Sheaves on support varieties and varieties of elementary subalgebras

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ABSTRACT

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We present several results about two closely related types of objects: the projectivized scheme $\mathbb{P}(G)$ of one parameter subgroups of an infinitesimal group scheme G and the variety $\mathbb{E}(\mathfrak{g})$ of maximal elementary subalgebras of a restricted Lie algebra $\mathfrak{g}$. We define and present background material for both objects. For $\mathbb{P}(G)$ we provide a partial answer to a question of Friedlander and Pevtsova on whether a certain sheaf $\mathcal{H}_{[1]}(M)$ constructed on $\mathbb{P}(G)$ from a representation M is zero if and only if M is projective. We also explicitly calculate the sheaves $\ker \Theta_M$ for all indecomposable $\mathfrak{sl}_2$-modules M and we calculate $\mathscr{F}_i(V(\lambda))$ where $V(\lambda)$ is a Weyl module and $i \neq p$. This extends work of Friedlander and Pevtsova who calculated $\mathscr{F}_i(V(\lambda))$ when $\lambda \leq 2p-2$. For $\mathbb{E}(\mathfrak{g})$ we explicitly calculate this variety when $\mathfrak{g}$ is the Lie algebra of a reductive algebraic group G and p is good and satisfies a separability condition with respect to G. This recovers work of Carlson, Friedlander, and Pevtsova who calculated $\mathbb{E}(\mathfrak{gl}_n)$, $\mathbb{E}(\mathfrak{sl}_n)$, and $\mathbb{E}(\mathfrak{sp}_{2n})$.

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Chapter 1

INTRODUCTION

1.1 Historical background

The cohomology ring $H^*(G;k)$ of a finite group scheme G has long been an important tool in modular representation theory over algebraically closed fields k of positive characteristic p. Unfortunately the cohomology rings of many important objects have proved resistant to computation. Consequently there has been a growing interest in what is known as the cohomology variety $|G|$ and the support variety $|G|_M$ of a G-module M. The *cohomology variety* $|G|$ is the affine variety defined by the maximal ideals of the commutative ring $H^\bullet(G;k) = \bigoplus_{n\in N} H^n(G;k)$, where $N = \mathbb{N}_0$ when p is even and $N = 2\mathbb{N}_0$ when p is odd. In the case of finite groups and, more generally, compact Lie groups (and their equivariant cohomology) the celebrated Quillen stratification theorem, proven in 1971 [43], allows one to determine basic geometric information about the cohomology variety even when an explicit description of the variety itself, and moreso the cohomology ring, is unknown. This was the first major result that showed that the cohomology variety may be an easier invariant to work with than the full cohomology ring.

To each representation M one may associate the $H^\bullet(G;k)$-module $\mathrm{Ext}^*_G(M,M)$. Its annihilator defines a subvariety $|G|_M \subseteq |G|$ of the cohomology variety called the *support variety* of M. Support varieties have become an important, and often explicitly calculable, invariant in modular representation theory. In the case of an elementary abelian group $(\mathbb{Z}/p\mathbb{Z})^n$ the cohomology ring is known hence the cohomology variety is known: One simply gets affine n-space $\mathbb{A}^n$. Thanks to work of Carlson [12] and Avrunin and Scott [2] in the early 80's the support variety of M has what is known as a rank variety description. One restricts M to various so called cyclic shifted subgroups (which are, in fact, subalgebras of the group ring and not subgroups of the group). These cyclic shifted subgroups are parameterized by $\mathbb{A}^n$ and $|G|_M$ consits of all points at which the restriction is not projective. The benefit of this description is that it is entirely non-cohomological and involves merely checking the ranks of various nilpotent matrices defined by the action of $(\mathbb{Z}/p\mathbb{Z})^n$ on M; an elementary task.

Three years later the rank variety description of a support variety was extended from elementary abelian groups to classical restricted Lie algebras when the prime is sufficiently large. Friedlander and Parshall [20, 21] showed that the cohomology variety of the restricted Lie algebra $\mathfrak{g}$ is homeomorphic to the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ of p-nilpotent elements in $\mathfrak{g}$. Moreover, the support variety of a $\mathfrak{g}$-module M may once again be determined non-cohomologically by looking at the ranks of various nilpotent matrices defined by the action of $\mathfrak{g}$ on M.

These results were further extended by Suslin et al. [58, 59] in 1997. They worked with an infinitesimal group scheme G of finite height and showed that the functor $R \mapsto \operatorname{Hom}(\mathbb{G}_{a(r),R}, G_R)$ of one-parameter subgroups of G is represented by an affine scheme $V(G)$ whose coordinate ring $k[V(G)]$ has a natural grading. The k-points of $V(G)$ yield an affine variety which they proved is homeomorphic to the cohomology variety $|G|$. Moreover, the subvariety $|G|_M$ of $|G|$ can be identified under this homeomorphism by looking at what is essentially the restriction of M under certain one-parameter subgroups and testing for projectivity. The infinitesimal group scheme $\mathbb{G}_{a(1)}^n$ has a representation theory which is equivalent to that of the elementary abelian group $(\mathbb{Z}/p\mathbb{Z})^n$ and $V(\mathbb{G}_{a(1)}^n) = \mathbb{A}^n$ recovers the rank variety description of Carlson, Avrunin, and Scott. Similarly, if $\mathfrak{g}$ is a restricted Lie algebra then there exists a height 1 infinitesimal group scheme $\underline{\mathfrak{g}}$ with representation theory equivalent to $\mathfrak{g}$. One gets $V(\underline{\mathfrak{g}}) = \mathcal{N}_1(\mathfrak{g})$ and we recover the results of Friedlander and Parshall but with no conditions on $\mathfrak{g}$ or on the prime p. Again we emphasize that this yields a non-cohomological description of $|G|_M$ which only requires one to check the ranks of various nilpotent matrices.

While there is both completed and ongoing research on extending the definition of $V(G)$

to a larger class of objects we are at this point interested instead in two closely related research directions that have spawned from the definition of $V(G)$. For the first direction, in 2011 Friedlander and Pevtsova [23] initiated the study of sheaves on $V(G)$ and on its projectivization, the scheme $\mathbb{P}(G) = \operatorname{Proj} k[V(G)]$. In particular, they take a representation M of G and define the associated global nilpotent operator Θ_M which essentially encodes as a sheaf homomorphism the nilpotent matrices whose ranks one checks when defining $|G|_M$. One can construct from Θ_M, in a functorial way, sheaves on $\mathbb{P}(G)$ whose geometric properties reflect representation theoretic properties of M. This raises questions about how faithfully these representation theoretic properties are reflected and what properties these sheaf functors posses.

In another direction, Carlson et al. [13] have defined the variety $\mathbb{E}(r, \mathfrak{g})$ whose points are simply the r-dimensional subalgebras contained in $\mathcal{N}_1(\mathfrak{g})$ (the so called *elementary* subalgebras) and whose topology arises from its natural inclusion as a closed subset of the Grassmannian of r-planes in $\mathfrak{g}$. As $\mathbb{E}(1, \mathfrak{g})$ is simply the projectivization of the variety $\mathcal{N}_1(\mathfrak{g})$ one thinks of $\mathbb{E}(r, \mathfrak{g})$ as a natural generalization of $\mathbb{P}(G)$. As in the $\mathbb{P}(G)$ case Carlson et al. [15, 14] have constructed global operators and, in a functorial way, sheaves on $\mathbb{E}(r, \mathfrak{g})$ arising from a representation of $\mathfrak{g}$. The variety $\mathbb{E}(r, \mathfrak{g})$ is also directly related to the much studied vareity of r-tuples of nilpotent commuting elements in $\mathfrak{g}$ and, consequently, to the cohomology of Frobenius kernels of algebraic groups. Further geometric properties of $\mathbb{E}(r, \mathfrak{g})$ were recently investigated by Warner [60].

1.2 Overview and main results

This thesis is divided into five chapters. This introduction is the first chapter. Chapters 2 - 4 concern the sheaf constructions of Friedlander and Pevtsova on $\mathbb{P}(G)$. The fifth chapter gives an explicit computation of the varieties $\mathbb{E}(r, \mathfrak{g})$ for maximal r and reductive $\mathfrak{g}$.

1.2.1 Chapter 2

We give background information required for chapters three and four, beginning with a brief reminder of what an infinitesimal group scheme G of finite height r is. In section 2.1 we summarize the definition and relevant results concerning the support scheme $V(G)$ and its projectivization $\mathbb{P}(G)$. We will construct the global operator Θ_M and give several relevant examples.

In section 2.2 we define and discuss the local Jordan type of M in terms of partitions and Young diagrams. Most notably we define the support $\mathbb{P}(G)_M \subseteq \mathbb{P}(G)$ as the set of points at which the global operator Θ_M does not yield a projective $K[t]/t^p$-module. This definition differs slightly from the definition given by Friedlander and Pevtsova [23] who define the local Jordan type over $V(G)$. We instead call this the affine local Jordan type and show in section 2.3 that these two notions of the Jordan type of a module are equivalent.

1.2.2 Chapter 3

In this chapter we investigate two open questions concerning G-modules and their associated sheaves: How the projectivity of a module M is reflected in a particular sheaf, $\mathcal{H}_{[1]}(M)$, and whether the support of $\mathcal{H}_{[1]}(M)$ can be given a representation theoretic description.

We begin in section 3.1 by defining the kernel, image, and cokernel sheaves associated to the global operator Θ_M and we define the sheaves

$$\mathscr{F}_i(M) = \frac{\ker \Theta_M \cap \operatorname{im} \Theta_M^{i-1}}{\ker \Theta_M \cap \operatorname{im} \Theta_M^{i}}$$

for $1 \le i \le p$. The $\mathscr{F}_i(M)$ were first defined by Benson and Pevtsova [6] for the case of an elementary abelian group of rank r. There we have $\mathbb{P}(G) = \mathbb{P}^{r-1}$ and they prove a realization theorem for vector bundles over this projective space; specifically, if $\mathscr{G}$ is a vector bundle of rank s then, up to a Frobenius twist, $\mathscr{G}$ can be realized as $\mathscr{F}_1(M)$ for some module M of constant Jordan type $[p]^n[1]^s$. When $p = 2$ the Frobenius twist is not present and Benson and Pevtsova note that their result can be thought of as a version of the Bernšteĭn, Gel'fand, Gel'fand (BGG) correspondence between modules for an exterior algebra and vector bundles over $\mathbb{P}^{r-1}$ (we discuss this connection in section 3.4).

The $\mathscr{F}_i(M)$ reflect many properties of the module M; in particular, we give a local version of the result by Benson and Pevtsova that a module M has constant Jordan type if and only if each $\mathscr{F}_i(M)$ is locally free and, moreover, that the ranks of the $\mathscr{F}_i(M)$ yield the Jordan type of M. In addition to this result, the $\mathscr{F}_i(M)$ are significant in that they are isomorphic to the quotients of filtrations of many of the sheaves that we are interested in.

The main theoretical results of this chapter are contained in section 3.2. Here we introduce the sheaves

$$\mathcal{H}_{[i]}(M) = \frac{\ker \Theta_M^i}{\operatorname{im} \Theta_M^{p-i}}$$

and ask the following two questions:

Question 3.2.2. Is M projective if and only if $\mathcal{H}_{[1]}(M) = 0$?

Question 3.2.7. Given a G-module M do we have $\mathbb{P}(G)_M = \operatorname{Supp} \mathcal{H}_{[1]}(M)$?

The sheaves $\mathcal{H}_{[i]}(M)$ were first defined by Friedlander and Pevtsova using the notation $\mathcal{M}^{[i]}$ [23, 5.14] and then later the notation $\mathcal{H}^{[i]}(M)$. They were motivated by work of Duflo and Serganova [19] on Lie superalgebras. Serganova has indicated in private correspondence that question 3.2.2 has an affirmative answer in the case of a Lie superalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ which satisfies the additional assumption that the self commuting elements of $\mathfrak{g}_1$ span $\mathfrak{g}_1$. This is a mild assumption satisfied, in particular, by any simple classical Lie superalgebra except $\mathfrak{osp}(1|2n)$. Duflo and Serganova note that in case $\mathfrak{g} = \mathfrak{osp}(1|2n)$ every finite-dimensional $\mathfrak{g}$-module is projective anyway [19, 3.6].

In theorem 3.2.6 we obtain a sizable list of conditions, any one of which implies that M is projective under the assumption $\mathcal{H}_{[1]}(M) = 0$. We show that if $\operatorname{Reg} \mathbb{P}(G)$ is the regular locus of $\mathbb{P}(G)$ then $\mathbb{P}(G)_M \cap \operatorname{Reg} \mathbb{P}(G) \subseteq \operatorname{Supp} \mathcal{H}_{[1]}(M)$. This answers question 3.2.7, and consequently question 3.2.2, in the affirmative at least in the case that $\mathbb{P}(G)$ is smooth:

Corollary 3.2.11. If $\mathbb{P}(G)$ is regular then $\mathbb{P}(G)_M = \operatorname{Supp} \mathcal{H}_{[1]}(M)$.

Corollary 3.2.12. If $\mathbb{P}(G)$ is regular then M is projective if and only if $\mathcal{H}_{[1]}(M) = 0$.

For the remainder of section 3.2 we consider the full subcategory $\ker(\mathcal{H}_{[1]})$ of the stable module category $\underline{\mathtt{mod}}(G)$ consisting of those objects M which satisfy $\mathcal{H}_{[1]}(M) = 0$. We show that this is a thick triangulated subcategory of $\underline{\mathtt{mod}}(G)$ and we give an example where $\ker(\mathcal{H}_{[1]})$ has infinite representation type, thus providing, in general, an answer to Questions 3.2.2 and 3.2.7 in the negative.

In section 3.3 we give additional explicit computations of $\mathbb{P}(G)$ for various G. In light of the results of section 3.2 we focus on examples of G such that $\mathbb{P}(G)$ is regular. We end, in section 3.4, by reminding the reader of the BGG correspondence. We give a generalization to DG-modules over a polynomial ring which is a slight modification of the correspondence obtained by Benson et al. [8, 5.5] (which was in turn inspired by a correspondence obtained by Avramov et al. [1, 7]) and we show that $\mathcal{H}_{[1]}$ factors through this correspondence. We use this to obtain further results on the functor $\mathcal{H}_{[1]}$ in the case of representations of an elementary abelian group.

1.2.3 Chapter 4

In the fourth chapter we explicitly compute examples of associated sheaves for the case of the Lie algebra $\mathfrak{sl}_2$. Descriptions of the finite dimensional indecomposable modules were given by Drozd [18] and Rudakov [46] in the early 80's and canonical realizations of these modules were given by Premet [42] in 1991. The representation type of $\mathfrak{sl}_2$ is tame; there are infinitely many isomorphism classes of indecomposable modules, so the category is rich enough to be interesting, but these occur in finitely many parameterized families which allow for direct computations. We also note that the variety $\mathbb{P}(\mathfrak{sl}_2)$ over which we wish to compute these sheaves is isomorphic to $\mathbb{P}^1$. By a theorem of Grothendieck we therefore know that locally free sheaves admit a strikingly simple description: They are all sums of twists of the structure sheaf. This makes $\mathfrak{sl}_2$ uniquely suited for such computations.

We begin in section 4.1 with the case of a general restricted Lie algebra $\mathfrak{g}$. We will review the definition of $\mathcal{N}_1(\mathfrak{g})$ and its projectivization $\mathbb{P}(\mathfrak{g})$. We use this to give a definition of the local Jordan type of a module M which has been simplified from the previous chapter for the case of a Lie algebra $\mathfrak{g}$. We also review a simplified definition of the global operator Θ_M associated to a $\mathfrak{g}$-module M.

In section 4.2 we begin the discussion of the category of $\mathfrak{sl}_2$-modules. Our computations are fundamentally based on having, for each indecomposable $\mathfrak{sl}_2$-module, an explicit basis and formulas for the $\mathfrak{sl}_2$ action. To this end we review Premet's description. There are four families and for each family we specify not only the explicit basis and $\mathfrak{sl}_2$-action, but also a graphical representation of the module and the local Jordan type of the module. For the Weyl modules $V(\lambda)$, dual Weyl modules $V(\lambda)^*$, and projective modules $Q(\lambda)$ this information was previously known (see for example Benkart and Osborn [4]) but for the non-constant modules $\Phi_\xi(\lambda)$ we do not know if such an explicit description has previously been given. Thus we give a proof that this description follows from Premet's definition of the modules $\Phi_\xi(\lambda)$. We also compute the Heller shifts $\Omega V(\lambda)$ of the Weyl modules for use in section 4.4.

In section 4.3 we digress from discussing Lie algebras and compute the kernels of four particular matrices. These matrices, with entries in the polynomial ring $k[s,t]$, will represent sheaf homomorphisms over $\mathbb{P}^1 = \operatorname{Proj} k[s,t]$ but in this section we do not work geometrically and instead consider these matrices to be maps of free $k[s,t]$-modules. This section contains the bulk of the computational effort of this chapter.

In section 4.4 we are finally ready to carry out the computations promised. Friedlander and Pevtsova have computed $\ker \Theta_{V(\lambda)}$ for the case $0 \leq \lambda \leq 2p-2$ [23]. We compute the sheaves $\ker \Theta_M$ for every indecomposable $\mathfrak{sl}_2$-module M. This computation is essentially the bulk of the work in the previous section; the four matrices in that section describe the global operators of the four families of $\mathfrak{sl}_2$-modules. We also compute $\mathscr{F}_i(V(\lambda))$ for $i \neq p$ and $V(\lambda)$ indecomposable using an inductive argument. The base case is that of a simple Weyl module ($\lambda < p$) and is done by noting that $\mathscr{F}_i(V(\lambda))$ is zero when $i \neq \lambda + 1$ and that $\mathscr{F}_{\lambda+1}(V(\lambda))$ can be identified with the kernel sheaf $\ker \Theta_{V(\lambda)}$. For the inductive step we use the Heller shift computation from section 4.2 together with a theorem of Benson and Pevtsova [6].

1.2.4 Chapter 5

In the final chapter we give a description of the variety $\mathbb{E}(\mathfrak{g}) = \mathbb{E}(r_{\max}, \mathfrak{g})$ for $\mathfrak{g} = \operatorname{Lie} G$ the Lie algebra of a reductive algebraic group and $r_{\max}$ the maximal dimension of an elementary subalgebra of $\mathfrak{g}$. The maximal dimension of an abelian nilpotent subalgebra of a complex simple Lie algebra $\mathfrak{g}$ is known thanks to the work of Malcev [36] while the general linear case was considered by Schur at the turn of the previous century [47]. Malcev has also classified such subalgebras up to automorphisms of the Lie algebra. It turns out that under a mild restriction on p the maximal dimension of an elementary subalgebra in the modular case agrees with Malcev's results. To compute the variety $\mathbb{E}(r_{\max}, \mathfrak{g})$ we need to consider elementary subalgebras of $\mathfrak{g}$ up to conjugation by G so the calculations and the end result differ from Malcev's, who classified abelian subalgebras up to automorphisms of $\mathfrak{g}$. Nonetheless, his linear algebraic approach is very useful.

We note that calculation of $\mathbb{E}(\mathfrak{g})$ in the special linear or symplectic case was done by Carlson et al. [13]. The arguments in that paper are based on induction on the dimension and are qualitatively different from the arguments presented here inspired by Malcev's approach.

We begin in section 5.1 by recalling various conditions on p such as good, very good, and torsion and define what it means to be separably good. We also state the combinatorial classification of maximal sets of commuting roots for irreducible root systems in Table 5.2. We refer to Malcev [36] for this classification but also give a detailed explanation in the Appendix motivated largely by the fact that Malcev's paper is very sketchy on details.

The classification of the maximal elementary subalgebras of the unipotent radical $\mathfrak{u} \subset \mathfrak{g}$ up to conjugation by G is settled in section 5.2 where we explicitly determine $\mathbb{E}(\mathfrak{u})$ for $\mathfrak{u} \subset \mathfrak{g}$ *as a set*. We define a map $\operatorname{Lie}\colon \max(\Phi) \to \mathbb{E}(\mathfrak{u})$ which sends a maximal set of commuting positive roots to an elementary subalgebra of maximal dimension in $\mathfrak{u}$ and show that there is an inverse map $\operatorname{LT}\colon \mathbb{E}(\mathfrak{u}) \to \max(\Phi)$ which splits Lie. The map Lie is not necessarily surjective but we show that for all irreducible root systems Φ except for G_2 and A_2 it is surjective up to conjugation by U. Hence, we effectively prove that the maximal elementary subalgebras in $\mathfrak{u}$ up to conjugation are given by the combinatorics of the root system of G. This calculation largely relies on the linear algebraic approach of Malcev and is split into several cases dictated by the existence of certain orderings on the corresponding root systems:

1. A_{2n+1}, B_2, B_3, C_n, E_7 (in these cases, there is a unique maximal elementary subalgebra in $\mathfrak{u}$ given by a maximal set of commuting positive roots),
2. D_n (three maximal elementary subalgebras for $n = 4$ and two for $n \geq 5$, all given by maximal sets of commuting positive roots),
3. B_n (three families of maximal elementary subalgebras for $n = 4$, two families for $n \geq 5$, only given by maximal sets of commuting positive roots up to conjugation by U),
4. G_2, A_2 (exceptional cases).

The cases of E_6, E_8 and F_4 are handled by a computer calculation for which we provide the code.

The calculation of $\mathbb{E}(\mathfrak{g})$ is carried out in section 5.3. We first reduce to the case of a simple algebraic group G with root system Φ. For such G, we compute $\mathbb{E}(\mathfrak{g})$ under the assumption that p is *separably good* for G (see definition 5.1.2). We rely on the result of Levy et al. [35] to show that any elementary subalgebra of $\mathfrak{g}$ can be conjugated into $\mathfrak{u} \subset \mathfrak{g}$, the Lie algebra of the unipotent radical U of the Borel subgroup $B \leq G$. With the exception of G_2 and A_2, every elementary subalgebra in $\mathbb{E}(\mathfrak{u})$ is G-conjugate to a subalgebra stabilized by the action of the Borel. The stabilizers of such subalgebras are parabolic subgroups which implies that the G-orbits in $\mathbb{E}(\mathfrak{g})$ are partial flag varieties. To finish the calculation in all types, except for G_2 and A_2, we prove in theorem 5.3.9 that for $\mathcal{E}$ with parabolic stabilizer the orbit map $G \twoheadrightarrow G \cdot \mathcal{E} \subset \mathbb{E}(\mathfrak{g})$ is separable.

When p is not separably good, such as $p = 2$ for $G = \mathrm{PGL}_2$, the answer for $\mathbb{E}(\mathfrak{g})$ can be somewhat surprising. In example 5.3.6 we utilize a construction from Levy et al. [35] to illustrate that. We also illustrate in example 5.3.10 that in general the orbit map $G \to G \cdot \epsilon \subset \mathbb{E}(\mathfrak{g})$ can fail to be separable.

The ultimate outcome is that the projective variety $\mathbb{E}(\mathfrak{g})$ is a product of $\mathbb{E}(\mathfrak{g}_i)$ where $\mathfrak{g}_i = \operatorname{Lie}(G_i)$ and the G_i range over the simple algebraic subgroups of the derived group of G. When G is simple $\mathbb{E}(\mathfrak{g})$ is a finite disjoint union of partial flag varieties *unless* G is of type G_2 or A_2. This is proved in theorem 5.3.11:

Theorem. *Let G be a simple algebraic group with root system Φ which is not of type A_2 or G_2, and let $\mathfrak{g} = \operatorname{Lie} G$. Assume that p is separably good for G. Then*

$$\mathbb{E}(\mathfrak{g}) = \coprod_{\substack{R \in \max(\Phi) \\ R \text{ an ideal}}} G/P_R,$$

where P_R is the parabolic stabilizer of the elementary Lie algebra associated with the root subset R.

Reinterpreted explicitly for each type, theorem 5.3.11 implies that $\mathbb{E}(\mathfrak{g})$ is a disjoint union of at most three copies of generalized Grassmannians in types A_n $(n \neq 2)$, C, D, E, whereas in types B and F the "two step" partial flag varieties appear (see Table 5.4). For types A_2, G_2 we show that $\mathbb{E}(\mathfrak{g})$ is an irreducible variety and compute its dimension and G-orbits. We find that this non-homogeneous answer partially justifies the fact that we have to resort to case-by-case considerations in our calculations.

In section 5.4 we apply our results to obtain information on conjugacy classes of maximal elementary abelian p-subgroups in Chevalley groups. An analogous classification of elementary abelian p-subgroups in a Chevalley group has been considered by several authors, for example Barry [3] and Milgram and Priddy [39], with the most complete account given by Gorenstein, Lyons and Solomon in [26, 3.3]. Using a particularly nice Springer isomorphism from the nullcone of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ to the unipotent variety of G (constructed by P. Sobaje [49]) we can quickly recover the known results on maximal dimensions of elementary abelian p-subgroups in (untwisted) Chevalley groups. We also obtain information on the conjugacy classes of elementary abelian p-subgroups of $G(\mathbb{F}_{p^r})$. Thanks to the Quillen stratification theorem this has an immediate application to mod-p group cohomology: The number of conjugacy classes of the elementary abelian p-subgroups gives the number of irreducible components of maximal dimension in $\operatorname{Spec} H^*(G(\mathbb{F}_{p^r}), \overline{\mathbb{F}}_p)$.

Throughout the paper, k will be an algebraically closed field of positive characteristic p.

Chapter 2

SUPPORT VARIETIES AND LOCAL JORDAN TYPE [51]

In this chapter we give background information required for the two following chapters, specifically we remind the reader what an infinitesimal group scheme of finite height is, and we discuss the support varieties and the notion of local Jordan type associated to such group schemes.

Recall that a group scheme G (over k) is *infinitesimal* of height at most r if the coordinate algebra $k[G]$ is a finite dimensional local ring and $x^{p^r} = 0$ for all x contained in the maximal ideal. To give G it suffices to designate a commutative Hopf algebra as the coordinate ring $k[G]$ or designate a cocommutative Hopf algebra as its linear dual: the group ring $kG = \mathrm{Hom}_k(k[G], k)$. The group ring is significant in particular because a the category of representations of G is equivalent to the category of kG-modules. Our two main examples, 2.0.1 and 2.0.4, will be height 1 group schemes whose representation theory is equivalent to the representation theory of a restricted Lie algebra and an elementary abelian group, respectively.

Example 2.0.1. Let $\mathfrak{g}$ be a restricted Lie algebra (always assumed to be finite dimensional) and $\mathcal{U}_p(\mathfrak{g})$ its restricted universal enveloping algebra. This is a cocommutative Hopf algebra whose primitive elements are exactly the elements of the Lie algebra $\mathfrak{g} \subseteq \mathcal{U}_p(\mathfrak{g})$. We define a group scheme $\underline{\mathfrak{g}}$ by designating the group ring $k\underline{\mathfrak{g}} = \mathcal{U}_p(\mathfrak{g})$; it is an infinitesimal height 1 scheme [31, I.8.5.b]. Now $\underline{\mathfrak{g}}$-modules are equivalent to $\mathcal{U}_p(\mathfrak{g})$-modules, i.e., representations of $\mathfrak{g}$ as a restricted Lie algebra.

Example 2.0.2. Let G be an algebraic group and $r \in \mathbb{N}$. The r^{th} *Frobenius kernel* of G is denoted $G_{(r)}$. It is the group scheme whose coordinate ring is the Hopf algebra

$$k[G_{(r)}] = \frac{k[G]}{\langle f^{p^r} \mid f \in I_1 \rangle},$$

where $I_1 \subseteq k[G]$ is the augmentation ideal. It is infinitesimal of height at most r. Note that if $\mathfrak{g} = \mathrm{Lie}(G)$ then the schemes $\underline{\mathfrak{g}}$ and $G_{(1)}$ can be identified; see Jantzen [31, I.9] for details.

Example 2.0.3. The r^{th} Frobenius kernel, $\mathbb{G}_{a(r)}$, of the additive group is an infinitesimal group scheme of height at most r. The coordinate ring is $k[\mathbb{G}_{a(r)}] = k[t]/t^{p^r}$ with Hopf structure given by designating t a primitive element (i.e. the comultiplcation map sends t to $t \otimes 1 + 1 \otimes t$). As a k-algebra the group ring of $\mathbb{G}_{a(r)}$ is

$$k\mathbb{G}_{a(r)} = \frac{k[u_1, \ldots, u_r]}{u_1^p, \ldots, u_r^p}$$

where u_i is dual to $t^{p^{i-1}}$ in the monomial basis. This is isomorphic to the group ring of the elementary abelian group $E = (\mathbb{Z}/p)^r$ but with a different Hopf structure. Consequently, the equivalence between modules over $\mathbb{G}_{a(r)}$ and modules over E does not extend to their tensor monoidal structures.

Example 2.0.4. The group scheme $\mathbb{G}_{a(1)}^r$ is infinitesimal of height 1. The coordinate ring is $k[\mathbb{G}_{a(1)}^r] = k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p)$ with Hopf structure given by designating each t_i a primitive element. As a k-algebra the group ring of $\mathbb{G}_{a(1)}^r$ is

$$k\mathbb{G}_{a(1)}^r = \frac{k[u_1, \ldots, u_r]}{u_1^p, \ldots, u_r^p}$$

where u_i is dual to t_i in the monomial basis. Once again, this group ring is isomorphic as a k-algebra to the group ring $k(\mathbb{Z}/p)^r$ but with a different Hopf structure: Each u_i is a primitive element. As with the previous example, the equivalence between $\mathbb{G}_{a(1)}^r$-modules and $(\mathbb{Z}/p)^r$-modules does not extend to their tensor monoidal structures.

Remark 2.0.5. The standard Hopf structure on the group ring of $E = (\mathbb{Z}/p)^r$ yields a group scheme which is finite but not infinitesimal. Our methods apply only to infinitesimal schemes so we will use $\mathbb{G}_{a(1)}^r$ when considering modules for an elementary abelian group of rank r.

2.1 The Global Operator of an Infinitesimal Group Scheme

A special role is played by the $\mathbb{G}_{a(r)}$ from example 2.0.3. Denote by G_R the base extension of G to a commutative k-algebra R; this is a group scheme over R with coordinate ring $R[G_R] = R \otimes_k k[G]$ and group ring $RG_R = R \otimes_k kG$. A 1-*parameter subgroup* of height r is a homomorphism $\mathbb{G}_{a(r),R} \to G_R$ of group schemes over some R. The collection of all such homomorphisms defines the functor of points of the *support scheme* $V(G)$ of G.

Theorem 2.1.1 ([58, 1.5, 1.12])**.** *Let G be an infinitesimal group scheme of height r. Then there exists an affine scheme $V(G) = \operatorname{Spec} k[V(G)]$ whose functor of points*

$$V(G)(-) = \operatorname{Hom}_{k\text{-}\mathtt{alg}}(k[V(G)], -)$$

is naturally isomorphic to the functor

$$R \mapsto \operatorname{Hom}_{R\text{-}\mathtt{grp}}(\mathbb{G}_{a(r),R}, G_R)$$

from commutative k-algebras to sets. Moreover, $k[V(G)]$ is a finitely generated connected graded k-algebra with homogeneous generators of degree p^i for $0 \leq i < r$.

Remark 2.1.2. For G infinitesimal of height 1 the coordinate ring $k[V(G)]$ is generated in degree 1, but for G of larger heights this need not be the case.

Example 2.1.3. Let $\mathfrak{g}$ be a restricted Lie algebra with basis $\{g_1, \ldots, g_n\}$ and dual basis $\{x_1, \ldots, x_n\}$. Given a commutative k-algebra R we extend the p-operation to $(-)^{[p]} \colon R \otimes_k \mathfrak{g} \to R \otimes_k \mathfrak{g}$ via $a \otimes v \mapsto a^p \otimes v^{[p]}$. Choose $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ such that

$$(x_1 \otimes g_1 + \cdots + x_n \otimes g_n)^{[p]} = f_1 \otimes g_1 + \cdots + f_n \otimes g_n.$$

We define the *restricted nullcone* of $\mathfrak{g}$ to be $\mathcal{N}_1(\mathfrak{g}) = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. This is the scheme whose functor of points is given by

$$\mathcal{N}_1(\mathfrak{g})(R) = \left\{ v \in R \otimes_k \mathfrak{g} \;\middle|\; v^{[p]} = 0 \right\}.$$

Note that the k-points $\mathcal{N}_1(\mathfrak{g})(k) \subseteq \mathfrak{g}$ of this scheme give the traditional definition of the restricted nullcone, but the scheme $\mathcal{N}_1(\mathfrak{g})$ need not be reduced.

The group $\underline{\mathfrak{g}}$ from example 2.0.1 has support scheme $\mathcal{N}_1(\mathfrak{g})$. The isomorphism $\mathcal{N}_1(\mathfrak{g})(R) \simeq \operatorname{Hom}_{R\text{-}\mathtt{grp}}(\mathbb{G}_{a(1),R}, \underline{\mathfrak{g}}_R)$ sends a p-nilpotent $v \in R \otimes_k \mathfrak{g}$ to the homomorphism $\mathbb{G}_{a(1),R} \to \underline{\mathfrak{g}}_R$ whose induced map on group rings, $R[x]/x^p \to R \otimes_k \mathcal{U}_p(\mathfrak{g})$, is defined by $x \mapsto v$.

Let $\exp\colon \mathcal{N}_1(\mathfrak{gl}_n) \to \mathrm{GL}_n$ be the matrix exponential defined on p-nilpotent matrices by the truncated exponential series

$$\exp(M) = 1 + M + \frac{1}{2}M^2 + \cdots + \frac{1}{(p-1)!}M^{p-1}.$$

Definition 2.1.4. Let G be an algebraic group and $\Phi\colon G \to \mathrm{GL}_n$ a closed embedding. If, for each p-nilpotent $x \in \mathrm{Lie}(G)$, the one-parameter subgroup $\mathbb{G}_a \to \mathrm{GL}_n$ defined by $t \mapsto \exp(\mathrm{d}\Phi(tx))$ factors through Φ then we say that Φ is an embedding of *exponential type*. If G has such an embedding then the group G is of exponential type.

By inspection the following groups are of exponential type: GL_n, SL_n, Sp_{2n}, B_n (upper triangular $n \times n$ matrices), U_n (strictly upper triangular $n \times n$ matrices), and the orthogonal group $O(\phi)$ associated to a non-degenerate bilinear form ϕ (see Suslin et al. [58, 1.8], McNinch [38], and Sobaje [50] for further discussion of the exponential type condition).

Example 2.1.5. Let G be of exponential type, $\mathfrak{g} = \mathrm{Lie}(G)$, and $r \in \mathbb{N}$. Define $\mathcal{N}_1^{[r]}(\mathfrak{g})$ to be the closed subscheme of $(\mathcal{N}_1(\mathfrak{g}))^r$ consisting of r-tuples whose elements pairwise commute; then the support scheme of $G_{(r)}$ is $V(G_{(r)}) = \mathcal{N}_1^{[r]}(\mathfrak{g})$. Given $(\alpha_1, \ldots, \alpha_r) \in \mathcal{N}_1^{[r]}(\mathfrak{g})$ the coordinate functions for the elements of the matrix α_i have degree p^{i-1}. See Suslin et al. [58, 1.7] and Friedlander and Pevtsova [23, 2.10] for more details.

