L}^{\pm}_{ff}[\chi_M,\lambda_M,B]}, \tag{3.76}$$

now with

$$\mathcal{L}^{\pm}_{ff}[\chi_M,\lambda_M,B] \equiv \lim_{|m_{\chi_M}|,|m_{\lambda_M}|\to\infty} \left(i\bar{\chi}^M \not{D}_B \chi_M \mp |m_{\chi_M}|\,\bar{\chi}^M\chi_M + i\bar{\lambda}^M \not{D}_B \lambda_M \mp |m_{\lambda_M}|\,\bar{\lambda}^M\lambda_M \right) \tag{3.77}$$

with χ_M and λ_M in the fundamental representation of $SU(N)$. The non-Abelian fiducial fermion prescription requires χ_M and λ_M be parametrically heavy N-component fields with $U(k)$ flavor symmetry.

As was done in [9], it will be useful to rewrite all BF terms as Chern-Simons terms in order to properly account for the edge theories. For example,

$$N\mathrm{CS}_k[\tilde{c}] + N\mathrm{BF}[\mathrm{Tr}_k(\tilde{c}); \tilde{A}_1 \mathbb{1}_k] = N\mathrm{CS}_k[\tilde{c} + \tilde{A}_1\mathbb{1}_k] - N\mathrm{CS}_k[\tilde{A}_1\mathbb{1}_k]. \tag{3.78}$$

The right-hand side makes the assignments of fiducial fermions clearer.

Global symmetries

The global symmetries manifest in the Lagrangian as three types of background Chern-Simons terms: (1) Abelian, namely $\tilde{A}_1$ and $\tilde{A}_2$, (2) non-Abelian, B and C, and (3) gravitational. All three of these global symmetries are related to a conserved current, which will allow us to put additional constraints on the fields.

First consider the Abelian symmetries. There is an identification between the currents which couple to $\tilde{A}_a$, found via

$$j^{\mu}_{U,a}(x) \equiv \frac{\delta S_U[\tilde{A}_a]}{\delta \tilde{A}_{a\mu}(x)}, \qquad \leftrightarrow \qquad j^{\mu}_{SU,a}(x) \equiv \frac{\delta S_{SU}[\tilde{A}_a]}{\delta \tilde{A}_{a\mu}(x)}. \tag{3.79}$$

As with the Abelian dualities, $\tilde{A}_1$ is associated with the flux current on U side and a particle current on the SU side. For example, when $N_s = 0$ and we set $\tilde{A}_1 = 0$ after variation,

$$j^{\mu}_{U,1} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu \mathrm{Tr}_k(\tilde{c}_\rho) \quad \leftrightarrow \quad j^{\mu}_{SU,1} = j^{\mu}_{\mathrm{fermion}}. \tag{3.80}$$

We will show below that the $\tilde{A}_2$ field plays a very similar role. Note that in the single species non-Abelian dualities the $\tilde{A}_2$ symmetry drops out [2], so it is only a feature of the master bosonization duality [68, 25].

The non-Abelian global flavor symmetries also give two currents related to the $SU(N_s)$ and $SU(N_f)$ symmetries on either side. These flavor currents are not just simply matter currents because there is also flux coupling to the background C_μ fields on the SU side of the duality.

Lastly, the equivalence of the gravitational currents simply identifies the stress-energy tensors on either side of the duality. We will not make use of this identification in what follows.

3.3.2 Non-Abelian U + scalars $\leftrightarrow$ SU + fermions

To start studying the non-Abelian dualities, we will consider $N_s = 0$. This reduces (1.27) to

$$SU(N)_{-k+\frac{N_f}{2}} \text{ with } N_f\ \psi \qquad \leftrightarrow \qquad U(k)_N \text{ with } N_f\ \Phi \tag{3.81}$$

with the mass identification $m_\psi \leftrightarrow -m_\Phi^2$. This duality is subject to the flavor bound $N_f \leq k$.[12]

Explicitly, the Lagrangians for the theories on either side of (3.81) are given by

$$\begin{aligned}\mathcal{L}_{SU} =& i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1}\psi - i\left((N_f - k)\mathrm{CS}_N[b'] + N\mathrm{CS}_{N_f}[C]\right) \\ &- i\left(-N(k-N_f)\mathrm{CS}_1[\tilde{A}_1] + 2NN_f\mathrm{CS}_{\mathrm{grav}}\right) \end{aligned} \tag{3.82}$$

$$\mathcal{L}_U = |D_{c+C}\Phi|^2 + \alpha_\varphi|\Phi|^4 - i\left(N\mathrm{CS}_k[c] + N\mathrm{BF}[\mathrm{Tr}_k(\tilde{c})\,;\tilde{A}_1] + 2Nk\mathrm{CS}_{\mathrm{grav}}\right). \tag{3.83}$$

[12] As discussed in Chapter 2, there are arguments that these flavor bounds can be extended slightly [86], but we will not consider such cases for the systems with boundaries discussed in this chapter.

Since the Lagrange multiplier term will not be important for this section, we have integrated out f as in (3.3a). Furthermore, in (3.83), we can split the $U(k)$ field, c, into its traceless $SU(k)$ part, c', field and non-zero trace, $\tilde{c}$, such that

$$\mathcal{L}_U = |D_{c+C}\Phi|^2 + \alpha_\varphi|\Phi|^4 - i\left(N\mathrm{CS}_k[c'] + N\mathrm{CS}_k[\tilde{c} + \tilde{A}_1\mathbb{1}_k] - Nk\mathrm{CS}_1[\tilde{A}_1] + 2Nk\mathrm{CS}_{\mathrm{grav}}\right) \tag{3.84}$$

Note that mass deformations in these theories correspond to phases I and II in Fig. 1.2. Specifically, $m_\psi < 0$ and $m_\Phi^2 > 0$ is Phase II, and $m_\psi > 0$ and $m_\Phi^2 < 0$ is Phase I. Also take note of the fact the duality has no $\tilde{A}_2$ dependence, since the $U(1)_{S,F}$ duality coupled to the fields associated with the $SU(N_s)$ symmetry.

Let us work through the counting of fiducial fermions in detail. First consider the U side of Phase II where $m_\Phi^2 > 0$. Integrating out the scalars when $m_\Phi^2 > 0$ is straightforward, they are simply gapped and cause no change in the Chern-Simons terms so we are left with

$$i\mathcal{L}_U^{II} = N\mathrm{CS}_k[c'] + N\mathrm{CS}_k[\tilde{c} + \tilde{A}_1\mathbb{1}_k] - Nk\mathrm{CS}_1[\tilde{A}_1] + 2Nk\mathrm{CS}_{\mathrm{grav}}. \tag{3.85}$$

We will start by assuming Neumann boundary conditions for the dynamical gauge fields; all the Chern-Simons terms are anomalous in the sense that they result in a non-vanishing current flowing onto the boundary. Fortunately, in Phase II it is straightforward to assign edge modes to compensate for the anomalies.

To start, N right-moving k-component fiducial fermions coupled to $c' + \tilde{c} + \tilde{A}_1\mathbb{1}_k$ will make $N\mathrm{CS}_k[c'] + N\mathrm{CS}_k[\tilde{c} + \tilde{A}_1\mathbb{1}_k]$ non-anomalous. Note that since we can shift away the $\tilde{A}_1$ factor, together these terms are equivalent to a $U(k)_N$ Chern-Simons term.

Next, Nk left-moving single-component fiducial fermions will make $Nk\mathrm{CS}_1[\tilde{A}_1]$ non-anomalous. The newly added Nk left and N right movers respectively generate gravitational Chern-Simons terms $+2Nk\mathrm{CS}_{\mathrm{grav}}$ and $-2Nk\mathrm{CS}_{\mathrm{grav}}$, and hence such terms cancel out.

Lastly, we need to make the remaining $+2Nk\mathrm{CS}_{\mathrm{grav}}$ term non-anomalous. We thus introduce Nk neutral right-moving single-component fiducial fermions. Moving forward, we note that a positive mass scalar does nothing to the Chern-Simons modes, and so, we will always use the $m_\Phi^2 > 0$ (or $m_\phi^2 > 0$) regime to determine the fiducial fermions on the scalar end of the dualities.

However, there is one subtlety we have not yet mentioned: introducing the fiducial fermions has given the theory additional symmetries on the boundary. For example, choosing to add $N \times \mathcal{L}_{ff}^{+}[c' + \tilde{c} + \tilde{A}_1 \mathbb{1}_k]$ introduces a new global $SU(N)$ symmetry on the boundary. We need to be careful with how we are assigning fiducial fermions on both sides of the duality so that their associated global symmetries match. While the global symmetries coming from the fiducial fermions for the background Chern-Simons terms trivially match, the fiducial fermions associated with dynamical gauge fields have no analog on the opposite side of the duality.

Taking care to assign the fiducial fermions for the dynamical gauge fields, recall that Dirichlet boundary conditions not only enhances the global symmetry on the boundary but also eliminates the need to make the dynamical gauge fields non-anomalous. This removes the need to assign fiducial fermions to the dynamical gauge fields for Dirichlet boundary conditions. In fact, the enhanced global symmetry from choosing Dirichlet boundary conditions on one side of the duality exactly match the additional global symmetry from introducing the dynamical fiducial fermions [50].

Let us demonstrate this mechanism explicitly in the present example. Table 3.4 summarizes all the fiducial fermions we had to add on both sides of the duality. We just explained this fiducial matter content on the U side and will turn to the SU side momentarily. There are common Nk left-moving fermions (charged under $\tilde{A}_1$) and Nk neutral right-moving fermions on both sides. They give rise to an extra $SU(Nk) \times U(Nk)$ global symmetry on both sides. In addition, there are N fiducial fermions on the $U(k)$ side that have no corresponding fiducial fermions on the $SU(N)$ side. We can account for the new $SU(N)$ global symmetry from these fiducial fermions by choosing Dirichlet boundary conditions for the dynamical $SU(N)$ gauge field b, which will produce a global $SU(N)$ symmetry on the boundary. More generally, choosing Neumann boundary conditions for the gauge fields on one side of the duality is *only* consistent with choosing Dirichlet boundary conditions on the other.

To complete the entries in Table 3.4, let us analyze the fiducial matter content on the SU side. Staying in Phase II and integrating out the N_f N-component dynamical fermions, we pick up additional Chern-Simons terms, which reduces (3.82) to

$$i\mathcal{L}^{II}_{SU} = -k\mathrm{CS}_N[b'] - kN\mathrm{CS}_1[\tilde{A}_1]. \tag{3.86}$$

At this point if we choose Dirichlet boundary conditions on b', the $-k\mathrm{CS}_N[b']$ term is non-anomalous on its own. First note that we have fermions on this side of the duality and so if we choose appropriate boundary conditions, DWFs can exists and potentially provide the necessary edge modes for (3.86) to be non-anomalous. However, with a bit of foresight we will choose the fermionic boundary condition which does not allow DWFs to exist in this phase, and hence all of our anomaly cancellation must come from fiducial fermions. This also turns out to be the right choice for matching global symmetries on the boundary.

Specifically, introducing k left-moving N-component fiducial fermions coupled to $\tilde{A}_1\mathbb{1}_N$ renders $-kN\mathrm{CS}_1[\tilde{A}_1]$ non-anomalous. To account for the gravitational Chern-Simons term $-2Nk\mathrm{CS}_{\mathrm{grav}}$ from the fiducial fermions, we should also introduce Nk right-moving neutral fermions. It is easy to see that the boundary global symmetries match the choice of Neumann boundary conditions on the U side above.

We have completed our first complete dual pair. As pointed out in [50], we have seen that the duality-consistent boundary conditions for dynamical gauge fields are Neumann on one side of the duality and Dirichlet on the other with the freedom to assign which side sees which boundary condition.

There is a second dual pair with the same gauge groups and matter content where we choose Dirichlet boundary conditions on the U side and Neumann boundary conditions on the SU side. We can work out the fiducial fermion content in this pair following the same logic as above.

Staying in Phase II, on the SU side we now need to assign fiducial fermions to make both terms in (3.86) non-anomalous. Fortunately this isn't much different from the case considered above, and simply requires the k left-moving N-component fiducial fermions be coupled to $b' + \tilde{A}_1 \mathbb{1}_N$ instead of just $\tilde{A}_1 \mathbb{1}_N$. This renders both $-k\text{CS}_N[b'] - kN\text{CS}_1[\tilde{A}_1]$ non-anomalous.

Now, consider imposing Dirichlet boundary conditions on the U side in Phase II. Above, we saw that Neumann boundary conditions required three types of fiducial fermions to render all the terms in (3.85) non-anomalous. However, chosing Dirichlet boundary conditions for c means that we no longer need to worry about canceling the anomaly associated with its Chern-Simons term. In this case the anomalies of $N\text{CS}_k[\tilde{c} + \tilde{A}_1 \mathbb{1}_k]$ and $-Nk\text{CS}_1[\tilde{A}_1]$ actually cancel, and this means that we only need the fiducial fermions that made the gravitational term non-anomalous.

Having established the fiducial fermion spectrum in Phase II, let's now check that the assignments work to make Phase I non-anomalous as well. For the SU + fermion theory, integrating out the fermions in (3.82) cancels the η-invariants, which leaves the Chern-Simons levels unaffected,

$$i\mathcal{L}^{I}_{SU} = (N_f - k)\text{CS}_N[b'] - (k - N_f)N\text{CS}_1[\tilde{A}_1] + N\text{CS}_{N_f}[C] + 2NN_f\text{CS}_{\text{grav}}. \tag{3.87}$$

However, it will be helpful to view $i\mathcal{L}^{I}_{SU}$ as coming from $i\mathcal{L}^{II}_{SU}$ in order to show that (3.87) is non-anomalous. Comparing to (3.86),

$$i\mathcal{L}^{I}_{SU} = i\mathcal{L}^{II}_{SU} + N_f\text{CS}_N[b'] + NN_f\text{CS}_1[\tilde{A}_1] + N\text{CS}_{N_f}[C] + 2NN_f\text{CS}_{\text{grav}}. \tag{3.88}$$

In order to get a non-anomalous theory, we can take advantage of the fact that the fiducial fermions that we have already assigned rendered $i\mathcal{L}^{II}_{SU}$ non-anomalous. It remains to be shown that the additional Chern-Simons terms in (3.88) are non-anomalous. Fortunately, we have chosen the boundary condition on the dynamical fermion such that we allow the DWFs to live for $m_\psi > 0$. From (3.3a), the dynamical fermions couple to $b'\mathbb{1}_{N_f} + C\mathbb{1}_N + \tilde{A}_1\mathbb{1}_{NN_f}$, and hence the DWFs are exactly the edge modes needed to cancel the residual anomalies of (3.88).

The cancellation of the edge modes happens analogously to the cancellation of the Chern-Simons terms. In the end, we have $k - N_f$ left-moving N-component fiducial fermions coupled to $b' + \tilde{A}_1 \mathbb{1}_N$, and N right-moving N_f-component fiducial fermions coupled to C to cancel the background $SU(N_f)$ and gravitational Chern-Simons terms.

To complete the duality in Phase I, we need to consider the scalar side in the Higgs regime ($m_\Phi^2 < 0$). Following [68, 25], we will assume that the N_f scalars maximally Higgs the $U(k)$. The breaking pattern is then $U(k)_{-N} \to U\left(k - N_f\right)_{-N} \times SU\left(N_f\right)_{-N}$; resulting in a Lagrangian

$$i\mathcal{L}_U^I = N\left(\text{CS}_{k-N_f}[c'] + \text{CS}_{k-N_f}[\tilde{c} + \tilde{A}_1 \mathbb{1}_{k-N_f}] + \text{CS}_{N_f}[C] - (k - N_f)\text{CS}_1[\tilde{A}_1] + 2k\text{CS}_{\text{grav}}\right). \tag{3.89}$$

Since there are fiducial fermions which couple to $U(k)$, the spontaneous breaking separates each of the k-component fiducial fermions into broken and unbroken parts, namely

$$N \times \mathcal{L}_{ff}^+[c' + \tilde{c} + \tilde{A}_1 \mathbb{1}_k] \to \begin{cases} N \times \mathcal{L}_{ff}^+[c' + \tilde{c} + \tilde{A}_1 \mathbb{1}_{k-N_f}] & \text{(unbroken)} \\ N \times \mathcal{L}_{ff}^+[C + \tilde{A}_1 \mathbb{1}_{N_f}] & \text{(broken)} \end{cases}. \tag{3.90}$$

Note that the $\tilde{A}_1$ part of the N N_f-component fiducial fermions from the broken sector combines with the opposite chirality Nk modes coupled to $\tilde{A}_1$; leaving a total of $N(k - N_f)$. The number of gravitational Chern-Simons terms is unchanged – we still have the same net number of modes. A straightforward check shows these edge modes render (3.89) non-anomalous.

Comparing the boundary spectra for mass deformations of the single species non-Abelian duality, we can match the degrees of freedom in kind. Thus, we see that

$$\mathcal{L}_U^I \leftrightarrow \mathcal{L}_{SU}^I \tag{3.91}$$

$$\mathcal{L}_U^{II} \leftrightarrow \mathcal{L}_{SU}^{II} \tag{3.92}$$

indicating a consistent duality in the bulk. We outline both instances of duality consistent boundary conditions and the additional edge modes in Table 3.4.

The last remaining question we have to address is how to identify boundary conditions for the scalar fields. Following a similar procedure used in [9], let us reinterpret the effect

	$SU(N)$ + **fermions**	$U(k)$ + **scalars**
Boundary Conditions	$\psi^-_{\alpha I}\vert_\partial = 0$	$\Phi_{\rho I}$: Dirichlet
	b': Dirichlet	c: Neumann
Additional Edge Modes	———	$N \times \mathcal{L}^+_{ff}[c' + \tilde{c} + \tilde{A}_1 \mathbb{1}_k]$
	$Nk \times \mathcal{L}^-_{ff}[\tilde{A}_1]$	$Nk \times \mathcal{L}^-_{ff}[\tilde{A}_1]$
	$Nk \times \mathcal{L}^+_{ff}[0]$	$Nk \times \mathcal{L}^+_{ff}[0]$

	$SU(N)$ + **fermions**	$U(k)$ + **scalars**
Boundary Conditions	$\psi^-_{\alpha I}\vert_\partial = 0$	$\Phi_{\rho I}$: Neumann
	b': Neumann	c: Dirichlet
Additional Edge Modes	$k \times \mathcal{L}^-_{ff}[b' + \tilde{A}_1 \mathbb{1}_N]$	———
	$Nk \times \mathcal{L}^+_{ff}[0]$	$Nk \times \mathcal{L}^+_{ff}[0]$

Table 3.4: The top (bottom) table counts the additional edge modes when choosing Neumann and Dirichlet boundary conditions on the dynamical gauge fields in U (SU) and SU (U) side respectively when $N_s = 0$.

of anomaly inflow when we choose Neumann boundary conditions on the U side. Alone, a Chern-Simons term is anomalous on the boundary due to a non-trivial current divergence. Since the associated current is not conserved, we can think of this as meaning the $U(1)_m$ symmetry is broken on the boundary. When we introduce edge modes on the boundary, there is a compensating term for the current flowing onto the boundary. In other words, if we identify the $U(1)$ axial symmetry on the boundary with the $U(1)_m$ symmetry in the bulk, we have a restored $U(1)$ symmetry everywhere. This is consistent with the SU side of the theory where there is no anomalous term and thus the $U(1)_b$ symmetry exists everywhere.

If we choose Neumann boundary conditions for c on the U side of the duality, this amounts to the constraint that $F_{yi}|_\partial = 0$, with F the field strength of c. Since the flux current is $j^\mu_{\text{flux}} \sim \epsilon^{\mu\nu\rho} F_{\nu\rho}$, Neumann boundary conditions automatically imply any flux current on the boundary must vanish. This is consistent with the $U(1)$ boundary symmetry being provided by the edge modes, rather than the flux current. The Neumann boundary condition on the gauge fields is also inconsistent with having a scalar current on the boundary since such a current is charged under the dynamical gauge field. Additionally, recall that the bulk equations of motion relate bulk flux and matter currents; schematically, $j^\mu_{\text{matter}} \sim j^\mu_{\text{flux}}$. Although such equations do not apply on the boundary, allowing for scalar current to flow on the boundary would be inconsistent with the continuity of the current and also have no compensating current on the SU side. Therefore, we choose Dirichlet boundary conditions for the scalar which kills off the scalar current on the boundary.

Now consider Dirichlet boundary conditions for c. Although $c_i|_\partial = 0$, this does not necessarily imply $F_{yi}|_\partial = 0$ since $\partial_i c_y|_\partial \neq 0$ (although it does imply $F_{ij}|_\partial = 0$). By the same reasoning above, this means we can have a nonzero flux current on the boundary. Such boundary conditions are consistent with there being matter charged under the dynamical gauge field on the boundary. The only boundary condition that is consistent with this is Neumann boundary conditions for the scalar. Again via the identification of global symmetry currents, we see that this is consistent with choosing Neumann boundary conditions for b'_μ since we have a nonzero edge modes coupling to $\tilde{A}_1$ on the SU end now.

Abelian Reduction

Let's apply a consistency check on our new non-Abelian prescription. We will take the limit $N_f = N = k = 1$, and choose Neumann boundary conditions for c on the U side to compare to the boundary analysis of the Abelian dualities [9]. Affecting this limit in (3.82) and (3.83) gives

$$\mathcal{L}_{SU} = i\bar{\psi}\not{D}_{\tilde{A}_1}\psi - i\left(2\mathrm{CS}_{\mathrm{grav}}\right) \tag{3.93}$$

$$\mathcal{L}_{U} = |D_{\tilde{c}}\Phi|^2 + \alpha_{\varphi}|\Phi|^4 - i\left(\mathrm{CS}_1[\tilde{c}+\tilde{A}_1] + \mathrm{CS}_1[\tilde{c}] + 2\mathrm{CS}_{\mathrm{grav}}\right), \tag{3.94}$$

which is similar to the Abelian "scalar + flux = fermion" considered in [9], up to the additional $2\mathrm{CS}_{\mathrm{grav}}$ terms.

Now, taking the Abelian limit of the tallied boundary modes in Table 3.4, we find that one fiducial fermion is needed on U side to be coupled to $\tilde{c}+\tilde{A}_1$ and, on both sides of the duality, we need one left-mover coupled to $\tilde{A}_1$ and a neutral right-mover.

Due to certain subtleties with the non-Abelian case, our convention has changed slightly as compared to [9] where the opposite boundary conditions on the dynamical fermions were chosen and gravitational Chern-Simons terms were absent. Without gravitational Chern-Simons terms present we do not need the right-moving neutral fiducial fermions on both sides of the duality. Choosing opposite boundary conditions on the dynamical fermions makes the $m_\Psi < 0$ regime consistent via a fiducial fermion rather than a dynamical fermion. This is why in the present analysis we find an additional left-moving fiducial fermion coupled to $\tilde{A}_1$ on the fermion side of the duality. Choosing Dirichlet boundary conditions on the scalar was also found for a similar reason. Thus, the number of edge modes is consistent modulo conventions.

Notice that the fermions couple to the background $U(1)_m$ spin$_c$ connection, $\tilde{A}_1$. The analysis in [9] requires that in order for the "scalar+flux = fermion" duality to be consistent in the presence of a boundary $\tilde{A}_1$ must be a spin$_c$ connection and not an ordinary $U(1)$. Meanwhile, $\tilde{c}$ was required to be an ordinary connection. Indeed, both of these requirements are consistent the Abelian limit.

3.3.3 Non-Abelian U + fermions ↔ SU + scalars

Now let us consider the other type of single species non-Abelian duality in [2] – rather its time reversed version – by setting $N_f = 0$ such that (1.24) reads

$$SU(N)_{-k} \, \text{with} \, N_s \, \phi \qquad \leftrightarrow \qquad U(k)_{N-\frac{N_s}{2}} \, \text{with} \, N_s \, \Psi. \tag{3.95}$$

with the mass identification $m_\phi^2 \leftrightarrow m_\Psi$. In this case the flavor bound is given by $N_s \leq N$ [25, 68]. The explicit Lagrangians for the theories on each side of the duality are given by

$$\begin{aligned}
\mathcal{L}_{SU} =& |D_{b+B}\phi|^2 + \alpha_\varphi |\phi|^4 - i\left(-k\mathrm{CS}_N[b] + \mathrm{BF}[f; \mathrm{Tr}_N(b) - N\tilde{B}]\right) \\
&- i\left(Nk\mathrm{CS}_1[\tilde{B}] - Nk\mathrm{CS}_1[\tilde{A}_1]\right),
\end{aligned} \tag{3.96}$$

$$\begin{aligned}
\mathcal{L}_U =& i\bar{\Psi} \not{D}_{c-\tilde{A}_2+B}\Psi - i\left(N\mathrm{CS}_k[c] + N\mathrm{BF}[\mathrm{Tr}_k(c); \tilde{A}_1] + 2Nk\mathrm{CS}_{\text{grav}}\right) \\
=& i\bar{\Psi} \not{D}_{c'+\tilde{a}+B}\Psi - i\left(N\mathrm{CS}_k[c'] + N\mathrm{CS}_k[\tilde{a} + \tilde{B}\mathbb{1}_k] - Nk\mathrm{CS}_1[\tilde{A}_1] + 2Nk\mathrm{CS}_{\text{grav}}\right).
\end{aligned} \tag{3.97}$$

Setting $N_f = 0$ has eliminated one of the gravitational Chern-Simons terms, and in the last line of (3.97) we defined the ordinary connection $\tilde{B} = \tilde{A}_1 + \tilde{A}_2$ and spin$_c$ connection $\tilde{a} = \tilde{c} - \tilde{A}_2\mathbb{1}_k$. Note that $\tilde{B}$ is now the background gauge field associated with the global $U(1)_{m,b}$ symmetry. We have also used

$$\mathrm{BF}[\tilde{A}_1; \tilde{A}_2] + \mathrm{CS}_1[\tilde{A}_2] = \mathrm{CS}_1[\tilde{B}] - \mathrm{CS}_1[\tilde{A}_1]. \tag{3.98}$$

For this dual pair, mass deformations correspond to Phase II ($m_\phi^2 > 0$ $m_\Psi > 0$) and Phase III ($m_\phi^2 < 0$ $m_\Psi < 0$) – see Fig. 1.2. As with the $N_s = 0$ case, we can find the fiducial fermion spectrum by looking at Phase II.

As with the last duality, we will find the boundary symmetries to be consistent only if we choose Neumann and Dirichlet boundary conditions for the dynamical gauge fields on opposite sides of the duality. Nevertheless, we will first proceed with the analysis for Neumann boundary conditions on both sides of the duality; generalizing to Dirichlet is straightforward. For the SU + scalar theory in (3.96) with Neumann boundary conditions for the dynamical gauge fields, integrating out the Lagrange multiplier gives

$$i\mathcal{L}_{SU}^{II} = -k\mathrm{CS}_N[b'] - kN\mathrm{CS}_1[\tilde{A}_1]. \tag{3.99}$$

k left-moving N-component fiducial fermions coupled to $b'+\tilde{A}_1\mathbb{1}_N$ compensate for the anomalies generated by $-k\text{CS}_N[b'] - kN\text{CS}_1[\tilde{A}_1]$. We also need Nk right-moving neutral fiducial fermions to cancel the gravitational term.

The U side of the duality is also easy to analyze with Neumann boundary conditions. Despite the new definitions of $\tilde{B}$ and $\tilde{a}$, the anomaly spectrum of (3.97) is identical to that of (3.85). We can choose exactly the same fiducial fermions for the U + fermion side of the duality that we did for U + scalar with Neumann conditions in Table 3.4.

Having quickly read off the fiducial fermions in Phase II, we should check that the assignment holds for Phase III. In Phase III for the U + fermion theory $m_\Psi < 0$, and so, integrating out the dynamical fermions shifts the Chern-Simons levels relative to their Phase II values:

$$i\mathcal{L}_U^{II} \to i\mathcal{L}_U^{II} - N_s\text{CS}_k[c] - N_s k\text{CS}_1[\tilde{A}_2] - k\text{CS}_{N_s}[B] - 2N_s k\text{CS}_{\text{grav}}. \tag{3.100}$$

Using the first line of (3.97), the Lagrangian for the U + fermion theory becomes

$$\begin{aligned} i\mathcal{L}_U^{III} =& \left(N-N_s\right)\text{CS}_k[c] - k\text{CS}_{N_s}[B] + N\text{BF}[\text{Tr}_k\left(c\right); \tilde{A}_1\mathbb{1}_k] \\ &- N_s k\text{CS}_1[\tilde{A}_2] + 2k\left(N-N_s\right)\text{CS}_{\text{grav}}. \end{aligned} \tag{3.101}$$

Rewriting the BF term as a sum of Chern-Simons terms, we find

$$\begin{aligned} i\mathcal{L}_U^{III} =& \left(N-N_s\right)\left(\text{CS}_k[c'] + \text{CS}_k\left[\tilde{c} + \frac{N}{N-N_s}\tilde{A}_1\mathbb{1}_k\right] + 2k\text{CS}_{\text{grav}}\right) \\ &- k\left(\text{CS}_{N_s}[B] + N\text{CS}_1[\tilde{A}_1] + \frac{NN_s}{(N-N_s)}\text{CS}_1[\tilde{B}]\right). \end{aligned} \tag{3.102}$$

So long as we choose the boundary condition such that dynamical DWFs are allowed for $m_\Psi < 0$, the U + fermion theory in Phase III non-anomalous theory. This follows for the same reason we saw in Phase I of the SU side in Sec. 3.3.2: from (3.100) $i\mathcal{L}_U^{II}$ is already non-anomalous due to the fiducial fermions and the dynamical DWF provides the rest of the edge modes to render the whole expression non-anomalous. Thus, the fiducial fermion assignment for Phase II works in Phase III, and the U + fermion theory is non-anomalous.

While it may be hard to see that (3.102) is non-anomalous, the cancelling of the edge modes can be seen directly from the cancellation of the Chern-Simons terms. Finally, note when one expands out $\tilde{B}$ in (3.102) this reproduces the stated background terms of (1.35c), as it should.

The SU + scalar theory in Phase III ($m_\phi^2 < 0$) is complicated slightly due to the Lagrange multiplier – which changes $SU(N) \to U(N) \times U(1)$ and makes the breaking pattern clearer. We do not want to treat the BF terms containing the Lagrange multiplier as additional Chern-Simons terms. We will be more concerned with analyzing the behavior of the edge modes after the breaking has occurred as above on the U side.

After spontaneously breaking $U(N) \to U(N - N_s) \times SU(N_s)$, $N - N_s$ scalars remain coupled to $b' + y\tilde{B}\mathbb{1}_{N-N_s}$. The N_s-components corresponding to the broken part of the gauge symmetry have no coupling to any part of b' but do couple to the $SU(N_s)$ flavor symmetry. The factor y is a rescaling of the Abelian coupling implemented by the Lagrange multiplier that is novel to this theory. Explicitly, the coupling of the $N - N_s$ modes now becomes

$$b' + y\tilde{B}\mathbb{1}_N \to b' + \frac{\sqrt{N}}{N - N_s}\tilde{B}\mathbb{1}_{N-N_s}. \tag{3.103}$$

Thus, when one integrates out the k fiducial fermions, they give

$$i\mathcal{L}_{SU}^{III} \supset -k\left(\mathrm{CS}_{N-N_s}[b'] + \frac{N}{N - N_s}\mathrm{CS}_1[\tilde{B}]\right), \tag{3.104}$$

which will combine with the existing background terms to reproduces the Abelian factor in (3.102).

Let us choose Neumann boundary conditions for b'. The dividing of the fiducial fermion that we would assign occurs analogously to the breaking of the Chern-Simons terms:

$$k \times \mathcal{L}_{ff}^{+}[b' + \tilde{B}] \to \begin{cases} k \times \mathcal{L}_{ff}^{+}\left[b' + \frac{\sqrt{N}}{N-N_s}\tilde{B}\mathbb{1}_{N-N_s}\right] & \text{(unbroken)} \\ k \times \mathcal{L}_{ff}^{+}[B] & \text{(broken)} \end{cases}. \tag{3.105}$$

There are still Nk total fermion components; $N_s k$ of which couple only to the flavor symmetry. Thus, we still have the same number of gravitational Chern-Simons terms as in Phase

II. The full Lagrangian for the SU side of Phase III is then

$$i\mathcal{L}_{SU}^{III} = -k\text{CS}_{N-N_s}[b'] - k\left(\text{CS}_{N_s}[B] - N\text{CS}_1[\tilde{A}_1] + \frac{NN_s}{N-N_s}\text{CS}_1[\tilde{B}]\right), \tag{3.106}$$

which is rendered non-anomalous by the edge modes from the fiducial fermions as assigned in Phase II.

Thus far, we have only considered Neumann boundary conditions for the dynamical gauge fields. To generalize these results to the Dirichlet case is straightforward: simply remove the coupling of the fiducial fermion to the dynamical field whose Chern-Simons terms is no longer anomalous on the boundary. Table 3.5 summarizes our results for this duality. Note once again a nice cancellation between anomalous terms occurs on the U side with Dirichlet boundary conditions.

Finally, consider the boundary conditions on the scalar fields. Again, we use fact that Neumann boundary conditions imply any flux current on the boundary must vanish and that the variation of $\tilde{A}_1$ relates the scalar matter current on the SU side to the flux current on the U side. Since there can be no flux current on the boundary, there can be no scalar current on the boundary as well. Hence we must choose Dirichlet boundary conditions in this case, $\phi_{\alpha M}|_\partial = 0$.

As we argued earlier, for Dirichlet boundary conditions on c we can have a nonzero flux current on the boundary. Again using the identification of global symmetry currents we can also have a nonzero scalar current on the SU side of the duality. Thus, we must choose scalar boundary conditions which allow for a nonzero boundary current, which means Neumann.

Abelian Reduction

Finally, let us check that this is consistent in the Abelian limit by setting $N = k = N_s = 1$, choosing Neumann boundary conditions for $\tilde{c}$, and moving all background terms to the

	$SU(N)$ + **scalars**	$U(k)$ + **fermions**
Boundary Conditions	$\phi_{\alpha M}$: Neumann b': Neumann	$\Psi^+_{\rho M}\vert_\partial = 0$ c: Dirichlet
Additional Edge Modes	$k \times \mathcal{L}^-_{ff}[b' + \tilde{A}_1 \mathbb{1}_N]$	————
	$Nk \times \mathcal{L}^+_{ff}[0]$	$Nk \times \mathcal{L}^+_{ff}[0]$

	$SU(N)$ + **scalars**	$U(k)$ + **fermions**
Boundary Conditions	$\phi_{\alpha M}$: Dirichlet b': Dirichlet	$\Psi^+_{\rho M}\vert_\partial = 0$ c: Neumann
Additional Edge Modes	————	$N \times \mathcal{L}^+_{ff}[c' + \tilde{a} + \tilde{B}\mathbb{1}_k]$ $= N \times \mathcal{L}^+_{ff}[c' + \tilde{c} + \tilde{A}_1\mathbb{1}_k]$
	$Nk \times \mathcal{L}^-_{ff}[\tilde{A}_1]$	$Nk \times \mathcal{L}^-_{ff}[\tilde{A}_1]$
	$Nk \times \mathcal{L}^+_{ff}[0]$	$Nk \times \mathcal{L}^+_{ff}[0]$

Table 3.5: The top (bottom) table counts the additional edge modes when choosing Neumann and Dirichlet boundary conditions on the dynamical gauge fields in SU (U) and U (SU) side respectively when $N_f = 0$.

fermion side. Affecting this limit, we find

$$\mathcal{L}_{SU} = |D_{\tilde{B}}\phi|^2 + \alpha_\varphi|\phi|^4 \tag{3.107}$$

$$\mathcal{L}_U = i\bar{\Psi}\not{D}_{\tilde{a}}\Psi - i\left(\mathrm{CS}_1[\tilde{a}+\tilde{B}] + 2\mathrm{CS}_{\mathrm{grav}}\right) \tag{3.108}$$

where we have canceled the two $-\mathrm{CS}_1[\tilde{A}_1]$ terms. This expression should be equivalent to the *time-reversed* fermion, but with the understanding that in [9] the time-reversed fermion came with an opposite sign Pauli-Villars regulator as well; our conventions for the η-invariant are different here. Accounting for this difference of convention, we pick up an overall shift by $-\mathrm{CS}_1[\tilde{a}] - 2\mathrm{CS}_{\mathrm{grav}}$ on the fermionic side of the duality and change the fermionic boundary condition. We end up with the dual theories being given by

$$\mathcal{L}_{SU} = |D_{\tilde{B}}\phi|^2 + \alpha_\varphi|\phi|^4, \tag{3.109}$$

$$\mathcal{L}_U = i\bar{\Psi}\not{D}_{\tilde{a}}\Psi - i\left(\mathrm{CS}_1[\tilde{a}+\tilde{B}] - \mathrm{CS}_1[\tilde{a}]\right). \tag{3.110}$$

Per our fiducial fermion choices shown in Table 3.5, we should have a single right-moving fiducial fermion coupled to $\tilde{a}+\tilde{B}$. Note the fermions associated to $\tilde{B}$ and neutral fiducial fermions on both ends of the duality cancel one another out.

Once more we see a nice consistency with our previous analysis: $\tilde{a} = \tilde{c} - \tilde{A}_2$ is a spin$_c$ connection, and the background field $\tilde{B} = \tilde{A}_1 + \tilde{A}_2$ is an ordinary $U(1)$ connection. Thus, we can start from the master bosonization duality, demand that a subset of Abelian factors be either ordinary or spin$_c$ connections, and consistently arrive at *both* known Abelian bosonization dualities with the correct coupling of gauge fields to matter. This is also consistent with the process of promoting background fields to dynamical and coupling to new background fields followed by integrating out the old dynamical fields [83, 112].

3.3.4 Discussion

Before turning back to the master bosonization duality, let us take stock of how the phases and edge modes changed when we moved to negative mass deformations for the fermions and scalars:

- **Fermion Deformations:** Given our choice of fermionic boundary conditions, the dynamical DWFs only existed when $m_\psi > 0$ or $m_\Psi < 0$. In the corresponding $m_\psi < 0$ and $m_\Psi > 0$ phases, we found that the additional Chern-Simons terms were rendered non-anomalous by the dynamical DWFs. Since the $m_\psi > 0$ and $m_\Psi < 0$ phases were non-anomalous due to the fiducial fermions, the resulting theory was non-anomalous. Furthermore, the same mechanism that rendered the Chern-Simons terms non-anomalous can be used to argue that – despite some simplified forms of the theories appearing to have extraneous edge modes – that edge modes are cancelled.

- **Scalar Deformations:** In the spontaneously broken phase, $m_\phi^2 < 0$ or $m_\Phi^2 < 0$, the dynamical gauge groups are split up into smaller dynamical groups and gave rise to new non-Abelian flavor symmetries. Additionally for SU + scalars, the background Abelian coupling was rescaled. The couplings of the fiducial fermions were changed according to the breaking pattern for the Chern-Simons terms. The fiducial fermions then split into parts, which couple to the broken and unbroken parts of the gauge group. The remaining dynamical and new flavor Chern-Simons terms are rendered non-anomalous by this set of fiducial fermions.

Although the master duality is slightly more complicated due to two independent mass deformations, we will see that the same mechanisms that lead to non-anomalous theories in both phases of the single-species non-Abelian cases completely generalize. Since the fiducial fermions make the positive mass phase non-anomalous and the fiducial/dynamical fermions – including the singlet – continue to work after Higgsing or integrating out negative mass fermions, all five phases of the master duality continue to be non-anomalous.

3.4 Master Duality with Boundaries

Now that we have firmly established how to derive the correct set of boundary conditions and assignments of fiducial fermions in order to render boundary theories non-anomalous in the single-species non-Abelian dualities, we can analyze the two-species master bosonization

	$SU(N)$ **Side**	$U(k)$ **Side**
Boundary Conditions	$\psi^-_{\alpha I}\vert_\partial = 0$	$\Psi^+_{\rho M}\vert_\partial = 0$
	$\phi_{\alpha M}$: Neumann	$\Phi_{\rho I}$: Neumann
	b': Neumann	c: Dirichlet
Additional Edge Modes	$k \times \mathcal{L}^-_{ff}[b' + \tilde{A}_1 \mathbb{1}_N]$	———
	$Nk \times \mathcal{L}^+_{ff}[0]$	$Nk \times \mathcal{L}^+_{ff}[0]$
	$SU(N)$ **Side**	$U(k)$ **Side**
Boundary Conditions	$\psi^-_{\alpha I}\vert_\partial = 0$	$\Psi^+_{\rho M}\vert_\partial = 0$
	$\phi_{\alpha M}$: Dirichlet	$\Phi_{\rho I}$: Dirichlet
	b': Dirichlet	c: Neumann
Additional Edge Modes	———	$N \times \mathcal{L}^+_{ff}[c' + \tilde{c} + \tilde{A}_1 \mathbb{1}_k]$
	$Nk \times \mathcal{L}^-_{ff}[\tilde{A}_1]$	$Nk \times \mathcal{L}^-_{ff}[\tilde{A}_1]$
	$Nk \times \mathcal{L}^+_{ff}[0]$	$Nk \times \mathcal{L}^+_{ff}[0]$

Table 3.6: The top (bottom) table counts the additional edge modes when choosing Neumann and Dirichlet boundary conditions on the dynamical gauge fields in SU (U) and U (SU) side respectively when $N_f \neq 0$ and $N_s \neq 0$.

duality. Having made the assignments in the common Phase II region, the fiducial fermions of the two single-species non-Abelian cases considered are consistent with one another – see Tables 3.4 and 3.5. We can then combine the two prescriptions and check their compatibility across all five mass deformed regions in Fig. 1.2.

We will analyze the phases on the U and SU sides roughly in order of increasing difficulty. The discussion will be kept brief for phases where cancellation is a straightforward generalization of what we have already observed in the single-species non-Abelian cases of Sec. 3.3. In the following analysis, we are interested in the assignments that render the theories non-anomalous, and so we will assume Neumann conditions on the dynamical gauge fields throughout. Although Neumann boundary conditions on both dynamical gauge fields does not yield a consistent duality, generalization to Dirichlet boundary conditions for one of the dynamical gauge fields is straightforward, see Sec. 3.3.3.

Phase II

This phase corresponds to $m_\psi < 0$ and $m_\phi^2 > 0$ on the SU side and $m_\Psi > 0$ and $m_\Phi^2 > 0$ on the U side. Starting from (3.4) and (3.3b), after integrating out all of the matter fields, we find that

$$i\mathcal{L}_{SU}^{II} = -k\mathrm{CS}_N[b] + \mathrm{BF}[f; \mathrm{Tr}_N(b) - N\tilde{A}_1 - N\tilde{A}_2] + Nk\mathrm{BF}[\tilde{A}_1; \tilde{A}_2] + Nk\mathrm{CS}_1[\tilde{A}_2], \quad (3.111)$$

After some simplification, (3.4) and (3.3b) reduce to

$$i\mathcal{L}_{SU}^{II} = -k\mathrm{CS}_N[b'] - Nk\mathrm{CS}_1[\tilde{A}_1], \quad (3.112)$$

$$i\mathcal{L}_{U}^{II} = N\mathrm{CS}_k[c'] + N\mathrm{CS}_k[\tilde{c} + \tilde{A}_1\mathbb{1}_k] - Nk\mathrm{CS}_1[\tilde{A}_1] + 2Nk\mathrm{CS}_{\mathrm{grav}}. \quad (3.113)$$

Since this was the phase of the duality where we chose all of our fiducial fermions such that the theory was non-anomalous, no further analysis is needed, and the assignments are listed in Table 3.6

Phase I

This phase corresponds to $m_\psi > 0$ and $m_\phi^2 > 0$ on the SU side and $m_\Psi > 0$ and $m_\Phi^2 < 0$ on the U side.

U Side

For $m_\Psi > 0$, the Chern-Simons levels are unaffected when integrating out the fermions. However because $m_\Phi^2 < 0$, the theory is in a spontaneously broken phase

$$i\mathcal{L}_U^I = N\left(\mathrm{CS}_{k-N_f}[c'] + \mathrm{CS}_{N_f}[C] + \mathrm{CS}_{k-N_f}[\tilde{c} + \tilde{A}_1\mathbb{1}_k] - (k - N_f)\,\mathrm{CS}_1[\tilde{A}_1] + 2k\mathrm{CS}_{\mathrm{grav}}\right) \quad (3.114)$$

As with the single-species non-Abelian case, the edge modes automatically split up to make the new Chern-Simons modes non-anomalous. The original N right-moving k-component fiducial fermions break in a manner completely analogous to (3.90). The modes coupling to the unbroken $U(k-N_f)$ render $N(\mathrm{CS}_{k-N_f}[c']+\mathrm{CS}_{k-N_f}[\tilde{c}+\tilde{A}_1\mathbb{1}_k])$ non-anomalous. The parts of the N_f-component modes coupling to $\tilde{A}_1$ can cancel with the fiducial fermions of opposite chirality which only couple to $\tilde{A}_1$, leaving only the C coupling. Hence, the $N\mathrm{CS}_{N_f}[C]$ and $-N\,(k-N_f)\,\mathrm{CS}_1[\tilde{A}_1]$ terms are also non-anomalous. Since the number of fiducial fermions hasn't changed at all, the gravitational Chern-Simons term is also still non-anomalous.

SU Side

On this side of the duality, we have $m_\psi > 0$ and $m_\phi^2 > 0$. Neither the scalar nor the fermion change the Chern-Simons terms when integrated out. Note that we have chosen the boundary conditions on the dynamical fermion such that we let the ψ DWFs exist in this phase.

The fact that the theory is non-anomalous, however, should be evident if we rewrite $\mathcal{L}_{SU}^{I}$ in terms of $\mathcal{L}_{SU}^{II}$,

$$\begin{aligned} i\mathcal{L}_{SU}^{I} &= i\mathcal{L}_{SU}^{II} + N_f\mathrm{CS}_N[b'] + N\mathrm{CS}_{N_f}[C] + NN_f\mathrm{CS}_1[\tilde{A}_1] + 2NN_f\mathrm{CS}_{\mathrm{grav}} \\ &= -(k-N_f)\mathrm{CS}_N[b'] + N\mathrm{CS}_{N_f}[C] - N(k-N_f)\mathrm{CS}_1[\tilde{A}_1] + 2NN_f\mathrm{CS}_{\mathrm{grav}}. \end{aligned} \tag{3.115}$$

We already have assigned the fiducial fermions so that the $i\mathcal{L}_{SU}^{II}$ is non-anomalous. Provided that the dynamical DWFs are enough to make the new Chern-Simons terms non-anomalous, the entire Lagrangian in (3.115) will be non-anomalous. Since the dynamical fermions couple to $b'+C+\tilde{A}_1$ this is indeed the case. The DWFs cancel with the existing N_f fiducial fermion edge modes, making (3.115) non-anomalous.

Phase III

This phase corresponds to $m_\psi < 0$ and $m_\phi^2 < 0$ on the SU side and $m_\Psi < 0$ and $m_\Phi^2 > 0$ on the U side.

SU Side

The gauge group is spontaneously broken in this phase, but since $m_\psi < 0$ we have no additional shift of Chern-Simons terms due to integrating out the fermion, relative to our fiducial fermion assignments of Phase II. Spontaneously breaking $SU(N)$ causes the Lagrangian to be modified to

$$i\mathcal{L}_{SU}^{III} = -\,k\mathrm{CS}_{N-N_s}[b] + \mathrm{BF}[f; \mathrm{Tr}_{N-N_s}(b) - N\tilde{A}_1 - N\tilde{A}_2] - k\mathrm{CS}_{N_s}[B] \tag{3.116}$$

$$+ Nk\mathrm{BF}[\tilde{A}_1; \tilde{A}_2] + Nk\mathrm{CS}_1[\tilde{A}_2], \tag{3.117}$$

After integrating out the Lagrange multiplier, we are left with

$$i\mathcal{L}_{SU}^{III} = -\,k\mathrm{CS}_{N-N_s}[b'] - k\mathrm{CS}_{N_s}[B] \tag{3.118}$$

$$- \frac{Nk}{N-N_s}\left(N\mathrm{CS}_1[\tilde{A}_1] + N_s\mathrm{BF}[\tilde{A}_1; \tilde{A}_2] + N_s\mathrm{CS}_1[\tilde{A}_2]\right). \tag{3.119}$$

The fact that we get such complicated Abelian Chern-Simons terms can be explained in a manner analogous to the non-Abelian SU Higgsing discussed earlier. Indeed, as we should expect, this expression matches (3.106). More precisely, the complicated breaking of the $SU(N)$ field can be simplified by transforming into a $U(N) \times U(1)$ field and breaking down the $U(N)$ field, and the Lagrange multiplier encodes a change in coupling to *both* Abelian factors $\tilde{A}_1$ and $\tilde{A}_2$. The splitting of the fiducial fermion modes once more occurs in a manner analogous to (3.105).

U Side

Since $m_\Phi^2 > 0$ the $U(k)$ symmetry remains unbroken, but the dynamical fermions change the Chern-Simons terms. The change in Chern-Simons terms and edge modes follows in a manner practically identical to (3.102).

Phase IVb

This phase corresponds to $m_\psi > 0$, $m_\phi^2 < 0$, $m_\Psi < 0$, and $m_\Phi^2 < 0$. Additionally, this will be the first phase where we have to worry about singlet fermions, and we have $m_s > 0$ in both theories.

SU Side

Similar to Phase III, the gauge group is spontaneously broken in this phase and this is slightly complicated by the fact this is the SU side. Additionally, the dynamical and singlet fermions contribute additional Chern-Simons terms relative to Phase II, but they also contribute dynamical DWFs which makes said terms automatically non-anomalous.

U Side

Here the $U(k)$ symmetry is spontaneously broken to $U(k-N_f)\times SU(N_f)$, but the dynamical fermion behavior is the same as that of Phase II. However, the singlet fermions have positive mass and thus shift a subset of the Chern-Simons level relative to that of Phase II. Although this is the first time we have seen the singlet fermion behaving differently from the dynamical fermions, there is nothing different about the way we end up at an non-anomalous theory. The singlet fermions give rise to DWFs which exactly compensate for their shift of the Chern-Simons levels in the bulk.

