Homological algebra of Stanley–Reisner rings and modules

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Abstract

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Associated to each simplicial complex Δ and each field $\Bbbk$ is the Stanley–Reisner ring $\Bbbk[\Delta]$. The answers to a multitude of questions related to simplicial complexes have historically been found through a thorough examination of the algebraic structure of $\Bbbk[\Delta]$. There is a rich pre-existing body of literature equating combinatorial and topological statements about the structure of a simplicial complex with statements about $\Bbbk[\Delta]$; this dissertation expands upon the dictionary translating such statements by examining algebraic structures derived from $\Bbbk[\Delta]$. In particular, we mainly focus on the local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ and the Ext modules $\mathrm{Ext}^i(\Bbbk, \Bbbk[\Delta])$.

Roughly speaking, a simplicial complex is called Buchsbaum if its geometric realization is similar to a manifold. In Chapter 2, we study the homological structure of $\Bbbk[\Delta]$ and some of its quotients by linear forms when Δ fails to be Buchsbaum in a way that may be considered "minimal." We obtain a large family of rings with interesting combinations of the (ring-theoretic) properties of Buchsbaumness and quasi-Buchsbaumness, while developing a geometric interpretation of their presence.

In Chapter 3, we turn our attention to complexes that exhibit some degree of symmetry via group actions. Here it is shown that the induced action on $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ can be described in a similar manner to the one induced on the simplicial cohomology modules of Δ and some of its subcomplexes. Some applications to the study of face numbers are provided.

If the definition of a simplicial complex is slightly relaxed, then one arrives at the notion of a simplicial poset. Chapter 4 is devoted to the study of these objects and their associated face rings. We provide extensions of well-known results describing the structure of the Ext and local cohomology modules of simplicial complexes to this larger class and further examine the Buchsbaum property.

In Chapter 5, we study the class of balanced triangulations of manifolds and obtain lower bounds on entries in the h-vector phrased in terms of topological invariants. This proves a conjecture of Klee and Novik.

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Chapter 1

INTRODUCTION

Given a simplicial complex Δ on vertex set V, one can view the Stanley–Reisner ring $\Bbbk[\Delta]$ as the collection of all monomials in the polynomial ring $\Bbbk[x_v : v \in V]$ whose support corresponds to a face of Δ. In principle, every property imaginable of a simplicial complex Δ can be rephrased as an algebraic property of $\Bbbk[\Delta]$; indeed, two simplicial complexes are isomorphic if and only if their Stanley–Reisner rings are isomorphic $\Bbbk$-algebras [14]. The main question that this dissertation aims to answer is the following: to what extent can algebraic methods applied to $\Bbbk[\Delta]$ be used to examine the topological structure of Δ? As it turns out, a range of derived functors that may appear on the surface to have no direct geometric meaning provide a deep connection between homological properties of $\Bbbk[\Delta]$ and the topology of Δ.

In one way or another, the bulk of results presented here (Chapters 2 through 4) may be traced back to two small but powerful collections of previous work. In [23], Gräbe provides a beautiful decomposition of the local cohomology modules of $\Bbbk[\Delta]$ into a direct sum of the simplicial cohomology modules of Δ and of the links of its faces. Additionally, the module structure is described in terms of obvious simplicial maps. Meanwhile, in [42] and [41], Miyazaki similarly lays out the structures of a wide range of Ext modules involving $\Bbbk[\Delta]$ along with their canonical maps to local cohomology modules. Both of these results have played some part in all of the chapters that follow, and a core theme to all of the chapters is attempting to leverage or extend these astounding theorems.

A natural class of complexes to restrict one's attention to is that of triangulations of manifolds or, more generally, Buchsbaum complexes (those in which the link of each non-empty face has potentially non-trivial reduced homology only in the top dimension). In this case, many properties of the local cohomology modules of generic Artinian reductions may be described simply and succinctly. These descriptions have proved to be a powerful tool in the study of the f- and h-vectors of triangulations of manifolds. Unfortunately, without the Buchsbaum assumption, this approach tends to break down completely. Chapter 2 aims to remedy this situation through a careful examination of the local cohomology modules of a Stanley–Reisner ring's quotient by a partial linear system of parameters. In particular, the foundational results due to Gräbe mentioned above are traced through successive quotients and we examine to what extent the geometric interpretation remains intact. This extends some results due to Miller, Novik, and Swartz in [38] and Novik and Swartz in [55].

In [65] and [1], Stanley and Adin (respectively) showed that a group action on a simplicial complex Δ naturally extends to a group action on $\Bbbk[\Delta]$ and that a refinement of standard Hilbert series arguments provides much stronger bounds on the h-vector when Δ is Cohen–Macaulay (e.g., a triangulation of a sphere). In Chapter 3, we consider to what degree a group action on Δ can be translated to representation-theoretic statements when the action is carried through to the local cohomology modules of $\Bbbk[\Delta]$. In particular, it is shown that the isotypic components of local cohomology modules are in direct correspondence with their topological counterparts. Once this connection has been established, we extend Stanley and Adin's results on h-vectors to Buchsbaum complexes admitting free group actions.

Simplicial complexes themselves are an example of a wider class of objects known as simplicial posets. The definition is simple: a poset is simplicial if it has a zero element and if every interval is a boolean lattice. Geometrically, a simplicial poset is formed by gluing together simplices with the restriction that the intersection of any two simplices is a union of simplices (in the case of a simplicial complex, this intersection is required to be a simplex). Associated to each simplicial poset is a face ring, which is once again formed as a certain quotient of a polynomial ring. In the case of a simplicial complex, the resulting ring is isomorphic to the standard Stanley–Reisner ring. For more general posets, the structure and properties of this ring are not only much less intuitive, but much less well-studied. Duval proved in [15] the $\Bbbk$-vector space isomorphism part of Gräbe's theorem for face rings of simplicial posets, but the module structure along with an analog of Miyazaki's results have not previously been considered. In Chapter 4 we extend both of these structural descriptions to face rings, allowing also for a quick characterization of Buchsbaumness of a poset.

Finally, a dissertation in this field would feel incomplete without an investigation of some depth into the growth properties of the h-vector of a simplicial complex. To this end, in Chapter 5 we examine the h-vectors of balanced homology manifolds; a $(d-1)$-dimensional simplicial complex is called balanced if its underlying graph may be properly colored with d colors. Such complexes arise naturally in many situations (for example, the barycentric subdivision of any simplicial poset is always balanced), and statements about more general complexes can often be greatly strengthened when restricting attention to the balanced case. As it turns out, the balanced condition allows for a rich graded structure to be placed on $\Bbbk[\Delta]$ that provides a direct link to the notion of the flag h-vector of a simplicial complex. By relating the flag h-vector of Δ to the standard h-vectors of various rank-selected subcomplexes, preexisting generalized lower bound theorems can be improved upon for the class of balanced manifolds. This chapter is based on the article [31] written with Martina Juhnke-Kubitzke, Satoshi Murai, and Isabella Novik.

Each chapter after the first is representative of an individual paper, all of which have been accepted for publication. Notation will be re-introduced in each chapter to suit the needs of its results, and each may be read independently.

Chapter 2

ALMOST BUCHSBAUMNESS

2.1 Introduction

Many combinatorial, algebraic, and topological statements about polytopes and triangulations of spheres or manifolds have been proven through the study of their Stanley–Reisner rings. These rings are well-understood, and the translation of their algebraic properties into combinatorial and topological invariants has a storied and celebrated past. The usefulness of this approach is made apparent by its presence in decades of continued progressive research (excellent surveys may be found in [67] and [33]).

In contrast, the main objects considered in this chapter are simplicial complexes with isolated singularities. A simplicial complex Δ has isolated singularities if it is pure and the link in Δ of every face of dimension at least 1 is Cohen-Macaulay (more precise definitions will be provided later). Common examples are provided by triangulations of a pinched torus, the suspension of a manifold, or more generally by pseudomanifolds that fail to be manifolds at finitely many points (see [18] for an in-depth discussion). The gap between pseudomanifolds and manifolds is well understood from a topological viewpoint, but there are powerful tools available to the Stanley–Reisner rings of triangulations of manifolds that presently lack any meaningful extension to the world of pseudomanifolds. For instance, unlike results due to Stanley ([62]) and Schenzel ([60]), we do not know the Hilbert series of a generic Artinian reduction of the Stanley–Reisner ring of such a complex, even when considering a triangulation of the suspension of a manifold that is not a homology sphere.

The central obstruction in extending the preexisting knowledge of triangulations of manifolds to the singular case is the Stanley–Reisner ring failing to be Buchsbaum. This roadblock may sometimes by circumvented by instead considering $\mathbb{k}[\Delta]/(\theta_1)\mathbb{k}[\Delta]$, the Stanley–Reisner ring modulo one generic linear form. This ring has some very nice properties when Δ has isolated singularities; for instance, Miller, Novik, and Swartz have shown that the local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta]/(\theta_1)\Bbbk[\Delta])$ have a simple geometric interpretation. It is rather immediate from their calculations that $\Bbbk[\Delta]/(\theta_1)\Bbbk[\Delta]$ is Buchsbaum if and only if $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(\Bbbk[\Delta]/(\theta_1)\Bbbk[\Delta]) = 0$, and it is shown in [38] that this condition is satisfied in the case that the singularities of Δ are *homologically isolated* (we will effectively show in Proposition 2.4.5 that the converse is true as well). Having established the Buchsbaum property, Novik and Swartz were able to calculate the Hilbert series of generic Artinian reductions of the Stanley–Reisner rings of complexes with homologically isolated singularities as well as prove singular analogs of the Dehn-Sommerville relations in [55].

Many natural questions arise following these pre-existing results, and principal among them is the following: If the singularities of Δ are not homologically isolated, then what nice properties may appear when the Stanley–Reisner ring is reduced by an additional generic linear form? Most optimistically, one may hope that $\Bbbk[\Delta]/(\theta_1, \theta_2)\Bbbk[\Delta]$ is always Buchsbaum in this situation. To this end, our primary motivation lies in examining this possibility through a precise investigation of how the topological properties of singular vertices manifest in the algebraic setting of Stanley–Reisner rings. This study leads to the notion of *generically isolated* singularities, defined in Section 2.2. This property plays a crucial role in our main results, and will allow us to characterize the degree to which $\Bbbk[\Delta]/(\theta_1, \theta_2)\Bbbk[\Delta]$ may fail to be Buchsbaum as follows.

Theorem 2.1.1. *Let Δ be a connected simplicial complex with isolated singularities on vertex set V, let $\Bbbk$ be an infinite field, and denote by A the polynomial ring $\Bbbk[x_v : v \in V]$ and by $\mathfrak{m}$ the irrelevant ideal of A. If θ_1, θ_2 is a generic regular sequence for the Stanley-Reisner ring $\Bbbk[\Delta]$, then:*

(i) The local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta]/(\theta_1, \theta_2)\Bbbk[\Delta])$ satisfy

$$\mathfrak{m} \cdot H^i_{\mathfrak{m}}(\Bbbk[\Delta]/(\theta_1, \theta_2)\Bbbk[\Delta]) = 0$$

for all i if and only if the singularities of Δ are generically isolated.

(ii) The canonical maps of graded modules

$$\varphi^i : \mathrm{Ext}^i_A(\Bbbk, \Bbbk[\Delta]/(\theta_1, \theta_2)\Bbbk[\Delta]) \to H^i_{\mathfrak{m}}(\Bbbk[\Delta]/(\theta_1, \theta_2)\Bbbk[\Delta])$$

are surjective in all degrees except (possibly) 0 *if and only if the singularities of* Δ *are generically isolated.*

A few notes on why the properties of Theorem 2.1.1 hold some degree of importance are in order. The condition $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(\Bbbk[\Delta]/(\theta_1, \theta_2)\Bbbk[\Delta]) = 0$ establishes a strong necessary condition for a ring to be Buchsbaum (see [21, Corollary 1.5]), known as quasi-Buchsbaumness. These rings were first introduced by Goto and Suzuki ([21]). As with Buchsbaum rings, the properties and various characterizations of quasi-Buchsbaum rings have long been of some interest (see, e.g., [69] and [75]). The usefulness of the property is evidenced, for example, by its equivalence to Buchsbaumness in some special cases ([68, Corollary 3.6]). On the other hand, the surjectivity of the maps φ^i is incredibly near to one characterization of Buchsbaumness (see Theorem 2.4.1).

As we will see in Corollary 2.4.7, if Γ is a Buchsbaum (but not Cohen-Macaulay) complex and Δ is an arbitrary triangulation of the geometric realization of the suspension of Γ, then $\Bbbk[\Delta]/(\theta_1, \theta_2)\Bbbk[\Delta]$ is often not Buchsbaum. Though Vogel ([72]) and Goto ([20]) provided initial examples of quasi-Buchsbaum but not Buchsbaum rings, here we exhibit an infinite family of quasi-Buchsbaum rings of arbitrary dimensions and varying depths which fail in a geometrically tangible way to be fully Buchsbaum in only the slightest sense.

The structure of this chapter is as follows. In Section 2.2 we provide definitions and foundational results, allowing for some initial computations in Section 2.3. We prove our main results in Section 2.4. Section 2.5 is devoted to applications: we will provide a wide range of examples, then use some properties of quasi-Buchsbaum rings to calculate the Hilbert series of a certain Artinian reduction of the Stanley–Reisner ring of a complex with generically isolated singularities. We will close with comments and open problems in Section 2.6.

2.2 Preliminaries

This chapter has been largely influenced by the works of Miller, Novik, and Swartz in [38] and Novik and Swartz in [55]. In order to retain consistency with these references, much of their notation will be adopted for our uses as well.

2.2.1 Combinatorics and topology

A **simplicial complex** Δ with vertex set V is a collection of subsets of V that is closed under inclusion. The elements of Δ are called **faces**, and the **dimension** of a face F is $\dim F := |F| - 1$. The 0-dimensional faces in Δ are called **vertices**, and the maximal faces under inclusion are called **facets**. We say that Δ is **pure** if all facets have the same dimension. The dimension of the complex Δ is $\dim \Delta := \max\{\dim F : F \in \Delta\}$. For the remainder of this chapter, unless stated otherwise we will assume that a simplicial complex Δ is pure of dimension $d-1$ with vertex set V.

The **link** of a face F is the subcomplex of Δ defined by

$$\operatorname{lk}_\Delta F = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\},$$

and the **contrastar** of a face F is defined by

$$\operatorname{cost}_\Delta F = \{G \in \Delta : F \not\subset G\}.$$

In the case that $F = \{v\}$ is a vertex, we write $\operatorname{lk}_\Delta v$ and $\operatorname{cost}_\Delta v$ instead of $\operatorname{lk}_\Delta\{v\}$ and $\operatorname{cost}_\Delta\{v\}$, respectively. If $W \subset V$, then the **induced subcomplex** Δ_W is the simplicial complex $\{F \in \Delta : F \subset W\}$. For $0 \leq i \leq d-1$, the complex $\Delta^{(i)} := \{F \in \Delta : |F| \leq i+1\}$ is the ***i*-skeleton** of Δ.

Given a field $\Bbbk$, denote by $H_i(\Delta)$ and $H^i(\Delta)$ the ith (simplicial) homology and cohomology groups of Δ computed over $\Bbbk$, respectively (definitions and further resources may be found in [26]). If F is a face of Δ, we denote by $H^i_F(\Delta)$ the relative (simplicial) cohomology group $H^i(\Delta, \operatorname{cost}_\Delta F)$ (in line with other conventions, we write $H^i_v(\Delta)$ for $H^i_{\{v\}}(\Delta)$). It will often be helpful to identify $H^i_F(\Delta)$ with $\tilde{H}^{i-|F|}(\mathrm{lk}_\Delta F)$ (see, e.g., [23, Section 1.3]); in particular, note that $H^i_\emptyset(\Delta) = \tilde{H}^i(\Delta)$, the reduced cohomology group of Δ. Finally, let

$$\iota^i_F : H^i_F(\Delta) \to H^i_\emptyset(\Delta)$$

be the map induced by the inclusion $(\Delta, \emptyset) \to (\Delta, \mathrm{cost}\, F)$.

If $H^0_\emptyset(\Delta) = 0$, then we call Δ **connected**. We say that a face F of Δ is **singular** if $H^i_F(\Delta) \neq 0$ for some $i < d-1$. Conversely, if $H^i_F(\Delta) = 0$ for all $i < d-1$, we call F a **nonsingular** face. We call Δ **Cohen-Macaulay** over $\Bbbk$ if every face of Δ (including $\emptyset$) is nonsingular, and we call Δ **Buchsbaum** over $\Bbbk$ if it is pure and every face aside from $\emptyset$ is nonsingular.

If Δ contains a singular face, then we define the **singularity dimension** of Δ to be $\max\{\dim F : F \in \Delta \text{ and } F \text{ is singular}\}$. If the singularity dimension of Δ is 0, then we say that Δ has **isolated singularities**. Such complexes will be our main objects of study. As a special case, if the images of the maps $\iota^i_v : H^i_v(\Delta) \to H^i_\emptyset(\Delta)$ are linearly independent as vector subspaces of $H^i_\emptyset(\Delta)$, then we call the singularities of Δ **homologically isolated**. Equivalently, the singularities of Δ are homologically isolated if the kernel of the sum of maps

$$\left(\sum_{v\in V} \iota^i_v\right) : \bigoplus_{v\in V} H^i_v(\Delta) \to H^i_\emptyset(\Delta)$$

decomposes as the direct sum

$$\bigoplus_{v\in V} \left(\mathrm{Ker}\,\iota^i_v : H^i_v(\Delta) \to H^i_\emptyset(\Delta)\right).$$

Lastly, we call the singularities of Δ **generically isolated** if for sufficiently generic choices of coefficients $\{\alpha_v : v \in V\}$ and $\{\gamma_v : v \in V\}$, the two maps θ_α and θ_γ defined by

$$\left(\sum_{v\in V} \alpha_v \iota^i_v\right) : \bigoplus_{v\in V} H^i_v(\Delta) \to H^i_\emptyset(\Delta) \quad \text{and} \quad \left(\sum_{v\in V} \gamma_v \iota^i_v\right) : \bigoplus_{v\in V} H^i_v(\Delta) \to H^i_\emptyset(\Delta),$$

respectively, satisfy

$$(\mathrm{Ker}\,\theta_\alpha) \cap (\mathrm{Ker}\,\theta_\gamma) = \bigoplus_{v\in V} \left(\mathrm{Ker}\,\iota^i_v : H^i_v(\Delta) \to H^i_\emptyset(\Delta)\right)$$

for all i.

2.2.2 The connection to algebra

For the remainder of the chapter, let $\Bbbk$ be a fixed infinite field. Define $A := \Bbbk[x_v : v \in V]$ and let $\mathfrak{m} = (x_v : v \in V)$ be the irrelevant ideal of A. If $F \subset V$, let

$$x_F = \prod_{v \in F} x_v$$

and define the **Stanley-Reisner ideal** I_Δ by

$$I_\Delta = (x_F : F \notin \Delta).$$

The ring $\Bbbk[\Delta] := A/I_\Delta$ is the **Stanley-Reisner ring** of Δ. We will usually consider $\Bbbk[\Delta]$ as an A-module that is graded with respect to $\mathbb{Z}$ by degree.

Given any $\mathbb{Z}$-graded A-module M of Krull dimension d, we denote by M_j the collection of homogeneous elements of M of degree j. A sequence $\Theta = (\theta_1, \theta_2, \ldots, \theta_d)$ of homogeneous elements of A is called a **homogeneous system of parameters** (or **h.s.o.p.**) for M if $M/\Theta M$ is a finite-dimensional vector space over $\Bbbk$. In the case that each θ_i is linear, we call Θ a **linear system of parameters** (or **l.s.o.p.**) for M. If

$$(\theta_1, \ldots, \theta_{i-1})M :_M \theta_i = (\theta_1, \ldots, \theta_{i-1})M :_M \mathfrak{m}$$

for $1 \leq i \leq d$ for any choice of l.s.o.p. Θ, then we call M **Buchsbaum**. The reasoning behind the geometric definition of a Buchsbaum simplicial complex is made apparent in the following theorem due to Schenzel ([60]):

Theorem 2.2.1. *A pure simplicial complex Δ is Buchsbaum over $\Bbbk$ if and only if $\Bbbk[\Delta]$ is a Buchsbaum A-module.*

Two collections of objects that are most vital to our results are the A-modules $\mathrm{Ext}^i_A(\Bbbk, \Bbbk[\Delta])$ and $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$. An excellent resource for their construction and basic properties is [29]. In the case of Stanley-Reisner rings, the $\mathbb{Z}$-graded structure of $\mathrm{Ext}^i_A(\Bbbk, \Bbbk[\Delta])$ is provided by [42, Theorem 1]:

Theorem 2.2.2 (Miyazaki)**.** *Let Δ be a simplicial complex. Then as vector spaces over $\Bbbk$,*

$$\operatorname{Ext}^i_A(\Bbbk, \Bbbk[\Delta])_j \cong \begin{cases} 0 & j < -1 \text{ or } j > 0 \\ H^{i-1}_{\emptyset}(\Delta) & j = 0 \\ \bigoplus_{v \in V} H^{i-2}_{\emptyset}(\operatorname{cost}_\Delta v) & j = -1. \end{cases}$$

The local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ may be computed as the direct limit of the $\operatorname{Ext}^i_A(A/\mathfrak{m}_l, \Bbbk[\Delta])$ modules (see [42, Corollary 1]), where $\mathfrak{m}_l = (x_1^l, x_2^l, \dots, x_n^l)$. However, we will also need a thorough understanding of the A-module structure of the local cohomology modules of $\Bbbk[\Delta]$. The necessary details are provided by the following special case of [23, Theorem 2]:

Theorem 2.2.3 (Gräbe)**.** *Let Δ be a $(d-1)$-dimensional simplicial complex with isolated singularities and let $0 \le i \le d$. Then*

$$H^i_{\mathfrak{m}}(\Bbbk[\Delta])_j \cong \begin{cases} 0 & j > 0 \\ H^{i-1}_{\emptyset}(\Delta) & j = 0 \\ \bigoplus_{v \in V} H^{i-1}_{v}(\Delta) & j < 0 \end{cases}$$

as $\Bbbk$-vector spaces. If $\alpha_v \in H^{i-1}_v(\Delta)$, then the A-module structure on $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ is given by

$$\begin{array}{rcc} \cdot x_u : H^i_{\mathfrak{m}}(\Bbbk[\Delta])_{j-1} & \longrightarrow & H^i_{\mathfrak{m}}(\Bbbk[\Delta])_j \\ \alpha_v & \longmapsto & \begin{cases} \alpha_v & j < 0 \text{ and } u = v \\ \iota^{i-1}_v(\alpha_v) & j = 0 \text{ and } u = v \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

The full statement of Theorem 2.2.3 shows that a pure simplicial complex Δ is Buchsbaum if and only if it is pure and $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ is concentrated in degree 0 for all $i \neq d$. We note at this point that an A-module M is called **quasi-Buchsbaum** if $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(M) = 0$ for all i. Additionally, the full statement implies that $\Bbbk[\Delta]$ is Buchsbaum if and only if it is quasi-Buchsbaum; this is certainly not the case for general modules, as we will see!

Elements and homomorphisms related to $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ will usually be represented and referenced according to the topological identifications above; these identifications will also be expanded as we progress. As a motivating example, consider the $\Bbbk$-vector space

$$K^i := \bigoplus_{v\in V} \left(\operatorname{Ker} \iota_v^{i-1} : H_v^{i-1}(\Delta) \to H_\emptyset^{i-1}(\Delta)\right).$$

This space is central to the definition of homological isolation of singularities. Sometimes, it will be easier to identify K^i with its counterpart in local cohomology. If we denote by $H^i_{\mathfrak{m}}(\Bbbk[\Delta])_v$ the direct summand of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])_{-1}$ corresponding to $H_v^{i-1}(\Delta)$ in Theorem 2.2.3, then K^i is identified with the submodule

$$\bigoplus_{v\in V} \operatorname{Ker}\left(\cdot x_v : H^i_{\mathfrak{m}}(\Bbbk[\Delta])_v \to H^i_{\mathfrak{m}}(\Bbbk[\Delta])_0\right)$$

of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$. In light of how intertwined these two objects are, we will use the notation K^i interchangeably between the two settings. In local cohomology, the equality

$$K^i = \operatorname{Ker}\left(\sum_{v\in V} \cdot x_v : \bigoplus_{v\in V} H^i_{\mathfrak{m}}(\Bbbk[\Delta])_v \to H^i_{\mathfrak{m}}(\Bbbk[\Delta])_0\right)$$

is equivalent to the singularities of Δ being homologically isolated. In the same way, if θ_1 and θ_2 are generic linear forms, then the generic isolation of the singularities of Δ is equivalent to the equality

$$\operatorname{Ker}\left(\cdot\theta_1 : \bigoplus_{v\in V} H^i_{\mathfrak{m}}(\Bbbk[\Delta])_v \to H^i_{\mathfrak{m}}(\Bbbk[\Delta])_0\right) \cap \operatorname{Ker}\left(\cdot\theta_2 : \bigoplus_{v\in V} H^i_{\mathfrak{m}}(\Bbbk[\Delta])_v \to H^i_{\mathfrak{m}}(\Bbbk[\Delta])_0\right) = K^i$$

holding for all i.

Remark 2.2.4. In their original statements, the above structure theorems are written with respect to a $\mathbb{Z}^{|V|}$-grading and are proved by examining the chain complex $\operatorname{Hom}_A(\mathcal{K}^l_\cdot, \Bbbk[\Delta])$, where $\mathcal{K}^l_\cdot$ is the Koszul complex of A with respect to $\mathfrak{m}_l$. When coarsening to a $\mathbb{Z}$-grading, this chain complex becomes much larger. However, an argument similar to Reisner's original proof that $H^i_{\mathfrak{m}}(k[\Delta])_0 \cong H_\emptyset^{i-1}(\Delta)$ (see [57], pp. 41-42) shows that the only potentially non-acyclic components of $\operatorname{Hom}_A(\mathcal{K}^l_\cdot, M)$ under a $\mathbb{Z}$-grading are those also appearing in the $\mathbb{Z}^{|V|}$-graded complex.

2.3 Auxiliary calculations

Unless stated otherwise, we will always assume that Δ is a connected $(d-1)$-dimensional simplicial complex with isolated singularities. Let V be the vertex set of Δ and set $A := \mathbb{k}[x_v : v \in V]$. We will always consider $R := \mathbb{k}[\Delta]$ as an A-module. If $\theta_1, \ldots, \theta_d$ is a homogeneous system of parameters for Δ, we denote $R^i := \mathbb{k}[\Delta]/(\theta_1, \ldots, \theta_i)\mathbb{k}[\Delta]$.

Since Δ is connected, the 1-skeleton $\Delta^{(1)}$ of Δ is Cohen-Macaulay and the depth of R is at least 2 (see [28, Corollary 2.6]). Hence, there exists a homogeneous system of parameters $\theta_1, \ldots, \theta_d$ for Δ in which θ_1, θ_2 are linear and form the beginning of a regular sequence for R, i.e., θ_1 is a non-zero-divisor on R and θ_2 is a non-zero-divisor on R^1. Generically, we may assume that θ_1 and θ_2 both have non-zero coefficients on all x_v's. Unless stated otherwise, we will always work with such a system of parameters for Δ.

Our results primarily depend upon an understanding of the A-modules $H^i_{\mathfrak{m}}(R^j)$. We begin by computing their dimensions over $\mathbb{k}$ when $j = 1$ or 2 and discussing some connections to the topology of Δ.

2.3.1 Local cohomology

Consider the exact sequence of A-modules

$$0 \to R \xrightarrow{\cdot\theta_1} R \xrightarrow{\pi} R^1 \to 0.$$

By looking at graded pieces of this sequence, there are exact sequences of vector spaces over $\mathbb{k}$ of the form

$$0 \to R_{j-1} \xrightarrow{\cdot\theta_1} R_j \xrightarrow{\pi} R^1_j \to 0. \tag{2.1}$$

These sequences induce the following long exact sequence in local cohomology, where $\theta_1^{i,j} : H^i_{\mathfrak{m}}(R)_{j-1} \to H^i_{\mathfrak{m}}(R)_j$ is the map induced by multiplication, π is the map induced by the projection $R \to R^1$, and δ is the connecting homomorphism:

$$H^i_{\mathfrak{m}}(R)_{j-1} \xrightarrow{\theta_1^{i,j}} H^i_{\mathfrak{m}}(R)_j \xrightarrow{\pi} H^i_{\mathfrak{m}}(R^1)_j \xrightarrow{\delta} H^{i+1}_{\mathfrak{m}}(R)_{j-1} \xrightarrow{\theta_1^{i+1,j}} H^{i+1}_{\mathfrak{m}}(R)_j.$$

In light of Theorem 2.2.3, we make the following conclusions. When $j > 1$, all terms are zero. When $j = 1$, δ is an isomorphism. When $j \leq -1$, each $\theta_1^{i,j}$ is an isomorphism (all coefficients of θ_1 are non-zero). When $j = 0$, we obtain the short exact sequence

$$0 \to \operatorname{Coker} \theta_1^{i,0} \to H^i_{\mathfrak{m}}(R^1)_0 \to \operatorname{Ker} \theta_1^{i+1,0} \to 0. \tag{2.2}$$

Hence, as $\Bbbk$-vector spaces,

$$H^i_{\mathfrak{m}}(R^1)_j \cong \begin{cases} H^i_{\emptyset}(\Delta) & j = 1 \\ \operatorname{Coker} \theta_1^{i,0} \oplus \operatorname{Ker} \theta_1^{i+1,0} & j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

It will be useful to keep in mind that $\operatorname{Coker} \theta_1^{i,0}$ is identified with a quotient of $H^{i-1}_{\emptyset}(\Delta)$ and that $\operatorname{Ker} \theta_1^{i+1,0}$ is identified with a submodule of $\oplus_{v \in V} H^i_v(\Delta)$. In Section 2.4, we will leverage the "geometric" A-module structures of $\operatorname{Coker} \theta_1^{i,0}$ and $\operatorname{Ker} \theta_1^{i+1,0}$ along with (2.2) to further analyze $H^i_{\mathfrak{m}}(R^1)$.

Now we repeat this argument; consider the short exact sequence

$$0 \to R^1_{j-1} \xrightarrow{\cdot \theta_2} R^1_j \xrightarrow{\pi} R^2_j \to 0 \tag{2.4}$$

of vector spaces over $\Bbbk$, giving rise to the long exact sequence

$$H^i_{\mathfrak{m}}(R^1)_{j-1} \xrightarrow{\theta_2^{i,j}} H^i_{\mathfrak{m}}(R^1)_j \xrightarrow{\pi} H^i_{\mathfrak{m}}(R^2)_j \xrightarrow{\delta} H^{i+1}_{\mathfrak{m}}(R^1)_{j-1} \xrightarrow{\theta_2^{i+1,j}} H^{i+1}_{\mathfrak{m}}(R^1)_j.$$

As in the previous computation, all terms are zero when $j < 0$ or $j > 2$, π is an isomorphism when $j = 0$, and δ is an isomorphism when $j = 2$. When $j = 1$, we have the exact sequence

$$0 \to \operatorname{Coker} \theta_2^{i,1} \to H^i_{\mathfrak{m}}(R^2)_1 \to \operatorname{Ker} \theta_2^{i+1,1} \to 0. \tag{2.5}$$

Hence, as vector spaces over $\Bbbk$,

$$H^i_{\mathfrak{m}}(R^2)_j \cong \begin{cases} H^{i+1}_{\emptyset}(\Delta) & j = 2 \\ \operatorname{Coker} \theta_2^{i,1} \oplus \operatorname{Ker} \theta_2^{i+1,1} & j = 1 \\ \operatorname{Coker} \theta_1^{i,0} \oplus \operatorname{Ker} \theta_1^{i+1,0} & j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

2.3.2 Local cohomology: suspensions

We will now briefly consider the special case of suspensions. Suppose Δ is an arbitrary triangulation of the suspension of a $(d-2)$-dimensional manifold that is not Cohen-Macaulay, with suspension points a and b (so that a and b are isolated singularities of Δ). In this context, the maps ι^i_a and ι^i_b from Section 2.2 are isomorphisms. If g^i is a generator for $H^i_\emptyset(\Delta)$, denote $g^i_a := (\iota^i_a)^{-1}(g) \in H^i_a(\Delta)$ and $g^i_b := (\iota^i_b)^{-1}(g) \in H^i_b(\Delta)$. As usual, we will consider these generators to be interchangeable with their corresponding elements in $H^{i+1}_{\mathfrak{m}}(R)$.

Examining the sequence in (2.2) for this special case, suppose $\theta_1 = \sum_{v\in V} x_v$ and $\theta_2 = \sum_{v\in V} c_v x_v$ with $c_a \neq c_b$ and $c_a, c_b \neq 0$. Given g^{i-1} a generator of $H^{i-1}_\emptyset(\Delta)$, the map $\theta^{i,0}_1$ acts via $\theta^{i,0}_1(g^{i-1}_a) = \theta^{i,0}_1(g^{i-1}_b) = g^{i-1}$. Hence, $\theta^{i,0}_1$ is a surjection whose kernel is generated as a direct sum by elements of the form $(g^{i-1}_a - g^{i-1}_b)$. In particular, $H^i_{\mathfrak{m}}(R^1)_0 \cong \mathrm{Ker}\,(\theta^{i+1,0}_1) \cong H^i_\emptyset(\Delta)$. In summary,

$$H^i_{\mathfrak{m}}(R^1)_j \cong \begin{cases} H^i_\emptyset(\Delta) & j = 1 \\ H^i_\emptyset(\Delta) & j = 0 \\ 0 & \text{otherwise,} \end{cases}$$

under the aforementioned isomorphisms.

Now repeat the process above using the sequence in (2.5). If g^i is a generator of $H^i_\emptyset(\Delta)$, then $\theta^{i+1,0}_2$ acts on $H^{i+1}_{\mathfrak{m}}(R)_{-1}$ via $\theta^{i+1,0}_2(g^i_a) = c_a g^i$ and $\theta^{i+1,0}_2(g^i_b) = c_b g^i$. In particular, identifying $H^{i+1}_{\mathfrak{m}}(R^1)_0$ with the subspace $\mathrm{Ker}\,(\theta^{i+1,0}_1)$ of $H^{i+1}_{\mathfrak{m}}(R)_1$, the induced action of $\theta^{i+1,1}_2$ is given by $\theta^{i+1,1}_2(g^i_a - g^i_b) = (c_a - c_b)g^i \in H^i_\emptyset(\Delta) \cong H^{i+1}_{\mathfrak{m}}(R^1)_1$. That is, $\theta^{i+1,1}_2$ is an isomorphism as long as $c_a \neq c_b$ (note also that the singularities of Δ are generically isolated); this means that $H^i_{\mathfrak{m}}(R^2)_1 = 0$. In summary:

$$H^i_{\mathfrak{m}}(R^2)_j \cong \begin{cases} H^{i+1}_\emptyset(\Delta) & j = 2 \\ H^i_\emptyset(\Delta) & j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Since $H^i_{\mathfrak{m}}(R^2)_1 = 0$, is is immediate that $H^i_{\mathfrak{m}}(R_2)$ is quasi-Buchsbaum for all i. The specific choice of θ_1 was made for the ease of calculation. For sufficiently generic choices of θ_1 and

θ_2, the same isomorphisms hold. In particular, it is evident that the singularities of Δ are generically isolated.

2.4 Results

2.4.1 Buchsbaumness

We now move on to showing whether or not certain modules are Buchsbaum. For this, the following theorem ([68, Theorem I.3.7]) is vital.

Theorem 2.4.1. *Let $\Bbbk$ be an infinite field, with M a noetherian graded A-module and $d := \dim M > 0$. Then M is a Buchsbaum module if and only if the natural maps $\varphi_M^i : \operatorname{Ext}_A^i(\Bbbk, M) \to H_{\mathfrak{m}}^i(M)$ are surjective for $i < d$.*

Thus far we know some limited information about $\operatorname{Ext}_A^i(\Bbbk, R^j)$ and $H_{\mathfrak{m}}^i(R^j)$ in terms of the simplicial cohomology of subcomplexes of Δ. Thankfully, Miyazaki has furthered this correspondence with an explicit description of φ_R^i in [41, Corollary 4.5].

Theorem 2.4.2. *The canonical map $\varphi_R^i : \operatorname{Ext}_A^i(\Bbbk, R) \to H_{\mathfrak{m}}^i(R)$ corresponds to the identity map on $H_\emptyset^{i-1}(\Delta)$ in degree zero and to the direct sum of maps*

$$\bigoplus_{v \in V} \left(\varphi_v^i : H_\emptyset^{i-2}(\operatorname{cost}_\Delta v) \to H_\emptyset^{i-2}(\operatorname{lk}_\Delta v)\right)$$

induced by the inclusions in degree -1.

For our purposes, an alternate expression for φ_R^i turns out to be even more powerful than the one above. For some fixed v, consider the long exact sequence in simplicial cohomology for the triple $(\Delta, \operatorname{cost}_\Delta v, \emptyset)$. In our notation, it is written as

$$\cdots \to H_\emptyset^{i-2}(\Delta) \to H_\emptyset^{i-2}(\operatorname{cost}_\Delta v) \xrightarrow{\delta} H_v^{i-1}(\Delta) \xrightarrow{\iota_v^{i-1}} H_\emptyset^{i-1}(\Delta) \to \cdots . \tag{2.8}$$

Under the isomorphism $H_v^{i-1}(\Delta) \cong H_\emptyset^{i-2}(\operatorname{lk}_\Delta v)$, a quick check shows that the connecting homomorphism δ in this sequence is equivalent to φ_v^i in the theorem above (this is also made apparent in examining its proof). On the other hand, if we consider the cohomology modules above as components of $H^i_{\mathfrak{m}}(R)$ as in Theorem 2.2.3, then the ι_v^{i-1} map in this sequence is the same as the "multiplication by x_v" map on $H^i_{\mathfrak{m}}(R)_v$. These equivalences along with the exactness of (2.8) allow us to deduce the following proposition.

Proposition 2.4.3. *The image of the $H^{i-2}_{\emptyset}(\operatorname{cost}_\Delta v)$ component of $\operatorname{Ext}^i_A(\mathbb{k}, R)_{-1}$ under the canonical map $\varphi^i_R : \operatorname{Ext}^i_A(\mathbb{k}, R) \to H^i_{\mathfrak{m}}(R)$ is precisely the kernel of $\iota_v^{i-1} : H^{i-1}_v(\Delta) \to H^{i-1}_{\emptyset}(\Delta)$. In particular,*

$$(\operatorname{Im}\varphi^i_R)_{-1} = K^i$$

through the identifications

$$\left(\operatorname{Im}\varphi^i_R\right)_{-1} = \bigoplus_{v\in V}\left(\operatorname{Im}\varphi^i_v\right) = \bigoplus_{v\in V}\left(\operatorname{Ker}\iota^{i-1}_v\right) = K^i.$$

Note that if θ is any linear form then the proposition immediately implies that $(\operatorname{Im}\varphi^i_R)_{-1} \subseteq \operatorname{Ker}\theta^{i,0}$. Examining this containment more closely provides a characterization of the Buchsbaumness of R^1. Before stating this characterization, we note that one commutative diagram in particular will be used repeatedly in proving many of our results. Here we explain its origin.

