

Heat Kernel Estimates for Markov Processes Associated with Time-Dependent Dirichlet Forms

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Abstract

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In this paper, time-inhomogeneous stable-like processes are investigated. We establish the relation between the transition operators and time-dependent parabolic equations, as well as upper heat kernel estimates.

1 Introduction

Let $d \geq 1$ and $\alpha \in (0, 2)$. Consider a family of Dirichlet forms $(E^{(t)}, F^{(t)})$ on $L^2(\mathbb{R}^d, dx)$ with

$$E^{(t)}(f, g) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x - y|^{d+\alpha}} (f(x) - f(y))(g(x) - g(y)) dx dy, \quad (1)$$

$$F^{(t)} = \{f \in L^2(\mathbb{R}^d, dx) : E^{(t)}(f, f) < \infty\}, \quad (2)$$

where $c(t, x, y)$ is a measurable function on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ that is bounded between two positive constants $c_1 < c_2$. $(E^{(t)}, F^{(t)})$ is associated with a family of self-adjoint operators L_t on $L^2(\mathbb{R}^d, dx)$ in the sense that

$$E^{(t)}(f, g) = -(L_t f, g)_{L^2(\mathbb{R}^d, dx)}.$$

Our goal is to study the behavior of the weak solutions to the parabolic equation

$$\frac{\partial u}{\partial t} = -L_t u \quad \text{on } \mathbb{R} \times \mathbb{R}^d \quad (3)$$

of divergence form. We point out that in the case of local Dirichlet forms $(E^{(t)}, F^{(t)})$ and local operators L_t , upper Gaussian estimates for the fundamental solutions have been long-established, see [1, 6, 10] and the references therein.

From a probabilistic point of view, it has been shown ([7]) that there exists a time-inhomogeneous Markov process X associated with $(E^{(t)}, F^{(t)})$. If we denote the transition operator of X by $P_{s,t}$, our estimates yield an upper bound for $\|P_{s,t}\|_{1 \rightarrow \infty}$ and prove the existence of a transition density function $p(s, t, x, y)$. When $c(t, x, y)$ is independent of t , Eq. (3) reduces to time-independent parabolic equation with non-local operator $L_t = L$ and X is a time-homogeneous stable-like process. Two sides estimates for such processes have been extensively studied in [3].

2 Time-Dependent Dirichlet form and associated Markov process

To begin with, let $(E^{(t)}, F^{(t)})$ be as defined in Eq. (1), (2). Since $c(t, x, y)$ is uniformly bounded, $(E^{(t)}, F^{(t)})$ share the common domain

$$F^{(t)} = F = \{f \in H : E^{(0)}(f, f) < \infty\} = H^{\alpha/2}(\mathbb{R}^d).$$

Here we write H for $L^2(\mathbb{R}^d, dx)$ and $H^{\alpha/2}(\mathbb{R}^d, dx)$ is the fractional Sobolev space. F is also a Hilbert space when equipped with the norm $\|f\|_F^2 = \|f\|_H^2 + E^{(0)}(f, f)$. Denote the dual of F by F' . Identifying H with its own dual space, we have

$$F \subset H \subset F'$$

with continuous and dense embeddings. In future, we will write (\cdot, \cdot) both for the inner product in H and the coupling between F and F' when this causes no confusions.

Fix an interval $I = [\sigma, \tau] \subset \mathbb{R}$ with the possibility that $\sigma = -\infty$ or $\tau = \infty$. For any function $u \in L^2(I \times \mathbb{R}^d)$, we view u as a function on I taking values in H . For simplicity, we will write u_t for $u(t)$. We denote the space of all such functions by $L^2(I \rightarrow H)$ and equip it with norm

$$\left(\int_I \|u_t\|_H^2 dt \right)^{1/2} = \|u\|_{L^2(I \times \mathbb{R}^d)}.$$

Similarly, let $L^2(I \rightarrow F)$ (resp. $L^2(I \rightarrow F')$) be the space of functions on I taking values in F (resp. F') with norm $(\int_I \|u_t\|_F^2 dt)^{1/2}$ (resp. $(\int_I \|u_t\|_{F'}^2 dt)^{1/2}$).

A few more terminologies need to be introduced before we define the time-dependent Dirichlet form. For any $u \in L^2(I \rightarrow F')$, let $\frac{\partial u}{\partial t}$ be its distributional derivative in t ; that is, for any test function $\phi : I \rightarrow \mathbb{R}$

$$\int_I \frac{\partial u}{\partial t} \phi_t dt = (u_\tau, \phi_\tau) - (u_\sigma, \phi_\sigma) - \int_I u_t \phi'_t dt \quad \text{in } F'.$$

Let $H^1(I \rightarrow F')$ be the Sobolev space of functions $u \in L^2(I \rightarrow F')$ with distributional derivative $\frac{\partial u}{\partial t} \in L^2(I \rightarrow F')$ equipped with norm $(\int_I \|u_t\|_{F'}^2 + \int_I \|\frac{\partial u}{\partial t}\|_{F'}^2 dt)^{1/2}$.

Finally, the space we mainly concern is

$$\mathcal{F}_I = L^2(I \rightarrow F) \cap H^1(I \rightarrow F')$$

equipped with norm $(\int_I \|u_t\|_F^2 + \int_I \|\frac{\partial u}{\partial t}\|_{F'}^2 dt)^{1/2}$.

The *time-dependent Dirichlet form* \mathcal{E} on $L^2(\mathbb{R} \times \mathbb{R}^d)$ is defined by

$$\mathcal{E}(u, v) = \begin{cases} -\int_{\mathbb{R}} \left(\frac{\partial u}{\partial t}, v_t \right) dt + \int_{\mathbb{R}} E^{(t)}(u_t, v_t) dt & \text{if } u \in \mathcal{F}_{\mathbb{R}} \text{ and } v \in L^2(\mathbb{R} \rightarrow F) \\ \int_{\mathbb{R}} \left(u_t, \frac{\partial u}{\partial t} \right) dt + \int_{\mathbb{R}} E^{(t)}(u_t, v_t) dt & \text{if } u \in L^2(\mathbb{R} \rightarrow F) \text{ and } v \in \mathcal{F}_{\mathbb{R}}. \end{cases} \quad (4)$$

In addition, we will write $\mathcal{E}_\beta(u, v) = \mathcal{E}(u, v) + \beta \int_{\mathbb{R}} (u_t, v_t) dt$ for $\beta > 0$.