Example 2.1.6. The support scheme of the group scheme $\mathbb{G}_{a(1)}^r$ from example 2.0.4 is $\mathbb{A}^r = \operatorname{Spec} k[x_1, \ldots, x_r]$. The isomorphism $\mathbb{A}^r(R) \simeq \operatorname{Hom}_{R\text{-}\mathtt{grp}}(\mathbb{G}_{a(1),R}, \mathbb{G}_{a(1),R}^r)$ sends the element $(a_1, \ldots, a_r) \in R^r$ to the homomorphism $\mathbb{G}_{a(1),R} \to \mathbb{G}_{a(1),R}^r$ whose induced map on coordinate rings, $R[t]/t^p \to R[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p)$, is defined by $t \mapsto \sum_i a_i t_i$.

Example 2.1.7. The support scheme of the group scheme $\mathbb{G}_{a(r)}$ from example 2.0.3 is $\mathbb{A}^r = \operatorname{Spec} k[x_1, \ldots, x_r]$. The isomorphism $\mathbb{A}^r(R) \simeq \operatorname{Hom}_{R\text{-}\mathtt{grp}}(\mathbb{G}_{a(r),R}, \mathbb{G}_{a(r),R})$ sends an element $(a_1, \ldots, a_r) \in R^r$ to the homomorphism $\mathbb{G}_{a(r),R} \to \mathbb{G}_{a(r),R}$ whose induced map on coordinate rings, $R[t]/t^{p^r} \to R[t]/t^{p^r}$, is defined by $t \mapsto \sum_i a_i t^{p^{i-1}}$.

The grading of $k[V(G)]$ is given as follows. From example 2.1.7 we have $V(\mathbb{G}_{a(r)}) = \mathbb{A}^r$ so we get a right monoid action $V(G) \times \mathbb{A}^r \to V(G)$ via composition of 1-parameter subgroups. Restrict to an action $V(G) \times \mathbb{A}^1 \to V(G)$ by including $\mathbb{A}^1 \subset \mathbb{A}^r$ as the first factor (these are the 1-parameter subgroups of $\mathbb{G}_{a(r)}$ whose maps on coordinate rings $k[t]/t^{p^r} \to k[t]/t^{p^r}$ are degree preserving). If $\phi\colon k[V(G)] \to k[V(G)] \otimes_k k[t]$ is the associated comorphism then the algebra $k[V(G)]$ is graded by $k[V(G)]_n = \phi^{-1}(k[V(G)] \otimes_k t^n)$ [58, 1.11].

Example 2.1.8. Consider $k[V(\mathbb{G}_{a(r)})] = k[x_1, \ldots, x_r]$ from example 2.1.7. To determine the grading note that the monoid action $\mathbb{A}^r \times \mathbb{A}^1 \to \mathbb{A}^r$ is given by

$$(a_1, a_2, \ldots, a_r) \cdot b = (a_1 b, a_2 b^p, \ldots, a_r b^{p^{r-1}}).$$

Its comorphism $k[x_1, \ldots, x_r] \to k[x_1, \ldots, x_r] \otimes_k k[t]$ is defined by $x_i \mapsto x_i \otimes t^{p^{i-1}}$ therefore $\deg x_i = p^{i-1}$.

Definition 2.1.9. Let $\mathbb{P}(G) = \operatorname{Proj} k[V(G)]/\mathfrak{N}$ where $\mathfrak{N}$ is the nilradical of $k[V(G)]$.

Remark 2.1.10. Reducing $V(G)$ before taking Proj isn't strictly necessary. Most of the results that follow work over $\operatorname{Proj} k[V(G)]$ as well, but for some directions in theorem 3.1.2 and theorem 3.2.6 we must assume that the open subscheme $U \subseteq \mathbb{P}(G)$ is reduced. As $V(G)$ is not, in general, reduced there are more applications of the theory if we reduce $V(G)$ here.

In section 3.1 we will construct sheaves on $\mathbb{P}(G)$ from kernels, images, and cokernels of certain global operators, Θ_M, which we now describe. The generic element of the functor of points of $V(G)$ is the homomorphism of group schemes

$$\mathcal{U}_G\colon \mathbb{G}_{a(r),k[V(G)]} \to G_{k[V(G)]}$$

in $V(G)(k[V(G)]) \simeq \operatorname{Hom}_{k\text{-}\mathtt{alg}}(k[V(G)], k[V(G)])$ corresponding to the identity map. This induces a Hopf $k[V(G)]$-algebra homomorphism between the corresponding group rings

$$\mathcal{U}_{G,*}\colon k[V(G)] \otimes_k \frac{k[u_1, \ldots, u_r]}{(u_1^p, \ldots, u_r^p)} \to k[V(G)] \otimes_k kG.$$

Given a kG-module M, its extension of scalars $k[V(G)] \otimes_k M$ is a module over $k[V(G)] \otimes_k kG$ and multiplication by $\Theta_G = \mathcal{U}_{G,*}(1 \otimes u_r)$ gives a homomorphism

$$\Theta_M\colon k[V(G)] \otimes_k M \to k[V(G)] \otimes_k M$$

of $k[V(G)]$-modules. This map is homogeneous of degree p^{r-1} [23, 2.11]; hence, after reducing, it induces a map of sheaves over $\mathbb{P}(G)$ which, by some abuse of notation, we will also denote Θ_M.

Definition 2.1.11. Given a G-module M we define $\widetilde{M} = \mathcal{O}_{\mathbb{P}(G)} \otimes_k M$. The *global operator* corresponding to M is the sheaf map

$$\Theta_M\colon \widetilde{M} \to \widetilde{M}(p^{r-1})$$

induced by the action of $\Theta_G = \mathcal{U}_{G,*}(1 \otimes u_r)$.

Remark 2.1.12. We caution the reader that $\mathcal{O}_{\mathbb{P}(G)}(1)$ need not be locally free when the ring $k[V(G)]$ is not generated in degree 1. Instead one has that for any integer $d \in \mathbb{Z}$ the sheaf $\mathcal{O}_{\mathbb{P}(G)}(dp^{r-1})$ is locally free of rank 1. This follows from the fact that the generators of $k[V(G)]$ have order dividing p^{r-1} (c.f. theorem 2.1.1 and Friedlander and Pevtsova [23, 4.5]). Consequently, $\widetilde{M}(dp^{r-1})$ is locally free of rank $\dim M$ and its specialization at a point $v \in \mathbb{P}(G)$ is $k(v) \otimes_{\mathcal{O}_{\mathbb{P}(G),v}} \widetilde{M}(dp^{r-1})_v \simeq k(v) \otimes_k M$.

Observe that $M \mapsto \widetilde{M}$ gives an exact functor from the category, $\mathtt{mod}(G)$, of finitely generated G-modules to the category, $\mathtt{Coh}_{\mathbb{P}(G)}$, of coherent sheaves over $\mathbb{P}(G)$. Global operators are natural with respect to this functor, i.e., given a module homomorphism $\phi\colon M \to N$ the diagram

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "\Theta_M"] \arrow[d, "\mathrm{id}\otimes\phi"'] & \widetilde{M}(p^{r-1}) \arrow[d, "\mathrm{id}\otimes\phi"] \\
\widetilde{N} \arrow[r, "\Theta_N"'] & \widetilde{N}(p^{r-1})
\end{tikzcd}$$

commutes.

Example 2.1.13. Recall when $G = \underline{\mathfrak{g}}$ we have $\mathbb{P}(G) = \operatorname{Proj} k[x_1, \dots, x_n]/\sqrt{f_1, \dots, f_n}$ from example 2.1.3. This is the reduced scheme corresponding to the projective variety given by the traditional restricted nullcone $\mathcal{N}_1(\mathfrak{g})(k) \subseteq \mathfrak{g}$. The global operator Θ_M is given by the action of $\Theta_{\underline{\mathfrak{g}}} = x_1 \otimes g_1 + \cdots + x_n \otimes g_n$.

Example 2.1.14. When $G = \mathbb{G}_{a(r)}$ example 2.1.8 gives that $\mathbb{P}(G)$ is the weighted projective space $\mathbb{P}(1, p, \dots, p^{r-1})$, i.e., we have $\mathbb{P}(G) = \operatorname{Proj} k[x_1, \dots, x_r]$ but with $\deg x_i = p^{i-1}$. Given a module M the global operator Θ_M is given by the action of $\Theta_{\mathbb{G}_{a(r)}} = x_1^{p^{r-1}} \otimes u_1 + x_2^{p^{r-2}} \otimes u_2 + \cdots + x_r \otimes u_r$. Note that the grading $\deg x_i = p^{i-1}$ implies $\deg x_i^{p^{r-i}} = p^{r-1}$ as required.

Example 2.1.15. When $G = \mathbb{G}_{a(1)}^r$ we have $\mathbb{P}(G) = \mathbb{P}^{r-1}$ from example 2.1.6. Given a module M the map Θ_M is given by the action of $\Theta_{\mathbb{G}_{a(1)}^r} = x_1 \otimes u_1 + \cdots + x_n \otimes u_n$. Note that $\Theta_{\mathbb{G}_{a(1)}^r}$ is the operator studied by Benson and Pevtsova [6].

2.2 Local Jordan Type

We now define the local Jordan type of a module by using the global operator Θ_M to associate a p-restricted partition to each point $v \in \mathbb{P}(G)$ in the underlying topological space of $\mathbb{P}(G)$, i.e., to each homogeneous prime ideal in $k[V(G)]$.

Definition 2.2.1. A *p-restricted partition* is a finite sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfying $p \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We call the λ_i the *parts* of λ and will also write such partitions in exponential notation where exponents denote repeated parts, for example $(5, 2, 2, 2, 1) = [5][2]^3[1]$.

Consider a finite dimensional vector space V. Given a linear operator $T\colon V \to V$ satisfying $T^p = 0$ we get a $k[t]/t^p$-module structure on V by letting t act via T. Each indecomposable $k[t]/t^p$-module is isomorphic to one of $k[t]/t^i$ where $1 \leq i \leq p$. If

$$V \simeq \bigoplus_{1 \leq i \leq p} \left(k[t]/t^i\right)^{a_i}$$

then we say the *Jordan type* of the operator T is $\mathrm{JType}(T) = [p]^{a_p}[p-1]^{a_{p-1}}\cdots[1]^{a_1}$. Note that a_i is the number of blocks of size i in the Jordan normal form of any matrix representation of T.

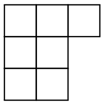
Definition 2.2.2. Let M be a G-module. The *local operator* at a point $v \in \mathbb{P}(G)$ is the linear operator $\theta_{v,M} = k(v) \otimes_{\mathcal{O}_{\mathbb{P}(G),v}} (\Theta_M)_v$ on $k(v) \otimes_k M$. Let $\mathscr{P}_p$ be the set of all p-restricted partitions. The *local Jordan type* of M is the function

$$\mathrm{JType}(-, M)\colon \mathbb{P}(G) \to \mathscr{P}_p$$

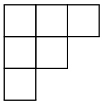
defined by $\mathrm{JType}(v, M) = \mathrm{JType}(\theta_{v,M})$.

The Young diagram of a partition will be a useful visualization tool. A *Young diagram* is a two dimensional array of finitely many boxes whose row lengths are weakly decreasing. Young diagrams correspond to partitions by reading row lengths from top to bottom and henceforth we will identify these two objects. We also remind the reader that the *conjugate* of a partition is the partition obtained by transposing the Young diagram.

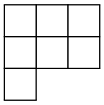
Example 2.2.3. The partitions $[3][2]^2$, $[3][2][1]$, and $[3]^2[1]$ are given by



,



, and



respectively and $[3][2]^2$ and $[3]^2[1]$ are conjugate.

The $k[t]/t^p$-module $k[t]/t^i$ corresponds to the partition $[i]$ given by 1 row of length i.

These i boxes correspond to the ordered basis $\{t^{i-1}, t^{i-2}, \ldots, t, 1\}$ of $k[t]/t^i$ so we can visualize the action of t as moving each box one step to the left and annihilating the leftmost box. More generally, given a linear operator $T\colon V \to V$ as above, the boxes in the Young diagram of $\mathrm{JType}(T)$ correspond to basis elements in the basis of V with respect to which T is in Jordan normal form. The action of T moves each box one step to the left and the boxes in the leftmost column of the Young diagram correspond to the basis elements which span the kernel of T.

Definition 2.2.4. Let $j \in \mathbb{N}$. The *local j-rank* of M is the function

$$\operatorname{rank}^j(-, M)\colon \mathbb{P}(G) \to \mathbb{N}_0$$

defined by $\operatorname{rank}^j(v, M) = \operatorname{rank} \theta^j_{v,M}$.

It is useful to know that we can reconstruct $\operatorname{JType}(v, M)$ if we know $\operatorname{rank}^j(v, M)$ for all j. Note, for our linear operator T, that $\operatorname{rank} T^j$ is exactly the number of boxes that do not lie in the first j columns of $\operatorname{JType}(T)$.

Lemma 2.2.5. *Let $T\colon V \to V$ be a p-nilpotent linear operator. Then* $\operatorname{JType}(T)$ *is conjugate to the partition*

$$(\operatorname{rank} T^0 - \operatorname{rank} T^1, \operatorname{rank} T^1 - \operatorname{rank} T^2, \ldots, \operatorname{rank} T^{p-1} - \operatorname{rank} T^p)$$

Proof. Follows immediately from the definition of a conjugate partition and the observation that $\operatorname{rank} T^{j-1} - \operatorname{rank} T^j$ is the number of boxes in the j^{th} column of $\operatorname{JType}(T)$. $\square$

Thus the local j-ranks of a module M encode the local Jordan type. The local Jordan type in turn encodes the data of whether or not M is projective.

Theorem 2.2.6 ([59, 7.6])**.** *A module M is projective if and only if its local Jordan type is the constant function $v \mapsto [p]^{\frac{\dim M}{p}}$.*

We say that M is *projective at $v \in \mathbb{P}(G)$* if the local Jordan type at v is $[p]^{\frac{\dim M}{p}}$, or more generally M is projective on $U \subseteq \mathbb{P}(G)$ if it is projective at every point of U. We say that M has *constant Jordan type* on a subset $U \subseteq \mathbb{P}(G)$ if the restriction of its local Jordan type to U is a constant function, and *constant j-rank* on U if the restriction of its local j-rank to U is a constant function. We note that the previous theorem can be interpreted as a statement about the support of the module M.

Definition 2.2.7. The *support*, $\mathbb{P}(G)_M$, of a module M is the set of points $v \in \mathbb{P}(G)$ at which M is not projective, i.e., at which $\operatorname{JType}(v, M) \neq [p]^{\frac{\dim M}{p}}$.

The support of M is a closed subset of $\mathbb{P}(G)$ [59, 6.1]. Theorem 2.2.6 says that a module M is projective (as a G-module) if and only if $\mathbb{P}(G)_M$ is the empty set or equivalently if and only if M is projective on $\mathbb{P}(G)$ (i.e. projective at every point of $\mathbb{P}(G)$).

2.3 Affine Local Jordan type

The definition of local Jordan type given above is slightly different from the one given by Friedlander and Pevtsova [23]. They work initially over $V(G)$ and define the local Jordan type as a function $V(G) \to \mathscr{P}_p$ out of $V(G)$ instead of $\mathbb{P}(G)$. Though not explicitly defined as such, they show that their definition is equivalent to the specialization of Θ_M when considered as a map of sheaves over $V(G)$. We will review this definition below under the name "affine local Jordan type" and show that it is equivalent to the local Jordan type defined above. Also see Friedlander and Pevtsova [23, Section 4] for the connection to the π-points definition of Jordan type used by Carlson et al. [16], Friedlander et al. [24], and Friedlander and Pevtsova [22].

We will need to think of $V(G)$ as both its functor of points $\operatorname{Hom}(k[V(G)], -)$ and as the spectrum of the ring $k[V(G)]$. When we say a point $v \in V(G)$ we will mean a prime ideal in $k[V(G)]$ and when we say a point $v \in V(G)(R)$ we will mean a homomorphism $k[V(G)] \to R$ corresponding to some homomorphism of group schemes $\mathbb{G}_{a(1),R} \to G_R$. There is some translation between these two types of points. Given a prime ideal $\mathfrak{p} \leq k[V(G)]$ we have a canonical homomorphism $k[V(G)] \to k(\mathfrak{p})$ where $k(\mathfrak{p})$ is the field of fractions of the integral domain $k[V(G)]/\mathfrak{p}$. Thus we associate to the point $\mathfrak{p} \in V(G)$ a point in $V(G)(k(\mathfrak{p}))$ and hence a group homomorphism $\mathbb{G}_{a(1),k(\mathfrak{p})} \to G_{k(\mathfrak{p})}$. Let $\mu_{\mathfrak{p}} \colon k(\mathfrak{p})[t]/t^p \to k(\mathfrak{p})G$ be the induced map on group rings.

As $k[V(G)]$ is a k-algebra the field $k(\mathfrak{p})$ is an extension of k. If M is a kG-module then its scalar extension $M_{k(\mathfrak{p})} = k(\mathfrak{p}) \otimes_k M$ is a $k(\mathfrak{p})G$ module. Let $\mu_{\mathfrak{p}}^*(M_{k(\mathfrak{p})})$ be the restriction of $M_{k(\mathfrak{p})}$ to $k(\mathfrak{p})[x]/x^p$; i.e., the $k(\mathfrak{p})[x]/x^p$-module defined by $a \cdot m = \mu_{\mathfrak{p}}(a)m$.

Definition 2.3.1. Let M be a finite dimensional G-module and $v \in V(G)$ a point in the support scheme. The *affine Jordan type of M at v*, $\mathrm{JType}(v, M)$, is defined to be the Jordan type of the restriction $\mu_{\mathfrak{p}}^*(M_{k(\mathfrak{p})})$, i.e., the Jordan type of a matrix representing the action of $\mu_v(t) \in k(v)G$ on $M_{k(v)}$ with respect to some basis. The function

$$\widehat{\mathrm{JType}}(-, M) \colon V(G) \to \mathscr{P}_p$$

is called the *affine local Jordan type* of M.

Recall that $\Theta_M \colon \widetilde{M} \to \widetilde{M}(p^{r-1})$ is defined by a homogeneous homomorphism $k[V(G)] \otimes_k M \to k[V(G)] \otimes_k M$ of graded $k[V(G)]$-modules. Let $\widehat{M} = \mathcal{O}_{V(G)} \otimes_k M$ and

$$\widehat{\Theta}_M \colon \widehat{M} \to \widehat{M}$$

be the map of sheaves defined by forgetting the grading on this homomorphism. One sees in Friedlander and Pevtsova [23, Definition 3.1] that the specialization $\widehat{k}(v) \otimes \Theta_M$ of $\widehat{\Theta}_M$ at $v \in V(G)$ is given by the action of $\mu_v(t)$, thus the affine local Jordan type may be defined by

$$\widehat{\mathrm{JType}}(v, M) = \mathrm{JType}\left(\widehat{\Theta}_M \otimes k(v)\right)$$

as in the projective case.

Let $k[V(G)]^+$ be the homogeneous prime ideal of $k[V(G)]$ generated by elements of positive degree. As a point in $V(G)$ we will consider $k[V(G)]^+$ to be the origin $0 \in V(G)$. Using [23, Proposition 2.11] one can show that we always have $\mu_0(t) = 0$. Thus for all finite dimensional modules M the affine Jordan type of M at 0 is $[1]^{\dim M}$. Thus the affine definition of constant Jordan type takes the following form.

Definition 2.3.2. Let M be a G-module. We say M has *constant affine Jordan type* if the restriction of the affine Jordan type of M to the open subset $V(G) \setminus \{0\}$ is a constant function.

To pass to the projectivization $\mathbb{P}(G)$ recall that we think (intuitively) of the geometric space $V(G)$ as being closed under scalar multiplication. As the various operators $\mu_v(t)$ are linked by $\widehat{\Theta}_M$ one then asks what does this tell us about how the Jordan type of a module varies over the support scheme? Multiplying by a nonzero constant does not change the Jordan type of a linear operator so we expect the affine local Jordan type of a module to be constant on the nonzero points of lines through the origin and thus pass to the projectivization.

Of course, $V(G)$ is a scheme so the above is only intuition, but the intuition is correct. We will show that the restriction of the affine local Jordan type to homogeneous ideals is exactly the local Jordan type defined from specializing Θ_M on $\mathbb{P}(G)$. Moreover, no information is lost in this restriction. Specifically, we can test whether a module is of constant Jordan type by checking the specializations of $\widehat{\Theta}_G$ instead of Θ_M.

We start by introducing the following notation. If A is a graded k-algebra and $\mathfrak{a} \subseteq A$ an ideal let $\mathfrak{a}^{\mathrm{gr}}$ be the largest homogeneous ideal contained in $\mathfrak{a}$. If $\mathfrak{p} \subseteq A$ is a prime ideal then so is $\mathfrak{p}^{\mathrm{gr}}$ [48, Proposition 2.9.1]. As usual $k(\mathfrak{p})$ will denote the residue field of $\operatorname{Spec} A$ at the prime $\mathfrak{p}$. In this section only we let $k(\mathfrak{p}^{\mathrm{gr}})_\circ$ denote the residue field of $\operatorname{Proj} A$ at the prime $\mathfrak{p}^{\mathrm{gr}}$. Further sections will only be concerned with $\mathbb{P}(G)$ and not with $V(G)$, hence after the current section we use $k(\mathfrak{p})$ for the residue field of $\mathbb{P}(G)$.

Lemma 2.3.3. *The field* $k(\mathfrak{p})$ *is an extension of* $k(\mathfrak{p}^{\mathrm{gr}})_\circ$.

Proof. As usual let $A_\mathfrak{p}$ be the localization consisting of fractions with denominator not contained in $\mathfrak{p}$. This is a local ring with maximal ideal consisting of the fractions whose numerator is contained in $\mathfrak{p}$ and residue field $k(\mathfrak{p})$. Let $A_{(\mathfrak{p}^{\mathrm{gr}})}$ be the localization consisting of fractions with homogeneous numerator and denominator of equal degree and denominator not contained in $\mathfrak{p}^{\mathrm{gr}}$. This is a local ring with maximal ideal consisting of the fractions whose numerator is not contained in $\mathfrak{p}^{\mathrm{gr}}$ and residue field $k(\mathfrak{p}^{\mathrm{gr}})_\circ$.

Observe that a homogeneous element is contained in $\mathfrak{p}$ if and only if it is contained in $\mathfrak{p}^{\mathrm{gr}}$. This means we have a natural map $A_{(\mathfrak{p}^{\mathrm{gr}})} \to A_\mathfrak{p}$ and it is a local homomorphism, i.e., an element $a \in A_{(\mathfrak{p}^{\mathrm{gr}})}$ maps into the maximal ideal of $A_\mathfrak{p}$ if and only if a is contained in the maximal ideal of $A_{(\mathfrak{p}^{\mathrm{gr}})}$. Factoring out the maximal ideals then gives a nonzero homomorphism

$k(\mathfrak{p}^{\mathrm{gr}})_\circ \to k(\mathfrak{p})$ as desired. $\square$

Now to simplify notation let $A = k[V(G)]$ and recall that it is graded connected so the underlying set of $\mathbb{P}(G)$ is the collection of all homogeneous prime ideals properly contained in A^+, the ideal generated by all homogeneous elements of positive degree. For any graded A-module N the graded localization of $N[p^{r-1}]$ at a prime is isomorphic to to the graded localization of N at that same prime. Explicitly, given a prime $\mathfrak{p} \in \mathbb{P}(G)$ we may choose a homogeneous degree p^{r-1} element $s \in A \setminus \mathfrak{p}$ [23, Proposition 2.10]. Then an isomorphism

$$\phi_{\mathfrak{p}}^s \colon N[p^{r-1}]_{(\mathfrak{p})} \to N_{(\mathfrak{p})}$$

is defined by $\frac{m}{b} \mapsto \frac{m}{bs}$. Now take $N = M \otimes_k A$ and let

$$\theta_{\mathfrak{p},M}^s \colon M_{k(\mathfrak{p}^{\mathrm{gr}})_\circ} \to M_{k(\mathfrak{p}^{\mathrm{gr}})_\circ}$$

be the local operator, i.e., the specialization of $\Theta_M \colon M \otimes_k A \to M \otimes_k A[p^{r-1}]$ as a map of graded modules at the prime $\mathfrak{p}$. The s in $\theta_{\mathfrak{p},M}^s$ indicates that we use $\phi_{\mathfrak{p}}^s$ to identify $M \otimes_k A[p^{r-1}]_{(\mathfrak{p})}$ with $M \otimes_k A_{(\mathfrak{p})}$ before factoring out maximal ideals.

On representative elements $\theta_{\mathfrak{p},M}^s$ is given by

$$\frac{m}{b} \mapsto \frac{\Theta_M(m)}{bs}.$$

Clearly $\theta_{\mathfrak{p},M}^s \otimes_{k(\mathfrak{p}^{\mathrm{gr}})_\circ} k(\mathfrak{p}) \simeq \frac{1}{s}(\Theta_M \otimes k(\mathfrak{p}))$ as the two maps are given by the same formula. Observe that if $r \in A \setminus \mathfrak{p}$ is another homogeneous degree p^{r-1} element then $\frac{r}{s}\theta_{\mathfrak{p},M}^s = \theta_{\mathfrak{p},M}^r$ so $\theta_{\mathfrak{p},M}^r$ and $\theta_{\mathfrak{p},M}^s$ differ by a nonzero constant, hence have the same Jordan type, kernel, image, and cokernel. This explains why we may write $\theta_{\mathfrak{p},M}$ and drop s from the notation: when we write $\theta_{\mathfrak{p},M}$ we mean $\theta_{\mathfrak{p},M}^s$ for any choice of s.

Proposition 2.3.4. *If* $\mathfrak{p} \neq k[V(G)]^+$ *is prime then* $\mathrm{JType}(M, \mathfrak{p}^{\mathrm{gr}}) = \widehat{\mathrm{JType}}(M, \mathfrak{p})$.

Proof. We have already seen that $s(\theta_{\mathfrak{p},M}^s \otimes_{k(\mathfrak{p}^{\mathrm{gr}})_\circ} k(\mathfrak{p})) \simeq \Theta_M \otimes k(\mathfrak{p}) = \mu_{\mathfrak{p}}(t)$. Extension of scalars and multiplication by a nonzero scalar do not change the Jordan type of a linear operator so the Jordan type of $\theta_{\mathfrak{p},M}$ is equal to the Jordan type of $\mu_{\mathfrak{p}}(t)$. $\square$

Simply put, this says that the affine local Jordan type at $v \neq 0$ in $V(G)$ equals the local Jordan type of the line through v and the origin in $\mathbb{P}(G)$. So affine local Jordan type is completely determined by local Jordan type, and in particular a module is of constant affine Jordan type if and only if it is of constant Jordan type.

Chapter 3

DETECTING PROJECTIVITY IN SHEAVES ASSOCIATED TO REPRESENTATIONS [55]

In this chapter we investigate two open questions concerning the associated sheaves on $\mathbb{P}(G)$ constructed by Friedlander and Pevtsova from a G-module M. Specifically, is it true that $\mathcal{H}_{[1]} = 0$ if and only if M is projective and can the support of $\mathcal{H}_{[1]}$ be given a representation theoretic description? We show that the answer is yes when $\mathbb{P}(G)$ is smooth but in general the answer is no. Finally, we investigate the kernel of $\mathcal{H}_{[1]}$ and show how $\mathcal{H}_{[1]}$ plays a role in a generalization of the BGG correspondence to DG-modules over a polynomial ring.

3.1 The Associated Sheaves of a Module

We start by constructing the *associated sheaves* of M, i.e., the kernel, image, and cokernel of powers of the global operator $\Theta_M \colon \widetilde{M} \to \widetilde{M}(p^{r-1})$. For $1 \leq i \leq p$ we also construct $\mathscr{F}_i(M)$ from these associated sheaves. In addition to being independently motivated, the $\mathscr{F}_i(M)$ are important because they are isomorphic, up to a shift, to the quotients of filtrations of $\widetilde{M}$, cokernels of the global operator, and the sheaves $\mathcal{H}_{[1]}(M)$ and $\mathcal{H}_{[p-1]}(M)$ of section 3.2.

In order to compose Θ_M with itself we must shift the degree of successive copies; hence we need a convention for which degrees we start and end at. Given $j \in \mathbb{N}$ we define

$$\begin{aligned}
\ker \Theta_M^j &= \ker \left[\Theta_M((j-1)p^{r-1}) \circ \cdots \circ \Theta_M(p^{r-1}) \circ \Theta_M\right], \\
\operatorname{im} \Theta_M^j &= \operatorname{im} \left[\Theta_M(-p^{r-1}) \circ \cdots \circ \Theta_M((1-j)p^{r-1}) \circ \Theta_M(-jp^{r-1})\right], \\
\operatorname{coker} \Theta_M^j &= \operatorname{coker} \left[\Theta_M(-p^{r-1}) \circ \cdots \circ \Theta_M((1-j)p^{r-1}) \circ \Theta_M(-jp^{r-1})\right].
\end{aligned}$$

Note that $\ker \Theta_M^j$ and $\operatorname{im} \Theta_M^j$ are subsheafs of $\widetilde{M}$, and $\operatorname{coker} \Theta_M^j$ is a quotient of $\widetilde{M}$. Also note that the canonical short exact sequence below now has a shift

$$0 \to \ker \Theta^j_M \to \widetilde{M} \to \operatorname{im} \Theta^j_M(jp^{r-1}) \to 0.$$

Properties of the local Jordan type of M, or more specifically the j-rank of M, are related to the property that these sheaves are locally free. We will detect this using the following lemma which is Exercise II.5.8 in Hartshorne [29] and a proof can be found in Friedlander and Pevtsova [23, 4.11]. We note that by *specialization* of a sheaf at a point $x \in \mathbb{P}(G)$ we mean the functor $\mathscr{F} \mapsto k(x) \otimes_{\mathcal{O}_{\mathbb{P}(G),x}} \mathscr{F}_x$ where one first takes the stalk at x and then factors out the maximal ideal. To ease the notation from here on out we will simply write $k(x) \otimes \mathscr{F}$ for the specialization of $\mathscr{F}$ at x.

Lemma 3.1.1. *Let X be a reduced Noetherian scheme and $\mathscr{F}$ a coherent sheaf over X. Then the function*

$$\phi(x) = \dim_{k(x)} k(x) \otimes \mathscr{F}$$

is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$ the set $\{x \in X \mid \phi(x) \geq n\}$ is closed. If X is connected then $\mathscr{F}$ is locally free if and only if ϕ is constant.

The following theorem is a corrected version of a theorem by Friedlander and Pevtsova [23, 4.13] which incorrectly equates constant rank with the image being locally free, as opposed to the cokernel.

Theorem 3.1.2. *Let $U \subseteq \mathbb{P}(G)$ be a connected open subscheme and j a positive integer. Given the following statements:*

1. *M has constant j-rank on U,*
2. *M has constant j-rank on the closed points of U,*
3. $\ker \Theta^j_M|_U$ *is locally free,*
4. $\operatorname{im} \Theta^j_M|_U$ *is locally free,*
5. $\operatorname{coker} \Theta^j_M|_U$ *is locally free,*
6. $\ker \theta^j_{v,M} \simeq k(v) \otimes \ker \Theta^j_M$ *for all $v \in U$ and* $\ker \Theta^j_M|_U$ *is locally free,*

7. $\operatorname{im}\theta^j_{v,M} \simeq k(v) \otimes \operatorname{im}\Theta^j_M$ *for all* $v \in U$*,*

8. $\operatorname{coker}\theta^j_{v,M} \simeq k(v) \otimes \operatorname{coker}\Theta^j_M$ *for all* $v \in U$*,*

we have that (8) *always holds,* (1)*,* (2)*,* (5)*,* (6)*, and* (7) *are all equivalent and imply* (4)*, and* (4) *implies* (3)*.*

Proof. First note that specialization is a right exact functor and therefore commutes with taking cokernels, hence (8) holds. We also get from lemma 3.1.1 that

$$\dim_{k(v)} \operatorname{coker}\theta^j_{v,M} = \dim M - \operatorname{rank}^j(v, M) \tag{$*$}$$

is upper semi-continuous as a function of v; hence $\operatorname{rank}^j(-, M)$ is lower semi-continuous. Now Hilberts Nullstellensatz says that the closed points contained in U are dense and a continuous function that is constant on a dense set is constant. Thus (1) $\Leftrightarrow$ (2).

Next recall that in any short exact sequence of sheaves of modules, if the middle and right sheaves are locally free then so is the left. The canonical short exact sequences then give (5) $\Rightarrow$ (4) $\Rightarrow$ (3). By lemma 3.1.1, (5) is equivalent to the statement that $(*)$ is independent of the choice of $v \in U$, hence is equivalent to (1).

Assume (5) holds; then (3) and (4) hold. Specialization is exact when restricted to short exact sequences of locally free sheaves. Specializing

$$\begin{tikzcd}
\ker\Theta^j_M\big|_U \arrow[r, hook] & \widetilde{M}\Big|_U \arrow[rr, "\Theta^j_M|_U"] \arrow[dr] & & \widetilde{M}(jp^{r-1})\Big|_U \arrow[r, two heads] & \operatorname{coker}\Theta^j_M(jp^{r-1})\big|_U \\
& & \operatorname{im}\Theta^j_M(jp^{r-1})\big|_U \arrow[ur, hook] & &
\end{tikzcd}$$

at $v \in U$ therefore gives

$$\begin{tikzcd}
k(v) \otimes \ker\Theta^j_M \arrow[r, hook] & k(v) \otimes_k M \arrow[rr, "\theta^j_{v,M}"] \arrow[dr] & & k(v) \otimes_k M \arrow[r, two heads] & \operatorname{coker}\theta^j_{v,M} \\
& & k(v) \otimes \operatorname{im}\Theta^j_M \arrow[ur, hook] & &
\end{tikzcd}$$

so (6) and (7) hold.

Assume (7) holds. Then by lemma 3.1.1 the local j-rank of M is both upper and lower semi-continuous on U. As U is connected it is therefore constant, hence (1) holds. Finally, assume (6) holds. Then lemma 3.1.1 and

$$\dim_{k(v)} \ker \theta^j_{v,M} = \dim M - \operatorname{rank}^j(v, M)$$

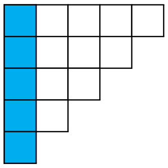
immediately gives (1). $\square$

Remark 3.1.3. For the Lie algebra $\mathfrak{sl}_2$ we will see in proposition 3.3.2 that $\mathbb{P}(\mathfrak{sl}_2) \simeq \mathbb{P}^1$ is a non-singular curve, so $\ker \Theta_M$, a subsheaf of $\widetilde{M}$, is locally free even when M does not have constant rank. This shows that (3) does not imply (1). Similarly one can check that the module $\Phi_{[0:1]}(4)$ from section 4.2 provides a counterexample to (4) $\Rightarrow$ (1).

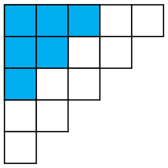
Consider a p-nilpotent linear operator $T\colon V \to V$. Looking at the Young diagram of $\operatorname{JType}(T)$ we see that $\ker T$ is a vector space whose dimension is the number of rows. As T^i annihilates any row of length less or equal to i we then get that the dimension of $\ker T \cap \operatorname{im} T^i$ gives the number of rows of length greater than i in the partition. Thus the dimension of the quotient space

$$\frac{\ker T \cap \operatorname{im} T^{i-1}}{\ker T \cap \operatorname{im} T^i}.$$

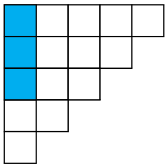
gives the number of rows of length exactly i. Motivated by this, Benson and Pevtsova [6] make the following definition.



