On the g_2-number of various classes of spheres and manifolds

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A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2017

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Program Authorized to Offer Degree:
UW Department of Mathematics

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Abstract

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For a $(d-1)$-dimensional simplicial complex Δ, we let $f_i = f_i(\Delta)$ be the number of i-dimensional faces of Δ for $-1 \leq i \leq d-1$. One classic problem in geometric combinatorics is the following: for a given class of simplicial complexes, find tight upper and lower bounds on the face numbers and characterize the complexes that attain these bounds. This dissertation studies these questions in various classes of simplicial complexes including balanced manifolds, flag manifolds and simplicial spheres.

A $(d-1)$-dimensional simplicial complex is called balanced if its graph is d-colorable. In Chapter 2, we determine the minimum number of vertices needed to provide balanced triangulations of $\mathbb{S}^{d-2}$-bundles over $\mathbb{S}^1$. Similar results apply to all balanced triangulated manifolds with $\beta_1 \neq 0$ and $\beta_2 = 0$.

In Chapter 3, we turn to the Upper Bound Conjecture for balanced simplicial spheres. We find the first two examples of non-octahedral balanced 2-neighborly spheres. Each construction is of dimension 3 and with 16 vertices. Along the way, we show that the rank-selected subcomplexes of a balanced simplicial sphere do not necessarily have an ear decomposition.

A simplicial complex is flag if it is the clique complex of its graph. In Chapter 4, we settle the Upper Bound Conjecture for flag 3-manifolds, establish a sharp upper bound on the number of edges of flag 5-manifolds and characterize the cases of equality.

In Chapter 5, we characterize homology manifolds with $g_2 \leq 2$. We prove that every

prime homology manifold with $g_2 = 2$ is obtained by centrally retriangulating a polytopal sphere with $g_2 \leq 1$ along a certain subcomplex. This implies that all homology spheres with $g_2 = 2$ are polytopal spheres.

In Chapter 6, we prove that for any prime homology $(d-1)$-sphere Δ $(d \geq 4)$ with $g_2(\Delta) \geq 1$ and any edge $e \in \Delta$, the graph $G(\Delta) - e$ is generically d-rigid. This confirms a conjecture of Nevo and Novinsky.

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ACKNOWLEDGMENTS

I would like to thank my advisor Professor Isabella Novik who guided and supported me in overcoming numerous obstacles I have been facing through my research. I am also grateful to the members of my committee, Steven Klee and Rekha Thomas, for their careful reading of the thesis. Finally, thanks are due to Michał Adamaszek, Moritz Firsching, Eran Nevo, Ed Swartz and many doctoral students in math department at UW for sharing their valuable ideas and insights.

Chapter 1

INTRODUCTION

A *simplicial complex* Δ on vertex set $V = V(\Delta)$ is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, and such that for every $v \in V$, $\{v\} \in \Delta$. There are many interesting classes of simplicial complexes, including graphs, the boundary complexes of simplicial polytopes and triangulations of manifolds. To study these simplicial complexes, it is helpful to find and understand some combinatorial invariants. Mathematicians noticed long time ago that the *f-numbers* of a simplicial complex, which count the number of faces in each dimension, capture rich geometric information from the associated topological space. For example, the Euler-Poincaré formula states that the alternating sum of the f-numbers of any d-polytope is exactly $1 - (-1)^d$.

Can we say more about the relations of the f-numbers for a given simplicial complex? In particular, can we characterize all possible f-vectors of a given class of simplicial complexes? Specifically, we will be interested in the classes of simplicial spheres and manifolds, that is, simplicial complexes whose geometric realization is homeomorphic to a sphere and a closed manifold, respectively. In general this problem is hard to solve, so instead we may attempt to find the tight upper and lower bounds on the face numbers and to characterize the complexes that attain these bounds. These extremal problems, known as the Upper Bound Conjecture and the Lower Bound Conjecture (UBC and LBC respectively, for short) can be traced back to the work of Brückner [14] even before the twentieth century.

The UBC for spheres gives an explicit upper bound $f_i(n, d)$ for the number of i-dimensional faces of a simplicial $(d-1)$-sphere with n vertices. The official formulation of the UBC is due to Motzkin [43], who conjectured that the bound $f_i(n, d)$ is achieved by cyclic d-polytopes with n vertices for all $2 \leq i \leq d-1$. In 1970 McMullen [42] used shellability to give a combinatorial proof of the UBC for simplicial polytopes, and in 1975 Stanley [54] extended this result to all simplicial spheres using methods from commutative algebra. Some significant generalizations of the UBC for several classes of (simplicial) homology manifolds were obtained by Novik [48], see also [29]. Along the way, new methods evolved from diverse areas such as graph theory, algebraic topology and commutative algebra, which greatly enriched our understanding of the topic.

A lot of progress has also been made on problems related to the LBC. This conjecture posited that in the class of all simplicial d-polytopes with n vertices, the stacked polytope (in other words, a connected sum of the boundary complexes of simplices) simultaneously minimizes all the face numbers. This conjecture was settled by Barnette for all simplicial polytopes [9] and even for simplicial manifolds [8]. His proofs were purely combinatorial. Later Kalai [34] used rigidity theory of frameworks to provide a different proof of this result; further, he showed that the lower bounds are attained only when the simplicial manifold is a stacked polytope. Kalai's insight was that if we consider the graph of a simplicial complex as a bar-and-joint structure, then g_2 represents the dimension of the stress space of the structure; furthermore, the lower bound on the number of edges can be simply rephrased as the condition $g_2 \geq 0$. Since then, the rigidity theory of frameworks has played a crucial role in the progress on lower-bound-related problems. One application is characterizing simplicial spheres with small g_2. Nevo and Novinsky [46] characterized all simplicial $(d-1)$-spheres with $g_2 = 1$: under an additional assumption that there are no missing facets, each such complex is either the join of two boundary complexes of simplices, or the join of a cycle and the boundary complex of a $(d-3)$-simplex.

We may further ask if it is possible to find the bounds on face numbers for simplicial spheres (or more generally, simplicial manifolds) with an additional structure. A $(d-1)$-dimensional simplicial complex Δ is called *balanced* if its graph is d-colorable. A simplicial complex is *flag* if it is the clique complex of its graph. Balanced complexes and flag complexes form important classes of simplicial complexes. For example, barycentric subdivisions of regular CW complexes, and more generally, order complexes of posets are both balanced and flag; so are Coxeter complexes.

The first part of this dissertation (Chapter 2-3) deals with the face numbers of balanced complexes. In Chapter 2, we focus on the minimal balanced triangulations of sphere bundles over the circle. For the non-balanced case, it is known that the minimal triangulation of the $(d-1)$-dimensional sphere bundle over the circle has either $2d$ or $2d+1$ vertices, depending on the parity of the dimension and whether the sphere bundle is orientable or not, see [38], [7] and [18]. Much more recently, Klee and Novik [35] gave explicit balanced triangulations of the $(d-1)$-dimensional sphere bundle over the circle with $3d$ or $3d+2$ vertices. Furthermore, they conjectured that the minimum number of vertices for balanced sphere bundles, similar to the non-balanced case, is either $3d$ or $3d+2$, depending on the parity of the dimension and whether the sphere bundle is orientable. In Chapter 2 we prove this conjecture and show that no balanced triangulations of sphere bundles with $3d+1$ vertices exist. One of the main ingredients in the proof is to show that the Lower Bound Theorem for balanced normal pseudomanifolds established by Klee and Novik [35] also holds for balanced Buchsbaum* complexes, which generalize doubly Cohen-Macaulay complexes.

In Chapter 3, we focus on the UBC for balanced spheres. Since k-neighborly spheres maximize the face numbers of dimension less than k among all simplicial spheres with the same number of vertices and dimension, it is natural to define an analogous notion of neighborliness for balanced spheres: a balanced simplicial complex is called *balanced k-neighborly* if every set of k or fewer vertices with distinct colors forms a face. However, from the current literature, we do not even know if there are balanced 2-neighborly spheres apart from the octahedral spheres. In Chapter 3 we give constructions of balanced neighborly spheres. We show that 1) every color-set of a balanced 2-neighborly 3-sphere must have the same size; that 2) no balanced 2-neighborly triangulations with 12 vertices exist; and that 3) there are at least two non-isomorphic constructions with 16 vertices.

In Chapter 4, we deal with the UBC for 3-dimensional flag manifolds. The additional structure of flag complexes makes their combinatorics quite different from that of general simplicial complexes. Since the graph of any flag $(d-1)$-dimensional complex does not contain cliques of size $d+1$, the upper bounds on face numbers given by the classical UBC are far from being sharp for flag $(d-1)$-spheres. In recent years, a plausible UBC for odd-dimensional flag spheres was gradually established based on the works of several people (see [47] and [40]). For $m \geq 1$, we let $J_m(n)$ be the $(2m-1)$-sphere on n vertices obtained as the join of m copies of the circle, each one a cycle with either $\lfloor \frac{n}{m} \rfloor$ or $\lceil \frac{n}{m} \rceil$ vertices. The conjecture posits that among all flag homology $(2m-1)$-spheres on n vertices, $J_m(n)$ simultaneously maximizes all the face numbers; moreover, $J_m(n)$ is the only maximizer of all face numbers. In dimension three, the inequality part of the conjecture was proved by Gal [25]. Later Adamaszek and Hladký [1] verified that the conjecture holds asymptotically for flag homology manifolds of any odd dimension. In Chapter 4 we verify that the flag UBC holds for all flag 3-manifolds. Furthermore, we show that the same upper bounds hold for all flag 3-dimensional Eulerian complexes, and establish an upper bound on the number of edges of flag 5-manifolds. The proof is based on properties of flag complexes and on an application of the inclusion-exclusion principle.

In the last two chapters of this dissertation, we use the rigidity theory of frameworks to study the graphs of simplicial spheres. As mentioned before, simplicial d-polytopes with $g_2 \leq 1$ were completely characterized by Kalai [34], Nevo and Novinsky [46]. In Chapter 5, we extend their characterizations to show that every simplicial sphere with $g_2 \leq 2$ can be obtained by centrally retriangulating a polytopal sphere with smaller g_2, and hence it is polytopal. In fact, we show that if the dimension of the complex in question is at least 4, then the same result holds in a much larger generality, namely, it holds for all normal pseudomanifolds. The main strategy is to use certain retriangulations of simplicial complexes that preserve the homeomorphism type of the original complex but result in a slight change of the g_2.

Chapter 6 is devoted to answering the following question: what edges in a homology sphere Δ with $g_2 \geq 1$ can be deleted without affecting the rigidity of the remaining graph? Our result shows that as long as Δ does not have a missing facet, the graph $G(\Delta) - e$ is always generically d-rigid. The main idea of the proof is to apply Whiteley's vertex splitting

[66] and to argue by induction on both the dimension and the value of g_2.

Each chapter is independent and can be read separately. Chapters 2 and 3 represent the material from two accepted papers [70] and [67]. The last three chapters are based on papers [68], [69] and [71] that are currently submitted for publication.

Chapter 2

MINIMAL BALANCED TRIANGULATIONS OF SPHERE BUNDLES OVER THE CIRCLE

2.1 Introduction

What is the minimum number of vertices needed to construct a triangulation of $\mathbb{S}^{d-2} \times \mathbb{S}^1$ or of the non-orientable $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$? This question was first studied by Kühnel in [38] for PL-triangulations, where he gave a construction with $2d+1$ vertices. Later Bagchi and Datta [7] proved, in the context of topological triangulations, that any non-simply connected $(d-1)$-dimensional closed manifold requires at least $2d+1$ vertices, and if it has $2d+1$ vertices, then it is isomorphic to one of Kühnel's minimal triangulations. In the same year, Chestnut, Sapir and Swartz [18] established a similar result. In fact, they characterized all pairs (f_0, f_1), where f_0 is the number of vertices and f_1 is the number of edges, that are possible for triangulations of $\mathbb{S}^{d-2}$-bundles over $\mathbb{S}^1$. Both papers [7] and [18] showed that if d is odd and the bundle is orientable, or if d is even and the bundle is non-orientable, then the minimum number of vertices needed is $2d+1$, while in the two other cases, the minimum is $2d+2$.

It is natural to ask the same question for the case of balanced triangulations. In [35], Klee and Novik gave an explicit construction of a $3d$-vertex balanced simplicial complex whose geometric realization is a sphere bundle over the circle (orientable or non-orientable depending on the parity of d). They also described similar constructions with any number $n \geq 3d+2$ of vertices that provide triangulations of both orientable and non-orientable $\mathbb{S}^{d-2}$-bundles over $\mathbb{S}^1$. However, they left open the question whether this $3d$-vertex construction is unique, and whether there exists a $(3d+1)$-vertex triangulation.

In this chapter, we answer these two questions by providing an affirmative answer to the conjecture raised in [35, Conjecture 6.8]. We show that the construction of a balanced $3d$-vertex triangulation in [35] is unique in the category of homology $(d-1)$-manifolds with $\beta_1 \neq 0$ and $\beta_2 = 0$, where Betti numbers are computed with coefficients in $\mathbb{Q}$. In particular, it applies to all $\mathbb{S}^{d-2}$-bundles over $\mathbb{S}^1$ for $d > 4$; and in the case $d = 4$, only the non-orientable $\mathbb{S}^2$-bundle is relevant, where in fact $\beta_2 = 0$. Besides that, we also show that there exist no balanced $(3d+1)$-vertex triangulations of $\mathbb{S}^{d-2}$-bundles over $\mathbb{S}^1$.

This chapter is structured as follows. In Section 2.2, we review the definitions and basic facts that will be necessary for our proofs. In Section 2.3, we establish the uniqueness of the balanced $3d$-vertex construction, see Theorem 2.3.6. In Section 2.4, we verify that no balanced $(3d+1)$-vertex triangulation exists, see Theorem 2.4.6.

2.2 Preliminaries

A *simplicial complex* Δ on vertex set V is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, and such that for every $v \in V$, $\{v\} \in \Delta$. For $\sigma \in \Delta$, let $\dim \sigma := |\sigma| - 1$ and define the *dimension* of Δ, $\dim \Delta$, as the maximum dimension of the faces of Δ. The *facets* of Δ are maximal under inclusion faces of Δ. We say that a simplicial complex Δ is *pure* if all of its facets have the same dimension.

We let $d = \dim \Delta + 1$ throughout. For $-1 \leq i \leq d-1$, the *f-number* $f_i = f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. It is often more convenient to study the *h-numbers* $h_i = h_i(\Delta)$, $0 \leq i \leq d$, defined by the relation $\sum_{j=0}^{d} h_j \lambda^{d-j} = \sum_{i=0}^{d} f_{i-1}(\lambda - 1)^{d-i}$.

If Δ is a simplicial complex and σ is a face of Δ, the *star* of σ in Δ is $\mathrm{st}_\Delta \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}$, and the *contrastar* of σ in Δ is $\mathrm{cost}_\Delta \sigma := \{\tau \in \Delta : \sigma \not\subseteq \tau\}$. We also define the *link* of σ in Δ as $\mathrm{lk}_\Delta \sigma := \{\tau - \sigma \in \Delta : \sigma \subseteq \tau \in \Delta\}$, the *deletion* of a subset of vertices W from Δ as $\Delta \backslash W := \{\sigma \in \Delta : \sigma \cap W = \emptyset\}$, and the *restriction* of Δ to a vertex set W as $\Delta[W] := \{\sigma \in \Delta : \sigma \subseteq W\}$. Finally, we recall that $F \subseteq V$ is a *missing face* if $F \notin \Delta$ but all proper subsets of F are faces of Δ; F is a *missing k-face* if it is a missing face and $|F| = k+1$.

A $(d-1)$-dimensional simplicial complex Δ is called *balanced* if the graph of Δ is d-

colorable, or equivalently, there is a coloring $\kappa : V(\Delta) \to [d]$, with $[d] = \{1, \cdots, d\}$, such that $\kappa(u) \neq \kappa(v)$ for all edges $\{u, v\} \in \Delta$. The *S-rank-selected subcomplex* of Δ is defined as $\Delta_S := \{\tau \in \Delta : \kappa(\tau) \subseteq S\}$ for $S \subseteq [d]$.

A simplicial complex Δ is a *simplicial manifold* if the geometric realization of Δ is homeomorphic to a manifold. We denote by $\tilde{H}_*(\Delta; \mathbf{k})$ the reduced homology with coefficients in a field $\mathbf{k}$, and denote the reduced Betti numbers of Δ with coefficients in $\mathbf{k}$ by $\beta_i(\Delta; \mathbf{k}) := \dim_{\mathbf{k}} \tilde{H}_i(\Delta; \mathbf{k})$. We say that Δ is a $(d-1)$-dimensional **k**-*homology manifold* if $\tilde{H}_*(\mathrm{lk}_\Delta \, \sigma; \mathbf{k}) \cong \tilde{H}_*(\mathbb{S}^{d-1-|\sigma|}; \mathbf{k})$ for every nonempty face $\sigma \in \Delta$. A $(d-1)$-simplicial complex Δ is *Buchsbaum* over $\mathbf{k}$ if Δ is pure and for every nonempty face σ in Δ, and every $i < d - 1 - \dim \sigma$, we have $\tilde{H}_i(\mathrm{lk}_\Delta \, \sigma; \mathbf{k}) = 0$. A $(d-1)$-dimensional simplicial complex Δ is *Buchsbaum** over $\mathbf{k}$ if it is Buchsbaum over $\mathbf{k}$, and for every pair of faces $\sigma \subseteq \tau$ of Δ, the map $i* : H_{d-1}(\Delta, \mathrm{cost}_\Delta \, \sigma; \mathbf{k}) \to H_{d-1}(\Delta, \mathrm{cost}_\Delta \, \tau; \mathbf{k})$ induced by injection, is surjective. (Here $H_{d-1}(\Delta, \Gamma; \mathbf{k})$ denotes the relative homology.) A simplicial manifold is a homology manifold as well as a Buchsbaum complex over any field $\mathbf{k}$. An orientable $\mathbf{k}$-homology manifold is Buchsbaum* over $\mathbf{k}$. The following lemma [13, Theorem 3.1] provides a basic property of balanced Buchsbaum* complex.

Lemma 2.2.1. *Let Δ be a $(d-1)$-dimensional balanced Buchsbaum* complex. Then the rank-selected subcomplex Δ_S is Buchsbaum* for every $S \subseteq [d]$.*

For more properties of balanced Buchsbaum* complexes, see [13] for a reference.

We will also need some basic facts from homology theory, such as the Mayer-Vietoris sequence, we refer to Hatcher's book [28] as a reference.

2.3 The $3d$-vertex Case

The main goal of this section is to prove Theorem 2.3.6, where we verify that the construction of the balanced $3d$-vertex triangulation of $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$ provided in [35] is unique. Our result then implies part 1 of Conjecture 6.8 in [35]. We begin with presenting this construction.

A d-dimensional cross-polytope is the convex hull of the set $\{u_1, \cdots, u_d, v_1, \cdots v_d\}$ in $\mathbb{R}^d$, where $u_1, \cdots, u_d$ are d linearly independent vectors in $\mathbb{R}^d$ and $v_i = -u_i$ for $1 \leq i \leq d$. The boundary complex of a d-dimensional cross-polytope is a balanced $(d-1)$-dimensional sphere with $\kappa(u_i) = \kappa(v_i) = i$ for all $i \in [d]$. Fix integers n and d with d a divisor of n, we define a stacked cross-polytopal sphere $\mathcal{ST}^{\times}(n, d-1)$ by taking the connected sum of $\frac{n}{d} - 1$ copies of the boundary complex of the d-dimensional cross-polytope. In each connected sum, we identify vertices of the same colors so that $\mathcal{ST}^{\times}(n, d-1)$ is a balanced $(d-1)$-sphere on n vertices.

From [35], we see that there is a balanced simplicial manifold, denoted BM_d, with $3d$ vertices that triangulates $\mathbb{S}^{d-2} \times \mathbb{S}^1$ if d is odd, and triangulates the non-orientable $\mathbb{S}^{d-1}$-bundle over $\mathbb{S}^1$ if d is even. This manifold is constructed in the following way: let Δ_1, Δ_2 and Δ_3 be the boundary complexes of d-dimensional cross-polytopes with $V(\Delta_1) = \{x_1, \cdots, x_d\} \cup \{y_1, \cdots, y_d\}$, $V(\Delta_2) = \{y'_1, \cdots, y'_d\} \cup \{z_1, \cdots, z_d\}$, and $V(\Delta_3) = \{z'_1, \cdots, z'_d\} \cup \{x'_1, \cdots, x'_d\}$, where each vertex with index i has color i. Then BM_d is exactly the complex we get after forming two connected sums followed by a handle addition that identifies x_i, y_i, z_i with x'_i, y'_i, z'_i respectively. Since the number of $(i-1)$-faces of a d-dimensional cross-polytope is $2^i\binom{d}{i}$ for $0 \leq i \leq d$, it follows immediately that

Lemma 2.3.1. *The number of $(i-1)$-faces of $\mathcal{ST}^{\times}(n, d-1)$ and BM_d are $[2^i(\frac{n}{d} - 1) - (\frac{n}{d} - 2)]\binom{d}{i}$ and $3(2^i - 1)\binom{d}{i}$, respectively, for $0 \leq i \leq d$.*

Now we establish a few other lemmas, the first of which is well-known.

Lemma 2.3.2 (Alexander Duality)**.** *Let Γ be a triangulation of a homology $(d-1)$-sphere over $\mathbb{Q}$ on vertex set V and W be a subset of V. Then $\beta_i(\Gamma[W]; \mathbb{Q}) = \beta_{d-i-2}(\Gamma[V - W]; \mathbb{Q})$ for all i.*

Lemma 2.3.3. *Let Δ be a balanced triangulation of a homology $(d-1)$-manifold over $\mathbb{Q}$ ($d \geq 4$), and W be a subset of vertices that all have the same color. Then $\tilde{H}_i(\Delta; \mathbb{Q}) = \tilde{H}_i(\Delta \backslash W; \mathbb{Q})$ for $1 \leq i \leq d-3$.*

Proof: Let $v \in W$. Since $\Delta = (\Delta \backslash \{v\}) \cup \mathrm{st}_\Delta\, v$ and $\mathrm{lk}_\Delta\, v = (\Delta \backslash \{v\}) \cap \mathrm{st}_\Delta\, v$, the Mayer-Vietoris sequence implies that

$$\cdots \to \tilde{H}_i(\mathrm{lk}_\Delta\, v; \mathbb{Q}) \to \tilde{H}_i(\Delta \backslash \{v\}; \mathbb{Q}) \oplus \tilde{H}_i(\mathrm{st}_\Delta\, v; \mathbb{Q}) \to \tilde{H}_i(\Delta; \mathbb{Q}) \to \tilde{H}_{i-1}(\mathrm{lk}_\Delta\, v; \mathbb{Q}) \to \cdots$$

is exact. The complex $\mathrm{st}_\Delta\, v$ is contractible, so $\tilde{H}_i(\mathrm{st}_\Delta\, v) = 0$ for all i. Since $\mathrm{lk}_\Delta\, v$ is a homology sphere of dimension $d-2$, $\tilde{H}_i(\mathrm{lk}_\Delta\, v) = 0$ for $0 \le i \le d-3$. Thus

$$0 \to \tilde{H}_i(\Delta \backslash \{v\}; \mathbb{Q}) \to \tilde{H}_i(\Delta; \mathbb{Q}) \to 0 \text{ is exact,}$$

which implies that $\tilde{H}_i(\Delta \backslash \{v\}; \mathbb{Q}) = \tilde{H}_i(\Delta; \mathbb{Q})$ for $1 \le i \le d-3$. Since all vertices in W have the same color, deleting some of them does not change the links of the remaining ones. Therefore the result follows by iterating this argument on other vertices in W. $\square$

Lemma 2.3.4. *Let G_1, G_2, G_3 be connected graphs on vertex set U, where $|U| = 2s-1 \ge 3$. Further assume that for $\{i, j, k\} = [3]$, every edge of G_i is also an edge of either G_j or G_k, and that every $G_i \cap G_j$ has s connected components. Then there exist distinct vertices u_1, u_2, u_3 such that the graph $G_i \backslash \{u_i\}$ is disconnected for $i = 1, 2, 3$.*

Proof: For $\{i, j, k\} = [3]$, since $G_i \cap G_j$ is a graph on $2s-1$ vertices and it has s connected components, one of the connected components must be a single vertex; we let it be u_k. We claim that u_1, u_2, u_3 are distinct. Otherwise, assume that $u_1 = u_2$. Since every edge of G_3 is an edge of either G_1 or G_2, it follows that $G_3 = (G_1 \cap G_3) \cup (G_2 \cap G_3)$. By the assumption, $\{u_1\} = \{u_2\}$ is a connected component in both $G_2 \cap G_3$ and $G_1 \cap G_3$. This, however, contradicts the fact that G_3 is connected.

Next consider $G_3 \backslash \{u_3\} = \big((G_1 \cap G_3) \backslash \{u_3\}\big) \cup \big((G_2 \cap G_3) \backslash \{u_3\}\big)$. Since $\{u_3\}$ is not a connected component in either $G_1 \cap G_3$ or $G_2 \cap G_3$, deleting u_3 from these two graphs will not reduce the number of components in the resulting graphs, and hence both $(G_1 \cap G_3) \backslash \{u_3\}$ and $(G_2 \cap G_3) \backslash \{u_3\}$ have at least s connected components. We claim that $G_3 \backslash \{u_3\}$ is disconnected. Indeed, if $G_3 \backslash \{u_3\}$ is connected, then there exist at least $s-1$ edges in $(G_2 \cap G_3) \backslash \{u_3\}$ so that these edges form a spanning tree on the connected components in

$(G_1 \cap G_3)\backslash\{u_3\}$. Since $(G_2 \cap G_3)\backslash\{u_3\}$ is a graph on $2s-2$ vertices, it implies that the number of connected components in $(G_2 \cap G_3)\backslash\{u_3\}$ is bounded by $(2s-2)-(s-1) = s-1$, which contradicts the fact that it is at least s. Hence, $G_3\backslash\{u_3\}$ is disconnected. Similarly, $G_1\backslash\{u_1\}$ and $G_2\backslash\{u_2\}$ are disconnected. □

Finally, we quote Theorem 6.6 of [35], which will serve as the main tool in proving our theorem.

Lemma 2.3.5. *Let Δ be a balanced triangulation of a homology $(d-1)$-manifold with $\beta_1(\Delta;\mathbb{Q}) \neq 0$.*

1. *If $d \geq 2$, then $f_{i-1}(\Delta) \geq f_{i-1}(BM_d)$ for all $0 < i \leq d$.*
2. *Moreover, if $d \geq 5$, and $(f_0(\Delta), f_1(\Delta), f_2(\Delta)) = (f_0(BM_d), f_1(BM_d), f_2(BM_d))$, then Δ is isomorphic to BM_d.*

Now we are in a position to prove the main result of this section.

Theorem 2.3.6. *If Δ is a balanced $3d$-vertex triangulation of a homology $(d-1)$-manifold over $\mathbb{Q}$ with $\beta_1(\Delta;\mathbb{Q}) \neq 0$ and $\beta_2(\Delta;\mathbb{Q}) = 0$, then Δ is isomorphic to BM_d.*

Proof: Since Δ is a homology manifold that is not a homology sphere, Δ is not a suspension. Therefore, Δ must have 3 vertices of each color. Since Δ is a balanced $3d$-vertex homology $(d-1)$-manifold, by part 1 of Lemma 2.3.5 and Lemma 2.3.1, $f_1(\Delta) \geq f_1(BM_d) = 9\binom{d}{2}$. However, since every vertex of Δ is adjacent to at most $3d-3$ vertices, $f_1(\Delta) \leq 9\binom{d}{2}$. Thus $f_1(\Delta) = f_1(BM_d) = 9\binom{d}{2}$, i.e., both of the graphs of Δ and BM_d are complete d-partite graphs.

To prove the theorem, first notice that the cases of $d=3$ and 4 is treated in Proposition 6.9 of [35] without the assumption $\beta_2(\Delta;\mathbb{Q}) = 0$. (In fact, their proposition has an additional assumption that the reduced Euler characteristic of Δ and BM_d are the same. However, in the case $d=3$, only the condition $f_i(\Delta) = f_i(BM_d)$ for $i = 0, 1$ is used in their proof; and in the case $d = 4$, $\tilde{\chi}(\Delta) = \tilde{\chi}(BM_d) = -1$ holds for any homology 3-manifold Δ.) Now assume that $d \geq 5$. The strategy is to show that Δ has the same f_2 as BM_d. The result will then follow from part 2 of Lemma 2.3.5.

We fix some notation here. Given a simplicial complex Γ, we denote the number of connected components in Γ by $c(\Gamma)$ and the graph of Γ by $G(\Gamma)$. We let $V_1 = \{v_1, v_2, v_3\}$ be the set of vertices of color 1. For every pair $\{i, j\} \subseteq [3]$, set $\Delta_{i,j} := \mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j$, $\Delta^{i,j} := \mathrm{lk}_\Delta v_i \cup \mathrm{lk}_\Delta v_j$, and $\Delta_{1,2,3} := \mathrm{lk}_\Delta v_1 \cap \mathrm{lk}_\Delta v_2 \cap \mathrm{lk}_\Delta v_3$. Since all codimension-1 faces of Δ are contained in exactly two facets of Δ, it follows that $\Delta^{i,j} = \Delta \backslash V_1$, and hence that for $\{i, j, k\} = \{1, 2, 3\}$,

$$\Delta_{i,j} \cup \mathrm{lk}_\Delta v_k = (\mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j) \cup \mathrm{lk}_\Delta v_k = \Delta^{i,k} \cap \Delta^{j,k} = \Delta \backslash V_1.$$

Below all homologies are computed with coefficients in $\mathbb{Q}$. We suppress $\mathbb{Q}$ from our notation. Applying the Mayer-Vietoris sequence to $\Delta \backslash V_1 = \mathrm{lk}_\Delta v_i \cup \mathrm{lk}_\Delta v_j$, we obtain

$$\cdots \to \tilde{H}_{m+1}(\mathrm{lk}_\Delta v_i) \oplus \tilde{H}_{m+1}(\mathrm{lk}_\Delta v_j) \to \tilde{H}_{m+1}(\Delta \backslash V_1) \to \tilde{H}_m(\Delta_{i,j}) \to \tilde{H}_m(\mathrm{lk}_\Delta v_i) \oplus \tilde{H}_m(\mathrm{lk}_\Delta v_j) \to \cdots .$$

If $d \geq 5$, then since all vertex links are $(d-2)$-dimensional homology spheres, $\tilde{H}_2(\mathrm{lk}_\Delta v_i) = \tilde{H}_1(\mathrm{lk}_\Delta v_i) = 0$ for all i. Taking $m = 1$, we conclude that

$$\tilde{H}_1(\Delta_{i,j}) = \tilde{H}_2(\Delta \backslash V_1) = \tilde{H}_2(\Delta) = 0.$$

(The second equality follows from Lemma 2.3.3.) Also taking $m = 0$ yields that $\dim \tilde{H}_0(\Delta_{i,j}) = \dim \tilde{H}_1(\Delta \backslash V_1) = \dim \tilde{H}_1(\Delta) > 0$. Thus $c(\Delta_{i,j}) \geq 2$ and it is independent of the pair i, j, so we set $s := c(\Delta_{i,j})$.

Similarly, applying the Mayer-Vietoris sequence to $\Delta \backslash V_1 = \Delta_{i,j} \cup \mathrm{lk}_\Delta v_k$, we infer that

$$\cdots \to \tilde{H}_1(\Delta_{i,j}) \oplus \tilde{H}_1(\mathrm{lk}_\Delta v_k) \to \tilde{H}_1(\Delta \backslash V_1) \to \tilde{H}_0(\Delta_{1,2,3}) \to \tilde{H}_0(\Delta_{i,j}) \oplus \tilde{H}_0(\mathrm{lk}_\Delta v_k) \to \tilde{H}_0(\Delta \backslash V_1) \to \cdots .$$

Hence

$$0 \to \tilde{H}_1(\Delta \backslash V_1) \to \tilde{H}_0(\Delta_{1,2,3}) \to \tilde{H}_0(\Delta_{i,j}) \to 0 \text{ is exact,}$$

which implies that $c(\Delta_{1,2,3}) = 2s - 1 \geq 3$.