Construction 2.4.4. The short exact sequence (2.1) induces the following commutative diagram of vector spaces with exact rows:

$$\begin{array}{ccccccccc}
\operatorname{Ext}^i_A(\mathbb{k},R)_{j-1} & \xrightarrow{\theta_1^{i,j}} & \operatorname{Ext}^i_A(\mathbb{k},R)_j & \xrightarrow{\pi} & \operatorname{Ext}^i_A(\mathbb{k},R^1)_j & \xrightarrow{\delta} & \operatorname{Ext}^{i+1}_A(\mathbb{k},R)_{j-1} & \xrightarrow{\theta_1^{i+1,j}} & \operatorname{Ext}^{i+1}_A(\mathbb{k},R)_j \\
\downarrow{\scriptstyle\varphi^i_R} & & \downarrow{\scriptstyle\varphi^i_R} & & \downarrow{\scriptstyle\varphi^i_{R^1}} & & \downarrow{\scriptstyle\varphi^{i+1}_R} & & \downarrow{\scriptstyle\varphi^{i+1}_R} \\
H^i_{\mathfrak{m}}(R)_{j-1} & \xrightarrow{\theta_1^{i,j}} & H^i_{\mathfrak{m}}(R)_j & \xrightarrow{\pi} & H^i_{\mathfrak{m}}(R^1)_j & \xrightarrow{\delta} & H^{i+1}_{\mathfrak{m}}(R)_{j-1} & \xrightarrow{\theta_1^{i+1,j}} & H^i_{\mathfrak{m}}(R)_j.
\end{array}$$

Since θ_1 is the beginning of a regular sequence for R, it acts trivially on $\operatorname{Ext}^i_A(A/\mathfrak{m}, R)_j$ for all i and j (see, e.g., [27, p. 272]). By the commutativity of the rightmost square, the image of $\operatorname{Ext}^{i+1}_A(\mathbb{k}, R)_{j-1}$ under φ^{i+1}_R must lie in $\operatorname{Ker}\theta^{i+1,j}_1$. On the other hand, the exactness of the bottom row tells us that $\pi : H^i_{\mathfrak{m}}(R)_j \to H^i_{\mathfrak{m}}(R^1)_j$ factors through the projection $H^i_{\mathfrak{m}}(R)_j \to \operatorname{Coker}\theta^{i,j}_1$. As this does not alter the commutativity of the diagram, we can now alter it so that the top and bottom rows are both short exact sequences as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R)_j & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R^1)_j & \longrightarrow & \operatorname{Ext}^{i+1}_A(\Bbbk, R)_{j-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \varphi^i_{R^1} & & \downarrow \varphi^{i+1}_R & & \\ 0 & \longrightarrow & \operatorname{Coker} \theta_1^{i,j} & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_j & \longrightarrow & \operatorname{Ker} \theta_1^{i+1,j} & \longrightarrow & 0, \end{array}$$

where the left vertical map is the composition of φ^i_R with the projection $H^i_{\mathfrak{m}}(R)_j \to \operatorname{Coker} \theta_1^{i,j}$. We can repeat this construction starting with the short exact sequence (2.4), yielding the same diagram as above with R, R^1, and θ_1 replaced by R^1, R^2, and θ_2, respectively.

Our first use of this construction will be in proving the following proposition (an alternate proof of the "if" direction also appears in [55, Lemma 4.3(2)]).

Proposition 2.4.5. *If Δ has isolated singularities, then R^1 is Buchsbaum if and only if the singularities of Δ are homologically isolated.*

Proof. Construction 2.4.4 provides the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R)_0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R^1)_0 & \longrightarrow & \operatorname{Ext}^{i+1}_A(\Bbbk, R)_{-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \varphi^i_{R^1} & & \downarrow \varphi^{i+1}_R & & \\ 0 & \longrightarrow & \operatorname{Coker} \theta_1^{i,0} & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_0 & \longrightarrow & \operatorname{Ker} \theta_1^{i+1,0} & \longrightarrow & 0. \end{array}$$

By definition, if Δ contains singularities that are not homologically isolated then there exists some i such that $K^{i+1} \subsetneq \operatorname{Ker} \theta_1^{i+1,0}$. By Proposition 2.4.3, this implies that the φ^{i+1}_R map appearing in the diagram above is not a surjection. Since $\varphi^i_R : \operatorname{Ext}^i_A(\Bbbk, R)_0 \to H^i_{\mathfrak{m}}(R)_0$ is an isomorphism, the left vertical map is always a surjection. Then the snake lemma applied to this diagram shows that $\varphi^i_{R^1}$ is not a surjection, so R^1 is not Buchsbaum by Theorem 2.4.1.

Conversely, if the singularities of Δ are homologically isolated, then $\operatorname{Ker} \theta_1^{i+1,0} = K^{i+1}$ for all i, so that the φ^{i+1}_R map in the diagram is always a surjection. The snake lemma now shows that $\varphi^i_{R^1}$ is a surjection in degree 0. In degree 1, we only need to raise the degrees in the previous diagram by one. The diagram simplifies to

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R^1)_1 & \longrightarrow & \operatorname{Ext}^{i+1}_A(\Bbbk, R)_0 & \longrightarrow & 0 \\ & & \downarrow \varphi^i_{R^1} & & \downarrow \varphi^{i+1}_R & & \\ 0 & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_1 & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R)_0 & \longrightarrow & 0, \end{array}$$

because $\operatorname{Ext}^i_A(\Bbbk, R)_1$ and $\operatorname{Coker}\theta_1^{i,1}$ are both zero. Since φ_R^{i+1} is an isomorphism in degree 0, this completes the proof. □

We have now seen that complexes with homologically isolated singularities are "close" to being Buchsbaum in that R^1 is Buchsbaum. It is natural to ask whether descending to R^2 could always provide a Buchsbaum module, even in the non-homologically-isolated case. This is not necessarily true, as exhibited by the following proposition.

Proposition 2.4.6. *Suppose Δ has non-homologically-isolated singularities and that there exists $i < d-2$ such that $H_\emptyset^{i-1}(\Delta) = 0$, while $\operatorname{Ker}\theta_1^{i+1,0} \neq 0$ and ι_v^i is injective for all v. Then R^2 is not Buchsbaum.*

Proof. The hypotheses combined with the exact sequence in (2.8) show that $H_\emptyset^{i-1}(\operatorname{cost} v) = 0$, so that $\operatorname{Ext}^{i+1}_A(\Bbbk, R)_{-1} = 0$. Also, $\operatorname{Coker}\theta_1^{i,0} = 0$ because $H_\emptyset^{i-1}(\Delta) = 0$. Then the diagram of Construction 2.4.4 can be filled in as follows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R)_0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R^1)_0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi^i_{R^1}} & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_0 & \longrightarrow & \operatorname{Ker}\theta_1^{i+1,0} & \longrightarrow & 0.
\end{array}$$

This demonstrates that $\varphi^i_{R^1}$ is the zero map in degree 0. Now repeat this argument using R^1 and R^2 instead of R and R^1. In this case, $\operatorname{Ext}^{i+1}_A(\Bbbk, R^1)_{-1} = 0$ because $\operatorname{Ext}^{i+1}_A(\Bbbk, R)_{-1} = 0$. Since $H^i_{\mathfrak{m}}(R^1)_{-1} = 0$, the diagram provided is

$$\begin{array}{ccc}
\operatorname{Ext}^i_A(\Bbbk, R^1)_0 & \xrightarrow{\sim} & \operatorname{Ext}^i_A(\Bbbk, R^2)_0 \\
\downarrow{\scriptstyle \varphi^i_{R^1}} & & \downarrow{\scriptstyle \varphi^i_{R^2}} \\
H^i_{\mathfrak{m}}(R^1)_0 & \xrightarrow{\sim} & H^i_{\mathfrak{m}}(R^2)_0,
\end{array}$$

showing that $\varphi^i_{R^2}$ is the zero map in degree 0 as well. Since $H^i_{\mathfrak{m}}(R^2)_0 \neq 0$ while i is strictly less than the dimension of R^2, Theorem 2.4.1 now completes the proof. □

The hypotheses of this proposition may seem fairly restrictive, but that is not necessarily the case. In fact, choosing i to be the least i such that $H^i_\emptyset(\Delta) \neq 0$ when Δ is a triangulation of the suspension of a certain type of manifold will always do the trick, yielding the following Corollary:

Corollary 2.4.7. *If* Δ *is the suspension of a Buchsbaum complex that satisfies* $H^i_\emptyset(\Delta) \neq 0$ *for some* $i < d-2$*, then* R^2 *is not Buchsbaum.*

2.4.2 Almost Buchsbaumness

Although these R^2 modules are not guaranteed to be Buchsbaum when Δ has non-homologically-isolated singularities, they are "close" to being Buchsbaum in some interesting ways and share some of the same properties. The examples above in which R^2 is not Buchsbaum fail the criterion of Theorem 2.4.1 in the degree 0 piece of $H^i_\mathfrak{m}(R^2)$. The second part of Theorem 2.1.1 asserts that (in the generically isolated case) this is the only possible obstruction to R^2 satisfying Theorem 2.4.1, and we now present its proof.

Proof of Theorem 2.1.1.(ii). By the calculations in Section 2.3, we only need to classify surjectivity in degrees 1 and 2. The last diagram in the proof of Proposition 2.4.5 holds regardless of whether or not the singularities of Δ are homologically isolated. Hence, $\varphi^i_{R^1}$ is an isomorphism in degree 1. Construction 2.4.4 then induces the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R^2)_2 & \longrightarrow & \operatorname{Ext}^{i+1}_A(\Bbbk, R^1)_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^i_{R^2}} & & \downarrow{\scriptstyle \varphi^{i+1}_{R^1}} & & \\ 0 & \longrightarrow & H^i_\mathfrak{m}(R^2)_2 & \longrightarrow & H^{i+1}_\mathfrak{m}(R^1)_1 & \longrightarrow & 0, \end{array}$$

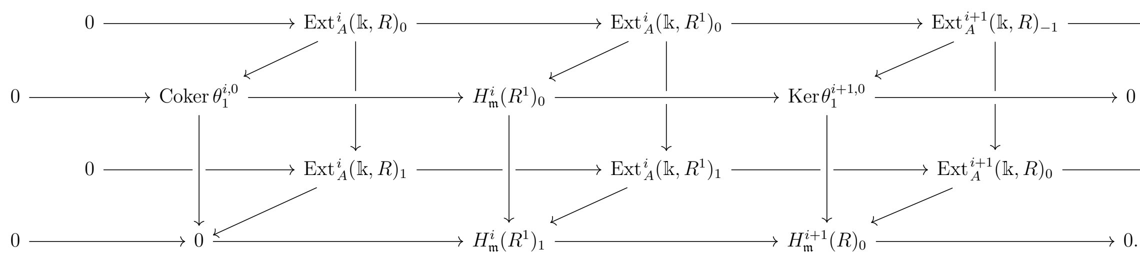
so that $\varphi^i_{R^2}$ is always an isomorphism in degree 2.

It remains to show that $\varphi^i_{R^2}$ is a surjection in degree 1. As usual, we have the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}^{i-1}_A(\Bbbk, R^1)_1 & \longrightarrow & \operatorname{Ext}^{i-1}_A(\Bbbk, R^2)_1 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R^1)_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \varphi^{i-1}_{R^2}} & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Coker}\theta_2^{i-1,1} & \longrightarrow & H^{i-1}_\mathfrak{m}(R^2)_1 & \longrightarrow & \operatorname{Ker}\theta_2^{i,1} & \longrightarrow & 0. \end{array}$$

According to the previous paragraph, the left vertical map must be a surjection. To complete the proof, we must show that the right vertical map is surjective if and only if the singularities of Δ are generically isolated. The right map is obtained by restricting the range of $\varphi^i_{R^1}$ to the subspace $\operatorname{Ker}\theta_2^{i,1}$ of $H^i_{\mathfrak{m}}(R^1)_0$. Note that the failure of $\varphi^i_{R^1}$ to be a surjection in this degree is precisely what made R^1 fail to be Buchsbaum in the non-homologically-isolated case.

Now consider a larger commutative diagram, all of whose rows are exact. All vertical maps are those induced by the action of θ_2, and all maps from the back "panel" to the front are induced by the canonical maps $\varphi^i_{R^j}$.



Now consider applying the snake lemma to both the front panel and the back panel. Note that the vertical maps on the back panel are all identically zero, since θ_2 acts trivially on all of the modules there. Denote by τ the restriction of $\theta_2^{i+1,0}$ to $\operatorname{Ker}\theta_1^{i+1,0}$, appearing as the right vertical map in the front panel of the diagram. By the naturality of the sequence induced by the snake lemma, we obtain maps from the "top" panel as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R)_0 & \longrightarrow & \operatorname{Ext}^i_A(\Bbbk, R^1)_0 & \longrightarrow & \operatorname{Ext}^{i+1}_A(\Bbbk, R)_{-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Coker}\theta_1^{i,0} & \longrightarrow & \operatorname{Ker}\theta_2^{i,0} & \longrightarrow & \operatorname{Ker}\tau & \longrightarrow & 0. \end{array}$$

Since the left vertical map is a surjection, we will be done if we can show that the right vertical map is a surjection precisely when the singularities of Δ are generically isolated. However, $\operatorname{Ker}\tau$ is simply

$$(\operatorname{Ker}\theta_1^{i+1,0}) \cap (\operatorname{Ker}\theta_2^{i+1,2}) := L^{i+1}.$$

Note that the singularities of Δ are generically isolated if and only if $L^{i+1} = K^{i+1}$. On the other hand, K^{i+1} is precisely the image of $\operatorname{Ext}^i_A(\Bbbk, R)_{-1}$ under φ^i_R, completing the proof. $\square$

The intersection L^{i+1} above is also central to the proof of Theorem 2.1.1.(i), which we now present as well.

Proof of Theorem 2.1.1.(i). Once more, since $H^i_{\mathfrak{m}}(R^2)$ may only have non-zero components in the graded degrees 0, 1, and 2, we only need to check that $\mathfrak{m}$ acts trivially on the degree 0 and degree 1 parts. We begin in degree 1. Let α_v, β_v, and γ_v all denote the map induced by multiplication by x_v in the diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Coker}\theta_2^{i,1} & \longrightarrow & H^i_{\mathfrak{m}}(R^2)_1 & \longrightarrow & \operatorname{Ker}\theta_2^{i+1,1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_v} & & \downarrow{\scriptstyle \beta_v} & & \downarrow{\scriptstyle \gamma_v} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H^i_{\mathfrak{m}}(R^2)_2 & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R^1)_1 & \longrightarrow & 0. \end{array}$$

The snake lemma provides the exact sequence

$$0 \to \operatorname{Coker}\theta_2^{i,1} \to \operatorname{Ker}\beta_v \to \operatorname{Ker}\gamma_v \to 0.$$

Comparing this to the top row of the previous diagram, $\operatorname{Ker}\gamma_v$ is all of $\operatorname{Ker}\theta_2^{i+1,1}$ precisely when $\operatorname{Ker}\beta_v$ is all of $H^i_{\mathfrak{m}}(R^2)_1$. Note that $\operatorname{Ker}\theta_2^{i+1,1}$ is a submodule of $H^{i+1}_{\mathfrak{m}}(R^1)_0$; to study this submodule, consider the following diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Coker}\theta_1^{i+1,0} & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R^1)_0 & \longrightarrow & \operatorname{Ker}\theta_1^{i+2,0} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_2^{i+1,1}} & & \downarrow{\scriptstyle \theta_2^{i+1,1}} & & \downarrow{\scriptstyle \tau} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R^1)_1 & \longrightarrow & H^{i+2}_{\mathfrak{m}}(R)_0 & \longrightarrow & 0. \end{array}$$

As in the previous proof, the rightmost vertical map τ is the restriction of $\theta_2^{i+2,0}$ to $\operatorname{Ker}\theta_1^{i+2,0}$. Once more, note that

$$\operatorname{Ker}\tau = (\operatorname{Ker}\theta_1^{i+2,0}) \cap (\operatorname{Ker}\theta_2^{i+2,2}) := L^{i+2}.$$

Through another application of the snake lemma, we get a short exact sequence fitting into the top row of the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Coker}\theta_1^{i+1,0} & \longrightarrow & \operatorname{Ker}\theta_2^{i+1,1} & \longrightarrow & L^{i+2} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot x_v} & & \downarrow{\scriptstyle \cdot x_v} & & \downarrow{\scriptstyle \cdot x_v} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R^1)_1 & \longrightarrow & H^{i+2}_{\mathfrak{m}}(R)_0 & \longrightarrow & 0. \end{array}$$

So, it now remains to show that the rightmost map is zero if and only if the singularities of Δ are generically isolated. But x_v acts trivially on L^{i+2} for all i and for all v if and only if

$$L^{i+2} = K^{i+2},$$

i.e., if and only if the singularities of Δ are generically isolated.

Now we show that $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(R^2)_0 = 0$, independent of the type of isolation of the singularities of Δ. Consider the diagram below:

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_0 & \longrightarrow & H^i_{\mathfrak{m}}(R^2)_0 & \longrightarrow & 0 \\ & & & & \downarrow{\cdot x_v} & & \downarrow{\cdot x_v} & & \\ H^i_{\mathfrak{m}}(R^1)_0 & \xrightarrow{\theta_2^{i,1}} & & & H^i_{\mathfrak{m}}(R^1)_1 & \longrightarrow & H^i_{\mathfrak{m}}(R^2)_1 & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R^1)_0. \end{array}$$

From the exactness of the rows of this diagram, we can conclude that $x_v \cdot H^i_{\mathfrak{m}}(R^2)_0 = 0$ if

$$x_v \cdot H^i_{\mathfrak{m}}(R^1)_0 \subseteq \theta_2 \cdot H^i_{\mathfrak{m}}(R^1)_0.$$

Generically, all coefficients of θ_2 are non-zero. Combining this with the structure of $H^i_{\mathfrak{m}}(R)$ outlined in Theorem 2.2.3, it is immediate that

$$x_v \cdot H^i_{\mathfrak{m}}(R)_{-1} \subseteq \theta_2 \cdot H^i_{\mathfrak{m}}(R)_{-1}.$$

The following diagram now completes the proof.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_1 & \xrightarrow{\delta} & H^{i+1}_{\mathfrak{m}}(R)_0 & \longrightarrow & 0 \\ & & \uparrow{\cdot x_v} & & \uparrow{\cdot x_v} & & \\ H^i_{\mathfrak{m}}(R)_0 & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_0 & \xrightarrow{\delta} & H^{i+1}_{\mathfrak{m}}(R)_{-1} & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R)_0 \\ & & \downarrow{\cdot \theta_2} & & \downarrow{\cdot \theta_2} & & \\ 0 & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_1 & \xrightarrow{\delta} & H^{i+1}_{\mathfrak{m}}(R)_0 & \longrightarrow & 0 \end{array}$$

□

2.4.3 Another surjection

By [68, Proposition I.3.4], the quasi-Buchsbaum property of some R^2 is equivalent to the fact that every homogeneous system of parameters of R^2 contained in $\mathfrak{m}^2$ is a weakly regular sequence. In light of the typical definition of the Buchsbaum property in terms of l.s.o.p.'s being weakly regular sequences (see [68]) along with the characterization appearing in Theorem 2.4.1 by surjectivity of the maps $\varphi^i_M : \mathrm{Ext}^i_A(\Bbbk, M) \to H^i_{\mathfrak{m}}(M)$, what happens when we consider the natural maps $\mathrm{Ext}^i_A(A/\mathfrak{m}_2, R^2)$ instead, where $\mathfrak{m}_2 = (x_1^2, \ldots, x_n^2)$? Our next result establishes another measure of the gap between the structure of R^2 and the Buchsbaum property:

Proposition 2.4.8. *Suppose Δ has isolated singularities. Then the canonical maps $\psi^i_{R^2}$: $\mathrm{Ext}^i_A(A/\mathfrak{m}_2, R^2) \to H^i_{\mathfrak{m}}(R^2)$ are surjective.*

Proof. Once more, surjectivity needs only to be demonstrated in degrees 0, 1, and 2. We will begin with the degree 2 piece. The exact sequence (2.1) with $j = 1$ and $l = 2$ gives rise to the commutative diagram below, where the horizontal maps are isomorphisms.

$$\begin{array}{ccc} \mathrm{Ext}^i_A(A/\mathfrak{m}_2, R^1)_1 & \xrightarrow{\delta} & \mathrm{Ext}^{i+1}_A(A/\mathfrak{m}_2, R)_0 \\ \downarrow \psi^i_{R^1} & & \downarrow \psi^{i+1}_R \\ H^i_{\mathfrak{m}}(R^1)_1 & \xrightarrow{\delta} & H^{i+1}_{\mathfrak{m}}(R)_0 \end{array}$$

By [41, Corollary 4.5], the map $\psi^{i+1}_R : \mathrm{Ext}^{i+1}_A(A/\mathfrak{m}_2, R)_0 \to H^{i+1}_{\mathfrak{m}}(R)_0$ is equivalent to the identity map on $H^i_{\emptyset}(\Delta)$. Hence, $\psi^{i+1}_{R^1}$ is an isomorphism in degree 1. By the same argument, the exact sequence (2.4) and the corresponding commutative diagram show that $\psi^i_{R^2} : \mathrm{Ext}^i_A(A/\mathfrak{m}_2, R^2)_2 \to H^i_{\mathfrak{m}}(R^2)_2$ is an isomorphism.

The arguments for further graded pieces are similar. First consider the following diagram with exact rows again induced by (2.1).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^i_A(A/\mathfrak{m}_2, R)_0 & \longrightarrow & \mathrm{Ext}^i_A(A/\mathfrak{m}_2, R^1)_0 & \longrightarrow & \mathrm{Ext}^{i+1}_A(A/\mathfrak{m}_2, R)_{-1} & \longrightarrow & 0 \\ & & \psi^i_R \downarrow & & \downarrow \psi^i_{R^1} & & \downarrow \psi^{i+1}_R & & \\ & & H^i_{\mathfrak{m}}(R)_0 & \longrightarrow & H^i_{\mathfrak{m}}(R^1)_0 & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R)_{-1} & & \end{array}$$

By [42, Corollary 2], the left and right vertical maps are isomorphisms. Exactness then implies that the middle vertical map is surjective. So, $\psi^i_{R^1}$ is an isomorphism in degree 1 and a surjection in degree 0. For the surjectivity of $\psi^i_{R^2}$ in degree 1, consider the following commutative diagram induced by (2.4) with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}^i_A(A/\mathfrak{m}_2, R^1)_1 & \longrightarrow & \operatorname{Ext}^i_A(A/\mathfrak{m}_2, R^2)_1 & \longrightarrow & \operatorname{Ext}^{i+1}_A(A/\mathfrak{m}_2, R^1)_0 & \longrightarrow & 0 \\ & & \downarrow \psi^i_{R^1} & & \downarrow \psi^i_{R^2} & & \downarrow \psi^{i+1}_{R^1} & & \\ & & H^i_{\mathfrak{m}}(R^1)_1 & \longrightarrow & H^i_{\mathfrak{m}}(R^2)_1 & \longrightarrow & H^{i+1}_{\mathfrak{m}}(R^1)_0 & & \end{array}$$

We have just demonstrated that the left and right vertical maps are at least surjections. Again by exactness, we now have that $\psi^i_{R^2}$ is a surjection in degree 1. Now consider one final commutative diagram.

$$\begin{array}{ccc} \operatorname{Ext}^i_A(A/\mathfrak{m}_2, R^1)_0 & \longrightarrow & \operatorname{Ext}^i_A(A/\mathfrak{m}_2, R^2)_0 \\ \downarrow \psi^i_{R^1} & & \downarrow \psi^i_{R^2} \\ H^i_{\mathfrak{m}}(R^1)_0 & \longrightarrow & H^i_{\mathfrak{m}}(R^2)_0 \end{array}$$

From Section 2.3, we know that the bottom map is an isomorphism. Since $\psi^i_{R^1}$ is a surjection in degree zero, $\psi^i_{R^2}$ must be as well. □

2.5 *Applications*

2.5.1 Examples

Up until now there has been no discussion of when R^2 may be a Buchsbaum ring. Our first family of examples shows that it is in fact possible for R^2 to be Buchsbaum even when the singularities of Δ are not homologically isolated.

Example 2.5.1. Let Δ be a 2-dimensional complex with generically isolated singularities. Since the Krull dimension of R^2 is one, we have $H^i_{\mathfrak{m}}(R^2) = 0$ for $i \neq 0, 1$. Furthermore, the depth of R^2 must be zero (otherwise R would be Cohen-Macaulay). In this case, quasi-Buchsbaumness is equivalent to Buchsbaumness by [68, Corollary I.3.6]. By Theorem 2.1.1, R^2 must then be Buchsbaum. For a concrete example in which R^1 is not Buchsbaum, consider the suspension of the disjoint union of two cycles.

On the other hand, combining Theorem 2.1.1 with Corollary 2.4.7 provides an infinite family of interesting examples of rings with some prescribed properties.

Example 2.5.2. Let M be a d-dimensional manifold with $d > 0$ satisfying $H^i_{\emptyset}(M) \neq 0$ for some $i < d-1$ and let Γ be an arbitrary triangulation of M. Suppose further that

$$\max\{i : \Gamma^{(i)} \text{ is Cohen-Macaulay}\} = r + 1.$$

Equivalently, the depth of $\Bbbk[\Gamma]$ is $r+1$. If we set Γ' to be the join of Γ with two points, then Γ' is a triangulation of the suspension of M and the depth of $\Bbbk[\Gamma']$ is $r+2$. Since the depth of the Stanley–Reisner ring is a topological invariant ([43, Theorem 3.1]), if Δ is an arbitrary triangulation of the suspension of M then R^2 has depth r. Since the singularities of Δ are generically isolated, R^2 is a quasi-Buchsbaum ring (Theorem 2.1.1) of Krull dimension d and depth r that is not Buchsbaum (Corollary 2.4.7); furthermore, the canonical maps $\varphi^i_{R^2} : \operatorname{Ext}^i_A(\Bbbk, R^2) \to H^i_{\mathfrak{m}}(R^2)$ are surjections in all degrees except 0.

Example 2.5.3. Suppose in the previous example that $H^i_{\emptyset}(M) = \Bbbk$, and let Γ' instead be the join of Γ with three points. In this case, K^{i+2} is trivial (recall its definition from Section 2.2.2) while each of $\operatorname{Ker} \theta_1^{i+2,0}$ and $\operatorname{Ker} \theta_2^{i+2,0}$ are two-dimensional subspaces of the three-dimensional space $H^{i+2}_{\mathfrak{m}}(R)_{-1}$. Hence, their intersection is non-trivial and the singularities of Δ are not generically isolated. Then R^2 is not even quasi-Buchsbaum, by Theorem 2.1.1.

2.5.2 An enumerative theorem

Although R^2 is not Buchsbaum, the quasi-Buchsbaum property does allow for a computation of the Hilbert series of the generic Artinian reduction of R by a h.s.o.p. of a particular type. Let Δ have generically isolated singularities and say $\Theta = \theta_1, \ldots, \theta_d$ is a homogeneous system of parameters for Δ such that θ_1, θ_2 is a linear regular sequence, while $\theta_3, \ldots, \theta_d$ are quadratic forms. For $2 \leq i \leq d-1$, there are exact sequences

$$0 \to (0 :_{R^i} \theta_{i+1})_{j-2} \to R^i_{j-2} \xrightarrow{\cdot \theta_{i+1}} R^i_j \to R^{i+1} \to 0.$$

Since R^2 is quasi-Buchsbaum, the sequence $\theta_3, \ldots, \theta_d$ is a weakly regular sequence by [68, Proposition I.2.1(ii)]. Furthermore, the proof of the proposition shows that $(0 :_{R^2} \theta_3) = H^0_{\mathfrak{m}}(R^2)$. On the other hand, [69, Theorem 3.6] states that R^i is quasi-Buchsbaum for $2 \leq i \leq d-1$. Hence, the sequence above can be re-written as

$$0 \to H^0_{\mathfrak{m}}(R^i)_{j-2} \to R^i_{j-2} \to R^i_j \to R^{i+1} \to 0.$$

for $2 \leq i \leq d-1$. If $\mathrm{Hilb}(M;t)$ denotes the Hilbert series of a $\mathbb{Z}$-graded A-module M, then these exact sequences imply

$$\mathrm{Hilb}(R^{i+1};t) = (1-t^2)\,\mathrm{Hilb}(R^i;t) + t^2\,\mathrm{Hilb}(H^0_{\mathfrak{m}}(R^i);t).$$

A standard calculation then shows

$$\mathrm{Hilb}(R^d;t) = (1+t)^{d-2}(1-t)^d\,\mathrm{Hilb}(R;t) + \sum_{i=2}^{d-1}\left[t^2(1-t^2)^{d-1-i}\,\mathrm{Hilb}(H^0_{\mathfrak{m}}(R^i);t)\right]. \quad (2.9)$$

The first term reduces to $(1+t)^{d-2}\sum_{i=0}^{d} h_i(\Delta)t^i$, following [62]. To analyze the sum, [69, Lemma 3.5] provides the exact sequence

$$0 \to H^j_{\mathfrak{m}}(R^i)_k \to H^j_{\mathfrak{m}}(R^{i+1})_k \to H^{j+1}_{\mathfrak{m}}(R^i)_{k-2} \to 0$$

for $2 \leq i \leq d-2$ and $0 \leq j \leq d-i-2$. So, as vectors spaces over $\Bbbk$, for $2 \leq i \leq d-1$ there are isomorphisms

$$H^0_{\mathfrak{m}}(R^i)_j \cong \bigoplus_{j=0}^{i-2}\left(\bigoplus_{\binom{i-2}{j}} H^j_{\mathfrak{m}}(R^2)_{-2j}\right).$$

That is,

$$\mathrm{Hilb}(H^0_{\mathfrak{m}}(R^i);t) = \sum_{j=0}^{i-2}\binom{i-2}{j}t^{2j}\,\mathrm{Hilb}(H^j_{\mathfrak{m}}(R^2);t). \quad (2.10)$$

Now define

$$\mu^i = \dim_{\Bbbk} H^i_{\mathfrak{m}}(R^2)_0 = \dim_{\Bbbk}\left(\mathrm{Coker}\,\theta_1^{i,0} \oplus \mathrm{Ker}\,\theta_1^{i+1,0}\right),$$

$$\nu^i = \dim_{\Bbbk} H^i_{\mathfrak{m}}(R^2)_1 = \dim_{\Bbbk}\left(\mathrm{Coker}\,\theta_2^{i,1} \oplus \mathrm{Ker}\,\theta_2^{i+1,1}\right),$$

and

$$\beta^i_\emptyset(\Delta) = \dim_{\Bbbk} H^i_\emptyset(\Delta),$$

so

$$\mathrm{Hilb}(H^i_{\mathfrak{m}}(R^2); t) = \mu^i + \nu^i t + \beta^{i+1}_\emptyset(\Delta) t^2.$$

Combining this equality with equations (2.9) and (2.10) implies the following theorem describing $\mathrm{Hilb}(R^d; t)$.

Theorem 2.5.4. *If $q = 2p$ is even, then*

$$\dim_{\Bbbk}(R^d_q) = \sum_{i=0}^{q} \binom{d-2}{q-i} h_i(\Delta) + (-1)^{p-1} \binom{d-2}{p} \sum_{k=0}^{p-1} \left[(-1)^k \left(\mu^k + \frac{p\beta^k_\emptyset(\Delta)}{d-1-p} \right) \right].$$

If $q = 2p+1$ is odd, then

$$\dim_{\Bbbk}(R^d_q) = \sum_{i=0}^{q} \binom{d-2}{q-i} h_i(\Delta) + (-1)^{p-1} \binom{d-2}{p} \sum_{k=0}^{p-1} (-1)^k \nu^k.$$

Remark 2.5.5. Note that μ^i is actually a topological invariant of Δ. If $\|\Delta\|$ is the geometric realization of Δ and Σ is the set of isolated singularities of Δ, then $\mu^i = \dim_{\Bbbk} H^{i-1}_\emptyset(\|\Delta\| \smallsetminus \Sigma)$ (see [55, Theorem 4.7]). At present there is no similar description for ν^i, as it is not clear how to trace the geometry of Δ all the way through to $H^i_{\mathfrak{m}}(R^2)_1$ in such a precise manner.

2.6 Comments

There are many possible abstractions of the results that have been presented. Perhaps the most immediate consideration is in introducing singularities of dimension greater than 0. In this case, the structure of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ outlined in Theorem 2.2.3 becomes more involved and hinders the calculations of Section 2.3. For instance, if Δ contains singular faces even of dimension 1, then $\theta_1^{i,j}$ will not, in general, be an isomorphism in degrees $j < 0$. This implies that there is some i such that $H^i_{\mathfrak{m}}(\Bbbk[\Delta]/\theta_1\Bbbk[\Delta])_j$ is non-zero for infinitely many values of j, i.e., $\Bbbk[\Delta]/\theta_1\Bbbk[\Delta]$ does not have finite local cohomology. In fact, Miller, Novik, and Swartz classified when this is the case for quotients of $\Bbbk[\Delta]$ by arbitrarily many generic linear forms ([38, Theorem 2.4]):

Theorem 2.6.1. *A simplicial complex Δ is of singularity dimension at most $m-1$ if and only if $\Bbbk[\Delta]/(\theta_1,\ldots,\theta_m)\Bbbk[\Delta]$ has finite local cohomology.*

In the case that $m=1$, we know that $\Bbbk[\Delta]/\theta_1\Bbbk[\Delta]$ not only has finite local cohomology, but it is also Buchsbaum if and only if the singularities of Δ are homologically isolated. So, one may pose the following question.

Question 2.6.2. *If Δ is of singularity dimension $m-1$, is there an analog of the homological isolation property for singularities of arbitrary dimension classifying when $\Bbbk[\Delta]/(\theta_1,\ldots,\theta_m)\Bbbk[\Delta]$ is Buchsbaum?*

A possible property could be that all pairs of images of maps of the form $H^i(\Delta, \operatorname{cost}_\Delta(F\cup\{u\}))\to H^i(\Delta,\operatorname{cost}_\Delta F)$ and $H^i(\Delta,\operatorname{cost}_\Delta(F\cup\{v\}))\to H^i(\Delta,\operatorname{cost}_\Delta F)$ occupy linearly independent subspaces of $H^i(\Delta,\operatorname{cost}_\Delta F)$ for all faces F and all vertices u and v in the appropriate dimensions.

On the other hand, when $m=1$ we know that $\Bbbk[\Delta]/\theta_1\Bbbk[\Delta]$ has finite local cohomology and that $\Bbbk[\Delta]/(\theta_1,\theta_2)\Bbbk[\Delta]$ is quasi-Buchsbaum if and only if the singularities of Δ are generically isolated. The leads to our next question.

Question 2.6.3. *If Δ is a simplicial complex of singularity dimension $m-2$ and $\Bbbk[\Delta]$ is of depth at least m with $\theta_1,\ldots,\theta_m$ a regular sequence on $\Bbbk[\Delta]$, is there an analog of the generic isolation property classifying when $\Bbbk[\Delta]/(\theta_1,\ldots,\theta_m)\Bbbk[\Delta]$ is quasi-Buchsbaum?*

Again, one candidate property would be that given $m+1$ generic linear forms, the pairwise intersections

$$\operatorname{Ker}\theta_i^{l,0}\cap\operatorname{Ker}\theta_j^{l,0}$$

are all trivially equal to K^l.

Lastly, the initial purpose of our investigation was to examine the possibility of $\Bbbk[\Delta]/(\theta_1,\theta_2)\Bbbk[\Delta]$ being Buchsbaum. For now, this question remains open:

Question 2.6.4. *If Δ has generically isolated singularities, what additional properties guarantee that $\Bbbk[\Delta]/(\theta_1,\theta_2)\Bbbk[\Delta]$ is Buchsbaum?*

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Chapter 3
GROUP ACTIONS

3.1 Introduction

Since the 1970's, an extensive dictionary translating the topological and combinatorial properties of a simplicial complex Δ into algebraic properties of the associated Stanley–Reisner ring $\Bbbk[\Delta]$ has been constructed. For instance, the local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ of $\Bbbk[\Delta]$ have a beautiful interpretation due to Hochster (and later Gräbe) as topological invariants of Δ (see [23, Theorem 2], [57], or [67, Section II.4]). The following formulation of their results is the starting point of this chapter (here $\operatorname{cost}_\Delta \sigma$ denotes the contrastar of a face σ of Δ, while $H^{i-1}(\Delta, \operatorname{cost}_\Delta \sigma)$ is the relative simplicial cohomology of the pair $(\Delta, \operatorname{cost}_\Delta \sigma)$ computed with coefficients in $\Bbbk$ and $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ is the local cohomology module of $\Bbbk[\Delta]$):

Theorem 3.1.1. *Let Δ be a simplicial complex on vertex set $\{1, \ldots, n\}$ and let $\Bbbk$ be a field. Then*

$$H^i_{\mathfrak{m}}(\Bbbk[\Delta])_{-j} \cong \bigoplus_{U \in T(\Delta)_j} H^{i-1}(\Delta, \operatorname{cost}_\Delta s(U))$$

as vector spaces over $\Bbbk$, where $T(\Delta)_j = \{U \in \mathbb{N}^n : s(U) \in \Delta \text{ and } |U| = j\}$, and for $U = (u_1, \ldots, u_n) \in \mathbb{N}^n$, $s(U) = \{k : u_k > 0\}$ and $|U| = \sum_{k=1}^n u_k$.

On the other hand, the effect on the Stanley–Reisner ring of a simplicial complex being endowed with a group action has also been studied in some depth (see [67, Section III.8] and [1]). Our primary goal is to bring these two topics together by studying the additional structure on $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ that appears when Δ admits a group action. As it turns out, the topological invariants introduced by the group action dictate a more detailed description of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$, providing our main result.

Our secondary goal is to apply this new decomposition of $H^i_{\mathfrak{m}}(\mathbb{k}[\Delta])$ to extend two separate classes of previously-existing results describing the h-vectors (perhaps the most widely-recognized and studied combinatorial invariants) of certain simplicial complexes. One of these classes of results examines Buchsbaum simplicial complexes: using Theorem 3.1.1, Schenzel was able to calculate the Hilbert series of the quotient of $\mathbb{k}[\Delta]$ by a linear system of parameters in the case that Δ is Buchsbaum. The culmination of this line of study was the following theorem appearing in [60], providing an algebraic interpretation of the h-vector of Δ.

Theorem 3.1.2 (Schenzel)**.** *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex, and let Θ be a linear system of parameters for Δ. Then*

$$\dim_{\mathbb{k}} \left(\mathbb{k}[\Delta]/\Theta\mathbb{k}[\Delta]\right)_i = h_i(\Delta) + \binom{d}{i}\sum_{j=0}^{i-1}(-1)^{i-j-1}\beta_{j-1}(\Delta),$$

where $\beta_{j-1}(\Delta)$ denotes the reduced simplicial Betti number of Δ computed over $\mathbb{k}$.

Since any dimension must be non-negative, the theorem provides lower bounds for the entries of $h(\Delta)$ in terms of some topological invariants of Δ. More recently, these bounds have been further lowered by the use of socles ([54]) and the sigma module $\Sigma(\Theta; \mathbb{k}[\Delta])$ (originally introduced by Goto in [19], and whose definition is provided in Section 3.2).