$\ker T$



$\operatorname{im} T^2$



$\ker T \cap \operatorname{im} T^2$

Definition 3.1.4. Given $1 \leq i \leq p$ we set

$$\mathscr{F}_i(M) = \frac{\ker \Theta_M \cap \operatorname{im} \Theta_M^{i-1}}{\ker \Theta_M \cap \operatorname{im} \Theta_M^i}.$$

Lemma 3.1.5. *Given $1 \leq i \leq p$ we have*

$$\mathscr{F}_i(M)((i-1)p^{r-1}) \simeq \frac{\ker \Theta_M^i + \operatorname{im} \Theta_M}{\ker \Theta_M^{i-1} + \operatorname{im} \Theta_M}$$

Proof. The map

$$\ker \Theta_M^i \longrightarrow (\ker \Theta_M \cap \operatorname{im} \Theta_M^{i-1})((i-1)p^{r-1})$$

induced by Θ_M^{i-1} is surjective with kernel $\ker \Theta_M^{i-1}$. Thus we have a commutative diagram

$$\begin{tikzcd}
(\ker \Theta_M \cap \operatorname{im} \Theta_M^i)((i-1)p^{r-1}) \arrow[r, "\mathrm{incl}"] & (\ker \Theta_M \cap \operatorname{im} \Theta_M^{i-1})((i-1)p^{r-1}) \\
\dfrac{\ker \Theta_M^{i+1}}{\ker \Theta_M^i}(-p^{r-1}) \arrow[u, "\Theta_M^i"] \arrow[r, "\Theta_M"'] & \dfrac{\ker \Theta_M^i}{\ker \Theta_M^{i-1}} \arrow[u, "\Theta_M^{i-1}"']
\end{tikzcd}$$

whose vertical arrows are sheaf isomorphisms, so the sheaf $\mathscr{F}_i(M)((i-1)p^{r-1})$ in question is isomorphic to the cokernel of the bottom arrow. The image of this bottom arrow is $(\ker \Theta_M^i \cap \operatorname{im} \Theta_M + \ker \Theta_M^{i-1}) / \ker \Theta_M^{i-1}$; therefore, the second and third isomorphism theorems, together with the modular law, give

$$\mathscr{F}_i(M)((i-1)p^{r-1}) \simeq \frac{\ker \Theta_M^i}{\ker \Theta_M^i \cap \operatorname{im} \Theta_M + \ker \Theta_M^{i-1}} \simeq \frac{\ker \Theta_M^i + \operatorname{im} \Theta_M}{\ker \Theta_M^{i-1} + \operatorname{im} \Theta_M}$$

as desired. ☐

We will need two results from Benson and Pevtsova [6]. For the first we note that their proof, for the case of an elementary abelian group ($r = 1$), goes through in our more general setting once the appropriate shifts have been added.

Proposition 3.1.6 ([6, 2.2, 2.3])**.** *The sheaf $\widetilde{M}$ has a filtration in which the filtered quotients are isomorphic to $\mathscr{F}_i(M)(jp^{r-1})$ for $0 \leq j < i \leq p$.*

As a corollary to the proof of this proposition we get the following.

Corollary 3.1.7. *The sheaf $\operatorname{coker} \Theta_M^\ell$ has a filtration in which the filtered quotients are isomorphic to $\mathscr{F}_i(M)(jp^{r-1})$ for $1 \leq i \leq p$ and $\max(0, i-\ell) \leq j < i$.*

Proof. In the proof of the previous proposition Benson and Pevtsova [6, 2.2] refine the kernel filtration

$$0 \subseteq \mathcal{K} \subseteq \mathcal{K}_2 \subseteq \cdots \subseteq \mathcal{K}_{p-1} \subseteq \widetilde{M},$$

where $\mathcal{K}_i = \ker \Theta_M^i$, by the image filtration

$$0 \subseteq \mathcal{I} \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_{p-1} \subseteq \widetilde{M},$$

where $\mathcal{I}_i = \operatorname{im} \Theta_M^{p-i}$, to obtain a filtration with quotients

$$\frac{(\mathcal{K}_{j+1} \cap \mathcal{I}_{i+1}) + \mathcal{K}_j}{(\mathcal{K}_{j+1} \cap \mathcal{I}_i) + \mathcal{K}_j} \simeq \frac{\mathcal{K}_{j+1} \cap \mathcal{I}_{i+1}}{(\mathcal{K}_{j+1} \cap \mathcal{I}_i) + (\mathcal{K}_j \cap \mathcal{I}_{i+1})}$$

which are isomorphic to $\mathscr{F}_{p-i+j}(M)(jp^{r-1})$. Note that this quotient is symmetric with respect to image vs. kernel so for the corollary we refine the image filtration by the kernel filtration and truncate below the term $\operatorname{im} \Theta_M^\ell$ to obtain a filtration of $\operatorname{coker} \Theta_M^\ell$. □

The conditions determining which $\mathscr{F}_i(M)(jp^{r-1})$ appear as filtered quotients become intuitively very clear if we accept the following maxim: *We should think of $\mathscr{F}_i(M)(jp^{r-1})$ as a sheafification of the j^{th} boxes in rows of length i in the Jordan type of M.* Here we mean for the boxes to be counted starting at 0 from left to right (see Figure 3.1). In $\widetilde{M}$ we need, in rows of length i, boxes $0, 1, \ldots, i-1$. On the other hand, in $\operatorname{coker} \Theta_M^\ell$ we only need the last ℓ boxes in a row. If $i \leq \ell$ this is the entire row, otherwise it is only boxes $i-\ell, i-\ell+1, \ldots, i-1$.

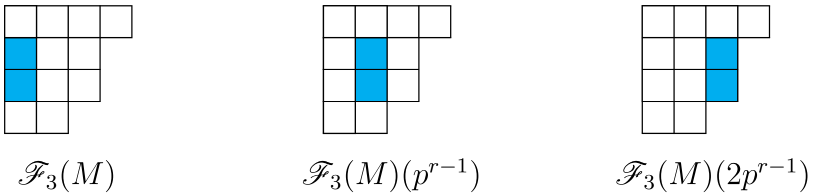


Figure 3.1: Visualizing $\mathscr{F}_i(M)(jp^{r-1})$

The second proposition needed from Benson and Pevtsova is proposition 3.1.8 below. We note that Benson and Pevtsova's proof appears to implicitly assume that the module is constant Jordan type when they appeal to a "block count" (this relies on a previous proposition [6, 2.1] which only applies to such modules). We give an altered version of their proof which appeals only to a diagram chase and hence applies in the case when M does not have constant Jordan type. Recall that $\Omega(M)$, the *Heller shift* of M, is defined to be the kernel of the map from the projective cover of M to M.

Proposition 3.1.8 ([6, 3.2])**.** *Let M be a G-module and $1 \leq i < p$. Then*

$$\mathscr{F}_i(M) \simeq \mathscr{F}_{p-i}(\Omega M)((p-i)p^{r-1}).$$

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\Omega_M} \arrow[r] \arrow[d, "\Theta_{\Omega_M}"] & \widetilde{P_M} \arrow[r] \arrow[d, "\Theta_{P_M}"] & \widetilde{M} \arrow[r] \arrow[d, "\Theta_M"] & 0 \\
0 \arrow[r] & \widetilde{\Omega(M)}(p^{r-1}) \arrow[r] \arrow[d, "\Theta^{p-i-1}_{\Omega(M)}"] & \widetilde{P_M}(p^{r-1}) \arrow[r] \arrow[d, "\Theta^{p-i-1}_{P_M}"] & \widetilde{M}(p^{r-1}) \arrow[r] \arrow[d, "\Theta^{p-i-1}_{M}"] & 0 \\
0 \arrow[r] & \widetilde{\Omega(M)}((p-i)p^{r-1}) \arrow[r] & \widetilde{P_M}((p-i)p^{r-1}) \arrow[r] & \widetilde{M}((p-i)p^{r-1}) \arrow[r] & 0.
\end{tikzcd}$$

Using $\Theta^p_{P_M} = 0$ and a diagram chase in the style of the snake lemma we get a well defined map

$$\delta\colon \ker \Theta_M \cap \operatorname{im} \Theta_M^{i-1} \longrightarrow \frac{\ker \Theta_{\Omega(M)} \cap \operatorname{im} \Theta^{p-i-1}_{\Omega(M)}}{\ker \Theta_{\Omega(M)} \cap \operatorname{im} \Theta^{p-i}_{\Omega(M)}}((p-i)p^{r-1}).$$

As P_M is a projective module it has constant Jordan type; therefore, lemma 3.2.3 gives $\operatorname{im} \Theta^j_{P_M} = \ker \Theta^{p-j}_{P_M}$ for all j. One then uses this in the diagram chase to show that δ is surjective and the kernel is exactly $\ker \Theta_M \cap \operatorname{im} \Theta^i_M$ so it induces the desired isomorphism. □

Finally we have the following proposition which is merely a local version of a global result of Benson and Pevtsova, which appears as corollary 3.1.10 below. The proof follows their original proof save that we avoid their specialization argument and instead use a short exact sequence to deduce that the $\mathscr{F}_i(M)|_U$ are locally free.

Proposition 3.1.9. *Let $U \subseteq \mathbb{P}(G)$ be open. The module M has constant Jordan type $[p]^{a_p}[p-1]^{a_{p-1}}\cdots[1]^{a_1}$ on U if and only if for each i the sheaf $\mathscr{F}_i(M)|_U$ is locally free of rank a_i.*

Proof. First assume M has constant Jordan type. There is a natural short exact sequence

$$0 \to \mathscr{F}_i(M)|_U \longrightarrow \frac{\operatorname{im}\Theta_M^{i-1}|_U}{\operatorname{im}\Theta_M^{i}|_U} \xrightarrow{\Theta_M} \frac{\operatorname{im}\Theta_M^{i}}{\operatorname{im}\Theta_M^{i+1}}(p^{r-1})|_U \to 0;$$

therefore, to prove that each $\mathscr{F}_i(M)|_U$ is locally free we need only show that for each i the quotient of $\operatorname{im}\Theta_M^{i-1}|_U$ by $\operatorname{im}\Theta_M^{i}|_U$ is locally free. For that consider the short exact sequence

$$0 \to \frac{\ker\Theta_M^{p-i}|_U}{\operatorname{im}\Theta_M^{i}|_U} \longrightarrow \frac{\operatorname{im}\Theta_M^{i-1}|_U}{\operatorname{im}\Theta_M^{i}|_U} \xrightarrow{\Theta_M} \operatorname{im}\Theta_M^{p-1}(p^{r-1})|_U \to 0.$$

By theorem 3.1.2 and lemma 3.2.3 (whose proof does not rely on this result) the outer two sheaves are locally free therefore the middle sheaf is as well.

Next assume each $\mathscr{F}_i(M)|_U$, hence each $\mathscr{F}_i(M)(jp^{r-1})|_U$, is locally free. Proposition 3.1.6 then says that $\operatorname{coker}\Theta_M^{\ell}|_U$ has a filtration with locally free filtered quotients. Inducting up this filtration we conclude that $\operatorname{coker}\Theta_M^{\ell}|_U$ is locally free. By theorem 3.1.2 the module M has constant ℓ-rank on U. This holds for all ℓ therefore M has constant Jordan type on U.

Finally we must show that the ranks of the $\mathscr{F}_i(M)|_U$ give the Jordan type of M. Let $v \in U$ be any point. By theorem 3.1.2 the rank of the sheaf $\operatorname{im}\Theta_M^{i}|_U$ is $\operatorname{rank}^i(v, M)$. The short exact sequence

$$0 \to \ker\Theta_M|_U \cap \operatorname{im}\Theta_M^{i}|_U \longrightarrow \operatorname{im}\Theta_M^{i-1}|_U \xrightarrow{\Theta_M} \operatorname{im}\Theta_M^{i}(p^{r-1})|_U \to 0$$

gives that $\ker\Theta_M|_U \cap \operatorname{im}\Theta_M^{i}|_U$ is locally free and its rank is $\operatorname{rank}^i(v, M) - \operatorname{rank}^{i-1}(v, M)$. But by lemma 2.2.5 this is exactly the number of rows in $\operatorname{JType}(v, M)$ of length greater or equal to i. Finally, the defining short exact sequence

$$0 \to \ker\Theta_M|_U \cap \operatorname{im}\Theta_M^{i}|_U \longrightarrow \ker\Theta_M|_U \cap \operatorname{im}\Theta_M^{i-1}|_U \longrightarrow \mathscr{F}_i(M)|_U \to 0$$

gives that $\operatorname{rank}\mathscr{F}_i(M)|_U$ is exactly the number of rows of length i. □

Corollary 3.1.10 ([7, 7.4.12], [6, 2.1])**.** *The module M has constant Jordan type $[p]^{a_p}[p-1]^{a_{p-1}}\cdots[1]^{a_1}$ if and only if for each i the sheaf $\mathscr{F}_i(M)$ is locally free of rank a_i.*

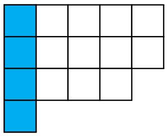
3.2 The Sheaf $\mathcal{H}_{[1]}(M)$

In this section we introduce the sheaf $\mathcal{H}_{[1]}(M)$ and are motivated to study the following question: Is it true that M is projective if and only if $\mathcal{H}_{[1]}(M) = 0$? We discuss and then improve upon previous results by Friedlander and Pevtsova, but we note that in general the statement is false. We prove that if $\mathcal{H}_{[1]}(M) = 0$ then the support of M is contained in the singular locus of $\mathbb{P}(G)$, hence the question has an affirmative answer in the smooth case. Finally, we show that the full subcategory of modules M such that $\mathcal{H}_{[1]}(M) = 0$ is a thick subcategory of the stable category.

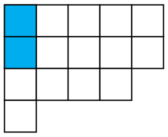
To motivate the definition of $\mathcal{H}_{[1]}(M)$ consider a p-nilpotent linear operator T. We have already noted that $\dim \ker T$ is the number of rows in $\operatorname{JType}(T)$ and $\ker T \cap \operatorname{im} T^{p-1} = \operatorname{im} T^{p-1}$ so $\dim \operatorname{im} T^{p-1}$ is the number of rows of size p. Thus the dimension of

$$H_T = \frac{\ker T}{\operatorname{im} T^{p-1}}$$

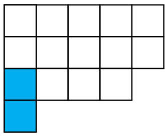
gives the number of rows of size less than p in $\operatorname{JType}(T)$. By theorem 2.2.6 a module M is projective if and only if its Jordan type at any $v \in \mathbb{P}(G)$ has only rows of size p, i.e., if and only if $H_{\theta_{v,M}} = 0$ for all $v \in \mathbb{P}(G)$. The global version of this space is the sheaf $\mathcal{H}_{[1]}(M)$.



$\ker T$



$\operatorname{im} T^4$



$\frac{\ker T}{\operatorname{im} T^4}$

Definition 3.2.1. Given $0 \leq i \leq p$ we define

$$\mathcal{H}_{[i]}(M) = \frac{\ker \Theta_M^i}{\operatorname{im} \Theta_M^{p-i}}.$$

Question 3.2.2. Is M projective if and only if $\mathcal{H}_{[1]}(M) = 0$?

As noted in the introduction Friedlander and Pevtsova have given a partial answer to this question. They show that a module M is projective if and only if it has constant rank, constant $(p-1)$-rank, and $\mathcal{H}_{[1]}(M) = 0$ [23, 5.19]. The forward direction of the next lemma shows that the assumption of constant $(p-1)$-rank is redundant. The converse direction is a new proof of Proposition 5.16 in Friedlander and Pevtsova [23].

Lemma 3.2.3. *Let $U \subseteq \mathbb{P}(G)$ be a connected open subscheme and M a G-module which has constant i-rank on U. Then $\mathcal{H}_{[i]}(M)|_U$ is locally free if and only if M has constant $(p-i)$-rank on U.*

Proof. We have $\widetilde{M}/\ker \Theta^i_M \simeq \operatorname{im} \Theta^i_M(ip^{r-1})$ and hence a short exact sequence

$$0 \to \mathcal{H}_{[i]}(M)\big|_U \to \operatorname{coker} \Theta^{p-i}_M\big|_U \to \operatorname{im} \Theta^i_M(ip^{r-1})\big|_U \to 0.$$

By theorem 3.1.2 the sheaf $\operatorname{im} \Theta^i_M|_U$, and hence the right sheaf in the sequence, is locally free. We conclude that the left sheaf is locally free if and only if the middle sheaf is, hence if and only if M has constant $(p-i)$-rank on U (using theorem 3.1.2 again). $\square$

Proposition 3.2.4. *The sheaves $\mathcal{H}_{[1]}(M)$ and $\mathcal{H}_{[p-1]}(M)$ each have a $(p-1)$-step filtration whose quotients are the sheaves $\mathscr{F}_i(M)$ and $\mathscr{F}_i(M)((i-1)p^{r-1})$, respectively, for $i = 1, 2, \ldots, p-1$.*

Proof. For $\mathcal{H}_{[1]}(M)$ the filtration is

$$\begin{aligned}
\operatorname{im} \Theta^{p-1}_M &= \ker \Theta_M \cap \operatorname{im} \Theta^{p-1}_M \\
&\subseteq \ker \Theta_M \cap \operatorname{im} \Theta^{p-2}_M \\
&\subseteq \cdots \\
&\subseteq \ker \Theta_M \cap \operatorname{im} \Theta_M \\
&\subseteq \ker \Theta_M \cap \operatorname{im} \Theta^0_M = \ker \Theta_M.
\end{aligned}$$

For $\mathcal{H}_{[p-1]}(M)$ the filtration is

$$\begin{aligned}
\operatorname{im}\Theta_M &= \operatorname{im}\Theta_M + \ker\Theta_M^0 \\
&\subseteq \operatorname{im}\Theta_M + \ker\Theta_M \\
&\subseteq \cdots \\
&\subseteq \operatorname{im}\Theta_M + \ker\Theta_M^{p-2} \\
&\subseteq \operatorname{im}\Theta_M + \ker\Theta_M^{p-1} = \ker\Theta_M^{p-1}
\end{aligned}$$

and we use lemma 3.1.5 to identify the quotients. □

Corollary 3.2.5. *For each point $v \in \mathbb{P}(G)$ the stalk $\mathcal{H}_{[1]}(M)_v$ is zero if and only if the stalk $\mathcal{H}_{[p-1]}(M)_v$ is zero.*

Proof. Localizing the filtrations from the previous proposition shows that $\mathcal{H}_{[1]}(M)_v$ and $\mathcal{H}_{[p-1]}(M)_v$ have filtrations with filtered quotients isomorphic to $\mathscr{F}_i(M)_v$ for $i = 1, 2, \ldots, p-1$, and a filtered object is zero if and only if each of its filtered quotients is. □

Theorem 3.2.6. *Let $U \subseteq \mathbb{P}(G)$ be a connected open subscheme and assume that $\mathcal{H}_{[1]}(M)|_U = 0$. Then the following conditions are equivalent.*

1. *M is projective on U,*
2. *M has constant rank on U,*
3. *M has constant $(p-1)$-rank on U,*
4. *$\operatorname{coker}\Theta_M^{p-1}|_U$ is locally free,*
5. *$\operatorname{coker}\Theta_M|_U$ is locally free,*
6. *$\operatorname{im}\Theta_M^{p-1}|_U$ is locally free,*
7. *$\operatorname{im}\Theta_M|_U$ is locally free,*
8. *$\ker\Theta_M^{p-1}|_U$ is locally free,*
9. *$\ker\Theta_M|_U$ is locally free.*

Proof. corollary 3.2.5 gives $\mathcal{H}_{[p-1]}M|_U = 0$ as well therefore $\ker \Theta_M|_U = \operatorname{im} \Theta_M^{p-1}|_U$ and $\ker \Theta_M^{p-1}|_U = \operatorname{im} \Theta_M|_U$. Now by theorem 3.1.2 condition (1) implies that all of the remaining conditions hold and each of the above conditions implies (9). All that is left is to show that (9) implies (1). For this we observe that proposition 3.2.4 gives $\mathscr{F}_i(M)|_U = 0$ for $1 \leq i < p$ and by definition $\mathscr{F}_p(M)|_U = \ker \Theta_M|_U$ so proposition 3.1.9 applies. $\square$

We have already defined the support, $\mathbb{P}(G)_M$, of a G-module M as the set of points at which M is not projective and the support of a sheaf $\mathscr{G}$ is the set of points at which the stalk $\mathscr{G}_v$ is non-zero. Thus question 3.2.2 can be interpreted as asking if it is true that $\mathbb{P}(G)_M$ is empty if and only if $\operatorname{Supp} \mathcal{H}_{[1]}(M)$ is empty. Both are known to be closed subsets so, more generally, we ask the following.

Question 3.2.7. Given a G-module M do we have $\mathbb{P}(G)_M = \operatorname{Supp} \mathcal{H}_{[1]}(M)$?

Friedlander and Pevtsova have given one inclusion and we can obtain the other on the regular locus of $\mathbb{P}(G)$ which we denote by $\operatorname{Reg} \mathbb{P}(G)$. We remind the reader that this is the set of points $v \in \mathbb{P}(G)$ such that the stalk, $\mathcal{O}_{\mathbb{P}(G),v}$, of the structure sheaf is a regular local ring.

Proposition 3.2.8 ([23, 5.25])**.** *Let M be a G-module. Then*

$$\operatorname{Supp} \mathcal{H}_{[1]}(M) \subseteq \mathbb{P}(G)_M.$$

Corollary 3.2.9. *If M is projective then $\mathcal{H}_{[1]}(M) = 0$.*

Theorem 3.2.10. *Let M be a G-module. Then*

$$\mathbb{P}(G)_M \cap \operatorname{Reg} \mathbb{P}(G) \subseteq \operatorname{Supp} \mathcal{H}_{[1]}(M).$$

Proof. We prove the contrapositive. Assume $v \in \operatorname{Reg} \mathbb{P}(G)$ and the stalk $\mathcal{H}_{[1]}(M)_v$ is zero, then we will show that M is projective in a neighborhood of v.

Consider the complex

$$\begin{tikzcd}
\cdots \arrow[r, "(\Theta_M)_v"] & \widetilde{M}_v \arrow[r, "(\Theta_M^{p-1})_v"] & \widetilde{M}_v \arrow[r, "(\Theta_M)_v"] & \widetilde{M}_v \arrow[r, "(\Theta_M^{p-1})_v"] & (\ker \Theta_M)_v \arrow[r] & 0.
\end{tikzcd}$$

As $\mathcal{H}_{[1]}(M)_v = 0$ we also have $\mathcal{H}_{[p-1]}(M)_v = 0$ by corollary 3.2.5; thus this is a resolution of $(\ker \Theta_M)_v$. Note that it is a 2-periodic resolution; the even syzygies are $(\ker \Theta_M)_v$. As v is regular $(\ker \Theta_M)_v$ has finite projective dimension so the syzygies of any projective resolution are eventually projective, hence free. We may therefore take a connected neighborhood $U \subseteq \mathbb{P}(G)$ of v such that $\ker \Theta_M|_U$ is locally free. By theorem 3.2.6, M is projective on U. □

Corollary 3.2.11. *If* $\mathbb{P}(G)$ *is regular then* $\mathbb{P}(G)_M = \operatorname{Supp} \mathcal{H}_{[1]}(M)$.

Corollary 3.2.12. *If* $\mathbb{P}(G)$ *is regular then* M *is projective if and only if* $\mathcal{H}_{[1]}(M) = 0$.

In general $\mathbb{P}(G)$ need not be regular; we will provide examples of when it is regular in the next section. In the singular case we still have corollary 3.2.9 so we will consider $\mathcal{H}_{[1]}$ as a functor out of the stable module category $\underline{\mathtt{mod}}(G)$. Recall that the objects of $\underline{\mathtt{mod}}(G)$ are exactly the objects of $\mathtt{mod}(G)$. For the hom-sets we define $\underline{\operatorname{Hom}}_G(M, N)$ to be the quotient of $\operatorname{Hom}_G(M, N)$ by the subspace of homomorphisms that factor through a projective. We will consider the kernel of $\mathcal{H}_{[1]}$ as a subcategory of $\underline{\mathtt{mod}}(G)$.

Definition 3.2.13. Let $\ker(\mathcal{H}_{[1]})$ be the kernel of the functor

$$\mathcal{H}_{[1]} \colon \underline{\mathtt{mod}}(G) \to \mathtt{Coh}_{\mathbb{P}(G)},$$

that is, $\ker(\mathcal{H}_{[1]})$ is the full subcategory of $\underline{\mathtt{mod}}(G)$ consisting of those modules M such that $\mathcal{H}_{[1]}(M) = 0$.

The stable module category has a triangulated structure where the triangles are those candidate triangles that are isomorphic to

$$N \to M \to M/N \to \Omega^{-1}(N)$$

for some short exact sequence $0 \to N \to M \to M/N \to 0$ in $\mathtt{mod}(G)$. We now show that $\ker(\mathcal{H}_{[1]})$ is a thick subcategory of $\underline{\mathtt{mod}}(G)$. Recall that this means $\ker(\mathcal{H}_{[1]})$ is closed under isomorphisms, shifts, and taking summands and it satisfies the 2-3 axiom: If

$$X \to Y \to Z \to \Omega^{-1}(X)$$

is a triangle and $X, Y \in \ker(\mathcal{H}_{[1]})$ then also $Z \in \ker(\mathcal{H}_{[1]})$.

Theorem 3.2.14. $\ker(\mathcal{H}_{[1]})$ *is a thick subcategory of* $\underline{\texttt{mod}}(G)$.

Proof. As $\mathcal{H}_{[1]}$ is functorial and clearly additive one has that $\ker(\mathcal{H}_{[1]})$ is closed under isomorphisms and taking summands. For closure under the shift operations note that $\mathcal{H}_{[1]}(M) = 0$ if and only if $\mathscr{F}_i(M) = 0$ for all $1 \leq i < p$. Proposition 3.1.8 then gives $\mathcal{H}_{[1]}(\Omega M) = 0$ if and only if $\mathcal{H}_{[1]}(M) = 0$. All that is left is the 2-3 axiom.

Using closure under isomorphism we see that it suffices to show that $N \subseteq M$ and $\mathcal{H}_{[1]}(N) = \mathcal{H}_{[1]}(M) = 0$ imply $\mathcal{H}_{[1]}(M/N) = 0$. As in the proof of theorem 3.2.10 the hypotheses $\mathcal{H}_{[1]}(N) = \mathcal{H}_{[1]}(M) = 0$ imply that the complexes

$$\cdots \xrightarrow{\Theta_N} \widetilde{N}((1-p)p^{r-1}) \xrightarrow{\Theta_N^{p-1}} \widetilde{N} \xrightarrow{\Theta_N} \widetilde{N}(p^{r-1}) \xrightarrow{\Theta_N^{p-1}} \cdots$$

and

$$\cdots \xrightarrow{\Theta_M} \widetilde{M}((1-p)p^{r-1}) \xrightarrow{\Theta_M^{p-1}} \widetilde{M} \xrightarrow{\Theta_M} \widetilde{M}(p^{r-1}) \xrightarrow{\Theta_M^{p-1}} \cdots$$

are acyclic. By naturality of global operators the inclusion $N \hookrightarrow M$ induces a map between these complexes and we find that

$$\cdots \xrightarrow{\Theta_{M/N}} \widetilde{M/N}((1-p)p^{r-1}) \xrightarrow{\Theta_{M/N}^{p-1}} \widetilde{M/N} \xrightarrow{\Theta_{M/N}} \widetilde{M/N}(p^{r-1}) \xrightarrow{\Theta_{M/N}^{p-1}} \cdots$$

is the cokernel of a quasi-isomorphism and hence is acyclic, giving $M/N \in \ker(\mathcal{H}_{[1]})$. $\square$

We now give an example where $\ker(\mathcal{H}_{[1]})$ has infinite representation type. This shows that in general the answer to Questions 3.2.2 and 3.2.7 is no. One might then hope that $\ker(\mathcal{H}_{[1]})$ has ideal closure, as the tensor ideal thick subcategories of $\underline{\texttt{mod}}(G)$ have been classified [22, 6.3]. Unfortunately, the example below will also show that in general $\ker(\mathcal{H}_{[1]})$ is not a tensor ideal. When G is unipotent, i.e., when k is the unique simple G-module, it is known that all thick subcategories are tensor ideals (c.f. Benson et al. [5, 3.5]) but in all known cases of unitary G one has that $\mathbb{P}(G)$ is smooth and so corollary 3.2.12 already identifies $\ker(\mathcal{H}_{[1]})$.

Consider the Lie algebra $\mathfrak{sl}_3$ with $p \neq 2$. The adjoint action of SL_3 induces an action on $\mathbb{P}(\mathfrak{sl}_3)$ with two orbits: The regular orbit, which is open and given as the set of lines through nilpotent matrices of rank 2 (matrices of Jordan type [3]), and the sub-regular orbit, which is closed and given as the set of lines through nilpotent matrices of rank 1 (matrices of Jordan type [2][1]). The regular locus of $\mathbb{P}(\mathfrak{sl}_3)$ is exactly the regular orbit [32, 7.14].

Example 3.2.15. Assume char $k = 3$ and recall that $\mathfrak{sl}_3$ can be considered as a height 1 infinitesimal group scheme. Let $M = k^3$ be the standard representation of $\mathfrak{sl}_3$. Using Macaulay2 [28] one can compute that $\mathcal{H}_{[1]}(M) = 0$. The support of M is exactly the sub-regular orbit of $\mathbb{P}(\mathfrak{sl}_3)$, so M is not projective. Moreover, the dimension of the support is well known to be the complexity of M. As M is indecomposable and the sub-regular orbit has dimension 4, the Heller shifts $\Omega^n(M)$ are indecomposable of increasing dimensions. These shifts therefore yield countably many non-isomorphic indecomposable modules contained in $\ker(\mathcal{H}_{[1]})$.

We can also see from this example that $\ker(\mathcal{H}_{[1]})$ is not closed under tensor products; in particular, it is not a tensor ideal thick subcategory. This is also done using Macaulay2. We first verify that $\mathcal{H}_{[1]}(M^*) = 0$ and then we compute the support of the sheaf $\mathcal{H}_{[1]}(M \otimes M^*)$ and observe that it is not empty.

The Macaulay2 code for these computations can be found in Figure 3.2. The variables `theta`, `thetaDual`, and `thetaTens` contain the module homomorphisms corresponding to Θ_M, Θ_{M^*}, and $\Theta_{M\otimes M^*}$ respectively.

3.3 Regular support varieties

Given the results of the previous section it is useful to have examples of groups G such that $\mathbb{P}(G)$ is regular. We give several such examples in this section, begining with some obvious examples of Lie algebras $\mathfrak{g}$ such that $\mathbb{P}(\mathfrak{g})$ is regular. Let $\mathfrak{u}_n$ and $\mathfrak{b}_n$ be the Lie algebras of U_n and B_n respectively. Also, let $\mathfrak{e}_n$ be the elementary abelian Lie algebra of dimension n, i.e., $\mathfrak{e}_n$ is a vector space of dimension n with trivial bracket and p-operation.

Proposition 3.3.1.

1. *If* $p \geq n$ *then* $\mathbb{P}(\mathfrak{u}_n) \simeq \mathbb{P}^{\frac{1}{2}(n+1)(n-2)}$.

```
i1 : R = ZZ/3[x1, x2, x3, y1, y2, y3, h7, h8]

i2 : M = matrix{{h7, x1, x3},{y1, h8-h7, x2},{y3, y2, -h8}}

i3 : kPG = R/(radical ideal M^3)

i4 : theta = map(kPG^3, kPG^3, sub(M, kPG), Degree => 1)

i5 : minimalPresentation(ker theta/image theta^2)

o5 = 0

i6 : thetaDual = -transpose theta

i7 : minimalPresentation(ker thetaDual/image thetaDual^2)

o7 = 0

i8 : thetaTens = (theta ** id_(kPG^3)) + (id_(kPG^3) ** thetaDual)

i9 : radical ann (ker thetaTens/image thetaTens^2)

o9 = ideal (x3*y3 + h7*h8, x2*y3 + y1*h8, x1*y3 - y2*h7, y1*y2 +
     -------------------------------------------------------------
                                                     2
     y3*h7 - y3*h8, x3*y2 + x1*h8, x2*y2 - h7*h8 + h8 , x3*y1 -
     -------------------------------------------------------------
     x2*h7, x1*x2 + x3*h7 - x3*h8)
```

Figure 3.2: Macaulay2 code for example 3.2.15

2. *$\mathbb{P}(\mathfrak{e}_n) \simeq \mathbb{P}^{n-1}$.*
3. *If $p \geq n$ then $\mathbb{P}(\mathfrak{b}_n) \simeq \mathbb{P}^{\frac{1}{2}(n+1)(n-2)}$.*

Proof. For (1) note that $p \geq n$ implies that every matrix in $\mathfrak{u}_n$ is p-nilpotent. Thus $\mathbb{P}(\mathfrak{u}_n)$ is the projective variety of lines in $\mathfrak{u}_n$ and

$$\dim_k \mathfrak{u}_n = \frac{1}{2}n(n-1) = \frac{1}{2}(n+1)(n-2) + 1.$$

A similar argument gives (2). For (3) note that an upper triangular matrix is nilpotent if and only if it's strictly upper triangular, so for any n the inclusion $\mathfrak{u}_n \to \mathfrak{b}_n$ induces an isomorphism of the k-points of their restricted nullcones. $\square$

Proposition 3.3.2. *The support variety of* $\mathfrak{sl}_2$ *is* $\mathbb{P}(\mathfrak{sl}_2) = \operatorname{Proj} \frac{k[x,y,z]}{xy+z^2} \simeq \mathbb{P}^1$.

Proof. A matrix $\left[\begin{smallmatrix} z & x \\ y & -z \end{smallmatrix}\right]$ with entries in the field k is nilpotent if and only if its square $(xy + z^2)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is zero. Thus the radical of the ideal generated by the relation $\left[\begin{smallmatrix} z & x \\ y & -z \end{smallmatrix}\right]^p = 0$ is $(xy+z^2)$ and immediately we get $\mathbb{P}(\mathfrak{sl}_2) = \operatorname{Proj} \frac{k[x,y,z]}{xy+z^2}$. The ring homomorphism

$$\begin{aligned} \frac{k[x,y,z]}{xy+z^2} &\to k[s,t] \\ (x,y,z) &\mapsto (s^2, -t^2, st). \end{aligned}$$

is easily seen to be injective and identifies $\frac{k[x,y,z]}{xy+z^2}$ with $k[s,t]^{[2]} = k[s^2,t^2,st]$. The induced map $\iota\colon \operatorname{Proj} k[s,t] \to \operatorname{Proj} \frac{k[x,y,z]}{xy+z^2}$ is therefore an isomorphism of schemes. □

The weighted projective space, $\mathbb{P}(1,p,\ldots,p^{r-1})$, with weight vector $(1,p,\ldots,p^{r-1})$ is defined by $\mathbb{P}(1,p,\ldots,p^{r-1}) = \operatorname{Proj} k[w_1,\ldots,w_r]$, with grading given by $\deg w_i = p^{i-1}$.