Since $G(\Delta)$ is a complete d-partite graph, for all $1 \leq i < j \leq 3$,

$$V(\mathrm{lk}_\Delta v_i) = V(\Delta_{i,j}) = V(\Delta_{1,2,3}) = V(\Delta) \backslash V_1.$$

Now let $\tilde{G}(\mathrm{lk}_\Delta v_i)$ be the graph obtained from $G(\mathrm{lk}_\Delta v_i)$ by identifying all the vertices in the same connected component in $\Delta_{1,2,3}$ as one vertex. We consider $V(\tilde{G}(\mathrm{lk}_\Delta v_i))$ as the vertex set U (hence each vertex in U represents a connected component in $\Delta_{1,2,3}$) and $\tilde{G}(\mathrm{lk}_\Delta v_i)$ as G_i from Lemma 2.3.4. Since $\tilde{G}(\mathrm{lk}_\Delta v_i)$ and $G(\mathrm{lk}_\Delta v_i)$ are both connected, and the argument above implies that G_1, G_2, G_3 satisfy all the conditions in Lemma 2.3.4, we conclude that there exists a connected component A_i in $\Delta_{1,2,3}$ such that $\tilde{G}(\mathrm{lk}_\Delta v_i) \backslash V(A_i)$ is not connected for $i = 1, 2, 3$. Therefore, the complex $\mathrm{lk}_\Delta v_i \backslash V(A_i)$, whose graph is $G(\mathrm{lk}_\Delta v_i) \backslash V(A_i)$, is also not connected.

Since $\mathrm{lk}_\Delta v_i$ is a homology $(d-2)$-sphere, by Alexander Duality,

$$\beta_0((\mathrm{lk}_\Delta v_i) \backslash V(A_i)) = \beta_{d-3}(\mathrm{lk}_\Delta v_i[V(A_i)]),$$

which implies that $\beta_{d-3}(\mathrm{lk}_\Delta v_i[V(A_i)])$ is also non-zero. Hence $f_0(A_i) \geq d-1$. Since $f_0(A_1 \cup A_2 \cup A_3) \leq f_0(\Delta_{1,2,3}) \leq 3(d-1)$, it follows that A_1, A_2 and A_3 are the only connected components in $\Delta_{1,2,3}$, and each of them has $d-1$ vertices. We obtain that

$$f_1(\mathrm{lk}_\Delta v_i) \leq \binom{3d-3}{2} - (d-1)^2 - 2(d-1) = 7\binom{d-1}{2},$$

where the "$-(d-1)^2$" on the right-hand side comes from the fact that no edges between A_j and A_k exist in $\mathrm{lk}_\Delta v_i$, and "$-2(d-1)$" comes from the fact that no vertex in A_i can be connected to the other two vertices of the same color. But the lower bound theorem for balanced connected homology manifolds [35, Theorem 3.2] implies that $f_1(\mathrm{lk}_\Delta v_i) \geq 7\binom{d-1}{2}$. Hence $f_1(\mathrm{lk}_\Delta v_i)$ is exactly $7\binom{d-1}{2}$ for all $i = 1, 2, 3$.

Applying the same argument to vertices of other colors, we obtain that for all $v \in V(\Delta)$, $f_1(\mathrm{lk}_\Delta v) = 7\binom{d-1}{2}$. Thus

$$f_2(\Delta) = \frac{1}{3} \sum_{v \in V(\Delta)} f_1(\mathrm{lk}_\Delta v) = 21\binom{d}{3},$$

which, by Lemma 2.3.1, is the number of 2-faces in BM_d. Then part 2 of Lemma 2.3.5 implies that Δ is isomorphic to BM_d. $\square$

2.4 The $(3d+1)$-vertex Case

The goal of this section is to show that no balanced $(3d+1)$-vertex triangulation of $\mathbb{S}^{d-2}$-bundles over $\mathbb{S}^1$ exists. In [35, Theorem 3.8], Klee and Novik proved that any balanced normal pseudomanifold Δ of dimension $d-1 \geq 2$ with $\beta_1(\Delta;\mathbb{Q}) \neq 0$ satisfies $2h_2(\Delta)-(d-1)h_1(\Delta) \geq 4\binom{d}{2}$. Our first step is to show that this result continues to hold for Buchsbaum* complexes. We begin with the following lemma.

Lemma 2.4.1. *Let Δ be a Buchsbaum* complex over a field* $\mathbf{k}$*. If Δ has a t-sheeted covering space Δ^t, then Δ^t is also Buchsbaum* over* $\mathbf{k}$*.*

Proof: First of all, Δ^t is Buchsbaum, since Δ is Buchsbaum and the links in Δ^t are isomorphic to the links in Δ. For every pair of faces $\sigma^t \subseteq \tau^t$ in Δ^t, their images form a pair of faces $\sigma \subseteq \tau$ in Δ. Let $\hat{\sigma^t}$ and $\hat{\tau^t}$ be the barycenters of $|\sigma^t|$ and $|\tau^t|$ respectively, and let $\hat{\sigma}$ and $\hat{\tau}$ denote their images in $|\sigma|$ and $|\tau|$ respectively. Below we suppress the coefficient field in the homology groups. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
H_{d-1}(|\Delta^t|, |\Delta^t| - \hat{\sigma^t}) & \xrightarrow{\cong} & H_{d-1}(\Delta^t, \operatorname{cost}_{\Delta^t} \sigma^t) & \xrightarrow{i_*^t} & H_{d-1}(\Delta^t, \operatorname{cost}_{\Delta^t} \tau^t) & \xrightarrow{\cong} & H_{d-1}(|\Delta^t|, |\Delta^t| - \hat{\tau^t}) \\
\downarrow{\scriptstyle p_*} & & & & & & \downarrow{\scriptstyle p'_*} \\
H_{d-1}(|\Delta|, |\Delta| - \hat{\sigma}) & \xrightarrow{\cong} & H_{d-1}(\Delta, \operatorname{cost}_{\Delta} \sigma) & \xrightarrow{i_*} & H_{d-1}(\Delta, \operatorname{cost}_{\Delta} \tau) & \xrightarrow{\cong} & H_{d-1}(|\Delta|, |\Delta| - \hat{\tau})
\end{array}$$

Since Δ is Buchsbaum*, the bottom horizontal map i_* is surjective. Also both p_* and p'_* are isomorphisms, since the covering map p is locally an isomorphism. Hence the top horizontal map i_*^t is surjective. Thus by the definition, Δ^t is Buchsbaum*. $\square$

Lemma 2.4.2. *Let Δ be a balanced Buchsbaum* (over a field* $\mathbf{k}$*) complex of dimension $d-1 \geq 3$. If $|\Delta|$ has a connected t-sheeted covering space, then* $2h_2(\Delta) - (d-1)h_1(\Delta) \geq$

$4\frac{t-1}{t}\binom{d}{2}$. *In particular, if* $\beta_1(\Delta;\mathbb{Q}) \neq 0$, *then* $2h_2(\Delta) - (d-1)h_1(\Delta) \geq 4\binom{d}{2}$, *or equivalently,* $f_1(\Delta) \geq \frac{3(d-1)}{2} f_0(\Delta)$.

Proof: The proof follows the same ideas as in [61, Theorem 4.3] and [35, Theorem 3.8]. Let $X = |\Delta|$ and let X^t be a connected t-sheeted covering space of X. The triangulation Δ of X lifts to a triangulation Δ^t of X^t, which is also balanced.

By the previous lemma and Theorem 4.1 in [13],

$$2h_2(\Delta^t) \geq (d-1)h_1(\Delta^t). \tag{2.4.1}$$

Also by Proposition 4.2 in [61], for $i = 1, 2$,

$$h_i(\Delta^t) = th_i(\Delta) + (-1)^{i-1}(t-1)\binom{d}{i}. \tag{2.4.2}$$

Substituting (2.4.2) for $i = 1, 2$ in (2.4.1) gives $2h_2(\Delta) - (d-1)h_1(\Delta) \geq 4\frac{t-1}{t}\binom{d}{2}$. The existence of a connected t-sheeted covering space of $|\Delta|$ with $\beta_1(\Delta;\mathbb{Q}) \neq 0$ for arbitrary large t implies the in-particular part. □

The previous lemma implies the following:

Lemma 2.4.3. *If* Δ *is a balanced* $(3d+1)$*-vertex triangulation of* $\mathbb{S}^{d-2}$*-bundle over* $\mathbb{S}^1$ $(d > 3)$*, whether orientable or non-orientable, then there is a unique color set* W *containing four vertices, and the graph of* $\Delta \backslash W$ *is complete* $(d-1)$*-partite.*

Proof: The existence of W follows from the same argument as in Theorem 2.3.6. Assume that $W = \{v_1, v_2, v_3, v_4\}$. First notice that by Lemma 2.3.3, $\beta_1(\Delta;\mathbb{Q}) = \beta_1(\Delta\backslash W;\mathbb{Q}) = 1$. Since Δ is a Buchsbaum* complex over $\mathbb{Z}/2\mathbb{Z}$, by Lemma 2.2.1, $\Delta\backslash W$ is also Buchsbaum* over $\mathbb{Z}/2\mathbb{Z}$. Thus by Lemma 2.4.2 and the fact that $\beta_1(\Delta\backslash W;\mathbb{Q}) \neq 0$, it follows that

$$f_1(\Delta\backslash W) \geq \frac{3(d-2)}{2} f_0(\Delta\backslash W) = 9\binom{d-1}{2}.$$

However, since every color set in $\Delta\backslash W$ is of cardinality 3, every vertex is connected to at most $3d-6$ vertices in $\Delta\backslash W$. By double counting,

$$f_1(\Delta\backslash W) = \frac{1}{2}\sum_{v\in V(\Delta)\backslash W} f_0(\mathrm{lk}_{\Delta\backslash W}\, v) \leq \frac{(3d-3)(3d-6)}{2} = 9\binom{d-1}{2}.$$

Hence $f_1(\Delta\backslash W) = 9\binom{d-1}{2}$ and $f_0(\mathrm{lk}_{\Delta\backslash W}\, v) = 3d-6$ for every vertex $v \in \Delta\backslash W$. This implies that the graph of $\Delta\backslash W$ is complete $(d-1)$-partite. $\square$

Lemma 2.4.4. *If Δ is a balanced $(3d+1)$-vertex triangulation of $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$ $(d > 5)$, whether orientable or non-orientable, and W is the unique color set containing four vertices, then $f_1(\mathrm{lk}_{\Delta\backslash W}\, v) \leq 7\binom{d-2}{2}$ for all $v \in V(\Delta)\backslash W$.*

Proof: The proof is very similar to the proof of the crucial upper bound for $f_1(\mathrm{lk}_\Delta\, v_i)$ in Theorem 2.3.6. However, since $\Delta\backslash W$ is not a homology manifold, we need to check a few things. Below we use the same notation as in the proof of Theorem 2.3.6. Given a simplicial complex Γ, we denote the number of connected components of Γ by $c(\Gamma)$. We write $\Delta\backslash W$ as $\bar{\Delta}$ and let $V_1 = \{v_1, v_2, v_3\}$ be one color set in $\bar{\Delta}$. For every pair $\{i,j\} \subseteq [3]$, set $\bar{\Delta}_{i,j} := \mathrm{lk}_{\bar{\Delta}}\, v_i \cap \mathrm{lk}_{\bar{\Delta}}\, v_j$, $\bar{\Delta}^{i,j} := \mathrm{lk}_{\bar{\Delta}}\, v_i \cup \mathrm{lk}_{\bar{\Delta}}\, v_j$ and $\bar{\Delta}_{1,2,3} := \mathrm{lk}_{\bar{\Delta}}\, v_1 \cap \mathrm{lk}_{\bar{\Delta}}\, v_2 \cap \mathrm{lk}_{\bar{\Delta}}\, v_3$. Since all codimension-1 faces are contained in *at least* two facets in $\bar{\Delta}$, $\bar{\Delta}^{i,j} = \bar{\Delta}\backslash V_1$ and $\bar{\Delta}_{i,j} \cup \mathrm{lk}_{\bar{\Delta}}\, v_k = \bar{\Delta}\backslash V_1$ still holds for $\{i,j,k\} = [3]$. Also for every $v \in \bar{\Delta}$, $\mathrm{lk}_{\bar{\Delta}}\, v = (\mathrm{lk}_\Delta\, v)\backslash W$. Hence by Lemma 2.3.3,

$$\tilde{H}_i(\mathrm{lk}_{\bar{\Delta}}\, v_i; \mathbb{Q}) = \tilde{H}_i\big((\mathrm{lk}_\Delta\, v_i)\backslash W; \mathbb{Q}\big) = \tilde{H}_i(\mathrm{lk}_\Delta\, v_i; \mathbb{Q}) = 0 \tag{2.4.3}$$

for $i < d-3$. Then applying the Mayer-Vietoris sequence, we obtain that for $i = 1, 2$,

$$0 = \tilde{H}_i(\mathrm{lk}_{\bar{\Delta}}\, v_i; \mathbb{Q}) \to \tilde{H}_i(\bar{\Delta}\backslash\{v_i\}; \mathbb{Q}) \oplus \tilde{H}_i(\mathrm{st}_{\bar{\Delta}}\, v_i; \mathbb{Q}) \to \tilde{H}_i(\bar{\Delta}; \mathbb{Q}) \to \tilde{H}_{i-1}(\mathrm{lk}_{\bar{\Delta}}\, v_i; \mathbb{Q}) = 0. \tag{2.4.4}$$

(In order for (2.4.4) to hold when $i = 2$, it is required that $d > 5$.) Since $\mathrm{st}_{\bar{\Delta}}\, v$ is contractible, by (2.4.3) and (2.4.4) it implies that $\tilde{H}_i(\bar{\Delta}\backslash\{v_i\}; \mathbb{Q}) = \tilde{H}_i(\bar{\Delta}; \mathbb{Q}) = \tilde{H}_i(\Delta; \mathbb{Q})$ for $i = 1, 2$. Iterating the argument on other vertices of V_1, it follows that $\tilde{H}_2(\bar{\Delta}\backslash V_1; \mathbb{Q}) = 0$ and $\tilde{H}_1(\bar{\Delta}\backslash V_1; \mathbb{Q}) \neq 0$. Hence by the proof of Theorem 2.3.6, we obtain that $c(\bar{\Delta}_{i,j}) = s \geq 2$ and $c(\bar{\Delta}_{1,2,3}) = 2s - 1 \geq 3$ for every $\{i,j\} \subseteq [3]$.

Next, by Lemma 2.4.3, for every $\{i,j\} \leq 3$ we also have

$$V(\mathrm{lk}_{\bar{\Delta}}\, v_i) = V(\bar{\Delta}_{i,j}) = V(\bar{\Delta}_{1,2,3}) = V(\bar{\Delta})\backslash V_1.$$

Hence applying the same argument that uses Lemma 2.3.4 in the proof Theorem 2.3.6, we conclude that there exist disjoint subcomplexes A_1, A_2, A_3 of $\mathrm{lk}_{\bar\Delta}\, v_1, \mathrm{lk}_{\bar\Delta}\, v_2, \mathrm{lk}_{\bar\Delta}\, v_3$ respectively such that $(\mathrm{lk}_{\bar\Delta}\, v_i)\backslash V(A_i)$ is not connected for $i = 1, 2, 3$. However, by Alexander Duality, this implies that

$$\tilde\beta_0\big((\mathrm{lk}_{\bar\Delta}\, v_i)\backslash V(A_i)\big) = \tilde\beta_0\big((\mathrm{lk}_\Delta\, v_i)\backslash(V(A_i)\cup W)\big) = \tilde\beta_{d-3}(\mathrm{lk}_\Delta\, v_i[V(A_i)\cup W]) \neq 0.$$

Hence the subcomplex $\mathrm{lk}_\Delta\, v_i[V(A_i)\cup W]$ is of dimension $\geq d-3$. Since every vertex in W has the same color, it follows that $|V(A_i)| \geq d-3$. However, if $|V(A_i)| = d-3$, then $\mathrm{lk}_\Delta\, v_i[V(A_i)]$ must be a $(d-4)$-simplex and thus $\tilde\beta_{d-3}(\mathrm{lk}_\Delta\, v_i[V(A_i)\cup W]) = 0$, a contradiction. So we conclude that $|V(A_i)| \geq d-2$. We proceed using the same argument as in Theorem 2.3.6, and the result follows. □

Lemma 2.4.5. *Neither the orientable nor the non-orientable $\mathbb{S}^3$-bundle over $\mathbb{S}^1$ has a balanced 16-vertex triangulation.*

Proof: Assume to the contrary that Δ is such a triangulation and V_i is the color set for $1 \leq i \leq 5$, with $V_5 = \{w_1, w_2, w_3, w_4\}$. Now take a vertex $u \in V_1$ and let $\Gamma = \mathrm{lk}_\Delta\, u$.

If $\Gamma \cap V_5 = V_5$, then by Lemma 2.4.3, $V(\Gamma) = V(\Delta)\backslash V_1$. Since Γ is a 3-sphere and each $\mathrm{lk}_\Gamma\, w_i$ is a 2-sphere, it follows that

$$f_1(\Gamma) - 13 = f_1(\Gamma) - f_0(\Gamma) = f_3(\Gamma) = \sum_{i=1}^{4} f_2(\mathrm{lk}_\Gamma\, w_i) = \sum_{i=1}^{4}(2f_0(\mathrm{lk}_\Gamma\, w_i) - 4). \qquad (2.4.5)$$

Take a vertex v of color other than 1 and 5. Since $\mathrm{lk}_\Gamma\, v$ is a 2-sphere, $\beta_1\big((\mathrm{lk}_\Gamma\, v)\backslash V_5\big) = |V_5| - 1 = 3$. Hence $(\mathrm{lk}_\Gamma\, v)\backslash V_5$ cannot be the bipartite graph on six vertices (otherwise its β_1 is 4), and $f_1(\mathrm{lk}_\Gamma\, v\backslash V_5) \leq 8$. On the other hand, since every edge of $\mathrm{lk}_\Gamma\, v\backslash V_5$ is contained in exactly two facets of $\mathrm{lk}_\Gamma\, v$, it is contained in two of the links $\mathrm{lk}_\Gamma\{v, w_i\}$. Hence $2f_1\big((\mathrm{lk}_\Gamma\, v)\backslash V_5\big) = \sum_{i=1}^{4} f_1(\mathrm{lk}_\Gamma\{v, w_i\}) \geq 16$. This implies $\mathrm{lk}_\Gamma\{v, w_i\}$ is a 4-cycle for every w_i and $v \in V(\Gamma)\backslash W$. Thus $\mathrm{lk}_\Gamma\, w_i$ is a cross-polytope. By (2.4.5), $f_1(\Gamma) = \sum_{i=1}^{4}(2\cdot 6-4)+13 = 45$. However, by the lower bound theorem for balanced spheres, $f_1(\Gamma) \geq \frac{1}{2}(9f_0(\Gamma) - 4\binom{4}{2}) = \frac{93}{2}$, a contradiction. Hence u is not connected to at least one vertex in V_5.

Similarly, every vertex in $\cup_{i=1}^4 V_i$ is not connected to at least one vertex in V_5. So there are at least 9 missing edges between the sets $\cup_{i=2}^4 V_i$ and W in Δ. Since $\Delta\backslash V_1$ is Buchsbaum*, by Lemma 2.4.2,

$$f_1(\Delta\backslash V_1) \geq \left\lceil \frac{3\cdot 3}{2} f_0(\Delta\backslash V_1) \right\rceil = 59.$$

The complete 4-partite graph on 13 vertices has 63 edges, so there are no more than 4 missing edges between $\cup_{i=2}^4 V_i$ and V_5. This leads to a contradiction and hence no such triangulation exists. □

We are now ready to state the theorem.

Theorem 2.4.6. *Neither the orientable nor the non-orientable $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$ has a balanced $(3d+1)$-vertex triangulation.*

Proof: The $d = 3, 4, 5$ cases are covered in [35, Proposition 6.10] and Lemma 2.4.5. Now assume that $d > 5$ and that Δ is such a triangulation. Let V_i be the set of vertices of color i and let V_1 be the unique set of four vertices. By Lemma 2.4.4, $f_1(\mathrm{lk}_{\Delta\backslash V_1} v) \leq 7\binom{d-2}{2}$ for all $v \in \Delta\backslash V_1$.

Since $d-2$ divides $f_0(\mathrm{lk}_{\Delta\backslash V_1} v) = 3d-6$, by Theorem 4.1 of [13],

$$f_j(\mathrm{lk}_{\Delta\backslash V_1} v) \geq f_j(\mathcal{ST}^\times(3d-6, d-3)) \text{ for all } j.$$

In particular, by Lemma 2.3.1, $f_1(\mathrm{lk}_{\Delta\backslash V_1} v) \geq (2^2\cdot 2-1)\binom{d-2}{2} = 7\binom{d-2}{2}$. Hence $f_1(\mathrm{lk}_{\Delta\backslash V_1} v) = 7\binom{d-2}{2}$. Since $\mathcal{ST}^\times(3d-6, d-3)$ has no missing k-faces for $1 < k < d-3$, it follows that $f_j(\mathrm{lk}_{\Delta\backslash V_1} v) = f_j(\mathcal{ST}^\times(3d-6, d-3))$ for all $j < d-3$. Thus $\mathrm{lk}_{\Delta\backslash V_1} v$ is either the stacked cross-polytopal sphere or the union of $\mathcal{ST}^\times(3d-6, d-3)$ with its missing facet σ_v.

On the other hand, the proof of Lemma 2.4.4 and Theorem 2.3.6 also implies that $\cap_{v\in V_2} \mathrm{lk}_{\Delta\backslash V_1} v$ has three connected components, where each component consists of $d-2$ vertices, all of different colors. Comparing with the structure of $\mathcal{ST}^\times(3d-6, d-3)$, we conclude that these three components are exactly the boundary complexes of the missing facets σ_v in $\mathrm{lk}_{\Delta\backslash V_1} v$, $v \in V_2$. Thus $\Delta\backslash V_1 = \cup_{v\in V_2} \mathrm{lk}_{\Delta\backslash V_1} v$ is the union of BM_{d-1} with its three missing facets, and hence by Lemma 2.3.1, $f_{d-2}(\Delta\backslash V_1) = f_{d-2}(BM_{d-1}) + 3 = 3\cdot 2^{d-1} + 3$.

Since for $w \in V_1$, $\mathrm{lk}_\Delta\, w$ is a homology sphere of dimension $d-2$ as well as a subcomplex of $\Delta \backslash V_1 = \cup_{v \in V_2} \mathrm{lk}_{\Delta \backslash V_1} v$, this link is either the cross-polytope or $\mathcal{ST}^\times(3d-3, d-2)$. Thus by Lemma 2.3.1, $f_{d-2}(\mathrm{lk}_\Delta\, w) \in \{2^{d-1}, 2^d - 1\}$. Therefore,

$$6 \cdot 2^{d-1} + 6 = 2f_{d-2}(\Delta \backslash V_1) = \sum_{w \in V_1} f_{d-2}(\mathrm{lk}_\Delta\, w) = (4+k)2^{d-1} - k, \text{ for some } k \in \{1, 2, 3, 4\},$$

where k is the number of vertices $w \in V_1$ such that $f_{d-2}(\mathrm{lk}_\Delta\, w) = 2^d - 1$. This leads to a contradiction and shows that no balanced $(3d+1)$-vertex triangulation of $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$ exists. □

Remark 2.4.7. The same proof also shows that in fact no $\mathbb{Q}$-homology manifold of dimension $d - 1 \geq 3$ and with $\beta_1(\Delta; \mathbb{Q}) \neq 0$ has a $(3d+1)$-vertex balanced triangulation.

Chapter 3

EAR DECOMPOSITION AND BALANCED SIMPLICIAL SPHERES

3.1 Introduction

A simplicial complex is called k-neighborly if every subset of vertices of size at most k is the set of vertices of one of its faces. Neighborly complexes, especially neighborly polytopes and spheres, are interesting objects to study. In the seminal work of McMullen [42] and Stanley [54], it was shown that in the class of polytopes and simplicial spheres, of a fixed dimension and with a fixed number of vertices, the cyclic polytope simultaneously maximizes all the face numbers. The d-dimensional cyclic polytope is $\lfloor \frac{d}{2} \rfloor$-neighborly. Since then, many other classes of neighborly polytopes have been discovered. We refer to [27], [52] and [49] for examples and constructions of neighborly polytopes. Meanwhile, the notion of neighborliness was extended to other classes of objects: for instance, neighborly cubical polytopes were defined and studied in [33], [32], and [51], and neighborly centrally symmetric polytopes and spheres were studied in [15], [31], [39], and [20].

One goal of this chapter is to discuss a similar notion for balanced simplicial complexes. Balanced complexes were defined by Stanley in [55], where they were called completely balanced. A $(d-1)$-dimensional simplicial complex is called balanced if its graph is d-colorable. We say that a balanced simplicial complex is *balanced k-neighborly* if every set of k or fewer vertices with *distinct* colors forms a face. It is natural to ask whether apart from cross-polytopes, balanced k-neighborly polytopes or balanced k-neighborly spheres exist. To the best of our knowledge, no examples of such objects appear in the current literature, even for $k = 2$, and there is not even a plausible sharp upper bound conjecture for balanced spheres.

In this chapter, we provide two constructions of balanced 2-neighborly 3-spheres with 16 vertices. In [50], Pfeifle, Pilaud and Santos, in answering the question of when a given graph is the graph of a polytope, studied the polytopality of Cartesian products of certain classes of non-polytopal graphs. As a consequence, our constructions show that the complete 4-partite graph on 16 vertices with parts of equal size is the graph of a 3-sphere.

In a different direction, it is also interesting to ask whether every rank-selected subcomplex of a balanced simplicial polytope or sphere has a convex ear decomposition. This statement, if true, would imply that rank-selected subcomplexes of balanced simplicial polytopes possess certain weak Lefschetz properties, see Theorem 3.9 in [59]. As a consequence, it would also provide an alternative proof of the balanced Generalized Lower Bound Theorem, see Theorem 3.3 and Remark 3.4 in [37]. Here we present examples giving a negative answer to this question for 3-dimensional spheres.

The structure of this chapter is as follows. In Section 3.2, after reviewing basic definitions, we establish basic properties of balanced neighborly spheres; in particular, we prove that for some values of f_0, such spheres cannot exist. In Section 3.3, we present our first construction of a balanced 2-neighborly 3-sphere with 16 vertices. We also show that several of its rank-selected subcomplexes do not have an ear decomposition, and find two balanced non-shellable 3-balls as its subcomplexes. In Section 3.4, we provide our second construction of a balanced 2-neighborly 3-sphere with 16 vertices. It is different from the first one since all of its rank-selected subcomplexes have ear decompositions. Several other examples are provided in Section 3.5.

3.2 Basic properties of balanced neighborly spheres

A *simplicial complex* Δ with vertex set V is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, and such that for every $v \in V$, $\{v\} \in \Delta$. For $\sigma \in \Delta$, let $\dim \sigma := |\sigma| - 1$ and define the *dimension* of Δ, $\dim \Delta$, as the maximum dimension of the faces of Δ. We say that a simplicial complex Δ is *pure* if all of its facets have the same dimension.

If Δ is a simplicial complex and σ is a face of Δ, the *star* of σ in Δ is $\mathrm{st}_\Delta\,\sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}$. We also define the *link* of σ in Δ as $\mathrm{lk}_\Delta\,\sigma := \{\tau - \sigma \in \Delta : \sigma \subseteq \tau \in \Delta\}$, and the *deletion* of a subset of vertices W from Δ as $\Delta\backslash W := \{\sigma \in \Delta : \sigma \cap W = \emptyset\}$. If Δ_1 and Δ_2 are simplicial complexes on disjoint vertex sets, then the join of Δ_1 and Δ_2, denoted $\Delta_1 * \Delta_2$, is the simplicial complex with vertex set $V(\Delta_1) \cup V(\Delta_2)$ whose faces are $\{\sigma_1 \cup \sigma_2 : \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$.

If Δ is a pure $(d-1)$-dimensional complex such that every $(d-2)$-dimensional face of Δ is contained in at most 2 facets, then the *boundary complex* of Δ consists of all $(d-2)$-dimensional faces that are contained in exactly one facet, as well as their subsets. A simplicial complex Δ is a *simplicial sphere* (resp. *simplicial ball*) if the geometric realization of Δ is homeomorphic to a sphere (resp. ball). The boundary complex of a simplicial d-ball is a simplicial $(d-1)$-sphere. A simplicial sphere is called *polytopal* if it is the boundary complex of a convex polytope. For instance, the boundary complex of an octahedron is a polytopal sphere; we will refer to it as an octahedral sphere.

For a fixed field $\mathbf{k}$, we say that Δ is a $(d-1)$-dimensional $\mathbf{k}$-*homology sphere* if $\tilde{H}_i(\mathrm{lk}_\Delta\,\sigma; \mathbf{k}) \cong \tilde{H}_i(\mathbb{S}^{d-1-|\sigma|}; \mathbf{k})$ for every face $\sigma \in \Delta$ (including the empty face) and $i \geq -1$. A *homology d-ball* (over a field $\mathbf{k}$) is a d-dimensional simplicial complex Δ such that (i) Δ has the same homology as the d-dimensional ball, (ii) for every face F, the link of F has the same homology as the $(d-|F|)$-dimensional ball or sphere, and (iii) the boundary complex, $\partial\Delta := \{F \in \Delta \mid \tilde{H}_i(\mathrm{lk}_\Delta F) = 0, \forall i\}$, is a homology $(d-1)$-sphere. The classes of simplicial $(d-1)$-spheres and homology $(d-1)$-spheres coincide when $d \leq 3$. From now on we fix $\mathbf{k}$ and omit it from our notation.

Next we define a special structure that exists in some pure simplicial complexes.

Definition 3.2.1. An *ear decomposition* of a pure $(d-1)$-dimensional simplicial complex Δ is an ordered sequence $\Delta_1, \Delta_2, \cdots, \Delta_m$ of pure $(d-1)$-dimensional subcomplexes of Δ such that:

1. Δ_1 is a simplicial $(d-1)$-sphere, and for each $j = 2, 3, \cdots, m$, Δ_j is a simplicial

$(d-1)$-ball.

2. For $2 \leq j \leq m$, $\Delta_j \cap (\cup_{i=1}^{j-1} \Delta_i) = \partial \Delta_j$.

3. $\cup_{i=1}^{m} \Delta_i = \Delta$.

We call Δ_1 the *initial complex*, and each Δ_j, $j \geq 2$, an *ear of this decompostion.* Notice that this definition is more general than Chari's original definition of a *convex ear decomposition*, see [16, Section 3.2], where the Δ_i's are required to be subcomplexes of the boundary complexes of polytopes. In particular, if a complex has no ear decomposition, then it has no convex ear decomposition. However, by the Steinitz theorem, all simplicial 2-spheres are polytopal, and hence also all simplicial 2-balls can be realized as subcomplexes of the boundary complexes of 3-dimensional polytopes. So for 2-dimensional simplicial complexes, the notion of an ear decomposition coincides with that of a convex ear decomposition.

A $(d-1)$-dimensional simplicial complex Δ is called *balanced* if the graph of Δ is d-colorable, or equivalently, there is a coloring map $\kappa : V \to [d]$ such that $\kappa(x) \neq \kappa(y)$ for any edge $\{x, y\} \in \Delta$. Here $[d] = \{1, 2, \cdots, d\}$ denotes the set of colors. A balanced simplicial complex is called *balanced k-neighborly* if every set of k or fewer vertices with distinct colors forms a face. For $S \subseteq [d]$, the subcomplex $\Delta_S := \{F \in \Delta : \kappa(F) \subseteq S\}$ is called the *rank-selected subcomplex* of Δ. We also define the *flag f-vector* $(f_S(\Delta) : S \subseteq [d])$ and the *flag h-vector* $(h_S(\Delta) : S \subseteq [d])$ of Δ, respectively, by letting $f_S(\Delta) := \#\{F \in \Delta : \kappa(F) = S\}$, where $f_\emptyset(\Delta) = 1$, and $h_S(\Delta) := \sum_{T \subseteq S} (-1)^{\#S - \#T} f_S(\Delta)$. The usual f-numbers and h-numbers can be recovered from the relations $f_{i-1}(\Delta) = \sum_{\#S=i} f_S(\Delta)$ and $h_i(\Delta) = \sum_{\#S=i} h_S(\Delta)$. The following symmetry of flag h-vectors of balanced spheres is well-known, see [10].