In [7], Y. Oshima proves that there exist resolvents $\{G_\beta\}$ and $\{\widehat{G}_\beta\}$ on $L^2(\mathbb{R} \rightarrow F')$ associated with the time-dependent Dirichlet form $(\mathcal{E}, \mathcal{F}_{\mathbb{R}})$. More precisely,

Theorem 1. *For any $u \in L^2(\mathbb{R} \rightarrow F')$ and $\beta > 0$, there exist uniquely the functions $G_\beta u \in \mathcal{F}_{\mathbb{R}}$ and $\widehat{G}_\beta u \in \mathcal{F}_{\mathbb{R}}$ such that*

$$\mathcal{E}_\beta(G_\beta u, v) = \mathcal{E}_\beta(\widehat{G}_\beta v, u) = \int_{\mathbb{R}} (u_t, v_t) dt.$$

for all $v \in L^2(\mathbb{R} \rightarrow F)$.

Secondly, there exists a Hunt process (Z_t, \mathbb{P}_z) on $\mathbb{R} \times \mathbb{R}^d$ associated with the time-dependent Dirichlet form $(\mathcal{E}, \mathcal{F}_{\mathbb{R}})$. If $Z_t = (\tau(t), X_{\tau(t)})$ is a decomposition of Z into processes τ on \mathbb{R} and X on \mathbb{R}^d , then $\tau(t) = \tau(0) + t$ is the uniform motion to the right and X is a time-inhomogenous Markov process.

We consider the transition operator T_h for the space-time process Z and the transition operator $P_{s,t}$ for its component X . Then

$$T_h u(s, x) = \mathbb{E}_{(s,x)}[f(\tau(h), X_{\tau(h)})] = (P_{s,s+h} u_{s+h})(x) \quad a.e. (s, x) \in \mathbb{R} \times \mathbb{R}^d$$

for any $u \in L^2(\mathbb{R} \rightarrow H)$. A similar identity holds for the dual operators. Let $\widehat{T}_h, \widehat{P}_{s,t}$ be the dual operators of T_h and $P_{s,t}$, respectively. For any $\phi \in L^2(\mathbb{R} \rightarrow H)$,

$$\begin{aligned} \int_{\mathbb{R}} ((\widehat{T}_h u)_s, \phi_s) ds &= \int_{\mathbb{R}} ((T\phi)_s, u_s) ds = \int_{\mathbb{R}} (P_{s,s+h} \phi_{s+h}, u_s) ds \\ &= \int_{\mathbb{R}} (P_{s-h,s} \phi_s, u_{s-h}) ds = \int_{\mathbb{R}} ((\widehat{P}_{s-h,s} u_{s-h}), \phi_s) ds. \end{aligned}$$

So the dual operators $\widehat{T}_h, \widehat{P}_{s,t}$ satisfy that

$$\widehat{T}_h u(s, x) = (\widehat{P}_{s-h,s} u_{s-h})(x) \quad a.e. (s, x) \in \mathbb{R} \times \mathbb{R}^d.$$

3 Parabolic Equations

Let L_t be the self-adjoint operator on $L^2(\mathbb{R}^d)$ associated with $(E^{(t)}, F)$. Fix an interval $I = [\sigma, \tau]$. We say that u is a weak solution to the parabolic equation

$$\frac{\partial u}{\partial t} = -L_t u \quad \text{on } I \times \mathbb{R}^d \quad (5)$$

if $u \in \mathcal{F}_I$ and

$$\int_I \left(\frac{\partial u}{\partial t}, \phi \right) dt - \int_I E^{(t)}(u_t, \phi_t) dt = 0 \quad \text{for all } \phi \in \mathcal{F}_I. \quad (6)$$

For any function $f \in H$, we say that u is a solution to the terminal value problem

$$\frac{\partial u}{\partial t} = -L_t u, \quad u_\tau = f \quad (7)$$

if u solves the parabolic equation (5) and $\lim_{t \uparrow \tau} u_t = f$ in H . It is established in [10, Proposition 1.2] that for every $f \in H$, there is a unique solution u that solves (7). This suggests an alternative way defining the transition operator, namely setting $\tilde{P}_{\sigma, \tau} f = u_\sigma$. Then we have the following basic properties of $\{\tilde{P}_{\sigma, \tau}\}$.

Proposition 1. $\{\tilde{P}_{\sigma, \tau}\}$ satisfies that

- (i) For any $0 \leq \sigma \leq \tau \leq \mu$, $\tilde{P}_{\sigma, \tau} \tilde{P}_{\tau, \mu} f = \tilde{P}_{\sigma, \mu} f$.
- (ii) $\sigma \mapsto \tilde{P}_{\sigma, \tau}$ is strongly continuous in H on $(-\infty, \tau]$.
- (iii) $\tilde{P}_{\sigma, \tau} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is a contraction operator.

Proof. Let u be the solution to the terminal value problem

$$\frac{\partial u}{\partial t} = -L_t u, \quad u_\mu = f.$$

Since $u_\tau = \tilde{P}_{\tau, \mu}$ by definition, u also solves the terminal value problem

$$\frac{\partial u}{\partial t} = -L_t u, \quad u_\tau = \tilde{P}_{\tau, \mu} f.$$

Therefore, $\tilde{P}_{\sigma, \tau} \tilde{P}_{\tau, \mu} f = u_\sigma = \tilde{P}_{\sigma, \mu} f$. Eq. (6) together with integration by parts yield that for any $v \in \mathcal{F}_I$, and $r < s \in I$,

$$(u_s, v_s) - (u_r, v_r) = \int_r^s \left(u_t, \frac{\partial v}{\partial t} \right) dt + \int_r^s E^{(t)}(u_t, v_t) dt. \quad (8)$$

In particular, when $v = u$, noting that $(u_t, \frac{\partial u}{\partial t}) = \frac{1}{2} \frac{\partial}{\partial t} \|u_t\|_H^2$, we have

$$\|u_s\|_H^2 - \|u_r\|_H^2 = 2 \int_r^s E^{(t)}(u_t, u_t) dt \geq 0. \quad (9)$$