Theorem 3.1.3 (Murai–Novik–Yoshida)**.** *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex and let Θ be a linear system of parameters for Δ. Then*

$$\dim_{\mathbb{k}} \left(\mathbb{k}[\Delta]/\Sigma(\Theta; \mathbb{k}[\Delta])\right)_i = h_i(\Delta) + \binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta).$$

Furthermore, the symmetry appearing in the h-vector described by Gräbe in [24] was reformulated by Novik and Murai in [48, Proposition 1.1], and using Theorem 3.1.3 it may be described as follows:

Theorem 3.1.4 (Murai–Novik–Yoshida)**.** *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex and let Θ be a linear system of parameters for Δ. Then*

$$\dim_{\mathbb{k}} \left(\mathbb{k}[\Delta]/\Sigma(\Theta; \mathbb{k}[\Delta])\right)_i = \dim_{\mathbb{k}} \left(\mathbb{k}[\Delta]/\Sigma(\Theta; \mathbb{k}[\Delta])\right)_{d-i}$$

for $i = 0, \ldots, d$.

The second class of results that we will enlarge deals with the h-vectors of complexes admitting specific group actions by $\mathbb{Z}/p\mathbb{Z}$ in the Cohen–Macaulay case. In particular, the following two theorems ([65, Theorem 3.2] and [1, Theorem 3.3], respectively) provide impressive bounds on the h-vector of such a complex and are ripe for extensions to more general complexes:

Theorem 3.1.5 (Stanley)**.** *Let* Δ *be a* $(d-1)$*-dimensional Cohen-Macaulay (over* $\mathbb{C}$*) simplicial complex admitting a very free action by* $\mathbb{Z}/p\mathbb{Z}$ *with* p *a prime. Then* $h_i(\Delta) \geq \binom{d}{i}$ *if* i *is even and* $h_i(\Delta) \geq (p-1)\binom{d}{i}$ *if* i *is odd.*

Theorem 3.1.6 (Adin)**.** *Let* Δ *be a* $(d-1)$*-dimensional Cohen-Macaulay (over* $\mathbb{C}$*) simplicial complex with a free action of* $\mathbb{Z}/p\mathbb{Z}$ *with* p *a prime such that* d *is divisible by* $p-1$*. Then*

$$\sum_{i=0}^{d} h_i(\Delta)\lambda^i \geq (1 + \lambda + \cdots + \lambda^{p-1})^{d/(p-1)},$$

where the inequality holds coefficient-wise.

In summary, our new results are the following (as much notation must be introduced before explicit statements may be provided, we present here a brief overview):

- *A new version of Hochster and Gräbe's theorem:* We examine how a group action leads to a special decomposition of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$, and we demonstrate how the isomorphism of Theorem 3.1.1 respects this decomposition (Theorem 3.3.1). Later, for the $G = \mathbb{Z}/p\mathbb{Z}$ case, we encounter a piece of the local cohomology of $\Bbbk[\Delta]$ that corresponds to the singular cohomology of the quotient $|\Delta|/G$.
- *Applications to h-vectors:* The dimension calculations of Theorems 3.1.2 and 3.1.3 are refined and specialized to the case of a Buchsbaum complex admitting a free action by a cyclic group of prime order (Theorems 3.4.2 and 3.4.8). Using this refinement, Theorems 3.1.5 and 3.1.6 are generalized to the setting of Buchsbaum complexes; in

some cases, the expressions are identical (Section 3.4.3). Lastly, in Theorems 3.4.12 and 3.4.15, we exhibit a symmetry in the dimensions of finely graded pieces of some Artinian reductions of Stanley–Reisner rings of certain orientable homology manifolds without boundary admitting a free group action by a cyclic group of prime order in the flavor of Theorem 3.1.4.

The chapter is organized as follows. In Section 3.2, we introduce notation, review classical results, and discuss in some depth the theory of graded rings and modules. In Section 3.3 we study the local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ and provide our main new result. Section 3.4 is devoted to calculating a finely-graded Hilbert series of an Artinian reduction of the Stanley–Reisner ring $\Bbbk[\Delta]$, providing many applications of our main theorem. We close with comments and questions in Section 3.5.

3.2 Preliminaries

3.2.1 Combinatorics and topology

A **simplicial complex** Δ on a finite vertex set V is a collection of subsets of V that is closed under inclusion. The elements of Δ are called **faces**, and the maximal faces (with respect to inclusion) are **facets**. The **dimension** of a face σ is defined by $\dim \sigma := |\sigma| - 1$, and the **dimension** of Δ is defined by $\dim \Delta := \max\{\dim \sigma : \sigma \in \Delta\}$. The faces of dimension zero are called **vertices**. If all facets of Δ have the same dimension, then we say that Δ is **pure**.

Given a face σ of Δ, we define the **contrastar** of σ in Δ by

$$\operatorname{cost}_\Delta \sigma := \{\tau \in \Delta : \sigma \not\subset \tau\}.$$

Similarly, the **link** of σ in Δ is

$$\operatorname{lk}_\Delta \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\}.$$

We define the ***f*-vector** $f(\Delta)$ of a $(d-1)$-dimensional simplicial complex Δ by

$$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta)),$$

where $f_i(\Delta)$ is the number of i-dimensional faces of Δ. The h-**vector** $h(\Delta)$ is then defined by $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$, where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}(\Delta).$$

Let $\Bbbk$ be a field, and let $\widetilde{H}^i(\Delta)$ be the i-th reduced simplicial cohomology group of Δ with coefficients in $\Bbbk$. If Γ is a subcomplex of Δ, then $H^i(\Delta, \Gamma)$ is the i-th relative cohomology group of the pair (Δ, Γ) with coefficients in $\Bbbk$. In the case that $\Gamma = \{\emptyset\}$, this is the same as the reduced cohomology group $\tilde{H}^i(\Delta)$. Denote the i**-th (reduced) Betti number** of Δ over $\Bbbk$ by

$$\beta_i(\Delta) := \dim_{\Bbbk} \widetilde{H}^i(\Delta).$$

We call a complex Δ **Cohen-Macaulay** if $\beta_i(\mathrm{lk}_\Delta \sigma) = 0$ for all faces σ and all $i < \dim \mathrm{lk}_\Delta \sigma$. Similarly, we call a complex **Buchsbaum** if Δ is pure and $\beta_i(\mathrm{lk}_\Delta \sigma) = 0$ for all faces $\sigma \neq \emptyset$ and all $i < \dim \mathrm{lk}_\Delta \sigma$. If Δ is $(d-1)$-dimensional and the link of each non-empty face σ of Δ has the homology of a $(d-|\sigma|-1)$-sphere, then we say that Δ is a $\Bbbk$**-homology manifold**; furthermore, if $\beta_{\dim \Delta}(\Delta)$ is equal to the number of connected components of Δ, then we say that Δ is **orientable**.

Now let G be a finite group and suppose G acts on a simplicial complex Δ (in particular, each $g \in G$ acts as a simplicial isomorphism on Δ). We say that G acts **freely** if $g \cdot \sigma \neq \sigma$ for all faces $\sigma \neq \emptyset$ and all non-identity elements $g \in G$ (we will always take our actions to be defined on the left). If G acts freely on Δ and, additionally, $(g \cdot \{v\}) \cup \{v\}$ is not an edge of Δ for every vertex $v \in V$ and every non-identity element $g \in G$, then we say that G acts **very freely**.

In the case that $G = \mathbb{Z}/2\mathbb{Z}$, free and very free actions are equivalent and we call Δ **centrally-symmetric** when such an action by G on Δ exists. When extending a free group action by G to the geometric realization $|\Delta|$ of Δ, the action obtained is a covering space action (see [26, Section 1.3]).

If Δ admits a group action by G, then this extends to to an action on the reduced (simplicial) chain complex $\tilde{C}.(\Delta)$ by permuting its basis elements in accordance with the group action. This extends further to the cochain complex $\tilde{C}^{\cdot}(\Delta) := \mathrm{Hom}_{\Bbbk}(\tilde{C}_{\cdot}(\Delta), \Bbbk)$ in the usual way: if $\hat{\sigma} \in \tilde{C}^i(\Delta)$ is the basis element defined by $\hat{\sigma}(\tau) = 0$ for $\sigma \neq \tau$ and $\hat{\sigma}(\sigma) = 1$, then

$$(g \cdot \hat{\sigma})(\tau) = \hat{\sigma}(g^{-1} \cdot \tau),$$

and hence $g \cdot \hat{\sigma} = \widehat{(g \cdot \sigma)}$.

Lastly, given a vector $U = (u_1, \ldots, u_n) \in \mathbb{N}^n$, let $s(U) = \{i : u_i > 0\}$ be the support of U and let $|U| = \sum_{i=1}^{n} u_j$ be the L^1-norm of U. If Δ is a simplicial complex on vertex set $[n]$, then denote

$$T(\Delta)_j = \{U \in \mathbb{N}^n : s(U) \in \Delta \text{ and } |U| = j\}.$$

3.2.2 Modules and group actions

Assume from now on that the field $\Bbbk$ is an extension of $\mathbb{C}$. If G is a finite abelian group and M is a $\Bbbk[G]$-module, then M admits a decomposition into isotypic components according to the action of G. That is,

$$M = \bigoplus_{\chi \in \hat{G}} N^{\chi}$$

where $\hat{G}$ is the group of irreducible characters χ of G and

$$M^{\chi} := \{m \in M : g \cdot m = \chi(g)m \text{ for all } g \in G\}.$$

Of vital importance will be the consideration of direct sums of simplicial cohomology modules of the form

$$\bigoplus_{U \in T(\Delta)_j} H^{i-1}(\Delta, \mathrm{cost}_{\Delta}\, s(U)),$$

which inherit a G-action from Δ as in the previous section. In particular, Theorem 3.3.1 will consider the isotypic components of modules of this form. In the $j = 0$ case, we set

$$\beta_i(\Delta)^{\chi} := \dim_{\Bbbk} \widetilde{H}^i(\Delta)^{\chi}. \tag{3.1}$$

These refined Betti numbers will be one of the main invariants considered in Sections 4 and 5 of this chapter.

Now let A be the polynomial ring $\Bbbk[x_1, \ldots, x_n]$, graded by degree, and let M be a finitely-generated $\mathbb{Z}$-graded A-module, written as an abelian group by $M = \bigoplus_i M_i$. If G acts on M and each M_i is G-invariant (i.e. $G \cdot M_i \subseteq M_i$), then each M_i can itself be thought of as a $\Bbbk[G]$-module. As above,

$$M_i = \bigoplus_{\chi \in \hat{G}} M_i^\chi \tag{3.2}$$

where

$$M_i^\chi = \{m \in M_i : g \cdot m = \chi(g)m \text{ for all } g \in G\}.$$

Suppose A is endowed with a degree-preserving action of G. Then decomposition in (3.2) makes A into a $(\mathbb{Z} \times \hat{G})$-graded ring, since $a_1 \in A_{i_1}^{\chi_1}$ and $a_2 \in A_{i_2}^{\chi_2}$ implies that $a_1a_2 \in A_{i_1+i_2}$ and that

$$g \cdot (a_1a_2) = (g \cdot a_1)(g \cdot a_2) = \chi_1(g)a_1\chi_2(g)a_2 = (\chi_1\chi_2)(g)a_1a_2$$

for all $g \in G$. If, additionally,

$$g \cdot (am) = (g \cdot a)(g \cdot m) \tag{3.3}$$

for all $a \in A$, $g \in G$, and $m \in M$, then the decomposition in (3.2) allows for a $(\mathbb{Z}\times\hat{G})$-grading for M, where M_i^χ is the component of M consisting of all elements of degree (i, χ).

If N is another $(\mathbb{Z} \times \hat{G})$-graded A-module satisfying the same conditions as M, then G acts on $\operatorname{Hom}_A(M, N)$ by defining

$$(g \cdot f)(m) := g \cdot f(g^{-1} \cdot m)$$

for all $g \in G$, $f \in \operatorname{Hom}_A(M, N)$, and $m \in M$. If $m \in M_j^{\chi_1}$ and $f \in \operatorname{Hom}_A(M, N)_i^{\chi_2}$, then for all $g \in G$,

$$\chi_2(g)f(m) = (g \cdot f)(m) = g \cdot f(g^{-1} \cdot m) = g \cdot f(\chi_1^{-1}(g)m) = \chi_1^{-1}(g)g \cdot f(m),$$

so $g \cdot f(m) = (\chi_1\chi_2)(g)f(m)$ and $f(m) \in N_{i+j}^{\chi_1\chi_2}$. On the other hand, if $f \in \operatorname{Hom}_A(M, N)$ is such that $f(m) \in N_{i+j}^{\chi_1\chi_2}$ for all choices of i, χ_1, and $m \in M_i^{\chi_1}$, then $g \cdot f = \chi_2(g)f$ for all $g \in G$. Hence,

$$\operatorname{Hom}_A(M, N)_i^{\chi_1} = \{f \in \operatorname{Hom}_A(M, N) : f(M_j^{\chi_2}) \subseteq N_{i+j}^{\chi_1\chi_2} \text{ for all } (j, \chi_2) \in \mathbb{Z} \times \hat{G}\}.$$

That is, the standard $(\mathbb{Z} \times \hat{G})$-grading on $\operatorname{Hom}_A(M, N)$ is consistent with the induced one above. In general, the set of graded A-module homomorphisms from M to N is a submodule of $\operatorname{Hom}_A(M, N)$. However, in our case of M being finitely generated, the two submodules are equal (see [8, Section II.11.6]). Unless stated otherwise, a $(\mathbb{Z} \times \hat{G})$-graded map of A-modules will refer to a homomorphism of degree 0.

Let $\mathfrak{m} = (x_1, \ldots, x_n)$ denote the irrelevant ideal of A, let $H^i_{\mathfrak{m}}(M)$ denote the i-th local cohomology module of M with support in $\mathfrak{m}$ (see [29] for some basic properties of these modules), and denote $\mathfrak{m}_j = (x_1^j, \ldots, x_n^j)$ (not to be confused with M_j, the j-th graded piece of M as a $\mathbb{Z}$-graded A-module). We can trace the extra grading of Hom modules through to some standard local cohomology results using the following proposition.

Proposition 3.2.1. *Let G be a finite abelian group, and let M be a $(\mathbb{Z} \times \hat{G})$-graded A-module with an action of G satisfying* (3.3)*. Then:*

(*a*) $\operatorname{Ext}^i_A(A/\mathfrak{m}_j, M)$ *and* $H^i_{\mathfrak{m}}(M)$ *are $(\mathbb{Z} \times \hat{G})$-graded A-modules for all i and all $j \geq 1$.*

(*b*) *The canonical maps* $\psi^M_{i,j} : \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M) \to H^i_{\mathfrak{m}}(M)$ *are $(\mathbb{Z} \times \hat{G})$-graded.*

(*c*) *If N satisfies the same conditions as M and $f : M \to N$ is a map of $(\mathbb{Z} \times \hat{G})$-graded A-modules, then the induced map $f^* : H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(N)$ is also $(\mathbb{Z} \times \hat{G})$-graded.*

(*d*) *If $0 \to L \to M \to N \to 0$ is a short exact sequence of $(\mathbb{Z} \times \hat{G})$-graded A-modules, then there is a long exact sequence*

$$\cdots \to H^{i-1}_{\mathfrak{m}}(N) \xrightarrow{\delta} H^i_{\mathfrak{m}}(L) \to H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(N) \xrightarrow{\delta} H^{i+1}_{\mathfrak{m}}(L) \to \cdots$$

of $(\mathbb{Z} \times \hat{G})$-graded A-modules.

Remark 3.2.2. While these statements are to be expected and follow the typical constructions, we could not find any all-encompassing reference for them. Since many of the modules and maps involved are vital to the proof of Theorems 3.3.1 and 3.4.2, the proofs have been included both for the sake of completeness and for a preview of what is to come.

Proof. Let $K^j_\bullet$ denote the Koszul complex of A with respect to the sequence $x^j_1, \ldots, x^j_n$. We view each term K_t of $K^j_\bullet$ as

$$K_t = \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_t \leq n} A(x^j_{i_1} \wedge x^j_{i_2} \wedge \cdots \wedge x^j_{i_t}),$$

and we can naturally extend the action of G on A to an action of G on $K^j_\bullet$ by defining

$$g \cdot \left(f x^j_{i_1} \wedge \cdots \wedge x^j_{i_t}\right) = (g \cdot f)((g \cdot x_{i_1})^j \wedge \cdots \wedge (g \cdot x_{i_t})^j).$$

Then equation (3.3) holds for each K_t, making them $(\mathbb{Z} \times \hat{G})$-graded A-modules. The differential map on $K^j_\bullet$ preserves the G action, so $\operatorname{Hom}_A(K^j_\bullet, M)$ is a cochain complex of $(\mathbb{Z} \times \hat{G})$-graded A-modules. Furthermore, the cohomology modules obtained from this cochain complex are also $(\mathbb{Z} \times \hat{G})$-graded A-modules (see [8, Proposition II.11.3.3]).

Since $K^j_\bullet$ provides a projective resolution of $A/\mathfrak{m}_j$,

$$H^i(\operatorname{Hom}_A(K^j_\bullet, M)) = \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M).$$

The maps $\varphi_j : K^{j+1}_\bullet \to K^j_\bullet$ defined by $\varphi_j(x^{j+1}_{i_1} \wedge \cdots \wedge x^{j+1}_{i_t}) = (x_{i_1} \cdots x_{i_t}) x^j_{i_1} \wedge \cdots \wedge x^j_{i_t}$ also preserve the action of G, so the pullbacks

$$\varphi^*_j : \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M) \to \operatorname{Ext}^i_A(A/\mathfrak{m}_{j+1}, M)$$

do as well. These maps provide a direct system of $(\mathbb{Z} \times \hat{G})$-graded A-modules in which

$$H^i_{\mathfrak{m}}(M) := \varinjlim \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M).$$

By [8, Remark II.11.3.3], $H^i_{\mathfrak{m}}(M)$ is a $(\mathbb{Z} \times \hat{G})$-graded A-module and the canonical maps $\psi^M_{i,j} : \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M) \to H^i_{\mathfrak{m}}(M)$ also preserve the action of G. This proves (a) and (b), from which (c) follows almost immediately; if $m \in H^i_{\mathfrak{m}}(M)$, choose j and m_j such that $m = \psi^M_{i,j}(m_j)$, and let

$$f_j : \operatorname{Ext}^i_A(A/\mathfrak{m}_j, M) \to \operatorname{Ext}^i_A(A/\mathfrak{m}_j, N)$$

be the map induced in Ext modules. It follows from the definitions that f_j is a $(\mathbb{Z} \times \hat{G})$-graded map, while $f^*(m) = (f \circ \psi^M_{i,j})(m_j) = (\psi^N_{i,j} \circ f_j)(m_j)$. Since all maps here are $(\mathbb{Z} \times \hat{G})$-graded, f^* is as well.

For part (d), all that remains to be checked is that the connecting homomorphism δ is $(\mathbb{Z}\times\hat{G})$-graded. This follows from its standard construction, as it is defined as a composition of certain $(\mathbb{Z}\times\hat{G})$-graded maps. □

For any integer a, denote by $M[a]$ the shifted $\mathbb{Z}$-graded A-module defined by $M[a]_i = M_{i+a}$. If we are also given $\chi_1 \in \hat{G}$, then we can define a new G-action $*$ on $M[a]$ by setting

$$g * m = \chi_1^{-1}(g)(g \cdot m).$$

for all $g \in G$. Denote by $M[a,\chi_1]$ the module $M[a]$ endowed with this new action of G. Then for all $g \in G$,

$$\{m \in M_{i+a} : g \cdot m = (\chi_1\chi_2)(g)m\} = \{m \in M[a]_i : g * m = \chi_2(g)m\},$$

hence $M[a,\chi_1]_i^{\chi_2} = M_{i+a}^{\chi_1\chi_2}$, and this makes $M[a,\chi_1]$ into a $(\mathbb{Z}\times\hat{G})$-graded A-module. Indeed, if $m \in M[a,\chi_1]_i^{\chi_2}$ and $f \in A_j^{\chi_3}$, then

$$g * (fm) = \chi_1^{-1}(g)(g \cdot fm) = \chi_1^{-1}(g)(\chi_1\chi_2\chi_3)(g)fm = (\chi_2\chi_3)(g)fm,$$

so that $fm \in M[a,\chi_1]_{i+j}^{\chi_2\chi_3}$. Moreover, if $f \in A_j^{\chi_2}$ then the multiplication map $\cdot f : M[-j,\chi_2^{-1}] \to M$ is a map of $(\mathbb{Z}\times G)$-graded A modules (of degree 0), since if $m \in M[-j,\chi_2^{-1}]_i^{\chi_1}$, then

$$f(g * m) = f\chi_2(g)(g \cdot m) = (g \cdot f)(g \cdot m) = g \cdot (fm).$$

For the rest of this section, we will only be considering groups G of the form $\mathbb{Z}/p\mathbb{Z}$ for a prime p. In this case, $G \cong \hat{G}$. Once we have fixed a generator g for G and a primitive p-th root of unity ζ, there is a group isomorphism mapping $\chi \in \hat{G}$ to $j \in \mathbb{Z}/p\mathbb{Z}$ such that

$$M^\chi = \{m \in M : g \cdot m = \zeta^j m\}.$$

Hence, we write

$$M = \bigoplus_{j=0}^{p-1} M^j$$

where

$$M^j := \{m \in M : g \cdot m = \zeta^j m\}.$$

With this identification, we view all of the modules constructed above as $(\mathbb{Z} \times G)$-graded modules, where M_i^j is the component of M in degree (i, j). As in our Definition (3.1), we set

$$\beta_i(\Delta)^j := \dim_{\Bbbk} \widetilde{H}^i(\Delta)^j. \tag{3.4}$$

Thinking now of M as a $(\mathbb{Z} \times G)$-graded vector space over $\Bbbk$, we can define the Hilbert series $\mathrm{Hilb}(M, \lambda, t)$ of M by

$$\mathrm{Hilb}(M, \lambda, t) = \sum_{(i,j)\in(\mathbb{Z}\times G)} (\dim_{\Bbbk} M_i^j)\lambda^i t^j$$

where λ and t are indeterminates with $t^p = 1$.

If M is of Krull dimension $d > 0$, we call a system $\Theta = \theta_1, \theta_2, \ldots, \theta_d$ of homogeneous elements in A a **homogeneous system of parameters** (or an h.s.o.p.) for M if $M/\Theta M$ is a finite-dimensional vector space over $\Bbbk$. If each $\theta_i \in A_1$, then we call Θ a **linear system of parameters** (or an l.s.o.p.) for M. We say that M is **Cohen-Macaulay** if every l.s.o.p. is a regular sequence on M, and M is **Buchsbaum** if every l.s.o.p. satisfies

$$(\theta_1, \ldots, \theta_{i-1})M :_M \theta_i = (\theta_1, \ldots, \theta_{i-1})M :_M \mathfrak{m}$$

for $i = 1, \ldots, d$. Given any h.s.o.p. Θ for M, we also have the notion of the sigma module $\Sigma(\Theta; M)$, defined by

$$\Sigma(\Theta; M) := \Theta M + \sum_{i=1}^{d} \Big((\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_d)M :_M \theta_i\Big).$$

3.2.3 Stanley–Reisner rings

Let Δ be a $(d-1)$-dimensional simplicial complex with vertex set $[n] := \{1, \ldots, n\}$. Given $\sigma \subset [n]$, write $x_\sigma = \prod_{i\in\sigma} x_i$. The **Stanley–Reisner ideal** of Δ is the ideal I_Δ of A defined by

$$I_\Delta = (x_\sigma : \sigma \subset [n], \sigma \notin \Delta).$$

The **Stanley–Reisner ring** of Δ (over $\Bbbk$) is

$$\Bbbk[\Delta] := A/I_\Delta.$$

We will always consider $\Bbbk[\Delta]$ as a module over A. The geometric notion of Buchsbaumness is tied algebraically to the Stanley–Reisner ring through the following vital result (found in [60]).

Theorem 3.2.3 (Schenzel)**.** *A pure simplicial complex Δ is Buchsbaum over $\Bbbk$ if and only if $\Bbbk[\Delta]$ is a Buchsbaum A-module.*

If Δ admits an action by the group $G = \mathbb{Z}/p\mathbb{Z}$, then this extends to an action on $\Bbbk[\Delta]$ and induces a $(\mathbb{Z} \times G)$-grading as detailed in the previous section. If this action is free, then for any j and $i \geq 1$ we have

$$\dim_\Bbbk \Bbbk[\Delta]_i^j = \frac{1}{p} \dim_\Bbbk \Bbbk[\Delta]_i.$$

Also, $\dim_\Bbbk \Bbbk[\Delta]_0^0 = 1$ and $\dim_\Bbbk \Bbbk[\Delta]_0^j = 0$ for $0 < j < p$. As in [65, Section 3], this implies the following expression for the $(\mathbb{Z} \times G)$-graded Hilbert series of $\Bbbk[\Delta]$.

Theorem 3.2.4. *Let Δ be a $(d-1)$-dimensional simplicial complex admitting a free action by the group $\mathbb{Z}/p\mathbb{Z}$. Then*

$$\mathrm{Hilb}(\Bbbk[\Delta], \lambda, t) = 1 + \frac{1}{p}\left[\frac{\sum_{i=1}^d h_i(\Delta)\lambda^i}{(1-\lambda)^d} - 1\right](1 + t + \cdots + t^{p-1}).$$

In fact, if Δ is centrally-symmetric (so $p = 2$) and Cohen-Macaulay, then a certain Hilbert series provides a strong inequality bounding the h-vector of Δ; see [67, Theorem III.8.1]. If Θ is an l.s.o.p. for Δ, then we denote by $\Bbbk(\Delta; \Theta)$ the quotient $\Bbbk[\Delta]/\Theta\Bbbk[\Delta]$.

Theorem 3.2.5 (Stanley)**.** *Let Δ be a $(d-1)$-dimensional centrally-symmetric Cohen-Macaulay simplicial complex, and suppose $\Theta = (\theta_1, \ldots, \theta_d)$ is a l.s.o.p. for $\Bbbk[\Delta]$ in which $\theta_i \in \Bbbk[\Delta]_1^1$ for $i = 1, \ldots, d$. Then*

$$\mathrm{Hilb}(\Bbbk(\Delta; \Theta), \lambda, t) = \frac{1}{2}\left[(1-t)(1+\lambda)^d + (1+t)\sum_{i=0}^d h_i(\Delta)\lambda^i\right].$$

3.3 The main theorem: a refinement of Hochster

For the rest of the chapter, fix Δ to be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$. Let $\Bbbk$ be an extension of $\mathbb{C}$, and let A be the polynomial ring $\Bbbk[x_1, \ldots, x_n]$. Recall that $T(\Delta)_j$ is the set $\{U \in \mathbb{N}^n : s(U) \in \Delta \text{ and } |U| = j\}$.

Theorem 3.3.1. *Let Δ be a simplicial complex with a (not necessarily free) action by a finite abelian group G. Then the isomorphisms*

$$H^i_{\mathfrak{m}}(\Bbbk[\Delta])_{-j} \cong \bigoplus_{U \in T(\Delta)_j} H^{i-1}(\Delta, \operatorname{cost}_\Delta s(U))$$

of Theorem 3.1.1 induce vector space isomorphisms

$$H^i_{\mathfrak{m}}(\Bbbk[\Delta])^\chi_{-j} \cong \left[\bigoplus_{U \in T(\Delta)_j} H^{i-1}(\Delta, \operatorname{cost}_\Delta s(U))\right]^\chi$$

for each irreducible character χ of G.

Remark 3.3.2. There are a multitude of proofs of Theorem 3.1.1, such as [23, Theorem 1] or [41, Corollary 4.4]. Our proof of the refinement above will be a modification of that which appears in [42].

Proof. As in Section 3.2.2, let $\mathfrak{m}_{j+1}$ denote the ideal $(x_1^{j+1}, \ldots, x_n^{j+1})$ and let $K_\bullet^{j+1}$ denote the Koszul complex of A with respect to the sequence $(x_1^{j+1}, \ldots, x_n^{j+1})$. If $C^\bullet(\Delta, \Gamma)$ denotes the relative simplicial cochain complex of the pair (Δ, Γ) with coefficients in $\Bbbk$, then we know from the proofs of Proposition 3.2.1 and [42, Theorem 1] that

$$H^i(\operatorname{Hom}_A(K_\bullet^{j+1}, \Bbbk[\Delta])_{-j}) = \operatorname{Ext}^i_A(A/\mathfrak{m}_{j+1}, \Bbbk[\Delta])_{-j} \cong \bigoplus_{U \in T(\Delta)_j} H^{i-1}(C^\bullet(\Delta, \operatorname{cost}_\Delta s(U))) \quad (3.5)$$

as vector spaces over $\Bbbk$. Note that while [42, Theorem 1] is stated with respect to a $\mathbb{Z}^n$-grading, Reisner's proof in [57, Theorem 2] shows that the only non-acyclic components of the cochain complex $\operatorname{Hom}_A(K_\bullet^{j+1}, \Bbbk[\Delta])$ of $\mathbb{Z}$-graded A-modules also occur in the $\mathbb{Z}^n$-graded case.

Our first goal is to show that this isomorphism inverts isotypic components in the sense that it induces isomorphisms of the form

$$\operatorname{Ext}^i_A(A/\mathfrak{m}_{j+1}, \Bbbk[\Delta])^{\chi}_{-j} \cong \left[\bigoplus_{U \in T(\Delta)_j} H^{i-1}(C^{\bullet}(\Delta, \operatorname{cost}_\Delta s(U)))\right]^{\chi} \tag{3.6}$$

for each irreducible character χ of G.

Given some set $\sigma = \{i_1, \dots, i_t\} \subset [n]$ with $i_1 < i_2 < \cdots < i_t$, define $\overline{x}_\sigma := x^{j+1}_{i_1} \wedge \cdots \wedge x^{j+1}_{i_t}$ in K^{j+1}_t and recall that $x_\sigma := x_{i_1} \cdots x_{i_t}$ in $\Bbbk[\Delta]$. Likewise, if $U = (i_1, \dots, i_n) \in \mathbb{N}^n$, define $x_U := x_1^{i_1} \cdots x_n^{i_n}$ in $\Bbbk[\Delta]$. If $\sigma \subset [n]$ is such that $|\sigma| = t$ and $U \in \mathbb{N}^n$ is such that $s(U) \subset \sigma$ and $|U| = j$, define

$$f^U_\sigma(\overline{x}_\tau) = \begin{cases} (x_\sigma)^{j+1}/x_U & \text{if } \tau = \sigma \\ 0 & \text{if } \tau \neq \sigma. \end{cases}$$

Then $\operatorname{Hom}_A(K^{j+1}_t, \Bbbk[\Delta])_{-j}$ has a vector subspace with basis given by

$$D^t = \{f^U_\sigma : |\sigma| = t,\ \sigma \in \Delta,\ |U| = j \text{ and } s(U) \subset \sigma\}.$$

For the right side of (3.5), given $\sigma \in \Delta$ with $|\sigma| = t$, as in Section 3.2.1 let $\hat{\sigma} : C_{t-1}(\Delta) \to \Bbbk$ be the homomorphism defined by $\hat{\sigma}(\tau) = 1$ if $\tau = \sigma$ and $\hat{\sigma}(\tau) = 0$ otherwise and define

$$\langle \sigma, U \rangle := \hat{\sigma} + C^{t-1}(\operatorname{cost}_\Delta s(U)).$$

Then a basis for

$$\bigoplus_{U \in T(\Delta)_j} C^{t-1}(\Delta, \operatorname{cost}_\Delta s(U))$$

is given by

$$B = \{\langle \sigma, U \rangle : \sigma \in \Delta,\ |\sigma| = t,\ U \in T(\Delta)_j, \text{ and } s(U) \subseteq \sigma\},$$

and Miyazaki's proof of Theorem 3.1.1 shows that the direct sum of maps

$$\psi : \bigoplus_{U \in T(\Delta)_j} C^{t-1}(\Delta, \operatorname{cost}_\Delta s(U)) \to \operatorname{Hom}_A(K^{j+1}_t, \Bbbk[\Delta])_{-j}$$

defined componentwise by

$$\psi(\langle \sigma, U \rangle) = f^U_\sigma$$

is the chain map inducing the isomorphism in cohomology in (3.5).

Now consider how G acts on each of these cochain complexes. First, $g \cdot f_\sigma^U = f_{g\cdot\sigma}^{g\cdot U}$ from the definition of the action on $\mathrm{Hom}_A(K_\bullet^{j+1}, \Bbbk[\Delta])$. On the other hand,

$$g \cdot \langle \sigma, U \rangle = \langle g \cdot \sigma, g \cdot U \rangle,$$

where G acts on $U \in \mathbb{N}^n$ by permuting indices in accordance with the action of G on the vertex set $[n]$. Hence, ψ is a G-equivariant isomorphism and (3.6) is established.

Finally, again referring to the proof of [42, Theorem 1], the canonical maps

$$\mathrm{Ext}_A^i(A/\mathfrak{m}_{j+1}, \Bbbk[\Delta]) \to H_\mathfrak{m}^i(\Bbbk[\Delta])$$

induce isomorphisms in $\mathbb{Z}$-graded components of degree at least $-j$. Since these are also maps of $(\mathbb{Z} \times \hat{G})$-graded A-modules by Proposition 3.2.1, we obtain the desired isomorphism in the statement of the theorem. □

3.4 Applications

3.4.1 Hilbert series of Artinian reductions and an analog of Schenzel's formula

From now on, Δ is fixed to be Buchsbaum and $G = \mathbb{Z}/p\mathbb{Z}$ for some prime p with a fixed generator g. We consider A to be $(\mathbb{Z} \times G)$-graded and $\Bbbk[\Delta]$ to be a $(\mathbb{Z} \times G)$-graded A-module as in Section 3.2. In a way similar to [67, Section 8], if the action of G is very free and $0 \le \delta_i \le p-1$ for $i = 1, \ldots, d$, then we can easily construct a l.s.o.p. $\Theta = \theta_1, \ldots, \theta_d$ for $\Bbbk[\Delta]$ in which $\theta_i \in A_1^{\delta_i}$ for $1 \le i \le d$ as follows.

First, choose one face from each G-orbit of Δ and collect them into the set Δ_G. Let W be the set of vertices of Δ that are in Δ_G, and choose functions $t_1, \ldots, t_d : W \to \Bbbk$ such that their restrictions to any subset of W of size d are linearly independent. Now extend t_i to all of $[n]$ by setting

$$t_i(g^k \cdot v) = \zeta^{-k\delta_i} t_i(v)$$

for all i and k, and let

$$\theta_i = \sum_{v \in [n]} t_i(v) x_v.$$

Then

$$g \cdot \theta_i = \sum_{v \in [n]} t_i(v) x_{g \cdot v} = \sum_{v \in [n]} t_i(g^{-1} \cdot v) x_v = \sum_{v \in [n]} \zeta^{\delta_i} t_i(v) x_v,$$

so $\theta_i \in A_1^{\delta_i}$ for $i = 1, \ldots, d$. Furthermore, since no facet of Δ contains $\{g^i \cdot v, g^j \cdot v\}$ for any v with $i \not\equiv j \mod p$, the system $\Theta = (\theta_1, \ldots, \theta_d)$ forms an l.s.o.p. that is homogeneous with respect to the $(\mathbb{Z} \times G)$-grading by [67, Lemma III.2.4(a)].

Remark 3.4.1. The construction above is included purely for the sake of concreteness in the case of a very free action. In fact, as Adin shows in his thesis ([1]), it is possible to construct an l.s.o.p. with the prescribed properties above even in the case of a free action. The construction involves changing the field $\Bbbk$ to a particular extension of $\mathbb{C}$. However, as we are now turning our focus to certain Hilbert series (which are invariant under field extensions), all of the results that follow hold for any extension of $\mathbb{C}$ and are stated with this understanding in mind.

Now that the existence of l.s.o.p.'s of the form above has been established, we may use them to prove the following theorem (recall the refined Betti number notation of definition (3.4) and that $\Bbbk(\Delta; \Theta) = \Bbbk[\Delta]/\Theta\Bbbk[\Delta]$ for an l.s.o.p. Θ).

Theorem 3.4.2. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free group action by $G = \mathbb{Z}/p\mathbb{Z}$, let $0 \leq m \leq p-1$ be some fixed degree, and let Θ be a G-homogeneous l.s.o.p. for $\Bbbk[\Delta]$ such that $\theta_i \in A_1^m$ for all i. Then the $(\mathbb{Z} \times G)$-graded Hilbert series of $\Bbbk(\Delta; \Theta)$ is given by*

$$\begin{aligned}\mathrm{Hilb}(\Bbbk(\Delta; \Theta), \lambda, t) = &\sum_{i=0}^{d} \left[(-1)^i \binom{d}{i} t^{mi} + \left(\frac{1}{p} \sum_{k=0}^{p-1} t^k \right) \left(h_i(\Delta) + (-1)^{i+1} \binom{d}{i} \right) \right] \lambda^i \\ &+ \sum_{i=0}^{d} \binom{d}{i} \lambda^i \sum_{j=0}^{i-1} (-1)^{i-j-1} \left(\sum_{k=0}^{p-1} t^k \beta_{j-1}(\Delta)^{k-im} \right).\end{aligned}$$

Remark 3.4.3. Recall that $t^p = 1$ in this Hilbert series, and note that by setting $t = 1$, we can recover Schenzel's Theorem 3.1.2. Furthermore, we conclude that $h_i(\Delta) \equiv (-1)^i \binom{d}{i}$ (mod p) for all i.