Lemma 3.2.2. *If Δ is a balanced homology $(d-1)$-sphere, then $h_S(\Delta) = h_{[d]\setminus S}(\Delta)$ for all $S \subseteq [d]$.*

In the remainder of this section, we establish some restrictions on the possible size of color sets of balanced neighborly spheres. In the following, V_i always denotes the set of vertices of color i.

Lemma 3.2.3. *Let Δ be a balanced k-neighborly homology $(2k-1)$-sphere. Then Δ has the same number of vertices of each color.*

Proof: Let W be an arbitrary subset of the set of colors, $[2k]$, of size k. Since Δ is balanced k-neighborly, Δ_W is also balanced k-neighborly, and hence Δ_W is the join of k color sets of colors in W, each considered as a 0-dimensional complex. By the fact that $h_k(\Delta_1 * \Delta_2) = \sum_{i=0}^{k} h_i(\Delta_1)h_{k-i}(\Delta_2)$ and by Lemma 3.2.2,

$$\prod_{i\in W}(|V_i|-1) = \prod_{i\in W} h_1(\Delta_{\{i\}}) = h_{|W|}(\Delta) = h_{|2k|-|W|}(\Delta) = \prod_{i\in[2k]\setminus W} h_1(\Delta_{\{i\}}) = \prod_{i\in[2k]\setminus W}(|V_i|-1).$$

Since $|[2k]\setminus W| = |W| = k$ and since $W \subseteq [2k]$ can be chosen arbitrarily, it follows that each color set in Δ must have the same size. $\square$

Lemma 3.2.4. *Let $d \geq 4$. If Δ is a balanced homology $(d-1)$-sphere and $V_d = \{v_1, v_2, v_3\}$ is the set of vertices of color d, then $\mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j$ is a homology $(d-2)$-ball for any $1 \leq i < j \leq 3$, and $\cap_{k=1}^{3} \mathrm{lk}_\Delta v_k$ is a homology $(d-3)$-sphere.*

Proof: Let $\Sigma = \mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j$ and $\Gamma = \cap_{k=1}^{3} \mathrm{lk}_\Delta v_k$. Every facet σ of Γ is a $(d-3)$-face whose link in Δ is a 6-cycle that contains the vertices v_1, v_2, v_3. Hence σ is contained in exactly one facet $\sigma \cup \{w\}$ of Σ, where w is the unique adjacent vertex to both v_i, v_j in $\mathrm{lk}_\Delta \sigma$. We conclude that Γ is the boundary complex of Σ and it is pure.

We first prove that Σ and Γ have the same homology as a $(d-2)$-ball and $(d-3)$-sphere respectively. Since each $(d-2)$-face of Δ is contained in exactly 2 facets, it follows that $\mathrm{lk}_\Delta v_i \cup \mathrm{lk}_\Delta v_j = \Delta_{[d-1]}$. By the Mayer-Vietoris sequence, for any $n \geq 0$,

$$\cdots \to H_{n+1}(\Delta_{[d-1]}) \to H_n(\Sigma) \to H_n(\mathrm{lk}_\Delta v_i) \oplus H_n(\mathrm{lk}_\Delta v_j) \to H_n(\Delta_{[d-1]}) \to \cdots . \quad (3.2.1)$$

Note that $\Delta_{[d-1]}$ is a deformation retract of Δ minus three points, hence $\beta_{d-2}(\Delta_{[d-1]}) = 2$ and $\beta_k(\Delta_{[d-1]}) = 0$ for $0 \leq k \leq d-3$. We conclude from (3.2.1) that $\beta_k(\Sigma) = 0$ for all $k \geq 0$. Again by the Mayer-Vietoris sequence and the fact that $\mathrm{lk}_\Delta v_{[3]-\{i,j\}} \cup \Sigma = \Delta_{[d-1]}$, we obtain

$$\cdots \to H_{n+1}(\Delta_{[d-1]}) \to H_n(\Gamma) \to H_n(\mathrm{lk}_\Delta v_{[3]-\{i,j\}}) \oplus H_n(\Sigma) \to H_n(\Delta_{[d-1]}) \to \cdots .$$

Hence $\beta_{d-3}(\Gamma) = 1$ and $\beta_k(\Gamma) = 0$ for $0 \leq k \leq d-4$.

Next, for any $\tau \in \Gamma$, we have $\mathrm{lk}_\Sigma \tau = \mathrm{lk}_{\mathrm{lk}_\Delta \tau} v_i \cap \mathrm{lk}_{\mathrm{lk}_\Delta \tau} v_j$ and $\mathrm{lk}_\Gamma \tau = \cap_{i=1}^{3} \mathrm{lk}_{\mathrm{lk}_\Delta \tau} v_i$. Since $\mathrm{lk}_\Delta \tau$ is a balanced homology $(d-1-|\tau|)$-sphere, using the same argument as above, we may show that $\mathrm{lk}_\Sigma \tau$ and $\mathrm{lk}_\Gamma \tau$ has the same homology as a $(d-2-|\tau|)$-ball and $(d-3-|\tau|)$-sphere respectively. Therefore Γ is a homology $(d-3)$-sphere. Finally, for any interior face σ of Σ, $\mathrm{lk}_\Sigma \sigma = \mathrm{lk}_{\mathrm{lk}_\Delta v_i} \sigma = \mathrm{lk}_{\mathrm{lk}_\Delta v_j} \sigma$, and hence $\mathrm{lk}_\Sigma \sigma$ is a homology sphere. By definition we conclude that Σ is a homology $(d-2)$-ball. □

Remark 3.2.5. The complex Γ in Lemma 3.2.4 is not balanced, since Γ is $(d-1)$-colorable instead of being $(d-2)$-colorable.

By Lemma 3.2.3, if balanced k-neighborly homology $(2k-1)$-spheres exist, then the number of vertices must be $2kl$ for some $l \geq 2$. However, as the following proposition shows, we cannot hope for the existence of such spheres for all values of $k \geq 2$ and $l \geq 2$.

Proposition 3.2.6. *No balanced 2-neighborly homology 3-spheres with 12 vertices exist.*

Proof: Assume that Δ is such a sphere. By Lemma 3.2.3, each color set of Δ has three

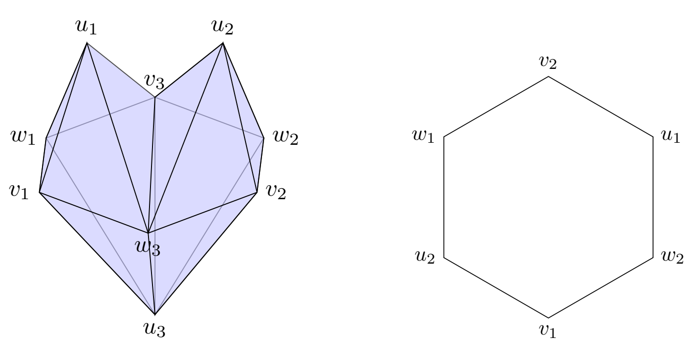


Figure 3.1: Left: triangulation of the vertex link $\mathrm{lk}_\Delta z_i$ for $z_i \in V_4$, where $\{u_1, u_2, u_3\}$, $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are the three other color sets. Right: the missing edges between vertices of different color of $\mathrm{lk}_\Delta z_i$.

vertices. We let $V_4 = \{z_1, z_2, z_3\}$ be the set of vertices of color 4. Since each $\mathrm{lk}_\Delta z_i$ is a

2-sphere with 9 vertices, its f-vector is (1,9,21,14). Furthermore, the balancedness of Δ implies that every vertex $v \in \mathrm{lk}_\Delta z_i$ has $\deg_{\mathrm{lk}_\Delta z_i} v = 4$ or 6. If x is the number of vertices of degree 6 in $\mathrm{lk}_\Delta z_i$, then

$$4(9-x)+6x = \sum_{v \in \mathrm{lk}_\Delta z_i} f_0(\mathrm{lk}_{\mathrm{lk}_\Delta z_i} v) = 2f_1(\mathrm{lk}_\Delta z_i) = 42,$$

and hence $x = 3$. A balanced 2-sphere with 9 vertices, 3 of which have degree 6, is unique up to an isomorphism, as shown in Figure 3.1. It is immediate that the missing edges between vertices of different colors in this sphere form a 6-cycle.

On the other hand, $\Sigma := \mathrm{lk}_\Delta z_1 \cap \mathrm{lk}_\Delta z_2$ is a triangulated 2-ball by Lemma 3.2.4. If we delete all of the boundary edges from Σ, the resulting complex Σ' is still contractible. However, Σ does not have interior vertices. (An interior vertex of Σ would not be in $V(\mathrm{lk}_\Delta v_3)$, which would contradict the 2-neighborliness of Δ.) Hence the missing edges of $\mathrm{lk}_\Delta z_3$ that form a 6-cycle are the only interior faces of Σ, i.e., Σ' is a 6-cycle. This contradicts that Σ' is contractible, so no such sphere exists. □

In fact, a stronger result holds.

Lemma 3.2.7. *Up to an isomorphism, there are only three triangulations of balanced 3-spheres with each color set of size 3.*

Proof: Let Δ be such a sphere and let $V_4 = \{z_1, z_2, z_3\}$. Each vertex link of Δ is a balanced 2-sphere with at most 9 vertices, hence it is either the octahedral sphere, the suspension of a 6-cycle, or the connected sum of two octahedral spheres. We denote these three 2-spheres as Σ_1, Σ_2 and Σ_3 respectively. By Lemma 3.2.4, $\Delta_{[3]}$ is the union of three triangulated 2-balls $B_i = \mathrm{lk}_\Delta z_j \cap \mathrm{lk}_\Delta z_k$, where $\{i,j,k\} = [3]$, glued along their common boundary complex c. Assume that $f_0(\mathrm{lk}_\Delta z_i) \leq f_0(\mathrm{lk}_\Delta z_j)$ when $i \leq j$. An easy counting leads to

$$f_0(\Delta_{[3]}) = f_0(c) + \sum_{i=1}^{3} f_0(B_i \backslash c) = 9, \quad f_0(\mathrm{lk}_\Delta z_i) = f_0(c) + f_0(B_j \backslash c) + f_0(B_k \backslash c) \in \{6, 8, 9\},$$

where $f_0(B_i \backslash c)$ counts the number of interior vertices of B_i. In what follows we enumerate all possible values of the triple $(f_0(\mathrm{lk}_\Delta z_1), f_0(\mathrm{lk}_\Delta z_3), f_0(\mathrm{lk}_\Delta z_3))$:

1. $(f_0(\mathrm{lk}_\Delta z_1), f_0(\mathrm{lk}_\Delta z_2), f_0(\mathrm{lk}_\Delta z_3)) = (6, 6, 9)$ or $(6, 8, 9)$. Since $\mathrm{lk}_\Delta z_1$ is combinatorially equivalent to Σ_1, it follows that $\mathrm{lk}_\Delta v_3$ is obtained from $\mathrm{lk}_\Delta v_2$ by a cross flip (see [30] for a reference). So either $\mathrm{lk}_\Delta v_2 \cong \Sigma_1$, $\mathrm{lk}_\Delta v_3 \cong \Sigma_3$, and the cross flip replaces a 2-face of $\mathrm{lk}_\Delta v_2$ with its complement in the octahedral sphere; or $\mathrm{lk}_\Delta v_2 \cong \Sigma_2$, $\mathrm{lk}_\Delta v_3 \cong \Sigma_3$, and the cross flip replaces the union of three 2-faces of $\mathrm{lk}_\Delta v_2$ with its complement in the octahedral sphere.

2. $(f_0(\mathrm{lk}_\Delta z_1), f_0(\mathrm{lk}_\Delta z_2), f_0(\mathrm{lk}_\Delta z_3)) = (8, 8, 8)$ or $(8, 8, 9)$. In the former case, c is a 6-cycle and $\Delta_{[3]} \backslash c$ consists of three disjoint vertices. It is easy to see that at least one of these vertices has degree 6. Then since $\mathrm{lk}_\Delta z_1 \cong \mathrm{lk}_\Delta z_2 \cong \Sigma_2$, the other two vertices must be of degree 6 as well, and hence $\Delta_{[3]}$ is the join of c and three disjoint vertices. In the latter case, Since the vertices of degree 6 in $\mathrm{lk}_\Delta z_3$ form a 3-cycle, the two disjoint vertices in $\Delta_{[3]} \backslash c$ cannot both have degree 6 or 4. However, if one vertex of $\Delta_{[3]} \backslash c$ is of degree 6, then since $\mathrm{lk}_\Delta z_1$ and $\mathrm{lk}_\Delta z_2$ are combinatorially equivalent to Σ_2 and c is a 7-cycle, B_3 must be the join of one vertex u and a path of length 6. Then u is not connected to any vertex of $\Delta_{[3]} - c$, a contradiction.

3. $(f_0(\mathrm{lk}_\Delta z_2), f_0(\mathrm{lk}_\Delta z_3)) = (9, 9)$. In this case, B_1 is a triangulated 2-ball with all of its 9 vertices on the boundary. There is only one balanced 2-sphere with 9 vertices that contains B_1 (it is isomorphic to Σ_3). Hence $\mathrm{lk}_\Delta z_2 = \mathrm{lk}_\Delta z_3$, a contradiction.

In sum, we obtain three balanced 3-spheres with 12 vertices: S_1, the connected sum of two octahedral 3-spheres; S_2, the join of two 6-cycles, and S_3, with $\mathrm{lk}_\Delta z_i \cong \Sigma_i$ for $1 \le i \le 3$. □

Proposition 3.2.8. *No balanced 2-neighborly homology* 4*-spheres with each color set of size 3 exist.*

Proof: Let Δ be such a sphere and let v_1, v_2, v_3 be the vertices of color 5. By Alexander Duality, $\tilde{H}_i(\Delta_{\{4,5\}}) \cong \tilde{H}_{3-i}(\Delta_{[3]})$. In particular, since $\Delta_{\{4,5\}}$ is balanced 2-neighborly,

$\beta_2(\Delta_{[3]}) = \beta_1(\Delta_{\{4,5\}}) = 4$ and $\beta_1(\Delta_{[3]}) = 0$. Hence

$$f_2(\Delta_{[3]}) = (f_1 - f_0 + \chi)(\Delta_{[3]}) = \frac{9 \cdot 6}{2} - 9 + 5 = 23.$$

By double counting, $\sum_{i=1}^{3} f_1(\mathrm{lk}_\Delta\, v_i) = \sum_{W=\{i,j,5\}\subseteq[5]} f_2(\Delta_W) = \binom{4}{2} f_2(\Delta_{[3]}) = 138$. Since $f_1(S_1) = 42$, $f_1(S_2) = 48$ and $f_1(S_3) = 46$, it follows that either $\mathrm{lk}_\Delta\, v_1 \cong S_1$ and $\mathrm{lk}_\Delta\, v_2, \mathrm{lk}_\Delta\, v_3 \cong S_2$, or $\mathrm{lk}_\Delta\, v_i \cong S_3$ for all i.

Consider the first case above. It can be checked that for any $W = \{i, j\}$, $f_1((\mathrm{lk}_\Delta\, v_1)_W) = 7$ and $f_1((\mathrm{lk}_\Delta\, v_2)_W) = 6$ or 9, depending on whether $(\mathrm{lk}_\Delta\, v_2)_W$ is a 6-cycle or not. Hence $f_2(\Delta_{W\cup\{5\}}) = \sum_{i=1}^{3} f_1((\mathrm{lk}_\Delta\, v_i)_W) \neq 23$, a contradiction. As for the second case, since $\mathrm{lk}_\Delta\, v_1 \cap \mathrm{lk}_\Delta\, v_2$ is a homology 3-ball with 12 vertices on the boundary, by Lemma 3.2.7 there is a unique balanced 3-sphere combinatorially equivalent to S_3 that contains $\mathrm{lk}_\Delta\, v_1 \cap \mathrm{lk}_\Delta\, v_2$ as a subcomplex. It follows that $\mathrm{lk}_\Delta\, v_1 = \mathrm{lk}_\Delta\, v_2$, a contradiction. Hence no balanced 2-neighborly homology 4-spheres with 15 vertices exist. □

Corollary 3.2.9. *The only balanced 3-neighborly homology 5-sphere with* ≤ 18 *vertices is the octahedral 5-sphere.*

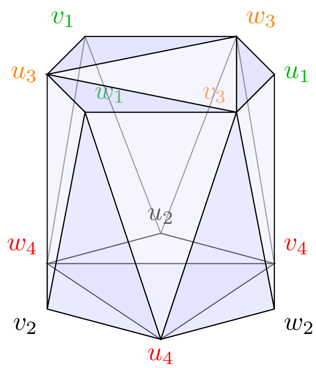
Proof: Let Δ be such a sphere. The vertex links of Δ are balanced 2-neighborly 4-spheres with ≤ 16 vertices. By Proposition 3.2.8, it must be the suspension of a balanced 2-neighborly 3-sphere with ≤ 14 vertices. Then the result follows from Lemma 3.2.3 and Proposition 3.2.6. □

We propose the following conjecture motivated by Proposition 3.2.6 and Corollary 3.2.9.

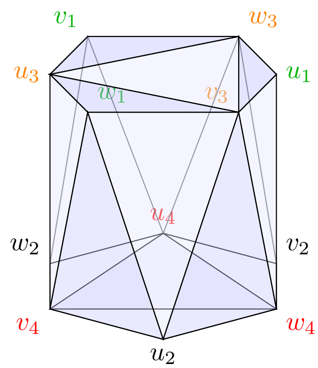
Conjecture 3.2.10. *For an arbitrary* $k \geq 2$*, there does not exist a balanced* k*-neighborly homology* $(2k-1)$*-sphere with* $6k$ *vertices.*

3.3 First Construction

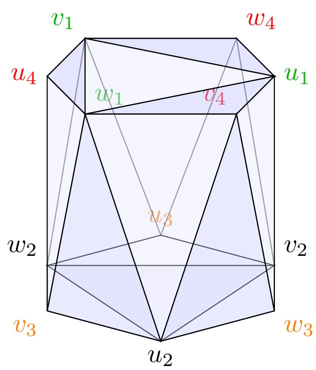
In this section we present our first construction of a balanced 2-neighborly 3-sphere with 16 vertices. We denote it by Δ. By Proposition 3.2.3, each color set of Δ has four vertices.



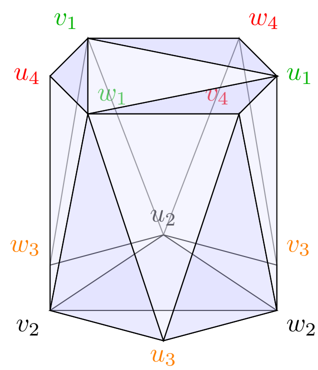
(a) $\mathrm{lk}_\Delta\, z_1$



(b) $\mathrm{lk}_\Delta\, z_2$



(c) $\mathrm{lk}_\Delta\, z_3$



(d) $\mathrm{lk}_\Delta\, z_4$

Figure 3.2: Four vertex links of Δ

Construction 3.3.1. Let the color sets of Δ be $V_1 = \{u_1, u_2, u_3, u_4\}$, $V_2 = \{v_1, v_2, v_3, v_4\}$, $V_3 = \{w_1, w_2, w_3, w_4\}$ and $V_4 = \{z_1, z_2, z_3, z_4\}$.

In Figure 2 we illustrate the construction of the vertex links $\mathrm{lk}_\Delta\, z_i$ for $i = 1, \ldots, 4$. All these links are realized as cylinders. Two links $\mathrm{lk}_\Delta\, z_1$ and $\mathrm{lk}_\Delta\, z_2$ share the same top and bottom, which are triangulated hexagons spanned by vertices $\{u_i, v_i, w_i : i = 1, 3\}$ and $\{u_i, v_i, w_i : i = 2, 4\}$, respectively. To construct $\mathrm{lk}_\Delta\, z_3$ from $\mathrm{lk}_\Delta\, z_1$, we switch the positions of vertices u_3, v_3, w_3 with vertices u_4, v_4, w_4 respectively and form a new cylinder. The new top and bottom hexagons contain the 2-faces $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$. Similarly, we construct the link $\mathrm{lk}_\Delta\, z_4$ from $\mathrm{lk}_\Delta\, z_2$ by switching the positions of vertices u_3, v_3, w_3 with vertices u_4, v_4, w_4 and letting $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$ be the 2-faces that appear in the triangulation of the top and bottom hexagons. It follows that $\mathrm{lk}_\Delta\, z_3$ and $\mathrm{lk}_\Delta\, z_4$ also share the same top and bottom.

Now since Δ is balanced 2-neighborly, by our construction, it only remains to show that Δ is a 3-sphere. The geometric realizations of $\mathrm{st}_\Delta\, z_1$ and $\mathrm{st}_\Delta\, z_2$ are filled cylinders that share top and bottom. So their union is a filled torus (that is, a genus-1 handlebody); so is the union of $\mathrm{st}_\Delta\, z_3$ and $\mathrm{st}_\Delta\, z_4$. Note that these two genus-1 handlebodies have identical boundary complexes, thus they provide a genus-1 Heegaard splitting of a 3-sphere, which is our Δ.

Remark 3.3.2. Our construction Δ has the following properties:

1. One can check that all vertex links are combinatorially equivalent. Furthermore, all 2-dimensional rank-selected subcomplexes of Δ are isomorphic.

2. The intersection of two vertex links, where the vertices are of the same color, always has at least two connected components. By the construction of Δ, $\mathrm{lk}_\Delta\, z_i \cap \mathrm{lk}_\Delta\, z_j$ has two connected components when $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$ (they are the top and bottom hexagons as shown in Figure 2); and it has three connected components when $i \in \{1, 2\}$ and $j \in \{3, 4\}$ (each component is the union of two facets along the side of the cylinders).

3. There are at least three group actions on the vertices of Δ:

   (a) Fix the subscript and rotate the corresponding vertices of color 1, 2 and 3 respectively. The generator is given by $(u_1v_1w_1)(u_2v_2w_2)(u_3v_3w_3)$.

   (b) Rotate vertices of the same color. The generator is

   $$(u_1u_3u_2u_4)(v_1v_3v_2v_4)(w_1w_3w_2w_4)(z_1z_3z_2z_4).$$

   (c) Exchange $\mathrm{lk}_\Delta\, z_1$ and $\mathrm{lk}_\Delta\, z_2$, $\mathrm{lk}_\Delta\, z_3$ and $\mathrm{lk}_\Delta\, z_4$, by exchanging v_i and w_i (or u_i and w_i, u_i and v_i) for all $i \in [4]$. The generators are $(z_1z_2)(z_3z_4)(v_1w_1)(v_2w_2)(v_3w_3)(v_4w_4)$, $(z_1z_2)(z_3z_4)(u_1w_1)(u_2w_2)(u_3w_3)(u_4w_4)$ and $(z_1z_2)(z_3z_4)(u_1v_1)(u_2v_2)(u_3v_3)(u_4v_4)$.

Hence the automorphism group of Δ is of size at least 96.

Proposition 3.3.3. *There exist balanced 3-spheres whose 2-dimensional rank-selected subcomplexes do not have an ear decomposition.*

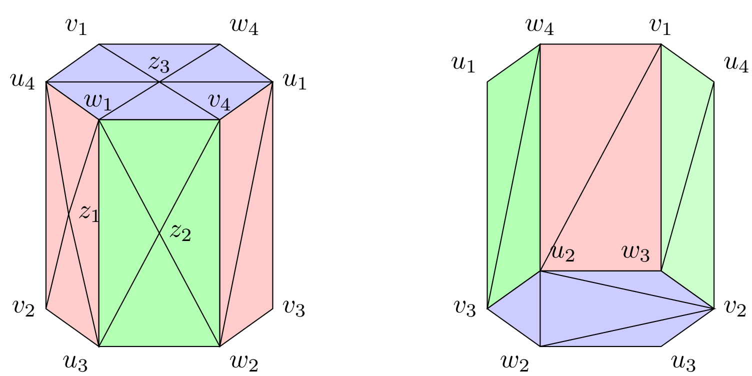
Proof: Consider the complex Δ in Construction 3.3.1 and its rank-selected subcomplex $\Delta_{[3]}$. Suppose $\Delta_{[3]}$ has an ear decomposition $(\Gamma_1, \Gamma_2, \cdots, \Gamma_k)$. Since $\Delta_{[3]}$ is the deformation retract of the sphere Δ minus 4 points, $\beta_2(\Delta_{[3]}) = 3$ and so k must be 3. Note that $\Delta_{[3]}$ is the union of three 3-balls $\mathrm{st}_\Delta\, z_i$ $(1 \leq i \leq 3)$ with three vertices z_1, z_2, z_3 removed. The only subcomplexes of $\Delta_{[3]}$ that can be realized as the boundary complexes of 3-balls must be the boundary complex of either $\mathrm{st}_\Delta\, z_i$, $\mathrm{st}_\Delta\, z_i \cup \mathrm{st}_\Delta\, z_j$ or $\cup_{i=1}^{3} \mathrm{st}_\Delta\, z_i$. Using the second property of Δ in Remark 3.3.2, we conclude that $\Gamma_1 = \mathrm{lk}_\Delta\, z_i$ for some $i \in [4]$. Furthermore, $\partial\Gamma_2$ divides Γ_1 into two 2-balls B_1 and B_2. Hence $B_1 \cup \Gamma_2$ is also a 2-sphere. Then the above argument yields that $B_1 \cup \Gamma_2 = \mathrm{lk}_\Delta\, z_j$ for some $j \neq i$. This, however, leads to a contradiction, since $\mathrm{lk}_\Delta\, z_i \cap \mathrm{lk}_\Delta\, z_j$ has at least two connected components. Hence $\Delta_{[3]}$ does not have an ear decomposition. $\square$

In the following we describe two balanced non-shellable 3-balls which are subcomplexes of Δ. Recall that a simplicial $(d-1)$-ball B is called *shellable* if its facets can be ordered $(F_1, F_2, \ldots, F_n)$ in such a way that the intersection of a facet with the union of previous facets is pure $(d-2)$-dimensional. Such an ordering of facets is called a *shelling*. A facet of B is defined to be *free* if its intersection with the boundary of B is a $(d-2)$-ball. A simplicial ball B is called *strongly non-shellable* if it has no free facet. Observe that if B is a shellable ball with more than one facet, and $(F_1, \ldots, F_n)$ is a shelling of B, then F_n must be a free facet. Hence if B is a strongly non-shellable ball, then B is non-shellable. For more discussion on shellability, see Ziegler's book [72].

Construction 3.3.4. Our construction begins with the balanced 3-sphere Δ in Construction 3.3.1. Remove the vertex z_4 from Δ and denote the resulting complex by B_1. Then B_1 is a balanced 3-ball with 15 vertices (z_1, z_2 and z_3 are the only interior vertices) and 60 facets.

The boundary of B_1 is exactly $\mathrm{lk}_\Delta z_4$. In particular, six 2-faces from $\mathrm{lk}_{B_1} z_1$ form three connected components, another six 2-faces of B_1 are from $\mathrm{lk}_{B_1} z_2$, and the remaining eight 2-faces that form two connected components are from $\mathrm{lk}_{B_1} z_3$ (see part 2 of Remark 3.3.2). We denote by a_1 the union of two adjacent faces $\{u_3v_2w_1\}$ and $\{u_4v_2w_1\}$ in $\mathrm{lk}_{B_1} z_1$, by a_2 the union of $\{u_3v_4w_1\}$ and $\{u_3v_4w_2\}$ in $\mathrm{lk}_{B_1} z_2$, and by a_3 the top triangulated hexagon in $\mathrm{lk}_{B_2} z_3 = \mathrm{lk}_\Delta z_3$, as shown in Figure 2(c).

From B_1 we construct another 3-ball B_2 by deleting the eight 3-faces in $\cup_{i=1}^3(a_i * z_i)$. The resulting complex, denoted as B_2, is a balanced 3-ball with 15 vertices (all of them are on the boundary of B_2) and 52 facets, see Figure 3.3 for a view of the boundary of B_2.



(a) ∂B_2, front view (b) ∂B_2, back view

Figure 3.3: The front view and back view of ∂B_2. The faces from $\mathrm{st}_\Delta z_1$, $\mathrm{st}_\Delta z_2$ and $\mathrm{st}_\Delta z_3$ are colored in pink, green and blue respectively.

Proposition 3.3.5. *The balanced 3-ball B_2 in Construction 3.3.4 is strongly non-shellable.*

Proof: Since every vertex of B_2 is on the boundary, the intersection of any facet σ and ∂B_2 must contain four vertices of different colors. Furthermore, if $\sigma \cap \partial B_2$ is pure 2-dimensional, then it contains at least two adjacent facets of ∂B_2. It can be checked that there are 18 pairs of adjacent 2-faces of ∂B_2 such that all four vertices are of different colors, but none of them can be realized as a subcomplex of a facet of B_2, a contradiction. Hence B_2 has no free facet, i.e., it is strongly non-shellable. $\square$

We proceed to show that B_1 is also non-shellable.

Proposition 3.3.6. *The balanced 3-ball B_1 in Construction 3.3.4 is non-shellable.*

Proof: Assume that B_1 has a shelling $\tau_1, \tau_2, \ldots, \tau_{60}$, and for $1 \leq j \leq 3$, τ_{n_j} is the last facet of $\text{st}_\Delta\, z_j$ that appears in the shelling; assume further that $\tau_{n_j} \backslash \{z_j\}$ belongs to the connected component a_j in $\text{lk}_\Delta\, z_j \cap \text{lk}_\Delta\, z_4$, where Δ is the balanced 3-sphere in Construction 3.3.1. Note that τ_{60} is a facet of $\Gamma = \cup_{j=1}^{3}(a_j * z_j)$. Now assume that $\tau_{60-i}, \ldots, \tau_{60}$ are facets of Γ for some $0 \leq i \leq 6$. If $\tau_{59-i} \notin \Gamma$, then by our assumption $n_j \geq 60 - i$ for any $1 \leq j \leq 3$, and so $\cup_{j=1}^{59-i}\tau_j$ is a 3-ball with no interior vertices. Moreover, the intersection $\tau_{59-i} \cap (\cup_{j=1}^{58-i}\tau_j)$ satisfies the following two conditions: 1) it contains all four vertices of τ_{59-i}, and 2) it is a subcomplex of $\tau_{59-i} \cap (B_1 \backslash \Gamma)$. However, a similar argument as in the proof of Proposition 3.3.5 yields that $\tau_{59-i} \cap (B_1 \backslash \Gamma)$ cannot be pure 2-dimensional. Hence $\tau_{59-i} \in \Gamma$.

By induction, the eight facets of Γ are exactly the last eight facets that appear in the shelling of B_1. This implies $B_1 \backslash \Gamma$ is shellable. However, the proof of Proposition 3.3.5 shows that $B_1 \backslash \Gamma$ is strongly non-shellable, a contradiction. □

Corollary 3.3.7. *The 3-sphere Δ in Construction 3.3.1 is non-polytopal.*

Proof: If Δ is polytopal, then there exists a shelling of Δ that ends with the facets of $\text{st}_\Delta\, z_4$, see [72, Lemma 8.10 and Theorem 8.12]. Hence $B_1 = \Delta \backslash \{z_4\}$ is shellable, which contradicts Proposition 3.3.6. □

Remark 3.3.8. At present, we do not know if the complex Δ is shellable or not.

3.4 Second Construction

In section 3 we constructed a balanced 2-neighborly 3-sphere all of whose 2-dimensional rank-selected subcomplexes have no ear decompositions. Now we provide a second construction that is not combinatorially equivalent to Construction 3.3.1.

