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# Some Linear and Nonlinear Geometric Inverse Problems

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**Abstract**

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Inverse problems is an area at the interface of several disciplines and has become a prominent research topic due to its potential applications. A wide range of these problems can be formulated under various geometric settings, and we call them *geometric inverse problems*. In this thesis, we study several linear and nonlinear geometric inverse problems. See Chapter 1 for a more detailed introduction.

Chapter 2 is devoted to an integral geometry problem regarding the invertibility of local weighted X-ray transforms of functions along general smooth curves. We extend the results on the local invertibility of geodesic ray transform proved by Uhlmann and Vasy [50] to X-ray transforms along general curves. In particular, our method shows that the geodesic nature of the curves does not play an essential role in this problem.

In Chapter 3, as a joint work with Yernat Assylbekov, we consider the boundary rigidity problem with respect to Hamiltonian systems involving both magnetic fields and potentials. We establish various rigidity results of such systems on compact manifolds with boundary. Unlike the cases of geodesic or magnetic systems, knowing boundary data of one energy level is insufficient for unique determination of our systems, we provide some counterexamples.

Given a bounded domain in  $\mathbb{R}^n$  with a conformally Euclidean metric, in Chapter 4, we develop an explicit reconstruction procedure for the inverse problem of recovering a semigeodesic neighborhood of the boundary of the domain and the conformal factor in this

neighborhood from some internal data. The key ingredient is the relation between the reconstruction procedure and a Cauchy problem of the conformal Killing equation. This is a joint work with Leonid Pestov and Gunther Uhlmann.

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## GLOSSARY

$M$ : a smooth manifold;

$(M, g)$ : a Riemannian manifold with metric  $g$ ;

$M^{int}$ : the interior of a smooth manifold  $M$ ;

$\partial M$ : the boundary of a smooth manifold  $M$ ;

$\overline{M}$ : the closure of a smooth manifold  $M$ ;

$TM$ : the tangent bundle of a smooth manifold  $M$ ;

$TM \setminus 0$ : the tangent bundle excluding the zero section;

$SM$ : the unit sphere bundle of a Riemannian manifold  $(M, g)$ ;

$\text{supp } f$ : the support of a smooth function  $f$ ;

$\text{grad } f$ : the gradient of a smooth function  $f$ ;

$\text{Hess } f$ : the Hessian of a smooth function  $f$ ;

$H^s(M)$ : Sobolev space of order  $s$  over the Riemannian manifold  $(M, g)$ ;

w.r.t.: an abbreviation for “with respect to”.

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## DEDICATION

*to my grandma Meiyun Zhu*

*&*

*my parents Ping Zhang and Guokang Zhou*

## Chapter 1

### INTRODUCTION

#### *1.1 An overview*

Inverse problems are routinely motivated by medical imaging, geophysics, quantum mechanics, astronomy, and so forth. This is an area at the interface of several disciplines and has become a prominent research topic due to its potential applications. Roughly speaking, inverse problems are concerned with the recovery of interior properties of a medium from boundary or remote measurements. Two typical examples of applications are:

**Medical imaging:** In modern medical imaging methods, one collects information outside the patient (e.g. electric currents, radiated energy) and attempts to reconstruct internal parameters (e.g. tissue's optical, conductive, elastic properties). Base on these internal parameters, the visual image and geometry of tissues and organs are reconstructed.

**Geophysics:** In geophysical imaging, one collects seismic measurements at the surface of the Earth and reconstruct the inner structure (e.g. density, refractive index) of the Earth. Another case is the oil exploration, where one creates shock waves that pass through hidden rock layers and interprets the waves that are reflected back to the surface.

The human body and the Earth can have complicated geometric structures, and they usually display many anisotropic properties. This motivates us to consider those inverse problems that are formulated under various geometric settings, namely the geometric inverse problems. Typical questions of geometric inverse problems are X-ray transforms, boundary or lens rigidity and spectral rigidity problems. In general, these problems are highly non-linear, demanding tools from many different areas of mathematics, including differential geometry, PDEs, microlocal analysis and functional analysis.

## 1.2 Boundary and lens rigidity problems

An outstanding inverse problem in geophysics consists in determining the inner structure of the Earth from measurements of travel times of seismic waves. From a mathematical point of view, the index of refraction of the Earth is modeled by a Riemannian metric, and the travel times by the lengths of unit speed geodesics between boundary points. This gives rise to the *boundary rigidity problem* or *travel time tomography* (also called *inverse kinematic problem* in the geophysics literature): is it possible to determine a Riemannian metric  $g$  on a compact smooth manifold  $M$  with smooth boundary, from its boundary distance function  $d_g$ ? Here the boundary distance function  $d_g : \partial M \times \partial M \rightarrow [0, +\infty)$  is the restriction of the usual distance function  $dist_g : M \times M \rightarrow [0, +\infty)$  w.r.t. the metric  $g$  to  $\partial M \times \partial M$ . Notice that the boundary distance function is unchanged under any isometry which fixes the boundary, thus one can only expect to recover the metric up to this natural obstruction.

The boundary rigidity problem is nonlinear and in general it is unsolvable without additional geometric assumptions on the metric. For example, any region of the interior of the manifold with a very large metric can not be reached by length minimizing geodesics connecting boundary points. Thus usually we need to put various geometric restrictions on the manifold, one such restriction is simplicity of the metric.

**Definition 1.2.1.** *A compact oriented Riemannian manifold  $(M, g)$  (or the metric  $g$ ) is called simple if the boundary  $\partial M$  is strictly convex and any two points  $x, y \in M$  are joined by a unique minimizing geodesic.*

Michel [28] proposed the conjecture that simple manifolds are boundary distance rigid. Pestov and Uhlmann [36] solved this conjecture for simple Riemannian surfaces. For dimensions greater than or equal to three, the conjecture is still open, it is known that boundary rigidity holds for a generic simple metric [43].

The boundary distance function takes into account only length minimizing geodesics, one can consider the behavior of all geodesics going through the manifold. This induces another type of geometric inverse problems: the *lens rigidity problem* and *scattering rigidity problem*,

which concerns the determination of a Riemannian metric up to the natural obstruction, from scattering relation or lens data. The scattering relation, introduced by Guillemin [14], is a map which sends the point and direction of entrance of a geodesic to point and direction of exit. The scattering relation together with information of lengths of geodesics gives the lens data. It is well known that on simple manifolds, boundary rigidity problem, lens rigidity problem and scattering rigidity problem are equivalent.

In Chapter 3, we study the boundary rigidity and scattering rigidity problems associated with more general Hamiltonian systems. Besides taking use of the boundary measurements, we show in Chapter 4 that one can expect to give explicit reconstruction procedure if some internal data is accessible.

### 1.3 X-Ray transforms and tensor tomography problem

Another important type of geometric inverse problems is *X-ray transforms*, and the most common case is the X-ray transform along geodesics. The *geodesic ray transform*, where one integrates a function or a tensor field along geodesics of a Riemannian metric, is closely related to the boundary rigidity problem. The integration of a function along geodesics is the linearization of the boundary rigidity problem in a fixed conformal class. The standard X-ray transform [15], where one integrates a function along straight lines, corresponds to the case of the Euclidean metric and is the basis of medical imaging techniques such as CT and PET. The case of integration along a general geodesic arises in geophysical and ultrasound imagings. The case of integrating tensors of order two along geodesics, also known as *deformation boundary rigidity*, is the linearization of the usual boundary rigidity problem.

Given a geodesic  $\gamma$  on a compact oriented Riemannian manifold  $(M, g)$  with boundary, let  $f$  be a symmetric tensor field of order  $m$ , the geodesic ray transform of  $f$  along  $\gamma$  is

$$If(\gamma) := \int f_{i_1 \dots i_m}(\gamma(t)) \dot{\gamma}^{i_1}(t) \dots \dot{\gamma}^{i_m}(t) dt,$$

where  $\dot{\gamma}$  is the velocity vector along  $\gamma$ . By the fundamental theorem of calculus, there is

a natural obstruction to the unique determination of  $f$  from  $If$ , namely  $I(dp) = 0$  for  $(m - 1)$ -tensor  $p$  with  $p|_{\partial M} = 0$  (here  $d$ , the symmetric differentiation, is the symmetric part of Levi-Civita connection). Thus in *tensor tomography problem* one would like to determine a symmetric tensor field up to natural obstruction (called potential tensors) from its integrals over geodesics [37]. Recently, Paternain, Salo and Uhlmann [34] settled the tensor tomography problem for simple surfaces. The tensor tomography problem on simple manifolds of dimension  $n \geq 3$  remains a major challenge, although substantial progress has been obtained in recent years. Injectivity of  $I$  acting on tensors of order two, up to natural obstruction, was proved for generic simple metrics [43] and manifolds of dimension  $\geq 3$  satisfying a global foliation condition [48]. The resolution of corresponding tensor tomography problem on Anosov surfaces is due to Guillarmou [13].

In this thesis, we will only consider X-ray transforms of functions. Uniqueness and stability of the geodesic ray transform was first shown by Mukhometov [29] on simple surfaces, and also for more general families of curves in two dimension. The case of geodesics was generalized also for simple manifolds to higher dimensions in [31, 3, 30]. Not much is known for non-simple manifolds, certain results are given in [7, 38, 39]. A microlocal analysis of the geodesic ray transform when the exponential map has fold type singularities was done in [46]. In dimension  $n \geq 3$ , the paper [12] proves injectivity and stability for the X-ray transform integrating over quite a general class of analytic curves with analytic weights on a class of non-simple manifolds with real-analytic metrics. A completely new approach was provided in [50] to the uniqueness and stability of the global geodesic ray transform in dimension  $n \geq 3$ , assuming the manifold is foliated by strictly convex hypersurfaces, an explicit reconstruction procedure was also given.

In Chapter 2, we extend the results of [50] to X-ray transforms of functions for a general family of curves by investigating the corresponding local problem. In particular, our method allows the appearance of a smooth non-vanishing weighted in the integrals, i.e. the weighted X-ray transform. There is also a short remark at the end of Chapter 3 regarding the X-ray transform on the simple Hamiltonian system under consideration.

## Chapter 2

# LOCAL WEIGHTED X-RAY TRANSFORM FOR GENERAL CURVES

The usual geodesic ray transform of functions on manifolds takes into account the integrals of functions along all possible geodesics joining boundary points. There is a local version of the geodesic ray transform problem: can one determine a smooth function in a neighborhood of a boundary point  $p$  from its integrals along geodesics near  $p$ ? Such problems arise naturally in applications since in many cases one doesn't have access to the whole boundary. The injectivity of local geodesic ray transform of functions on manifolds of dimensions  $n \geq 3$  was proven by Uhlmann and Vasy [50] recently assuming the boundary is strictly convex near  $p$ , with stability estimates and a reconstruction algorithm. In this chapter we consider this local inverse problem of (weighted) X-ray transform along general smooth curves.

### 2.1 Introduction

Given a Riemannian manifold  $(M, g)$  with boundary  $\partial M$  of dimension  $\geq 3$ , we consider smooth curves  $\gamma$  (with some parameter) on  $M$ ,  $|\dot{\gamma}| \neq 0$ , that satisfy the following equation

$$\nabla_{\dot{\gamma}} \dot{\gamma} = G(\gamma, \dot{\gamma}), \tag{2.1}$$

where  $\nabla$  is the Levi-Civita connection with respect to metric  $g$ ,  $G(z, v) \in T_z M$  is smooth on  $TM$ . We call the set of such smooth curves on  $M$ , denoted by  $\mathcal{G}$ , a *general family of curves*. Generally  $(\gamma, \dot{\gamma})$ ,  $\gamma \in \mathcal{G}$  need not necessarily induce a Hamiltonian flow. Note that if  $G \equiv 0$ ,  $\mathcal{G}$  is the set of usual geodesics. The X-ray transform of a smooth function  $f \in C^\infty(M)$  for a

general family of curves is defined by

$$(If)(\gamma) = \int f(\gamma(t)) dt, \quad \gamma \in \mathcal{G}.$$

More generally, given a smooth non-vanishing function  $w$  on  $TM \setminus 0$ , one can consider the following *weighted* X-ray transform for a general family of curves

$$(I_w f)(\gamma) = \int w(\gamma(t), \dot{\gamma}(t)) f(\gamma(t)) dt, \quad \gamma \in \mathcal{G}.$$

So the usual X-ray transform  $I$  corresponds to the case  $w \equiv 1$ . Since  $w$  is non-vanishing, without loss of generality, we assume  $w > 0$ . For the sake of simplicity, we assume  $\gamma \in \mathcal{G}$  are parameterized by arclength, and our arguments also work for X-ray transforms along curves with non-constant speed. Notice that given  $(z, v) \in SM$ , (2.1) determines a unique curve  $\gamma = \gamma_{z,v}$  with  $(\gamma(0), \dot{\gamma}(0)) = (z, v)$ , and  $\gamma_{z,v}$  depends smoothly on  $(z, v)$ .

For an open set  $O \subset \bar{M}$ ,  $O \cap \partial M \neq \emptyset$ , we call segments  $\gamma \in \mathcal{G}$  which are contained in  $O$  with endpoints at  $\partial X$  *O-local curves*; we denote the set of these by  $\mathcal{G}_O$ . Thus  $\mathcal{G}_O$  is an open subset of the smooth manifold  $\mathcal{G}$  (with some proper metric). We then define the *local X-ray transform for a general family of curves*  $\mathcal{G}$  of a function  $f$  defined on  $M$  as the collection  $(I_w f)(\gamma)$  of integrals of  $f$  along  $\gamma \in \mathcal{G}_O$ , i.e. as the restriction of the X-ray transform to  $\mathcal{G}_O$ . In this chapter, we consider the following question:

*Can we determine  $f|_O$ , the restriction of  $f$  on the open set  $O$ , by knowing  $(I_w f)(\gamma)$ , for all  $\gamma \in \mathcal{G}_O$ ?*

In order to state our main theorem in concrete terms, we introduce some concepts below. We embed  $M$  into some neighborhood  $\widetilde{M}$  and extend  $g, G$  smoothly onto  $\widetilde{M}$ , so  $\gamma \in \mathcal{G}$  is extended to a smooth curve on  $\widetilde{M}$ . Let  $\rho \in C^\infty(\widetilde{M})$  be a defining function of  $\partial M$ , so that  $\rho > 0$  in  $M \setminus \partial M$  and  $\rho < 0$  on  $\widetilde{M} \setminus \bar{M}$ , vanishes non-degenerately at  $\partial M$ . We generalized the concept of usual convexity (w.r.t. the metric  $g$ ) to one w.r.t. the general family of curves  $\mathcal{G}$ .

**Definition 2.1.1.** *Let  $p \in \partial M$ ,  $\partial M$  is convex (concave) at  $p$  w.r.t.  $\mathcal{G}$  if for all  $\gamma \in \mathcal{G}$  with  $\gamma(0) = p$ ,  $\dot{\gamma}(0) = v \in T_p(\partial M)$ , we have*

$$\frac{d^2}{dt^2} \rho(\gamma(t))|_{t=0} \leq 0 (\geq 0).$$

If the inequality is always strict, then we say  $\partial M$  is strictly convex (strictly concave) at  $z$  w.r.t.  $\mathcal{G}$ .

It is easy to see that the geometric meaning of our definition is similar to the usual convexity w.r.t. the metric (geodesics).

Our main theorem is an invertibility result for the local X-ray transform on neighborhoods of  $p \in \partial M$  in  $\overline{M}$  of the form  $\{\tilde{x} > -c\}$ , for sufficiently small  $c > 0$ , where  $\tilde{x}$  is a function with  $\tilde{x}(p) = 0$ ,  $d\tilde{x}(p) = -d\rho(p)$ , see Section 2.2 for the detailed definition of  $\tilde{x}$ . The statement of our theorem is similar to the main theorem in [50] on the invertibility of local geodesic ray transform.

**Theorem 2.1.2.** *Assume  $\partial M$  is strictly convex at  $p \in \partial M$  w.r.t. a general family of curves  $\mathcal{G}$ . There exists a function  $\tilde{x} \in C^\infty(\overline{M})$  with  $O_p = \{\tilde{x} > -c\} \cap \overline{M}$  for sufficiently small  $c > 0$ , the local weighted X-ray transform for  $\mathcal{G}$  is injective on  $H^s(O_p)$ ,  $s \geq 0$ .*

Further, let  $H^s(\mathcal{G}_{O_p})$  denote the restriction of elements of  $H^s(\mathcal{G})$  to  $\mathcal{G}_{O_p}$ , and for  $F > 0$  let

$$H_F^s(O_p) = e^{F/(\tilde{x}+c)} H^s(O_p) = \{f \in H_{loc}^s(O_p) : e^{-F/(\tilde{x}+c)} f \in H^s(O_p)\},$$

then for  $s \geq 0$  there exists  $C > 0$  such that for all  $f \in H_F^s(O_p)$ ,

$$\|f\|_{H_F^{s-1}(O_p)} \leq C \|I_w f|_{\mathcal{G}_{O_p}}\|_{H^s(\mathcal{G}_{O_p})}.$$

Here  $F > 0$  can be taken small, but non-vanishing. This large weight  $e^{F/(\tilde{x}+c)}$  means that the control over  $f$  in terms of  $I_w f$  is weak at  $\tilde{x} = -c$ . There are also reconstruction methods in the form of Neumann series.

Krishnan [22] studied the local geodesic ray transform under the assumption that the metric is real-analytic. The only result so far in the smooth category is for the local geodesic ray transform proved by Uhlmann and Vasy [50]. They defined an operator  $A$  which is essentially a ‘microlocal normal operator’ for the geodesic ray transform, such that  $Af$  only depends on the integral of  $f$  on the elements of  $\mathcal{G}_{O_p}$  (the set of  $O_p$ -local geodesics). The operator  $A$  is an elliptic pseudodifferential operator only in  $\tilde{x} > -c$ . Moreover, it turns

out that the exponential conjugate  $A_F$  of  $A$  is a scattering pseudodifferential operator in Melrose's scattering calculus [27] on  $\tilde{x} \geq -c$ , see also [50, Section 2]. They showed that  $A_F$  is a Fredholm operator and it is invertible for  $c$  near 0, which induces both uniqueness and stability estimates for local geodesic transform. The key ingredient from the geodesic nature in their paper is that for  $z \in \widetilde{M}$  near point  $p$ , and geodesics  $\gamma$  with  $\gamma(0) = z$ ,  $\frac{d}{dt}(\tilde{x} \circ \gamma)|_{t=0} = 0$ , then  $\frac{d^2}{dt^2}(\tilde{x} \circ \gamma)|_{t=0}$  induces a positive definite quadratic form  $\alpha$  near  $p$ . This quadratic form  $\alpha$  plays a crucial role in showing the ellipticity of the principal symbol of  $A_F$  at the artificial boundary  $\tilde{x} = -c$ , however for a general family of curves, one no longer has such special structure.

Theorem 2.1.2 is proved along the lines of [50]. We prove the invertibility of  $A_F$  for general smooth curves without the aid of the existence of some positive definite quadratic form, by using the polar coordinates to analyze the principal symbol of  $A_F$  restricted to  $\tilde{x} = -c$ .