Proof. Suppose that Θ is an arbitrary l.s.o.p. with $\theta_i \in A_1^{\delta_i}$ for $i = 1, \dots, d$. For $s = 1, \dots, d$, denote

$$\Bbbk_s[\Delta] := \Bbbk[\Delta]/(\theta_1, \dots, \theta_s)\Bbbk[\Delta].$$

Then (recall the shifted module discussion from Section 3.2.2) we have exact sequences of $(\mathbb{Z} \times G)$-graded A-modules with degree-preserving maps of the form

$$0 \to Q(s)[-1, -\delta_s] \to \Bbbk_{s-1}[\Delta][-1, -\delta_s] \xrightarrow{\cdot\theta_s} \Bbbk_{s-1}[\Delta] \to \Bbbk_s[\Delta] \to 0,$$

where

$$Q(s) = 0 :_{\Bbbk_{s-1}[\Delta]} \theta_s = 0 :_{\Bbbk_{s-1}[\Delta]} \mathfrak{m} = H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]).$$

The second equality above follows from the definition of Buchsbaumness, and the third follows from the proof of [68, Proposition I.2.1]. On the level of Hilbert series, a standard argument yields the following equation:

$$\begin{aligned}\operatorname{Hilb}(\Bbbk(\Delta;\Theta), \lambda, t) = \operatorname{Hilb}(\Bbbk[\Delta], \lambda, t) \prod_{i=1}^{d} (1 - \lambda t^{\delta_i}) \\ + \sum_{s=1}^{d} \lambda t^{\delta_i} \operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]), \lambda, t) \prod_{j=s+1}^{d} (1 - \lambda t^{\delta_j}).\end{aligned}$$

In the case $\theta_s \in A_1^m$ for $s = 1, \dots, d$, the equation above simplifies to

$$\begin{aligned}\operatorname{Hilb}(\Bbbk(\Delta;\Theta), \lambda, t) = (1 - \lambda t^m)^d \operatorname{Hilb}(\Bbbk[\Delta], \lambda, t) \\ + \sum_{s=1}^{d} \lambda t^m (1 - \lambda t^m)^{d-s} \operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]), \lambda, t).\end{aligned}$$

Analyzing the first term yields the following, using Theorem 3.2.4:

$$\begin{aligned}&(1 - \lambda t^m)^d \operatorname{Hilb}(\Bbbk[\Delta], \lambda, t) \\ &= (1 - \lambda t^m)^d \left[1 + \frac{1}{p}\left(\frac{\sum_{i=0}^{d} h_i(\Delta)\lambda^i}{(1-\lambda)^d} - 1\right)(1 + t + \cdots + t^{p-1})\right] \\ &= (1 - \lambda t^m)^d + \frac{(1 + t + \cdots + t^{p-1})}{p}\left[\sum_{i=0}^{d} h_i(\Delta)\lambda^i - (1 - \lambda t^m)^d\right] \\ &= \sum_{i=0}^{d}\left[(-1)^i \binom{d}{i} t^{mi} + \frac{1}{p}\sum_{k=0}^{p-1} t^k \left(h_i(\Delta) + (-1)^{i+1}\binom{d}{i}\right)\right]\lambda^i.\end{aligned}$$

For the second term,

$$H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]) \cong \bigoplus_{i=0}^{s-1}\left(\bigoplus_{\binom{s-1}{i}} H^i_{\mathfrak{m}}(\Bbbk[\Delta])[-i,-im]\right)$$

by [68, Proposition II.4.14'] (though the stated result is for $\mathbb{Z}$-graded A-modules, the exact same proof works in the $(\mathbb{Z}\times G)$-graded case using Proposition 3.2.1). By our Theorem 3.3.1, $H^i_{\mathfrak{m}}(\Bbbk[\Delta])[-i,-im]$ is concentrated in $\mathbb{Z}$-degree i and has dimension $\beta_{i-1}(\Delta)^{k-im}$ in degree (i,k). Hence,

$$\operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]),\lambda,t) = \sum_{i=0}^{s-1}\binom{s-1}{i}\lambda^i\left(\sum_{k=0}^{p-1} t^k\beta_{i-1}(\Delta)^{k-im}\right),$$

and so the $\lambda^i t^k$ coefficient of $\lambda t^m \operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]),\lambda,t)$ is $\binom{s-1}{i-1}\beta_{i-2}(\Delta)^{k-im}$. Then the $\lambda^i t^k$ coefficient of $(1-\lambda t^m)^{d-s}\lambda t^m \operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk_{s-1}[\Delta]),\lambda,t)$ is given by

$$\sum_{r=0}^{i}(-1)^r\binom{d-s}{r}\binom{s-1}{i-r-1}\beta_{i-r-2}(\Delta)^{k-im}.$$

Now we sum over all values of s, setting $f(s)=\binom{d-s}{r}\binom{s-1}{i-r-1}$:

$$\begin{aligned}\sum_{s=1}^{d}\sum_{r=0}^{i}(-1)^r f(s)\beta_{i-r-2}(\Delta)^{k-im} &= \sum_{r=0}^{i}\left[(-1)^r\beta_{i-r-2}(\Delta)^{k-im}\sum_{s=1}^{d} f(s)\right]\\ &= \binom{d}{i}\sum_{r=0}^{i}(-1)^r\beta_{i-r-2}(\Delta)^{k-im}\\ &= \binom{d}{i}\sum_{j=0}^{i-1}(-1)^{i-j-1}\beta_{j-1}(\Delta)^{k-im}.\end{aligned}$$

Here the second equality follows from the identity $\sum_{s=1}^{d}\binom{d-s}{r}\binom{s-1}{i-r-1}=\binom{d}{i}$ and the third follows from setting $j=i-r-1$. □

Of particular interest is the $G=\mathbb{Z}/2\mathbb{Z}$ case, in which many simplifications can be made to the expression in Theorem 3.4.2. This results in the following corollary.

Corollary 3.4.4. *Let Δ be a centrally-symmetric Buchsbaum complex with $\Theta = (\theta_1, \dots, \theta_d)$ an l.s.o.p. for Δ such that $\theta_i \in A_1^m$ for all i and some fixed m. Then*

$$\dim_{\Bbbk} \Bbbk(\Delta; \Theta)_i^k = \frac{1}{2}\left(h_i(\Delta) + (-1)^{i+k+mi}\binom{d}{i}\right) + \binom{d}{i}\sum_{j=0}^{i-1}(-1)^{i-j-1}\beta_{j-1}(\Delta)^{k-im}.$$

3.4.2 The sigma module

When only considering a $\mathbb{Z}$-grading on $\Bbbk[\Delta]$, it is possible to mod $\Bbbk(\Delta; \Theta)$ out by an additional submodule in order to get an even tighter bound on the h-vector of Δ. In particular, the sigma module can be used to great effect as follows ([49, Theorem 1.2]).

Theorem 3.4.5 (Murai–Novik–Yoshida)**.** *Let Δ be a Buchsbaum simplicial complex of dimension $d-1$ and let Θ be an l.s.o.p. for Δ. Then*

$$\dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_i = h_i(\Delta) + \binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta).$$

Of course, we would like for analogous statements to hold for the $(\mathbb{Z} \times G)$-graded Hilbert series of $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])$. In order to even consider this series, we must first verify that this module is in fact $(\mathbb{Z} \times G)$-graded by establishing that the sigma module is G-invariant. This is accomplished by the following lemma, whose proof is nearly immediate and has been omitted.

Lemma 3.4.6. *Let Δ be a Buchsbaum simplicial complex with a free action by G. If $\Theta = (\theta_1, \dots, \theta_d)$ is an l.s.o.p. for Δ and each θ_i is homogeneous with respect to the $(\mathbb{Z} \times G)$-grading of A, then the sigma module $\Sigma(\Theta; \Bbbk[\Delta])$ is G-invariant.*

Fortunately, the proof of our needed aspects of [49, Theorem 2.3] goes through essentially verbatim in the $(\mathbb{Z} \times G)$-graded case using Proposition 3.2.1. One aspect not covered directly by the proposition (though it is considered in the proof) is that images and kernels of maps of $(\mathbb{Z} \times \hat{G})$-graded A-modules are themselves $(\mathbb{Z} \times \hat{G})$-graded; this follows from [8, Proposition II.11.3.3]. Hence, we obtain the following proposition.

Proposition 3.4.7. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free action by G and let $\Theta = (\theta_1, \ldots, \theta_d)$ be an l.s.o.p. for Δ with $\theta_i \in A_1^m$ for all i. Then*

$$\Sigma(\Theta; \Bbbk[\Delta])/\Theta\Bbbk[\Delta] \cong \bigoplus_{i=0}^{d-1} \left(\bigoplus_{\binom{d}{i}} H_{\mathfrak{m}}^i(\Bbbk[\Delta])[-i, -im] \right)$$

as $(\mathbb{Z} \times G)$-graded A-modules. In particular,

$$\dim_{\Bbbk} \left(\Sigma(\Theta; \Bbbk[\Delta])/\Theta\Bbbk[\Delta]\right)_i^k = \binom{d}{i} \beta_{i-1}(\Delta)^{k-im}.$$

This immediately establishes the following extension of Theorem 3.4.2:

Theorem 3.4.8. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free action by G and let Θ be a G-homogeneous l.s.o.p. for $\Bbbk[\Delta]$ such that $\theta_i \in A_1^m$ for all i. Then*

$$\begin{aligned} &\mathrm{Hilb}(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta]), \lambda, t) = \\ &\quad \sum_{i=0}^{d} \left[(-1)^i \binom{d}{i} t^{mi} + \left(\frac{1}{p} \sum_{k=0}^{p-1} t^k \right) \left(h_i(\Delta) + (-1)^{i+1} \binom{d}{i} \right) \right] \lambda^i \\ &\quad + \sum_{i=0}^{d} \binom{d}{i} \lambda^i \sum_{j=0}^{i} (-1)^{i-j-1} \left(\sum_{k=0}^{p-1} t^k \beta_{j-1}(\Delta)^{k-im} \right). \end{aligned}$$

3.4.3 Inequalities

Theorem 3.4.5 implies the following inequality bounding the h-vector of any Buchsbaum simplicial complex Δ:

$$h_i(\Delta) \geq \binom{d}{i} \sum_{j=0}^{i} (-1)^{i-j} \beta_{j-1}(\Delta). \tag{3.7}$$

For Cohen-Macaulay complexes admitting a very free action by G, bounds on the h-vector may be obtained through Theorem 3.1.5. In this section we present some similar inequalities by exploiting the $(\mathbb{Z} \times G)$-graded Hilbert series of Buchsbaum complexes admitting free group actions by G. To begin, setting $m = 0$ and examining the $\lambda^i t^k$ coefficients in Theorem 3.4.8

immediately allows us to bound the h-vector of Δ by

$$h_i(\Delta) \geq (p-1)(-1)^{i+1}\binom{d}{i} + p\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta)^0 \tag{3.8}$$

and

$$h_i(\Delta) \geq (-1)^i\binom{d}{i} + p\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta)^k \tag{3.9}$$

for $k \not\equiv 0 \mod p$.

The inequalities above provide an immediate "permutable" version of (3.7) that is worth recognizing.

Corollary 3.4.9. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free action by the group $\mathbb{Z}/p\mathbb{Z}$. If $\mathcal{M}$ is a multiset of size $p-1$ on $\{1,\ldots,p-1\}$, then*

$$h_i(\Delta) \geq \binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta)^0 + \binom{d}{i}\sum_{m\in\mathcal{M}}\sum_{j=0}^{i}(-1)^{i-j}\beta_{j-1}(\Delta)^m.$$

Proof. For each $m \in \mathcal{M}$, consider the corresponding inequality (3.9). Add all of the inequalities together along with (3.8), then divide by p. □

Note that by taking $\mathcal{M} = [p-1]$, we obtain the original bound (3.7). On the other hand, by considering the degrees of the group representations produced by G acting on $H^i(\Delta)$, we can also obtain some extensions of Theorem 3.1.5.

Corollary 3.4.10. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex admitting a free action by G. Then:*

(a) *If $\beta_i(\Delta) = 0$ for $i \leq j$, then $h_i(\Delta) \geq \binom{d}{i}$ for $i \leq j+2$.*

(b) *If $\beta_i(\Delta) < (p-1)$ for $i \leq j$, then $h_i(\Delta) \geq (-1)^i\binom{d}{i}$ for $i \leq j+2$.*

Proof. The statement in (a) is an immediate consequence of the inequalities in (3.8) and (3.9). For (b), consider the reduced cohomology group $\tilde{H}^i(\Delta;\mathbb{Q})$ with coefficients in $\mathbb{Q}$, which satisfies

$$\beta_i(\Delta) = \dim_{\mathbb{Q}} \tilde{H}^i(\Delta;\mathbb{Q}).$$

Furthermore,

$$\tilde{H}^i(\Delta; \Bbbk) \cong \tilde{H}^i(\Delta; \mathbb{Q}) \otimes_{\mathbb{Q}} \Bbbk$$

as representations of G by taking $\Bbbk$ to have a trivial action.

There are only two irreducible representations of G over $\mathbb{Q}$: the 1-dimensional trivial representation and a $(p-1)$-dimensional non-trivial representation. Hence, if $\dim_{\mathbb{Q}} \tilde{H}^i(\Delta; \mathbb{Q}) < p-1$ then G acts trivially on $\tilde{H}^i(\Delta; \mathbb{Q})$. From the above isomorphism, G acts trivially on $\tilde{H}^i(\Delta; \Bbbk)$ as well. That is, $\beta_i(\Delta)^r = 0$ for $1 \leq r \leq p-1$ and $i \leq j$, and the result follows from inequality (3.9). □

It is also worth examining the properties of the quotient Δ/G. This quotient has elements corresponding to orbits of the faces of Δ under the action of G, and it can always be made into a poset with zero element $\{\emptyset\}$ (assuming $\Delta \neq \emptyset$) under the ordering defined by setting $x \preceq y$ if $\sigma \subseteq \tau$ for some $\sigma \in x$ and $\tau \in y$. This poset is ranked by setting the rank of an orbit to be the cardinality of any of its members.

In some cases, Δ/G is itself isomorphic to the face poset of a simplicial complex (for instance, when $\mathbb{Z}/2\mathbb{Z}$ acts by rotation on the boundary of a hexagon, the quotient is isomorphic to the face poset of the boundary of a triangle). When this happens, we naturally view Δ/G as the corresponding simplicial complex. Less optimistically, Δ/G may have the property that every interval $[\{\emptyset\}, x]$ is isomorphic to a boolean lattice. In this case, we say that Δ/G is a simplicial poset. Many invariants of simplicial complexes have natural extensions to simplicial posets (see [66] for a nice overview). In particular, the h-vector of a simplicial poset is a well-studied object that our next proposition examines.

Proposition 3.4.11. *Let Δ be a Buchsbaum simplicial complex admitting a free group action by G. If Δ/G is a simplicial poset, then*

$$h_i(\Delta/G) \geq (-1)^i \binom{d}{i} + \binom{d}{i} \sum_{j=0}^{i} (-1)^{i-j} \beta_{j-1}(\Delta)^k$$

for $k \not\equiv 0 \mod p$.

Proof. A straightforward calculation shows that

$$h_i(\Delta) = (p-1)(-1)^{i+1}\binom{d}{i} + ph_i(\Delta/G). \tag{3.10}$$

Now combine (3.9) with (3.10) and divide by p. □

As examples, Δ/G is always a simplicial poset when Δ is the order complex of a poset ([17, Section 6]), or, more generally, when Δ is balanced and the action of G is color-preserving ([56]). Note that this proposition provides a "Buchsbaum-like" bound on the h-vector of Δ/G in the style of equation (3.9) without any direct consideration of whether or not Δ/G is itself Buchsbaum.

3.4.4 Dehn-Sommerville relations

Here we present a new results akin to Theorem 1.4, relating the symmetric values of the dimensions $\dim_{\Bbbk}(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta]))_i^j$. First, let $\tilde{\chi}(\Delta)$ denote the reduced Euler characteristic of a simplicial complex Δ, defined by

$$\tilde{\chi}(\Delta) = \sum_{j=-1}^{d-1}(-1)^j\beta_j(\Delta),$$

where $d-1$ is the dimension of Δ. We will also denote by $\chi(\Delta) := \tilde{\chi}(\Delta)+1$ the (unreduced) Euler characteristic of Δ. If Δ is an orientable homology manifold without boundary, then Klee's formula

$$h_{d-i}(\Delta) - h_i(\Delta) = (-1)^i\binom{d}{i}((-1)^{d-1}\tilde{\chi}(\Delta) - 1) \tag{3.11}$$

(found in [34]) provides a relation between the symmetric entries in the h-vector of Δ in terms of $\tilde{\chi}(\Delta)$, abstracting the Dehn-Sommerville relations for simplicial polytopes. We will first examine the symmetric values of the dimensions of $\dim_{\Bbbk}(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta]))_i^0$, in the case that $\Theta \subset A_1^0$.

Theorem 3.4.12. *Let Δ be a $(d-1)$-dimensional triangulation of a connected orientable manifold without boundary admitting a free group action by G. If $\Theta \subset A_1^0$ and $\widetilde{H}^{d-1}(\Delta) =$*

$\widetilde{H}^{d-1}(\Delta)^0$, *then*

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i^0 = \dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_{d-i}^0.$$

Proof. If $\Delta = \{\emptyset\}$, then the result is immediate. Assume that $\Delta \neq \{\emptyset\}$. Applying Theorem 3.4.8, we can write

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i^0 = \frac{h_i(\Delta)}{p} + \frac{(-1)^i(p-1)}{p}\binom{d}{i} + \binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta)^0.$$

Using (3.11),

$$\begin{aligned}\frac{h_i(\Delta)}{p} - \frac{h_{d-i}(\Delta)}{p} &= \frac{(-1)^{i+1}}{p}\binom{d}{i}((-1)^{d-1}\tilde{\chi}(\Delta)-1)\\ &= \frac{(-1)^{d-i}}{p}\binom{d}{i}\tilde{\chi}(\Delta) + \frac{(-1)^i}{p}\binom{d}{i}\\ &= \frac{(-1)^{d-i}}{p}\binom{d}{i}\chi(\Delta) + \frac{(-1)^{d-i-1}}{p}\binom{d}{i} + \frac{(-1)^i}{p}\binom{d}{i}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left[\frac{h_i(\Delta)}{p} + \frac{(-1)^i(p-1)}{p}\binom{d}{i}\right] - \left[\frac{h_{d-i}(\Delta)}{p} + \frac{(-1)^{d-i}(p-1)}{p}\binom{d}{i}\right]\\ &\quad= \frac{(-1)^{d-i}}{p}\binom{d}{i}\chi(\Delta) + (-1)^i\binom{d}{i} + (-1)^{d-i-1}\binom{d}{i}. \qquad (3.12)\end{aligned}$$

Identify Δ with its geometric realization $|\Delta|$, and set $\Gamma = |\Delta|/G$. Since G acts freely on Δ, the projection $\pi : |\Delta| \to \Gamma$ is a covering space map. By [26, Proposition 3G.1],

$$H^i(|\Delta|)^0 = H^i(|\Delta|)^G \cong H^i(\Gamma).$$

Since $\widetilde{H}^{d-1}(\Delta) = \widetilde{H}^{d-1}(\Delta)^0$ by assumption, Γ is itself an orientable manifold. Hence, Poincaré duality applies and $\beta_{j-1}(\Delta)^0 = \beta_{d-j}(\Delta)^0$ for $j > 1$ while $\beta_0(\Delta)^0 = \beta_{d-1}(\Delta)^0 - 1$. Furthermore, $\beta_{-1}(\Delta)^0 = 0$ since $\Delta \neq \{\emptyset\}$. Then

$$\begin{aligned}\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta)^0 &= (-1)^{i+1}\binom{d}{i} + \binom{d}{i}\sum_{j=1}^{i}(-1)^{i-j-1}\beta_{d-j}(\Delta)^0\\ &= (-1)^{i+1}\binom{d}{i} + \binom{d}{i}\sum_{\ell=i+1}^{d}(-1)^{d-i-\ell}\beta_{\ell-1}(\Delta)^0.\end{aligned}$$

Thus,

$$\begin{aligned}\binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta)^0 - \binom{d}{i}\sum_{j=0}^{d-i}(-1)^{d-i-j-1}\beta_{j-1}(\Delta)^0& \\ = (-1)^{i+1}\binom{d}{i} + \binom{d}{i}\sum_{j=0}^{d}(-1)^{d-i-j}\beta_{j-1}(\Delta)^0& \\ = (-1)^{i+1}\binom{d}{i} + (-1)^{d-i-1}\binom{d}{i}\tilde{\chi}(\Gamma).&\end{aligned}$$

Now note that Δ is a p-sheeted covering space for Γ. This implies that $\chi(\Delta) = p\chi(\Gamma)$ (see [26, Section 2.2]). Hence, we can re-write the difference as

$$\begin{aligned}(-1)^{i+1}\binom{d}{i}+&(-1)^{d-i-1}\binom{d}{i}\tilde{\chi}(\Gamma) \\ &= (-1)^{i+1}\binom{d}{i} + (-1)^{d-i-1}\binom{d}{i}\chi(\Gamma) + (-1)^{d-i}\binom{d}{i} \\ &= (-1)^{i+1}\binom{d}{i} + \frac{(-1)^{d-i-1}}{p}\binom{d}{i}\chi(\Delta) + (-1)^{d-i}\binom{d}{i}.\end{aligned}$$

Adding this difference to (3.12) completes the proof. □

Remark 3.4.13. In the case that $G = \mathbb{Z}/p\mathbb{Z}$ with p odd, the hypothesis $\widetilde{H}^{d-1}(\Delta) = \widetilde{H}^{d-1}(\Delta)^0$ of Theorem 3.4.12 is always true. Indeed, the fundamental class of Δ generating $\widetilde{H}^{d-1}(\Delta)$ is of the form

$$z = \sum_{\widehat{\sigma}\in C^{d-1}(\Delta)} a_\sigma \widehat{\sigma}$$

with $a_\sigma = \pm 1$ for all σ. If $h \in G$ acts on this class, then it can only flip the sign of some coefficients; it cannot introduce a factor of ζ^i across all coefficients for some non-trivial i.

In fact, the theorem above can be greatly strengthened by appealing to algebraic properties of $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])$. First, we will need the following theorem (see [53, Theorem 1.4] or [49, Remark 3.8]).

Theorem 3.4.14. *Let Δ be a triangulation of a $(d-1)$-dimensional connected manifold without boundary that is orientable over $\Bbbk$, and let Θ be a linear system of parameters for $\Bbbk[\Delta]$. Then $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])$ is an Artinian Gorenstein $\Bbbk$-algebra.*

With this fact, our next theorem quickly follows.

Theorem 3.4.15. *Let Δ be a triangulation of a $(d-1)$-dimensional connected manifold without boundary that is orientable over $\Bbbk$ and admits a free group action by G, and let Θ be a $(\mathbb{Z}\times G)$-homogeneous linear system of parameters for $\Bbbk[\Delta]$. If s is such that $\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_d$ in concentrated in $(\mathbb{Z}\times G)$-degree (d,s), then*

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i^j = \dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_{d-i}^{s-j}$$

for $i = 0,\dots,d$ and any j, where we interpret $s-j$ modulo p.

Before starting the proof of this theorem, note that $\dim_{\Bbbk}\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_d = 1$ because $\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])$ is Gorenstein and hence such an s always exists.

Proof. Since $\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])$ is Gorenstein, the product map

$$\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_i \times \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_{d-i} \to \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_d$$

is a perfect pairing (see [25, Theorem 2.79]). Furthermore, the finely-graded product maps

$$\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_i^j \times \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_{d-i}^k \to \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_d^{j+k}$$

only land in a non-zero component of $\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_d$ when $j+k\equiv s \pmod p$. That is,

$$\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_i^j \times \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_{d-i}^{s-j} \to \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_d^s$$

is a perfect pairing for each i,j and hence

$$\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_i^j \cong \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_{d-i}^{s-j}$$

as vector spaces over $\Bbbk$. □

By comparing 3.4.12 with 3.4.15, we obtain the following corollary.

Corollary 3.4.16. *Let Δ be a $(d-1)$-dimensional triangulation of a connected orientable manifold without boundary admitting a free group action by G. If $\Theta\subset A_1^0$ and $\widetilde{H}^{d-1}(\Delta) = \widetilde{H}^{d-1}(\Delta)^0$, then $\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_d$ in concentrated in $(\mathbb{Z}\times G)$-degree $(d,0)$ and*

$$\dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i^j = \dim_{\Bbbk}\left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_{d-i}^{p-j}$$

for $i = 0,\dots,d$ and $j = 0,\dots,p-1$.

3.5 Comments

It was shown by Duval in [15] that a version of of Theorem 3.1.1 also holds when considering the face ring of a simplicial poset. Our main result then begs the following question.

Question 3.5.1. *Does Theorem 3.3.1 extend to simplicial posets admitting free group actions?*

An answer to this question will likely require a careful examination of a $(\mathbb{Z} \times G)$-grading imposed on the face ring of a simplicial poset, an object that at times is considerably more complicated than the Stanley–Reisner ring.

Stanley was able to examine some implications of equality being attained in his inequalities of Theorem 3.1.5 for the $G = \mathbb{Z}/2\mathbb{Z}$ case (see [67, Proposition III.8.2]). In light of these results, our next question naturally arises.

Question 3.5.2. *If equality is attained in either inequality* (3.8) *or* (3.9)*, what can be said about the other entries in the h-vector of Δ?*

Lastly, this chapter would feel incomplete without some mention of the g-conjecture. To this end, let Δ be a $(d-1)$-dimensional orientable $\Bbbk$-homology manifold and denote

$$h''_i(\Delta) = \dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])\right)_i = h_i(\Delta) + \binom{d}{i}\sum_{j=0}^{i}(-1)^{i-j-1}\beta_{j-1}(\Delta).$$

Kalai's manifold g-conjecture asserts in part that $h''_0(\Delta) \leq h''_1(\Delta) \leq \cdots \leq h''_{\lfloor d/2 \rfloor}(\Delta)$ (in the case that Δ is a sphere, this is the same as the corresponding part of the usual g-conjecture). There is much evidence in favor of this statement, and it has been shown to be true in some special cases (see [47, Theorem 5.3], [48, Theorem 1.5], and [53, Theorem 1.6]). The inequality is sometimes demonstrated by exhibiting a Lefschetz element $\omega \in A_1$ such that the multiplication map

$$\cdot\omega : \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_{i-1} \to \Bbbk[\Delta]/\Sigma(\Theta;\Bbbk[\Delta])_i$$

is an injection for $1 \leq i \leq \lfloor d/2 \rfloor$. As has been our theme, we would like to show that similar inequalities hold under the finer grading. Indeed, if such an ω can in fact be chosen to be an element of A_1^m for some m, then we immediately have the inequalities

$$\dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_{i-1}^{j} \leq \dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_{i}^{j+m}$$

for any choice of j. Although the Lefschetz elements that have been found thus far cannot be specified to reside in some fixed homogeneous $(\mathbb{Z} \times G)$-degree of A_1, the strength of the finer grading on $\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])$ prompts the following conjecture.

Conjecture 3.5.3. *Let Δ be an orientable $\Bbbk$-homology manifold admitting a free group action by $\mathbb{Z}/p\mathbb{Z}$. Then there exists m such that*

$$\dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_{i-1}^{j} \leq \dim_{\Bbbk} \left(\Bbbk[\Delta]/\Sigma(\Theta; \Bbbk[\Delta])\right)_{i}^{j+m}$$

for $1 \leq i \leq \lfloor d/2 \rfloor$ and $0 \leq j \leq p-1$.

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Chapter 4

SIMPLICIAL POSETS

4.1 Introduction

The Stanley–Reisner ring of a simplicial complex has been one of the most useful tools in the study of combinatorial properties of simplicial polytopes, spheres, and manifolds for decades. From the upper bound theorem to the g-theorem, this algebraic construction has played a central role in many of the main results in this area of research. Its structure is well-understood, and the translation of geometric properties of a simplicial complex into algebraic statements about the associated ring is often clear and elegant. For an excellent overview of the historical advances allowed by these rings, the reader is referred to [67, Sections II and III].

When shifting from simplicial complexes to simplicial posets, the situation changes rather dramatically. Stanley defined the face ring of a simplicial poset as a generalization of the Stanley–Reisner ring ([66]); in the case that the poset in question is the face lattice of a simplicial complex, the two rings are isomorphic. Unfortunately, in this broader scope the structure of the face ring becomes much less tangible. Though many classification theorems for simplicial complexes have analogs in the world of simplicial posets (see [66, Theorem 3.10] for an analog of the g-theorem, or [10], [36], and [46] for further abstractions), many algebraic statements about Stanley–Reisner rings do not have generalizations to face rings of simplicial posets.

The purpose of this chapter is to provide such generalizations. In particular, we wish to extend the homological properties of Stanley–Reisner rings established by Gräbe, Hochster, Miyazaki, Reisner, and Schenzel to the world of simplicial posets and their face rings, in the combinatorial spirit of the work begun by Duval in [15]. It is worth noting that these face rings are also squarefree modules; these objects have been studied from a various viewpoints by Yanagawa (see, e.g., [77] and [76]). Furthermore, similar topics have been investigated from a sheaf-theoretic viewpoint by Brun and Römer in [13] and by Brun, Bruns, and Römer in [12].

Before stating our main results, we will fix some notation. Let P be a simplicial poset with vertex set $V = \{x_1, \ldots, x_n\}$ and let $\operatorname{supp}(z) = \{i : x_i \preceq z\}$ denote the support of an element $z \in P$. Define $A = \Bbbk[x_1, \ldots, x_n]$, let $\mathfrak{m}$ be the irrelevant ideal $(x_1, \ldots, x_n)$, and let $\mathfrak{m}_\ell$ be the ideal $(x_1^\ell, \ldots, x_n^\ell)$. Lastly, let A_P be the face ring of P; we consider A_P as a $\mathbb{Z}^n$-graded A-module (we defer all definitions until the next section). Our first new result is an extension of Miyazaki's calculation of the graded pieces of the Ext-modules of a Stanley–Reisner ring ([42, Theorem 1]):

Theorem 4.3.1. *Let P be a simplicial poset with vertex set V, and let $\alpha \in \mathbb{Z}^n$. Set $B = \{i : -\ell < \alpha_i < 0\}$, $C = \{i : \alpha_i = -\ell\}$, and $D = \{i : \alpha_i > 0\}$. If $-\ell \leq \alpha_i$ for all i, then*

$$\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha \cong \bigoplus_{\operatorname{supp}(z)=B\cup D} \widetilde{H}^{i-|B|-|C|-1}\left([\hat{0}, z_D] \times \operatorname{lk}_P(z)_{V\smallsetminus C}\right)$$

and $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha = 0$ otherwise. In particular, if $D \neq \emptyset$ then $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha = 0$.

By considering $H^i_\mathfrak{m}(A_P)$ as the direct limit of the $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)$ modules, we will reprove the result of Duval ([15, Theorem 5.9]) that calculates the dimensions of the graded pieces of $H^i_\mathfrak{m}(A_P)$. By tracing the A-module structure of $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)$ through this direct limit, we will further be able to establish an extension of Gräbe's result detailing the corresponding structure of $H^i_\mathfrak{m}(A_P)$ ([23, Theorem 2]). More precisely, our second main contribution is:

Theorem 4.4.4. *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$. Then*

$$H^i_\mathfrak{m}(A_P)_\alpha \cong \bigoplus_{\operatorname{supp}(w)=\{j:\alpha_j\neq 0\}} H^{i-1}(P, \operatorname{cost}_P(w))$$

if $\alpha \in \mathbb{Z}^n_{\leq 0}$ and $H^i_\mathfrak{m}(A_P)_\alpha = 0$ otherwise. Under these isomorphisms, the A-module structure of $H^i_\mathfrak{m}(A_P)$ is given as follows. Let $\gamma = \alpha + \deg(x_j)$. If $\alpha_j < -1$, then $\cdot x_j : H^i_\mathfrak{m}(A_P)_\alpha \rightarrow$

$H^i_{\mathfrak{m}}(A_P)_\gamma$ *corresponds to the direct sum of identity maps*

$$\bigoplus_{\operatorname{supp}(w)=\{j:\alpha_j\neq 0\}} H^{i-1}(P, \operatorname{cost}_P(w)) \to \bigoplus_{\operatorname{supp}(w)=\{j:\alpha_j\neq 0\}} H^{i-1}(P, \operatorname{cost}_P(w)).$$

If $\alpha_j = -1$*, then* $\cdot x_j$ *corresponds to the direct sum of maps*

$$\bigoplus_{\operatorname{supp}(w)=\{j:\alpha_j\neq 0\}} H^{i-1}(P, \operatorname{cost}_P(w)) \to \bigoplus_{\operatorname{supp}(z)=\{j:\gamma_j\neq 0\}} H^{i-1}(P, \operatorname{cost}_P(z))$$

induced by the inclusions of pairs $(P, \operatorname{cost}_P(w \smallsetminus \{x_j\})) \to (P, \operatorname{cost}_P(w))$*. If* $\alpha_j \geq 0$*, then* $\cdot x_j$ *is the zero map.*

The structure of this chapter is as follows. In Section 4.2 we will review notation and some foundational results, then examine the structure of A_P in preparation for later computations. In Section 4.3 and Section 4.4 we will prove Theorems 4.3.1 and 4.4.4, respectively. In Section 4.5 we will apply these results in a discussion of Cohen-Macaulay and Buchsbaum properties. Finally, we will close with some additional comments in Section 4.6.

4.2 Preliminaries

4.2.1 Combinatorics and topology

A **simplicial poset** is a partially ordered finite set $(P, \preceq)$ satisfying the following two properties:

i There exists an element $\hat{0} \in P$ satisfying $\hat{0} \preceq y$ for all $y \in P$.

ii For every element $y \in P$, the interval $[\hat{0}, y]$ is a Boolean lattice.

The **rank** of an element y of P is the maximal length among chains from $\hat{0}$ to y in P and is denoted $\operatorname{rk}_P(y)$. When the poset P is understood, we sometimes abbreviate this to $\operatorname{rk}(y)$. The **atoms** of P are the elements of rank 1.

If R is a subset of P such that $x \in R$ whenever $y \in R$ and $x \preceq y$ in P, then we call R an **order ideal** of P and view it as a poset under the partial order inherited from P. If y is an element of a poset P, then we define the **link** of y in P by

$$\mathrm{lk}_P(y) := \{w \in P : y \preceq w\}.$$

Note that y plays the role of $\hat{0}$ in $\mathrm{lk}_P(y)$, and, more generally, if $w \in \mathrm{lk}_P(y)$ then $\mathrm{rk}_{\mathrm{lk}_P(y)}(w) = \mathrm{rk}_P(w) - \mathrm{rk}_P(y)$. Similarly, we define the **contrastar** of y in P by

$$\mathrm{cost}_P(y) := \{w \in P : y \not\preceq w\}.$$

If the atoms of P are labeled $x_1, \ldots, x_n$, then we define the **support** of an element $y \in P$ to be the set

$$\mathrm{supp}(y) := \{i : x_i \preceq y\}.$$

Given $B \subseteq \mathrm{supp}(y)$, we denote by y_B the unique face of P satisfying $y_B \preceq y$ and $\mathrm{supp}(y_B) = B$ (the existence and uniqueness of y_B follows from the fact that $[\hat{0}, y]$ is a Boolean lattice). Similarly, if $W \subseteq \{1, \ldots, n\}$, then define

$$P_W := \{y \in P : \mathrm{supp}(y) \subseteq W\}.$$

If w and y are two elements in P, then we define

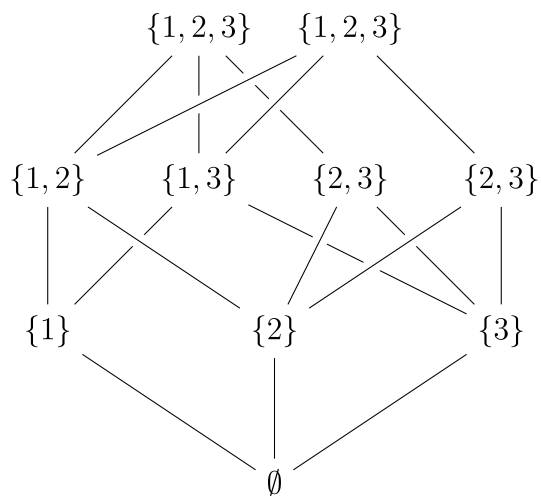
$$\mathcal{M}(w, y) := \{z \in P : z \text{ is a minimal upper bound for } w \text{ and } y\}.$$

Note that if P is a simplicial poset and $\mathcal{M}(w, y) \neq \emptyset$, then for any $z \in \mathcal{M}(w, y)$ the interval $[\hat{0}, z]$ is a boolean lattice containing both w and y. Hence, w and y have a unique greatest lower bound; we call this the **meet** of w and y and denote it by $w \wedge y$.

The properties defining simplicial posets are preserved under some important operations and constructions, as outlined in the following proposition ([7, Proposition 2.6]). Recall that if P and Q are posets, then a partial order may be placed on the product $P \times Q$ by defining $(p_1, q_1) \preceq (p_2, q_2)$ if $p_1 \preceq p_2$ in P and $q_1 \preceq q_2$ in Q.

Proposition 4.2.1. *Let P and Q be simplicial posets, and let $y \in P$. Then $P \times Q$ and $\mathrm{lk}_P(y)$ are both simplicial posets. If R is an order ideal of P, then R is a simplicial poset as well; in particular, $\mathrm{cost}_P(y)$ is a simplicial poset.*

There is a wealth of geometric meaning behind simplicial posets. In particular, every simplicial poset may be realized as the face poset of a regular CW complex ([7, Section 3]). Such a complex is much like a simplicial complex in that the cells are simplices. However, these more general objects may differ from simplicial complexes in that the intersection of two faces needs only to be a (possibly empty) subcomplex of each of the two faces, instead of a single simplex. For example, the following figure depicts the face poset of a cone formed by two triangles with two pairs of edges identified and vertices labeled 1, 2, and 3, with the apex of the cone being the vertex 1. The elements of the poset have been labeled according to their support.



From this geometric viewpoint, we refer to the atoms of P as **vertices** and to all elements of P as **faces**. The **dimension** of a face $y \in P$ is $\dim(y) := \mathrm{rk}(y) - 1$, and the dimension of P is $\dim(P) := \max\{\dim(y) : y \in P\}$. If all maximal faces of P have the same dimension, then we say that P is **pure**. We denote the regular CW complex associated to the poset P by $|P|$, and we call it the **geometric realization** of P.

In addition to the geometric realization of P, there is another associated geometric object known as the **order complex**. First, let $\overline{P}$ denote the poset $P \setminus \{\hat{0}\}$. The order complex of $\overline{P}$ is the simplicial complex $\Delta(\overline{P})$ whose elements are the chains of $\overline{P}$. In fact, $\Delta(\overline{P})$ can be realized as the face poset of the barycentric subdivision of $|P|$ ([35, Theorem 2.1.7]). Furthermore, $\Delta(\overline{P \times Q})$ isomorphic as a simplicial complex to the join $\Delta(\overline{P}) * \Delta(\overline{Q})$.

Let P be a simplicial poset with vertices $\{x_1, \ldots, x_n\}$, and let $\Bbbk$ be a field. Denote by $\widetilde{C}^i(P)$ the vector space over $\Bbbk$ whose basis elements correspond to elements of rank $i+1$ in P. Next let $\mathcal{O}$ be a partial ordering of the indices $\{1, \ldots, n\}$ of the vertices of P such that the restriction of $\mathcal{O}$ to the support of any one face is a total order; we call such an ordering an **orientation**. Define a coboundary operator $\delta : \widetilde{C}^i(P) \to \widetilde{C}^{i+1}(P)$ by

$$\delta(y) = \sum_{i=1}^{n} (-1)^{u_i} \sum_{z \in \mathcal{M}(x_i, y)} z$$

where $u_i = |\{j \in \operatorname{supp}(y) : i <_{\mathcal{O}} j\}|$. We define the **reduced cohomology groups** $\widetilde{H}^i(P)$ of P with coefficients in $\Bbbk$ to be the homology of the chain complex $\widetilde{C}^\bullet(P)$ with respect to the coboundary δ. If Q is a non-empty order ideal in P, then we define the **relative cohomology groups** $H^i(P,Q)$ of the pair (P,Q) with coefficients in $\Bbbk$ to be the homology of the chain complex $C^\bullet(P,Q) := \widetilde{C}^\bullet(P)/\widetilde{C}^\bullet(Q)$ with respect to the coboundary induced by δ. This definition of cohomology is identical to the cellular cohomology of the CW complex $|P|$ (see, e.g., [26, Sections 2.2 and 3.1]). From this fact (along with its relative analog), the equivalence of cellular, singular, and simplicial (co)homology imply the following theorem.