Proposition 3.3.3. *Let* $n, r \in \mathbb{N}$. *Then*

$$V(\mathrm{B}_{n(r)})_{\mathrm{red}} = V(\mathrm{U}_{n(r)})_{\mathrm{red}}$$

and

$$\mathbb{P}(\mathrm{B}_{2(r)}) = \mathbb{P}(\mathrm{U}_{2(r)}) = \mathbb{P}(1,p,\ldots,p^{r-1}).$$

Proof. As in the proof of proposition 3.3.1 observe that the restricted nullcones of $\mathfrak{b}_n$ and $\mathfrak{u}_n$ are identical therefore $V(\mathrm{B}_{n(r)})_{\mathrm{red}} = V(\mathrm{U}_{n(r)})_{\mathrm{red}}$ follows immediately from the description in example 2.1.5. In particular we get $\mathbb{P}(\mathrm{B}_{2(r)}) = \mathbb{P}(\mathrm{U}_{2(r)})$. For $\mathbb{P}(\mathrm{U}_{2(r)}) = \mathbb{P}(1,p,\ldots,p^{r-1})$, the map $\left[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right] \mapsto a$ yields an isomorphism $\mathrm{U}_2 \simeq \mathbb{G}_a$, hence $\mathbb{P}(\mathrm{U}_{2(r)}) \simeq \mathbb{P}(\mathbb{G}_{a(r)})$, and the result follows from the description in example 2.1.8. □

Corollary 3.3.4. *The support varieties* $\mathbb{P}(\mathrm{B}_{2(r)})$ *and* $\mathbb{P}(\mathrm{U}_{2(r)})$ *are regular if and only if* $r = 1, 2$.

Proof. This follows from the previous proposition once we know that the weighted projective space $\mathbb{P}(1,p,\ldots,p^{r-1})$ is regular if and only if $r \in \{1,2\}$. To see this we first observe that

$\mathbb{P}(1) = \operatorname{Proj} k[w_1] \simeq \operatorname{Spec} k$ is regular and the p^{th} truncation of $k[w_1, w_2]$ is $k[w_1^p, w_2] \simeq k[s,t]$ therefore $\mathbb{P}(1,p) \simeq \mathbb{P}^1$ is regular. For $r > 2$ one checks directly that the non-vanishing locus of w_3 is the spectrum of the sub-ring

$$k[\varpi_0, \varpi_1, \ldots, \varpi_p, \sigma_4, \sigma_5, \ldots, \sigma_r] \subseteq k[w_1, \ldots, w_r]_{w_3},$$

where $\varpi_a = \frac{w_1^{ap} w_2^{p-a}}{w_3}$ and $\sigma_a = \frac{w_a}{w_3^{p^{a-3}}}$. We can think of this as a polynomial ring, with each ϖ_a and σ_a an indeterminant, modulo relations. These relations are generated by $\varpi_a \varpi_b - \varpi_c \varpi_d$ when $a+b = c+d$; hence they are homogeneous and the spectrum of this ring has a singular point at the origin. Thus $\mathbb{P}(1, p, \ldots, p^{r-1})$ has non-zero singular locus when $r > 2$. □

3.4 DG-algebras and BGG correspondence

Let k be a field of characteristic 2 and define

$$\Lambda = \frac{k[y_1, \ldots, y_n]}{(y_1^2, \ldots, y_n^2)} \qquad \text{and} \qquad S = k[x_1, \ldots, x_n],$$

graded via $|x_i| = |y_i| = 1$ so that $\mathbb{P}^n = \operatorname{Proj} S$. We refer the reader to Appendix A of Okonek et al. [40] for a detailed discussion of the classical Bernšteĭn, Gel'fand, Gel'fand (BGG) correspondence. It is routine to check that the results therein hold in characteristic 2 and give equivalences:

$$\underline{\mathtt{grmod}}(\Lambda) \equiv \mathsf{D}^b(\mathtt{grmod}(\Lambda))/\mathtt{Perf}(\Lambda) \equiv \mathsf{D}^b(\mathtt{grmod}(S))/\mathtt{fdim}(S) \equiv \mathsf{D}^b(\mathtt{Coh}(\mathbb{P}^{\mathtt{n}})).$$

Here $\underline{\mathtt{grmod}}(\Lambda)$ is the graded stable module category of graded modules modulo projectives, $\mathsf{D}^b(-)$ indicates the bounded derived category, $\mathtt{Perf}(\Lambda) \subseteq \mathsf{D}^b(\mathtt{grmod}(\Lambda))$ are the perfect complexes (i.e. those complexes that are isomorphic to a bounded complex of projective modules), and $\mathtt{fdim}(S) \subseteq \mathsf{D}^b(\mathtt{grmod}(S))$ are the complexes that are isomorphic to a bounded complex of finite dimensional modules. In this section we give a generalization of these equivalences to DG-modules. We indicate how the functor $\mathcal{H}_{[1]}$ factors through the resulting equivalence and use this to obtain further results.

Consider S to be a DG-algebra with zero differential (we will use a subscript to indicate the grading of DG-S-modules and parenthesized superscripts to indicate the position in a chain complex of Λ-modules). By $\mathsf{D}(S)$ we mean the derived category of DG-S-modules and let $\mathsf{D}(\mathtt{Mod}(\Lambda))$ be the unbounded derived category of all Λ-modules (not necessarily finitely generated). Given a chain complex M of Λ-modules we define $G(M) = S \otimes_k M$. We give $G(M)$ the structure of a DG-S-module via the grading $S_i \otimes M^{(j)} \subseteq G(M)_{i+j}$ and differential $\partial_{G(M)} = \mathrm{id}_S \otimes \partial_M + \sum_\ell x_\ell \otimes y_\ell$.

Lemma 3.4.1. *$G\colon \mathsf{D}(\mathtt{Mod}(\Lambda)) \to \mathsf{D}(S)$ is a well defined functor.*

Proof. We clearly have a well defined functor from the category of chain complexes of Λ-modules to the category of DG-S-modules, we need only show that it preserves quasi-isomorphisms. A simple diagram chase shows that a map of chain complexes of Λ-modules, or a map of DG-S-modules, is a quasi-isomorphism if and only if both its kernel and cokernel are acyclic. As G is given by tensoring over a field it is exact therefore we reduce to showing that G takes acyclic complexes to acyclic DG-S-modules.

Let $J = S \otimes_k \Lambda$ be the DG-S-module with grading $J_n = S_n \otimes \Lambda$ and differential $\partial_J = \sum_\ell x_\ell \otimes y_\ell$ and assume that M is an acyclic complex of Λ-modules. We observe that both J and $G(M)$ have the structure of a complex of Λ-modules and one easily sees that $G(M)$ is the direct sum total complex of the double complex obtained by tensoring J and M over Λ. As J is positively graded and M is exact the Acyclic Assembly Lemma [61, Lemma 2.7.3] gives that $G(M)$ is also exact. $\square$

Given an object X in a triangulated category T we let $\mathsf{thick}_{\mathsf{T}}(X)$ (or just $\mathsf{thick}(X)$ when there is no confusion) be the intersection of all thick subcategories that contain X, i.e., the smallest thick subcategory containing X.

Theorem 3.4.2. *G restricts to an equivalence $\mathsf{D}^b(\mathtt{mod}(\Lambda)) \equiv \mathsf{thick}_{\mathsf{D}(S)}(S)$.*

Proof. Consider $I = S^* \otimes_k \Lambda$, where by S^* we mean the graded dual defined by $(S^*)_n = \mathrm{Hom}_k(S_{-n}, k)$. This is a DG-$S$-module with grading $I_n = (S^*)_n \otimes_k \Lambda$ and differential $\partial_I = \sum_\ell x_\ell \otimes y_\ell$. Note that I also has the structure of a complex of Λ-modules, so if M is also a complex of Λ-modules we may define $\mathrm{Hom}^d_\Lambda(I, M)$ to be the set of all Λ-module homomorphisms of degree d from I to M (not necessarily commuting with differentials). Using the S-module structure of I and the regular representation we get that $\operatorname{Hom}^{\bullet}_{\Lambda}(I,M) = \bigoplus_d \operatorname{Hom}^d_{\Lambda}(I,M)$ is a graded S-module; in fact it is a DG-S-module with differential $\phi \mapsto \partial_M\phi + \phi\partial_I$.

Let $\mathsf{K}(\operatorname{Inj}\Lambda)$ be the homotopy category of complexes of injective Λ-modules and, as in the proof of lemma 3.4.1, let $J = S \otimes_k \Lambda$. Consider $\operatorname{Hom}^{\bullet}_{\Lambda}(I,-)\colon \mathsf{K}(\operatorname{Inj}\Lambda) \to \mathsf{D}(S)$. Benson et al. [8, Theorem 5.5] prove that the analogous functor $\operatorname{Hom}^{\bullet}_{\Lambda}(J,-)\colon \mathsf{K}(\operatorname{Inj}\Lambda) \to \mathsf{D}(S)$ is an equivalence of categories. With only minor modifications the same proof shows that $\operatorname{Hom}^{\bullet}_{\Lambda}(I,-)\colon \mathsf{K}(\operatorname{Inj}\Lambda) \to \mathsf{D}(S)$ is an equivalence as well. The natural inclusion $\mathsf{K}(\operatorname{Inj}\Lambda) \to \mathsf{D}(\mathtt{Mod}(\Lambda))$ restricts to an equivalence of categories $\mathsf{K}(\operatorname{Inj}\Lambda)^{\mathsf{c}} \equiv \mathsf{D}^b(\mathtt{mod}(\Lambda))$, where for any triangulated category T we let T^{c} denote the full subcategory of compact objects in T [34, Proposition 2.3]. The complex I is a bounded above complex of projective modules, therefore $\operatorname{Hom}^{\bullet}_{\Lambda}(I,-)$ preserves quasi-isomorphisms and consequently factors through this inclusion to give an equivalence $\operatorname{Hom}^{\bullet}_{\Lambda}(I,-)\colon \mathsf{D}^b(\mathtt{mod}(\Lambda)) \to \mathsf{D}(S)^{\mathsf{c}}$.

$$\begin{tikzcd}
\mathsf{D}(S) & \mathsf{K}(\operatorname{Inj}\Lambda) \arrow[l, "\equiv", "\operatorname{Hom}^{\bullet}_{\Lambda}(I,-)"'] \arrow[r] & \mathsf{D}(\mathtt{Mod}(\Lambda)) \\
\mathsf{D}(S)^{\mathsf{c}} \arrow[u, hook] & \mathsf{K}(\operatorname{Inj}\Lambda)^{\mathsf{c}} \arrow[u, hook] \arrow[l, "\equiv"] \arrow[r, "\equiv"] & \mathsf{D}^b(\mathtt{mod}(\Lambda)) \arrow[u, hook] \arrow[ll, bend left, "\operatorname{Hom}^{\bullet}_{\Lambda}(I,-)"]
\end{tikzcd}$$

All that's left is to identify $\mathsf{D}(S)^{\mathsf{c}}$ with $\mathsf{thick}(S)$ and show that $\operatorname{Hom}^{\bullet}_{\Lambda}(I,-)$ is naturally isomorphic to G. The first claim is well known, see for example Benson et al. [8, Theorem 2.2]. For the second it suffices to show that $\operatorname{Hom}^{\bullet}_{\Lambda}(I,M) \simeq G(M)$ when M is a bounded complex of finitely generated Λ-modules. For this we observe that the d^{th} graded piece of $\operatorname{Hom}^{\bullet}_{\Lambda}(I,M)$ is

$$\operatorname{Hom}^d_{\Lambda}(S^* \otimes_k \Lambda, M) = \prod_i \operatorname{Hom}_k((S^*)_i, M_{d+i}) = \prod_i S_{-i} \otimes_k M_{d+i}.$$

As M is bounded, M_{d+i} is nonzero for only finitely many i. The product is therefore a direct sum and we get $\operatorname{Hom}^d_{\Lambda}(I,M) \simeq (S \otimes_k M)_d = G(M)_d$. One easily checks that this respects the differentials and is natural in M. $\square$

There is an equivalence $\underline{\mathtt{mod}}(\Lambda) \equiv \mathsf{D}^b(\mathtt{mod}(\Lambda))/\mathtt{Perf}(\Lambda)$ defined by sending a Λ-module M to the complex

$$\cdots \to 0 \to M \to 0 \to \cdots$$

with M in the zeroth position and a zero in all other positions [45, Theorem 2.1]. Using the natural inclusion $\underline{\mathtt{grmod}}(\Lambda) \hookrightarrow \underline{\mathtt{mod}}(\Lambda)$ we get the commutative diagram below.

$$\begin{tikzcd}
\underline{\mathtt{grmod}}(\Lambda) \arrow[r, "\equiv"] \arrow[d, hook] & \frac{\mathsf{D}^b(\mathtt{grmod}(\Lambda))}{\mathtt{Perf}(\Lambda)} \arrow[r, "\equiv"] \arrow[d, hook] & \frac{\mathsf{D}^b(\mathtt{grmod}(S))}{\mathtt{fdim}(S)} \arrow[r, "\equiv"] & \mathsf{D}^b(\mathtt{Coh}(\mathbb{P}^n)) \\
\underline{\mathtt{mod}}(\Lambda) \arrow[r, "\equiv"] & \frac{\mathsf{D}^b(\mathtt{mod}(\Lambda))}{\mathtt{Perf}(\Lambda)} & &
\end{tikzcd}$$

By $\Lambda \in \mathsf{D}^b(\mathtt{mod}(\Lambda))$ we mean its image under the equivalence above. The perfect complexes are then exactly the elements of $\mathsf{thick}(\Lambda) \subseteq \mathsf{D}^b(\mathtt{mod}(\Lambda))$ [8, Theorem 2.2] and $G(\Lambda) = J$, the DG-S-module from the proof of lemma 3.4.1. One can show that J is quasi-isomorphic to the trivial DG-S-module k (c.f. the map η in Theorem 5.5 of Benson et al. [8]) hence $G(\Lambda) \simeq k$. This implies that G restricts to an equivalence $\mathtt{Perf}(\Lambda) \equiv \mathsf{thick}(k)$ and consequently induces an equivalence between the quotients $\mathsf{D}^b(\mathtt{mod}(\Lambda))/\mathtt{Perf}(\Lambda) \equiv \mathsf{thick}(S)/\mathsf{thick}(k)$. Thus there is an induced $\mathsf{D}^b(\mathtt{grmod}(S))/\mathtt{fdim}(S) \hookrightarrow \mathsf{thick}(S)/\mathsf{thick}(k)$. One can check that this is defined by mapping a complex M to the total complex $\bigoplus_i M^{(i)}$ with grading given by declaring the elements of $M_j^{(i)}$ to be homogeneous of degree $i+j$. The diagram is now as follows.

$$\begin{tikzcd}
\underline{\mathtt{grmod}}(\Lambda) \arrow[r, "\equiv"] \arrow[d, hook] & \frac{\mathsf{D}^b(\mathtt{grmod}(\Lambda))}{\mathtt{Perf}(\Lambda)} \arrow[r, "\equiv"] \arrow[d, hook] & \frac{\mathsf{D}^b(\mathtt{grmod}(S))}{\mathtt{fdim}(S)} \arrow[r, "\equiv"] \arrow[d, hook] & \mathsf{D}^b(\mathtt{Coh}(\mathbb{P}^n)) \\
\underline{\mathtt{mod}}(\Lambda) \arrow[r, "\equiv"] & \frac{\mathsf{D}^b(\mathtt{mod}(\Lambda))}{\mathtt{Perf}(\Lambda)} \arrow[r, "\equiv"] & \frac{\mathsf{thick}(S)}{\mathsf{thick}(k)} &
\end{tikzcd}$$

If M is a DG-S-module then the homology, $H(M)$, of M is a graded module for the homology of S, which is just S. A DG-S-module is contained in $\mathsf{thick}(S) \subseteq \mathsf{D}(S)$ (resp. $\mathsf{thick}(k)$) if and only if its homology is finitely generated (resp. finite dimensional) as an S-module [1, 7.5]. Thus homology induces a functor $\mathsf{thick}(S)/\mathsf{thick}(k) \to \mathtt{grmod}(S)/\mathtt{fdim}(S) \equiv \mathtt{Coh}(\mathbb{P}^n)$.

One can check that the functor $\mathsf{D}^b(\mathtt{Coh}(\mathbb{P}^n)) \to \mathtt{Coh}(\mathbb{P}^n)$ defined by $\mathscr{G} \mapsto \bigoplus_i H^i(\mathscr{G})(-i)$ completes the diagram below.

$$\begin{tikzcd}
\underline{\mathtt{grmod}}(\Lambda) \arrow[r, "\equiv"] \arrow[d, hook] & \frac{\mathsf{D}^b(\mathtt{grmod}(\Lambda))}{\mathtt{Perf}(\Lambda)} \arrow[r, "\equiv"] \arrow[d, hook] & \frac{\mathsf{D}^b(\mathtt{grmod}(S))}{\mathtt{fdim}(S)} \arrow[r, "\equiv"] \arrow[d, hook] & \mathsf{D}^b(\mathtt{Coh}(\mathbb{P}^n)) \arrow[d] \\
\underline{\mathtt{mod}}(\Lambda) \arrow[r, "\equiv"] & \frac{\mathsf{D}^b(\mathtt{mod}(\Lambda))}{\mathtt{Perf}(\Lambda)} \arrow[r, "\equiv"] & \frac{\mathsf{thick}(S)}{\mathsf{thick}(k)} \arrow[r, "H"] & \mathtt{Coh}(\mathbb{P}^n)
\end{tikzcd}$$

Proposition 3.4.3. *The functor* $\underline{\mathtt{mod}}(\Lambda) \to \mathtt{Coh}(\mathbb{P}^n)$ *along the bottom row of the above diagram is identical to* $\mathcal{H}_{[1]}\colon \underline{\mathtt{mod}}(\mathbb{G}^n_{a(1)}) \to \mathtt{Coh}(\mathbb{P}^n)$.

Proof. We have $k\mathbb{G}^r_{a(1)} \simeq \Lambda$ so $\mathbb{G}^r_{a(1)}$-modules are equivalent to Λ-modules. If M is any such module then applying G to the complex

$$\cdots \to 0 \to M \to 0 \to \cdots$$

yields the DG-S-module $S \otimes_k M$ with grading $|s \otimes m| = |s|$ and differential $\partial_{S\otimes_k M} = \sum_\ell x_\ell \otimes y_\ell$. As an S-module this is exactly the graded module corresponding to the sheaf $\widetilde{M}$ and $\partial_{S\otimes_k M}$ is the homomorphism corresponding to Θ_M. Thus taking homology and then the associated sheaf yields $\mathcal{H}_{[1]}(M)$. $\square$

Observe that this gives an alternate proof of corollary 3.2.12 in this case, for if $\mathcal{H}_{[1]}(M) = 0$ then $G(M)$ has finite dimensional homology and therefore is an element of $\mathsf{thick}(k)$, hence the equivalence gives $M \simeq 0$ in the stable module category. We also deduce the following results about $\mathcal{H}_{[1]}$. Note that in characteristic 2 we have $\mathscr{F}_1 = \mathcal{H}_{[1]}$ so Benson and Pevtsova have proven a stronger version of item 1 [6].

Theorem 3.4.4. *Let k have characteristic 2 and consider the group $G = \mathbb{G}^n_{a(1)}$. The corresponding functor $\mathcal{H}_{[1]}$ is...*

1. *essentially surjective on objects,*
2. *essentially surjective on maps, i.e., for all sheaf homomorphisms ϕ there is a module homomorphism ψ such that $\mathcal{H}_{[1]}(\psi)$ equals ϕ up to isomorphisms in the domain and codomain,*

3. not a faithful functor.

Proof. Both item 1 and item 2 follow immediately from the diagram and the fact that these properties obviously hold for the functor $\bigoplus_i H^i(-)(-i)\colon \mathsf{D}^b(\mathtt{Coh}(\mathbb{P}^\mathtt{n})) \to \mathtt{Coh}(\mathbb{P}^\mathtt{n})$. For item 3 consider the trivial module k and its first Heller shift $\Omega(k)$. As k is simple the stable homomorphisms from k to $\Omega(k)$ are exactly the homomorphisms $\mathrm{Hom}_G(k, \Omega(k)) \simeq k$. But $\mathcal{H}_{[1]}(\Omega(k)) = \mathcal{O}_{\mathbb{P}^n}(-1)$ and $\mathcal{H}_{[1]}(k) = \mathcal{O}_{\mathbb{P}^n}$ [6, 3.5] and $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-1)) = \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)) = 0$. $\square$

Chapter 4

COMPUTATIONS OF SHEAVES ASSOCIATED TO $\mathfrak{sl}_2$ [53]

In this chapter we explicitly compute examples of the associated sheaves of the previous chapter for the case of the Lie algebra $\mathfrak{sl}_2$. We begin by reviewing the definition of the restricted nullcone of a Lie algebra $\mathfrak{g}$ and of the local Jordan type of a $\mathfrak{g}$-module M, the global operator associated to a $\mathfrak{g}$-module M, and the sheaves associated to such an operator.

4.1 Jordan type and global operators for Lie algebras

Let $\mathfrak{g}$ be a restricted Lie algebra over an algebraically closed field k of positive characteristic p. Recall that this means $\mathfrak{g}$ is a Lie algebra equipped with an additional *p-operation* $(-)^{[p]}\colon \mathfrak{g} \to \mathfrak{g}$ satisfying certain axioms (see Strade and Farnsteiner [57] for details). Here we merely note that for the classical subalgebras of $\mathfrak{gl}_n$ the p-operation is given by raising a matrix to the p^{th} power.

Definition 4.1.1. The restricted nullcone of $\mathfrak{g}$ is the set

$$\mathcal{N}_1(\mathfrak{g}) = \left\{x \;\middle|\; x^{[p]} = 0\right\}$$

of p-nilpotent elements. This is a conical irreducible subvariety of the affine space $\mathfrak{g}$. By example 2.1.3 and definition 2.1.9 the support variety $\mathbb{P}(\mathfrak{g})$ is the projective variety whose points are lines through the origin in $\mathcal{N}_1(\mathfrak{g})$.

Example 4.1.2. Let $\mathfrak{g} = \mathfrak{sl}_2$ and take the usual basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $\{x, y, z\}$ be the dual basis so that $\mathfrak{sl}_2$, as an affine space, can be identified with $\mathbb{A}^3$ and has coordinate ring $k[x, y, z]$. A 2×2 matrix over a field is nilpotent if and only if its square

$$\begin{bmatrix} z & x \\ y & -z \end{bmatrix}^2 = (xy + z^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is zero therefore, independent of p, we get that $\mathcal{N}_1(\mathfrak{sl}_2)$ is the zero locus of $xy + z^2$.

Now $\mathbb{P}(\mathfrak{sl}_2)$ is the projective variety defined by the homogeneous polynomial $xy + z^2$. Let $\mathbb{P}^1$ have coordinate ring $k[s, t]$ and define a map $\iota\colon \mathbb{P}^1 \to \mathbb{P}(\mathfrak{sl}_2)$ via $[s, t] \mapsto [s^2 : -t^2 : st]$. One then checks that the maps $[1 : y : z] \mapsto [1 : z]$ and $[x : 1 : z] \mapsto [-z : 1]$ defined on the open sets $x \neq 0$ and $y \neq 0$, respectively, glue to give an inverse to ι. Thus $\mathbb{P}(\mathfrak{sl}_2) \simeq \mathbb{P}^1$. Note that this is simply proposition 3.3.2 without the scheme-theoretical language.

Recall, from example 2.1.13, that $\Theta_{\mathfrak{g}} = x_1 \otimes g_1 + \cdots + x_n \otimes g_n$ where $\{g_1, \ldots, g_n\}$ is a basis for $\mathfrak{g}$ with corresponding dual basis $\{x_1, \ldots, x_n\}$. As an element of $\mathfrak{g}^* \otimes_k \mathfrak{g} \simeq \operatorname{Hom}_k(\mathfrak{g}, \mathfrak{g})$ this is just the identity map and is therefore independent of the choice of basis. We have that $\Theta_{\mathfrak{g}}$ acts on $k[\mathcal{N}_1(\mathfrak{g})] \otimes_k M \simeq k[\mathcal{N}_1(\mathfrak{g})]^{\dim M}$ as a degree 1 endomorphism of graded $k[\mathcal{N}_1(\mathfrak{g})]$-modules (where $\deg x_i = 1$). The map of sheaves corresponding to this homomorphism is the global operator Θ_M.

Example 4.1.3. We have $\Theta_{\mathfrak{sl}_2} = x \otimes e + y \otimes f + z \otimes h$. Consider the 3-dimensional Weyl module $V(2)$ where e, f, and h act via

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

respectively. The global operator corresponding to $V(2)$ is the sheaf map

$$\begin{bmatrix} 2z & 2x & 0 \\ y & 0 & x \\ 0 & 2y & -2z \end{bmatrix} : \mathcal{O}_{\mathbb{P}(\mathfrak{sl}_2)}^3 \to \mathcal{O}_{\mathbb{P}(\mathfrak{sl}_2)}(1)^3.$$

Taking the pullback through the map $\iota\colon \mathbb{P}^1 \to \mathbb{P}(\mathfrak{sl}_2)$ from example 4.1.2 we get that $\Theta_{V(2)}$ is the sheaf map

$$\begin{bmatrix} 2st & 2s^2 & 0 \\ -t^2 & 0 & s^2 \\ 0 & -2t^2 & -2st \end{bmatrix} : \mathcal{O}_{\mathbb{P}^1}^3 \to \mathcal{O}_{\mathbb{P}^1}(2)^3.$$

For the local Jordan type of a $\mathfrak{g}$-module, specialization of $\Theta_{\mathfrak{g}}$ corresponds to substituting, up to a scalar, the coordinates of a point of $\mathfrak{g}$ for the x_i therefore the local Jordan type has the following simple description: The local Jordan type $\mathrm{JType}(v, M)$ of M at a point $v \in \mathbb{P}(\mathfrak{g})$ is the Jordan type of any linear endomorphism of M given as multiplication by a nonzero point $x \in \mathfrak{g}$ on the line v.

Example 4.1.4. Assume $p > 2$ and consider the Weyl module $V(2)$ from example 4.1.3. The global operator $\Theta_{V(2)}$ is given by the matrix

$$A = \begin{bmatrix} 2z & 2x & 0 \\ y & 0 & x \\ 0 & 2y & -2z \end{bmatrix}.$$

For the purposes of computing the local Jordan type we consider $[x : y : z]$ to be an element in the projective space $\mathbb{P}(\mathfrak{g})$. If $y = 0$ then $xy + z^2 = 0$ implies $z = 0$ and we can scale to $x = 1$. This immediately gives Jordan type $[3]$. If $y \neq 0$ then we can scale to $y = 1$ and therefore $x = -z^2$. This gives

$$A = \begin{bmatrix} 2z & -2z^2 & 0 \\ 1 & 0 & -z^2 \\ 0 & 2 & -2z \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 2z^2 & -4z^3 & 2z^4 \\ 2z & -4z^2 & 2z^3 \\ 2 & -4z & 2z^2 \end{bmatrix}$$

therefore $\operatorname{rank} A = 2$ and $\operatorname{rank} A^2 = 1$. Using lemma 2.2.5 we conclude that the Jordan type is the conjugate of $(3-2, 2-1, 1) = [1]^3$, which is $[3]$. Thus the local Jordan type of $V(2)$ is the constant function $v \mapsto [3]$.

Note that $V(2)$ is a rational $\mathfrak{sl}_2$-module, i.e., the $\mathfrak{sl}_2$-module structure of $V(2)$ comes from an SL_2-module structure. When $\mathfrak{g}$ is the Lie algebra of an algebraic group G, the adjoint action of G on $\mathfrak{g}$ induces an action of G on the restricted nullcone and hence on $\mathbb{P}(\mathfrak{g})$. One can show that the local Jordan type of a G-module (when considered as a $\mathfrak{g}$-module) is constant on the G-orbits of this action. The adjoint action of SL_2 on $\mathbb{P}(\mathfrak{sl}_2)$ is transitive so we get the following.

Theorem 4.1.5 ([16, 2.5])**.** *Every rational $\mathfrak{sl}_2$-module has constant Jordan type.*

4.2 The category of $\mathfrak{sl}_2$-modules

The calculations in section 4.4 will be based on detailed information about the category of $\mathfrak{sl}_2$-modules, which we develop in this section. The indecomposable $\mathfrak{sl}_2$-modules have been classified, each is one of the following four types: a Weyl module $V(\lambda)$, the dual of a Weyl module $V(\lambda)^*$, an indecomposable projective module $Q(\lambda)$, or a non-constant module $\Phi_\xi(\lambda)$. Explicit bases for the first three types are known; we will remind the reader of these formulas and develop similar formulas for the $\Phi_\xi(\lambda)$. We will also calculate the local Jordan type $\mathrm{JType}(-, M)\colon \mathbb{P}^1 \to \mathscr{P}_p$ for each indecomposable M. Finally we will calculate the Heller shifts $\Omega(V(\lambda))$.

We begin by stating the results for each of the four types and the classification. Recall the standard basis for $\mathfrak{sl}_2$ is $\{e, f, h\}$ where

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let λ be a non-negative integer and write $\lambda = rp + a$ where $0 \leq a < p$ is the remainder of λ modulo p. Each type is parametrized by the choice of λ, with the parametrization of $\Phi_\xi(\lambda)$ requiring also a choice of $\xi \in \mathbb{P}^1$. The four types are as follows:

- The **Weyl modules** $V(\lambda)$.

Basis: $\{v_0, v_1, \ldots, v_\lambda\}$

Action:
$$\begin{aligned} ev_i &= (\lambda - i + 1)v_{i-1} \\ fv_i &= (i+1)v_{i+1} \\ hv_i &= (\lambda - 2i)v_i \end{aligned}$$

Graph: Figure 4.1

Local Jordan type: Constant Jordan type $[p]^r[a+1]$

- The **dual Weyl modules** $V(\lambda)^*$.

Basis: $\{\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_\lambda\}$

Action:
$$\begin{aligned} e\hat{v}_i &= i\hat{v}_{i-1} \\ f\hat{v}_i &= (\lambda - i)\hat{v}_{i+1} \\ h\hat{v}_i &= (\lambda - 2i)\hat{v}_i \end{aligned}$$

Graph: Figure 4.1

Local Jordan type: Constant Jordan type $[p]^r[a+1]$

- The **projectives** $Q(\lambda)$.

Define $Q(p-1) = V(p-1)$. For $0 \leq \lambda < p-1$ we define $Q(\lambda)$ via

Basis: $\{v_0, v_1, \ldots, v_{2p-\lambda-2}\} \cup \{w_{p-\lambda-1}, w_{p-\lambda}, \ldots, w_{p-1}\}$

Action:
$$\begin{aligned} ev_i &= -(\lambda + i + 1)v_{i-1} \\ fv_i &= (i+1)v_{i+1} \\ hv_i &= -(\lambda + 2i + 2)v_i \\ ew_i &= -(\lambda + i + 1)w_{i-1} + \tfrac{1}{i}v_{i-1} \\ fw_i &= (i+1)w_{i+1} - \tfrac{1}{\lambda+1}\delta_{-1,i}v_p \\ hw_i &= -(\lambda + 2i + 2)w_i \end{aligned}$$

Graph: Figure 4.2

Local Jordan type: Constant Jordan type $[p]^2$

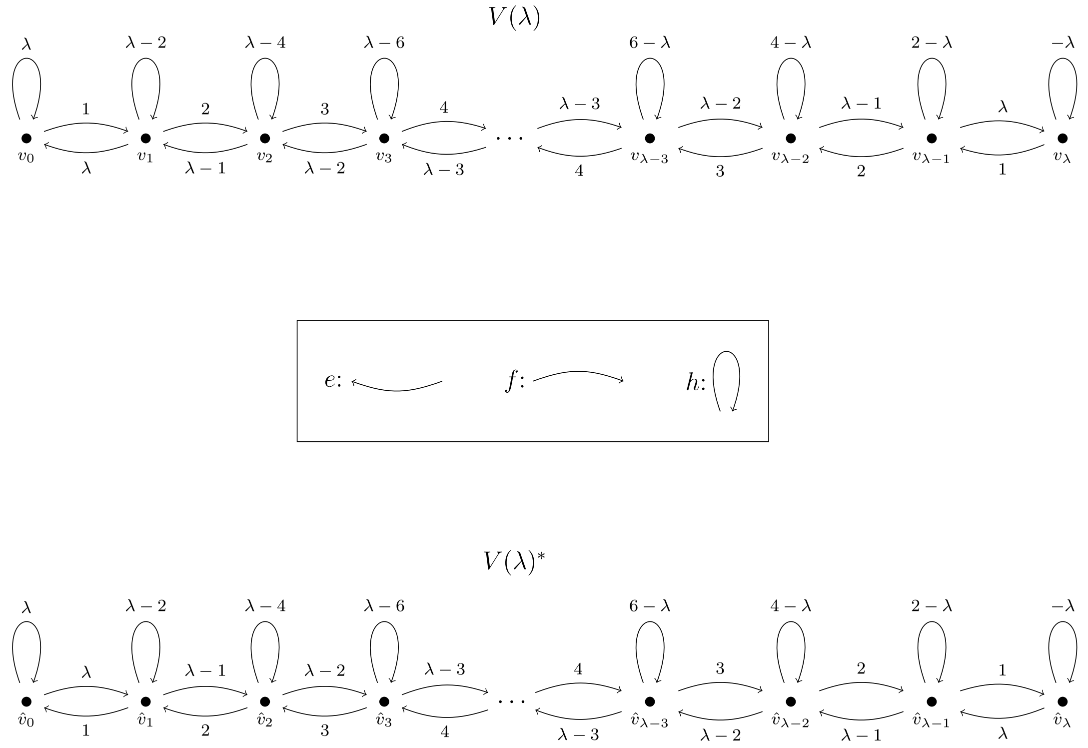


Figure 4.1: Graphs of $V(\lambda)$ and $V(\lambda)^*$

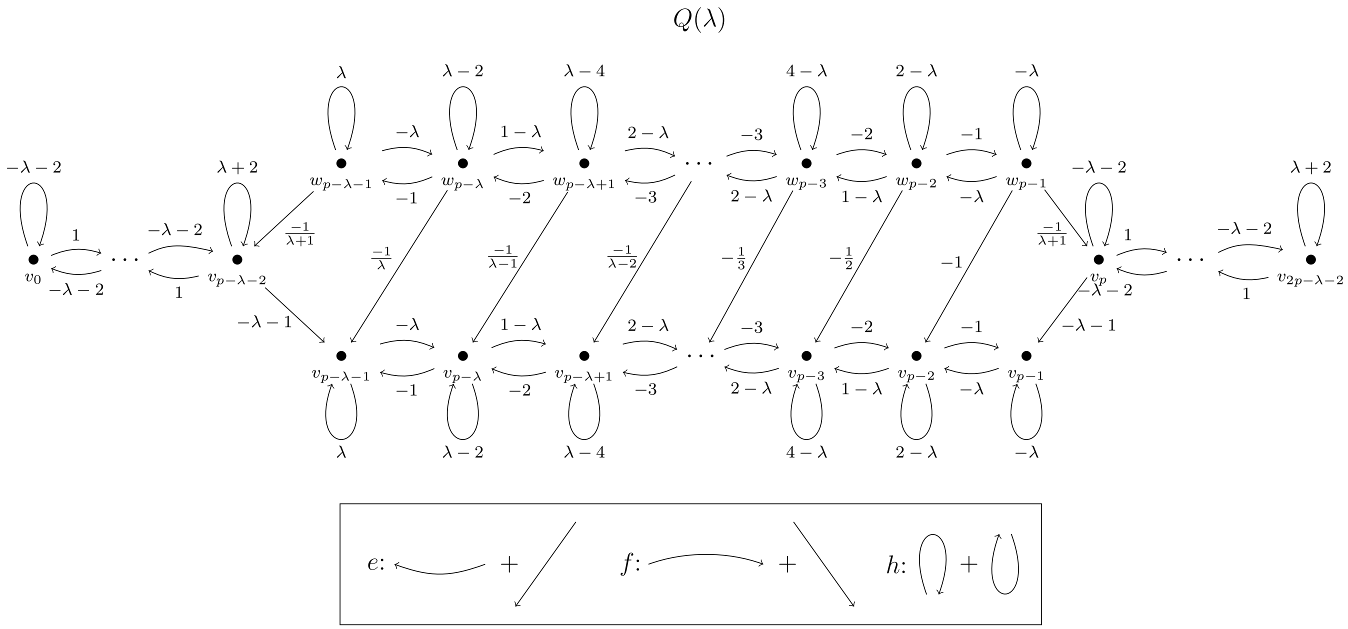


Figure 4.2: Graph of $Q(\lambda)$

- The **non-constant modules** $\Phi_\xi(\lambda)$.