## 2.2 Detailed settings for the inversion problem

Suppose that  $M$  is strictly convex at  $p \in \partial M$  w.r.t.  $\mathcal{G}$  (hence near  $p$ ; the convexity assumption will guarantee that locally near  $p$  in  $\overline{M}$ , every two points are joined by a unique  $\gamma \in \mathcal{G}$  lying entirely in  $M$  with possible exception of the endpoints). Let  $\rho$  be a non-degenerate boundary defining function of  $\overline{M}$ , thus for any  $\gamma = \gamma_{p,v} \in \mathcal{G}$  with  $v \in S_p(\partial M)$  (since we assume  $\gamma \in \mathcal{G}$  has unit speed, we only consider the unit sphere bundle), we have

$$\frac{d^2}{dt^2}(\rho \circ \gamma)(0) < 0.$$

Notice that

$$v \in S_p(\partial M) \iff \frac{d}{dt}(\rho \circ \gamma_{p,v})(0) = 0,$$

by the compactness of the unit sphere, there is a neighborhood  $U_0$  of  $p$  in  $\widetilde{M}$  and  $\delta > 0, C_0 > 0$  such that for vectors  $(z, v) \in S_{U_0}\widetilde{M}$ ,

$$\left| \frac{d}{dt}(\rho \circ \gamma_{z,v})(0) \right| < \delta \implies \frac{d^2}{dt^2}(\rho \circ \gamma_{z,v})(0) \leq -C_0.$$

By shrinking  $U_0$  if necessary, we may assume that  $U_0$  is a coordinate neighborhood of  $p$ . We then define the function, for  $\epsilon > 0$  to be decided,

$$\tilde{x}(z) = -\rho(z) - \epsilon|z - p|^2$$

near  $p$ , so  $\tilde{x}(p) = 0$ , here  $|\cdot|$  is the Euclidean norm. Then  $\tilde{x} \geq -c$  gives  $\rho + \epsilon|\cdot - p|^2 \leq c$  and thus  $\rho \leq c$ ; further, with  $\rho \geq 0$  this gives  $|z - p|^2 \leq c/\epsilon$ . Thus for  $c/\epsilon$  sufficiently small (so  $c > 0$  small), the region  $\tilde{x} \geq -c, \rho \geq 0$ , is compactly contained in  $U_0$ . Further, for  $(z, v) \in S_{U_0}\widetilde{M}$ ,  $\frac{d}{dt}(\tilde{x} \circ \gamma_{z,v})(0) = -\frac{d}{dt}(\rho \circ \gamma_{z,v})(0) - \epsilon\frac{d}{dt}|\gamma_{z,v}(t) - p|^2(0)$ , so

$$\frac{d}{dt}(\tilde{x} \circ \gamma_{z,v})(0) = 0 \implies \left| \frac{d}{dt}(\rho \circ \gamma_{z,v})(0) \right| < C'\epsilon,$$

so with  $\delta > 0$  as above there is  $\epsilon' > 0$  such that for  $\epsilon \in (0, \epsilon')$ ,

$$\frac{d}{dt}(\tilde{x} \circ \gamma_{z,v})(0) = 0 \implies \left| \frac{d}{dt}(\rho \circ \gamma_{z,v})(0) \right| < \delta.$$

Then for  $\epsilon < \epsilon'$ ,

$$\frac{d^2}{dt^2}(\tilde{x} \circ \gamma_{z,v})(0) = -\frac{d^2}{dt^2}(\rho \circ \gamma_{z,v})(0) - \epsilon\frac{d^2}{dt^2}|\gamma_{z,v}(t) - p|^2(0) \geq C_0 - C''\epsilon,$$

which implies the existence of  $\epsilon_0 > 0$  such that for  $\epsilon \in (0, \epsilon_0)$ ,

$$\frac{d^2}{dt^2}(\tilde{x} \circ \gamma_{p,v})(0) \geq C_0/2$$

at  $v \in S_p\widetilde{M}$  when  $\frac{d}{dt}(\tilde{x} \circ \gamma_{p,v})(0) = 0$ . Hence for  $c_0 > 0$  sufficiently small (corresponding to  $\epsilon_0$ ), the function  $\tilde{x}$  is defined on a neighborhood  $U_0$  of  $p$  with concave (w.r.t.  $\mathcal{G}$ ) level sets (from the side of the super-level sets of  $tx$ ), such that for  $0 \leq c \leq c_0$ ,

$$O_c = \{\tilde{x} > -c\} \cap \{\rho \geq 0\}$$

has compact closure in  $U_0 \cap \widetilde{M}$  (Here  $O_c$  is exactly the  $O_p$  in Theorem 2.1.2).

We remark that the actual boundary  $\rho = 0$  only plays a role at the end since ellipticity properties only hold in  $U_0$  and the function  $f$  is supported in  $\rho \geq 0$  to ensure localization. Thus for most of the following discussion we completely ignore the actual boundary

For fixed  $c \in [0, c_0]$  we work with  $x = \tilde{x} + c$ , which is the boundary defining function of the region  $\{x \geq 0\}$ . We complete  $x$  to a coordinate system  $(x, y)$  on a neighborhood  $U_1 \subset U_0$  of  $p$ . Letting  $V$  be a vector field orthogonal with respect to  $g$  to the level sets of  $x$  with  $Vx = 1$ , and using  $\{x = 0\}$  as the initial hypersurface, the flow of  $V$  (locally) identifies a neighborhood of  $\{x = 0\}$  with  $(-\epsilon, \epsilon)_x \times \{x = 0\}$ , with the first coordinate being exactly the function  $x$ . By choosing local coordinates  $y_j$  on  $x = 0$ , we obtain coordinates on this neighborhood such that  $\partial_{y_j}$  and  $\partial_x$  are orthogonal, i.e. the metric is of the form  $f(x, y)dx^2 + h(x, y, dy)$  with  $f > 0$ . For each point  $z = (x, y)$  we can parameterize curves  $\gamma \in \mathcal{G}$  through this point by the unit sphere  $S_z \widetilde{M}$ ; the relevant ones for us are ‘almost tangent’ to level sets of  $x$ , i.e. we are interested in those  $\gamma_{z,v}$  with unit tangent vector  $v = k(\lambda\partial_x + \omega\partial_y)$ ,  $k(x, y, \lambda, \omega) > 0$ ,  $\omega \in \mathbb{S}^{n-2}$  and  $\lambda$  relatively small.

Now, for a given curve  $\gamma = \gamma_{x,y,k\lambda,k\omega}$  with  $\gamma(0) = (x, y)$ ,  $\dot{\gamma}(0) = k(\lambda\partial_x + \omega\partial_y)$  (where  $\omega \in \mathbb{S}^{n-2}$  and  $\lambda$  relatively small) that is parameterized by arclength, it is obvious that  $k$  is close to 1 for sufficiently small  $\lambda$  (in the region of interest), thus we can totally omit  $k$  in the following arguments. From now on we use  $(x, y, \lambda, \omega)$ , instead of  $(x, y, k\lambda, k\omega)$ , to parameterize the curves  $\gamma$  for sufficiently small  $\lambda$ .

*Remark:* Indeed by removing  $k$ , we are doing a reparametrization of the curve  $\gamma$  and it has a (non-constant) speed close to 1 under the new parameter (we denote the curve under the new parameter by  $\tilde{\gamma}$ ). Consequently, the X-ray transform  $I_w$  is converted to a new weighted ray transform  $I_{\tilde{w}}$  whose weight  $\tilde{w}$  is sufficiently close to  $w$ . So our argument below is actually proving the invertibility of  $I_{\tilde{w}}$ . However, since  $(I_w f)(\gamma) = (I_{\tilde{w}} f)(\tilde{\gamma})$ , this implies the invertibility of original transform  $I_w$ . Please see the analysis in the end of Section 5 for more details.

Given  $(x, y, \lambda, \omega)$  and  $\gamma = \gamma_{x,y,\lambda,\omega}$ , we define

$$\alpha(x, y, \lambda, \omega, t) := \frac{1}{2} \frac{d^2}{dt^2} (x \circ \gamma)(t).$$

By convexity assumption, we have

$$\alpha(x, y, 0, \omega, 0) = \frac{1}{2} \frac{d^2}{dt^2} (x \circ \gamma)(0) > 0.$$

*Remark:* In the case of the geodesic flow, the function  $\alpha$  defined above gives a positive definite quadratic form on the tangent space of the level sets of  $x$ , which plays an important role in [50]. However, for a general family of curves,  $\alpha$  no longer has such a special structure, this force us to use a different way to analyze the invertibility later.

Similar to [50], we will work in the following general setting. We consider integrals along a family of curves  $\gamma_{x,y,\lambda,\omega}$  in  $\mathbb{R}^n$ ,  $(x, y, \lambda, \omega) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-2}$ , depending smoothly on the parameters. Here  $\mathbb{R}_y^{n-1}$  could be replaced by an arbitrary manifold and below we make  $x$  small, so we are working in a tubular neighborhood of a codimension one submanifold of  $\widetilde{M}$ . Since the changes in the manifold setting are essentially just notational and we are working on a single coordinate chart, for the sake of clarity we work with  $\mathbb{R}^n$ . Moreover, below we will work with neighborhood of a compact subset  $\{0\} \times K \subset \mathbb{R}_x \times \mathbb{R}_y^{n-1}$ ;  $\gamma_{x,y,\lambda,\omega}(t)$  would only need to be defined for  $(x, y)$  in a fixed neighborhood  $U$  of  $\{0\} \times K$  for  $|\lambda|, |t| < \delta_0$ , where  $\delta_0 > 0$  a fixed small constant.

The basic geometric feature we need is that for  $x \geq 0$  and for  $\lambda$  sufficiently small, depending on  $x$ , the curves  $\gamma_{x,y,\lambda,\omega}$  will stay in  $[0, \infty) \times \mathbb{R}^{n-1}$ . In particular for  $x = 0$ , only  $\lambda = 0$  is allowed. We assume that

$$\gamma_{x,y,\lambda,\omega}(0) = (x, y), \quad \dot{\gamma}_{x,y,\lambda,\omega}(0) = (\lambda, \omega),$$

$$\ddot{\gamma}_{x,y,\lambda,\omega}(t) = 2(\alpha(x, y, \lambda, \omega, t), \beta(x, y, \lambda, \omega, t)),$$

with  $\alpha, \beta$  smooth and

$$\alpha(0, y, 0, \omega, 0) \geq 2C > 0.$$

This implies that if  $K \subset \mathbb{R}^{n-1}$  is compact, then for a sufficiently small neighborhood  $U$  of  $\{0\} \times K$  in  $\mathbb{R}^n$  (with compact closure), and  $|\lambda|, |t| < \delta_0$ , where  $\delta_0 > 0$  small, we have

$$\alpha(x, y, \lambda, \omega, t) \geq C > 0.$$

One may assume that  $x < \delta_0$  on  $U$ . Thus for  $\gamma(t) = (x'(t), y'(t))$ ,

$$x' = x + \lambda t + t^2 \int_0^1 (1 - \sigma)\alpha(x, y, \lambda, \omega, \sigma) d\sigma \geq x + \lambda t + Ct^2/2,$$

so if  $|\lambda|, |t| < \delta_0, (x, y) \in U$ ,

$$x' \geq \frac{C}{2} \left(t + \frac{\lambda}{C}\right)^2 + \left(x - \frac{\lambda^2}{2C}\right).$$

Thus, for  $|\lambda| \leq \sqrt{2Cx}$  (and  $|\lambda| < \delta_0$ ), the curves remain in the half-space  $x' \geq 0$  at least for  $|t| < \delta_0$ . Further, if we fix  $x_0 > 0$ , then  $x' \geq x_0$  provided  $|t + \frac{\lambda}{C}| > \sqrt{2x_0/C}$  and  $|t| < \delta_0$ , thus when  $|\lambda| \leq C_0 x_0$  and  $|\lambda| < \delta_0$ , then  $x' \geq x_0$  provided  $|t| > \frac{C_0 x_0}{C} + \sqrt{2x_0/C}, |t| < \delta_0$ . Assuming  $x \leq x_0$  and taking  $x_0$  sufficiently small so that  $\frac{C_0}{C} x_0 + \sqrt{2x_0/C} < \delta_0$ , above argument implies that the curve segments  $\gamma_{x,y,\lambda,\omega}|_{(-\delta_0,\delta_0)}$  are outside the region  $\{x' < x_0\}$  for  $t$  outside a fixed compact subinterval of  $(-\delta_0, \delta_0)$ . From now on, by  $\gamma$  we mean the restriction  $\gamma_{x,y,\lambda,\omega}|_{(-\delta_0,\delta_0)}$ , and we assume that the functions we integrate along  $\gamma$  are supported in  $\{x' \leq x_0/2\}$ , so all integrals are on a compact subinterval.

Similar to the facts in [12] and [50], the maps

$$\Gamma_+ : S\widetilde{M} \times [0, \infty) \rightarrow [\widetilde{M} \times \widetilde{M}; \text{diag}], \quad \Gamma_+(z, v, t) = \left(z, |z' - z|, \frac{z' - z}{|z' - z|}\right)$$

and

$$\Gamma_- : S\widetilde{M} \times (-\infty, 0] \rightarrow [\widetilde{M} \times \widetilde{M}; \text{diag}], \quad \Gamma_-(z, v, t) = \left(z, -|z' - z|, -\frac{z' - z}{|z' - z|}\right)$$

are two diffeomorphisms near  $S\widetilde{M} \times \{0\}$ , here  $z' = \gamma_{z,v}(t)$  and  $[\widetilde{M} \times \widetilde{M}; \text{diag}]$  is the blow-up of the diagonal  $\Delta(\widetilde{M} \times \widetilde{M})$ . We actually work with (locally, in the region of interest) the set of tangent vectors of the form  $\lambda \partial_x + \omega \partial_y$ , where  $\omega \in \mathbb{S}^{n-2}$  and  $\lambda$  relatively small, which can be regarded as a small perturbation of a portion of the sphere bundle  $S\widetilde{M}$ . Thus we reduce  $\delta_0$  if necessary so that  $\Gamma_+$  is a diffeomorphism on  $U_{x,y} \times (-\delta_0, \delta_0)_\lambda \times \mathbb{S}_\omega^{n-2} \times [0, \delta_0)_t$ , and analogously for  $\Gamma_-$ .

Our inversion problem is recovering  $f$  from weighted integrals

$$(I_w f)(x, y, \lambda, \omega) = \int_{\mathbb{R}} w(\gamma_{x,y,\lambda,\omega}(t), \dot{\gamma}_{x,y,\lambda,\omega}(t)) f(\gamma_{x,y,\lambda,\omega}(t)) dt.$$

Recall our convention from above, indeed the integral is over  $(-\delta_0, \delta_0)$ , and  $f(x', y')$  vanishes for  $x' \geq x_0/2$ . Different from the usual geodesics, generally the curves defined by  $G$  is not

time reversible, i.e.

$$\gamma_{x,y,-\lambda,-\omega}(-t) \neq \gamma_{x,y,\lambda,\omega}(t).$$

Thus we will have two curves associated with a given tangent line at each  $(x, y)$ , so having the integral of functions along both.

We consider the pseudodifferential operator introduced in [50], namely to consider for  $x > 0$

$$(Af)(x, y) = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} (I_w f)(x, y, \lambda, \omega) \chi(\lambda/x) d\lambda d\omega, \quad (2.2)$$

where  $\chi$  is compactly supported (for sufficiently small  $x$ ). Thus when  $x$  is small, only  $(I_w f)(x, y, \lambda, \omega)$  with  $\lambda$  sufficiently small (which is exactly the case we expect) will contribute to the operator. We can allow  $\chi$  to depend smoothly on  $y$  and  $\omega$ ; over compact sets such a behavior is uniform.

For any  $s \geq 0$ , in view of the diffeomorphism property of  $\Gamma_{\pm}$ ,  $I$  is bounded

$$I_w : H^s([0, \infty) \times \mathbb{R}^{n-1}) \rightarrow H^s([0, \infty) \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-2}).$$

If we let

$$(Lu)(x, y) := \int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} u(x, y, \lambda, \omega) \chi(\lambda/x) d\lambda d\omega,$$

it is shown in [50] that  $L$  is bounded

$$L : H^s([0, \infty) \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-2}) \rightarrow x^{-s} H^s([0, \infty) \times \mathbb{R}^{n-1}).$$

Thus for  $s \geq 0$ ,

$$A = L \circ I_w : H^s([0, \infty) \times \mathbb{R}^{n-1}) \rightarrow x^{-s} H^s([0, \infty) \times \mathbb{R}^{n-1}) \quad (2.3)$$

is bounded. If we show that  $A$  is invertible as a map between above spaces of functions supported near  $x = 0$ , we obtain an estimate for  $f$  in terms of  $I_w f$ .

### 2.3 Scattering calculus

Before giving the proof of our main theorem, we explain the role of the so-called scattering calculus (which are typically used to study phenomena ‘at infinity’) in our problem. The

reason we introduce these concepts is that the region we are interested in is a manifold with boundary, while as will be proved later, the operator  $A$  is an elliptic pseudodifferential operator only in the interior. However, this ellipticity is not sufficient for Fredholm properties or stability estimates, due to the boundary  $\{x = 0\}$  and the functions we consider generally are non-zero near the boundary. Thus we need the help of scattering calculus, which is related to the uniform behavior of an operator up to the boundary. We refer to [50, Section 2] for a more thorough exposition. Scattering calculus was introduced by Melrose [27], in a geometric setting, but on  $\mathbb{R}^n$  this actually corresponds to a special case of Hörmander's Weyl calculus [19], also studied earlier by Shubin [40] and Parenti [33].

We start with the definition of scattering pseudodifferential operators on  $\mathbb{R}^n$ . *Scattering symbols* of order  $(m, l)$  are defined to be smooth functions  $a$  on  $\mathbb{R}_z^n \times \mathbb{R}_\zeta^n$  satisfying

$$|D_z^\alpha D_\zeta^\beta a(z, \zeta)| \leq C_{\alpha, \beta} \langle z \rangle^{l - |\alpha|} \langle \zeta \rangle^{m - |\beta|},$$

i.e. they are 'product type' symbols in  $z$  and  $\zeta$ . The set of such symbols is denoted by  $S^{m, l}(\mathbb{R}^n, \mathbb{R}^n)$  or simply  $S^{m, l}$ . One then defines *scattering pseudodifferential operators*,  $\Psi_{sc}^{m, l}(\mathbb{R}^n)$ , to be the left quantizations of such symbols, i.e. operators of the form

$$Au(z) = (2\pi)^{-n} \int e^{i(z-z') \cdot \zeta} a(z, \zeta) u(z') dz, d\zeta.$$

Further, we define the *principal symbol* of  $A$  to be the equivalence class of the amplitude  $a$  in  $S^{m, l} \setminus S^{m-1, l-1}$ . In particular, if  $A$  is *elliptic*, i.e. its principal symbol  $a$  is invertible in the sense that there is  $b \in S^{-m, -l}$  such that  $ab - 1 \in S^{-1, -1}$ , then there is a parametrix  $B \in \Psi_{sc}^{-m, -l}(\mathbb{R}^n)$  such that  $BA - Id \in \Psi_{sc}^{-\infty, -\infty}(\mathbb{R}^n)$ , the smoothing operators. Moreover, we define the corresponding *polynomially weighted Sobolev spaces*  $H^{s, r}(\mathbb{R}^n) = \langle z \rangle^{-r} H^s(\mathbb{R}^n)$ , then  $A \in \Psi_{sc}^{m, l}(\mathbb{R}^n)$  is bounded  $H^{s, r} \rightarrow H^{s-m, r-l}$ , and if  $A$  is elliptic then  $A$  is Fredholm.

Now, we focus on our setting and consider the reciprocal spherical coordinate map,  $(0, \epsilon)_x \times \mathbb{S}_\theta^{n-1} \rightarrow \mathbb{R}^n$  by  $(x, \theta) \rightarrow x^{-1}\theta \in \mathbb{R}^n$ . This map is a diffeomorphism onto its range, and it provides a compactification of  $\mathbb{R}^n$  by adding  $\{0\}_x \times \mathbb{S}_\theta^{n-1}$  as infinity to  $\mathbb{R}^n$ , to obtain  $\overline{\mathbb{R}^n}$ , which is diffeomorphic to a ball. Then one can generalize the concepts of

scattering pseudodifferential operators and weighted Sobolev spaces onto  $\overline{\mathbb{R}^n}$  to get  $\Psi_{sc}^{m,l}(\overline{\mathbb{R}^n})$  and  $H^{s,r}(\overline{\mathbb{R}^n})$  respectively. In view of the coordinates  $(x, y)$  defined in previous section, we may regard locally  $\{x = 0\}$  as a coordinate chart in  $\mathbb{S}^{n-1}$ , and thus obtain an identification of  $\overline{M} = \{x \geq 0\}$  with a region intersecting  $\overline{\mathbb{R}^n}$ . Thus the artificial boundary  $\{x = 0\}$  corresponds to the boundary of  $\overline{\mathbb{R}^n}$ ,  $\partial\overline{\mathbb{R}^n}$ . Then one can define the so-called *scattering Sobolev spaces*  $H_{sc}^{s,r}(\overline{M})$ , which is locally, under the above identification, the weighted Sobolev spaces  $H^{s,r}(\overline{\mathbb{R}^n})$ , while  $\Psi_{sc}^{m,l}(\overline{M})$  is Melrose's scattering pseudodifferential algebra, which locally under the same identification, corresponds to  $\Psi_{sc}^{m,l}(\overline{\mathbb{R}^n})$ . It is also useful to relate the standard Sobolev spaces  $H^s(\overline{M})$  to  $H_{sc}^{s,r}(\overline{M})$ , for  $s \geq 0$ ,

$$H^s(\overline{M}) \subset H_{sc}^{s,r}(\overline{M}), \quad r \leq -\frac{n+1}{2},$$

with continuous inclusion map. For the converse direction, we have

$$H_{sc}^{s,r}(\overline{M}) \subset H^s(\overline{M}), \quad r \geq -\frac{n+1}{2} + 2s,$$

with continuous inclusion map. By duality,

$$H_{sc}^{-s,-r}(\overline{M}) \subset H^{-s}(\overline{M}), \quad -r \geq \frac{n+1}{2}.$$

Here we give some description of the behavior of the Schwartz kernel of elements of  $\Psi_{sc}^{m,l}(\overline{M})$  on  $\overline{M} \times \overline{M}$ . Using local coordinate  $y$  on  $\mathbb{S}^{n-1}$ , and corresponding coordinates  $(x, y, x', y')$  on  $\overline{M} \times \overline{M}$ , in the *scattering coordinates* [50, Section 2]

$$x, y, X = \frac{x' - x}{x^2}, \quad Y = \frac{y' - y}{x},$$

valid for  $x > 0$ , so the diagonal is  $X = 0, Y = 0$  when  $x > 0$ , the Schwartz kernel of an element of  $\Psi_{sc}^{m,l}(\overline{M})$  is of the form  $x^{-l}K$ , where  $K$  is smooth in  $(x, y)$  down to  $x = 0$  with values in conormal distributions on  $\mathbb{R}_{X,Y}^n$ , conormal to  $\{X = 0, Y = 0\}$ , which are Schwartz at infinity. Moreover, generally given the principal symbol  $a$  of a scattering pseudodifferential operator, one can define the so-called *boundary principal symbol*  $\tilde{a}$ , which is the restriction of  $a$  at the boundary, i.e.  $\tilde{a} = a|_{\partial(M \times M)}$ . Thus locally near the boundary, ellipticity is simply

equivalent to the non-vanishingness of this function  $\tilde{a}$ . So the boundary principal symbol is simply  $x^{-l}$  times the Fourier transform in  $(X, Y)$  of  $\tilde{K} = K|_{x=0}$ .