It follows immediately that $\tilde{P}_{\sigma, \tau}$ is a contraction operator and $t \mapsto \|u_t\|_{L^2(\mathbb{R}^d)}$ is continuous and decreasing on \bar{I} (this in fact has been observed in [10, (1.19)]). At last, for any $\phi \in F$, taking $v = \phi$ in Eq. (8), we have

$$(u_s, \phi) - (u_r, \phi) = \int_r^s E^{(t)}(u_t, \phi) dt$$

and consequently, $t \mapsto (u_t, \phi)$ is continuous on $(-\infty, \tau]$. Since F is dense in H , $s \mapsto \tilde{P}_{s, t}$ is weakly continuous on $(-\infty, t]$. As $\|u_t\|_{L^2(\mathbb{R}^d)}$ is continuous on $(-\infty, \tau]$, it follows that $t \mapsto u_t$ is strongly continuous in H on $(-\infty, \tau]$. \square

The next proposition shows that $\tilde{P}_{\sigma, \tau}$ is Markovian.

Proposition 2. Let $f \in H$. If $0 \leq f \leq 1$, then $0 \leq \tilde{P}_{\sigma, \tau} f \leq 1$.

Proof. This proposition has been established in [10, Lemma 1.4]. For reader's convenience, we spell out the details. Let $u_t = \tilde{P}_{t,\tau}f$ and $u^\# = (u \vee 0) \wedge 1$. By Eq. (8),

$$(u_\tau, u_\tau - u_\tau^\#) - (u_\sigma, u_\sigma - u_\sigma^\#) = \int_\sigma^\tau \left(u_t, \frac{\partial}{\partial t}(u_t - u_t^\#) \right) dt + \int_\sigma^\tau E^{(t)}(u_t, u_t - u_t^\#) dt.$$

Since

$$\left(u_t, \frac{\partial}{\partial t}(u_t - u_t^\#) \right) = \begin{cases} 0 & \text{if } 0 \leq u_t \leq 1 \\ \frac{1}{2} \frac{\partial}{\partial t} \|u_t\|_H^2 & \text{otherwise,} \end{cases}$$

and $P_{\sigma,\tau}$ is a contraction operator, we have $\int_\sigma^\tau \left(u_t, \frac{\partial}{\partial t}(u_t - u_t^\#) \right) dt \geq 0$. It is easy to verify that unit contractions operate on $E^{(t)}$ and $E^{(t)}(u_t, u_t - u_t^\#) \geq 0$. Noting that $(u_\tau, u_\tau - u_\tau^\#) = 0$, we have $(u_\sigma, u_\sigma - u_\sigma^\#) \leq 0$. So $u_\sigma^\# = u_\sigma$ or equivalently $0 \leq u_\sigma \leq 1$. \square

We can also define the dual operators using the same scheme. Consider the parabolic equation

$$\frac{\partial u}{\partial t} = L_t u \quad \text{on } I \times \mathbb{R}^d \quad (10)$$

We say that u is a solution to Eq. (10) if $u \in \mathcal{F}_I$ and

$$\int_I \left(\frac{\partial u}{\partial t}, \phi \right) dt + \int_I E^{(t)}(u_t, \phi_t) dt = 0. \quad (11)$$

For any function $f \in H$, we say that u is a solution to the initial value problem

$$\frac{\partial u}{\partial t} = L_t u, \quad u_\sigma = f, \quad (12)$$

if u solves Eq. (10) and $\lim_{t \downarrow \sigma} u_t = f$ in H . Now define $\tilde{P}_{\sigma,\tau}f = u_\tau$. We show that $\tilde{P}_{\sigma,\tau}$ and $\tilde{\tilde{P}}_{\sigma,\tau}$ are indeed dual operators.

Proposition 3. For any $f, g \in H$,

$$(\tilde{P}_{\sigma,\tau}f, g) = (f, \tilde{\tilde{P}}_{\sigma,\tau}g).$$

Proof. Let $u_t = \tilde{P}_{t,\tau}f$ and $v_t = \tilde{\tilde{P}}_{\sigma,t}g$. Since u_t and v_t solves Eq. (5) and Eq. (10) respectively, it follows from Eq. (8) and (11) that

$$(u_\tau, v_\tau) - (u_\sigma, v_\sigma) = \int_I \left(u_t, \frac{\partial v}{\partial t} \right) dt + \int_I E^{(t)}(u_t, v_t) dt = 0.$$

However, $(u_\tau, v_\tau) = (f, \tilde{\tilde{P}}_{\sigma,\tau}g)$ and $(u_\sigma, v_\sigma) = (\tilde{P}_{\sigma,\tau}f, g)$, we arrive at our conclusion. \square

Proposition 4. For any $p \geq 1$, $\tilde{P}_{\sigma,\tau}$ is a contraction operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.

Proof. By Proposition 2, $\tilde{P}_{\sigma,\tau}$ is a contraction from $L^\infty(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$. With a minor modification one can prove that Proposition 2 also holds for the dual operator $\tilde{\tilde{P}}_{\sigma,\tau}$. Thus $\tilde{\tilde{P}}_{\sigma,\tau}$ is a contraction from $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. The rest follows from interpolation. \square

Next, we show that the transition operators defined through probabilistic approach $\{P_{s,t}; s < t\}$ and analytic approach $\{\tilde{P}_{s,t}, s < t\}$ are identical.

Proposition 5. For any $s < t$ and $f \in H$, $P_{s,t}f = \tilde{P}_{s,t}f$.