Theorem 4.2.2. *Let P be a simplicial poset, and let Q be a non-empty order ideal of P. If $\widetilde{H}^i(|P|)$ (resp. $H^i(|P|,|Q|)$) denotes the singular cohomology of $|P|$ (resp. $(|P|,|Q|)$) and $\widetilde{H}^i(\Delta(\overline{P}))$ (resp. $H^i(\Delta(\overline{P}), \Delta(\overline{Q}))$)denotes the simplicial cohomology of $\Delta(\overline{P})$ (resp. $(\Delta(\overline{P}), \Delta(\overline{Q}))$), then*

$$\widetilde{H}^i(P) \cong \widetilde{H}^i(|P|) \cong \widetilde{H}^i(\Delta(\overline{P}))$$

and

$$H^i(P,Q) \cong H^i(|P|,|Q|) \cong H^i(\Delta(\overline{P}), \Delta(\overline{Q})).$$

4.2.2 Algebra

Let P be a simplicial poset with atoms (or vertices) $x_1, \ldots, x_n$. Fix $\Bbbk$ to be a field and define A to be the polynomial ring over $\Bbbk$ whose variables are the atoms of P:

$$A := \Bbbk[x_1, \ldots, x_n].$$

Next, define $\Bbbk[P]$ to be the polynomial ring whose variables are the elements of P:

$$\Bbbk[P] := \Bbbk[y : y \in P].$$

Let I_P be the ideal of $\Bbbk[P]$ defined as follows. For each pair of elements w and y in P, if $\mathcal{M}(w, y) = \emptyset$ then add wy as a generator of I_P. If $\mathcal{M}(w, y) \neq \emptyset$, then add

$$wy - (w \wedge y) \sum_{z \in \mathcal{M}(w,y)} z$$

as a generator of I_P (as noted above, if $\mathcal{M}(w, y) \neq \emptyset$ then $w \wedge y$ always exists). Finally, add $\hat{0} - 1$ to I_P. Define the **face ring** A_P to be the quotient $\Bbbk[P]/I_P$.

Given some elements $y_1, \ldots, y_k$ in P and positive integers $p_1, \ldots, p_k$, let m be the monomial $\prod_{i=1}^{k} y_i^{p_i}$ in A_P. We define the **support** of m to be the set

$$\operatorname{supp}(m) := \{i : x_i \preceq y_j \text{ for some } j\}.$$

We say that the monomial m is **standard** if $y_1 \succ y_2 \succ \cdots \succ y_k$. In this case, we call y_1 the **leading variable** of m. Of vital importance for our arguments will be the following fact ([66, Lemma 3.4]).

Lemma 4.2.3. *If P is a simplicial poset, then A_P is an algebra with straightening law (ASL) on P. In particular, the standard monomials form a basis for A_P as a vector space over $\Bbbk$.*

Note that A is naturally $\mathbb{Z}^n$-graded by setting $\deg(x_i) = \mathbf{e}_i$, the i-th standard basis vector of $\mathbb{Z}^n$. We may extend this grading to $\Bbbk[P]$ by defining the degree of a variable y to be

$$\deg(y) := \sum_{i \in \operatorname{supp}(y)} \mathbf{e}_i.$$

Since I_P is homogeneous with respect to this grading, the quotient A_P is $\mathbb{Z}^n$-graded as well. We will usually view A_P as a $\mathbb{Z}^n$-graded A-module.

Given some degree $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{Z}^n$, we will also speak of its **support**, defined by

$$\operatorname{supp}(\alpha) := \{i : \alpha_i \neq 0\}.$$

At times it will be necessary to focus only on the "negative" part of α. This negative part is the degree $\alpha^- \in \mathbb{Z}^n$ defined by

$$\alpha_i^- := \begin{cases} \alpha_i & \alpha_i < 0 \\ 0 & \alpha_i \geq 0. \end{cases}$$

4.2.3 The structure of A_P

Although the face ring of a simplicial poset has many subtleties and does not generally behave as well as the Stanley–Reisner ring of a simplicial complex, it does still have some nice properties. We will now highlight two of these properties with the following lemmas that will be critical to later computations.

Lemma 4.2.4. *Let P be a simplicial poset, let $w, y \in P$, and let q be a positive integer. Then in A_P,*

$$(wy)^q = (w \wedge y)^q \sum_{z \in \mathcal{M}(w,y)} z^q.$$

Proof. Set $\mathcal{M}(w, y) = \{z_1, \ldots, z_k\}$. If $q = 1$ or $k \leq 1$, then the statement is immediate. If $k > 1$ and $q > 1$, then write

$$\begin{aligned} (wy)^q &= (w \wedge y)^q \left(\sum_{i=1}^{k} z_i \right)^q \\ &= (w \wedge y)^q \sum_{i=1}^{k} z_i^q + (w \wedge y)^q \sum_{\ell} f_\ell \end{aligned}$$

for some monomials f_ℓ, where each f_ℓ contains at least one product $z_i z_j$ with $z_i \neq z_j$. Suppose that z_i and z_j have some upper bound b in P. Then $[\hat{0}, b]$ would be a Boolean lattice containing w and y, in which w and y have at least two distinct minimal upper bounds. This is a contradiction, so $z_i z_j = 0$ in A_P and each $f_\ell = 0$ as well. □

Our next lemma shows that the product of a standard monomial and a variable corresponding to a single vertex in A_P may be expressed as a sum of "nice" standard monomials.

Lemma 4.2.5. *If $m = \prod_{i=1}^{k} y_i^{p_i}$ is a standard monomial in A_p with $y_1 \succ y_2 \succ \cdots \succ y_k$ and ℓ is a positive integer, then $(x_j)^\ell m$ can be written in A_p as*

$$(x_j)^\ell m = \sum_{z \in \mathcal{M}(x_j, y_1)} m_z$$

where m_z is a standard monomial with leading variable z.

Proof. Write $q_r = \sum_{i=1}^{r} p_i$ for $0 \leq r \leq k$. We have three cases.

Case 1: $j \notin \operatorname{supp}(y_1)$ and $\ell \geq q_k$. We can re-write $x_j^\ell m$ as

$$x_j^\ell m = \left(\prod_{i=1}^{k} (x_j y_i)^{p_i} \right) x_j^{\ell - q_k}. \tag{4.1}$$

By the definition of A_P,

$$x_j y_1 = (x_j \wedge y_1) \sum_{z \in \mathcal{M}(x_j, y_1)} z = \sum_{z \in \mathcal{M}(x_j, y_1)} z,$$

since $j \notin \operatorname{supp}(y_1)$. If $\mathcal{M}(x_j, y_1) = \emptyset$, then the entire expression (4.1) is zero in A_P. If $\mathcal{M}(x_j, y_1) \neq \emptyset$, then for each $z \in \mathcal{M}(x_j, y_1)$ let z_i be the unique element of $[\hat{0}, z]$ with vertices consisting of those in y_i along with x_j (note $z = z_1$). By Lemma 4.2.4, we have

$$x_j^\ell m = \sum_{z \in \mathcal{M}(x_j, y_1)} \left(x_j^{\ell - q_k} \prod_{i=1}^{k} z_i^{p_i} \right) + \sum_{z \in \mathcal{M}(x_j, y_1)} \left(z^{p_1} x_j^{\ell - q_k} \prod_{i=2}^{k} \left(\sum_{\substack{w_i \in \mathcal{M}(x_j, y_i) \\ w_i \neq z_i}} w_i^{p_i} \right) \right).$$

However, all terms in the second summand on the right are zero in A_P. Indeed, if z and some $w_i \neq z_i$ were to have some upper bound z', then both z_i and w_i would be a minimal upper bound for x_j and y_i in the boolean lattice $[\hat{0}, z']$, a contradiction.

Writing m_z for the standard monomial $z^{p_1} \left(\prod_{i=2}^{k} z_i^{p_i} \right) x_j^{\ell - q_k}$, we see that

$$(x_j)^\ell m = \sum_{z \in \mathcal{M}(x_j, y_1)} m_z$$

as desired.

Case 2: $j \notin \operatorname{supp}(y_1)$ and $\ell < q_k$. Let r be such that $q_r \leq \ell$ while $q_{r+1} > \ell$. Now re-write $x_j^\ell m$ as

$$x_j^\ell m = \left(\prod_{i=1}^{r} (x_j y_i)^{p_i} \right) (x_j y_{r+1})^{\ell - q_r} y_{r+1}^{p_{r+1} - (l - q_r)} \left(\prod_{i=r+2}^{k} y_i^{p_i} \right).$$

After replacing the products x_jy_i using the definition of A_P and using the notation of Case 1, our "cross terms" cancel again and this simplifies to

$$x_j^\ell m = \sum_{z\in\mathcal{M}(x_j,y)} \left(\left(\prod_{i=1}^{r} z_i^{p_i} \right) z_{r+1}^{\ell-q_r} y_{r+1}^{p_{r+1}-(\ell-q_r)} \left(\prod_{i=r+2}^{k} y_i^{p_i} \right) \right)$$

Now if

$$m_z = z^{p_1} \left(\prod_{i=2}^{r} z_i^{p_i} \right) z_{r+1}^{\ell-q_r} y_{r+1}^{p_{r+1}-(\ell-q_r)} \left(\prod_{i=r+2}^{k} y_i^{p_i} \right)$$

then again we have

$$(x_j)^\ell m = \sum_{z\in\mathcal{M}(x_j,y_1)} m_z.$$

Case 3: $j \in \operatorname{supp}(y_1)$. If $j \in \operatorname{supp}(y_i)$ for all i, then $m(x_j)^\ell$ is already a standard monomial and we are done. So, suppose that $j \in \operatorname{supp}(y_i)$ for $1 \leq i \leq r$ while $j \notin \operatorname{supp}(y_i)$ for $i > r$. Then by Cases 1 and 2,

$$(x_j^\ell) \prod_{i=r+1}^{k} y_i^{p_i} = \sum_{z\in\mathcal{M}(x_j,y_{r+1})} m_z$$

for some standard monomials m_z with leading coefficient z. Note that there is one unique $z \in \mathcal{M}(x_j, y_{r+1})$ such that $z \prec y_r$. Now write

$$(x_j^\ell)m = \left(\prod_{i=1}^{r} y_i^{p_i} \right) m_z + \left(\prod_{i=1}^{r} y_i^{p_i} \right) \left(\sum_{\substack{z'\in\mathcal{M}(x_j,y_{r+1}) \\ z'\neq z}} m_{z'} \right).$$

The first term is a standard monomial with leading coefficient y_1, the unique element of $\mathcal{M}(x_j, y)$, while all other terms are zero as in Case 1. □

4.3 Ext *modules*

4.3.1 Koszul complexes and an important basis

All of our results will crucially depend upon calculating the A-modules $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)$, where $\mathfrak{m}_\ell$ is the ideal $(x_1^\ell, \ldots, x_n^\ell)$. Let $K_\bullet^\ell$ denote the Koszul complex of A with respect to the sequence $(x_1^\ell, \ldots, x_n^\ell)$ (for the construction of $K_\bullet^\ell$ and some of its associated properties, the reader is referred to [27, Section A.3]). We will view each K_t^ℓ as the direct sum

$$K_t^\ell = \bigoplus_{1 \le i_1 < i_2 < \cdots < i_t \le n} A(x_{i_1}^\ell \wedge x_{i_2}^\ell \wedge \cdots \wedge x_{i_t}^\ell),$$

where $\wedge$ denotes the exterior product. Though this notation conflicts with that for the meet in a poset, we will rarely refer to exterior products directly and the meaning should be apparent from context. Through the above decomposition, a free A-module basis for K_t^ℓ is indexed by sets $F = \{i_1, \ldots, i_t\}$ with $1 \le i_1 < i_2 < \cdots < i_t \le n$. Given such an F, define $\hat{x}_F^\ell$ to be the corresponding basis element $x_{i_1}^\ell \wedge x_{i_2}^\ell \wedge \cdots \wedge x_{i_t}^\ell$.

Since $K_\bullet^\ell$ provides a projective resolution for $A/\mathfrak{m}_\ell$, we can calculate $\operatorname{Ext}_A^i(A/\mathfrak{m}_\ell, A_P)$ as the homology of the chain complex $\operatorname{Hom}_A(K_\bullet^\ell, A_P)$. Using the $\mathbb{Z}^n$-grading of A and A_P, this complex splits into a direct sum of chain complexes of the form $\operatorname{Hom}_A(K_\bullet^\ell, A_P)_\alpha$, where $\alpha \in \mathbb{Z}^n$ (it is easily verified that the differential preserves the graded pieces). Recall (Lemma 4.2.3) that the standard monomials form a basis for A_P as a vector space over $\Bbbk$. Then $\operatorname{Hom}_A(K_t^\ell, A_P)_\alpha$ has a basis consisting of homomorphisms $f_{F,m}$ defined by

$$f_{F,m}(\hat{x}_G^\ell) = \begin{cases} m, & \text{if } G = F, \\ 0, & \text{if } G \neq F, \end{cases}$$

where m is a standard monomial that satisfies

$$\deg(m) = \deg(\hat{x}_F^\ell) + \alpha.$$

4.3.2 The main theorem

Having examined the structure of A_P as well as that of $\operatorname{Hom}_A(K_\bullet^\ell, A_P)$, we are now prepared to prove the first of our main results: a topological description of the graded pieces of the A-modules $\operatorname{Ext}_A^i(A/\mathfrak{m}_\ell, A_P)$.

Theorem 4.3.1. *Let P be a simplicial poset with vertex set V and let $\alpha \in \mathbb{Z}^n$. Set $B = \{i : -\ell < \alpha_i < 0\}$, $C = \{i : \alpha_i = -\ell\}$, and $D = \{i : \alpha_i > 0\}$. If $-\ell \le \alpha_i$ for all i, then*

$$\operatorname{Ext}_A^i(A/\mathfrak{m}_\ell, A_P)_\alpha \cong \bigoplus_{\operatorname{supp}(z) = B \cup D} \widetilde{H}^{i - |\operatorname{supp}(\alpha^-)| - 1}\left([\hat{0}, z_D] \times \operatorname{lk}_P(z)_{V \smallsetminus C}\right)$$

and $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha = 0$ *otherwise. In particular, if* $D \neq \emptyset$ *then* $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha = 0$.

Proof. Fix α, and without loss of generality assume that

$$D = \{1, \ldots, r\} \quad \text{and} \quad C = \{s, \ldots, n\} \tag{4.2}$$

for some r and s. Let $K^\ell_\bullet$ denote the Koszul complex of A with respect to the sequence $(x_1^\ell, \ldots, x_n^\ell)$. As in Section 4.3.1, we will compute $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha$ by computing the homology of the chain complex $\operatorname{Hom}_A(K^\ell_\bullet, A_P)_\alpha$.

Suppose first that $\alpha_i < -\ell$ for some i. Then there cannot exist $F = \{i_1, \ldots, i_t\}$ and a standard monomial m satisfying

$$\deg(m) = \deg(\hat{x}^\ell_F) + \alpha.$$

That is, $\operatorname{Hom}_A(K^\ell_t, A_P)_\alpha = 0$ and hence $\operatorname{Ext}^t_A(A/\mathfrak{m}^\ell, A_P)_\alpha = 0$. Assume from now on that $\alpha_i \geq -\ell$ for all i.

If $f_{F,m}$ is a basis element of $\operatorname{Hom}_A(K^\ell_t, A_P)_\alpha$, let y be the leading variable of m and set $w = y_{D\cap F}$. Then we can consider $f_{F,m}$ as an element of the poset

$$[\hat{0}, y_D] \times \operatorname{lk}_P(y_{B\cup D})_{V\smallsetminus C}$$

where $f_{F,m}$ corresponds to the element (w, y), since $B \cup D \subset \operatorname{supp}(y)$ and $\operatorname{supp}(y) \cap C = \emptyset$. Note that the rank of (w, y) in this poset is given by

$$\begin{aligned}
\operatorname{rk}(w, y) &= \operatorname{rk}_{[\hat{0}, y_D]}(w) + \operatorname{rk}_{\operatorname{lk}_P(y_{B\cup D})}(y) \\
&= |\operatorname{supp}(w)| + |\operatorname{supp}(y)| - |B| - |D| \\
&= |D \cap F| + |F \smallsetminus C| + |D \smallsetminus F| - |B| - |D| \\
&= |D \cap F| + t - |\operatorname{supp}(\alpha^-)| - |D \cap F| \\
&= t - |\operatorname{supp}(\alpha^-)|,
\end{aligned}$$

where the fourth line follows because $|F \smallsetminus C| - |B| = t - |\operatorname{supp}(\alpha^-)|$ (recall that $|F| = t$) and $|D \smallsetminus F| - |D| = -|D \cap F|$. So, we can define a map

$$\varphi : \operatorname{Hom}_A(K^\ell_t, A_P)_\alpha \to \bigoplus_{\operatorname{supp}(z) = B\cup D} \widetilde{C}^{t - |\operatorname{supp}(\alpha^-)| - 1}\left([\hat{0}, z_D] \times \operatorname{lk}_P(z)_{V\smallsetminus C}\right)$$

by $\varphi(f_{F,m}) = (w, y)$ under the correspondence above.

To prove that φ is actually an isomorphism of vector spaces, we would like to construct an inverse. Given some fixed z with $\operatorname{supp}(z) = B \cup D$, let (w, y) be an element of $[0, z_D] \times \operatorname{lk}_P(z)_{V \smallsetminus C}$ of rank $t - \operatorname{supp}(\alpha^-)$. First, set

$$F = C \cup (\operatorname{supp}(y) \smallsetminus (D \smallsetminus \operatorname{supp}(w))) .$$

Then it is immediate that

$$y_{F \cap D} = y_{\operatorname{supp}(w)} = w.$$

Furthermore, since $\operatorname{rk}(w, y) = t - |\operatorname{supp}(\alpha^-)| = t - |B| - |C|$, we have

$$\begin{aligned} \operatorname{rk}(w, y) &= \operatorname{rk}_{[\hat{0}, y_D]}(w) + \operatorname{rk}_{\operatorname{lk}_P(y_{B \cup D})}(y) \\ t - |B| - |C| &= |\operatorname{supp}(w)| + |\operatorname{supp}(y)| - |B| - |D| \\ t &= |C| + |\operatorname{supp}(y)| - (|D| - |\operatorname{supp}(w)|), \end{aligned}$$

from which it follows that $|F| = t$. So, if we can construct a monomial m such that the leading variable of m is y while $\deg(m) = \deg(\hat{x}_F^\ell) + \alpha$, then we will have constructed the desired inverse.

Let $\deg(\hat{x}_F^\ell) + \alpha = \delta = (\delta_1, \ldots, \delta_n)$, and define $p_0 = \min\{\delta_i : \delta_i > 0\}$ and $S_0 = \{i : \delta_i \geq p_0\}$. If $j > 0$ and $p_{j-1} \neq \max\{\delta_i : 1 \leq i \leq n\}$, then define

$$p_j = \min\{\delta_i : \delta_i > p_{j-1}\} \quad \text{and} \quad S_j = \{i : \delta_i \geq p_j\}.$$

Eventually, this process terminates at some point and $p_\ell = \max\{\delta_i : 1 \leq i \leq n\}$ for some $\ell \geq 0$. Now define

$$m = y^{p_0} \prod_{i=1}^{\ell} y_{S_i}^{p_i - p_{i-1}}.$$

We will verify that $\deg(m) = \delta$. Fix i. Then $\delta_i = p_j$ for some j, which means that $i \in S_k$ for $k \leq j$ and $i \notin S_k$ for $k > j$. Then $x_i \preceq y_{S_k}$ for $k \leq j$, while $x_i \not\preceq y_{S_k}$ for $k > j$, so $\deg(y_{S_k})_i = 1$ for $k \leq j$ and $\deg(y_{S_k}) = 0$ for $k > j$. Hence,

$$\deg(m)_i = p_0 \deg(y)_i + \sum_{m=1}^{\ell} (p_m - p_{m-1}) \deg(y_{S_m})_i = p_0 + \sum_{m=1}^{j} (p_m - p_{m-1}) = p_j.$$

We have now shown that $f_{F,m} \in \operatorname{Hom}_A(K_t^\ell, A_P)_\alpha$, and it is clear by construction that $\varphi(f_{F,m}) = (w, y)$. Hence, φ is a vector space isomorphism.

It remains to show that φ provides an isomorphism of chain complexes by verifying that the differentials in each complex are preserved under φ. Let $f_{F,m}$ be a basis element in $\operatorname{Hom}_A(K_\bullet^\ell, A_P)_\alpha$ with $m = y^p \prod_{i=1}^k y_i^{p_i}$. The differential on $\operatorname{Hom}_A(K_\bullet^\ell, A_P)_\alpha$ sends the homomorphism $f_{F,m}$ to a homomorphism $df_{F,m}$ that maps $\hat{x}_{F\cup\{j\}} \in K_{t+1}^\ell$ to $(-1)^{u_j} x_j^\ell m \in A_P$ for any $j \notin F$, where $u_j = |\{i \in F : i < j\}|$. By Lemma 4.2.5, we can write each $(-1)^{u_j} x_j^\ell m$ as

$$(-1)^{u_j} x_j^\ell m = (-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, y)} m_z,$$

where m_z is a standard monomial with leading variable z. Hence, each value of j contributes to $df_{F,m}$ a term of the form

$$\left((-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, y)} f_{F\cup\{j\}, m_z} \right),$$

where m_z is a standard monomial with leading variable z. Then we may write $d(f_{F,m})$ as

$$\begin{aligned} d(f_{F,m}) &= \sum_{j \notin F} \left((-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, y)} f_{F\cup\{j\}, m_z} \right) \\ &= \sum_{j \in (D \smallsetminus F)} \left((-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, y)} f_{F\cup\{j\}, m_z} \right) + \sum_{j \notin (D \cup F)} \left((-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, y)} f_{F\cup\{j\}, m_z} \right). \end{aligned}$$

Choose $z \in \mathcal{M}(x_j, y)$. If $j \in D$, then $x_j \preceq y$ already because $D \subset \operatorname{supp}(y)$ and thus $z = y$. Furthermore, $z_{D\cap F} = y_{D\cap F}$ regardless of the value of j and $z_{(D\cap F)\cup\{j\}} = y_{(D\cap F)\cup\{j\}}$ if $j \in D$. Hence, applying φ to this expression allows for the following simplifications:

$$\begin{aligned} (\varphi \circ d)(f_{F,m}) &= \sum_{j \in (D \smallsetminus F)} \left((-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, y)} (z_{(D\cap F)\cup\{j\}}, z) \right) + \sum_{j \notin (D \cup F)} \left((-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, y)} (z_{D\cap F}, z) \right) \\ &= \sum_{j \in (D \smallsetminus F)} (-1)^{u_j} (y_{(D\cap F)\cup\{j\}}, y) + \sum_{j \notin (D \cup F)} \left((-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, y)} (y_{D\cap F}, z) \right). \end{aligned} \tag{4.3}$$

Our final step is to show that the differential on $(y_{D\cap F}, y)$ in $\widetilde{C}^\bullet([\hat{0}, y_D] \times \mathrm{lk}_P(y_{B\cup D})_{V\smallsetminus C})$ acts the same way. First note that if w is an atom of $\mathrm{lk}_P(y_{B\cup D})_{V\smallsetminus C}$, then w is an element of $\mathcal{M}(x_j, y_{B\cup D})$ for some $j \notin C$. Arbitrarily label the elements of $\mathcal{M}(x_j, y_{B\cup D})$ by $x_{j^1}, x_{j^2}, \ldots, x_{j^{|\mathcal{M}(x_j, y_{B\cup D})|}}$ for each j. Then by definition, we can write

$$(d \circ \varphi)(f_{F,m}) = \sum_{j\in(D\smallsetminus F)} (-1)^{v_j} (y_{(D\cap F)\cup\{j\}}, y) + \sum_{\substack{j\notin \mathrm{supp}(y)\\ j\notin C}} \sum_{k=1}^{|\mathcal{M}(x_j,y_{B\cup D})|} \left((-1)^{w_{j^k}} \sum_{z\in\mathcal{M}(x_{j^k},y)} (y_{D\cap F}, z) \right), \quad (4.4)$$

where v_j and w_{j^i} depend upon some choice of orientations for $[\hat{0}, y_D]$ and $\mathrm{lk}_P(y_{B\cup D})_{V\smallsetminus C}$, respectively. Since $\mathrm{supp}(y) \cup C = D \cup F$, the right-hand sums in (4.3) and (4.4) run over the same values of j. On the other hand,

$$\bigcup_{j=1}^{n} M(x_j, y) = \bigcup_{j=1}^{n} \left(\bigcup_{k=1}^{|\mathcal{M}(x_j,y)|} M(x_{j^k}, y) \right),$$

so if the characteristic of $\Bbbk$ is 2 then we are done. The construction of consistent orientations for fields of other characteristics is tedious, but unavoidable. If $\mathcal{O}_1$ is an ordering of the vertices in $[\hat{0}, y_D]$ and $\mathcal{O}_2$ is an ordering of the vertices in $\mathrm{lk}_P(y_{B\cup D})_{V\smallsetminus C}$, then

$$v_j = |\{i \in \mathrm{supp}(y_{D\cap F}) : i <_{\mathcal{O}_1} j\}|$$

and

$$w_{j^k} = |\{i \in \mathrm{supp}(y) : i <_{\mathcal{O}_2} j^k\}|.$$

In $[\hat{0}, y_D]$ we will choose $\mathcal{O}_1$ to be the natural ordering of the vertices. To show that $u_j = v_j$ in the left sums of (4.3) and (4.4), first choose $j \in D \smallsetminus F$. If $i \in D \cap F$ satisfies $i <_{\mathcal{O}_1} j$, then $i < j$ in the natural order and $i \in F$. On the other hand, if $i \in F$ satisfies $i < j$, then $i \in D$ by our assumption (4.2) because $j \in D$. Hence, $u_j = v_j$ as desired.

We now wish to construct an ordering of the j^i's such that $w_{j^i} = w_{j^k}$ for any pair i, k and such that $u_j = w_{j^i}$ for all $j \notin (D \cup F)$. Once this is done, we will have that

$$\sum_{\substack{j\notin \mathrm{supp}(y)\\ j\notin C}} \sum_{i=1}^{|\mathcal{M}(x_j,y)|} \left((-1)^{w_{j^i}} \sum_{z\in\mathcal{M}(x_{j^i},y)} (y_{D\cap F}, z) \right) = \sum_{\substack{j\notin \mathrm{supp}(y)\\ j\notin C}} \left((-1)^{u_j} \sum_{z\in\mathcal{M}(x_j,y)} (y_{D\cap F}, z) \right)$$

as desired. We choose the partial ordering $\mathcal{O}_2$ of the vertices of $\operatorname{lk}_P(y_{B\cup D})_{V\smallsetminus C}$ as follows. First, order the indices of the vertices of $V \smallsetminus C$ as

$$r+1 < r+2 < \cdots < s-1 < 1 < 2 < \cdots < n,$$

and call this order $\mathcal{O}_3$. Next order the vertices of $\operatorname{lk}_P y$ such that $(j_1)^{i_1} < (j_2)^{i_2}$ if $j_1 < j_2$ under the order $\mathcal{O}_3$. In P, we have $\operatorname{supp}(y) = \{j_1, j_2, \dots, j_{|\operatorname{supp}(y)|}\}$ for some j_i's with $j_i \in (V \smallsetminus C)$. Then in $\operatorname{lk}_P(y_{B\cup D})$, we have $\operatorname{supp}(y) = \{j_1^{i_1}, j_2^{i_2}, \dots, j_{|\operatorname{supp}(y)|}^{i_{|\operatorname{supp}(y)|}}\}$ for some $j_k^{i_k}$'s with $j_k \in (V \smallsetminus C)$. In particular, no pair $j_{k_1}^{i_{k_1}}, j_{k_2}^{i_{k_2}} \in \operatorname{supp}(y)$ has $k_1 = k_2$. Hence, the restriction of $\mathcal{O}_2$ to any face of $\operatorname{lk}_P(y_{B\cup D})_{V\smallsetminus C}$ is a linear order and $\mathcal{O}_2$ provides a valid orientation for $\operatorname{lk}_P(y_{B\cup D})_{V\smallsetminus C}$. Under this orientation,

$$w_{j^i} = |\{k^m \in \operatorname{supp}(y) : k^m <_{\mathcal{O}_2} j^i\}| = |\{k^m \in \operatorname{supp}(y) : k <_{\mathcal{O}_3} j\}|.$$

Now choose $j \notin D \cup F$. If $i \in F$ is such that $i < j$, then i cannot be in C on account of our assumption (4.2) and thus $i \in \operatorname{supp}(y)$ in P. Then in $\operatorname{lk}_P(y_{B\cup D})_{V\smallsetminus C}$, $i^k \in \operatorname{supp}(y)$ for some single value k. Furthermore, $i <_{\mathcal{O}_3} j$, so $i^k <_{\mathcal{O}_2} j^m$ for all m. That is, $u_j \leq w_{j^m}$ for all m.

On the other hand, if $i^k \in \operatorname{supp}(y)$ (in $\operatorname{lk}_P(y_{B\cup D})_{V\smallsetminus C}$) is such that $i^k <_{\mathcal{O}_2} j^m$, then $i <_{\mathcal{O}_3} j$ and hence i cannot be in D. Then $i \in F$, and $i < j$ in the natural order. Hence, $w_j^m \leq u_j$ for all m, so $w_j^m = u_j$ for all j and m and φ provides an isomorphism of chain complexes.

Lastly, if $D \neq \emptyset$, then the CW complex corresponding to $[\hat{0}, z_D] \times \operatorname{lk}_P(z)_{V\smallsetminus C}$ is the join of a simplex with the CW complex corresponding to $\operatorname{lk}_P(z)_{V\smallsetminus C}$ ([7, Section 3]). In particular, it is contractible; hence, $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha = 0$. □

4.4 Local cohomology modules

4.4.1 Reproving Duval's result

Let $\mathfrak{m} = (x_1, \dots, x_n)$ denote the irrelevant ideal of A. From Theorem 4.3.1 we can quickly obtain the dimensions of the graded pieces of the local cohomology modules $H^i_{\mathfrak{m}}(A_P)$, a calculation originally completed by Duval in [15, Theorem 5.9]. Here we will reprove the result using the techniques employed by Miyazaki in the simplicial complex case in [42], allowing for an important corollary.

Theorem 4.4.1. *If P is a simplicial poset and $\alpha \in \mathbb{Z}^n$ is such that $\alpha_i \leq 0$ for all i then*

$$H^i_{\mathfrak{m}}(A_P)_\alpha \cong \bigoplus_{\operatorname{supp}(w)=\operatorname{supp}(\alpha)} \widetilde{H}^{i-|\operatorname{supp}(\alpha)|-1}\left(\operatorname{lk}_P(w)\right),$$

and $H^i_{\mathfrak{m}}(A_P)_\alpha = 0$ otherwise.

Proof. Since the chain of ideals $\mathfrak{m}, \mathfrak{m}_2, \mathfrak{m}_3, \ldots$ is cofinal with the chain $\mathfrak{m}, \mathfrak{m}^2, \mathfrak{m}^3, \ldots$ (that is, for all p, there exists q and r such that $\mathfrak{m}^r \subseteq \mathfrak{m}_p$ and $\mathfrak{m}_s \subseteq \mathfrak{m}^p$), we can compute $H^i_{\mathfrak{m}}(A_P)_\alpha$ as the direct limit of the modules $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha$ (see [29, Chapter 7]). Since $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha = 0$ for all ℓ if $\alpha_i > 0$ for some i, it follows that $H^i_{\mathfrak{m}}(A_P)_\alpha = 0$ if $\alpha_i > 0$ for some i.

Suppose now that $\alpha_i \leq 0$ for all i, and choose ℓ such that $1 - \ell \leq \alpha_i$ for all i. Then it suffices to show that the following diagram commutes, in which the diagonal maps are the isomorphisms of Theorem 4.3.1 and $\widetilde{\pi}$ is induced by the canonical projection $\pi : A/\mathfrak{m}_{\ell+1} \to A/\mathfrak{m}_\ell$:

$$\begin{array}{ccc}
\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha & \xrightarrow{\widetilde{\pi}} & \operatorname{Ext}^i_A(A/\mathfrak{m}_{\ell+1}, A_P)_\alpha \\
\searrow & & \swarrow \\
& \displaystyle\bigoplus_{\operatorname{supp}(w)=\operatorname{supp}(\alpha)} \widetilde{H}^{i-|\operatorname{supp}(\alpha)|-1}\left(\operatorname{lk}_P(w)\right). &
\end{array}$$

To begin, define a map $\hat{\pi} : K^{\ell+1}_\bullet \to K^\ell_\bullet$ by

$$\hat{\pi}(x^{\ell+1}_{i_1} \wedge \cdots \wedge x^{\ell+1}_{i_t}) = (x_{i_1} \cdots x_{i_t})(x^\ell_{i_1} \wedge \cdots \wedge x^\ell_{i_t}).$$

Then the following diagram of projective resolutions commutes

$$\begin{array}{ccccc}
K^{\ell+1}_\bullet & \longrightarrow & A/\mathfrak{m}_{\ell+1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} & & \\
K^\ell_\bullet & \longrightarrow & A/\mathfrak{m}_\ell & \longrightarrow & 0,
\end{array}$$

so dualizing produces another commutative diagram:

$$\begin{array}{ccccc} \operatorname{Hom}_A(K_\bullet^{\ell+1}, A_P)_\alpha & \longleftarrow & \operatorname{Hom}_A(A/\mathfrak{m}_{\ell+1}, A_P)_\alpha & \longleftarrow & 0 \\ {\scriptstyle \hat{\pi}^*}\uparrow & & {\scriptstyle \pi^*}\uparrow & & \\ \operatorname{Hom}_A(K_\bullet^{\ell}, A_P)_\alpha & \longleftarrow & \operatorname{Hom}_A(A/\mathfrak{m}_{\ell}, A_P)_\alpha & \longleftarrow & 0. \end{array}$$

When taking homology to compute $\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha$ and $\operatorname{Ext}^i_A(A/\mathfrak{m}_{\ell+1}, A_P)_\alpha$, the map $\hat{\pi}^*$ may be used to compute $\widetilde{\pi}$. That is, we only need to check that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Hom}_A(K_t^{\ell}, A_P)_\alpha & \xrightarrow{\hat{\pi}^*} & \operatorname{Hom}_A(K_t^{\ell+1}, A_P)_\alpha \\ & \searrow \quad \swarrow & \\ & \bigoplus_{\operatorname{supp}(w)=\operatorname{supp}(\alpha)} \widetilde{C}^{t-|\operatorname{supp}(\alpha)|-1}\left(\operatorname{lk}_P(w)\right). & \end{array} \tag{4.5}$$

Let $f_{F,m}$ be a basis element for $\operatorname{Hom}_A(K_t^\ell, A_P)_\alpha$, where $F = \{i_1, \ldots, i_t\}$ and $m = y^p \prod_{i=1}^k y_i^{p_i}$ with $y \succ y_1 \succ y_2 \succ \cdots \succ y_k$. Then $f_{F,m}$ corresponds to y in $\widetilde{C}^{t-|\operatorname{supp}(\alpha)|-1}(\operatorname{lk}_P(y_{\operatorname{supp}(\alpha)}))$ under the map φ of Theorem 4.3.1.

On the other hand, $f_{F,m} \circ \hat{\pi}$ is the homomorphism that sends $\hat{x}_F^{\ell+1}$ to $(x_{i_1} \cdots x_{i_t})m$ and all other basis elements of $K_t^{\ell+1}$ to zero. Note that $x_{i_j} \in \operatorname{supp}(y)$ for $1 \le j \le t$ because $1 - \ell \le \alpha_i \le 0$ for all i. By iterating Lemma 4.2.5 t times, we have $(x_{i_1} \cdots x_{i_t})m = m'$ where m' is a standard monomial in A_P with leading variable y. Hence, $\varphi(f_{F,m}) = (\varphi \circ \hat{\pi}^*)(f_{F,m})$ and Diagram (4.5) commutes. □

The proof of Theorem 4.4.1 immediately provides the following corollary, which we will use in Section 4.5 to characterize Buchsbaum simplicial posets.

Corollary 4.4.2. *If $\alpha \in \mathbb{Z}^n$ is such that $1 - \ell \le \alpha_i$ for all i, then the canonical map*

$$\operatorname{Ext}^i_A(A/\mathfrak{m}_\ell, A_P)_\alpha \to H^i_{\mathfrak{m}}(A_P)_\alpha$$

is an isomorphism for all i.

4.4.2 The A-module structure of $H^i_{\mathfrak{m}}(A_P)$

The simplicial complex analog of Theorem 4.4.1 counting the dimensions of the local cohomology modules $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ of a Stanley–Reisner ring $\Bbbk[\Delta]$ was proved first by Hochster ([67, Section II.4]). Later, Gräbe ([23, Theorem 2]) examined the graded module structure of $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$. Here we will prove a similar result for $H^i_{\mathfrak{m}}(A_P)$. First, we will need the following simplicial poset analog of [23, Lemma, p.273].

Lemma 4.4.3. *Let P be a simplicial poset and let $y \in P$. Then*

$$\widetilde{H}^{i-|\operatorname{supp}(y)|-1}(\operatorname{lk}_P(y)) \cong H^{i-1}(P, \operatorname{cost}_P(y)).$$

Proof. We clearly have an isomorphism of vector spaces $\widetilde{C}^{i-|\operatorname{supp}(y)|}(\operatorname{lk}_P(y)) \cong C^i(P, \operatorname{cost}_P(y))$ under the map $\varphi : w \mapsto w$. To show that this is an isomorphism of chain complexes, we must proceed as in the proof of Theorem 4.3.1.

Label the vertices of P by $x_1, \ldots, x_n$, and let $\mathcal{O}_1$ be any ordering of the vertices of P in which the vertices contained in $\operatorname{supp}(y)$ come last. If w is a vertex of $\operatorname{lk}_P(y)$, then w is an element of $\mathcal{M}(x_j, y)$ for some j with $j \notin \operatorname{supp}(y)$. Arbitrarily label the elements of $\mathcal{M}(x_j, y)$ by $x_{j^1}, x_{j^2}, \ldots, x_{j^{|\mathcal{M}(x_j,y)|}}$. Let $\mathcal{O}_2$ be the partial ordering of the vertices of $\operatorname{lk}_P(y)$ given by setting $j_1^{k_1} < j_2^{k_2}$ if $j_1 <_{\mathcal{O}_1} j_2$.

If $w \in (P, \operatorname{cost}_P(y))$, then write $\operatorname{supp}(w) = \{j_1, \ldots, j_t\}$. In $\operatorname{lk}_P(y)$, we have $\operatorname{supp}(w) = \{j_1^{k_1}, \ldots, j_t^{k_t}\}$ for some k_i's. That is, $\mathcal{O}_2$ is a total order when restricted to $\operatorname{supp}(w)$ for any $w \in \operatorname{lk}_P(y)$, so $\mathcal{O}_2$ provides a valid orientation for $\operatorname{lk}_P(y)$.

In $C^t(P, \operatorname{cost}_P(y))$,

$$\delta(w) = \sum_{j \notin \operatorname{supp}(w)} \left((-1)^{u_j} \sum_{z \in \mathcal{M}(x_j, w)} z \right)$$

where

$$u_j = |\{i \in \operatorname{supp}(w) : i <_{\mathcal{O}_1} j\}|.$$

On the other hand, in $\widetilde{C}^{t-|\operatorname{supp}(y)|}(\operatorname{lk}_P(y))$,

$$\delta(w) = \sum_{j \notin \operatorname{supp}(w)} \left(\sum_{k=1}^{|\mathcal{M}(x_j,w)|} (-1)^{v_{j^k}} \sum_{z \in \mathcal{M}(x_{j^k}, w)} z \right)$$

where

$$v_{j^k} = |\{i \in \operatorname{supp}(w) : i <_{\mathcal{O}_2} j^k\}|.$$

Note that

$$\bigcup_{j \notin \operatorname{supp}(w)} M(x_j, w) = \bigcup_{j \notin \operatorname{supp}(w)} \left(\bigcup_{k=1}^{|\mathcal{M}(x_{j^k}, w)|} \mathcal{M}(x_{j^k}, w) \right)$$

and that $u_j = v_{j^k}$ for fixed j and arbitrary k under our choice of ordering for $\mathcal{O}_1$. The result now follows. □

With this lemma in hand, we are now prepared to prove our second main result.