  Assume $\lambda \geq p$ and let $\xi \in \mathbb{P}^1$. If $\xi = [1 : \varepsilon]$ then $\Phi_\xi(\lambda)$ is defined by

$$\begin{array}{rl} \text{Basis:} & \{w_{a+1}, w_{a+2}, \ldots, w_\lambda\} \\ \text{Action:} & ew_i \; = (i+1)(w_{i+1} - \binom{d}{i}\varepsilon^{i-a}\delta_{\lambda,i}w_{a+1}) \\ & fw_i \; = (\lambda - i + 1)w_{i-1} \\ & hw_i \; = (2i - \lambda)w_i \\ \text{Graph:} & \text{Figure 4.3} \\ \text{Local Jordan type:} & [p]^{r-1}[p-a-1][a+1] \text{ at } \xi \text{ and } [p]^r \text{ elsewhere} \end{array}$$

  If $\xi = [0 : 1]$ then $\Phi_\xi(\lambda)$ is defined to be the submodule of $V(\lambda)$ spanned by the basis elements $\{v_{a+1}, v_{a+2}, \ldots, v_\lambda\}$. It is also depicted in Figure 4.3 and has the same local Jordan type as above.

Theorem 4.2.1 ([42])**.** *Each of the following modules are indecomposable:*

- $V(\lambda)$ *and* $Q(\lambda)$ *for* $0 \leq \lambda < p$.
- $V(\lambda)$ *and* $V(\lambda)^*$ *for* $\lambda \geq p$ *such that* $p \nmid \lambda + 1$.
- $\Phi_\xi(\lambda)$ *for* $\xi \in \mathbb{P}^1$ *and* $\lambda \geq p$ *such that* $p \nmid \lambda + 1$.

Moreover, these modules are pairwise non-isomorphic, save $Q(p-1) = V(p-1)$*, and give a complete classification of the indecomposable restricted* $\mathfrak{sl}_2$*-modules.*

As stated before the explicit bases for $V(\lambda)$, $V(\lambda)^*$, and $Q(\lambda)$ are known; see, for example, Benkart and Osborn [4]. For the local Jordan type of $V(\lambda)$ and $V(\lambda)^*$ the matrix that describes the action of e with respect to the given basis is almost in Jordan normal form, one needs only to scale the basis elements appropriately, so we can immediately read off the local Jordan type at the point $ke \in \mathbb{P}(\mathfrak{sl}_2)$, and theorem 4.1.5 gives that these modules have constant Jordan type. As theorem 2.2.6 gives the local Jordan type of the $Q(\lambda)$ all that is left is to justify the explicit description of $\Phi_\xi(\lambda)$ and its local Jordan type.

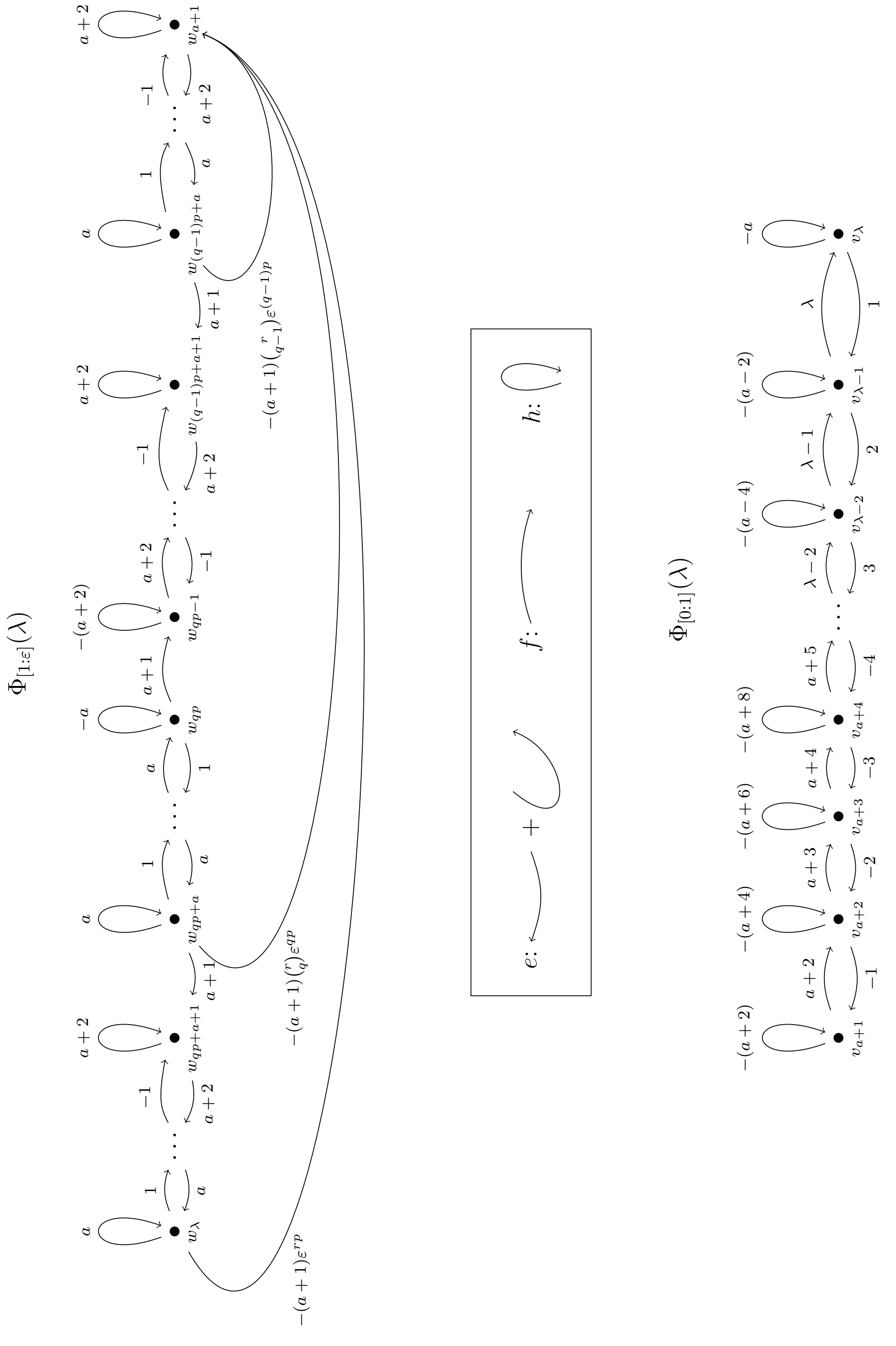


Figure 4.3: Graphs of $\Phi_\xi(\lambda)$

First we recall the definition of $\Phi_\xi(\lambda)$. Let $V = k^2$ be the standard representation of SL_2, then the dual V^* is a 2-dimensional representation with basis $\{x, y\}$ (dual to the standard basis for V). The induced representation on the symmetric product $S(V^*)$ is degree preserving and the dual $S^\lambda(V^*)^*$ of the degree λ subspace is the Weyl module $V(\lambda)$. Specifically, we let $v_i \in V(\lambda)$ be dual to $x^{\lambda-i}y^i$.

Let $B_2 \subseteq \mathrm{SL}_2$ be the Borel subgroup of upper triangular matrices and recall that the homogeneous space SL_2/B_2 is isomorphic to $\mathbb{P}^1$ as a variety; the map $\phi\colon \mathbb{P}^1 \to \mathrm{SL}_2$ given by

$$[1:\varepsilon] \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix} \qquad \text{and} \qquad [0:1] \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

factors to an explicit isomorphism $\mathbb{P}^1 \xrightarrow{\sim} \mathrm{SL}_2/B_2$.

Definition 4.2.2 ([42])**.** Let $\Phi(\lambda)$ be the $\mathfrak{sl}_2$-submodule of $V(\lambda)$ spanned by the vectors $\{v_{a+1}, v_{a+2}, \ldots, v_\lambda\}$. Given $\xi \in \mathbb{P}^1$ we define $\Phi_\xi(\lambda)$ to be the $\mathfrak{sl}_2$-module $\phi(\xi)\Phi(\lambda)$.

Observe first that $\Phi_{[0:1]}(\lambda) = \Phi(\lambda)$ so in this case we have the desired description. Now assume $\xi = [1:\varepsilon]$ where $\varepsilon \in k$. As $\phi(\xi)$ is invertible multiplication by it is an isomorphism so $\Phi_\xi(\lambda)$ has basis $\{\phi(\xi)v_i\}$. Our basis for $\Phi_\xi(\lambda)$ will be obtained by essentially a row reduction of this basis, so to proceed we now compute the action of SL_2 on $V(\lambda)$. Observe:

$$\begin{aligned}
\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} v_i\right)(x^{\lambda-j}y^j) &= v_i\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} x^{\lambda-j}y^j\right) \\
&= v_i\left(\sum_{s=0}^{\lambda-j}\sum_{t=0}^{j}\binom{\lambda-j}{s}\binom{j}{t}a^k b^{\lambda-j-k}c^t d^{j-t}x^{s+t}y^{\lambda-k-s}\right) \\
&= \sum\binom{\lambda-j}{s}\binom{j}{t}a^s b^{\lambda-j-s}c^t d^{j-t}
\end{aligned}$$

where the sum is over pairs $(s,t) \in \mathbb{N}^2$ such that $0 \le s \le \lambda - j$, $0 \le t \le j$, and $s + t = \lambda - i$. Such pairs come in the form $(\lambda - i - t, t)$ where t ranges from $\max(0, j-i)$ to $\min(j, \lambda - i)$ therefore

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} v_i = \sum_{j=0}^{r}\sum_{t=\max(0,j-i)}^{\min(j,\lambda-i)}\binom{\lambda-j}{\lambda-i-t}\binom{j}{t}a^{\lambda-i-t}b^{t+i-j}c^t d^{j-t}v_j.$$

For computing $\Phi_\xi(\lambda)$ we will need only the following special case:

$$\phi(\xi)v_i = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix} v_i = \sum_{j=\lambda-i}^{\lambda} (-1)^j \binom{j}{\lambda - i} \varepsilon^{i+j-\lambda} v_j.$$

Proposition 4.2.3. *Given $i = qp + b$, $0 \leq b < p$, define*

$$w_i = \begin{cases} v_{\lambda-i} - \binom{r}{q}\varepsilon^{qp} v_{\lambda-b} & \text{if } \ b \leq a \\ v_{\lambda-i} & \text{if } \ b > a. \end{cases}$$

Then the vectors $w_{a+1}, w_{a+2}, \ldots, w_\lambda$ form a basis of $\Phi_\xi(\lambda)$.

Proof. We will prove by induction that that for all $a + 1 \leq i \leq \lambda$ we have

$$\operatorname{span}_k \{\phi(\xi)v_{a+1}, \ldots, \phi(\xi)v_i\} = \operatorname{span}_k \{w_{a+1}, \ldots, w_i\}.$$

For the base case the formula above gives

$$\begin{aligned} \phi(\xi)v_{a+1} &= \sum_{j=rp-1}^{\lambda} (-1)^j \binom{j}{rp-1} \varepsilon^{j-rp+1} v_j \\ &= (-1)^{rp-1} \binom{rp-1}{rp-1} v_{rp-1} \\ &= (-1)^{rp-1} w_{a+1} \end{aligned}$$

so clearly the statement is true.

For the inductive step we assume the statement holds for integers less than i. Then by hypothesis we have

$$\operatorname{span}_k \{\phi(\xi)v_{a+1}, \ldots, \phi(\xi)v_i\} = \operatorname{span}_k \{w_{a+1}, \ldots, w_{i-1}, \phi(\xi)v_i\}$$

and can replace $\phi(\xi)v_i$ with the vector

$$w' = (-1)^{\lambda-i}\phi(\xi)v_i - \sum_{j=a+1}^{i-1} (-1)^{i-j} \binom{\lambda - j}{\lambda - i} \varepsilon^{i-j} w_j$$

to get another spanning set. We then show that $w' = w_i$ by checking that the coordinates of each vector are equal. Note that for $j < \lambda - i$ the coefficient of v_j in each of the factors of w' is zero, as it is in w_i. The coefficient of $v_{\lambda-i}$ in $(-1)^{\lambda-i}\phi(\xi)v_i$ is 1 and in each w_j, $a+1 \leq j < i$ it is zero, hence the coefficient in w' is 1, as it is in w_i.

Next assume $\lambda - i < j < rp$. Then only $\phi(\xi)v_i$ and $w_{\lambda-j}$ contribute a v_j term therefore the coefficient of v_j in w' is

$$(-1)^{\lambda-i+j}\binom{j}{\lambda-i}\varepsilon^{i+j-\lambda} - (-1)^{i+j-\lambda}\binom{j}{\lambda-i}\varepsilon^{i+j-\lambda} = 0$$

which again agrees with w_i. Now all that's left is to check the coefficients of $v_{rp}, v_{rp+1}, \ldots, v_\lambda$. Note that w_t has a v_{rp+j} term only if

$$t = p+a-j, 2p+a-j, \ldots, \lambda - j$$

and the coefficient of v_{rp+j} in w_{tp+a-j}, for $1 \leq t \leq r$, is

$$-\binom{r}{t}\varepsilon^{tp}.$$

Thus the coefficient of v_{rp+j} in w' is

$$(-1)^{a-i-j}\binom{rp+j}{\lambda-i}\varepsilon^{i+j-a} + \sum(-1)^{i+j-tp-a}\binom{r}{t}\binom{(r-t)p+j}{\lambda-i}\varepsilon^{i+j-a}$$

where the sum is over those t such that $1 \leq t \leq r$ and $tp + a \leq i + j - 1$.

From here there are several cases. Assume first that $a < b$. Then, from $b < p$ we get $p + a - b > a \geq j$ hence any binomial coefficient those top number equals $a - b$ in k and whose bottom number equals j in k will be zero because there will be a carry. Both $\binom{\lambda-t}{\lambda-i}$ and $\binom{(r-t)p+j}{\lambda-i}$ satisfy this condition therefore the coefficient of v_{rp+j} in w' is zero. Thus if $a < b$ then we have $w' = w_i$.

Next assume that $b \leq a$. Then the formula above for the coefficient of v_{rp+j} in w' becomes

$$\begin{aligned}
&(-1)^{a-i-j}\left[\binom{r}{q} + \sum(-1)^{tp}\binom{r}{t}\binom{r-t}{r-q}\right]\binom{j}{a-b}\varepsilon^{i+j-a} \\
&= (-1)^{a-i-j}\left[\binom{r}{q} + \sum(-1)^{t}\binom{r}{q}\binom{q}{t}\right]\binom{j}{a-b}\varepsilon^{i+j-a} \\
&= (-1)^{a-i-j}\left[1 + \sum(-1)^{t}\binom{q}{t}\right]\binom{r}{q}\binom{j}{a-b}\varepsilon^{i+j-a}
\end{aligned}$$

where the sum is over the same t from above. If $j < a-b$ then clearly this is zero. If $j > a-b$ then the sum is over $t = 1, 2, \ldots, q$ and

$$1 + \sum_{t=1}^{q} (-1)^t \binom{q}{t} = \sum_{t=0}^{q} (-1)^t \binom{q}{t} = 0$$

so the only $v_{\lambda-b}$ occurs as a term in w'. In that case the sum is over $t = 1, 2, \ldots, q-1$ and

$$1 + \sum_{t=1}^{q-1} (-1)^t \binom{q}{t} = (-1)^{q+1} + \sum_{t=0}^{q} (-1)^t \binom{q}{t} = (-1)^{q+1}$$

so the coefficient is

$$(-1)^{a-i-(a-b)+q+1} \binom{r}{q} \varepsilon^{i+(a-b)-a} = -\binom{r}{q} \varepsilon^{qp}$$

as desired. Thus $w' = w_i$ and the proof is complete. $\square$

Now that we have a basis it's trivial to determine the action.

Proposition 4.2.4. *Let $i = qp + b$, $a + 1 \le i \le \lambda$. Then*

$$ew_i = \begin{cases} (i+1)\left(w_{i+1} - \binom{\lambda}{i}\varepsilon^{qp} w_{b+1}\right) & \text{if } a = b \\ (i+1) w_{i+1} & \text{if } a \neq b \end{cases}$$

$$fw_i = (\lambda - i + 1) w_{i-1}$$

$$hw_i = (2i - \lambda) w_i$$

where $w_a = w_{\lambda+1} = 0$.

Proof. The proof is just a case by case analysis. We start with $e \in \mathfrak{sl}_2$. If $b < a$ then

$$ew_i = ev_{\lambda-i} - \binom{r}{q} \varepsilon^{qp} ev_{\lambda-b} = (i+1)v_{\lambda-i-1} - (b+1)\binom{r}{q}\varepsilon^{qp} v_{\lambda-b-1} = (i+1)w_{i+1}.$$

If $b = a$ then

$$ew_i = (i+1)v_{\lambda-i-1} - (b+1)\binom{r}{q}\varepsilon^{qp} v_{\lambda-b-1} = (i+1)\left(w_{i+1} - \binom{\lambda}{i}\varepsilon^{qp} w_{b+1}\right).$$

If $p - 1 > b > a$ then

$$ew_i = ev_{\lambda-i} = (i+1)v_{\lambda-i-1} = (i+1)w_{i+1}.$$

Finally if $b = p - 1$ then

$$ew_i = (i+1)v_{\lambda-i-1} = 0.$$

All the above cases fit the given formula so we are done with e. Next consider $f \in \mathfrak{sl}_2$. If $0 < b \leq a$ then

$$\begin{aligned} fw_i &= fv_{\lambda-i} - \binom{r}{q}\varepsilon^{qp} f v_{\lambda-b} \\ &= (\lambda - i + 1)v_{\lambda-i+1} - (\lambda - b + 1)\binom{r}{q}\varepsilon^{qp} v_{\lambda-b+1} \\ &= (\lambda - i + 1)w_{i-1}. \end{aligned}$$

If $b = 0$ then

$$fw_i = fv_{\lambda-i} - \binom{r}{q}\varepsilon^{qp} f v_{\lambda} = (\lambda+1)v_{\lambda-i+1} = (\lambda - i + 1)w_{i-1}.$$

If $a + 1 = b$ then

$$fw_i = fv_{\lambda-i} = (\lambda - i + 1)v_{\lambda-i+1} = spv_{\lambda-i+1} = 0.$$

Finally if $b > a + 1$ then

$$fw_i = (\lambda - i + 1)v_{\lambda-i+1} = (\lambda - i + 1)w_{i-1}.$$

All the above cases fit the given formula so we are done with f. Last but not least consider $h \in \mathfrak{sl}_2$. If $b \leq a$ then

$$hw_i = hv_{\lambda-i} - \binom{r}{q}\varepsilon^{qp} h v_{\lambda-b} = (2i - \lambda)v_{\lambda-i} - (2b - \lambda)\binom{r}{q}\varepsilon^{qp} v_{\lambda-b} = (2i - \lambda)w_i.$$

If $b > a$ then

$$hw_i = hv_{\lambda-i} = (2i - \lambda)v_{\lambda-i} = (2i - \lambda)w_i.$$

$\square$

Lastly we calculate that the Jordan type is as stated: $[p]^{r-1}[p-a-1][a+1]$ at ξ and $[p]^r$ elsewhere. First note that the result holds for $\Phi_{[0:1]}(\lambda)$ by lemma 4.3.5; furthermore, that the point $[0:1] \in \mathbb{P}^1$ at which the Jordan type is $[p]^{r-1}[p-a-1][a+1]$ corresponds to the line through $f \in \mathcal{N}_1(\mathfrak{sl}_2)$ under the map

$$\begin{aligned} \iota\colon \mathbb{P}^1 &\to \mathbb{P}(\mathfrak{sl}_2) \\ [s:t] &\mapsto \begin{bmatrix} st & s^2 \\ -t^2 & -st \end{bmatrix} \end{aligned}$$

from example 4.1.2. Let

$$\mathrm{ad}\colon\ \mathrm{SL}_2 \to \mathrm{End}(\mathfrak{sl}_2)$$

be the adjoint action of SL_2 on $\mathfrak{sl}_2$. As $V(\lambda)$ is a rational SL_2-module this satisfies

$$\mathrm{ad}(g)(E) \cdot m = g \cdot (E \cdot (g^{-1} \cdot m))$$

for all $g \in \mathrm{SL}_2$, $E \in \mathfrak{sl}_2$, and $m \in V(\lambda)$. Along with $\Phi_\xi(\lambda) = \phi(\xi)\Phi_{[0:1]}(\lambda)$ this implies commutativity of the following diagram:

$$\begin{tikzcd}
\Phi_{[0:1]}(\lambda) \arrow[r, "\phi(\xi)"] \arrow[d, "E"'] & \Phi_\xi(\lambda) \arrow[d, "\mathrm{ad}(\phi(\xi))(E)"] \\
\Phi_{[0:1]}(\lambda) \arrow[r, "\phi(\xi)"'] & \Phi_\xi(\lambda)
\end{tikzcd}$$

As multiplication by $\phi(\xi)$ is an isomorphism, letting E range over $\mathcal{N}_1(\mathfrak{sl}_2)$ we see that the module $\Phi_\xi(\lambda)$ has Jordan type $[p]^{r-1}[p-a-1][a+1]$ at $\mathrm{ad}(\phi(\xi))(f)$ and $[p]^r$ elsewhere. Then we simply calculate

$$\begin{aligned} \mathrm{ad}(\phi(\xi))(f) &= \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix}^{-1} \\ &= \begin{bmatrix} -\varepsilon & -1 \\ \varepsilon^2 & \varepsilon \end{bmatrix} \end{aligned}$$

and observe that, as an element of $\mathbb{P}(\mathfrak{sl}_2)$, this is $\iota([1:\varepsilon])$.

Thus we now have a complete description of the indecomposable $\mathfrak{sl}_2$-modules. We finish this section with one more computation that will be needed in section 4.4: the computation of the Heller shifts $\Omega(V(\lambda))$ for indecomposable $V(\lambda)$. Note that $V(p-1) = Q(p-1)$ is projective so $\Omega(V(p-1)) = 0$. For other $V(\lambda)$ we have the following.

Proposition 4.2.5. *Let $\lambda = rp + a$ be a non-negative integer and $0 \leq a < p$ its remainder modulo p. If $a \neq p-1$ then $\Omega(V(\lambda)) = V((r+2)p - a - 2)$.*

Proof. This will be a direct computation. We will determine the projective cover $f\colon P \to V(\lambda)$ and then set $f(x) = 0$ for an arbitrary element $x \in P$. This will give us the relations determining $\ker f = \Omega(V(\lambda))$ which we will convert into a basis and identify with $V((r+2)p - a - 2)$.

The indecomposable summands of P are in bijective correspondence with the indecomposable summands (all simple) of the top of $V(\lambda)$, i.e. $V(\lambda)/\operatorname{rad} V(\lambda)$. This correspondence is given as follows, if $\pi_q\colon V(\lambda) \to V(a)$ is the projection onto a summand of the top of $V(\lambda)$ then the projective cover $\phi_a\colon Q(a) \to V(a)$ factors through π_q.

$$\begin{tikzcd}
Q(a) \arrow[rr, "\phi_a"] \arrow[dr, dashed, "f_q"'] & & V(a) \\
& V(\lambda) \arrow[ur, "\pi_q"'] &
\end{tikzcd}$$

The map $f_q\colon Q(a) \to V(\lambda)$ so defined is the restriction of $f\colon P \to V(\lambda)$ to the summand $Q(a)$ of P.

The module $V(\lambda)$ fits into a short exact sequence

$$0 \to V(p-a-2)^{\oplus r+1} \to V(\lambda) \xrightarrow{\pi} V(a)^{\oplus r+1} \to 0$$

where π has components π_q for $q = 0, 1, \ldots, r$. Each $\pi_q\colon V(\lambda) \to V(a)$ is given by

$$v_i \mapsto \begin{cases} v_{i-qp} & \text{if } qp \leq i \leq qp + a \\ 0 & \text{otherwise.} \end{cases}$$

Hence the top of $V(\lambda)$ is $V(a)^{\oplus r+1}$ and $P = Q(a)^{\oplus r+1}$. Recall that $Q(a)$ has basis

$$\{v_0, v_1, \ldots, v_{2p-a-2}\} \cup \{w_{p-a-1}, w_{p-a}, w_{p-1}\}.$$

The map f_q is uniquely determined up to a nonzero scalar and is given by

$$\begin{aligned}
&v_i \mapsto -(a+1)^2\binom{p-i-1}{a+1}v_{(q-1)p+a+i+1} && \text{if } 0 \leq i \leq p-a-2,\\
&w_i \mapsto (-1)^{i+a}\binom{a}{i+a+1-p}^{-1}v_{(q-1)p+a+i+1} && \text{if } p-a-1 \leq i \leq p-1,\\
&v_i \mapsto 0 && \text{if } p-a-1 \leq i \leq p-1,\\
&v_i \mapsto (-1)^{a+1}(a+1)^2\binom{i+a+1-p}{a+1}v_{(q-1)p+a+i+1} && \text{if } p \leq i \leq 2p-a-2.
\end{aligned}$$

This gives $f = [f_0\ f_1\ \cdots\ f_r]$. To distinguish elements from different summands of $Q(a)^{\oplus r+1}$ let $\{v_{q,0}, v_{q,1}, \ldots, v_{q,2p-a-2}\} \cup \{w_{q,p-a-1}, w_{q,p-a}, w_{q,p-1}\}$ be the basis of the q^{th} summand of $Q(a)^{\oplus r+1}$. Then any element of the cover can be written in the form

$$x = \sum_{q=0}^{r}\left[\sum_{i=0}^{2p-a-2} c_{q,i}v_{q,i} + \sum_{i=p-a-1}^{p-1} d_{q,i}w_{q,i}\right].$$

for some $c_{q,i}, d_{q,i} \in k$. Applying f gives

$$\begin{aligned}
f(x) = \sum_{q=0}^{r}\Bigg[&-(a+1)^2\sum_{i=0}^{p-a-2}\binom{p-i-1}{a+1}c_{q,i}v_{(q-1)p+a+i+1}\\
&+(-1)^{a+1}(a+1)^2\sum_{i=p}^{2p-a-2}\binom{i+a+1-p}{a+1}c_{q,i}v_{(q-1)p+a+i+1}\\
&+\sum_{i=p-a-1}^{p-1}(-1)^{a+i}\binom{a}{i+a+1-p}^{-1}d_{q,i}v_{(q-1)p+a+i+1}\Bigg]
\end{aligned}$$

Observe that $0 \leq i \leq p-a-2$ and $p \leq i \leq 2p-a-2$ give $a+1 \leq a+i+1 \leq p-1$ and $p+a+1 \leq a+i+1 \leq 2p-1$ respectively, whereas $p-a-1 \leq i \leq p-1$ gives $p \leq a+i+1 \leq p+a$. Looking modulo p we see that the basis elements $v_{(q-1)p+a+i+1}$, for $0 \leq q \leq r$ and $p-a-1 \leq i \leq p-1$, are linearly independent. Thus $f(x) = 0$ immediately yields $d_{q,i} = 0$ for all q and i.

Now rearranging we have

$$\begin{aligned}
f(x) =& \sum_{q=0}^{r}\sum_{i=0}^{p-a-2}\Bigg[(-1)^{a+1}(a+1)^2\binom{i+a+1}{i}c_{q,i+p}v_{qp+a+1+i}\\
&-(a+1)^2\binom{p-i-1}{a+1}c_{q,i}v_{(q-1)p+a+1+i}\Bigg]\\
=&-(a+1)^2\sum_{i=0}^{p-a-2}\Bigg[\sum_{q=0}^{r-1}(-1)^a\binom{i+a+1}{i}c_{q,i+p}v_{qp+a+1+i}\\
&+\sum_{q=1}^{r}\binom{p-i-1}{a+1}c_{q,i}v_{(q-1)p+a+1+i}\Bigg]\\
=&\sum_{q=0}^{r-1}\sum_{i=0}^{p-a-2}\left[(-1)^a\binom{i+a+1}{i}c_{q,i+p}+\binom{p-i-1}{a+1}c_{q+1,i}\right]v_{qp+a+1+i}
\end{aligned}$$

so the kernel is defined by choosing $c_{q,i}$, for $0 \le q \le r-1$ and $0 \le i \le p-a-2$, such that

$$(-1)^a\binom{i+a+1}{i}c_{q,i+p}+\binom{p-i-1}{a+1}c_{q+1,i}=0.$$

Note that

$$\frac{\binom{p-i-1}{a+1}}{\binom{i+a+1}{i}}=\frac{\binom{p-1}{i+a+1}}{\binom{p-1}{i}}=\frac{(-1)^{i+a+1}}{(-1)^i}=(-1)^{a+1}$$

so the above equation simplifies to

$$c_{q,i+p}=c_{q+1,i}.$$

Thus for $0 \le i \le (r+2)p-a-2$ the vectors

$$v_i' = \begin{cases} v_{0,i} & \text{if } 0 \le i < p,\\ v_{q,b}+v_{q-1,p+b} & \text{if } 1 \le q \le r,\ 0 \le b \le p-a-2,\\ v_{q,b} & \text{if } 1 \le q \le r,\ p-a-1 \le b < p,\\ v_{r,b} & \text{if } q = r+1,\ 0 \le b \le p-a-2. \end{cases}$$

form a basis for the kernel, where $i = qp+b$ with $0 \le b < p$ the remainder of i modulo p. It is now trivial to check that the $\mathfrak{sl}_2$-action on this basis is identical to that of $V((r+2)p-a-2)$. □

4.3 Matrix Theorems

In this section we determine the kernel of four particular maps between free $k[s,t]$-modules. While these maps are used to represent sheaf homomorphisms in section 4.4 we do not approach this section geometrically. Instead we carry out these computations in the category of $k[s,t]$-modules.

The first map is given by the matrix $M_\varepsilon(\lambda) \in \mathbb{M}_{rp}(k[s,t])$ shown in Figure 4.4. For convenience we index the rows and columns of this matrix using the integers $a{+}1, a{+}2, \ldots, \lambda$. Then we can say more precisely that the $(i,j)^{\text{th}}$ entry of this matrix is

$$M_\varepsilon(\lambda)_{ij} = \begin{cases} is^2 & \text{if } \ i = j+1 \\ (2i-a)st & \text{if } \ i = j \\ (i-a)t^2 & \text{if } \ i = j-1 \\ -(a+1)\binom{r}{q}\varepsilon^{qp}s^2 & \text{if } \ (i,j) = (0, qp+a) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.3.1. *The kernel of $M_\varepsilon(\lambda)$ is a free $k[s,t]$-module (ungraded) of rank r whose basis elements are homogeneous of degree $p-a-2$.*

Proof. The strategy is as follows: First we will determine the kernel of $M_\varepsilon(\lambda)$ when considered as a map of $k[s,\frac{1}{s},t]$-modules. We do this by exhibiting a basis $H_1,\ldots,H_r$ via a direct calculation. Then by clearing the denominators from these basis elements we get a linearly independent set of vectors in the $k[s,t]$-kernel of $M_\varepsilon(\lambda)$. We conclude by arguing that these vectors in fact span, thus giving an explicit basis for the kernel of $M_\varepsilon(\lambda)$ considered as a map of $k[s,t]$-modules.

To begin observe that s is a unit in $k[s,\frac{1}{s},t]$, thus over this ring the kernel of $M_\varepsilon(\lambda)$ is

$$\begin{bmatrix}
(a+2)st & t^2 & & & -(a+1)\binom{r}{1}\varepsilon^{p}s^2 & & & -(a+1)\binom{r}{2}\varepsilon^{2p}s^2 & & & \cdots & & -(a+1)\binom{r}{r}\varepsilon^{rp}s^2 \\
(a+2)s^2 & (a+4)st & 2t^2 & & & & & & & & & & \\
 & (a+3)s^2 & \ddots & \ddots & & & & & & & & & \\
 & & \ddots & \ddots & -t^2 & & & & & & & & \\
 & & & as^2 & ast & 0 & & & & & & & \\
 & & & & (a+1)s^2 & \ddots & \ddots & & & & & & \\
 & & & & & \ddots & \ddots & -t^2 & & & & & \\
 & & & & & & as^2 & ast & 0 & & & & \\
 & & & & & & & (a+1)s^2 & \ddots & \ddots & & & \\
 & & & & & & & & \ddots & \ddots & -3t^2 & & \\
 & & & & & & & & & (a-2)s^2 & (a-4)st & -2t^2 & \\
 & & & & & & & & & & (a-1)s^2 & (a-2)st & -t^2 \\
 & & & & & & & & & & & as^2 & ast
\end{bmatrix}$$

Figure 4.4: $M_\varepsilon(\lambda)$

$$\begin{bmatrix}
(a+2)x & x^2 & & & -(a+1)\binom{r}{1}\varepsilon^{p} & & & -(a+1)\binom{r}{2}\varepsilon^{2p} & & & \cdots & & -(a+1)\binom{r}{r}\varepsilon^{rp} \\
a+2 & (a+4)x & 2x^2 & & & & & & & & & & \\
 & a+3 & \ddots & \ddots & & & & & & & & & \\
 & & \ddots & \ddots & -x^2 & & & & & & & & \\
 & & & a & ax & 0 & & & & & & & \\
 & & & & a+1 & \ddots & \ddots & & & & & & \\
 & & & & & \ddots & \ddots & -x^2 & & & & & \\
 & & & & & & a & ax & 0 & & & & \\
 & & & & & & & a+1 & \ddots & \ddots & & & \\
 & & & & & & & & \ddots & \ddots & -3x^2 & & \\
 & & & & & & & & & a-2 & (a-4)x & -2x^2 & \\
 & & & & & & & & & & a-1 & (a-2)x & -x^2 \\
 & & & & & & & & & & & a & ax
\end{bmatrix}$$

Figure 4.5: $\frac{1}{s^2}M_{\varepsilon}(\lambda)$

equal to the kernel of the matrix $\frac{1}{s^2}M_\varepsilon(\lambda)$ (shown in Figure 4.5) with $(i,j)^{\text{th}}$ entries

$$\frac{1}{s^2}M_\varepsilon(\lambda)_{ij} = \begin{cases} i & \text{if } \; i = j+1 \\ (2i-a)x & \text{if } \; i = j \\ (i-a)x^2 & \text{if } \; i = j-1 \\ -(a+1)\binom{r}{q}\varepsilon^{qp} & \text{if } \; (i,j) = (0, qp+a) \\ 0 & \text{otherwise.} \end{cases}$$

where $x = \frac{t}{s}$. Let

$$f = \begin{bmatrix} f_{a+1} \\ f_{a+2} \\ \vdots \\ f_\lambda \end{bmatrix}$$

be an arbitrary element of the kernel. Given $i = qp + b$ where $0 \le b < p$ and $a+1 \le i \le \lambda$ we claim that

$$f_i = (-1)^{\lambda-i}x^{\lambda-i}f_\lambda + (-1)^b\binom{p+a-b}{p-b-1}x^{p-b-1}h_{q+1} \tag{4.3.1.1}$$

for some choice of $h_1, \ldots, h_r \in k[s, \frac{1}{s}, t]$ and $h_{r+1} = 0$. Moreover, any such choice defines an element $f \in k[s, \frac{1}{s}, t]^{rp}$ such that

$$\frac{1}{s^2}M_\varepsilon(\lambda)f = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4.3.1.2}$$

holds.