Proof. For any $u \in L^2(\mathbb{R} \rightarrow H)$, we define analogues of the transition operator T_h and the resolvent G_β ; that is, for $h > 0$, let

$$\tilde{T}_h u(s, x) = (\tilde{P}_{s, s+h} u_{s+h})(x)$$

and for $\beta > 0$, let

$$\tilde{G}_\beta u(s, x) = \int_0^\infty e^{-\beta h} \tilde{T}_h f(s, x) dh. \quad (13)$$

A simple change of variable in (13) gives

$$(\tilde{G}_\beta u)_s = \int_0^\infty e^{-\beta h} (\tilde{P}_{s, s+h} u_{s+h}) dh = e^{\beta s} \int_s^\infty e^{-\beta t} (\tilde{P}_{s, t} u_t) dt.$$

Therefore, for any $v \in L^2(\mathbb{R} \rightarrow F)$,

$$\begin{aligned} \left(\frac{\partial}{\partial s} (\tilde{G}_\beta u), v_s \right) &= \left(\beta e^{\beta s} \int_s^\infty e^{-\beta t} (\tilde{P}_{s, t} u_t) dt - u_s + e^{\beta s} \int_s^\infty e^{-\beta t} \frac{\partial}{\partial s} (\tilde{P}_{s, t} u_t) dt, v_s \right) \\ &= \beta \left((\tilde{G}_\beta u)_s, v_s \right) - (u_s, v_s) + E^{(s)} \left((\tilde{G}_\beta u)_s, v_s \right). \end{aligned}$$

Integrating over \mathbb{R} , we obtain that $\mathcal{E}_\beta(\tilde{G}_\beta u, v) = \int_{\mathbb{R}} (u_s, v_s) ds$. By the uniqueness part of Theorem 1, \tilde{G}_β coincides with G_β . It then follows from Hille-Yosida theorem that $T_h = \tilde{T}_h$. At last, for any $f \in H$, taking $u(s, x) = 1_{[-q, q]}(s) f(x)$ and then letting $q \rightarrow \infty$ via rational numbers, we have

$$P_{s, s+h} f = \tilde{P}_{s, s+h} f \quad a.e. s \in \mathbb{R}.$$

Since both $P_{s, t}$ and $\tilde{P}_{s, t}$ are strongly continuous, we have $P_{s, t} = \tilde{P}_{s, t}$ for all $s < t$. \square

Therefore, the properties of $\tilde{P}_{s, t}$ also hold for $P_{s, t}$. In particular, $P_{s, t}$ and the dual operator $\hat{P}_{s, t}$ satisfy the following backward and forward equations, respectively.

Corollary 1. *Let $I = [\sigma, \tau]$.*

(i) *For any $f \in H$, $P_{\cdot, \tau} \in \mathcal{F}_I$. Moreover, $P_{t, \tau} f$ satisfies the backward equation*

$$\left(\frac{\partial}{\partial t} P_{t, \tau} f, g \right) = E^{(t)}(P_{t, \tau} f, g) \quad \forall t \in I, g \in F. \quad (14)$$

(ii) *For any $f \in H$, $\hat{P}_{\sigma, \cdot} f \in F_I$. Moreover, $\hat{P}_{\sigma, t}$ satisfies the forward equation*

$$\left(\frac{\partial}{\partial t} \hat{P}_{\sigma, t} f, g \right) = -E^{(t)}(\hat{P}_{\sigma, t} f, g) \quad \forall t \in I, g \in F. \quad (15)$$

4 On-Diagonal Estimates

In this section, we establish the on-diagonal upper bound for $P_{s, t}$. We need the following Nash type inequality.

Theorem 2. *There exists a constant $c_3 = c_3(d, \alpha, c_1)$, such that for any $t > 0$ and $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have*

$$\|f\|_2^{2+2\alpha/d} \leq c_3 E^{(t)}(f, f) \|f\|_1^{2\alpha/d}. \quad (16)$$

Proof. This has been established in [3]. For reader's convenience, we present a proof here, which is modified from that of [4, Theorem 3.1]. First, observe that $E^{(t)}(f, f) = 0$ implies that $f = 0$ a.e.. Clearly the

inequality holds in this case. So without loss of generality we may assume that $0 < E^{(t)}(f, f) < \infty$. Fix a positive constant r . Consider the average of function f over the ball $B_r(x)$

$$f_r(x) = \frac{1}{\omega_d r^d} \int_{B_r(x)} f(y) dy,$$

where ω_d is the volume of d -dimensional unit ball. We have

$$\|f\|_2^2 \leq 2\|f - f_r\|_2^2 + 2\|f_r\|_2^2.$$

The first term on the right side is bounded by

$$\begin{aligned} \|f - f_r\|_2^2 &= \int_{\mathbb{R}^d} \left(\frac{1}{\omega_d r^d} \int_{B_r(x)} (f(x) - f(y)) dy \right)^2 dx \\ &\leq \frac{1}{\omega_d r^d} \int_{\mathbb{R}^d} \int_{B_r(x)} (f(x) - f(y))^2 dy dx \\ &\leq \frac{r^\alpha}{\omega_d c_1} \int_{\mathbb{R}^d} \int_{B_r(x)} \frac{c(t, x, y)}{|x - y|^{d+\alpha}} (f(x) - f(y))^2 dy dx \\ &\leq \frac{r^\alpha}{\omega_d c_1} E^{(t)}(f, f). \end{aligned}$$

For the second term, we have the naive bounds

$$|f_r(x)| \leq \frac{1}{\omega_d r^d} \|f\|_1, \quad \|f_r\|_1 \leq \|f\|_1,$$

and hence

$$\|f_r\|_2^2 = \|f_r\|_1 \|f_r\|_\infty \leq \frac{1}{\omega_d r^d} \|f\|_1^2.$$

Combining the preceding inequalities, we have

$$\|f\|_2^2 \leq \frac{2}{\omega_d c_1} E^{(t)}(f, f) r^\alpha + \frac{2}{\omega_d} \|f\|_1^2 r^{-d}.$$

To minimize the right side, we take $r = E^{(t)}(f, f)^{-1/(d+\alpha)} \|f\|_1^{2/(d+\alpha)}$, obtaining the desired inequality. \square

Theorem 3. *The transition operator $P_{s,t}$ possesses a kernel $p(s, t, x, y)$. Moreover, there exists a positive constant $c_4 = c_4(d, \alpha, c_1)$ such that*

$$p(s, t, x, y) \leq c_4 (t - s)^{-d/\alpha} \tag{17}$$

for all $t > s \geq 0$ and $x, y \in \mathbb{R}^d$.