Phase IVa

This phase corresponds to $m_\psi > 0$, $m_\phi^2 < 0$, and $m_s < 0$ on the SU side and $m_\Psi < 0$, $m_\Phi^2 < 0$, and $m_s < 0$ on the U side. Again, this phase is a repeat of what we have already looked at in Phase IVb but with negative mass singlet fermions. For the U side, the singlet fermions have the same sign mass as the dynamical dynamical fermions and hence both contribute a shift to the Chern-Simons terms, but the *different* masses break the flavor symmetry between the two.

Lastly, let us comment on the scalar boundary conditions for the master duality. As with the single species non-Abelian cases considered above, we can deduce whether ϕ and Φ obey Neumann or Dirichlet boundary conditions by comparing the global symmetry currents.

Recall that when $N_s = 0$ the $\tilde{A}_2$ coupling vanished and the $\tilde{A}_1$ global symmetry could be attributed to the $U(1)_{m,b}$ symmetry. Meanwhile, when we took $N_f = 0$ in Sec. 3.3.3, $\tilde{A}_1$ and $\tilde{A}_2$ could be combined into a new background field $\tilde{B}$ which was then associated with its own $U(1)_{m,b}$ symmetry. For the case when both N_f and $N_s \neq 0$, the $\tilde{A}_1$ and $\tilde{A}_2$ background fields play the same roles. The combinations $\tilde{A}_1$ and $\tilde{A}_1 + \tilde{A}_2$ are associated with two $U(1)_{m,b}$ symmetries, one whose $U(1)_b$ part is the ψ matter, and the other, the ϕ matter. As such, all arguments of identifying global symmetries on either side of the duality to impose scalar boundary conditions still hold for the master duality, and so we find the same results, as shown in Table 3.6.

3.4.1 Generalization to SO and USp

Finally, we will briefly comment on the generalization of our methods to the versions of the master duality for the SO and USp groups in the presence of a boundary.

Accounting for the change to real fermions and scalars, there are half as many matter degrees of freedom as compared to the U/SU dualities, which can most easily be understood by starting with complex scalars and Dirac fermions and imposing a reality condition [3]. Explicitly for the USp duality, we will take ψ to be a Dirac fermion but require that $\psi_{\alpha I}\Omega^{\alpha\beta}\tilde{\Omega}^{IJ} = (\psi^{\beta J})^c$; with ψ^c the charge conjugate of ψ and $\Omega^{\alpha\beta}$ ($\tilde{\Omega}^{IJ}$) symplectic invariant tensor of $USp(2N)$ ($USp(2N_f)$). Hence, integrating out real fermions provides half the change in Chern-Simons level as that of a full Dirac fermion.

As with the U/SU case, the Chern-Simons terms are anomalous in the presence of a boundary. Fortunately, the fiducial fermion prescription used above can be generalized to be used with Majorana fermions. Alternatively, the fiducial Dirac fermions can still be used with the reality conditions discussed above. Thus, the SO and USp dualities can be rendered non-anomalous by rewriting Chern-Simons terms as fiducial Majorana fermions. Deriving the boundary conditions and DWFs for Majorana fermions follows similarly.

The global symmetries on either side of the master dualities also change slightly. For instance, the flavor symmetries of the fermions of the SO (USp) duality are now $SO(N_f)$

($USp(2N_f)$) on the left-hand side of (1.36) and (1.37), respectively. The fiducial Majorana fermions for a given SO or USp Chern-Simons term have an analogous "flavor" symmetry whose rank scales with the Chern-Simons level. Thus, when one chooses Dirichlet boundary conditions for the dynamical gauge field on one end of the duality, the fiducial fermions on the Neumann end once again share the same global symmetry on the boundary.

3.5 Discussion

Physical samples that we can drive to criticality and probe in a laboratory setting have boundaries, and too often conjectured dualities do not or cannot make explicit the role of boundary conditions. In order to understand what – if any – role dualities such as 2+1 dimensional master bosonization duality or any of its single species non-Abelian and Abelian limit cases play in describing physical critical systems, we must carefully analyze the admissible boundary theories consistent with bulk duality.

The relative simplicity of the gauge sector in the Abelian dualities hid an important aspect of the choice of boundary conditions for the dynamical gauge fields. In this chapter, we have reconciled the Abelian fiducial fermion prescription with those subtle aspects that are necessarily present in all non-Abelian bosonization dualities in 2+1 dimensions regardless of the types of fundamental matter considered. The important takeaway is that the additional complication of having dynamical gauge fields on both sides of the duality necessitated an alternating prescription of boundary condition such that Neumann conditions are mapped to Dirichlet conditions across the duality. As first observed in [50] and later elaborated in [45], the reason this change in boundary conditions is due to emergent global symmetries in the boundary theories that must match in order to be duality-compatible.

Beyond simply analyzing the gauge sectors, in the preceding sections, we have constructed the necessary duality-compatible boundary conditions and additional edge modes for the master bosonization duality for Chern-Simons-matter theories in [68, 25]. A non-trivial check on the analysis in this chapter has been the consistent reduction of the duality-consistent boundary conditions in the master bosonization duality to the Abelian case. The check furnished by the Abelian reduction also resolved a subtlety not addressed in [68, 25] regarding whether the Abelian gauge fields $U(1)_{m,b}$ and $U(1)_{F,S}$ were ordinary $U(1)$ or spin$_c$ connections. Further, the motivation of the boundary conditions on the scalar sector of the U side of the non-Abelian single species and master bosonization dualities discussed in 3.3 provides a more satisfying picture than the Abelian analysis in [9] had suggested. Lastly, the novel extension of the fiducial fermion prescription to SO and USp dualities filled out the spectrum of 2+1 dimensional bosonization dualities in the presence of a boundary.

That being said, there are further questions to ask in the context of 2+1 dimensional dualities involving Chern-Simons-matter theories in the presence of a boundary. As noted at the start of this chapter, at the core of all of the bosonization dualities sits the basic level-rank duality familiar from WZW theories. In the non-Abelian dualities, we *cannot* integrate out the non-Abelian Chern-Simons terms for dynamical fields in the massive phases. Since the dynamical fields are related by the level-rank duality rather than simply being the same, this has resulted in slightly different boundary theories. One could then wonder whether WZW-matter theories participate in other non-trivial level-rank dualities. To our knowledge, there has been little work done on the effects of level-rank duality for WZW theories with non-trivial matter sectors.

Chapter 4

BUILDING BOSONIC QUIVERS WITH 3D BOSONIZATION

In this chapter we discuss work where we use the master duality and methods similar to those developed in [69] to derive novel Bose-Bose dualities between non-Abelian linear quivers. We argue that a subset of these dualities can be viewed as a natural generalization of the bosonic particle-vortex duality to non-Abelian gauge groups since the quivers share many of the qualitative features present in the particle-vortex duality.

Of particular interest is the application of these dualities to $2+1$-dimensional defects in QCD theory on $\mathbb{R}^4$, which will be the focus of the latter half of this chapter. It has recently been shown that there is a mixed 't Hooft anomaly between time-reversal symmetry and center symmetry at $\theta = \pi$ [51]. This is rooted in the fact that $SU(N)$ YM theory is believed to have N distinct vacua associated to N branches of the theory. Such branches are individually $2\pi N$ periodic and correspond to $SU(N)/\mathbb{Z}_N$ gauge theories. This seems to contradict the long held belief that θ is 2π periodic in $SU(N)$ YM theory, but the conflict is resolved since the vacua interchange roles under a 2π transformation. More specifically, if one tracks the true ground state of the theory, one changes branches in a single 2π period. Thus, as theta is varied from, say $\theta = 0$ to $2\pi n$, the theory traverses several vacua. However, this changes when one couples the one-form center symmetry to a background (two-form) gauge field. In this case one cannot consistently choose the coefficient of the counterterm, sometimes referred to as the "discrete theta angle", to make the theory non-anomalous. Since this counterterm changes as one traverses branches, a spatially varying θ angle gives rise to domain walls separating regions with distinct discrete theta angle. Using anomaly inflow arguments, the effective field theory living on the interface is found to be a Chern-Simons gauge theory (see [51, 52] for more details).

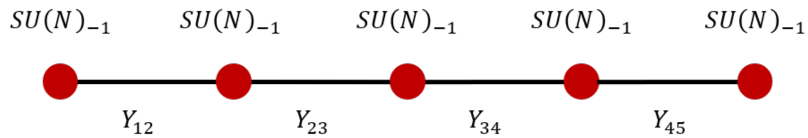

Figure 4.1: Parent Chern-Simons matter theory at the phase transition for the special case $n = 5$. Nodes represent gauge theories with the associated Chern-Simons term and links represent matter bifundamentally charged under the gauge groups on the adjacent nodes.

Although anomaly considerations require a non-trivial theory to live on the interface, they alone do not fully fix the theory. Among others, $[SU(N)_{-1}]^n$ or $SU(N)_{-n}$ would be consistent choices.[1] The authors of [52] argue that, at least at $n \ll N$, $[SU(N)_{-1}]^n$ is the appropriate description for slowly varying theta (meaning that $|\nabla\theta| \ll \Lambda$ where Λ is the strong coupling scale of the confining gauge theory), whereas $SU(N)_{-n}$ is appropriate for a sharp interface such as a discrete jump by $2\pi n$ at a given location. If these are indeed the correct descriptions this suggests that there is a phase transition as one smooths out a given jump in θ. If this phase transition is second order, the transition point would be governed by a CFT which is most easily realized as a Chern-Simons-matter theory. In any case, this CFT can serve as a parent theory from which topological field theories, describing either the slowly varying as well as the sharp step, can be realized as massive deformation.

The conjectured CFT between the two extreme phases is schematically

$$[SU(N)_{-1}]^n + \text{bifundamental scalars} \tag{4.1}$$

which was used in ref. [52] to explain the transition between two different vacua of $(3+1)$-dimensional Yang-Mills. This parent CFT is based on a quiver gauge theory as displayed in Fig. 4.1. Each node depicts a $SU(N)_{-1}$ Chern-Simons gauge theory, the links connecting them represent bifundamental scalar fields, Y. The theory has two obvious massive defor-

[1]Note that we are changing the direction of the θ gradient relative to [52] and so have negative levels for our Chern-Simons theories. This is in order to conform to the conventions of [70] for the stringy embeddings.

mations: we can give all the scalars a positive or a negative mass squared. In the former case the scalars simply decouple and we are left with the $[SU(N)_{-1}]^n$ TFT appropriate for slowly varying theta, in the latter case the gauge group factors get Higgsed down to the diagonal subgroup and we find the $SU(N)_{-n}$ associated with the steep defect. There are also mixed phases, where some of the Y have negative and some positive mass squared.

In this chapter, we propose a theory dual to (4.1) which is supported by both 3d bosonization of non-Abelian linear quivers and holographic duality. The proposed "theta wall" duality is

$$[SU(N)_{-1}]^n + \text{bifundamental scalars} \qquad \leftrightarrow \qquad U(n)_N + \text{adjoint scalars.} \tag{4.2}$$

We will see that this is a special case of the more general quiver dualities derived in Sec. 4.1 which do not include matter in the adjoint. This is a special feature of (4.2), owed to the fact that when all ranks of the SU quiver theory are equal, the U quiver contains nodes which are confining. With the careful addition of interactions in the proposed theories, mass deformations on either side of the duality yield TFTs which are level-rank dual to each other.

This chapter is outlined as follows. Sec. 4.1 contains our derivation of the non-Abelian linear quiver dualities, including the details of how such dualities should be viewed as generalization of the particle-vortex duality. We then specialize to quivers applicable to theta interfaces in $3+1$-dimensional $SU(N)$ Yang-Mills theory in Sec. 4.2. Subsections 4.2.1 and 4.2.2 contain the 3d bosonization and holographic support for such dualities, respectively. In Sec. 4.5 we discuss our results and conclude. The appendix contains several details of our construction of the non-Abelian quivers.

As we were finalizing this work, we were made aware of [17] which studies domain walls in different phases of the Witten-Sakai-Sugimoto model. This has some overlap with Sec. 4.2.2, particularly regarding the nature of domain walls in the pure YM sector.

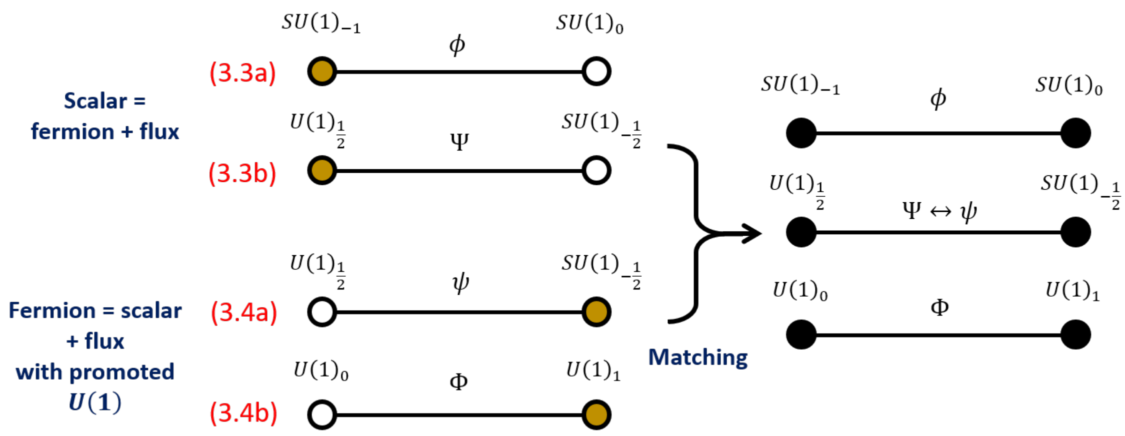

Figure 4.2: Derivation of the bosonic particle-vortex duality as a duality between two-node linear quiver theories. On the left-hand side, we have represented each side of Aharony's Abelian dualities as a two-node quiver. The filled yellow circle represents the color gauge group while the empty circle represents promoted global symmetries (which for the case of $SU(1)$ are placeholders). The equation numbers corresponding to the two-node quivers are shown in red. Since the two fermionic theories are the same, one can perform a matching to arrive at a duality between three two-node quiver theories, the top and bottom of which are the XY and Abelian Higgs models, respectively.

4.1 Non-Abelian Linear Quiver Dualities

We now turn to constructing linear quivers using the master duality. As explained in the introduction, we are ultimately motivated by the theta wall construction that leads to (4.2), but we will derive dualities for a far more general case. We will begin with recasting the 3d bosonization derivation of bosonic particle-vortex duality in a way that highlights the relation to the non-Abelian quivers.

4.1.1 Bosonic Particle-Vortex Duality

To derive the bosonic particle-vortex duality we will use 3d bosonization techniques similar to those used in refs. [83, 112]. We then show how one can reinterpret the derivation in terms of a two-node quiver. This will be the simplest non-trivial case of the far more general quivers we derive in Sec. 4.1.2. We will drop tildes from Abelian gauge fields in this subsection since the distinction is not necessary.

Recall the bosonic particle-vortex duality states that, at low energies, the XY model is dual to the Abelian Higgs model [105, 40],

$$\mathcal{L}_{\text{XY}} = |D_{A_1}\phi|^2 \qquad \leftrightarrow \qquad \mathcal{L}_{\text{AH}} = |D_c\Phi|^2 - i\left[-\frac{1}{2\pi}cdA_1\right]. \tag{4.3}$$

The mapping of the phases is such that positive mass deformations on one end maps to a negative deformation on the other end, $m_\Phi^2 \leftrightarrow -m_\phi^2$.

In order to derive (4.3) we start by taking the Abelian limit of Aharony's dualities, (1.16). In particular, take the $N = k = N_f = 1$ and $N_s = 0$ limit of (1.28), which yields the "scalar + $U(1)_1 \leftrightarrow$ free fermion" duality,

$$\mathcal{L}_{SU} = i\bar{\psi}\not{D}_{A_1}\psi \tag{4.4a}$$

$$\mathcal{L}_{U} = |D_c\Phi|^2 - i\left[\frac{1}{4\pi}cdc - \frac{1}{2\pi}cdA_1\right], \tag{4.4b}$$

with $m_\psi \leftrightarrow -m_\Phi^2$. Meanwhile, the "fermion + $U(1)_{-1/2} \leftrightarrow$ WF scalar" duality is obtained by taking the $N = k = N_s = 1$ and $N_f = 0$ limit,

$$\mathcal{L}_{SU} = |D_{A_1}\phi|^2 - i\left[-\frac{1}{4\pi}A_1dA_1\right], \tag{4.5a}$$

$$\mathcal{L}_{U} = i\bar{\Psi}\not{D}_c\Psi - i\left[\frac{1}{4\pi}cdc - \frac{1}{2\pi}cdA_1\right], \tag{4.5b}$$

with $m_\phi^2 \leftrightarrow m_\Psi$.

Deriving the bosonic particle-vortex duality from the above two dualities is straightforward. Note that we already have the XY model in (4.5a) up to the additional background Chern-Simons term. Hence, we should look for another bosonic theory dual to (4.5b). To do so, add $-i\left[\frac{1}{4\pi}A_1 dA_1 - \frac{1}{2\pi}A_1 dB_1\right]$ to each side of (4.4) and promote the $U(1)$ background field to be dynamical, $A_1 \to a_1$. This gives the dual theories

$$\mathcal{L}'_{SU} = i\bar{\psi}\not{D}_{a_1}\psi - i\left[\frac{1}{4\pi}a_1 da_1 - \frac{1}{2\pi}a_1 dB_1\right] \tag{4.6a}$$

$$\mathcal{L}'_{U} = |D_c\Phi|^2 - i\left[\frac{1}{4\pi}cdc - \frac{1}{2\pi}cda_1 + \frac{1}{4\pi}a_1 da_1 - \frac{1}{2\pi}a_1 dB_1\right]. \tag{4.6b}$$

Since the action is quadratic in the newly promoted a_1 field we can integrate it out, which imposes the constraint $a_1 = c + B_1$. Plugging this in, we find

$$\mathcal{L}'_{U} = |D_c\Phi|^2 - i\left[-\frac{1}{2\pi}cdB_1 - \frac{1}{4\pi}B_1 dB_1\right]. \tag{4.7}$$

After relabeling the dynamical field in (4.6a) as $a_1 \to c$ and changing the background field $B_1 \to A_1$, we see (4.6a) matches (4.5b), and thus (4.7) is dual to (4.5a). Canceling the common background Chern-Simons term, we arrive at the usual particle-vortex duality, (4.3). Note we get the relative mass flipping between the two ends of the duality since there is only a relative sign flip between ψ and Φ masses.

We would now like to recast the derivation we just performed to motivate generalization to a two-node linear quiver. Fig. 4.2 schematically shows how we would like to view the derivation. Each of our dual theories in (4.5) and (4.6) can be viewed as a two-node linear quiver, with the matter bifundamentally charged under the two nodes which it connects.

This is motivated by the fact that in Aharony's dualities (1.13), each matter field is fundamentally charged under both a dynamical gauge field and background global flavor symmetry. If we were to promote said flavor symmetry to be dynamical, the matter becomes a bifundamental and thus admits a natural description as a two-node quiver. This looks rather trivial since $SU(N)$ gauge groups for $N = 1$ are nonsensical, but will generalize nicely for $N \geq 2$. For the Abelian case we will use $SU(1)$ as a placeholder for symmetries that can be gauged in the more general case.

To see this on the Abelian Higgs side, we will first shift the dynamical gauge field,

$c \to c + a_1$, so that (4.6b) becomes

$$\mathcal{L}''_U = |D_{c+a_1}\Phi|^2 - i\left[\frac{1}{4\pi}cdc - \frac{1}{2\pi}a_1 dB_1\right]. \tag{4.8}$$

In this form the scalar is bifundamentally charged under two $U(1)$ gauge groups, which represent the two nodes in the quiver theory. The dual to the Abelian Higgs model, (4.6a), couples to a single dynamical $U(1)$ gauge field, a_1. This was previously the flavor symmetry but was promoted to a gauge symmetry in moving from (4.4) to (4.6). As mentioned above, the gauge field belonging to the second node is absent only because we are working in the Abelian limit of Aharony's dualities. On the XY model end of the duality (4.5a), ϕ couples to two $SU(1)$ fields, so it has no gauge couplings at all.[2]

The upshot of recasting the derivation in this form is that it readily generalizes to more complicated two-node quivers. One can use the more general Aharony's dualities to perform very similar steps as was done in the Abelian case. We'll see particle-vortex duality generalizes to a duality of the form

$$\begin{aligned} &SU(N_1)_{-k_1} \times SU(N_2)_{-k_2} + \text{bifundamental scalar} \\ &\qquad\qquad \leftrightarrow \qquad U(k_1)_{N_1-N_2} \times U(k_1+k_2)_{N_2} + \text{bifundamental scalar}. \end{aligned} \tag{4.9}$$

The bosonic particle-vortex duality is then just the $N_1 = N_2 = k_1 = 1$ and $k_2 = 0$ case. In Sec. 4.1.3 we present further evidence of this interpretation by matching the spectrum of particles and vorticies in (4.9) in a manner similar to the Abelian case. Before we do this we demonstrate how we can systematically construct the non-Abelian quivers for an arbitrary number of nodes. This requires the use of the master duality when the number of nodes is greater than two.

4.1.2 Building Non-Abelian Linear Quiver Dualities

Following the discussion in the previous subsection, our strategy in deriving dual descriptions of quiver gauge theories is to start with the master duality and gauge global symmetries on

[2]For our purposes here, we are ignoring the possibility of gauging the $U(1)$ global symmetry since its properties are well established in the particle-vortex duality as a global symmetry.

both sides of the duality in order to arrive at a duality for the resulting product gauge group. Since in a quiver gauge theory the gauge group associated with a given node sees the gauge groups associated with the neighboring nodes as global flavor symmetries, this roughly speaking amounts to dualizing the quiver one node at a time. While not a proof, this procedure suggests the resulting theories are dual. This basic idea had previously been pursued in ref. [69] using Aharony's duality, but the flavor bounds put severe limitations on the quivers that were amendable to this analysis. In particular, the most interesting case with equal rank gauge groups on each node was out of reach. We will see that the master duality will help overcome many of these limitations.

To streamline the derivation it is helpful to follow ref. [69] and rearrange BF terms to group the $SU(N)$ and $U(1)$ global symmetries together. Additionally, a key ingredient in matching this analysis to the existing particle-vortex duality will be the global $U(1)$ symmetries on either side of the duality. As such, we will be especially careful in keeping track of the global symmetries at every step.

We start by recalling how ref. [69] derived their quiver transformations and generalize their method to the master duality. Starting from (1.23b), one can use the fact the $U(k)$ field can be separated into its Abelian and non-Abelian parts, i.e. $c = c' + \tilde{c}\mathbb{1}_k$, to perform a shift on the Abelian portion, $\tilde{c} \to \tilde{c} + \tilde{A}_1$. This allows one to rewrite $\mathcal{L}_U$ as

$$\mathcal{L}_U = i\bar{\Psi}\not{D}_{c+B+\tilde{A}_1}\Psi - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right) - \frac{Nk}{4\pi}\tilde{A}_1 d\tilde{A}_1\right]. \tag{4.10}$$

Canceling the overall factor of $i\frac{Nk}{4\pi}\tilde{A}_1 d\tilde{A}_1$ on either side of the duality and defining the new $U(N_s)$ background field $G_\mu \equiv B_\mu + \tilde{A}_{1\mu}\mathbb{1}_{N_s}$, (1.23b) becomes

$$\mathcal{L}_{SU} = |D_{b'+G}\phi|^2 - i\left[-\frac{k}{4\pi}\mathrm{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right)\right] \tag{4.11a}$$

$$\mathcal{L}_U = i\bar{\Psi}\not{D}_{c+G}\Psi - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right)\right]. \tag{4.11b}$$

The procedure used in [69] to derive new dualities is to promote the non-Abelian $U(N_s)$ global symmetry to be dynamical. Since both the ϕ and Ψ matter is charged under G, this turns the matter into bifundamentals. Schematically, we denote the promoted duality as

$$SU(N)_{-k} \times U(N_s)_0 \qquad \leftrightarrow \qquad U(k)_{N-N_s/2} \times U(N_s)_{-k/2}. \tag{4.12}$$

This is subject to the flavor bound $N \geq N_s$.

In promoting the $U(1)$ global symmetry to a gauge symmetry, we get another $U(1)$ global symmetry which couples to the new gauge current on either side of the duality. If we wanted to make the coupling to the new background gauge field $\tilde{B}_1$ explicit, we would add a $-i\frac{1}{2\pi}\tilde{A}_1 d\tilde{B}_1$ term to each side of the duality. This is completely analogous to the procedure preformed in ref. [83], where a new BF term was included with each promotion to represent the new $U(1)$-monopole symmetry on each side of the duality.

Below, we will sometimes apply this duality to strictly SU gauge fields, in which case it is advantageous to only gauge the SU part of the flavor symmetry, so that (4.12) becomes

$$SU(N)_{-k} \times SU(N_s)_0 \qquad \leftrightarrow \qquad U(k)_{N-N_s/2} \times SU(N_s)_{-k/2}. \tag{4.13}$$

Note that in this form of the duality each side retains the original global $U(1)$ symmetries and we do not obtain the additional global $U(1)$ as above.

We could apply the same procedures to the case where the SU side contains the fermion and the U side contains the scalar, where we would then find

$$SU(N)_{-k+N_f/2} \times U(N_f)_{N/2} \qquad \leftrightarrow \qquad U(k)_N \times U(N_f)_0, \tag{4.14}$$

which matches the result found in ref. [69] up to an overall shift in the level of the background term. This case is considered in more detail in Appendix D.1.

Now let us perform similar manipulations to the master duality in (1.28). Since the Chern-Simons terms on the U side are identical to (1.19b), performing the same manipulations, (1.28b) becomes

$$\mathcal{L}_U = \left|D_{c+\tilde{A}_1+C}\Phi\right|^2 + i\bar{\Psi}\not{D}_{c+\tilde{A}_1+B+\tilde{A}_2}\Psi + \mathcal{L}'_{\text{int}} - i\left[\frac{N_1}{4\pi}\text{Tr}_{k_1}\left(cdc - i\frac{2}{3}c^3\right) - \frac{N_1k_1}{4\pi}\tilde{A}_1 d\tilde{A}_1\right]. \tag{4.15}$$

Again, we cancel the common $\tilde{A}_1$ Chern-Simons terms on either side of the duality. It will also be convenient to perform a shift to move the $\tilde{A}_2$ fields onto the ψ and Φ matter, so we take $\tilde{A}_1 \to \tilde{A}_1 - \tilde{A}_2$ on either side of the duality. We could now combine the $U(1)$ and $SU(N_f)$ global symmetries into the definition of $E_\mu = C_\mu + \tilde{A}_{1\mu}\mathbb{1}_{N_f}$ as we did in (4.12). However, we will hold off on doing this since it is more convenient to keep the two global symmetries separate for our purposes. This leaves us with a duality of the form

$$\mathcal{L}_{SU} = \left|D_{b'+B+\tilde{A}_1}\phi\right|^2 + i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1-\tilde{A}_2}\psi + \mathcal{L}_{\text{int}} - i\left[\frac{N_f-k}{4\pi}\text{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right)\right]$$
$$- i\left[\frac{N}{4\pi}\text{Tr}_{N_f}\left(CdC - i\frac{2}{3}C^3\right) + \frac{NN_f}{4\pi}(\tilde{A}_1-\tilde{A}_2)d(\tilde{A}_1-\tilde{A}_2)\right], \tag{4.16a}$$

$$\mathcal{L}_U = \left|D_{c+C+\tilde{A}_1-\tilde{A}_2}\Phi\right|^2 + i\bar{\Psi}\not{D}_{c+B+\tilde{A}_1}\Psi + \mathcal{L}'_{\text{int}} - i\left[\frac{N}{4\pi}\text{Tr}_k\left(cdc - i\frac{2}{3}c^3\right)\right]. \tag{4.16b}$$

At this point, we have two choices with how to treat the global symmetry associated with $\tilde{A}_2$. The first choice is to simply leave it as a global symmetry and gauge only the $SU(N_f)$ flavor symmetry associated with C. In this form, each side of the master duality retains the $U(1)$ global symmetries associated with $\tilde{A}_1$ and $\tilde{A}_2$. Alternatively, we could also gauge the global symmetry associated with $\tilde{A}_2$. The latter of these cases will be useful for our purposes in this chapter, so we define a $U(N_f)$ gauge field $G_\mu \equiv C_\mu - \tilde{A}_{2\mu}\mathbb{1}_{N_f}$ to which ψ and Φ couple. After gauging the $U(N_f)$ and $SU(N_s)$ global symmetries, this leaves us with

$$SU(N)_{-k+N_f/2} \times U(N_f)_{N/2} \times SU(N_s)_0 \quad \leftrightarrow \quad U(k)_{N-N_s/2} \times U(N_f)_0 \times SU(N_s)_{-k/2}. \tag{4.17}$$

Similar to Aharony's duality, in gauging the $U(N_f)$ symmetry which is associated with G, we pick up an additional monopole $U(1)$ symmetry on either side of the duality which couples to the newly gauged $\tilde{A}_2$ field. We will denote the background gauge field associated with said symmetry by $\tilde{B}_2$. For completeness, we consider the master duality with all global symmetries gauged in Appendix D.1.

Note that we can modify either of the above dualities by adding additional background flavor levels to either side of the duality before promotion. As a reminder, these dualities are subject to the flavor bound $k \geq N_f$ and $N \geq N_s$, but $(k, N) \neq (N_f, N_s)$. In the $N_f = 0$ and $N_s = 0$ limits, (4.17) reduces to (4.13) and (4.14), respectively, with appropriate relabeling.

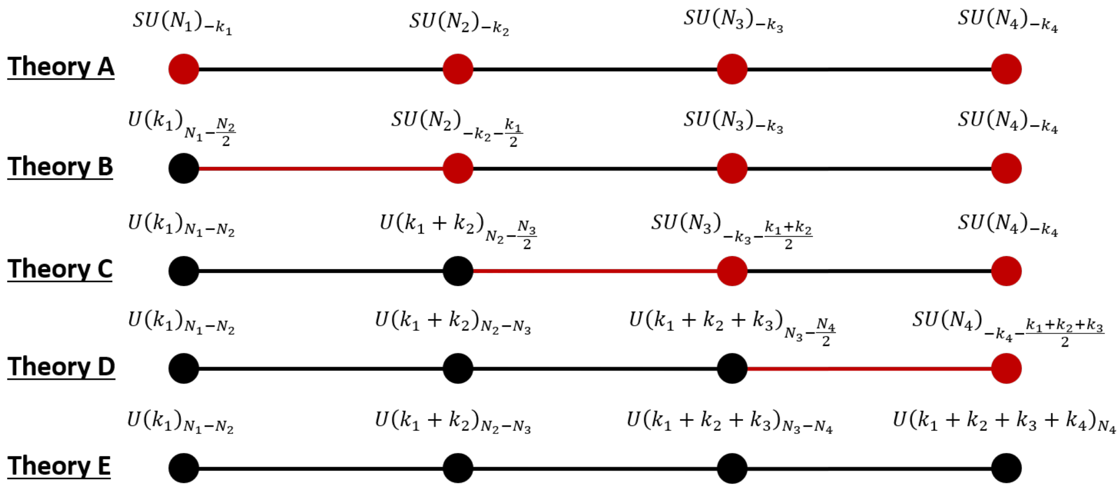

Figure 4.3: Dualizing a linear quiver. Red nodes are SU gauge groups and black nodes are U groups. Black (red) links are bifundamental bosons (fermions). Applying Aharony's duality to the leftmost link turns the scalar into a fermion. Then, applying the master duality repeatedly moves said fermion across the quiver until it reaches the final link where Aharony's duality can again be used to turn the fermion into a boson.

Four-Node Example

Let us now use the dualities we've defined to dualize a four-node quiver. Walking through this construction will make generalization to the n-node case straightforward. We begin with the SU side of the theory

$$\text{Theory A:} \qquad SU\left(N_1\right)_{-k_1} \times SU\left(N_2\right)_{-k_2} \times SU\left(N_3\right)_{-k_3} \times SU\left(N_4\right)_{-k_4}. \tag{4.18}$$

This theory has the bifundamental scalars which have charges as given in Table 4.1. In what follows we will assume that $N_1 \geq N_2 \geq N_3 \geq N_4$ as well as $k_i \geq 0$ for $i = 1, 2, 3, 4$. Although this is not the most general case, below we will find this is required to avoid flavor bounds to get to the desired U theory. Each of the bifundamentals is charged under a global $U(1)$ symmetry which rotates its overall phase, giving this side of the duality a $[U(1)]^3$ global symmetry.

Theory A	$SU(N_1)_{-k_1}$	$SU(N_2)_{-k_2}$	$SU(N_3)_{-k_3}$	$SU(N_4)_{-k_4}$
$Y_{1,2}$	$\square$	$\square$	1	1
$Y_{2,3}$	1	$\square$	$\square$	1
$Y_{3,4}$	1	1	$\square$	$\square$

Table 4.1: Charges of the bifundamental matter in our linear quivers. $\square$ denotes the matter transforms in the fundamental representation of the corresponding gauge group.

We will denote the bifundamental scalars living between nodes j and $j+1$ by $Y_{j,j+1}$ and $X_{j,j+1}$ on the SU and U side of the duality, respectively. The masses of the U bifundamentals are denoted by $m_{j,j+1}$ while we will use $M_{j,j+1}$ for those on the SU side.

Before embarking on deriving the duality, note that the uniform mass deformations of (4.18) are given by

$$\text{(A1)} \qquad M^2_{i,i+1} > 0: \qquad SU(N_1)_{-k_1} \times SU(N_2)_{-k_2} \times SU(N_3)_{-k_3} \times SU(N_4)_{-k_4} \tag{4.19a}$$

$$\begin{aligned} \text{(A2)} \qquad M^2_{i,i+1} < 0: \qquad & SU(N_1 - N_2)_{-k_1} \times SU(N_2 - N_3)_{-k_1-k_2} \times SU(N_3 - N_4)_{-k_1-k_2-k_3} \\ & \times SU(N_4)_{-k_1-k_2-k_3-k_4} \, . \end{aligned} \tag{4.19b}$$

Here we have been careful to account for which gauge group is Higgsed by each bifundamental scalar. The bifundamental scalar $Y_{i,i+1}$ has $N_i \times N_{i+1}$ components. Below we will always view the smaller of the two gauge groups to be associated with the "flavor" symmetry of the bifundamentals. As such, if we assume the Higgsing to be maximal, the Higgsing can be thought of as acting on the "color" gauge group, i.e. the group with the larger rank, while leaving the flavor group unchanged.[3] Since to meet flavor bounds below we have assumed

[3]For example, for a bifundamental coupled to $SU(N_1)_c$ and $SU(N_2)_f$ with $N_1 \geq N_2$, what occurs can be best understood by first splitting $SU(N_1)_c \times SU(N_2)_f \to SU(N_1 - N_2)_c \times SU(N_2)_c \times SU(N_2)_f$. Since the bifundamental is maximally Higgsed in the $SU(N_2)$ subgroup, the unbroken part of the two $SU(N_2)$ factors is their diagonal, leaving $SU(N_1 - N_2)_c \times SU(N_2)_{\text{diag}}$. Thus, saying the flavor group is unchanged is a merely a convenient relabeling. Also note that the Chern-Simons level of the new flavor group will be the sum of the original flavor and color Chern-Simons levels.

$N_1 \geq N_2 \geq N_3 \geq N_4$, this means vacuum expectation value takes the form

$$\langle Y_{i,i+1}^{a_i a_{i+1}} \rangle \propto \begin{pmatrix} \mathbb{1}_{N_{i+1}} \\ 0 \end{pmatrix}^{a_i a_{i+1}} \tag{4.20}$$

with a_i, b_i and a_{i+1}, b_{i+1} gauge indices to $SU(N_i)$ and $SU(N_{i+1})$, respectively. Reassuringly, no gauge group acquires a negative rank with this Higgsing pattern.

Returning to our derivation of the non-Abelian linear quiver dualities, we will now show five theories are dual to one another,

$$\text{Theory A} \quad \leftrightarrow \quad \text{Theory B} \quad \leftrightarrow \quad \cdots \quad \leftrightarrow \quad \text{Theory E}, \tag{4.21}$$

by sequentially dualizing each node from left to right, see Fig. 4.3. To begin, we apply Aharony's duality (4.13) to the first node to obtain

$$\text{Theory B:} \qquad U\left(k_1\right)_{N_1-N_2+\frac{N_2}{2}} \times SU\left(N_2\right)_{-k_2-k_1+\frac{k_1}{2}} \times SU\left(N_3\right)_{-k_3} \times SU\left(N_4\right)_{-k_4}. \tag{4.22}$$

Flavor bounds are satisfied so long as $N_1 \geq N_2$. This turns the bifundamental scalar on the link between nodes one and two into a bifundamental fermion. The $U(1)$ global symmetry of the first bifundamental becomes a monopole symmetry for the Abelian part of the gauge field which lives on the first node. This will be a common theme as we sequentially step through nodes and the details are shown in Appendix D.2.

Dualizing the $SU(N_2)$ node is where we will need to use something new. We could try applying Aharony's dualities (4.12) or (4.14) to the second node. However, one will inevitably run into flavor bound issues since nodes with links on two sides require a $SU(N_{i-1}+N_{i+1})$ flavor symmetry, which exceeds the $SU(N_i)$ color symmetry for the cases we are interested in here.

Notice that since the master duality has two types of matter it has two *separate* flavor symmetries, each subject to its own flavor bound. This is useful for dualizing the nodes with two links and, furthermore, has the correct matter content since node two in Theory B has both a bifundamental scalar and fermion attached to it. However, the master duality is quite a bit different from Aharony's original dualities in that it requires additional interactions terms between the scalars and fermions on a given side of the duality, as given in (1.30). Let us consider how we could introduce such interactions terms and how they affect the theories we are considering.

Including Bifundamental Interactions

The interaction we need in order to apply the master duality in theory B is

$$\text{Theory B:} \qquad C_2^{(B)} \left(\bar{\psi}_{1,2}^{a_2 a_1} Y_{a_2 a_3}^{2,3}\right) \left(Y_{2,3}^{\dagger b_2 b_3} \psi_{b_2 a_1}^{1,2}\right). \tag{4.23}$$

Here $C_2^{(B)}$ is the coefficient of the interaction on the second node in theory B and we have not yet committed to its sign or magnitude, but we will do so later by matching TFTs. In what follows, it will be useful to associate each interaction term with one of the interior nodes of the quiver.

In order to give rise to (4.23), we must backtrack slightly since a similar interaction should also then be present in theory A in its dualized form. The exact matching of the interactions between theories A and B is quite subtle and requires auxiliary field techniques that were originally introduced in the large N and k literature [96]. Here we will only give a schematic overview. The full details of this matching are given in Appendix D.3.

Recall that the purpose of the interaction term in the master duality is to ensure that when the scalars acquire a vacuum expectation value we also gain an additional mass term for the fermions [25, 68]. This was vital for matching the phases and TFTs on either side of the duality. Importantly, regardless of the sign of the mass deformation, the fermions never condense and thus there is no opportunity for a fermion condensate to influence the mass of the scalars through the same interaction term. If the fermions did condense, this would yield a very different looking phase diagram than that found in refs. [25, 68].

Here we identify the fermions with scalars, which *can* condense when their quadratic term goes negative. In order to match the TFT phase diagrams of theories A and B under mass deformations, we need to make sure the interaction term does not allow the $Y_{1,2}$ condensate to influence the $Y_{2,3}$ mass. In Appendix D.3 we derive an interaction term which has the desired properties: this term will cause a nonzero vacuum expectation value for the $Y_{2,3}$ bifundamental to give a positive or negative mass to $Y_{1,2}$, depending on the sign of the coefficients $C_2^{(A)}$. However, the opposite effect cannot occur: $Y_{1,2}$ acquiring a vacuum expectation value *cannot* influence the mass of $Y_{2,3}$. More generally, for the SU side of the duality, the vacuum expectation value of a link can only affect nodes/links to its *left*. The interaction term is unidirectional as it is for the original master duality.

We will *schematically* denote the interactions we add to Theory A as

$$\text{Theory A:} \qquad C_2^{(A)} \left(Y_{1,2}^{\dagger a_2 a_1} Y_{a_2 a_3}^{2,3} \right) \left(Y_{2,3}^{\dagger b_2 a_3} Y_{b_2 a_1}^{1,2} \right) \tag{4.24}$$

with the understanding that the true interaction is as given in (D.26). Eq. (4.24) is equivalent to (D.26) if we simply ignore the fact that when the $Y_{1,2}$ acquires a vacuum expectation value the interaction term gives a mass to $Y_{2,3}$, so we will do so henceforth for brevity. An analogous interaction term is added to node three as well since it will be needed when stepping from theory C to D.

Having introduced the necessary interaction term, we subsequently apply the master duality (4.17) to the second and third nodes, this gives

$$\text{Theory C:} \qquad U(k_1)_{N_1-N_2} \times U(k_1+k_2)_{N_2-\frac{N_3}{2}} \times SU(N_3)_{-k_3-k_1-k_2+\frac{k_1+k_2}{2}} \times SU(N_4)_{-k_4} \tag{4.25}$$

$$\begin{aligned} \text{Theory D:} \qquad & U(k_1)_{N_1-N_2} \times U(k_1+k_2)_{N_2-N_3} \times U(k_1+k_2+k_3)_{N_3-\frac{N_4}{2}} \\ & \times SU(N_4)_{-k_4-k_1-k_2-k_3+\frac{k_1+k_2+k_3}{2}} . \end{aligned} \tag{4.26}$$

These in turn require the flavor bounds $N_2 \geq N_3, k_2 \geq 0$ and $N_3 \geq N_4, k_3 \geq 0$, respectively.[4] Each application of the master duality changes a boson link to a fermion link and vice versa, effectively driving the single fermion link down the quiver, see Fig. 4.3. As with the duality

[4]More precisely, this should exclude the double saturation cases where $N_2 = N_3$ and $k_2 = 0$ or $N_3 = N_4$ and $k_3 = 0$. We will not make note of such special cases henceforth since they will not be relevant for our purposes.

relating theories A and B, the application of the master duality above changes the global $U(1)$ symmetry across the duality. Specifically, it changes the $U(1)$ global symmetry under which $Y_{2,3}$ was charged to a monopole-like symmetry which couples to the Abelian part of gauge field on the second node. A completely analogous transformation occurs for the baryon number symmetry of $Y_{3,4}$. The details of how this occurs are shown in Appendix D.2.

Finally, to arrive at the desired dual theory we again apply Aharony's duality (4.14) to the last node. Flavor bounds require $k_4 \geq 0$. This ultimately gives

$$\text{Theory E:} \qquad U(k_1)_{N_1-N_2} \times U(k_1+k_2)_{N_2-N_3} \times U(k_1+k_2+k_3)_{N_3-N_4} \times U(k_1+k_2+k_3+k_4)_{N_4}\,. \tag{4.27}$$

Note that the fourth node does not pick up a monopole-like global symmetry for its Abelian gauge field. This is related to the fact we have one more node than bifundamentals and is also a feature of the dualities in ref. [69] and ABJM theory [4].

Following the mass identifications through the dualities, we see $M^2_{i,i+1} \leftrightarrow -m^2_{i,i+1}$. The uniform mass deformations of Theory E are

$$\text{(E1)} \qquad m^2_{i,i+1} < 0: \qquad U(k_1)_{N_1} \times U(k_2)_{k_2} \times U(k_3)_{N_3} \times U(k_4)_{N_4} \tag{4.28a}$$

$$\text{(E2)} \qquad m^2_{i,i+1} > 0: \qquad U(k_1)_{N_1-N_2} \times U(k_1+k_2)_{N_2-N_3} \times U(k_1+k_2+k_3)_{N_3-N_4} \times U(k_1+k_2+k_3+k_4)_{N_4} \tag{4.28b}$$

which, reassuringly, are level-rank dual to the phases considered above in (4.19). Here once again some care is required for the Higgs phase. Since we are assuming all $k_i \geq 0$ in order to meet the flavor bounds above, the maximal Higgsing vacuum expectation value is, using block matrix notation,

$$\langle X^{a_i a_{i+1}}_{i,i+1} \rangle \propto \begin{pmatrix} \mathbb{1}_{K_i} & 0 \end{pmatrix}^{a_i a_{i+1}}. \tag{4.29}$$

where a_i, a_{i+1} are gauge indices to $U(K_i)$ and $U(K_{i+1})$, respectively and we have defined the shorthand

$$K_j \equiv \sum_{i=1}^{j} k_i. \tag{4.30}$$

Of course, as we apply all the aforementioned dualities the matter interaction terms are changing as well. We also end up with the interaction term (D.25) between adjacent bifundamental scalars for theory E. The interaction is such that a bifundamental scalar vacuum expectation can only affect nodes/links to its *right* now. [5] The analog of (4.24), which is a schematic stand-in for (D.25), is

$$\text{Theory E:} \qquad C_2^{(E)} \left(X_{1,2}^{\dagger a_2 a_1} X_{a_2 a_3}^{2,3} \right) \left(X_{2,3}^{\dagger b_2 a_3} X_{b_2 a_1}^{1,2} \right). \tag{4.31}$$

where now we ignore the fact that when $X_{2,3}$ acquires a vacuum expectation value it gives a mass to $X_{1,2}$.

Effect of Interactions

We would now like to show that these interaction terms are vital for a matching the mass deformed TFTs. Although we found a matching between phases for the completely gapped/Higgsed phases above, these were very special cases. In order to observe the expected partial gapping/Higgsing behavior to apply to theta walls, we need to carefully treat the interactions.

First it will be helpful to specialize to a particular sign and magnitude of interaction terms coefficients, $C_I^{(A)}$ and $C_I^{(E)}$ for $I = 2, 3$ (i.e. all internal nodes). Specifically for the purposes of matching onto the phases of (4.1), we take $C_I^{(A)} < 0$ and $C_I^{(E)} < 0$. [6] We also assume $|C_I^{(E)}| \to \infty$ such that $|C_I^{(E)}| \gg |m_{i,i+1}^2|$. Although not considered in [25, 68], it is straightforward to check that a very large interaction term on one side of the master duality implies a very small interaction term on the other side of the duality.[7] Hence, $C_I^{(A)} \to 0$ so

[5] This is most obvious to see when moving from Theory C to Theory D. In theory C, the $X_{1,2}$ bifundamental can influence the mass of the fermion on the $2, 3$ link, but not vice versa. Hence, in Theory D the interaction between $X_{1,2}$ and $X_{2,3}$ should obey the same rule to get a matching of TFTs.

[6] We must choose the coefficients of the interactions to be the same sign for a matching of TFTs. To see this, first note that we use the master duality once on each internal node, and under the master duality the interaction term flips sign [25, 68]. An additional sign flip comes from the application of the master duality for the node to the left of an internal node, which sign flip when changing the fermions in the interaction to bosons.

[7] To see this, let us specialize to the notation used in [68]. Here we saw that by changing the sign of c_4

the hierarchy $M^2_{i,i+1} \gg C_I^{(A)} > 0$ holds for all mass deformations. In such a limit we can effectively ignore the interaction terms on the SU side of the duality.

The choice of magnitudes above has the added effect of simplifying the analysis of the interaction terms and the TFT structure. It is possible to derive quiver theories for more general interaction coefficients and still find matching TFTs, but we leave such analysis for future work.

Let us now consider the effect of the interaction terms on the U side of the duality. The two interaction terms of theory E are given by

$$C_2^{(E)} \left(X_{1,2}^{\dagger a_2 a_1} X_{a_2 a_3}^{2,3} \right) \left(X_{2,3}^{\dagger b_2 a_3} X_{b_2 a_1}^{1,2} \right) + C_3^{(E)} \left(X_{2,3}^{\dagger a_3 a_2} X_{a_3 a_4}^{3,4} \right) \left(X_{3,4}^{\dagger b_3 a_4} X_{b_3 a_2}^{2,3} \right). \tag{4.32}$$

Consider the case when $X_{1,2}$ acquires a nonzero vacuum expectation value as in (4.29). This term breaks $U(K_2)$ down to $U(K_2 - K_1)$, which is the usual effect of the Higgsing. Additionally, the first interaction term (4.32) becomes

$$- \left(X_{1,2}^{\dagger a_2 a_1} X_{a_2 a_3}^{2,3} \right) \left(X_{2,3}^{\dagger b_2 a_3} X_{b_2 a_1}^{1,2} \right) \propto - \left(\begin{array}{cc} \mathbb{1}_{K_1} & 0 \\ 0 & 0 \end{array} \right)_{b_2}^{a_2} X_{2,3}^{\dagger b_2 a_3} X_{a_2 a_3}^{2,3}. \tag{4.33}$$

Hence the vacuum expectation value of $X_{1,2}$ shows up as a negative mass deformation for the first K_1 components of $X_{2,3}$ and thus *also* breaks the $U(K_3)$ to $U(K_3 - K_1)$. As such, except for the $X_{1,2}$ link, each bifundamental can acquire a mass deformation from two different sources: its explicit mass term as well as the interaction terms to its left. As an example, let us assume $m^2_{2,3} = m^2_{3,4} = 0$ but $m^2_{1,2} < 0$. Then $X_{2,3}$ acquires a vacuum expectation value from (4.33). The interaction term between $X_{2,3}$ and $X_{3,4}$ also means $X_{3,4}$ gets a negative mass shift,

$$- \left(X_{2,3}^{\dagger a_3 a_2} X_{a_3 a_4}^{3,4} \right) \left(X_{3,4}^{\dagger b_3 a_4} X_{b_3 a_2}^{2,3} \right) \propto - \left(\begin{array}{cc} \mathbb{1}_{K_1} & 0 \\ 0 & 0 \end{array} \right)_{b_3}^{a_3} X_{3,4}^{\dagger b_3 a_4} X_{a_3 a_4}^{3,4}. \tag{4.34}$$

and c_4', one could change the location of the "singlet critical line". Smoothly changing $c_4 \to -c_4$ causes the line to move from phase IV to phase III. Meanwhile, changing $c_4' \to -c_4'$ to move from phase IV' to phase I'. Thus, for example, we can shrink the size of region IVb and IVb' by decreasing the magnitude of c_4' and increasing the magnitude of c_4. In the limit $c_4' \to 0$, we must take $|c_4| \to \infty$.

breaking $U(K_4)$ to $U(K_4 - K_1)$ and giving the first K_1 components an additional negative mass deformation. If there are no other mass term deformations, this effect cascades to the *right* across the entire quiver.