The proof of this claim is by completely elementary methods, we simply induct up the rows of $\frac{1}{s^2}M_\varepsilon(\lambda)$ observing that the condition imposed by each row in Equation 4.3.1.2 either determines the next f_i or is automatically satisfied allowing us to introduce a free parameter (the h_i).

For the base case plugging $i = \lambda$ into Equation 4.3.1.1 gives $f_\lambda = f_\lambda$. The condition imposed by the last row in Equation 4.3.1.2 is $af_{\lambda-1} + axf_\lambda = 0$ so if $a \neq 0$ then $f_{\lambda-1} = -xf_\lambda$

and if $a = 0$ then this condition is automatically satisfied. The formula, when $a = 0$, gives $f_{rp-1} = -xf_\lambda + h_r$ so we take this as the definition of h_r.

Assume the formula holds for all f_j with $j > i \geq a + 1$ and that these f_j satisfy the conditions imposed by rows $i + 2, i + 3, \ldots, \lambda$ of $\frac{1}{s^2}M_\varepsilon(\lambda)$. Row $i + 1$ has nonzero entries $i+1$, $(2i - a + 2)x$, and $(i - a + 1)x^2$ in columns i, $i + 1$, and $i + 2$ respectively. First assume $i + 1 \neq 0$ in k or equivalently $b \neq p - 1$ where $i = qp + b$ and $0 \leq b < p$. Then the condition

$$(i + 1)f_i + (2i - a + 2)xf_{i+1} + (i - a + 1)x^2 f_{i+2} = 0$$

imposed by row $i + 1$ can be taken as the definition of f_i. Observe that

$$\begin{aligned}
&\frac{-1}{i+1}\big((-1)^{\lambda-i-1}(2i - a + 2) + (-1)^{\lambda-i-2}(i - a + 1)\big) \\
&\quad = \frac{(-1)^{\lambda-i}}{i+1}\left((2i - a + 2) - (i - a + 1)\right) \\
&\quad = \frac{(-1)^{\lambda-i}}{i+1}(i + 1) \\
&\quad = (-1)^{\lambda-i}
\end{aligned}$$

so $f_i = (-1)^{\lambda-i}x^{\lambda-i}f_\lambda +$ (terms involving h_j). For the h_j terms there are two cases. First assume $b < p - 2$. Then using $\frac{c}{e}\binom{c-1}{e-1} = \binom{c}{e}$ we see that

$$\begin{aligned}
&\frac{-1}{i+1}\left((-1)^{b+1}(2i - a + 2)\binom{p + a - b - 1}{p - b - 2} + (-1)^{b+2}(i - a + 1)\binom{p + a - b - 2}{p - b - 3}\right) \\
&\quad = \frac{(-1)^b}{b+1}\left((2b - a + 2)\binom{p + a - b - 1}{p - b - 2} + (p + a - b - 1)\binom{p + a - b - 2}{p - b - 3}\right) \\
&\quad = \frac{(-1)^b}{b+1}\left((2b - a + 2)\binom{p + a - b - 1}{p - b - 2} + (p - b - 2)\binom{p + a - b - 1}{p - b - 2}\right) \\
&\quad = \frac{(-1)^b}{b+1}(b - a)\binom{p + a - b - 1}{p - b - 2} \\
&\quad = \frac{(-1)^b}{b+1}(b + 1)\binom{p + a - b}{p - b - 1} \\
&\quad = (-1)^b\binom{p + a - b}{p - b - 1}.
\end{aligned}$$

Putting these together we get that

$$\begin{aligned} f_i &= \frac{-1}{i+1}((2i-a+2)xf_{i+1} + (i-a+1)x^2 f_{i+2}) \\ &= (-1)^{\lambda-i}x^{\lambda-i}f_\lambda + (-1)^b \binom{p+a-b}{p-b-1} x^{p-b-1}h_{q+1} \end{aligned}$$

as desired. Next assume $b = p-2$ so that $f_{i+2} = f_{(q+1)p}$. The coefficient of h_{q+2} in $f_{(q+1)p}$ involves the binomial $\binom{p+a}{p-1}$. As $0 \leq a < p-1$ there is a carry when the addition $(p-1)+(a+1) = p+a$ is done in base p, thus this coefficient is zero and the h_j terms of f_i are

$$\begin{aligned} \frac{(-1)^p}{i+1}(2i-a+2)\binom{a+1}{0} xh_{q+1} &= \frac{(-1)^{p-2}}{2}(a+2)xh_{q+1} \\ &= (-1)^b \binom{a+2}{1} xh_{q+1} \end{aligned}$$

as desired. Thus the induction continues when $i+1 \neq 0$ in k.

Now assume $i+1 = 0$ in k; equivalently, $b = p-1$. Then the condition imposed by row $i+1 = (q+1)p$ is

$$-axf_{(q+1)p} - ax^2 f_{(q+1)p+1} = 0.$$

If $a = 0$ then this is automatic. If $a > 0$ then there is a base p carry in the addition $(p-2)+(a+1) = p+a-1$, hence

$$\begin{aligned} &xf_{(q+1)p} + x^2 f_{(q+1)p+1} \\ &\quad = (-1)^{(r-q-1)p+a}x^{(r-q-1)p+a+1}f_\lambda + (-1)^{(r-q-1)p+a-1}x^{(r-q-1)p+a+1}g \\ &\quad\quad - \binom{p+a-1}{p-2} x^p h_{q+2} \\ &\quad = 0 \end{aligned}$$

because the f_λ terms cancel and the binomial is zero. So in either case the condition above is automatic. The formula for f_i when $i = qp + (p-1)$ is

$$f_i = (-1)^{(r-q-1)p+a+1}x^{(r-q-1)p+a+1}g + h_{q+1}$$

so we take this as the definition of h_{q+1} and the induction is complete.

Now f must have the given form for some choice of $h_1, \dots, h_r$ and any such choice gives an element f such that $\frac{1}{s^2}M_\varepsilon(\lambda)f$ is zero in all coordinates save the top $(a+1)$. All that is left is to impose the condition that $\frac{1}{s^2}M_\varepsilon(\lambda)f$ is zero in the $(a+1)^{\text{th}}$ coordinate as well. This condition is

$$(a+2)xf_{a+1} + x^2 f_{a+2} - (a+1)\sum_{q=1}^{r}\binom{r}{q}\varepsilon^{qp}f_{qp+a} = 0.$$

In $(a+2)xf_{a+1} + x^2 f_{a+2}$ the h_j terms are

$$\begin{aligned}
&(-1)^{a+1}\left((a+2)\binom{p-1}{p-a-2} - \binom{p-2}{p-a-3}\right)x^{p-a-1}h_1 \\
&= (-1)^{a+1}\left((a+2)\binom{p-1}{p-a-2} + (p-a-2)\binom{p-1}{p-a-2}\right)x^{p-a-1}h_1 \\
&= 0
\end{aligned}$$

and the coefficient of the h_j term in f_{qp+a} involves the binomial $\binom{p}{p-a-1}$ which is zero. Thus the top row imposes a condition only on f_λ, and this condition is

$$\begin{aligned}
0 &= (-1)^{rp-1}(a+2)x^{rp}f_\lambda + (-1)^{rp-2}x^{rp}f_\lambda \\
&\quad - (a+1)\sum_{q=1}^{r}(-1)^{(r-q)p}\binom{r}{q}\varepsilon^{qp}x^{(r-q)p}f_\lambda \\
&= (-1)^{rp-1}(a+1)\left[\sum_{q=0}^{r}(-1)^{qp}\binom{r}{q}\varepsilon^{qp}x^{(r-q)p}\right]f_\lambda.
\end{aligned}$$

Note that $x = \frac{t}{s}$ is algebraically independent over k in $k[s, \frac{1}{s}, t]$ and by hypothesis $a+1 \neq 0$ in k. The localization of an integral domain is again an integral domain therefore if f is in the kernel then we must have $f_\lambda = 0$.

As the $h_1, \dots, h_r$ can be chosen arbitrarily this completes the determination of the kernel of $M_\varepsilon(\lambda)$, considered as a map of $k[s, \frac{1}{s}, t]$-modules. It is free of rank r and the basis elements are given by taking the coefficients of these h_q in Equation 4.3.1.2. Let H_q be the basis element that corresponds to h_q; it is shown in Figure 4.6. I claim that $s^{p-a-2}H_q$, for $1 \leq q \leq r$, is a basis for the kernel of $M_\varepsilon(\lambda)$, considered as a map of $k[s,t]$-modules.

$$\begin{array}{cc}\begin{matrix} a+1 \\ \vdots \\ qp-1 \\ qp \\ qp+1 \\ \vdots \\ qp+a \\ qp+a+1 \\ \vdots \\ (q+1)p-2 \\ (q+1)p-1 \\ (q+1)p \\ \vdots \\ \lambda \end{matrix} & \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \binom{a+p}{p-1}x^{p-1} \\ -\binom{a+p-1}{p-2}x^{p-2} \\ \vdots \\ (-1)^{p-a-1}\binom{p}{p-a-1}x^{p-a-1} \\ (-1)^{p-a-2}\binom{p-1}{p-a-2}x^{p-a-2} \\ \vdots \\ -\binom{a+2}{1}x \\ \binom{a+1}{0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}\end{array} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ (-1)^{p-a-2}\binom{p-1}{p-a-2}x^{p-a-2} \\ \vdots \\ -(a+2)x \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \xrightarrow{s^{p-a-2}} \begin{array}{cc}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{p-a-2}\binom{p-1}{p-a-2}t^{p-a-2} \\ \vdots \\ -(a+2)s^{p-a-3}t \\ s^{p-a-2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{matrix} a+1 \\ \vdots \\ qp+a \\ qp+a+1 \\ \vdots \\ (q+1)p-2 \\ (q+1)p-1 \\ (q+1)p \\ \vdots \\ \lambda \end{matrix}\end{array}$$

Figure 4.6: $H_q \to s^{p-a-2}H_q$

First note that H_q is supported in coordinates $(q+1)p-1$ through $qp+a+1$. These ranges are disjoint for different H_q therefore the $s^{p-a-2}H_q$ are clearly linearly independent. Let $f \in k[s,t]^{rp}$ be an element of the kernel of $M_\varepsilon(\lambda)$. Then as an element of $k[s,\frac{1}{s},t]$ we have that f is in the kernel of $\frac{1}{s^2}M_\varepsilon(\lambda)$ and can write

$$f = \sum_{q=1}^{r} c_q H_q.$$

where $c_q \in k[s,\frac{1}{s},t]$. The $((q+1)p-1)^{\text{th}}$ coordinate of f is c_q hence $c_q \in k[s,t]$. Also the $(qp+a+1)^{\text{th}}$ coordinate of f is

$$(-1)^{p-a-2}\binom{p-1}{p-a-2}c_q x^{p-a-2}$$

and the binomial coefficient in that expression is nonzero in k so $c_q x^{p-a-2} \in k[s,t]$. In particular, s^{p-a-2} must divide c_q so write $c_q = s^{p-a-2}c'_q$ for some $c'_q \in k[s,t]$. We now have

$$f = \sum_{q=1}^{r} c'_q s^{p-a-2} H_q$$

so the $s^{p-a-2}H_q$ span and are therefore a basis. Each H_q is homogeneous of degree 0 so each $s^{p-a-2}H_q$ is homogeneous of degree $p-a-2$. □

The second map we wish to consider is given by the matrix $B(\lambda) \in \mathbb{M}_{\lambda+1}(k[s,t])$ shown in Figure 4.7. Index the rows and columns of this matrix using the integers $0,1,\ldots,\lambda$. Then the entries of $B(\lambda)$ are

$$B(\lambda)_{ij} = \begin{cases} -it^2 & \text{if } i = j+1 \\ (\lambda-2i)st & \text{if } i = j \\ (\lambda-i)s^2 & \text{if } i = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.3.2. *The kernel of $B(\lambda)$ is a free $k[s,t]$-module of rank $r+1$. There is one basis element that is homogeneous of degree λ and the remaining are homogeneous of degree $p-a-2$.*

$$\begin{bmatrix}
\lambda st & \lambda s^2 & & & & & & & & & & \\
-t^2 & (\lambda-2)st & (\lambda-1)s^2 & & & & & & & & & \\
 & -2t^2 & (\lambda-4)st & (\lambda-2)s^2 & & & & & & & & \\
 & & -3t^2 & \ddots & \ddots & & & & & & & \\
 & & & \ddots & \ddots & (\lambda+2)s^2 & & & & & & \\
 & & & & t^2 & (\lambda+2)st & (\lambda+1)s^2 & & & & & \\
 & & & & & 0 & \lambda st & \lambda s^2 & & & & \\
 & & & & & & -t^2 & \ddots & \ddots & & & \\
 & & & & & & & \ddots & \ddots & 3s^2 & & \\
 & & & & & & & & -(\lambda-2)t^2 & -(\lambda-4)st & 2s^2 & \\
 & & & & & & & & & -(\lambda-1)t^2 & -(\lambda-2)st & s^2 \\
 & & & & & & & & & & -\lambda t^2 & -\lambda st
\end{bmatrix}$$

Figure 4.7: $B(\lambda)$

Proof. The proof is very similar to the proof of proposition 4.3.1. We start by finding the kernel of the matrix $\frac{1}{s^2}B(\lambda)$ shown in Figure 4.8 whose entries are given by

$$\frac{1}{s^2}B(\lambda)_{ij} = \begin{cases} -ix^2 & \text{if } \ i = j+1 \\ (\lambda - 2i)x & \text{if } \ i = j \\ \lambda - i & \text{if } \ i = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

with $x = \frac{t}{s}$. Let

$$f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_\lambda \end{bmatrix}$$

be an arbitrary element of the kernel. We induct down the rows of the matrix to show that if $i = qp + b$, where $0 \leq b < p$ then

$$f_{\lambda-i} = (-1)^{\lambda-i}x^{\lambda-i}g + (-1)^b \binom{p+a-b}{p-b-1} x^{p-b-1}h_q$$

where $h_r = 0$.

For the base case $i = \lambda$ in the formula gives $f_0 = g$ so we take this as the definition of g. The condition imposed by the first row is $axg + af_1 = 0$ so if $a \neq 0$ then $f_1 = -xg$. The formula gives $f_1 = -xg + (-1)^a - 1\binom{p+1}{p-a}x^{p-a}h_r = -xg$ so these agree. If $a = 0$ then the condition is automatically satisfied and the formula gives $f_1 = -xg + h_{r-1}$ so we take this as the definition of h_{r-1}.

For the inductive step assume the formula holds for $f_0, f_1, \ldots, f_{i-1}$ and that these f_j satisfy the conditions imposed by rows $0, \ldots, \lambda - i - 2$. The three nonzero entries in row $\lambda - i - 1$ are $(b - a + 1)x^2$, $(2b - a + 2)x$, and $b + 1$ in columns $\lambda - i - 2$, $\lambda - i - 1$, and $\lambda - i$ respectively, thus the condition imposed is

$$(b - a + 1)x^2 f_{\lambda-i-2} + (2b - a + 2)x f_{\lambda-i-1} + (b + 1)f_{\lambda-i} = 0.$$

$$\begin{bmatrix}
\lambda x & \lambda & & & & & & & & & & \\
-x^2 & (\lambda-2)x & \lambda-1 & & & & & & & & & \\
 & -2x^2 & (\lambda-4)x & \lambda-2 & & & & & & & & \\
 & & -3x^2 & \ddots & \ddots & & & & & & & \\
 & & & \ddots & \ddots & \lambda+2 & & & & & & \\
 & & & & x^2 & (\lambda+2)x & \lambda+1 & & & & & \\
 & & & & & 0 & \lambda x & \lambda & & & & \\
 & & & & & & -x^2 & \ddots & \ddots & & & \\
 & & & & & & & \ddots & \ddots & 3 & & \\
 & & & & & & & & -(\lambda-2)x^2 & -(\lambda-4)x & 2 & \\
 & & & & & & & & & -(\lambda-1)x^2 & -(\lambda-2)x & 1 \\
 & & & & & & & & & & -\lambda x^2 & -\lambda x
\end{bmatrix}$$

Figure 4.8: $\frac{1}{s^2}B(\lambda)$

If $b < p-2$ then we can solve this for $f_{\lambda-i}$ and we find that it agrees with the formula above (for the h_j terms the computation is identical to the one shown in proposition 4.3.1). If $b = p-2$ we get

$$\begin{aligned} f_{\lambda-i} &= -(a+1)x^2 f_{\lambda-i-2} - (a+2)x f_{\lambda-i-1} \\ &= (-1)^{\lambda-i-1}(a+1)x^{\lambda-i}g + (-1)^{\lambda-i}(a+2)x^{\lambda-i}g - (a+2)h_q \\ &= (-1)^{\lambda-i}x^{\lambda-i}g - \binom{a+2}{1} x h_q \end{aligned}$$

as desired. Finally if $b = p-1$ then $b+1 = 0$ in k so the condition is

$$-ax^2 f_{\lambda-i-2} - axf_{\lambda-i-1} = 0$$

and this is automatically satisfied (the formulas are the same as in proposition 4.3.1 again). Thus no condition is imposed on $f_{\lambda-i}$ so we take the formula

$$f_{\lambda-i} = (-1)^{\lambda-i}x^{\lambda-i}g + h_q$$

as the definition of h_q. This completes the induction.

Note that the final row to be $\lambda - i - 1$ we must choose $i = -1$ and therefore are in the case where $b+1 = 0$ and the condition is automatically satisfied. The rest of the proof goes as in proposition 4.3.1, except that there is no final condition forcing $g = 0$. If we let G and $H_0, \ldots, H_{r-1}$ be the basis vectors corresponding to g and $h_0, \ldots, h_{r-1}$ then the H_q are linearly independent as before. The first (0^{th}) coordinate of G is 1 while the first coordinate of each H_q is 0 therefore G can be added and this gives a basis for the kernel. The largest power of x in G is λ in the last coordinate and the largest power of x in H_q is $p-a-2$ in the $(\lambda - qp - a + 1)^{\text{th}}$ coordinate. These basis vectors lift to basis vectors of the kernel as a $k[s,t]$-module and are in degrees λ and $p-a-2$ as desired. □

Before we move on to the third map, let us first prove the following lemma which will be needed in proposition 4.4.2.

Lemma 4.3.3. *Assume $0 \le \lambda < p$. Then the $(i,j)^{th}$ entry of $B(\lambda)^{\lambda}$ is contained in the one dimensional space $ks^{\lambda+j-i}t^{\lambda-j+i}$.*

Proof. Let b_{ij} be the $(i,j)^{\text{th}}$ entry of $B(\lambda)$. By definition the $(i,j)^{\text{th}}$ entry of $B(\lambda)^\lambda$ is given by

$$(B(\lambda)^\lambda)_{ij} = \sum_{n_1,n_2,\ldots,n_{\lambda-1}} b_{in_1}b_{n_1n_2}\cdots b_{n_{\lambda-1}j}.$$

From the definition of $B(\lambda)$ we have

$$\begin{aligned} b_{ij} &\in ks^2 && \text{if } j-i=1, \\ b_{ij} &\in kst && \text{if } j-i=0, \\ b_{ij} &\in kt^2 && \text{if } j-i=-1, \\ b_{ij} &= 0 && \text{otherwise.} \end{aligned}$$

so any given term $b_{in_1}b_{n_1n_2}\cdots b_{n_{\lambda-1}j}$ in the summation can be nonzero only if the $(\lambda+1)$-tuple $(n_0,n_1,\ldots,n_\lambda)$ is a *walk* from $n_0=i$ to $n_\lambda=j$, i.e. each successive term of the tuple must differ from the last by at most 1. For such a walk we now show by induction that $b_{n_0n_1}b_{n_1n_2}\cdots b_{n_{m-1}n_m} \in ks^{m+n_m-n_0}t^{m-n_m+n_0}$. For the base case $m=1$ we have the three cases above for $b_{n_0n_1}$ and one easily checks that the formula gives kt^2, kst, or ks^2 as needed.

Now assume the statement holds for $m-1$ so that

$$b_{n_0n_1}\cdots b_{n_{m-2}n_{m-1}}b_{n_{m-1}n_m} \in ks^{m-1+n_{m-1}-n_0}t^{m-1-n_{m-1}+n_0}b_{n_{m-1}n_m}.$$

There are three cases. First if $n_m = n_{m-1}+1$ then $b_{n_{m-1}n_m} \in ks^2$ and the set becomes

$$ks^{m-1+n_{m-1}-n_0}t^{m-1-n_{m-1}+n_0}\cdot s^2 = ks^{m-n_m+n_0}t^{m+n_m-n_0}$$

as desired. Next if $n_m = n_{m-1}$ then $b_{n_{m-1}n_m} \in kst$ and the set becomes

$$ks^{m-1+n_{m-1}-n_0}t^{m-1-n_{m-1}+n_0}\cdot st = ks^{m+n_m-n_0}t^{m-n_m+n_0}$$

as desired. Finally if $n_m = n_{m-1}-1$ then $b_{n_{m-1}n_m} \in kt^2$ and the set becomes

$$ks^{m-1+n_{m-1}-n_0}t^{m-1-n_{m-1}+n_0}\cdot t^2 = ks^{m+n_m-n_0}t^{m-n_m+n_0}$$

as desired. Thus the induction is complete and for $m=\lambda$ this gives

$$b_{n_0n_1}b_{n_1n_2}\cdots b_{n_{\lambda-1}n_\lambda} \in ks^{\lambda+n_\lambda-n_0}t^{\lambda-n_\lambda+n_0} = ks^{\lambda+j-i}t^{\lambda-j+i}$$

and completes the proof. □

Moving on, the third map we wish to consider is $B'(\lambda) \in \mathbb{M}_{rp}(k[s,t])$ defined to be the rp^{th} trailing principal minor of $B(\lambda)$, i.e., the minor of $B(\lambda)$ consisting of rows and columns $a+1, a+2, \ldots, \lambda$.

Proposition 4.3.4. *The kernel of* $B'(\lambda)$ *is a free* $k[s,t]$*-module (ungraded) of rank* r *whose basis elements are homogeneous of degree* $p-a-2$.

Proof. The induction from the proof of proposition 4.3.2 applies giving

$$f_{\lambda-i} = (-1)^{\lambda-i}x^{\lambda-i}g + (-1)^b\binom{p+a-b}{p-b-1}x^{p-b-1}h_q$$

for $0 \leq i < rp$. All that is left is the condition

$$-(a+2)xf_{a+1} - f_{a+2} = 0$$

from the first row of $\frac{1}{s^2}B'(\lambda)$. Substituting in the formulas we get

$$(-1)^{a+1}(a+1)x^{a+2}g = 0$$

which forces $g = 0$. Thus as a basis for the kernel we get $H_0, \ldots, H_{r-1}$. □

Before we move on to the final map, let us first prove the following lemma which was needed in section 4.2

Lemma 4.3.5. *Let* $s, t \in k$ *so that* $B'(\lambda) \in \mathbb{M}_{rp}(k)$.

$$\text{JType}(B'(\lambda)) = \begin{cases} [1]^{rp} & \textit{if } s = t = 0, \\ [p]^{r-1}[p-a-1][a+1] & \textit{if } s = 0, t \neq 0, \\ [p]^r & \textit{if } s \neq 0. \end{cases}$$

Proof. If $(s,t) = (0,0)$ then $B'(\lambda)$ is the zero matrix, hence the Jordan type is $[1]^{rp}$. If $s = 0$ and $t \neq 0$ then $B'(\lambda)$ only has non-zero entries on the sub-diagonal. Normalizing these entries to 1 gives the Jordan form of the matrix from which we read the Jordan type. If we use the row numbering from $B(\lambda)$ (i.e. the first row is $a+1$, the second $a+2$, etc.) then the zeros on the sub-diagonal occur at rows $p, 2p, \ldots, rp$. Thus the first block is size $p-a-1$, followed by $r-1$ blocks of size p, and the last block is size $a+1$. Hence the Jordan type is $[p]^{r-1}[p-a-1][a+1]$.

Now assume $s \neq 0$. There are exactly $r(p-1)$ non-zero entries on the super-diagonal and no non-zero entries above the super-diagonal therefore $\operatorname{rank} B'(\lambda) \geq r(p-1)$. But this is the maximal rank that a nilpotent matrix can achieve and such a matrix has Jordan type consisting only of blocks of size p. Hence the Jordan type is $[p]^r$. $\square$

The final map we wish to consider is given by the matrix $C(\lambda) \in \mathbb{M}_{\lambda+1}(k[s,t])$ shown in Figure 4.7. Index the rows and columns of this matrix using the integers $0, 1, \ldots, \lambda$. Then the entries of $C(\lambda)$ are

$$C(\lambda)_{ij} = \begin{cases} (i-\lambda-1)t^2 & \text{if } \ i = j+1 \\ (\lambda - 2i)st & \text{if } \ i = j \\ (i+1)s^2 & \text{if } \ i = j-1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.3.6. *The kernel of $C(\lambda)$ is a free $k[s,t]$-module (ungraded) of rank $r+1$ whose basis elements are homogeneous of degree a.*

Proof. Let

$$f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_\lambda \end{bmatrix}$$

be an arbitrary element of the kernel of $\frac{1}{s^2}C(\lambda)$ shown in Figure 4.10 whose entries are given

$$\begin{bmatrix}
\lambda st & s^2 & & & & & & & & & & \\
-\lambda t^2 & (\lambda-2)st & 2s^2 & & & & & & & & & \\
 & -(\lambda-1)t^2 & (\lambda-4)st & 3s^2 & & & & & & & & \\
 & & -(\lambda-2)t^2 & \ddots & \ddots & & & & & & & \\
 & & & \ddots & \ddots & -s^2 & & & & & & \\
 & & & & -(\lambda+2)t^2 & (\lambda+2)st & 0 & & & & & \\
 & & & & & -(\lambda+1)t^2 & \lambda st & s^2 & & & & \\
 & & & & & & -\lambda t^2 & \ddots & \ddots & & & \\
 & & & & & & & \ddots & \ddots & (\lambda-2)s^2 & & \\
 & & & & & & & & -3t^2 & -(\lambda-4)st & (\lambda-1)s^2 & \\
 & & & & & & & & & -2t^2 & -(\lambda-2)st & \lambda s^2 \\
 & & & & & & & & & & -t^2 & -\lambda st
\end{bmatrix}$$

Figure 4.9: $C(\lambda)$

$$\begin{bmatrix}
\lambda x & 1 & & & & & & & & & & \\
-\lambda x^2 & (\lambda-2)x & 2 & & & & & & & & & \\
 & -(\lambda-1)x^2 & (\lambda-4)x & 3 & & & & & & & & \\
 & & -(\lambda-2)x^2 & \ddots & \ddots & & & & & & & \\
 & & & \ddots & \ddots & -1 & & & & & & \\
 & & & & -(\lambda+2)x^2 & (\lambda+2)x & 0 & & & & & \\
 & & & & & -(\lambda+1)x^2 & \lambda x & 1 & & & & \\
 & & & & & & -\lambda x^2 & \ddots & \ddots & & & \\
 & & & & & & & \ddots & \ddots & \lambda-2 & & \\
 & & & & & & & & -3x^2 & -(\lambda-4)x & \lambda-1 & \\
 & & & & & & & & & -2x^2 & -(\lambda-2)x & \lambda \\
 & & & & & & & & & & -x^2 & -\lambda x
\end{bmatrix}$$

Figure 4.10: $\frac{1}{s^2}C(\lambda)$

by

$$\frac{1}{s^2}C(\lambda)_{ij} = \begin{cases} (i-\lambda-1)x^2 & \text{if } \ i=j+1 \\ (\lambda-2i)x & \text{if } \ i=j \\ i+1 & \text{if } \ i=j-1 \\ 0 & \text{otherwise.} \end{cases}$$

with $x = \frac{t}{s}$. We show by induction that if $i = qp+b$ and $0 \leq b < p$ then

$$f_i = (-1)^b \binom{\lambda}{b} x^b h_q.$$

For the base case the formula gives $f_0 = h_0$ so we take this as the definition of h_0. The condition imposed by row 1 is $-\lambda x f_0 + f_1 = 0$ which gives $f_1 = -xh_0$ as desired.

For the inductive step assume the formula holds for indices less then i and the condition imposed by all rows of index less than $i-1$ is satisfied. Row $i-1$ has nonzero entries $(i-\lambda-2)x^2$, $(\lambda-2i+2)x$, and i in columns $i-2$, $i-1$, and i respectively so the condition is

$$(i-\lambda-2)x^2 f_{i-2} + (\lambda-2i+2)xf_{i-1} + if_i = 0$$

First assume $i \neq 0, 1$ in k. Then we have

$$\begin{aligned} f_i &= \frac{-1}{i}\left((-1)^{b-2}(i-\lambda-2)\binom{\lambda}{b-2}x^b h_q + (-1)^{b-1}(\lambda-2i+2)\binom{\lambda}{b-1}x^b h_q\right) \\ &= \frac{(-1)^b}{b}\left((\lambda-b+2)\binom{\lambda}{b-2} + (\lambda-2b+2)\binom{\lambda}{b-1}\right)x^b h_q \\ &= \frac{(-1)^b}{b}((b-1)+(\lambda-2b+2))\binom{\lambda}{b-1}x^b h_q \\ &= (-1)^b\frac{\lambda-b+1}{b}\binom{\lambda}{b-1}x^b h_q \\ &= (-1)^b\binom{\lambda}{b}x^b h_q \end{aligned}$$

as desired. Next assume $i = 0$ in k. Then

$$\begin{aligned} &(i-\lambda-2)x^2 f_{i-2} + (\lambda-2i+2)xf_{i-1} \\ &\qquad = (\lambda+2)\binom{\lambda}{p-2}x^p h_{q-1} + (\lambda+2)\binom{\lambda}{p-1}x^p h_{q-1}. \end{aligned}$$

But $a+1 \neq 0$ so $\binom{\lambda}{p-1} = 0$ and if $a+1 \neq 0$ then $\binom{\lambda}{p-2} = 0$ otherwise $\lambda + 2 = 0$. In any case the above expression is 0 so the condition imposed by row $i-1$ is automatically satisfied. The formula gives $f_i = h_q$ so we take this as the definition of h_q. Finally assume $i = 1$ in k. Then we have

$$\begin{aligned} f_i &= (\lambda+1)\binom{\lambda}{p-1}x^{p+1}h_{q-1} - \lambda x h_q \\ &= -\binom{\lambda}{1}xh_q \end{aligned}$$

as desired. This completes the induction. We know that the given formulas for f_i satisfy the conditions imposed by all rows save the last, whose condition is

$$-x^2 f_{\lambda-1} - \lambda x f_\lambda = 0.$$

We have

$$\lambda x f_\lambda = (-1)^a \lambda \binom{\lambda}{a} x^{a+1} h_r = (-1)^a \lambda x^{a+1} h_r.$$

If $a = 0$ then

$$x^2 f_{\lambda-1} = (-1)^{p-1}\binom{\lambda}{p-1}x^{p+1}h_{r-1} = 0$$

and $\lambda = 0$ so this conditions is satisfied. If $a \neq 0$ then

$$x^2 f_{\lambda-1} = (-1)^{a-1}\binom{\lambda}{a-1}x^{a+1}h_r = (-1)^{a-1}ax^{a-1}h_r$$

so

$$\begin{aligned} & x^2 f_{\lambda-1} + \lambda x f_\lambda \\ &\qquad = (-1)^{a-1}ax^{a-1}h_r + (-1)^a \lambda x^{a+1} h_r \\ &\qquad = (-1)^a(\lambda - a)x^{a+1}h_r \\ &\qquad = 0 \end{aligned}$$

and the condition is again satisfied so we have found a basis. If H_q is the basis vector associated to h_q then the smallest and largest powers of x in H_q are 0 in coefficient qp and a in coefficient $qp + a$. By the usual arguments the H_q lift to a basis for the kernel of $C(\lambda)$ that is homogeneous of degree a. □

The final map we want to consider is parametrized by $0 \leq a < p-1$. Given such an a, let $D(a) \in \mathbb{M}_{2p}(k[s,t])$ be the block matrix

$$D(a) = \begin{bmatrix} B(2p-a-2) & D'(a) \\ 0 & B(a)^\dagger \end{bmatrix}$$

where $D'(a)$ and $B(a)^\dagger$ are as follows. The matrix $D'(a)$ is a $(2p-a-1) \times (a+1)$ matrix whose $(i,j)^{\text{th}}$ entry is

$$D'(a)_{ij} = \begin{cases} \frac{1}{i+1}s^2 & \text{if } i-j = p-a-2 \\ \frac{1}{a+1}t^2 & \text{if } (i,j) = (p,a) \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $B(a)^\dagger$ is produced from $B(a)$ by taking the transpose and then swapping the variables s and t.

$$B(a)^\dagger = \begin{bmatrix} ast & -s^2 & & & \\ at^2 & (a-2)st & -2s^2 & & \\ & (a-1)t^2 & \ddots & \ddots & \\ & & \ddots & -(a-2)st & -as^2 \\ & & & t^2 & -ast \end{bmatrix}$$

Proposition 4.3.7. *The inclusion of* $k[s,t]^{2p-a-1}$ *into* $k[s,t]^{2p}$ *as the top* $2p-a-1$ *coordinates of a column vector induces an isomorphism* $\ker B(2p-a-2) \simeq \ker D(a)$*.*

Proof. As $D(a)$ is block upper-triangular with $B(2p-a-2)$ the top most block on the diagonal it suffices to show that every element of $\ker D(a)$ is of the form $\left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right]$ with respect to this block decomposition. That is, we must show that if

$$f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{2p-1} \end{bmatrix}$$

is an element of $\ker D(a)$ then $f_i = 0$ for all $2p - a - 1 \leq i \leq 2p - 1$. Obviously it suffices to prove this for $\frac{1}{t^2}D(a)$ over $k[s, t, \frac{1}{t}]$ so let $x = \frac{s}{t}$.