Finally when ellipticity holds only locally, thus we suppose  $A \in \Psi_{sc}^{m,l}(\overline{M})$  is elliptic on  $O$ , the open subset of  $\overline{M}$ , and  $K \subset O$  is compact. Notice that local ellipticity of  $A$  implies the existence of a local parametrix  $P \in \Psi_{sc}^{-m,-l}(\overline{M})$  and some compact operator  $E$  such that  $PA = Id + E$ . In particular, the norm of  $E$  is small due to the localness. Then for sufficiently small  $c$ ,

$$\text{Ker } A \cap \{f \in H_{sc}^{s,r}(\overline{M}) : \text{supp } f \subset K \cap \overline{M}\} = \{0\}$$

and for  $f \in H_{sc}^{s,r}(\overline{M})$  supported in  $K$ , one has the stability estimate

$$\|f\|_{H_{sc}^{s,r}(\overline{M})} \leq C \|Af\|_{H_{sc}^{s-m,r-l}(\overline{M})},$$

see [50, Section 2] for more details.

## 2.4 Proof of the main theorem

We want to study the uniform behavior of  $A$  as  $x \rightarrow 0$ . It turns out that  $A$  is an elliptic pseudodifferential operator in  $x > 0$  for proper choice of  $\chi$ ; while the conjugates of  $A$  by exponential weights are scattering pseudodifferential operators on  $x \geq 0$ .

**Proposition 2.4.1.** *Suppose  $\chi \in C_c^\infty(\mathbb{R})$ ,  $\chi \geq 0$  with  $\chi > 0$  near 0, then  $A \in \Psi^{-1}(x > 0)$  is elliptic, and the operator  $A_F = x^{-2}e^{-F/x}Ae^{F/x}$  is in  $\Psi_{sc}^{-1,0}(x \geq 0)$  for  $F > 0$ .*

*Proof.* The proof is similar to the one of [50, Proposition 3.3]. First we work in  $x > 0$ , ignoring the boundary  $\{x = 0\}$ . The diffeomorphism property of  $\Gamma_\pm$  allows us to rewrite, with  $|dv|$  denoting a smooth measure on the transversal such as  $|d\lambda||d\omega|$ ,

$$Af(z) = \sum_{\bullet=+,-} \int f(z') \chi(\Gamma_{\bullet}^{-1}(z, z')) (\Gamma_{\bullet}^{-1})^*(|dv|dt)$$

in terms of  $z, z'$  as

$$\int f(z') |z' - z|^{-n+1} \left\{ b(z, |z' - z|, \frac{z' - z}{|z' - z|}) + b(z, -|z' - z|, -\frac{z' - z}{|z' - z|}) \right\} dz', \quad (2.4)$$

$$b(z, 0, v) = w(z, v)\chi(z, v)\sigma(z, v),$$

where  $\sigma > 0$  is bounded below (it is actually the Jacobian factor). Thus  $A$  is a pseudodifferential operator with principal symbol given by the Fourier transform in  $Z = z' - z$  of

$$|z' - z|^{-n+1}\left\{(w\chi\sigma)\left(z, \frac{z' - z}{|z' - z|}\right) + (w\chi\sigma)\left(z, -\frac{z' - z}{|z' - z|}\right)\right\}.$$

By a standard calculation, we can get that the principal symbol of  $A$  at  $(z, \zeta)$  is of the form

$$c_n|\zeta|^{-1} \int_{\mathbb{S}^{n-2}} (w\chi\sigma)(z, \hat{Z}^\perp) + (w\chi\sigma)(z, -\hat{Z}^\perp) d\hat{Z}^\perp = 2c_n|\zeta|^{-1} \int_{\mathbb{S}^{n-2}} (w\chi\sigma)(z, \hat{Z}^\perp) d\hat{Z}^\perp, \quad (2.5)$$

where  $\hat{Z}^\perp$  is the parameter for the sphere  $\mathbb{S}^{n-2}$  which is orthogonal to  $\zeta$ . In particular, under our assumption of  $\chi$  and  $n > 2$  we can always find some  $\hat{Z}^\perp$  orthogonal to  $\zeta$  such that  $\chi(z, \hat{Z}^\perp) > 0$ . Together with the assumption  $w > 0$ , the integral on the right hand side of (2.5) is always positive,  $A$  is an elliptic pseudodifferential operator of order  $-1$ .

We then turn to the scattering behavior, i.e. as  $x \rightarrow 0$ . We apply the scattering coordinates  $(x, y, X, Y)$ ,

$$X = \frac{x' - x}{x^2}, Y = \frac{y' - y}{x},$$

with  $K$  denoting the Schwartz kernel of  $A$ , to get that  $A_F$  has Schwartz kernel

$$K_F(x, y, X, Y) = x^{-2}e^{-FX/(1+xX)}K(x, y, X, Y), \quad (2.6)$$

here  $K$  has polynomial bounds in terms of  $X, Y$  in view of (2.4). Our main claim is that  $K_F$  and its derivatives have exponential decay for  $F > 0$ , and  $K_F$  is smooth down to  $x = 0$  for  $(X, Y)$  finite, non-zero, conormal to  $(X, Y) = 0$ . This will imply that  $A_F$  is a scattering pseudodifferential operator.

One actually can use

$$x, y, |y' - y|, \frac{x' - x}{|y' - y|}, \frac{y' - y}{|y' - y|}$$

as the local coordinates on  $\Gamma_+(\text{supp } \chi \times [0, \delta_0])$ ; and analogously for  $\Gamma_-(\text{supp } \chi \times (-\delta_0, 0])$

the coordinates are

$$x, y, -|y' - y|, -\frac{x' - x}{|y' - y|}, -\frac{y' - y}{|y' - y|}.$$

Indeed, this corresponds to the fact that we are using  $(x, y, \lambda, \omega)$  with  $\omega \in \mathbb{S}^{n-2}$ , instead of  $S\widetilde{M}$ , to parameterize curves, when  $|y' - y|$  is large relative to  $x' - x$ , i.e. in our region of interest. Now, under the scattering coordinates,

$$|y' - y| = x|Y|, \quad \frac{x' - x}{|y' - y|} = \frac{xX}{|Y|}, \quad \frac{y' - y}{|y' - y|} = \hat{Y}.$$

Thus we get following expansions in terms of  $x$ ,

$$\begin{aligned} \lambda(\Gamma_+^{-1}) &= x \frac{X}{|Y|} + x|Y| \tilde{\Lambda}(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}), \\ \lambda(\Gamma_-^{-1}) &= -x \frac{X}{|Y|} - x|Y| \tilde{\Lambda}(x, y, -x|Y|, -\frac{xX}{|Y|}, -\hat{Y}). \end{aligned} \quad (2.7)$$

Similarly,

$$\begin{aligned} \omega(\Gamma_+^{-1}) &= \hat{Y} + x|Y| \tilde{\Omega}(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}), \\ \omega(\Gamma_-^{-1}) &= -\hat{Y} - x|Y| \tilde{\Omega}(x, y, -x|Y|, -\frac{xX}{|Y|}, -\hat{Y}) \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} t(\Gamma_+^{-1}) &= x|Y| + x^2|Y|^2 \tilde{T}(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}), \\ t(\Gamma_-^{-1}) &= -x|Y| - x^2|Y|^2 \tilde{T}(x, y, -x|Y|, -\frac{xX}{|Y|}, -\hat{Y}). \end{aligned} \quad (2.9)$$

Here  $\tilde{\Lambda}$ ,  $\tilde{\Omega}$  and  $\tilde{T}$  are smooth in terms of their arguments. Thus,

$$\begin{aligned} dt d\lambda d\omega(\Gamma_{\pm}^{-1}) &= J(x, y, \pm|Y|, \pm \frac{X}{|Y|}, \pm \hat{Y}) x^2 |Y|^{-1} dX d|Y| d\hat{Y} \\ &= J(x, y, \pm|Y|, \pm \frac{X}{|Y|}, \pm \hat{Y}) x^2 |Y|^{-n+1} dX dY \end{aligned} \quad (2.10)$$

where the density factor  $J$  is smooth and positive, and  $J|_{x=0} = 1$ .

We write

$$x' = x + \lambda t + \alpha(x, y, \lambda, \omega) t^2 + O(t^3), \quad y' = y + \omega t + O(t^2), \quad (2.11)$$

where  $O(t^3)$  and  $O(t^2)$  have coefficients which are smooth in  $(x, y, \lambda, \omega)$ . Correspondingly

$$\lambda' = \frac{dx'}{dt} = \lambda + 2\alpha(x, y, \lambda, \omega)t + O(t^2), \quad \omega' = \frac{dy'}{dt} = \omega + O(t).$$

Thus,

$$X = \frac{x' - x}{x^2} = \frac{\lambda t}{x^2} + \frac{\alpha t^2}{x^2} + \frac{t^3}{x^2} \Upsilon(x, y, \lambda, \omega, t),$$

with  $\Upsilon$  a smooth function of its arguments, so by (2.9)

$$X = \frac{\lambda(\Gamma_+^{-1})}{x} |Y| (1 + x|Y|\tilde{T}) + \alpha(\Gamma_+^{-1}) |Y|^2 (1 + x|Y|\tilde{T})^2 + x|Y|^3 \Upsilon(\Gamma_+^{-1}),$$

also

$$X = \frac{\lambda(\Gamma_-^{-1})}{x} |Y| (-1 - x|Y|\tilde{T}) + \alpha(\Gamma_-^{-1}) |Y|^2 (-1 - x|Y|\tilde{T})^2 - x|Y|^3 \Upsilon(\Gamma_-^{-1}).$$

Thus

$$\begin{aligned} \frac{\lambda(\Gamma_+^{-1})}{x} &= \frac{X - \alpha(\Gamma_+^{-1})|Y|^2}{|Y|} + O(x), \\ \frac{\lambda(\Gamma_-^{-1})}{x} &= \frac{-X + \alpha(\Gamma_-^{-1})|Y|^2}{|Y|} + O'(x), \end{aligned} \tag{2.12}$$

where  $O(x)$  and  $O'(x)$  have smooth coefficients in terms of  $x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}$ . The weight  $w(x', y', \lambda', \omega')$  is defined on  $TM \setminus 0$ , so we need to express the arguments in terms of  $x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}$ , by (2.12), (2.8) and (2.9)

$$x'(\Gamma_{\pm}^{-1}) = x + O_{\pm}(x^2), \quad y'(\Gamma_{\pm}^{-1}) = y + O'_{\pm}(x); \tag{2.13}$$

and

$$\begin{aligned} \lambda'(\Gamma_+^{-1}) &= x \frac{X - \alpha(\Gamma_+^{-1})|Y|^2}{|Y|} + 2x|Y|\alpha(\Gamma_+^{-1}) + O_+(x^2), \quad \omega'(\Gamma_+^{-1}) = \hat{Y} + O'_+(x); \\ \lambda'(\Gamma_-^{-1}) &= x \frac{-X + \alpha(\Gamma_-^{-1})|Y|^2}{|Y|} - 2x|Y|\alpha(\Gamma_-^{-1}) + O_-(x^2), \quad \omega'(\Gamma_-^{-1}) = \hat{Y} + O'_-(x). \end{aligned} \tag{2.14}$$

All the remainders in (2.13) and (2.14) have smooth coefficients in terms of  $x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}$ . Note that on the blow-up of the scattering diagonal,  $\{X = 0, Y = 0\}$ , in the region  $|Y| > \epsilon|X|$ , thus on the support of  $\chi$  in view of (2.7),

$$(x, y, |Y|, \frac{X}{|Y|}, \hat{Y}) \text{ and } (x, y, -|Y|, -\frac{X}{|Y|}, -\hat{Y})$$

are valid coordinates. We deduce that the kernel  $K_F$  is given by

$$e^{-FX/(1+xX)}|Y|^{-n+1} \left\{ w(x'(\Gamma_+^{-1}), y'(\Gamma_+^{-1}), \lambda'(\Gamma_+^{-1}), \omega'(\Gamma_+^{-1})) \chi\left(\frac{\lambda(\Gamma_+^{-1})}{x}\right) J(x, y, |Y|, \frac{X}{|Y|}, \hat{Y}) \right. \\ \left. + w(x'(\Gamma_-^{-1}), y'(\Gamma_-^{-1}), \lambda'(\Gamma_-^{-1}), \omega'(\Gamma_-^{-1})) \chi\left(\frac{\lambda(\Gamma_-^{-1})}{x}\right) J(x, y, -|Y|, -\frac{X}{|Y|}, -\hat{Y}) \right\}, \quad (2.15)$$

so in particular it is conormal to  $Y = 0$ , the front face on the blow-up of the scattering diagonal, of the form  $\rho^{-n+1}b$ , where  $b$  is smooth up to the front face, and without the first exponential factor it, together with its derivatives has polynomial growth as  $(X, Y) \rightarrow \infty$ . We decompose  $K_F$  into two pieces supported in  $|(X, Y)| < 2$  and  $|(X, Y)| > 1$  by a partition of unity. For the first term, supported in  $|(X, Y)| < 2$ , similar calculations as in (2.4) shows that this term is indeed the Schwartz kernel of an element of  $\Psi_{sc}^{-1,0}$ .

About the second term supported in  $|(X, Y)| > 1$ , notice that for  $\frac{\lambda_{\pm}}{x} \in (-c, c)$ ,  $-c|Y| < X - \alpha(\Gamma_{\pm}^{-1})|Y|^2 < c|Y|$ , this implies (by the positivity of  $\alpha$ ) that  $X \rightarrow +\infty$  on  $\text{supp}\chi$  as  $|Y| \rightarrow \infty$ , in particular  $X > C|Y|^2$  for some  $C > 0$  for  $|Y|$  large enough. Now for all  $N$  the exponential factor in (2.15) is  $\leq C|(X, Y)|^{-N}$  for suitable  $C$  on the support of  $\chi$ , it follows that  $K_F$  is smooth in  $(x, y)$ , Schwartz in  $(X, Y)$  for  $(X, Y) \neq 0$ , and conormal to  $(X, Y) = 0$ , which is exactly the characterization of the Schwartz kernel of an element in  $\Psi_{sc}^{-1,0}$ . This finishes the proof of the Proposition.  $\square$

Here the additional information is the behavior of  $A_F$  at  $x = 0$ , the information is encoded in the properties of its boundary principal symbol. The restriction of (2.15) at  $x = 0$  (the scattering front face) gives the Schwartz kernel  $\tilde{K}(y, X, Y)$  of  $A_F$ . We allow that  $\chi$  also depends on  $y$  and  $\omega$ , thus we have the following Lemma:

**Lemma 2.4.2.** *The boundary principal symbol of  $A_F = x^{-2}e^{-F/x}Ae^{F/x}$  is the  $(X, Y)$ -Fourier transform of*

$$\tilde{K}(y, X, Y) = e^{-FX}|Y|^{-n+1} \left\{ w(0, y, 0, \hat{Y}) \chi\left(\frac{X - \alpha(0, y, 0, \hat{Y})|Y|^2}{|Y|}\right), y, \hat{Y} \right. \\ \left. + w(0, y, 0, -\hat{Y}) \chi\left(\frac{-X + \alpha(0, y, 0, -\hat{Y})|Y|^2}{|Y|}\right), y, -\hat{Y} \right\}. \quad (2.16)$$

So the desired invertibility of  $A_F$  amounts to the Fourier transform of the kernel,  $\tilde{K}(y, \cdot, \cdot)$  being bounded below in absolute value by  $c\langle(\xi, \eta)\rangle^{-1}$ ,  $c > 0$  (here  $(\xi, \eta)$  are the Fourier dual variables of  $(X, Y)$ ). Thus our job now is to find a proper  $\chi$  such that the Fourier transform of  $\tilde{K}(y, \cdot, \cdot)$  is elliptic.

**Proposition 2.4.3.** *For  $F > 0$  there exists  $\chi \in C_c^\infty(\mathbb{R})$ ,  $\chi \geq 0$ ,  $\chi(0) = 1$ , such that for the corresponding operator  $A_F = x^{-2}e^{-F/x}Ae^{F/x}$  the boundary principal symbol is elliptic.*

*Proof.* In order to find a suitable  $\chi$  to make  $A_F$  invertible, we take use of the strategy of [50], namely we first do calculation for  $\chi(s) = e^{-s^2/(2F^{-1}\alpha)}$  with  $F > 0$  ( here we need the positivity of  $\alpha$ ), so  $\hat{\chi}(\cdot) = c\sqrt{F^{-1}\alpha}e^{-F^{-1}\alpha|\cdot|^2/2}$  for appropriate  $c > 0$ . As the first step, the Fourier transform of  $\tilde{K}$  in  $X$ ,  $\mathcal{F}_X\tilde{K}(y, \xi, Y)$ , is a non-zero multiple of

$$|Y|^{2-n} \left\{ w_+(F^{-1}\alpha_+)^{\frac{1}{2}} e^{-F^{-1}(\xi^2+F^2)\alpha_+|Y|^2/2} + w_-(F^{-1}\alpha_-)^{\frac{1}{2}} e^{-F^{-1}(\xi^2+F^2)\alpha_-|Y|^2/2} \right\}, \quad (2.17)$$

where  $w_\pm = w(0, y, 0, \pm\hat{Y})$ ,  $\alpha_\pm = \alpha(0, y, 0, \pm\hat{Y})$ .

Now we estimate the Fourier transform of (2.17) in  $Y$  upto some non-zero multiple. As remarked previously, not like the geodesic ray transform in [50], generally our  $\alpha$  may contain terms other than a quadratic form in  $\hat{Y}$ , which means the exponential term of  $\mathcal{F}_X\tilde{K}(y, \xi, Y)$  is not Gaussian like in  $Y$ , thus we use polar coordinates to compute the Fourier transform in  $Y$ . We denote  $\frac{F^{-1}(\xi^2+F^2)}{2}$  by  $b$ , then

$$\begin{aligned} \mathcal{F}_Y(\mathcal{F}_X\tilde{K})(y, \xi, \eta) &\simeq \int e^{-i\eta\cdot Y} |Y|^{2-n} (w_+\alpha_+^{1/2} e^{-b\alpha_+|Y|^2} + w_-\alpha_-^{1/2} e^{-b\alpha_-|Y|^2}) dY \\ &= \int_0^{+\infty} \int_{\mathbb{S}^{n-2}} e^{-i\eta\cdot\hat{Y}|Y|} |Y|^{2-n} (w_+\alpha_+^{1/2} e^{-b\alpha_+|Y|^2} + w_-\alpha_-^{1/2} e^{-b\alpha_-|Y|^2}) |Y|^{n-2} d|Y| d\hat{Y} \\ &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} e^{-i\eta\cdot\hat{Y}t} (w_+\alpha_+^{1/2} e^{-b\alpha_+t^2} + w_-\alpha_-^{1/2} e^{-b\alpha_-t^2}) dt d\hat{Y} \\ &\simeq \frac{1}{2} \int_{\mathbb{S}^{n-2}} b^{-1/2} (w_+ e^{-|\eta\cdot\hat{Y}|^2/4b\alpha_+} + w_- e^{-|\eta\cdot\hat{Y}|^2/4b\alpha_-}) d\hat{Y} \\ &= b^{-1/2} \int_{\mathbb{S}^{n-2}} w(0, y, 0, \hat{Y}) e^{-|\eta\cdot\hat{Y}|^2/4b\alpha(0, y, 0, \hat{Y})} d\hat{Y}, \end{aligned}$$

here  $\simeq$  means equal up to some multiple.

To estimate the Fourier transform of (2.16), we need to study the joint  $(\xi, \eta)$ -behavior of  $\mathcal{F}_{X,Y}\tilde{K}(y, \xi, \eta)$ , i.e. when  $\langle(\xi, \eta)\rangle$  is going to infinity, where we need lower bounds. We denote  $(\xi^2 + F^2)^{1/2}$  by  $\langle\xi\rangle$ , then  $\mathcal{F}_{X,Y}\tilde{K}(y, \xi, \eta)$  is a constant multiple of

$$\langle\xi\rangle^{-1} \int_{\mathbb{S}^{n-2}} w(0, y, 0, \hat{Y}) e^{-|\frac{\eta}{\langle\xi\rangle} \cdot \hat{Y}|^2 / 4c\alpha(0, y, 0, \hat{Y})} d\hat{Y}$$

with  $c = c(F) > 0$ . Under our assumption  $\alpha(0, y, 0, \hat{Y}) > 0$ , thus there exist positive  $c_1, c_2$  that depend on  $y$  and are locally uniform such that  $0 < c_1 \leq \alpha \leq c_2$ . Since the weight  $w$  is positive too, similar bounds hold for  $w$ , i.e.  $0 < c'_1(y) \leq w(0, y, 0, \hat{Y}) \leq c'_2(y)$ .