Theorem 4.4.4. *Let $\alpha \in \mathbb{Z}^n$. Then*

$$H^i_{\mathfrak{m}}(A_P)_\alpha \cong \bigoplus_{\operatorname{supp}(w)=\operatorname{supp}(\alpha)} H^{i-1}(P, \operatorname{cost}_P(w))$$

if $\alpha \in \mathbb{Z}^n_{\leq 0}$ and $H^i_{\mathfrak{m}}(A_P)_\alpha = 0$ otherwise. Under these isomorphisms, the A-module structure of $H^i_{\mathfrak{m}}(A_P)$ is given as follows. Let $\gamma = \alpha + \deg(x_j)$. If $\alpha_j < -1$, then $\cdot x_j : H^i_{\mathfrak{m}}(A_P)_\alpha \to H^i_{\mathfrak{m}}(A_P)_\gamma$ corresponds to the direct sum of identity maps

$$\bigoplus_{\operatorname{supp}(w)=\operatorname{supp}(\alpha)} H^{i-1}(P, \operatorname{cost}_P(w)) \to \bigoplus_{\operatorname{supp}(w)=\operatorname{supp}(\gamma)} H^{i-1}(P, \operatorname{cost}_P(w)).$$

If $\alpha_j = -1$, then $\cdot x_j$ corresponds to the direct sum of maps

$$\bigoplus_{\operatorname{supp}(w)=\operatorname{supp}(\alpha)} H^{i-1}(P, \operatorname{cost}_P(w)) \to \bigoplus_{\operatorname{supp}(z)=\operatorname{supp}(\gamma)} H^{i-1}(P, \operatorname{cost}_P(z))$$

induced by the inclusions of pairs $(P, \operatorname{cost}_P(w \setminus \{x_j\})) \to (P, \operatorname{cost}_P(w))$. If $x_j \notin \operatorname{supp}(\alpha)$, then $\cdot x_j$ is the zero map.

Proof. The first statement is immediate, as it is a reformulation of Theorem 4.4.1 using the isomorphisms of Lemma 4.4.3. Let ℓ be such that $1 - \ell \leq \alpha_i \leq 0$ for all i. By Corollary 4.4.2, we only need to show that the multiplication maps above are valid for $\operatorname{Ext}^t_A(A/\mathfrak{m}_\ell, A_P)_\alpha$, as the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Ext}^t_A(A/\mathfrak{m}_\ell, A_P)_\alpha & \xrightarrow{\sim} & H^t_{\mathfrak{m}}(A_P)_\alpha \\ \downarrow{\cdot x_j} & & \downarrow{\cdot x_j} \\ \operatorname{Ext}^t_A(A/\mathfrak{m}_\ell, A_P)_\gamma & \xrightarrow{\sim} & H^t_{\mathfrak{m}}(A_P)_\gamma. \end{array}$$

First note that if $x_j \notin \operatorname{supp}(\alpha)$, then $\gamma_j = 1$ so that $\operatorname{Ext}^t_A(A/\mathfrak{m}_\ell, A_P)_\gamma = H^t_{\mathfrak{m}}(A_P)_\gamma = 0$. Assume now that $x_j \in \operatorname{supp}(\alpha)$ and let $f_{F,m}$ be a basis element for $\operatorname{Hom}^t_A(A/\mathfrak{m}_\ell, A_P)_\alpha$ in which $m = y^p \prod_{i=1}^k y_i^{p_i}$ with $y \succ y_1 \succ y_2 \succ \cdots \succ y_k$.

By the proof of Theorem 4.3.1, $f_{F,m}$ corresponds to y in $\widetilde{C}^{t-|\operatorname{supp}(\alpha)|-1}(\operatorname{lk}_P(y_{\operatorname{supp}(\alpha)}))$. On the other hand, $x_j \cdot f_{F,m} = f_{F,x_j m}$ and $x_j \prec y$, so by Lemma 4.2.5 we can write $x_j m$ as a standard monomial m' in which the leading variable of m' is still y. Then $f_{F,m'}$ corresponds to the face y of $\widetilde{C}^{t-1}(P, \operatorname{cost}_P(y_{\operatorname{supp}(\gamma)}))$.

Suppose first that $\gamma_j < 0$. Then $\operatorname{supp}(\alpha) = \operatorname{supp}(\gamma)$, so

$$\widetilde{C}^{t-1}(P, \operatorname{cost}_P(y_{\operatorname{supp}(\alpha)})) = \widetilde{C}^{t-1}(P, \operatorname{cost}_P(y_{\operatorname{supp}(\gamma)}))$$

and $\cdot x_v$ corresponds to the identity map, as desired.

If $\gamma_j = 0$, then we have a map

$$\widetilde{C}^{t-1}(P, \operatorname{cost}_P(y_{\operatorname{supp}(\alpha)})) \xrightarrow{\cdot x_v} \widetilde{C}^{t-1}(P, \operatorname{cost}_P(y_{\operatorname{supp}(\alpha) \smallsetminus \{x_j\}}))$$

taking y to y, which is identical to the map induced by the inclusion of pairs $(P, \operatorname{cost}_P(y_{\operatorname{supp}(\alpha) \smallsetminus \{x_j\}})) \to (P, \operatorname{cost}_P(y_{\operatorname{supp}(\alpha)}))$. □

4.5 Characterizing Buchsbaumness

Central to the historical study of simplicial complexes have been the notions of a complex being **Cohen-Macaulay** or **Buchsbaum**. In analogy to the results of Reisner ([57]) and Schenzel ([60]) for simplicial complexes, we will say that a poset P is Cohen-Macaulay (over $\Bbbk$) if its order complex is Cohen-Macaulay (over $\Bbbk$) in the topological sense; that is,

$$\widetilde{H}^i(\operatorname{lk}_{\Delta(\overline{P})}(F)) = 0$$

for all faces $F \in \Delta(\overline{P})$ and all $i < \dim(\operatorname{lk}_{\Delta(\overline{P})}(F))$. We say that P is Buchsbaum (over $\Bbbk$) if it is pure and the link of every vertex of $\Delta(\overline{P})$ is Cohen-Macaulay (over $\Bbbk$).

There are also algebraic notions of Cohen-Macaulayness and Buchsbaumness. Let R be a ring with maximal ideal $\mathfrak{m}$ and let M be a noetherian R-module of dimension $d > 0$. We call a sequence of elements $\theta_1, \ldots, \theta_d \in \mathfrak{m}$ a **system of parameters** if $M/(\theta_1, \ldots, \theta_d)M$ is a finite-dimensional vector space over $\Bbbk$. If

$$(\theta_1, \ldots, \theta_{i-1})M :_M \theta_i = (\theta_1, \ldots, \theta_{i-1})M$$

for $i = 1, \ldots, d$, then we say that $\theta_1, \ldots, \theta_d$ is an **M-sequence**. If every system of parameters for M is an M-sequence, then we say that M is a **Cohen-Macaulay** R-module. If

$$(\theta_1, \ldots, \theta_{i-1})M : \theta_i = (\theta_1, \ldots, \theta_{i-1})M : \mathfrak{m}$$

for $i = 1, \ldots, d$, then we say that $\theta_1, \ldots, \theta_d$ is a **weak M-sequence**. If every system of parameters for M is a weak M-sequence, then we call M a **Buchsbaum** R-module. In the case that M is a graded A-module, we say that M is **(graded) Buchsbaum** if the localization $M_{\mathfrak{m}}$ is a Buchsbaum module over $R_{\mathfrak{m}}$.

We say that the ring R is Cohen-Macaulay (resp. Buchsbaum) if it is Cohen-Macaulay (resp. Buchsbaum) as a module over itself. Stanley showed ([6, Theorem 3.4, p. 314]) that the topological notion of Cohen-Macaulayness is tied to this algebraic property (proofs of the forward direction appear frequently in the literature, but we have had trouble finding a proof of the reverse; however, it does follow quickly from Duval's results, e.g., [15, Corollary 6.1]):

Theorem 4.5.1. *Let P be a simplicial poset. Then P is Cohen-Macaulay as a poset over $\Bbbk$ if and only if A_P is a Cohen-Macaulay ring.*

In fact, an additional characterization is that A_P is a Cohen-Macaulay A-module. Our next goal is obtaining an analogous characterization for the Buchsbaum property. To prove our result, we will need the following homological characterization of Buchsbaumness ([68, Theorem I.3.5]):

Theorem 4.5.2. *Let M be a graded A-module with $\dim(M) > 0$. Then M is a (graded) Buchsbaum A-module if and only if the canonical maps*

$$\operatorname{Ext}^i_A(\Bbbk, M) \to H^i_{\mathfrak{m}}(M)$$

are surjective for $i < \dim(M)$*.*

Along with Theorem 4.4.4 and Corollary 4.4.2, this result will provide a quick proof of the following generalization of [68, Theorem II.2.4], the forward direction of which appeared first as [54, Proposition 6.2]:

Theorem 4.5.3. *Let* P *be a simplicial poset with* $\dim(P) \geq 0$*. Then* P *is Buchsbaum over* $\Bbbk$ *as a poset if and only if* A_P *is a (graded) Buchsbaum* A*-module.*

Proof. Suppose first that P is a Buchsbaum poset. Under the isomorphisms

$$\widetilde{H}^i(\mathrm{lk}_{\Delta(\overline{P})}(F)) \cong \widetilde{H}^i(\Delta(\overline{\mathrm{lk}_P(w)})) \cong \widetilde{H}^i(\mathrm{lk}_P(w)) \tag{4.6}$$

for F a saturated chain in $(\hat{0}, w]$, by Theorem 4.4.1 along with the purity of P, $H^i_{\mathfrak{m}}(A_P)$ is concentrated in degree zero for $i < \dim(A_P)$. By Corollary 4.4.2, the canonical map $\mathrm{Ext}^i_A(\Bbbk, A_P)_0 \to H^i_{\mathfrak{m}}(A_P)_0$ is always an isomorphism. Hence, A_P satisfies the conditions of Theorem 4.5.2 and A_P is a (graded) Buchsbaum A-module.

Now suppose that A_P is a (graded) Buchsbaum A-module, and for the moment assume that P is pure. This condition ensures that every homogeneous system of parameters for A_P is a weak A_P-sequence (see [68, Definition I.3.1] and the surrounding discussion). Then by [68, Proposition I.3.4], $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(A_P) = 0$ for all $i \neq \dim A_P$. By Theorem 4.4.4 and purity, this is only possible if

$$\widetilde{H}^i(\mathrm{lk}_P(y)) = 0$$

for all $y \neq \emptyset$ in P and all $i < \dim(\mathrm{lk}_P(y))$. Then once more, the isomorphisms in (4.6) imply that $\widetilde{H}^i(\mathrm{lk}_{\Delta(\overline{P})}(F)) = 0$ for all faces $F \neq \emptyset$ of $\Delta(\overline{P})$ and all $i < \dim(\mathrm{lk}_{\Delta(\overline{P})}(F))$. In particular, $\mathrm{lk}_{\Delta(\overline{P})}(v)$ is Cohen-Macaulay for each vertex v of $\Delta(\overline{P})$.

It remains to show that P is pure. Suppose to the contrary that there exists some maximal face $y \in P$ with $\dim(y) = k$, while $\dim(P) = d-1$ and $k < d-1$. Note first that $\widetilde{H}^{-1}(\mathrm{lk}_P(y)) \neq 0$.

Let $\alpha \in \mathbb{Z}^n$ be such that $\mathrm{supp}(\alpha) = \mathrm{supp}(\alpha) + \mathbf{e}_i = \mathrm{supp}(y)$ for some $\mathbf{e}_i$. By Theorem 4.4.1, $\widetilde{H}^{-1}(\mathrm{lk}_P(y))$ is a submodule of $H^{k+1}_{\mathfrak{m}}(A_P)_\alpha$. Furthermore, the multiplication map

$\cdot x_i : H^{k+1}_{\mathfrak{m}}(A_P)_\alpha \to H^{k+1}_{\mathfrak{m}}(A_P)_{\alpha+\mathbf{e}_i}$ is the identity map on this summand by Theorem 4.4.4. Then $\mathfrak{m} \cdot H^{k+1}_{\mathfrak{m}}(A_P) \neq 0$ and $k+1 < \dim(A_P)$, which contradicts [68, Corollary I.2.4]. Hence, P must be pure. □

Remark 4.5.4. As noted in this proof, if A_P is (graded) Buchsbaum then every linear system of parameters for A_P is a weak A_P-sequence. In the case that $\Bbbk$ is infinite, a linear system of parameters can always be found. As in [54, Proposition 6.3], combining this fact with the characterization in Theorem 4.5.3 allows for some analysis of the well-studied h-vector of a Buchsbaum poset P using the Hilbert functions of A_P and its local cohomology modules.

4.6 Comments and further applications

Our Theorem 4.4.4 detailing the A-module structure of $H^i_{\mathfrak{m}}(A_P)$ has the potential to extend many results about simplicial complexes relying upon Gräbe's theorem to the simplicial poset case. For example, many of the theorems for face rings of simplicial complexes with isolated singularities found in [38], [55], and [58] appear likely to be true in this greater generality.

On the other hand, Theorems 4.3.1 and 4.4.4 may be worth further specializing in their own right. Simplicial posets come about frequently as quotients of simplicial complexes by free group actions (see [17, Section 6]). It has been shown in [59] that the Ext- and local cohomology modules of a simplicial complex admitting a free action by an abelian group inherit some nice symmetry properties. It would be interesting to see to what extent these results continue to hold in the simplicial poset case.

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Chapter 5

LOWER BOUND FOR BALANCED MANIFOLDS

5.1 Introduction

At the intersection of geometry, algebra, and combinatorics is the study of the face numbers of simplicial complexes. If $f_i(\Delta)$ denotes the number of i-dimensional faces of a $(d-1)$-dimensional simplicial complex Δ, then the h-numbers $h_i(\Delta)$ are defined by $h_i(\Delta) = \sum_{j=0}^{i}(-1)^{i-j}\binom{d-i}{j-i}f_{j-1}(\Delta)$. Of the most important results in the study of face numbers of simplicial complexes, many have been elegantly phrased in the language of the h-numbers. Principal among these are the Dehn–Sommerville relations, the lower and upper bound theorems, and their culmination – the g-theorem. Our starting point is the following generalized lower bound theorem (or GLBT) conjectured by McMullen and Walkup [37] and proved by Stanley [64], and Murai and Nevo [44]:

Theorem 5.1.1. *Let P be a d-dimensional simplicial polytope. Then*

$$h_0(P) \leq h_1(P) \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor}(P);$$

also the equality $h_{i-1}(P) = h_i(P)$ occurs for a certain $i \leq \lfloor \frac{d}{2} \rfloor$ if and only if P is $(i-1)$-stacked.

It is natural to ask to what extent these inequalities can be specialized. In particular, are there classes of simplicial polytopes whose successive h-numbers satisfy more drastic inequalities? Of recent interest have been balanced simplicial complexes (those complexes whose underlying graphs have a "minimal" coloring), introduced by Stanley in [63]. Examples of balanced simplicial complexes include barycentric subdivisions of regular CW complexes, Coxeter complexes, and Tits buildings. The following strengthening of Theorem 5.1.1 for balanced simplicial polytopes was conjectured in [32] and proved by Juhnke-Kubitzke and Murai in [30].

Theorem 5.1.2. *Let P be a d-dimensional balanced simplicial polytope. Then*

$$\frac{h_0(P)}{\binom{d}{0}} \leq \frac{h_1(P)}{\binom{d}{1}} \leq \cdots \leq \frac{h_i(P)}{\binom{d}{i}} \leq \cdots \leq \frac{h_{\lfloor d/2 \rfloor}(P)}{\binom{d}{\lfloor d/2 \rfloor}}.$$

Our goal is to examine extensions of this result to more general complexes. In particular, the complexes considered in this chapter are balanced $\mathbb{F}$-homology manifolds with and without boundary, where $\mathbb{F}$ is a field. (We defer most of the definitions until the following sections.) When confining our attention to this class of simplicial complexes, the natural analog of the h-numbers turns out to be the h''-numbers (for polytopes, these are one and the same): for a $(d-1)$-dimensional complex Δ and $i < d$, $h''_i(\Delta)$ is defined by $h_i(\Delta) - \binom{d}{i} \sum_{j=1}^{i} (-1)^{i-j} \tilde{\beta}_{j-1}(\Delta)$, where $\tilde{\beta}_{j-1}(\Delta)$, $1 \leq j \leq d$, are the reduced Betti numbers computed over $\mathbb{F}$. Specifically, the manifold GLBT asserts that if Δ is a $(d-1)$-dimensional $\mathbb{F}$-homology manifold with or without boundary whose vertex links have the weak Lefschetz property, then $h''_i(\Delta, \partial\Delta) \geq h''_{i-1}(\Delta, \partial\Delta) + \binom{d}{i-1} \tilde{\beta}_{i-1}(\Delta, \partial\Delta)$ for all $i \leq \lfloor d/2 \rfloor$; see [52, eq. (9)] and [48, Theorem 1.5]. In view of this result, it seems plausible that the statement of Theorem 5.1.2 can be appropriately extended to balanced $\mathbb{F}$-homology manifolds. Indeed, the following is one of our main results.

Theorem 5.1.3. *Let Δ be a $(d-1)$-dimensional balanced $\mathbb{F}$-homology manifold with or without boundary and let $1 \leq \ell \leq \lfloor d/2 \rfloor$ be an integer. Assume further that the link of each codimension-$(2\ell - 1)$ face of Δ has the weak Lefschetz property.*

(i) If Δ has no boundary, then $\dfrac{h''_\ell(\Delta)}{\binom{d}{\ell}} \geq \dfrac{h''_{\ell-1}(\Delta)}{\binom{d}{\ell-1}} + \tilde{\beta}_{\ell-1}(\Delta)$.

(ii) If Δ is an ***orientable*** *$\mathbb{F}$-homology manifold with non-empty boundary, then* $\dfrac{h''_\ell(\Delta, \partial\Delta)}{\binom{d}{\ell}} \geq \dfrac{h''_{\ell-1}(\Delta, \partial\Delta)}{\binom{d}{\ell-1}} + \tilde{\beta}_{\ell-1}(\Delta, \partial\Delta)$.

Recall that by [64], the boundary complexes of all simplicial polytopes even have the strong Lefschetz property over $\mathbb{Q}$. Thus Theorem 5.1.3(i) holds for all balanced triangulations of $\mathbb{Q}$-homology manifolds with polytopal vertex links and all ℓ. Moreover, according to [40, Corollary 3.5] and [74], triangulations of 2-spheres have the weak Lefschetz property over *any* field $\mathbb{F}$. Hence, the case $\ell = 2$ of Theorem 5.1.3(i) is valid for *any* balanced $\mathbb{F}$-homology manifold without boundary. We prove the following stronger result.

Theorem 5.1.4. *Let Δ be a $(d-1)$-dimensional balanced simplicial complex. If Δ is an $\mathbb{F}$-homology manifold without boundary and $d \geq 4$, then*

$$\frac{h''_2(\Delta)}{\binom{d}{2}} \geq \frac{h''_1(\Delta)}{\binom{d}{1}} + \tilde{\beta}_1(\Delta).$$

Equivalently, $2h_2(\Delta) - (d-1)h_1(\Delta) \geq 4\binom{d}{2}(\tilde{\beta}_1(\Delta) - \tilde{\beta}_0(\Delta))$. Furthermore, if $d \geq 5$, then this inequality holds as equality if and only if each connected component of Δ is in the balanced Walkup class.

This result provides a balanced analog of [54, Theorem 5.2] (see also [47, Theorem 5.3]) and settles Conjecture 4.14 of [32] (see also [32, Remark 3.8]). It is worth mentioning that for $d-1 \geq 4$, the condition that Δ is in the balanced Walkup class is equivalent to all vertex links of Δ being stacked cross-polytopal spheres (see [32, Corollary 4.12]).

We also extend Theorem 5.1.4 to the class of Buchsbaum* simplicial complexes introduced by Athanasiadis and Welker [4] as well as discuss extensions of Theorem 5.1.3(i) to this generality, under an additional assumption that proper links of the Buchsbaum* complex in question satisfy a certain conjecture of Björner and Swartz.

Our proofs combine techniques from [30] along with recent results on Buchsbaum complexes, most notably those from [49]. In particular, we extend the exploitation of $\mathbb{N}^m$-gradings (rather than the usual $\mathbb{N}$-grading) to (certain quotients of) the canonical modules of Stanley–Reisner rings of balanced Buchsbaum complexes. For most of the proofs we need to work in the generality of $\mathbf{a}$-*balanced* simplicial complexes with $\mathbf{a} \in \mathbb{N}^m$. (As m varies, this class of complexes interpolates between the class of balanced simplicial complexes and that of all simplicial complexes.) We introduce the notions of *flag h'*- and *flag h''-vectors* for $\mathbf{a}$-balanced simplicial complexes as flag analogs of the usual h'- and h''-vectors, and develop basic properties of these vectors from the viewpoint of the Stanley–Reisner ring theory.

The layout of the rest of the chapter is as follows. In Section 2 we recall several notions pertaining to balanced simplicial complexes and their Stanley–Reisner rings. In Section 3 we introduce flag h'- and h''-vectors and develop their basic properties. Section 4 is devoted to the proof of both parts of Theorem 5.1.3 for the case of orientable homology manifolds with and without boundary. In Section 5, we review some known results on canonical modules as well as develop new techniques for studying Stanley–Reisner rings via the canonical modules of the links. In Section 6 we provide a proof of Theorem 5.1.3(i), and hence also of the inequality part of Theorem 5.1.4 for all (closed) homology manifolds. Section 7 settles the equality part of Theorem 5.1.4. We finish with some remarks and open problems in Section 8.

Initially, the main result of this paper was proved by the team of Juhnke-Kubitzke and Murai, and by the team of Novik and Sawaske. We decided to combine our efforts in a joint paper.

5.2 *Algebraic properties and combinatorics of simplicial complexes*

Here we review several notions and results that are used in the rest of the paper.

5.2.1 Combinatorics of simplicial complexes

We start with several definitions. An excellent reference to this material is Stanley's book [67]. Let V be a finite set. An (abstract) **simplicial complex** Δ on the vertex set V is a collection of subsets of V that is closed under inclusion and contains all singletons $\{v\}$ with $v \in V$. Throughout this paper, we assume that all simplicial complexes are finite. Elements of Δ are called **faces** of Δ and maximal faces (with respect to inclusion) are called **facets** of Δ. The **dimension** of a face $\sigma \in \Delta$ is its cardinality minus one, and the **dimension** of Δ is the maximal dimension of its faces. The 0-dimensional faces are called **vertices**, and we denote by $V(\Delta)$ the set of vertices of Δ. We say that a simplicial complex Δ is **pure** if all facets of Δ have the same dimension.

If $\dim \Delta = d-1$, then the f**-vector** of Δ is $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_{d-1}(\Delta))$, where $f_i(\Delta)$ denotes the number of i-dimensional faces of Δ, and the h**-vector** of Δ is $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$, where $h_i(\Delta)$ is defined by

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}(\Delta).$$

When P is a d-dimensional simplicial polytope, $f(P)$ and $h(P)$ refer to the f-vector and the h-vector of the boundary complex of P, respectively.

Given a fixed field $\mathbb{F}$, denote by $\tilde{\beta}_i(\Delta) = \dim_{\mathbb{F}} \tilde{H}_i(\Delta; \mathbb{F})$ the i^{th} reduced Betti number of Δ computed over $\mathbb{F}$. We define the h''**-numbers** of Δ by

$$h''_i(\Delta) = \begin{cases} h_i(\Delta) - \binom{d}{i} \sum_{j=1}^{i} (-1)^{i-j} \tilde{\beta}_{j-1}(\Delta), & \text{for } 0 \le i \le d-1, \\ \tilde{\beta}_{d-1}(\Delta), & \text{for } i = d. \end{cases}$$

The Betti numbers and the h''-numbers depend on $\mathbb{F}$, but $\mathbb{F}$ is usually understood from the context and is omitted from our notation.

A $(d-1)$-dimensional simplicial complex Δ is called **balanced** if its underlying graph is d-colorable, that is, there exists a map $\pi : V(\Delta) \to [d] = \{1, \ldots, d\}$ such that $\pi(v) \neq \pi(w)$ if $\{v, w\} \in \Delta$. As an example, consider the d-dimensional cross-polytope, i.e., the convex hull of the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_d, -\mathbf{e}_1, \ldots, -\mathbf{e}_d\}$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ is the standard basis of $\mathbb{R}^d$. Assigning vertices $\mathbf{e}_i$ and $-\mathbf{e}_i$ color i for all $1 \le i \le d$, makes the boundary complex of this polytope into a balanced sphere, denoted C_d^*.

As a generalization of balanced simplicial complexes, we now recall the definition of **a**-balanced simplicial complexes. Let $\mathbb{N}$ denote the set of non-negative integers, and as above let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denote the standard basis for $\mathbb{Z}^m$. For $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{N}^m$, let $|\mathbf{b}| = b_1 + \cdots + b_m$. When $\mathbf{b} = (b_1, \ldots, b_m)$, $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{N}^m$, we say that $\mathbf{b} \le \mathbf{c}$ if $b_i \le c_i$ for all i; in such a case, we define

$$\binom{\mathbf{c}}{\mathbf{b}} := \prod_{i=1}^{m} \binom{c_i}{b_i}.$$

Let $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$. An **a-balanced** simplicial complex is a tuple (Δ, π), where

(i) Δ is a simplicial complex of dimension $|\mathbf{a}| - 1$; and

(ii) π is a map from $V(\Delta)$ to $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ such that for every face $\sigma \in \Delta$, $\pi(\sigma) = \sum_{v\in\sigma} \pi(v) \leq \mathbf{a}$.

For convenience, we also say that Δ is **a-balanced** if (Δ, π) is $\mathbf{a}$-balanced for some π. In this paper, π will often be referred to as a **coloring** of Δ. Note that a $(d-1)$-dimensional simplicial complex Δ is $(1, 1, \ldots, 1)$-balanced, if and only if its 1-skeleton is d-colorable, which happens if and only if Δ is balanced. Also, any $(d-1)$-dimensional simplicial complex can be seen as a monochromatic balanced simplicial complex (that is, a (d)-balanced simplicial complex).

For an $\mathbf{a}$-balanced simplicial complex (Δ, π) and $\mathbf{b} \in \mathbb{N}^m$, we denote by $f_{\mathbf{b}}(\Delta, \pi)$ the number of faces $\sigma \in \Delta$ with $\pi(\sigma) = \mathbf{b}$, and we define

$$h_{\mathbf{b}}(\Delta, \pi) = \sum_{\mathbf{c}\leq\mathbf{b}} (-1)^{|\mathbf{b}|-|\mathbf{c}|} \binom{\mathbf{a}-\mathbf{c}}{\mathbf{b}-\mathbf{c}} f_{\mathbf{c}}(\Delta, \pi).$$

The vectors $(f_{\mathbf{b}}(\Delta, \pi) : \mathbf{b} \leq \mathbf{a})$ and $(h_{\mathbf{b}}(\Delta, \pi) : \mathbf{b} \leq \mathbf{a})$ are called the **flag *f*-vector** and the **flag *h*-vector** of (Δ, π), respectively. These vectors refine the usual f- and h-vectors, as it is easily seen that $f_{i-1}(\Delta) = \sum_{\mathbf{b}\leq\mathbf{a},\ |\mathbf{b}|=i} f_{\mathbf{b}}(\Delta, \pi)$ and $h_i(\Delta) = \sum_{\mathbf{b}\leq\mathbf{a},\ |\mathbf{b}|=i} h_{\mathbf{b}}(\Delta, \pi)$ for all $0 \leq i \leq d$.

5.2.2 Stanley–Reisner rings of balanced simplicial complexes

In this subsection, we recall some basic properties of Stanley–Reisner rings of $\mathbf{a}$-balanced simplicial complexes, originally proved by Stanley in [63]. In the following, let $\mathbb{F}$ be an infinite field and let Δ be a simplicial complex on vertex set $V = V(\Delta)$. Let A be the polynomial ring $\mathbb{F}[x_v : v \in V]$ and let $\mathfrak{m} = (x_v : v \in V)$ be the graded maximal ideal of A. For $\sigma \subseteq V$, we write $x_\sigma = \prod_{v\in\sigma} x_v$.

The **Stanley–Reisner ideal** I_Δ of Δ is the ideal of A defined by

$$I_\Delta = (x_\sigma \ : \ \sigma \subseteq V,\ \sigma \notin \Delta).$$

The **Stanley–Reisner ring** $\mathbb{F}[\Delta]$ of Δ (over $\mathbb{F}$) is the quotient ring

$$\mathbb{F}[\Delta] = A/I_\Delta.$$

If (Δ, π) is an $\mathbf{a}$-balanced simplicial complex (where $\mathbf{a} \in \mathbb{N}^m$), the rings A and $\mathbb{F}[\Delta]$ have the following $\mathbb{N}^m$-graded structure induced by the coloring π:

$$\deg x_v = \pi(v) \in \mathbb{N}^m \quad \text{for} \quad v \in V.$$

For an $\mathbb{N}^m$-graded A-module M and $\mathbf{b} \in \mathbb{N}^m$, we denote by $M_{\mathbf{b}}$ the submodule of M consisting of all homogeneous elements of degree $\mathbf{b}$, and we write $M(-\mathbf{b})$ for the module M with the grading defined by $M(-\mathbf{b})_{\mathbf{a}} = M_{\mathbf{a}-\mathbf{b}}$, where $\mathbf{a} \in \mathbb{N}^m$. We will also make use of the submodules

$$M_{\geq \mathbf{a}} := \bigoplus_{\mathbf{b} \geq \mathbf{a}} M_{\mathbf{b}}.$$

The ($\mathbb{N}^m$-graded) **Hilbert series** of M is the formal power series in variables $t_1, \ldots, t_m$ defined by

$$\mathrm{Hilb}(M; t_1, \ldots, t_m) := \sum_{\mathbf{b} \in \mathbb{N}^m} (\dim_{\mathbb{F}} M_{\mathbf{b}}) \mathbf{t}^{\mathbf{b}}, \quad \text{where } \mathbf{t}^{\mathbf{b}} = t_1^{b_1} \cdots t_m^{b_m}.$$

Theorem 5.2.1 (Stanley [63, Section 3]). *If (Δ, π) is an $\mathbf{a}$-balanced simplicial complex, then*

$$\mathrm{Hilb}(\mathbb{F}[\Delta]; t_1, \ldots, t_m) = \frac{\sum_{\mathbf{b} \leq \mathbf{a}} h_{\mathbf{b}}(\Delta, \pi) \mathbf{t}^{\mathbf{b}}}{(1-t_1)^{a_1} \cdots (1-t_m)^{a_m}}.$$

For a finitely generated graded A-module M of Krull dimension d, a **homogeneous system of parameters** for M is a sequence $\Theta = \theta_1, \ldots, \theta_d$ of d homogeneous elements in $\mathfrak{m}$ such that $\dim_{\mathbb{F}} M/\Theta M < \infty$. Such a system is called a **linear system of parameters** (or l.s.o.p. for short) if it consists of linear forms. By the Noether normalization lemma, if $\mathbb{F}$ is infinite, then an l.s.o.p. always exists. A system of parameters Θ for M is $\mathbb{N}^m$**-graded** if each θ_i is homogeneous w.r.t. the $\mathbb{N}^m$-grading of A. In this case, each θ_i is a linear combination of variables x_v of the same color (i.e., $\pi(v) = \pi(w)$ for any x_v and x_w that occur in θ_i with non-zero coefficients).

Theorem 5.2.2 (Stanley [63, Theorem 4.1]). *Let (Δ, π) be an $\mathbf{a}$-balanced simplicial complex. Then $\mathbb{F}[\Delta]$ admits an $\mathbb{N}^m$-graded l.s.o.p. Θ. Moreover, $(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\mathbf{b}} = 0$ for any $\mathbf{b} \in \mathbb{N}^m$ with $\mathbf{b} \not\leq \mathbf{a}$.*

We note that if $\Theta = \theta_1, \ldots, \theta_{|\mathbf{a}|}$ is an $\mathbb{N}^m$-graded l.s.o.p. for the Stanley–Reisner ring of an **a**-balanced simplicial complex, then Θ contains exactly a_i linear forms of degree $\mathbf{e}_i$ for each i (this follows, for instance, from [67, Lemma III.2.4]).

5.2.3 Buchsbaum and Cohen–Macaulay simplicial complexes

For a simplicial complex Δ and a face $\sigma \in \Delta$, the simplicial complexes

$$\mathrm{st}_\Delta(\sigma) = \{\tau \in \Delta \ : \ \tau \cup \sigma \in \Delta\} \ \text{ and } \ \mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta \ : \ \tau \cup \sigma \in \Delta, \ \tau \cap \sigma = \emptyset\}$$

are called the **star** and the **link** of σ in Δ, respectively. We say that the link of σ, $\mathrm{lk}_\Delta(\sigma)$, is **proper** if $\sigma \neq \emptyset$. If (Δ, π) is a pure **a**-balanced simplicial complex, then so is $(\mathrm{st}_\Delta(\sigma), \pi)$; furthermore, $(\mathrm{lk}_\Delta(\sigma), \pi)$ is an $(\mathbf{a} - \pi(\sigma))$-balanced simplicial complex. (Here, π is identified with its restriction to the vertex sets of $\mathrm{st}_\Delta(\sigma)$ and $\mathrm{lk}_\Delta(\sigma)$, respectively.)

Recall that a finitely generated graded A-module M of Krull dimension d is **Buchsbaum** if for every homogeneous system of parameters $\Theta = \theta_1, \ldots, \theta_d$ of M,

$$(\theta_1, \ldots, \theta_{i-1})M :_M \theta_i = (\theta_1, \ldots, \theta_{i-1})M :_M \mathfrak{m} \quad \text{for all } 1 \leq i \leq d.$$

If, additionally, the above colon module is zero for all $1 \leq i \leq d$, then M is said to be **Cohen–Macaulay**.

We call a simplicial complex Δ **Buchsbaum** or **Cohen–Macaulay** (over $\mathbb{F}$) if $\mathbb{F}[\Delta]$ is Buchsbaum or Cohen–Macaulay considered as an A-module. It is known that a simplicial complex Δ of dimension $d-1$ is Cohen–Macaulay over $\mathbb{F}$ if and only if, for every face $\sigma \in \Delta$ (including the empty face), $\tilde{\beta}_i(\mathrm{lk}_\Delta(\sigma)) = 0$ for all $i \neq d-1-|\sigma|$ (see [67, Corollary II.4.2]). Similarly, a simplicial complex is Buchsbaum over $\mathbb{F}$ if and only if it is pure and all of its vertex links are Cohen–Macaulay over $\mathbb{F}$ (see [67, Theorem II.8.1]).

A pure $(d-1)$-dimensional simplicial complex is an **$\mathbb{F}$-homology manifold without boundary** (or a **closed $\mathbb{F}$-homology manifold**) if every proper link of Δ, $\mathrm{lk}_\Delta(\sigma)$, has the homology of a $(d-1-|\sigma|)$-dimensional sphere (over $\mathbb{F}$). Similarly, a pure $(d-1)$-dimensional simplicial complex Δ is an **$\mathbb{F}$-homology manifold with boundary** if (i) every proper link of Δ, $\mathrm{lk}_\Delta(\sigma)$, has the homology of a $(d-1-|\sigma|)$-dimensional ball or a sphere (over $\mathbb{F}$), and (ii) the boundary complex of Δ, i.e.,

$$\partial(\Delta) = \{\sigma \in \Delta \ : \ \tilde{H}_*(\mathrm{lk}_\Delta(\sigma); \mathbb{F}) = 0\} \cup \{\emptyset\},$$

is a $(d-2)$-dimensional $\mathbb{F}$-homology manifold without boundary. An **$\mathbb{F}$-homology $(d-1)$-sphere** is an $\mathbb{F}$-homology manifold without boundary that has the same homology as the $(d-1)$-dimensional sphere, and an **$\mathbb{F}$-homology $(d-1)$-ball** is an $\mathbb{F}$-homology manifold with boundary whose homology is trivial and whose boundary complex is an $\mathbb{F}$-homology $(d-2)$-sphere. Thus, every proper link of an $\mathbb{F}$-homology manifold with or without boundary is either an $\mathbb{F}$-homology sphere or an $\mathbb{F}$-homology ball. In particular, if Δ is an $\mathbb{F}$-homology manifold with or without boundary, then Δ is Buchsbaum over $\mathbb{F}$.

We will often say that (Δ, π) is Cohen–Macaulay or Buchsbaum or an $\mathbb{F}$-homology manifold if Δ has that property.

5.2.4 Weak Lefschetz property

Let Δ be a $(d-1)$-dimensional Cohen–Macaulay simplicial complex. We say that Δ has the **weak Lefschetz Property** (or **WLP**) over $\mathbb{F}$ if there is an l.s.o.p. Θ for $\mathbb{F}[\Delta]$ and a linear form ω such that the multiplication map

$$\cdot\omega : (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\lfloor \frac{d}{2} \rfloor} \to (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\lfloor \frac{d}{2} \rfloor + 1}$$

is surjective. Similarly, we say that Δ has the **dual WLP** if there is an l.s.o.p. Θ for $\mathbb{F}[\Delta]$ and a linear form ω such that the multiplication map

$$\cdot\omega : (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\lfloor \frac{d+1}{2} \rfloor - 1} \to (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\lfloor \frac{d+1}{2} \rfloor}$$

is injective. While, in general, the above definitions differ from the usual definitions of the weak Lefschetz property (see [25]), for homology spheres, our WLP coincides with the usual definition of the WLP; furthermore, in this case the WLP and the dual WLP are equivalent to each other (see Remark 5.5.4).

The boundary complex of any simplicial d-polytope has the WLP over $\mathbb{Q}$, and so does any triangulated $(d-1)$-ball that is a subcomplex of the boundary complex of a simplicial d-polytope ([64] and [61, Lemma 2.2]). It was repeatedly conjectured that all homology spheres and balls have the WLP. While this conjecture is wide open at present, the following special case is well-known to be true.

Lemma 5.2.3. *All $\mathbb{F}$-homology 2-spheres and all $\mathbb{F}$-homology 2-balls have the WLP over $\mathbb{F}$.*

Indeed, in dimension 2, the class of $\mathbb{F}$-homology spheres coincides with the class of triangulations of the (topological) 2-sphere. The fact that triangulated 2-spheres have the WLP over any field follows from [40, Corollary 3.5] and [74]. For $\mathbb{F}$-homology 2-balls, the lemma is then derived exactly as in [61, Lemma 2.2].

5.3 Flag h'- and h''-vectors

Over the last few decades, several refinements and modifications of h-vectors of simplicial complexes have been introduced and studied. On one hand, already in 1979, Stanley [63] introduced **a**-balanced simplicial complexes together with their flag h-vectors as a refinement of the classical h-vectors. Subsequently, these vectors have played an important role in the study of f-vectors of simplicial polytopes and simplicial complexes. On the other hand, in order to study face numbers of homology manifolds, one often considers their h'- and h''-vectors as certain modifications of the classical h-vector (cf. Section 5.2.1; also see [33, 70] for various applications of these combinatorial invariants). Here we (i) combine these two approaches — this results in the notions of **flag** h'**-** and **flag** h''**-vectors** of balanced simplicial complexes, and (ii) initiate the study of basic properties of these vectors. Most results in this section are natural extensions of known results on h'-, h''- and flag h-vectors, and so some details of proofs are omitted.