We start by proving that $f_{2p-1} = 0$. There are two cases, first assume that $a + 2 = 0$ in k. Then row p of $\frac{1}{t^2}D(a)$ has only one nonzero entry, a $\frac{1}{a+1}$ in column $2p - 1$. Thus $f \in \ker \frac{1}{t^2}D(a)$ gives $\frac{1}{a+1}f_{2p-1} = 0$ in $k[s, t,]$, hence $f_{2p-1} = 0$. Next assume that $a + 2 < p$. Then the induction from proposition 4.3.2 applies to rows $p + 1, \ldots, 20 - a - 2$ and gives

$$f_i = (-1)^{a+i}x^{2p-a-2-i}f_{2p-a-2}$$

for $p \leq i \leq 2p - a - 2$. The condition imposed by row p is

$$-(a + 2)xf_p - (a + 2)x^2 f_{p+1} + \frac{1}{a + 1}f_{2p-1} = 0.$$

But note that the induction gave us $f_p = -xf_{p+1}$ so this simplifies to $\frac{1}{a+1}f_{2p-1} = 0$ and again we have $f_{2p-1} = 0$.

Now the condition imposed by the last row of $D(a)$ gives $f_{2p-2} = axf_{2p-1} = 0$. By induction the i^{th} row gives $-if_{i-1} = (2i + a + 2)xf_i + (i + a + 2)x^2 f_{i+1} = 0$, hence $f_{i-1} = 0$, for $p - a \leq i \leq 2p - 2$ and this completes the proof. □

4.4 *Explicit computation of* $\ker \Theta_M$ *and* $\mathscr{F}_i(V(\lambda))$

In this final section we carry out the explicit computations of the sheaves $\ker \Theta_M$, for every indecomposable $\mathfrak{sl}_2$-module M, and $\mathscr{F}_i(V(\lambda))$ for $i \neq p$. Friedlander and Pevtsova [23, Proposition 5.9] have calculated the sheaves $\ker \Theta_{V(\lambda)}$ for Weyl modules $V(\lambda)$ such that $0 \leq \lambda \leq 2p-2$. Using the explicit descriptions of these modules found in section 4.2 we can do the calculation for the remaining indecomposable modules in the category.

Proposition 4.4.1. *Let $\lambda = rp + a$ with $0 \leq a < p$ the remainder of λ modulo p. The kernel* *bundles associated to the indecomposable $\mathfrak{sl}_2$-modules from theorem 4.2.1 are*

$$\begin{aligned}
\ker \Theta_{\Phi_\xi(\lambda)} &\simeq \mathcal{O}_{\mathbb{P}^1}(a+2-p)^{\oplus r} \\
\ker \Theta_{V(\lambda)} &\simeq \mathcal{O}_{\mathbb{P}^1}(-\lambda) \oplus \mathcal{O}_{\mathbb{P}^1}(a+2-p)^{\oplus r} \\
\ker \Theta_{V(\lambda)^*} &\simeq \mathcal{O}_{\mathbb{P}^1}(-a)^{\oplus r+1} \\
\ker \Theta_{Q(a)} &\simeq \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(a+2-2p)
\end{aligned}$$

Proof. Assume first that $\xi = [1 : \varepsilon]$. Then using the basis from section 4.2 we get that the matrix defining $\Theta_{\Phi_\xi(\lambda)}$ has entries

$$(\Theta_{\Phi_\xi(\lambda)})_{ij} = \begin{cases} ix & \text{if } \ i = j+1 \\ (2i-a)z & \text{if } \ i = j \\ (a-i)y & \text{if } \ i = j-1 \\ -(a+1)\binom{r}{q}\varepsilon^{qp}x & \text{if } \ (i,j) = (0, qp+a) \\ 0 & \text{otherwise.} \end{cases}$$

Pulling back along the map $\iota \colon \mathbb{P}^1 \to \mathbb{P}(\mathfrak{sl}_2)$ from example 4.1.2 corresponds with extending scalars through the homomorphism

$$\begin{aligned}
\frac{k[x,y,z]}{(xy+z^2)^n} &\to k[s,t] \\
(x,y,z) &\mapsto (s^2, -t^2, st).
\end{aligned}$$

Thus the matrix of $\Theta_{\Phi_\xi(\lambda)}$ becomes the matrix $M_\varepsilon(\lambda)$ from proposition 4.3.1 and we see that the kernel is free. A basis element, homogeneous of degree m, spans a summand of the kernel isomorphic to $k[s,t][-m]$. By definition the $\mathcal{O}_{\mathbb{P}^1}$-module corresponding to $k[s,t][-m]$ is $\mathcal{O}_{\mathbb{P}^1}(-m)$ so the description of the kernel translates directly to the description of the sheaf above.

The remaining cases are all identical. The modules $V(\lambda)$, $\Phi_{[0:1]}(\lambda)$, $V(\lambda)^*$, and $Q(a)$ give the matrices $B(\lambda)$, $B'(\lambda)$, $C(\lambda)$, and $D(a)$ whose kernels are calculated in Propositions proposition 4.3.2, proposition 4.3.4, proposition 4.3.6, and proposition 4.3.7 respectively. $\square$

Next we compute $\mathscr{F}_i(V(\lambda))$ for any $i \neq p$ and any indecomposable $V(\lambda)$. The proof is by induction on r in the expression $\lambda = rp + a$. For the base case we start with $V(\lambda)$ a simple module, i.e., $r = 0$. Note that for the base case we do indeed determine $\mathscr{F}_p(V(\lambda))$, it is during the inductive step that we lose $i = p$.

Proposition 4.4.2. *If* $0 \leq \lambda < p$ *then*

$$\mathscr{F}_i(V(\lambda)) = \begin{cases} \ker \Theta_{V(\lambda)} & \textit{if } i = \lambda + 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First note that $V(\lambda)$ has constant Jordan type $[\lambda + 1]$ so proposition 3.1.9 tells us that when $i \neq \lambda + 1$ the sheaf $\mathscr{F}_i(V(\lambda))$ is locally free of rank 0, hence is the zero sheaf.

For $i = \lambda + 1$ recall from the previous proof that the map $\Theta_{V(\lambda)}$ of sheaves is given in the category of $k[s,t]$-modules by the matrix $B(\lambda)$ in Figure 4.7. The $(\lambda + 1)^{\text{th}}$ power of a matrix of Jordan type $[\lambda + 1]$ is zero so the entries of $B(\lambda)^{\lambda+1}$ are polynomials representing the zero function. We assume that k is algebraically closed so this means $B(\lambda)^{\lambda+1} = 0$ and therefore $\Theta_{V(\lambda)}^{\lambda+1} = 0$. In particular

$$\operatorname{im} \Theta_{V(\lambda)}^{\lambda} \subseteq \ker \Theta_{V(\lambda)}$$

so the definition of $\mathscr{F}_{\lambda+1}(V(\lambda))$ gives

$$\mathscr{F}_{\lambda+1}(V(\lambda)) = \frac{\ker \Theta_{V(\lambda)} \cap \operatorname{im} \Theta_{V(\lambda)}^{\lambda}}{\ker \Theta_{V(\lambda)} \cap \operatorname{im} \Theta_{V(\lambda)}^{\lambda+1}} = \operatorname{im} \Theta_{V(\lambda)}^{\lambda}. \tag{4.4.2.1}$$

We have a short exact sequence of $k[s,t]$-modules

$$0 \to \operatorname{im} B(\lambda)^{\lambda} \to \ker B(\lambda) \to \frac{\ker B(\lambda)}{\operatorname{im} B(\lambda)^{\lambda}} \to 0.$$

If we show that the quotient $\ker B(\lambda)/\operatorname{im} B(\lambda)^{\lambda}$ is finite dimensional then by Serre's theorem and Equation 4.4.2.1 this gives a short exact sequence of sheaves

$$0 \to \mathscr{F}_i(V(\lambda)) \to \ker \Theta_{V(\lambda)} \to 0 \to 0$$

and completes the proof.

To show that $\ker B(\lambda)/\operatorname{im} B(\lambda)^\lambda$ is finite dimensional note that from $B(\lambda)^{\lambda+1} = 0$ we get that the columns of $B(\lambda)^\lambda$ are contained in the kernel of $B(\lambda)$ which, in proposition 4.3.2 we determined is a free $k[s,t]$-module with basis element

$$G = \begin{bmatrix} s^\lambda \\ -s^{\lambda-1}t \\ \vdots \\ (-1)^\lambda t^\lambda \end{bmatrix}.$$

We also know by lemma 4.3.3 that the first entry in the j^{th} column of $B(\lambda)^\lambda$ is $c_j s^{\lambda+j} t^{\lambda-j}$ for some $c_j \in k$, so the j^{th} column must therefore be $c_j s^j t^{\lambda-j} G$. The columns of $B(\lambda)^\lambda$ range from $j = 0$ to $j = \lambda$ so this shows that G times any monomial of degree λ is contained in the image of $B(\lambda)^\lambda$. Thus the quotient $\ker B(\lambda)/\operatorname{im} B(\lambda)^\lambda$ is spanned, as a vector space, by the set of vectors of the form G times a monomial of degree strictly less than λ. There are only finitely many such monomials therefore $\ker B(\lambda)/\operatorname{im} B(\lambda)^\lambda$ is finite dimensional and the proof is complete. $\square$

Now for the inductive step we will make use of proposition 3.1.8, but in a slightly different form. Note that the shift in proposition 3.1.8 is given by tensoring with the sheaf $\mathcal{O}_{\mathbb{P}(\mathfrak{sl}_2)}(1)$ associated to the shifted module $\frac{k[x,y,z]}{xy+z^2}[1]$. Likewise we consider $\mathcal{O}_{\mathbb{P}^1}(1)$ to be the sheaf associated to $k[s,t][1]$. Pullback through the isomorphism $\iota\colon \mathbb{P}^1 \to \mathbb{P}(\mathfrak{sl}_2)$ of example 4.1.2 yields $\iota^*\mathcal{O}_{\mathbb{P}(\mathfrak{sl}_2)}(1) = \mathcal{O}_{\mathbb{P}^1}(2)$. Consequently, proposition 3.1.8 has the following corollary.

Corollary 4.4.3. *Let M be an $\mathfrak{sl}_2$-module and $1 \le i < p$. With twist coming from $\mathbb{P}^1$ we have*

$$\mathscr{F}_i(M) \simeq \mathscr{F}_{p-i}(\Omega M)(2p-2i).$$

Observe that $i \ne p$ in the theorem; this is why our calculation of $\mathscr{F}_p(V(\lambda))$ for $\lambda < p$ does not induce a calculation of $\mathscr{F}_p(V(\lambda))$ when $\lambda \ge p$.

Proposition 4.4.4. *If $V(\lambda)$ is indecomposable and $i \neq p$ then*

$$\mathscr{F}_i(V(\lambda)) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-\lambda) & \textit{if } i = \lambda + 1 \pmod{p} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $\lambda = rp + a$ where $0 \leq a < p$ is the remainder of λ modulo p. We prove the result by induction on r. The base case $r = 0$ follows from Propositions 4.4.1 and 4.4.2. For the inductive step assume $r \geq 1$. By hypothesis the formula holds for $rp - a - 2$ and by proposition 4.2.5 we have $\Omega V(rp - a - 2) = V(\lambda)$. Applying corollary 4.4.3 we get

$$\mathscr{F}_i(V(\lambda)) = \mathscr{F}_{p-i}(V(rp - a - 2))(-2i).$$

If $i = a + 1$ then

$$\begin{aligned} \mathscr{F}_{a+1}(V(\lambda)) &= \mathscr{F}_{p-a-1}(V(rp - a - 2))(-2a - 2) \\ &= \mathcal{O}_{\mathbb{P}^1}(a + 2 - rp)(-2a - 2) \\ &= \mathcal{O}_{\mathbb{P}^1}(-a - rp) \\ &= \mathcal{O}_{\mathbb{P}^1}(-\lambda) \end{aligned}$$

whereas if $i \neq a + 1$ then $p - i \neq p - a - 1$ so $\mathscr{F}_{p-i}(V(rp - a - 2)) = 0$. This completes the proof. □

Chapter 5

VARIETIES OF ELEMENTARY SUBALGEBRAS OF MAXIMAL DIMENSION [41]

In this chapter we compute the variety $\mathbb{E}(r, \mathfrak{g})$ when $\mathfrak{g}$ is the Lie algebra of a reductive group and r is the maximal dimension of an elementary subalgebra of $\mathfrak{g}$. We follow the convention in Jantzen [31] and assume that our reductive algebraic k-groups are defined and split over $\mathbb{Z}$. We begin with some standard notational conventions about reductive groups and their Lie algebras, and we discuss conditions on the characteristics of the fields over which we work.

5.1 *Preliminaries*

5.1.1 Notations and conventions

Let $G_{\mathbb{Z}}$ be a split connected reductive algebraic $\mathbb{Z}$-group. Set $G_R = (G_{\mathbb{Z}})_R$ for any ring R and $G = G_k$. Let $T_{\mathbb{Z}} \subseteq G_{\mathbb{Z}}$ be a split maximal torus and define T_R and T as we did with G. Fix a Borel subgroup B containing T and let U be the unipotent radical of B. Let $X(T) = \operatorname{Hom}(T, \mathbb{G}_m)$ be the character group and let $\Phi \subseteq X(T)$ be the root system associated to G with respect to T. Let $\Lambda_r(\Phi) = \mathbb{Z}\Phi$ and $\Lambda(\Phi)$ be the root and integral weight lattices of Φ respectively. The quotient $\Lambda(\Phi)/\Lambda_r(\Phi)$ is called the fundamental group of the root system Φ. If Φ is irreducible then its fundamental group is cyclic, except for type D_n when n is even, in which case one gets the Klein 4-group.

If G is semisimple then the quotient $\Lambda(\Phi)/X(T)$ is called the fundamental group of G. We say that G is simply connected if its fundamental group is trivial. For reductive G we denote by G_{sc} the simply connected semisimple group of the same type as G. For any subgroup A of the fundamental group of Φ there is a semisimple group G with root system

Φ, fundamental group A, and a central isogeny $G_{\text{sc}} \to G$ (see Knus et al. [33, 25] for more details). The following lemma clarifies the significance of the fundamental group of G for our calculations.

Lemma 5.1.1 ([56, 2.4])**.** *Let G be a semisimple algebraic group with root system Φ. If p does not divide the order of the fundamental group of G then the isogeny $G_{\text{sc}} \to G$ is separable.*

Proof. Let H be the scheme-theoretic kernel of the isogeny $G_{\text{sc}} \to G$. Then H is a diagonalizable group scheme associated to the finite abelian group $\Lambda(\Phi)/X(T)$ (see, for example, Jantzen [31, II.1.6]). In particular, the dimension of the coordinate ring $k[H]$ is equal to the order of $\Lambda(\Phi)/X(T)$. The assumption on p implies that $(p, \dim k[H]) = 1$, hence, H is an étale group scheme, $k[H]$ is a separable algebra, and the map $G_{\text{sc}} \to G$ is separable. $\square$

If $\beta = \sum_i m_i \alpha_i$ is the highest root written as a linear combination of simple roots then p is *bad* for Φ if $p = m_i$ for some i. Similarly we may write the dual of this root as a linear combination of dual simple roots $\beta^\vee = \sum_i m_i' \alpha_i^\vee$ and p is *torsion* for Φ if $p = m_i'$ for some i. A prime is *good* (respectively *non-torsion*) if it is not bad (respectively torsion). We say p is *very good* for Φ if p is good for Φ and p does not divide the order of the fundamental group of Φ. Some authors include this condition in the definition of non-torsion but we will not. We instead say that a prime is *very non-torsion* if it is non-torsion and p does not divide the order of the fundamental group of Φ.

Definition 5.1.2. If G is a semisimple algebraic group we say that p is *separably good* for G if

1. p is good for G,
2. the isogeny $G_{\text{sc}} \to G$ is separable.

If G is a connected reductive group we say that p is *separably good* for G if it is separably good for its derived group $[G, G]$.

Note that very good implies separably good by lemma 5.1.1 but in type A the separably good condition is less restrictive. The simple groups of type A_{n-1} are SL_n/μ_d for $d \mid n$ and by lemma 5.1.1 if $p \nmid d$ then p is separably good. In particular, p is always separably good if $d = 1$ so the Special Linear group SL_n is covered by our results for all p.

Let $\mathfrak{g}$ and $\mathfrak{u}$ be the Lie algebras of G and U respectively. For any ring R let $\mathfrak{g}_R = \operatorname{Lie} G_R$ so that $\mathfrak{g} = \mathfrak{g}_k$. Note that if $R \subseteq S$ are rings then one has $\mathfrak{g}_S = \mathfrak{g}_R \otimes_R S$.

Definition 5.1.3. Let $R \subseteq k$ be a subring. An element $x \in \mathfrak{g}$ is *defined over* R if there exists an $x' \in \mathfrak{g}_R$ such that $x = x' \otimes 1$. A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *defined over* R if there exists a Lie subalgebra $\mathfrak{h}' \subseteq \mathfrak{g}_R$ such that $\mathfrak{h} = \mathfrak{h}' \otimes_R k$. We will call x' and $\mathfrak{h}'$ R*-forms* of x and $\mathfrak{h}$ respectively.

For a simply connected semisimple group G the Lie algebra $\mathfrak{g}$ has a Chevalley basis $\{x_\alpha, h_i \mid \alpha \in \Phi,\ 1 \le i \le \operatorname{rank} \Phi\}$ defined over $\mathbb{Z}$. In particular, this means $[x_\alpha, x_\beta] = N_{\alpha,\beta} x_{\alpha+\beta}$, where $N_{\alpha_\beta} = 0$ if $\alpha + \beta \notin \Phi$ and when $\alpha + \beta \in \Phi$ we have $N_{\alpha,\beta} = \pm(r+1)$ where $-r\alpha + \beta, \ldots, s\alpha + \beta$ is the α-string through β. The sign can be inductively determined depending on a choice of ordering for the roots and our choice that $N_{\alpha,\beta} = +(r+1)$ when (α, β) is an extraspecial pair defined by this ordering (see Carter [17, 4.2] for details).

For a general reductive group there exists a central isogeny $G_{\mathrm{sc}} \times D \to G$ where D is some torus. The preimage of the Borel $B \subseteq G$ is a Borel in the reductive group $G_{\mathrm{sc}} \times D$ and the isogeny restricts to an isomorphism between the unipotent parts of these Borels. We may therefore define $x_\alpha \in \mathfrak{g}$ and $h_i = [x_{\alpha_i}, x_{-\alpha_i}]$ as the image under the isogeny of the corresponding elements of $\operatorname{Lie} G_{\mathrm{sc}}$. It may be the case that the h_i do not span $\operatorname{Lie}(T)$, nevertheless, $\{x_\alpha \mid \alpha \in \Phi^+\}$ is still a basis for $\mathfrak{u}$ and satisfies the same relations as in the simply connected case. We say that the set $\{x_\alpha \mid \alpha \in \Phi\}$ is induced from a Chevalley basis.

Definition 5.1.4. A subalgebra $\mathcal{E} \subseteq \mathfrak{g}$ is called *Chevalley* if it is spanned by some subset of the $\{x_\alpha \mid \alpha \in \Phi\}$ induced from a Chevalley basis.

Note that the non-uniqueness of a Chevalley basis is up to scaling of the basis vectors, so the property of being a Chevalley subalgebra does not depend on the choice of Chevalley basis from which the x_α are induced.

Finally, we note that good primes are greater than or equal to the length of the longest root string in the root system Φ. In particular, this implies that the structure constants $N_{\alpha,\beta} \in \mathbb{Z}$ for the Chevalley basis are not divisible by p. Thus in $\mathfrak{g}$ one has $[x_\alpha, x_\beta] \neq 0$ if and only if $\alpha + \beta \in \Phi$.

Table 5.1: Bad primes, torsion primes, fundamental group orders, and maximal root string lengths. [25, 2.13] [37, 9.2]

Type	A_n	B_n $(n \geq 2)$	C_n $(n \geq 3)$	D_n $(n \geq 4)$	E_6	E_7	E_8	F_4	G_2
Bad	none	2	2	2	$2,3$	$2,3$	$2,3,5$	$2,3$	$2,3$
Torsion	none	2	none	2	$2,3$	$2,3$	$2,3,5$	$2,3$	2
$\lvert\Lambda/\Lambda_r\rvert$	$n+1$	2	2	4	3	2	1	1	1
Longest root string length	2	3	3	2	2	2	2	3	4

5.1.2 Maximal sets of commuting roots

Let Φ be an irreducible root system with positive roots Φ^+, corresponding base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, and the Weyl group W. Throughout the paper, we follow the labeling in Bourbaki [11, 6, §4]. Let $S \subseteq \Delta$ be a set of simple roots and $\overline{S} = \Delta \setminus S$ its complement. Then we define

$$\Phi_S^{\text{rad}} = \Phi^+ \setminus \mathbb{N}\overline{S}$$

to be the positive roots that cannot be written as a linear combination of the simple roots not in S. Note that $S \mapsto \Phi_S^{\text{rad}}$ commutes with unions and intersections. If $S = \{\alpha_i\}$ then we will write Φ_i^{rad} instead of $\Phi_{\{\alpha_i\}}^{\text{rad}}$.

For any $I \subset \Delta$ define the parabolic subgroup W_I and its corresponding root system Φ_I as in Humphreys [30, 1.10]. For $I = \Delta \setminus S$, we have $\Phi_S^{\text{rad}} = \Phi^+ \setminus \Phi_I^+$. If $P_I = L_I \ltimes U_I$ is the standard parabolic determined by the subset $I = \Delta \setminus S$ with the Levi factor L_I and the unipotent radical U_I, then the root subgroups U_α with $\alpha \in \Phi_S^{\mathrm{rad}}$ are precisely the ones generating U_I.

Lemma 5.1.5. *Let* $S \subseteq \Delta$. *Then* $\mathrm{Stab}_W(\Phi_S^{\mathrm{rad}}) = W_{\Delta\setminus S}$.

Proof. Let $I = \Delta \setminus S$. As W_I stabilizes Φ and Φ_I, it also stabilizes $\Phi \setminus \Phi_I = \Phi_S^{\mathrm{rad}} \cup -\Phi_S^{\mathrm{rad}}$. Now for any element $w \in W_I$ its length as an element of W equals its length as an element of W_I. Length is characterized by the number of positive roots that are sent to negative roots so such roots are in Φ_I. In particular, $w\Phi_S^{\mathrm{rad}} \subseteq \Phi_S^{\mathrm{rad}} \cup -\Phi_S^{\mathrm{rad}}$ are positive so $w\Phi_S^{\mathrm{rad}} \subseteq \Phi_S^{\mathrm{rad}}$. This proves that W_I stabilizes Φ_S^{rad}.

Conversely assume $w \in W \setminus W_I$ is of minimal length stabilizing Φ_S^{rad}. Then w stabilizes Φ_I and for any $\alpha \in I$ we have $\ell(ws_\alpha) = \ell(w) + 1$ so $w(\alpha)$ is positive. This means w stabilizes Φ_I^+, so it stabilizes all of Φ^+, but the identity is the only such element in W. ☐

Lemma 5.1.6. *The set* Φ_S^{rad} *is not conjugate to any other set of positive roots.*

Proof. If w does not stabilize Φ_S^{rad} then it may be written in the form $w = us_{\alpha_i}v$ where $v \in W_{\Delta\setminus S}$ and $s_{\alpha_i} \notin W_{\Delta\setminus S}$, i.e., $\alpha_i \in S$. By Humphreys [30, 1.7] we have $us_{\alpha_i}(\alpha_i) < 0$ so w sends $v^{-1}(\alpha_i) \in \Phi_S^{\mathrm{rad}}$ to a negative root. ☐

Two positive roots *commute* if their sum is not a root. A set of commuting roots is a set of positive roots which pairwise commute and $R \subseteq \Phi^+$ is a *maximal* set of commuting roots if it is maximal with respect to order, i.e., if R' is any other set of commuting roots then $|R'| \leq |R|$.

Notation 5.1.7. Let $\max(\Phi)$ be the set of all maximal sets of commuting roots in Φ. Let $m(\Phi)$ be the order of a maximal set of commuting roots in Φ^+. If Φ is irreducible of type T then we may write $\max(T)$ and $m(T)$ instead.

To formulate the theorem on the maximal sets of commuting roots, we need to introduce additional notation for type B_n. We first recall notation from Bourbaki [11, 6, §4.5]:

Notation 5.1.8. Type B_n.

$$\begin{aligned} \epsilon_i &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_n, && 1 \leq i \leq n \\ \epsilon_i + \epsilon_j &= (\alpha_i + \alpha_{i+1} + \cdots + \alpha_n) + (\alpha_j + \alpha_{j+1} + \cdots + \alpha_n), && 1 \leq i < j \leq n \\ \epsilon_i - \epsilon_j &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}, && 1 \leq i < j \leq n \end{aligned}$$

Define the following subsets of positive roots, as in Malcev [36].

$$\begin{aligned} S_t &= \{\epsilon_t, \epsilon_i + \epsilon_j \,|\, 1 \leq i < j \leq n\}, && t = 1, 2, \ldots, n \\ S_t^* &= \{\epsilon_t, \epsilon_i + \epsilon_j, \epsilon_{i'} - \epsilon_n \,|\, 1 \leq i < j < n, 1 \leq i' < n\}, && t = 1, 2, \ldots, n \end{aligned}$$

We present what is known about the maximal sets of positive roots in Table 5.2. The proofs can be found in Malcev [36], see also Appendix A.

Remark 5.1.9 (On cominuscule roots)**.** Note that in types A, C, D, E_6, E_7, the sets of maximal commuting roots are given by Φ_i^{rad} with α_i a simple cominuscule root (see Billey and Lakshmibai [9] or Richardson et al. [44] for more on cominuscule roots). One of the equivalent definitions of a simple cominuscule root α_i is that the unipotent radical of the corresponding parabolic is abelian [44, Lem 2.2] so the set Φ_i^{rad} is a natural candidate to be in $\max(T)$. As one sees from the table, the sets Φ_i^{rad} defined by a simple cominuscule root do have maximal dimension in almost all cases when they exist *except* for the most mysterious case of B_n.

Definition 5.1.10. We say that $R \subseteq \Phi^+$ is an *ideal* if $\alpha + \beta \in R$ whenever $\alpha \in \Phi^+$, $\beta \in R$, and $\alpha + \beta \in \Phi^+$.

Note, for example, that the sets Φ_i^{rad} are always ideals. The computations below require knowing which maximal sets of commuting roots are ideals and for such sets R what is the stabilizer $\mathrm{Stab}_W(R) = \{w \in W \mid wR = R\}$. This information can be found in Table 5.3 (see Appendix A for the calculation).

Note that $\max(\Phi)$ contains non-ideals only in types $B_n (n \geq 4)$, E_8, F_4, and G_2. One can check in types B_n, E_8, and F_4 that every set in $\max(\Phi)$ is W-conjugatate to an ideal

Table 5.2: Maximal sets of commuting roots.

Type T	Restrictions on rank	$\max(T)$	$\#\max(T)$	$m(T)$
A_{2n}	$n \geq 1$	Φ^{rad}_{n+1}, Φ^{rad}_{n}	2	$n(n+1)$
A_{2n+1}	$n \geq 0$	Φ^{rad}_{n+1}	1	$(n+1)^2$
B_n	$n = 2, 3$	Φ^{rad}_{1}	1	$2n-1$
	$n = 4$	Φ^{rad}_{1}, S_1, S_2, S_3, S_4, S^*_1, S^*_2, S^*_3	8	7
	$n \geq 5$	$S_t, 1 \leq t \leq n$ $S^*_t, 1 \leq t < n$	$2n-1$	$\frac{1}{2}n(n-1)+1$
C_n	$n \geq 3$	Φ^{rad}_{n}	1	$\frac{1}{2}n(n+1)$
D_n	$n = 4$	$\Phi^{\text{rad}}_{1}, \Phi^{\text{rad}}_{3}, \Phi^{\text{rad}}_{4}$	3	6
	$n \geq 5$	$\Phi^{\text{rad}}_{n-1}, \Phi^{\text{rad}}_{n}$	2	$\frac{1}{2}n(n-1)$
E_6		$\Phi^{\text{rad}}_{1}, \Phi^{\text{rad}}_{6}$	2	16
E_7		Φ^{rad}_{7}	1	27
E_8		none of the form Φ^{rad}_{i}	134	36
F_4		none of the form Φ^{rad}_{i}	28	9
G_2		none of the form Φ^{rad}_{i}	5	3

in $\max(\Phi)$. In any type there is at most one ideal that is not of the form Φ^{rad}_{i} for some i, therefore in all types except G_2 lemma 5.1.6 gives that each set in $\max(\Phi)$ is conjugate to a *unique* ideal in $\max(\Phi)$. The exceptional case is G_2, where one finds that there are two orbits under the partial action of W on $\max(\Phi)$ and only one contains an ideal.

Table 5.3: Maximal commuting ideals and their stabilizers

Type T	Ideal R	$\mathrm{Stab}_W(R)$
Any	Φ_i^{rad}	$W_{\Delta\setminus\{\alpha_i\}}$
B_n, $n \geq 4$	S_1	$W_{\Delta\setminus\{\alpha_1,\alpha_n\}}$
E_8	R is unique	$W_{\Delta\setminus\{\alpha_2\}}$
F_4	R is unique	$W_{\{\alpha_1,\alpha_3\}}$
G_2	R is unique	$W_{\{\alpha_2\}}$

5.1.3 Varieties $\mathbb{E}(r, \mathfrak{g})$

Definition 5.1.11 (Carlson et al. [13]). An elementary subalgebra $\mathcal{E} \subset \mathfrak{g}$ of dimension r is a p-restricted Lie subalgebra of dimension r which is commutative and has p-restriction equal to 0. We define

$$\mathbb{E}(r, \mathfrak{g}) = \{\mathcal{E} \subset \mathfrak{g} \mid \mathcal{E} \text{ elementary subalgebra of dimension } r\}$$

So defined, $\mathbb{E}(r, \mathfrak{g})$ is a closed subset of the Grassmannian of r-planes in the vector space $\mathfrak{g}$ and hence has a natural structure of a projective algebraic variety. In this paper, we are concerned with the varieties of elementary subalgebras of maximal dimension. We set

$$\begin{aligned} r_{\max} &= \max\{r \mid \mathbb{E}(r, \mathfrak{g}) \text{ is nonempty}\}, \\ \mathbb{E}(\mathfrak{g}) &= \mathbb{E}(r_{\max}, \mathfrak{g}). \end{aligned}$$

Theorem 5.1.12. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be restricted Lie algebras whose maximal elementary abelians have dimensions r and s respectively. Then $r + s$ is the dimension of a maximal elementary abelian in $\mathfrak{g} \oplus \mathfrak{h}$ and $\mathbb{E}(\mathfrak{g} \oplus \mathfrak{h}) \simeq \mathbb{E}(\mathfrak{g}) \times \mathbb{E}(\mathfrak{h})$.*

Proof. If $\mathcal{E} \subseteq \mathfrak{g} \oplus \mathfrak{h}$ is maximal and $\mathcal{E}_1 \subseteq \mathfrak{g}$ and $\mathcal{E}_2 \subseteq \mathfrak{h}$ are its images under the projections to $\mathfrak{g}$ and $\mathfrak{h}$ respectively then $\mathcal{E}_1 \oplus \mathcal{E}_2 \subseteq \mathfrak{g} \oplus \mathfrak{h}$ is elementary abelian and contains $\mathcal{E}$, hence equals $\mathcal{E}$. This proves that every maximal elementary abelian is a sum of necessarily maximal elementary abelians from $\mathfrak{g}$ and $\mathfrak{h}$. Thus $r+s$ is the maximal dimension of an elementary abelian in $\mathfrak{g} \oplus \mathfrak{h}$ and there is a bijection $\mathbb{E}(\mathfrak{g}) \times \mathbb{E}(\mathfrak{h}) \to \mathbb{E}(\mathfrak{g} \oplus \mathfrak{h})$. One then checks on the standard affine open sets of the Grassmannian that this is an isomorphism. $\square$

5.2 *Unipotent case*

In this section we assume that G is a simple algebraic group and we compute $\mathbb{E}(\mathfrak{u})$ as a *set*. We do this by defining the leading terms associated to a particular subalgebra as a subset of the root system and showing that such a subset must be a maximal set of commuting roots. We then perform a case by case analysis based on the information in Table 5.2 which shows that in most cases $\mathbb{E}(\mathfrak{u}) = \mathrm{Lie}(\max(\Phi))$, with $\mathrm{Lie}(\max(\Phi))$ defined in Equation 5.2.1.1. The exceptions are type A_2 where $\mathbb{E}(\mathfrak{u}) \simeq \mathbb{P}^1$, type B_n for $n \geq 4$ where $\mathbb{E}(\mathfrak{u})$ includes the lie subalgebras $B(a_1, \dots, a_n)$ and $\exp(\mathrm{ad}(a_n x_{\alpha_n}))(C(a_1, \dots, a_{n-1}))$ defined below, and the exceptional types of which only G_2 is calculated explicitly. For types F and E we verify later that under the adjoint action one has $\mathbb{E}(\mathfrak{u}) \subseteq G \cdot \mathrm{Lie}(\max(\Phi))$. We note that the condition $\mathbb{E}(\mathfrak{u}) \subseteq G \cdot \mathrm{Lie}(\max(\Phi))$ holds in all types except G_2.

Though we are interested in the case when p is separably good, the results in this section are valid so long as p is greater than or equal to the length of the longest root string in Φ.

5.2.1 *Correspondence with sets of commuting roots*

We must first choose a total ordering $\preceq$ on Φ^+ which respects addition of positive roots, that is, if $\beta, \gamma, \lambda, \beta+\lambda, \gamma+\lambda \in \Phi^+$ and $\beta \preceq \gamma$ then $\beta + \lambda \preceq \gamma + \lambda$. In examples the choice of ordering will depend on the root system but we note here that such a total ordering always exists: The standard ordering $\leq$ on Φ respects addition, as does a reverse lexicographical ordering with respect to any ordering of the simple roots. This ordering will define the extraspecial pairs in our root system and consequently the signs in the structure constants of the Chevalley basis which induces the x_α. We will often construct such orderings through refinement.

Definition 5.2.1. Let $\succeq_1, \succeq_2, \ldots, \succeq_n$ be relations on a set X which define $\succ_i$ and $=_i$ in the obvious way. The relation

$$\succeq\; = (\succeq_1, \succeq_2, \ldots, \succeq_n)$$

is defined as follows: $x \succeq y$ if either

1. For some $1 \leq i \leq n$ we have $x \succ_i y$ and $x =_j y$ for all $j < i$, or
2. $x =_i y$ for all $1 \leq i \leq n$.

We say that $\succeq$ is given by *refining* $\succeq_1$, first by $\succeq_2$, then by $\succeq_3$, and so on.

In simple terms we compare by first trying $\succeq_1$ and inductively trying $\succeq_{i+1}$ if $\succeq_i$ gives equality. We will always choose the $\succeq_i$ to be preorders which respect addition of positive roots and we will choose $\succeq_n$ to be, moreover, a total order. Then $\succeq$ is a total order which respects addition.

Now for each set $R \subseteq \Phi^+$ of commuting roots there is an abelian Chevalley subalgebra $\mathrm{Lie}(R) = \mathrm{span}_k \{x_\alpha \mid \alpha \in R\}$ associated to R with $\dim \mathrm{Lie}(R) = |R|$. As the elements x_α induced from a Chevalley basis are always p-nilpotent this is in fact an elementary abelian subalgebra. We wish to show that this induces a map

$$\mathrm{Lie}\colon \max(\Phi) \longrightarrow \mathbb{E}(\mathfrak{u}), \tag{5.2.1.1}$$

that is, that the Lie subalgebra associated to a maximal commuting set of roots has maximal dimension among all elemenatry subalgebras. We do this using an argument of Malcev [36] which shows that there exists a surjection

$$\mathrm{LT} : \mathbb{E}(\mathfrak{u}) \twoheadrightarrow \max(\Phi) \tag{5.2.1.2}$$

that splits Lie.