When  $\frac{|\eta|}{\langle\xi\rangle}$  is bounded from above, then  $\langle\xi\rangle^{-1}$  is equivalent to  $\langle(\xi, \eta)\rangle^{-1}$  in this region in terms of decay rates,

$$\int_{\mathbb{S}^{n-2}} w(0, y, 0, \hat{Y}) e^{-|\frac{\eta}{\langle\xi\rangle} \cdot \hat{Y}|^2 / 4c\alpha(0, y, 0, \hat{Y})} d\hat{Y} \geq C' \int_{\mathbb{S}^{n-2}} e^{-c'|\frac{\eta}{\langle\xi\rangle}|^2} d\hat{Y} \geq C' \int_{\mathbb{S}^{n-2}} e^{-C''} d\hat{Y} = C.$$

Thus  $\mathcal{F}_{X,Y}\tilde{K}(y, \xi, \eta) \geq C\langle\xi\rangle^{-1} \simeq C\langle(\xi, \eta)\rangle^{-1}$ .

When  $\frac{|\eta|}{\langle\xi\rangle}$  is bounded from below, in which case  $\langle(\xi, \eta)\rangle^{-1}$  is equivalent to  $|\eta|^{-1}$  in terms of decay rates, we write  $\hat{Y} = (\hat{Y}^\parallel, \hat{Y}^\perp)$  according to the orthogonal decomposition of  $\hat{Y}$  relative to  $\frac{\eta}{|\eta|}$ , where  $\hat{Y}^\parallel = \hat{Y} \cdot \frac{\eta}{|\eta|}$ , and  $d\hat{Y}$  is of the form  $a(\hat{Y}^\parallel) d\hat{Y}^\parallel d\theta$  with  $\theta = \frac{\hat{Y}^\perp}{|\hat{Y}^\perp|}$ ,  $a(0) = 1$  then

$$\begin{aligned} \int_{\mathbb{S}^{n-2}} w(0, y, 0, \hat{Y}) e^{-|\frac{\eta}{\langle\xi\rangle} \cdot \hat{Y}|^2 / 4c\alpha(0, y, 0, \hat{Y})} d\hat{Y} &\geq C' \int_{\mathbb{S}^{n-2}} e^{-c'|\frac{\eta}{\langle\xi\rangle} \cdot \hat{Y}|^2} d\hat{Y} \\ &= C' \int_{\mathbb{R}} \int_{\mathbb{S}^{n-3}} e^{-c'(\hat{Y}^\parallel \frac{|\eta|}{\langle\xi\rangle})^2} a(\hat{Y}^\parallel) d\theta d\hat{Y}^\parallel \\ &= C' \frac{\langle\xi\rangle}{|\eta|} \int_{\mathbb{R}} \left\{ \frac{|\eta|}{\langle\xi\rangle} e^{-c'(\hat{Y}^\parallel \frac{|\eta|}{\langle\xi\rangle})^2} \right\} a(\hat{Y}^\parallel) d\hat{Y}^\parallel \int_{\mathbb{S}^{n-3}} d\theta. \end{aligned} \quad (2.18)$$

Since  $\frac{|\eta|}{\langle\xi\rangle} e^{-c'(\hat{Y}^\parallel \frac{|\eta|}{\langle\xi\rangle})^2} \rightarrow \delta_0$  in distributions as  $\frac{|\eta|}{\langle\xi\rangle} \rightarrow \infty$ , (2.18) is equal to  $C' \frac{\langle\xi\rangle}{|\eta|} \int_{\mathbb{S}^{n-3}} d\theta = 2C \frac{\langle\xi\rangle}{|\eta|}$  ( $C > 0$ ) modulo terms decaying faster as  $\frac{|\eta|}{\langle\xi\rangle} \rightarrow \infty$ . In particular, there is  $N > 0$ , such that

$$\int_{\mathbb{R}} \left\{ \frac{|\eta|}{\langle\xi\rangle} e^{-c'(\hat{Y}^\parallel \frac{|\eta|}{\langle\xi\rangle})^2} \right\} a(\hat{Y}^\parallel) d\hat{Y}^\parallel \int_{\mathbb{S}^{n-3}} d\theta \geq C$$

for  $\frac{|\eta|}{\langle\xi\rangle} \geq N$ . (Notice that the integral on  $\mathbb{S}^{n-3}$  uses very strongly the assumption  $n \geq 3$ ; when  $n = 3$ ,  $d\theta$  is the point measure) Thus  $\mathcal{F}_{X,Y}\tilde{K}(y, \xi, \eta) \geq C \frac{1}{\langle\xi\rangle} \frac{\langle\xi\rangle}{|\eta|} = C|\eta|^{-1} \simeq C\langle(\xi, \eta)\rangle^{-1}$ .

Therefore we deduce that  $\mathcal{F}_{X,Y}\tilde{K}(y, \xi, \eta) \geq c\langle(\xi, \eta)\rangle^{-1}$  for some  $c > 0$ , i.e the ellipticity claim for the case that  $\chi$  is a Gaussian. Now we pick a sequence  $\chi_n \in C_c^\infty(\mathbb{R})$  which converges to the Gaussian in Schwartz functions, then the Fourier transforms  $\hat{\chi}_n$  converge to  $\hat{\chi}$ . One concludes that for some large enough  $n$ , if we use  $\chi_n$  to define the operator  $A$ , then the Fourier transform of  $\tilde{K}$ , i.e. the boundary principal symbol, still has lower bounds  $c'\langle(\xi, \eta)\rangle^{-1}$ ,  $c' > 0$ , as desired.  $\square$

*Proof of Theorem 2.1.2:* By Proposition 2.4.1 and Proposition 2.4.3, we have that

$$A_F = x^{-2}e^{-F/x}Ae^{F/x} \in \Psi_{sc}^{-1,0}$$

is elliptic both in the sense of the standard principal symbol (in the set of interest which is a neighborhood of  $O_c$  in  $M_c = \{\tilde{x} > -c\}$  for  $c$  small), and the scattering principal symbol, which is at  $\tilde{x} = -c$ . In particular the estimates of the last paragraph of Section 2.3 on elliptic scattering pseudodifferential operators are applicable. Thus, if we denote  $K = \{\tilde{x} \geq -c\} \cap \{\rho \geq 0\}$ ,  $H_{sc}^{s,r}(\overline{M}_c)_K = \{f \in H_{sc}^{s,r}(\overline{M}_c) : \text{supp } f \subseteq K\}$ ,

$$A = x^2e^{F/x}A_Fe^{-F/x} : e^{F/x}H_{sc}^{s,r}(\overline{M}_c)_K \rightarrow e^{F/x}H_{sc}^{s+1,r+2}(\overline{M}_c)$$

satisfies the estimates

$$\|f\|_{e^{F/x}H_{sc}^{s,r}(\overline{M}_c)_K} \leq C\|Af\|_{e^{F/x}H_{sc}^{s+1,r+2}(\overline{M}_c)}.$$

Finally, we apply an argument similar to the very end of [50]. Recall the facts from Section 2.3, namely for  $s \geq 0$ ,  $H^s(\overline{M}_c) \subset H_{sc}^{s,r}(\overline{M}_c)$  for  $r \leq -\frac{n+1}{2}$  while for  $r \geq -\frac{n+1}{2} + 2s$ ,  $H_{sc}^{s,r}(\overline{M}_c) \subset H^s(\overline{M}_c)$ , with continuous inclusion maps. Moreover, one can remove the polynomial weights of the scattering Sobolev spaces by adding an exponential weight  $e^{\delta/x}$  in the front, for some  $\delta > 0$ . Thus we have

$$\|f\|_{e^{(F+\delta)/x}H^s(\overline{M}_c)_K} \leq C\|Af\|_{e^{F/x}H^{s+1}(\overline{M}_c)} \quad (2.19)$$

Notice the decomposition of  $A$  in (2.3), and the boundedness of  $L$ , we have for all  $F > 0$ ,

$$\|Af\|_{e^{F/x}H^{s+1}(\overline{M}_c)} \leq C'\|Lwf\|_{H^{s+1}(\mathcal{G}_{\overline{M}_c})} \quad (2.20)$$

for  $f \in H^{s+1}(\overline{M}_c)_K$ . We thus deduce for  $s \geq 0, \delta > 0$

$$\|f\|_{e^{(F+\delta)/x}H^s(\overline{M}_c)_K} \leq C'' \|I_w f\|_{H^{s+1}(\mathcal{G}_{\overline{M}_c})}.$$

Since (2.20) is valid for  $s+1 \geq 0$ , i.e.  $s \geq -1$ , further study shows indeed above inequality is true for  $s \geq -1$ , by the inclusions  $H^{s+1}(\overline{M}_c) \hookrightarrow e^{F/x}H_{sc}^{s+1,r}(\overline{M}_c) \hookrightarrow e^{F/x}H_{sc}^{s,r}(\overline{M}_c)$  for  $s \geq -1$  and  $e^{F/x}H_{sc}^{s,c}(\overline{M}_c) \hookrightarrow e^{(F+\delta)/x}H^s(\overline{M}_c)$  for  $-1 \leq s \leq 0$  (similar to the arguments for inequality (2.19)). With  $F+\delta$  replaced by  $F$  (since both  $F > 0$  and  $\delta > 0$  are arbitrary), this completes the proof of the main theorem. Thus we obtain the local injectivity and stability estimates for the weighted X-ray transform along a general family of curves.  $\square$

## 2.5 Applications

As an immediate consequence and application of our main theorem, we consider the global weighted X-ray transform for  $\mathcal{G}$ . Assume  $M$  has compact closure equipped with a boundary defining function  $\rho : \overline{M} \rightarrow [0, \infty)$  whose level sets  $\rho^{-1}(t), 0 \leq t < T$  for some  $T > 0$ , are strictly convex w.r.t.  $\mathcal{G}$  (viewed from  $\rho^{-1}((t, \infty))$ ), and  $d\rho$  is non-zero on these level sets.  $w$  is a smooth positive function on  $TM \setminus 0$ . We obtain the following corollary on the global injectivity of the weighted X-ray transform along general curves, which is an analog of the geodesic case in [50].

**Corollary 2.5.1.** *For  $M, \rho$  and  $w$  defined as above, if the set  $\rho^{-1}([T, \infty))$  has 0 measure, the global weighted X-ray transform for  $\mathcal{G}$  is injective on  $L^2(M)$ , while if  $\rho^{-1}([T, \infty))$  has empty interior, the global weighted X-ray transform for  $\mathcal{G}$  is injective on  $H^s(M), s > n/2$ .*

The proof of the corollary is similar to that in [50], for the completeness, we include it here.

*Proof.* Assume  $I_w f = 0$  and  $f \in H^s, s > n/2, f \neq 0$ , by Sobolev embedding theorem  $f$  is continuous, then  $\text{supp } f$  has non-empty interior; while if  $f \in L^2, f \neq 0$ , then  $\text{supp } f$  has non-zero measure. Let  $\tau = \inf_{\text{supp } f} \rho$ , if  $\tau \geq T$ , then  $\text{supp } f \subset M \setminus \cup_{t \in [0, T)} \rho^{-1}(t)$  and we are done. Thus we assume  $\tau < T$ , so  $f = 0$  on  $\rho^{-1}(t)$  for  $t < \tau$ , but there exists  $q \in \rho^{-1}(\tau) \cap \text{supp } f$

(since  $\text{supp } f$  is closed and  $\overline{M}$  is compact). By applying the main theorem on  $\rho^{-1}(\tau, \infty)$  one concludes that  $f$  is constantly zero in a neighborhood of  $q$  which is a contradiction.  $\square$

We point out that one can derive global reconstruction methods and stability estimates by a layer stripping algorithm as that in [50].

It is also worthy to point out that Corollary 2.5.1 provides a new approach to the uniqueness of the global problem for the X-ray transform along general curves. The only method up to now, except in the real analytic category [12], [45], has been the use of energy type equalities introduced by Mukhometov [29] which are now called ‘‘Pestov identities’’. The global geometric condition imposed here is a natural analog of the condition  $\frac{d}{dr}(r/c(r)) > 0$  proposed by Herglotz [16] and Wiechert and Zoeppritz [52] for an isotropic radial sound speed  $c(r)$ , and the condition of being foliated by strictly convex hypersurfaces in [50].

Finally we make a remark about X-ray transforms along curves with non-constant speed. As already mentioned in the beginning, the whole machinery exhibited in Section 2 and 4 works for curves with non-constant speed by a minor modification. However, one can always convert the inversion problem for curves with non-constant speed to the corresponding problem with constant speed (say of unit speed), and vice versa.

If  $\gamma(t) = \gamma_{z,v}(t)$  is a smooth curve of non-constant speed,  $|\dot{\gamma}(t)| \neq 0$ , with  $z = \gamma(0)$ ,  $v = \dot{\gamma}(0)$ . We define an increasing function  $s : \mathbb{R} \rightarrow \mathbb{R}$  by

$$s(t) = \int_0^t |\dot{\gamma}(\sigma)| d\sigma, \quad t \in \mathbb{R},$$

which satisfies  $s(0) = 0$  and  $\frac{ds}{dt} = |\dot{\gamma}(t)| \neq 0$ . Thus there exists an inverse function  $t(s)$  with  $\frac{dt}{ds} = |\dot{\gamma}(t(s))|^{-1}$ . Now we reparametrize the curve  $\gamma$  by defining  $\tilde{\gamma}(s) = \gamma(t(s))$  with

$$|\dot{\tilde{\gamma}}(s)| = |\dot{\gamma}(t(s)) \cdot \frac{dt}{ds}| = |\dot{\gamma}(t(s))| \cdot |\dot{\gamma}(t(s))|^{-1} = 1,$$

i.e.  $\tilde{\gamma}$  represents the same curve (unparameterized) as  $\gamma$ , but parameterized by arclength. Given a smooth positive weight  $w$ , the weighted X-ray transform of a smooth function  $f$  along  $\gamma$  is

$$(I_w f)(\gamma) = \int_{\mathbb{R}} w(\gamma(t), \dot{\gamma}(t)) f(\gamma(t)) dt.$$

After the reparametrization, we obtain

$$\int_{\mathbb{R}} w(\gamma(t), \dot{\gamma}(t)) f(\gamma(t)) dt = \int_{\mathbb{R}} w(\tilde{\gamma}(s), |\dot{\gamma}(t(s))| \dot{\tilde{\gamma}}(s)) f(\tilde{\gamma}(s)) |\dot{\gamma}(t(s))|^{-1} ds := (I_{\tilde{w}} f)(\tilde{\gamma}),$$

where  $\tilde{w}(\tilde{\gamma}, \dot{\tilde{\gamma}}) := w(\gamma, \dot{\gamma}) |\dot{\gamma}|^{-1}$  is a new positive weight ( $\tilde{w}$  is first defined on  $SM$ , then one can extend it onto  $TM \setminus 0$  smoothly). In particular, if  $\mathcal{G} = \{\gamma\}$  is a smooth manifold, then  $\tilde{\mathcal{G}} = \{\tilde{\gamma}\}$  is a smooth manifold too whose elements are smooth curves with unit speed, thus  $\tilde{w}$  is a smooth weight on  $TM \setminus 0$ . Above analysis shows that we can obtain the information of weighted X-ray transform  $I_{\tilde{w}} f$  on  $\tilde{\mathcal{G}}$  from the information of  $I_w f$  on  $\mathcal{G}$ . Once we achieve the injectivity of  $I_{\tilde{w}}$ , then injectivity of  $I_w$  follows. By reversing above argument, injectivity of  $I_w$  implies the injectivity of  $I_{\tilde{w}}$  too.

## Chapter 3

# BOUNDARY RIGIDITY IN THE PRESENCE OF MAGNETIC FIELDS AND POTENTIALS

Typical boundary rigidity problem is related to geodesic flows  $\phi = (\gamma, \dot{\gamma})$  on the unit sphere bundle, where  $\gamma$  are geodesics with unit speed. It is natural to consider inverse problems associated with more general dynamical flows. Several boundary rigidity results for geodesic case were generalized by Dairbekov, Paternain, Stefanov and Uhlmann [8] to magnetic flows. In current chapter, we study the boundary rigidity problem for Hamiltonian flows involving both magnetic fields and potentials.

### 3.1 Introduction

Given a compact Riemannian manifold  $(M, g)$  of dimension  $n \geq 2$  with boundary, endowed with a magnetic field  $\Omega$ , that is a closed 2-form, we consider the law of motion described by the Newton's equation

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Y(\dot{\gamma}) - \nabla U(\gamma), \quad (3.1)$$

where  $U$  (the potential) is a smooth function on  $M$ ,  $\nabla$  is the Levi-Civita connection of  $g$  and  $Y : TM \rightarrow TM$  is the *Lorentz force* associated with  $\Omega$ , i.e., the bundle map uniquely determined by

$$\Omega_x(\xi, \eta) = \langle Y_x(\xi), \eta \rangle \quad (3.2)$$

for all  $x \in M$  and  $\xi, \eta \in T_x M$ . A curve  $\gamma : [a, b] \rightarrow M$ , satisfying (3.1) is called an *MP-geodesic*. The equation (3.1) defines a flow  $\phi_t$  on  $TM$  that we call an *MP-flow*. These are not standard terms in general. Note that time is not reversible on the *MP-geodesics*, unless  $\Omega = 0$ .

When  $\Omega = 0$  the flow is called a *potential flow*; while if  $U = 0$  we obtain a *magnetic flow*. Therefore, the equation (3.1) describes the motion of a particle on a Riemannian manifold under the influence of a magnetic field  $\Omega$  in a potential field  $U$ .  $\mathcal{MP}$ -flows are related to dynamical systems, symplectic geometry, classical mechanics and mathematical mechanics.

When  $\Omega$  is exact, i.e.  $\Omega = d\alpha$  for some magnetic potential  $\alpha$ , the  $\mathcal{MP}$ -flow also arises as the Hamiltonian flow of  $H(x, p) = \frac{1}{2}|p + \alpha|_{g(x)}^2 + U(x)$  with respect to the canonical symplectic form of  $T^*M$ .

Defining the energy at  $(x, \xi) \in TM$  by

$$E(x, \xi) = \frac{1}{2}|\xi|^2 + U(x),$$

by the Law of Conservation of Energy, the energy  $E$  is constant along  $\mathcal{MP}$ -flows. Unlike the geodesic flow, where the flow is the same (up to time scale) on any energy levels,  $\mathcal{MP}$ -flow depends essentially on the choice of the energy level. Throughout the paper we assume the energy level  $k > \sup_{x \in M} U(x)$  with  $S^k M = E^{-1}(k)$ , *the bundle of energy  $k$* . Note that it is necessary for  $k$  to be strictly greater than the supremum of  $U$ , otherwise we would get that at some  $x \in M$  every vector  $\xi \in S_x^k M$  has non-positive length.

We will define some action  $\mathbb{A}(x, y)$  between boundary points as a minimizer of the appropriate action functional, see (3.4) and Appendix A.1 for details. In the case  $\Omega = 0$  and  $U = 0$ , the function  $\mathbb{A}(x, y)$  coincides with the boundary distance function  $d_g(x, y)$ . In this case, we cannot expect to recover  $g$  from  $d_g$  up to isometry, unless some additional assumptions are imposed on  $g$ , see, e.g., [5]. One such assumption is the simplicity of the metric, see, e.g., [28, 37, 42, 43]. We consider below the analog of simplicity for  $\mathcal{MP}$ -systems.

**Definition 3.1.1.** *Let  $\Lambda$  denote the second fundamental form of  $\partial M$ , and  $\nu(x)$  the inward unit vector normal to  $\partial M$  at  $x$ . We say that  $\partial M$  is strictly  $\mathcal{MP}$ -convex if*

$$\Lambda(x, \xi) > \langle Y_x(\xi), \nu(x) \rangle - d_x U(\nu(x)) \tag{3.3}$$

for all  $(x, \xi) \in S^k(\partial M)$ .

For  $x \in M$ , we define the  $\mathcal{MP}$ -exponential map at  $x$  to be the partial map  $\exp_x^{\mathcal{MP}} : T_x M \rightarrow M$  given by

$$\exp_x^{\mathcal{MP}}(t\xi) = \pi \circ \phi_t(\xi), \quad t \geq 0, \quad \xi \in S_x^k M,$$

where  $\pi : TM \rightarrow M$  is the canonical projection. It is not hard to show that, for every  $x \in M$ ,  $\exp_x^{\mathcal{MP}}$  is a  $C^1$ -smooth partial map on  $T_x M$  which is  $C^\infty$ -smooth on  $T_x M \setminus \{0\}$ .

**Definition 3.1.2.** *A compact manifold  $M$  with boundary is simple w.r.t.  $(g, \Omega, U)$  if  $\partial M$  is strictly  $\mathcal{MP}$ -convex and the  $\mathcal{MP}$ -exponential map  $\exp_x^{\mathcal{MP}} : (\exp_x^{\mathcal{MP}})^{-1}(M) \rightarrow M$  is a diffeomorphism for every  $x \in M$ .*

In this case,  $M$  is diffeomorphic to the unit ball of  $\mathbb{R}^n$ . Thus  $\Omega$  is exact, and there is a 1-form  $\alpha$  on  $M$  such that

$$d\alpha = \Omega,$$

$\alpha$  is called a magnetic potential. Henceforth we call  $(g, \alpha, U)$  a *simple  $\mathcal{MP}$ -system* on  $M$ . We will also say that  $(M, g, \alpha, U)$  is a simple  $\mathcal{MP}$ -system. It is easy to see that the simplicity is stable under a small perturbation of the energy level.