Now let us consider how this changes if $X_{2,3}$ also had a mass deformation. A negative mass deformation would further break down the $U(K_3)$ subgroup to $U(K_3 - K_2)$, and this could also propagate down the quiver, as we just discussed. Positive mass deformations are slightly more tricky since we need to consider them in two regimes. First consider the case where the negative mass deformation from (4.33) is larger than that of the mass term for $X_{2,3}$. Then the results considered above are unchanged, $X_{1,2}$ is still partially broken. When the mass term has a larger positive mass contribution than that of (4.33), $X_{2,3}$ is completely gapped. This means none of the components have a nonzero vacuum expectation value, and thus the interaction (4.34) contributes no mass to $X_{3,4}$. In other words, if the mass term for $X_{2,3}$ is large enough, it can stop of propagation of $X_{1,2}$'s breaking down the quiver. We have avoided this case by assuming $\left|C_I^{(E)}\right| \gg \left|m_{i,i+1}^2\right|$, thereby forbidding large positive mass deformations from blocking the propagation of the breaking down the quiver.

What about when $X_{2,3}$ acquires a negative mass deformation but the $X_{1,2}$ and $X_{3,4}$ mass terms are untouched? By the same reasoning above, $X_{3,4}$ will also acquire a negative mass shift to its first K_2 components via the interaction terms, causing the breaking of $U(K_3)$ and $U(K_4)$ to $U(K_3 - K_2)$ and $U(K_4 - K_2)$, respectively. However, as we have been careful to argue in Appendix D.3, $X_{2,3}$'s vacuum expectation value should not be able to influence nodes/links to its left. Hence $X_{1,2}$ is unaffected.

We are now in the position to consider mass deformations which are partially Higgsed. That is, not all bifundamental masses are taken to be the same sign. Specifically, consider the case where the bifundamentals $Y_{1,2}$ and $Y_{3,4}$ are Higgsed and $Y_{2,3}$ is gapped (corresponding via mass identifications to $X_{1,2}$ and $X_{3,4}$ being gapped and $X_{2,3}$ being Higgsed). This yields the phases

$$\text{(A3)} \quad : \quad SU(N_1-N_2)_{-k_1} \times SU(N_2)_{-k_1-k_2} \times SU(N_3-N_4)_{-k_3} \times SU(N_4)_{-k_3-k_4} \tag{4.35a}$$

$$\text{(E3)} \quad : \quad U(k_1)_{N_1-N_2} \times U(k_1+k_2)_{N_2} \times U(k_3)_{N_3-N_4} \times U(k_3+k_4)_{N_4} \tag{4.35b}$$

which are level-rank dual to one another! Note that the interactions are vital for us to reach this conclusion. We have used the fact that $X_{2,3}$ acquiring a vacuum expectation value breaks the $U(K_2)$ subgroup of $U(K_3) \to U(k_3)_{N_3-N_4} \times U(K_2)_{N_3-N_4}$ and also, due to the interaction term, $U(K_4)_{N_4} \to U(k_3+k_4)_{N_4} \times U(K_2)_{N_4}$. Without such terms we would have found the $U(K_4)_{N_4}$ group unbroken, yielding the TFT

$$(\text{E}\bar{3}) \quad : \quad U(k_1)_{N_1-N_2} \times U(k_1+k_2)_{N_2-N_4} \times U(k_3)_{N_3-N_4} \times U(K_4)_{N_4} \tag{4.36}$$

which is clearly not level-rank dual to (4.35a).

Generalization to n Nodes

Now let us generalize this prescription to an arbitrary number of nodes. For n nodes, there is a duality between the following two theories:

$$\text{Theory A:} \quad SU(N_1)_{-k_1} \times \prod_{i=2}^{n} \left[SU(N_i)_{-k_i} \times \text{bifundamental } Y_{i-1,i}\right] \tag{4.37a}$$

$$\text{Theory B:} \quad \prod_{i=1}^{n-1} \left[U(K_i)_{N_i-N_{i+1}} \times \text{bifundamental } X_{i,i+1}\right] \times U(K_n)_{N_n} \tag{4.37b}$$

where flavor bounds require $k_i \geq 0$ and $N_1 \geq N_2 \geq \ldots \geq N_n$. As with the above case, these theories can be shown to be dual by systematically applying Aharony's duality (1.13b) to the first node, the master duality to every two-link node, and then Aharony's other duality (1.13a) to the last node. The master duality is not needed for the two-node case.

Implied above are interaction terms on the U side of the duality of the form of (D.25). Equivalently, we can use the schematic interaction

$$\mathcal{L}^{(B)} \supset \sum_{I=2}^{n-1} C_I^{(B)} \left(X_{I-1,I}^{\dagger a_I a_{I-1}} X_{a_I a_{I+1}}^{I,I+1}\right) \left(X_{I,I+1}^{\dagger b_I a_{I+1}} X_{b_I a_{I-1}}^{I-1,I}\right) \tag{4.38}$$

with the understanding that such interactions can only give mass terms to the link on their left. Here, a_i, b_i the gauge indices of the ith node and $C_I^{(B)} \to -\infty$ so that $|C_I^{(B)}| \gg m_{i,i+1}^2$. In this limit, on the SU side of the duality the interaction terms are very small and have no effect on the mass deformed phases, so we ignore them.

To summarize the interaction behavior on the U side: a bifundamental scalar $X_{j,j+1}$ acquiring a nonzero vacuum expectation value affects nodes/links to the right but *not* to the left. Namely, it causes all bifundamental scalars (i.e $X_{i,i+1}$ with $i > j$) to acquire a similar vacuum expectation for the first K_j components. This in turn causes a breaking of all gauge groups nodes $i > j$ to $U(K_i - K_j)$. Note this effect can compound, so if $X_{j,j+1}$ and $X_{\ell,\ell+1}$ acquire a vacuum expectation value from their respective mass deformations, the gauge group on node $i > j > \ell$ undergoes breaking $U(K_i) \to U(K_i - K_j) \times U(K_j - K_\ell) \times U(K_\ell)$.

As mentioned earlier, other dualities which flow to other TFTs can be constructed by changing the sign/magnitude of the interaction terms, but such considerations are left for future work.

4.1.3 Self-Consistency Checks

Returning to the bosonic particle-vortex duality, it should now be clear the derivation we outlined in Sec. 4.1.1 is the two-node case of the more general non-Abelian linear quivers with values

$$N_1 = N_2 = k_1 = 1, \qquad k_2 = 0, \tag{4.39}$$

which is shown in Fig. 4.2. Note this saturates all flavor bounds and carries the minimum value of parameters without being completely trivial, so the particle-vortex duality can be thought of as the simplest case of an infinite class of $2+1$ dimensional Bose-Bose dualities. Additionally, it is clear that the derivation of the two-node quiver requires no master duality since there are no nodes connected to two links.

Another helpful tool in analyzing the more general non-Abelian quiver dualities as well as comparing them to the holographic dualities in Sec. 4.2.2 will be comparing the spectrum of the two theories. To this end, let us briefly review how the spectra of the particle-vortex duality match on either side of the duality.

First consider the case when Φ acquires a vacuum expectation value in (4.7) through a negative mass deformation. It is well known the breaking of the $U(1)$ gauge symmetry gives rise to vortex solutions of finite mass charged under B_1 flux. Since there is no dynamical Chern-Simons term on this end, there is no funny business with flux attachment or alternative vortex solutions. These vortices carry flux charge under the broken $U(1)$ gauge group which can be seen by looking at the asymptotic behavior of the gauge field.

Now consider the Abelian-Higgs model but instead in the form which is more amenable to matching onto the non-Abelian quivers (i.e. (4.8)). In this form we have two $U(1)$ gauge fields, one of which is redundant and can be integrated out. When $\langle\Phi\rangle \sim v$, it forces the breaking of $U(1) \times U(1) \to U(1)_A$, where $U(1)_A$ is the subgroup where the two $U(1)$ transformations act oppositely on Φ, leaving it invariant. Again, the breaking of a $U(1)$ symmetry ensures that there are vortex solutions which are charged under the flux of the broken symmetry. In this case, it corresponds to a nonzero winding of both a_1 and c at spatial infinity, since the broken $U(1)$ group is where they are set equal to one another (i.e. $U(1)_{\text{diag}}$). For the vortex solution where $a_1 = c$ energy contributions from the Chern-Simons terms drop out, as they should since they weren't present in (4.7). Since there is nonzero a_1 flux the vortex is charged under the background B_1 field. Also note that the vortex has finite mass proportional to the vacuum expectation value of the scalar. As expected, we reach the same conclusions when working from (4.7), albeit in a slightly more complicated manner.

Due to the mass identification, the phase where Φ has a vacuum expectation value should be identified with the phase where ϕ is simply gapped. The $U(1)$ global symmetry is unbroken and ϕ excitations of (4.5a) are charged under the B_1 field, which are identified with the vortices on the opposite side of the duality.

Meanwhile, for mass deformations where ϕ obtains a vacuum expectation value and Φ is gapped, the $U(1)$ global symmetry on both sides of the duality is broken. This is straightforward to see on the ϕ side of the duality and is made clear on the Φ side by rewriting the photon using the Abelian duality, $F_{\mu\nu} \sim \epsilon_{\mu\nu\rho}\partial^\rho\sigma$. Since the $U(1)$ global symmetry is broken, we expect Goldstone bosons on either side of the duality. For the ϕ field, we have massless angular excitations. In this phase the photon remains gapless and it is identified with the Goldstone boson of the Φ side.

We claim the above results completely generalize to the n-node quiver case. Across the duality we have established the fact that the global $U(1)$ symmetry of the $X_{i,i+1}$ bifundamental is identified with the monopole number symmetry of the ith node. We begin with the side of the duality where a $U(1)$ global symmetry is unbroken, which corresponds to a positive mass deformation on the SU side and a negative mass deformation on the U side. We will focus on the behavior of a single bifundamental since generalization is straightforward.

When we gap the $Y_{i,i+1}$ bifundamental on the SU side, on the U side this should cause the $X_{i,i+1}$ bifundamental to acquire a vacuum expectation value as a result of the $M^2_{i,i+1} \leftrightarrow -m^2_{i,i+1}$ mass mapping.

Let's take a closer look at the breaking term to account for degrees of freedom. Schematically the interaction term can be written in the form

$$V(X_{i,i+1}) \sim \mathrm{Tr}\left[(X_{i,i+1})_{a_i a_{i+1}}\left(X^\dagger_{i,i+1}\right)^{b_i a_{i+1}} - v^2\delta^{b_i}_{a_i}\right]^2. \tag{4.40}$$

With the gauge freedom we can take $\langle X_{i,i+1}\rangle$ to be of the from of (4.29). This causing the breaking of

$$U(K_i) \times U(K_{i+1}) \to U(K_i)_{\mathrm{diag}} \times U(k_{i+1}), \tag{4.41}$$

corresponding to an overall broken $U(K_i)$ gauge symmetry. Each $X_{i,i+1}$ field has $2K_iK_{i+1}$ total degrees of freedom. Within the $U(K_i)$ subspace, there are K_i^2 flat directions corresponding to "angular" excitations, which are consumed by the broken $U(K_i)$ gauge fields to become (two-component) massive "W-bosons". The remaining K_i^2 scalar degrees of freedom represent "modulus" excitations in directions of the potential which are not flat and are thus analogous to Higgs bosons. Additionally, these modes are now adjoint particles since they are charged under $U(K_i)_{\mathrm{diag}}$. This leaves $2K_ik_{i+1}$ degrees of freedom, which acquire a mass through the double trace-like interaction term that is present for Wilson-Fisher scalars. Hence all bifundamental scalar particles are gapped, as they should be.

As with the particle-vortex duality, we would like to show vortices on the U side should be identified with the gapped particles on the SU side of the duality. The gapped $Y_{i,i+1}$ particles are charged under the unbroken $U(1)$ symmetry and carry baryon number. Meanwhile, when the $X_{i,i+1}$ particles acquire a vacuum expectation value, the breaking of the corresponding $U(1)$ subgroup mean vortices associated to that link now have finite mass and are topologically stable since [8]

$$\pi_1 \left(U(K_i) \times U(K_{i+1})/U(K_i)_{\text{diag}} \right) \simeq \mathbb{Z}. \tag{4.42}$$

Specifically, the vortex configurations correspond to a winding of the broken $U(1)_A \subset U(K_i) \times U(K_{i+1})/U(K_i)_{\text{diag}}$ gauge group as well as the phase of $\langle X_{i,i+1} \rangle$ at spatial infinity. Since the broken subgroup $U(1)_A$ contains the ith node's $U(1)$ factor, through the BF term of that node it coupled to the $\tilde{B}_2$ field.[9] This is the same symmetry the gapped $Y_{i,i+1}$ couple to, and thus the two modes should be identified in a manner analogous to what we saw for the particle-vortex duality.

Unlike the particle-vortex duality, the presence of nonzero Chern-Simons terms in the mass deformed phases means variation with respect to dynamical gauge groups imposes a flux attachment condition on the excitations. That is, particles charged under the respective

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[8]One might worry that we may be able to form other vortex solutions by winding the other broken subsets, say $U(1)_A \subset U(K_{i+1}) \times U(K_{i+2})/U(K_i)_{\text{diag}}$. Note however that the interactions force the vacuum expectation value of the associated $U(K_i)$ subgroup of the $X_{i+1,i+2}$ bifundamental to be effectively infinite. This is distinct from $\langle X_{i,i+1} \rangle$ which is presumed to be proportional to the mass deformation and finite. Thus such vortices are significantly heavier than the vortex formed from a winding of the $X_{i,i+1}$ bifundamental and its corresponding gauge groups. Note for the $X_{i,i+1}$ vortex, no matter the mass deformations of bifundamentals to its left, its $U(k_i)$ subgroup will always have a finite vacuum expectation value, and thus the topologically stable vortex solutions can always have finite energy via a winding of this corresponding subgroup.

[9]Since the $X_{i,i+1}$ vortices contain winding under $U(1)_A$, which is a subgroup of the $U(1) \times U(1)$ gauge symmetries of the ith and $(i+1)$th nodes, one might worry that such a vortex also carries flux under the $(i+1)$th gauge group and is thus charged under the $U(1)$ symmetry of the $(i+1)$th node. However, as explained in Appendix D.2, the BF coupling is such that nodes to the right of a bifundamental are only coupled via the unbroken gauge group. Hence, although the vortices carry $U(1)$ flux of the $(i+1)$th node, they are only charged under global symmetry of the ith node.

symmetry must be attached to the vortex excitations. This might be modified slightly due to the breaking of the U gauge group, since the broken gauge degrees of freedom will become massive giving an extra term when varying with respect to the corresponding massive gauge degrees of freedom. We leave such analysis for future projects.

4.2 Theta Wall Dualities

In this section we consider duals to the Chern-Simons theories found on defects in $3+1$-dimensional $SU(N)$ Yang Mills theory when the θ angle varies as a function of location. Specifically, we look for a dual to (4.1).

We begin by reviewing a few essential facts about the expected theta dependence in pure $SU(N)$ gauge theories. Such gauge theories are believed to have multiple vacua related to the physics of the theta angle. In each vacuum, physical quantities are not periodic in theta with period 2π but instead with period $2\pi N$. The physical properties of the system are nevertheless 2π periodic. As we change theta by a single 2π period, the true vacuum of the system changes and the physics in the new vacuum at $\theta = \theta_0 + 2\pi$ is the same as the physics in the original vacuum at $\theta = \theta_0$.

This picture can be most rigourously established at large N. In this limit the vacuum energy as a function of θ is expected to scale as [128, 132]

$$E(\theta) = N^2 h(\theta/N) \tag{4.43}$$

for some to be determined function h. This appears to be inconsistent with the periodicity requirement

$$E(\theta) = E(\theta + 2\pi). \tag{4.44}$$

As claimed above, a single vacuum with energy of the form (4.43) is expected to be $2\pi N$ periodic, not 2π periodic. This conundrum can easily be solved by postulating that the theory has a family of N vacua labeled by an integer K. In this case the vacuum energy in the Kth vacuum is given by

$$E_K(\theta) = N^2 h((\theta + 2\pi K)/N). \tag{4.45}$$

Most of these vacua are meta-stable, the truly stable vacuum for any given θ is given by minimizing over K:

$$E(\theta) = N^2 \min_K h((\theta + 2\pi K)/N). \tag{4.46}$$

The resulting function $E(\theta)$ has the expected 2π periodicity. While the energy of (say) the 0-th vacuum keeps increasing as we increase theta from 0 towards 2π, the energy of the $K = -1$ vacuum at $\theta = 2\pi$ is exactly the same as the energy of the 0-th vacuum was at $\theta = 0$. One expects that a transition from the 0-th to the (-1)-th vacuum is triggered at $\theta = \pi$. While physics in any given vacuum is $2\pi N$ periodic, the system as a whole, in its true vacuum, is 2π periodic.

We are now in a position to discuss the physics of theta interfaces and domain walls. Let us first turn to the case of interfaces. Starting with a confining gauge theory (pure Yang-Mills in this case), one can introduce interfaces across which the theta angle changes by an integer multiple of 2π,

$$\Delta\theta = 2\pi n. \tag{4.47}$$

The theory is assumed to be everywhere in the true ground state. This means, in particular, that the index labeling the local vacuum state changes by $-n$ units as the theta angle changes by $2\pi n$. Since the theta angle is a parameter in the Lagrangian, translation invariance is explicitly broken in this theory and we do not expect any Goldstone bosons corresponding to fluctuations of the position of the interface.

As we explained in the introduction, a spatially varying theta gives rise to domain walls on which Chern-Simons theories live. However, anomaly inflow does not constrain the exact Chern-Simons theory. Ref. [112] has argued that for $|\nabla\theta| \ll \Lambda$ and $|\nabla\theta| \gg \Lambda$, one should expect the TFTs $[SU(N)_{-1}]^n$ and $SU(N)_{-n}$, respectively. Assuming a smooth transition, (4.1) was proposed as a possible CFT to describe the transition between these two extreme cases.

The generic phase of (4.1) is characterized by a partition $\{n_i\}$ of n, that is integers n_i with the property that $\sum_i n_i = n$. Each n_i denotes the number of gauge group factors along the quiver that have been Higgsed down to their diagonal subgroup before we encounter a positive mass squared scalar. For example, $n_1 = n$ corresponds to the completely Higgsed $SU(N)_{-n}$ phase associated with the steep wall, $n_i = 1$ for $i = 1, \ldots, n$ corresponds to the shallow wall with $[SU(N)_{-1}]^n$. The generic phase is given by a TFT based on

$$\text{Phase } \{n_i\}: \qquad \prod_i SU(N)_{-n_i}. \tag{4.48}$$

One extra subtlety that arises concerns global symmetries. The scalar fields are bifundamentals under neighboring $SU(N)_{-1}$ gauge group factors. This leaves an overall phase rotation of every single scalar as a global symmetry, for a combined $U(1)^{n-1}$ extra global symmetry from the $n-1$ scalar fields. If these indeed were global symmetries of the parent theory this would lead to unexpected consequences. Most notably, in the fully broken phase the low energy theory on the interface would not just be the topological $SU(N)_{-n}$ Chern-Simons theory we expect, but would in addition contain $n-1$ massless Goldstone bosons as these extra global symmetries are spontaneously broken in the condensed phase. The proposal of [52] is to add extra terms to the action that break these extra global symmetries so that there are no Goldstone bosons. The simplest option to do so is a $\det(Y)$ term for each link[10], which is indeed gauge invariant under all $SU(N)_{-1}$ gauge group factors but is charged under overall phase rotations of Y. The quiver gauge theories we discussed in the last section do not have these determinant terms added to the potential. The dualities we derive will most naturally apply to the theory without the determinant terms. To connect to the theory of the theta interfaces we will have to add the extra determinant term as a deformation.

In addition to interfaces a second type of co-dimension one defect we can discuss are domain walls. These are already present in a theory with constant theta. They govern the decay of one of the meta-stable vacua of the theory to the true vacuum. In the idealized case, we can consider the theory in a state where we interpolate between two metastable vacua

[10] Of course any power of $\det(Y)$ would do the job in that it is gauge invariant but charged under the global symmetry. For small values of N we need to make use of this freedom. For example, for $N = 1$ $\det(Y) = Y$ and we would simply add linear potentials, whereas for $N = 2$ we would be adding mass terms. Instead we should add $\det(Y)^4$ and $\det(Y)^2$ respectively in those two cases.

as we move along a single direction, which we once more chose to be the x_3 direction. For simplicity we are only interested in configurations which preserve $2+1$ -dimensional Lorentz invariance, that is we focus on flat domain walls. If the theory starts in the 0-th vacuum as $x_3 \to \infty$, we can interpolate to the n-th vacuum at $x_3 \to -\infty$. While the state of the system at large negative x_3 is not in the true local ground state, this configuration is meta-stable. The false vacuum has to decay via bubble formation, which is governed by the domain wall tension. It has been argued [132] that the tension of the wall is of order N, a fact that is obvious in the holographic realization of these walls which we will turn to in Sec. 4.2.2. At large N this means that decay of the meta-stable vacuum is e^{-N} suppressed. In addition the difference in vacuum energies will exert a pressure on the domain wall generically causing the wall to move. But since the pressure difference is order N^0, whereas the domain wall tension is order N, the domain wall can be treated as static in the large N limit.

As far as the anomalies are concerned, the analysis of [52] generalizes to the case of walls: the gauge theory on the defect should be the same whether we are forced to jump n vacua because of a $2\pi n$ jump in θ, or whether we study a dynamical wall that interpolates between two n-separated vacua in a theory with fixed θ. The main difference appears to be that this time the wall is dynamical with a finite tension. Most notably, this implies that we should have (at large N) a massless scalar living on the wall whose expectation value gives the location of the wall. It being the Goldstone boson of broken translations, the scalar has an exact shift symmetry that protects it from becoming massive. While interfaces were characterized by a free function $\theta(x_3)$ and were only loosely characterized into shallow and steep, for walls it is much easier to characterize the moduli space of allowed configurations. We have a total of n discrete jumps from one vacuum to the next. When the walls are widely separated, we should have n separate walls connecting two neighboring vacua each. In this limit, we should have a total of n translational modes as the different basic walls can presumably move independently. The gauge theory living on these widely separated walls should be $[SU(N)_{-1}]^n$ as above together with these decoupled light translational modes. This is indeed what follows from the analysis of Acharya and Vafa in the closely related case of $\mathcal{N}=1$ supersymmetric gauge theories [1] (see also [44]). The other extreme is when all n walls coincide and we have a single wall across which we jump by n vacua, presumably governed by a single $SU(N)_{-n}$ gauge theory and a single translational mode.

To summarize, note that we are still characterizing the phases by partitions $\{n_i\}$ of n and the gauge theory on the wall is once again governed by the topological field theory (4.48). In addition we have the decoupled translational modes. At finite N the walls no longer correspond to static configurations as they will be pushed around by the pressure differences, making them generically much harder to study than the case of interfaces. The reason we discuss them at all as that, at large N, they have a very simple holographic realization which we will employ in what follows to check our dualities.

4.2.1 Theta Wall Dualities via 3d Bosonization

We now consider possible duals to (4.1) via 3d bosonization. Fortunately, such a theory can easily be constructed from the non-Abelian linear quiver dualities.[11] Consider the n node linear quiver, (4.37), and take

$$1 = k_1 = k_2 = \cdots = k_n \tag{4.49a}$$

$$N = N_1 = N_2 = \cdots = N_n. \tag{4.49b}$$

This satisfies all flavor bounds of the derivation given above since $k_i \geq 0$ and $N = N_j \geq N_{j+1} = N$. In this case the dual quiver theories become

$$\text{Theory A:} \qquad [SU(N)_{-1}]^n \times \prod_{p=1}^{n-1} \text{bifundamental } Y_{p,p+1} \tag{4.50a}$$

$$\text{Theory B:} \qquad \prod_{p=1}^{n-1} [U(p)_0 \times \text{bifundamental } X_{p,p+1}] \times U(n)_N \tag{4.50b}$$

[11] As touched upon earlier, if one tried to derive such a quiver using only Aharony's dualities, one would inevitably run into violations of the flavor bound. Thus it appears the interactions between links which come from the master duality are a necessity.

and the relevant mass deformations for all bifundamentals taken positive/negative are given by

$$\begin{aligned}
&\text{(A1)} \qquad M^2_{i,i+1} > 0: \qquad [SU(N)_{-1}]^n && \text{(4.51a)}\\
&\text{(A2)} \qquad M^2_{i,i+1} < 0: \qquad SU(N)_{-n} \times \prod_{p=1}^{n-1} [SU(0)_{-p}] && \text{(4.51b)}\\
&\text{(B1)} \qquad m^2_{i,i+1} < 0: \qquad [U(1)_N]^n && \text{(4.51c)}\\
&\text{(B2)} \qquad m^2_{i,i+1} > 0: \qquad U(n)_N \times \prod_{p=1}^{n-1} [U(p)_0]\,. && \text{(4.51d)}
\end{aligned}$$

The topological sector of Theory A matches the TFTs we set to find at the outset, $SU(N)_{-n}$ and $[SU(N)_{-1}]^n$. In addition both sides have decoupled massless modes that also match. We assume that the non-Abelian part of $U(p)_0$ confines at low energies and is therefore gapped. This implies that the dynamics of the confining gauge group should have no effect on the physics at scales well below the gap. The $U(1)$ part however gives rise to a light photon for every level 0 unitary gauge group. On the SU side these light photons map to Goldstone bosons. In a theory with $N_s < N$ scalars charged under a $SU(N)$ gauge symmetry a full global $U(N_s)$ flavor symmetry is unbroken as the gauge group is broken to $SU(N - N_s)$ by a scalar vacuum expectation value. The broken gauge generators can be used to compensate any flavor rotation. In the special case of $N_s = N$, which is of interest to us here, the $U(1)$ part of the flavor symmetry however is broken and so we will get a corresponding Goldstone boson. In order to keep track of these light scalars we denote the Goldstone bosons as $SU(0)_{-p}$ theories, which continue to be "level-rank" dual to the $U(p)_0$ factors of Theory (B2); either theory denotes a decoupled light scalar mode. Including these factors we see that there is a perfect matching both between the topological sector and the decoupled light modes.

One should note that these extra massless Goldstone bosons are exactly the ones that in the theory of theta interfaces have been eliminated by the $\det(Y)$ potentials. As it stands, our quiver duality applies to (4.1) without these extra determinant terms. Since the global $U(1)$

baryon number symmetries under which $\det(Y)$ is charged map to monopole symmetries on the U side, the corresponding dual operator is a monopole operator. Adding this monopole operator to the theory should lead to confinement of the $U(1)_0$ factors together with their non-Abelian counterparts and hence remove the massless photons associated to these factors from the spectrum, just as we removed their dual Goldstone bosons on the SU side.

Given the fact that most of the gauge group factors on the U side confine, we can further simplify the low energy description of this side of the duality. The confining groups cause the bifundamental matter and antimatter to form "mesons". If the matter/antimatter is still charged under some gauge group with nonzero Chern-Simons level, the meson transforms as an adjoint under said gauge group as conjectured in (4.2). For phase (B2), there are the adjoints formed from $X^\dagger_{n-1,n}X_{n-1,n}$ since the $(n-1)$th node confines. It is difficult to say if bound states such as $X^\dagger_{n-1,n}X^\dagger_{n-2,n}X_{n-2,n}X_{n-1,n}$, which are also adjoints under the $U(n)_N$ gauge group, would be stable or if it would split into separate particles $X^\dagger_{n-2,n}X_{n-2,n}$ and $X^\dagger_{n-1,n}X_{n-1,n}$. If we assume the latter, there is only a single light adjoint scalar charged under the $U(n)$ considered above. [12] We also would want to conjecture that there are no additional neutral mesons that become light together with the adjoint; such extra light matter is not accounted for on the SU side of the duality. With these dynamical assumptions our quiver duality boils down to the one we advertised in the introduction

$$[SU(N)_{-1}]^n + \text{bifundamental scalars} \qquad \leftrightarrow \qquad U(n)_N + \text{adjoint scalars} \tag{4.52}$$

with a $\det(Y)$ potential for all the bifundamental scalars on the SU side implied.

Unlike the quiver dualities, which we derived from gauging global symmetries, the duality (4.2) only follows upon making extra dynamical assumptions regarding the confining mechanism. We can give extra evidence for this duality by, once again, looking at the phase structure. On the U side the various massive phases are realized by adding mass squared

[12]Note that when the bifundamental scalars on the SU side acquire a negative vacuum expectation value, by assumption this breaks their gauge symmetry down to the common diagonal symmetry group and causes the Higgsed bifundamentals to become adjoint particles. Thus we get gapped adjoint particles on both sides of the duality for phase 2 considered above.

terms that give expectation values to the adjoint scalar (or remove it completely) together with $\mathrm{Tr}\, X^k$ terms in the potential. We can always chose a gauge in which the scalar expectation value is diagonal, so the generic expectation values is characterized by the n eigenvalues of the scalar expectation value. Due to the presence of the interaction terms, it is possible to have none of the eigenvalues coincide, in which case the gauge group is $[U(1)_N]^n$. But whenever two or more eigenvalues coincide, we do get an enhanced unbroken subgroup. Once again the most general phase is encoded in partitions $\{n_i\}$ of n, where each integer n_i denotes the multiplicity of a given eigenvalue. The generic phase is given by $n_i = 1$ for all i, whereas the case of n coincident eigenvalues with a single $U(n)_N$ gauge group factor corresponds to $n_1 = n$. The generic partition corresponds to

$$\text{Phase } \{n_i\} : \qquad \prod_i U(n_i)_N. \tag{4.53}$$

Reassuringly, this is exactly the level-rank dual gauge group of what we found for the quiver theory, (4.48). In the next section we will give further support for the validity of this duality, at least in the large N limit, using holography.

4.2.2 Theta Wall Dualities via Holography

Now we turn to the holographic proof of the duality. Our work will follow closely the stringy embedding of bosonization presented in [70] based on the earlier string theory realization of level-rank duality in [49]. In this construction the holographic duality between field theory and supergravity becomes, at low energies, the purely field theoretic bosonization duality.

One starts with a well known holographic pair. The work of [49, 70] employs the original holographic duality [92] between $\mathcal{N} = 4$ super-Yang Mills (SYM) and type IIB string theory on $\mathrm{AdS}_5 \times S^5$. We then deform the theory in such a way that all, or at least most, degrees of freedom gap out and one is left with a non-trivial topological field theory (in the case of level-rank) or conformal field theory (in the case of bosonization) in the infrared. Following the same deformations in the dual gravity solution one finds that the spectrum of most supergravity excitations also gets gapped out. The only remaining low energy excitations are localized on a probe brane. These probe degrees of freedom in the bulk are found to be related to the boundary degrees of freedom by the desired field theory duality.

Review of Holography Applied to 3d Bosonization

Let us first briefly review the case of level-rank. Starting with $\mathcal{N} = 4$ SYM one can go to $2 + 1$ dimensions via compactifying the theory on a circle of radius R. With anti-periodic boundary conditions for the fermions in the theory, all fermionic Kaluza Klein modes pick up masses of order $1/R$ and the scalars then pick up masses of the same order via loop corrections. At energies below $1/R$ we are left with pure Yang-Mills in $2 + 1$ dimensions, which is believed to confine. The theory is gapped with gap of order $1/R$. This is not quite yet the theory we want, the IR is trivial rather than a non-trivial Chern-Simons TFT.

To produce the desired Chern-Simons terms we need to introduce the theta angle. Like all coupling constants in the Lagrangian, the theta angle in $3+1$ dimensional gauge theories is usually introduced as a position independent constant, but it can be promoted to a non-trivial background field. What we need here is a theta angle that linearly changes by $2\pi n$ as we walk around the circle once. Since theta is only well defined modulo 2π this is consistent as long as n is an integer. The $\theta F \wedge F$ term in the Lagrangian with constant theta gradient can be integrated by parts to turn into a $2+1$ dimensional Chern-Simons term with level $-n$. So in short, $\mathcal{N} = 4$ SYM with anti-periodic boundary conditions for fermions and a constant theta gradient gives rise to a gapped $2 + 1$ dimensional theory which, at low energies, is well described by an $SU(N)_{-n}$ Chern-Simons theory.

These deformations are easily repeated in the holographic dual. The compactification with anti-periodic boundary conditions for fermions is dual to the cigar geometry of [130], that is a doubly-Wick rotated planar Schwarzschild black-hole where the compact time direction of the Euclidean black hole plays the role of the compact spatial directions, whereas one of the directions along the planar “horizon” becomes the new time direction. Most importantly, the radial coordinate in this cigar geometry truncates at a finite value $r = r_*$ where the compact circle contracts. Consequently this geometry acts as a finite box and so indeed all supergravity fluctuations exhibit a gapped spectrum [130] with a mass gap of order $1/R$. In order to retain a non-trivial topological sector we still need to implement the spatially varying theta angle. The theta angle is set by the near boundary behavior of the bulk axion field, so we are looking for a supergravity solution where the axion asymptotes to $a \sim ny/R$. Here y denotes the coordinate along the circle direction and $a = ny/R$ is an exact solution to the axion equation of motion in the cigar background. As long as we are only interested in the $n \ll N$ limit we can ignore the backreaction of the axion on the background geometry and $a = ny/R$ appears to be the full solution to the problem. The only remaining issue is that the axion field strength $f_y = \partial_y a = n/R$ in the bulk has to be supported by a source. This source can be introduced by locating n D7 branes, wrapping the entire internal S^5, at the tip of the cigar at $r = r_*$. This stack of D7 brane introduces new degrees of freedom in the bulk. The scalar fields corresponding to fluctuations of the D7 away from the tip are massive due to the geometry of the cigar. Like all other geometric fluctuations they have mass of order $1/R$. The only other degree of freedom introduced by the n D7 branes is the worldvolume gauge field. The latter acquires a Chern-Simons term of level N from the Wess-Zumino coupling to the N units of background 5-form flux through the S^5. Lo and behold, the low energy description of the holographic bulk dual is simply a $U(n)_N$ Chern-Simons gauge theory living on the D7 branes. Comparing low energy descriptions on both sides, AdS/CFT boiled down to level-rank duality for the emerging TFTs.

The last step in order to derive 3d bosonization rather than level-rank from this construction is to add extra light matter into the theory. This can be easily accomplished using flavor probe branes [81]. In the construction put forward in [70] an extra probe D5 adds fermionic matter localized on $2+1$ dimensional defects in the $3+1$ dimensional theory. These defects live at points in the circle direction, so at low energies they simply become light fermions coupled to the $SU(N)_{-n}$ Chern-Simons gauge fields. The same probe branes can be argued, from the bulk point of view, to add scalar matter to the dual $U(n)_N$ Chern-Simons gauge theory. Instead of simply giving us level-rank, in this case holography, at low energies, reduces to the basic non-Abelian 3d bosonization duality.

Holographic Realization of Theta Walls

To holographically realize the field theory theta domain walls we just reviewed we need to start with a holographic duality for a confining $3+1$ dimensional theory and then simply once again follow the field theory deformation corresponding to turning on theta in the bulk. The simplest realization of a confining $3+1$ gauge theory with a gravity dual is Witten's black hole [130]. This is almost the same construction we employed previously, but lifted one dimension up. We start with a 5d gauge theory, maximally supersymmetric YM with gauge group $SU(N)$, and compactify it on a circle with anti-periodic boundary conditions. The dual geometry has once again the basic shape of a cigar, and the explicit supergravity solution is given by

$$ds^2 = \left(\frac{u}{L}\right)^{3/2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + f(u)dy^2\right) + \left(\frac{L}{u}\right)^{3/2}\left(\frac{du^2}{f(u)} + u^2 d\Omega_4^2\right),$$

$$e^\phi = g_s\left(\frac{u}{L}\right)^{3/4}, \quad F_4 = dC_3 = \frac{2\pi N}{V_4}\epsilon_4, \quad f(u) = 1 - \frac{u_*^3}{u^3}. \tag{4.54}$$

Here x^μ are the 4 coordinates of $3+1$ dimensional Minkowski space, y is the circle direction we compactified to go from $4+1$ to $3+1$ dimensions. ϕ is the dilaton field, F_4 the RR 4-form field strength. Ω_4 is the internal 4-sphere, with $d\Omega_4^2$, ϵ_4 and $V_4 = 8\pi^2/3$ its line element, volume form and volume respectively. The string coupling g_s and the string length l_s are the parameters of the underlying type IIA super-string theory. L sets the curvature radius of the solution, it is determined by Einstein's equations to be $L^3 = \pi g_s N l_s^3$. Last but not least u_* is the location of the tip of the cigar, it is related to the periodicity $2\pi R$ of the compactification circle by $R = \frac{2}{3}L^{3/2}u_*^{-3/2}$.

The holographic realization of turning on a constant theta angle has been worked out in [132]. The theta angle is dual to the Wilson line of the bulk RR 1-form C_μ along the compact y direction:

$$\int_{S^1} C = \theta + \dots \tag{4.55}$$

where the ellipses denote terms with negative powers of u, that is terms that vanish near the boundary. The Wilson line is gauge invariant modulo $2\pi\mathbb{Z}$, so theta is indeed an angle.

Using Stokes's law, we can rewrite the condition (4.55) as

$$\int_D F = \theta + 2\pi K. \tag{4.56}$$

Here $F = dC$ is the field strength associated with the RR one-form and D is the cigar geometry, which has the topology of a disc. Since $\int_D F$ is a well-defined real number whereas θ is an angle, we have a $2\pi K$ ambiguity in F where K is an integer. For a given theta there is more than one bulk solution for F, characterized by K. This is responsible for the multi-branched structure of the allowed ground states which we expect to find. Physics in any given one of the branches is only periodic in $2\pi N$, the actual periodicity of θ is 2π as it should be. We simply jump to a different branch.

For generic theta it is non-trivial to solve the supergravity solutions subject to the constraint (4.56). But a very simple solution can once more be found [132, 22] in the probe limit $(\theta + 2\pi K) \ll N$, or in other words $K/N \ll 1$. In this limit one can neglect the backreaction of the axion on the background geometry. Newton's constant is of order $1/N^2$ in units where the curvature scale $L = 1$, whereas the axion action and hence its stress tensor is of order 1 in the large N counting. The only non-trivial equation left to solve is Maxwell's equation for C_1 in the background geometry (4.54) subject to the boundary condition (4.56). The solution is

$$C_1 = \frac{f(u)}{2\pi R}(\theta + 2\pi K)dy. \tag{4.57}$$

The integer K is the bulk manifestation of the K-th vacuum. In fact, plugging the solution (4.57) back into the action we find that the vacuum energy density of the K-th vacuum has exactly the expected form from (4.45) with [22]

$$h(\theta/N) = -\frac{2N^2\lambda}{3^7\pi^2 R^4}\left[1 - 3\left(\frac{\lambda}{4\pi^2}\right)^2\left(\frac{\theta + 2\pi K}{N}\right)^2\right] \tag{4.58}$$

where $\lambda = g_{YM}^2 N = 2\pi g_s l_s N/R$ is the 't Hooft coupling.

While it is not obvious to us how to realize interfaces in this setup, the holographic dual for a domain wall has already been proposed in [132]. A jump in vacuum, according to

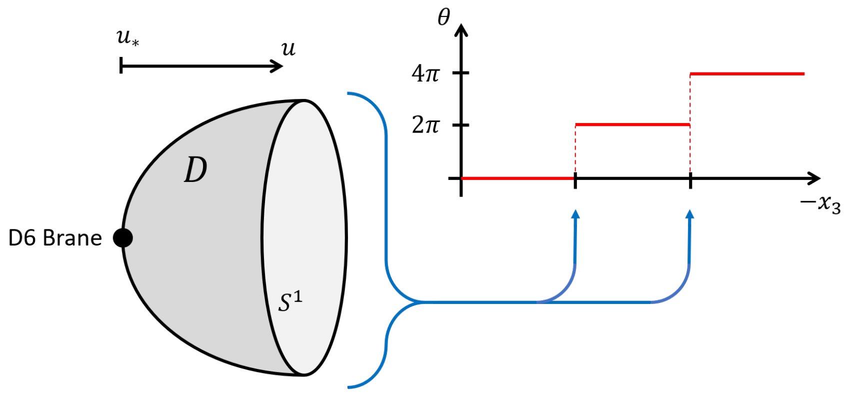

Figure 4.4: Configuration of D6 branes.

(4.56), requires a jump in $\int_D F$, which in turn requires a source magnetically charged under the RR 1-form. The naturally stringy object carrying the appropriate RR charge is a D6 brane. The D6 brane needs to wrap the entire internal S^4 as well as the 3d Minkowski space spanned by t, x_1 and x_2. It is localized in the x_3 direction as well as on the cigar geometry D. From the induced metric of a D6 sitting at a fixed position u and wrapping $M^{2,1} \times S^4$ is we can infer that the D6 Lagrangian density $e^{-\phi}\sqrt{-g_I}$ reads

$$\mathcal{L} \propto u^{5/2} \tag{4.59}$$

meaning that the D6 brane experience a potential pulling it to smaller values of u: the D6 brane will sink to the tip of the cigar, see Fig. 4.4.

Let us first discuss the case of a single D6 brane. Without loss of generality, we can place the D6 at $x_3 = 0$. If we denote by D_- the cigar/disc spanned by (y, u) at a fixed negative x_3 and D_+ the cigar/disc at a fixed positive x_3, then the analog of the magnetic Gauss' law for the D6 brane reads

$$\int_{D_+} F - \int_{D_-} F = 2\pi. \tag{4.60}$$

Comparing with (4.56) we see that this means that $(\theta + 2\pi K)$ jumps by 2π as we cross, in the field theory, the bulk x_3 location of the D6-brane. This also implies that the vacuum energy of the theory jumps across the D6. Furthermore, the D6 brane is clearly dynamical.

The x_3 position of the D6 brane is a dynamical field. Since the metric is independent of x_3 the corresponding worldvolume scalar is massless. These facts together clearly identify the D6 brane as the domain wall between the j-th and $(j+1)$-th vacuum [132] at a fixed theta angle. The general wall in which we jump from the j-th vacuum to the $(j+K)$-th simply corresponds to K coincident D6 branes. We can pull apart the stack of K D6 branes to obtain a configuration of walls where the vacuum jumps one unit at a time at well-separated locations in the x_3 direction.

It is fairly straightforward to determine the low energy physics in the bulk. The background geometry once more truncates at a finite radial position, $u = u_*$. Correspondingly all supergravity modes are gapped. The only degrees of freedom surviving are the ones localized on the D6 branes. For a stack of n coincident D6 branes, these worldvolume degrees of freedom are a $U(n)$ gauge field as well as 3 adjoint scalars corresponding to motion of the stack into the u, y and x_3 direction. The u and y fluctuations are massive due to the cigar geometry just as we reviewed above in section 4.2.2. The x_3 scalar, however, is massless. The worldvolume gauge field picks up a Chern-Simons term of level N from the Wess-Zumino coupling of the worldvolume gauge field to the N units of 4-form flux. So the low energy dynamics in the bulk is governed by a $U(n)_N$ gauge theory with a single massless adjoint representation scalar. Holographic duality implies that this is an equivalent representation of the quiver gauge theory with the additional $n-1$ translational modes associated with the domain walls at least in the large N limit.

Note that this way we almost landed on the duality (4.2). There is however a small difference. On the quiver side, we have the extra light modes corresponding to the translational motion of the domain walls. As we argued before, we expect these to be present at large N. At any finite N the domain walls would no longer be static. The phases of the quiver theory are still given by (4.48) as long as one accounts for the extra decoupled translational modes. The analogous statement on the U side of the duality is that the adjoint scalar governing the position of the stack of probe branes this time corresponds to a flat direction. The various phases are still parametrized by the n eigenvalues of the scalar matrix $\langle x_3 \rangle$. But this time instead of having to add deformations to the potential we have a moduli space of vacua where we can freely dial the expectation values of x_3. The eigenvalues of $\langle x_3 \rangle$ simply correspond to D6 positions and the enhanced gauge symmetries we encounter for coincident eigenvalues simply arise from coincident D6 branes. The gauge groups of the various phases once again are given by (4.53), but since the scalar potential was exactly flat this time each gauge group factor comes with an extra massless adjoint. In the generic case where the gauge group is $[U(1)_N]^n$ these extra massless adjoints map exactly to the n translational modes we identified on the quiver side. Surely the exactly flat potential for the probe scalar is also an artifact of the large N limit and any bulk quantum corrections would lift this flat direction. Furthermore, the phase where the adjoint gets a positive mass squared is not easily realized in the brane picture. Modulo these extra light scalars, matching on both sides, the holographic construction exactly reproduces our conjectured duality (4.2).

4.3 Adding Flavors to Quivers

we will focus on a particular application of said dualities: flavored quiver dualities applicable to domain walls of QCD_4 with N_F fundamental fermions [52].[13] This is a natural extension of previous work [8], where some of the present authors proposed a new $2+1$ dimensional duality relating quiver gauge theories to field theory with adjoint matter. Such quiver gauge theories provide a $2+1$ dimensional effective description domain walls and interfaces in $3+1$-dimensional $SU(N)$ Yang-Mills, i.e. the $N_F = 0$ case of QCD_4.

One thing we do in this chapter is to include fundamental flavors. In [52] domain walls in QCD_4 were considered not just for the case of $N_F = 0$ (i.e. pure YM), but also for the cases $N_F = 1$ and $N_F > 1$, which appear somewhat distinct. The quiver theory described above gets augmented with extra fundamental matter on each node (see Fig. 4.6). Once more, we can derive a dual via node-by-node dualization. We will see the $N_F > 1$ and $N_F = 1$

[13] A quick clarification on notation: throughout this chapter N_F will be used when referring to the number of flavors of fundamental fermions in QCD_4 while N_f will be used when referring to the parameter in the 3d bosonization dualities.

cases require use of two distinct regimes of the flavor-violated master duality we proposed in Chapter 2.

Holographically, the inclusion of flavor can be accomplished by adding probe D8 and $\overline{\text{D8}}$ branes as in the Sakai-Sugimoto model [110]. Holographic theta walls in this context have been discussed recently in [17]. While both constructions have their own subtleties, in the end both give closely related conjectured duals for flavored quivers.

A second generalization is to extend the original construction to gauge theories based on orthogonal and symplectic groups. The master duality is known for these gauge groups as well, so once more we can employ a node-by-node dualization. On the holographic side, the projection to orthogonal and symplectic groups can be enforced by orientifolds. Again we see consistency between the node-by-node dualization and the holographic construction.

We now turn our attention to the construction and dualization of flavored quiver gauge theories. These purely $2+1$ dimensional theories will serve as an effective description of interfaces in 3+1 dimensional QCD when one varies the θ angle along a particular coordinate direction. Interestingly anomaly considerations alone aren't sufficient to pin down the $2+1$ dimensional theory. While it is difficult to prove, it has been argued in [52] that different theories govern the steep versus the shallow interface. Concretely, let us focus on the case where the θ angle experiences a net jump of $2\pi n$ with an integer n. In this case the shallow interface in a 3+1 dimensional gauge theory with gauge group $SU(N)$ is believed to be described by a 2+1 dimensional $[SU(N)_{-1}]^n$ gauge theory, whereas the steep wall is described by a single $SU(N)_{-n}$ theory.

At least for the shallow interface this can easily be argued based on the general expectations for the θ dependence in $SU(N)$ gauge theories, at least at large N, as described in [132]. The vacuum energy in any given vacuum at large N can be shown to be $2\pi N$ periodic. In order to reconcile this with the expected 2π periodicity of QCD one postulates that the theory has N different vacua as depicted in figure 4.5. At any given θ the vacuum energy is lowest in one of the N vacua. As θ increases one finds that whenever it reaches an odd multiple of π two of the vacua are degenerate. Further increasing θ past this point triggers

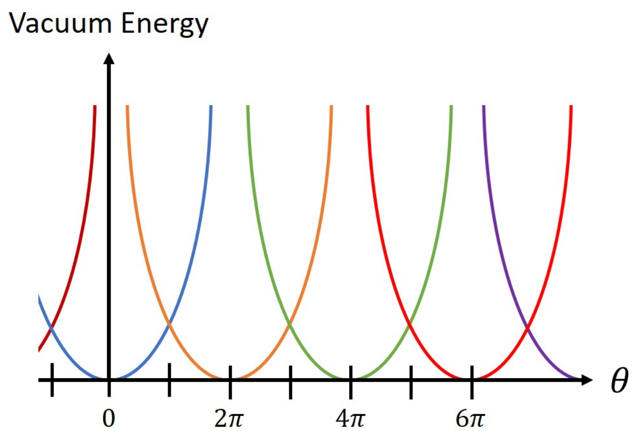

Figure 4.5: Expected behavior of the vacuum energy as a function of θ in a large N gauge theory. Distinct branches are shown in different colors.

a transition to a different vacuum and by the time we shifted θ by 2π one indeed is back to the same physics, but in a different vacuum.

This allows a very simple description of the shallow interface, where the gradient of θ is small compared to the strong coupling scale of the theory. All the interesting physics is localized at the points where θ passes odd multiples of π. Since the gradient of θ is small, these loci are widely separated and we expect the total topological field theory to be simply n copies of the theory living on a single interface, which then can be argued to be $SU(N)_{-1}$ based on anomalies [1, 44]. For a steep interface there is another theory that can carry the correct anomaly – a single $SU(N)_{-n}$. If the transition between the steep and shallow interfaces is second order, it should be described by a conformal field theory. This is the quiver gauge theory of (4.1). If we give all the scalars a large positive mass, we simply remove all bifundamental scalars leaving behind the product gauge group of the shallow interface. On the other hand, giving all the scalars a large negative mass drives the gauge groups into the Higgs phase, breaking them down to the single gauge group of the steep interface. We wish to see how this picture changes with the addition fundamental fermions.[14] This will require the use of the flavor violated master duality developed in the previous section.

To construct the flavorless quivers, we used the fact that one can identify and gauge the $SU(N_f) \times SU(N_s)$ flavor symmetries of (multiple copies of) the master duality to get bifundamental matter charged under various gauge symmetries. Adding flavors to the nodes can be achieved if we only gauge a *part* of the flavor symmetry instead of the entire global flavor symmetry. The leftover global symmetry and the corresponding field components then become additional matter on each node.