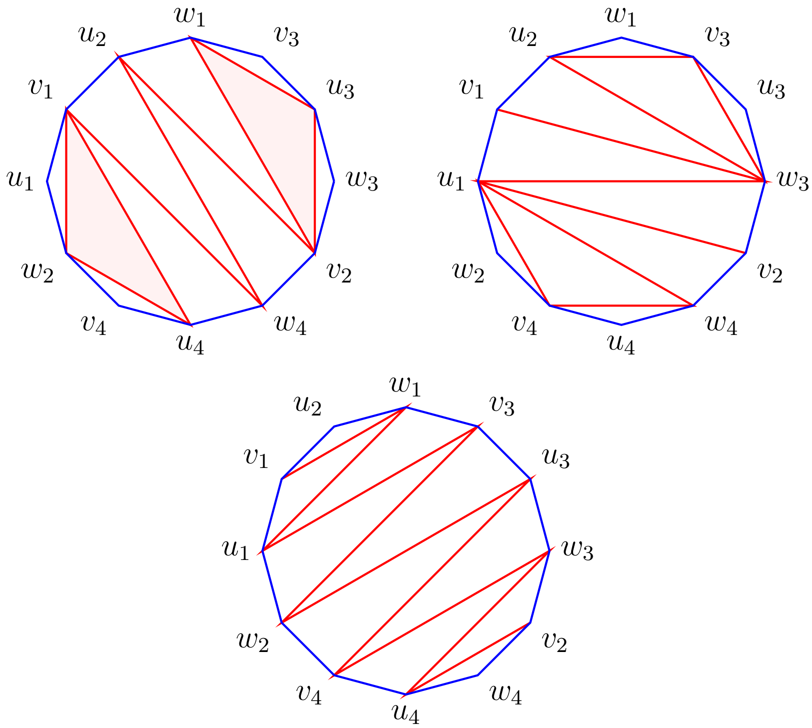


Figure 3.4: Discs A, B and C (from left to right)

Construction 3.4.1. Let $V_1 = \{u_1, u_2, u_3, u_4\}$, $V_2 = \{v_1, v_2, v_3, v_4\}$, $V_3 = \{w_1, w_2, w_3, w_4\}$ and $V_4 = \{z_1, z_2, z_3, z_4\}$ be the four color sets of a balanced 3-sphere Γ. We let $\mathrm{lk}_\Gamma\, z_1 = A \cup_{\partial A \sim \partial C} C$ and $\mathrm{lk}_\Gamma\, z_3 = B \cup_{\partial B \sim \partial C} C$, where A, B and C are triangulated 2-balls sharing the same boundary as shown in Figure 3.4. All possible edges that do not appear in A, B and C are shown in Figure 3.5 as solid red edges in disc D'. Notice that the dashed edges in D' are edges in discs A and B, so we may rearrange the boundary of D by switching the positions of vertices v_1 and v_2, and then replacing the edges containing v_1 or v_2 in $\partial D'$ by the dashed edges. In this way, we obtain a triangulation of a 12-gon D as shown in Figure 3.5. Furthermore, $\partial D \subseteq A \cup B$, and ∂D divides the sphere $= A \cup_{\partial A \sim \partial B} B$ into two discs A' and B' as shown in Figure 6.

We let $\mathrm{lk}_\Gamma\, z_2 = A' \cup_{\partial A' \sim \partial D} D$ and $\mathrm{lk}_\Gamma\, z_4 = B' \cup_{\partial B' \sim \partial D} D$. It is not hard to see that the complex Γ with the given four links is indeed a balanced 2-neighborly 3-sphere.

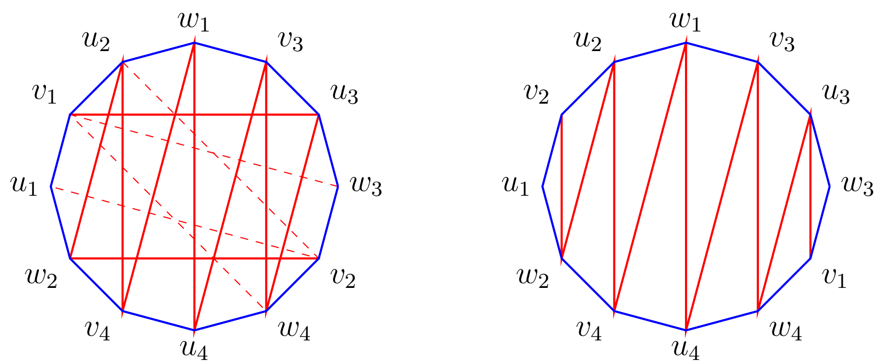


Figure 3.5: Left: disc D'. Right: disc D obtained after rearranging the boundary of D'.

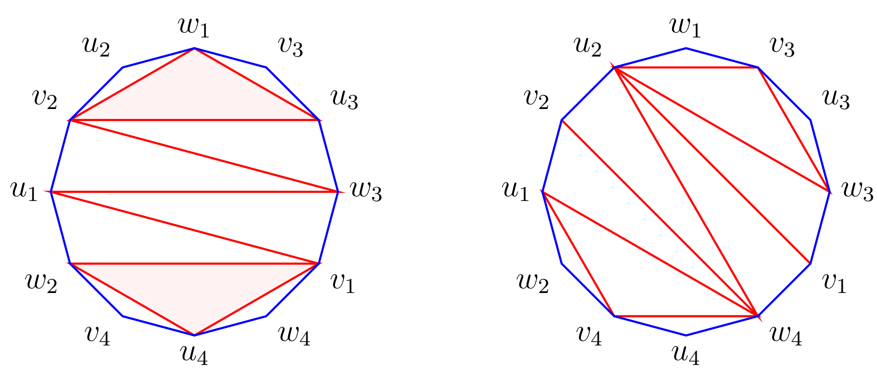


Figure 3.6: Left: disc A'. Right: disc B'. Notice that $\partial A' = \partial B' = \partial D$.

Here we list some properties of Γ in Construction 3.4.1. First, $(A \cup B, C, D)$ is an ear decomposition of $\Gamma_{[3]}$. Since property 1 of Remark 3.3.2 and Proposition 3.3.3 implies that all 2-dimensional rank-selected subcomplexes of the 3-sphere in Construction 3.3.1 do not have an ear decomposition, we conclude that Γ is not combinatorially equivalent to Construction 3.3.1. Alternatively, we may prove non-equivalence by inspecting the vertex links. In Construction 3.3.1, all vertex links are combinatorially equivalent; however, there are two types of vertex links of Γ: $\mathrm{lk}_\Gamma\, z_i$ and $\mathrm{lk}_\Gamma\, v_i$ for $i = 1, 2$ are combinatorially equivalent to the triangulated spheres in Figure 2 (to see this, note that the light red triangles that appear in discs A and A' correspond to the middle triangles in the top and bottom hexagons of the cylinders in Figure 2); the other 12 vertex links are combinatorially equivalent to the balanced 2-sphere with 12 vertices such that exactly two of its vertices have degree 8.

Second, the automorphism group of Γ has at least two generators

$$(u_1u_3u_2u_4)(v_1z_2v_2z_1)(v_3z_4v_4z_3)(w_1w_4w_2w_3), (z_1v_1)(z_2v_2)(z_3v_3)(z_4v_4)(u_1w_1)(u_2w_2)(u_3w_3)(u_4w_4).$$

(The second generator is given by switching vertices of color 1 and 3, and color 2 and 4, but with the same subscript.) Hence $\text{Aut}(\Delta)$ has at least 8 elements.

Remark 3.4.2. The Construction 3.4.1 is shellable. For $\text{lk}_\Gamma z_1 = A \cup_{\partial A \sim \partial C} C$, there exist two shellings $c_1, \ldots, c_{10}, a_1, \ldots, a_{10}$ and $a'_1, \ldots, a'_{10}, c'_1, \ldots, c'_{10}$ such that for any $1 \leq i \leq 10$, c_i, c'_i are facets from C and a_i, a'_i are facets from A. Similarly, there exist two shellings $c_1, \ldots, c_{10}, b_1, \ldots, b_{10}$ and $b'_1, \ldots, b'_{10}, c'_1, \ldots, c'_{10}$ for $\text{lk}_\Gamma z_3 = B \cup_{\partial B \sim \partial C} C$, where b_i, b'_i are facets from B. Then

$$a'_1 * z_1, \ldots, a'_{10} * z_1, c'_1 * z_1, \ldots, c'_{10} * z_1, c_1 * z_3, \ldots, c_{10} * z_3, , b_1 * z_3 \ldots, b_{10} * z_3$$

gives a shelling of $\text{st}_\Gamma z_1 \cup \text{st}_\Gamma z_3$. We may extend this shelling into a complete shelling of Γ by constructing two similar shellings of $\text{lk}_\Gamma z_2$ and $\text{lk}_\Gamma z_4$. However, we have not been able to check whether Γ is polytopal or not.

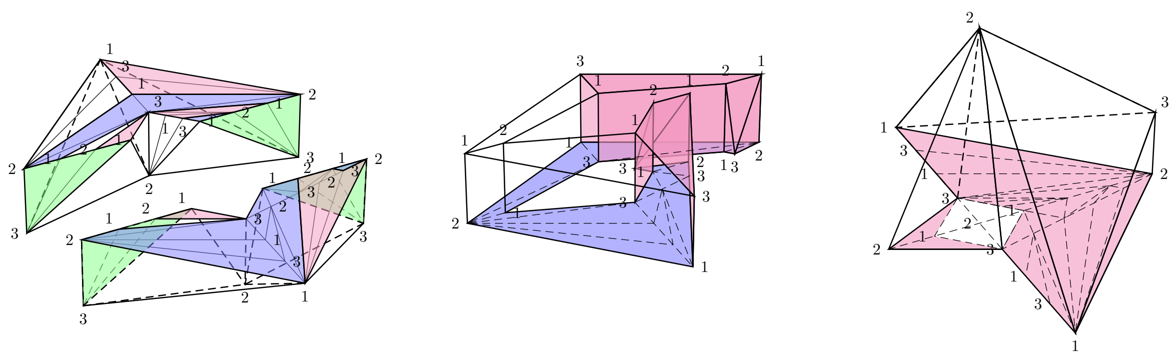
Remark 3.4.3. It is easy to see that if Δ_1 is a balanced 2-neighborly $(d_1 - 1)$-sphere and Δ_2 is a balanced 2-neighborly $(d_2 - 1)$-sphere, then $\Delta_1 * \Delta_2$ is a balanced 2-neighborly $(d_1 + d_2 - 1)$-sphere. Hence by taking joins, we find balanced 2-neighborly $(4k - 1)$-spheres with $16k$ vertices for any $k \geq 1$.

Question 3.4.4. *Let $d \geq 4$ and $m \geq 5$ be arbitrary integers. Is there a balanced 2-neighborly simplicial $(d - 1)$-sphere all of whose color sets have the same size m? Is there a polytopal sphere with these properties?*

3.5 Third Construction

In general, there are many balanced 3-spheres, not necessarily 2-neighborly, such that some of their rank-selected subcomplexes have no ear decompositions. Here we present an example different from Construction 3.3.1. Its rank-selected subcomplex can be embedded in $\mathbb{R}^3$.

The strategy is that we want to construct a balanced 3-sphere Δ so that for a fixed color set $W = \{v_1, \cdots, v_5\}$, the intersection of any two vertex links $\mathrm{lk}_\Delta\, v_i \cap \mathrm{lk}_\Delta\, v_j$ always has at least two connected components. In Figure 3.7 we show the links $\mathrm{lk}_\Delta\, v_1, \cdots, \mathrm{lk}_\Delta\, v_4$. Every label represents the color of the vertex. Also each connected component of $\mathrm{lk}_\Delta\, v_1 \cap \mathrm{lk}_\Delta\, v_2$ is colored in green, $\mathrm{lk}_\Delta\, v_i \cap \mathrm{lk}_\Delta\, v_3$ is colored in blue for $i = 1, 2$, and $\mathrm{lk}_\Delta\, v_j \cap \mathrm{lk}_\Delta\, v_4$ is colored in pink for $j = 1, 2, 3$. Immediately we check that all these intersections of vertex links have 2 or 3 connected components.

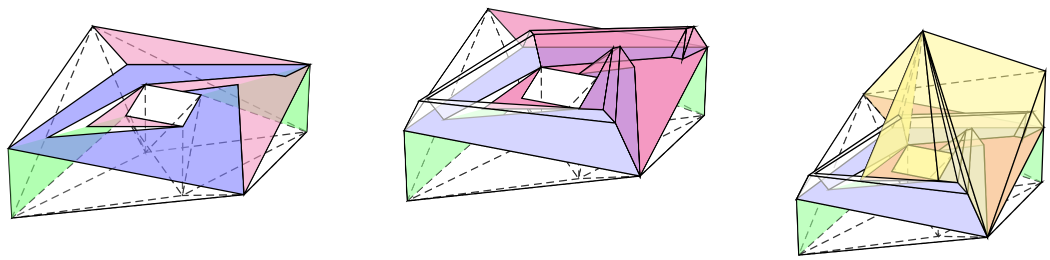


(a) $\mathrm{lk}_\Delta\, v_1$ and $\mathrm{lk}_\Delta\, v_2$ (b) $\mathrm{lk}_\Delta\, v_3$ (c) $\mathrm{lk}_\Delta\, v_4$

Figure 3.7: Four vertex links as triangulated 2-spheres. For simplicity's sake, we omit some diagonal edges in the quadrilaterals in (b), and some labels of vertices in (c).

Figure 3.8 shows how $\Delta \backslash W$ is formed from these links. First we glue $\mathrm{lk}_\Delta\, v_1$ and $\mathrm{lk}_\Delta\, v_2$ along two green triangles. The resulting complex $\mathrm{lk}_\Delta\, v_1 \cup \mathrm{lk}_\Delta\, v_2$ is shown in Figure 3.8a. Then we place $\mathrm{lk}_\Delta\, v_3$ on top of $\mathrm{lk}_\Delta\, v_1 \cup \mathrm{lk}_\Delta\, v_2$. As we see from Figure 3.8b, the boundary complex of $\cup_{i=1}^3 \mathrm{st}_\Delta\, v_i$ is a triangulated torus. Finally, we place $\mathrm{lk}_\Delta\, v_4$ on top of $\cup_{i=1}^3 \mathrm{lk}_\Delta\, v_i$ so that $\mathrm{st}_\Delta\, v_4$ "covers the 1-dimensional hole" in $\cup_{i=1}^3 \mathrm{st}_\Delta\, v_i$, see Figure 3.8c. We denote the subspace of $\mathbb{R}^3$ enclosed by $\mathrm{lk}_\Delta\, v_i$ as S_i for $1 \leq i \leq 4$, and let $S_5 := \cup_{i \leq 4} S_i$. From our construction it follows that the boundary complex of S_5 is a 2-sphere; we let it be $\mathrm{lk}_\Delta\, v_5$.

Since each $\mathrm{lk}_\Delta\, v_i \cap \mathrm{lk}_\Delta\, v_j$ has at least two connected components for $1 \leq i \neq j \leq 4$, the Mayer-Vietoris sequence implies that $S_i \cup S_j$ is not contractible for all $1 \leq i \neq j \leq 4$.



(a) $\mathrm{lk}_\Delta\, v_1 \cup \mathrm{lk}_\Delta\, v_2$ (b) $\cup_{i=1}^{3} \mathrm{lk}_\Delta\, v_i$ (c) $\cup_{i=1}^{4} \mathrm{lk}_\Delta\, v_4$

Figure 3.8: how the links are glued together.

A similar inspection of $\mathrm{lk}_\Delta\, v_i \cup \mathrm{lk}_\Delta\, v_j \cup \mathrm{lk}_\Delta\, v_k$ also implies that the boundary complexes of $S_i \cup S_j \cup S_k$'s cannot be triangulated 2-spheres for distinct $1 \leq i, j, k \leq 4$.

Now we prove that $\Delta \backslash W$ does not have an ear decomposition by using a similar argument to the one in the proof of Proposition 3.3.3. In the following we denote the union of interior faces of a complex τ by $\mathrm{int}\, \tau$. Suppose $\Delta \backslash W$ has an ear decomposition $(\Gamma_1, \Gamma_2, \cdots, \Gamma_k)$. Since $|W| = 5$ and $\beta_2(\Delta \backslash W) = 4$, k must be 4. Notice first that $\cup_{i \leq 4} \mathrm{lk}_\Delta\, v_i$ divides $\mathbb{R}^3$ into five subspaces, namely, $S_1, \cdots, S_4$ and the complement of S_5, each having $\mathrm{lk}_\Delta\, v_i$ as the boundary complex for $1 \leq i \leq 5$ respectively. Since $\Gamma_1 \cup \Gamma_2 - \mathrm{int}(\Gamma_1 \cap \Gamma_2)$ must be a triangulated 2-sphere, by the Jordan theorem, it separates $\mathbb{R}^3$ into two connected components, hence the bounded component must be either $S_i \cup S_j$ or $S_i \cup S_j \cup S_k$ for some $1 \leq i, j, k \leq 4$. (We may assume that it is not S_i, since otherwise we may consider the 2-sphere $\cup_{i \leq 3} \Gamma_i - \cup_{1 \leq i \neq j \leq 3} \mathrm{int}(\Gamma_i \cap \Gamma_j)$ instead of $\Gamma_1 \cup \Gamma_2 - \mathrm{int}(\Gamma_1 \cap \Gamma_2)$, where the subset enclosed by this sphere in $\mathbb{R}^3$ cannot be S_i anymore.) This contradicts the fact that the boundaries of $S_i \cup S_j$ or $S_i \cup S_j \cup S_k$ are not 2-spheres.

Remark 3.5.1. One can think of all the figures illustrated above as projections of a subcomplex of $\Delta - \mathrm{st}_\Delta\, v_5$ onto $\mathbb{R}^3$. However, we do not know whether the complex provided in this section can be realized as the boundary of a 4-polytope.

Chapter 4

THE FLAG UPPER BOUND THEOREM FOR 3- AND 5-MANIFOLDS

4.1 Introduction

One of the classical problems in geometric combinatorics deals with the following question: for a given class of simplicial complexes, find tight upper bounds on the number of i-dimensional faces as a function of the number of vertices and the dimension. Since Motzkin [43] proposed the upper bound conjecture (UBC, for short) for polytopes in 1957, this problem has been solved for various families of complexes. In particular, McMullen [42] and Stanley [54] proved that neighborly polytopes simultaneously maximize all face numbers in the class of polytopes and simplicial spheres. However, it turns out that, apart from cyclic polytopes, many other classes of neighborly spheres or even neighborly polytopes exist, see [52] and [49] for examples and constructions of neighborly polytopes.

A simplicial complex Δ is *flag* if all of its minimal non-faces have cardinality two, or equivalently, Δ is the clique complex of its graph. Flag complexes form a beautiful and important class of simplicial complexes. For example, barycentric subdivisions of simplicial complexes, order complexes of posets, and Coxeter complexes are flag complexes. Despite a lot of effort that went into studying the face numbers of flag spheres, in particular in relation with the Charney-Davis conjecture [17], and its generalization given by Gal's conjecture [25], a flag upper bound theorem for spheres is still unknown. The upper bounds on face numbers for general simplicial $(d-1)$-spheres are far from being sharp for flag $(d-1)$-spheres, since the graph of any flag $(d-1)$-dimensional complex is K_{d+1}-free. In [25], Gal confirmed that the real rootedness conjecture [19] does hold for flag homology spheres of dimension less than five, and thus the upper bounds on the face numbers of these flag spheres were established.

However, starting from dimension five, there are only conjectural upper bounds. For $m \geq 1$, we let $J_m(n)$ be the $(2m-1)$-sphere on n vertices obtained as the join of m copies of the circle, each one a cycle with either $\lfloor \frac{n}{m} \rfloor$ or $\lceil \frac{n}{m} \rceil$ vertices. The following conjecture is due to Nevo and Petersen [47, Conjecture 6.3].

Conjecture 4.1.1. *If Δ is a flag homology sphere, then $\gamma(\Delta)$ satisfies the Frankl-Füredi-Kalai inequalities. In particular, if Δ is of dimension $2m-1$, then $f_i(\Delta) \leq f_i(J_m(n))$ for all $1 \leq i \leq 2m-1$.*

Here we denote by $f_i(\Delta)$ the number of i-dimensional faces of Δ; the entries of the vector $\gamma(\Delta)$ are certain linear combinations of the f-numbers of Δ. For Frankl-Füredi-Kalai inequalities, see [24].

As for the case of equality, Lutz and Nevo [40, Conjecture 6.3] posited that, as opposed to the case of all simplicial spheres, for a fixed dimension $2m-1$ and the number of vertices n, there is *only* one maximizer of the face numbers.

Conjecture 4.1.2. *Let $m \geq 2$ and let Δ be a flag simplicial $(2m-1)$-sphere on n vertices. Then $f_i(\Delta) = f_i(J_m(n))$ for some $1 \leq i \leq m$ if and only if $\Delta = J_m(n)$.*

Recently, Adamaszek and Hladký [1] proved that Conjecture 4.1.1 and 4.1.2 hold asymptotically for flag homology manifolds. Several celebrated theorems from extremal graph theory served as tools for their work. As a result, the proof simultaneously gives upper bounds on f-numbers, h-numbers, g-numbers and γ-numbers, but it only applies to flag homology manifolds with an extremely large number of vertices.

Our first main result is that Conjecture 4.1.1 and 4.1.2 hold for *all* flag 3-manifolds. In particular, we show that the balanced join of two circles is the unique maximizer of face numbers in the class of flag 3-manifolds. We also establish an analogous result on the number of edges of flag 5-manifolds. The proof only relies on simple properties of flag complexes and Eulerian complexes.

In 1964, Klee [36] proved that Motzkin's UBC for d-polytopes holds for a much larger class of Eulerian complexes as long as they have sufficiently many vertices (in fact, d^2 vertices), and conjectured that the UBC holds for *all* Eulerian complexes. Our second main result deals with flag Eulerian complexes, and asserts that Conjecture 4.1.1 continues to hold for all flag 3-dimensional Eulerian complexes. This provides supporting evidence to a question of Adamaszek and Hladký [1, Problem 17(i)] in the case of dimension 3, where they proposed that Conjecture 4.1.1 holds for all odd-dimensional flag weak pseudomanifolds with sufficiently many vertices. We also give constructions of the maximizers of face numbers in this class and show that they are the only maximizers. Our proof is based on an application of the inclusion-exclusion principle and double counting.

This chapter is organized as follows. In Section 4.2, we discuss basic facts on simplicial complexes and flag complexes. In Section 4.3, we provide the proof of our first main result asserting that given a number of vertices n, the maximum face numbers of a flag 3-manifold are achieved only when this manifold is the join of two circles of length as close as possible to $\frac{n}{2}$. In Section 4.4, we apply an analogous argument to the class of flag 5-manifolds. In Section 4.5, we show that the same upper bounds continue to hold for the class of flag 3-dimensional Eulerian complexes, and discuss the maximizers of the face numbers in this class. Finally, we close in Section 4.6 with some concluding remarks.

4.2 Preliminaries

A *simplicial complex* Δ on a vertex set $V = V(\Delta)$ is a collection of subsets $\sigma \subseteq V$, called faces, that is closed under inclusion. For $\sigma \in \Delta$, let $\dim \sigma := |\sigma| - 1$ and define the *dimension* of Δ, $\dim \Delta$, as the maximal dimension of its faces. A *facet* in Δ is a maximal face under inclusion, and we say that Δ is *pure* if all of its facets have the same dimension. We will denote by $\sqcup$ the disjoint union of simplicial complexes, and by $\uplus$ the disjoint union of sets.

If Δ is a simplicial complex and σ is a face of Δ, the *link* of σ in Δ is $\mathrm{lk}_\Delta\, \sigma := \{\tau - \sigma \in \Delta : \sigma \subseteq \tau \in \Delta\}$, and the *deletion* of a vertex set W from Δ is $\Delta \backslash W := \{\sigma \in \Delta : \sigma \cap W = \emptyset\}$. The *restriction* of Δ to a vertex set W is defined as $\Delta[W] := \{\sigma \in \Delta : \sigma \subseteq W\}$. If Δ and Γ are two simplicial complexes on disjoint vertex sets, then the *join* of Δ and Γ, denoted as $\Delta * \Gamma$, is the simplicial complex on vertex set $V(\Delta) \uplus V(\Gamma)$ whose faces are $\{\sigma \cup \tau : \sigma \in \Delta, \tau \in \Gamma\}$.

A simplicial complex Δ is a *simplicial manifold* (resp. *simplicial sphere*) if the geometric realization of Δ is homeomorphic to a manifold (resp. sphere). We denote by $\tilde{H}_*(\Delta;\mathbf{k})$ the reduced homology of Δ computed with coefficients in a field $\mathbf{k}$, and by $\tilde{\beta}_i(\Delta;\mathbf{k}) := \dim_{\mathbf{k}} \tilde{H}_i(\Delta;\mathbf{k})$ the reduced Betti numbers of Δ with coefficients in $\mathbf{k}$. We say that Δ is a $(d-1)$-dimensional $\mathbf{k}$-*homology manifold* if $\tilde{H}_*(\mathrm{lk}_\Delta\,\sigma;\mathbf{k}) \cong \tilde{H}_*(\mathbb{S}^{d-1-|\sigma|};\mathbf{k})$ for every nonempty face $\sigma \in \Delta$. A $\mathbf{k}$-*homology sphere* is a $\mathbf{k}$-homology manifold that has the $\mathbf{k}$-homology of a sphere. Every simplicial manifold (resp. simplicial sphere) is a homology manifold (resp. homology sphere). Moreover, in dimension two, the class of homology 2-spheres coincides with that of simplicial 2-spheres, and hence in dimension three, the class of homology 3-manifolds coincides with that of simplicial manifolds.

For a $(d-1)$-dimensional complex Δ, we let $\tilde{\chi}(\Delta) := \sum_{i=0}^{d-1}(-1)^i\tilde{\beta}_i(\Delta;\mathbf{k})$ be the *reduced Euler characteristic* of Δ. A simplicial complex Δ is called an *Eulerian* complex if Δ is pure and $\tilde{\chi}(\mathrm{lk}_\Delta\,\sigma) = (-1)^{\dim \mathrm{lk}_\Delta \sigma}$ for every $\sigma \in \Delta$, including $\sigma = \emptyset$. In particular, it follows from the Poincaré duality theorem that all odd-dimensional orientable homology manifolds are Eulerian. As all simplicial manifolds are orientable over $\mathbb{Z}/2\mathbb{Z}$, all odd-dimensional simplicial manifolds are Eulerian.

A $(d-1)$-dimensional simplicial complex Δ is called a *weak* $(d-1)$-*pseudomanifold* if it is pure and every $(d-2)$-face (called *ridge*) of Δ is contained in exactly two facets. A weak $(d-1)$-pseudomanifold Δ is called a *normal* $(d-1)$-*pseudomanifold* if it is connected, and the link of each face of dimension at most $d-3$ is also connected. Every Eulerian complex is a weak pseudomanifold, and every connected homology manifold is a normal pseudomanifold. In fact, every normal 2-pseudomanifold is also a homology 2-manifold. However, for $d > 3$, the class of normal $(d-1)$-pseudomanifolds is much larger than the class of homology $(d-1)$-manifolds. It is well-known that if Δ is a weak (resp. normal) $(d-1)$-pseudomanifold and σ is a face of Δ of dimension at most $d-2$, then the link of σ is also a weak (resp. normal) pseudomanifold. The following lemma gives another property of normal pseudomanifolds, see [7, Lemma 1.1].

Lemma 4.2.1. *Let Δ be a normal $(d-1)$-pseudomanifold, and let W be a subset of vertices of Δ such that the induced subcomplex $\Delta[W]$ is a normal $(d-2)$-pseudomanifold. Then the induced subcomplex of Δ on vertex set $V(\Delta)\backslash W$ has at most two connected components.*

For a $(d-1)$-dimensional complex Δ, we let $f_i = f_i(\Delta)$ be the number of i-dimensional faces of Δ for $-1 \le i \le d-1$. The vector $(f_{-1}, f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ. Since the graph of any simplicial 2-sphere is a maximal planar graph, it follows that the f-vector of a simplicial 2-sphere is uniquely determined by f_0. For a 3-dimensional Eulerian complex, the following lemma indicates that its f-vector is uniquely determined by f_0 and f_1.

Lemma 4.2.2. *The f-vector of a 3-dimensional Eulerian complex satisfies*

$$(f_0, f_1, f_2, f_3) = (f_0, f_1, 2f_1 - 2f_0, f_1 - f_0).$$

Proof: Let Δ be a 3-dimensional Eulerian complex. Since Δ is Eulerian, $\tilde{\chi}(\Delta) + 1 = f_0 - f_1 + f_2 - f_3 = 0$. Also since every ridge of an Eulerian complex is contained in exactly two facets, by double counting, we obtain that $2f_2 = 4f_3$. Hence the result follows. □

A simplicial complex Δ is *flag* if all minimal non-faces of Δ, also called missing faces, have cardinality two; equivalently, Δ is the clique complex of its graph. The following lemma [47, Lemma 5.2] gives a basic property of flag complexes.

Lemma 4.2.3. *Let Δ be a flag complex on vertex set V. If $W \subseteq V$, then $\Delta[W]$ is also flag. Furthermore, if σ is a face in Δ, then $\mathrm{lk}_\Delta\, \sigma = \Delta[V(\mathrm{lk}_\Delta\, \sigma)]$. In particular, all links in a flag complex are also flag.*

Finally, we recall some terminology from graph theory. A graph G is a path graph if the set of its vertices can be ordered as $x_1, x_2, \ldots, x_n$ in such a way that $\{x_i, x_{i+1}\}$ is an edge for all $1 \le i \le n-1$ and there are no other edges. Similarly, a cycle graph is a graph obtained from a path graph by adding an edge between the endpoints of the path.

4.3 The Proof of the Lutz-Nevo Conjecture for flag 3-manifolds

The goal of this section is to prove Conjecture 4.1.2 for 3-dimensional manifolds. We start by setting up some notation and establishing several lemmas that will be used in the proof. Recall that in the Introduction, we defined $J_m(n)$ to be the $(2m-1)$-sphere on n vertices obtained as the join of m circles, each one of length either $\lfloor \frac{n}{m} \rfloor$ or $\lceil \frac{n}{m} \rceil$. In the following, we define $J^*_m(n)$ as the suspension of $J_m(n-2)$, and denote by $\mathcal{C}^*_d$ the $(d-1)$-dimensional octahedral sphere, i.e., the boundary complex of the d-dimensional cross-polytope. Equivalently, C^*_d is a d-fold join of disjoint copies of the 0-dimensional sphere $\mathbb{S}^0$, and thus $\mathcal{C}^*_{2m} = J_m(4m)$ and $\mathcal{C}^*_{2m+1} = J^*_m(4m+2)$. The following lemma [47, Lemma 5.3], originally stated for the class of flag homology spheres, gives a sufficient condition for a flag normal pseudomanifold to be an octahedral sphere. (As the proof is identical to that of [47, Lemma 5.3], we omit it.)

Lemma 4.3.1. *Let Δ be a $(d-1)$-dimensional flag normal pseudomanifold on vertex set V such that for any $v \in V$, $\mathrm{lk}_\Delta\, v$ is an octahedral sphere. Then Δ is an octahedral sphere.*

Next we characterize all flag normal $(d-1)$-pseudomanifolds on no more than $2d+1$ vertices.

Lemma 4.3.2. *Let Δ ba a flag normal $(d-1)$-pseudomanifold, where $d \geq 2$. Then*

(a) $f_0(\Delta) \geq 2d$. Equality holds if and only if $\Delta = \mathcal{C}^*_d$.

(b) $f_0(\Delta) = 2d+1$ if and only if $\Delta = J_1(5) * \mathcal{C}^*_{d-2}$, or equivalently, $\Delta = J_m(4m+1)$ if $d = 2m$, and $\Delta = J^*_m(4m+3)$ if $d = 2m+1$.

Proof: Part (a) is well-known, see the proof of Proposition 2.2 in [4]. For part (b), we use induction on the dimension. If $d = 2$ and $f_0(\Delta) = 5$, then Δ is a circle of length 5, that is, $J_1(5)$. Now assume that the claim holds for $d < k$, and consider a flag normal $(k-1)$-pseudomanifold Δ on $2k+1$ vertices. Since $\mathrm{lk}_\Delta\, v$ is a flag normal pseudomanifold for any vertex $v \in \Delta$, by part (a), $f_0(\mathrm{lk}_\Delta v) \geq 2k-2$. Moreover, if equality holds, then $\mathrm{lk}_\Delta v$ is an octahedral sphere. However, if $f_0(\mathrm{lk}_\Delta v) = 2k-2$ for every vertex $v \in \Delta$, then by Lemma 4.3.1, Δ must be the octahedral sphere of dimension $k-1$, contradicting that $f_0(\Delta) = 2k+1$. Hence there is at least one vertex $u \in \Delta$ such that $f_0(\mathrm{lk}_\Delta u) \geq 2k-1$. Since Δ is a normal pseudomanifold, for any facet $\sigma \in \mathrm{lk}_\Delta u$, $\mathrm{lk}_\Delta \sigma$ consists of two vertices. On the other hand, since Δ is flag, $\mathrm{lk}_\Delta u$ is also flag by Lemma 4.2.3, and hence $\mathrm{lk}_\Delta \sigma$ does not contain any vertex in $V(\mathrm{lk}_\Delta u)$. It follows that $f_0(\mathrm{lk}_\Delta u) \leq (2k+1)-2 = 2k-1$. Therefore, $f_0(\mathrm{lk}_\Delta u) = 2k-1$, and the links of all facets of $\mathrm{lk}_\Delta u$ are the same two vertices. This implies that Δ is the suspension of $\mathrm{lk}_\Delta u$. By induction, $\mathrm{lk}_\Delta u = J_1(5) * \mathcal{C}^*_{d-3}$, and hence $\Delta = J_1(5) * \mathcal{C}^*_{d-2}$. □

Now we estimate the number of edges in a flag normal 3-pseudomanifold on n vertices.