Now we state the boundary rigidity problem. Given  $x, y \in M$ , let

$$\mathcal{C}(x, y) = \{\gamma : [0, T] \rightarrow M : T > 0, \gamma(0) = x, \gamma(T) = y, \gamma \text{ is absolutely continuous}\}.$$

The *time free action* of a curve  $\gamma \in \mathcal{C}(x, y)$  w.r.t.  $(g, \alpha, U)$  is defined as

$$\mathbb{A}(\gamma) = \frac{1}{2} \int_0^T |\dot{\gamma}(t)|^2 dt + kT - \int_\gamma (\alpha + U).$$

For a simple  $\mathcal{MP}$ -system,  $\mathcal{MP}$ -geodesics with energy  $k$  minimize the time free action (see Appendix A.1)

$$\mathbb{A}(x, y) := \inf_{\gamma \in \mathcal{C}(x, y)} \mathbb{A}(\gamma) = 2kT_{x, y} - \int_{\gamma_{x, y}} (\alpha + 2U), \quad (3.4)$$

where  $\gamma_{x, y} : [0, T_{x, y}] \rightarrow M$  is the unique  $\mathcal{MP}$ -geodesic with constant energy  $k$  from  $x$  to  $y$ . The function  $\mathbb{A}(x, y)$  is referred to as Mañé's action potential (of energy  $k$ ), and we call the restriction  $\mathbb{A}|_{\partial M \times \partial M}$  the *boundary action function*.

We say that two  $\mathcal{MP}$ -systems  $(g, \alpha, U)$  and  $(g', \alpha', U')$  are *gauge equivalent* if there is a diffeomorphism  $f : M \rightarrow M$ , which is the identity on the boundary, and a smooth function  $\varphi : M \rightarrow \mathbb{R}$ , vanishing on the boundary, such that  $g' = f^*g$ ,  $\alpha' = f^*\alpha + d\varphi$  and  $U' = U \circ f$ . Observe that given two gauge equivalent  $\mathcal{MP}$ -systems, if one is simple, then the other one is also simple. Moreover, if two simple  $\mathcal{MP}$ -systems are gauge equivalent, then they have the same boundary action function.

The *boundary rigidity problem in the presence of a magnetic field and a potential* studies that to which extend an  $\mathcal{MP}$ -system  $(g, \alpha, U)$  on  $M$  is determined by the boundary action functions. By the above observation, one can only expect to obtain the uniqueness up to gauge equivalence. For the zero potential, i.e.  $U = 0$ , we obtain the boundary rigidity problem for the magnetic systems that was considered by N. Dairbekov, G. Paternain, P. Stefanov and G. Uhlmann in [8]. In the absence of both magnetic fields and potentials, i.e.  $\Omega = 0$  and  $U = 0$ , we come to the ordinary boundary rigidity problem for the Riemannian metrics. For recent surveys on the ordinary boundary rigidity problem see [6, 44]. It also worths to mention that recently P. Stefanov, G. Uhlmann and A. Vasy [47] proved in dimension  $n \geq 3$  the boundary rigidity with partial data for metrics in a given conformal class, this is so far the only local boundary rigidity result.

It is well-known that on simple manifolds, boundary rigidity problem is equivalent to another important inverse problem, called scattering rigidity problem. Next, we define a scattering relation and state the *scattering rigidity problem in the presence of a magnetic field and a potential*. Let  $\partial_+ S^k M$  and  $\partial_- S^k M$  denote the bundles of inward and outward vectors of energy  $k$  over  $\partial M$

$$\partial_{\pm} S^k M = \{(x, \xi) \in S^k M : x \in \partial M, \pm \langle \xi, \nu(x) \rangle \geq 0\}$$

where  $\nu$  is the inward unit vector normal to  $\partial M$ . For  $(x, \xi) \in \partial_+ S^k M$  let  $\tau(x, \xi)$  be the time when the  $\mathcal{MP}$ -geodesic  $\gamma_{x, \xi}$ , such that  $\gamma_{x, \xi}(0) = x$ ,  $\dot{\gamma}_{x, \xi}(0) = \xi$ , exits  $M$ . By Lemma A.3.1 the function  $\tau : \partial_+ S^k M \rightarrow \mathbb{R}$  is smooth.

The scattering relation  $\mathcal{S} : \partial_+ S^k M \rightarrow \partial_- S^k M$  of an  $\mathcal{MP}$ -system  $(M, g, \alpha, U)$  is defined

as

$$\mathcal{S}(x, \xi) = (\phi_{\tau_+(x, \xi)}(x, \xi)) = (\gamma_{x, \xi}(\tau_+(x, \xi)), \dot{\gamma}_{x, \xi}(\tau_+(x, \xi))).$$

Observe that two gauge equivalent  $\mathcal{MP}$ -systems have the same scattering relation. Is this the only type of nonuniqueness? In other words, the scattering rigidity problem studies whether a simple  $\mathcal{MP}$ -system  $(M, g, \alpha, U)$ , up to gauge equivalence, is uniquely determined by the scattering relations. In the Euclidean space this problem was considered by R. G. Novikov [32], in the absence of magnetic field, and by A. Jollivet [20]. On Riemannian manifolds endowed with magnetic fields, scattering rigidity problem was studied by N. Dairbekov, G. Paternain, P. Stefanov and G. Uhlmann [8], P. Herreros [17], P. Herreros and J. Vargo [18]. The reconstruction of both the Riemannian metrics and magnetic fields from the scattering relations was considered by N. Dairbekov and G. Uhlmann [10] for simple two-dimensional magnetic systems.

For simple  $\mathcal{MP}$ -systems, the boundary rigidity and the scattering rigidity problems are equivalent, see Lemma 3.4.2 and Theorem 3.4.3. Therefore, we formulate all rigidity results in terms of the boundary rigidity problem. However, unlike the boundary rigidity problems for simple manifolds or simple magnetic systems (with energy  $1/2$ ), the boundary rigidity problem for simple  $\mathcal{MP}$ -systems needs the information of the boundary action functions for two different energy levels. See the counterexamples in Section 3.3 and the proofs of the main results in Section 3.5 for details.

We consider these problems under various natural restrictions: simple  $\mathcal{MP}$ -systems with metrics in a given conformal class, simple real-analytic  $\mathcal{MP}$ -systems and simple two-dimensional  $\mathcal{MP}$ -systems.

The rest of this chapter is organized as follows. In Section 3.2, we show that by changing the metric, one can reduce a simple  $\mathcal{MP}$ -system to a simple magnetic system with the same boundary action function. Section 3.3 provides counterexamples which show that knowing the boundary action function for only one energy level is insufficient for solving the boundary rigidity problem, even under the assumption that the restriction of the system on the boundary  $\partial M$  is known. In Section 3.4, we demonstrate the equivalence between the bound-

ary rigidity problem and the scattering rigidity problem for a simple  $\mathcal{MP}$ -system. Section 3.5 is devoted to the proofs of the boundary rigidity for various systems, namely, simple  $\mathcal{MP}$ -systems with metrics in a given conformal class, simple real-analytic  $\mathcal{MP}$ -systems and simple two-dimensional  $\mathcal{MP}$ -systems. We give some remarks on the case that we only know the boundary action function for one energy level and the corresponding linear problem in Section 3.6.

### 3.2 Relations between $\mathcal{MP}$ -systems and magnetic systems

For a fixed energy level  $k > \sup_{x \in M} U(x)$ , let  $\sigma(t)$  be an  $\mathcal{MP}$ -geodesic with the constant energy  $k$ . Consider the time change

$$s(t) = \int_0^t 2(k - U(\sigma)) dt.$$

Then  $s$  is the arclength of  $\gamma(s) = \sigma(t(s))$  under the metric  $G = 2(k - U)g$ . The following version of Maupertuis' principle says that  $\gamma(s) = \sigma(t(s))$  is a unit speed magnetic geodesic of the magnetic system  $(G, \alpha)$ .

**Proposition 3.2.1.** *Let  $(g, \alpha, U)$  be an  $\mathcal{MP}$ -system on  $M$  and let  $k$  be a constant such that  $k > \sup_M U$ . Suppose  $\sigma(t)$  is an  $\mathcal{MP}$ -geodesic of energy  $k$ . Then  $\gamma(s) = \sigma(t(s))$  is a unit speed magnetic geodesic of the magnetic system  $(G, \alpha)$ .*

*Proof.* It is immediate to check that  $\gamma$  has unit speed with respect to  $G$ . Let  $\rho$  denote the arclength of the metric  $g$ . Since we fix the energy to be  $k$ , the parameter  $t$  of  $\sigma$  must be proportional to the length, i.e.  $dt = d\rho/\sqrt{2(k - U)}$ . We denote by  $\dot{\gamma}$  the derivative of  $\gamma$  with respect to  $s$  and by  $\dot{\sigma}$  the derivative of  $\sigma$  with respect to  $t$ . We define the Lagrangian  $\mathcal{L}(x, \xi) = |\xi|_{G(x)} - \alpha_x(\xi)$ , by the Maupertuis' principle, the  $\mathcal{MP}$ -geodesic is an extremal of the action

$$\begin{aligned} \int_{\sigma} \frac{\partial \mathcal{L}}{\partial \xi^i}(\sigma, \dot{\sigma}) \dot{\sigma}^i dt &= \int_{\sigma} \sqrt{2(k - U(\sigma))} d\rho - \int_{\sigma} \alpha \left( \sigma, \frac{d\sigma}{d\rho} \right) d\rho \\ &= \int_{\gamma} ds - \int_{\gamma} \alpha(\gamma, \dot{\gamma}) ds. \end{aligned}$$

Hence  $\mathcal{L}$  satisfies the Euler-Lagrange equation with respect to  $s$  which has the form

$$\frac{d}{ds} \left( \frac{G_{ki} \dot{\gamma}^i}{|\dot{\gamma}|_G} - \alpha_k \right) = \frac{1}{2} \frac{1}{|\dot{\gamma}|_G} \frac{\partial G_{ij}}{\partial x^k} \dot{\gamma}^i \dot{\gamma}^j - \frac{\partial \alpha_i}{\partial x^k} \dot{\gamma}^i.$$

Since  $s$  is the arclength of  $G$ , for which  $|\dot{\gamma}|_G = 1$ , this equation takes the form

$$\frac{d}{ds} (G_{ki} \dot{\gamma}^i - \alpha_k) = \frac{1}{2} \frac{\partial G_{ij}}{\partial x^k} \dot{\gamma}^i \dot{\gamma}^j - \frac{\partial \alpha_i}{\partial x^k} \dot{\gamma}^i.$$

Taking the derivative with respect to  $s$  and multiplying by  $G^{mk}$  we have

$$\ddot{\gamma}^m + G^{mk} \left( \frac{\partial G_{ki}}{\partial x^j} - \frac{1}{2} \frac{\partial G_{ij}}{\partial x^k} \right) \dot{\gamma}^i \dot{\gamma}^j = G^{mk} \left( \frac{\partial \alpha_k}{\partial x^i} - \frac{\partial \alpha_i}{\partial x^k} \right) \dot{\gamma}^i,$$

which is the equation of magnetic geodesics of the magnetic system  $(G, \alpha)$ .  $\square$

We give an alternative proof of Proposition 3.2.1 based on the flow equation itself.

*Proof.* Given an  $\mathcal{MP}$ -geodesic  $\sigma(t)$  with energy  $k$  and a positive smooth function  $\phi$ , let  $G = \phi g$ ,  $ds = \sqrt{2\phi(k - U)} dt$ , so  $s$  will be the arclength of  $\gamma(s) = \sigma(t(s))$  under the metric  $G$ . If we denote the Christoffel symbols and the covariant derivative under the new metric  $G$  by  $\tilde{\Gamma}_{jk}^i$  and  $\tilde{D}$  respectively, then

$$\tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + \frac{1}{2} \phi^{-1} (\delta_k^i \frac{\partial \phi}{\partial x^j} + \delta_j^i \frac{\partial \phi}{\partial x^k} - \phi^i g_{jk}),$$

where  $\phi^i = g^{il} \frac{\partial \phi}{\partial x^l}$ . So

$$\begin{aligned} \frac{\tilde{D}\dot{\gamma}}{ds} &= \ddot{\gamma}^i \frac{\partial}{\partial x^i} + \dot{\gamma}^j \dot{\gamma}^k \tilde{\Gamma}_{jk}^i \frac{\partial}{\partial x^i} \\ &= \ddot{\sigma}^i \left( \frac{dt}{ds} \right)^2 \frac{\partial}{\partial x^i} + \dot{\sigma}^i \frac{d^2 t}{ds^2} \frac{\partial}{\partial x^i} + \left( \frac{dt}{ds} \right)^2 \dot{\sigma}^j \dot{\sigma}^k (\Gamma_{jk}^i + \frac{1}{2} \phi^{-1} (\delta_k^i \frac{\partial \phi}{\partial x^j} + \delta_j^i \frac{\partial \phi}{\partial x^k} - \phi^i g_{jk})) \frac{\partial}{\partial x^i} \\ &= \left( \frac{dt}{ds} \right)^2 \frac{D\dot{\sigma}}{dt} + \left( \frac{dt}{ds} \right)^2 (\phi^{-1} \langle \nabla \phi, \dot{\sigma} \rangle_g \dot{\sigma} - \frac{1}{2} \phi^{-1} |\dot{\sigma}|_g^2 \nabla \phi) + \frac{d^2 t}{ds^2} \dot{\sigma}. \end{aligned}$$

Notice that

$$\begin{aligned} \frac{d^2 t}{ds^2} &= -\frac{1}{2} \frac{1}{(2\phi(k - U))^2} \frac{d(2\phi(k - U))}{dt} = -\frac{1}{2} \left( \frac{dt}{ds} \right)^2 \frac{1}{2\phi(k - U)} \frac{d(2\phi(k - U))}{dt} \\ &= -\left( \frac{dt}{ds} \right)^2 \left\{ \frac{1}{2} \phi^{-1} \langle \nabla \phi, \dot{\sigma} \rangle_g + \frac{1}{2(k - U)} \langle \nabla(k - U), \dot{\sigma} \rangle_g \right\}, \end{aligned}$$

thus

$$\begin{aligned} \frac{\tilde{D}\dot{\gamma}}{ds} &= \left(\frac{dt}{ds}\right)^2 \{Y(\dot{\sigma}) - \nabla U(\sigma) + \phi^{-1} \langle \nabla \phi, \dot{\sigma} \rangle_g \dot{\sigma} - \phi^{-1} (k - U) \nabla \phi \\ &\quad - \left(\frac{1}{2} \phi^{-1} \langle \nabla \phi, \dot{\sigma} \rangle_g + \frac{1}{2(k - U)} \langle \nabla(k - U), \dot{\sigma} \rangle_g\right) \dot{\sigma}\}. \end{aligned}$$

Now we let  $\phi = 2(k - U)$ , so  $ds = 2(k - U)dt$  (Actually all  $\phi = c(k - U)$ ,  $c \in \mathbb{R}^+$  will work for our argument), we get

$$\frac{\tilde{D}\dot{\gamma}}{ds} = \left(\frac{dt}{ds}\right)^2 Y(\dot{\sigma}) = \frac{dt}{ds} Y(\dot{\gamma}).$$

This indeed gives us the magnetic flow with the Lorentz force  $Y_G = \frac{1}{2(k-U)}Y$ . Moreover, one can see that the magnetic potential  $\tilde{\alpha}$  associated to  $(G, Y_G)$  is  $\alpha$  too, i.e. the new magnetic system is  $(G, \alpha)$ .  $\square$

The next result says that the simplicity of  $(g, \alpha, U)$  implies the simplicity of  $(G, \alpha)$ , and vice versa. A simple magnetic system is a special case of simple  $\mathcal{MP}$ -systems by assuming the potential  $U = 0$  and the energy  $k = \frac{1}{2}$ , see [8] for more details.

**Proposition 3.2.2.** *The  $\mathcal{MP}$ -system  $(g, \alpha, U)$  on  $M$  (of energy  $k$ ) is simple if and only if so is the magnetic system  $(G, \alpha)$  (of energy  $\frac{1}{2}$ ).*

*Proof.* Since the trajectories of these two systems coincide, for every  $x \in M$  the  $\mathcal{MP}$ -exponential map  $\exp_x^{\mathcal{MP}} : (\exp_x^{\mathcal{MP}})^{-1}(M) \rightarrow M$  is a diffeomorphism if and only if the magnetic exponential map  $\exp_x^\mu : (\exp_x^\mu)^{-1}(M) \rightarrow M$  is a diffeomorphism (The definition of  $\exp_x^\mu$  is similar to  $\exp_x^{\mathcal{MP}}$  by replacing the  $\mathcal{MP}$ -flow with a magnetic flow of energy  $\frac{1}{2}$ ). Hence, it is sufficient to prove that  $\partial M$  is strictly  $\mathcal{MP}$ -convex if and only if it is strictly magnetic convex with respect to  $(G, \alpha)$ .

First, we introduce some notations. The inward unit vector normal to  $\partial M$  with respect to the metric  $G$  is indicated as  $n$ , thus  $n = (2(k - U))^{-\frac{1}{2}}\nu$  ( $G$  is conformal to  $g$ ). The unit sphere bundle of the metric  $G$  is denoted by  $SM$ . By  $\Lambda_G$  we denote the second fundamental form of  $\partial M$  with respect to metric  $G$ . From the definition of the second fundamental form and using the formula for connection of  $G$  in terms of connection of  $g$ , we obtain the following

formula:

$$\Lambda_G(x, \xi) = \sqrt{2(k - U(x))} \Lambda(x, \xi) + \frac{d_x U(\nu)}{\sqrt{2(k - U(x))}} |\xi|_g^2, \quad x \in \partial M, \xi \in T_x \partial M. \quad (3.5)$$

The Lorentz force of the magnetic field  $d\alpha$  with respect to the metric  $G$  is indicated as  $Y_G$ . The next formula is obvious,

$$\langle Y_G(\xi), n \rangle_{G(x)} = \frac{1}{\sqrt{2(k - U(x))}} \langle Y(\xi), \nu \rangle, \quad x \in \partial M, \xi \in T_x \partial M. \quad (3.6)$$

Now, suppose that  $\partial M$  is strictly  $\mathcal{MP}$ -convex. Take any  $x \in \partial M$  and  $v \in S_x(\partial M)$  for the metric  $G$ . We substitute the vector  $\xi = 2(k - U)v \in S_x^k(\partial M)$  for the metric  $g$  in formulas (3.5–3.6) to obtain

$$\begin{aligned} \Lambda_G(x, v) &= (2(k - U(x)))^{-\frac{3}{2}} (\Lambda(x, \xi) + d_x U(\nu)), \\ \langle Y_G(v), n \rangle_{G(x)} &= (2(k - U(x)))^{-\frac{3}{2}} \langle Y(\xi), \nu \rangle, \end{aligned}$$

which implies that  $\partial M$  is strictly magnetic convex with respect to  $(G, \alpha)$  by (3.3).

By similar arguments one can show that  $\partial M$  is strictly  $\mathcal{MP}$ -convex whenever it is strictly magnetic convex with respect to  $(G, \alpha)$ .  $\square$

Here we show that the boundary action functions of the two simple systems  $(g, \alpha, U)$  and  $(G, \alpha)$  coincide. Assuming the potential  $U = 0$  and the energy  $k = \frac{1}{2}$ , the corresponding boundary action function of (3.4) is the one for a simple magnetic system.

**Proposition 3.2.3.** *Let  $\mathbb{A}$  be the Mañé's action potential (of energy  $k$ ) for a simple  $\mathcal{MP}$ -system  $(g, \alpha, U)$  and  $\mathbb{A}_G$  be the Mañé's action potential (of energy  $1/2$ ) for the simple magnetic system  $(G, \alpha)$ , then  $\mathbb{A}|_{\partial M \times \partial M} = \mathbb{A}_G|_{\partial M \times \partial M}$ .*

*Proof.* Take  $x, y \in \partial M$  and consider the unique  $\mathcal{MP}$ -geodesic  $\sigma$  from  $x$  to  $y$ . Then Proposition 3.2.1 implies that  $\gamma(s) = \sigma(t(s))$  is a unit speed magnetic geodesic (from  $x$  to  $y$ ) of the

system  $(G, \alpha)$  and  $s$  is the arclength of  $\gamma$  under the metric  $G$ . Thus,

$$\begin{aligned} \mathbb{A}(x, y) &= 2kT(x, y) - \int_{\sigma} (\alpha(\sigma, \dot{\sigma}) + 2U(\sigma)) dt \\ &= \int_{\sigma} 2(k - U(\sigma)) dt - \int_{\sigma} \alpha(\sigma, \dot{\sigma}) dt = \int_{\gamma} ds - \int_{\gamma} \alpha(\gamma, \dot{\gamma}) ds \\ &= T_G(x, y) - \int_{\gamma} \alpha(\gamma, \dot{\gamma}) ds = \mathbb{A}_G(x, y). \end{aligned}$$

□

### 3.3 Counterexamples

Before moving to the detailed study of the boundary and scattering rigidity problems of simple  $\mathcal{MP}$ -systems, we provide some counterexamples which show that knowing the boundary action function for only one energy level is insufficient for solving the boundary rigidity problem, even under the assumption that we know the restriction of the system on the boundary  $\partial M$ . More precisely, there are simple  $\mathcal{MP}$ -systems  $(g, \alpha, U)$  and  $(g', \alpha', U')$  with the same boundary action function for some energy level  $k$ , whose restrictions onto the boundary are the same (i.e.  $g|_{\partial M} = g'|_{\partial M}, \alpha|_{\partial M} = \alpha'|_{\partial M}, U|_{\partial M} = U'|_{\partial M}$ ), but are not gauge equivalent. This makes one turn to considering boundary action functions of two different energy levels.