We employ the aforementioned procedure in order to engineer a quiver with SU gauge theories and scalar matter on the links. Node-by-node duality in general turns this into an equal length quiver with U gauge theories on the nodes and once again scalar bifundamental matter. We add extra fundamental representation fermions on the original SU side, which we will see necessitate extra scalar matter on the dual U side. Intermediate steps involve theories with bifundamental fermions as well, but they are present neither in the initial nor in the final theory. The U side can be argued to collapse to a single gauge group when the ranks of the gauge groups on all nodes are equal in the original SU type theory. This pattern is what we previously found in the case of the unflavored quivers, and we will see it again once flavors are included.

While we can construct these quiver dualities for generic ranks and levels on the nodes (subject to certain bounds) there are special values of these parameters for which certain nodes confine. While in this case uncontrolled strong coupling dynamics is important, holography suggests that in this case a much simpler duality emerges. We will argue that

$$[SU(N)_{-1+N_F/2}]^n + \text{bifundamental scalars} + N_F \text{ fundamental fermions per site} \tag{4.61}$$

with certain potential terms is dual to

$$U(n)_N + \text{adjoint scalar} + N_F \text{ fundamental scalars.} \tag{4.62}$$

[14]For a discussion on interfaces in QCD_4, see Appendix A of [52].

Clearly this is a intuitive generalization of (4.1) and (4.2). Let us justify these results again using node-by-node duality and holography.

4.3.1 Node by Node Duality

Before deriving the dual description for flavored quivers, let's remind ourselves of some notation. We will index the nodes by $i = 1, \ldots, n$. The bifundamental scalars between the ith and $(i+1)$th node in the SU and U theories are labeled by $Y_{i,i+1}$ and $X_{i,i+1}$, respectively. The bifundamental fermions of the intermediate theories will be labeled by $\psi_{i,i+1}$. We'll call the flavored fermions belonging to the ith node ψ_i. Meanwhile, the scalar flavor degrees of freedom which are the dualized ψ_i are denoted by ϕ_i. All levels used in these notes are equivalent to the "bare" levels used in [86].

The most general 3-node quiver with flavor degrees of freedom on each node is shown in Fig. 4.6. Its dual can be derived as in Ref. [8] using the master duality and its $N_f = 0$ and $N_s = 0$ limits.[15] In particular, if one gauges the flavor symmetries of the master duality, one arrives at the following two and three-node dualities,

$$SU(N)_{-k} \times [SU(N_s)_0] \quad \leftrightarrow \quad U(k)_{N-N_s/2} \times \left[SU(N_s)_{-k/2}\right] \tag{4.63}$$

$$SU(N)_{-k+N_f/2} \times \left[U(N_f)_{N/2}\right] \quad \leftrightarrow \quad U(k)_N \times [U(N_f)_0] \tag{4.64}$$

$$SU(N)_{-k+N_f/2} \times \left[U(N_f)_{N/2} \times SU(N_s)_0\right] \quad \leftrightarrow \quad U(k)_{N-N_s/2} \times \left[U(N_f)_0 \times SU(N_s)_{-k/2}\right]. \tag{4.65}$$

Each of these is subject to particular flavor bounds, but let us ignore them for a moment. Stepping from Theory A to Theory B we use (4.65) with the $U(N_f)$ symmetry ungauged (i.e. the master duality with only the $SU(N_s)$ symmetry promoted to be dynamical). From Theory B and Theory C, again use (4.65) but gauge only part of the background flavor symmetry such that the $U(k_1 + F_2)$ background fermion flavor symmetry becomes a $U(k_1)$

[15] In Ref. [8] the $N_F = 0$ quiver was derived with no regard to distinguishing between ordinary and spin$_c$ connections. In Appendix D.4 we elaborate on how such quivers can be consistently formulated on spin$_c$ manifolds.

gauge symmetry and the remaining $SU(F_2) \times U(1)$ are still global symmetries. Finally, use (4.64) to go from Theory C to Theory D, again with a split flavor symmetry with a part which is gauged and another which is untouched.

It is straightforward to see how this pattern generalize to the n-node quiver. Let N_1, ..., N_n denote the number of colors on the nodes and k_1, ..., k_n the levels of the corresponding Chern-Simons terms. The duality for the n-node quiver reads

$$\left[SU(N_1)_{-k_1+\frac{F_1}{2}} + \text{fund } \psi_1\right] \times \prod_{i=2}^{n} \left[SU(N_i)_{-k_i+\frac{F_i}{2}} + \text{bifund } Y_{i-1,i} + \text{fund } \psi_i\right] \tag{4.66a}$$

$$\leftrightarrow \qquad \prod_{i=1}^{n-1} \left[U(K_i)_{N_i-N_{i+1}} + \text{bifund } X_{i-1,i} + \text{fund } \phi_i\right] \times \left[U(K_n)_{N_n} + \text{fund } \phi_n\right]. \tag{4.66b}$$

The ranks of the gauge groups in the U quiver are given by

$$K_n \equiv \sum_{i=1}^{n} k_i. \tag{4.67}$$

This is very similar to the relation as was found in the unflavored case [8]. As in the 3-node case depicted in Fig. 4.6 we can also keep track of the level of the flavor groups. These can always be shifted by an overall background Chern-Simons term added on both sides, but if we chose the levels to be $SU(F_i)_{N_i/2}$ on the SU side, they end up being $SU(F_i)_0$ on the U side.

Now let us specialize to the case where $N_1 = \ldots = N_n = N$, $k_1 = \ldots = k_n = 1$, and $F_1 = \ldots = F_n = N_F$. If, in addition, we add a $\det X_{i,i+1}$ potential term on each link, this is the theory that describes the physics of interfaces in QCD$_4$. Recall, the determinant term was needed in order to eliminate the additional $U(1)^{n-1}$ global symmetry present on the quivers [8, 52], and its U side equivalent is a monopole operator. It is easy to see that only the $N_F = 0$ case does not violate any flavor bounds. This is because all the levels must satisfy $k_i \geq F_i$ and in addition we must avoid the double-saturation limit of the master duality. Since we have already saturated the $N_i \geq N_{i+1}$ flavor bound, $N_F = 0$ is the only way we can avoid violating such bounds.

The flavor-extended master duality and double saturated master we laid out in the previous section allows us to proceed. The $N_F = 1$ case corresponds to use of the double-saturated

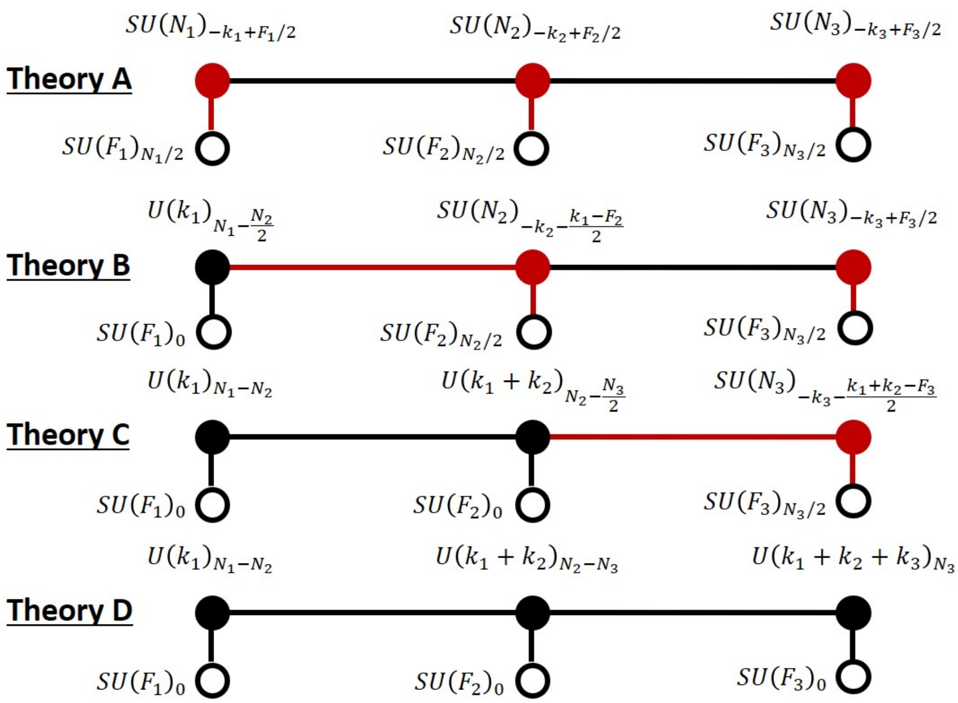

Figure 4.6: Flavored quiver with fermions on the SU side and scalars on the U side. The red/black nodes correspond to U and SU gauge groups, respectively. The red and black links are fermions and scalars. The white nodes are background flavor symmetries.

master duality. As discussed in Sec. 2.3, the introduction of the determinant terms allow extend the validity of the master duality to this limit. We will discuss the $N_F > 1$ in detail in what follows below, but let us summarize here the essential points for general $N_F \geq 1$. We saw that for one sign of the mass the flavor extended duality is just the same as we had when the flavor bounds were obeyed (see (2.7c) and (2.7d)) , so we can just continue to use the duality above if we stay in such phases. On the SU side, each of the nodes has a

$$SU(N)_{-1+N_F/2} \text{ with } N_F\ \psi \tag{4.68}$$

theory on it. We also see that in this special case of equal N_i we get a dual quiver on the U side where all but the last node have a level 0 Chern-Simons term. This is once again completely identical to the case of the un-flavored quivers considered in [8]. The non-Abelian factors with level 0 confine. Lo and behold, all but the last node disappear from the low energy spectrum. To pin down the remaining charged matter under the last gauge group, we need to make some dynamical assumptions for this confinement mechanism. In [8] it was argued that the only light remnant of the bifundamental matter is an adjoint "meson" made from the bifundamentals on the last link. This assumption gave rise to a duality conjecture that agreed with the holographic construction and passed several non-trivial consistency tests. If we make the same assumption here, we are lead to the duality conjecture of eqs. (4.61) and (4.62). Once again, we will see that this is also what the holographic construction tells us.

Similar dualities can also be derived if the extra flavors on the SU side are taken to be scalars, but the patterns that emerge are more complicated than in the fermionic case and do not appear to yield any simple interesting new dualities.

Constructing the $N_F > 1$ Quiver

Let us now use the flavor extended master duality to construct quivers. This will allow us to extend the validity of our dualities to $N_F > 1$. As we have seen, this will be complicated slightly by the fact one needs two theories to describe the entire phase diagram of the U

side. Since there are even more mass deformed phases in the quivers, we should expect the number of theories needed to ultimately describe all possible mass deformations of the U side to be quite large. Every time one uses a flavor violated duality the number of scalar theories needed to describe the entire phase diagram doubles. For our three-node case we can consider $2^3 = 8$ separate theories to capture all possible mass deformations of the original fermionic theory. However, all eight theories should be equivalent descriptions of the same Grassmannian manifold.

In this section we will only consider two of the eight possible descriptions of the U side of the quiver. These will be the cases which are valid when all masses of the quivers are deformed such that we are in phases III and IV or I and II of the flavor-extended master duality. These correspond to the extreme cases in the original SU theory where all the fermions are tuned to large positive and large negative mass, respectively. It is possible construct quivers for mixed phases, but we do not consider them here.

In what follows, we will first apply this general strategy to the special case of $n = 3$ nodes. After considering the three-node case, we will generalize such extreme cases to n-nodes.

$m_{\psi_i} < 0$ *Duality*

The special case where all masses are negative is especially easy since in this case the duality map is formally still given by (4.65). That is, the entire derivation shown in Fig. 4.6 in principle still holds since the initial theory is the same and the extended master/Aharony dualities are also the same. Thus much of the same conclusions we made there can be applied here. The primary difference is that the identification between the two sides of the phase diagrams is shifted from the flavorless case. Following the mass mappings through the derivation, for the bifundamentals we still have $m^2_{Y_{1,2}} \leftrightarrow m^2_{\psi_{1,2}} \leftrightarrow -m^2_{X_{1,2}}$. For the flavor degrees of freedom we have

$$m_{\psi_i} + m_* \leftrightarrow -m^2_{\phi}. \tag{4.69}$$

Zero mass for the fermion no longer maps to zero mass for the boson. In particular, the $m_{\psi_i} = 0$ region now corresponds to the $m^2_{\Phi_i} < 0$ region, and this "mass offset" will introduce certain complications.

Let us discuss in detail how the analysis of interfaces changes in the presence of the additional flavors due to this mass offset. Once more we specialize to the values relevant for QCD_4, namely $N_i = N$, $k_i = 1$, and $F_i = N_F > 1$. We'll begin by assuming no deformations of the matter on the SU side, i.e. $m_{\psi_i} = 0$ and $m^2_{X_{i,i+1}} = 0$. First off, note that because $N_F > 1$ each node in the initial SU theory has Grassmannians. Thus, this should ultimately map to something on the U side of the duality which also produces Grassmannians. Since the U side only has scalars, in order to produce said Grassmannians some of said scalars must be in their Higgs phase. This is already very different than the flavorless case where theories with no mass deformations were mapped to one another.

It will be useful to keep in mind which matter is "responsible" for the flavor violation and presence of the Grassmannian description. Recall, per the conjectured of [86], this can occur from either having too many light fermions or too many scalars with a negative mass. The culprit is obvious in the SU theory: without the additional flavors on each node we have the flavorless case which we know meets all flavor bounds. Thus, it is the additional fermions of each node which are responsible.

This is complicated in (say) Theory B, where we now have two subgroups of fermions connected to the second node, corresponding to $SU(F_2)$ and $SU(k_1)$ (see Fig. 4.6). The former are the additional flavors that were on that node to begin with, the latter are the bifundamental fermions which were previously scalars. If all such fermions were light for the portion of the phase diagrams we are interested in, the Grassmannian manifold of the second node may have changed, and this would be problematic since we claim it is dual to Theory A.

To resolve this issue, first note that in Theory B the flavor degrees of freedom on the first node have been converted into scalars, and because the mass offset these scalars have a negative mass. This is consistent with there being a Grassmannian on the first node.

What we have neglected are the interactions introduced by using the master duality that are now present in Theory B. Since the scalar flavors on the first node are now Higgs'd, the interactions cause the $\psi_{1,2}$ fermions to be gapped. Meanwhile, the ψ_2 remain light. Hence the only light fermionic degrees of freedom are those which were already present in Theory A, so the Grassmannian on the second node also remains the same.

Generalizing this argument down the quiver, we see that it is always either the ψ_i or ϕ_i (i.e. the flavor degrees of freedom on each node) which break the nodes down to Grassmannian descriptions. This might have been expected, since these are the new ingredients relative to the flavorless case, but it is nice that the quivers get it right.

As in Ref. [8], the phase corresponding to $m^2_{X_{i,i+1}} < 0$ causes a breaking of the Chern-Simons terms in the theory to $[U(1)_N]^3$. Thus on the U side of the theory we ultimately end up with

$$U(1)_N \text{ with } N_F \ \phi_i \tag{4.70}$$

on each node, with $m^2_{\phi_i} < 0$, as we would have expected.

Critical Theory Formally the duality map (2.6) is only valid when we tune the fermion mass to it's critical value, the scalar mass to zero and apply the map (2.9). In this case the dualized scalars ϕ_i are massless and the interactions terms introduced through the master duality do nothing to the mass of the fermions. This is okay since we would not expect Grassmannians at this location in phase space. Additionally, since the number of bifundamental fermions is less than the level of the corresponding node, we do not get any Grassmannians. The duality thus states

$$\begin{aligned} &[SU(N)_{-1} + N_F \text{ fundamental fermions with } m_{\psi_i} = -m_*]^n + \text{bifundamental scalars} \\ &\qquad \leftrightarrow \qquad U(n)_N + N_F \text{ fundamental scalars} + \text{bifundamental scalars.} \end{aligned} \tag{4.71}$$

It is interesting to consider deformations of the bifundamentals in this phase. Luckily, almost all of the analysis performed in [8] holds here as well. That is, on the SU side gapping the bifundamentals simply removes them from the spectrum. Higgsing the bifundamentals leaves the gauge transformations which transform two adjacent nodes equally unbroken. This causes the levels of the two nodes to add. The mass identifications of the bifundamentals on the U side are still opposite of those on the SU side. Thus for gapped scalars on the SU side we have Higgsing of the U groups down to $U(1)$. Meanwhile, the first $n-1$ nodes on the U side are confining for full Higgsing on the SU side.

Let us consider this latter case in a little more detail. Consider sitting at the critical point where all $m_{\psi_i} = -m_*$ and all $m^2_{Y_{i,i+1}} = 0$. Now give all $Y_{i,i+1}$ a negative mass, so the Chern-Simons term on the SU side is $SU(N)_{-n}$. All bifundamentals are gapped on the U side, and the only non-confining node is all the way on the right, given by a $U(n)_N$ Chern-Simons theory. Now consider tuning the masses of the flavor degrees of freedom. Move the fermion on the last node, ψ_n, to a slightly larger mass (but smaller in magnitude), $-m_* + \epsilon$, which causes us to get a $\mathcal{M}(N_F, n)$ Grassmannian. On the U side this causes the dual scalar ϕ_n to acquire a negative mass, breaking down the node flavor symmetry to $\mathcal{M}(N_F, n)$. Thus both sides are consistent.

One needs to be slightly more careful when tuning the masses of any other matter, say ψ_j for $j < n$. Moving to a larger mass $m_{\psi_j} = -m_* + \epsilon$ again causes the corresponding ϕ_j scalar to get a negative mass, which due to interaction terms present on the U side gives the $X_{j,j+1}$ bifundamental a *negative* mass. Although the interactions are finite, we will only focus on the case when they are stronger than mass deformations. The scalar then breaks the corresponding the $U(j+1)_0$ gauge group and this propagates down the quiver similar to the flavorless case. Ultimately this causes a breaking of $U(n)_N \to U(n-j)_N \times U(j)_N$. On the SU side of things this is reflected in the fermion condensing, causing a breaking of the gauge group and a $\mathcal{M}(N_f, j)$ Grassmannian on the jth node, and a leftover $SU(N)_{-n+j}$. This is consistent since the breaking from the negative mass bifundamental scalar, $Y_{j,j+1}$, is inconsequential for corresponding node since it is instead now broken by the fermion condensate. To summarize, tuning the masses of flavor degrees of freedom on nodes $j < n$ undoes the effects of Higgsing the bifundamental scalars on the SU side/gapping the bifundamental scalars on the U side.

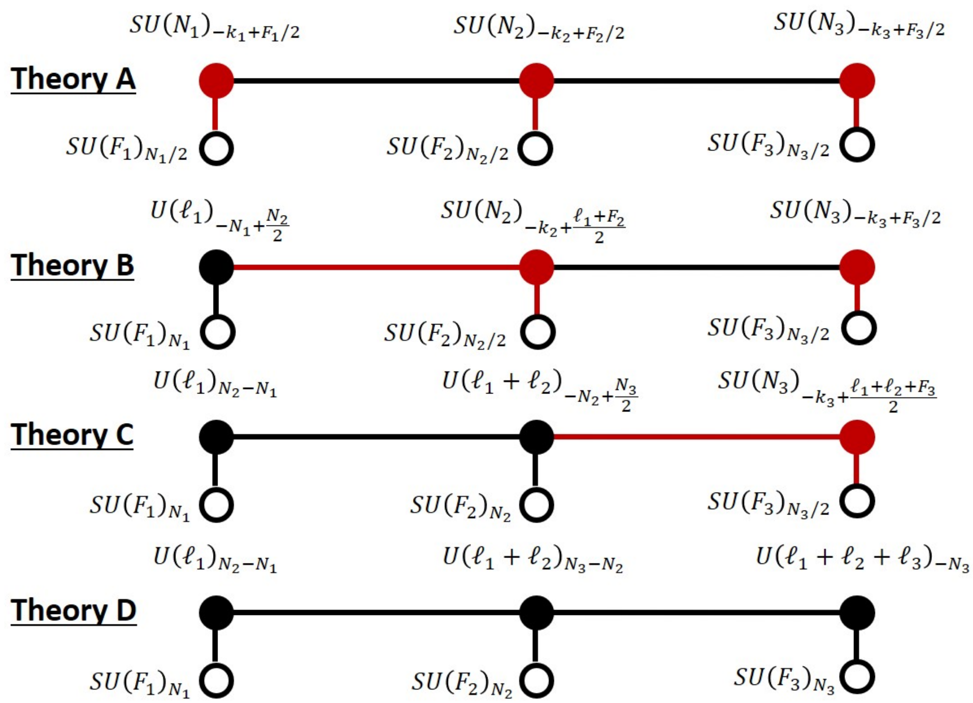

Figure 4.7: Flavored quiver with fermions on the SU side and scalars on the U side. This is an alternate theory in the flavor extended case. Here we have introduced the notation $\ell_i \equiv F_i - k_i$.

$m_{\psi_i} > 0$ *Duality*

Now let us instead look at the case where all masses are positive. In this case we need to use the $m_\psi = m_*$ theory of (2.6), which is spelled out explicitly in (C.12), to derive the flavor extended quiver. We consider again the three node quiver for the most general set of parameters. It will be useful to introduce a new notation $\ell_i \equiv F_i - k_i$. The derivation is shown in Fig. 4.7.

Note since we are using the $m_\psi = m_*$ duality, the interaction in this phase is slightly different. Namely, the interactions are now in the $m_\Phi^2 < 0$ and $m_\Psi > 0$ phase and prevent the fermions from getting a positive mass. That means that the interaction term comes with a negative sign out front.

4.3.2 Holographic Construction

Having generalized the node-by-node construction in order to motivate the duality between (4.61) and (4.62), let us now turn to the holographic derivation. The holographic construction used in [8] is in spirit similar to the one first introduced in [70] for the study of 2+1 dimensional bosonization dualities, which is based on the even earlier holographic realization of level/rank duality in [49]. The key idea here is to embed the Chern-Simons matter theory of interest inside a larger gauge theory with a known holographic dual. In the UV we have the full set of degrees of freedom of the larger gauge theory, dual to a theory of gravity in the bulk. Upon suitable deformation on both sides the gauge theory and its gravitational dual develop a gap for most degrees of freedom, only a small subset survives in the infrared. The original field theory is engineered to reduce to a Chern-Simons matter theory in 2+1 dimensions. In the bulk most degrees of freedom are gapped out as well, including all the fluctuations of the graviton and its superpartners. The only degrees of freedom that survive are localized on a probe D-brane.

More specifically, in [70] the basic bosonization dualities were reproduced by compactifying $\mathcal{N} = 4$ super Yang-Mills on a circle with antiperiodic boundary conditions for the fermions and a θ angle wrapping non-trivially around the circle direction. The low energy theory yielded a gapped Chern-Simons theory, with extra light matter added via probe branes. In order to obtain quiver gauge theories one needs to start with a slightly more complicated construction. Fortunately, most of this framework has been laid out in [132] in the context of the simplest holographic dual for a confining gauge theory: Witten's black hole [130]. Witten's black hole describes the gravitational dual of $4 + 1$ dimensional super Yang-Mills compactified on a circle with anti-periodic boundary conditions for the fermions. This procedure gives masses to all matter fields and so, at low energies, one is left with pure Yang-Mills in $3 + 1$ dimensions, which itself is gapped. Topologically Witten's black hole solution is $\mathbb{R}^{3,1} \times D_2 \times S^4$, where the first factor is $3 + 1$ dimensional Minkowski space and the disc or "cigar" D_2 contains the radial direction of the holographic dual as well as a compact circle direction which smoothly shrinks to zero size at a critical value r_* of the radial coordinate. The relevant holographic dual for the interfaces we are seeking are D6 branes sitting at the bottom of the cigar, wrapping all of the internal S^4 and being localized in one of the three spatial directions of the $\mathbb{R}^{3,1}$ factor.

One subtlety here is that the D6 branes were argued in [132] to be dual to a domain walls between two vacua at a fixed θ rather than a θ interface. At a given θ we still have N vacua, with all but one of them being only meta-stable. The energy difference between vacua however is of order 1 in the large N counting, whereas the tension of the domain wall is of order N. So in the large N limit these domain walls are long lived and approximately stationary. To truly describe an interface, we should turn on a θ gradient in addition to the D6 branes, which corresponds to a RR 1-form in the bulk that depends on the radial direction in the bulk as well as the spatial direction orthogonal to the domain walls, call it x. The simplest such supergravity solution is an RR 1-form that grows linearly along x. In this case the RR 1-form equation of motion is solved without giving the 1-form any dependence on the holographic radial coordinate. This solution as it stands corresponds to increasing θ along a spatial direction without adjusting vacua, but instead staying on a meta-stable branch. This is clearly not the physical vacuum of the theory. In order to truly correspond to the interfaces of [52] we need to still include D6 branes in addition to the spatially varying theta angle in order to ensure that the field theory always is living in the locally true vacuum. At least in the simple case of a linearly varying θ angle it is easy to see that the D6 branes experience a potential that will automatically localize them where θ crosses odd multiples of π. To see that this is the case, recall that the backreaction of the D6 branes induces an RR 1-form field $\sim (\theta + 2\pi K)$ which gives rise to an energy density of order $(\theta + 2\pi K)^2$. Here K labels the different vacua. Let us look at a spatial region in which θ varies over a 2π range. Let us chose this range to be 0 to 2π with $K = 0$. This can always be done by the way we label vacua in terms of K. Let us assume that the change is θ is taking place over an interval of length L with $\theta = 2\pi x/L$. There is a single D6 brane located at x_0 with $0 \leq x_0 \leq L$ across which the vacuum jumps to $K = -1$. The vacuum energy associated to this configuration is

$$E \sim \int_0^l dx \, (\frac{2\pi}{L}x - \Theta(x-x_0)2\pi)^2 = \frac{4\pi^2(L^2 - 3Lx_0 + 3x_0^2)}{3L} \tag{4.72}$$

where Θ is the step function that is 1 when its argument is positive and 0 otherwise. Minimizing with respect to x_0 we indeed find $x_0 = L/2$. The D6 brane wants to sit at the middle of the interval where $\theta = \pi$. Note that these energies are all of order 1 in the large N counting. This is due to the fact that they involve the brane backreaction. When determining the leading order in N physics on the brane, we can neglect these potentials. The upshot is that irrespective on whether we describe interfaces or domain walls, the holographic dual description is given in terms of a stack of n D6 branes at the bottom of the cigar as long as we jump n vacua across the co-dimension one object.

Interestingly, this holographic dual at low energies does not give back the quiver gauge theory (4.1), but its dual incarnation (4.2): The theory on a stack of n D6 branes is a $U(n)$ super-symmetric gauge theory. Due to the Wess-Zumino terms on the worldvolume coupling to the 4-form flux supporting Witten's black hole the gauge field picks up a Chern-Simons term of level N. The worldvolume fermions get mass from the compactification on the internal S^4. The worldvolume scalars correspond to geometric fluctuations of the D6 branes. The two scalars corresponding to fluctuations of the D6 on the disc are massive due to the warped geometry of the spacetime: there is a non-vanishing potential energy cost associated with fluctuating up the cigar. The only light matter on the D6 branes is a single massless adjoint scalar corresponding to the fluctuations in the x directions. To leading order in N it is massless, even though we've already seen above that a non-trivial potential will surely be generated at order 1. Lo and behold, the gauge theory on the D6 branes exactly realizes (4.2) as advertised.

The inclusion of extra flavors is now conceptually straightforward but the details are somewhat daunting. Many aspects of this construction have been discussed nicely in [17]. On the field theory side, our flavored quiver gauge theory from (4.61) indeed already appeared in the study of interfaces alongside its flavorless cousin. If instead of studying θ interfaces in pure Yang-Mills one studies them in a confining $SU(N)$ gauge theory with fundamental fermions, the correct gauge theory on the domain walls is exactly the flavored quiver [52]. In the holographic dual, we need to augment Witten's black hole with N_F flavor D8 branes in order to describe holographic QCD [110]. The D8 branes are localized on the compact circle but their worldvolume extends in all other directions. In particular, the D6 branes dual to the domain walls are entirely embedded inside the D8 worldvolume. That is, the D6/D8 system constitutes a Dp/Dp+2 brane system with 2ND directions. Correspondingly, the 6-8 strings connecting the two gives rise to extra scalar matter in the fundamental representation of the $U(n)_N$ gauge theory on the D6 worldvolume. Indeed we have found that the flavored quiver of (4.61) has a holographic dual description in terms of the theory in (4.62). When asking more detailed questions, lots of open problems emerge. The scalar at a 2ND brane intersection is tachyonic. So in order to find the CFT dual to the flavored quiver, we need to let the scalars condense. The result is outside of the range of perturbative string theory, so the outcome is somewhat inspired guesswork. The fact that this condensation is happening is presumably related to the fact that in the field theory we passed the flavor bound. The extension of the duality into this regime moved occurrence of the conformal field theory in the phase diagram from zero mass to the edge of the chirally broken regime with its Grassmannian pion Lagrangian as already argued in [17]. So qualitatively holography supports our duality conjecture.

4.3.3 Phase Matching

As a final check we want to confirm that the possible phases match in the duality of (4.61) and (4.62). Recall how the matching of phases worked in the flavor-less case. On the SU side the bifundamental scalars could either acquire an expectation value or a mass. In the latter case they just disappear from the low energy spectrum, in the former they break two neighboring nodes to the diagonal subgroup. The corresponding levels add. The allowed phases were hence given by partitions $\{n_I\}$ of n, that is integers n_I with $\sum_I n_I = n$. Each n_I specified how many consecutive nodes were broken down to the diagonal subgroup before encountering a scalar that became massive. So $n_1 = n$ corresponds to the case were all bifundamentals acquired an expectation value, whereas $n_I = 1$ for $I = 1, \ldots, n$ is the case where all bifundamentals get a mass. The generic partition led to a phase governed by a topological field theory based on a gauge group

$$\prod_I SU(N)_{-n_I}. \tag{4.73}$$

The phases of the $U(N)_n$ gauge theory were parametrized by the expectation values of the adjoint scalar. Latter can always be diagonalized, so we have to specify N eigenvalues. Enhanced unbroken gauge groups arise whenever eigenvalues coincide. Using again the partition $\{n_I\}$ to denote to multiplicities of repeated eigenvalues the phases of the $U(N)_n$ plus adjoint theory gave

$$\prod_I U(n_I)_N. \tag{4.74}$$

Eqs. (4.73) and (4.74) are level-rank duals, so both sides have the same phase diagram.

Now let us see what happens in the presence of flavors. Let us first take a look at the U side of the duality. Under the gauge symmetry breaking of (4.74) triggered by the adjoint scalar the fundamental matter multiplets decompose into fundamental matter under each of the product factors, so that we obtain a theory

$$\prod_I \left[U(n_I)_N + N_F \text{ fund scalars}\right]. \tag{4.75}$$

For each gauge group factor we now can, as usual, drive the fundamental scalars to condense or to become heavy and decouple. We know that the dual description of all these phases is captured by:

$$\prod_I \left[SU(N)_{-n_I+N_F/2} + N_F \text{ fund fermions}\right]. \tag{4.76}$$

This is indeed a theory we can get out of the flavored quiver, but it requires a non-trivial potential. Note that in the quiver without potential we would find that whenever two nodes with their $SU(N)_{-1+N_F/2}$ gauge groups and N_F fundamental fermions each get broken down to their diagonal subgroup, we would get a $SU(N)_{-2+N_F}$ and $2N_F$ fundamental fermions and an enhanced flavor symmetry. In order to get (4.76) we need a quartic potential which gives one of the two sets of N_F flavors a negative mass so they decouple and shift the Chern-Simons level back to $N_F/2$. Generally, when we break down n_I nodes to their diagonal factor, we need $n_I - 1$ sets of N_F flavors to get a negative mass from the $n_I - 1$ bifundamentals. This can be accomplished by the appropriate quartic potential, but it is crucial that this is included. Apparently the limit of no interactions on the SU side that we have mostly been working with in the flavorless case is inconsistent with the confinement scenario where we are left with the theory of (4.62) on the U side.

4.3.4 Enhanced Flavor Symmetries

A crucial feature of the new flavor-violated master duality is the presence of finite interactions on the SU side of the duality. These interactions ensure that when one Higgses the bifundamental scalars they give a gap to all but the very last node's flavor degrees of freedom. Explicitly, the interactions enter in the form[16]

$$\mathcal{L} \subset c' \sum_{i=1}^{n-1} \left(Y^{\dagger}_{i,i+1} \psi_i \right) \left(\bar{\psi}_i Y_{i,i+1} \right) . \tag{4.77}$$

Thus, when we Higgs the $Y_{i,i+1}$ bifundamentals, the ψ_j for $j = 1, \ldots, n-1$ all acquire a finite positive mass. Since only the ψ_n remains light, we effectively have an $SU(N)_{-n+N_F/2}$ theory coupled to only N_F fermions. If these interaction terms were not present, all ψ_i would remain light. This would lead to an $SU(N)_{-n+N_F/2}$ theory coupled to nN_F fermion, which gives a Grassmannian which does not agree with the pure $3+1$ dimensional analysis [52].

Although the ψ_j acquire a mass from interaction terms, this can be canceled by an *explicit* mass deformation for all the ψ_j. This is what occurs on critical lines corresponding to II-III, II-VI, and III-VI. If this is done for all ψ_j, then we once more have nN_F light fermions, which can lead to enhanced symmetries and strange looking Grassmannians. For consistency, these

[16] For consistency we must also include finite interactions for scalars on adjacent links, however we are only focusing on the part of phase space where these interactions are small compared to any given mass scalar mass deformation. We leave a full mapping of the phase diagram with finite interactions for future work.

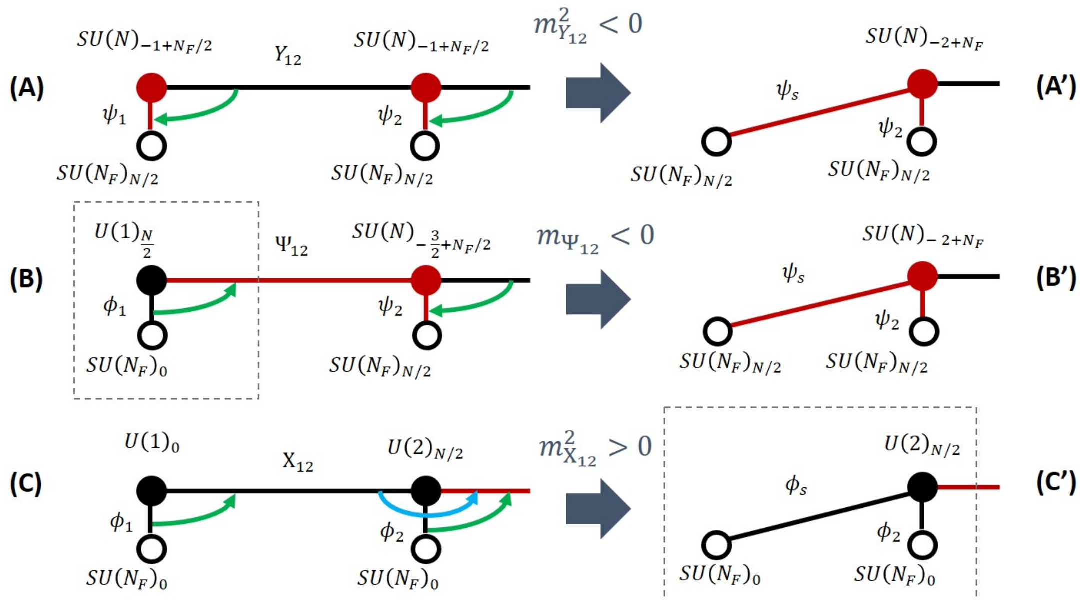

Figure 4.8: Explanation of interaction and enhanced flavor symmetries which arise due to the flavor-violated master duality and its finite interaction terms. We are using green/blue arrows to denote the (unidirectional) interactions. The green arrows represent finite interactions between the flavor degrees of freedom and adjacent bifundamentals. The light blue arrows represent interactions between bifundamental scalars, which were also present in the pure YM case of [8]. We assume they take the very same form they did there. The right-hand side shows how one arrives at enhanced flavor symmetries on the adjacent node at each step of the duality.

flavor enhanced Grassmannians should be present on the U side of the duality as well. This is indeed what occurs. The procedure is summarized in Fig. 4.8.

First let us review what occurs in the fully SU theory, denoted by A in Fig. 4.8. When one Higgses the Y_{12} bifundamental, this effectively ties the gauge theories of the first and second node together. We have denoted this mass deformation in A' by representing said nodes as a single node. From the point of view of Fig. 2.4, this corresponds to moving along the II-III critical line. Now, when one gauges the $SU(N_s)$ global symmetry, one gets a Grassmannian around where this line was previously. This is because the non-Abelian background terms have Chern-Simons levels which violate the flavor bounds. This Grassmannian will be $\mathcal{M}\left(2N_F, 2\right)$, where the $2N_F$ comes from the N_F fermion flavors which were already tied to the second node, and another N_F flavors which previously belonged to the first node.

Now let us discuss how this is matched in B of Fig. 4.8. To do so, it is helpful to reference what occurs in Fig. 2.4 along the II-III critical line when the $SU(N_s)$ global symmetry is gauged. Note that, prior to the gauging of said symmetry, we also get N_sN_f light fermions along said line, which we will call ψ_s [17]. From Fig. 2.4, we get the same behavior when we give Ψ_{12} a negative mass deformation. The ψ_s lead to an enhanced flavor symmetry on the second node, shown in B' of Fig. 4.8. If we took the ψ_s and ψ_2 to be light, we would again arrive at the Grassmannian $\mathcal{M}(2N_F, 2)$. Everything seems consistent so far, which shouldn't be too surprising since we have only used the flavor-violated master duality once, and it can be confirmed this is consistent when gauging the $SU(N_s)$ symmetry.

In moving from B to C we have used the flavor-violated master duality on the second node. This changes the fermions Ψ_{12} and ψ_2 to scalars. Strictly looking at C and gapping the X_{12} scalars, which via mass mapping is analogous to Higgsing the Y_{12}, it is difficult to see how we again get the enhanced flavor symmetry on the second node. This isn't too

[17] As a quick aside, notice that it is difficult to see from the point of view of just the blue scalar theory where these light fermions come from (as opposed to the III-VI critical line, where all kN_s fermions modes are simply light). Said light fermions are needed to explain the change in background Chern-Simons terms between phases II-III, which gives us confidence their number is correct. Abstractly, they can be viewed as the two critical lines II-VI and III-VI merging together, but these singlet fermions are described by the two distinct scalar theories (i.e. the red and blue ones).

surprising, because even in moving from B to B' it was difficult to explain where the $N_s N_f$ ψ_s came from. Instead, consider applying the master duality to B' (i.e. B with Ψ_{12} already deformed to large negative mass). This would change the N_F light ψ_2 and N_F light ψ_s to corresponding light scalars in C'. Thus from this point of view, it makes sense that we get N_F light ϕ_2, but also N_F light ϕ_s, leading once more to an enhanced flavor symmetry. Similar to ψ_s, the we can conjecture the ϕ_s are light by some cancellation between the interaction terms and the explicit mass deformations. Since we now have $2N_F$ light scalars, if we take them to negative mass we once more find the $\mathcal{M}(2N_F, 2)$ Grassmannian.

To see this continues to occur as we move to more general quivers, note that we can continue to use these very same arguments used above to see enhanced flavor symmetries in U theories. In particular, in Fig. 4.8, the part of the quiver in the dashed box of C' is identical to that of B, except for the fact the former has $2N_F$ light scalars instead of only N_F light scalars. Thus, repeating the very same arguments used in moving from B to B' above, we come to the conclusion that as one takes Ψ_{23} to negative mass, we should again get light ψ_s fermions bifundamentally charged under the scalar and fermion flavor symmetries. However, now those symmetries are $SU(N_s) \times SU(2N_F)$. This would then lead to an $SU(3N_F)$ symmetry on the third node.

Lastly, let us propose where the extra light ψ_s and ϕ_s we saw above in theories B' and C' arise. There are very natural bound states which have the exact quantum numbers to match these bifundamentals, namely

$$\psi_s = \phi_1 \Psi_{12}, \qquad \phi_s = \phi_1 X_{12}. \tag{4.78}$$

Furthermore, the corresponding U node under which each component of these bound states is charged (i.e. node one) is confining. Thus we conjecture that some combination of confinement and mass deformations somehow makes said bound states light. This procedure generalizes as one moves down the quiver. For example, the additional light scalars one gets on the third node for certain mass deformations could correspond to bound states $\phi_1 X_{12} X_{23}$ and $\phi_2 X_{23}$. Along with ϕ_3, said scalars can lead to an enhanced flavor symmetry.

4.4 Orthogonal and Symplectic Gauge Groups

4.4.1 Node-by-node

We can generalize the duality for quiver gauge theories to the case of symplectic and orthogonal gauge groups. The node-by-node construction generalizes in a straightforward fashion. According to [3] the basic bosonization dualities in this case are

$$SO(N)_k + N_f \text{ real scalars} \quad \leftrightarrow \quad SO(k)_{-N+N_f/2} + N_f \text{ real fermions} \tag{4.79a}$$

$$Sp(N)_k + N_f \text{ real scalars} \quad \leftrightarrow \quad Sp(k)_{-N+N_f/2} + N_f \text{ real fermions.} \tag{4.79b}$$

Both of these can be extended to the case of "master dualities" with fermions and scalars on both sides [25, 68]; in any case, note that the assignments of the matrix size (which is now twice the rank) as well as the levels on the dual side exactly match the corresponding assignments in the unitary group. So going through the exercise of dualizing node by node we get exactly the same answers as in the unitary case, except for the fact that SU and U both get replaced with all SO or all Sp respectively. Let us focus here for simplicity on the case of unflavored quivers, the flavored case follows straightforwardly. We get the following two dualities in direct analogy with the unitary case:

$$\prod_{i=1}^{n} SO(N_i)_{-k_i} \qquad \text{is dual to} \qquad \left[\prod_{i=1}^{n-1} SO(K_i)_{N_i-N_{i+1}}\right] \times SO(K_n)_{N_n} \tag{4.80}$$

and

$$\prod_{i=1}^{n} Sp(N_i)_{-k_i} \qquad \text{is dual to} \qquad \left[\prod_{i=1}^{n-1} Sp(K_i)_{N_i-N_{i+1}}\right] \times Sp(K_n)_{N_n}. \tag{4.81}$$

Once again we can use these node-by node duality chains to argue for the special case that all $N_i = N$ and all ranks are $k_i = 1$. In this case the original theories become quivers with equal rank and level nodes analogous to 4.1. In the dual theory we get all but the last node to be at level 0. Appealing to a confinement scenario as in the unitary case, we would argue that the last bifundamental gets replaced by a meson gauge invariant under the second to last node. This is a symmetric combination in the case of orthogonal gauge groups and an anti-symmetric combination in the symplectic case. Correspondingly we should conclude that

$$[SO(N)_{-1}]^n + \text{bifundamental scalars} \qquad \leftrightarrow \tag{4.82a}$$

$$SO(n)_N + \text{symmetric rank-2 scalar} \tag{4.82b}$$

and

$$[Sp(N)_{-1}]^n + \text{bifundamental scalars} \qquad \leftrightarrow \tag{4.83a}$$

$$Sp(n)_N + \text{antisymmetric rank-2 scalar.} \tag{4.83b}$$

When we add flavors, we face same the complications as the unitary case. Fortunately these dualities can be extended into the flavor violated regime in exactly the same manner as the SU theories. The main difference is the Grassmannians become

$$\frac{SO(N_f)}{SO(k)\times SO(N_f-k)} \qquad \text{and} \qquad \frac{Sp(N_f)}{Sp(k)\times Sp(N_f-k)}. \tag{4.84}$$

These changes introduces a whole slew of interesting physics which is discussed in great detail in [86]. For our purposes, we simply note that the story of flavored quivers goes through in the same manner as in the SU case.

4.4.2 Phase Matching

The phase matching in the symplectic and orthogonal cases also directly mimics their unitary counterparts. Let us focus first on the orthogonal case. In the quiver we can once again express the phases by partitions of n, determining whether the bifundamental scalars get positive or negative mass squareds. The generic phase of the quiver is given by

$$\prod_i SO(N)_{-n_i} \tag{4.85}$$

in direct analogy with (4.73) of the SU quiver. The phases of the $U(N)_n$ gauge theory were parametrized by the expectation values of the adjoint scalar, a hermitian matrix that we were able to diagonalize with the unitary gauge transformation. This time we are having a scalar in a symmetric matrix, exactly the object that can be diagonalized by our orthogonal gauge transformations. So once again the phases of the dual theory can be parametrized by the eigenvalues of the matrix and enhanced unbroken gauge groups arise whenever eigenvalues coincide. The phases of the dual theory hence become

$$\prod_i SO(n_i)_N, \tag{4.86}$$

indeed a level/rank dual representation of (4.85). In the symplectic case the antisymmetric matrix plays exactly the role of the symmetric matrix in the orthogonal case. It can be brought into normal form by symplectic transformations, breaking the gauge group down into products of smaller symplectic groups.

4.4.3 Holographic Construction

Holographic QCD was generalized to the orthogonal and symplectic case in [62]. An orientifold O6 plane is introduced into the D4/D8 system in order to project the original unitary group down to either an orthogonal or symplectic subgroup. Both O6 and D8 are localized on the circle; if both are present they need to be offset from each other by an angle $\pi/4$ so that the stack of D8 branes gets mirrored onto the antipodal stack of anti-D8 branes by the orientifold projection. The two options for the gauge group are distinguished by exactly what type of O6 we add, the two options are usually denoted O6$^-$ for orthogonal and O6$^+$ for symplectic groups, the superscript indicating the sign of the RR charge associated with the orientifold plane. The full brane content both in the flat embedding space picture as well as in the dual geometry is enumerated in Tables 4.2a and 4.2b. Included in Table 4.2b is also the D6 brane that acts as the domain wall.

In order to understand the theory in the bulk, we simply need to determine the effect of the O6 plane on the gauge theory living on the domain wall D6s. As is apparent from Tables 4.2a and 4.2b the O6 planes have 4 relative ND directions with both the D4s of the flat space embedding and the D6s in the black hole geometry. What this means is that the

	0	1	2	3	4	5	6	7	8	9
D4	x	x	x	x	x	o	o	o	o	o
D8	x	x	x	x	o	x	x	x	x	x
O6	x	x	x	x	o	x	x	x	o	o

(a) Brane realization of the gauge theory in 10d flat space. X_4 is a compact direction.

	0	1	2	3	r	θ	S^4
D8	x	x	x	x	x	o	S^4
O6	x	x	x	x	x	o	S^2
D6	x	x	x	o	o	o	S^4

(b) Embedding of the probe branes in Witten's black hole.

Table 4.2: Brane realization of holographic QCD with symplectic or orthogonal gauge group. In a) N color D4 branes intersect an orientifold O6 to project to a real gauge group. In b) the same branes are embedded in the dual $\mathbb{R}^{3,1} \times D_2 \times S^4$ geometry, together with D6 branes dual to the θ-interfaces. Also indicated are flavor D8 branes that would be needed to add additional flavors. t, $x_{1,2,3}$ are the coordinates on $\mathbb{R}^{3,1}$, and r and θ the coordinates on D_2.

worldvolume theory on both D4s and D6s experience the same type of projection: the gauge groups is either orthogonal on both or symplectic on both. This is exactly as we would expect from our node-by-node procedure.

Last but not least we need to determine what happens to the formerly adjoint scalar corresponding to motion of the stack of D6s in the x_3 direction. Let us start with the case of an orthogonal gauge group. For a scalar describing motion transverse to the orientifold one finds that branes have to move off in mirror pairs, with a $U(1)$ gauge theory on each pair that can enhance to $U(n_i)$ for n_i coincident branes on one side. This is the breaking pattern we would expect from an antisymmetric rank 2 tensor. On the other hand, a scalar corresponding to motion inside the orientifold allows the stack of n branes to completely separate into n individual branes - since they are inside the orientifold they need no mirror partner. Each brane has an $SO(1)$ gauge group on its worldvolume that can get enhanced to $SO(n_i)$ for coincident eigenvalues. Since the motion of the D6s into the x_3 direction is inside the O6, this corresponds to a symmetric rank-2 tensor. Reassuringly this is exactly what we are supposed to find according to our duality conjecture in (4.82b). For symplectic gauge groups, the role of symmetric and antisymmetric rank-2 tensors is reversed. So the bulk physics yields indeed (4.83b) in that case.

In order to fully realize our dualities we would need to find (4.82a) and (4.83a) to be the corresponding field theories living on the boundary. For the case of symplectic gauge groups this appears indeed to be the obvious guess for the theory on the domain walls of a confining $Sp(N)$ gauge theory generalizing the discussion of [52] in the unitary case. In the orthogonal case this is certainly incorrect for the case of $SO(N)$ gauge groups, but it appears reasonable if the gauge group is instead $Spin(N)$. Since the stringy realizations of these gauge groups always involve heavy spinors, it is entirely reasonable to assume that the relevant gauge theories living on the branes were actually $Spin(N)$ groups all along. Unfortunately, these global issues are often not considered carefully in the orientifold literature. Somewhat confusingly, the node-by-node dualization above seems to be literally working with SO groups, not $Spin$ groups. So while a coherent and self-consistent duality story seems to emerge on the symplectic side, a complete understanding of the orthogonal case would require solid control of these global issues.

4.5 Discussion

In this chapter, we have developed the methodology for dualizing linear quiver gauge theories with bifundamental scalars and argued that they can be viewed as the non-Abelian generalization of particle/vortex duality. Crucial to this is the interaction terms, which couple scalars living on adjacent links and propagate the symmetry breaking pattern down the quiver in a unidirectional manner. This is required to ensure the mass deformed phases are level-rank dual to each other.

We then specialize this general framework to the study of domain walls that arise in $3+1$ dimensional Yang-Mills theory with a spatially varying theta angle. In addition, we embed this special case in string theory and study the duality holographically. We find a novel duality between a theory with bifundamental matter and one with adjoint matter, schematically given by (4.2). Let us comment on the similarities and differences between these two approaches.