Proof. Suppose that $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ with $\|f\|_1 = 1$. For any $t > s \geq 0$, let $u_s = P_{s,t} f$. By the backward equation and Nash inequality,

$$\frac{\partial}{\partial s} \|u_s\|_2^2 = 2E^{(s)}(u_s, u_s) \geq \frac{2}{c_3} \|u_s\|_2^{2+2\alpha/d}.$$

It implies that $\frac{\partial}{\partial s} \left(\|u(s)\|_2^{-2\alpha/d} \right) \leq -2\alpha/dc_3$ and hence $\|u_s\|_2 \leq c(t - s)^{-d/2\alpha}$. Since $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we have $\|P_{s,t}\|_{1 \rightarrow 2} \leq c(t - s)^{-d/2\alpha}$.

On the other hand, fix $s \geq 0$ and consider the function $\hat{u}_t = \hat{P}_{s,t} f$. By the forward equation and again Nash inequality, we can derive in a similar way that $\|\hat{u}_t\|_2 \leq c(t - s)^{-d/2\alpha}$. It follows that $\|\hat{P}_{s,t}\|_{1 \rightarrow 2} \leq c(t - s)^{-d/2\alpha}$. Combining these two estimates we have $\|P_{s,t}\|_{1 \rightarrow \infty} = \|P_{s,r}\|_{1 \rightarrow 2} \|P_{r,t}\|_{2 \rightarrow \infty} \leq c(t - s)^{-d/\alpha}$, where $r = (t + s)/2$. This proves the existence of a kernel $p(s, t, x, y)$ and that $p(s, t, x, y)$ satisfies the on-diagonal estimate (17). \square

5 Off-Diagonal Estimates: Davies Methods

To establish the off-diagonal estimates, we follow the classic approach introduced by E. B. Davies. Consider transition operator $P_{s,t}^\psi$ defined by

$$P_{s,t}^\psi f(x) = e^{\psi(x)} P_{s,t}(e^{-\psi} f)(x), \quad (18)$$

where ψ is a function to be determined later. We establish an on-diagonal estimate for $P_{s,t}^\psi$ with the aid of energy measures, then vary ψ over a large class of functions to obtain off-diagonal estimates for $P_{s,t}$.

5.1 Energy Measures

For any $f, g \in F$, the *energy measure* of $E^{(t)}(f, g)$ is defined by

$$d\Gamma^{(t)}(f, g) = \left(\int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x - y|^{d+\alpha}} (f(x) - f(y))(g(x) - g(y)) dy \right) dx \quad (19)$$

Apparently the definition above can be extended to functions in $\widehat{F} = \{f + c : f \in F, c \text{ is a constant}\}$. We define \mathcal{S} to be the set of functions $\psi \in \widehat{F}$ such that

$$\Gamma^{(t)}(\psi)^2 = \left\| \frac{d}{dx} \left(e^{-2\psi} \Gamma^{(t)}(e^\psi, e^\psi) \right) \right\|_\infty \vee \left\| \frac{d}{dx} \left(e^{2\psi} \Gamma^{(t)}(e^{-\psi}, e^{-\psi}) \right) \right\|_\infty < \infty, \quad (20)$$

for all $t \geq 0$. We can verify by direct calculations that the following two inequalities hold.

Theorem 4. *For any $f \in F$,*

$$E^{(t)}(e^\psi f, e^{-\psi} f) \geq E^{(t)}(f, f) - \Gamma^{(t)}(\psi)^2 \|f\|_2^2. \quad (21)$$

Moreover, for any $f \in F$, $f > 0$ and $p \geq 1$,

$$E^{(t)}(e^\psi f^{2p-1}, e^{-\psi} f) \geq \frac{1}{2p} E^{(t)}(f^p, f^p) - 9p \Gamma^{(t)}(\psi) \|f\|_{2p}^{2p}. \quad (22)$$

Proof. To prove (21), we use the fact that

$$\begin{aligned} (e^{\psi(x)} f(x) - e^{\psi(y)} f(y))(e^{-\psi(x)} f(x) - e^{-\psi(y)} f(y)) - (f(x) - f(y))^2 \\ = f(x)f(y)(e^{\psi(x)} - e^{\psi(y)})(e^{-\psi(x)} - e^{-\psi(y)}). \end{aligned}$$

It follows that

$$\begin{aligned} E^{(t)}(e^\psi f, e^{-\psi} f) - E^{(t)}(f, f) \\ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x - y|^{d+\alpha}} f(x)f(y)(e^{\psi(x)} - e^{\psi(y)})(e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\ \geq - \left(\int_{\mathbb{R}^d} f^2 e^{-2\psi} d\Gamma^{(t)}(e^\psi, e^\psi) \right)^{1/2} \left(\int_{\mathbb{R}^d} f^2 e^{2\psi} d\Gamma^{(t)}(e^{-\psi}, e^{-\psi}) \right)^{1/2} \\ \geq - \Gamma^{(t)}(\psi) \|f\|_2^2. \end{aligned}$$

For $p \geq 1$, noticing that

$$\begin{aligned} (e^{\psi(x)} f(x)^{2p-1} - e^{\psi(y)} f(y)^{2p-1})(e^{-\psi(x)} f(x) - e^{-\psi(y)} f(y)) - (f(x)^{2p-1} - f(y)^{2p-1})(f(x) - f(y)) \\ = f^{2p-1}(x)f(y)e^{\psi(x)}(e^{-\psi(x)} - e^{-\psi(y)}) - f(x)f^{2p-1}(y)e^{\psi(y)}(e^{-\psi(x)} - e^{-\psi(y)}), \end{aligned}$$

we have

$$\begin{aligned}
& E^{(t)}(e^\psi f^{2p-1}, e^{-\psi} f) - E^{(t)}(f^{2p-1}, f) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f^{2p-1}(x) f(y) e^{\psi(x)} (e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\
&\quad - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f(x) f^{2p-1}(y) e^{\psi(y)} (e^{-\psi(x)} - e^{-\psi(y)}) dx dy.
\end{aligned}$$

To estimate the first integral, we use the fact that

$$f^{2p-1}(x)f(y) = f^p(x)f^p(y) + f^p(x)(f^p(x) - f^p(y)) - f^{2p-1}(x)(f(x) - f(y)).$$