**Counterexamples:** Given some simple magnetic system  $(g, \alpha)$  on a compact manifold  $M$  with boundary, we define two  $\mathcal{MP}$ -systems  $(\frac{1}{4}g, \alpha, U_1)$  and  $(\frac{1}{2}g, \alpha, U_2)$ , where  $U_1 \equiv 1$  on  $M$  and  $U_2 \equiv 2$  on  $M$ . We fix the energy  $k = 3$ , then it is easy to see that these two  $\mathcal{MP}$ -systems reduce to the same magnetic system  $(g, \alpha)$ . Since  $(g, \alpha)$  is simple, Proposition 3.2.2 implies that both  $(\frac{1}{4}g, \alpha, U_1)$  and  $(\frac{1}{2}g, \alpha, U_2)$  are simple  $\mathcal{MP}$ -systems. Moreover, applying Proposition 3.2.3, we conclude that they have the same boundary action function for the energy  $k = 3$ . Obviously these two  $\mathcal{MP}$ -systems are not gauge equivalent, since the metrics  $\frac{1}{4}g$  and  $\frac{1}{2}g$ , potentials  $U_1$  and  $U_2$  are even not equal on the boundary  $\partial M$ .

Next, by modifying the two  $\mathcal{MP}$ -systems near the boundary, we can make them equal on the boundary. Let  $\varphi$  and  $\psi$  be two smooth functions on  $M$ , and  $\varphi \equiv \psi \equiv 1$  for points away from a small tubular neighborhood of the boundary  $\partial M$ . We assume  $1 \leq \varphi < \frac{3}{2}$  in the

interior of  $M$  and  $\varphi = \frac{3}{2}$  on  $\partial M$ ;  $\frac{3}{4} < \psi \leq 1$  in the interior of  $M$  and  $\psi = \frac{3}{4}$  on  $\partial M$ . Then  $\varphi = \varphi U_1 < \frac{3}{2} < \psi U_2 = 2\psi$  in the interior of  $M$ . We define  $\tilde{g} = \frac{1}{2(3-\varphi)}g$  and  $\tilde{g}' = \frac{1}{2(3-2\psi)}g$ . Then it is easy to check that the  $\mathcal{MP}$ -systems  $(\tilde{g}, \alpha, \varphi)$  and  $(\tilde{g}', \alpha, 2\psi)$  reduce to the same magnetic system  $(g, \alpha)$  for the energy  $k = 3$ . Applying Proposition 3.2.2 and 3.2.3 again, these two  $\mathcal{MP}$ -systems  $(\tilde{g}, \alpha, \varphi)$  and  $(\tilde{g}', \alpha, 2\psi)$  are simple with the same boundary action function for energy  $k = 3$ . Moreover,  $\tilde{g}|_{\partial M} = \tilde{g}'|_{\partial M} = \frac{1}{3}g, \varphi|_{\partial M} = 2\psi|_{\partial M} = \frac{3}{2}$ , i.e. the boundary restrictions of the two systems are the same on the boundary. However, they are still not gauge equivalent, there is no diffeomorphism  $f : M \rightarrow M$  such that  $2\psi = \varphi \circ f$  (since  $\varphi < \frac{3}{2} < 2\psi$  in the interior of  $M$ ).

### 3.4 Boundary determination and scattering relation

Here we show that up to gauge equivalence the boundary action functions of two different energy levels completely determine the Riemannian metric, magnetic potential and potential on the boundary of the manifold under study. As mentioned in Section 3.3, the boundary action function of one energy level is insufficient for determining the restriction of the system on the boundary.

**Lemma 3.4.1.** *If  $(g, \alpha, U)$  and  $(g', \alpha', U')$  are simple  $\mathcal{MP}$ -systems on  $M$  with the same boundary action functions for both energy  $k_1$  and  $k_2$ , then*

$$i^*g = i^*g', \quad i^*\alpha = i^*\alpha', \quad U \circ i = U' \circ i, \quad (3.7)$$

where  $i : \partial M \rightarrow M$  is the embedding map.

*Proof.* Given  $x \in \partial M$  and  $\xi \in T_x(\partial M)$ , let  $\tau(s)$ ,  $-\varepsilon < s < \varepsilon$ , be a curve on  $\partial M$  with  $\tau(0) = x$  and  $\dot{\tau}(0) = \xi$ . Let  $G = 2(k_1 - U)g$ , by Proposition 3.2.1 and Proposition 3.2.2,  $(G, \alpha)$  is a simple magnetic system of energy  $\frac{1}{2}$ . Applying Proposition 3.2.3 it is easy to see that

$$\lim_{s \rightarrow 0} \frac{\mathbb{A}(x, \tau(s))}{s} = \lim_{s \rightarrow 0} \frac{\mathbb{A}_G(x, \tau(s))}{s} = |\xi|_G - \alpha(\xi) = \sqrt{2(k_1 - U)}|\xi|_g - \alpha(\xi).$$

A similar equality holds for the system  $(g', \alpha', U')$ . Therefore,

$$\sqrt{2(k_1 - U)}|\xi|_g - \alpha(\xi) = \sqrt{2(k_1 - U')}|\xi|_{g'} - \alpha'(\xi).$$

Changing  $\xi$  to  $-\xi$ , we get

$$\sqrt{2(k_1 - U)}|\xi|_g + \alpha(\xi) = \sqrt{2(k_1 - U')}|\xi|_{g'} + \alpha'(\xi),$$

whence we infer the second equation in (3.7). Notice that we also get that

$$(k_1 - U)|\xi|_g^2 = (k_1 - U')|\xi|_{g'}^2.$$

Similarly, for energy  $k_2$ , we obtain

$$(k_2 - U)|\xi|_g^2 = (k_2 - U')|\xi|_{g'}^2.$$

Since  $k_1 \neq k_2$ , by taking the difference of above two equations, we have  $|\xi|_g = |\xi|_{g'}$ , thus  $\iota^*g = \iota^*g'$ . This also implies that  $U \circ \iota = U' \circ \iota$ .  $\square$

Now we prove that the boundary action functions of two different energy levels actually determine the full jets of the metric  $g$ , magnetic potential  $\alpha$  and potential function  $U$  on the boundary.

**Lemma 3.4.2.** *If  $(g, \alpha, U)$  and  $(g', \alpha', U')$  are simple  $\mathcal{MP}$ -systems on  $M$  with the same boundary action functions for both energy  $k_1$  and  $k_2$ , then  $(g', \alpha', U')$  is gauge equivalent to some simple  $\mathcal{MP}$ -system  $(\bar{g}, \bar{\alpha}, \bar{U})$  such that in any local coordinate system we have  $\partial^m g|_{\partial M} = \partial^m \bar{g}|_{\partial M}$ ,  $\partial^m \alpha|_{\partial M} = \partial^m \bar{\alpha}|_{\partial M}$  and  $\partial^m U|_{\partial M} = \partial^m \bar{U}|_{\partial M}$  for every multi-index  $m$ .*

*Proof.* Let  $G_i = 2(k_i - U)g$  and  $G'_i = 2(k_i - U')g'$ ,  $i = 1, 2$  by Proposition 3.2.1 and Proposition 3.2.2,  $(G_i, \alpha)$  and  $(G'_i, \alpha')$  are simple magnetic systems of energy  $\frac{1}{2}$ . Let  $\mathbb{A}_{G_i}$  and  $\mathbb{A}_{G'_i}$  denote the Mañé's action potentials (of energy  $1/2$ ) for  $(G_i, \alpha)$  and  $(G'_i, \alpha')$  respectively. Then by Proposition 3.2.3 we have  $\mathbb{A}_{G_i}|_{\partial M \times \partial M} = \mathbb{A}_{G'_i}|_{\partial M \times \partial M}$ . Then [8, Theorem 2.2] implies that there is  $(\bar{G}_i, \bar{\alpha}_i)$ , gauge equivalent to  $(G'_i, \alpha')$ , such that in any local coordinate system  $\partial^m G_i|_{\partial M} = \partial^m \bar{G}_i|_{\partial M}$  and  $\partial^m \alpha|_{\partial M} = \partial^m \bar{\alpha}_i|_{\partial M}$  for every multi-index  $m$ . Thus there is some

diffeomorphism  $f_i$  with  $f_i|_{\partial M} = Id$ , and some smooth function  $\varphi_i$  with  $\varphi_i|_{\partial M} = 0$ , such that  $\bar{G}_i = f_i^* G'_i = 2(k_i - U' \circ f_i) f_i^* g'$  and  $\bar{\alpha}_i = f_i^* \alpha' + d\varphi_i$ . Actually by Lemma 3.4.1  $U|_{\partial M} = U'|_{\partial M}$ , the proof of [8, Theorem 2.2] shows that near the boundary  $\partial M$ , one can choose  $f_1 = f_2 = \exp_{\partial M} \circ (\exp'_{\partial M})^{-1}$  where  $\exp_{\partial M}$  and  $\exp'_{\partial M}$  are the ‘‘usual’’ boundary exponential maps w.r.t.  $g$  and  $g'$  respectively. Thus  $f_1^* g' = f_2^* g'$ ,  $U' \circ f_1 = U' \circ f_2$  near the boundary. We define  $\bar{g} = f_1^* g'$ ,  $\bar{\alpha} = \bar{\alpha}_1$ ,  $\bar{U} = U' \circ f_1$ , by Lemma 3.4.1,  $U|_{\partial M} = U'_{\partial M} = \bar{U}|_{\partial M}$ . Thus  $G_1|_{\partial M} = \bar{G}_1|_{\partial M}$  implies  $g|_{\partial M} = \bar{g}|_{\partial M}$ .

Now we prove the equality of derivatives on the boundary by introducing boundary normal coordinates  $(x', x^n)$  w.r.t.  $g$  near arbitrary  $x_0 \in \partial M$ . Since  $g|_{\partial M} = \bar{g}|_{\partial M}$ , the same coordinates are boundary normal coordinates w.r.t.  $\bar{g}$ . Thus locally the metrics are of the form

$$g = g_{ij} dx'_i dx'_j + dx_n^2,$$

$$\bar{g} = \bar{g}_{ij} dx'_i dx'_j + dx_n^2,$$

where  $i, j$  vary from 1 to  $n - 1$ . It suffices to prove that the normal derivatives are equal, i.e.

$$\partial_n^m g_{ij}|_{x=x_0} = \partial_n^m \bar{g}_{ij}|_{x=x_0}, \quad \partial_n^m U|_{x=x_0} = \partial_n^m \bar{U}|_{x=x_0} \quad \forall m = 0, 1, \dots; i, j = 1, \dots, n - 1.$$

We prove above equalities by induction, the case  $m = 0$  is granted. Assume for some nonnegative integer  $l$  and all  $0 \leq m \leq l$ ,  $\partial_n^m g_{ij}|_{x=x_0} = \partial_n^m \bar{g}_{ij}|_{x=x_0}$ ,  $\partial_n^m U|_{x=x_0} = \partial_n^m \bar{U}|_{x=x_0}$ . Since  $\partial_n^{l+1} G_1|_{x=x_0} = \partial_n^{l+1} \bar{G}_1|_{x=x_0}$ , then

$$(-\partial_n^{l+1} U) g_{ij} + (k_1 - U) \partial_n^{l+1} g_{ij} = (-\partial_n^{l+1} \bar{U}) \bar{g}_{ij} + (k_1 - \bar{U}) \partial_n^{l+1} \bar{g}_{ij} \quad (3.8)$$

at  $x_0$ . Similarly for energy  $k_2$ , since  $\bar{g} = f_2^* g'$ ,  $\bar{U} = U' \circ f_2$  near  $\partial M$ , we have at  $x_0$

$$(-\partial_n^{l+1} U) g_{ij} + (k_2 - U) \partial_n^{l+1} g_{ij} = (-\partial_n^{l+1} \bar{U}) \bar{g}_{ij} + (k_2 - \bar{U}) \partial_n^{l+1} \bar{g}_{ij}. \quad (3.9)$$

Taking difference of above two equalities, we arrive

$$(k_1 - k_2) \partial_n^{l+1} g_{ij}|_{x=x_0} = (k_1 - k_2) \partial_n^{l+1} \bar{g}_{ij}|_{x=x_0}.$$

Since  $k_1 \neq k_2$ , we obtain  $\partial_n^{l+1} g_{ij}|_{x=x_0} = \partial_n^{l+1} \bar{g}_{ij}|_{x=x_0}$ . Now return to the equation (3.8), since  $g|_{\partial M} = \bar{g}|_{\partial M}$  is positive definite,  $U|_{\partial M} = \bar{U}|_{\partial M}$ , we eventually get  $\partial_n^{l+1} U|_{x=x_0} = \partial_n^{l+1} \bar{U}|_{x=x_0}$ . This finishes the proof.  $\square$

Now we show that for simple  $\mathcal{MP}$ -systems, the boundary rigidity problem is equivalent to the problem of restoring a Riemannian metric, a magnetic potential and a potential from the scattering relations. Thus we will formulate all rigidity results in terms of the boundary rigidity problem in the next Section.

**Theorem 3.4.3.** *Suppose that  $(g, \alpha, U)$  and  $(g', \alpha', U')$  are simple  $\mathcal{MP}$ -systems on  $M$  of the same energy  $k$  such that  $g|_{\partial M} = g'|_{\partial M}$ ,  $U|_{\partial M} = U'|_{\partial M}$  and  $i^* \alpha = i^* \alpha'$ . Then the boundary action functions  $\mathbb{A}|_{\partial M \times \partial M}$  and  $\mathbb{A}'|_{\partial M \times \partial M}$  of both the systems coincide if and only if the scattering relations  $\mathcal{S}$  and  $\mathcal{S}'$  of these systems coincide.*

*Proof.* First, we introduce some notations. Let  $G = 2(k - U)g$ ,  $G' = 2(k - U')g'$ , by Proposition 3.2.1 and Proposition 3.2.2,  $(G, \alpha)$  and  $(G', \alpha')$  are simple magnetic systems of energy  $\frac{1}{2}$ . We denote by  $\mathcal{S}_G$  and  $\mathcal{S}_{G'}$  the scattering relations of  $(G, \alpha)$  and  $(G', \alpha')$  respectively (The definition of the scattering relation for a simple magnetic system is similar to that for a simple  $\mathcal{MP}$ -system by considering the magnetic flow of energy  $\frac{1}{2}$ ). The notation  $\partial_+ SM$  denotes the bundle of inward unit vectors at  $\partial M$  with respect to the metric  $G$  (and also of  $G'$ , since  $G|_{\partial M} = G'|_{\partial M}$ ).

Suppose  $\mathbb{A}|_{\partial M \times \partial M} = \mathbb{A}'|_{\partial M \times \partial M}$ , then by Proposition 3.2.3 we have

$$\mathbb{A}_G|_{\partial M \times \partial M} = \mathbb{A}_{G'}|_{\partial M \times \partial M}.$$

Then [8, Lemma 2.5] implies that  $\mathcal{S}_G = \mathcal{S}_{G'}$ . Now we prove that this implies  $\mathcal{S} = \mathcal{S}'$ . Since the trajectories of  $(g, \alpha, U)$  and  $(G, \alpha)$  coincide, for any  $(x, \xi) \in \partial_+ S^k M$  the scattering relation  $\mathcal{S}$  can be expressed in terms of  $\mathcal{S}_G$  in the following way

$$\mathcal{S}(x, \xi) = 2 \left[ k - U \left( \mathfrak{s}_G \left( x, \frac{\xi}{2(k - U(x))} \right) \right) \right] \mathcal{S}_G \left( x, \frac{\xi}{2(k - U(x))} \right),$$

where  $\mathfrak{s}_G = \pi \circ \mathcal{S}_G$  ( Here we define  $c(x, v) \doteq (x, cv)$  ). Exactly in the same way  $\mathcal{S}'$  can be expressed in terms of  $\mathcal{S}_{G'}$ . Since  $\mathcal{S}_G = \mathcal{S}_{G'}$ , these expressions imply that  $\mathcal{S} = \mathcal{S}'$ .

Conversely, assume that  $\mathcal{S} = \mathcal{S}'$ . Since the trajectories of these two systems coincide, for any  $(x, \xi) \in \partial_+ SM$  the scattering relation  $\mathcal{S}_G$  can be expressed in terms of  $\mathcal{S}$  in the following way

$$\mathcal{S}_G(x, \xi) = \frac{\mathcal{S}(x, 2(k - U(x))\xi)}{2(k - U(\mathfrak{s}(x, 2(k - U(x))\xi)))},$$

where  $\mathfrak{s} = \pi \circ \mathcal{S}$ . Exactly in the same way  $\mathcal{S}_{G'}$  can be expressed in terms of  $\mathcal{S}'$ . Since  $\mathcal{S} = \mathcal{S}'$ , these expressions imply that  $\mathcal{S}_G = \mathcal{S}_{G'}$ . Then [8, Lemma 2.6] implies that

$$\mathbb{A}_G|_{\partial M \times \partial M} = \mathbb{A}_{G'}|_{\partial M \times \partial M}.$$

Applying Proposition 3.2.3 we come to  $\mathbb{A}|_{\partial M \times \partial M} = \mathbb{A}'|_{\partial M \times \partial M}$ .  $\square$

*Remark:* Theorem 3.4.3 together with the counterexamples of the previous section shows that for generally a simple  $\mathcal{MP}$ -system, knowing the scattering relation of only one energy level is also insufficient for solving the scattering rigidity problem.

### 3.5 Main results

Here we give the proof of our first main result which is a rigidity theorem in a fixed conformal class of a metric. The theorem below generalizes the corresponding well-known results for the ordinary boundary rigidity problem, see [5, 30, 31], and for the magnetic boundary rigidity problem, see [8].

**Theorem 3.5.1.** *Let  $(g, \alpha, U)$  and  $(g', \alpha', U')$  be simple  $\mathcal{MP}$ -systems on  $M$  with the same boundary action functions for both energy  $k_1$  and  $k_2$ . If  $g'$  is conformal to  $g$ , then  $g' = g$ ,  $\alpha' = \alpha + d\varphi$  and  $U' = U$  for some smooth function  $\varphi$  on  $M$  vanishing on  $\partial M$ , hence  $(g', \alpha', U')$  is gauge equivalent to  $(g, \alpha, U)$ .*

*Proof.* Let  $G_i = 2(k_i - U)g$ ,  $G'_i = 2(k_i - U')g'$ ,  $i = 1, 2$ , by Proposition 3.2.1 and Proposition 3.2.2,  $(G_i, \alpha)$  and  $(G'_i, \alpha')$ , for  $i = 1, 2$ , are all simple magnetic systems of energy  $\frac{1}{2}$ . Let  $\mathbb{A}_{G_i}$

and  $\mathbb{A}_{G'_i}$  denote the Mañé's action potentials (of energy 1/2) for  $(G_i, \alpha)$  and  $(G'_i, \alpha')$  respectively. Then by Proposition 3.2.3 we have  $\mathbb{A}_{G_i}|_{\partial M \times \partial M} = \mathbb{A}_{G'_i}|_{\partial M \times \partial M}$ . By the assumption  $g' = \omega g$  for some strictly positive function  $\omega \in C^\infty(M)$ , therefore

$$G'_i = \omega(k_i - U')(k_i - U)^{-1}G_i.$$

Applying [8, Theorem 6.1], we get  $G'_i = G_i$ , i.e.

$$\omega(k_i - U')(k_i - U)^{-1} \equiv 1, \quad (3.10)$$

and that there are  $\varphi_i \in C^\infty(M)$ , with  $\varphi_i|_{\partial M} = 0$ , such that  $\alpha' = \alpha + d\varphi_i$ . But  $\omega(k_i - U')(k_i - U)^{-1} \equiv 1$ ,  $i = 1, 2$  also implies that

$$\frac{k_1 - U'}{k_1 - U} \equiv \frac{k_2 - U'}{k_2 - U}.$$

Thus  $U = U'$  on  $M$  (since  $k_1 \neq k_2$ ), together with (3.10) this gives  $\omega \equiv 1$ . On the other hand,  $d\varphi_1 = d\varphi_2$  with  $\varphi_1|_{\partial M} = \varphi_2|_{\partial M} = 0$  implies  $\varphi_1 = \varphi_2 = \varphi$ , thus  $\alpha' = \alpha + d\varphi$  for some  $\varphi \in C^\infty(M)$  with  $\varphi|_{\partial M} = 0$ .  $\square$

*Remark:* In Jollivet's paper on the scattering rigidity problem [20], the metrics  $g$  and  $g'$  are the same, namely the Euclidean metric, which means  $\omega \equiv 1$  under the setting of Theorem 3.5.1. Thus we have  $(k - U')(k - U)^{-1} \equiv 1$ , which implies  $U = U'$  on  $M$ . That's why one fixed energy level is sufficient for Euclidean case. However, for general simple  $\mathcal{MP}$ -systems we need the information of two energy levels, as can be seen from the counterexamples and the proof above.