Lemma 4.3.3. *Let Δ be a flag normal 3-pseudomanifold on n vertices. Then $f_1(\Delta) \leq f_1(J_2(n)) + 3 - 3\min_{v\in\Delta} \tilde{\chi}(\mathrm{lk}_\Delta v)$.*

Proof: Let v be a vertex of maximum degree in $V(\Delta)$. We let $a = f_0(\mathrm{lk}_\Delta v)$, $W_1 = V(\mathrm{lk}_\Delta v)$ and $W_2 = V(\Delta)\backslash V(\mathrm{lk}_\Delta v)$. Since Δ is a normal 3-pseudomanifold, $\mathrm{lk}_\Delta v$ is a normal 2-pseudomanifold, i.e., a simplicial 2-manifold. Furthermore, since Δ is flag, by Lemma 4.2.3, $\mathrm{lk}_\Delta v$ is the restriction of Δ to W_1. Thus, by Lemma 4.2.1, the induced subcomplex $\Delta[W_2]$ has at most two connected components. Since v is not connected to any vertices in $W_2\backslash\{v\}$, it follows that $\{v\}$ and $\Delta[W_2\backslash\{v\}]$ are the two connected components in $\Delta[W_2]$.

We now count the edges of Δ. They consist of the edges of $\Delta[W_1] = \mathrm{lk}_\Delta v$, the edges of $\Delta[W_2]$ and the edges between these two sets. In addition, $\sum_{w\in W_2} f_0(\mathrm{lk}_\Delta w)$ counts the edges of $\Delta[W_2]$ twice. Thus,

$$
\begin{aligned}
f_1(\Delta) &= f_1(\Delta[W_1]) + \Big(\sum_{w\in W_2} f_0(\mathrm{lk}_\Delta w)\Big) - f_1(\Delta[W_2]) \\
&\overset{(*)}{\leq} f_1(\mathrm{lk}_\Delta v) + |W_2| \cdot \max_{w\in W_2} f_0(\mathrm{lk}_\Delta w) - (f_0(\Delta[W_2\backslash\{v\}]) - 1) \\
&\overset{(**)}{=} \big(3a - 6 + 3(1 - \tilde{\chi}(\mathrm{lk}_\Delta v))\big) + (n-a)a - (n-a-2) \\
&= -a^2 + a(n+4) - (n+4) + 3(1 - \tilde{\chi}(\mathrm{lk}_\Delta v)) \\
&\overset{(***)}{\leq} \left\lfloor \frac{n^2}{4} \right\rfloor + n + 3(1 - \tilde{\chi}(\mathrm{lk}_\Delta v)) \\
&= f_1(J_2(n)) + 3(1 - \tilde{\chi}(\mathrm{lk}_\Delta v)).
\end{aligned}
\tag{4.3.1}
$$

Here in (*) we used that $\Delta[W_2\backslash\{v\}]$ is connected and hence has at least $f_0(\Delta[W_2\backslash\{v\}]) - 1$ edges. Equality (**) follows from the fact that $\mathrm{lk}_\Delta v$ is a 2-manifold with a vertices, and (***) is obtained by optimizing the function $p(a) = -a^2 + a(n+4)$. Hence the result follows. □

Theorem 4.3.4. *Let Δ be a flag 3-manifold on n vertices. Then $f_i(\Delta) \leq f_i(J_2(n))$. If equality holds for some $1 \leq i \leq 3$, then $\Delta = J_2(n)$.*

Proof: We use the same notation as in the proof of Lemma 4.3.3. That is, we let v be a vertex of maximum degree in $V(\Delta)$. We let $a = f_0(\mathrm{lk}_\Delta v)$, $W_1 = V(\mathrm{lk}_\Delta v)$ and $W_2 = V(\Delta)\backslash V(\mathrm{lk}_\Delta v)$. Since Δ is a flag 3-manifold, $\tilde{\chi}(\mathrm{lk}_\Delta w) = 1$ for every $w \in \Delta$. Hence by Lemma 4.3.3, $f_1(\Delta) \leq f_1(J_2(n))$. Furthermore, it follows from steps (*) and (***) in equality (4.3.1) that $f_1(\Delta) = f_1(J_2(n))$ holds only if $f_0(\mathrm{lk}_\Delta w) = a = \lceil \frac{n+4}{2} \rceil$ or $\lfloor \frac{n+4}{2} \rfloor$ for all $w \in W_2$, and $\Delta[W_2\backslash\{v\}]$ is a tree.

We claim that if $f_1(\Delta) = f_1(J_2(n))$, then $\Delta = J_2(n)$. This indeed holds if $n = 8$ or 9, since by Lemma 4.3.2, the only flag 3-manifolds on 8 or 9 vertices are $J_2(8)$ and $J_2(9)$. Next we assume that $n \geq 10$, where $|W_2| = n - a \geq \lceil \frac{n}{2} \rceil - 2 > 2$. Hence the tree $\Delta[W_2\backslash\{v\}]$ has at least one edge, and thus there is a vertex $u_1 \in W_2$ such that $\deg_{\Delta[W_2]} u_1 = 1$. Let u_2 be the unique vertex in W_2 that is connected to u_1. Since $f_0(\mathrm{lk}_\Delta u_1) = a$, the vertex u_1 must be connected to all vertices in W_1 except for one vertex. We let z_1 be this vertex and denote the circle $\mathrm{lk}_{\mathrm{lk}_\Delta v}\, z_1$ by C_1. Since Δ is flag, $\mathrm{lk}_\Delta u_1 \supseteq \Delta[W_1\backslash\{z_1\}] = \mathrm{lk}_\Delta v - \{z_1\} * C_1$, and hence

$$\mathrm{lk}_\Delta u_1 = (\mathrm{lk}_\Delta v - \{z_1\} * C_1) \cup (\{u_2\} * C_1).$$

If $\{z_1\} \in \mathrm{lk}_\Delta u_2$, then $\mathrm{lk}_\Delta u_2 \supseteq C_1 * \{u_1, z_1\}$. Since $C_1 * \{u_1, z_1\}$ is a 2-sphere, it follows that $\mathrm{lk}_\Delta u_2 = C_1 * \{u_1, z_1\}$ and $f_0(C_1) = a-2$. Hence $W_2 = \{u_1, u_2\}$ and $W_1 = V(C_1) \cup \{z_1\} \cup \{z_2\}$ for some vertex $z_2 \in W_1$, so that $\mathrm{lk}_\Delta v = \{z_1, z_2\} * C_1$. Now assume that $\{z_1\} \notin \mathrm{lk}_\Delta u_2$ and u_2 is connected to vertices $u_3, u_4, \ldots, u_k$ in $\Delta[W_2]$. Since C_1 is a circle in the 2-sphere $\mathrm{lk}_\Delta u_2$, the subcomplex $\mathrm{lk}_\Delta u_2 \backslash V(C_1)$ has two contractible connected components. If there is a vertex u_i such that $\mathrm{lk}_{\mathrm{lk}_\Delta u_2} u_i = C_1$, then $\mathrm{lk}_\Delta u_2 \supseteq C_1 * \{u_1, u_i\}$ and hence this link is exactly $C_1 * \{u_1, u_i\}$. This implies that $\deg_{\Delta[W_2]} u_2 = 2$. Otherwise, if $\mathrm{lk}_{\mathrm{lk}_\Delta u_2} u_i \neq C_1$ for all $3 \leq i \leq k$, then each u_i is connected to at least one vertex in $\mathrm{lk}_\Delta v \backslash (V(C_1) \cup \{z_1\})$. Since $\mathrm{lk}_\Delta u_1 \supseteq \mathrm{lk}_\Delta v \backslash \{z_1\}$, it follows that the vertices u_1 and $u_3, \ldots, u_k$ are in the same connected component, and hence $\mathrm{lk}_\Delta u_2 \backslash V(C_1)$ is connected, a contradiction.

By applying the above argument inductively, we obtain that $\Delta[W_2\backslash\{v\}]$ is a path graph $(u_1, u_2, \ldots, u_{n-a-1})$, and there is a vertex z_2 in W_1 such that $\mathrm{lk}_\Delta u_1 = \{z_2, u_2\} * C_1$ and $\mathrm{lk}_\Delta v = C_1 * \{z_1, z_2\}$. Furthermore, $C_1 \subseteq \mathrm{lk}_\Delta u_i$ for all $u_i \in W_2$. Then we let C_2 be the cycle graph $(v, z_2, u_1, u_2, \ldots, u_{n-a-1}, z_1)$. It follows that $\Delta = C_1 * C_2$. Since $a = |C_1| + 2 = \lfloor \frac{n+4}{2} \rfloor$ or $\lceil \frac{n+4}{4} \rceil$, C_1 and C_2 must be cycles of length $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. This implies $\Delta = J_2(n)$.

By Lemma 4.2.2, the f-vector of Δ is uniquely determined by $f_0(\Delta) = n$ and $f_1(\Delta)$. This yields the result. □

4.4 Counting edges of flag 5-manifolds

For even-dimensional flag simplicial spheres, we have the following weaker form of the flag upper bound conjecture, see Conjecture 18 in [1]:

Conjecture 4.4.1. *Fix $m \geq 1$. For every flag $2m$-sphere Δ on n vertices, we have $f_1(M) \leq f_1(J^*_m(n))$.*

Using almost the same argument as in the proof of Theorem 4.3.4, we establish the following proposition.

Proposition 4.4.2. *Let Δ be a flag $(2m+1)$-manifold on n vertices. If Conjecture 4.4.1 holds for all flag $2i$-spheres with $1 \leq i \leq m$, then $f_1(\Delta) \leq f_1(J_{m+1}(n))$. Equality holds only when $\Delta = J_{m+1}(n)$.*

Proof: The proof is by induction on m. The case $m = 1$ is confirmed by Theorem 4.3.4. Now assume that $m \geq 2$ and Δ is a flag $(2m+1)$-manifold on n vertices. If $n \equiv q \mod m$ $(0 \leq q < m)$, then a simple computation shows that

$$f_1(J_m(n)) = \frac{m-1}{2m}n^2 + n + \frac{q(q-m)}{2m}, \; f_1(J^*_m(n)) = \frac{m-1}{2m}(n-2)^2 + 3(n-2) + \frac{q(q-m)}{2m}. \tag{4.4.1}$$

As in the proof of Lemma 4.3.3, we let v be a vertex of maximum degree in $V(\Delta)$, $a = f_0(\mathrm{lk}_\Delta v)$, $W_1 = V(\mathrm{lk}_\Delta v)$ and $W_2 = V(\Delta)\backslash V(\mathrm{lk}_\Delta v)$. Following the same argument as in Lemma 4.3.3, we obtain the following analog of (4.3.1):

$$\begin{aligned} f_1(\Delta) &\leq f_1(J^*_m(a)) + |W_2| \cdot \max_{w \in W_2} f_0(\mathrm{lk}_\Delta w) - (f_0(\Delta[W_2\backslash\{v\}]) - 1) \\ &\overset{(\diamond)}{\leq} \frac{m-1}{2m}(a-2)^2 + 3(a-2) + \frac{q(q-m)}{2m} + (n-a)a - (n-a-2) \\ &= -\frac{m+1}{2m}\left(a - (\frac{mn}{m+1} + 2)\right)^2 + \frac{m}{2m+2}n^2 + n + \frac{q(q-m)}{2m}, \end{aligned}$$

where in $(\diamond)$ we used our assumption that Conjecture 4.4.1 holds for flag $2m$-spheres. By (4.4.1),

$$f_1(J_{m+1}(n)) = \frac{m}{2m+2}n^2 + n + \frac{q(q-m-1)}{2m+2}.$$

Hence we conclude that $f_1(\Delta) \leq f_1(J_{m+1}(n))$. Moreover, equality holds when $a = \lceil \frac{mn}{m+1} \rceil + 2$ or $a = \lfloor \frac{mn}{m+1} \rfloor + 2$, and $\Delta[W_2\backslash\{v\}]$ is a tree. If $n = 4m+4$ or $4m+5$, then by Lemma 4.3.2, Δ is either $J_{m+1}(4m+4)$ or $J_{m+1}(4m+5)$. If $n > 4m+5$, then the tree $\Delta[W_2\backslash\{v\}]$ has at least one edge. We proceed with the same argument as in Theorem 4.3.4 to show that $\mathrm{lk}_\Delta v$ must be the suspension of a $(2m-1)$-sphere on either $\lceil \frac{mn}{m+1} \rceil$ or $\lfloor \frac{mn}{m+1} \rfloor$ vertices. By induction, this sphere is the join of m circles, each having length either $\lceil \frac{n}{m+1} \rceil$ or $\lfloor \frac{n}{m+1} \rfloor$. Hence $\Delta = J_{m+1}(n)$. □

Theorem 4.4.3. *Let Δ be a flag 5-manifold on n vertices. Then $f_1(\Delta) \leq f_1(J_3(n))$. Equality holds if and only if $\Delta = J_3(n)$.*

Proof: The result follows from Proposition 4.4.2 and the fact that Conjecture 4.4.1 is known to hold in the case of dimension four (see [25], Theorem 3.1.3). □

4.5 The face numbers of flag 3-dimensional Eulerian complexes

In Lemma 4.3.3, we established an upper bound on the number of edges for all flag normal 3-pseudomanifolds. In this section, we find sharp upper bounds on the face numbers for all flag 3-dimensional Eulerian complexes. The proof relies on the following three lemmas.

Lemma 4.5.1. *Let Δ be a flag $(d-1)$-dimensional simplicial complex.*

(a) If σ_1 and σ_2 are two ridges that lie in the same facet σ in Δ, then the links of σ_1 and σ_2 are disjoint.

(b) If $\sigma = \tau_1 \uplus \tau_2$ is a face in Δ, then $V(\mathrm{lk}_\Delta \tau_1) \cap V(\mathrm{lk}_\Delta \tau_2) = V(\mathrm{lk}_\Delta \sigma)$. In particular, if σ is a facet, then $f_0(\mathrm{lk}_\Delta \tau_1) + f_0(\mathrm{lk}_\Delta \tau_2) \leq f_0(\Delta)$.

Proof: For part (a), if v is a common vertex of $\mathrm{lk}_\Delta \sigma_1$ and $\mathrm{lk}_\Delta \sigma_2$, then v must be adjacent to each vertex of $\sigma_1 \cup \sigma_2 = \sigma$. Thus, since Δ is flag, $\{v\} \cup \sigma \in \Delta$, which contradicts our assumption that σ is a facet.

For part (b), the inclusion $V(\mathrm{lk}_\Delta \tau_1) \cap V(\mathrm{lk}_\Delta \tau_2) \supseteq V(\mathrm{lk}_\Delta \sigma)$ holds for any simplicial complex. If $v \in V(\mathrm{lk}_\Delta \tau_1) \cap V(\mathrm{lk}_\Delta \tau_2)$, then $v \cup \tau_1, v \cup \tau_2 \in \Delta$. Since Δ is flag, it follows that $v \cup \sigma \in \Delta$. If σ is not a facet, then $v \in \mathrm{lk}_\Delta \sigma$. This implies $V(\mathrm{lk}_\Delta \tau_1) \cap V(\mathrm{lk}_\Delta \tau_2) \subseteq V(\mathrm{lk}_\Delta \sigma)$. However, if σ is a facet, then $v \cup \sigma$ cannot be a facet in Δ. In this case, $V(\mathrm{lk}_\Delta \tau_1) \cap V(\mathrm{lk}_\Delta \tau_2) = V(\mathrm{lk}_\Delta \sigma) = \emptyset$, and so $f_0(\mathrm{lk}_\Delta \tau_1) + f_0(\mathrm{lk}_\Delta \tau_2) \leq f_0(\Delta)$. □

Lemma 4.5.1 part (b) implies that if Δ is a flag 3-dimensional simplicial complex and $\sigma \in \Delta$ is a facet, then $\sum_{e\subseteq\sigma} f_0(\mathrm{lk}_\Delta e) \leq 3f_0(\Delta)$, where the sum is over the edges of σ. The following lemma suggests a better estimate on $\sum_{e\subseteq\sigma} f_0(\mathrm{lk}_\Delta e)$ if Δ is a flag weak 3-pseudomanifold.

Lemma 4.5.2. *Let Δ be a flag weak 3-pseudomanifold on n vertices. Then for any facet $\sigma = \{v_1, v_2, v_3, v_4\}$ of Δ, $\sum_{e\subseteq\sigma} f_0(\mathrm{lk}_\Delta e) \leq n + 16$, where the sum is over the edges of σ. If equality holds, then $\cup_{w\in\tau} V(\mathrm{lk}_\Delta w) = V(\Delta)$ for any ridge $\tau \subseteq \sigma$.*

Proof: Let $V_i = V(\mathrm{lk}_\Delta v_i)$ for $1 \leq i \leq 4$. By Lemma 4.5.1 part (b), for any distinct $1 \leq i, j \leq 4$, we have $V_i \cap V_j = V(\mathrm{lk}_\Delta\{v_i, v_j\})$ and $V_1 \cap V_2 \cap V_3 \cap V_4 = V(\mathrm{lk}_\Delta \sigma) = \emptyset$. Also since Δ is a weak 3-pseudomanifold, any ridge of Δ is contained in exactly two facets. Hence $V_i \cap V_j \cap V_k = V(\mathrm{lk}_\Delta\{v_i, v_j, v_k\})$ is a set of cardinality two. By the inclusiong-exclusion principle, we obtain that

$$\begin{aligned}
\sum_{1\leq i<j\leq 4} |V_i \cap V_j| &= -|V_1 \cup V_2 \cup V_3 \cup V_4| + \sum_{1\leq i\leq 4} |V_i| + \sum_{1\leq i<j<k\leq 4} |V_i \cap V_j \cap V_k| - |V_1 \cap V_2 \cap V_3 \cap V_4| \\
&= \sum_{1\leq i\leq 4} |V_i| - |V_1 \cup V_2 \cup V_3 \cup V_4| + \binom{4}{3} \cdot 2 \\
&= (|V_1| + |V_2| - |V_1 \cup V_2|) + (|V_3| + |V_4| - |V_3 \cup V_4|) + |(V_1 \cup V_2) \cap (V_3 \cup V_4)| + 8 \\
&= |V_1 \cap V_2| + |V_3 \cap V_4| + |(V_1 \cup V_2) \cap (V_3 \cup V_4)| + 8.
\end{aligned} \tag{4.5.1}$$

For simplicity, we denote the set $(V_1 \cup V_2) \cap (V_3 \cup V_4)$ as $\bar{V}$. Notice that by Lemma 4.5.1 part (b), any vertex $v \in \Delta$ belongs to at most one of the sets $V_1 \cap V_2$ and $V_3 \cap V_4$. We split the vertices of Δ into the following three types.

1. If $v \in V_1 \cap V_2$ and $v \notin V_3 \cup V_4$, or if $v \in V_3 \cap V_4$ and $v \notin V_1 \cup V_2$, then $v \notin \bar{V}$. Each of these vertices contributes 1 to the right-hand side of (4.5.1).

2. If $v \in V_i \cap V_j \cap V_k$ for some triple $\{i, j, k\} \subseteq [4]$, then v belongs to either $V_1 \cap V_2$ or $V_3 \cap V_4$, and $v \in \bar{V}$. By Lemma 4.5.1 part (a), every pair of ridges in σ has disjoint

links. Since $|V_i \cap V_j \cap V_k| = 2$, the number of such vertices is exactly 8, and each of them contributes 2 to the right-hand side of (4.5.1).

3. If $v \notin V_1 \cap V_2$ and $v \notin V_3 \cap V_4$, then v contributes to the right-hand side of (4.5.1) at most 1. This case occurs only when $v \in \bar{V}$, that is, when v belongs to one of V_1 and V_2, and one of V_3 and V_4.

Hence $\sum_{\{i,j\}\subseteq[4]} |V_i \cap V_j| \leq n+8+8 = n+16$. Furthermore, if equality holds, then for every vertex v in Δ, either $v \in V_1 \cap V_2$, or $v \in V_3 \cap V_4$, or $v \in \bar{V}$. This implies that every vertex in Δ belongs to at least two of the four links $\mathrm{lk}_\Delta v_1, \ldots, \mathrm{lk}_\Delta v_4$. This proves the second claim. $\square$

Lemma 4.5.3. *Let Δ be a flag weak 3-pseudomanifold on n vertices, and let $\sigma = \{v_1, v_2, v_3, v_4\}$ be an arbitrary facet of Δ. Then $\sum_{1\leq i\leq 4} f_0(\mathrm{lk}_\Delta v_i) \leq 2n+8$. If equality holds, then $\cup_{w\in\tau} V(\mathrm{lk}_\Delta w) = V(\Delta)$ for any ridge $\tau \subseteq \sigma$.*

Proof: As in Lemma 4.5.2, we let $V_i = V(\mathrm{lk}_\Delta v_i)$. By the inclusion-exclusion principle,

$$\begin{aligned}\sum_{1\leq i\leq 4} |V_i| &= \sum_{1\leq i<j\leq 4} |V_i \cap V_j| - \sum_{1\leq i<j<k\leq 4} |V_i \cap V_j \cap V_k| + |V_1 \cap V_2 \cap V_3 \cap V_4| + |V_1 \cup V_2 \cup V_3 \cup V_4| \\ &= \sum_{1\leq i<j\leq 4} |V_i \cap V_j| + |V_1 \cup V_2 \cup V_3 \cup V_4| - 8 \\ &\leq n+16+n-8 = 2n+8.\end{aligned}$$

The last inequality follows from Lemma 4.5.2 and the fact that $|V_1 \cup V_2 \cup V_3 \cup V_4| \leq |V(\Delta)| = n$. The second claim also follows from Lemma 4.5.2. $\square$

Now we are ready to prove the main result in this section.

Theorem 4.5.4. *Let Δ be a flag 3-dimensional Eulerian complex on n vertices. Then $f_1(\Delta) \leq f_1(J_2(n))$.*

Proof: We denote the vertices of Δ by $v_1, v_2, \ldots, v_n$, and we let $a_i = f_0(\mathrm{lk}_\Delta\, v_i)$. Since $\mathrm{lk}_\Delta\, v_i$ is a 2-dimensional Eulerian complex, the f-numbers of $\mathrm{lk}_\Delta\, v_i$ satisfy the relations

$$f_2 - f_1 + f_0 = 2,\;\; 3f_2 = 2f_1.$$

Hence $f_2(\mathrm{lk}_\Delta\, v_i) = 2a_i - 4$. By double counting, we obtain that

$$\sum_{\sigma\in\Delta,|\sigma|=4}\sum_{v\in\sigma} f_0(\mathrm{lk}_\Delta\, v) = \sum_{i=1}^{n} f_0(\mathrm{lk}_\Delta\, v_i)\cdot\#\{\sigma : v_i \in \sigma, |\sigma| = 4\} = \sum_{i=1}^{n} a_i(2a_i-4) = 2\sum_{i=1}^{n} a_i^2 - 4\sum_{i=1}^{n} a_i. \tag{4.5.2}$$

By Lemma 4.5.3, the left-hand side of (4.5.2) is bounded above by $f_3(\Delta)(2n+8)$, which also equals $(f_1(\Delta) - n)(2n + 8)$ by Lemma 4.2.2. However, since $2f_1(\Delta) = \sum_{i=1}^{n} f_0(\mathrm{lk}_\Delta\, v_1) = \sum_{i=1}^{n} a_i$, by the Cauchy-Schwartz inequality the right-hand side of (4.5.2) is bounded below by $\frac{8f_1(\Delta)^2}{n} - 8f_1(\Delta)$, and equality holds only if $a_i = \frac{2f_1(\Delta)}{n}$ for all $1 \le i \le n$. Hence,

$$(f_1(\Delta) - n)(2n + 8) \ge \frac{8f_1(\Delta)^2}{n} - 8f_1(\Delta).$$

We simplify this inequality to get

$$(f_1(\Delta) - n)\Big(\frac{8}{n}f_1(\Delta) - (2n + 8)\Big) \le 0.$$

Since $f_1(\Delta) > n$, it follows that $f_1(\Delta) \le \left\lfloor \frac{n^2}{4} \right\rfloor + n$, that is, $f_1(\Delta) \le f_1(J_2(n))$. □

The following corollary provides some properties of the maximizers of the face numbers in the class of flag 3-dimensional Eulerian complexes.

Corollary 4.5.5. *Let Δ be a flag 3-dimensional Eulerian complex on n vertices. Then $f_1(\Delta) = f_1(J_2(n))$ if and only if (i) $\left\lceil \frac{n}{2} \right\rceil$ vertices of Δ satisfy $f_0(\mathrm{lk}_\Delta\, v) = \left\lfloor \frac{n}{2} \right\rfloor + 2$ while $\left\lfloor \frac{n}{2} \right\rfloor$ vertices satisfy $f_0(\mathrm{lk}_\Delta\, v) = \left\lceil \frac{n}{2} \right\rceil + 2$, and (ii) Δ and all of its vertex links are connected.*

Proof: Part (i) of the claim follows from the proof of Theorem 4.5.4. (Notice that the a_i's in the proof of Theorem 4.5.4 must be integers, so they are either $\left\lfloor \frac{2f_1(\Delta)}{n} \right\rfloor$ or $\left\lceil \frac{2f_1(\Delta)}{n} \right\rceil$.) Also by Theorem 4.5.4, if $f_1(\Delta) = f_1(J_2(n))$, then $\sum_{v\in\sigma} f_0(\mathrm{lk}_\Delta\, v) = 2n + 8$ for every facet $\sigma = \{v_1, v_2, v_3, v_4\} \in \Delta$. By Lemma 4.5.3, every vertex of Δ belongs to $\cup_{1\le i\le 3} V(\mathrm{lk}_\Delta\, v_i)$, and hence Δ is connected. If there is a vertex v such that $\mathrm{lk}_\Delta v$ is not connected, then we let $\tau_1 = \{u_1, u_2, u_3\}$ and $\tau_2 = \{w_1, w_2, w_3\}$ be two 2-faces in distinct connected components of $\mathrm{lk}_\Delta v$. Since Δ is flag, by Lemma 4.2.3, $\mathrm{lk}_\Delta v = \Delta[V(\mathrm{lk}_\Delta v)]$. Therefore no edges exist between τ_1 and τ_2. However, Theorem 4.5.4 and Lemma 4.5.3 also imply that $\cup_{1\leq i\leq 3} V(\mathrm{lk}_\Delta u_i) = V(\Delta)$, contradicting the fact that $\{w_1, w_2, w_3\} \not\subseteq V(\mathrm{lk}_\Delta u_i)$ for all $1 \leq i \leq 3$. Hence every vertex link in Δ is connected. $\square$

The next lemma, which might be of interest in its own right, provides a sufficient condition for a flag complex to be the join of two of its links.

Lemma 4.5.6. *Let Δ be a flag $(d-1)$-dimensional simplicial complex. If $\sigma = \tau_1 \uplus \tau_2$ is a facet of Δ, where τ_1 is an i-face of Δ and τ_2 is a $(d-i-2)$-face of Δ, then $V(\mathrm{lk}_\Delta \tau_1) \cup V(\mathrm{lk}_\Delta \tau_2) = V(\Delta)$ implies that $\Delta \subseteq \mathrm{lk}_\Delta \tau_1 * \mathrm{lk}_\Delta \tau_2$. Moreover, if Δ is a flag normal $(d-1)$-pseudomanifold, then $V(\mathrm{lk}_\Delta \tau_1) \cup V(\mathrm{lk}_\Delta \tau_2) = V(\Delta)$ if and only if $\Delta = \mathrm{lk}_\Delta \tau_1 * \mathrm{lk}_\Delta \tau_2$.*

Proof: Since Δ is flag, $\Delta[V(\mathrm{lk}_\Delta \tau_j)] = \mathrm{lk}_\Delta \tau_j$ for $j = 1, 2$. Hence for every $(d-i-2)$-face τ_2' in $\mathrm{lk}_\Delta \tau_1$, the link $\mathrm{lk}_\Delta \tau_2'$ does not contain any vertex in $\mathrm{lk}_\Delta \tau_1$. This implies $\mathrm{lk}_\Delta \tau_2' \subseteq \mathrm{lk}_\Delta \tau_2$. Similarly, for every $(d-i-2)$-face $\tau_1' \in \mathrm{lk}_\Delta \tau_2$, we have $\mathrm{lk}_\Delta \tau_1' \subseteq \mathrm{lk}_\Delta \tau_1$. Thus, $\Delta \subseteq \mathrm{lk}_\Delta \tau_1 * \mathrm{lk}_\Delta \tau_2$.

If Δ is a normal pseudomanifold, then both $\mathrm{lk}_\Delta \tau_2$ and $\mathrm{lk}_\Delta \tau_2'$ are normal pseudomanifolds. Since no proper subcomplex of a normal pseudomanifold can be a normal pseudomanifold of the same dimension, it follows that $\mathrm{lk}_\Delta \tau_2' = \mathrm{lk}_\Delta \tau_2$. Similarly, for every i-face $\tau_1' \in \mathrm{lk}_\Delta \tau_2$, $\mathrm{lk}_\Delta \tau_1' = \mathrm{lk}_\Delta \tau_1$. Hence $\Delta = \mathrm{lk}_\Delta \tau_1 * \mathrm{lk}_\Delta \tau_2$. $\square$

Remark 4.5.7. The second result in Lemma 4.5.6 does not hold for flag weak pseudomanifolds, even assuming connectedness. Indeed, let $L_1, \ldots, L_4$ be four distinct circles of length ≥ 4. Then $\Delta = (L_1 * L_3) \cup (L_2 * L_3) \cup (L_1 * L_4)$ is a flag weak 3-pseudomanifold. If τ_1 and τ_2 are edges in L_1 and L_3 respectively, then $\mathrm{lk}_\Delta \tau_1 = L_3 \sqcup L_4$ and $\mathrm{lk}_\Delta \tau_2 = L_1 \sqcup L_2$. Hence $V(\mathrm{lk}_\Delta \tau_1) \cup V(\mathrm{lk}_\Delta \tau_2) = V(\Delta)$. However, Δ is a proper subcomplex of $\mathrm{lk}_\Delta \tau_1 * \mathrm{lk}_\Delta \tau_2$.

In Theorem 4.3.4 we proved that the maximizer of the face numbers is unique in the class of flag 3-manifolds on n vertices. Is this also true for flag 3-dimensional Eulerian complexes? Corollary 4.5.5 implies that if the case of equality is not a join of two circles, then some of its edge links are not connected. Motivated by the example in Remark 4.5.7, we construct a family of flag 3-dimensional Eulerian complexes on n vertices that have the same f-numbers as those of $J_2(n)$.