Splitting the integral into three parts, we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f^{2p-1}(x) f(y) e^{\psi(x)} (e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\
&\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f^p(x) f^p(y) e^{\psi(x)} (e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\
&\quad - \left(\int_{\mathbb{R}^d} f^{2p} e^{2\psi} d\Gamma^{(t)}(e^{-\psi}, e^{-\psi}) \right)^{1/2} E^{(t)}(f^p, f^p)^{1/2} \\
&\quad - \left(\int_{\mathbb{R}^d} f^{2p} e^{2\psi} d\Gamma^{(t)}(e^{-\psi}, e^{-\psi}) \right)^{1/2} \left(\int_{\mathbb{R}^d} f^{2p-2} d\Gamma^{(t)}(f, f) \right)^{1/2} \\
&\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f^p(x) f^p(y) e^{\psi(x)} (e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\
&\quad - \Gamma^{(t)}(\psi) \|f\|_{2p}^p \left(E^{(t)}(f^p, f^p)^{1/2} + \left(\int_{\mathbb{R}^d} f^{2p-2} d\Gamma^{(t)}(f, f) \right)^{1/2} \right).
\end{aligned}$$

A similar estimate holds for the second integral,

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f(x) f^{2p-1}(y) e^{\psi(y)} (e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\
&\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f^p(x) f^p(y) e^{\psi(y)} (e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\
&\quad - \Gamma^{(t)}(\psi) \|f\|_{2p}^p \left(E^{(t)}(f^p, f^p)^{1/2} + \left(\int_{\mathbb{R}^d} f^{2p-2} d\Gamma^{(t)}(f, f) \right)^{1/2} \right).
\end{aligned}$$

Combining the preceding two inequalities, we have

$$\begin{aligned}
& E^{(t)}(e^\psi f^{2p-1}, e^{-\psi} f) - E^{(t)}(f^{2p-1}, f) \\
&\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f^p(x) f^p(y) (e^{\psi(x)} - e^{\psi(y)}) (e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\
&\quad - 2\Gamma^{(t)}(\psi) \|f\|_{2p}^p \left(E^{(t)}(f^p, f^p)^{1/2} + \left(\int_{\mathbb{R}^d} f^{2p-2} d\Gamma^{(t)}(f, f) \right)^{1/2} \right).
\end{aligned}$$

Similar to the first case when $p = 1$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} f^p(x) f^p(y) (e^{\psi(x)} - e^{\psi(y)}) (e^{-\psi(x)} - e^{-\psi(y)}) dx dy \\
&\geq - \left(\int_{\mathbb{R}^d} f^{2p} e^{-2\psi} d\Gamma^{(t)}(e^\psi, e^\psi) \right)^{1/2} \left(\int_{\mathbb{R}^d} f^{2p} e^{2\psi} d\Gamma^{(t)}(e^{-\psi}, e^{-\psi}) \right)^{1/2} \geq -\Gamma^{(t)}(\psi)^2 \|f\|_{2p}^{2p}.
\end{aligned}$$

To bound the second term, we use the facts that

$$E^{(t)}(f^{2p-1}, f) \geq \frac{2p-1}{p^2} E^{(t)}(f^p, f^p),$$

$$\int_{\mathbb{R}^d} f^{2p-2} d\Gamma^{(t)}(f, f) \geq \frac{1}{p^2} E^{(t)}(f^p, f^p).$$

Finally, we have

$$\begin{aligned} E^{(t)}(e^\psi f^{2p-1}, e^{-\psi} f) - E^{(t)}(f^{2p-1}, f) \\ \geq \frac{2p-1}{p^2} E^{(t)}(f^p, f^p) - \frac{2p+2}{p} \Gamma^{(t)}(\psi) E^{(t)}(f^p, f^p)^{1/2} \|f\|_{2p}^p - \Gamma^{(t)}(\psi)^2 \|f\|_{2p}^{2p} \\ \geq \frac{1}{2p} E^{(t)}(f^p, f^p) - 9p \Gamma^{(t)}(\psi) \|f\|_{2p}^{2p}. \end{aligned}$$

□

5.2 Off-diagonal Estimates

We also need the following technical lemma for the off-diagonal estimates. It is a backward analogue of Lemma (3.21) in [2].

Lemma 1. *Suppose that $u : [0, t] \rightarrow (0, \infty)$ is absolutely continuous and satisfies the following differential inequality*

$$u'(s) \geq \frac{1}{cp} u(s)^{1+\beta p} v(s)^{-\beta p} - \lambda p u(s), \quad (23)$$

where $v : [0, t] \rightarrow (0, \infty)$ is a bounded measurable function, and c, β, λ, p are positive constants with $p \geq 2$. Then we have

$$u(s) \leq \left(\frac{\beta}{2cp^2} \right)^{-1/\beta p} (t-s)^{-\frac{1}{\beta} \cdot \frac{p-1}{p}} e^{\lambda(t-s)/p} w(s) \quad (24)$$

for all $s \in [0, t)$. Here

$$w(s) = \operatorname{ess\,sup}_{r \in [s, t]} (t-r)^{\frac{1}{\beta} \cdot \frac{p-2}{p}} v(r).$$

Proof. First, we set $f(s) = e^{\lambda p s} u(s)$. f satisfies the differential inequality

$$f'(s) \geq \frac{1}{cp} e^{\lambda p s} f(s)^{1+\beta p} v(s)^{-\beta p}.$$

It follows that

$$(f^{-\beta p}(s))' \leq -\frac{\beta}{c} e^{-\lambda \beta p^2 s} v(s)^{-\beta p}.$$

Integrating over $[s, t]$ gives

$$e^{-\lambda \beta p^2 s} u(s)^{-\beta p} \geq \frac{\beta}{c} \int_s^t e^{-\lambda \beta p^2 r} v(r)^{-\beta p} dr.$$

Next, let w be defined as (1). Since $w(s)$ is non-increasing, we have

$$\begin{aligned} u(s)^{-\beta p} &\geq \frac{\beta}{c} w(s)^{-\beta p} \int_s^t (t-r)^{p-2} e^{-\lambda \beta p^2 (r-s)} dr. \\ &\geq \frac{\beta}{c} (t-s)^{p-1} w(s)^{-\beta p} \int_0^1 (1-r)^{p-2} e^{-\lambda \beta p^2 (t-s)r} dr. \end{aligned}$$

The integral in the last expression is bounded from below by

$$\int_0^{1/p^2} (1-r)^{p-2} e^{-\lambda\beta p^2(t-s)r} \geq e^{-\lambda\beta(t-s)} \cdot \frac{1}{p-1} (1 - (1 - 1/p^2)^{p-1}) > e^{-\lambda\beta(t-s)}/2p^2.$$

Combining the last two inequalities, we obtain (24). \square

Now we're ready to establish the off-diagonal estimates.