Our next result says that rigidity also holds in a class of real-analytic simple  $\mathcal{MP}$ -systems. This generalizes the corresponding result for the magnetic boundary rigidity problem in [8].

**Theorem 3.5.2.** *If  $M$  is a real-analytic compact manifold with boundary, and  $(g, \alpha, U)$  and  $(g', \alpha', U')$  are simple real-analytic  $\mathcal{MP}$ -systems on  $M$  with the same boundary action functions for both energy  $k_1$  and  $k_2$ , then these systems are gauge equivalent.*

*Proof.* Let  $G = 2(k_1 - U)g$ ,  $G' = 2(k_1 - U')g'$ , by Proposition 3.2.1 and Proposition 3.2.2,  $(G, \alpha)$  and  $(G', \alpha')$  are simple real-analytic magnetic systems of energy  $\frac{1}{2}$ . Let  $\mathbb{A}_G$  and  $\mathbb{A}_{G'}$  denote the Mañé's action potentials (of energy  $1/2$ ) for  $(G, \alpha)$  and  $(G', \alpha')$  respectively. Then by Proposition 3.2.3 we have  $\mathbb{A}_G|_{\partial M \times \partial M} = \mathbb{A}_{G'}|_{\partial M \times \partial M}$ . Then [8, Theorem 6.2] implies that  $(G, \alpha)$  and  $(G', \alpha')$  are gauge equivalent, i.e. there are some real-analytic diffeomorphism  $f : M \rightarrow M$  with  $f|_{\partial M} = Id$  and some real-analytic function  $\varphi$  on  $M$  with  $\varphi|_{\partial M} = 0$ , such that  $G' = f^*G = 2(k_1 - U \circ f)f^*g$ ,  $\alpha' = f^*\alpha + d\varphi$ . In particular,  $g'$  is conformal to  $f^*g$ .

Now we consider the systems  $(g', \alpha', U')$  and  $(f^*g, \alpha', U \circ f)$ . Let  $\mathbb{A}_i$  and  $\mathbb{A}'_i$ ,  $i = 1, 2$ , denote the Mañé's action potentials (of energy  $k_i$ ) for simple real-analytic  $\mathcal{MP}$ -systems  $(g, \alpha, U)$  and  $(g', \alpha', U')$  respectively. By our assumption,

$$\mathbb{A}'_i|_{\partial M \times \partial M} = \mathbb{A}_i|_{\partial M \times \partial M} = \bar{\mathbb{A}}_i|_{\partial M \times \partial M}, \quad i = 1, 2,$$

where  $\bar{\mathbb{A}}_i|_{\partial M \times \partial M}$  is the boundary action function of  $(f^*g, \alpha', U \circ f)$  for energy  $k_i$ . (the second equality comes from the fact that  $(g, \alpha, U)$  and  $(f^*g, \alpha', U \circ f)$  are gauge equivalent) Then, Theorem 3.5.1 implies that  $U' = U \circ f$  and  $g' = f^*g$ .  $\square$

We show that two-dimensional simple  $\mathcal{MP}$ -systems are always rigid. Our result generalizes the boundary rigidity theorem for simple Riemannian surfaces [36] and for simple two-dimensional magnetic systems [8].

**Theorem 3.5.3.** *If  $\dim M = 2$  and  $(g, \alpha, U)$  and  $(g', \alpha', U')$  are simple  $\mathcal{MP}$ -systems on  $M$  with the same boundary action functions for both energy  $k_1$  and  $k_2$ , then these systems are gauge equivalent.*

*Proof.* Let  $G = 2(k_1 - U)g$ ,  $G' = 2(k_1 - U')g'$ , by Proposition 3.2.1 and Proposition 3.2.2,  $(G, \alpha)$  and  $(G', \alpha')$  are simple magnetic systems. Let  $\mathbb{A}_G$  and  $\mathbb{A}_{G'}$  denote the Mañé's action potentials (of energy  $1/2$ ) for  $(G, \alpha)$  and  $(G', \alpha')$  respectively. Then by Proposition 3.2.3 we have  $\mathbb{A}_G|_{\partial M \times \partial M} = \mathbb{A}_{G'}|_{\partial M \times \partial M}$ . Applying [8, Theorem 7.1] we find some diffeomorphism  $f : M \rightarrow M$  with  $f|_{\partial M} = Id$ , and a smooth function  $\varphi : M \rightarrow \mathbb{R}$ , with  $\varphi|_{\partial M} = 0$ , such that  $g' = (k_1 - U \circ f)(k_1 - U')^{-1}f^*g$  (i.e.  $g'$  is conformal to  $f^*g$ ) and  $\alpha' = f^*\alpha + d\varphi$ . Let  $\mathbb{A}_i$  and

$\mathbb{A}'_i$  denote the Mañé's action potentials (of energy  $k_i$ ) for simple  $\mathcal{MP}$ -systems  $(g, \alpha, U)$  and  $(g', \alpha', U')$  respectively. By our assumption,

$$\mathbb{A}'_i|_{\partial M \times \partial M} = \mathbb{A}_i|_{\partial M \times \partial M} = \bar{\mathbb{A}}_i|_{\partial M \times \partial M}, \quad i = 1, 2,$$

where  $\bar{\mathbb{A}}_i|_{\partial M \times \partial M}$  is the boundary action function of  $(f^*g, \alpha', U \circ f)$  for energy  $k_i$ . (the second equality comes from the fact that  $(g, \alpha, U)$  and  $(f^*g, \alpha', U \circ f)$  are gauge equivalent) Then, Theorem 3.5.1 implies that  $U' = U \circ f$  and  $g' = f^*g$ .  $\square$

### 3.6 Final remarks

Our main results and the counterexamples have shown that it's necessary to consider two different energy levels for the boundary and scattering rigidity problems of simple  $\mathcal{MP}$ -systems. However, assuming the boundary action functions  $\mathbb{A} = \mathbb{A}'$  for some fixed energy  $k$ , we still can obtain some weak version of boundary rigidity.

After reviewing the proof of the main results, if under some additional assumption (fixed conformal classes, real analyticity or dimension 2) two simple  $\mathcal{MP}$ -systems  $(g, \alpha, U)$  and  $(g', \alpha', U')$  have the same boundary action function for some energy  $k$ , then there exists a diffeomorphism  $f : M \rightarrow M$  with  $f|_{\partial M} = Id$ , and a smooth function  $\varphi : M \rightarrow \mathbb{R}$  with  $\varphi|_{\partial M} = 0$ , such that  $g' = (k - U')^{-1}(k - U \circ f)f^*g$  and  $\alpha' = f^*\alpha + d\varphi$ . Thus at least we can show that the magnetic potentials of these two  $\mathcal{MP}$ -systems are gauge equivalent, and the metrics of the two  $\mathcal{MP}$ -systems are gauge equivalent up to some conformal factor  $(k - U')^{-1}(k - U \circ f)$ , which is determined by the potentials of the two systems. In particular,  $f = Id$  when  $g$  is conformal to  $g'$ , and for the real-analytic  $\mathcal{MP}$ -systems,  $f$  and  $\varphi$  are both real-analytic. In some sense, this can be regarded as a weak boundary rigidity result, but the two systems may have different boundary action functions for energy levels other than  $k$ . However, if two simple  $\mathcal{MP}$ -systems are gauge equivalent, they must have the same boundary action functions for all  $k > \sup_M U = \sup_M U'$ . Similar situation occurs for the scattering rigidity problem of simple  $\mathcal{MP}$ -systems.

One can also consider the linearized problem of the boundary rigidity problem for simple

$\mathcal{MP}$ -systems, namely the X-ray transforms along  $\mathcal{MP}$ -geodesics. Let  $\sigma(t)$  be an  $\mathcal{MP}$ -geodesic connecting boundary points  $\sigma(0)$  and  $\sigma(T)$  with energy  $k > \sup_M U$ ,  $f \in C^\infty(M)$ , the X-ray transform of  $f$  along  $\sigma$  is

$$(I_{\mathcal{MP}}f)(\sigma) = \int_0^T f(\sigma(t)) dt.$$

We are interested in inverting the operator  $I_{\mathcal{MP}}$ , in particular we would like to know that whether it is injective, i.e. if  $(I_{\mathcal{MP}}f)(\sigma) = 0$  for all such  $\sigma$  that connect boundary points, then is  $f = 0$ ?

Recall the Proposition 3.2.1 that once we change the parameter by

$$s(t) = \int_0^t 2(k - U(\sigma)) dt,$$

the curve  $\gamma(s) = \sigma(t(s))$  is a unit speed magnetic geodesic of the magnetic system  $(G, \alpha)$ .

Then

$$\int_0^T f(\sigma(t)) dt = \int_0^{s(T)} f(\gamma(s)) \frac{dt}{ds} ds = \int_0^{s(T)} f(\gamma(s)) \frac{1}{2(k - U(\gamma(s)))} ds = (I_\mu \frac{f}{2(k - U)})(\gamma),$$

$I_\mu$  is the magnetic ray transform w.r.t.  $(G, \alpha)$ .

Now if  $(g, \alpha, U)$  is a simple  $\mathcal{MP}$ -system, by Proposition 3.2.2,  $(G, \alpha)$  is a simple magnetic system, thus the unique  $\mathcal{MP}$ -geodesic  $\sigma(t)$  (of energy  $k$ ) connecting  $x, y \in \partial M$  induces the unique magnetic geodesic  $\gamma(s)$  of unit speed. By [8, Theorem 5.3] on the injectivity of  $I_\mu$  on simple magnetic systems,

$$0 = (I_{\mathcal{MP}}f)(\sigma) = (I_\mu \frac{f}{2(k - U)})(\gamma)$$

implies that  $\frac{f}{2(k-U)} = 0$ , thus  $f = 0$ .

Similarly by converting the problem to magnetic case, we can consider the ray transform of symmetric tensor fields.

## Chapter 4

# AN INVERSE KINEMATIC PROBLEM WITH INTERNAL SOURCES

Usually the data for inverse problems are some boundary measurements, one can also use internal measurements. This chapter is devoted to the study of an inverse kinematic problem with data from internal sources, in particular an explicit reconstruction procedure is derived.

### 4.1 Introduction

The inverse kinematic problem arises in geophysics in an attempt to determine the substructure of the Earth by measuring at the surface of the Earth the travel times of seismic waves. The Earth is generally isotropic, thus one can assume that the geometry of the Earth is conformally Euclidean. In applications the conformal factor corresponds to  $1/c^2(x)$ , where  $c(x)$  is the sound speed (index of refraction). This problem goes back to [16, 52] who considered the case of a radial metric conformal to the Euclidean metric. The geometric version of the problem is related to the boundary rigidity problem and its linearization, namely the geodesic ray transform, see [45, 51] for recent surveys.

The usual inverse kinematic problem takes into account only the travel times between boundary points. In the current chapter we also make use of the internal data (travel times from internal sources to the boundary), so that we can reconstruct the geometry explicitly. The problem of determining a sound speed of a medium from the corresponding Dirichlet-to-Neumann map has been solved using the boundary control method, see [21]. An ingredient in the method is to recover the distance function between interior points and the boundary. Our arguments give an explicit reconstruction in the case conformal to the Euclidean metric.

The determination of the boundary distance function from the Dirichlet-to-Neumann map for the wave equation was considered earlier in [49]. We describe now the mathematical setting of the problem.

Let  $(M, g)$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$  with smooth boundary  $\partial M$ . We assume  $g$  is conformal to the Euclidean metric, i.e.  $g = \rho e$ , where  $\rho$  is a positive smooth function on  $M$  and  $e = (dx^1)^2 + \cdots + (dx^n)^2$  is the Euclidean metric. Let  $\Gamma$  be a domain in  $\partial M$  (in particular  $\Gamma$  could be  $\partial M$ ), from each  $x' \in \Gamma$ , there is a unique geodesic  $\gamma_{x'}(t)$  with  $\gamma_{x'}(0) = x'$ ,  $\dot{\gamma}_{x'}(0) = \nu(x')$ , where  $\nu(x')$  is the inward unit normal vector to  $\partial M$  at  $x'$  w.r.t. the metric  $g$ . Moreover, since  $\gamma$  is a geodesic of unit speed w.r.t. the metric  $g$ , we have  $\rho(\gamma)|\dot{\gamma}|^2 = 1$ , where  $|\cdot|$  is the Euclidean norm.

There is a positive smooth function  $T(x')$  on  $\Gamma$  such that for each  $x' \in \Gamma$ , the geodesic  $\gamma_{x'}$ , which is orthogonal to  $\partial M$  at  $x'$ , is defined on the interval  $[0, T(x')]$ . Let

$$D := \{(x', t) : x' \in \Gamma, 0 \leq t < T(x')\},$$

we consider the map  $\gamma : D \rightarrow \gamma(D)$ ,  $\gamma(x', t) := \gamma_{x'}(t) = x$ . Generally such a map is not a diffeomorphism, for example given the Euclidean disk with radius  $r$ , let  $\Gamma$  be a domain of the boundary, then  $\gamma$  is not a diffeomorphism if  $T(x') > r$ . Thus we modify  $T(x')$ , i.e.  $D$ , so that  $\gamma : D \rightarrow \gamma(D)$  is a diffeomorphism. Under this assumption,  $D$  actually provides a semigeodesic coordinate system (or boundary normal coordinates) for  $\gamma(D)$ .

Now given a point  $x \in \gamma(D)$ , if  $x' \in \Gamma$  such that  $x = \gamma(x', t)$  for some  $t \in [0, T(x')]$ , let  $U(x') \subset \Gamma$  be a neighborhood of  $x'$ . We assume that  $U(x')$  depends on  $x'$  smoothly. Moreover, we fix  $U(x')$  for all  $x \in \gamma_{x'}([0, T(x')])$ . The definition of our travel time data depends on a priori knowledge of  $x \in \gamma(D)$ , i.e. we need to know the geodesic projection  $x'(x) \in \Gamma$  of any  $x \in \gamma(D)$ . We define the *travel time data* with respect to  $D$  by

$$\Omega(D) := \{(\tau(x, x''), x'(x)) : x \in \gamma(D), x'' \in U(x')\},$$

where  $\tau(x, x'') := \text{dist}_g(x, x'')$  is the distance between points  $x$  and  $x''$ . In this paper, we consider the problem of recovering the neighborhood  $\gamma(D)$  and the conformal factor  $\rho$  in

$\gamma(D)$  from the travel time data  $\Omega(D)$ . To solve the problem, we need some extra inverse data, namely we assume the Cartesian coordinates of  $\Gamma$  is known. This is a reasonable assumption, since any rigid transformation of the domain  $M$  does not change the travel time data. We call the problem the *inverse kinematic problem with internal sources*. It's worth pointing out that we do not make any assumption on the convexity of the boundary (or  $\Gamma$ ).

**Theorem 4.1.1.** *Let  $M$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$  and  $g = \rho e$  be a conformally Euclidean metric on  $M$ . Let  $\Gamma$  be a domain in  $\partial M$ , then there exists a semigeodesic coordinate system  $D$  such that  $\gamma : D \rightarrow \gamma(D) \subset \mathbb{R}^n$  is a diffeomorphism and  $\Gamma = \gamma(\{t = 0\})$ . From the travel time data  $\Omega(D)$  and the Cartesian coordinates of  $\Gamma$  one can recover the diffeomorphism  $\gamma$  and the conformal factor  $\rho$  in  $\gamma(D)$ .*

Uniqueness for this inverse problem was proved by Anikonov [1]. In this paper we give a reconstruction procedure, based on conformal Killing vector fields.

Generally one cannot expect to reconstruct  $\rho$  on the whole manifold, as the necessary assumption that  $\gamma$  is a diffeomorphism. However, if  $M \setminus \gamma(D)$  has empty interior, we reconstruct the domain  $M$  and the conformal factor  $\rho$  globally since points in  $M \setminus \gamma(D)$  are limit points of  $\gamma(D)$ . The example mentioned above satisfies the assumption, and it is easy to see that  $M \setminus \gamma(D)$  is the center of the disk if  $\Gamma = \partial M$ ,  $T(x') \equiv r$ .

In [23, 21] an isometric copy of a compact Riemannian manifold was recovered from the set of boundary distance functions  $\{r_x(y) := \text{dist}(x, y) : x \in M, y \in \partial M\}$ , which is also internal data. Such internal data is also related to the broken geodesic flow which consists of two geodesic segments sharing a common end point inside the manifold, see e.g. [21, 24]. A related reconstruction problem with different assumptions was considered in [11] by reducing the travel time data to measurements of the shape operators of the wave fronts of waves diffracted from interior points. Different from our method, their approach treated the case of two dimensions and the case of three and higher dimensions separately.

Notice that the statement of Theorem 4.1.1 actually shows that this is a local problem, i.e. for a point  $x = \gamma(x', t)$ , we only need the travel time data from  $x$  to an arbitrarily small

neighborhood  $U(x') \subset \Gamma$ . If the function  $T(x')$  is also uniformly small, the problem can be formulated just near one boundary point. Similar to the local version of the problem, the local boundary rigidity problem and local geodesic ray transform were considered in [50, 47], and a generalization to local ray transforms along general smooth curves was studied in Chapter 2.

As mentioned above, our arguments give a reconstruction procedure for the diffeomorphism  $\gamma$  and the conformal factor  $\rho$ . The reconstruction procedure consists of two steps: step 1 (Section 4.2) is devoted to the recovery of a semigeodesic (isometric) copy of the metric  $g$  and the boundary restriction of the conformal factor from the inverse data. In step 2 (Section 4.3), we reconstruct  $\gamma$  and  $\rho$  by studying the relation between  $\gamma$  and conformal Killing vector fields on the semigeodesic copy  $D$ .

As noticed that the inverse dynamical problem for the wave equations (with boundary data) may be reduced by the boundary control method to the inverse kinematic problem with internal sources and then to the Yamabe problem [4]. It gives in our case the Cauchy problem for the Laplace operator. We take another approach by using conformal Killing vector fields, which are determined by a linear system of first order partial differential equations.

## 4.2 Recovery of the semigeodesic copy of the metric

Given  $x \in \gamma(D)$ , there is a unique geodesic  $\gamma_{x'}$  (normal to  $\partial M$ ) and  $0 \leq t < T(x')$  such that  $x = \gamma(x', t)$ . Here  $x'(x)$  is the geodesic projection of  $x$  on  $\Gamma$ , so  $t(x) = \tau(x, x'(x))$  is the distance from  $x$  to the boundary. Thus the travel time data  $\Omega(D)$  uniquely determines the semigeodesic coordinates of points in  $\gamma(D)$ .

Let the pair  $(x'(x), t(x))$  be the semigeodesic coordinate of the point  $x$ . Thus for any point  $y = (x', t) \in D$ , the function  $\tau(\gamma(x', t), x'')$  is known. Define

$$\lambda((x', t), x'') := \tau(\gamma(x', t), x''),$$

then  $\lambda(y, x'')$ ,  $y = (x', t)$  is the distance between points  $y \in D$  and  $x'' \in U(x')$  in the metric  $\tilde{g} := \gamma^*(g)$ . Note that we identify  $\Gamma$  with  $\Gamma \times \{0\}$ . We call  $\tilde{g}$  the *semigeodesic copy* of the

metric  $g$ .

We first recover the metric  $\tilde{g}$ . Notice that  $\gamma$ , now as an isometry, sends geodesics to geodesics, this implies

$$\tilde{g}_{kn}(x', 0) = \delta_{kn}, \quad x' \in \Gamma, \quad 1 \leq k \leq n,$$

where  $\delta_{ij}$  is the Kronecker delta. In local coordinates,  $y = (y^1, \dots, y^n)$ , where  $y^n = t$ , one has the following eikonal equation

$$1 = |\nabla\tau(x, x'')|_g^2 = |\tilde{\nabla}\lambda(y, x'')|_{\tilde{g}}^2 = \tilde{g}^{ij}(y) \frac{\partial\lambda(y, x'')}{\partial y^i} \frac{\partial\lambda(y, x'')}{\partial y^j}, \quad x'' \in U(x'). \quad (4.1)$$

Note that if  $y'$  is close enough to  $y$ , and  $x'' \in U(x'(y))$  sufficiently close to  $x'(y)$ , then  $x'' \in U(x'(y'))$  too. (4.1) gives a family of linear algebraic equations with respect to the contravariant components  $\tilde{g}^{ij}(y)$  of the metric  $\tilde{g}$ . For  $y \in D$  with  $y^n > 0$ , it is known that  $\tilde{g}^{ij}(y)$  can be recovered by the knowledge of  $\tilde{g}^{ij}(y)\partial_{y^i}\lambda(y, x''_k)\partial_{y^j}\lambda(y, x''_k)$  for  $N = n(n+1)/2$  “generic” points  $x''_k \in U(x'(y))$ ,  $k = 1, 2, \dots, N$ , see e.g. [37, 41]. Such  $N$  generic points always exist in a neighborhood of  $x'(y)$  in  $\Gamma$ . Thus  $\tilde{g}^{ij}(y)$  is determined for  $y \in D$ ,  $y^n > 0$ . By the smoothness of  $\tilde{g}$ , it also recovers  $\tilde{g}^{\alpha\beta}(x', 0)$ ,  $1 \leq \alpha, \beta \leq n-1$ ,  $x' \in \Gamma$ . So the metric  $\tilde{g}$  is uniquely determined.