From the setup in ref. [52], we expect the bifundamental scalars on the SU side of the duality to interpolate between a smoothly varying and a sharp domain wall/interface. However, the pure field theoretic quiver approach of Sec. 4.1.2 makes opaque the geometric interpretation of a physical wall located in space. The complementary geometric approach of Sec. 4.2.2 makes this manifest: Higgsing a bifundamental is literally removing a D6 brane (i.e. domain wall) from a stack and moving it to a different physical location in space. Widely separated D6s correspond to the smoothly varying phase, reinforcing our intuition of the theory at small $|\nabla\theta|$.

When the walls on top of one another in the holographic duality we have new light matter one both sides of the duality, but it manifests itself in a very different manner. On the U side the extra matter enhances the gauge symmetry. Meanwhile, on the SU side the bifundamentals just become additional massless scalars. This may seem peculiar given the fact the bifundamental degrees of freedom match quite nicely when the walls are separated (albeit by construction). But this is precisely the behavior we would expect in our 3d bosonization duality. As we learned from the particle-vortex duality generalization, it is not actually the bifundamental degrees of freedom which should match on either side of the duality but rather particles and vortices. The very same mismatch of particle degrees of freedom is present in the bosonic particle-vortex duality as well.

In fact, the matching of the particle and vortex degrees of freedom is very nicely realized in the holographic duality. The dual of the baryons in the bulk are based on the standard holographic construction of the baryon vertex [131], very similar to what was found in [70]. Namely, they are D4 branes wrapping the S^4 and also extended along the time direction. In order to be neutral, fundamental strings run from the D4 branes to the D6 branes on which they live. Furthermore, the D6 branes dissolve the D4 branes turning them into magnetic flux (it is more energetically favorable and has the same quantum numbers). The attachment of N fundamental strings is analogous to particle/flux attachment. It can be argued that the N fundamental strings cannot end on the same D6 brane. Hence when the D6 branes get separated, the monopoles must also pick up a mass since the fundamental strings must stretch from one D6 brane to another – providing more evidence that lines of flux can end on domain walls. This is also in nice agreement with the behavior which occurs on the SU side where the bifundamentals are interpreted as strings which stretch from one brane to another and thus both acquire a mass proportional to the separation between branes.

One may wonder whether our quiver dualities can be useful in the context of deconstruction, following the recent work of [14]. There it was shown that Abelian quiver dualities can be lifted to dualities in 3+1 dimensions. It would be very interesting to do this in the non-Abelian case. One important ingredient in this construction is the use of "all scale" versions of the duality, following the construction of [79] in the supersymmetric case. We'd like to point out that at least for two-node quivers, our method of gauging global flavor symmetries does allow us to give all scale versions of the non-Abelian duality. Say we want a dual for $SU(N)_k$ with N_f fermionic flavors with a finite gauge coupling. Since the gauge coupling is dimensionful we are describing a theory with a non-trivial RG running. It interpolates between a free theory in the UV and a strongly coupled CFT in the IR. We can obtain this theory by starting with NN_f free fermions and gauging a $SU(N)$ subgroup of the global $SU(NN_f)$ flavor symmetry. At this stage we can add both the Chern-Simons as well as the Maxwell kinetic terms. The original theory of NN_f free fermions has dual descriptions in terms of a $U(K)_1$ gauge theory coupled to NN_f scalars. Modulo flavor bounds K is a free parameter. The global flavor symmetry simply rotates the scalar flavors in this dual. Promoting a $SU(N)$ subgroup to be dynamical we end up with a $U(K)_1 \times SU(N)_k$ gauge theory with N_f bi-fundamentals. While the $U(K)$ factor has infinite coupling, the $SU(N)_k$ factor has a finite Maxwell term which maps directly to the Maxwell term of the same $SU(N)_k$ factor on the dual side. This way we did construct a non-Abelian all scale dual to $SU(N)_k$ with fermions. Unfortunately it is not yet clear how to generalize this construction to more interesting quivers.

Using the flavor violated master duality, we constructed a dual for a flavored quiver with bifundamental scalars on the links and fundamental fermions coupled to each node via node-by-node dualization. Such a theory describes interfaces in $3+1$ dimensional QCD as discussed in Ref. [52]. The resulting dual theories were qualitatively supported by a holographic construction with D6 branes in the Sakai-Sugimoto model. We then extended our work to orthogonal and symplectic gauge groups, both in the node-by-node dualization and in holography. More interesting directions would be exploring other gauge theories with different matter contents. It is widely believed that infrared phase of three-dimensional theories describes effective worldvolume theory of walls/interfaces in corresponding four-dimensional theories [56, 1, 86, 52, 8, 17, 34, 23, 80]. It would be nice to understand what kind of quiver gauge theories with its phase diagram and stringy constructions would emerge when analyzing phase transitions of multiple walls/interfaces in various cases similar to what we've done in the case of QCD_4.

There are some limitations to our approach, however. First, the analysis in Appendix A of [52] seems to indicate a periodicity of the Grassmannian as a function of the number of nodes, n. That is, the most general Grassmannian is not $\mathcal{M}(N_f, n)$ but instead $\mathcal{M}(N_f, n \bmod N_f)$. Our quiver theory seems to indicate that this is not the case – if the number of nodes n is greater than N_f, the full Higgsed regime gives $SU(N)_{-n+\frac{N_f}{2}}$ with N_f ψ. This is flavor bounded and so does not exhibit a quantum regime. The case examined in [52] is valid for $n \leq N_f$. The analysis just seems to point to this periodicity, but an explicitly demonstration of it has not yet been completed. Perhaps there is some subtle strong coupling dynamics in the $3+1$ dimensional theory that causes the Grassmannian to disappear. Or maybe our quiver theory is only valid for $n \leq N_f$ and there is some extension of our work that can bring us into the $n > N_f$ regime. We hope to report on this interesting question in future work.

We have also found that the quivers corresponding to $N_F = 1$ and $N_F > 1$ require the use of two very different versions of the master duality – the double-flavor saturated and flavor violated cases, respectively. In Ref. [52], it was found these two cases also separated themselves quite distinctly due to the lack of an enhanced symmetry when the $3+1$-dimensional fermion mass disappeared in the $N_f = 1$ case. It is not obvious to the authors if these two facts are somehow related.

The holographic side of things is ripe with interesting puzzles. For instance, it is still unknown what the exact form of the orthogonal duality is we derived in this manner. As briefly commented on in the end of Sec. 4.4, the duality derived from the orientifold projection onto the orthogonal subgroup seems to be insensitive to the global properties of the gauge group. That is, we know that there are multiple different types of orthogonal gauge groups in 2+1 d that differ by the required background terms and the gauging of various $\mathbb{Z}_2$ global symmetries [37]. Can these global issues be understood in string theory? Moreover, how does one even see a quantum phase in the holographic construction of $SU(N)$ gauge theories with fundamental fermions in holography? We leave these questions for future work.

Chapter 5

FERMIONIC QUIVERS

Thus far, we have worked primarily with 3d bosonization dualities involving fundamental or antifundamental representations, i.e. rank-one matter. Schematically, these dualities are given by (1.13). There also exists conjectures for dualities very much analogous to (1.13), but instead involving adjoint, symmetric, or antisymmetric representations, i.e. rank-two matter [56, 33].

Using the master duality, the single-species dualities of (1.13), and the Abelian limit of (1.13), a large web of rank-one dualities has been constructed, see Fig. 5.1 [83, 112, 69, 8, 7]. Although the rank-two dualities bear a strong resemblance to the rank-one 3d bosonization dualities, they have so far remained disconnected from aforementioned web.

The usual duality web construction involving adding a background term/decoupled theory and gauging global symmetries was explored within the context of rank-one dualities in the work of [69, 8, 7] (and Chapter 4) to construct *bosonic* quivers. That is, by gauging non-Abelian global symmetries the rank-one dualities can be used to derive dualities between theories with product gauge groups coupled to each other with bifundamental scalar matter. Such quiver dualities were generalized to an arbitrary number of nodes. The two-node case was shown to be a generalization of the bosonic particle-vortex duality [105, 40]. A special case of the general quiver was shown to have application for dualities of interfaces in QCD_4 [52].

In this chapter, we follow a similar methodology but instead construct two-node quiver dualities which have *fermionic* matter on both ends. We will start by constructing said dualities using only the flavor-bounded 3d bosonization dualities. As one might expect from previous work [8], such quivers provide a generalization of Son's fermionic particle-vortex

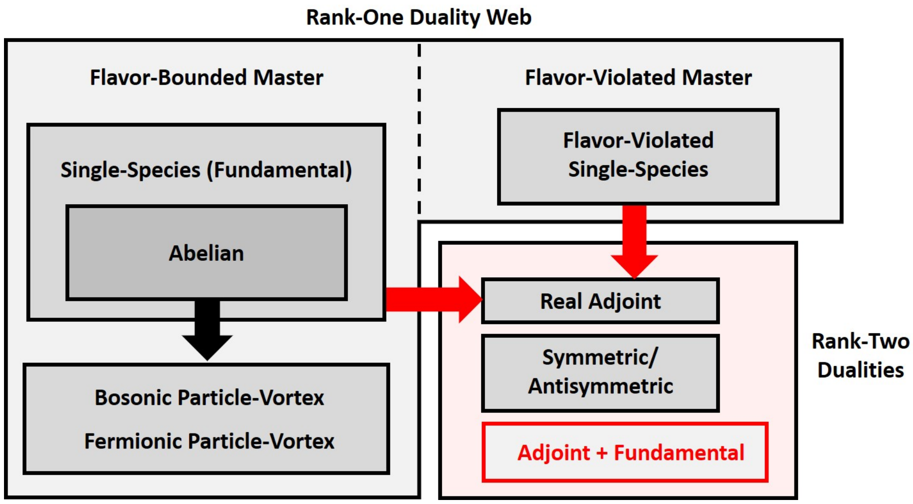

Figure 5.1: Web of bosonization dualities in $2+1$-dimensions. The contributions discussed in this chapter are shaded in red. Specifically, we show that the rank-two adjoint matter dualities can be connected to the rest of the web of dualities (red arrows). We also conjecture several new rank-two dualities including a duality with both fundamental and adjoint representation matter.

duality [119]. Nevertheless, such quivers are much more strongly constrained compared to their bosonic counterparts as already noted in Ref. [69].

This motivates us to consider the construction of such two-node dualities using the flavor-violated 3d bosonization duality. This relaxes constraints on parameters at the cost of a more complicated phase diagram. A special case of these flavor-violated two-node quivers exhibits a $\mathbb{Z}_2$ symmetry under interchange of its two nodes.

Our main focus of this chapter will be a novel procedure of deriving new and old fermionic dualities using this $\mathbb{Z}_2$-symmetric quiver. In particular, we combine said quivers with an orbifolding procedure, which allows us to connect the rank-one 3d bosonization duality to dualities with rank-two matter, see Fig. 5.1. This links the rank-two dualities to the rank-one bosonization dualities [83, 112], meaning the web of 3d bosonization dualities is larger than previously thought.

After demonstrating we can recover the aforementioned rank-two dualities, we derive new dualities involving rank-two matter. The first of these closely resembles the known rank-two dualities with the $U(1)$ factors shuffled around. We also explore the possibility of adding fundamental matter on each of the nodes, which gives rise to a new duality involving both adjoint and fundamental matter.

The chapter is outlined as follows. In Sec. 5.1 we first construct the flavor-bounded two-node quiver, and then the flavor-violated version. Sec. 5.2 introduces the orbifolding procedure to derive new and old fermionic dualities. Finally, we discuss possible extensions of these constructions in Sec. 5.3.

Many details of this chapter can be found in the appendices. Specifically, Appendix E.1 contains the details of the two-node derivations. Appendix E.2 discusses how the flavor-bounded two-node is connected to the fermionic particle-vortex duality of Ref. [119]. In Appendix E.3, we summarize the generalization of the flavor-violated fermionic quiver dualities for orthogonal and symplectic gauge groups. Finally, we illustrate the gravitational counterterm matching as a consistency check for the two-node quiver dualities in Appendix E.4.

5.1 Two-Node Fermionic Quivers

It has been shown in Refs. [8, 7] that we can use two-node quivers to create generalizations of the bosonic particle-vortex duality. In this section we construct the fermionic equivalent. Flavor constraints will imposes tighter bounds on the possible theories we can create, since the intermediate quivers contain scalars. This will motivate us to also consider two-node fermionic quivers which use the flavor-violated version of 3d bosonization. In this section we will mostly stick to schematic notation for brevity. Details of this derivation at the Lagrangian level can be found in Appendix E.1.

Notation

In this chapter it will be important to be careful about the representation of matter fields under various gauge groups, both dynamical and background. We will thus slightly change our notation for covariant derivatives such that the covariant derivatives terms in, say, (1.23) are rewritten as

$$\left(D_{b'-C+\tilde{A}_1}\right)_\mu \psi = \left[\partial_\mu - i\left(b'_\mu \mathbb{1}_{N_f} + C_\mu \mathbb{1}_N + \tilde{A}_{1\mu}\mathbb{1}_{NN_f}\right)\right]\psi, \tag{5.1a}$$

$$\left(D_{c-C}\right)_\mu \Phi = \left[\partial_\mu - i\left(c_\mu \mathbb{1}_{N_f} + C_\mu \mathbb{1}_k\right)\right]\Phi, \tag{5.1b}$$

where b'_μ and c_μ are fundamental representations and C_μ is in the anti-fundamental representation of their respective gauge groups. That is, antifundamental representations will be denoted by a minus sign in the subscript of the covariant derivative.

5.1.1 Flavor-Bounded

The general procedure for constructing two-node quiver dualities involves promoting the $SU(N_f)$ and/or $U(1)$ global symmetries after adding appropriate background Chern-Simons terms [69]. For example, promoting the background flavor symmetries to be dynamical, the schematic form of the 3d bosonization dualities in (1.13) is given by

$$U(k)_N \times SU(N_f)_0 + \Phi \quad \leftrightarrow \quad SU(N)_{-k+N_f/2} \times SU(N_f)_{N/2} + \psi, \tag{5.2a}$$

$$SU(k)_N \times U(N_f)_0 + \phi \quad \leftrightarrow \quad U(N)_{-k+N_f/2} \times U(N_f)_{N/2} + \Psi. \tag{5.2b}$$

Each side either has a single bifundamentally charged fermion or boson, e.g. the right-hand side of (5.2b) has fermion in the $(\mathbf{N}, \bar{\mathbf{N}}_f)$ representation. In the latter of these dualities we have also promoted the $U(1)$ global symmetry. When we do this we introduce a new $U(1)$ global symmetry whose corresponding background field couples to the promoted $\tilde{A}_1$ via a BF term.

Relabeling parameters and adding background terms (before promotion), the duality

(5.2a) conjectures the following two theories are dual

$$\text{Theory A:} \qquad SU(N_1)_{-k_1+N_2/2} \times SU(N_2)_{-k_2+N_1/2} + \psi, \tag{5.3}$$

$$\text{Theory B':} \qquad U(k_1)_{N_1} \times SU(N_2)_{-k_2} + \Phi, \tag{5.4}$$

which is subject to the flavor bound $k_1 \geq N_2$. Now also rewrite the second duality, (5.2b), again with a special choice of added background terms and relabeling,

$$\text{Theory B":} \qquad U(k_1)_{N_1} \times SU(N_2)_{-k_2} + \phi, \tag{5.5}$$

$$\text{Theory C:} \qquad U(k_1)_{N_1-k_2/2} \times U(k_2)_{N_2-k_1/2} + \Psi, \tag{5.6}$$

subject to the flavor bound $N_2 \geq k_1$.

Theories B' and B" are identical, so we will collectively call them Theory B. Since Theory A and Theory C are both dual to Theory B, they must also be dual to one another. That is, we have a fermion-fermion duality between (5.3) and (5.6). Each side has a bifundamental fermion coupled to two gauge fields with Chern-Simons terms. Note simultaneously satisfying both of the flavor bounds requires $N_2 = k_1$, but N_1 and k_2 are unbounded. At the Lagrangian level, this is a duality between the theories

$$\begin{aligned}\mathcal{L}_{\mathrm{A}} &= i\bar{\psi}\not{D}_{b'-c'+\tilde{A}_1}\psi - i\left[\frac{N_2-k_1}{4\pi}\mathrm{Tr}_{N_1}\left(b'db' - i\frac{2}{3}b'^3\right)\right] \\ &\quad - i\left[\frac{N_1-k_2}{4\pi}\mathrm{Tr}_{N_2}\left(c'dc' - i\frac{2}{3}c'^3\right) + \frac{N_1N_2}{4\pi}\tilde{A}_1 d\tilde{A}_1\right],\end{aligned} \tag{5.7a}$$

$$\mathcal{L}_{\mathrm{C}} = i\bar{\Psi}\not{D}_{-c+g+\tilde{A}_1}\Psi - i\left[\frac{N_1}{4\pi}\mathrm{Tr}_{k_1}\left(gdg - i\frac{2}{3}g^3\right) + \frac{N_2}{4\pi}\mathrm{Tr}_{k_2}\left(cdc - i\frac{2}{3}c^3\right)\right], \tag{5.7b}$$

where $b' \in su(N_1)$, $c' \in su(N_2)$, $g \in u(k_1)$, and $c \in u(k_2)$. The quiver construction is summarized in Fig. 5.2.

In Appendix E.2, we will discuss more details about this fermion-fermion duality constructed from the flavor-bounded bosonization duality. This includes showing Son's fermionic particle-vortex duality [119] is simply the $N_1 = N_2 = k_1 = k_2 = 1$ limit. For now, we move onto deriving a two-node fermionic duality without such tight flavor constraints.

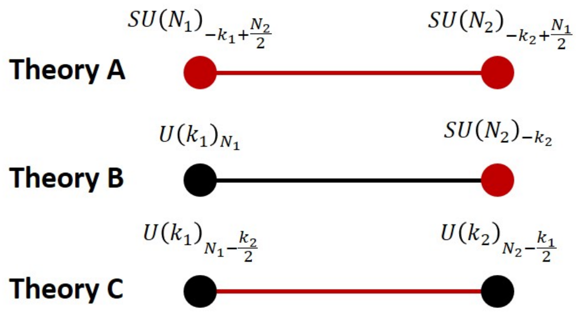

Figure 5.2: Generalized fermion quiver duality of (5.7) with flavor-bounded dualities. Theory B represents both Theories B' and B" since they are identical. Our quiver notation is such that we use filled red (black) nodes to represent dynamical SU (U) gauge groups. The red (black) links between said circles represent fermions (scalars) charged under the corresponding gauge groups. This duality is subject to the flavor bound $N_2 = k_1$.

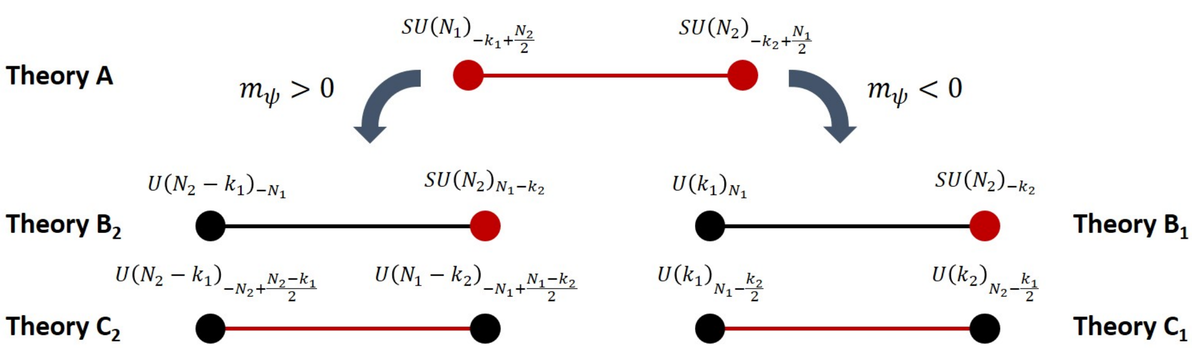

Figure 5.3: Generalized fermion quiver duality with flavor-violated bosonization duality. The first step uses the flavor-violated 3d bosonization duality, while the second step uses the flavor-bounded version. The diagram is valid for the $k_1 < N_2 \leq N_*(N_2, k_1)$, $k_2 < N_1 \leq N_*(N_1, k_2)$.

5.1.2 Flavor-Violated

Above, we saw the derivation of the two-node fermions was constrained by the flavor-bound of the 3d bosonization dualities. It is also possible to derive two-node quivers using the flavor-violated duality proposed by Ref. [86]. This comes at the cost of the full phase diagram of Theory B being described by two distinct dual theories. We then apply a flavor-bounded bosonization duality to Theory B, which means Theory C's phase diagram requires two fermionic theories. This construction is summarized in Fig. 5.3.

In more detail, we start with the same Theory A as we did for the flavor-bounded version above, (5.3). We now use the flavor-violated bosonization duality, which lifts the earlier flavor-bound constraint, $k_1 \geq N_2$, to $k_1 < N_2 \leq N_*(N_2, k_1)$ instead. Here, N_* is some unknown function of N and k as discussed in Ref. [86]. This yields a Theory B which must be described by two distinct scalar duals. One of these scalar duals, corresponding to the $m_\psi < 0$ side, is identical to (5.4). The $m_\psi > 0$ side is in general distinct. Schematically, the full phase diagram is described by the theories

$$\text{Theory A:} \qquad SU(N_1)_{-k_1+N_2/2} \times SU(N_2)_{-k_2+N_1/2} + \psi, \tag{5.8}$$

$$\text{Theory B}_1\text{:} \qquad U(k_1)_{N_1} \times SU(N_2)_{-k_2} + \phi_1, \tag{5.9}$$

$$\text{Theory B}_2\text{:} \qquad U(N_2 - k_1)_{-N_1} \times SU(N_2)_{N_1-k_2} + \phi_2, \tag{5.10}$$

where subscripts $1, 2$ correspond to the $m_\psi < 0$ and $m_\psi > 0$ theories, respectively.

Next, we can use flavor-bounded 3d bosonization to find a fermionic theory which matches the intermediate scalar theory. Since there are two scalar duals in the intermediate theory, we will also need two fermion theories to describe the full phase diagram. Once more, the $m_\psi < 0$ side is identical to what we found above in (5.6). A subtlety is that, due to the flipping of the sign of the level, on the $m_\psi > 0$ side we must use the time-reversed version of (5.2a). This yields the theories

$$\text{Theory C}_1\text{:} \qquad U(k_1)_{N_1-k_2/2} \times U(k_2)_{N_2-k_1/2} + \Psi_1, \tag{5.11}$$

$$\text{Theory C}_2\text{:} \qquad U(N_1 - k_2)_{-N_2+(N_2-k_1)/2} \times U(N_2 - k_1)_{-N_1+(N_1-k_2)/2} + \Psi_2. \tag{5.12}$$

Thus we again arrive at two-node fermionic quiver theories on either end. Once again they are conjectured to be dual to one another, but the flavor bound of $N_2 = k_1$ has been replaced by $k_1 < N_2 < N_*(N_2, k_1)$ and $k_2 < N_1 < N_*(N_1, k_2)$, which allows us a little more freedom in choosing parameters.[1]

The explicit form of the Lagrangians for Theories B_2 and C_2 are given in (E.5). Note the presence of the mixed BF term between c and g, which arises from the fact that dual fermion is charged under both $U(1)$ groups. Since the $m_\psi < 0$ side is identical to the flavor-bounded case, $\mathcal{L}_{\mathrm{C}_1}$ is given by (5.7b). The BF term between dynamical gauge fields c and g is also present in (5.7b), but is hidden due to our convention of η-invariant terms (see footnote 1).

Finally, it is slightly subtle to see how the quantum phases described by Theories C_1 and C_2 are dual to one another. We denote a level ± 1 BF term between two unitary gauge fields c and g, e.g. $\pm\frac{1}{2\pi}\mathrm{Tr}_{k_1}(c)d\mathrm{Tr}_{k_2}(g)$, as "$\pm$BF". The intermediate quantum phase is then described by the exact duality,

$$U(k_1)_{N_1-k_2} \times U(k_2)_{N_2-k_2} - \mathrm{BF} \quad \Leftrightarrow \quad U(N_1-k_2)_{-k_1} \times U(N_2-k_1)_{-k_2} + \mathrm{BF}. \tag{5.13}$$

This is obtained from gauging the diagonal $U(1)$ symmetry of two copies of the level-rank duality (1.11), which is described in more detail in Appendix E.1.

5.2 Adjoint Dualities via Orbifolding

We now consider the two-node quiver we derived in Sec. 5.1.2, with $N_1 = N_2 = N$ and $k_1 = k_2 = k$ where $N > k$ due to the flavor bound. These parameters are more constrained for the flavor-bounded case where $k_1 = N_2$ was required, which would imply $N = k$. What is special about these particular values is that the fermionic theories of this quiver have an explicit $\mathbb{Z}_2$ symmetry, which can be thought of as interchanging the two nodes. In this section we will argue this special class of quivers allows us to re-derive the rank-two bosonization

[1] An important point is that we need further condition $k_2 < N_1 \leq N_*(N_1, k_2)$ to have mutually non-local dual descriptions as the ungauged version. The main reason for this additional constraint is that since we have two dynamical gauge groups, we can equivalently view the first gauge group as the flavor symmetry group and second as originally dynamical gauge group before constructing the quiver.

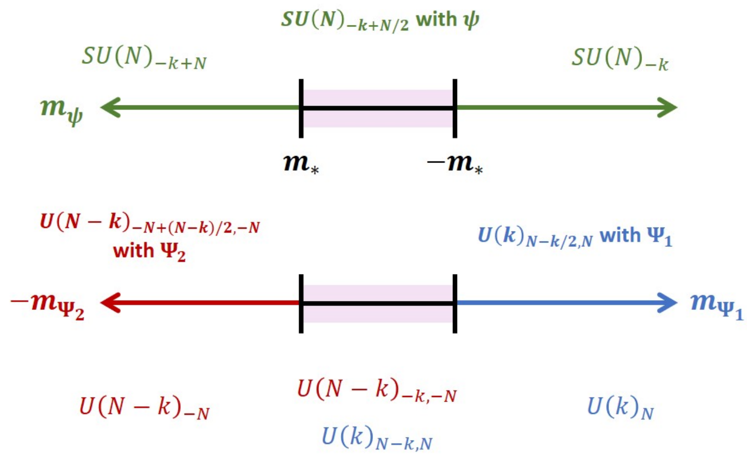

Figure 5.4: Phase diagram of the 3d bosonization duality with real adjoint matter [56]. Distinct theories are shown in different colors. The region shaded in purple corresponds to the quantum phase.

duality with real adjoint matter on each side [56]. First, let us briefly review the adjoint matter dualities.

5.2.1 Adjoint 3d Bosonization Duality

Ref. [56] conjectured the phases of QCD$_3$ adjoint.[2] Schematically, the fermion side is

$$SU(N)_{-k+N/2} \text{ with adjoint } \psi, \tag{5.14}$$

with $N > k$. Mass deforming the adjoint fermion gives $SU(N)_{-k}$ and $SU(N)_{-k+N}$. When $N \leq k$, these two TFTs are conjectured to have no quantum region in between them. However, when $N > k$ an intermediate Grassmannian phase is conjectured to exist between the asymptotic semi-classical regions. The claim is that once again two different dual theories

[2]Ref. [56] uses a slightly different notation than we use here. Their results can be recovered by $k \to k - N/2$ with the overall time-reversal which flips the signs of the Chern-Simons levels.

describe the fixed points at the edge of the Grassmannian. These U side theories are

$$SU(N)_{-k+N/2} \text{ with } \psi^{\text{adj}} \qquad \leftrightarrow \qquad \begin{cases} U(k)_{N-\frac{k}{2},N} \text{ with } \Psi_1^{\text{adj}} & m_{\psi^{\text{adj}}} = -m_* \\ U(N-k)_{-\frac{N}{2}-\frac{k}{2},-N} \text{ with } \Psi_2^{\text{adj}} & m_{\psi^{\text{adj}}} = m_*. \end{cases} \tag{5.15}$$

This is qualitatively similar to the flavor-violated bosonization duality and is summarized in Fig. 5.4. For the quantum phase, we find the TFTs $U(k)_{N-k,N}$ and $U(N-k)_{-k,-N}$, which as expected are level-rank dual by (1.26). Note the quantum phases on the SU side has no good description in terms of the SU theory – it is best described by the U side of the theory. This is also a feature of the flavor-violated duality (2.2) and we will see similar features below.

5.2.2 Adjoint Duality via Orbifolding

As mentioned in the introduction, what is special about the $N_1 = N_2 = N$ and $k_1 = k_2 = k$ subclass of two-node quivers is that they present a possible means of constructing the rank-two matter dualities. More explicitly, since both sides of the dualities exhibit "theory space" $\mathbb{Z}_2$ symmetries, we can orbifold said symmetries on both sides of the duality to obtain a new conjectured duality. In this section, we show the rank-two dualities involving adjoint matter can be obtained by orbifolding[3] different two-node quiver theories.

Fig. 5.5 summarizes the full phase diagram of the $\mathbb{Z}_2$ symmetric quiver, including the

[3]Orbifolding is a technique which projects a theory with a certain symmetry onto an invariant subspace. It is perhaps most familiar in the string theory literature, but such techniques have also been adapted to quantum field theory [75, 90] in the context of planar equivalence between mother and daughter theories [27, 87] or in the context of constructing manifestly supersymmetric lattice gauge theories [35]. In a quantum field theory orbifolding is simply a procedure to produce a new theory (the "daughter") from an old one (the "mother") by projecting out all degrees of freedom not invariant under a specified discrete group. If two theories flow to one and the same IR fixed point and this fixed point possesses a $\mathbb{Z}_2$ symmetry, then obviously after orbifolding we once again obtain a unique IR theory and so we inherit a duality of the daughters from the duality of the mothers. The only way this could fail is if the $\mathbb{Z}_2$ symmetry of the IR is not manifest in the UV theories or different from the apparent UV symmetries.

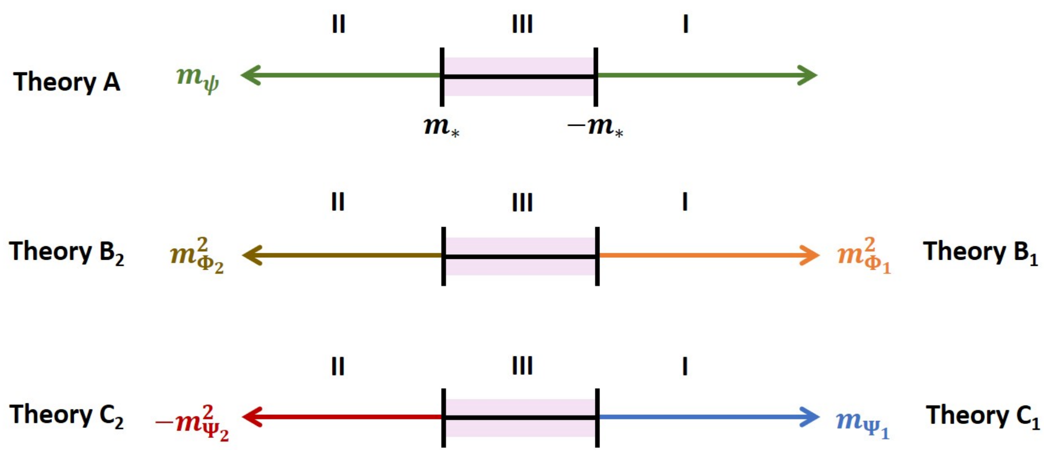

Figure 5.5: Phase diagram of the two-node quiver with explicit $\mathbb{Z}_2$ symmetry. Distinct theories are shown in different colors. Note the splitting of the bottom two phase diagrams which results from the use of the flavor-violated 3d bosonization duality.

intermediate scalar theories. The phases of Theory A are given by

$$\text{(I)}: \qquad SU(N)_{-k} \times SU(N)_{-k} \tag{5.16a}$$

$$\text{(II)}: \qquad SU(N)_{-k+N} \times SU(N)_{-k+N} \tag{5.16b}$$

$$\text{(III)}: \qquad \text{Better described by } U \text{ side.} \tag{5.16c}$$

Meanwhile, theories C_1 and C_2 describe the U side, given by

$$\text{(I)}: \qquad U(k)_N \times U(k)_N \tag{5.17a}$$

$$\text{(II)}: \qquad U(N-k)_{-N} \times U(N-k)_{-N} \tag{5.17b}$$

$$\text{(III C}_1\text{)}: \qquad U(k)_{N-k} \times U(k)_{N-k} + \mathrm{BF} \tag{5.17c}$$

$$\text{(III C}_2\text{)}: \qquad U(N-k)_{-k} \times U(N-k)_{-k} - \mathrm{BF}. \tag{5.17d}$$

Similar to what we saw in Sec. 5.1.2, the BF terms in the phase III make the phases (III C_1) and (III C_2) level/rank dual to one another, as shown in Appendix E.1. Observe that these phases already resemble "two copies" of the phases of the adjoint duality, up to certain subtleties in the qunatum phase, see Fig. 5.4. Schematically, orbifolding causes the bifundamental fermions to become Majorana adjoint fermions and takes the TFTs from $G_P \times G_P \to G_P$, where G_P is a gauge group G with Chern-Simons level P, up to subtleties with Abelian levels. Orbifolding a $\mathbb{Z}_2$ symmetry to change a bifundamental to an adjoint representation has been previously explored in Ref. [88, 122].

More explicitly, the $\mathbb{Z}_2$ symmetric theory describing the $m_\psi < 0$ half of the phase diagram is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{A}} = i\bar{\psi}\not{D}_{b_1-b_2+\tilde{A}_1}\psi - i\left[\frac{N-k}{4\pi}\mathrm{Tr}_N\left(b_1db_1 - i\frac{2}{3}b_1^3\right)\right] \\ - i\left[\frac{N-k}{4\pi}\mathrm{Tr}_N\left(b_2db_2 - i\frac{2}{3}b_2^3\right) + \frac{N^2}{4\pi}\tilde{A}_1d\tilde{A}_1\right],\end{aligned} \tag{5.18a}$$

$$\mathcal{L}_{\mathrm{C}_1} = i\bar{\Psi}_1\not{D}_{c_1-c_2+\tilde{A}_1}\Psi_1 - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(c_1dc_1 - i\frac{2}{3}c_1^3\right) + \frac{N}{4\pi}\mathrm{Tr}_k\left(c_2dc_2 - i\frac{2}{3}c_2^3\right)\right], \tag{5.18b}$$

where we have relabeled the gauge fields belonging to the first and second node of the SU side as b^1_μ and b^2_μ, respectively. On the U side, c^1_μ and c^2_μ are the gauge fields of the first and second nodes. $\tilde{A}_1$ is the background gauge field associated with the $U(1)$ global symmetry.

The bifundamental fermion is in the representation $(\mathbf{N}, \bar{\mathbf{N}})$. The $\mathbb{Z}_2$ symmetry acts as

$$\mathbb{Z}_2: \qquad \psi \to -\psi^\dagger, \qquad b^1_\mu \to b^2_\mu, \qquad b^2_\mu \to b^1_\mu. \tag{5.19}$$

Similarly, on the U side, the theory is invariant under the symmetry transformation

$$\mathbb{Z}_2: \qquad \Psi_a \to -\Psi_a^\dagger, \qquad c^1_\mu \to c^2_\mu, \qquad c^2_\mu \to c^1_\mu \tag{5.20}$$

for $a = 1, 2$ corresponding to Theories C_1 and C_2. If we orbifold with respect to these $\mathbb{Z}_2$ symmetries on either side of the duality, the resulting daughter theories should continue to be dual.

We can arrive at the new Lagrangians describing the daughter theories by projecting the terms in (5.18) to those invariant under (5.19) and (5.20). On the SU side, this effectively ties the b_1 and b_2 fields together, meaning the bifundamental now transforms as an adjoint, as desired. Note that this orbifold also breaks the $U(1)$ global symmetry associated with the phase of the matter to $\mathbb{Z}_2$, i.e. $\psi \to -\psi$.

The behavior of the Chern-Simons terms under the orbifold is slightly subtle. Since the action of orbifold projections on matter is well-known, it will be helpful to view the Chern-Simons term as arising from integrating out heavy fermions. This is directly analogous to the "fiducial fermion" presecription used to analyze the 3d bosonization dualities in the presence of a boundary [9, 15]. For example, the $U(k)_N$ Chern-Simons term of a given node can arise from N "fiducial" fermions with a large negative mass. In this case, the Chern-Simons term in the Lagrangian is replaced with

$$-i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc-i\frac{2}{3}c^3\right)\right] \qquad \Leftrightarrow \qquad \lim_{|m_\chi|\to\infty} i\bar{\chi}^M \not{D}_c\chi_M + m_\chi\bar{\chi}^M\chi_M \tag{5.21}$$

where χ_M are the fiducial fermions with $M=1,\ldots,N$.[4]

Adopting this prescription for the Lagrangians in (5.18), each of the two nodes on the SU (U) side picks up an extra $N-k$ (N) fundamental-represention fermions, which we will collectively denote χ_i for $i=1,2$. The $\mathbb{Z}_2$ symmetry transformations of (5.19) and (5.20) must then be supplemented with $\chi_1\to\chi_2$ and $\chi_2\to\chi_1$. The orbifold projection reduces these additional matter fields to a single group of fundamental fermions under the invariant gauge group. If we then restore the Chern-Simons term one gets from integrating out the fiducial fermions, we see that we simply lose one of the Chern-Simons terms under our orbifold, i.e. $U(k)_N\times U(k)_N\to U(k)_N$. Note this is fairly different than naively setting $c_1=c_2$ in (5.18b), in which case one would arrive at a $U(k)_{2N}$ Chern-Simons term.

Another important subtlety is one needs to orbifold the Pauli-Villars regulators as well. Hence, what was effectively a bifundamental Pauli-Villars regulator under $SU(N)\times SU(N)$ becomes a real adjoint PV regulator under $SU(N)$. All this means is that the η-invariant terms obey a similar relation to the Chern-Simons levels discussed above.

Thus the orbifolding reduces the "two copies" of the Chern-Simons terms to a Chern-Simons matter theory with a single dynamical gauge field resembling the adjoint duality. However, in the adjoint bosonization duality the TFTs found in the quantum region have

[4]For brevity, we have neglected the Pauli-Villars regulators which also accompany fiducial fermions. See Refs. [9, 15] for more details regarding this construction.

the non-Abelian levels shifted relative to the Abelian levels, see Fig. 5.4. For example, for the $m_{\psi^{\text{adj}}} < 0$ side, as we mass deform Ψ_1^{adj} we move from $U(k)_N$ to $U(k)_{N-k,N}$. This is contrast to the $\mathbb{Z}_2$ symmetric quiver, where we see no such effect because fundamental representation fermions shift Abelian and non-Abelian levels on equal footing. To see how such a change can arise from orbifolding, consider the bifundamental fermion on the U side. One can clearly see that the bifundamental fermions remain invariant under the subgroup where $\text{Tr}\,(c_1) = \text{Tr}\,(c_2)$. Thus, since we a projecting to precisely this subgroup via our orbifold, the remaining $U(1)$ part should decouple.

To see this at the Lagrangian level, note the coupling of said fermion is giving by $i\bar{\Psi}\not{D}_{-c+g+\tilde{A}_1}\Psi$, and so for negative mass deformation we get the Chern-Simons term

$$\begin{aligned} i\mathcal{L}_{\text{shift}} &= \frac{-1}{4\pi}\text{Tr}_{k^2}\left[\left(c_1\mathbb{1}_k - \mathbb{1}_k c_2 + \mathbb{1}_{k^2}\tilde{A}_1\right) d\left(c_1\mathbb{1}_k - \mathbb{1}_k c_2 + \mathbb{1}_{k^2}\tilde{A}_1\right) + \ldots\right] \\ &= -\frac{k^2}{4\pi}c_1 dc_1 - \frac{k^2}{4\pi}c_2 dc_2 + \frac{1}{2\pi}\text{Tr}_k\,(c_1)\,d\text{Tr}_k\,(c_2) + \ldots \end{aligned} \tag{5.22}$$

where "..." represents additional terms not relevant for the cancellation. Absolutely vital to the story is the additional BF term we get from the cross term of the two dynamical $U(k)$ gauge fields. Under the orbifolding procedure, we have $c_1 = c_2$. Notice that when this is enforced the BF term precisely cancels the Abelian Chern-Simons terms. The net result is that the Abelian part of the Chern-Simons level gets no overall shift. The same effect occurs in the PV regulator. This means the quantum phase description

$$U(k)_{N-k} \times U(k)_{N-k} \quad \xrightarrow{\text{orbifold}} \quad U(k)_{N-k,N} \tag{5.23}$$

This exactly matches the quantum phase of an adjoint fermion described in Sec. 5.2.1.

Putting this altogether, we arrive at the orbifolded Lagrangian description

$$\mathcal{L}_\psi = i\bar{\psi}\not{D}_b\psi - i\left[\frac{N-k}{4\pi}\text{Tr}_N\left(bdb - i\frac{2}{3}b_1^3\right)\right] \tag{5.24a}$$

$$\mathcal{L}_\Psi = i\bar{\Psi}\not{D}_c\Psi - i\left[\frac{N}{4\pi}\text{Tr}_k\left(cdc - i\frac{2}{3}c^3\right)\right] \tag{5.24b}$$

where we have defined the $\mathbb{Z}_2$ invariant subgroup as $b_1 = b_2 \equiv b$ and $c_1 = c_2 \equiv c$. We have

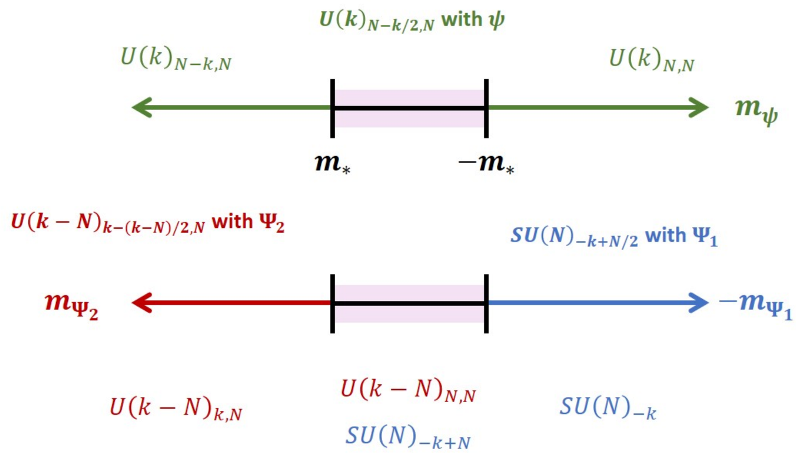

Figure 5.6: Phase diagram for the novel real adjoint bosonization duality.

dropped the $\tilde{A}_1$ background field, because as mentioned above, the orbifolding breaks the $U(1)$ global symmetry down to $\mathbb{Z}_2$.[5]

5.2.3 Novel Adjoint Dualities

We saw above that we can recover the rank-two duality of Ref. [56] by considering the two-node fermionic quiver when one of the 3d bosonization dualities is in the flavor-violated regime. In particular, in Sec. 5.1.2 we considered the case where the Theory A to B duality was flavor-violated, while the Theory B to C duality was flavor-bounded, which brought us into the regime of $k < N < N_*$. A natural next step is to consider the opposite case, where now the Theory B to C duality is flavor-violated and the Theory A to B duality is flavor-bounded. This corresponds to the regime $N < k < N_*$ and gives a new $\mathbb{Z}_2$ symmetric quiver.

Working though the same construction we performed in Sec. 5.1.2, but instead begin-

[5]Because there is a decoupled $U(1)$ Chern-Simons theory, it may look like there still should be a $U(1)$ global symmetry associated with its monopole current. However, this is not the case because there is no matter charged under said $U(1)$ Chern-Simons theory. Said another way, it is not possible to create gauge-invariant monopoles because there is no matter to cancel the GNO charge.

ning on the U-U side, we can construct a general flavor-violated two-node quiver and then specialize to the $\mathbb{Z}_2$ symmetric case. The fact that we are starting with a U-U theory on one end complicates things slightly because of the presence of mixed BF terms coming from the η-invariants in the mass deformed phases. The A, C_1, and C_2 theory again exhibit a manifest $\mathbb{Z}_2$ symmetry. If one again orbifolds with respect to said $\mathbb{Z}_2$ symmetry, we arrive at a new adjoint duality, given by

$$U(k)_{N-k/2,N} \text{ with } \psi^{\text{adj}} \qquad \leftrightarrow \qquad \begin{cases} SU(N)_{-k+N/2} \text{ with } \Psi_1^{\text{adj}} & m_{\psi^{\text{adj}}} = m_* \\ U(k-N)_{k-(k-N)/2,N} \text{ with } \Psi_2^{\text{adj}} & m_{\psi^{\text{adj}}} = -m_* \end{cases} \tag{5.25}$$

where now we require $k > N$. The phase diagram for said duality is shown in Fig. 5.6. Once more, the $\mathbb{Z}_2$ projection causes some of the BF terms to cancel $U(1)$ Chern-Simons terms, making the $U(1)$ Chern-Simons theories decoupled from the adjoint matter.

What is unique about this adjoint duality is that one side of the phase diagram is described by both an SU and U theory. One can see this is necessary from the start. Mass deforming the original U theory gives the TFTs $U(k)_{N,N}$ and $U(k)_{N-k,N}$. Via the level-rank dualities, (1.11) and (1.26), the first of these is dual to an SU theory while the second is dual to a U theory. Despite this difference, the two theories agree in the quantum region, since the U theory yields a TFT which is level-rank dual to an SU TFT.

There is an alternative way to arrive at this duality which also lends insight into the relationship between (5.15) and (5.25). The $\mathbb{Z}_2$-symmetric quiver of Fig. 5.5 has a $U(1)$ global symmetry at every step, which we call $\tilde{A}_1$. If we gauge said symmetry, we would pick up a new $U(1)$ global monopole symmetry which couples to $\tilde{A}_1$ through a BF term, which we call $\tilde{B}_1$. If we *also* gauge this $U(1)$ symmetry, we claim we arrive at the very same $\mathbb{Z}_2$-symmetric quiver which produced Fig. 5.6, up to a time-reversal and label interchange $N \leftrightarrow k$.

How this occurs is straightforward to see qualitatively. On the $SU(N)_{-k+N/2} \times SU(N)_{-k+N/2}$ side of the duality, the two additional $U(1)$ gauge symmetries allow us to rewrite the theory as $U(N)_{-k+N/2} \times U(N)_{-k+N/2}$. Similarly, for the $U(k)_N \times U(k)_N$ theory, integrating out the two additional $U(1)$ gauge symmetries cancels the existing $U(1) \subset U(k)$ gauge groups, giving an $SU(k)_N \times SU(k)_N$ theory. Theory C_2 follows in a similar manner.[6]

Having considered $\mathbb{Z}_2$-symmetric quivers in the $N < k$ and $k < N$ regimes, the very last case we can consider consists of taking $N = k$. This corresponds to using two flavor-bounded dualities to derive the two-node quiver, and thus brings us back into the regime of the flavor-bounded duality of Sec. 5.1.1. Again, we can $\mathbb{Z}_2$-orbifold, and we find

$$SU(N)_{-N/2} \text{ with } \psi^{\text{adj}} \qquad \leftrightarrow \qquad U(N)_{N/2,N} \text{ with } \Psi^{\text{adj}}. \tag{5.26}$$

Note there is no quantum phase for these theories, which is expected because no flavor-violated duality was needed in the construction of the relevant $\mathbb{Z}_2$ symmetric quiver.

5.2.4 Dualities with Extra Flavors

We might try to use this procedure to see if we can derive qualitatively different dualities. Because we required $\mathbb{Z}_2$ symmetry we have considered all relative values of N and k, we need to look at new quiver configurations. Following Ref. [7], a natural generalization is to add flavors on each of the two nodes. We do this in such a way to maintain the $\mathbb{Z}_2$ symmetry, but this could of course be done more generally. Starting with fermion flavors on each node, the quiver diagram for this procedure is summarized in Fig. 5.7.

On the SU side, we begin with the theory

$$SU(N)_{-k+(N+F)/2} \times SU(N)_{-k+(N+F)/2} + \text{bifund. } \psi + F\,\psi_1 + F\,\psi_2, \tag{5.27}$$

where we have denoted by the extra flavor degrees of freedom belonging to node i by ψ_i. This can then be dualized using a flavor-violated version of Aharony's duality. The application of said duality changes one of the nodes to a U group and both the bifundamental and corresponding flavor degree of freedom into scalars.

[6]Here, we remark that the adjoint duality with unitary gauge group (5.25) can also be achieved by consistent S and T operations along the phase diagram of $SU(N)_k$ + adjoint theory in Ref. [56]. Although there is no $U(1)$ global symmetry acting faithfully, we could still introduce the $U(1)$ gauge field and couple it only to the gauge fields. Consistency of the result supports that the mapping of the $\mathbb{Z}_2$ symmetries along the orbifolding quiver is correct.

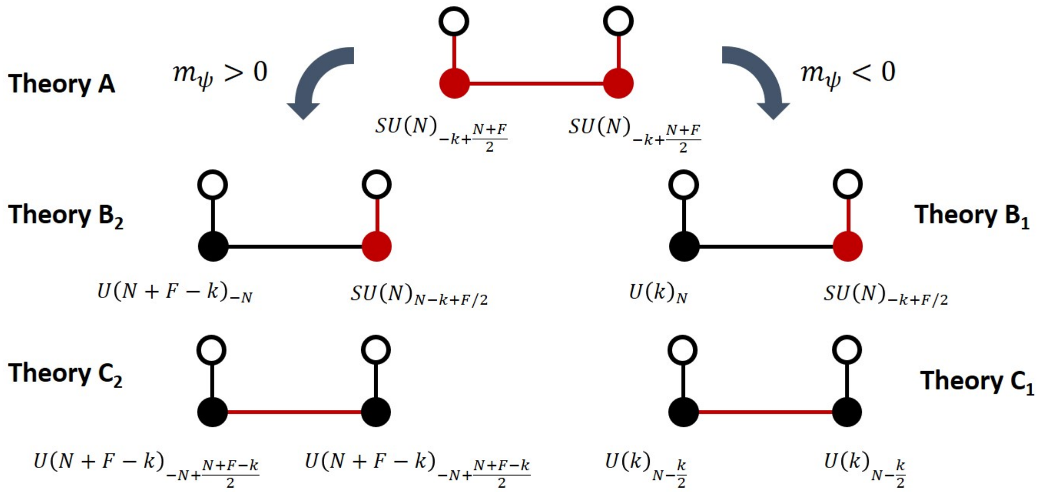

Figure 5.7: Phase diagram of the two-node quiver with extra flavors on each node and explicit $\mathbb{Z}_2$ symmetry. Here the white nodes represent *global* flavor symmetries of the extra matter. Once more, the splitting of the bottom two phase diagrams results from the use of the flavor-violated 3d bosonization duality. We assume $N > k$ and $F < k$.