Theorem 5. *There exists a positive constant $c_5 = c_5(d, \alpha, c_1)$ such that for any $\psi \in \mathcal{S}$ with $\Gamma^{(r)}(\psi) \leq \gamma$ for all r , we have*

$$p(s, t, x, y) \leq c_5(t-s)^{-d/\alpha} \exp(-|\psi(x) - \psi(y)| + 10\gamma(t-s)) \quad (25)$$

where $t > s \geq 0$ and $x, y \in \mathbb{R}^d$.

Proof. We first use the backward equation to obtain an upper bound for $\|P_{s,t}^\psi\|_{2 \rightarrow \infty}$. Let $f \in F$ and $f > 0$. Fix $t > 0$. We set $f_s = P_{s,t}^\psi f$ for $s \in [0, t]$. By the backward equation and Eq. (21), we have

$$\frac{\partial}{\partial s} \|f_s\|_2^2 = 2 \left(\frac{\partial}{\partial s} (P_{s,t} e^{-\psi} f), e^\psi f_s \right) = 2E^{(s)}(e^{-\psi} f_s, e^\psi f_s) \geq -\gamma \|f_s\|_2^2. \quad (26)$$

For $k \in \mathbb{N}$ we put $p_k = 2^k$ and $u_k(s) = \|f_s\|_{p_k}$. Eq. (26) implies that

$$u_1(s) \leq \|f\|_2 e^{\gamma(t-s)}. \quad (27)$$

Similarly, for any $p \geq 1$, by the backward equation, Eq. (22) and the Nash inequality, we have

$$\begin{aligned} \frac{\partial}{\partial s} \|f_s\|_{2p}^{2p} &= 2p \left(\frac{\partial}{\partial s} (P_{s,t} e^{-\psi} f), e^\psi f_s^{2p-1} \right) \\ &= 2pE^{(s)}(e^{-\psi} f_s, e^\psi f_s^{2p-1}) \\ &\geq E^{(s)}(f_s^p, f_s^p) - 18p^2\gamma \|f_s\|_{2p}^{2p} \\ &\geq \frac{1}{c_3} \|f_s\|_{2p}^{2p(1+\alpha/d)} \|f_s\|_p^{-2p\alpha/d} - 18p^2\gamma \|f_s\|_{2p}^{2p}. \end{aligned}$$

Then by taking $p = p_k$, we obtain the recursive inequality

$$u'_{k+1}(s) \geq \frac{1}{2p_k c_3} u_{k+1}^{1+2p_k\alpha/d}(s) u_k^{-2p_k\alpha/d}(s) - 9p_k\gamma u_{k+1}(s).$$

Now we apply Lemma 1 to u_{k+1} with $v = u_k$, $c = 2c_3$, $\beta = 2\alpha/d$, $\lambda = 9\gamma$ and $p = p_k$, obtaining

$$u_{k+1}(s) \leq \left(\frac{\alpha}{2dc_3 p_k^2} \right)^{-d/2\alpha p_k} (t-s)^{-\frac{d}{2\alpha} \cdot \frac{p_k-1}{p_k}} e^{9\gamma(t-s)/p_k} w_k(s),$$

where w_k is defined as in the lemma

$$w_k(s) = \max_{r \in [s, t]} (t-r)^{\frac{d}{2\alpha} \cdot \frac{p_k-2}{p_k}} u_k(r).$$

Multiplying two sides by $(t-s)^{-\frac{d}{2\alpha} \cdot \frac{p_k-1}{p_k}}$ and then maximizing over $[s, t]$, we have

$$w_{k+1}(s) \leq \left(\frac{\alpha}{2dc_3 p_k^2} \right)^{-d/2\alpha p_k} e^{9\gamma(t-s)/p_k} w_k(s).$$

It follows that $\limsup_{k \rightarrow \infty} w_k(s) \leq ce^{9\gamma(t-s)}w_1(s)$, where $c = c(d, \alpha, c_1)$ is a positive constant. Since $w_1(s) = \max_{r \in [0, t]} u_1(r) \leq \|f\|_2 e^{\gamma(t-s)}$ by (27), we have

$$(t-s)^{d/2\alpha} \|f_s\|_\infty \leq ce^{10\gamma(t-s)} \|f\|_2.$$

So $\|P_{s,t}^\psi\|_{2 \rightarrow \infty} \leq c(t-s)^{-d/2\alpha} e^{10\gamma(t-s)}$.

On the other hand, fixing $s > 0$ and setting $\widehat{f}_t = \widehat{P}_{s,t}^\psi f$ for $t \in [s, \infty)$, with the aid of the forward equation, we can prove that

$$\|\widehat{P}_{s,t}^\psi\|_{2 \rightarrow \infty} \leq c(t-s)^{-d/2\alpha} e^{10\gamma(t-s)}.$$

Combining these two estimates we have

$$\|P_{s,t}^\psi\|_{1 \rightarrow \infty} \leq \|P_{s,r}^\psi\|_{1 \rightarrow 2} \|P_{r,t}^\psi\|_{2 \rightarrow \infty} \leq c(t-s)^{-d/\alpha} e^{10\gamma(t-s)},$$

where $r = (t+s)/2$. So the density function $p^\psi(s, t, x, y)$ of $P_{s,t}^\psi$ exists. Since

$$p^\psi(s, t, x, y) = e^{\psi(x)} p(s, t, x, y) e^{-\psi(y)},$$

we have

$$p(s, t, x, y) \leq c(t-s)^{-d/\alpha} \exp[-(\psi(x) - \psi(y)) + 10\gamma(t-s)]$$

Since $\Gamma^{(t)}(-\psi) = \Gamma^{(t)}(\psi)$ for all $t \geq 0$, the same upper bound holds for $-\psi$ and we obtain the off-diagonal estimate as in Eq. (25). \square

5.3 Choice of ψ

We point out that \mathcal{S} contains functions of the following form.