Notice that the boundary restriction of the isometry  $\gamma$  is given, i.e.  $x' = x'(y^1, \dots, y^{n-1}, 0) = (x'^1, \dots, x'^n)$  as a point in  $\mathbb{R}^n$  is known, we recover the conformal factor  $\rho$  on  $\Gamma$ . Since  $\tilde{g} = \gamma^*(g)$  is the pullback, we have

$$\tilde{g}_{\alpha\beta}(x', 0) = \frac{\partial\gamma^k}{\partial y^\alpha} \frac{\partial\gamma^l}{\partial y^\beta} \rho(x') \delta_{kl} = \rho(x') \sum_{k=1}^n \frac{\partial x'^k}{\partial y^\alpha} \frac{\partial x'^k}{\partial y^\beta}, \quad \alpha, \beta = 1, \dots, n-1. \quad (4.2)$$

Thus equation (4.2) and the knowledge of  $\tilde{g}|_{t=0}$  together determine  $\rho|_\Gamma$ .

### 4.3 Recovery of the isometry and the conformal factor

To recover the conformal factor  $\rho$ , we need to solve the *pullback problem*, i.e. to find the map  $\gamma$ . To this end, we need some knowledge of conformal Killing vector fields. Recall that a vector field  $u$  is called a conformal Killing vector field if it satisfies the conformal Killing

equation ( in covariant form )

$$Ku := \sigma \nabla u - g \delta u / n = 0,$$

where  $\sigma \nabla$  is the symmetric part of the covariant derivative  $\nabla$ ,  $\delta$  is the divergence. They are exactly those vector fields whose flows preserve the conformal structures of the manifolds. In local coordinates, the conformal Killing equation has the form ( for covariant components )

$$\frac{1}{2} \left( \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - 2\Gamma_{ij}^k u_k \right) - \frac{1}{n} g_{ij} \left( g^{kl} \frac{\partial u_l}{\partial x^k} - g^{kl} u_m \Gamma_{kl}^m \right) = 0.$$

Thus the conformal Killing equation is equivalent to a linear system of first order partial differential equations.

However, not all of the metrics admit conformal Killing vector fields, actually for  $n \geq 3$ , a “generic” metric does not possess any non-trivial conformal Killing vector fields, see e.g. [2, 25]. On the other hand, note that the metric  $g$  is conformal to the Euclidean metric, thus they share the same set of conformal Killing vector fields. When  $n = 2$ , in the Cartesian coordinates  $(x^1, x^2)$  all conformal Killing vector fields of the Euclidean metric have the form  $u = (u^1, u^2)$ , where  $u^1$  and  $u^2$  are conjugate harmonic functions. In the case  $n > 2$ , the contravariant components of  $u$  in the Cartesian coordinates  $(x^1, \dots, x^n)$  are given by

$$u^i(x) = a_0 x^i + (Ax)^i - b^i |x|^2 + 2x^i(b, x) + c^i,$$

where  $a_0$  is a real constant,  $A$  is an  $n \times n$  skew-symmetric constant matrix,  $b$  and  $c$  are vectors in  $\mathbb{R}^n$ .

Now we are in a position to recover the map  $\gamma$  and the conformal factor  $\rho$ . Let  $e_{(j)} = \frac{\partial}{\partial x^j}$ ,  $j = 1, \dots, n$  be the standard basis vectors in  $\mathbb{R}^n$ . It is easy to see that they are conformal Killing vector fields in Euclidean metric, thus also conformal Killing vector fields for  $g$ . Then  $u_{(j)} = \gamma^* e_{(j)}$ ,  $j = 1, \dots, n$  are conformal Killing vector fields for the metric  $\tilde{g} = \gamma^* g$ . This implies  $u_{(j)}$ ,  $j = 1, \dots, n$  satisfy the conformal Killing equation

$$Ku_{(j)}(y) = 0, \quad y \in D, \quad j = 1, \dots, n.$$

It is known that  $u_{(j)}$  is uniquely determined by the Cauchy data  $\{u_{(j)}(x', 0) : x' \in \Gamma\}$ , see e.g. [26, 9]. Thus we calculate the Cauchy data of  $u_{(j)}$  first.

Since we have already recovered the semigeodesic copy  $\tilde{g}$ , we denote the dual vector of  $u_{(j)}$  by  $u^{(j)}$ . In the mean time, we denote the dual vector of  $e_{(j)}$  under the metric  $g = \rho e$  by  $w^{(j)}$ , then  $w^{(j)} = \rho dx^j$ . In local coordinates, the equality  $u_{(j)} = \gamma^* e_{(j)}$  means ( for covariant components )

$$u_i^{(j)}(y) = w_k^{(j)} \frac{\partial x^k}{\partial y^i} = \rho \frac{\partial \gamma^j(y)}{\partial y^i}.$$

This observation is crucial in our reconstruction procedure, it relates the isometry  $\gamma$  to the conformal Killing vector fields on the semigeodesic copy  $D$ . Thus at  $y^n = t = 0$ ,

$$u_\alpha^{(j)}(x', 0) = \rho(x') \frac{\partial \gamma^j}{\partial y^\alpha}(x', 0) = \rho(x') \frac{\partial x'^j}{\partial y^\alpha}, \quad \alpha = 1, \dots, n-1.$$

Since  $\rho(x')$  and  $\gamma(x', 0)$  are known for  $x' \in \Gamma$ ,  $u_\alpha^{(j)}(x', 0)$  are determined. To determine the value of  $u_n^{(j)}$  at  $t = 0$ , notice that  $\dot{\gamma}_{x'}(0) = \nu(x')$ , so

$$u_n^{(j)}(x', 0) = \rho(x') \frac{\partial \gamma^j}{\partial t}(x', 0) = \rho(x') \nu^j(x').$$

However,  $\nu$  is a unit vector w.r.t. metric  $g = \rho e$ , if we denote the inward unit normal vector on  $\partial M$  w.r.t. the Euclidean metric by  $\nu_0$ , then  $\nu = \frac{1}{\sqrt{\rho}} \nu_0$ , i.e.

$$u_n^{(j)}(x', 0) = \sqrt{\rho(x')} \nu_0^j(x').$$

Fortunately, the Cartesian coordinates of the hypersurface  $\Gamma$  are given, thus  $\nu_0$  as the normal vector to  $\Gamma$  is known. Together with the knowledge of  $\rho|_\Gamma$ , we recover  $u_n^{(j)}|_{t=0}$  too.

From the Cauchy data, we uniquely recover the conformal Killing vector fields  $u_{(j)} = \gamma^* e_{(j)}$ , equivalently the dual vector  $u^{(j)}$ . Now by defining  $v = (v^1, \dots, v^n)$ ,

$$v^j(x', t) := u_n^{(j)}(x', t) = \rho(\gamma(x', t)) \frac{\partial \gamma^j(x', t)}{\partial t},$$

we have  $v = \rho(\gamma) \dot{\gamma}$ . In the mean time, notice that

$$|v|^2 = \rho^2(\gamma) |\dot{\gamma}|^2 = \rho(\gamma) \quad (\text{since } |\dot{\gamma}|_g = 1), \quad (4.3)$$

we obtain

$$\dot{\gamma} = \frac{v}{|v|^2}.$$

This implies

$$\gamma(x', t) = \int_0^t \dot{\gamma}_{x'}(t) dt + x' = \int_0^t \frac{v}{|v|^2}(x', t) dt + x',$$

i.e. we recover the geodesics  $\gamma_{x'}(t)$ , therefore the diffeomorphism  $\gamma : D \rightarrow \gamma(D)$ , namely the range  $\gamma(D)$ . Moreover, by (4.3)

$$\rho(\gamma(x', t)) = |v(x', t)|^2,$$

the conformal factor  $\rho|_{\gamma(D)}$  is determined, and this finishes the proof of the main theorem.

## Appendix A

### GEOMETRIC PROPERTIES OF $\mathcal{MP}$ -SYSTEMS

#### A.1 Mañé's critical values

Here we adapt a certain part of the theory of convex superlinear Lagrangians to the case of manifolds with boundary, see also [8, Appendix A.1].

Let  $M$  be a compact Riemannian manifold with boundary and let  $L : TM \rightarrow \mathbb{R}$  be a  $C^\infty$  Lagrangian satisfying the following hypotheses:

- *Convexity:* For all  $x \in M$  the restriction of  $L$  to  $T_x M$  has everywhere positive definite Hessian.
- *Superlinear growth:*

$$\lim_{|\xi| \rightarrow \infty} \frac{L(x, \xi)}{|v\xi|} = +\infty$$

uniformly on  $x \in M$ .

The action of  $L$  on an absolutely continuous curve  $\gamma : [a, b] \rightarrow M$  is

$$\mathbb{A}_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt.$$

For each  $\lambda \in \mathbb{R}$ , the *Mañé action potential*  $\mathbb{A}_\lambda : M \times M \rightarrow \mathbb{R} \cup \{-\infty\}$  is defined by

$$\mathbb{A}_\lambda(x, y) = \inf_{\gamma \in \mathcal{C}(x, y)} \mathbb{A}_{L+\lambda}(\gamma),$$

where  $\mathcal{C}(x, y) = \{\gamma : [0, T] \rightarrow M : \gamma(0) = x, \gamma(T) = y, \gamma \text{ is absolutely continuous}\}$ .

The *critical level*  $c = c(L)$  is defined as

$$\begin{aligned} c(L) &= \sup\{\lambda \in \mathbb{R} : \mathbb{A}_{L+\lambda}(\gamma) < 0 \text{ for some closed curve } \gamma\} \\ &= \inf\{\lambda \in \mathbb{R} : \mathbb{A}_{L+\lambda}(\gamma) \geq 0 \text{ for every closed curve } \gamma\}. \end{aligned}$$

Recall that the *energy function*  $E : TM \rightarrow \mathbb{R}$  for  $L$  is defined by

$$E(x, \xi) = \frac{\partial L}{\partial \xi}(x, \xi) \cdot \xi - L(x, \xi),$$

and it is constant on every solution  $x(t)$  of the Euler–Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \xi}(x(t), \dot{x}(t)) = \frac{\partial L}{\partial x}(x(t), \dot{x}(t)). \quad (\text{A.1})$$

Let  $\psi^t : TM \rightarrow TM$  be the Euler–Lagrange flow, defined by  $\psi^t(x, \xi) = (\gamma(t), \dot{\gamma}(t))$ , where  $\gamma$  is the solution of (A.1) with  $\gamma(0) = x$  and  $\dot{\gamma}(0) = \xi$ . For  $x \in M$  and  $k \in \mathbb{R}$ , the *exponential map* at  $x$  of energy  $\lambda$  is defined to be the partial map  $\exp_x^\lambda : T_x M \rightarrow M$  given by

$$\exp_x^\lambda(t\xi) = \pi \circ \psi^t(\xi), \quad t \geq 0, \quad \xi \in T_x M, \quad E(x, \xi) = \lambda.$$

Then  $\exp_x^\lambda$  is a  $C^1$ -smooth partial map on  $T_x M$  which is  $C^\infty$ -smooth on  $T_x M \setminus \{0\}$ .

The next two propositions were proved in [8, Appendix A.1].

**Proposition A.1.1.** *If  $\exp_x^\lambda : (\exp_x^\lambda)^{-1}(M) \rightarrow M$  is a diffeomorphism for every  $x \in M$ , then  $\lambda \geq c(L)$ .*

**Proposition A.1.2.** *If  $\lambda > c(L)$  and  $x, y \in M$   $x \neq y$ , then there is  $\gamma \in \mathcal{C}(x, y)$  such that*

$$\mathbb{A}_\lambda(x, y) = \mathbb{A}_{L+\lambda}(\gamma).$$

Moreover, the energy of  $\gamma$  is  $E(\gamma, \dot{\gamma}) \equiv \lambda$ .

Now, we apply the above to the case of  $\mathcal{MP}$ -systems. For a simple  $\mathcal{MP}$ -system  $(M, g, \alpha, U)$ , the  $\mathcal{MP}$ -flow can also be obtained as the Euler–Lagrange flow with the corresponding Lagrangian defined by

$$L(x, \xi) = \frac{1}{2}|\xi|_g^2 - \alpha_x(\xi) - U(x).$$

**Lemma A.1.3.** *Let  $(g, \alpha, U)$  be a simple  $\mathcal{MP}$ -system on  $M$ . For  $x, y \in M$ ,  $x \neq y$ ,*

$$\mathbb{A}_k(x, y) = \mathbb{A}_{L+k}(\gamma_{x,y}) = 2kT_{x,y} - \int_{\gamma_{x,y}} (\alpha + 2U),$$

where  $\gamma_{x,y} : [0, T_{x,y}] \rightarrow M$  is the  $\mathcal{MP}$ -geodesic with constant energy  $k$  from  $x$  to  $y$ .

*Proof.* It is easy to see that the simplicity assumption implies that for this Lagrangian the assumptions of Proposition A.1.1 hold for all  $\lambda$  sufficiently close to  $k$ . Therefore, the proposition gives  $k > c(L)$ . Then Proposition A.1.2 shows that, given  $x \neq y$  in  $M$ , there is  $\gamma \in \mathcal{C}(x, y)$  with energy  $k$  such that  $\mathbb{A}(x, y) = \mathbb{A}(\gamma)$ . Using simplicity, one can then prove that  $\gamma$  is an  $\mathcal{MP}$ -geodesic with constant energy  $k$ , i.e.,  $\gamma = \gamma_{x,y}$ .  $\square$

## A.2 $\mathcal{MP}$ -convexity

Let  $M$  be a compact manifold with boundary, endowed with a Riemannian metric  $g$ , a closed 2-form  $\Omega$  and a smooth function  $U$ . Consider an open manifold  $\widetilde{M}$  such that  $\widetilde{M} \supset M$ , we extend  $g$ ,  $\Omega$  and  $U$  to  $\widetilde{M}$  smoothly, preserving the former notation for extensions. We say that  $M$  is  $\mathcal{MP}$ -convex at  $x \in \partial M$  if there is a neighborhood  $O$  of  $x$  in  $\widetilde{M}$  such that all  $\mathcal{MP}$ -geodesics of constant energy  $k$  in  $O$ , passing through  $x$  and tangent to  $\partial M$  at  $x$ , lie in  $\widetilde{M} \setminus M^{\text{int}}$ . If, in addition, these geodesics do not intersect  $M$  except for  $x$ , we say that  $M$  is *strictly*  $\mathcal{MP}$ -convex at  $x$ . It is not hard to show that these definitions depend neither on the choice of  $\widetilde{M}$  nor on the way we extend  $g$ ,  $\Omega$  and  $U$  to  $\widetilde{M}$ .

As before, we let  $\Lambda$  denote the second fundamental form of  $\partial M$  and  $\nu(x)$  the inward unit vector normal to  $\partial M$  at  $x$ .

**Lemma A.2.1.** *If  $M$  is  $\mathcal{MP}$ -convex at  $x \in \partial M$ , then*

$$\Lambda(x, \xi) \geq \langle Y_x(\xi), \nu(x) \rangle - d_x U(\nu(x)) \quad \text{for all } \xi \in S_x^k(\partial M). \quad (\text{A.2})$$

*If the inequality is strict, then  $M$  is strictly  $\mathcal{MP}$ -convex at  $x$ .*

*Proof.* Suppose  $M$  is  $\mathcal{MP}$ -convex at  $x$ . Choosing a smaller  $O$  if necessary, we may assume that there is a smooth function  $\rho$  on  $O$  such that  $|\text{grad } \rho| = 1$ ,  $M \cap O = \{x : \rho(x) \geq 0\}$  and  $\partial M \cap O = \rho^{-1}(0)$ . Further we may assume that all the above  $\mathcal{MP}$ -geodesics lie in  $O^- = \{x : \rho(x) \leq 0\}$ .

Let  $\xi \in S_x^k(\partial M)$  and  $\gamma(t)$  be the  $\mathcal{MP}$ -geodesic with  $\gamma(0) = x$ ,  $\dot{\gamma}(0) = \xi$ . By our assumption,  $\rho \circ \gamma(t) \leq 0$  for all small  $t$ . Therefore,

$$\left. \frac{d^2}{dt^2} \rho \circ \gamma(t) \right|_{t=0} \leq 0.$$

Since

$$\begin{aligned} \frac{d^2}{dt^2} \rho \circ \gamma(t) &= \frac{d}{dt} \langle \text{grad } \rho(\gamma(t)), \dot{\gamma}(t) \rangle \\ &= \langle \nabla_{\dot{\gamma}(t)} \text{grad } \rho(\gamma(t)), \dot{\gamma}(t) \rangle + \langle \text{grad } \rho(\gamma(t)), \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) \rangle \\ &= \text{Hess}_{\gamma(t)} \rho(\dot{\gamma}(t), \dot{\gamma}(t)) + \langle \text{grad } \rho(\gamma(t)), Y(\dot{\gamma}(t)) - \nabla U(\gamma(t)) \rangle \end{aligned}$$

and since  $\Lambda(x, \xi) = -\text{Hess}_x \rho(\xi, \xi)$  and  $\text{grad } \rho(x) = \nu(x)$  when  $(x, \xi) \in S^k(\partial M)$ , we obtain (A.2).

Now, assume that (A.2) is strict, then there is  $\delta > 0$  such that for every  $\mathcal{MP}$ -geodesic  $\gamma$  in  $\widetilde{M}$  with  $\gamma(0) = x$  and  $\dot{\gamma}(0) = \xi \in S_x^k(\partial M)$ ,

$$\left. \frac{d^2}{dt^2} \rho \circ \gamma(t) \right|_{t=0} \leq -\delta.$$

Thus, there is a small  $\varepsilon > 0$  such that

$$\rho \circ \gamma(t) \leq -\frac{1}{4} \delta t^2 \quad \text{for all } t \in (-\varepsilon, \varepsilon).$$

This proves the second statement.  $\square$

### A.3 Scattering relation

For  $(x, \xi) \in \partial_+ S^k M$ , let  $\tau(x, \xi)$  be the time when the  $\mathcal{MP}$ -geodesic  $\gamma_{x, \xi}$ , such that  $\gamma_{x, \xi}(0) = x$ ,  $\dot{\gamma}_{x, \xi}(0) = \xi$ , exists. Clearly, the function  $\tau(x, \xi)$  is continuous and, using the implicit function theorem, it is easy to see that  $\tau$  is smooth near a point  $(x, \xi)$  such that the  $\mathcal{MP}$ -geodesic  $\gamma_{x, \xi}(t)$  meets  $\partial M$  transversally at  $t = \tau(x, \xi)$ . By (3.3) and Lemma A.2.1, the last condition holds everywhere on  $\partial_+ S^k M \setminus S^k(\partial M)$ . Thus,  $\tau$  is a smooth function on  $\partial_+ S^k M \setminus S^k(\partial M)$ .

**Lemma A.3.1.** *For a simple  $\mathcal{MP}$ -system, the function  $\tau : \partial_+ S^k M \rightarrow \mathbb{R}$  is smooth.*

*Proof.* Let  $\rho$  be a smooth nonnegative function on  $M$  such that  $\partial M = \rho^{-1}(0)$  and  $|\text{grad } \rho| = 1$  in some neighborhood of  $\partial M$ . Put  $h(x, \xi, t) = \rho(\gamma_{x, \xi}(t))$  for  $(x, \xi) \in \partial_+ S^k M$ . Then

$$\begin{aligned} h(x, \xi, 0) &= 0, \\ \frac{\partial h}{\partial t}(x, \xi, 0) &= \langle \nu(x), \xi \rangle, \\ \frac{\partial^2 h}{\partial t^2}(x, \xi, 0) &= \text{Hess}_x \rho(\xi, \xi) + \langle \nu(x), Y(\xi) - \nabla U(x) \rangle. \end{aligned}$$

Therefore, for some smooth function  $R(x, \xi, t)$ ,

$$h(x, \xi, t) = \langle \nu(x), \xi \rangle t + \frac{1}{2} (\text{Hess}_x \rho(\xi, \xi) + \langle \nu(x), Y(\xi) - \nabla U(x) \rangle) t^2 + R(x, \xi, t) t^3.$$

Since  $h(x, \xi, \tau(x, \xi)) = 0$ , it follows that  $T = \tau(x, \xi)$  is a solution of the equation

$$F(x, \xi, T) := \langle \nu(x), \xi \rangle + \frac{1}{2} (\text{Hess}_x \rho(\xi, \xi) + \langle \nu(x), Y(\xi) - \nabla U(x) \rangle) T + R(x, \xi, T) T^2 = 0. \quad (\text{A.3})$$

By (3.3), for  $(x, \xi) \in S^k(\partial M)$

$$\begin{aligned} \frac{\partial F}{\partial T}(x, \xi, 0) &= \frac{1}{2} (\text{Hess}_x \rho(\xi, \xi) + \langle \nu(x), Y(\xi) - \nabla U(x) \rangle) \\ &= \frac{1}{2} (-\Lambda(x, \xi) + \langle \nu(x), Y(\xi) - \nabla U(x) \rangle) < 0. \end{aligned}$$

Now, the implicit function theorem yields smoothness of  $\tau(x, \xi)$  in a neighborhood of  $S^k(\partial M)$ .

Since  $\tau$  is also smooth on  $\partial_+ S^k M \setminus S^k(\partial M)$ , we conclude that  $\tau$  is smooth on  $\partial_+ S^k M$ .  $\square$

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