Since the second node has both fermionic and bosonic degrees of freedom, one can make use of the master duality to dualize.[7] Before doing so, a subtle point is that having both scalars and fermions charged under the same node naturally gives rise to scalar-fermion interactions. These are identical in form to the interaction of the master bosonization duality [68, 25]. Specifically, these are interactions of the form

$$\mathcal{L}_{\text{int}} \supset \left(\phi^\dagger_{A\alpha_2}\psi_2^{\alpha_1\alpha_2}\right)\left(\bar{\psi}^2_{\alpha_1\beta_2}\phi^{A\beta_2}\right). \tag{5.28}$$

with α_1, β_1 indices of the $SU(N)$ gauge symmetry associated with the first node, α_2, β_2 indices of the second node, and A, B flavor indices.

Dualizing the second node also turns it into a U group, and changes the scalars (fermions) degrees of freedom to fermions (scalars). Hence, we ultimately end up with a two-node quiver coupled via a bifundamental fermion with each of the nodes having scalar flavor degrees of

[7]See Refs. [8, 7] for more details on how to dualize quivers using the master duality.

freedom. Once more, the scalar and fermion degrees of freedom of each node have interactions of the form

$$\mathcal{L}_{\text{int}} \supset \left(\bar{\psi}_{\alpha_1\alpha_2}\Phi_1^{\alpha_1 A}\right)\left(\Phi^{1\dagger}_{\beta_1 A}\psi^{\beta_1\alpha_2}\right) + \left(\bar{\psi}_{\alpha_1\alpha_2}\Phi_2^{A\alpha_2}\right)\left(\Phi^{2\dagger}_{A\beta_2}\psi^{\alpha_1\beta_2}\right) \tag{5.29}$$

with the same convention for coefficients used above. In order for the asymptotic phases to match the interaction must come with a positive coefficient. The net effect of this term is that when the scalars acquire a vacuum expectation value, a subgroup of the bifundamental fermion acquires a positive mass. In particular, with the assumption of maximal Higgsing, we have a breaking of $U(k) \to U(k-F) \times U(F)$ (or $U(N+F-k) \to U(N-k) \times U(F)$ on the $m_\psi > 0$ side) . The mass term is of the form

$$\mathcal{L}_{\text{int}} \supset \begin{pmatrix} \mathbb{1}_F & \\ & 0 \end{pmatrix}^{\alpha_1}_{\beta_1} \bar{\psi}_{\alpha_1\alpha_2}\psi^{\beta_1\alpha_2} + \begin{pmatrix} \mathbb{1}_F & \\ & 0 \end{pmatrix}^{\alpha_2}_{\beta_2} \bar{\psi}_{\alpha_1\alpha_2}\psi^{\alpha_1\beta_2} \tag{5.30}$$

so that it is the $U(F)$ subgroup which acquires a mass. Note from the Higgsing pattern above, these fermions are no longer charged under the dynamical symmetry and thus are sometimes referred to as "singlets". Such interactions are necessary to get a matching of the phase diagrams across the duality, see Fig. 5.8.

We now orbifold with respect to the very same $\mathbb{Z}_2$ symmetry we used above of the adjoint case, with the additional identifications $\psi_1 \Leftrightarrow \psi_2$ and $\Phi_1 \Leftrightarrow \Phi_2$. This is analogous to the orbifolding of the fiducial fermions used earlier. We arrive at a duality which has both adjoint and fundamental representation matter on each side,

$$\begin{aligned} &SU(N)_{-k+(N+F)/2} \text{ with } \psi^{\text{adj}} \text{ and } F\ \psi' \qquad \leftrightarrow \\ &\qquad\qquad \begin{cases} U(k)_{N-k/2,N} \text{ with } \Psi_1^{\text{adj}} \text{and } F\ \phi_1 & m_{\psi^{\text{adj}}} = -m_*, \\ U(N+F-k)_{-N+(N+F-k)/2,-N} \text{ with } \Psi_2^{\text{adj}} \text{and } F\ \phi_2 & m_{\psi^{\text{adj}}} = m_*. \end{cases} \end{aligned} \tag{5.31}$$

Note the interactions between the adjoint matter and the scalars is still present, e.g. $\left(\bar{\Psi}^{\alpha}_{1\beta}\phi_1^{\gamma A}\right)\times\left(\phi^\dagger_{1\alpha A}\Psi^{\beta}_{1\gamma}\right)$, with α, β, γ the $U(k)$ gauge indices and A the $U(F)$ flavor index.

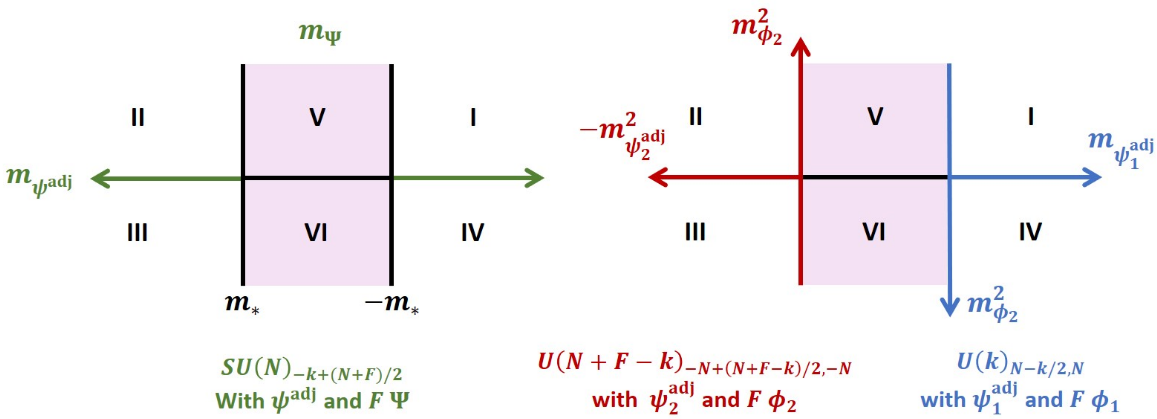

Figure 5.8: Phase diagram of the 3d bosonization duality with both adjoint and fundamental representation matter. Here we assume $N > k$ and $F < k$.

The phase diagram for this duality is shown in Figure 5.8. The phases on the SU side are given by

$$\text{(I)}: \quad SU(N)_{-k+F} \tag{5.32a}$$

$$\text{(II)}: \quad SU(N)_{-k+F+N} \tag{5.32b}$$

$$\text{(III)}: \quad SU(N)_{-k+N} \tag{5.32c}$$

$$\text{(IV)}: \quad SU(N)_{-k} \tag{5.32d}$$

$$\text{(V), (VI)}: \quad \text{Better described by } U \text{ side.} \tag{5.32e}$$

Meanwhile, the theories on the U side are given by

$$\text{(I)}: \quad U(k-F)_N \tag{5.33a}$$

$$\text{(II)}: \quad U(N+F-k)_{-N} \tag{5.33b}$$

$$\text{(III)}: \quad U(N-k)_{-N} \tag{5.33c}$$

$$\text{(IV)}: \quad U(k)_N \tag{5.33d}$$

$$\text{(V)}: \quad U(k-F)_{N-k+F,N} \quad \Leftrightarrow \quad U(N+F-k)_{F-k,-N} \tag{5.33e}$$

$$\text{(VI)}: \quad U(k)_{N-k,N} \quad \Leftrightarrow \quad U(N-k)_{-k,-N}. \tag{5.33f}$$

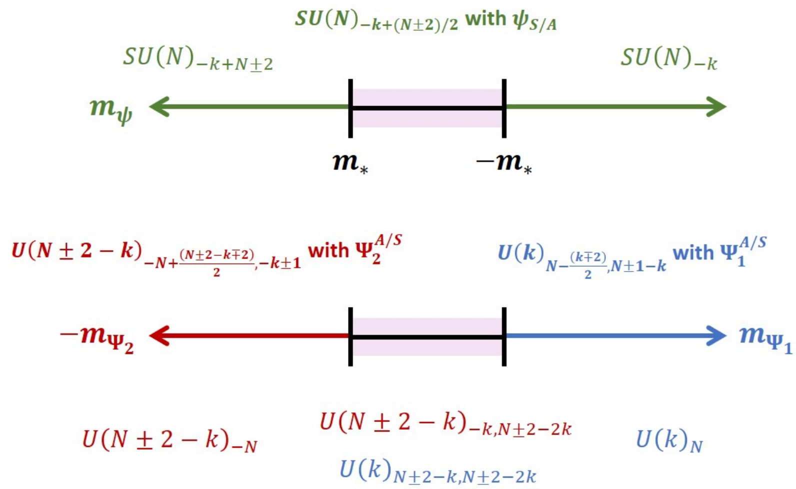

Figure 5.9: Symmetric and antisymmetric representation phase diagram.

5.3 *Discussion and Further Work*

In this chapter we showed that we can connect the rank-one matter bosonization dualities to many dualities which contain adjoint matter for the unitary group case. It turns out that generalization of the two-node quiver dualities to the orthogonal and symplectic group is straightforward using both flavor-bounded and flavor-violated bosonization duality in Refs. [3, 86]. We describe these results in the Appendix E.3. The dualities of adjoint fermions in the orthogonal/symplectic group cases are distinguished from the unitary group case since the matter representation of the dual theory is not adjoint but symmetric-traceless/antisymmetric-traceless, respectively. Thus the natural next question is whether or not we can connect them to the rank-two matter dualities which contain symmetric and antisymmetric matter.

Ref. [33] conjectured a similar duality between symmetric and antisymmetric matter for the unitary group case, the phase diagram of which is summarized in Fig. 5.9. Note the difference in the levels on the SU is determined by (twice) the Dynkin index, $T(\text{S/AS}) = \frac{1}{2}(N \pm 2)$. In order for all the levels to match on the U side, one must introduce the opposite type of matter, giving the dualities

$$SU(N)_{-k+\frac{1}{2}(N\pm 2)} \text{ with } \psi^{\text{S/AS}} \;\leftrightarrow\; \begin{cases} U(k)_{N-\frac{1}{2}(k\mp 2),N-(k\mp 1)} \text{ with } \Psi_1^{\text{AS/S}} & m_\psi = -m_* \\ U(N\pm 2-k)_{-N+\frac{1}{2}(N\pm 2+k\mp 2),\pm 1-k} \text{ with } \Psi_2^{\text{AS/S}} & m_\psi = m_* \end{cases} \tag{5.34}$$

where the superscript S/AS denotes symmetric/antisymmetric representation under the gauge group.

The analog of orbifold producing daugther theory with symmetric or antisymmetric matter is called an "orientifold". For the bifundamental fermion in the $(\mathbf{N},\mathbf{N})$ representation, corresponding $\mathbb{Z}_2$ symmetries of the orientifold projection is given by[8] [122]

$$\mathbb{Z}_2^{\text{S}}: \qquad \psi\to\psi^T, \qquad b_\mu^1\to b_\mu^2, \qquad b_\mu^2\to b_\mu^1, \tag{5.36}$$

$$\mathbb{Z}_2^{\text{AS}}: \qquad \psi\to-\psi^T, \qquad b_\mu^1\to b_\mu^2, \qquad b_\mu^2\to b_\mu^1. \tag{5.37}$$

Constructing two-node quiver dualities with $(\mathbf{N},\mathbf{N})$ representation matter is straightforward. Much of the derivation of the $(\mathbf{N},\bar{\mathbf{N}})$ case is mirrored here, but now the matter will simply be fundamentally charged under both gauge groups at every step of the way.

The naive application of the above procedure is not successful. The difficulty lies in realizing $\mathcal{O}(1)$ shifts of the rank of the gauge groups which are required for the daughter dualities. We have not yet been able to find where the required finite shifts in the gauge group come from. Similar subtleties arise in the case of the SO/Sp gauge theory with rank-two fermions. It would be interesting to analyze the orbifold/orientifold equivalence in the Chern-Simons matter theories up to the subleading $\mathcal{O}(1)$ order to resolve the issue. Another interesting question is why the unitary group rank-two dualities map representations as

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[8] Note that even in the $(\mathbf{N},\bar{\mathbf{N}})$ case there exists another $\mathbb{Z}_2$ symmetry

$$\mathbb{Z}_2^{\prime\text{adj}}: \qquad \psi\to\psi^\dagger, \qquad b_\mu^1\to b_\mu^2, \qquad b_\mu^2\to b_\mu^1. \tag{5.35}$$

In this respect both cases are on an equal footing. The difference here is the fact that in the $(\mathbf{N},\bar{\mathbf{N}})$ case it doesn't matter which $\mathbb{Z}_2$ we orbifold with respect to on either side; both lead to an adjoint rank-two tensor.

adjoint to adjoint and symmetric to antisymmetric.[9] It would be nice to understand the origin of this mapping carefully from the orbifolding point of view. We leave these puzzles for future work.

Despite the difficulty in connecting such dualities to the two-node quivers, we can follow a similar procedure as we did above for constructing new adjoint dualities to slightly generalize the symmetric/antisymmetric dualities. For instance, we can conjecture the new duality

$$\begin{aligned} &U(k)_{N-\frac{1}{2}(k\pm 2),N-k\mp 1} \text{ with } \psi^{\mathrm{S/AS}} \\ &\qquad\leftrightarrow \begin{cases} SU(N)_{-k+\frac{1}{2}(N\mp 2)} \text{ with } \Psi_1^{\mathrm{AS/S}} & m_\psi = -m_* \\ U(k\pm 2-N)_{k-\frac{1}{2}(k\pm 2-N\mp 2),k\pm 1} \text{ with } \Psi_2^{\mathrm{AS/S}} & m_\psi = m_* \end{cases} \end{aligned} \tag{5.38}$$

which must obey $k > N \pm 2$. This passes many of the same consistency checks as the original symmetric/antisymmetric duality. Furthermore, limiting case $k = T(R)$ gives the new symmetric/antisymmetric duality with no quantum phase,

$$SU(N)_{-(N\pm 2)/2} \text{ with } \psi^{\mathrm{S/A}} \qquad \leftrightarrow \qquad U(N\pm 2)_{N/2,\mp 1} \text{ with } \Psi^{\mathrm{A/S}}, \tag{5.39}$$

which is qualitatively similar to the new adjoint duality found in (5.26).

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[9]The symplectic (orthogonal) dualities follow this latter pattern since in this case the adjoint is the symmetric (antisymmetric) to begin with.

Chapter 6

MOVING FROM 3D TO 4D

Thus far in this work, we have discussed several aspects of the 3d bosonization dualities. While some potential applications arise in the context of quantum Hall physics [119], the impact of these dualities on our understanding of nature is still somewhat hampered by the fact that these low-dimensional dualities do not describe the 3+1 dimensional world we live in. Unfortunately, as far as 3+1 dimensional dualities in interesting gauge theories are concerned, our understanding has not evolved much beyond the realm of supersymmetry.

One may wonder whether some of the recent progress in 2+1 dimensions can be lifted to 3+1 dimensions. One promising avenue to pursue is the idea of deconstruction [19]. Deconstruction allows one to write a product gauge theory in d dimensions which, in an intermediate energy regime, mimics a gauge theory in $d+1$ dimensions. The Lagrangian of the d dimensional gauge theory is essentially the $d+1$ dimensional theory put on a circle of radius R with the compact direction discretized on a one dimensional lattice of lattice spacing a with N sites so that $2\pi R = Na$. At energies $a^{-1} \gg E \gg R^{-1}$ the theory behaves as a $d+1$ dimensional continuum field theory. In the UV the lattice spacing becomes visible and the theory reverts to the d dimensional product gauge group. At energies below R^{-1} the theory of course also becomes d dimensional due to the circle compactification.

In this chapter we will deconstruct maximally supersymmetric electromagnetism that is a simple $\mathcal{N} = 4$ supersymmetric $U(1)$ gauge theory. The 2+1 dimensional field theory accomplishing this is a $U(1)^N$ quiver gauge theory with bifundamental matter and $\mathcal{N} = 4$ supersymmetry.[1] Such supersymmetric 2+1 dimensional quivers are well known to have a

[1]Note that $\mathcal{N} = 4$ supersymmetry in 2+1 dimensions is half the supersymmetry of the 3+1 dimensional theory we are aiming for, 8 instead of 16 supercharges. This is a well known phenomena in deconstruction: together with the broken translation invariance along the circle direction due to the discretization we lose

dual description [64]: Mirror symmetry maps the quiver to supersymmetric QED with N flavors. Given that the quiver gauge theory realizes 3+1 dimensional maximally supersymmetric electromagnetism, mirror symmetry implies that QED with N flavors also has to have a 3+1 dimensional window, at least when the parameters are dialed appropriately. In this chapter we will demonstrate that this indeed the case. The mirror theory in 2+1 dimensions realizes the electromagnetic dual, or S-dual, of the 3+1 dimensional theory we started with. A summary of the various theories and their relations is shown in figure 6.1.

Many of the techniques used in deconstructing 3+1 dimensional S-duality easily import into the non-supersymmetric case. From a base duality, we derive new all scale quiver dualities consisting of Wilson-Fisher scalars (WF) on one side and QED with N flavors on the other; both theories containing Chern-Simons terms. The quiver now deconstructs a non-supersymmetric $U(1)$ gauge theory in 3+1 dimensions, which implies QED with N flavors must also have a 3+1 dimensional description. Up to order one factors hidden by strong coupling, this again appears to be the case, and it further motivates the conjecture that 2+1 dimensional bosonization realizes non-supersymmetric electromagnetic duality of the original 3+1 dimensional theory.

The organization of this note is as follows. In section 6.1 we will review the essentials of deconstruction and, in particular, exhibit the 2+1 dimensional deconstruction of maximally supersymmetric 3+1 dimensional electromagnetism. In section 6.2 we recall the basics of mirror symmetry. We will see that in order to construct the mirror of deconstructed $\mathcal{N} = 4$ electromagnetism we need its all scale version [79]. In section 6.3 we present our main result: we demonstrate that the mirror theory indeed reproduces the S-dual realization of 3+1 dimensional electromagnetism in the continuum limit. In section 6.4, we consider a non-supersymmetric version of deconstructing 3+1 dimensional Abelian dualities through the all scale version of bosonization. We conclude in section 6.5.

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half of the supersymmetry for every compact dimension. The full supersymmetry of the $3 + 1$ dimensional theory only emerges in the continuum limit, that is at energies below a^{-1}.

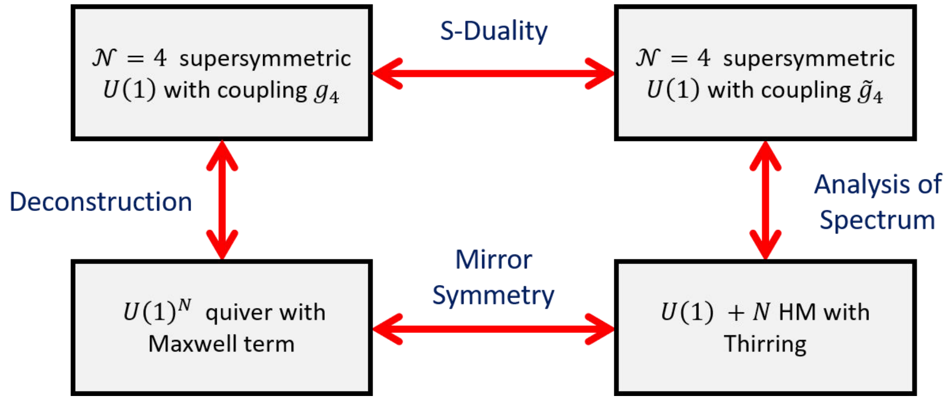

Figure 6.1: Summary of the relationship between the various supersymmetric theories we consider.

6.1 Deconstruction

The action for a pure 3+1 dimensional gauge theory with one of the spatial dimensions compacitified on a circle of circumference $2\pi R$ and discretized into N lattice sites of lattice spacing a reads [19]

$$S = \int d^3x \left(-\frac{1}{4G^2} \sum_{i=1}^{N} F_i^2 + \frac{1}{2G^2 a^2} \sum_{i=1}^{N} (D_\mu U_{i,i+1})^\dagger (D^\mu U_{i,i+1}) + \dots \right). \tag{6.1}$$

Here i labels the N lattice sites. There is a 2+1 dimensional gauge theory with field strength $F_i = db_i$ living on each site with the same gauge group as the 3+1 dimensional parent. In this chapter we will be mostly concerned with the Abelian case, but deconstruction works equally well for any gauge group. From the 2+1 dimensional point of view we are describing a $U(1)^N$ gauge theory. G is the 2+1 dimensional gauge coupling on each site with its standard dimension of Length$^{-1/2}$. The diagonal $U(1)$ decouples completely since no matter is charged under it, but the corresponding photon is part of our dynamical fields. This is different from how quiver gauge theories are typically treated in discussions of mirror symmetry, where the diagonal $U(1)$ gauge boson is removed from the spectrum completely and one only keeps the non-trivial $U(1)^N/U(1)$ part of the gauge theory.

The group valued matrices U live on the links, so they are labeled by $(i, i+1)$, the two sites they connect. From the 3+1 dimensional point of view the kinetic term for U simply is a discretization of $-F_{\mu 3}F^{\mu 3}/(4G^2)$ and so the two terms combine into the standard gauge kinetic term. From the 2+1 dimensional point of view the U fields can be thought of as "non-linear sigma model fields" in the bifundamental representation. Under a gauge transformation $\mathfrak{g}_i$ the U fields transform as

$$U_{i,i+1} \to \mathfrak{g}_i^{-1} U_{i,i+1} \mathfrak{g}_{i+1}. \tag{6.2}$$

The covariant derivatives of $U_{i,i+1}$ are given by

$$D_\mu U_{i,i+1} \equiv \partial_\mu U_{i,i+1} - ib^i_\mu U_{i,i+1} + iU_{i,i+1} b^{i+1}_\mu. \tag{6.3}$$

For our Abelian case we can think of the U's as pure phase variables,

$$U_{i,i+1} = e^{ia\phi_{i,i+1}}. \tag{6.4}$$

ϕ can be interpreted as b_4, the component of the gauge field along the circle direction. The non-linear sigma models described by the U fields are not UV complete theories in $d = 2+1$ dimensions all by themselves. The non-linear sigma model has a strong coupling scale, $f_\pi^{-1/2} = aG$. But the non-linear sigma models can be UV completed entirely within the 2+1 dimensional theory. The original work of [19] envisioned two scenarios to do so. One option is to introduce an extra quiver node in the middle of every link. Furthermore, the new link fields will just be bifundamental fermions. As long as all the extra nodes are dialed to confine at an energy scale $\Lambda \gg a^{-1}$, the relevant degrees of freedom at scale a^{-1} are simply the pions of that confining gauge group, which are well described as non-linear sigma model fields. The alternate option is to replace the non-linear sigma model fields with linear sigma model fields, that is with un-constrained bifundamental scalar fields $X_{i,i+1}$. As long as we force the scalar fields to have a vacuum expectation value

$$\langle X_{i,i+1} \rangle = X_0 \tag{6.5}$$

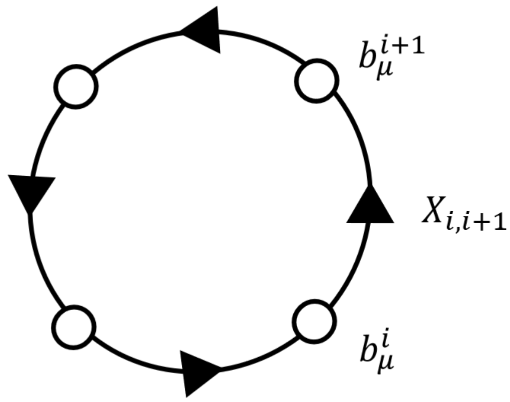

Figure 6.2: Quiver diagram for the 2+1 dimensional deconstruction of maximally supersymmetric electromagnetism. Nodes (white circles) correspond to $U(1)$ gauge groups, links (black arrows) to bifundamental hypermultiplets.

the low energy degrees of freedom are once more given by the phase variables, $X_{i,i+1} = X_0 e^{ia\phi_{i,i+1}}$. In general this second option requires one to carefully choose quartic potentials to ensure that we get the desired X_0. However, in supersymmetric gauge theories, especially those with extended supersymmetry, this second option is in fact very easy to implement. Such theories often have moduli spaces of vacua: the scalar potential is exactly flat and we simply can dial the vacuum expectation value X_0 as a free parameter.

Using this second purely 2+1 dimensional UV completion, we can present the deconstruction of a pure $\mathcal{N} = 4$ supersymmetric $U(1)$ gauge theory in 3+1 dimensions. The relevant theory is a 2+1 dimensional $\mathcal{N} = 4$ quiver gauge theory with N nodes connected according to the quiver diagram in figure 6.2. Each node corresponds to a $U(1)$ vector multiplet: a dynamical photon together with two adjoint Dirac fermions and three adjoint real scalars. Since we are considering only $U(1)$ gauge groups, adjoint here simply means neutral. Each link corresponds to a bifundamental hypermultiplet, that is 2 complex scalars and 2 Dirac fermions with charge $(+1, -1)$ under the two $U(1)$ factors they connect.

Besides N, the basic parameters are G, the coupling of each $U(1)$ factor, and X_0, the vacuum expectation value for scalar fields in the bifundamental hypermultiplets. This theory indeed has a moduli space of vacua, so X_0 can freely be dialed and different values of X_0 correspond to different super-selection sectors. More precisely, X_0 is a parameter along the Higgs branch of the theory, where the gauge group gets broken down to the diagonal subgroup,

$$U(1)^N \to U(1). \tag{6.6}$$

The moduli space in principle also contains a Coulomb branch where X_0 vanishes but instead the scalars in the vectormultiplets living on the nodes get a vacuum expectation value. The Coulomb branch will make an appearance in the dual theory, since mirror symmetry maps the Higgs branch of one theory to the Coulomb branch of the other. But to use the standard deconstruction procedure we need to be on the Higgs branch.

This quiver theory deconstructs a 3+1 dimensional pure $\mathcal{N} = 4$ gauge theory with $U(1)$ gauge group on a circle of radius R with parameters of the 3+1 theory given by

$$g_4^2 = \frac{G}{X_0}, \quad 2\pi R = \frac{N}{GX_0}, \quad a = 2\pi\frac{R}{N} = \frac{1}{GX_0}. \tag{6.7}$$

The identification of a comes from looking at the kinetic term for the scalar. The mass term the gauge bosons pick up from the Higgs mechanism reads $X_0^2(b_\mu^{i+1} - b_\mu^i)^2$. Comparing this to the desired action (6.1) tells us that we need to identify $X_0^2 = (aG)^{-2}$. The circumference $2\pi R$ is Na, the number of lattice sites times the lattice spacing. To identify the 4d gauge coupling we can simply look at the massless mode of the gauge field. From the 3+1 dimensional point of view this is the constant zero mode along the circle, so its coupling is given by performing the integral over the circle in the action and hence $g_{ZM}^{-2} = (2\pi R)g_4^{-2}$. From the 2+1 dimensional point of view the zero mode is the unbroken $U(1)$ that remains after breaking $U(1)^N \to U(1)$, so its coupling is given by $g_{ZM}^{-2} = NG^{-2}$. Equating the two expressions for the zero mode gives us the identification of g_4 in terms of G and X_0. This construction follows exactly the one outlined in [20] where maximally supersymmetric gauge theories in 4+1 and 5+1 dimensions were deconstructed in terms of 3+1 dimensional gauge theories.

As described in the introduction, the theory behaves 3+1 dimensional in the energy window $a^{-1} \gg E \gg R^{-1}$. To take a genuine continuum limit we want to send $a \to 0$ while keeping g_4 and R fixed. So we need:

$$G, X_0, N \to \infty, \quad \frac{G}{X_0} \text{ fixed }, \quad \frac{N}{GX_0} \text{ fixed.} \tag{6.8}$$

One important aspect of this limit is that we can not take G to infinity from the get go. Since the ratio of G and X_0 is fixed, $a = (GX_0)^{-1}$ is really the only basic length scale in the problem. If we work at strictly infinite coupling, this scale vanishes identically and so we never see a finite radius circle. We only want to take a to zero as we take N to infinity. Infinite G corresponds to the IR limit of the theory. In order to understand what happens to our theory as we take the limit we need to be able to control excitations away from the strict IR limit.

Since the 3+1 dimensional theory lives on a circle, the only massless gauge boson is the zero mode, in full agreement with the fact that at low energies we only have a single 2+1 dimensional $U(1)$ gauge theory. Note that this low energy theory is actually $\mathcal{N} = 8$ supersymmetric since it is just the circle reduction of maximally supersymmetric electromagnetism. The fact that we are really studying a 3+1 dimensional theory on a circle is encoded in the spectrum of Kaluza-Klein (KK) modes. Since X_0 is the vacuum expectation value for bifundamental matter we see that the mass matrix for the massive W-bosons takes the standard balls-and-springs form

$$M_W^2 = G^2 X_0^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & 2 & -1 & & \vdots \\ \vdots & 0 & -1 & \ddots & & 0 \\ 0 & & & & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{6.9}$$

with eigenvalues

$$m_n^2 = 4G^2 X_0^2 \sin^2 \frac{\pi n}{N} \tag{6.10}$$

where n is an integer with $-N/2 < n \leq N/2$. These are in fact the expected momentum modes on a discretized circle. Our classical calculation of the W-masses unfortunately is only valid in the weak coupling limit, whereas our continuum limit involves a strong coupling limit. In a supersymmetric theory one might hope that the W-masses could be protected by supersymmetry. As pointed out in [20] this is almost true here: while the W-bosons are *not* BPS states in the full quiver gauge theory, they do become BPS states from the point of view of the low energy $\mathcal{N} = 8$ gauge theory. What this means is that while we can not guarantee that the full spectrum of W-boson states remains to be given by (6.10) at strong coupling, we can trust the formula for the low lying states with $n \ll N$ where $m_n^2 \approx n^2/R^2$ exactly as we expect in the continuum limit. We have in fact successfully deconstructed maximally supersymmetric electromagnetism in 3+1 dimensions.

Let us close this section with noting that there is an alternate way of thinking of the parameter X_0. We can force the bifundamental scalar fields to have a non-vanishing expectation value by adding an Fayet-Iliopoulos (FI) terms to the theory.[2] A FI is a term linear in the auxiliary D field in the vector multiplet, $\mathcal{L}_D = \xi D/\pi$. After integrating out D it corresponds to a new scalar term in the potential together with the fermionic partners that are needed for supersymmetry. In terms of the moduli space, the FI term imposes a "D-term" constraint that ensures that the modified potential vanishes. A FI term ξ_i on any of the individual nodes would give a D-term demanding that the expectation values of the 2 complex scalar fields $X_{i,i+1}$ and $\tilde{X}_{i,i+1}$ on each site obey

$$|X_{i,i+1}|^2 - |\tilde{X}_{i,i+1}|^2 - |X_{i-1,i}|^2 + |\tilde{X}_{i-1,i}|^2 - \frac{\xi_i}{\pi} = 0. \tag{6.11}$$

This is not what we want. It would force a non-vanishing difference between the expectation values of the incoming and outgoing link at each node. We do want to work with $\xi_i = 0$ for all i so that all scalars can have the same expectation value X_0. Fortunately there is a different $U(1)$ symmetry we can use in the problem. Rotating all matter fields by an overall

[2]In our conventions we follow [79], where all normalizations are clearly spelled out. The Maxwell term has a prefactor of $(4G^2)^{-1}$, Chern-Simons like terms have an integer coefficient times $(4\pi)^{-1}$, whereas FI terms have a coefficient of π^{-1}.

phase,

$$X_{i,i+1} \to e^{i\alpha/N} X_{i,i+1}, \qquad \tilde{X}_{i,i+1} \to e^{-i\alpha/N} \tilde{X}_{i,i+1} \tag{6.12}$$

gives a global symmetry rather than a gauge symmetry. We can introduce a background FI term for this global symmetry as a parameter in our Lagrangian, giving us a D-term

$$\frac{1}{N} \sum_i (|X_{i,i+1}|^2 - |\tilde{X}_{i,i+1}|^2) - \frac{\xi}{\pi} = 0. \tag{6.13}$$

For $\xi = \pi X_0^2$ this gives us the desired $X_{i,i+1} = X_0$ as the simplest solution. Thinking of X_0 as being imposed on us by a FI term will help us with the parameter mapping to the mirror.

6.2 All-Scale Mirror Symetry

6.2.1 Mirror Symmetry

Mirror symmetry as introduced in [64] is an IR duality, that is it is only valid at infinite G. Mirror symmetry relates two different supersymmetric gauge theories that flow to the same conformal field theory in the infrared. In terms of deformations due to giving scalar fields vacuum expectation values, it maps the Coulomb branch of one theory to the Higgs branch of the other and vice versa. While mirror symmetry is well established also for non-Abelian gauge groups [64, 42, 41], the general dual pair is easiest to construct in the Abelian case. For Abelian mirror symmetry all possible dual pairs can be derived from a single base pair [79]. This is in complete parallel with the derivation of the non-supersymmetric duality web [83, 112] from a single seed duality. In fact, it was the supersymmetric insights that served as a template for the non-supersymmetric web. In the $\mathcal{N} = 4$ supersymmetric case the base pair demands the following equivalence (HM=hymermultiplet):

Base pair (BP): $U(1)$ + 1 charged HM = free HM

Both sides of the base pair have a global $U(1)$ symmetry and the corresponding conserved current can be coupled to a background source B_μ. Here and in what follows we will denote dynamical gauge fields with lower case letters and background fields with upper case letters.[3] We use a_μ for the dynamical gauge field on the left hand side of (BP) and denote its field strength by $f_{\mu\nu}$.

On the right hand side of (BP) we see a free hypermultiplet. The global symmetry is simply particle or baryon number, counting the number of scalars and fermions. The corresponding background field couples via the standard minimal coupling in the kinetic term. On the left hand side baryon number is gauged, so it is not a global symmetry. Instead we have a topological global symmetry, monopole number, whose conserved current is $j_\mu = \epsilon^{\mu\nu\rho} f_{\nu\rho}$. j^μ is identically conserved due to the Bianchi identity. The background field couples to the theory via a BF term

$$S_{BF}[a, B] = \frac{1}{2\pi} \int d^3x \, \epsilon^{\mu\nu\rho} a_\mu \partial_\nu B_\rho. \tag{6.14}$$

Once again, this is a story that has recently resurfaced in the context of non-supersymmetric dualities but had been well appreciated in the supersymmetric context long ago.

Last but not least let us recall how we can use (BP) to derive mirror duals for arbitrary Abelian gauge theories. Starting with N copies of the right hand side we can realize any Abelian gauge theory by promoting r of the N global baryon number symmetries to gauge symmetries. Which linear combination we gauge is up to us, so we can chose our N matter fields to have charges R_i^a under the r gauge groups. This construction has been nicely laid out in [120]. We end up with

Theory A: $\mathcal{N} = 4$ $U(1)^r$ gauge theory with N hypermultiplets of charge R_i^a.

[3] Readers familiar with the non-supersymmetric duality web may wonder whether our gauge and background fields are genuine $U(1)$ connections or rather spin$_c$ connections. For the supersymmetric duality all our background fields are in fact $U(1)$ connections and not spin connections. Symmetries under which a whole multiplet is charged manifestly violate the spin/charge relation that is required to hold for spin$_c$ connections. Both fermions and bosons in the multiplet carry identical charges. The only exception to this rule are R-symmetries, under which the supercharges themselves are charged and so bosons and fermions within a given multiplet transform differently. The $\mathcal{N} = 4$ supersymmetric gauge theories we consider have an $SU(2)_L \times SU(2)_R$ R-symmetry and treating background fields for these symmetries as spin$_c$ connections would allow us to study the theories on manifolds which aren't spin but only spin$_c$. For our purposes here, turning on background fields for R-symmetries is not needed and so all our dynamical and background fields are $U(1)$ connections.

Here i runs from 1 to N as before, whereas a runs from 1 to r. The dual theory starts with N copies of the left hand side, that is N copies of SUSY QED with 1 flavor coupled to background fields via BF couplings. Promoting r of the background gauge fields to dynamical gauge fields gives masses to both the r background gauge fields and the r dynamical gauge fields they couple to via the BF couplings. We are left with the $N-r$ gauge groups (labeled by p running from 1 to $N-r$), that is with

Theory B: $\mathcal{N}=4$ $U(1)^{N-r}$ gauge theory with N hypermultiplets of charge S_i^p with $\sum_i R_i^a S_i^p = 0$ for all a and p.

The quiver is a special case of this general duality. Usually it is considered to be the $r = N-1$ case with charges assigned according to the quiver of figure 6.2 with the overall $U(1)$ modded out (so that the N nodes only give rise to $N-1$ gauge groups). The overall $U(1)$ completely decouples from the theory. It can be added back in on both sides in the end to obtain our deconstructing $U(1)^N$ theory of figure 6.2. The dual is simply a single $U(1)$ gauge group with N flavors, that is $\mathcal{N} = 4$ supersymmetric QED. This duality between the Abelian quiver theory with N nodes and QED with N flavors was in fact the very first example of mirror symmetry uncovered in [64].

This particular instance of mirror symmetry can also be very easily understood in terms of Hanany-Witten style brane setups [59]. Introducing branes as displayed in table 6.1, we can suspend snippets of D3 branes between NS5 branes in the x_6 direction. As long as this direction is compactified on a circle, we can realize the quiver gauge theory of figure 6.2 via a single D3 brane wrapping the entire x_6 circle with N NS5 branes along the circle as displayed in the left panel of figure 6.3.

Each D3 segment connecting neighboring NS5 branes corresponds to a $U(1)$ gauge group factor. The bifundamental hypermultiplets arise from strings connecting D3 brane segments across the NS5 branes. The coupling constants G_i of the individual $U(1)$ factors is encoded in the length L_i of the D3 segments, since the 2+1 dimensional gauge theories arise from compactifying the 3+1 dimensional gauge theory living on the D3 branes on the segment,

	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x	o	o	o	o
D3	x	x	x	o	o	o	x	o	o	o
D5	x	x	x	o	o	o	o	x	x	x

Table 6.1: Brane realization of 2+1 dimensional quiver gauge theories and their mirrors.

so $G_i^{-2} = L_i g_4^{-2}$. The configuration in figure 6.3 corresponds to the theory which we need for deconstruction with the NS5 branes equally spaced along the circle so that all gauge couplings are equal, $G_i = G$ for all i. This is not the point at which mirror symmetry is valid. The conformal field theory which has the two mirror symmetric descriptions is only realized when all N NS5 branes sit on top of each other. The equally spaced configuration can be viewed as a deformation of the CFT by the irrelevant Maxwell terms corresponding to the NS5 brane separations. Once again we see that for the application to deconstruction we would like to understand the theory away from the strict IR limit.

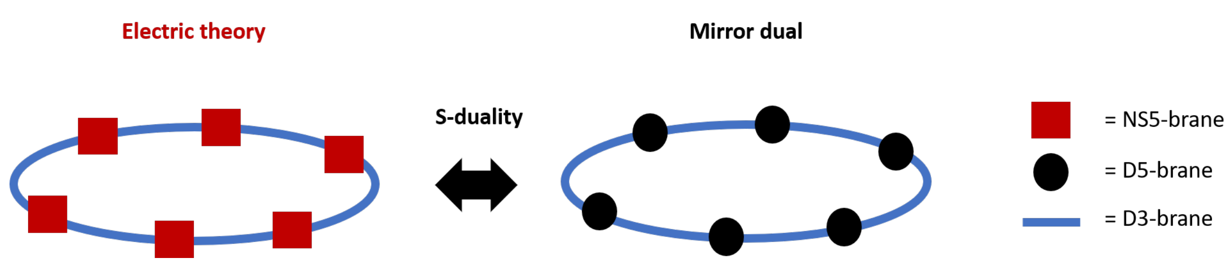

Figure 6.3: Brane construction of the quiver gauge theory and its mirror dual.

To find the mirror dual from the brane setup we simply perform S-duality in type IIB string theory. This takes the D3 brane back into itself, but exchanges NS5 branes with D5 branes. To stay within the configurations made out of the objects displayed in table 6.1 we need to combine this transformation with a relabeling of the $x_{3,4,5}$ and $x_{7,8,9}$ directions. This relabelling captures the fact that mirror symmetry exchanges the two $SU(2)$ R-symmetry factors, which correspond to rotations in 789 and 345 respectively. In the field theory the two R-symmetries act on Higgs and Coulomb branch respectively and so exchanging them also means we are exchanging the two branches. The dual of the quiver gauge theory is displayed in the right panel of figure 6.3. This time there are no NS5 branes present, so we have a single $U(1)$ gauge group. The N D5 branes give rise to N charged hypermultiplets. So indeed we see that the mirror is QED with N flavors.[4] We saw that in the quiver gauge theory, the CFT point corresponds to coincident NS5 branes and the configuration relevant for deconstruction involves turning on the irrelevant deformations corresponding to the NS5 separations. In the electric quiver theory these corresponded to the gauge couplings, the concrete identification of the corresponding "magnetic couplings" on the mirror side has long been known to be non-trivial. Fortunately in the Abelian case a full resolution is known, as we will explain in the next subsection.

Let us close this discussion with noting that the brane pictures we just introduced immediately generalize to the non-Abelian case. If instead of a single D3 brane we use K D3 branes, the quiver gauge theory will be $U(K)^N$ with bifundamental matter. The overall $U(1)$ again decouples, but this time all N $SU(K)$ factors are interacting. This quiver theory nicely deconstructs $\mathcal{N} = 4$ SYM with $U(K)$ gauge group. The mirror dual theory is $U(K)$ with N flavors. The main reason we restrict ourselves to the Abelian case in this chapter is the identification of the magnetic couplings which we are about to review.

[4] As noted before, the $U(1)$ gauge theory living on the D3 brane is in fact an $\mathcal{N} = 8$ gauge theory, supersymmetry is only broken to $\mathcal{N} = 4$ by the charged matter hypermultiplets. That is, there is an additional adjoint (=neutral) hypermultiplet present. In the $U(1)$ case, this adjoint hypermultiplet is completely decoupled. A free hypermultiplet is equivalent to a free vector multiplet (after dualizing the photon into a dual scalar) and it is this extra hypermultiplet that, in the brane picture, is dual to the decoupled overall $U(1)$ on the electric side. The non-trivial $U(1)^N/U(1)$ quiver gauge theory is dual to $\mathcal{N} = 4$ QED with N flavors.

6.2.2 All-Scale Duality

Mirror symmetry as it stands is only valid in the deep IR, that is when the $U(1)$ factors are infinitely coupled. We have argued before that in order to properly understand the dual of deconstructed maximally supersymmetric electromagnetism, we seem to require being able to keep track of both theories even at finite coupling. Fortunately this is easy to do, at least in the Abelian case, using the same techniques that we used to derive new mirrors from a single base pair. We can start with (BP), gauge the global $U(1)$ baryon number on the right hand side, but this time add a $F^2/(4G^2)$ Maxwell term to it, with F the field strength of B_μ. Since B_μ started out as a background gauge field, we should feel free to add such contact terms before we promote it to a dynamical gauge field, $B_\mu \to b_\mu$, as long as we do so consistently on both sides of the duality. This gives us the theory on the left hand side of (BP) but with an extra Maxwell term. Whatever this same procedure does on the original left hand side of (BP) has got to be the new right hand side, that is the all-scale dual to $U(1)$ + 1 charged HM + Maxwell term. On the original left hand side of (BP) the dual monopole $U(1)$ couples via a BF term, so we have, schematically

$$S_{\text{lhs}} = \text{Hyper}[a] + S_{BF}[a,b] - \tfrac{1}{4G^2} \int F^2.$$

Hyper$[a]$ denotes the action of the hypermultiplet minimally coupled to a_μ and S_{BF} is the BF coupling of (6.14). Integrating out a_μ and b_μ gives a (supersymmetric version) of the Thirring model: a hyper with a quartic fermion interaction. This is irrelevant in 2+1 dimensions, but so is a Maxwell term. The Thirring interaction is the dual to the Maxwell term. After exchanging left and right hand side, this gives an all-scale version of (BP):

all scale Base pair (aBP): $U(1)$ + 1 charged HM + Maxwell = free HM + Thirring

While (aBP) is useful as a mnemonic to understand what all scale mirror symmetry does, we will find it more useful in what follows to keep both the massive gauge fields around on the Thirring side of the duality in order to correctly capture the physics of this irrelevant deformation.

Applying this philosophy to the $U(1)^N$ quiver gauge theory we want for deconstruction, we need to start with N free hypermultiplets and gauge N linear combinations of the global baryon number symmetries. Denoting the original baryon number symmetries by B_i, we form the bifundamental linear combinations

$$B_i = b_{i+1} - b_i, \tag{6.15}$$

and also promote the b_i to dynamical fields. Here the index i is defined modulo N, that is $b_{N+1} \equiv b_1$. At this step we can introduce a Maxwell term with coupling G for each of the $U(1)$ factors b_i. Repeating the same steps as in the case of a single $U(1)$ above, we can find the all scale dual for the quiver, including the irrelevant coupling G. The dual has N QED gauge bosons a_i from the N decoupled factors of QED with 1 flavor we started out with. These couple to the same N dynamical gauge fields $B_i = b_{i+1} - b_i$ we newly gauged. Somewhat schematically, the mirror action (M) reads

$$S_{\mathrm{M}} = \sum_i \ \mathrm{Hyper}_i[a_i] \ + \sum_i \ S_{BF}[a_i, b_{i+1} - b_i] \ - \sum_i \frac{1}{4G^2} \int F_i^2. \tag{6.16}$$

Once again, we could integrate out the massive gauge fields a_i and b_i to reduce to QED with N flavors (and a decoupled center of mass factor). The coupling constant G would imprint itself as a current-current Thirring like interaction. But for the purpose at hand it will prove more convenient to directly work with (M).

6.3 S-duality from mirror symmetry

In the previous section we identified the mirror (M) of the quiver gauge theory responsible for deconstructing maximally supersymmetric electromagnetism in 3+1 dimensions. We found that at the conformal point the non-trivial part of the dual is just QED with N flavors. Turning on the irrelevant Maxwell coupling G induces a particular pattern of current-current interactions which is best captured by the introduction of an extra $2N - 2$ massive gauge fields. There is one more step we need to take in order to implement the theory described by the matching in (6.7): we need to turn on the vacuum expectation value X_0 for the scalars

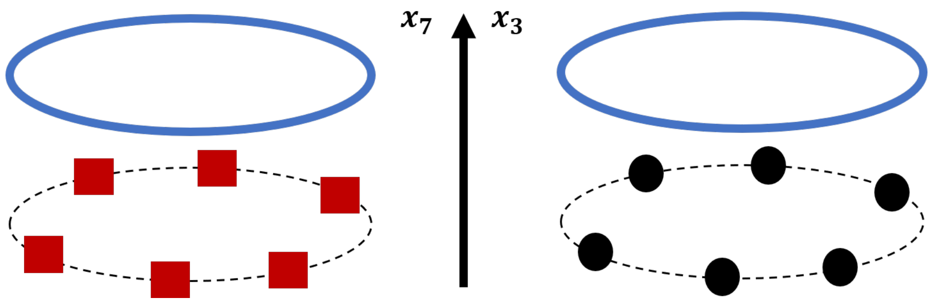

Figure 6.4: Higgs branch deformation of the quiver gauge theory and the dual Coulomb branch deformation of its mirror.

in the quiver theory. Recall that in the original quiver this expectation value took us out onto the Higgs branch, where $U(1)^N$ is broken to the diagonal $U(1)$. In the dual theory (M) this same expectation value takes us out onto the Coulomb branch. If we integrate out the massive vectors and simply look at QED with N flavors going on the Coulomb branch amounts to giving masses to all flavors. Writing $\mathcal{N}=4$ in $\mathcal{N}=2$ notation, the adjoint chiral multiplet X that is part of the $\mathcal{N}=4$ vector multiplet couples to the chiral multiplets $\tilde{Q}_i$ and Q_i in the hypermultiplet via a superpotential coupling

$$W \sim X \sum_i \tilde{Q}_i Q_i. \tag{6.17}$$

That is giving a vacuum expectation value $\langle X \rangle = X_0$ to the vector multiplet scalar is equivalent to adding mass X_0 to all N matter fields.

The effects of moving out onto this branch, both in the original and in the mirror theory, can once again be best understood from a brane picture as displayed in figure 6.4.

Turning on X_0 to move onto the Higgs branch in the original quiver corresponds to moving the D3 brane off from the NS5 branes into an orthogonal direction (say x_7). This brane picture makes it manifest that the low energy physics is just the 3+1 dimensional $\mathcal{N}=4$ $U(1)$ gauge theory living on the D3 brane. The continuum limit is exactly achieved by taking the limit where the D3 brane gets infinitely removed. The KK modes living on the D3 brane appear as W-bosons from the UV field theory. In the S-dual picture we see the same low energy physics: a single D3 brane far removed (this time in the x_3 direction) from a stack of D5 branes. Since the two pictures are related by type IIB S-duality, which acts as electro-magnetic duality on the D3, the $U(1)$ on the D3 brane this time is the S-dual of the theory we originally deconstructed.

The D-brane realization of the gauge theory also allows us to make one more important point. In addition to the W-bosons we expect from the deconstructed circle, our theory also has BPS vortices. In the string theory construction they correspond to D1 strings connecting the D3-brane to one of the NS5 branes. In the mirror theory they map to F1 fundamental strings connecting the D3 to one of the D5s. The mass of these states is proportional to the length of the string, so it grows linearly with X_0. In the continuum limit, where $X_0 \to \infty$, these states become infinitely heavy and decouple.