Let (Δ, π) be an **a**-balanced simplicial complex. We define the **flag** h'**-vector** $h'(\Delta, \pi) =$

$(h'_{\mathbf{b}}(\Delta,\pi)\ :\ \mathbf{b}\le\mathbf{a})$ of (Δ,π) by

$$h'_{\mathbf{b}}(\Delta,\pi) = h_{\mathbf{b}}(\Delta,\pi) - \binom{\mathbf{a}}{\mathbf{b}}\left(\sum_{j=1}^{|\mathbf{b}|-1}(-1)^{|\mathbf{b}|-j}\tilde{\beta}_{j-1}(\Delta)\right).$$

Schenzel [60] proved that for a $(d-1)$-dimensional Buchsbaum complex Δ and an integer $0\le j\le d$, $\dim_{\mathbb{F}}(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_j = h_j(\Delta) - \binom{d}{j}\sum_{i=1}^{j-1}(-1)^{j-i}\tilde{\beta}_{i-1}(\Delta)$ (see also [67, Theorem II.8.2]). The following theorem establishes a flag analog of Schenzel's formula.

Theorem 5.3.1. *Let (Δ,π) be an $\mathbf{a}$-balanced simplicial complex and let $\Theta=\theta_1,\dots,\theta_{|\mathbf{a}|}$ be an $\mathbb{N}^m$-graded l.s.o.p. for $\mathbb{F}[\Delta]$. If Δ is Buchsbaum, then*

$$\mathrm{Hilb}(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta];t_1,\dots,t_m) = \sum_{\mathbf{b}\le\mathbf{a}} h'_{\mathbf{b}}(\Delta,\pi)\mathbf{t}^{\mathbf{b}}.$$

Proof. We only sketch the proof since it is essentially the same as the proof of [60, Theorem 4.3]. For a finitely generated graded A-module M, we denote by $H^i_{\mathfrak{m}}(M)$ the i^{th} local cohomology module of M. Let $R=\mathbb{F}[\Delta]$ and, for $S\subseteq\{1,2,\dots,|\mathbf{a}|\}$, let $\boldsymbol{\delta}_S := \sum_{i\in S}\deg\theta_i$.

Since R is a Buchsbaum ring, we have the following exact sequences

$$\begin{aligned} 0 \ &\longrightarrow H^0_{\mathfrak{m}}(R/(\theta_1,\dots,\theta_{j-1})R)(-\deg\theta_j) \longrightarrow (R/(\theta_1,\dots,\theta_{j-1})R)(-\deg\theta_j) \\ &\xrightarrow{\times\theta_j} R/(\theta_1,\dots,\theta_{j-1})R \longrightarrow R/(\theta_1,\dots,\theta_j) \longrightarrow 0 \end{aligned} \tag{5.1}$$

for $1\le j\le|\mathbf{a}|$. Moreover, it follows from [68, Lemma II.4.14′(i′)] that

$$H^0_{\mathfrak{m}}(R/(\theta_1,\dots,\theta_{j-1})R) \cong \bigoplus_{S\subseteq[j-1]} H^{|S|}_{\mathfrak{m}}(R)(-\boldsymbol{\delta}_S). \tag{5.2}$$

(While the statement given in [68] assumes the $\mathbb{Z}$-grading, the proof carries verbatim to the $\mathbb{N}^m$-graded setting.) Finally, since $R=\mathbb{F}[\Delta]$ is Buchsbaum,

$$H^i_{\mathfrak{m}}(R) = (H^i_{\mathfrak{m}}(R))_0 \cong \tilde{H}_{i-1}(\Delta;\mathbb{F}) \quad \text{for } i\le\dim\Delta \tag{5.3}$$

(see [68, Corollary II.4.13 and Lemma II.2.5(ii)]). Combining Theorem 5.2.1 with (5.1), (5.2), and (5.3), we inductively obtain that

$$\mathrm{Hilb}(R/(\theta_1,\dots,\theta_j)R;t_1,\dots,t_m) =$$
$$\frac{\sum_{\mathbf{b}\le\mathbf{a}} h_{\mathbf{b}}(\Delta,\pi)\mathbf{t}^{\mathbf{b}}}{\prod_{i=j+1}^{|\mathbf{a}|}(1-\mathbf{t}^{\deg\theta_i})} - \sum_{\mathbf{b}\le\boldsymbol{\delta}_{[j]}}\binom{\boldsymbol{\delta}_{[j]}}{\mathbf{b}}\left(\sum_{k=1}^{|\mathbf{b}|-1}(-1)^{|\mathbf{b}|-k}\tilde{\beta}_{k-1}(\Delta)\right)\mathbf{t}^{\mathbf{b}}$$

for $1 \le j \le |\mathbf{a}|$. This proves the desired equation. $\square$

We define the **flag h''-vector** $h''(\Delta, \pi) = (h''_{\mathbf{b}}(\Delta, \pi) \;:\; \mathbf{b} \le \mathbf{a})$ of an $\mathbf{a}$-balanced simplicial complex (Δ, π) by

$$h''_{\mathbf{b}}(\Delta, \pi) = \begin{cases} h_{\mathbf{b}}(\Delta, \pi) - \binom{\mathbf{a}}{\mathbf{b}}\left(\sum_{j=1}^{|\mathbf{b}|}(-1)^{|\mathbf{b}|-j}\tilde{\beta}_{j-1}(\Delta)\right), & \text{if } \mathbf{b} \neq \mathbf{a}, \\ \tilde{\beta}_{|\mathbf{a}|-1}(\Delta), & \text{if } \mathbf{b} = \mathbf{a}. \end{cases}$$

We will see that the flag h''-vector is intimately related to the following ideal defined by Goto [19]. This ideal plays a crucial role in the paper. For a finitely generated graded A-module M of Krull dimension d and an l.s.o.p. $\Theta = \theta_1, \ldots, \theta_d$ for M, let

$$\Sigma(\Theta; M) := \Theta M + \sum_{i=1}^{d}\Big((\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_d)M :_M \theta_i\Big).$$

Here, $\hat{\theta}_i$ indicates that θ_i is omitted from Θ. Note that if M and Θ are $\mathbb{N}^m$-graded, then so are $M/\Theta M$ and $M/\Sigma(\Theta; M)$.

The following property was essentially proved by Goto in the setting of local rings. The proof for the $\mathbb{N}$-graded case can be found in [49, Theorem 2.3] and it extends easily to the $\mathbb{N}^m$-graded setting.

Theorem 5.3.2 (Essentially Goto [19])**.** *Let M be a finitely generated $\mathbb{N}^m$-graded Buchsbaum A-module of Krull dimension d and let $\Theta = \theta_1, \ldots, \theta_d$ be an $\mathbb{N}^m$-graded l.s.o.p. for M. Then there is an isomorphism of A-modules*

$$\Sigma(\Theta; M)/\Theta M \cong \bigoplus_{S \subsetneq [d]} H_{\mathfrak{m}}^{|S|}(M)(-\textstyle\sum_{k \in S} \deg \theta_k).$$

In particular, $\Sigma(\Theta; M)/\Theta M$ is contained in the socle of $M/\Theta M$. In other words, $\mathfrak{m} \cdot (\Sigma(\Theta; M)/\Theta M) = 0$.

Note that the latter statement of the previous theorem follows from the fact that $\mathfrak{m} \cdot H_{\mathfrak{m}}^i(M) = 0$ holds for any Buchsbaum A-module M and $i < \dim M$ (see [68, Proposition I.2.1 (iii)]). Since $H_{\mathfrak{m}}^i(\mathbb{F}[\Delta]) = (H_{\mathfrak{m}}^i(\mathbb{F}[\Delta]))_0 \cong \tilde{H}_{i-1}(\Delta; \mathbb{F})$ for $i \le \dim \Delta$ when Δ is Buchsbaum, Theorem 5.3.2 implies the following result.

Corollary 5.3.3. *Let (Δ, π) be an $\mathbf{a}$-balanced simplicial complex and let Θ be an $\mathbb{N}^m$-graded l.s.o.p. for $\mathbb{F}[\Delta]$. If Δ is Buchsbaum over $\mathbb{F}$, then*

$$\dim_{\mathbb{F}} \left(\Sigma(\Theta; \mathbb{F}[\Delta])/\Theta\mathbb{F}[\Delta]\right)_{\mathbf{b}} = \binom{\mathbf{a}}{\mathbf{b}} \tilde{\beta}_{|\mathbf{b}|-1}(\Delta) \quad \text{for any } \mathbf{b} \lneq \mathbf{a}.$$

Observe that for $\mathbf{b} \neq \mathbf{a}$, $h''_{\mathbf{b}}(\Delta, \pi) = h'_{\mathbf{b}}(\Delta, \pi) - \binom{\mathbf{a}}{\mathbf{b}}\tilde{\beta}_{|\mathbf{b}|-1}(\Delta)$ and that $h''_{\mathbf{a}}(\Delta) = h'_{\mathbf{a}}(\Delta)$. Since $\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]$ and $\mathbb{F}[\Delta]/\Sigma(\Theta; \mathbb{F}[\Delta]) \oplus \Sigma(\Theta; \mathbb{F}[\Delta])/\Theta\mathbb{F}[\Delta]$ are isomorphic as graded vector spaces, Theorem 5.3.1 and Corollary 5.3.3 lead to the following algebraic interpretation of flag h''-vectors.

Theorem 5.3.4. *Let (Δ, π) be an $\mathbf{a}$-balanced simplicial complex and let Θ be an $\mathbb{N}^m$-graded l.s.o.p. for $\mathbb{F}[\Delta]$. If Δ is Buchsbaum over $\mathbb{F}$, then*

$$\mathrm{Hilb}(\mathbb{F}[\Delta]/\Sigma(\Theta; \mathbb{F}[\Delta]); t_1, \ldots, t_m) = \sum_{\mathbf{b} \leq \mathbf{a}} h''_{\mathbf{b}}(\Delta, \pi)\mathbf{t}^{\mathbf{b}}.$$

Remark 5.3.5. The flag h'- and h''-numbers refine the usual h'- and h''-numbers:

$$h'_i(\Delta) = \sum_{\mathbf{b} \leq \mathbf{a}, |\mathbf{b}|=i} h'_{\mathbf{b}}(\Delta, \pi) \quad \text{and} \quad h''_i(\Delta) = \sum_{\mathbf{b} \leq \mathbf{a}, |\mathbf{b}|=i} h''_{\mathbf{b}}(\Delta, \pi).$$

Remark 5.3.6. All results on flag h'- and h''-vectors in this section are natural extensions of known results on the usual h'- and h''-vectors. In the $\mathbb{N}$-graded setting, the formula for h'-vectors in Theorem 5.3.1 was proved in [60], and the formula for h''-vectors in Theorem 5.3.4 was given in [49]. When Δ is a Cohen–Macaulay simplicial complex, the flag h'- and h''-vectors coincide with the usual flag h-vectors. In this case the formula for the Hilbert series in Theorem 5.3.1 is due to Stanley [63].

The results in this section continue to hold in the generality of Stanley–Reisner modules of relative simplicial complexes. Here we quickly review some relevant notions.

For a simplicial complex Δ with the vertex set V and a subcomplex Γ of Δ, the A-module

$$\mathbb{F}[\Delta, \Gamma] = I_\Gamma / I_\Delta$$

is called the **Stanley–Reisner module** of the pair (Δ, Γ), where we consider $I_\Gamma = (x_\sigma \,:\, \sigma \subseteq V, \sigma \notin \Gamma)$ as an ideal of A. The faces of (Δ, Γ) are the elements of $\Delta \setminus \Gamma$. With this convention in hand, we define the f-, h-, h'- and h''-vector of the pair (Δ, Γ) as well as the flag f-, h-, h'- and h''-vectors in the same way as for a single simplicial complex. In particular, for an $\mathbf{a}$-balanced simplicial complex (Δ, π) and its subcomplex Γ, $f_{\mathbf{b}}(\Delta, \Gamma, \pi)$ is the number of faces $\sigma \in \Delta \setminus \Gamma$ with $\pi(\sigma) = \mathbf{b}$ and

$$h''_{\mathbf{b}}(\Delta, \Gamma, \pi) = \begin{cases} h_{\mathbf{b}}(\Delta, \Gamma, \pi) - \binom{\mathbf{a}}{\mathbf{b}}\left(\sum_{j=1}^{|\mathbf{b}|}(-1)^{|\mathbf{b}|-j}\tilde{\beta}_{j-1}(\Delta, \Gamma)\right), & \text{if } \mathbf{b} \neq \mathbf{a}, \\ \tilde{\beta}_{|\mathbf{a}|-1}(\Delta, \Gamma), & \text{if } \mathbf{b} = \mathbf{a}, \end{cases}$$

where $\tilde{\beta}_i(\Delta, \Gamma) := \dim_{\mathbb{F}} \tilde{H}_i(\Delta, \Gamma; \mathbb{F})$.

Theorem 5.3.1, Corollary 5.3.3, and Theorem 5.3.4, in fact, hold for all Buchsbaum Stanley–Reisner modules. (We omit the proofs since they are identical to the proofs above, except that the notation becomes somewhat more cumbersome). Specifically, if Δ is an $\mathbb{F}$-homology manifold with boundary, then $\mathbb{F}[\Delta, \partial\Delta]$ is Buchsbaum, and hence the statements of Theorem 5.3.1, Corollary 5.3.3, and Theorem 5.3.4 continue to hold in this setting but with Δ replaced throughout by $(\Delta, \partial\Delta)$.

5.4 *Proofs in the orientable case*

The purpose of this section is to prove Theorem 5.1.3 for *orientable* homology manifolds with and without boundary, see Theorem 5.4.4. If Δ is a homology manifold without boundary, we will often identify Δ with the pair $(\Delta, \partial\Delta)$ where we let $\partial\Delta = \emptyset$. We say that an $\mathbb{F}$-homology manifold Δ with or without boundary is **orientable** if the top Betti number of $(\Delta, \partial\Delta)$ computed over $\mathbb{F}$ is equal to the number of connected components of Δ.

Let (Δ, π) be a balanced $(d-1)$-dimensional $\mathbb{F}$-homology manifold. If Δ has no boundary, then the link of each codimension-1 face of Δ consists of two vertices. Thus, for each color i, in each connected component there exist at least two vertices of color i, so that $f_0(\Delta) \geq 2d(1+\tilde{\beta}_0(\Delta))$. Hence $h''_1(\Delta) = f_0(\Delta) - d - d\tilde{\beta}_0(\Delta) \geq d(1+\tilde{\beta}_0(\Delta))$. Since $h''_0(\Delta) = 1$, the inequality $\frac{h''_1(\Delta)}{d} \geq \frac{h''_0(\Delta)}{1} + \tilde{\beta}_0(\Delta)$ follows. Similarly, if Δ has non-empty boundary, then

$h''_0(\Delta, \partial\Delta) = 0$, $\tilde{\beta}_0(\Delta, \partial\Delta)$ equals the number of connected components of Δ that have no boundary, and each component with non-empty boundary has at least d vertices. The same computation as above then shows that $h''_1(\Delta, \partial\Delta) \geq d\tilde{\beta}_0(\Delta, \partial\Delta)$. We conclude that Theorem 5.1.3 holds for $\ell = 1$, and from now on, assume that $1 < \ell \leq \lfloor d/2 \rfloor$.

Moreover, it suffices to prove Theorem 5.1.3 for connected $\mathbb{F}$-homology manifolds. Indeed, a straightforward computation shows that if Δ is disconnected with connected components $\Delta^1, \dots, \Delta^s$, then $h''_j(\Delta, \partial\Delta) = \sum_{k=1}^{s} h''_j(\Delta^k, \partial\Delta^k)$ for all $j \geq 1$; in addition, $\tilde{\beta}_j(\Delta, \partial\Delta) = \sum_{k=1}^{s} \tilde{\beta}_j(\Delta^k, \partial\Delta^k)$ for $j \geq 1$. (Furthermore, if Δ is orientable, then so is each connected component of Δ.) Therefore, if each connected component of Δ satisfies the inequality in Theorem 5.1.3, then so does Δ.

For a graded A-module N, let $N^\vee$ denote the Matlis dual of N. One crucial ingredient in the proof of Theorem 5.1.3 is the following result established in [49, Corollary 1.4]. Although, in [49], only the $\mathbb{N}$-graded case is treated, the same proof works in the $\mathbb{N}^m$-graded setting.

Theorem 5.4.1 (Murai–Novik–Yoshida)**.** *Let* $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}^m$*, let* (Δ, π) *be an* $\mathbf{a}$*-balanced, connected, orientable* $\mathbb{F}$*-homology manifold with or without boundary, and let* Θ *be an* $\mathbb{N}^m$*-graded l.s.o.p. for* $\mathbb{F}[\Delta]$*. Then*

$$\big(\mathbb{F}[\Delta, \partial\Delta]/\Sigma(\Theta; \mathbb{F}[\Delta, \partial\Delta])\big) \cong \big(\mathbb{F}[\Delta]/\Sigma(\Theta; \mathbb{F}[\Delta])\big)^\vee(-\mathbf{a}).$$

In particular, $h''_{\mathbf{b}}(\Delta, \partial\Delta, \pi) = h''_{\mathbf{a}-\mathbf{b}}(\Delta, \pi)$ *for all* $\mathbf{b} \in \mathbb{N}^m$ *with* $\mathbf{b} \leq \mathbf{a}$*.*

The proof of Theorem 5.1.3 will essentially follow from the next proposition.

Proposition 5.4.2. *Let* $1 < \ell \leq \lfloor d/2 \rfloor$*,* $\mathbf{a} = (2\ell - 1, 1, \dots, 1) \in \mathbb{N}^{d-2\ell+2}$*, and* $\mathbf{b} = \mathbf{a} - (2\ell - 1)\mathbf{e}_1$*. Let* (Δ, π) *be an* $\mathbf{a}$*-balanced, connected, orientable* $\mathbb{F}$*-homology manifold with or without boundary and suppose that for every face* $\sigma \in \Delta$ *with* $\pi(\sigma) = \mathbf{b}$*, the link of* σ *in* Δ *has the WLP. Then*

$$h''_{\ell\mathbf{e}_1}(\Delta, \partial\Delta, \pi) - h''_{(\ell-1)\mathbf{e}_1}(\Delta, \partial\Delta, \pi) \geq \binom{2\ell - 1}{\ell} \tilde{\beta}_{\ell-1}(\Delta, \partial\Delta).$$

Proof. Let $\Theta = \theta_1, \ldots, \theta_d$ be an $\mathbb{N}^{d-2\ell+2}$-graded l.s.o.p. for $\mathbb{F}[\Delta]$ and let $\Theta' := \{\theta_i \ : \ \deg \theta_i = \mathbf{e}_1\}$. Consider the following modules:

$$L = \bigoplus_{\sigma \in \Delta, \pi(\sigma) = \mathbf{b}} \mathbb{F}[\mathrm{lk}_\Delta(\sigma)]/\Theta'\mathbb{F}[\mathrm{lk}_\Delta(\sigma)] \ \text{ and } \ M = (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\geq \mathbf{b}}.$$

In analogy to [30, Lemma 2.3 (i)], there exists a surjection $\psi : L \to M(\mathbf{b})$. Thus for any linear form $\omega \in \mathbb{F}[\Delta]$ with $\deg \omega = \mathbf{e}_1$, there is the following commutative diagram:

$$\begin{array}{ccl} L_{\ell\mathbf{e}_1} & \xrightarrow{\psi} & M(\mathbf{b})_{\ell\mathbf{e}_1} = (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\ell\mathbf{e}_1+\mathbf{b}} \\ \cdot\omega \uparrow & & \uparrow \cdot\omega \\ L_{(\ell-1)\mathbf{e}_1} & \xrightarrow{\psi} & M(\mathbf{b})_{(\ell-1)\mathbf{e}_1} = (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{(\ell-1)\mathbf{e}_1+\mathbf{b}}. \end{array}$$

Note that all links, $\mathrm{lk}_\Delta(\sigma)$, in the above diagram are monochromatic (indeed they are $(2\ell-1)$-balanced). Since $\mathrm{lk}_\Delta(\sigma)$ has the WLP for all $\sigma \in \Delta$ with $\pi(\sigma) = \mathbf{b}$ by assumption, it follows that the left multiplication map $\cdot\omega$ is surjective for a generic choice of Θ and ω. This fact and the surjectivity of the horizontal maps ψ implies that the multiplication map $\cdot\omega$ on the right is also surjective. Furthermore, since $\Sigma(\Theta; \mathbb{F}[\Delta])/\Theta\mathbb{F}[\Delta]$ is contained in the socle of $\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]$ (see Theorem 5.3.2), we infer that the map

$$\cdot\omega : (\mathbb{F}[\Delta]/\Sigma(\Theta; \mathbb{F}[\Delta]))_{(\ell-1)\mathbf{e}_1+\mathbf{b}} \to (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\ell\mathbf{e}_1+\mathbf{b}}$$

is well-defined and surjective. Consequently,

$$\dim_{\mathbb{F}} (\mathbb{F}[\Delta]/\Sigma(\Theta; \mathbb{F}[\Delta]))_{(\ell-1)\mathbf{e}_1+\mathbf{b}} \geq \dim_{\mathbb{F}} (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\ell\mathbf{e}_1+\mathbf{b}}. \tag{5.4}$$

To finish the proof, we compute both sides of (5.4). Theorems 5.3.4 and 5.4.1 imply

$$\dim_{\mathbb{F}} (\mathbb{F}[\Delta]/\Sigma(\Theta; \mathbb{F}[\Delta]))_{(\ell-1)\mathbf{e}_1+\mathbf{b}} = h''_{(\ell-1)\mathbf{e}_1+\mathbf{b}}(\Delta, \pi) = h''_{\ell\mathbf{e}_1}(\Delta, \partial\Delta, \pi), \tag{5.5}$$

while by Theorem 5.3.1,

$$\begin{aligned} \dim_{\mathbb{F}} (\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta])_{\ell\mathbf{e}_1+\mathbf{b}} =& h'_{\ell\mathbf{e}_1+\mathbf{b}}(\Delta, \pi) \\ =& h''_{\ell\mathbf{e}_1+\mathbf{b}}(\Delta, \pi) + \binom{2\ell-1}{\ell-1}\tilde{\beta}_{d-\ell}(\Delta) \\ =& h''_{(\ell-1)\mathbf{e}_1}(\Delta, \partial\Delta, \pi) + \binom{2\ell-1}{\ell-1}\tilde{\beta}_{\ell-1}(\Delta, \partial\Delta). \end{aligned} \tag{5.6}$$

Here the last step follows from Theorem 5.4.1 and Poincaré–Lefschetz duality asserting that $\tilde{\beta}_{\ell-1}(\Delta, \partial\Delta) = \tilde{\beta}_{d-\ell}(\Delta)$. Substituting (5.5) and (5.6) in (5.4) yields the result. □

For a balanced simplicial complex (Δ, π) of dimension $d-1$, a subcomplex Γ of Δ, and $S \subseteq [d]$, define

$$h_S(\Delta, \Gamma, \pi) = h_{\mathbf{e}_S}(\Delta, \Gamma, \pi) \text{ and } h''_S(\Delta, \Gamma, \pi) = h''_{\mathbf{e}_S}(\Delta, \Gamma, \pi).$$

We also define the **normalized h''_i-number** of the pair (Δ, Γ), $\overline{h}''_i(\Delta, \Gamma)$, by

$$\overline{h}''_i(\Delta, \Gamma) = \frac{h''_i(\Delta, \Gamma)}{\binom{d}{i}} = \frac{\sum_{S \subseteq [d], |S|=i} h''_S(\Delta, \Gamma, \pi)}{\binom{d}{i}}.$$

The following lemma is an easy consequence of [30, Lemma 3.6]; we omit the proof.

Lemma 5.4.3. *Let (Δ, π) be a $(d-1)$-dimensional balanced simplicial complex and Γ a subcomplex of Δ. Then for any $1 \leq \ell \leq d/2$,*

$$\overline{h}''_\ell(\Delta, \Gamma) - \overline{h}''_{\ell-1}(\Delta, \Gamma) =$$
$$\frac{1}{\binom{2\ell-1}{\ell}\binom{d}{2\ell-1}} \left(\sum_{S \subseteq [d], |S|=2\ell-1} \left(\sum_{T \subseteq S, |T|=\ell} h''_T(\Delta, \Gamma, \pi) - \sum_{T \subseteq S, |T|=\ell-1} h''_T(\Delta, \Gamma, \pi) \right) \right).$$

We are now in a position to prove the main result of this section.

Theorem 5.4.4. *Let $1 < \ell \leq \lfloor d/2 \rfloor$. Let (Δ, π) be a balanced orientable $\mathbb{F}$-homology manifold with or without boundary of dimension $d-1$. Suppose that for all faces $\sigma \in \Delta$ of codimension-$(2\ell-1)$, the link of σ has the WLP. Then*

(i) for any $S \subseteq [d]$ with $|S| = 2\ell - 1$,

$$\sum_{T \subseteq S, |T|=\ell} h''_T(\Delta, \partial\Delta, \pi) - \sum_{T \subseteq S, |T|=\ell-1} h''_T(\Delta, \partial\Delta, \pi) \geq \binom{2\ell-1}{\ell} \tilde{\beta}_{\ell-1}(\Delta, \partial\Delta).$$

(ii) Consequently, $\overline{h}''_\ell(\Delta, \partial\Delta) - \overline{h}''_{\ell-1}(\Delta, \partial\Delta) \geq \tilde{\beta}_{\ell-1}(\Delta, \partial\Delta)$.

Proof. First note that part (ii) of the statement is an immediate consequence of part (i) and Lemma 5.4.3. To prove part (i), we may assume that Δ is connected since for $T \neq \emptyset$, the number $h''_T(\Delta, \partial\Delta, \pi)$ is the sum of the corresponding statistics of connected components of Δ. Furthermore, by relabeling the vertices, we may assume that $S = \{1, d-2\ell+3, d-2\ell+4, \dots, d\}$. Define $\tilde{\pi} : V(\Delta) \to \{\mathbf{e}_1, \dots, \mathbf{e}_{d-2\ell+1}\}$ by $\tilde{\pi}(v) = \pi(v)$ if $\pi(v) \notin \{\mathbf{e}_i \ : \ i \in S\}$ and $\tilde{\pi}(v) = \mathbf{e}_1$ if $\pi(v) \in \{\mathbf{e}_i \ : \ i \in S\}$. Then $(\Delta, \tilde{\pi})$ is $(2\ell-1, 1, \dots, 1)$-balanced. As

$$h''_{\ell\mathbf{e}_1}(\Delta, \partial\Delta, \tilde{\pi}) = \sum_{T \subseteq S, |T|=\ell} h''_T(\Delta, \partial\Delta, \pi)$$

and

$$h''_{(\ell-1)\mathbf{e}_1}(\Delta, \partial\Delta, \tilde{\pi}) = \sum_{T \subseteq S, |T|=\ell-1} h''_T(\Delta, \partial\Delta, \pi),$$

the claim follows from Proposition 5.4.2. □

Since all proper links of a homology manifold with or without boundary are homology spheres or homology balls, we infer from Lemma 5.2.3 the following result.

Corollary 5.4.5. *Let Δ be a balanced orientable $\mathbb{F}$-homology manifold with non-empty boundary. If Δ has dimension ≥ 3, then $\overline{h}''_2(\Delta, \partial\Delta) - \overline{h}''_1(\Delta, \partial\Delta) \geq \tilde{\beta}_1(\Delta, \partial\Delta)$.*

5.5 Canonical modules

Our proof of Theorem 5.1.3(i) for non-orientable homology manifolds relies on canonical modules. This requires a few auxiliary results on canonical modules, some of which are discussed in this section.

Recall that if M is a finitely generated graded A-module of Krull dimension d, then the **canonical module** of M is the module

$$\Omega(M) := H^d_{\mathfrak{m}}(M)^\vee.$$

In particular, for an $\mathbf{a}$-balanced simplicial complex (Δ, π) with $\mathbf{a} \in \mathbb{N}^m$, the canonical module of $\mathbb{F}[\Delta]$ is $\mathbb{N}^m$-graded.

We start by reviewing some dualities that are exhibited by canonical modules of Buchsbaum rings. The following is an algebraic generalization of Theorem 5.4.1 above, proved in [49, Theorem 1.3].

Theorem 5.5.1 (Murai–Novik–Yoshida). *Let (Δ, π) be an $\mathbf{a}$-balanced Buchsbaum simplicial complex with $|\mathbf{a}| \geq 2$. Let $\Theta = \theta_1, \ldots, \theta_{|\mathbf{a}|} \in \mathbb{F}[\Delta]$ be an $\mathbb{N}^m$-graded l.s.o.p. for $\mathbb{F}[\Delta]$. If Δ is connected, then*

$$\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta; \Omega(\mathbb{F}[\Delta])) \cong \big(\mathbb{F}[\Delta]/\Sigma(\Theta; \mathbb{F}[\Delta])\big)^{\vee}(-\mathbf{a}).$$

We note that, in the above statement, Θ is automatically also an l.s.o.p. for $\Omega(\mathbb{F}[\Delta])$.

Remark 5.5.2. If M is a Cohen–Macaulay A-module and Θ is an l.s.o.p. for M, then $\Sigma(\Theta; M) = \Theta M$ by definition of the Cohen–Macaulay property. Thus, if Δ is Cohen–Macaulay, then Theorem 5.5.1 gives an isomorphism

$$\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta]) \cong \big(\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]\big)^{\vee}(-\mathbf{a}),$$

a fact that is well-known in commutative algebra.

The above duality (for the monochromatic case) implies the following equivalent formulation of the dual WLP; we will use it in the next section.

Lemma 5.5.3. *Let Δ be a Cohen–Macaulay simplicial complex of dimension $d-1$. Then Δ has the dual WLP if and only if there is an l.s.o.p. Θ for $\mathbb{F}[\Delta]$ and a linear form ω such that the multiplication map*

$$\cdot\omega : (\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta]))_{\lfloor d/2 \rfloor} \to (\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta]))_{\lfloor d/2 \rfloor + 1}$$

is surjective.

Remark 5.5.4. If Δ is an $\mathbb{F}$-homology sphere, then $\Omega(\mathbb{F}[\Delta]) \cong \mathbb{F}[\Delta]$. Hence, in the case of $\mathbb{F}$-homology spheres, having the WLP is equivalent to having the dual WLP.

We will also use the following duality result due to Schenzel [68, Theorem II.4.9].

Theorem 5.5.5 (Schenzel). *Let R be a finitely generated graded $\mathbb{F}$-algebra of Krull dimension $d > 0$. If R is Buchsbaum, then $\Omega(R)$ is also Buchsbaum and*

$$H^i_{\mathfrak{m}}(\Omega(R)) \cong (H^{d-i+1}_{\mathfrak{m}}(R))^{\vee} \quad \text{for all } 2 \leq i \leq d-1.$$

One of the key properties used in the proof of Theorem 5.4.4 (see the proof of Proposition 5.4.2) was the existence of a surjection from L to $M(\mathbf{b})$. The goal of the rest of this section is to establish an analogous surjection for canonical modules.

In the rest of this section, we assume that (Δ, π) is an $\mathbf{a}$-balanced simplicial complex on the vertex set $[n]$, and we consider $\mathbb{F}[\Delta]$, $\mathbb{F}[\mathrm{st}_\Delta(\sigma)]$ and $\mathbb{F}[\mathrm{lk}_\Delta(\sigma)]$ as modules over the polynomial ring $A = \mathbb{F}[x_1, \ldots, x_n]$. We utilize two different fine gradings of A: the $\mathbb{N}^m$-grading induced by the coloring π and the $\mathbb{N}^n$-grading defined by $\deg x_i = \mathfrak{e}_i$, where $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ is the standard basis for $\mathbb{Z}^n$. To avoid confusion, we use bold letters for elements in $\mathbb{N}^m$ whereas we use letters in Fraktur for elements in $\mathbb{N}^n$. For $\sigma \subseteq [n]$, we set $\mathfrak{e}_\sigma = \sum_{i \in \sigma} \mathfrak{e}_i$.

Lemma 5.5.6. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^m$ with $\mathbf{b} \leq \mathbf{a}$. Let (Δ, π) be an $\mathbf{a}$-balanced simplicial complex and let $\Theta = \theta_1, \ldots, \theta_{|\mathbf{a}|}$ be an $\mathbb{N}^m$-graded l.s.o.p. for $\mathbb{F}[\Delta]$. Then, there is a surjection*

$$\bigoplus_{\sigma \in \Delta, \pi(\sigma) = \mathbf{b}} \Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) \to \Omega(\mathbb{F}[\Delta])_{\geq \mathbf{b}}.$$

Proof. Let $|\mathbf{a}| = d$ and let $\sigma \in \Delta$ be any face. The long exact sequence of local cohomology modules induced by the natural surjection $\mathbb{F}[\Delta] \to \mathbb{F}[\mathrm{st}_\Delta(\sigma)]$ provides us with a surjection

$$H^d_{\mathfrak{m}}(\mathbb{F}[\Delta]) \to H^d_{\mathfrak{m}}(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]).$$

By taking the Matlis dual of both sides, we obtain an injection

$$\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) \to \Omega(\mathbb{F}[\Delta]) \quad \text{for all } \sigma \in \Delta. \tag{5.7}$$

When $\mathfrak{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$, we let $s(\mathfrak{u}) = \{i \,:\, u_i \neq 0\}$ be the support of $\mathfrak{u}$. If $s(\mathfrak{u}) \supseteq \sigma$, then Hochster's formula for the Hilbert series of local cohomology modules [67, Theorem II.4.1] shows that

$$\Omega(\mathbb{F}[\Delta])_{\mathfrak{u}} \cong \tilde{H}_{d-1-|s(\mathfrak{u})|}\big(\mathrm{lk}_\Delta(s(\mathfrak{u})); \mathbb{F}\big) = \tilde{H}_{d-1-|s(\mathfrak{u})|}\big(\mathrm{lk}_{\mathrm{st}_\Delta(\sigma)}(s(\mathfrak{u})); \mathbb{F}\big) \cong \Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)])_{\mathfrak{u}}.$$

If $s(\mathfrak{u}) \not\supseteq \sigma$, then $\mathrm{lk}_{\mathrm{st}_\Delta(\sigma)}(s(\mathfrak{u}))$ is a cone over $\sigma \setminus s(\mathfrak{u})$, and hence has trivial homology. It hence follows from Hochster's formula that $\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)])$ is equal to $\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)])_{\geq \mathfrak{e}_\sigma}$. Thus, for any $\sigma \in \Delta$, the injection in (5.7) induces an isomorphism

$$\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) = \Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)])_{\geq \mathfrak{e}_\sigma} \to \Omega(\mathbb{F}[\Delta])_{\geq \mathfrak{e}_\sigma}. \tag{5.8}$$

Let

$$\mathcal{L}_{\mathbf{b}} = \{\mathfrak{u} \in \mathbb{N}^n \ : \ \mathfrak{u} \geq \mathfrak{e}_\sigma \text{ for some } \sigma \in \Delta \text{ with } \pi(\sigma) = \mathbf{b}\}.$$

Taking the following sum of the maps in (5.8) yields the desired surjection

$$\bigoplus_{\sigma \in \Delta, \pi(\sigma) = \mathbf{b}} \Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) \to \bigoplus_{\mathfrak{u} \in \mathcal{L}_{\mathbf{b}}} \Omega(\mathbb{F}[\Delta])_{\mathfrak{u}} = \Omega(\mathbb{F}[\Delta])_{\geq \mathbf{b}}.$$

□

The following modification of Lemma 5.5.6 provides an appropriate analog of a surjection from L to $M(\mathbf{b})$ on the level of canonical modules.

Lemma 5.5.7. *Let $0 \leq \ell \leq m$, $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{N}^m$, and $\mathbf{b} = (a_1, \ldots, a_\ell, 0, \ldots, 0) \in \mathbb{N}^m$. Let (Δ, π) be an $\mathbf{a}$-balanced simplicial complex, $\Theta = \theta_1, \ldots, \theta_{|\mathbf{a}|}$ an $\mathbb{N}^m$-graded l.s.o.p. for $\mathbb{F}[\Delta]$, and $\Theta' = (\theta_i \ : \ \deg \theta_i \notin \{\mathbf{e}_1, \ldots, \mathbf{e}_\ell\})$. Then there is a surjection*

$$\bigoplus_{\sigma \in \Delta, \pi(\sigma) = \mathbf{b}} \big(\Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])/\Theta'\Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])\big)(-\mathbf{b}) \to \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\geq \mathbf{b}}.$$

Proof. By Lemma 5.5.6 there exists a surjection

$$\bigoplus_{\sigma \in \Delta, \pi(\sigma) = \mathbf{b}} \Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) \to (\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta]))_{\geq \mathbf{b}}.$$

Thus, in order to prove the claim, it is enough to show that for any $\sigma \in \Delta$ with $\pi(\sigma) = \mathbf{b}$, there is an isomorphism

$$\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) \cong \big(\Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])/\Theta'\Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])\big)(-\mathbf{b}). \tag{5.9}$$

Fix a face $\sigma \in \Delta$ with $\pi(\sigma) = \mathbf{b}$. Since the variables x_v with $v \in \sigma$ form a regular sequence of $\mathbb{F}[\mathrm{st}_\Delta(\sigma)]$ and since $\mathbb{F}[\mathrm{st}_\Delta(\sigma)]/(x_v : v \in \sigma)\mathbb{F}[\mathrm{st}_\Delta(\sigma)] \cong \mathbb{F}[\mathrm{lk}_\Delta(\sigma)]$, it follows from [11, Theorem 3.3.5] that

$$\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)])/(x_v : v \in \sigma)\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) \cong \Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])(-\mathbf{b}). \tag{5.10}$$

Also, since Θ contains exactly a_i linear forms of color $\mathbf{e}_i \in \mathbb{N}^m$ for all i and since $\mathrm{st}_\Delta(\sigma)$ contains exactly a_i vertices of color $\mathbf{e}_i \in \mathbb{N}^m$ for all $i \leq \ell$, we obtain that

$$\Theta\mathbb{F}[\mathrm{st}_\Delta(\sigma)] = \big((\Theta') + (x_v : v \in \sigma)\big)\mathbb{F}[\mathrm{st}_\Delta(\sigma)].$$

Then, since $\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)])$ is an $\mathbb{F}[\mathrm{st}_\Delta(\sigma)]$-module, we conclude that

$$\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) = \Theta'\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]) + (x_v : v \in \sigma)\Omega(\mathbb{F}[\mathrm{st}_\Delta(\sigma)]). \tag{5.11}$$

Combining (5.10) and (5.11) yields the desired isomorphism (5.9). □

5.6 Proofs for non-orientable manifolds

In this section we prove Theorem 5.1.3(i) for non-orientable homology manifolds, see Theorem 5.6.2. As in Section 4, we may assume that $\ell > 1$ and that Δ is connected. We start with the following result (cf. Proposition 5.4.2).