Example 4.5.8. We write C_i to denote a circle of length i. For a fixed number $n \geq 8$, let $a_1, a_2, \ldots, a_s,\ b_1, b_2, \ldots, b_t \geq 4$ be integers such that

$$\sum_{1\leq i\leq s} a_i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } \sum_{1\leq j\leq t} b_j = \left\lceil \frac{n}{2} \right\rceil.$$

We claim that $\Delta = \bigcup_{1\leq i\leq s, 1\leq j\leq t}(C_{a_i} * C_{b_j})$ is flag and Eulerian, where C's have disjoint vertex sets. Since the circles C_{a_i} and C_{b_j} are of length ≥ 4, they are flag and hence Δ is also flag. Also any ridge τ in Δ can be expressed as $\tau = v \cup e$, where v is a vertex of C_{a_i} and e is an edge of C_{b_j} (or $v \in C_{b_j}$ and $e \in C_{a_i}$) for some i, j. By the construction of Δ, the ridge τ is contained in exactly two facets $\{v, v'\} \cup e$ and $\{v, v''\} \cup e$ of Δ, where v' and v'' are neighbors of v in the circle C_{a_i} (or C_{b_j}). Hence the links of ridges in Δ are Eulerian. Since every edge link in Δ is either a circle or a disjoint unions of circles, and every vertex link in Δ is the suspension of a disjoint union of circles, these links are also Eulerian. Finally, the vertices in C_{a_i} have degree $\lceil \frac{n}{2} \rceil + 2$ and the vertices in C_{b_j} have degree $\lfloor \frac{n}{2} \rfloor$, and thus $f_1(\Delta) = f_1(J_2(n))$. A simple computation also shows that $f_2(\Delta) = f_2(J_2(n))$ and $f_3(\Delta) = f_3(J_2(n))$. Hence $\chi(\Delta) = \chi(J_2(n))$, and Δ is Eulerian.

We denote the set of all complexes on n vertices constructed in Example 4.5.8 as $GJ(n)$. It turns out that $GJ(n)$ is exactly the set of maximizers of the face numbers in the class of flag 3-dimensional Eulerian complex on n vertices. To prove this, we begin with the following lemma.

Lemma 4.5.9. *Let Δ be a flag 3-dimensional Eulerian complex on n vertices. If $f_1(\Delta) = f_1(J_2(n))$, then every vertex link is the suspension of a disjoint union of circles.*

Proof: Assume that v is an arbitrary vertex of Δ and denote $\mathrm{lk}_\Delta v$ by Γ. Since $f_1(\Delta) = f_1(J_2(n))$, by the proof of Theorem 4.5.4 and Lemma 4.5.3, it follows that for every 2-face $\{v_1, v_2, v_3\} \in \Gamma$, we have $\cup_{1\le i\le 3} V(\mathrm{lk}_\Delta v_i) = V(\Delta)$. In particular,

$$\cup_{1\le i\le 3} V(\mathrm{lk}_\Gamma v_i) = \cup_{1\le i\le 3} V(\mathrm{lk}_\Delta v_i[V(\Gamma)]) = V(\Gamma).$$

Since $f_0(\mathrm{lk}_\Gamma v_i \cap \mathrm{lk}_\Gamma v_j) = 2$ for $1 \le i < j \le 3$ and $f_0(\cap_{i=1}^3 \mathrm{lk}_\Gamma v_i) = 0$, by the inclusion-exclusion principle,

$$\sum_{i=1}^{3} f_0(\mathrm{lk}_\Gamma v_i) = f_0(\Gamma) + \sum_{1\le i<j\le 3} f_0(\mathrm{lk}_\Gamma v_i \cap \mathrm{lk}_\Gamma v_j) - f_0(\cap_{i=1}^3 \mathrm{lk}_\Gamma v_i) = f_0(\Gamma) + 6.$$

Also since Γ is Eulerian, the f-vector of Γ is $(f_0(\Gamma), 3f_0(\Gamma) - 6, 2f_0(\Gamma) - 4)$. Moreover, every vertex link in Γ is a disjoint union of circles, and hence $f_0(\mathrm{lk}_\Gamma v) = f_1(\mathrm{lk}_\Gamma v)$. By double counting,

$$\sum_{\sigma\in\Gamma, |\sigma|=3} \sum_{v\in\sigma} f_0(\mathrm{lk}_\Gamma v) = \sum_{v\in\Gamma} f_0(\mathrm{lk}_\Gamma v) \cdot \#\{\sigma : v \in \sigma, |\sigma| = 3\} = \sum_{v\in\sigma} f_0(\mathrm{lk}_\Gamma v)^2. \quad (4.5.3)$$

The left-hand side of (4.5.3) equals $f_2(\Gamma)(f_0(\Gamma) + 6) = 2(f_0(\Gamma) - 2)(f_0(\Gamma) + 6)$. However, since $\sum_{v\in\sigma} f_0(\mathrm{lk}_\Gamma v) = 2f_1(\Gamma) = 6f_0(\Gamma) - 12$, and every vertex link in Γ has at least four vertices, the right-hand side of (4.5.3) is bounded above by $2(f_0(\Gamma) - 2) + (f_0(\Gamma) - 2) \cdot 4^2 = 2(f_0(\Gamma) - 2)(f_0(\Gamma) + 6)$. Then the equality forces the right-hand side to obtain its maximum. Therefore, there exist two vertices $u_1, u_2 \in \Gamma$ whose vertex links in Γ have $f_0(\Gamma) - 2$ vertices and the other vertex links have 4 vertices. If $f_0(\Gamma) = 6$, then Γ is the cross-polytope. Else if $f_0(\Gamma) > 6$, then $f_0(\mathrm{lk}_\Gamma u_1) > 4$. Since Γ is flag, by Lemma 4.2.3, $\Gamma[V(\mathrm{lk}_\Gamma u_1)] = \mathrm{lk}_\Gamma u_1$, and hence every vertex of $\mathrm{lk}_\Gamma u_1$ is not connected to $f_0(\mathrm{lk}_\Gamma u_1) - 3 > 1$ vertices in $\mathrm{lk}_\Gamma u_1$. This implies that $u_2 \notin \mathrm{lk}_\Gamma u_1$. Hence $\mathrm{lk}_\Gamma u_2 = \Gamma[V(\Gamma)\backslash\{u_1, u_2\}] = \mathrm{lk}_\Gamma u_1$, and Γ is the join of $\mathrm{lk}_\Gamma u_1$ and two vertices u_1, u_2. □

Theorem 4.5.10. *Let Δ be a flag 3-dimensional Eulerian complex on n vertices. If $f_1(\Delta) = f_1(J_2(n))$, then $\Delta \in GJ(n)$.*

Proof: By Lemma 4.5.9, we may assume that the link of vertex $v_1 \in \Delta$ is the join of C and two other vertices v_2, v_3, where C is a disjoint union of circles. Then again by Lemma 4.5.9, the link of vertex v_2 is also the suspension of C. If v_1' is any vertex of C and its adjacent vertices in C are v_2', v_3', then by Lemma 4.2.3, $\Delta[V(C)] = C$, and it follows that $f_0(\mathrm{lk}_\Delta v_i' \cap C) = 2$ for $i = 1, 2$. Hence for $1 \le i, j \le 2$,

$$f_0(\mathrm{lk}_\Delta\{v_i, v_j'\}) = f_0(\mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j') \le f_0(C \cap \mathrm{lk}_\Delta v_j') + 2 = 4.$$

Furthermore, $V(\mathrm{lk}_\Delta\{v_1', v_2'\})$ is disjoint from $V(\mathrm{lk}_\Delta\{v_1, v_2\})$. So we obtain that

$$\sum_{e \subseteq \{v_1', v_2', v_1, v_2\}} f_0(\mathrm{lk}_\Delta e) \le n + 4 \cdot 4 = n + 16,$$

where the sum is over the edges of $\{v_1', v_2', v_1, v_2\}$. Since $f_1(\Delta) = f_1(J_2(n))$, by the proof of Theorem 4.5.4 and Lemma 4.5.2, it follows that this sum is exactly $n + 16$. Hence $V(\mathrm{lk}_\Delta\{v_1, v_2\}) \uplus V(\mathrm{lk}_\Delta\{v_1', v_2'\}) = V(\Delta)$. By Lemma 4.5.6, $\Delta \subseteq \mathrm{lk}_\Delta\{v_1, v_2\} * \mathrm{lk}_\Delta\{v_1', v_2'\}$. We count the number of edges in Δ to get

$$f_1(J_2(n)) = f_1(\Delta) \le f_1(\mathrm{lk}_\Delta\{v_1, v_2\} * \mathrm{lk}_\Delta\{v_1', v_2'\}) = f_0(\mathrm{lk}_\Delta\{v_1, v_2\}) \cdot f_0(\mathrm{lk}_\Delta\{v_1', v_2'\}) + n \le f_1(J_2(n)).$$

Thus $f_1(\Delta) = f_1(\mathrm{lk}_\Delta\{v_1, v_2\} * \mathrm{lk}_\Delta\{v_1', v_2'\})$, and the edge links $\mathrm{lk}_\Delta\{v_1, v_2\}, \mathrm{lk}_\Delta\{v_1', v_2'\}$ must be disjoint unions of circles on $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ vertices respectively. Since the flag complex Δ is determined by its graph, it follows that $\Delta = \mathrm{lk}_\Delta\{v_1, v_2\} * \mathrm{lk}_\Delta\{v_1', v_2'\}$, i.e., $\Delta \in GJ(n)$. □

Remark 4.5.11. Theorem 4.5.10 implies Theorem 4.3.4. This is because every 3-manifold is Eulerian and the only complex in $GJ(n)$ that is also a 3-manifold is $J_2(n)$.

Remark 4.5.12. The complexes from Example 5.8 form asymptotically the complete list of maximizers of the edge number in the class of $K_{1,3,3}$-free graphs, see [53, Theorem 5]. (Here $K_{r_1,r_2,\ldots,r_m}$ denotes the complete m-partite graph with r_i vertices of color i.) A more general result on extremal graphs not containing $K_{r_1,r_2,\ldots,r_m}$ can be found in [21]. Studying these extremal graphs is the main tool of Adamaszek and Hladký's work [1] on asymptotic upper bounds.

4.6 Concluding Remarks

We close this chapter with a few remarks and open problems.

As mentioned in the introduction, Klee [36] verified that the Motzkin's UBC for polytopes holds for Eulerian complexes with sufficiently many vertices, and conjectured it holds for all Eulerian complexes. Can the upper bound conjecture for flag spheres also be extended to flag Eulerian complexes? Motivated by Theorem 4.3.4 and Theorem 4.5.4, we posit the following conjecture in the same spirit as Problem 17(i) from [1]:

Conjecture 4.6.1. *Let Δ be a flag $(2m-1)$-dimensional complex, where $m \geq 2$. Assume further that Δ is an Eulerian complex on n vertices. Then $f_i(\Delta) \leq f_i(J_m(n))$ for all $i = 1, \ldots, 2m-1$.*

Theorem 4.5.4 gives an affirmative answer in the case of $m = 2$ and $1 \leq i \leq 3$. The next case is $i = 1$ and $m = 3$. In this case, Theorem 4.4.3 verifies Conjecture 4.6.1 for flag 5-manifolds. At present other cases are completely open.

The above results and conjectures discuss odd-dimensional flag complexes. What happens in the even-dimensional cases? To this end, we pose the following strengthening of Conjecture 18 from [1].

Let $J_m^*(n) := \mathbb{S}^0 * C_1 * \cdots * C_m$, where each C_i is a circle of length either $\lceil \frac{n-2}{m} \rceil$ or $\lfloor \frac{n-2}{m} \rfloor$, and the total number of vertices of $J_m^*(n)$ is $n \geq 4m+2$. Now we let $\mathcal{S}_n$ denote the set of flag 2-spheres on n vertices, and define

$$\mathcal{J}_m^*(n) := \{S * C_2 * \cdots * C_m \mid S \in \mathcal{S}_{V(C_1)+2}\}.$$

It is not hard to see that every element in $\mathcal{J}_m^*(n)$ is a flag $2m$-sphere.

Conjecture 4.6.2. *Let Δ be a flag homology $2m$-sphere on n vertices. Then $f_i(\Delta) \leq f_i(J_m^*(n))$ for all $i = 1, \ldots, 2m$. If equality holds for some $1 \leq i \leq 2m$, then $\Delta \in \mathcal{J}_m^*(n)$.*

Chapter 5

A CHARACTERIZATION OF HOMOLOGY MANIFOLDS WITH $G_2 \leq 2$

5.1 Introduction

Characterizing face-number related invariants of a given class of simplicial complexes has been a central topic in topological combinatorics. One of the most well-known results is the g-theorem (see [12], [11], and [56]), which completely characterizes the g-vectors of simplicial d-polytopes. It follows from the g-theorem that for every simplicial d-polytope P, the g-numbers of P, $g_0, g_1, \cdots, g_{\lfloor d/2 \rfloor}$, are non-negative. This naturally leads to the question of when equality $g_i = 0$ is attained for a fixed i. While it is easy to see that $g_1(P) = 0$ holds if and only if P is a d-dimensional simplex, the question of which polytopes satisfy $g_2 = 0$ is already highly non-trivial. This question was settled by Kalai [34], using rigidity theory of frameworks, in the generality of simplicial manifolds; his result was then further extended by Tay [63] to all normal pseudomanifolds.

To state these results, known as the lower bound theorem, recall that a *stacking* is the operation of building a shallow pyramid over a facet of a given simplicial polytope, and a *stacked* $(d-1)$*-sphere* on n vertices is the $(n-d)$-fold connected sum of the boundary complex of a d-simplex, denoted as $\partial\sigma^d$, with itself.

Theorem 5.1.1. *Let Δ be a normal pseudomanifold of dimension $d \geq 3$. Then $g_2(\Delta) \geq 0$. Furthermore, if $d \geq 4$, then equality holds if and only if Δ is a stacked sphere.*

Continuing this line of research, Nevo and Novinsky [46] characterized all homology spheres with $g_2 = 1$. Their main theorem is quoted below.

Theorem 5.1.2. *Let $d \geq 4$, and let Δ be a homology $(d-1)$-sphere without missing facets. Assume that $g_2(\Delta) = 1$. Then Δ is combinatorially isomorphic to either the join of $\partial\sigma^i$ and*

$\partial\sigma^{d-i}$, where $2 \leq i \leq d-2$, or the join of $\partial\sigma^{d-2}$ and a cycle. Hence every homology $(d-1)$-sphere with $g_2 = 1$ is combinatorially isomorphic to a homology $(d-1)$-sphere obtained by stacking over any of these two types of spheres.

Their result implies that all homology spheres with $g_2 = 1$ are polytopal. The proof is based on rigidity theory for graphs.

In this chapter, we characterize all homology manifolds with $g_2 \leq 2$. Our main strategy is to use three different retriangulations of simplicial complexes with the properties that (1) the homeomorphism type of the complex is preserved under these retriangulations; and (2), the resulting changes in g_2 are easy to compute. Specifically, for a large subclass of these retriangulations, g_2 increases or decreases exactly by one. We use these properties to show that every homology manifold with $g_2 \leq 2$ is obtained by centrally retriangulating a polytopal sphere of the same dimension but with a smaller g_2. As a corollary, every homology sphere with $g_2 \leq 2$ is polytopal. Incidentally, this implies a result of Mani [41] that all triangulated spheres with $g_1 \leq 2$ are polytopal.

This chapter is organized as follows. In Section 5.2 we recall basic definitions and results pertaining to simplicial complexes, polytopes and framework rigidity. In Section 5.3 we define three retriangulations of simplicial complexes that serve as the main tool in later sections. In Section 5.4 and Section 5.5 we use these retriangulations to characterize normal pseudomanifolds with $g_2 = 1$ (of dimension at least four) and homology manifolds with $g_2 = 2$ (of dimension at least three), respectively, see Theorems 5.4.4, 5.5.3 and 5.5.4.

5.2 Preliminaries

5.2.1 Basic definitions

We begin with basic definitions. A *simplicial complex* Δ on vertex set $V = V(\Delta)$ is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, and such that for every $v \in V$, $\{v\} \in \Delta$. The *dimension* of a face σ is $\dim(\sigma) = |\sigma| - 1$, and the *dimension* of Δ is $\dim(\Delta) = \max\{\dim(\sigma) : \sigma \in \Delta\}$. The *facets* of Δ are maximal faces of Δ under inclusion.

We say that a simplicial complex Δ is *pure* if all of its facets have the same dimension. A *missing* face of Δ is any subset σ of $V(\Delta)$ such that σ is not a face of Δ but every proper subset of σ is. A missing i-face is a missing face of dimension i. A pure simplicial complex Δ is *prime* if it does not have any missing facets.

The *link* of a face σ is $\mathrm{lk}_\Delta\, \sigma := \{\tau - \sigma \in \Delta : \sigma \subseteq \tau \in \Delta\}$, and the *star* of σ is $\mathrm{st}_\Delta\, \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}$. If $W \subseteq V(\Delta)$ is a subset of vertices, then we define the *restriction* of Δ to W to be the subcomplex $\Delta[W] = \{\sigma \in \Delta : \sigma \subseteq W\}$. The *antistar* of a vertex is $\mathrm{ast}_\Delta(v) = \Delta[V - \{v\}]$. We also define the *i-skeleton* of Δ, denoted as $\mathrm{Skel}_i(\Delta)$, to be the subcomplex of all faces of Δ of dimension at most i. If Δ and Γ are two simplicial complexes on disjoint vertex sets, their *join* is the simplicial complex $\Delta * \Gamma = \{\sigma \cup \tau : \sigma \in \Delta, \tau \in \Gamma\}$. When Δ consists of a single vertex, we write the *cone* over Γ as $u * \Gamma$.

A *polytope* is the convex hull of a finite set of points in some $\mathbb{R}^e$. It is called a d-polytope if it is d-dimensional. A polytope is *simplicial* if all of its facets are simplices. A *simplicial sphere* (resp. ball) is a simplicial complex whose geometric realization is homeomorphic to a sphere (resp. ball). The boundary complex of a simplicial polytope is called a *polytopal sphere*. We usually denote the d-simplex by σ^d and its boundary complex by $\partial\sigma^d$. For a fixed field $\mathbf{k}$, we say that Δ is a $(d-1)$-dimensional $\mathbf{k}$-*homology sphere* if $\tilde{H}_i(\mathrm{lk}_\Delta\, \sigma; \mathbf{k}) \cong \tilde{H}_i(\mathbb{S}^{d-1-|\sigma|}; \mathbf{k})$ for every face $\sigma \in \Delta$ (including the empty face) and $i \geq -1$. (Here we denote by $\tilde{H}_*(\Delta, \mathbf{k})$ the reduced homology with coefficients in a field $\mathbf{k}$.) Similarly, Δ is a $(d-1)$-dimensional $\mathbf{k}$-*homology manifold* if all of its vertex links are $(d-2)$-dimensional $\mathbf{k}$-homology spheres. A $(d-1)$-dimensional simplicial complex Δ is called a *normal* $(d-1)$-*pseudomanifold* if (i) it is pure and connected, (ii) every $(d-2)$-face of Δ is contained in exactly two facets and (iii) the link of each face of dimension $\leq d-3$ is also connected. For a fixed d, we have the following hierarchy:

$$\text{polytopal } (d-1)\text{-spheres} \subseteq \text{homology } (d-1)\text{-spheres} \subseteq \text{connected homology } (d-1)\text{-manifolds} \subseteq \text{normal } (d-1)\text{-pseudomanifolds}.$$

When $d = 3$, the first two classes and the last two classes of complexes above coincide;

however, starting from $d = 4$, all of the inclusions above are strict.

For a $(d-1)$-dimensional simplicial complex Δ, the *f-number* $f_i = f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. The vector $(f_{-1}, f_0, \cdots, f_{d-1})$ is called the *f-vector* of Δ. We also define the *h-vector* $h(\Delta) = (h_0, \cdots, h_d)$ by the relation $\sum_{j=0}^{d} h_j \lambda^{d-j} = \sum_{i=0}^{d} f_{i-1}(\lambda - 1)^{d-i}$. If Δ is a homology $(d-1)$-sphere, then by the Dehn-Sommerville relations, $h_i(\Delta) = h_{d-i}(\Delta)$ for all $0 \leq i \leq d$. Hence it is natural to consider the successive differences between the h-numbers: we form a vector called the *g-vector*, whose entries are given by $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $1 \leq i \leq \lfloor d/2 \rfloor$. The f-vector and h-vector of any homology sphere are determined by its g-vector. The following lemma, which was first stated by McMullen [42] for shellable complexes and later generalized to all pure complexes by Swartz [58, Proposition 2.3], is a useful fact for face enumeration.

Lemma 5.2.1. *If Δ is a pure $(d-1)$-dimensional simplicial complex, then for $k \geq 1$,*

$$\sum_{v \in V(\Delta)} g_k(\mathrm{lk}_\Delta v) = (k+1)g_{k+1}(\Delta) + (d+1-k)g_k(\Delta).$$

5.2.2 The generalized lower bound theorem for polytopes

Theorem 5.1.1 provides a full description of normal pseudomanifolds with $g_2 = 0$. To characterize the simplicial polytopes with $g_i = 0$ for $i \geq 3$, we need to generalize stackedness. Following Murai and Nevo [44], given a simplicial complex Δ and $i \geq 1$, we let

$$\Delta(i) := \{\sigma \subseteq V(\Delta) \mid \mathrm{Skel}_i(2^\sigma) \subseteq \Delta\},$$

where 2^σ is the power set of σ. (In other words, we add to Δ all simplices whose i-dimensional skeleton is contained in Δ.)

A *homology d-ball* (over a field $\mathbf{k}$) is a d-dimensional simplicial complex Δ such that (i) Δ has the same homology as the d-dimensional ball, (ii) for every face F, the link of F has the same homology as the $(d-|F|)$-dimensional ball or sphere, and (iii) the boundary complex, $\partial\Delta := \{F \in \Delta \mid \tilde{H}_i(\mathrm{lk}_\Delta F) = 0, \forall i\}$, is a homology $(d-1)$-sphere. If Δ is a homology d-ball, the faces of $\Delta - \partial\Delta$ are called the *interior* faces of Δ. If furthermore Δ has no interior

k-faces for $k \leq d - r$, then Δ is said to be $(r-1)$-*stacked*. An $(r-1)$-*stacked* homology sphere (resp. simplicial sphere) is the boundary complex of an $(r-1)$-stacked triangulation of a homology ball (resp. simplicial ball). It is easy to see that being stacked is equivalent to being 1-stacked. The following theorem is a part of the generalized lower bound theorem established by Murai and Nevo, see [44, Theorem 1.2 and Lemma 2.1].

Theorem 5.2.2. *Let Δ be a polytopal $(d-1)$-sphere and $2 \leq r \leq d/2$. Then $g_r(\Delta) = 0$ if and only if Δ is $(r-1)$-stacked. Furthermore, if that happens, then $\Delta(d-r) = \Delta(r-1)$ is a simplicial d-ball.*

5.2.3 Rigidity Theory

We give a short presentation of rigidity theory that will be used in later sections. Let $G = (V, E)$ be a graph. A *d-embedding* is a map $\phi : V \to \mathbb{R}^d$. It is called *rigid* if there exists an $\epsilon > 0$ such that if $\psi : V \to \mathbb{R}^d$ satisfies $\text{dist}(\phi(u), \psi(u)) < \epsilon$ for every $u \in V$ and $\text{dist}(\psi(u), \psi(v)) = \text{dist}(\phi(u), \phi(v))$ for every $\{u, v\} \in E$, then $\text{dist}(\psi(u), \psi(v)) = \text{dist}(\phi(u), \phi(v))$ for every $u, v \in V$. (Here dist denotes the Euclidean distance.) A graph G is called *generically d-rigid* if the set of rigid d-embeddings of G is open and dense in the set of all d-embeddings of G.

Given a graph G and a d-embedding ϕ of G, we define the matrix $\text{Rig}(G, \phi)$ associated with a graph G as follows: it is an $f_1(G) \times df_0(G)$ matrix with rows labeled by edges of G and columns grouped in blocks of size d, with each block labeled by a vertex of G; the row corresponding to $\{u, v\} \in E$ contains the vector $\phi(u) - \phi(v)$ in the block of columns corresponding to u, the vector $\phi(v) - \phi(u)$ in columns corresponding to v, and zeros everywhere else. It is easy to see that for a generic ϕ the dimensions of the kernel and image of $\text{Rig}(G, \phi)$ are independent of ϕ. Hence we define the *rigidity matrix* of G as $\text{Rig}(G, d) = \text{Rig}(G, \phi)$ for a generic ϕ. Given a d-embedding $f : V \to \mathbb{R}^d$, a stress with respect to f is a function $w : E \to \mathbb{R}$ such that for every vertex $v \in V$

$$\sum_{u : \{v, u\} \in E} w(\{v, u\})(f(v) - f(u)) = 0.$$

We say that an edge $\{u, v\}$ participates in a stress w if $w(\{u, v\}) \neq 0$, and that a vertex v participates in w if there is an edge that participates in w and contains v. The following three lemmas summarize a few basic results of rigidity theory. For a simplicial complex Δ, we denote the graph of Δ (equivalently, the 1-skeleton of Δ) by $G(\Delta)$. We say a simplicial complex Δ is generically d-rigid if $G(\Delta)$ is generically d-rigid.

Lemma 5.2.3. *Let Δ be a simplicial complex.*

1. (Cone lemma, [65], [34] and [63]) For an arbitrary $v \in V(\Delta)$ and any d, $\operatorname{lk}_\Delta v$ is generically $(d-1)$-rigid if and only if $\operatorname{st}_\Delta v$ is generically d-rigid.

2. ([23]) If Δ is a normal $(d-1)$-pseudomanifold, then Δ is generically d-rigid.

3. If Δ is generically d-rigid, then $g_2(\Delta) = \dim \operatorname{LKer}(\operatorname{Rig}(\Delta, d))$, where $\operatorname{LKer}(M)$ is the left null space of a matrix M.

The next lemma was originally stated in [46] for the class of homology spheres. Since the proof given in [46] only uses the fact that vertex links of these complexes are generically $(d-1)$-rigid and that the facet-ridge graph of the antistar of any vertex is connected, part 2 of Lemma 5.2.3 allows us to generalize the statement to the class of normal pseudomanifolds. (For details about facet-ridge graphs of normal pseudomanifolds and their connectivity, see, for instance, [6, Section 2].)

Lemma 5.2.4. ([46, Proposition 2.10]) Let $d \geq 4$ and let Δ be a prime normal $(d-1)$-pseudomanifold. Then every vertex $u \in \Delta$ participates in a generic d-stress of the graph of Δ.

The following result is proved in Kalai's paper [34, Theorem 7.3].

Lemma 5.2.5. *Let $d \geq 4$. For any generically d-rigid pure $(d-1)$-dimensional simplicial complex Δ, $g_2(\operatorname{lk}_\Delta v) \leq g_2(\Delta)$.*

5.3 Retriangulations of simplicial complexes

A *triangulation* of a topological space M is any simplicial complex Δ such that the geometric realization of Δ is homeomorphic to M. In this section, we introduce three operations that produce new triangulations of the original topological space. We will use these operations extensively to characterize homology manifolds with $g_2 \leq 2$. The first one is called the central retriangulation, see [61, Section 5].

Definition 5.3.1. Let Δ be a d-dimensional simplicial complex and B be a subcomplex of Δ; assume also that B is a simplicial d-ball. The *central retriangulation* of Δ along B, denoted as $\mathrm{crtr}_B(\Delta)$, is the new complex we obtain after removing all of the interior faces of B and replacing them with the interior faces of the cone on the boundary of B, where the cone point is a new vertex u.

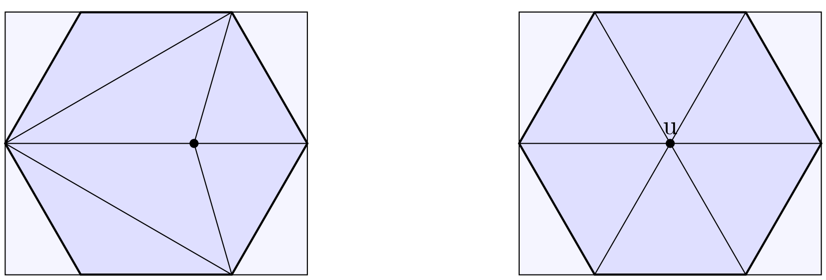


Figure 5.1: Central retriangulation along a subcomplex B (the darker blue region), where B has six interior edges, one interior vertex, and ∂B is a 6-cycle.

Recall that the *stellar subdivision* of a simplicial complex Δ at the face τ is

$$\mathrm{sd}_\tau(\Delta) = (\Delta \backslash \tau) \cup (u * \partial(\mathrm{st}_\Delta\, \tau)),$$

where u is the newly added vertex. It immediately follows from the definition that $\mathrm{crtr}_{\mathrm{st}_\Delta \tau}(\Delta) = \mathrm{sd}_\tau(\Delta)$. In this chapter, we will mainly discuss central retriangulations of Δ along an $(r-1)$-stacked $(2 \leq r \leq d/2)$ subcomplex. The following lemma indicates how the g-vector changes under central retriangulations.

Lemma 5.3.2. *Let Δ be a d-dimensional simplicial complex and $B \subseteq \Delta$ be an $(r-1)$-stacked d-dimensional ball, where $2 \leq r \leq d/2$. Then $g_i(\mathrm{crtr}_B(\Delta)) = g_i(\Delta) + g_{i-1}(\partial B)$ for $1 \leq i \leq d/2$.*

Proof: Since B is $(r-1)$-stacked, B has no interior faces of dimension $\leq d-r$. Hence by the definition of central retriangulation, $f_i(\mathrm{crtr}_B(\Delta)) = f_i(\Delta) + f_{i-1}(\partial B)$ for $0 \leq i \leq d-r$. Now use the formula $g_j(\Gamma) = \sum_{i=0}^{j}(-1)^{j-i}\binom{d+2-i}{j-i} f_{i-1}(\Gamma)$, where $d = \dim \Gamma + 1$, together with an observation that $\dim \partial B = \dim \Delta - 1$ to obtain $g_i(\mathrm{crtr}_B(\Delta)) = g_i(\Delta) + g_{i-1}(\partial B)$. $\square$

If P is a d-polytope, H a supporting hyperplane of P such that H^+ is the closed half-space determined by H that contains P, and $v \in \mathbb{R}^d \backslash H$, then we say that v is *beneath* H (with respect to P) if $v \in H^+$ and v is *beyond* H if $v \notin H^+$. In the following lemma we denote the set of missing k-faces of Δ by $M_k(\Delta)$.

Lemma 5.3.3. *Let Δ be a homology d-manifold and τ be an i-face of Δ. Then the following holds:*

1. *If $i > d/2$, then $\mathrm{st}_\Delta \tau$ is a $(d-i)$-stacked homology ball.*

2. $M_k(\mathrm{sd}_\tau(\Delta)) = \big(M_k(\Delta) - \{F \in M_k(\Delta) \mid \tau \subseteq F\}\big) \cup \{u * F \mid F \in \Delta,\ F \in M_{k-1}(\mathrm{st}_\Delta \tau)\}$. *Here u is the new vertex of the retriangulation.*

3. *If Δ is a polytopal d-sphere, then $\mathrm{sd}_\tau(\Delta)$ is also a polytopal d-sphere.*

Proof: Part 1 and 2 follow from the definitions. For part 3, we let P be a d-polytope whose boundary complex coincides with Δ and we let H_F be the supporting hyperplane of a facet F. There exists a point $p \in \mathbb{R}^d$ such that p is beyond all hyperplanes H_F for $\tau \subseteq F$, and beneath all H_F for $\tau \notin F$. Then by [27, Theorem 1 in Section 5.2], $\mathrm{sd}_\tau(\Delta)$ is the boundary complex of $\mathrm{conv}(V(\Delta) \cup \{p\})$, which is a polytope. $\square$

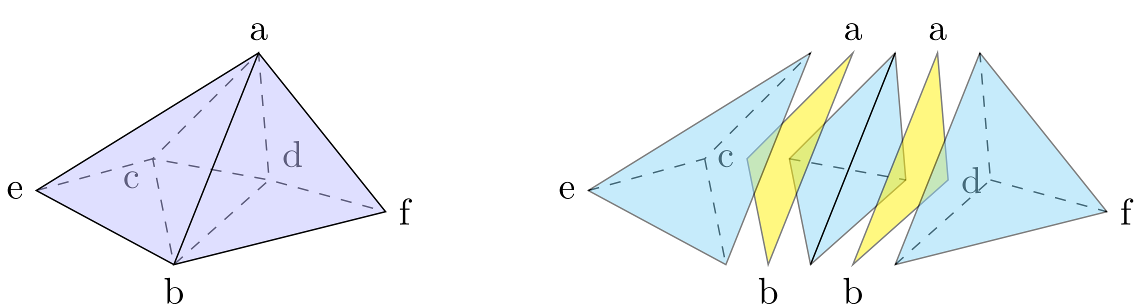
Next we introduce the second retriangulation, which in a certain sense is the inverse of central retriangulation along an $(r-1)$-stacked subcomplex.