To confirm that the lessons the brane picture teach us about the mirror are indeed realized in the field theory we need to confirm that the low energy physics of theory (M) along its Coulomb branch indeed realizes a tower of KK modes. Instead of integrating out the massive vector bosons in (M) to study the spectrum of the massive large N Thirring-like model that is left behind, we find it more convenient to integrate out the N massive matter fields instead. After all, the limit we are taking is the one in which the flavor fields get infinitely heavy. Integrating out the flavor fields in the action (6.16) leaves behind, at low energies, $2N$ vector bosons a_i and b_i with masses induced from the topological BF terms. The effects of the massive flavors is to modify the kinetic terms for the gauge fields they couple to, that is the a_i fields. This is encoded in an effective metric on the Coulomb branch, which fortunately is tightly constrained due to the high supersymmetry in the system and we can directly borrow the results from [113, 111] to read off the answer.

To understand exactly how X_0 is related to the hypermultiplet mass it is easiest to use the FI terms to force the expectation value. ξ is an FI term for the D-term of the background gauge field coupling to the baryon number global symmetry. On the magnetic side this symmetry is mapped to the monopole number of the overall $U(1)$ field

$$\tilde{a} = \frac{1}{N}\sum_i a_i. \tag{6.18}$$

$\tilde{a}$ couples to the background field via a BF term whose supersymmetric completion couples the scalar superpartner $\tilde{x}$ of $\tilde{a}$ to the auxiliary field D of the background multiplet. Together with the FI term this means that the terms in the action containing D simply read $D(\tilde{x}-\xi)/\pi$; so D acts as a Lagrange multiplier that forces $\tilde{x}$ to get expectation value equal to ξ. Or in other words, the hypermultiplets pick up mass $m = \xi$. This is the standard parameter mapping in mirror symmetry: FI terms map to mass terms. So our task is to determine the correction to the gauge couplings of the gauge fields under which the hypermultiplets are charged due to integrating out a massive hypermultiplet of mass $m = \xi = \pi X_0^2$. The hypermultiplets are neutral under the b_i gauge fields, so the latter are unaffected and their gauge coupling remains G. Each a_i has exactly one charged hypermultiplet. The quantum corrections to QED with 1-flavor are indeed well understood [113, 111]. This theory has no Higgs branch (quartic potentials prevent the hypermultiplets from getting expectation values) but has a one (quaternionic) dimensional Coulomb branch. Corrections are usually phrased in terms of the metric on this branch which encodes the kinetic terms in the low energy effective action along the branch. Classically the Coulomb branch is simply $\mathbb{R}^3 \times S^1$, parametrized by the 3 scalars in the vector multiplet together with the dual photon. Quantum mechanically the Coulomb branch picks up a 1-loop exact correction turning it into Taub-NUT [113]. This metric has no singularities, which would signal extra massless particles, in accordance with string theory expectations [111]. But the correction can be interpreted as including a shift in the effective gauge coupling which has been shown in [109] to read (in the normalization of [79] we have been using all along):

$$g_{a_i}^{-2}(X) = g_{cl}^{-2} + \frac{1}{4\pi|\vec{X}-\vec{m}|} \tag{6.19}$$

where $\vec{X}$ is the triplet of the scalars contained in the hypermultiplet and $\vec{m}$ is the triplet of the allowed supersymmetric mass terms. Note that the classical value of the gauge coupling,

g_{cl}, is not G. In the derivation of all-scale mirror symmetry adding a Maxwell term on the electric side with coupling G induced a Thirring term or equivalently a Maxwell term for b_i, but not for a_i. The latter remained infinitely strongly coupled throughout, $g_{cl}^{-2} = 0$. The $U(1)$ gauge couplings of the a_i fields is entirely due to integrating out the hypermultiplets of mass $m = \pi X_0^2$:

$$g_{a_i} = 2\pi X_0. \tag{6.20}$$

With this the mirror action (M') on the Coulomb branch reads

$$S_{\text{M'}} = -\sum_i \frac{1}{4(2\pi X_0)^2} \int f_i^2 + \sum_i S_{BF}[a_i, b_{i+1} - b_i] - \sum_i \frac{1}{4G^2} \int F_i^2 \tag{6.21}$$

where f_i is the field strength of a_i. To calculate the spectrum of massive fluctuations we can rescale $f_i \to f_i' = \frac{2\pi X_0}{G} f_i$ so that all $2N$ gauge fields have the same kinetic term. The newly rescaled fields of course would not be properly quantized, but for the purposes of determining the spectrum of small fluctuations this is irrelevant. Maxwell's equations for the $2N$ gauge fields pick up an effective mass matrix from the BF terms, which can be written in block diagonal form

$$M_M = GX_0 \begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix}. \tag{6.22}$$

0 denotes an N by N block vanishing entries and C is the N by N matrix given by

$$C = \begin{pmatrix} +1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & +1 & -1 & 0 & & 0 \\ 0 & 0 & +1 & -1 & & \vdots \\ \vdots & & 0 & \ddots & & 0 \\ 0 & & & & +1 & -1 \\ -1 & 0 & \cdots & 0 & 0 & +1 \end{pmatrix}. \tag{6.23}$$

Note, in particular, that comparing to the W-boson mass matrix of (6.9) we have

$$C^T C = \frac{M_W^2}{G^2 X_0^2}. \tag{6.24}$$

So the eigenvalues of the magnetic mass matrix M_M indeed are just the square root of the W-boson mass squareds from (6.10), that is

$$m_n^2 = 4G^2 X_0^2 \sin^2 \frac{\pi n}{N} \tag{6.25}$$

where once again n in an integer with $-N/2 \leq n \leq N/2$. Note that in M_M each eigenvalue shows up twice. That is due to the fact that we are really working out the masses of a single polarization of a gauge field, whereas M_W^2 was already fully accounting for the 2 polarizations of the massive gauge boson after "eating" the corresponding scalars. So the spectrum of massive KK modes completely agrees, the mirror Thirring theory indeed grew an extra dimension!

Looking at the massless modes, we see two massless vectors on the mirror side. One of those has all $a_i = \tilde{a}$ with $b_i = 0$ and the other one has all $b_i = \tilde{b}$ with $a_i = 0$. On the electric side we had a massless vector and a massless hyper (the "Higgs boson") combining into an $\mathcal{N} = 8$ vector multiplet. As we alluded to before, a decoupled massless vector can be rewritten as a massless hyper and, indeed, the two sides are completely equivalent. As we pointed out above, the fact that we needed to dualize a single free hyper into a vector is also manifest in the brane picture.

Last but not least let us look at the coupling constants in the magnetic theory. The coupling constant G plays the role of a magnetic coupling in the mirror theory, as expected. The electric coupling is what governs the Maxwell term of the zero mode $\tilde{a}$ from (6.18), which was induced from the massive hypermultiplets. Since each a_i had coupling X_0, we have that the corresponding coupling of this zero mode is given by

$$\tilde{g}_{ZM}^{-2} = N(2\pi X_0)^{-2} \tag{6.26}$$

from which we can obtain the gauge coupling of the deconstructed 3+1 dimensional theory as before:

$$\tilde{g}_4^2 = (2\pi R)\tilde{g}_{ZM}^2 = (Na)\frac{(2\pi)^2 X_0^2}{N} = (2\pi)^2 \frac{X_0}{G} = \frac{(2\pi)^2}{g_4^2}. \tag{6.27}$$

As expected[5] [129], the coupling constant of the dual theory is $(2\pi)^2$ times the inverse of the coupling of the electric theory! We successfully deconstructed S-duality.

6.4 Deconstruction with Abelian bosonization

Let us take above analysis and apply it to a more conjectural setting: non-supersymmetric Abelian dualities. Taking stock of the key ingredients needed in the supersymmetric case, we are motivated to consider the Abelian duality

$$\mathbf{BP'}:\quad \text{WF Scalar} = \text{Fermion} + U(1)_{-\frac{1}{2}}$$

as a starting point. Quivers valid in the IR limit can be constructed in a manner similar to those found in [82]. The philosophy behind deriving the all-scale Base Pairs remains intact. That is, all of the quantities appearing in the process of arriving at the aBP relation used in deconstruction are simple supersymmetric generalizations of terms that can be easily identified in the non-supersymmetric context: The $\mathcal{N} = 4$ $U(1)$ gauge theories nodes with hypermultiplet links of the quiver become Chern-Simons theories coupled by bifundamental Wilson-Fisher (WF) scalars, and the dual picture becomes N flavors of Dirac fermions with flux attachment in the form of a $U(1)_{-\frac{N}{2}}$ Chern-Simons theory. That is, the single node non-supersymmetric version of aBP reads

$$\mathbf{aBP'}:\quad \text{WF} + (U(1) + \text{Maxwell}) = U(1)_{-\frac{1}{2}} + \text{Fermion} + (U(1) + \text{Maxwell}).$$

The main concern is what becomes of the Chern-Simons terms in the quiver theory in the 3+1 dimensional limit. Naïvely, one would guess that they become some form of linearly varying θ-angle [52]. To understand what happens, let us consider the case where all of the scalars acquire the same vacuum expectation value $\langle X_{i,i+1}\rangle = X_0$ such that we again arrive a linear sigma model for the phase variable, $U_{i,i+1}$, i.e. $X_{i,i+1} = X_0 U_{i,i+1}$. The discretized

[5] Note that the g^2 as defined in [129] is twice the coupling squared we have been using here.

scalar theory becomes

$$S_{\text{scalar}} = k^{ij} S_{CS}[b_i; b_j] + \int d^3x \left(-\frac{1}{4G^2} \sum_{i=1}^{N} F_i^2 + \frac{1}{2G^2 a^2} \sum_{i=1}^{N} \left(D_\mu U_{i,i+1}\right)^\dagger \left(D^\mu U_{i,i+1}\right) + \dots \right), \tag{6.28}$$

where we use

$$S_{CS}[A_i, A_j] \equiv \frac{1}{4\pi} \int d^3x \; \epsilon_{\mu\nu\rho} A_i^\mu \partial^\nu A_j^\rho \tag{6.29}$$

to denote both Chern-Simons and BF terms and the coefficients take on a form reminiscent of the balls-and-springs mass matrix we saw earlier

$$k^{ij} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & & 0 \\ 0 & -1 & 2 & -1 & & \vdots \\ \vdots & 0 & -1 & \ddots & & 0 \\ 0 & & & & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{6.30}$$

The form of (6.28) is very similar to the supersymmetric case with the exception of the Chern-Simons terms, which we will now argue have no effect on the continuum deconstructed theory. The discretized Chern-Simons term take the form of a combination of forward and backward hopping terms on the lattice:

$$k^{ij} S_{CS}[b_i; b_j] = \frac{a}{4\pi} \sum_{i=1}^{N} \left(\frac{b_i - b_{i-1}}{a} \right) db_i - \frac{a}{4\pi} \sum_{i=1}^{N} \left(\frac{b_{i+1} - b_i}{a} \right) db_i, \tag{6.31}$$

where periodicity $b_i \equiv b_{i+N}$ has been imposed. What is immediately obvious is that in the continuum limit the two hopping terms become equivalent and cancel one another. Thus despite being important for the functioning of the quiver duality, the Chern-Simons and BF terms do not affect the duality in the limit to a continuum theory in 3+1 dimensions.[6]

[6] One might also worry that the additional Chern-Simons terms in the 2+1 dimensional theory will give

Indeed, if the bifundamental scalars all pick up the same vacuum expectation value, then the dynamical gauge bosons acquire the same mass matrix as in (6.9), and taking limit as in (6.8), we find simply that the scalar side of the duality has deconstructed pure 3+1 dimensional Maxwell theory compactified on an S^1

$$S = -\frac{1}{4g_4^2}\int_{\mathbb{R}^3\times S^1} d^4x\, F_b^2 \tag{6.32}$$

with F_b the 3+1 dimensional field strength. Note that our non-supersymmetric theory no longer has a moduli space, so X_0 is not a parameter we can dial. We expect that adding a large negative mass squared to all WF scalars will force them to pick up a large expectation value, but due to the the strong coupling limit involved we have no control over the order one factors.

The question now is what becomes of the fermionic side of the quiver dual picture in the continuum limit, which also comes with additional Chern-Simons terms,

$$S_{\text{fermion}} = -\sum_i S_f[\psi_i, a_i] - \frac{\delta^{ij}}{2} S_{CS}[a_i; a_j] - \delta^{ij} S_{CS}\left[a_i; b_j - b_{j+1}\right] - \sum_i \frac{1}{4G^2}\int F_i^2. \tag{6.33}$$

On the scalar side, the continuum limit corresponds to a large negative mass squared for the matter fields and, through the identification of mass deformations in the single node context $m_\psi \leftrightarrow -m_X^2$, we expect that continuum limit in the fermionic theory corresponds to opening a large gap in the fermion spectrum. Integrating out heavy fermions in QED$_3$ not only cancels out their respective η-invariants but also gives to one loop [109, 39] the coupling of the 3+1 dimensional theory

$$g_{a_i}^{-2} = g_{cl}^{-2} + \frac{1}{12\pi m_\psi}. \tag{6.34}$$

We note again that clasically $g_{cl}^{-2} \to 0$ as Abelian bosonization is, like BP, a relation in the

competing mass contributions to b_i. Although the Chern-Simons induced masses scale as $G^2 \sim GX_0$ in the continuum limit, the Chern-Simons mass matrix also comes with two factors of k^{ij}. Diagonalizing each factor of k^{ij} gives $\sin^2(\pi n/N) \sim n^2/N^2$ for $n \ll N$. Thus, the Chern-Simons masses are $1/N^2$ suppressed relative to those due to Higgsing. This is consistent with the 3+1 dimensional point of view that effects due to the Chern-Simons terms are suppressed in the continuum limit.

deep IR, and so $g_{a_i} = \sqrt{12\pi m_\psi}$. The gapped theory is then described by

$$S'_{\text{fermion}} = -\sum_i \frac{1}{4g_{a_i}^2} \int f_i^2 - \delta^{ij} S_{CS}\left[a_i; b_j - b_{j+1}\right] - \sum_i \frac{1}{4G^2} \int F_i^2. \tag{6.35}$$

Canonicalizing the kinetic terms, diagonalizing the mass matrix for the gauge fields, and computing the mass spectrum of Kaluza-Klein modes gives

$$m_n^2 = \frac{G^2 g_{a_i}^2}{\pi^2} \sin^2 \frac{\pi n}{N}. \tag{6.36}$$

Analogous to the supersymmetric case, we can postulate $m_\psi \sim X_0^2$.[7] Then, with (6.34), $g_{a_i}^2 \sim X_0^2$ and hence (6.36) is the same as (6.25), up to an $\mathcal{O}(1)$ factor. Similar arguments follow for the non-supersymmetric generalization of (6.27), i.e. $\tilde{g}_4^2 \sim (2\pi)^2 g_4^{-2}$. So while the basic construction seems to work just as in the supersymmetric case, it is difficult to confirm this duality quantitatively. It's not even clear that the spectrum of KK modes survives the strong coupling limit unless one can find a large N argument that suppresses corrections.

6.5 Discussion

Let us take stock of what has been accomplished. We used a well established yet still somewhat conjectural duality between strongly coupled field theories in 2+1 dimensions in order to rederive a somewhat trivial statement about a free theory in 3+1 dimensions. What is this good for? For one, the procedure we followed is a nice additional check of mirror symmetry, in particular its somewhat less well studied all-scale version. But, of course, more interesting is that this demonstration can serve as the existence proof that potentially new dualities in 3+1 dimensions can indeed be derived from established dualities in 2+1 dimensions.

A fairly straightforward generalization of the work we have done here is to study the non-Abelian case in order to establish the less trivial S-duality of $\mathcal{N} = 4$ SYM. The brane construction is identical to the one we used here and so we expect exactly the same pattern of

[7]At least for negative scalar mass squared this is the more basic version of the parameter map in bosonization, where the fermion mass is directly identified with the scalar expectation value.

masses in this case. The new complication is that this time we do not know of a Lagrangian description of the magnetic coupling. Unlike the Abelian $U(1)$ monopole number symmetry, which is visible at all scales and can be coupled to via a BF term, the corresponding non-Abelian symmetries only arise as accidental symmetries in the IR. As long as we take a formal definition of the magnetic coupling as an irrelevant deformation of the IR fixed point, the brane picture demonstrates that the mirror theory will again grow an extra dimensions if we chose the electric theory to deconstruct $SU(K)$ SYM in 3+1 dimensions.

One other obvious generalization would be to include a θ term in the gauge theory; one would hope that it should be possible to find the full $SL(2,\mathbb{Z})$ invariance of the 3+1 dimensional theory and not just S-duality.

More interesting will be to return to the recent progress in our understanding of 3d bosonization dualities. While the original set of dualities were established for theories with single gauge groups [2] the generalization to Abelian [82] and even non-Abelian product gauge groups [69], including those based on quivers, appears to be straightforward. In either case, the 2+1 dimensional dualities typically include Chern-Simons terms which aren't present in the straightforward application of deconstruction and their role needs to be clarified. For the Abelian case we were able to demonstrate that the basic construction carries through, even though the lack of control over the strong coupling regime makes it difficult to make quantitative comparisons in this case. The non-Abelian case presents extra complications. Beyond the problems with the magnetic couplings we already encountered in the supersymmetric case, the flavor bounds inherent in the dualities of [2] make it difficult to derive dualities for some of the most interesting non-Abelian quivers we might want to study. Nevertheless, the fact that in the simplest, most supersymmetric example deconstruction allowed us to derive a well established 3+1 dimensional duality from 2+1 dimensional mirror symmetry makes us hopeful that progress along these lines can be made.

Appendix A

REVIEW OF CHERN-SIMONS THEORIES

In this appendix we collect information about Chern-Simons theories which will be useful throughout the main text. Much of the material here follows Refs. [46, 121] and the reader can refer to said references for a significantly more detailed discussion of such theories.

A Chern-Simons term in Minkowski space[1] is defined to be

$$\mathcal{L}_{\rm CS} = \frac{k}{4\pi}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho = \frac{k}{4\pi}AdA\,, \tag{A.1}$$

where in the second equality we have written the term in form notation, which is used throughout the body of this work. For now, the gauge field A is Abelian and can either be a dynamical or background gauge field. The coefficient k is referred to as the "level" of the Chern-Simons term. Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$ a Chern-Simons term changes by a total derivative

$$\delta\mathcal{L}_{\rm CS} = \frac{k}{4\pi}\partial_\mu\left(\Lambda\epsilon^{\mu\nu\rho}\partial_\nu A_\rho\right)\,. \tag{A.2}$$

Thus, so long as we can neglect boundary terms, then the action with a Chern-Simons term is gauge invariant.

A Chern-Simons term can be defined in any dimension, but in $2+1$-dimensions they are on special footing because the operator is dimension 3 and is thus relevant. Since the Maxwell term in $2+1$-dimensions is irrelevant, Chern-Simons terms are the most natural Lorentz- and gauge-invariant term that is quadratic and local in the gauge fields.

The level of the Chern-Simons theories must be quantized, i.e. $k \in \mathbb{Z}$. This is easiest to see in the non-Abelian case that we discuss below. The argument can be generalized to

[1] In the body of this text we will work in Euclidean space. This amounts to including an extra factor of $-i$ with every Chern-Simons term in a Lagrangian.

Abelian Chern-Simons terms with a little more effort, see Ref. [121].

If we introduce a source term to (A.1), i.e. $\mathcal{L}_{\text{CS}} \to \mathcal{L}_{\text{CS}} - A_\mu J^\mu$, the equations of motion are

$$\frac{k}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} = J^\mu\,. \tag{A.3}$$

Using the usual definitions of electric and magnetic fields, this reduces to

$$\rho = \frac{k}{2\pi}B\,, \qquad J^i = \frac{k}{2\pi}\epsilon^{i,j}E_j\,, \tag{A.4}$$

with $J^\mu = (\rho, \vec{J})$. The first of these equations states that wherever there is charge there must be nonzero magnetic field, inversely proportional to the Chern-Simons level. The latter equation simply ensures the former relation does not change under time evolution. For this reason, Chern-Simons terms are said to exhibit “flux attachment”.

A very important consequence of flux attachment comes into play via the Aharonov-Bohm effect. Consider a particle of charge 1 moving along some contour C around an identical particle in the presence of a level-k Chern-Simons term. Due to the flux of each particle, the nonrelativistic wavefunction picks up an Aharonov-Bohm phase,

$$\exp\left(i\oint_C d\vec{x}\cdot\vec{A}\right) = \exp\left(\frac{2\pi i}{k}\right)\,. \tag{A.5}$$

If we were, say, considering with the statistics of such particles under interchange, we can take C to be a half-circle, in which case the additional phase due to flux attachment is $\Delta\theta_{\text{flux attachment}} = \pi/k$.

For the case of either $k = \pm1$, the additional phase is the same one would pick up from interchanging two fermions. If we call this exchange operation $\mathcal{O}$, we then have the schematic relation

$$\mathcal{O}\left|\text{boson} + \text{flux}\right\rangle = \mathcal{O}\left|\text{fermion}\right\rangle\,, \tag{A.6}$$

$$\mathcal{O}\left|\text{fermion} + \text{flux}\right\rangle = \mathcal{O}\left|\text{boson}\right\rangle\,. \tag{A.7}$$

Thus, the statistics of bosons and fermions can be interchanged in the presence of a Chern-Simons term.

Of course, there is no reason why we need to consider a theory of gauge bosons with only a Chern-Simons term. A Chern-Simons term can be combined with a Maxwell term, in which case the Lagrangian is

$$\mathcal{L}_{\mathrm{MCS}} = -\frac{1}{4G^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho \,. \tag{A.8}$$

The equation of motion is now given by

$$\partial_\mu F^{\mu\nu} + \frac{kG^2}{4\pi}\epsilon^{\nu\mu\rho}F_{\mu\rho} = 0 \,. \tag{A.9}$$

It is straightforward to show this expression describes a single (transverse) degree of freedom with mass $m_{\mathrm{MCS}} = kG^2/2\pi$ [46]. Thus, in the IR limit where $G^2 \to \infty$, the gauge degrees of freedom are infinitely massive and do not propagate.

The generalization of Chern-Simons terms no non-Abelian gauge fields is given by

$$\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\epsilon_{\mu\nu\rho}\left(A^\mu\partial^\nu A^\rho - i\frac{2}{3}A^\mu A^\nu A^\rho\right) = \frac{k}{4\pi}\left(AdA - i\frac{2}{3}A^3\right) \,, \tag{A.10}$$

where once more in the second equality we have adopted form notation. One can see that if A is Abelian, the second term vanishes. Much of the results for the non-Abelian Chern-Simons terms are directly analogous to what was found above for their Abelian counterparts. This includes the flux attachment condition we found above.

Under a gauge transformation, $A_\mu \to g^{-1}A_\mu g + g^{-1}\partial_\mu g$, the Chern-Simons term transforms as [46]

$$\mathcal{L}_{\mathrm{CS}} \to \mathcal{L}_{\mathrm{CS}} + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\partial_\mu \mathrm{tr}\left(\partial_\nu g g^{-1}A_\rho\right) - \frac{k}{12\pi}\epsilon^{\mu\nu\rho}\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g\right) . \tag{A.11}$$

Similar to the Abelian case, the first term vanishes when we can neglect the boundary. The second term is a well-known topological invariant known as the winding number density of the group element g, defined as

$$w(g) \equiv \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho}\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g\right) \,. \tag{A.12}$$

It can be shown that with proper boundary conditions $w(g)$ is integer valued [46]. As such, for a transformation with winding number N, we have under a gauge transformation

$$S_{\text{CS}} \to S_{\text{CS}} - 2\pi k N, \tag{A.13}$$

which ensures the amplitude in the path integral remains unchanged under gauge transformations so long as $k \in \mathbb{Z}$. Thus we have arrived at our quantization condition.

A slight generalization of the Chern-Simons terms is one for which the two gauge fields are not the same, e.g.

$$\mathcal{L}_{\text{BF}} = \frac{k}{2\pi}\epsilon_{\mu\nu\rho}B^{\mu}\partial^{\nu}A^{\rho} = \frac{k}{2\pi}BdA. \tag{A.14}$$

If one rewrites $F = dA$, one can see why this term is often referred to as a “BF term”, which is the vocabulary we adopt in the main text.

Appendix B

3D BOSONIZATION DUALITY SUPPLEMENTARY

Alternatively, one can think of the Wilson-Fisher scalar by introducing an auxiliary scalar (Hubbard-Stratonovich) field, σ, such that

$$S_{WF}[\varphi,\,\sigma,\,B]=\int d^3x\,\left(|(\partial_\mu-iB_\mu)\varphi|^2-\sigma|\varphi|^2+\frac{\sigma^2}{2\alpha}\right).\tag{B.1}$$

Integrating out σ produces (3.7a). Treating σ as a background field, it functions as a mass-term source. Relating the operator insertion sourced by σ through either of the dualities yields the map: $\sigma\leftrightarrow-\bar{\psi}\psi$. The way that we will interpret this map for mass deformed theories is that the scalar and fermion mass terms are mapped into one another under the duality as

$$\pm\,m_\varphi^2|\varphi|^2\qquad\leftrightarrow\qquad\mp m_\psi\bar{\psi}\psi.\tag{B.2}$$

Consistency of the dualities (3.5) and (3.6) for positive and negative mass deformations will be a guiding principle in what follows.

B.1 Spin Considerations

A large portion of the subtleties involved in extending these dualities to include manifolds with boundaries comes from the differences between spin and spin$_c$ valued $U(1)$ connections. We will now take a brief detour to review some of these concepts. The discussion here will be largely heuristic, while more mathematically oriented treatments can be found in [114, 112, 94].

Consider an arbitrary manifold, $\mathcal{M}$, and turn on a background gauge field, i.e. a $U(1)$ connection $\mathcal{A}$. Suppose that we want to ask questions about the dynamics of a system of fermions on $\mathcal{M}$ that couple to $\mathcal{A}$. We first must ensure that it is sensible to define the Dirac operator on $\mathcal{M}$. This requires us to define an appropriate connection, ${\omega_\mu}^{ab}$, that consistently parallel transports a local Lorentz frame over all of $\mathcal{M}$, allowing us to meaningfully talk about placing a spinor anywhere on $\mathcal{M}$. An $\mathcal{M}$ that admits a global definition of ${\omega_\mu}^{ab}$ is called a spin manifold. On a spin manifold the full covariant Dirac operator is given by

$$D_\mu = \partial_\mu + \frac{1}{4}{\omega_\mu}^{ab}\gamma_a\gamma_b + i\mathcal{A}_\mu. \tag{B.3}$$

However, certain topological constraints imply that not every manifold admits a global definition of ${\omega_\mu}^{ab}$. The topological obstruction to defining ω everywhere on $\mathcal{M}$ can be compensated by a non-standard choice for quantization of $\mathcal{A}$:

$$\frac{1}{2\pi}\int_\Sigma d\mathcal{A} \in 2\mathbb{Z}, \tag{B.4}$$

where Σ is an oriented co-dimension 2 surface in $\mathcal{M}$. Within this quantization scheme, the covariant Dirac operator

$$D_\mu^{(n)} = \partial_\mu + \frac{1}{4}{\omega_\mu}^{ab}\gamma_a\gamma_b + in\mathcal{A}_\mu \tag{B.5}$$

is well defined for odd n. An $\mathcal{M}$ whose topological obstruction to a global definition of the Dirac operator is compensated by the unusual quantization of $\mathcal{A}$ is called a spin$_c$ manifold, and the $\mathcal{A}$ obeying (B.4) will be referred to as a spin$_c$ valued connection.

Further, we can impose (B.4) even if the manifold admits a global definition of ${\omega_\mu}^{ab}$, which implies that spin and spin$_c$ valued connections can be defined spin manifolds. Thus since $\mathbb{R}^{2,1}$ and $\mathbb{R}^{2,1}_+$ are spin manifolds, the distinction that we must make is at the level of fermions being coupled to either a spin or spin$_c$ valued connection.

The restriction to odd n gives rise to the spin-charge relation of condensed matter physics; particles with integer spin have even charge and half-integer spin have odd charge. While this does not appear to be a fundamental law of nature, it is believed to be valid for systems made up of protons, electrons and other charged (quasi-)particles. This motivates the distinction between spin and spin$_c$ valued connections in our notation and further implies that our background field appearing in (3.5) is spin$_c$ [114].

As an example of how this distinction can enter into the seed dualities, consider pure a $U(1)_1$ theory with spin valued connection, b, on $\mathcal{M} = T^3$.[1] Further, consider that $\mathcal{M}$ is the boundary of a four-dimensional manifold X. Upon quantization, we find that there is just one state such that the path integral is

$$\mathcal{Z} = \int \mathcal{D}b \; e^{iS_{CS}[b]} = e^{-i\Omega} \tag{B.6}$$

where Ω denotes "framing anomaly"

$$\Omega = 2\int_{\partial X} \mathrm{CS}_{\mathrm{grav}} = \frac{1}{96\pi}\int_X \mathcal{R}\wedge\mathcal{R}, \tag{B.7}$$

and CS_{g} is the gravitational Chern-Simons term.

If the same $U(1)$ theory with dynamical gauge field b is defined with respect to a background spin$_c$ structure with connection A on $\mathcal{M}$, we must couple our dynamical $U(1)$ field b to the background connection A through a BF term. As with the previous case, there is only one state, and the theory is uniquely determined. The difference is that the partition function evaluates to

$$\mathcal{Z}[A] = \int \mathcal{D}b \; e^{iS_{CS}[b]+iS_{BF}[b,A]} = e^{-i\Omega - i\frac{1}{4\pi}\int AdA}. \tag{B.8}$$

Accounting for these extra terms will prove to be a useful guiding principle for keeping track of edge modes across the duality.

[1]This discussion follows Appendix B of [112] where more details can be found.

Appendix C

FLAVOR-VIOLATED MASTER DUALITY DETAILS

In this appendix we work out the Lagrangians of the flavor extended master duality coupled to background fields, which provide important consistency checks for the phase diagram of the extended master duality proposed in section 2.2. We mostly analyze the case of $N > N_s$ (Fig. 2.3) and briefly comment for the case of $N = N_s$ (Fig. 2.4). Finally we discuss gauging of flavor symmetry and description of flavor-violated quivers which are used extensively in the section 4.3.

SU side

First, we discuss the SU side of the duality. The Lagrangian corresponding to the SU side of flavor-violated master duality (2.6) is given by

$$\begin{aligned}\mathcal{L}_{SU} = & \left|D_{b'+B+\tilde{A}_1+\tilde{A}_2}\phi\right|^2 + i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1}\psi + \mathcal{L}_{\text{int}} - i\left[\frac{N_f-k}{4\pi}\text{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right)\right] \\ & - i\left[\frac{N}{4\pi}\text{Tr}_{N_f}\left(CdC - i\frac{2}{3}C^3\right) + \frac{N(N_f-k)}{4\pi}\tilde{A}_1 d\tilde{A}_1 + 2NN_f\text{CS}_{\text{grav}}\right] \qquad \text{(C.1)}\end{aligned}$$

which we denote using

$$SU\left(N\right)_{-k+N_f/2} \times \left[SU\left(N_f\right)_{N/2} \times SU\left(N_s\right)_0 \times U\left(1\right)_{N(N_f-k)/2} \times U\left(1\right)_0\right] + N_f\psi + N_s\phi \quad \text{(C.2)}$$

where again the terms in the $[\cdots]$ are global symmetries. In the mass deformed regime where $|m_\psi| \gg m_*$, phases are straightforwardly obtained from SU side as follows:[1]

$$\text{(I)}: \qquad SU(N)_{-k+N_f} \times \left[SU\left(N_f\right)_N \times SU\left(N_s\right)_0 \times J_I\right] \tag{C.3a}$$

$$\text{(II)}: \qquad SU(N-N_s)_{-k+N_f} \times \left[SU\left(N_f\right)_N \times SU\left(N_s\right)_{-k+N_f} \times J_{II}\right] \tag{C.3b}$$

$$\text{(III)}: \qquad SU(N-N_s)_{-k} \times \left[SU\left(N_f\right)_0 \times SU\left(N_s\right)_{-k} \times J_{III}\right] \tag{C.3c}$$

$$\text{(IV)}: \qquad SU(N)_{-k} \times \left[SU\left(N_f\right)_0 \times SU\left(N_s\right)_0 \times J_{IV}\right]. \tag{C.3d}$$

where

$$J_i \equiv J_i^{ab} \frac{1}{4\pi} \tilde{A}_a d\tilde{A}_b \tag{C.4}$$

is the Chern-Simons term for the Abelian background gauge fields in the ith phase in for $a, b = 1, 2$. For the asymptotic mass phases, the J_i^{ab} are

$$J_{\text{I}}^{ab} = \begin{pmatrix} N(N_f - k) & 0 \\ 0 & 0 \end{pmatrix} \tag{C.5a}$$

$$J_{\text{II}}^{ab} = \frac{N(N_f - k)}{N - N_s} \begin{pmatrix} N & N_s \\ N_s & N_s \end{pmatrix} \tag{C.5b}$$

$$J_{\text{III}}^{ab} = \frac{-Nk}{N - N_s} \begin{pmatrix} N & N_s \\ N_s & N_s \end{pmatrix} \tag{C.5c}$$

$$J_{\text{IV}}^{ab} = \begin{pmatrix} -Nk & 0 \\ 0 & 0 \end{pmatrix}. \tag{C.5d}$$

For the region of small fermion mass $|m_\psi| < m_*$, corresponding to phases V to VIII, we are in the quantum regime and flavor symmetry is expected to be spontaneously broken by the non-perturbative fermion condensate,

$$U(N_f) \to U(N_f - k) \times U(k). \tag{C.6}$$

[1]We are suppressing gravitational Chern-Simons terms for brevity. They are straightforward to restore using level-rank duality.

We now move on to the U side of master duality, which describes quantum regime of SU side as well as the semiclassical regimes

U side

The Lagrangians for the U side of the flavor-violated master duality can be fixed by demanding that they yield the same TFTs and background Chern-Simons terms as the SU case for phases I to IV. As mentioned in the main text, this is achieved via two different scalar theories, one of which is identical to the flavor-bounded case. Explicitly, the Lagrangians corresponding to (2.6) are given by

$$\begin{aligned}
\mathcal{L}_U^{m_\psi<0} &= \left|D_{c+C}\Phi_1\right|^2 + i\bar{\Psi}_1 \not{D}_{c+B+\tilde{A}_2}\Psi_1 + \mathcal{L}_{\text{int}}^{\prime(1)} \\
&\quad - i\left[\frac{N}{4\pi}\text{Tr}_k\left(cdc - i\frac{2}{3}c^3\right) - \frac{N}{2\pi}\text{Tr}_k(c)d\tilde{A}_1 + 2Nk\text{CS}_{\text{grav}}\right] \qquad \text{(C.7a)} \\
\mathcal{L}_U^{m_\psi>0} &= \left|D_{c+C}\Phi_2\right|^2 + i\bar{\Psi}_2 \not{D}_{c+B-\tilde{A}_2}\Psi_2 + \mathcal{L}_{\text{int}}^{\prime(2)} \\
&\quad - i\left[\frac{-N+N_s}{4\pi}\text{Tr}_{N_F-k}\left(cdc - i\frac{2}{3}c^3\right) + 2\left(Nk - N_s(N_f-k)\right)\text{CS}_{\text{grav}}\right] \\
&\quad - i\left[\frac{N}{4\pi}\text{Tr}_{N_f}\left(CdC - i\frac{2}{3}C^3\right) + \frac{N_f-k}{4\pi}\text{Tr}_{N_s}\left(BdB - i\frac{2}{3}B^3\right)\right] \\
&\quad - i\left[-\frac{N}{2\pi}\text{Tr}_{N_f-k}\left(c\right)d\tilde{A}_1 - \frac{N_s}{2\pi}\text{Tr}_{N_f-k}(c)d\tilde{A}_2 + \frac{N_s(N_f-k)}{4\pi}\tilde{A}_2 d\tilde{A}_2\right]. \qquad \text{(C.7b)}
\end{aligned}$$

In general, dynamical gauge groups for these two theories are distinct, so the number of degrees of freedom in the matter fields is different. The reason the $m_\psi > 0$ Lagrangian looks significantly more complicated is largely a result of our convention for η-invariant terms (see footnote 2). Schematically, these theories can be denoted by

$$\begin{aligned}
m_\psi < 0: \qquad & U(k)_{N-N_s/2} \times \left[SU(N_f)_0 \times SU(N_s)_{-k/2} \times U(1)_{NN_f/2} \times U(1)_{-NN_s/2}\right] \\
& + N_s\Psi_1 + N_f\Phi_1 \qquad \text{(C.8a)} \\
m_\psi > 0: \qquad & U(N_f-k)_{-N+N_s/2} \times \left[SU(N_f)_N \times SU(N_s)_{(-k+N_f)/2} \times U(1)_0 \times U(1)_{NN_s/2}\right] \\
& + N_s\Psi_2 + N_f\Phi_2. \qquad \text{(C.8b)}
\end{aligned}$$

Reassuringly, the procedure laid out in [68] gives background terms for the asymptotic region which match those from the SU side. The next complication comes in determining the $U(1)$ levels in the quantum phase. On the U side, the monopole current couples to $\tilde{A}_1$ which couples to the Kahler-form w

$$\mathcal{L}_U \supset \frac{N}{2\pi} w d\tilde{A}_1. \tag{C.9}$$

We can then integrate out w in a procedure analogous to the semiclassical regions discussed above.

As mentioned in the main text, moving into the quantum regions we find an additional $U(1)$ background symmetry which was a subgroup of the original $SU(N_f)$ global symmetry. Being careful with said breaking pattern results in the TFTs described in (2.12).

$N = N_s$

We remark that all the above analysis of $N > N_s$ case is directly extended to $N = N_s$ with two important differences.

First, in the phases II and III there is a spontaneous breaking of the diagonal $\tilde{A}_1 + \tilde{A}_2$ background field. This is straightforward to see on the SU side where the dynamical gauge symmetry is completely Higgs when $N = N_s$. To see this on the U side, one can move to a dual photon description to make the shift symmetry explicit. The breaking signals the naive divergence of background terms proportional to $\frac{N}{4\pi}(\tilde{A}_1 + \tilde{A}_2)d\tilde{(}\tilde{A}_1 + \tilde{A}_2)$ in (C.5) for $N = N_s$.

Second, as it's clear from the figure 2.4, phases VII and VIII have finite regions in contrast to the $N > N_s$ case. Thus we have extra critical line II-III with $U(1)_0 + NN_f\ \tilde{\psi}_s$, and it is reassuringly consistent with all the background terms calculated in phases II and III.

Quiver description

Since the flavor-violated SU theory is master dual to two different U theories which describe certain patches of the SU phase diagram respectively, it will become useful to adopt a notation similar to (4.63) to denote the appropriate theory located in the common region

$m_\psi = 0$ and $m_\phi^2 = 0$ of phase space, corresponding to a Grassmannian manifold. One can choose either of the U theories to describe phases V and VI, which contain the SU theory. Since the two scalar theories are uniquely related to one another, later we will adopt a method for converting from one scalar theory to the other. As such, one should still be able to start with the U theory and determine all mass deformations. Choosing the $m_\psi < 0$ theory, we have a duality between the two theories

$$SU(N)_{-k+N_f/2} \times \left[SU(N_f)_{N_f/2} \times SU(N_s)_0\right] \quad \leftrightarrow \quad U(k)_{N-N_s/2} \times \left[SU(N_f)_0 \times SU(N_s)_{-k/2}\right]. \tag{C.10}$$

As with the flavorless case, it will be useful to have a version of this theory where the SU side has its fermion flavor symmetry and a $U(1)$ global symmetry combined into a single $U(N_f)$ symmetry. This is achieved by grouping the C and $\tilde{A}_1$ fields together(we do not include the additional $\tilde{B}$ symmetry to make contact with 4.63 in addition to include $N = N_s$). The above duality becomes

$$SU(N)_{-k+N_f/2} \times \left[U(N_f)_{N/2} \times SU(N_s)_0\right] \quad \leftrightarrow \quad U(k)_{N-N_s/2} \times \left[U(N_f)_0 \times SU(N_s)_{-k/2}\right]. \tag{C.11}$$

Immediately note that this duality is identical to that of (4.65).

Alternatively, if we had chosen the $m_\psi > 0$ theory on the U side, we would have arrive at something a little different, namely,

$$SU(N)_{-k+N_f/2} \times \left[U(N_f)_{N/2} \times SU(N_s)_0\right] \quad \leftrightarrow \quad U(N_f - k)_{-N+N_s/2} \times \left[U(N_f)_N \times SU(N_s)_{(-k+N_f)/2}\right]. \tag{C.12}$$

Appendix D

BOSONIC QUIVER SUPPLEMENTARY

Here we provide further details of our construction of the bosonic non-Abelian linear quivers.

D.1 Other Forms of the Duality

The second of Aharony's dualities is given by taking the $N_s = 0$ of the master duality, (1.28), which gives

$$\begin{aligned}
\mathcal{L}_{SU} &= i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1}\psi - i\left[\frac{N_f-k}{4\pi}\mathrm{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right) + \frac{N}{4\pi}\mathrm{Tr}_{N_f}\left(CdC - i\frac{2}{3}C^3\right)\right] \\
&\quad - i\left[\frac{N\left(N_f-k\right)}{4\pi}\tilde{A}_1 d\tilde{A}_1\right], && \text{(D.1a)} \\
\mathcal{L}_U &= \left|D_{c+C}\phi\right|^2 - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right) - \frac{N}{2\pi}\mathrm{Tr}_k\left(c\right)d\tilde{A}_1\right]. && \text{(D.1b)}
\end{aligned}$$

Performing the using $\tilde{c} \to \tilde{c} + \tilde{A}_1$ shift, canceling common factors on either side of the duality, and defining $E_\mu \equiv C_\mu + \tilde{A}_{1\mu}$, we end up with

$$\begin{aligned}
\mathcal{L}_{SU} &= i\bar{\psi}\not{D}_{b'+E}\psi - i\left[\frac{N_f-k}{4\pi}\mathrm{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right) + \frac{N}{4\pi}\mathrm{Tr}_{N_f}\left(EdE - i\frac{2}{3}E^3\right)\right] && \text{(D.2a)} \\
\mathcal{L}_U &= \left|D_{c+E}\phi\right|^2 - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right)\right]. && \text{(D.2b)}
\end{aligned}$$

This yields the duality (4.14) which we use in going from theory D to theory E. Note that in promoting $\tilde{A}_1$ into a dynamical field, we again introduce a new global symmetry. We will call the background gauge field associated with said symmetry $\tilde{B}_1$.

Returning to the master duality, in the main text we could have combined the $U(1)$ and $SU(N_f)$ global symmetries into the definition of $E_\mu = C_\mu + \tilde{A}_{1\mu}\mathbb{1}_{N_f}$. For the purposes of deriving the non-Abelian linear quivers, this was not necessary. For completeness, we show the form of the duality here, because it gives the explicit master duality with all background fields in its most succinct form. It is also convenient to define the $U(N_s)$ gauge field $H_\mu \equiv B_\mu + \tilde{A}_{1\mu} + \tilde{A}_{2\mu}$. This leaves us with a duality of the form

$$\begin{aligned}\mathcal{L}_{SU} &= |D_{b'+H}\phi|^2 + i\bar{\psi}\not{D}_{b'+E}\psi + \mathcal{L}_{\text{int}} - i\left[\frac{N_f-k}{4\pi}\text{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right)\right] \\ &\quad - i\left[\frac{N}{4\pi}\text{Tr}_{N_f}\left(EdE - i\frac{2}{3}E^3\right)\right], \end{aligned} \tag{D.3a}$$

$$\mathcal{L}_U = |D_{c+E}\Phi|^2 + i\bar{\Psi}\not{D}_{c+H}\Psi + \mathcal{L}'_{\text{int}} - i\left[\frac{N}{4\pi}\text{Tr}_k\left(cdc - i\frac{2}{3}c^3\right)\right]. \tag{D.3b}$$

This makes the $U(N_s) \times U(N_f)$ global symmetry explicit.

D.2 Global Symmetries

Here we discuss the matching of the global symmetries across the dualities in more detail.

In our four node example of Sec. 4.1.2, when stepping from theory A to B and subsequently from theory B to C, there is an implicit matching that occurs between the two equivalent Lagrangians we call theory B. To follow the global symmetries all the way through from theory A to E, it is necessary to look at this matching more carefully. This implicit matching also occurs for every intermediate theory as well (i.e. theories C and D for the four node case). We will call these two equivalent descriptions theory B' and B" and use a similar notation to describing the matching of C and D as well.

To start, let us explicitly consider the matching of theory B. Specifically, theory B' is what we get from using (4.13) with theory A. Meanwhile, theory B" is what we would like to apply the master duality in the form of (4.17) to get out what we now call theory C'. When matching these theories to one another, what was the color gauge symmetry of theory B' gets mapped to the (promoted) $U(k_3)$ flavor symmetry of theory B". Meanwhile, the flavor symmetry of theory B' becomes the color symmetry of theory B".

It will be helpful to consider in closer detail how we are matching all the gauge fields to which the fermions couple. The fermion couplings for the two theories are given by

$$\text{(B')}: \qquad i\bar{\Psi}\not{D}_{c+B+\tilde{A}_1}\Psi \tag{D.4}$$

$$\text{(B")}: \qquad i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1-\tilde{A}_2}\psi. \tag{D.5}$$

With a slight abuse of notation, the dynamical/background gauge fields of theory B' and B" denoted above are completely distinct and must be matched. The matching of the gauge fields associated with gauge and global symmetries are shown in Table D.1. Note the $\tilde{A}_1$ field belonging to Ψ is matched to $\tilde{B}_2$ field of the subsequent node.

This careful matching allows us to focus on how the global $U(1)$ symmetry gets transferred through the dualities. In the duality relating theory A and B', the $U(1)$ global symmetry is associated with $\tilde{A}_1$. For theory A this shows up as a baryon-number symmetry and in theory B' it appears as monopole-like symmetry which couples to the U field flux. From Table D.1, we identify this symmetry with the $U(1)$ monopole symmetry of theory B", where the associated background gauge field is $\tilde{B}_2$. Recall, this is the new global $U(1)$ symmetry which couples to the newly gauged $\tilde{A}_2$ field associated with the U symmetry of the first node. Since this new monopole symmetry is the same on both sides of the duality relating B" and C', when we ultimately arrive at theory C' we have a monopole-like symmetry which still couples to the newly dynamical $\tilde{A}_2$. From theory C' onward, the nodes and links associated with such a symmetry are untouched. Thus the $U(1)$ global symmetry which coupled to the $Y_{1,2}$ bifundamental in theory A becomes a monopole symmetry coupled to the Abelian part of the first U node in theory E.

Similarly matching must occur for theory C. Fortunately, we don't need to work very hard because the matching between C' and C" is identical to what occurs above for B' and B" and is schematically shown in Fig. D.1. The additional global symmetry associated with $\tilde{A}_1$ is identified with the global $U(1)$ symmetry on the next node.

We can again follow a global symmetry through from theory A to E for any of the global $U(1)$ symmetries which couple to bifundamentals higher up the quiver. We find results identical to those of the first node/link above: each $U(1)$ global symmetry which couples to

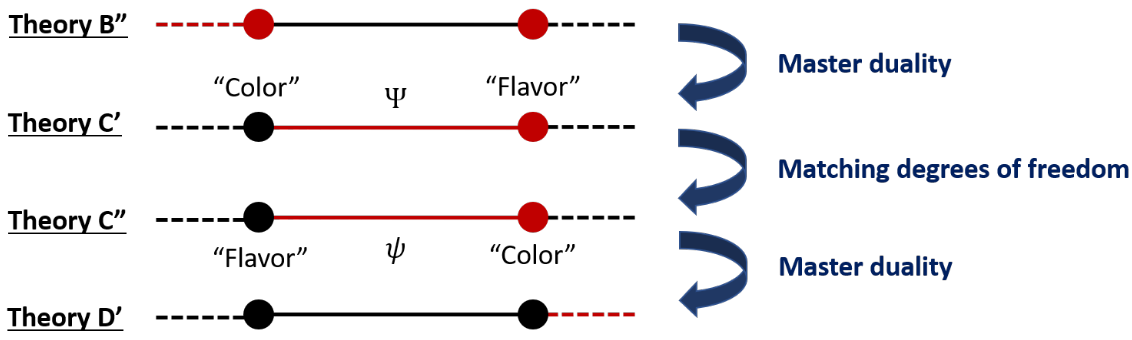

Figure D.1: How the matching occurs on a generic internal link, which for the purposes of concreteness we have labeled C' and C".

the $Y_{i,i+1}$ bifundamental becomes a $U(1)$ global symmetry associated with monopole number for the ith node.

Finally, consider how the matching occurs for the last node. In going from theory D" to theory E in the example we considered in the main text, we used (4.14). Since for this duality we need to promote the entire $U(k_2)$ flavor symmetry to dynamical, we once more acquired an additional global symmetry which couples to the newly promoted $\tilde{A}_1$ whose associated background field we call $\tilde{B}_1$. The matching of symmetries is slightly different and is shown in Table D.1. Since Aharony's dualities only have one $U(1)$ global symmetry, there is no monopole-like symmetry associated with the right-most node.

We should also point out a special feature of the global symmetries that occurs for certain mass deformed phases. Note that the $\tilde{A}_1$ coupling of (1.28b) only couples to the unbroken part of the dynamical gauge field c. That is, when the U gauge group is broken down to say $U(k_1 - k_2)$, the coupling changes from $\mathrm{Tr}_{k_1}(c)dA_1 \to \mathrm{Tr}_{k_1-k_2}(c)dA_1$. This is important because when we are in certain mass deformed phases, we must be careful what gauge components are coupled to the $\tilde{A}_1$ charge. Of particular concern in the main text is whether or not vortices couple to certain global symmetries. Since certain finite mass vortices are charged under the broken part of certain gauge fields, they will not couple to particular

$\bullet$	B' Side	B" Side	C' Side	C" Side	D' Side	D" Side
U Node	c	$C - \tilde{A}_2$	c	$C - \tilde{A}_2$	c	$C + \tilde{A}_1$
SU Node	B	b'	B	b'	B	b'
$U(1)_m$ Global Symmetry	$\tilde{A}_1$	$\tilde{B}_2$	$\tilde{A}_1$	$\tilde{B}_2$	$\tilde{A}_1$	$\tilde{B}_1$

Table D.1: Matching of the intermediate theories. Note the matching of theory B' and B" generalizes for any internal matching, except for the very last. Here, $\tilde{B}_2$ is the background gauge field associated with the new global symmetry we get from gauging $\tilde{A}_2$.