Proposition 5.6.1. *Let $1 < \ell \leq \lfloor d/2 \rfloor$, $\mathbf{a} = (2\ell - 1, 1, \ldots, 1) \in \mathbb{N}^{d-2\ell+2}$, and $\mathbf{b} = \mathbf{a} - (2\ell - 1)\mathbf{e}_1$. Let (Δ, π) be an $\mathbf{a}$-balanced, connected, Buchsbaum simplicial complex and suppose that for every face $\sigma \in \Delta$ with $\pi(\sigma) = \mathbf{b}$, the link of σ in Δ has the dual WLP. Then*

$$h''_{\ell\mathbf{e}_1}(\Delta, \pi) - h''_{(\ell-1)\mathbf{e}_1}(\Delta, \pi) \geq \binom{2\ell - 1}{\ell}\tilde{\beta}_{\ell-1}(\Delta).$$

Proof. The proof is similar to that of Proposition 5.4.2. For any $\mathbb{N}^m$-graded l.s.o.p. $\Theta = \theta_1, \ldots, \theta_{|\mathbf{a}|}$ for $\mathbb{F}[\Delta]$ and for any linear form ω with $\deg \omega = \mathbf{e}_1$, there is the following commutative diagram:

$$\begin{array}{ccc} \bigoplus\limits_{\sigma\in\Delta,\pi(\sigma)=\mathbf{b}} \big(\Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])/\Theta'\Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])\big)_{\ell\mathbf{e}_1} & \xrightarrow{\varphi} & \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\ell\mathbf{e}_1+\mathbf{b}} \\ \cdot\omega\uparrow & & \uparrow\cdot\omega \\ \bigoplus\limits_{\sigma\in\Delta,\pi(\sigma)=\mathbf{b}} \big(\Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])/\Theta'\Omega(\mathbb{F}[\mathrm{lk}_\Delta(\sigma)])\big)_{(\ell-1)\mathbf{e}_1} & \xrightarrow{\varphi} & \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{(\ell-1)\mathbf{e}_1+\mathbf{b}}, \end{array}$$

where $\Theta' = (\theta_i \ : \ \deg\theta_i = \mathbf{e}_1)$ and φ is the surjection guaranteed by Lemma 5.5.7.

Since $\mathrm{lk}_\Delta(\sigma)$ has the dual WLP over $\mathbb{F}$ for all $\sigma \in \Delta$ with $\pi(\sigma) = \mathbf{b}$, we conclude from Lemma 5.5.3 that for a generic choice of Θ' and a generic linear form ω with $\deg\omega = \mathbf{e}_1$, the left vertical map is surjective. Hence, the multiplication map

$$\cdot\omega : \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{(\ell-1)\mathbf{e}_1+\mathbf{b}} \to \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\ell\mathbf{e}_1+\mathbf{b}}$$

is also surjective. Furthermore, since $\mathfrak{m}\cdot\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))/\Theta\Omega(\mathbb{F}[\Delta]))$ is zero by Theorem 5.3.2, the above surjection gives rise to a surjection

$$\cdot\omega : \big(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))\big)_{(\ell-1)\mathbf{e}_1+\mathbf{b}} \to \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\ell\mathbf{e}_1+\mathbf{b}}. \tag{5.12}$$

Therefore,

$$\dim_{\mathbb{F}} \big(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))\big)_{(\ell-1)\mathbf{e}_1+\mathbf{b}} \geq \dim_{\mathbb{F}} \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\ell\mathbf{e}_1+\mathbf{b}}. \tag{5.13}$$

The right-hand-side of (5.13) can be rewritten as

$$\begin{aligned} &\dim_{\mathbb{F}} \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\ell\mathbf{e}_1+\mathbf{b}} \\ =&\dim_{\mathbb{F}} \big(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))\big)_{\ell\mathbf{e}_1+\mathbf{b}} + \dim_{\mathbb{F}} \big(\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\ell\mathbf{e}_1+\mathbf{b}} \\ =&\dim_{\mathbb{F}} \big(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))\big)_{\ell\mathbf{e}_1+\mathbf{b}} + \binom{2\ell-1}{\ell}\tilde{\beta}_{\ell-1}(\Delta), \end{aligned}$$

where the last equality follows from Theorems 5.3.2 and 5.5.5. In addition, it follows from Theorems 5.3.4 and 5.5.1 that for all $\mathbf{b} \leq \mathbf{a}$,

$$\dim_{\mathbb{F}} \big(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))\big)_{\mathbf{a}-\mathbf{b}} = \dim_{\mathbb{F}} \big(\mathbb{F}[\Delta]/\Sigma(\Theta;\mathbb{F}[\Delta])\big)_{\mathbf{b}} = h''_{\mathbf{b}}(\Delta,\pi).$$

Substituting these formulas in (5.13), we infer that

$$h''_{\ell\mathbf{e}_1}(\Delta,\pi) \geq h''_{(\ell-1)\mathbf{e}_1}(\Delta,\pi) + \binom{2\ell-1}{\ell}\tilde{\beta}_{\ell-1}(\Delta),$$

as desired. □

Proposition 5.6.1 implies the following theorem exactly in the same way as Proposition 5.4.2 implied Theorem 5.4.4.

Theorem 5.6.2. *Let $1 < \ell \leq \lfloor d/2 \rfloor$. Let (Δ,π) be a balanced Buchsbaum simplicial complex of dimension $d-1$. Suppose that for all faces $\sigma \in \Delta$ of codimension $2\ell-1$, the link of σ has the dual WLP. Then*

(i) for any $S \subseteq [d]$ with $|S| = 2\ell - 1$,

$$\sum_{T\subseteq S, |T|=\ell} h''_T(\Delta,\pi) - \sum_{T\subseteq S,|T|=\ell-1} h''_T(\Delta,\pi) \geq \binom{2\ell-1}{\ell}\tilde{\beta}_{\ell-1}(\Delta).$$

(ii) Consequently, $\overline{h}''_\ell(\Delta) - \overline{h}''_{\ell-1}(\Delta) \geq \tilde{\beta}_{\ell-1}(\Delta)$.

Remark 5.6.3. The inequality in part (ii) of Theorem 5.6.2 (Theorem 5.4.4, resp.) holds as equality if and only if the inequality in part (i) holds as equality for **all** $S \subseteq [d]$ with $|S| = 2\ell - 1$.

Since all proper links of a closed $\mathbb{F}$-homology manifold are $\mathbb{F}$-homology spheres, the above theorem implies our Theorem 5.1.3(i). Moreover, since all homology 2-spheres have the (dual) WLP, the inequality part of Theorem 5.1.4 also follows:

Corollary 5.6.4. *Let Δ be a balanced $\mathbb{F}$-homology manifold without boundary. If Δ has dimension ≥ 3, then $\overline{h}''_2(\Delta) - \overline{h}''_1(\Delta) \geq \tilde{\beta}_1(\Delta)$.*

5.7 *The equality part of Theorem 5.1.4*

In this section, we complete the proof of Theorem 5.1.4, see Theorem 5.7.4. We first recall some results on stacked cross-polytopal spheres verified in [32].

Let Δ and Γ be pure simplicial complexes of the same dimension with disjoint vertex sets. Let $\sigma \in \Delta$ and $\tau \in \Gamma$ be facets and let $\varphi : \sigma \to \tau$ be a bijection. The **connected sum** $\Delta \#_{\varphi} \Gamma$ of Δ and Γ is the simplicial complex obtained from $(\Delta \setminus \{\sigma\}) \cup (\Gamma \setminus \{\tau\})$ by identifying v with $\varphi(v)$ for all $v \in \sigma$. If Δ and Γ are balanced, then so is $\Delta \#_{\varphi} \Gamma$. A **stacked cross-polytopal sphere** of dimension $d-1$ is the connected sum of several copies of the boundary complex of the d-dimensional cross-polytope.

The following result was established in [32, Theorem 4.1 and Lemma 4.2]. Below, $\overline{h}_i(\Delta) := \frac{h_i(\Delta)}{\binom{d}{i}}$ denotes the **normalized h_i-number** of Δ.

Theorem 5.7.1 (Klee–Novik)**.** *Let (Δ, π) be a balanced $\mathbb{F}$-homology manifold of dimension $d-1 \geq 3$. Then the following conditions are equivalent:*

(i) Δ is a stacked cross-polytopal sphere.

(ii) $\overline{h}_2(\Delta) - \overline{h}_1(\Delta) = 0$.

(iii) For any $S \subseteq [d]$ with $|S| = 3$, $\sum_{T \subseteq S, |T|=2} h_T(\Delta, \pi) = \sum_{T \subseteq S, |T|=1} h_T(\Delta, \pi)$.

Let (Δ, π) be a pure balanced simplicial complex. Let σ and τ be facets of Δ and let $\varphi : \sigma \to \tau$ be a bijection with $\pi(v) = \pi(\varphi(v))$ for all $v \in \sigma$. Such a bijection φ is called **admissible** if $\mathrm{lk}_{\Delta}(v) \cap \mathrm{lk}_{\Delta}(\varphi(v)) = \{\emptyset\}$ for all $v \in \sigma$. For an admissible bijection φ, define Δ^{φ} as the simplicial complex obtained from $\Delta \setminus \{\sigma, \tau\}$ by identifying v with $\varphi(v)$ for all $v \in \sigma$. We say that Δ^{φ} is obtained from Δ by a **balanced handle addition**. The **balanced Walkup class** $\mathcal{BH}^d$ is the set of all balanced simplicial complexes obtained from the boundary complexes of d-dimensional cross-polytopes by successively applying the operations of connected sums and balanced handle additions.

The following result is [32, Corollary 4.12].

Theorem 5.7.2 (Klee–Novik)**.** *Let Δ be a balanced $\mathbb{F}$-homology manifold of dimension $d-1 \geq 4$. Then $\Delta \in \mathcal{BH}^d$ if and only if all vertex links of Δ are stacked cross-polytopal spheres.*

Our proof of the equality part of Theorem 5.1.4 relies on the following lemma.

Lemma 5.7.3. *Let Δ be a $(d-1)$-dimensional connected $\mathbb{F}$-homology manifold without boundary and let Θ be an l.s.o.p. for $\mathbb{F}[\Delta]$. If $d \geq 2$, then for any vertex v of Δ with $x_v \notin (\Theta)$, there is an injection*

$$\varphi : \Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)]) \to \Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])$$

such that its composition with the natural surjection from $\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])$ to $\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))$ is also an injection.

Proof. We assume that $V(\Delta) = \{1, 2, \ldots, n\}$ and $A = \mathbb{F}[x_1, \ldots, x_n]$. As shown in the proof of Lemma 5.5.6 (see (5.7)), there is an injection

$$\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)]) \to \Omega(\mathbb{F}[\Delta]). \tag{5.14}$$

This injection gives rise to an A-homomorphism

$$\varphi : \Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)]) \to \Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta]).$$

Composing this A-homomorphism with the natural surjection

$$\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta]) \to \Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta])),$$

leads to an A-homomorphism

$$\varphi' : \Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)]) \to \Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta])). \tag{5.15}$$

Thus to prove the desired statement, it suffices to show that the map φ' is injective.

Note that Θ is an l.s.o.p. for $\mathbb{F}[\mathrm{st}_\Delta(v)]$ since $\mathrm{st}_\Delta(v)$ is a full-dimensional subcomplex of Δ. Taking the Matlis dual of modules in (5.15) and using Theorem 5.5.1, leads to an A-homomorphism

$$\mathbb{F}[\Delta]/\Sigma(\Theta;\mathbb{F}[\Delta]) \to \mathbb{F}[\mathrm{st}_\Delta(v)]/\Theta\mathbb{F}[\mathrm{st}_\Delta(v)].$$

Since for all graded ideals I and J of A, any A-homomorphism from A/I to A/J of degree 0 must be either zero or surjective, the map φ' is either zero or injective. We prove that φ' is non-zero.

Since $\mathrm{st}_\Delta(v)$ is a cone over an $\mathbb{F}$-homology sphere $\mathrm{lk}_\Delta(v)$, it follows that $\mathbb{F}[\mathrm{st}_\Delta(v)]$ is a Gorenstein ring that is isomorphic to $\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])(+\mathfrak{e}_v)$; in particular, $\mathbb{F}[\mathrm{st}_\Delta(v)]_0 \cong \Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])_{\mathfrak{e}_v}$. Let α be a non-zero element of $\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])_{\mathfrak{e}_v}$. We claim that $\varphi'(\alpha)$ is non-zero. Since $H^i_{\mathfrak{m}}(\Omega(\mathbb{F}[\Delta])) = 0$ for $i \leq 1$ (see [3, Lemma 1]), Theorems 5.3.2 and 5.5.5 imply that

$$\big(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta;\Omega(\mathbb{F}[\Delta]))\big)_1 = \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_1.$$

Thus, to prove that $\varphi'(\alpha) \neq 0$, it is enough to show that $\varphi(\alpha) \neq 0$.

If Δ is orientable, then $\mathbb{F}[\Delta]$ is isomorphic to $\Omega(\mathbb{F}[\Delta])$ by a result of Gräbe [22]. Since the map in (5.14) preserves the $\mathbb{N}^n$-grading and α has degree $\mathfrak{e}_v$, $\varphi(\alpha) \in \Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])$ can be identified with a non-zero scalar multiple of x_v in $\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]$, which is a non-zero element by our assumption that $x_v \notin (\Theta)$. Suppose that Δ is non-orientable. Then $\Omega(\mathbb{F}[\Delta])_0 \cong \tilde{H}_{d-1}(\Delta;\mathbb{F})$ is zero by Hochster's formula, and so

$$\big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_1 = \Omega(\mathbb{F}[\Delta])_1.$$

In this case, the fact that $\varphi(\alpha)$ is non-zero follows from the injectivity of (5.14). □

We now turn to the proof of the main result of this section which completes the proof of Theorem 5.1.4.

Theorem 5.7.4. *Let (Δ,π) be a balanced, connected, $\mathbb{F}$-homology manifold without boundary of dimension $d-1 \geq 4$. Then $\overline{h}''_2(\Delta) - \overline{h}''_1(\Delta) = \tilde{\beta}_1(\Delta)$ if and only if $\Delta \in \mathcal{BH}^d$.*

Proof. As noted at the end of Section 4 in [32], the "if"-part is easy. We prove the "only if"-part. Let $S \subseteq [d]$ with $|S| = 3$ and let v be a vertex of Δ with $\pi(v) \notin \{\mathbf{e}_i \;:\; i \in S\}$. By Theorems 5.7.1 and 5.7.2, it suffices to check that

$$\sum_{T\subseteq S,\ |T|=2} h_T(\mathrm{lk}_\Delta(v),\pi) = \sum_{T\subseteq S,\ |T|=1} h_T(\mathrm{lk}_\Delta(v),\pi). \tag{5.16}$$

We may assume that $S = \{1, d-1, d\}$. Let $\mathbf{a} = (3, 1, \ldots, 1) \in \mathbb{N}^{d-2}$. Define $\tilde{\pi} : V(\Delta) \to \{\mathbf{e}_1, \ldots, \mathbf{e}_{d-2}\}$ by $\tilde{\pi}(u) = \pi(u)$ if $\pi(u) \notin \{\mathbf{e}_i : i \in S\}$ and $\tilde{\pi}(u) = \mathbf{e}_1$ if $\pi(u) \in \{\mathbf{e}_i : i \in S\}$. Then $(\Delta, \tilde{\pi})$ is $\mathbf{a}$-balanced, $h''_{\mathbf{e}_1}(\Delta, \tilde{\pi}) = \sum_{T \subseteq S, |S|=1} h''_T(\Delta, \pi)$, and $h''_{2\mathbf{e}_1}(\Delta, \tilde{\pi}) = \sum_{T \subseteq S, |S|=2} h''_T(\Delta, \pi)$. The proof of Proposition 5.6.1 (see (5.12)) shows that there is an $\mathbb{N}^m$-graded l.s.o.p. Θ for $\mathbb{F}[\Delta]$ and a linear form ω with $\deg \omega = \mathbf{e}_1$ such that

$$\cdot\omega : \big(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta; \Omega(\mathbb{F}[\Delta]))\big)_{\mathbf{a}-2\mathbf{e}_1} \to \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\mathbf{a}-\mathbf{e}_1}$$

is surjective. Since, by our assumption, $\overline{h}''_2(\Delta) - \overline{h}''_1(\Delta) = \tilde{\beta}_1(\Delta)$, it follows from Remark 5.6.3 that $h''_{2\mathbf{e}_1}(\Delta, \tilde{\pi}) - h''_{\mathbf{e}_1}(\Delta, \tilde{\pi}) = 3\tilde{\beta}_1(\Delta)$. The proof of Proposition 5.6.1 then implies that the above map is, in fact, an isomorphism.

We have the following commutative diagram

$$\begin{array}{ccc} \big(\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])\big)_{\mathbf{a}-\mathbf{e}_1} & \xrightarrow{\varphi'} & \big(\Omega(\mathbb{F}[\Delta])/\Theta\Omega(\mathbb{F}[\Delta])\big)_{\mathbf{a}-\mathbf{e}_1} \\ \uparrow \cdot\omega & & \uparrow \cdot\omega \\ \big(\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])\big)_{\mathbf{a}-2\mathbf{e}_1} & \xrightarrow{\varphi} & \big(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta; \Omega(\mathbb{F}[\Delta]))\big)_{\mathbf{a}-2\mathbf{e}_1}, \end{array}$$

where φ and φ' are injections given in Lemma 5.7.3. Since the right vertical map and the lower horizontal map are injective, we conclude that the left vertical map is injective. This implies that

$$\dim_{\mathbb{F}}(\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)]))_{\mathbf{a}-\mathbf{e}_1} \geq \dim_{\mathbb{F}}(\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)]))_{\mathbf{a}-2\mathbf{e}_1},$$

and, since the star, $\mathrm{st}_\Delta(v)$, is Cohen–Macaulay, we infer from Remark 5.5.2 that

$$\dim_{\mathbb{F}} \big(\mathbb{F}[\mathrm{st}_\Delta(v)]/\Theta\mathbb{F}[\mathrm{st}_\Delta(v)]\big)_{\mathbf{e}_1} \geq \dim_{\mathbb{F}} \big(\mathbb{F}[\mathrm{st}_\Delta(v)]/\Theta\mathbb{F}[\mathrm{st}_\Delta(v)]\big)_{2\mathbf{e}_1}.$$

As the flag h-vectors of $\mathrm{st}_\Delta(v)$ and $\mathrm{lk}_\Delta(v)$ coincide, the above inequality shows that $h_{\mathbf{e}_1}(\mathrm{lk}_\Delta(v), \tilde{\pi}) \geq h_{2\mathbf{e}_1}(\mathrm{lk}_\Delta(v), \tilde{\pi})$. On the other hand, since $\tilde{\pi}(v) \neq \mathbf{e}_1$ and $d-1 \geq 4$, it follows that the link, $\mathrm{lk}_\Delta(v)$, satisfies the assumptions of Proposition 5.6.1 for $\ell = 2$. Hence, by this proposition, $h_{\mathbf{e}_1}(\mathrm{lk}_\Delta(v), \tilde{\pi}) \leq h_{2\mathbf{e}_1}(\mathrm{lk}_\Delta(v), \tilde{\pi})$. We conclude that $h_{\mathbf{e}_1}(\mathrm{lk}_\Delta(v), \tilde{\pi}) = h_{2\mathbf{e}_1}(\mathrm{lk}_\Delta(v), \tilde{\pi})$. The result follows, since according to the definition of $\tilde{\pi}$, this equality is equivalent to the desired statement (5.16). $\square$

Remark 5.7.5. Let $\ell \leq \lfloor (d-1)/2 \rfloor$ and let Δ be a balanced $(d-1)$-dimensional $\mathbb{Q}$-homology manifold such that all vertex links of Δ are polytopal. In the same way as in the proof of Theorem 5.7.4, one can show that if $\overline{h}''_\ell(\Delta) - \overline{h}''_{\ell-1}(\Delta) = \tilde{\beta}_{\ell-1}(\Delta)$, then $\overline{h}_\ell(\mathrm{lk}_\Delta(v)) = \overline{h}_{\ell-1}(\mathrm{lk}_\Delta(v))$ for all $v \in V(\Delta)$. Along with a recent result of Adiprasito [2, Section 9], this, in turn, implies that all vertex links of Δ have the balanced $(\ell-1)$-stacked property. (A balanced analog of the $(\ell-1)$-stacked property for homology spheres and manifolds was defined in [32, Definition 5.3].)

5.8 Closing remarks and open problems

We close with several remarks as well as some problems related to this paper that we left unsolved.

5.8.1 Buchsbaum simplicial complexes*

Our proof for the non-orientable case (see Theorem 5.6.2) applies not only to homology manifolds but also to Buchsbaum* complexes. A simplicial complex Δ of dimension $d-1$ is **Buchsbaum*** (over $\mathbb{F}$) if it is Buchsbaum (over $\mathbb{F}$) and, in addition, $\tilde{H}_{d-2}(|\Delta| - p; \mathbb{F}) \cong \tilde{H}_{d-2}(|\Delta|; \mathbb{F})$ for every point p in the geometric realization $|\Delta|$ of Δ. A simplicial complex Δ of dimension $d-1$ is called **doubly Cohen–Macaulay** (over $\mathbb{F}$) if it is Cohen–Macaulay, and, in addition, $\Delta \setminus v = \{F \in \Delta \ : \ v \notin F\}$ is Cohen–Macaulay of dimension $d-1$ for every vertex $v \in V(\Delta)$.

It was shown in [73, Theorem 9.8] that being doubly Cohen–Macaulay is equivalent to being both Buchsbaum* and Cohen–Macaulay, while according to [4, Corollary 2.12], every proper link of a Buchsbaum* simplicial complex is doubly Cohen–Macaulay. Furthermore, Björner and Swartz suggested the following conjecture, see [71, Problem 4.2].

Conjecture 5.8.1 (Björner–Swartz)**.** *All doubly Cohen–Macaulay simplicial complexes have the dual WLP.*

Thus, if true, Conjecture 5.8.1 would imply that the conclusions of Theorem 5.6.2 hold for all Buchsbaum* simplicial complexes. Recall that Conjecture 5.8.1 does hold in dimension two. (Indeed, since by [51] all 2-dimensional doubly Cohen–Macaulay complexes are minimal cycles in the sense of [16], the statement in characteristic zero follows from the main result of [16]. For nonzero characteristic, see the discussion and references in [52, Section 5].) Hence we obtain the following result that strengthens the result of Browder and Klee [9, Theorem 4.1].

Theorem 5.8.2. *Let Δ be a balanced Buchsbaum* simplicial complex of dimension ≥ 3, then $\overline{h}''_2(\Delta) - \overline{h}''_1(\Delta) \geq \tilde{\beta}_1(\Delta)$.*

In [4, Question 5.7(ii)], Athanasiadis and Welker asked whether for a $(d-1)$-dimensional Buchsbaum* simplicial complex, the vector given by the successive differences of the first half of the h''-vector is an M-sequence (that is, the Hilbert function of some standard graded $\mathbb{F}$-algebra). Our next result shows that the validity of Conjecture 5.8.1 would provide an affirmative answer to this question.

Proposition 5.8.3. *Let Δ be a connected Buchsbaum* simplicial complex of dimension $d-1$. If all vertex links of Δ have the dual WLP, then the vector*

$$(h''_0(\Delta), h''_1(\Delta) - h''_0(\Delta), \ldots, h''_{\lfloor d/2 \rfloor}(\Delta) - h''_{\lfloor d/2 \rfloor - 1}(\Delta)) \tag{5.17}$$

is an M-sequence.

Proof. (Sketch) We start with two observations. First, by Lemma 5.5.6, there is a surjection

$$N := \bigoplus_{v \in V(\Delta)} \Omega(\mathbb{F}[\mathrm{st}_\Delta(v)])/\Theta\Omega(\mathbb{F}[\mathrm{st}_\Delta(v)]) \to \bigoplus_{k \geq 1} \left(\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta; \Omega(\mathbb{F}[\Delta])\right)_k,$$

where Θ is a generic l.s.o.p. Second, since x_v is a non-zero divisor on $\mathbb{F}[\mathrm{st}_\Delta(v)]$ and $\mathbb{F}[\mathrm{st}_\Delta(v)]/(x_v)\mathbb{F}[\mathrm{st}_\Delta(v)] \cong \mathbb{F}[\mathrm{lk}_\Delta(v)]$ while $\mathrm{lk}_\Delta(v)$ is a $(d-2)$-dimensional complex that has the dual WLP, it follows that for a generic linear form ω, the multiplication map $\cdot\omega : N_k \to N_{k+1}$ is surjective for $k = \lfloor (d-1)/2 \rfloor$. These two observations imply that multiplication by ω on $\Omega(\mathbb{F}[\Delta])/\Sigma(\Theta; \Omega(\mathbb{F}[\Delta]))$ from degree $\lfloor (d-1)/2 \rfloor$ to degree $\lfloor (d-1)/2 \rfloor + 1$ is also surjective.

Thus, by Theorem 5.5.1, multiplication by ω on $\mathbb{F}[\Delta]/\Sigma(\Theta;\mathbb{F}[\Delta])$ from degree $\lfloor d/2\rfloor-1$ to degree $\lfloor d/2\rfloor$ is injective. Then, since $\mathbb{F}[\Delta]/\Sigma(\Theta;\mathbb{F}[\Delta])$ is a level algebra by [50], it follows from Proposition 2.1(b) in [39] that the map $\cdot\omega : (\mathbb{F}[\Delta]/\Sigma(\Theta;\mathbb{F}[\Delta]))_k \to (\mathbb{F}[\Delta]/\Sigma(\Theta;\mathbb{F}[\Delta]))_{k+1}$ is injective for all $k \le \lfloor d/2\rfloor - 1$, and so the vector in (5.17) is the Hilbert function of $\mathbb{F}[\Delta]/(\Sigma(\Theta;\mathbb{F}[\Delta]) + \omega\mathbb{F}[\Delta])$. □

5.8.2 Open problems

Theorem 5.1.4 is stated for the class of $\mathbb{F}$-homology manifolds. However, an analogous "non-balanced" result is known to hold in a larger generality: the inequality $h''_2(\Delta) \ge h''_1(\Delta) + d\tilde{\beta}_1(\Delta)$ (for $d \ge 4$) is proved in [47, Theorem 5.3] for *all* normal pseudomanifolds. Thus, it is tempting to conjecture that the statement of Theorem 5.1.4 remains valid for all balanced normal pseudomanifolds.

Bagchi and Datta [5] introduced the notion of μ-numbers. These numbers satisfy the following Morse-type inequalities: for any simplicial complex Δ,

$$\sum_{k=0}^{j}(-1)^{j-k}\mu_k(\Delta) \ge \sum_{k=0}^{j}(-1)^{j-k}\tilde{\beta}_k(\Delta) + (-1)^j \quad \text{for all } j \ge 0.$$

In light of [48, Theorem 6.5] and [48, Theorem 7.3], we conjecture that the statements of Theorem 5.1.3 and Corollary 5.1.4 can be strengthened by replacing the Betti numbers with the μ-numbers. Furthermore, we conjecture that the resulting inequality in fact holds for all (not necessary orientable) homology manifolds with boundary if we replace Δ with the pair $(\Delta, \partial\Delta)$.

We now turn our discussion towards characterizing the cases of equality in Theorem 5.1.3(i) and Theorem 5.1.4. Theorem 5.1.4 provides such a characterization when $d \ge 5$ but leaves the case of $d = 4$ open. However, in view of [47, Theorem 5.3], it is plausible that the same characterization continues to hold in the $d = 4$ case. As for Theorem 5.1.3(i), Remark 5.7.5 along with [45, Theorem 4.6 & Corollary 5.8] leads us to conjecture that if Δ satisfies the assumptions of Theorem 5.1.3(i) and $\ell < d/2$, then

$$\overline{h}''_\ell(\Delta) = \overline{h}''_{\ell-1}(\Delta) + \tilde{\beta}_{\ell-1}(\Delta)$$

if and only if Δ has the balanced $(\ell - 1)$-stacked property.

Finally, it would be very interesting to find out if Theorems 5.1.2–5.1.4 have analogs for $\mathbf{a}$-balanced complexes for an arbitrary $\mathbf{a}$.

BIBLIOGRAPHY

[1] Ron Adin. *Combinatorial structure of simplicial complexes with symmetry.* PhD thesis, Hebrew University, Jerusalem, 1991.

[2] Karim Adiprasito. Toric chordality. *J. Math. Pures Appl. (9)*, 108(5):783–807, 2017.

[3] Y. Aoyama. On the depth and the projective dimension of the canonical module. *Japan. J. Math. (N.S.)*, 6(1):61–66, 1980.

[4] C. A. Athanasiadis and V. Welker. Buchsbaum* complexes. *Math. Z.*, 272(1-2):131–149, 2012.

[5] B. Bagchi and B. Datta. On stellated spheres and a tightness criterion for combinatorial manifolds. *European J. Combin.*, 36:294–313, 2014.

[6] T. Bisztriczky, P. McMullen, R. Schneider, and A. Ivić Weiss, editors. *Polytopes: abstract, convex and computational*, volume 440 of *NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences.* Kluwer Academic Publishers Group, Dordrecht, 1994.

[7] A. Björner. Posets, regular CW complexes and Bruhat order. *European J. Combin.*, 5(1):7–16, 1984.

[8] Nicolas Bourbaki. *Algebra. I. Chapters 1–3.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1989. Translated from the French, Reprint of the 1974 edition.

[9] J. Browder and S. Klee. Lower bounds for Buchsbaum* complexes. *European J. Combin.*, 32(1):146–153, 2011.

[10] Jonathan Browder and Steven Klee. A classification of the face numbers of Buchsbaum simplicial posets. *Math. Z.*, 277(3-4):937–952, 2014.

[11] W. Bruns and J. Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1996. Revised edition.

[12] Morten Brun, Winfried Bruns, and Tim Römer. Cohomology of partially ordered sets and local cohomology of section rings. *Adv. Math.*, 208(1):210–235, 2007.

[13] Morten Brun and Tim Römer. On algebras associated to partially ordered sets. *Math. Scand.*, 103(2):169–185, 2008.

[14] W. Bruns and J. Gubeladze. Combinatorial invariance of Stanley-Reisner rings. *Georgian Math. J.*, 3(4):315–318, 1996.

[15] Art M. Duval. Free resolutions of simplicial posets. *J. Algebra*, 188(1):363–399, 1997.

[16] A. L. Fogelsanger. *The generic rigidity of minimal cycles*. ProQuest LLC, Ann Arbor, MI, 1988. Thesis (Ph.D.)–Cornell University.

[17] A. M. Garsia and D. Stanton. Group actions of Stanley-Reisner rings and invariants of permutation groups. *Adv. in Math.*, 51(2):107–201, 1984.

[18] Mark Goresky and Robert MacPherson. Intersection homology theory. *Topology*, 19(2):135–162, 1980.

[19] Shiro Goto. On the associated graded rings of parameter ideals in Buchsbaum rings. *J. Algebra*, 85(2):490–534, 1983.

[20] Shiro Goto. A note on quasi-Buchsbaum rings. *Proc. Amer. Math. Soc.*, 90(4):511–516, 1984.

[21] Shiro Goto and Naoyoshi Suzuki. Index of reducibility of parameter ideals in a local ring. *J. Algebra*, 87(1):53–88, 1984.

[22] H-G. Gräbe. Über den Stanley-Reisner-Ring von Quasimannigfaltigkeiten. *Math. Nachr.*, 117:161–174, 1984.

[23] Hans-Gert Gräbe. The canonical module of a Stanley-Reisner ring. *J. Algebra*, 86(1):272–281, 1984.

[24] Hans-Gert Gräbe. Generalized Dehn-Sommerville equations and an upper bound theorem, 1987.

[25] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe. *The Lefschetz properties*, volume 2080 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2013.

[26] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[27] Jürgen Herzog and Takayuki Hibi. *Monomial ideals*, volume 260 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[28] Takayuki Hibi. Quotient algebras of Stanley-Reisner rings and local cohomology. *J. Algebra*, 140(2):336–343, 1991.

[29] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther. *Twenty-four hours of local cohomology*, volume 87 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007.

[30] Martina Juhnke-Kubitzke and Satoshi Murai. Balanced generalized lower bound inequality for simplicial polytopes. *Selecta Math. (N.S.)*, 24(2):1677–1689, 2018.

[31] Martina Juhnke-Kubitzke, Satoshi Murai, Isabella Novik, and Connor Sawaske. A generalized lower bound theorem for balanced manifolds. *Mathematische Zeitschrift*, Nov 2017.

[32] S. Klee and I. Novik. Lower bound theorems and a generalized lower bound conjecture for balanced simplicial complexes. *Mathematika*, 62:441–477, 2016.

[33] Steven Klee and Isabella Novik. Face enumeration on simplicial complexes. In *Recent trends in combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 653–686. Springer, [Cham], 2016.

[34] Victor Klee. A combinatorial analogue of Poincaré's duality theorem. *Canad. J. Math.*, 16:517–531, 1964.

[35] A.T. Lundell and S. Weingram. The topology of CW complexes. The University Series in Higher Mathematics. New York etc.: Van Nostrand Reinhold Company. VIII, 216 p. (1969)., 1969.

[36] Mikiya Masuda. h-vectors of Gorenstein* simplicial posets. *Adv. Math.*, 194(2):332–344, 2005.

[37] P. McMullen and D. W. Walkup. A generalized lower-bound conjecture for simplicial polytopes. *Mathematika*, 18:264–273, 1971.

[38] Ezra Miller, Isabella Novik, and Ed Swartz. Face rings of simplicial complexes with singularities. *Math. Ann.*, 351(4):857–875, 2011.

[39] J. C. Migliore, R. M. Miró-Roig, and U. Nagel. Monomial ideals, almost complete intersections and the weak Lefschetz property. *Trans. Amer. Math. Soc.*, 363(1):229–257, 2011.

[40] S. Murai. Algebraic shifting of strongly edge decomposable spheres. *J. Combin. Theory Ser. A*, 117(1):1–16, 2010.

[41] Mitsuhiro Miyazaki. On the canonical map to the local cohomology of a Stanley-Reisner ring. *Bull. Kyoto Univ. Ed. Ser. B*, (79):1–8, 1991.

[42] Mitsuhiro Miyazaki. Characterizations of Buchsbaum complexes. *Manuscripta Math.*, 63(2):245–254, 1989.

[43] James R. Munkres. Topological results in combinatorics. *Michigan Math. J.*, 31(1):113–128, 1984.

[44] S. Murai and E. Nevo. On the generalized lower bound conjecture for polytopes and spheres. *Acta Math.*, 210(1):185–202, 2013.

[45] S. Murai and E. Nevo. On r-stacked triangulated manifolds. *J. Algebraic Combin.*, 39(2):373–388, 2014.

[46] Satoshi Murai. Face vectors of simplicial cell decompositions of manifolds. *Israel J. Math.*, 195(1):187–213, 2013.

[47] Satoshi Murai. Tight combinatorial manifolds and graded Betti numbers. *Collect. Math.*, 66(3):367–386, 2015.

[48] Satoshi Murai and Isabella Novik. Face numbers of manifolds with boundary. *Int. Math. Res. Not. IMRN*, (12):3603–3646, 2017.

[49] Satoshi Murai, Isabella Novik, and Ken-ichi Yoshida. A duality in Buchsbaum rings and triangulated manifolds. *Algebra Number Theory*, 11(3):635–656, 2017.

[50] U. Nagel. Level algebras through Buchsbaum* manifolds. *Collect. Math.*, 62(2):187–196, 2011.

[51] E. Nevo. Rigidity and the lower bound theorem for doubly Cohen-Macaulay complexes. *Discrete Comput. Geom.*, 39(1-3):411–418, 2008.

[52] Isabella Novik and Ed Swartz. Applications of Klee's Dehn-Sommerville relations. *Discrete Comput. Geom.*, 42(2):261–276, 2009.

[53] Isabella Novik and Ed Swartz. Gorenstein rings through face rings of manifolds. *Compos. Math.*, 145(4):993–1000, 2009.

[54] Isabella Novik and Ed Swartz. Socles of Buchsbaum modules, complexes and posets. *Adv. Math.*, 222(6):2059–2084, 2009.

[55] Isabella Novik and Ed Swartz. Face numbers of pseudomanifolds with isolated singularities. *Math. Scand.*, 110(2):198–222, 2012.

[56] Victor Reiner. Quotients of Coxeter complexes and P-partitions. *Mem. Amer. Math. Soc.*, 95(460):vi+134, 1992.

[57] Gerald Allen Reisner. Cohen-Macaulay quotients of polynomial rings. *Advances in Math.*, 21(1):30–49, 1976.

[58] Connor Sawaske. Almost Buchsbaumness of some rings arising from complexes with isolated singularities. *J. Commut. Algebra*, 2017. to appear.

[59] Connor Sawaske. Stanley-Reisner rings of simplicial complexes with a free action by an abelian group. *Algebraic Combinatorics*, 2018. to appear.

[60] Peter Schenzel. On the number of faces of simplicial complexes and the purity of Frobenius. *Math. Z.*, 178(1):125–142, 1981.

[61] R. P. Stanley. A monotonicity property of h-vectors and h^*-vectors. *European J. Combin.*, 14(3):251–258, 1993.

[62] Richard P. Stanley. The upper bound conjecture and Cohen-Macaulay rings. *Studies in Appl. Math.*, 54(2):135–142, 1975.

[63] Richard P. Stanley. Balanced Cohen-Macaulay complexes. *Trans. Amer. Math. Soc.*, 249(1):139–157, 1979.

[64] Richard P. Stanley. The number of faces of a simplicial convex polytope. *Adv. in Math.*, 35(3):236–238, 1980.

[65] Richard P. Stanley. On the number of faces of centrally-symmetric simplicial polytopes. *Graphs Combin.*, 3(1):55–66, 1987.

[66] Richard P. Stanley. f-vectors and h-vectors of simplicial posets. *J. Pure Appl. Algebra*, 71(2-3):319–331, 1991.

[67] Richard P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1996.

[68] J. Stückrad and W. Vogel. *Buchsbaum rings and applications*, volume 21 of *Mathematische Monographien [Mathematical Monographs]*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1986. An interaction between algebra, geometry, and topology.

[69] Naoyoshi Suzuki. On quasi-Buchsbaum modules. An application of theory of FLC-modules. In *Commutative algebra and combinatorics (Kyoto, 1985)*, volume 11 of *Adv. Stud. Pure Math.*, pages 215–243. North-Holland, Amsterdam, 1987.

[70] E. Swartz. Thirty-five years and counting. `http://arxiv.org/pdf/1411.0987.pdf`, 2014.

[71] Ed Swartz. g-elements, finite buildings and higher Cohen-Macaulay connectivity. *J. Combin. Theory Ser. A*, 113(7):1305–1320, 2006.

[72] Wolfgang Vogel. A non-zero-divisor characterization of Buchsbaum modules. *Michigan Math. J.*, 28(2):147–152, 1981.

[73] J. W. Walker. *Topology and Combinatorics of ordered sets*. ProQuest LLC, Ann Arbor, MI, 1981. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[74] W. Whiteley. Vertex splitting in isostatic frameworks, 1990. Dual French-English text.

[75] Kikumichi Yamagishi. Buchsbaumness of certain generalization of the associated graded modules in the equi-$\mathbb{I}$-invariant case. *J. Algebra*, 322(8):2861–2885, 2009.

[76] Kohji Yanagawa. Dualizing complex of the face ring of a simplicial poset. *J. Pure Appl. Algebra*, 215(9):2231–2241, 2011.

[77] Kohji Yanagawa. Higher Cohen-Macaulay property of squarefree modules and simplicial posets. *Proc. Amer. Math. Soc.*, 139(9):3057–3066, 2011.