Definition 5.3.4. Let Δ be a d-dimensional simplicial complex. Assume that there is a vertex $v \in V(\Delta)$ such that $\mathrm{lk}_\Delta\, v$ is an $(r-1)$-stacked homology $(d-1)$-sphere. If no interior face of $(\mathrm{lk}_\Delta\, v)(r-1)$ is a face of Δ, then define the *inverse stellar retriangulation* on vertex v by

$$\mathrm{sd}_v^{-1}(\Delta) = (\Delta \backslash \{v\}) \cup (\mathrm{lk}_\Delta\, v)(r-1).$$

In other words, we replace the star of v with the ball $(\mathrm{lk}_\Delta\, v)(r-1)$. It is easy to see that $\mathrm{sd}_v^{-1}(\Delta)$ is PL-homeomorphic to Δ. Using the same argument as in Lemma 5.3.2, we prove the following result.

Lemma 5.3.5. *Let Δ be a d-dimensional simplicial complex. If $\mathrm{lk}_\Delta\, v$ is an $(r-1)$-stacked homology $(d-1)$-sphere for some $2 \leq r \leq \frac{d+1}{2}$ and no interior face of $(\mathrm{lk}_\Delta\, v)(r-1)$ is a face of Δ, then $\mathrm{sd}_v^{-1}(\Delta)$ is well-defined and $g_i(\mathrm{sd}_v^{-1}(\Delta)) = g_i(\Delta) - g_{i-1}(\mathrm{lk}_\Delta\, v)$ for $1 \leq i \leq d/2$.*



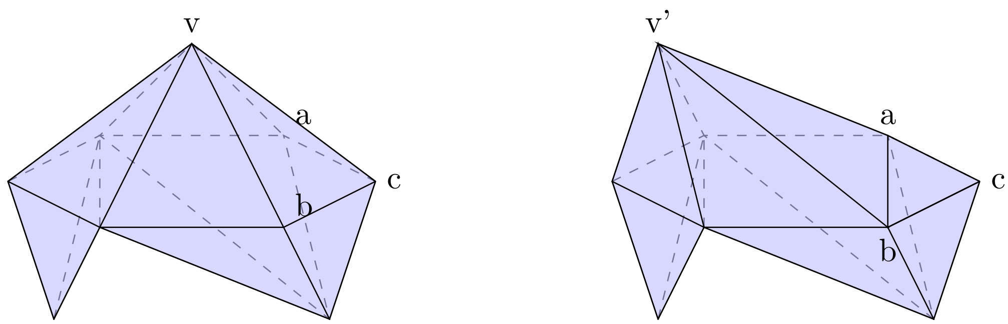
(a) A stacked vertex link, $\mathrm{lk}_\Delta\, v$, in a 3-dimensional complex Δ

(b) Two missing faces $\{a, b, c\}$ and $\{a, b, d\}$ in $\mathrm{lk}_\Delta\, v$, and three missing facets in $(\mathrm{lk}_\Delta\, v) \cup \{\{a, b, c\} \cup \{a, b, d\}\}$

Figure 5.2: Constructing $\mathrm{sd}_v^{-1}(\Delta)$ from Δ: remove the vertex v and add all five missing faces above to Δ.

A similar retriangulation that reduces g_2 was introduced by Swartz [60]. In contrast with the inverse stellar retriangulation, the number of vertices, or equivalently g_1, is not necessarily reduced in Swartz's operation.

Definition 5.3.6. Let Δ be a d-dimensional simplicial complex such that one of the vertex links, $\mathrm{lk}_\Delta\, v$, is a homology $(d-1)$-sphere. If a missing facet τ of $\mathrm{lk}_\Delta\, v$ is also a missing face of Δ, then we define the *Swartz operation* on (v, τ) of Δ by first removing v, next adding τ, then coning off two remaining homology spheres S_1, S_2 with two new vertices v_1, v_2. (Here S_1, S_2 are the two homology spheres such that their connected sum by identifying the face τ is $\mathrm{lk}_\Delta\, v$.) If one of the two spheres, say S_1, forms the boundary of a missing facet of $\Delta \cup \{\tau\}$, then we simply add this missing facet to $\Delta \cup \{\tau\}$ instead of coning off S_1 with v_1. The resulting complex is denoted by $\mathrm{so}_{v,\tau}(\Delta)$. If the dimension of Δ is at least three, then iterating this process, we are able to add all missing facets of $\mathrm{lk}_\Delta\, v$ to Δ. The resulting complex is denoted by $\mathrm{so}_v(\Delta)$.



(a) A 2-sphere Δ: $\mathrm{lk}_\Delta\, v$ is a 6-cycle and $\tau = \{a, b\}$ is a missing edge of Δ.

(b) A retriangulation $\Delta' = \mathrm{so}_{v,\tau}(\Delta)$: add $\{a, b\}$ to Δ' and replace $\mathrm{st}_\Delta\, v$ with $\mathrm{st}_{\Delta'}\, v' \cup \{a, b, c\}$

Figure 5.3: The Swartz operation on a 2-sphere

Note that $\mathrm{so}_v(\Delta)$ is indeed well-defined since the construction is independent of the order of missing facets of $\mathrm{lk}_\Delta\, v$ chosen. Also $\mathrm{so}_v(\Delta)$ is PL-homeomorphic to Δ, and if $\mathrm{lk}_\Delta\, v$ is a stacked sphere of dimension ≥ 3, then $\mathrm{so}_v(\Delta) = \mathrm{sd}_v^{-1}(\Delta)$. If Δ is of dimension ≥ 3, then since $g_2(\mathrm{so}_{v,\tau}(\Delta)) = g_2(\Delta) - 1$ by [60], we obtain that

$$g_2(\mathrm{so}_v(\Delta)) = g_2(\Delta) - \#\{\text{missing facets of } \mathrm{lk}_\Delta\, v\}.$$

Lemma 5.3.7. *Let Δ be a normal $(d-1)$-pseudomanifold for $d \geq 4$. If a vertex link, $\mathrm{lk}_\Delta v$, is a homology $(d-2)$-sphere and there are k missing facets of $\mathrm{lk}_\Delta v$ that are not faces of Δ, then $g_2(\Delta) \geq k$.*

5.4 From $g_2 = 0$ to $g_2 = 1$

The goal of this section is to provide an alternative proof of Theorem 5.1.2 for the case of $d \geq 5$ but in a much larger class – that of normal pseudomanifolds, see Theorem 5.4.4.

If Δ is a stacked $(d-1)$-sphere, and τ is a face of Δ with the property that $\mathrm{lk}_\Delta \tau$ is the boundary complex of a simplex, then $\partial(\mathrm{st}_\Delta \tau) = \partial\tau * \mathrm{lk}_\Delta \tau$ is a join of two boundary complexes of simplices, and hence has $g_1 = 1$. Therefore by Lemma 5.3.2, centrally retriangulating Δ along $\mathrm{st}_\Delta \tau$ results in a $(d-1)$-sphere with $g_2 = 1$. However, the resulting complex is not necessarily prime. In the rest of the chapter, we denote by $\mathcal{G}_d$ the set of complexes that is either the join of $\partial\sigma^i$ and $\partial\sigma^{d-i}$, where $2 \leq i \leq d-2$, or the join of $\partial\sigma^{d-2}$ and a cycle. The following lemma is a special case of Theorem 1(a) in [6]. We give a proof for the sake of completeness.

Lemma 5.4.1. *If Δ is a normal $(d-1)$-pseudomanifold on $d+2$ vertices, then Δ is the join of two boundary complexes of simplices.*

Proof: Let σ be a missing i-face of Δ, $1 \leq i \leq d-1$. Then for any $(i-1)$-face $\tau \subseteq \partial\sigma$, $\mathrm{lk}_\Delta \tau$ is a normal $(d-i-1)$-pseudomanifold, and $d-i+1 \leq f_0(\mathrm{lk}_\Delta \tau) \leq f_0(\Delta) - f_0(\sigma) = d-i+1$. Hence $\mathrm{lk}_\Delta \tau$ is the boundary complex of a $(d-i)$-simplex. Furthermore, $V(\sigma) \sqcup V(\mathrm{lk}_\Delta \tau) = V(\Delta)$ and the link of every $(i-1)$-face of σ must be exactly $\mathrm{lk}_\Delta \tau$. This implies $\Delta \supseteq \partial\sigma * \mathrm{lk}_\Delta \tau$. Since $\partial\sigma * \mathrm{lk}_\Delta \tau$ is a normal $(d-1)$-pseudomanifold and no proper subcomplex of Δ can be a normal $(d-1)$-pseudomanifold, it follows that $\Delta = \partial\sigma * \mathrm{lk}_\Delta \tau$. □

Proposition 5.4.2. *Let Δ be a stacked $(d-1)$-sphere and let τ be a ridge. If $\mathrm{sd}_\tau(\Delta)$ is prime, then $\mathrm{sd}_\tau(\Delta)$ is the join of $\partial\sigma^{d-2}$ and a cycle. In particular, $\mathrm{sd}_\tau(\Delta) \in \mathcal{G}_d$.*

Proof: Assume that $\Delta = \Delta_1 \# \Delta_2 \# \cdots \# \Delta_{n+1}$, where $\Delta_1, \cdots, \Delta_{n+1}$ are boundary complexes of d-simplices. Assume further that $\tau_1, \cdots, \tau_n$ are the missing facets of Δ. Since $\mathrm{sd}_\tau(\Delta)$ is prime, by part 2 of Lemma 5.3.3, every missing facet of Δ must contain τ. So there exist distinct vertices $v_0, \cdots, v_{n+1}$ of $V(\Delta)$ such that $\tau_i = \tau \cup \{v_i\}$, $\Delta_1 = \partial(v_0 * \tau_1)$ and $\Delta_{n+1} = \partial(v_{n+1} * \tau_n)$. It follows that $\Delta = \partial(\tau * P)$, where P is the path $(v_0, v_1, \cdots, v_{n+1})$. Hence $\mathrm{sd}_\tau(\Delta) = \partial\tau * \tilde{P} \in \mathcal{G}_d$, where $\tilde{P}$ is the (graph) cycle obtained by adding the new vertex in $\mathrm{sd}_\tau(\Delta)$ and connecting it to the endpoints of P. □

The next lemma ([62, Corollary 1.8]) places restrictions on the first few g-numbers for normal pseudomanifolds. (See page 56 in [57] for definition of $g_i^{<k>}$ and M-vector.)

Lemma 5.4.3. *Let Δ be a normal $(d-1)$-pseudomanifold with $d \geq 4$. Then $g_3 \leq g_2^{<2>}$. In particular, if $g_3 \geq 0$, then $(1, g_1, g_2, g_3)$ is an M-vector.*

Now we are ready to give an alternative proof of Theorem 5.1.2 for dimension $d-1 \geq 4$.

Theorem 5.4.4. *Let Δ be a prime normal $(d-1)$-pseudomanifold with $g_2(\Delta) = 1$ and $d \geq 5$. Then Δ is the stellar subdivision of a stacked $(d-1)$-sphere at a face of dimension i, where $0 < i < d-1$. Furthermore, $\Delta \in \mathcal{G}_d$.*

Proof: By Lemma 5.4.3, $g_3(\Delta) \leq g_2(\Delta)^{<2>} = 1$. Also by Lemma 5.2.3, since $\mathrm{lk}_\Delta v$ and Δ are normal pseudomanifolds of dimension $d-2$ and $d-1$ respectively, they are generically $(d-1)$- and d-rigid respectively. Hence by Lemma 5.2.5, $0 \leq g_2(\mathrm{lk}_\Delta v) \leq g_2(\Delta) \leq 1$. Using Lemma 5.2.1, we obtain

$$\sum_{v \in V(\Delta)} g_2(\mathrm{lk}_\Delta v) = d - 1 + 3g_3(\Delta) \leq d + 2.$$

If $g_2(\mathrm{lk}_\Delta v) = 1$ for every vertex v of Δ, then the above inequality implies that $f_0(\Delta) \leq d+2$. However, $f_0(\Delta) \geq d+2$ and so Δ has exactly $d+2$ vertices. Hence by Lemma 5.4.1, $\Delta = \partial\sigma^i * \partial\sigma^{d-i}$ for some $2 \leq i \leq d-2$, i.e., $\Delta \in \mathcal{G}_d$. It is easy to see that in this case Δ is the stellar subdivision of $\partial\sigma^d$ at an i-face.

Otherwise, there exists a vertex v such that $g_2(\mathrm{lk}_\Delta v) = 0$. By Theorem 5.1.1, $\mathrm{lk}_\Delta v$ is a stacked sphere. We claim that every missing facet τ of $\mathrm{lk}_\Delta v$ is not a face of Δ; otherwise, $\tau \in \Delta$ and $v*\tau$ is a missing facet of Δ, contradicting the fact that Δ is prime. Hence we may apply the inverse stellar retriangulation on the vertex v to obtain a new normal pseudomanifold $\mathrm{sd}_v^{-1}(\Delta)$. By Theorem 5.1.1 and Lemma 5.3.5, $0 \leq g_2(\mathrm{sd}_v^{-1}(\Delta)) = g_2(\Delta) - g_1(\mathrm{lk}_\Delta v) \leq 0$, which implies that $\mathrm{sd}_v^{-1}(\Delta)$ is a stacked sphere. Furthermore, $g_1(\mathrm{lk}_\Delta v) = 1$, so $\mathrm{lk}_\Delta v$ is the connected sum of two boundary complexes of simplices. This implies that Δ is the stellar subdivision of $\mathrm{sd}_v^{-1}(\Delta)$ at a ridge (the unique missing facet of $\mathrm{lk}_\Delta v$). This proves the first claim. Finally, the second claim follows immediately from Proposition 5.4.2. □

5.5 From $g_2 = 1$ to $g_2 = 2$

In this section, we find all homology $(d-1)$-manifolds with $g_2 = 2$ for $d \geq 4$. Our strategy, as in the previous section, is to apply certain central retriangulations to homology $(d-1)$-spheres with $g_2 = 1$ and show that in this way we obtain all homology manifolds with $g_2 = 2$, apart from one exception in dimension 3. We begin with a few lemmas.

Lemma 5.5.1. *Let $d \geq 5$ and let Δ be a prime normal $(d-1)$-pseudomanifold with $g_2(\Delta) = 2$. Furthermore, assume that $g_2(\mathrm{lk}_\Delta v) \geq 1$ for every vertex $v \in V(\Delta)$. Then every vertex link of Δ with $g_2 = 1$ is prime.*

Proof: Assume by contradiction that $g_2(\mathrm{lk}_\Delta u) = 1$ and $\mathrm{lk}_\Delta u$ is not prime for some vertex $u \in V(\Delta)$. Then by Theorem 5.1.2, $\mathrm{lk}_\Delta u$ can be written as $\Delta_1 \# \Delta_2 \# \cdots \# \Delta_k$, where $k \geq 2$, $\Delta_1 \in \mathcal{G}_{d-1}$ and the other Δ_i's are boundary complexes of simplices. First we claim that every missing facet τ of $\mathrm{lk}_\Delta u$ is not a face of Δ. Otherwise, $\tau * u$ is a missing facet of Δ, contradicting that Δ is prime. Applying the Swartz operation on vertex u (with a new vertex u'), we obtain a new normal $(d-1)$-pseudomanifold $\Delta' := \mathrm{so}_u(\Delta)$ and $g_2(\Delta') = g_2(\Delta) - (k-1) = 3-k$. Since $g_2(\Delta') \geq 0$, it follows that $k \leq 3$.

Since $\mathrm{st}_{\Delta'} u'$ is generically d-rigid and $g_2(\mathrm{st}_{\Delta'} u') = g_2(\mathrm{lk}_{\Delta'} u') = 1$, there is a nontrivial stress of Δ' supported on $\mathrm{st}_{\Delta'} u'$, and so $k \neq 3$. Next if $k = 2$, then the link of the vertex

$w = V(\Delta_2 \backslash \Delta_1)$ has $g_2(\mathrm{lk}_{\Delta'} w) = g_2(\mathrm{lk}_\Delta w) \geq 1$. Hence there exists a generic stress of Δ' supported on $\mathrm{st}_{\Delta'} w$, and w participates in this stress. Since $w \notin \mathrm{st}_{\Delta'} u'$, we must have $g_2(\Delta') \geq 2$, contradicting the fact that $g_2(\Delta') = 1$. We conclude that $k = 1$ and $\mathrm{lk}_\Delta u$ is prime. □

Lemma 5.5.2. *Let $d \geq 5$ and let Δ be a prime normal $(d-1)$-pseudomanifold with $g_2(\Delta) = 2$. Furthermore, assume that $g_2(\mathrm{lk}_\Delta v) \geq 1$ for every vertex $v \in V(\Delta)$. Then the following holds:*

1. *If $g_2(\mathrm{lk}_\Delta u) = 2$ for some vertex u, then $V(\mathrm{st}_\Delta u) = V(\Delta)$.*

2. *If $g_2(\mathrm{lk}_\Delta u) = 1$ and $G(\Delta[V(\mathrm{lk}_\Delta u)]) = G(\mathrm{lk}_\Delta u) \cup \{e\}$ for some vertex u and edge e, then $\Delta = \partial\sigma^1 * \partial\sigma^2 * \partial\sigma^{d-3}$.*

3. *If every vertex u with $g_2(\mathrm{lk}_\Delta u) = 1$ also satisfies $G(\Delta[V(\mathrm{lk}_\Delta u)]) = G(\mathrm{lk}_\Delta u)$, then at least one of such vertex links is the join of two boundary complexes of simplices.*

Proof: For part 1, note that $g_2(\mathrm{lk}_\Delta u) = g_2(\mathrm{st}_\Delta u) = 2$. If $V(\mathrm{st}_\Delta u) \neq V(\Delta)$, then by Lemma 5.2.4, there is a vertex not in $V(\mathrm{st}_\Delta u)$ that participates in a generic d-stress of $G(\Delta)$. Hence $g_2(\Delta) \geq g_2(\mathrm{st}_\Delta u) + 1 = 3$, contradicting $g_2(\Delta) = 2$.

For part 2, note that $g_2(\Delta[V(\mathrm{st}_\Delta u)]) = g_2(\mathrm{st}_\Delta u)+1 = 2$, and so using the same argument as in part 1 we obtain that $V(\mathrm{st}_\Delta u) = V(\Delta)$. Since $G(\mathrm{lk}_\Delta u)$ is not a complete graph (it misses e) and $\mathrm{lk}_\Delta u$ is prime by Lemma 5.5.1, it follows from Theorem 5.1.2 that $\mathrm{lk}_\Delta u$ is the join of a cycle C and $\partial\sigma^{d-3}$. Hence $V(e) \subseteq V(C)$. For every vertex $v \in V(C) - V(e)$, its degree in $\Delta[V(C)] = C \cup \{e\}$ is exactly 2, and thus $V(\mathrm{st}_\Delta v) \subsetneq V(\Delta)$. By part 1, $g_2(\mathrm{lk}_\Delta v) = 1$. Then as $V(\mathrm{st}_\Delta v)$ is strictly contained in $V(\Delta)$, $f_0(\mathrm{lk}_\Delta v) \leq 3 + f_0(\partial\sigma^{d-3}) = d+1$ yields that $\mathrm{lk}_\Delta v$ is the join of a 3-cycle and $\partial\sigma^{d-3}$, which further implies that $e \in \mathrm{lk}_\Delta v$. Hence $\Delta[V(C)]$ is a triangulated 2-ball whose boundary is the 4-cycle C. Since $V(\mathrm{st}_\Delta u) = V(\Delta)$, $\Delta[V(\mathrm{lk}_\Delta u)]$ is a homology ball. Hence it follows that $\Delta[V(\mathrm{lk}_\Delta u)] = \Delta[V(C)] * \partial\sigma^{d-3}$, and so Δ is the suspension of $\partial(e * u) * \partial\sigma^{d-3}$.

For part 3, by Lemma 5.5.1 and Theorem 5.1.2, either $\mathrm{lk}_\Delta u = C * \partial F$ for some cycle C of length ≥ 4 and missing $(d-3)$-face F of $\mathrm{lk}_\Delta u$, or $\mathrm{lk}_\Delta u = \partial\sigma^i * \partial\sigma^{d-1-i}$ for some $2 \leq i \leq d-3$. If every vertex link with $g_2 = 1$ is of the former type, then since $G(\Delta[V(C)]) = G(C)$, it follows that $V(\mathrm{st}_\Delta a) \subsetneq V(\Delta)$ for every vertex $a \in V(C)$. Hence by part 1, $g_2(\mathrm{lk}_\Delta a) = 1$, and it is the join of ∂F and a cycle. Also every vertex from $\Delta - \mathrm{st}_\Delta u$ is not connected to u, so again by part 1 the links of these vertices have $g_2 = 1$. On the other hand, the link of every vertex $b \in F$ contains a subcomplex $u * \partial(F - \{b\}) * C$ yet $(F - \{b\}) * C \nsubseteq \mathrm{lk}_\Delta b$. Hence $g_2(\mathrm{lk}_\Delta b) \neq 1$ by Theorem 5.1.2. By Lemma 5.2.5, $g_2(\mathrm{lk}_\Delta b) = 2$, and by part 1, $V(\mathrm{st}_\Delta v) = V(\Delta)$ for every $b \in F$.

We claim that F is a missing face of Δ. Otherwise, let w be a vertex in $\mathrm{lk}_\Delta F$. Since $g_2(\mathrm{lk}_\Delta w) = 1$, the previous argument shows that $\mathrm{lk}_\Delta w$ must be the join of ∂F and a cycle. However, $F \in \mathrm{lk}_\Delta w$, a contradiction. Now $\mathrm{lk}_\Delta u = \Delta[V(\mathrm{lk}_\Delta u)]$, so we apply the inverse stellar retriangulation on vertex u to obtain a new complex $\mathrm{sd}_u^{-1}(\Delta)$ with $0 \leq g_2(\mathrm{sd}_u^{-1}(\Delta)) = g_2(\Delta) - g_1(\mathrm{lk}_\Delta u) \leq 2 - 2 = 0$. Hence $\mathrm{sd}_u^{-1}(\Delta)$ is stacked. On the other hand, there exists a vertex z in $V(\mathrm{sd}_u^{-1}(\Delta)) - V(\mathrm{lk}_\Delta u)$ and its link in $\mathrm{sd}_u^{-1}(\Delta)$ is also stacked. But then $g_2(\mathrm{lk}_\Delta z) = g_2(\mathrm{lk}_{\mathrm{sd}_u^{-1}(\Delta)} z) = 0$, which contradicts our assumption $g_2(\mathrm{lk}_\Delta z) \geq 1$. The result follows. □

Theorem 5.5.3. *Let $d \geq 5$. Every prime normal $(d-1)$-pseudomanifold with $g_2 = 2$ can be obtained from a polytopal $(d-1)$-sphere with $g_2 = 0$ or 1, by centrally retriangulating along some stacked subcomplex.*

Proof: Let Δ be the normal $(d-1)$-pseudomanifold with $g_2(\Delta) = 2$. By Lemma 5.4.3, $g_3(\Delta) \leq g_2^{<2>}(\Delta) = 2$. Also by Lemma 5.2.1,

$$\sum_{v \in V(\Delta)} g_2(\mathrm{lk}_\Delta v) = 2(d-1) + 3g_3(\Delta) \leq 2d + 4.$$

In the following we consider two different cases.

Case 1: $g_2(\mathrm{lk}_\Delta v) \geq 1$ for every vertex $v \in V(\Delta)$. First notice that there exists a vertex $u \in V(\Delta)$ with $g_2(\mathrm{lk}_\Delta u) = 1$. Otherwise, $2f_0(\Delta) = \sum_{v \in V(\Delta)} g_2(\mathrm{lk}_\Delta v) \leq 2d + 4$, and by

Lemma 5.4.1 $g_2(\Delta) \leq 1$, a contradiction. Then Lemma 5.5.1 and Lemma 5.5.2 imply that either $\Delta = \partial\sigma^1 * \partial\sigma^2 * \partial\sigma^{d-3}$, or there exist a vertex u such that $\mathrm{lk}_\Delta u = \partial\sigma^i * \partial\sigma^{d-i-1}$ for some i. In the former case, Δ is exactly the complex $\mathrm{crtr}_{\mathrm{st}_\Delta \tau}(\partial\sigma^2 * \partial\sigma^{d-2})$, where τ is a facet of $\partial\sigma^{d-2}$. Now we deal with the latter case by first determining the g_2-numbers of all vertex links. If $w \in V(\mathrm{lk}_\Delta u)$, then $\mathrm{lk}_\Delta w$ contains either the subcomplex $u * \partial\sigma^i * \partial\sigma^{d-i-2}$ or $u * \partial\sigma^{i-1} * \partial\sigma^{d-i-1}$. Hence $\mathrm{lk}_\Delta w \notin \mathcal{G}_{d-1}$ and we conclude that $g_2(\mathrm{lk}_\Delta w) = 2$. On the other hand, every vertex $w' \in V(\Delta - \mathrm{st}_\Delta u)$ is not connected to u, so by part 1 of Lemma 5.5.2, $g_2(\mathrm{lk}_\Delta w') = 1$. Hence

$$f_0(\Delta) + (d+1) = \sum_{v \in V(\Delta)} g_2(\mathrm{lk}_\Delta v) = 2(d-1) + 3g_3(\Delta) \leq 2d + 4,$$

which implies that $f_0 \leq d+3$. Hence $|V(\Delta) - V(\mathrm{st}_\Delta u)| = 1$, and Δ is the suspension of $\mathrm{lk}_\Delta u$. Note that for any $2 \leq i \leq d-3$, the complex $\partial\sigma^1 * \partial\sigma^i * \partial\sigma^{d-i-1}$ is obtained from $\partial\sigma^i * \partial\sigma^{d-i}$ by central retriangulation along the star of a facet of $\partial\sigma^{d-i}$.

Case 2: $g_2(\mathrm{lk}_\Delta u) = 0$ for some vertex u. As Δ is prime, every missing facet τ of $\mathrm{lk}_\Delta u$ is also a missing face of Δ. We apply the inverse stellar retriangulation on vertex u to obtain a new normal $(d-1)$-pseudomanifold $\mathrm{sd}_u^{-1}(\Delta)$. Since by Lemma 5.3.5, $g_2(\mathrm{sd}_u^{-1}(\Delta)) = g_2(\Delta) - g_1(\mathrm{lk}_\Delta u) \geq 0$, either $\mathrm{sd}_u^{-1}(\Delta)$ is a stacked $(d-1)$-sphere and we let $B := (\mathrm{lk}_\Delta u)(1)$ is the union of three adjacent facets of $\mathrm{sd}_u^{-1}(\Delta)$, or $\mathrm{sd}_u^{-1}(\Delta)$ is a polytopal sphere with $g_2 = 1$ and we let B be the union of two adjacent facets of $\mathrm{sd}_u^{-1}(\Delta)$. In both cases, Δ is obtained by centrally retriangulating the polytopal sphere $\mathrm{sd}_u^{-1}(\Delta)$ along B. □

It is left to treat the case of dimension 3.

Theorem 5.5.4. *Let Δ be a prime homology 3-manifold with $g_2(\Delta) = 2$. Then Δ is either the octahedral 3-sphere, or the stellar subdivision of a 3-sphere with $g_2 = 1$ at a ridge.*

Proof: Let u be a vertex of minimal degree in $V(\Delta)$. Since $g_2(\Delta) = \frac{1}{2}\sum_{v \in V(\Delta)} f_0(\mathrm{lk}_\Delta v) - 4f_0(\Delta) + 10 = 2$, it follows that $4 < \deg u \leq 7$. We have the following cases.

Case 1: $\deg u = 5$. Then $\mathrm{lk}_\Delta u$ is the connected sum of two boundary complexes of 3-simplices. As before, $\mathrm{sd}_u^{-1}(\Delta)$ is well-defined, and $g_2(\mathrm{sd}_u^{-1}(\Delta)) = 1$. In this case Δ is the stellar subdivision of a 3-sphere with $g_2 = 1$ at a ridge.

Case 2: $\deg u = 6$ or 7 and $\mathrm{lk}_\Delta u$ is not prime. Then $\mathrm{lk}_\Delta u$ is either the connected sum of three or four boundary complexes of simplices, or it is obtained by stacking over an octahedral 2-sphere. In the former case, $g_2(\mathrm{sd}_u^{-1}(\Delta)) = g_2(\Delta) - g_1(\mathrm{lk}_\Delta u) = 0$ or -1. So by the lower bound theorem, $\mathrm{sd}_u^{-1}(\Delta)$ must be stacked. Hence there exists a vertex $w \in \mathrm{sd}_u^{-1}(\Delta)$ of degree 4 ($w \neq u$). Then $\deg_\Delta w \leq \deg_{\mathrm{sd}_u^{-1}(\Delta)} w + 1 \leq 5$, a contradiction. If $\mathrm{lk}_\Delta u$ is obtained by stacking over the octahedral 2-sphere, then by applying the Swartz operation on the vertex u, we obtain a new complex Δ' with $g_2(\Delta') = 1$ (u' is the new vertex). Note that $\deg_{\Delta'} v \geq \deg_\Delta v - 1 \geq 5$ for every vertex $v \in \Delta'$, so Δ' is prime. Since $\mathrm{lk}_{\Delta'} u'$ is the octahedral 2-sphere, by Theorem 5.1.2 it follows that Δ' must be the join of a 3-cycle and 4-cycle. However, $f_0(\Delta') = f_0(\Delta) \geq \deg u + 1 = 8$, a contradiction. Hence case 2 is impossible.

Case 3: $\mathrm{lk}_\Delta u$ is the octahedral 2-sphere. Since $\mathrm{st}_\Delta u$ is genericially 4-rigid, $g_2(\mathrm{st}_\Delta u) = 0$, and $g_2(\Delta[V(\mathrm{st}_\Delta u)]) \leq g_2(\Delta) = 2$, it follows that at least one pair of antipodal vertices in $\mathrm{lk}_\Delta u$ forms a missing edge. Let this missing edge be $\{a, b\}$. We remove u and replace $\mathrm{st}_\Delta u$ with the 3-ball $(\mathrm{lk}_\Delta u \cup \{a, b\})(1)$ to obtain a new complex Δ'. We have $g_2(\Delta') = 1$. Moreover, $\deg_{\Delta'} v = \deg_\Delta v - 1 \geq 5$ if $v \in V(\mathrm{lk}_\Delta u) \backslash \{a, b\}$, and else $\deg_{\Delta'} v = \deg_\Delta v \geq 6$. Hence Δ' is prime. By Theorem 5.1.2, Δ' is the join of a 3-cycle and a $(f_0(\Delta') - 3)$-cycle. Since every vertex in the $(f_0(\Delta') - 3)$-cycle has degree 5, it follows that this $(f_0(\Delta') - 3)$-cycle is the 4-cycle $\mathrm{lk}_\Delta\{a, u\}$. Hence $f_0(\Delta) = 8$ and Δ is the octahedral 3-sphere.

Case 4: $\mathrm{lk}_\Delta u = (a * C) \cup (b * C)$ for a 5-cycle C and two vertices a and b. Then a similar argument as in case 3 implies that $\Delta[V(C)]$ has at least one missing edge e. As $C \cup e$ is the union of a 4-cycle and a 3-cycle, $(\mathrm{lk}_\Delta u \cup e)(1)$ is the union of a octahedral sphere S and a 3-ball B (which is the suspension of a triangle). We construct a new complex Δ' by removing u, adding a new vertex u' and the edge e, and replacing $\mathrm{st}_\Delta u$ with $(S * u') \cup B$. Then Δ' is a homology 3-manifold with $g_2(\Delta') = 2$. Furthermore, the degree of every vertex of Δ' is at least 6, and so Δ' is prime. By case 2 and 3, Δ' is the octahedral 3-sphere. However, this implies the vertex a has $\deg_\Delta a = \deg_{\Delta'} a = 6 < 7$, contradicting that u is of minimal degree. □

Corollary 5.5.5. *Let $d \geq 4$ and let Δ be a homology $(d-1)$-manifold with $g_2 = 2$. Then Δ is either the connected sum of two polytopal $(d-1)$-spheres with $g_2 = 1$ (not necessarily prime) given by Theorem 5.1.2, or it is obtained by stacking over the complexes indicated in Theorem 5.5.3 and 5.5.4.*

Remark 5.5.6. Note that except for the octahedral 3-sphere (which by definition is polytopal), all prime homology $(d-1)$-spheres with $g_2 = 2$ are obtained by centrally retriangulating a polytopal sphere with $g_2 = 0$ or 1 along the star of a face or the union of three adjacent facets. Therefore by Lemma 5.3.3, all such homology spheres are polytopal. Since the connected sum of polytopes is a polytope, it follows that all homology $(d-1)$-spheres with $g_2 \leq 2$ are polytopal.