Lemma 2. *Let*

$$\psi(x) = k(|x - x_0| \wedge R) \quad x \in \mathbb{R}^d, \quad (28)$$

where k and R are positive constants, $x_0 \in \mathbb{R}^d$. Then $\psi \in \mathcal{S}$. In fact, there exists a positive constant $c_6 = c_6(d, \alpha, c_2)$ such that

$$\Gamma^{(t)}(\psi) \leq c_6 e^{3kR} R^{-\alpha} \quad \text{for all } t \geq 0.$$

Proof. Note that $|1 - e^x| \leq |x|e^{|x|}$ for all $x \in \mathbb{R}$. We have

$$\begin{aligned} \left| \frac{d}{dx} (e^{-2\psi} \Gamma(e^\psi, e^\psi)) \right| &= e^{-2\psi(x)} \int_{\mathbb{R}^d} \frac{c(t, x, y)}{|x-y|^{d+\alpha}} (e^{\psi(x)} - e^{\psi(y)})^2 dy \\ &\leq c_2 \int_{\mathbb{R}^d} |x-y|^{-d-\alpha} (1 - e^{\psi(y)-\psi(x)})^2 dy \\ &\leq c_2 \int_{B_R(x_0)} |x-y|^{-d-\alpha} |\psi(x) - \psi(y)|^2 e^{2|\psi(x)-\psi(y)|} dy \\ &\quad + c_2 \int_{B_R(x_0)^c} |x-y|^{-d-\alpha} e^{2|\psi(x)-\psi(y)|} dy \\ &\leq c_2 k^2 e^{2kR} \int_{B_R(x_0)} |x-y|^{-d-\alpha+2} dy + c_2 e^{2kR} \int_{B_R(x_0)^c} |x-y|^{-d-\alpha} dy \\ &\leq c_2 \left(\frac{\omega_d}{2-\alpha} k^2 e^{2kR} R^{2-\alpha} + \frac{\omega_d}{\alpha} e^{2kR} R^{-\alpha} \right) \leq ce^{3kR} R^{-\alpha}. \end{aligned}$$

Clearly, the same estimate holds for $\frac{d}{dx} (e^{2\psi} \Gamma(e^{-\psi}, e^{-\psi}))$. Therefore $\psi \in \mathcal{S}$ and $\Gamma^{(t)}(\psi) \leq ce^{3kR} R^{-\alpha}$. \square

By choosing an appropriate $\psi \in \mathcal{S}$, we have the following upper bound for the transition density $p(s, t, x, y)$.

Theorem 6. *There exists a constant $c_7 = c_7(d, \alpha, c_1, c_2)$ such that*

$$p(s, t, x, y) \leq c_7 \left((t-s)^{-d/\alpha} \wedge \frac{t-s}{|x-y|^{d+\alpha}} \right) \quad (29)$$

for all $t > s \geq 0$ and $x, y \in \mathbb{R}^d$.

Proof. For $|x-y| \leq (t-s)^{1/\alpha}$, (29) follows from the on-diagonal estimate (17). Hence we may assume that $|x-y| < (t-s)^{1/\alpha}$. Fix $x, y \in \mathbb{R}^d$. For simplicity, we write $r = |x-y|$. Let ψ be as defined in (28). We choose $x_0 = y$ and apply Theorem 5 with $\gamma = c_6 e^{3kR} R^{-\alpha}$, obtaining

$$p(s, t, x, y) \leq c_5 (t-s)^{-d/\alpha} \exp(-kr + 10c_6 e^{3kR} R^{-\alpha} (t-s)).$$

Set $k = \frac{1}{r}(d+\alpha) \log \frac{r}{(t-s)^{1/\alpha}}$, $R = \lambda r$ with $\lambda = \alpha/3(d+\alpha)$. The right side above simplifies to

$$c_5 (t-s)^{-d/\alpha} \left(\frac{(t-s)^{1/\alpha}}{r} \right)^{d+\alpha} \exp \left(10c_6 \left(\frac{r}{(t-s)^{1/\alpha}} \right)^{3\lambda(d+\alpha)} \frac{t-s}{(\lambda r)^\alpha} \right) = c_5 e^{10c_6/\lambda^\alpha} \frac{t-s}{r^{d+\alpha}}.$$

□

References

- [1] D. G. Aronson, Non-negative solutions of linear parabolic equations. *Annali della Scuola Normale Superiore di Pisa* **22** (1968), 607-694.
- [2] E. A. Carlen, S. Kusuoka and D. W. Stroock, Upper bounds for symmetric Markov transition functions. *Ann. Inst. Henri Poincaré-Probab. Statist.*, **23** (1987), 245-287.
- [3] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d-sets. *Stochastic Process Appl.*, **108** (2003), 27-62.
- [4] Z.-Q. Chen and T. Kumagai, heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Relat. Fields* **140** (2008), 277-317.
- [5] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*. de Gruyter, Berlin, 1994.
- [6] J. R. Norris and D. W. Stroock, Estimates on the fundamental solution to heat flows with uniformly elliptic coefficients. *Proc. London Math. Soc.* **62** (1991), 373-402.
- [7] Y. Oshima, On a construction of Markov processes associated with time dependent Dirichlet spaces, *Forum Math.* **4** (1992), 395-415.
- [8] Y. Oshima, Some Properties of Markov processes associated with time dependent Dirichlet forms, *Osaka J. Math.* **29** (1992), 103-127.
- [9] K-T. Sturm, Analysis on local Dirichlet spaces I: Recurrence, conservativeness and L^p -Liouville properties, *J. Reine Angew. Math.* **456** (1994), 173-196.
- [10] K-T. Sturm, Analysis on local Dirichlet spaces II. Upper Gaussian estimates for fundamental solutions of parabolic equations, *Osaka J. Math.* **32** (1995), 275-312.
- [11] K-T. Sturm, Analysis on local Dirichlet spaces III. The parabolic Harnack inequality, *J. Math. Pures Appl.*, **75** (1996), 273-297.