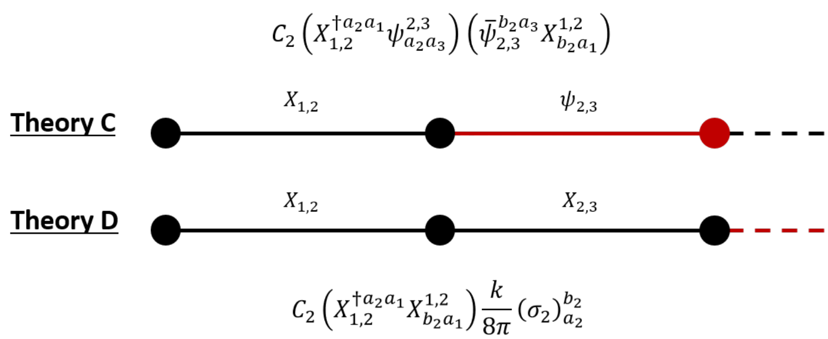

Figure D.2: Example case of properly dualizing interaction terms. This is for the second node of the four node case considered in the main text in Sec. 4.1.2 and also pictures in Figure 4.3.

global symmetries and this will be important for matching excitations.

D.3 Bosonizing Interaction Terms

In this appendix, our goal is to justify the proper mapping of the interaction term present in the master duality. We take as an example what occurs on the second node of our four node example in Sec. 4.1.2, which has a $U(K_2)$ gauge symmetry on it. Specifically, we look at how the interaction term changes when moving from Theory C to Theory D (see Fig. D.2). With the proper transformation in hand, we will then generalize to interaction terms on the

SU and U sides of the duality.

The term from the master duality we would like to apply the duality toward is

$$C_2^{(C)}\left(\bar{\psi}_{2,3}^{a_2a_3}X_{a_2a_1}^{1,2}\right)\left(X_{1,2}^{\dagger b_2a_1}\psi_{b_2a_3}^{2,3}\right). \tag{D.6}$$

Although we commit to a magnitude for $C_2^{(C)}$ in the main text, to keep this appendix general we only assume $C_2^{(C)} < 0$ in what follows. From the master duality, the purpose of this interaction term is that when the $X_{1,2}$ field obtains a vacuum expectation value this should cause a subset of the fermions to get a mass [25, 68].

One might guess the proper transformation of the term between $\psi_{2,3}$ and $X_{2,3}$ from theory C to theory D is simply a generalization of the $m_\psi\bar{\psi}\psi \leftrightarrow -m_X^2\left|X\right|^2$ mass identification. Hence a naive generalization and the shorthand used in the main text is

$$C_2^{(C)}\left(\bar{\psi}_{2,3}^{a_2a_3}X_{a_2a_1}^{1,2}\right)\left(X_{1,2}^{\dagger b_2a_1}\psi_{b_2a_3}^{2,3}\right) \quad \leftrightarrow \quad -C_2^{(D)}\left(X_{2,3}^{a_2a_3}X_{a_2a_1}^{1,2}\right)\left(X_{1,2}^{\dagger b_2a_1}X_{b_2a_3}^{2,3}\right). \tag{D.7}$$

This term has the correct behavior when $X_{1,2}$ acquires a vacuum expectation value. Namely, it gives $X_{2,3}$ a color-breaking vacuum expectation value. Unfortunately this suffers from the fact that the reverse procedure can happen as well. That is, when $X_{2,3}$ acquires a vacuum expectation, this gives a mass to $X_{1,2}$. This is because $X_{1,2}$ and $X_{2,3}$ enter on symmetric footing in (D.7), so it is difficult to see where the desired unidirectional behavior that was present in (D.6) will come from. This means the TFT deformations do not match, so we must look for a better generalization.

To do so, it will be helpful to go back to the large N and k studies where the duality map between operators is better established. We will assume such mappings continue to hold for finite values of N and k, at least up to order one factors, which will not affect the results of our analysis.

Ultimately we'd like to find the true bosonic dual of $\bar{\psi}_{2,3}^{a_2a_3}\psi_{b_2a_3}^{2,3}$ with a_2 and b_2 promoted flavor indices. To do so, we'll look for the bosonic dual of $\bar{\psi}\psi$ and assume our results generalize to arbitrary flavor indices. We follow ref. [96] which represents the most general fermion Lagrangian describing the fixed point as,[1]

$$\begin{aligned}\mathcal{L}_F = i\bar{\psi}\not{D}_{b'}\psi - i\left[\frac{N_f-k}{4\pi}\mathrm{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right)\right] \\ + \sigma_F\left(\bar{\psi}\psi - y_2^2\frac{N_f-k}{4\pi}\right) - y_4\frac{N_f-k}{4\pi}\sigma_F^2 + y_6\frac{N_f-k}{4\pi}\sigma_F^3.\end{aligned} \tag{D.8}$$

where σ_F is an auxiliary field and y_2^2, y_4, and y_6 are arbitrary coefficients of the relevant and marginal operators of the UV Lagrangian. For the coefficients we choose the values

$$y_4 = x, \qquad y_2^2 = -2xm_\psi, \qquad y_6 = 0. \tag{D.9}$$

Eventually we would like to flow to the deep IR, which amounts to taking $x \to \infty$, where the fermions pick up a finite mass m_ψ, but for now we will assume it to be finite. The dual of $\mathcal{L}_F$ on the bosonic side is

$$\begin{aligned}\mathcal{L}_B = |D_c\phi|^2 - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right)\right] \\ + m_B^2\phi^\dagger\phi + b_4\frac{4\pi}{N}\left(\phi^\dagger\phi\right)^2 + x_6\frac{(2\pi)^2}{N^2}\left(\phi^\dagger\phi\right)^3\end{aligned} \tag{D.10}$$

where m_B^2, b_4, and x_6 are coefficient of marginal/relevant operators (the dual of the fermionic parameters) which are then

$$b_4 = x, \qquad m_B^2 = 2x\left(\frac{N-k}{N}\right)m_\phi^2, \qquad x_6 = 0. \tag{D.11}$$

Unfortunately, in the analysis of ref. [96], there is no clear dual to $\bar{\psi}\psi$. We can however express $\bar{\psi}\psi$ in terms of operators which do have a well-defined dual. Integrating out σ_F in S_F enforces the equations of motion

$$x\frac{N_f-k}{2\pi}\sigma_F = \bar{\psi}\psi + x\frac{N_f-k}{2\pi}m_\psi. \tag{D.12}$$

We can rearrange this expression to solve for $\bar{\psi}\psi$,

$$\bar{\psi}\psi = x\frac{N_f-k}{2\pi}\left(\sigma_F - m_\psi\right). \tag{D.13}$$

[1]We use a different regularization convention for the Chern-Simons-matter theories than that in [96]. See [2] for a nice discussion on the differences in regularization conventions.

Fortunately, we know how to dualize the RHS from the maps in [96]. We obtain

$$\bar{\psi}\psi = 2x\frac{N_f - k}{4\pi}\left(\sigma_F - m_\psi\right) \leftrightarrow -2x\left(\phi^\dagger\phi + \frac{N-k}{4\pi}m_\phi^2\right). \tag{D.14}$$

Now let's see how this is implemented for the WF scalar. We know we can rewrite the WF scalar using auxiliary fields, namely, we write the quadratic and quartic terms so that S_B then contains the terms

$$S_B \supset \int d^3x \left[\sigma_B\left(\phi^\dagger\phi + \frac{N-k}{4\pi}m_\phi^2\right) - \frac{N}{16\pi x}\sigma_B^2\right]. \tag{D.15}$$

Again, σ_B is an auxiliary field and its corresponding equation of motion is

$$\phi^\dagger\phi + \frac{N-k}{4\pi}m_\phi^2 = \frac{N}{16\pi x}\sigma_B. \tag{D.16}$$

Notably, the LHS of this expression matches the term on the RHS of (D.14), so we can establish the duality

$$\bar{\psi}\psi \leftrightarrow -\frac{N}{8\pi}\sigma_B. \tag{D.17}$$

Note all factors of x have dropped from this expression.

Hence, under the bosonization of the fermion end of this interaction, a naive generalization of (D.17) for (D.6) would be

$$C_2^{(C)}\left(\bar{\psi}_{2,3}^{a_2a_3}X_{a_2a_1}^{1,2}\right)\left(X_{1,2}^{\dagger b_2a_1}\psi_{b_2a_3}^{2,3}\right) \quad \leftrightarrow \quad -C_2^{(D)}\left(X_{1,2}^{\dagger b_2a_1}X_{a_2a_1}^{1,2}\right)\frac{N}{8\pi}\left(\sigma_{2,3}\right)_{b_2}^{a_2} \tag{D.18}$$

where we have introduced the $K_2 \times K_2$ auxiliary fields $\sigma_{2,3}$. Importantly, there is an inherent asymmetry here because $\sigma_{2,3}$ belongs to the $X_{2,3}$ link. Grouping this together with the other terms linear in $\sigma_{2,3}$ and taking the $x \to \infty$ limit, the bosonic action of the link to the right of the node is a generalization of (D.15) is

$$S_B \supset \int d^3x \, \left(\sigma_{2,3}\right)_{b_2}^{a_2}\left(X_{2,3}^{\dagger b_2a_3}X_{a_2a_3}^{2,3} + \frac{N-k}{4\pi}m_{2,3}^2\delta_{a_2}^{b_2} - C_2^{(D)}\frac{N}{8\pi}X_{1,2}^{\dagger b_2a_1}X_{a_2a_1}^{1,2}\right) \tag{D.19}$$

where we have assumed that we have used the $U(K_2)$ symmetry such that $\left(m^2\right)_{a_2}^{b_2}$ is always diagonal.

Before proceeding, let us briefly comment on the need to introduce $(K_2)^2$ σ fields. Had we introduced only K_2 sigma fields, all self-interactions would have been of the form

$$\sum_{a_2}\left(X_{2,3}^{\dagger a_2 a_3}X^{2,3}_{a_2 a_3}\right)^2. \tag{D.20}$$

In contrast, (D.19) implies self-interaction terms of the form $\left(X_{2,3}^{\dagger b_2 a_3}X^{2,3}_{a_2 a_3}\right)^2$ with *no sum* over a_2 and b_2. These can be combined into a perfect square, $\left(\sum_{a_2}X_{2,3}^{\dagger a_2 a_3}X^{2,3}_{a_2 a_3}\right)^2$. This potential is needed to realize the full "flavor" symmetry, i.e. $SU(N_s)$. Meanwhile, (D.20) only preserves the diagonal $U(1)^{N_s}$ subgroup. For example, in the $N_s = 2$ the two distinct choices for the interaction term are

$$|\phi_1|^4 + |\phi_2|^4 \qquad \text{or} \qquad \left(|\phi_1|^2 + |\phi_2|^2\right)^2. \tag{D.21}$$

The former is invariant under a $U(1)^2$ symmetry while the latter is invariant under $SU(2)$. While it doesn't appear that anyone has been careful enough to distinguish the two possibilities in the context of Non-Abelian 3d bosonization dualities[2], it appears that for symmetry matching of the global flavor symmetries we do need (D.19) with its $(K_2)^2$ auxiliary σ fields.

To analyze the effect of the interaction terms in (D.19) , let us recall how the original WF term without the additional interactions, $\mathcal{L}_B \supset \sigma_B\left(\phi^\dagger\phi + \frac{N-k}{4\pi}m_\phi^2\right)$. We know that we should expect

$$m_\phi^2 > 0: \qquad \left\langle\phi^\dagger\phi\right\rangle = 0 \tag{D.22a}$$

$$m_\phi^2 < 0: \qquad \left\langle\phi^\dagger\phi\right\rangle \sim m_\phi^2. \tag{D.22b}$$

We can use this to make conclusions about (D.19). Note that we are assuming the U groups on the nodes are of unequal rank and thus $\left\langle X_{1,2}^{\dagger a_1 a_2}X^{1,2}_{a_1 b_2}\right\rangle = (v_{1,2}^2)^{a_2}_{b_2} \neq v^2\delta^{a_2}_{b_2}$ as in (4.29) for some constant v. We'll also assume maximal Higgsing so that the a negative deformation of

[2] In the Abelian case, dualities which differ only by the structure of the quartic interactions have been discussed in [124].

$m^2_{2,3}$ produces a vacuum expectation value of $\left\langle X^{\dagger a_2 a_3}_{2,3} X^{2,3}_{b_2 a_3} \right\rangle = (v^2_{2,3})^{a_2}_{b_2} \sim \delta^{a_2}_{b_2}$. We then find

$$
\begin{aligned}
&m^2_{2,3} > 0, && \left\langle X^{\dagger a_1 a_2}_{1,2} X^{1,2}_{a_1 b_2} \right\rangle > 0 : && \left\langle X^{\dagger a_2 a_3}_{2,3} X^{2,3}_{b_2 a_3} \right\rangle \sim C^{(D)}_2 \left(v_{1,2}\right)^{a_2}_{b_2} && \text{(D.23a)} \\
&m^2_{2,3} > 0, && \left\langle X^{\dagger a_1 a_2}_{1,2} X^{1,2}_{a_1 b_2} \right\rangle = 0 : && \left\langle X^{\dagger a_2 a_3}_{2,3} X^{2,3}_{b_2 a_3} \right\rangle = 0 && \text{(D.23b)} \\
&m^2_{2,3} < 0, && \left\langle X^{\dagger a_1 a_2}_{1,2} X^{1,2}_{a_1 b_2} \right\rangle > 0 : && \left\langle X^{\dagger a_2 a_3}_{2,3} X^{2,3}_{b_2 a_3} \right\rangle \sim \left(C^{(D)}_2 v_{1,2} + v_{2,3}\right)^{a_2}_{b_2} . && \text{(D.23c)} \\
&m^2_{2,3} < 0, && \left\langle X^{\dagger a_1 a_2}_{1,2} X^{1,2}_{a_1 b_2} \right\rangle = 0 : && \left\langle X^{\dagger a_2 a_3}_{2,3} X^{2,3}_{b_2 a_3} \right\rangle \sim \left(v_{2,3}\right)^{a_2}_{b_2} . && \text{(D.23d)}
\end{aligned}
$$

This is precisely the behavior that we would expect for a term dual to (D.6). That is, we see that a nonzero vacuum expectation value of the $X_{1,2}$ fields can cause certain components of $X_{2,3}$ to also get a vacuum expectation value. Importantly, if $X_{2,3}$ has a vacuum expectation value but $X_{1,2}$ does not, the interaction term does not work in reverse. To see this, note that the equation of motion for $(\sigma_2)^{a_2}_{b_2}$ would require

$$(v^2_{2,3})^{a_2}_{b_2} + (N-k)\frac{m^2_{2,3}}{4\pi} = C^{(D)}_2 \frac{N}{8\pi} X^{\dagger b_2 a_1}_{1,2} X^{1,2}_{a_2 a_1}. \tag{D.24}$$

Similar equations of motion hold for the $X_{3,4}$ link, but importantly the $X_{1,2}$ link contains no such interaction term. To determine the vacuum expectation values, one then just needs to consistently solve the three equations of motion. To do so, it is helpful to start with the one corresponding to the $X_{1,2}$ link. Since there is no interaction term for the $X_{1,2}$ link, $X_{2,3}$ has no influence on $v_{1,2}$ and its value follows in a manner completely analogous to (D.22). When $X_{1,2}$ acquires a vacuum expectation value, it then affects the $X_{2,3}$ equation of motion via (D.24). Hence we get our desired unidirectional influence. This behavior was always difficult to achieve using the naive interaction generalization in (D.7) because both the $X_{2,3}$ and $X_{1,2}$ fields appeared in the term symmetrically. Thus the effect of one field acquiring a nonzero vacuum expectation value was always the same as the other field, up to flavor structure.

More generally, for interactions on the U side we have

$$C_I \left(\bar{\psi}^{a_I a_{I+1}}_{I,I+1} X^{I-1,I}_{a_I a_{I-1}}\right)\left(X^{\dagger b_I a_{I-1}}_{I-1,I} \psi^{I,I+1}_{b_I a_{I+1}}\right) \quad \leftrightarrow \quad -C_I \left(X^{\dagger b_I a_{I-1}}_{I-1,I} X^{I-1,I}_{a_I a_{I-1}}\right) \frac{K_{I-1}}{8\pi} \left(\sigma_{I,I+1}\right)^{a_I}_{b_I} \tag{D.25}$$

while for interactions on the SU side, which are unidirectional to the left,

$$C_I \left(\bar{\psi}^{a_{I-1} a_I}_{I-1,I} Y^{I,I+1}_{a_I a_{I+1}} \right) \left(Y^{\dagger b_{I+1} a_I}_{I,I+1} \psi^{I-1,I}_{b_{I+1} a_I} \right) \quad \leftrightarrow \quad -C_I \left(Y^{\dagger b_{I+1} a_I}_{I,I+1} Y^{I,I+1}_{a_{I+1} a_I} \right) \frac{N_I}{8\pi} \left(\sigma_{I-1,I} \right)^{a_I}_{b_I} \tag{D.26}$$

where the $\sigma_{I,I+1}$ are auxiliary fields belonging to the $(I, I+1)$-th bifundamental and I runs over all internal nodes (e.g. $I = 2, 3$ for the four node case).

D.4 Quivers and Spin$_c$

In [15] it was discussed how the master duality is consistent with being put on a spin$_c$ manifold. Unfortunately, we weren't particularly careful with ordinary and spin$_c$ connections in [8], so we should clarify the consistency here.

In [8] a modified version of the master duality is used and various shifts were performed. Specifically, a shift on the Abelian portion of c was performed, $\tilde{c} \to \tilde{c} + \tilde{A}_1$ and the common $\tilde{A}_1$ Chern-Simons term was canceled. Being more careful with connections, this is equivalent to redefining a new spin$_c$ gauge field which we will call $\tilde{a} = \tilde{c} - \tilde{A}_1$ and ordinary gauge field $\tilde{B} = \tilde{A}_1 + \tilde{A}_2$, so that the U side is now

$$\begin{aligned} \mathcal{L}_U = & \left| D_{c' + \tilde{a} + \tilde{A}_1 + C} \Phi \right|^2 + i \bar{\Psi} \not{D}_{c' + \tilde{a} + B + \tilde{B}} \Psi + \mathcal{L}'_{\text{int}} \\ & - i \left[\frac{N}{4\pi} \text{Tr}_k \left(c' dc' - i \frac{2}{3} c'^3 \right) + \frac{Nk}{4\pi} \tilde{a} d\tilde{a} + 2Nk \text{CS}_{\text{grav}} \right]. \end{aligned} \tag{D.27}$$

We also shifted the $\tilde{A}_2$ fields into the ψ and Φ matter by taking $\tilde{A}_1 \to \tilde{A}_1 - \tilde{A}_2$. Instead, define a new ordinary connection $\tilde{G} = \tilde{A}_1 + \tilde{A}_2$(corresponds to the $\tilde{B}$ in the U side above) so the SU side of the duality now reads

$$\begin{aligned} \mathcal{L}_{SU} = & \left| D_{b' + B + \tilde{G}} \phi \right|^2 + i \bar{\psi} \not{D}_{b' + C + \tilde{A}_1} \psi + \mathcal{L}_{\text{int}} - i \left[\frac{N_f - k}{4\pi} \text{Tr}_N \left(b' db' - i \frac{2}{3} b'^3 \right) \right] \\ & - i \left[\frac{N}{4\pi} \text{Tr}_{N_f} \left(C dC - i \frac{2}{3} C^3 \right) + N N_f \left(\frac{1}{4\pi} \tilde{A}_1 d\tilde{A}_1 \right) + 2 N N_f \text{CS}_{\text{grav}} \right]. \end{aligned} \tag{D.28}$$

This is the same as Ref. [8] with the replacements, $\tilde{G} \to \tilde{A}_1$, $\tilde{A}_1 \to \tilde{A}_1 - \tilde{A}_2$, $c' + \tilde{a} \to c$, $\tilde{B} \to \tilde{A}_1$. We also promote the field $\tilde{A}_2$ which introduces a new background symmetry which we call $\tilde{B}_2$. In our new notation this is equivalent to gauging $\tilde{A}_1 \to \tilde{a}_1$, and we will called the new ordinary connection to which it couples $\tilde{B}_1$.

The part we need to be most careful is in the match of the two theories as we move across the quiver construction. This is spelled out in the most detail in Appendix A.3 of [8] become. There we make certain identifications based on fermion interactions which are now equivalent to

$$i\bar{\Psi}\not{D}_{c'+\tilde{a}+B+\tilde{B}}\Psi \qquad \leftrightarrow \qquad i\bar{\psi}\not{D}_{b'+C+\tilde{A}_1}\psi. \tag{D.29}$$

The field level identification which are in Table 3 of [8] become

$$c' + \tilde{a} \qquad \leftrightarrow \qquad C + \tilde{a}_1 \tag{D.30a}$$

$$B \qquad \leftrightarrow \qquad b' \tag{D.30b}$$

$$\tilde{B} \qquad \leftrightarrow \qquad \tilde{B}_1. \tag{D.30c}$$

Since we are matching ordinary connections to ordinary connections and spin$_c$ connections to spin$_c$ connections, everything is consistent and it does appear these quivers are consistent with being put on spin$_c$ manifolds.

Appendix E

FERMIONIC QUIVER SUPPLEMENTARY

Here we provide further details of our construction of the fermionic non-Abelian linear quivers.

E.1 Deriving Two-Node Fermionic Dualities

In this appendix we give the details of the two-node quivers constructed in Sec. 5.1.

We begin with the flavor-bounded case. In the main text, we showed a two-node fermionic quiver can be found by matching the scalar sides of Aharony's dualities (1.13), with appropriate relabeling and background terms. To achieve this matching, it is useful to work with the charge conjugated version of (1.23b), which at the Lagrangian level is given by

$$\mathcal{L}_{SU} = \left|D_{-b'+B+\tilde{A}_1}\phi\right|^2 - i\left[-\frac{k}{4\pi}\mathrm{Tr}_N\left(b'db' - i\frac{2}{3}b'^3\right) - \frac{Nk}{4\pi}\tilde{A}_1 d\tilde{A}_1\right], \tag{E.1a}$$

$$\mathcal{L}_U = i\bar{\Psi}\not{D}_{-c+B}\Psi - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right) + \frac{N}{2\pi}\mathrm{Tr}_k\left(c\right)d\tilde{A}_1\right], \tag{E.1b}$$

where $B \in su(N_f)$ and the duality is subject to the flavor bound $N \geq N_f$ and mass mapping $m_\Psi \leftrightarrow m_\phi^2$.

To perform the matching, we rearrange the U sides of (1.19) from (1.23) by shifting the dynamical field to eliminate the BF terms and add appropriate background Chern-Simons terms. For (1.19) we take $\tilde{c} \to \tilde{c} + \tilde{A}_1$, giving

$$\mathcal{L}_U = \left|D_{c-C+\tilde{A}_1}\Phi\right|^2 - i\left[\frac{N}{4\pi}\mathrm{Tr}_k\left(cdc - i\frac{2}{3}c^3\right) - \frac{Nk}{4\pi}\tilde{A}_1 d\tilde{A}_1\right]. \tag{E.2}$$

Additionally, we add $-i\left[\frac{Nk}{4\pi}\tilde{A}_1 d\tilde{A}_1\right]$ to both sides of (1.23) to cancel the $\tilde{A}_1$ background Chern-Simons term. For (E.1b), we take $\tilde{c} \to \tilde{c} - \tilde{A}_1$ and then add $-i\left[\frac{Nk}{4\pi}\tilde{A}_1 d\tilde{A}_1\right]$ to both sides.

At this point one can promote the background flavor symmetries in both dualities so all the matter is bifundmentally charged. If one compares the scalar sides of the two dualities, an explicit matching can be achieved by identifying the fields

$$SU \text{ field:} \qquad C \quad \Leftrightarrow \quad b', \tag{E.3a}$$

$$U \text{ field:} \qquad c \quad \Leftrightarrow \quad G \equiv B + \tilde{A}_1 \mathbb{1}_{N_s}, \tag{E.3b}$$

$$U(1) \text{ background field:} \qquad \tilde{A}_1 \quad \Leftrightarrow \quad \tilde{B}_1. \tag{E.3c}$$

Thus we also arrive at a duality between the respective fermion dualities, which is our desired flavor-bounded two-node quiver. The mass mapping is such that m_ψ is identified with $-m_\Psi$.

A self-consistency check can be performed by making sure the two sides of the duality still match under mass deformations. We find

$$\text{(A)} \qquad m_\psi < 0: \qquad SU(N_1)_{-k_1} \times SU(N_2)_{-k_2} \tag{E.4a}$$

$$\text{(C)} \qquad m_\Psi > 0: \qquad U(k_1)_{N_1} \times U(k_2)_{N_2} \tag{E.4b}$$

$$\text{(A)} \qquad m_\psi > 0: \qquad SU(N_1)_{-k_1+N_2} \times SU(N_2)_{-k_2+N_1} = SU(N_2)_{-k_2+N_1} \tag{E.4c}$$

$$\text{(C)} \qquad m_\Psi < 0: \qquad U(k_1)_{N_1-k_2} \times U(k_2)_{N_2-k_1} = SU(N_2)_{-k_2+N_1} \tag{E.4d}$$

where we have used the fact $k_1 = N_2$ is required by the flavor constraints. Clearly (E.4a) and (E.4b) are level-rank dual to one another. Eqs. (E.4c) and (E.4d) take slightly more work. Since $k_1 = N_2$, each of these theories has one of the gauge field's Chern-Simons terms vanish. In (E.4c), all degrees of freedom of the $SU(N_1)_0$ theory are gapped out in the IR limit, and we can thus drop this factor. For (E.4d), one can integrate out the $U(1)$ subgroups of $U(k_1)$ and $U(k_2)$. When one does this, both $U(1)$ factors get eliminated and the theory reduces to $SU(N_2)_{N_1-k_2} \times SU(k_2)_0$, which matches (E.4c) after dropping the second term.

If we want to work in the regime where $k_1 \neq N_2$, we would need to use the flavor-violated 3d bosonization duality in one of these steps. In the main text, we choose to replace (1.23) with its flavor-violated equivalent. This extends the flavor bounds to $k_1 < N_2 < N_*$ but means the full phase diagram of the U side of the duality is described by two separate scalar theories, corresponding to the $m_\psi > 0$ and $m_\psi < 0$ halves of the SU phase diagram (this is analogous to the layout in Fig. 5.4).

Fortunately, the $m_\psi < 0$ theory is identical at the Lagrangian level to the flavor-bounded duality, i.e. (1.19). Thus the very same matching that was performed above for the flavor-bounded case can be repeated for this end of the phase diagram. The $m_\psi > 0$ side requires one to perform the matching once more. The explicit form the Lagrangian for Theories B_2 and C_2 are given by

$$\begin{aligned}\mathcal{L}_{\text{B}_2} &= \left|D_{c-C-\tilde{A}_1}\Phi_2\right|^2 - i\left[\frac{-N_1}{4\pi}\text{Tr}_{N_2-k_1}\left(cdc - i\frac{2}{3}c^3\right)\right] \\ &\quad - i\left[\frac{N_1-k_2}{4\pi}\text{Tr}_{N_2}\left(CdC - i\frac{2}{3}C^3\right) + \frac{N_1N_2}{4\pi}\tilde{A}_1 d\tilde{A}_1\right], \qquad \text{(E.5a)} \\ \mathcal{L}_{\text{C}_2} &= i\bar{\Psi}\not{D}_{-c+g+\tilde{A}_1}\Psi - i\left[-\frac{k_1}{4\pi}\text{Tr}_{N_1-k_2}\left(cdc - i\frac{2}{3}c^3\right) - \frac{k_2}{4\pi}\text{Tr}_{N_2-k_1}\left(gdg - i\frac{2}{3}g^3\right)\right] \\ &\quad - i\left[\frac{1}{2\pi}\text{Tr}_{N_1-k_2}\left(c\right)d\text{Tr}_{N_2-k_1}\left(g\right) + \frac{N_1N_2}{4\pi}\tilde{A}_1 d\tilde{A}_1 + \frac{\left(N_1-k_2\right)\left(N_2-k_1\right)}{4\pi}\tilde{A}_1 d\tilde{A}_1\right], \quad \text{(E.5b)}\end{aligned}$$

where we have defined the field $g \in U(N_2 - k_1)$.

In order for the flavor-violated quiver to match in the intermediate phases, we detail the generalized level/rank duality between the two node quiver with BF term pointed out in (5.13). The main point as illustrated in Sec. 5.1 is that the gauging of diagonal $U(1)$ symmetry of the two copies of level/rank duality $SU(N_1)_{k_1} \times SU(N_2)_{k_2} \leftrightarrow U(k_1)_{-N_1} \times U(k_2)_{-N_2}$.

The explicit Lagrangian with two $U(1)$ background terms is given by [60],

$$\begin{aligned}&- i\left[\frac{k_1}{4\pi}\text{Tr}_{N_1}(ada - i\frac{2}{3}a^3) + \frac{1}{2\pi}ed\text{Tr}_{N_1}(a+B) + \frac{k_2}{4\pi}\text{Tr}_{N_2}(bdb - i\frac{2}{3}b^3) + \frac{1}{2\pi}fd\text{Tr}_{N_2}(b+C)\right] \Leftrightarrow \\ &- i\left[-\frac{N_1}{4\pi}\text{Tr}_{k_1}(udu - i\frac{2}{3}u^3) + \frac{1}{2\pi}\text{Tr}_{k_1}(u)dB - \frac{N_2}{4\pi}\text{Tr}_{k_2}(vdv - i\frac{2}{3}v^3) + \frac{1}{2\pi}\text{Tr}_{k_2}(v)dC\right]. \qquad \text{(E.6)}\end{aligned}$$

Now if we add mixed counter-term $-i[-\frac{1}{2\pi}BdC]$ along the duality and gauge the diagonal

$U(1)$ as $B = C \to c$, we get following duality after solving the equation of motion

$$\begin{aligned}
&- i\left[\frac{k_1}{4\pi}\mathrm{Tr}_{N_1}(ada - i\frac{2}{3}a^3) + \frac{k_2}{4\pi}\mathrm{Tr}_{N_2}(bdb - i\frac{2}{3}b^3) - \frac{1}{2\pi}\mathrm{Tr}_{N_1}(a)d\mathrm{Tr}_{N_2}(b)\right] \\
&\Leftrightarrow -i\left[-\frac{N_1}{4\pi}\mathrm{Tr}_{k_1}(udu - i\frac{2}{3}u^3) - \frac{N_2}{4\pi}\mathrm{Tr}_{k_2}(vdv - i\frac{2}{3}v^3) + \frac{1}{2\pi}\mathrm{Tr}_{k_1}(u)d\mathrm{Tr}_{k_2}(v)\right].
\end{aligned} \tag{E.7}$$

After change of variables, the above becomes equivalent to (5.13).

E.1.1 Consistency Checks

Another non-trivial consistency check of the quantum phase diagram of the flavor-violated two-node quiver can be established using similar approaches to the one in the literature, e.g. Refs. [60, 37, 25, 33]. A useful check is the consistency of the background counterterms along the phase diagram, which we discuss in more detail in Appendix E.4 for the case of gravitational counterterms.

Here, we show the non-trivial matching of the quantum phase of $SU(2)_k \times SU(2)_k + \psi^{\text{bifund}}$ and its isomorphic expression $Spin(4)_k + 2\ \psi^{\text{vec}}$ analyzed in Ref. [37]. We will use explicit superscripts in this section to avoid confusion between vector and bifundamental matter. Since both of these theories use different dual descriptions – the former with bifundamental scalar or fermion and the latter a vector scalar – the matching of the intermediate phase mutually supports the two-node construction we have laid out above.

First, note the quantum phase exists only when $k = 0$. From the viewpoint of the orthogonal gauge group, $Spin(4)_k + 2\ \psi^{\text{vec}}$ at small fermion mass flows to a non-linear sigma model with target space S^1 [37]. On the other hand, the two-node fermionic quiver predicts $SU(2)_0 \times SU(2)_0 + \psi^{\text{bifund}}$ flows to

$$U(1)_1 \times U(1)_1 - \mathrm{BF} \quad \leftrightarrow \quad U(1)_{-1} \times U(1)_{-1} + \mathrm{BF}. \tag{E.8}$$

In terms of a K-matrix description, the phase is described by the 2 by 2 matrix

$$SU(2)_0 \times SU(2)_0 + \psi^{\text{bifund}} \quad \xrightarrow{m_\psi \simeq 0} \quad K_{CS} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{E.9}$$

It turns out that $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \simeq U(1)_1 \times U(1)_0$ under the $SL(2,\mathbb{Z})$ transformation (see [43] for a recent discussion) and, since $U(1)_1$ is trivial, $U(1)_0$ is isomorphic to a compact boson. We see that the intermediate quantum phase is isomorphic to the one obtained in the $Spin(4) + 2\ \psi^{\text{vec}}$ theory.

It is quite interesting to see the global symmetry matching between these two distinct constructions. On the $Spin(4)$ side, the $SO(2)$ flavor symmetry rotating two vector fermions lead to a nonlinear σ-model with S^1. The scalar dual description of it preserves the $SO(2)$ flavor symmetry and the symmetry breaking is realized through the condensation of the scalar. On the $SU(2) \times SU(2)$ side, the $SO(2)$ flavor symmetry is translated to the $U(1)$ baryon symmetry, and either of the scalar or fermion dual descriptions have a $U(1)$ monopole symmetry which doesn't act on the matter field. When we deform the mass of the dual description to flow into the quantum phase, the monopole symmetry in the ultraviolet flows to the $U(1)$ shift symmetry of the compact boson, where the symmetry breaking is triggered by the non-trivial monopole flux.

E.2 Generalized Fermion Particle-Vortex Duality

In this appendix, we elaborate on how the flavor-bounded quiver we constructed in Sec. 5.1.1 is a generalization the fermionic particle-vortex duality proposed by Son [119]. As a reminder, this duality also lives in $2+1$-dimensions and conjectures a free Dirac fermion is dual to a $U(1) \times U(1)$ Chern-Simons matter theory, which we refer to as "QED$_3$". The generalization is a straightforward extension of previous work [8], where it was shown that the bosonic two-node quiver duality can be viewed as a generalization of the bosonic particle-vortex duality. We begin by emphasizing the special case which reduces to Son's duality, which corresponds to taking $N_1 = N_2 = k_1 = k_2 = 1$.

E.2.1 Son's Particle-Vortex Duality

Let us begin by reviewing how Son's duality is constructed from the Abelian limits of (1.13) as it was done in Ref. [112]. We will then show this is simply the Abelian limit of the two-node quiver construction we performed above. The "scalar + flux = fermion" duality at the Lagrangian level is

$$\mathcal{L}_{SU} = i\bar{\psi} \not{D}_{\tilde{A}_1} \psi \tag{E.10a}$$

$$\mathcal{L}_{U} = |D_c\Phi|^2 - i\left[\frac{1}{4\pi}cdc - \frac{1}{2\pi}cd\tilde{A}_1\right] \tag{E.10b}$$

with mass mapping $m_\psi \leftrightarrow -m_\Phi^2$. The (time-reversed) "fermion + flux = scalar" is given by

$$\mathcal{L}_{SU} = \left|D_{\tilde{A}_1}\phi\right|^2 - i\left[-\frac{1}{4\pi}\tilde{A}_1 d\tilde{A}_1\right] \tag{E.11a}$$

$$\mathcal{L}_{U} = i\bar{\Psi}\not{D}_c\Psi - i\left[\frac{1}{4\pi}cdc - \frac{1}{2\pi}cd\tilde{A}_1\right] \tag{E.11b}$$

which has a mass mapping $m_\Psi \leftrightarrow m_\phi^2$.

In the above expressions, we have a free fermion in (E.10a), which automatically gives one side of Son's duality. Our goal should be to match the scalar side, (E.10b), using the fermion + flux duality. We can take (E.11) and add $-i\left[\frac{2}{4\pi}\tilde{A}_1 d\tilde{A}_1 - \frac{1}{2\pi}\tilde{A}_1 d\tilde{B}_1\right]$ to both ends and promote the background gauge field, $\tilde{A}_1 \to \tilde{a}_1$, which yields

$$\mathcal{L}_{SU} = |D_{\tilde{a}_1}\phi|^2 - i\left[\frac{1}{4\pi}\tilde{a}_1 d\tilde{a}_1 - \frac{1}{2\pi}\tilde{a}_1 d\tilde{B}_1\right] \tag{E.12a}$$

$$\mathcal{L}_{U} = i\bar{\Psi}\not{D}_c\Psi - i\left[\frac{1}{4\pi}cdc - \frac{1}{2\pi}cd\tilde{a}_1 + \frac{2}{4\pi}\tilde{a}_1 d\tilde{a}_1 - \frac{1}{2\pi}\tilde{a}_1 d\tilde{B}_1\right]. \tag{E.12b}$$

The scalar end of these dualities are identical under the identification $\phi \Leftrightarrow \Phi$, $\tilde{a}_1 \Leftrightarrow c$, and $\tilde{A}_1 \Leftrightarrow \tilde{B}_1$. Thus we arrive at the duality

$$i\bar{\psi}\not{D}_{\tilde{A}_1}\psi \qquad \leftrightarrow \qquad i\bar{\Psi}\not{D}_c\Psi - i\left[\frac{1}{4\pi}cdc - \frac{1}{2\pi}cd\tilde{a}_1 + \frac{2}{4\pi}\tilde{a}_1 d\tilde{a}_1 - \frac{1}{2\pi}\tilde{a}_1 d\tilde{A}_1\right] \tag{E.13}$$

with the mass identification $m_\psi \leftrightarrow -m_\Psi$. Recall, we are using the notation where the fermion comes with a default level $-1/2$, so it may look slightly different than what was is

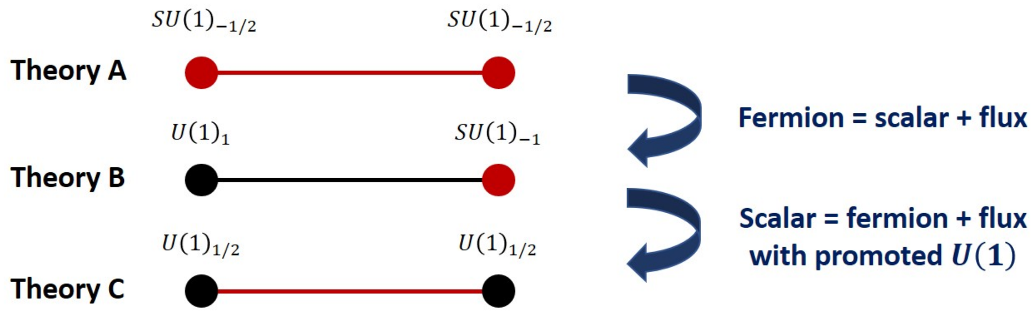

Figure E.1: Fermion particle-vortex duality as a two-node quiver theory.

sometimes in the literature. This is the properly quantized form of the duality also found in Ref. [112]. This construction is essentially the time-reversed version of the derivation performed there. As discussed in Ref. [112], integrating out the dynamical gauge field c brings the right-hand side of (E.13) to its simpler and more familiar form, at the cost of violating Dirac quantization.

To match the duality to two-node quivers we shift the gauge field on the right-hand-side of (E.13) to $c \to c + \tilde{a}_1 + \tilde{A}_1$, which then gives

$$\mathcal{L}_U = i\bar{\Psi}\not{D}_{c+\tilde{a}_1+\tilde{A}_1}\Psi - i\left[\frac{1}{4\pi}cdc + \frac{1}{4\pi}\tilde{a}_1 d\tilde{a}_1 + \frac{1}{4\pi}\tilde{A}_1 d\tilde{A}_1\right]. \tag{E.14}$$

One can check that, after a relabeling of dynamical gauge fields and rearranging the background Chern-Simons terms, this is simply the $N_1 = N_2 = k_1 = k_2 = 1$ limit of the Lagrangians in (5.7). The derivation is summarized in Fig. E.1.

E.2.2 Theories Dual to the Free Dirac Fermion

We have just shown the fermion particle-vortex duality admits generalizations in the form of the flavor-bounded two-node quiver we constructed in Sec. 5.1.1. Generalizations can then be constructed by tuning the three independent parameters: N_1, N_2 $(= k_1)$, and k_2.

For instance, there is an infinite class of dualities which continue to just have a fermion on one end of the duality, so long as $N_1 = N_2 = 1$. We can vary k_2 and get a theory which would is still dual to the free Dirac fermion,

$$\mathcal{L}_{SU} = i\bar{\psi}\not{D}_{\tilde{A}_1}\psi - i\left[\frac{1}{4\pi}\tilde{A}_1 d\tilde{A}_1\right], \tag{E.15a}$$

$$\mathcal{L}_{U} = i\bar{\Psi}\not{D}_{c+g+\tilde{A}_1}\Psi - i\left[\frac{1}{4\pi}gdg + \frac{1}{4\pi}\mathrm{Tr}_{k_2}\left(cdc - i\frac{2}{3}c^3\right)\right], \tag{E.15b}$$

where now $c \in U(k_2)$. On the SU side changing k_2 only effects the level of the non-existent Chern-Simons term. This also opens up the possibility of taking k_2 to be large and possibility doing some sort of $1/k_2$ expansion [61].

E.2.3 Theories Similar to "QED$_3$"

Also interesting is the case of $N_2 = k_1 = k_2 = 1$ with N_1 kept arbitrary, where we still get two Abelian fields on the U side of the duality so the theory is similar to QED$_3$. However, what will change is the relative levels of the Chern-Simons terms on this end of the duality. This would be dual to a fermion coupled only to an SU field. Again, it would be interesting to take N_1 to be large and do some $1/N_1$ expansion. Explicitly,

$$\mathcal{L}_{SU} = i\bar{\psi}\not{D}_{b'+\tilde{A}_1}\psi - i\left[\frac{N_1}{4\pi}\tilde{A}_1 d\tilde{A}_1\right], \tag{E.16a}$$

$$\mathcal{L}_{U} = i\bar{\Psi}\not{D}_{c+g+\tilde{A}_1}\Psi - i\left[\frac{N_1}{4\pi}gdg + \frac{1}{4\pi}cdc\right]. \tag{E.16b}$$

First off, note that the presence of the level-1/2 b' Chern-Simons term on the ψ side means this cannot be particle-hole symmetric. That's okay if we're not attempting to describe $\nu = 1/2$.

We can make this look more similar to the properly-quantized version of Son's duality by shifting $c \to c - g + \tilde{A}_1$, which makes the $\mathcal{L}_\Psi$ becomes

$$\mathcal{L}_{U} = i\bar{\Psi}\not{D}_{c}\Psi - i\left[\frac{N_1+1}{4\pi}gdg - \frac{1}{2\pi}gdc + \frac{1}{4\pi}cdc - \frac{N_1}{2\pi}gd\tilde{A}_1 + \frac{N_1}{4\pi}\tilde{A}_1 d\tilde{A}_1\right]. \tag{E.17}$$

This is the same as QED$_3$ (with a relabeling of $g \to \tilde{a}_1$), except now the g and $\tilde{A}_1$ Chern-Simons terms have an arbitrary integer level. This reduces to Son's duality in the $N_1 = 1$ limit.

E.3 Two-Node Fermionic Quivers for the SO/Sp gauge group

Here we generalize the two-node fermionic quiver dualities analyzed in Sec. 5.1 to the case of SO and Sp gauge groups. We only list the main results since the approach is largely similar to the unitary case.

First, we write down the flavor bounded two-node quiver dualties for the SO/Sp gauge group. The construction parallels that of Sec. 5.1.1, but instead using the version of 3d bosonization dualities for the SO/Sp gauge group appeared in Ref. [3]. First, the boson-fermion duality is obtained from 3d bosonization by gauging the full flavor symmetry subject to the flavor-bound $k_1 \geq N_2$,

$$SO(N_1)_{-k_1+\frac{N_2}{2}} \times SO(N_2)_{-k_2+\frac{N_1}{2}} + \psi \quad \leftrightarrow \quad SO(k_1)_{N_1} \times SO(N_2)_{-k_2} + \Phi, \tag{E.18a}$$

$$Sp(N_1)_{-k_1+\frac{N_2}{2}} \times Sp(N_2)_{-k_2+\frac{N_1}{2}} + \psi \quad \leftrightarrow \quad Sp(k_1)_{N_1} \times Sp(N_2)_{-k_2} + \Phi. \tag{E.18b}$$

From the above dualities, there exists fermion-fermion dualities similar to (5.7) for the special case of $k_1 = N_2$,

$$SO(N_1)_{-\frac{N_2}{2}} \times SO(N_2)_{-k_2+\frac{N_1}{2}} + \psi \quad \leftrightarrow \quad SO(N_2)_{N_1-\frac{k_2}{2}} \times SO(k_2)_{\frac{N_2}{2}} + \Psi, \tag{E.19a}$$

$$Sp(N_1)_{-\frac{N_2}{2}} \times Sp(N_2)_{-k_2+\frac{N_1}{2}} + \psi \quad \leftrightarrow \quad Sp(N_2)_{N_1-\frac{k_2}{2}} \times Sp(k_2)_{\frac{N_2}{2}} + \Psi. \tag{E.19b}$$

The flavor-violated version of boson-fermion two-node quiver dualities for the SO/Sp gauge group are built from the extension of bosonization dualities for the SO/Sp gauge group beyond the flavor bound analyzed in Ref. [86]. The results are similar to (5.8) and valid for $k_1 < N_2,\ k_2 < N_1$,

$$\begin{aligned}
&SO(N_1)_{-k_1+N_2/2}\times SO(N_2)_{-k_2+N_1/2}+\psi \\
&\qquad\leftrightarrow\quad \begin{cases} SO(k_1)_{N_1}\times SO(N_2)_{-k_2}+\phi_1 & m_\psi=-m_*, \\ SO(N_2-k_1)_{-N_1}\times SO(N_2)_{N_1-k_2}+\phi_2 & m_\psi=m_* \end{cases} \\
&Sp(N_1)_{-k_1+N_2/2}\times Sp(N_2)_{-k_2+N_1/2}+\psi \\
&\qquad\leftrightarrow\quad \begin{cases} Sp(k_1)_{N_1}\times Sp(N_2)_{-k_2}+\phi_1 & m_\psi=-m_* \\ Sp(N_2-k_1)_{-N_1}\times Sp(N_2)_{N_1-k_2}+\phi_2 & m_\psi=m_*. \end{cases}
\end{aligned} \tag{E.20}$$

Finally, the new fermionic two-node quiver dual descriptions, which equivalently describe the above phase transitions, for the SO/Sp cases are

$$\begin{aligned}
&SO(N_1)_{-k_1+N_2/2}\times SO(N_2)_{-k_2+N_1/2}+\psi \\
&\qquad\leftrightarrow\quad \begin{cases} SO(k_1)_{N_1-k_2/2}\times SO(k_2)_{N_2-k_1/2}+\Psi_1 & m_\psi=-m_* \\ SO(N_1-k_2)_{-N_2/2-k_1/2}\times SO(N_2-k_2)_{-N_1/2-k_2/2}+\Psi_2 & m_\psi=m_*, \end{cases} \\
&Sp(N_1)_{-k_1+N_2/2}\times Sp(N_2)_{-k_2+N_1/2}+\psi \\
&\qquad\leftrightarrow\quad \begin{cases} Sp(k_1)_{N_1-k_2/2}\times Sp(k_2)_{N_2-k_1/2}+\Psi_1 & m_\psi=-m_* \\ Sp(N_1-k_2)_{-N_2/2-k_1/2}\times Sp(N_2-k_2)_{-N_1/2-k_2/2}+\Psi_2 & m_\psi=m_*. \end{cases}
\end{aligned} \tag{E.21}$$

We comment that SO/Sp case is simpler than the unitary case since the corresponding level/rank duality preserves the type of Lie group. Similar subtleties as the unitary case will arise if one constructs the two-node quiver for the various modification of the orthogonal or symplectic group with discrete gauging/extension (e.g. $Spin, Pin^{\pm}, ...$), as analyzed for the orthogonal case in Ref. [37].

E.4 Gravitational Counterterm Matching

Here we show that the gravitational counterterms are consistent along the phase diagram of the two-node quiver discussed in Sec. 5.1. In our convention, we define the gravitational counterterm as the coeffcient of twice the gravitational Chern-Simons term $2\mathrm{CS}_{\mathrm{grav}}$, where $\int_{M=\partial X} \mathrm{CS}_{\mathrm{grav}} = \frac{1}{192\pi}\int_X \mathrm{tr} R\wedge R$. Refs. [25, 15] contain many of the 3d bosonization dualities used throughout this work with gravitational Chern-Simons terms made explicit.

First, let's discuss the boson-fermion dualities between bifundamental matter in (5.8). Consistency of the phase diagram requires that following a non-trivial closed path in phase space should have net zero change of the gravitational counterterm

$$\Delta c\,[(\mathrm{A\ I}) \to (\mathrm{B_1\ I}) \to (\mathrm{B_1\ III}) \to (\mathrm{B_2\ III}) \to (\mathrm{B_2\ II}) \to (\mathrm{A\ II}) \to (\mathrm{A\ I})] = 0 \tag{E.22}$$

where (A I) represents phase I of Theory A, etc. It turns out that there are two kinds of contributions to the gravitational counterterm along the path above. First, there is a contribution coming from the difference of the one-loop determinant of the fermion coupled to the background metric between negative or positive mass, which is equivalent to the complex dimension of the representation. Second, there is one coming from the compensating gravitational Chern-Simons term along the level-rank duality in the various phases, which is explained in Ref. [60]. Namely, $\Delta c = -Nk$ along the level-rank duality from $SU(N)_k$ to $U(k)_{-N}$. It is important to point out that scalar doesn't contribute to the gravitational counterterm and the middle phases described by the two scalar dual descriptions B_1 and B_2 are equivalent without the need of any duality transformation. After tracking down the non-trivial closed path in (E.22), one can show that the gravitational counterterm is consistent, i.e. $\Delta c|_{\text{closed path}} = 0$.

Now the above test could also be done with the two-node fermionic dual description described in (5.11). Using a similar procedure and formulas as above, consistency of the gravitational counterterms can be established.

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