Remark 5.5.7. In [46, Example 6.2], it was shown that there exist non-polytopal spheres of any dimension ≥ 5 with $g_3 = 0$. The Barnette sphere S is an example of a non-polytopal 3-sphere with 8 vertices and 19 facets, so $g_2(S) = f_1(S) - 4f_0(S) + 10 = 19 + 8 - 4 \cdot 8 + 10 = 5$. (The construction can be found in [22].) Also in [2], all non-polytopal 3-spheres with nine vertices are classified and it turns out that $g_2 \geq 5$ in this case as well. The minimum value of g_2 for non-polytopal $(d-1)$-spheres appears to be unknown at present. On the other hand, in dimension three, $g_2 < 10$ implies the manifold is a sphere, as was originally proved by Walkup [64]. It was shown in [5] that for all $d \geq 3$ there are triangulations of $\mathbb{RP}^2 * \mathbb{S}^{d-4}$ ($\mathbb{S}^{-1}$ is the complex consisting of the empty set) that have $g_2 = 3$. This raises the question of whether $\mathbb{RP}^2 * \mathbb{S}^{d-4}$ is the only non-sphere pseudomanifold with $g_2 = 3$ triangulations.

Remark 5.5.8. For a simplicial ball Δ, one can compute, in addition to $g_2(\Delta)$, the relative g_2-number $g_2(\Delta, \partial\Delta)$. It is known that $g_2(\Delta, \partial\Delta) \geq 0$. The case of equality (for $d \geq 3$) was characterized in [45]. It would be interesting to characterize simplicial balls with $g_2(\Delta, \partial\Delta) = 1$.

Chapter 6

THE RIGIDITY OF GRAPHS OF HOMOLOGY SPHERES MINUS ONE EDGE

6.1 Introduction

The main object of this chapter is the notion of generic rigidity. We now briefly mention a few relevant definitions, defering the rest until later sections. Recall that a *d-embedding* of a graph $G = (V, E)$ is a map $\psi : V \to \mathbb{R}^d$. This embedding is called *rigid* if there exists an $\epsilon > 0$ such that if $\psi : V \to \mathbb{R}^d$ satisfies $\mathrm{dist}(\phi(u), \psi(u)) < \epsilon$ for every $u \in V$ and $\mathrm{dist}(\psi(u), \psi(v)) = \mathrm{dist}(\phi(u), \phi(v))$ for every $\{u, v\} \in E$, then $\mathrm{dist}(\psi(u), \psi(v)) = \mathrm{dist}(\phi(u), \phi(v))$ for every $u, v \in V$. A graph G is called *generically d-rigid* if the set of rigid d-embeddings of G is open and dense in the set of all d-embeddings of G.

The first substantial mathematical result concerning rigidity can be dated back to 1813, when Cauchy proved that any bijection between the vertices of two convex 3-polytopes that induces a combinatorial isomorphism and an isometry of the facets, induces an isometry of the two polytopes. Based on Cauchy's theorem and on later results by Dehn and Alexandrov, in 1975 Gluck [26] gave a complete proof of the fact that the graphs of all simplicial 3-polytopes are generically 3-rigid. Later Whiteley [65] extended this result to the graphs of simplicial d-polytopes for any $d \geq 3$. Many other generalizations have been made since, including, for example, the following theorem proved by Fogelsanger.

Theorem 6.1.1. [23] *Let $d \geq 3$. The graph of a minimal $(d-1)$-cycle complex is generically d-rigid. In particular, the graphs of all homology $(d-1)$-spheres are generically d-rigid.*

The rigidity theory of frameworks is a very useful tool for tackling the lower bound conjectures. For a $(d-1)$-dimensional simplicial complex Δ, we define $g_2(\Delta) := f_1(\Delta) - df_0(\Delta) + \binom{d+1}{2}$, where f_1 and f_0 are the numbers of edges and vertices of Δ, respectively.

By interpreting $g_2(\Delta)$ as the dimension of the left kernel of the rigidity matrix of Δ, Kalai [34] proved that the g_2-number of an arbitrary triangulated manifold Δ of dimension at least three is nonnegative (thus reproving the Lower Bound Theorem due to Barnette [9], [8]). Furthermore, Kalai showed that $g_2(\Delta) = 0$ is attained if and only if Δ is a stacked sphere. Kalai's theorem was then extended to the class of normal pseudomanifolds by Tay [63], where Theorem 1.1 served as a key ingredient in the proof. We refer to [35] for another application of the rigidity theory to the Balanced Lower Bound Theorem.

It might be tempting to conjecture that the graph of a non-stacked homology sphere Δ minus any edge of Δ is also generically d-rigid. This is not true in general; for example, let Δ be obtained by stacking over a facet of any $(d-1)$-sphere Γ, and let e be any edge not in Γ. In this case the graph of $\Delta - e$ is not generically d-rigid. However, Nevo and Novinsky [46] showed that this statement does hold if, in addition, one requires that Δ is *prime* (i.e., Δ has no missing facets) and $g_2(\Delta) = 1$. They raised the following question.

Problem 6.1.2. [46, Problem 2.11] *Let $d \geq 4$ and let Δ be a prime homology $(d-1)$-sphere. Is it true that for any edge e in Δ, the graph $G(\Delta) - e$ is generically d-rigid?*

In this chapter we give an affirmative answer to the above problem. The proof is based on the rigidity theory of frameworks. Specifically, we first verify the base cases $d = 4$ and $g_2 = 1$, and then prove the result by inducting on both the dimension and the value of g_2.

The chapter is organized as follows. In Section 6.2 after reviewing some preliminaries on simplicial complexes, we introduce the rigidity theory of frameworks and summerize several well-known results in this field. We then prove our main result (Theorem 6.3.4) in Section 6.3.

6.2 Preliminaries

A *simplicial complex* Δ on vertex set $V = V(\Delta)$ is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, and such that for every $v \in V$, $\{v\} \in \Delta$. The *dimension* of a face σ is $\dim(\sigma) = |\sigma| - 1$, and the *dimension* of Δ is $\dim(\Delta) = \max\{\dim(\sigma) : \sigma \in \Delta\}$.

The *facets* of Δ are maximal faces of Δ under inclusion. We say that a simplicial complex Δ is *pure* if all of its facets have the same dimension. A *missing* face of Δ is any subset σ of $V(\Delta)$ such that σ is not a face of Δ but every proper subset of σ is. A pure simplicial complex Δ is *prime* if it does not have any missing facets.

For a simplicial complex Δ, we denote the graph of Δ by $G(\Delta)$. If $G = (V, E)$ is a graph and $U \subseteq V$, then the *restriction* of G to U is the subgraph $G|_U$ whose vertex set is U and whose edge set consists of all of the edges in E that have both endpoints in U. We denote by $C(G)$ the graph of the cone over a graph G, and by $K(V)$ the complete graph on the vertex set V. For brevity of notation, in the following we will use $G+e$ (resp. $G-e$) to denote the graph obtained by adding an edge e to (resp. deleting e from) G.

In this chapter we focus on the graphs of a certain class of simplicial complexes. Given an edge $e = \{a, b\}$ of a simplicial complex Δ, the contraction of e to a new vertex v in Δ is the simplicial complex

$$\Delta^{\downarrow e} := \{F \in \Delta : a, b \notin F\} \cup \{F \cup \{v\} : F \cap \{a, b\} = \emptyset \text{ and either } F \cup \{a\} \in \Delta \text{ or } F \cup \{b\} \in \Delta\}.$$

A simplicial complex Δ is a *simplicial sphere* if the geometric realization of Δ, denoted as $||\Delta||$, is homeomorphic to a sphere. Let $\tilde{H}_*(\Gamma, \mathbf{k})$ denote the reduced singular homology of $||\Gamma||$ with coefficients in $\mathbf{k}$. The *link* of a face σ is $\mathrm{lk}_\Delta\, \sigma := \{\tau - \sigma \in \Delta : \sigma \subseteq \tau \in \Delta\}$, and the *star* of σ is $\mathrm{st}_\Delta\, \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}$. For a pure $(d-1)$-dimensional simplicial complex Δ and a field $\mathbf{k}$, we say that Δ is a homology sphere over $\mathbf{k}$ if $\tilde{H}_*(\mathrm{lk}_\Delta\, \sigma; \mathbf{k}) \cong \tilde{H}_*(\mathbb{S}^{d-1-|\sigma|}; \mathbf{k})$ for every face $\sigma \in \Delta$, including the empty face. We have the following inclusion relations:

$$\begin{aligned} \text{boundary complexes of simplicial } d\text{-polytopes} &\subseteq \text{simplicial } (d-1)\text{-spheres} \\ &\subseteq \text{homology } (d-1)\text{-spheres}. \end{aligned}$$

It follows from Steinitz's theorem that when $d = 3$, all three classes above coincide. When $d \geq 4$, all three inclusions are strict.

We are now in a position to review basic definitions of rigidity theory of frameworks. Given a graph G and a d-embedding ϕ of G, we define the matrix $\mathrm{Rig}(G, \phi)$ associated with a graph G as follows: it is an $f_1(G)\times df_0(G)$ matrix with rows labeled by edges of G and columns grouped in blocks of size d, with each block labeled by a vertex of G; the row corresponding to $\{u,v\}\in E$ contains the vector $\phi(u)-\phi(v)$ in the block of columns corresponding to u, the vector $\phi(v)-\phi(u)$ in columns corresponding to v, and zeros everywhere else. It is easy to see that for a generic ϕ the dimensions of the kernel and image of $\mathrm{Rig}(G,\phi)$ are independent of ϕ. Hence we define the *rigidity matrix* of G as $\mathrm{Rig}(G,d)=\mathrm{Rig}(G,\phi)$ for a generic ϕ. It follows from [3] that G is generically d-rigid if and only if $\mathrm{rank}(\mathrm{Rig}(G,d))=df_0(G)-\binom{d+1}{2}$. The following lemmas summerize a few additional results on framework rigidity.

Lemma 6.2.1 (Cone Lemma, [65]). *G is generically $(d-1)$-rigid if and only if $C(G)$ is generically d-rigid.*

Since the star of any face σ in a homology sphere is the join of σ with the link of σ, and since the link of σ is a homology sphere, Theorem 6.1.1 along with the cone lemma implies the following corollary.

Corollary 6.2.2. *Let $d\geq 4$ and let Δ be a homology $(d-1)$-sphere. Then the graph of $\mathrm{st}_\Delta\,\sigma$ is generically d-rigid for any face σ with $|\sigma|\leq d-3$.*

Lemma 6.2.3 (Gluing Lemma, [3]). *Let G_1 and G_2 be generically d-rigid graphs such that $G_1\cap G_2$ has at least d vertices. Then $G_1\cup G_2$ is also generically d-rigid.*

Lemma 6.2.4 (Replacement Lemma, [34]). *Let G be a graph and U a subset of $V(G)$. If both $G|_U$ and $G\cup K(U)$ are generically d-rigid, then G is generically d-rigid.*

Finally we state a variation of the gluing lemma.

Lemma 6.2.5. *Let G_1 and G_2 be two graphs, and assume that $a,b\in U=V(G_1\cap G_2)$. Assume further that G_1 and G_2 satisfy the following conditions: 1) the set U contains at least d vertices, including a and b, 2) both G_1 and $G_2+\{a,b\}$ are generically d-rigid, and 3) $G_1|_U=G_2|_U$. Then $G_1\cup G_2$ is also generically d-rigid.*

Proof: The second condition implies that $G_1+\{a,b\}$ is generically d-rigid. Since $G_i+\{a,b\}$ are generically d-rigid for $i=1,2$ and their intersection contains at least d vertices, by the gluing lemma, $G:=(G_1\cup G_2)+\{a,b\}$ is generically d-rigid. Note that by condition 3), the restriction of G to $V(G_1)$ is $G_1+\{a,b\}$. Replacing $G_1+\{a,b\}$ by the generically d-rigid graph G_1, we obtain the graph $G_1\cup G_2$, which is also generically d-rigid by the replacement lemma. □

6.3 Proof of the main theorem

In this section we will prove our main result, Theorem 6.3.4. We begin with the following lemma that is originally due to Kalai. We give a proof here for the sake of completeness.

Lemma 6.3.1. *Let $d\geq 4$ and let Δ be a homology $(d-1)$-sphere. If σ is a missing k-face in Δ and $2\leq k\leq d-2$, then $G(\Delta)-e$ is generically d-rigid for any edge $e\subseteq\sigma$.*

Proof: Let $\tau=\sigma\backslash e$. The dimension of $\mathrm{lk}_\Delta\,\tau$ is

$$\dim\mathrm{lk}_\Delta\,\tau=d-1-|\tau|=(d-1)-(|\sigma|-2)\geq d+1-(d-1)=2,$$

so $\mathrm{lk}_\Delta\,\tau$ is generically $(d-|\tau|)$-rigid. By Corollary 6.2.2, $\mathrm{st}_\Delta\,\tau$ is generically d-rigid. Note that $e\notin\mathrm{st}_\Delta\,\tau$, and the induced subgraph of $G(\Delta)$ on $W=V(\mathrm{st}_\Delta\,\tau)$ contains a generically d-rigid subgraph $G(\mathrm{st}_\Delta\,\tau)$. Applying the replacement lemma on W (that is, replacing $G(\Delta)|_W$ with $G(\Delta)|_W-e$), we conclude that the resulting graph $G(\Delta)-e$ is also generically d-rigid. □

The following proposition was mentioned in [46] without a proof.

Proposition 6.3.2. *Let Δ be a prime homology $(d-1)$-sphere with $g_2(\Delta)=1$, where $d\geq 4$. Then for any edge $e\in\Delta$, the graph $G(\Delta)-e$ is generically d-rigid.*

Proof: By Theorem 1.3 in [46], $\partial\Delta=\sigma_1*\sigma_2$, where σ_1 is the boundary complex of an i-simplex for some $i\geq\frac{d+1}{2}$, and σ_2 is either the boundary complex of a $(d+1-i)$-simplex, or a cycle graph $(c_1, \ldots, c_k)$ when $i = d-2$. If $e \in \sigma_1$, then $G(\Delta) - e$ is generically d-rigid by Lemma 6.3.1. Now assume that e contains a vertex v in σ_2. Note that $\sigma_2 \backslash \{v\}$ is either a simplex or a path graph. In the former case, the graph of $\Delta \backslash \{v\}$ is the complete graph on $d+1$ vertices, and hence it is generically d-rigid. In the latter case, since the graph of $\sigma_1 * \{c_i, c_{i+1}\}$ is also the complete graph on $d+1$ vertices, by the gluing lemma, $G(\Delta \backslash \{v\})$ is generically d-rigid. Finally, the graph $G(\Delta) - e$ is obtained by adding to $G(\Delta \backslash \{v\})$ the vertex v and $\deg v - 1 \geq d$ edges containing v. Hence $G(\Delta) - e$ is generically d-rigid. □

Proposition 6.3.3. *Let Δ be a prime homology 3-sphere. For any edge $e \in \Delta$, the graph $G(\Delta) - e$ is generically 4-rigid.*

Proof: The proof has a similar flavor to the proof of Proposition 1 in [66]. If e is an edge in a missing 2-face of Δ, then by Lemma 6.3.1, $G(\Delta) - e$ is generically 4-rigid. Now assume that $e = \{a, b\}$ does not belong to any missing 2-face of Δ. We claim that $\mathrm{lk}_\Delta\, e = \mathrm{lk}_\Delta\, a \cap \mathrm{lk}_\Delta\, b$. If $v \in \mathrm{lk}_\Delta\, a \cap \mathrm{lk}_\Delta\, b$, then $e = \{a, b\}$, $\{a, v\}$ and $\{b, v\}$ are edges of Δ. Hence, by our assumption, $\{a, b, v\} \in \Delta$, and so $v \in \mathrm{lk}_\Delta\, e$. Also if $e' = \{c, d\} \in \mathrm{lk}_\Delta\, a \cap \mathrm{lk}_\Delta\, b$, then $e' * \partial e \subseteq \Delta$. Since e does not belong to any missing 2-face of Δ, it follows that $c, d \in \mathrm{lk}_\Delta\, e$. Hence $(e' * \partial e) \cup (e * \partial e') \subseteq \Delta$, which by the primeness of Δ implies that $e * e' \subseteq \Delta$, i.e., $e' \in \mathrm{lk}_\Delta\, e$. Finally, if $\mathrm{lk}_\Delta\, a \cap \mathrm{lk}_\Delta\, b$ contains a 2-dimensional face τ whose boundary edges are e_1, e_2 and e_3, then the above argument implies that $e_i \cup \{b\} \in \mathrm{lk}_\Delta\, a$ for $i = 1, 2, 3$. Hence $\partial(\tau \cup \{b\}) \subseteq \mathrm{lk}_\Delta\, a$, and so $\mathrm{lk}_\Delta\, a$ is the boundary complex of a 3-simplex. This contradicts the fact that Δ is prime. We conclude that both $\mathrm{lk}_\Delta\, e$ and $\mathrm{lk}_\Delta\, a \cap \mathrm{lk}_\Delta\, b$ are 1-dimensional. Furthermore, $\mathrm{lk}_\Delta\, a \cap \mathrm{lk}_\Delta\, b \subseteq \mathrm{lk}_\Delta\, e$. However, it is obvious that the reverse inclusion also holds. This proves the claim.

If $\mathrm{lk}_\Delta\, e$ is a 3-cycle, then the filled-in triangle τ determined by $\mathrm{lk}_\Delta\, e$ is not a face of Δ. Otherwise, by the fact that $\tau \cup (\mathrm{lk}_\Delta\, e * \{a\})$ and $\tau \cup (\mathrm{lk}_\Delta\, e * \{b\})$ are subcomplexes of Δ and by the primeness of Δ, we obtain that $\tau \cup \{a\}, \tau \cup \{b\} \in \Delta$. Then since $\mathrm{lk}_\Delta\, e = \mathrm{lk}_\Delta\, a \cap \mathrm{lk}_\Delta\, b$, we conclude that $\tau \in \mathrm{lk}_\Delta\, e$, contradicting that $\mathrm{lk}_\Delta\, e$ is 1-dimensional. Hence we are able to construct a new sphere Γ from Δ by replacing $\mathrm{st}_\Delta\, e$ with the suspension of τ (indeed, Γ

and Δ differ in a bistellar flip), and therefore $G(\Delta) - e = G(\Gamma)$ is generically 4-rigid. Next we assume that $\mathrm{lk}_\Delta\, e$ has at least 4 vertices. By [46, Proposition 2.3], the edge contraction $\Delta^{\downarrow e}$ of Δ is also a homology sphere. Assume that in a d-embedding ψ of $G(\Delta)$, both a and b are placed at the origin, $V(\mathrm{lk}_\Delta\, e) = \{u_1, \dots, u_l\}$, $V(\mathrm{lk}_\Delta\, a) - V(\mathrm{lk}_\Delta\, e) = \{v_1, \dots, v_m\}$ and $V(\mathrm{lk}_\Delta\, b) - V(\mathrm{lk}_\Delta\, e) = \{w_1, \dots, w_n\}$. The rigidity matrix of $G(\Delta) - e$ can be written as a block matrix

$$M := \mathrm{Rig}(G(\Delta) - e, \psi) = \begin{pmatrix} A & B \\ 0 & R \end{pmatrix},$$

where the columns of B and R correspond to the vertices in $\mathrm{st}_\Delta\, a \cup \mathrm{st}_\Delta\, b$, and the rows of R correspond to the edges containing either a or b but not both. For convenience, we write $\mathbf{v_i}$ (resp. $\mathbf{u_i}$, $\mathbf{w_i}$) to represent $\psi(v_i)$ (resp. $\psi(u_i)$ and $\psi(w_i)$). Then

$$R = \begin{array}{c} \begin{matrix} v_1 & \dots & v_m & u_1 & \dots & u_\ell & w_1 & \dots & w_n & a & b \end{matrix} \\ \begin{pmatrix} \mathbf{v_1} & & & & & & & & & -\mathbf{v_1} & \\ & \ddots & & & & & & & & \vdots & \\ & & \mathbf{v_m} & & & & & & & -\mathbf{v_m} & \\ & & & \mathbf{u_1} & & & & & & -\mathbf{u_1} & \\ & & & & \ddots & & & & & \vdots & \\ & & & & & \mathbf{u_\ell} & & & & -\mathbf{u_\ell} & \\ & & & & & & \mathbf{w_1} & & & & -\mathbf{w_1} \\ & & & & & & & \ddots & & & \vdots \\ & & & & & & & & \mathbf{w_n} & & -\mathbf{w_n} \\ & & & \mathbf{u_1} & & & & & & & -\mathbf{u_1} \\ & & & & \ddots & & & & & & \vdots \\ & & & & & \mathbf{u_\ell} & & & & & -\mathbf{u_\ell} \end{pmatrix} \begin{matrix} \\ \\ \\ (*_1) \\ \vdots \\ (*_\ell) \\ \\ \\ \\ (**_1) \\ \vdots \\ (**_\ell) \end{matrix} \end{array},$$

where the rest of the entries not indicated above are 0. We apply the following row and column operations to matrix M: first add the last four columns, i.e. columns corresponding to b to the corresponding columns of a, then substract row $(*_i)$ from the row $(**_i)$ for

$i = 1, \ldots, \ell$. This gives

$$M'(\psi') = \begin{pmatrix} \mathrm{Rig}(G(\Delta^{\downarrow e}), \psi') & * \\ 0 & -\mathbf{u_1} \\ \vdots & \vdots \\ 0 & -\mathbf{u_l} \end{pmatrix},$$

where ψ' is the 4-embedding of $G(\Delta^{\downarrow e})$ induced by ψ, where $\psi'(v) = \psi(a) = \psi(b)$ for the new vertex v, and $\psi'(x) = \psi(x)$ for all other vertices $x \neq a, b$. Since $\ell = |V(\mathrm{lk}_\Delta\, e)| \geq 4$, it follows that the last four columns of $M'(\psi')$ are linearly independent. Hence for a generic ψ',

$$\mathrm{rank}(M) = \mathrm{rank}(M'(\psi')) = \mathrm{rank}(\mathrm{Rig}(G(\Delta^{\downarrow e}), 4)) + 4 = (4f_0(\Delta^{\downarrow e}) - 10) + 4 = 4f_0(\Delta) - 10.$$

Since $4f_0(\Delta) - 10$ is the maximal rank that the rigidity matrix of a 4-dimensional framework with $f_0(\Delta)$ vertices can have, and a small generic perturbation of a and b preserves the rank of the rigidity matrix, we conclude that $\mathrm{rank}(\mathrm{Rig}(G(\Delta) - e, 4)) = \mathrm{rank}(M) = 4f_0(\Delta) - 10$. Hence $G(\Delta) - e$ is generically 4-rigid. □

In the following we generalize the previous proposition to the case of $d > 4$ by inducting on the dimension and the value of g_2. We fix some notation here. If a homology $(d-1)$-sphere Δ is the connected sum of n prime homology spheres $S_1, \cdots, S_n$, then each S_i is called a *prime factor* of Δ. In particular, Δ is called *stacked* if each S_i is the boundary complex of a d-simplex. For every stacked $(d-1)$-sphere Δ with $d \geq 3$, there exists a unique simplicial d-ball with the same vertex set as Δ and whose boundary complex is Δ; we denote it by $\Delta(1)$. We refer to such a ball as a *stacked ball.*

Theorem 6.3.4. *Let $d \geq 4$ and let Δ be a prime homology $(d-1)$-sphere with $g_2(\Delta) > 0$. Then for any edge $e \in G(\Delta)$, the graph $G(\Delta) - e$ is generically d-rigid.*

Proof: The two base cases $g_2(\Delta) = 1, d \geq 4$ and $d = 4, g_2(\Delta) \geq 1$ are proved in Proposition 6.3.2 and 6.3.3 respectively. Now we assume that the statement is true for every prime homology $(d_0 - 1)$-sphere S with $5 \le d_0 \le d$ and $1 \le g_2(S) < g_2(\Delta)$ and every edge $e \in S$. The result follows from the following two claims. □

Claim 6.3.5. Under the above assumptions, if, furthermore, $g_2(\mathrm{lk}_\Delta u) = 0$ for some vertex $u \in V(\Delta)$, then $G(\Delta) - e$ is generically d-rigid for any edge $e \in \Delta$.

Proof: Since $\mathrm{lk}_\Delta u$ is at least 3-dimensional and since $g_2(\mathrm{lk}_\Delta u) = 0$, it follows that $\mathrm{lk}_\Delta u$ is a stacked sphere. Also since Δ is prime, the interior faces of the stacked ball $(\mathrm{lk}_\Delta u)(1)$ are not faces of Δ (or otherwise such a face together with u will form a missing facet of Δ). Let

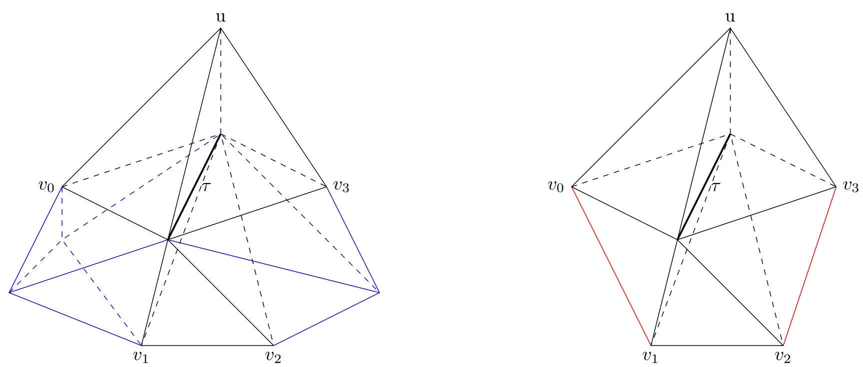
$$\Gamma := (\Delta \backslash \{u\}) \cup (\mathrm{lk}_\Delta u)(1).$$

Then Γ is a homology $(d-1)$-sphere but not necessarily prime. (For more details on this and similar operations, see [68].) Also by the primeness of Δ, every missing facet σ of Γ must contain a missing facet of $\mathrm{lk}_\Delta u$. Pick a missing facet τ of $\mathrm{lk}_\Delta u$ and assume that there are k prime factors of Γ that contain τ. We first find two facets of $(\mathrm{lk}_\Delta u)(1)$ that contain τ and say they are $\{v_0\} \cup \tau$ and $\{v_k\} \cup \tau$. Now assume that the k prime factors of Γ are $S_1, S_2, \ldots, S_k$, and each of them satisfies $S_i \cap S_{i+1} = \tau \cup \{v_i\}$ for some other vertices $v_1, \ldots, v_{k-1} \in \Delta$ and $1 \le i \le k-1$. Furthermore, $S_j \cap (\mathrm{lk}_\Delta u)(1) = \tau \cup \{v_j\}$ for $j = 0, k$. Let $G_\tau := E \cup (\cup_{i=1}^k G(S_i))$, where E is the set of edges connecting u and the vertices in $\tau \cup \{v_0, v_k\}$. Since an arbitrary edge e of $G(\Delta)$ either contains u or belongs to one of S_i's, it follows that $G(\Delta) - e = \cup(G_\tau - e)$, where the union is taken over all missing facets τ of $\mathrm{lk}_\Delta u$. By the gluing lemma, it suffices to show that $G_\tau - e$ is generically d-rigid for any τ and edge $e \in G_\tau$. We consider the following two cases:

Case 1: $e \in S_i$ for some i, S_i is not the boundary complex of the d-simplex, and $e \notin S_j$ for any other $j \neq i$. Since $G(S_i)$ is a generically d-rigid subgraph of Γ, it follows that

$$g_2(S_i) \le g_2(\Gamma) = g_2(\Delta) - f_0(\mathrm{lk}_\Delta u) + d < g_2(\Delta).$$

Furthemore, by the inductive hypothesis on g_2, $G(S_i) - e$ is generically d-rigid for any edge $e \in \Delta$. Also since $G(S_i)$ is the induced subgraph of G_τ on $V(S_i)$, by the replacement lemma, $G_\tau - e$ is generically d-rigid.



(a) The subgraph G_τ: fixing a missing facet τ of $\mathrm{lk}_\Delta u$, there are three corresponding prime factors S_1, S_2, S_3. Here S_2 is the boundary complex of the 3-simplex.

(b) G'_τ, obtained from G_τ by replacing all the blue edges in G_τ with the red edges $\{v_0, v_1\}$ and $\{v_2, v_3\}$.

Figure 6.1: The corresponding graphs G_τ and G'_τ, given the graph $G = G(\Delta)$ and a missing facet in a vertex link.

Case 2: either $e \in S_i$ for some i and S_i is the boundary complex of the d-simplex (in this case the edge $\{v_{i-1}, v_i\} \in S_i$), or $e \in \mathrm{lk}_\Delta u$, or $u \in e$. Hence $e \in G'_\tau := G(\tau * C)$, where C is the cycle graph $(u, v_0, \ldots, v_k)$. By Lemma 6.3.2, $G'_\tau - e$ is generically d-rigid for any edge e. The graph $G_\tau - e$ can be recovered from $G'_\tau - e$ by replacing each edge $\{v_{i-1}, v_i\}$ with the edges in $S_i \backslash G'_\tau$ whenever S_i is not the boundary complex of the d-simplex. Note that nothing needs to be done when S_i is the boundary complex of a simplex, since S_i is already a subcomplex of G'_τ. (See Figure 1 for an illustration in a lower dimension case.) Repeatedly applying Lemma 6.2.5 with $\{a, b\} = \{v_{i-1}, v_i\}$, $G_1 = G(S_i) - e$ and $G_2 + \{a, b\} = G'_\tau - e$, we conclude that $G_\tau - e$ is also generically d-rigid. □

Claim 6.3.6. Under the above assumption, if, furthermore, every vertex link of Δ has $g_2 \geq 1$, then $G(\Delta) - e$ is generically d-rigid for any edge $e \in \Delta$.

Proof: Assume that there is a vertex $u \in \Delta$ such that $\mathrm{lk}_\Delta u$ is the connected sum of prime factors $S_1, \dots, S_k$ and $e = \{v, w\} \in S_i$ for some i. If e is an edge in a missing facet of $\mathrm{lk}_\Delta u$ (which is also a missing $(d-2)$-face of Δ), then by Lemma 6.3.1, $G(\Delta) - e$ is generically d-rigid.

Otherwise, assume first that $g_2(S_i) \neq 0$. Then $G(S_i) - e$ is generically $(d-1)$-rigid by the inductive hypothesis on the dimension. Hence by the gluing lemma and cone lemma, we obtain that $G(\mathrm{st}_\Delta u) - e$ is generically d-rigid. By the replacement lemma, $G(\Delta) - e$ is generically d-rigid.

Finally, assume that S_i is the boundary complex of a $(d-1)$-simplex, or equivalently, $\mathrm{lk}_\Delta\{u, v, w\}$ is the boundary complex of a $(d-3)$-simplex. If furthermore for any vertex $x \in \mathrm{lk}_\Delta e$, the link of $\{x, v, w\}$ in Δ is the boundary complex of $(d-3)$-simplex, then $\mathrm{lk}_\Delta e$ must be the boundary complex of a $(d-2)$-simplex. Hence $\mathrm{lk}_\Delta v$ is obtained by adding a pyramid over a facet σ of some $(d-2)$-sphere, and w is the apex of the pyramid. Now we construct a new homology $(d-1)$-sphere Δ' as follows: first delete the edge e from Δ, then add the faces σ, $\sigma \cup \{v\}$ and $\sigma \cup \{w\}$ to Δ. It follows that $G(\Delta') = G(\Delta) - e$, which implies that $G(\Delta) - e$ is generically d-rigid.

Otherwise, there exists a vertex x such that $\mathrm{lk}_\Delta\{x, v, w\}$ is not the boundary complex of $(d-3)$-simplex. Then we may show that $G(\Delta) - e$ is generically d-rigid by applying the same argument as above on $\mathrm{lk}_\Delta x$. This proves the claim. $\square$

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