Let $\mathcal{E} \subseteq \mathfrak{u}$ be an elementary subalgebra. The ordering $\succeq$ on Φ^+ gives an ordering on the basis elements x_β of $\mathfrak{u}$. Choose the unique basis of $\mathcal{E}$ which is in reduced echelon form with respect to this ordering and let $\mathrm{LT}(\mathcal{E})$ be the set of roots β such that the corresponding x_β

are the leading terms in this reduced basis. Observe that if β and γ are the leading terms of $b_1 = x_\beta + \langle\text{lower terms}\rangle$ and $b_2 = x_\gamma + \langle\text{lower terms}\rangle$ respectively, and if $\beta + \gamma \in \Phi^+$ then $[x_\beta, x_\gamma] = N_{\beta,\gamma} x_{\beta+\gamma}$ is the leading term of $[b_1, b_2]$. Thus if $[b_1, b_2] = 0$ then β and γ commute. This proves that $\mathrm{LT}(\mathcal{E})$ is a commuting set of roots. Clearly $\mathrm{LT}(\mathrm{Lie}(R)) = R$ so LT splits Lie and both maps preserve maximality.

5.2.2 Types A_{2n+1}, B_2, B_3, C_n, and E_7

In these types there is a unique maximal set of commuting roots of the form Φ_i^{rad} for some i. Let $\succeq$ be the reverse lexicographic ordering given by $\alpha_i < \alpha_1 < \alpha_2 < \cdots$, so $\beta \succeq \gamma$ if when written as a linear combination of simple roots the coefficient of α_i in γ is larger than the coefficient of α_i in β, or if those coefficients are equal then coefficient of α_1 in γ is larger, and so on. With this ordering the roots in Φ_i^{rad}, which all have nonzero α_i coefficient, are strictly smaller than the roots in $\Phi^+ \setminus \Phi_i^{\mathrm{rad}}$, which have 0 as the α_i coefficient. The following lemma then gives that $\mathrm{Lie}(\Phi_i^{\mathrm{rad}})$ is the only possible maximal elementary subalgebra.

Lemma 5.2.2. *If* $\Phi^+ \setminus \mathrm{LT}(\mathcal{E}) \succ \mathrm{LT}(\mathcal{E})$ *then* $\mathcal{E} = \mathrm{Lie}(\mathrm{LT}(\mathcal{E}))$.

Proof. Choose $\beta \in \mathrm{LT}(\mathcal{E})$ and let b the element in the reduced echelon form basis of $\mathcal{E}$ with leading term x_β. The terms in $b - x_\beta$ are of the form cx_γ with $\beta \succeq \gamma$, equivalently, $\gamma \in \mathrm{LT}(\mathcal{E})$. So $c = 0$ because x_γ is the leading term of some other basis element. Thus we have $b = x_\beta$ and our reduced basis is $\{x_\beta \mid \beta \in \mathrm{LT}(\mathcal{E})\}$. □

5.2.3 Type A_2

Along with type G_2 this case will be exceptional in that it is not true that every elementary subalgebra is conjugate to a subalgebra in $\mathrm{Lie}(\max(\Phi))$ and it is not true that every Weyl group orbit in $\max(\Phi)$ contains an ideal. Consequently we will find below that the variety $\mathbb{E}(\mathfrak{g})$ for a simple algebraic group of type A_2 is not a disjoint union of flag varieties.

The highest root $\alpha_1 + \alpha_2$ commutes with all positive roots so $x_{\alpha_1+\alpha_2}$ is contained in any maximal elementary $\mathcal{E}$. As the dimension of such $\mathcal{E}$ is 2 this means $\mathcal{E}$ may be generated by

$x_{\alpha_1+\alpha_2}$ and an element of the form $ax_{\alpha_1} + bx_{\alpha_2}$ where $a, b \in k$ are not both 0. One can check that no other conditions on a and b are needed to get an elementary abelian therefore we have a bijection

$$\begin{aligned} \mathbb{P}^1 &\simeq \mathbb{E}(\mathfrak{u}) \\ [a:b] &\mapsto \langle ax_{\alpha_1} + bx_{\alpha_2}, x_{\alpha_1+\alpha_2} \rangle. \end{aligned}$$

We note that each of these subalgebras is fixed under conjugation by $U < G$, conjugation by the torus allows us to scale the constants a and b, and no two of these subalgebras are conjugate via a representative of a Weyl group element. Thus up to conjugation there are three subalgebras:

$$\begin{aligned} &\langle x_{\alpha_1}, x_{\alpha_1+\alpha_2} \rangle, \\ &\langle x_{\alpha_2}, x_{\alpha_1+\alpha_2} \rangle, \\ &\langle x_{\alpha_1} + x_{\alpha_2}, x_{\alpha_1+\alpha_2} \rangle. \end{aligned}$$

5.2.4 Type A_{2n}, $n \geq 2$

Consider positive roots β and γ written as a linear combination of the simple roots α_i. Define $\beta \succeq_1 \gamma$ if the sum of the coefficients of α_n and α_{n+1} in the expression for γ is greater or equal the sum for β. Define $\succeq_2$ to be the reverse lexicographic ordering given by $\alpha_{n+1} \prec \alpha_n \prec \alpha_1 \prec \alpha_2 \prec \cdots$. Finally let $\succeq$ be the refinement of $\succeq_1$ by $\succeq_2$. One can now check that

$$\Phi^+ \setminus (\Phi_n^{\mathrm{rad}} \cup \Phi_{n+1}^{\mathrm{rad}}) \;\succ\; \Phi_n^{\mathrm{rad}} \setminus \Phi_{n+1}^{\mathrm{rad}} \;\succ\; \Phi_{n+1}^{\mathrm{rad}} \setminus \Phi_n^{\mathrm{rad}} \;\succ\; \Phi_n^{\mathrm{rad}} \cap \Phi_{n+1}^{\mathrm{rad}}.$$

Let $\mathcal{E} \subseteq \mathfrak{u}$ be an elementary subalgebra. If $\mathrm{LT}(\mathcal{E}) = \Phi_{n+1}^{\mathrm{rad}}$ then lemma 5.2.2 gives $\mathcal{E} = \mathrm{Lie}(\Phi_{n+1}^{\mathrm{rad}})$. Assume $\mathrm{LT}(\mathcal{E}) = \Phi_n^{\mathrm{rad}}$. Then there is a basis whose leading terms are contained in either $\Phi_n^{\mathrm{rad}} \setminus \Phi_{n+1}^{\mathrm{rad}}$ or $\Phi_{n+1}^{\mathrm{rad}} \cap \Phi_n^{\mathrm{rad}}$. For the latter the argument in the proof of lemma 5.2.2 applies and we get that those basis elements are just the x_β for $\beta \in \Phi_{n+1}^{\mathrm{rad}} \cap \Phi_n^{\mathrm{rad}}$. We must show that the same is true for basis elements whose leading term is in $\Phi_n^{\mathrm{rad}} \setminus \Phi_{n+1}^{\mathrm{rad}}$.

We have

$$\Phi_n^{\mathrm{rad}} \setminus \Phi_{n+1}^{\mathrm{rad}} = \{\epsilon_i - \epsilon_{n+1} \mid 1 \leq i < n+1\}$$
$$\Phi_{n+1}^{\mathrm{rad}} \setminus \Phi_n^{\mathrm{rad}} = \{\epsilon_{n+1} - \epsilon_j \mid n+1 < j \leq 2n+1\}$$

(where the notation follows Bourbaki [11, 4.7]) so the remaining basis elements are of the form

$$b_i = x_{\epsilon_i - \epsilon_{n+1}} + \sum_{n+1<j\leq 2n+1} a_{ij} x_{\epsilon_{n+1}-\epsilon_j}$$

for $1 \leq i < n+1$. Now we compute

$$[b_i, b_{i'}] = \sum_{n+1<j\leq 2n+1} a_{i'j} N_{\epsilon_i-\epsilon_{n+1},\epsilon_{n+1}-\epsilon_j} x_{\epsilon_i-\epsilon_j} + a_{ij} N_{\epsilon_{n+1}-\epsilon_j,\epsilon_{i'}-\epsilon_{n+1}} x_{\epsilon_{i'}-\epsilon_j}.$$

As $n \geq 2$ we may choose $j \neq j'$. That this expression must equal 0 consequently gives $a_{ij} = 0$ for all i and j. Thus $b_j = x_{\epsilon_{n+1}-\epsilon_j}$ and $\mathcal{E} = \Phi_{n+1}^{\mathrm{rad}}$ as desired.

5.2.5 *Type B_n, $n \geq 5$*

Recall that we define

$$\epsilon_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n$$

where the α_i are the simple roots ordered as in Bourbaki [11, 6, §4]. Let $\succeq$ be the reverse lexicographical ordering given by

$$\alpha_1 \succ \cdots \succ \alpha_{n-1} \succ \alpha_n.$$

Specifically, we have

$$\epsilon_r - \epsilon_s \succeq \epsilon_t \succeq \epsilon_i + \epsilon_j$$

for all i, j, t, r, s and if $i < j$ and $r < s$ then

$$\begin{aligned} &\epsilon_r \succeq \epsilon_i && \text{if } i < r, \\ &\epsilon_r + \epsilon_s \succeq \epsilon_i + \epsilon_j && \text{if } j < s \text{ or if } j = s \text{ and } i < r, \\ &\epsilon_r - \epsilon_s \succeq \epsilon_i - \epsilon_j && \text{if } s < j \text{ or if } j = s \text{ and } i < r. \end{aligned}$$

One can also compute that if $i < j < n$ then $N_{\epsilon_i+\epsilon_n,\epsilon_j-\epsilon_n} = -N_{\epsilon_j+\epsilon_n,\epsilon_i-\epsilon_n} = 1$

Now if we define

$$\begin{aligned} R_1 &= \{\epsilon_i + \epsilon_j \mid 1 \le i < j < n\} \\ R_2 &= \{\epsilon_i + \epsilon_n \mid 1 \le i < n\} \\ R_3 &= \{\epsilon_i - \epsilon_n \mid 1 \le i < n\} \end{aligned}$$

then the set of positive roots, Φ^+, of B_n is the union of the sets

$$\{\epsilon_i - \epsilon_j \mid 1 \le i < j < n\} \succ R_3 \succ \{\epsilon_i \mid 1 \le i \le n\} \succ R_2 \succ R_1.$$

and

$$\begin{aligned} S_t &= R_1 \cup R_2 \cup \{\epsilon_t\} \\ S_t^* &= R_1 \cup R_3 \cup \{\epsilon_t\} \end{aligned}$$

(see notation 5.1.8 for the definition of S_t, S_t^*).

One can check that the following subalgebras are elementary and are maximal if not all a_i are zero.

$$\begin{aligned} B(a_1, \ldots, a_n) &= \operatorname{span}_k \left\{ x_\beta, \sum_{i=1}^{n} a_i x_{\epsilon_i} \;\middle|\; \beta \in R_1 \cup R_2 \right\} \\ C(a_1, \ldots, a_{n-1}) &= \operatorname{span}_k \left\{ x_\beta, \sum_{i=1}^{n-1} a_i x_{\epsilon_i} \;\middle|\; \beta \in R_1 \cup R_3 \right\}. \end{aligned}$$

Theorem 5.2.3. *If $n \ge 4$ and $\mathcal{E} \in \max(B_n)$ satisfies $\mathrm{LT}(\mathcal{E}) = S_t$ or S_t^* then there exist $a_1, \ldots, a_n$ such that $\mathcal{E} = B(a_1, \ldots, a_n)$ or $C(a_1, \ldots, a_{n-1})^{\exp(\mathrm{ad}(a_n x_{\alpha_n}))}$ respectively.*

Proof. If $\mathrm{LT}(\mathcal{E}) = S_t$ for some t then the argument of lemma 5.2.2 immediately gives $\mathcal{E} = B(0, \ldots, 0, 1, a_{t+1}, \ldots, a_n)$ for some $a_{t+1}, \ldots, a_n$. Now assume that $\mathrm{LT}(\mathcal{E}) = S_t^*$ for some $1 \le t < n$. The reduced echelon form basis of $\mathcal{E}$ then consists of the elements $x_{\epsilon_i+\epsilon_j}$ where $1 \le i < j < n$ and for $1 \le i < n$ the elements

$$x = x_{\epsilon_t} + \sum_{s=1}^{t-1} a_s x_{\epsilon_s} + \sum_{s=1}^{n-1} b_s x_{\epsilon_s+\epsilon_n} \quad \text{and} \quad y_i = x_{\epsilon_i-\epsilon_n} + \sum_{s=1}^{n} c_{is} x_{\epsilon_s} + \sum_{s=1}^{n-1} d_{is} x_{\epsilon_s+\epsilon_n}$$

for some a_s, b_s, c_{sk}, and d_{sk}. That it's reduced means $c_{it} = 0$ for all i. Notice that $\exp(\mathrm{ad}(\lambda x_{\alpha_n}))$ is upper triangular with respect to $\preceq$ so $\mathrm{LT}(\exp(\mathrm{ad}(\lambda x_{\alpha_n}))(\mathcal{E})) = \mathrm{LT}(\mathcal{E})$ and the element in the reduced basis of $\exp(\mathrm{ad}(\lambda x_{\alpha_n}))(\mathcal{E})$ with leading term x_{ϵ_t} is $\exp(\mathrm{ad}(\lambda x_{\alpha_n}))(x)$. When $\lambda = -b_t N^{-1}_{\epsilon_n,\epsilon_t}$ we have

$$\exp(\mathrm{ad}(\lambda x_{\alpha_n}))(x) = x_{\epsilon_t} + \sum_{s=1}^{t-1} a_s x_{\epsilon_s} + \sum_{s=1}^{t-1}(b_s - a_s b_t N^{-1}_{\epsilon_n,\epsilon_t} N_{\epsilon_n,\epsilon_s}) x_{\epsilon_s+\epsilon_n} + \sum_{s=t+1}^{n-1} b_s x_{\epsilon_s+\epsilon_n}.$$

As $x_{\epsilon_t+\epsilon_n}$ is not a term in this basis element it suffices to show that $\mathcal{E}$ is the subalgebra $C(0,\ldots,0,1,a_{t+1},\ldots,a_{n-1})$ when $b_t = 0$. We do this by showing that the b_s, c_{ij}, and d_{ij} must all be zero.

One can check that the coefficient of x_{ϵ_i} in $[y_i, y_j]$ is $N_{\epsilon_i-\epsilon_n,\epsilon_n} c_{jn}$ so $c_{jn} = 0$ for all j. Also if $j \neq t$ then the coefficient of $x_{\epsilon_j+\epsilon_t}$ in $[x, y_i]$ is $N_{\epsilon_t,\epsilon_j} c_{ij}$ so $c_{ij} = 0$ for all i, j. For $i \neq t$ the coefficient of $x_{\epsilon_i+\epsilon_t}$ in $[x, y_t]$ is $N_{\epsilon_i+\epsilon_n,\epsilon_t-\epsilon_n} b_i$ so $b_i = 0$ for all i.

If $i, j, t < n$ are distinct then the coefficients of $x_{\epsilon_j+\epsilon_t}$ and $x_{\epsilon_i+\epsilon_j}$ in $[y_i, y_t]$ are $N_{\epsilon_j+\epsilon_n,\epsilon_t-\epsilon_n} d_{ij}$ and $N_{\epsilon_i-\epsilon_n,\epsilon_j+\epsilon_n} d_{tj}$ respectively. As n is at least 4 this gives $d_{ij} = 0$ for all $i \neq j$.

Finally, if $i, j < n$ are distinct then $N_{\epsilon_i-\epsilon_n,\epsilon_j+\epsilon_n} d_{jj} + N_{\epsilon_i+\epsilon_n,\epsilon_j-\epsilon_n} d_{ii}$ is the coefficient of $x_{\epsilon_i+\epsilon_j}$ in $[y_i, y_j]$. For $i < j < t < n$ we thus get a system of equations

$$\begin{aligned} N_{\epsilon_i-\epsilon_n,\epsilon_j+\epsilon_n} d_{jj} + N_{\epsilon_i+\epsilon_n,\epsilon_j-\epsilon_n} d_{ii} &= d_{jj} + d_{ii} = 0 \\ N_{\epsilon_i-\epsilon_n,\epsilon_t+\epsilon_n} d_{tt} + N_{\epsilon_i+\epsilon_n,\epsilon_t-\epsilon_n} d_{ii} &= d_{tt} + d_{ii} = 0 \\ N_{\epsilon_j-\epsilon_n,\epsilon_t+\epsilon_n} d_{tt} + N_{\epsilon_j+\epsilon_n,\epsilon_t-\epsilon_n} d_{jj} &= d_{tt} + d_{jj} = 0 \end{aligned}$$

whose unique solution is $d_{ii} = d_{jj} = d_{tt} = 0$. This gives $d_{ii} = 0$ for all i and completes the proof of the theorem. □

We note that $\mathbb{E}(\mathfrak{u}) = \mathrm{Lie}(\max(\Phi))$ does not hold in type B_n, $n \geq 5$. In the next proposition we show that any elementary subalgebra in $\mathbb{E}(\mathfrak{u})$ is G-conjugate to a subalgebra in $\mathrm{Lie}(\max(\Phi))$.

Proposition 5.2.4. *Let $F/\mathbb{F}_p$ be a field extension. Any F-point of $\mathbb{E}(\mathfrak{u})$ is $G(F)$-conjugate to an elementary subalgebra in* $\mathrm{Lie}(\max(\Phi))$.

Proof. We show that any elementary subalgebra in $\mathbb{E}(\mathfrak{u})$ defined over F is $G(F)$-conjugate to $\mathrm{Lie}(S_1)$ where S_1 is as defined in notation 5.1.8.

The simple reflection s_n acts by negating ϵ_n and fixing the remaining ϵ_i therefore any representative $\dot{s_n} \in N_G(T)$ conjugates $C(a_1, \ldots, a_{n-1})$ to $B(a_1, \ldots, a_{n-1}, 0)$. Similarly s_i, where $i < n$, swaps ϵ_i with ϵ_{i+1} and fixes the remaining ϵ_j so by conjugation we may assume our elementary subalgebra is of the form $B(a_1, \ldots, a_{n-1}, 1)$. Finally conjugation by $\exp(\mathrm{ad}(a_i N^{-1}_{\epsilon_i-\epsilon_n,\epsilon_n} x_{\epsilon_i-\epsilon_n}))$ lets us assume $a_i = 0$ and does not alter the remaining a_j, thus we have conjugated our subalgebra to $B(0, \ldots, 0, 1)$. Using simple reflections we conjugate to $B(1, 0, \ldots, 0) = \mathrm{Lie}(S_1)$ and are done. Note that if the variables $(a_1, \ldots, a_{n-1}, a_n)$ belong to the field F then all conjugations we have to perform to reduce $C(a_1, \ldots, a_{n-1})$ or $B(a_1, \ldots, a_{n-1}, a_n)$ to $B(1, 0, \ldots, 0)$ are by elements in $G(F)$. □

5.2.6 Type B_4

We keep the ordering and choice of basis from the last section. As theorem 5.2.3 applies here as well, all that is left is to prove the following.

Theorem 5.2.5. *If* $\mathrm{LT}(\mathcal{E}) = \Phi_1^{\mathrm{rad}}$ *then* $\mathcal{E} = \mathrm{Lie}(\Phi_1^{\mathrm{rad}})$.

Proof. The reduced echelon form basis of $\mathcal{E}$ is of the form

$$x_i = x_{\epsilon_1+\epsilon_i} + \sum_{\substack{3\le r\le i\\ 2\le s<r}} c_{isr} x_{\epsilon_s+\epsilon_r}, \quad \text{and}$$

$$y_j = x_{\epsilon_1-\epsilon_j} + \sum_{\substack{j\le r\le 4\\ 2\le s<r}} d_{jsr} x_{\epsilon_s-\epsilon_r} + \sum_{\substack{r=3,4\\ 2\le s<j}} e_{jsr} x_{\epsilon_s+\epsilon_r}$$

for $i = 2, 3, 4$ and $j = 1, 2, 3, 4$. These 7 basis elements can be formed into 21 possible commutators which must equal zero. Setting their coefficients equal to zero gives a system of equations which can be solved by hand and whose unique solution is $c_{isr} = d_{jsr} = e_{jsr} = 0$ as desired. □

As above we do not get $\mathbb{E}(\mathfrak{u}) = \mathrm{Lie}(\max(\Phi))$. Instead we get $\mathbb{E}(\mathfrak{u}) \subseteq G \cdot \mathrm{Lie}(S_1) \cup \{\mathrm{Lie}(\Phi_1^{\mathrm{rad}})\}$.

5.2.7 Type D_n, $n \geq 5$

Define

$$\epsilon_i = \begin{cases} \alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2} + \frac{1}{2}(\alpha_{n-1} + \alpha_n) & \text{if } i \leq n-2 \\ \frac{1}{2}(\alpha_{n-1} + \alpha_n) & \text{if } i = n-1 \\ \frac{1}{2}(\alpha_n - \alpha_{n-1}) & \text{if } i = n. \end{cases}$$

where the α_i are the simple roots ordered as in Bourbaki [11, 6, §4] and let $\succeq$ be the reverse lexicographical ordering given by

$$\alpha_{n-1} \succ \cdots \succ \alpha_2 \succ \alpha_1 \succ \alpha_{n-1} \succ \alpha_n.$$

One can check that for $i < j < n$ this gives $N_{\epsilon_i - \epsilon_n, \epsilon_j + \epsilon_n} = -N_{\epsilon_j - \epsilon_n, \epsilon_i + \epsilon_n} = 1$.

If we define

$$R = \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n-1\}$$

then

$$\Phi_n^{\text{rad}} = R \cup \{\epsilon_i + \epsilon_n \mid 1 \leq i \leq n-1\}$$
$$\Phi_{n-1}^{\text{rad}} = R \cup \{\epsilon_i - \epsilon_n \mid 1 \leq i \leq n-1\}$$

and Φ^+ is the union of the sets

$$\Phi^+ \setminus \Phi^{\text{rad}}_{\{\alpha_1, \alpha_{n-1}, \alpha_n\}} \succ \Phi_1^{\text{rad}} \setminus \Phi^{\text{rad}}_{\{\alpha_{n-1}, \alpha_n\}} \succ \Phi_{n-1}^{\text{rad}} \setminus R \succ \Phi_n^{\text{rad}} \setminus R \succ R.$$

Theorem 5.2.6. *If $n \geq 4$ and $\mathcal{E} \in \max(D_n)$ satisfies $\operatorname{LT}(\mathcal{E}) = \Phi_n^{\text{rad}}$ or Φ_{n-1}^{rad} then $\mathcal{E} = \operatorname{Lie}(\Phi_n^{\text{rad}})$ or $\operatorname{Lie}(\Phi_{n-1}^{\text{rad}})$ respectively.*

Proof. If $\operatorname{LT}(\mathcal{E}) = \Phi_n^{\text{rad}}$ then lemma 5.2.2 immediately gives $\mathcal{E} = \operatorname{Lie}(\Phi_n^{\text{rad}})$, so assume $\operatorname{LT}(\mathcal{E}) = \Phi_{n-1}^{\text{rad}}$. The reduced basis of $\mathcal{E}$ then consists of the x_α corresponding to $\alpha \in R$ and the elements

$$y_i = x_{\epsilon_i - \epsilon_n} + \sum_{s=1}^{n-1} a_{is} x_{\epsilon_s + \epsilon_n}$$

for $1 \leq i \leq n-1$ and we want to show that $a_{ij} = 0$ for all i, j.

If $i, j, t < n$ are distinct then the coefficient of $x_{\epsilon_i+\epsilon_t}$ in $[y_i, y_j]$ is $N_{\epsilon_i-\epsilon_n,\epsilon_t+\epsilon_n}a_{jt}$. As $n \geq 4$ this gives $a_{jt} = 0$ for all $j \neq t$. Now for $i \neq j$ the coefficient of $x_{\epsilon_i+\epsilon_j}$ in $[y_i, y_j]$ is $N_{\epsilon_i-\epsilon_n,\epsilon_j+\epsilon_n}a_{jj} + N_{\epsilon_i+\epsilon_n,\epsilon_j-\epsilon_n}a_{ii}$. Thus if $i < j < t < n$ we get a system of equations

$$N_{\epsilon_i-\epsilon_n,\epsilon_j+\epsilon_n}a_{jj} + N_{\epsilon_i+\epsilon_n,\epsilon_j-\epsilon_n}a_{ii} = a_{jj} + a_{ii} = 0$$

$$N_{\epsilon_i-\epsilon_n,\epsilon_t+\epsilon_n}a_{tt} + N_{\epsilon_i+\epsilon_n,\epsilon_t-\epsilon_n}a_{ii} = a_{tt} + a_{ii} = 0$$

$$N_{\epsilon_j-\epsilon_n,\epsilon_t+\epsilon_n}a_{tt} + N_{\epsilon_j+\epsilon_n,\epsilon_t-\epsilon_n}a_{jj} = a_{tt} + a_{jj} = 0$$

whose unique solution is $a_{ii} = a_{jj} = a_{tt} = 0$. This gives $a_{ii} = 0$ for all i and completes the proof of the theorem. □

5.2.8 Type D_4

We keep the ordering from the last section. As theorem 5.2.6 applies here as well, all that is left is to prove that if $\mathrm{LT}(\mathcal{E}) = \Phi_1^{\mathrm{rad}}$ then $\mathcal{E} = \mathrm{Lie}(\Phi_1^{\mathrm{rad}})$. The reduced basis of $\mathcal{E}$ is of the form

$$x_{\epsilon_1-\epsilon_2} + a_{11}x_{\epsilon_2-\epsilon_3} + a_{12}x_{\epsilon_3-\epsilon_4} + a_{13}x_{\epsilon_2-\epsilon_4} + a_{14}x_{\epsilon_3+\epsilon_4} + a_{15}x_{\epsilon_2+\epsilon_4} + a_{16}x_{\epsilon_2+\epsilon_3}$$

$$x_{\epsilon_1-\epsilon_3} + a_{21}x_{\epsilon_3-\epsilon_4} + a_{22}x_{\epsilon_2-\epsilon_4} + a_{23}x_{\epsilon_3+\epsilon_4} + a_{24}x_{\epsilon_2+\epsilon_4} + a_{25}x_{\epsilon_2+\epsilon_3}$$

$$x_{\epsilon_1-\epsilon_4} + a_{31}x_{\epsilon_3+\epsilon_4} + a_{32}x_{\epsilon_2+\epsilon_4} + a_{33}x_{\epsilon_2+\epsilon_3}$$

$$x_{\epsilon_1+\epsilon_4} + a_{41}x_{\epsilon_2+\epsilon_3}$$

$$x_{\epsilon_1+\epsilon_3}$$

$$x_{\epsilon_1+\epsilon_2}$$

and we wish to show that the a_{ij} are zero. As in the B_4 case, setting commutators equal to zero yields a system of equations for the a_{ij} which can be solved by hand and whose unique solution is $a_{ij} = 0$ for all i, j.

5.2.9 Type G_2

As with type A_2 this case is exceptional in that it is not true that every elementary subalgebra is conjugate to a subalgebra in Lie(max(Φ)) and it is not true that every Weyl group orbit in max(Φ) contains an ideal.

Recall that we assume the characteristic p is good for Φ, so explicitly we assume $p \neq 2, 3$. Let α_1 be the short root and α_2 the long root in the basis, with $s_1, s_2 \in W$ the corresponding simple reflections. We choose the reverse graded lexicographic ordering on Φ with $\alpha_1 \succ \alpha_2$ so that the positive roots are ordered as follows:

$$\alpha_1 \succ \alpha_2 \succ \alpha_1 + \alpha_2 \succ 2\alpha_2 + \alpha_2 \succ 3\alpha_1 + \alpha_2 \succ 3\alpha_1 + 2\alpha_2.$$

The s_i act via

$$\begin{aligned} s_i(\alpha_i) &= -\alpha_i, \\ s_1(\alpha_2) &= 3\alpha_1 + \alpha_2, \\ s_2(\alpha_1) &= \alpha_1 + \alpha_2, \end{aligned}$$

and W is the dihedral group of order 12 generated by the reflection s_1 and rotation $s_1 s_2$. It is simple to check that the maximal sets of commuting positive roots fall into two orbits:

$$\begin{aligned} C_1 &= \{\alpha_1, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ C_2 &= \{\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ C_3 &= \{\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \end{aligned}$$

and

$$\begin{aligned} C_4 &= \{\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \\ C_5 &= \{2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}. \end{aligned}$$

Only C_5 is an ideal.

Theorem 5.2.7. *Every* $\mathcal{E} \in \mathbb{E}(\mathfrak{u})$ *is* G*-conjugate to one of* $\mathrm{Lie}(C_3)$*,* $\mathrm{Lie}(C_5)$*, or* $L = \langle x_{\alpha_2} + x_{3\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2}\rangle$*. Moreover, these subalgebras are pairwise non-conjugate.*

Proof. As C_3 and C_5 are representatives of the two orbits we must show that every $\mathcal{E} \in \mathbb{E}(\mathfrak{g})$ is conjugate to either L or $\mathrm{Lie}(C_i)$ for any i. We handle each of the five possible leading terms separately.

An abelian subalgebra with leading terms C_1 is generated by elements of the form

$$x_{\alpha_1} + a_1 x_{\alpha_1+\alpha_2} + a_2 x_{2\alpha_1+\alpha_3},$$
$$x_{3\alpha_1+\alpha_2},$$
$$x_{3\alpha_1+2\alpha_2}.$$

for some $a_1, a_2 \in k$. Conjugating by $\exp(\mathrm{ad}(\frac{1}{2}a_2 x_{\alpha_1+\alpha_2}))\exp(\mathrm{ad}(a_1 x_{\alpha_2}))$ gives $\mathrm{Lie}(C_1)$.

An abelian subalgebra with leading terms C_2 is generated by elements of the form

$$x_{\alpha_1+\alpha_2} + a_1 x_{2\alpha_1+\alpha_2},$$
$$x_{3\alpha_1+\alpha_2},$$
$$x_{3\alpha_1+2\alpha_2},$$

for some $a_1 \in k$. Conjugating by $\exp(\mathrm{ad}(-\frac{1}{2}a_1 x_{\alpha_1}))$ gives $\mathrm{Lie}(C_2)$.

An abelian subalgebra with leading terms C_3 is generated by elements of the form

$$x_{\alpha_2} + a_1 x_{\alpha_1+\alpha_2} + a_2 x_{3\alpha_1+\alpha_2},$$
$$x_{2\alpha_1+\alpha_2} + 3a_1 x_{3\alpha_1+\alpha_2},$$
$$x_{3\alpha_1+2\alpha_2}.$$

for some $a_1, a_2 \in k$. Conjugation by $\exp(\mathrm{ad}(-a_1 x_{\alpha_1}))$ allows us to assume $a_1 = 0$, giving

$$x_{\alpha_2} + a_2 x_{3\alpha_1+\alpha_2},$$
$$x_{2\alpha_1+\alpha_2},$$
$$x_{3\alpha_1+2\alpha_2}.$$

If $a_2 = 0$ this is $\mathrm{Lie}(C_3)$. If $a_2 \neq 0$ then conjugation by $\alpha_2^\vee(\sqrt[3]{a_2})$ sends the subalgebra above to L.

The only subalgebra with leading terms C_5 is $\mathrm{Lie}(C_5)$ so all that is left is C_4. An abelian subalgebra with leading terms C_4 is generated by elements of the form

$$\begin{aligned} &x_{\alpha_2} + a_1 x_{2\alpha_1+\alpha_2} + a_2 x_{3\alpha_1+\alpha_2}, \\ &x_{\alpha_1+\alpha_2} + a_3 x_{2\alpha_1+\alpha_2} - 3a_1 x_{3\alpha_1+\alpha_2}, \\ &x_{3\alpha_1+2\alpha_2}. \end{aligned}$$

for some $a_1, a_2, a_3 \in k$. Conjugation by $\exp(\mathrm{ad}(-\frac{1}{2}a_3 x_{\alpha_1}))$ allows us to assume $a_3 = 0$. If $a_1 = 0$ then conjugation by $\dot{s_1}$ yeilds a subalgebra generated by elements of the form

$$\begin{aligned} &a x_{\alpha_2} + x_{3\alpha_1+\alpha_2}, \\ &x_{2\alpha_1+\alpha_2}, \\ &x_{3\alpha_1+2\alpha_2}, \end{aligned}$$

and the leading terms are either C_3 or C_5, so assume $a_1 \neq 0$. Then conjugation by $\alpha_2^\vee(\sqrt[3]{a_1})$ allows us to assume $a_1 = 1$. Our generators are now of the form

$$\begin{aligned} &x_{\alpha_2} + x_{2\alpha_1+\alpha_2} + a x_{3\alpha_1+\alpha_2}, \\ &x_{\alpha_1+\alpha_2} - 3x_{3\alpha_1+\alpha_2}, \\ &x_{3\alpha_1+2\alpha_2}. \end{aligned}$$

For any choice of $u, v \in k$ we may conjugate by $\exp(\mathrm{ad}(u x_{\alpha_1}))\dot{s_1}\exp(\mathrm{ad}(v x_{\alpha_1}))$ to get a subalgebra generated by terms of the form

$$\begin{aligned} &\lambda_1 x_{\alpha_2} + ((u^3 + 3u + a)v + u^2 + 1)x_{\alpha_1+\alpha_2} + \text{ lower terms}\ldots, \\ &\lambda_2 x_{\alpha_2} + (3(u^2 - 1)v + 2u)x_{\alpha_1+\alpha_2} + \text{lower terms}\ldots, \\ &x_{3\alpha_1+2\alpha_2}. \end{aligned}$$

From the constant term one sees that for any a the polynomial $u^4 - 6u^2 - 2ua - 3$ must always have a root distinct from ± 1 and necessarily nonzero. Choose u to be such a root.

Then

$$\frac{4u+a}{u^2-1} = \frac{u^2+3}{2u}$$

holds and, moreover, implies that the equations

$$3(u^2-1)v + 2u = 0,$$
$$3(4u+a)v + u^2 + 3 = 0$$

determine a unique v. Finally

$$u[3(u^2-1)v+2u] + [3(4u+a)v+u^2+3] = 3[(u^3+3u+a)v+u^2+1]$$

so $(u^3+3u+a)v+u^2+1=0$. Thus with this choice of u, v we have conjugated the subgroup to one for whom $\alpha_1+\alpha_2$ is not a leading term. This means we are no longer in the case of leading terms C_4 and our previous arguments apply. Hence we have shown that every $\mathcal{E} \in \mathbb{E}(\mathfrak{u})$ is G-conjugate to one of $\mathrm{Lie}(C_3)$, $\mathrm{Lie}(C_5)$, or L.

All that is left is to show that these three subalgebras are not conjugate. For this we simply observe that their normalizers

$$\begin{aligned} N_{\mathfrak{g}}(\mathrm{Lie}(C_3)) &= \langle h_1, h_2, x_\beta \mid \beta \in \Phi^+ \setminus \{\alpha_1\}\rangle, \\ N_{\mathfrak{g}}(\mathrm{Lie}(C_5)) &= \langle h_1, h_2, x_\beta \mid \beta \in \Phi^+ \cup \{-\alpha_2\}\rangle, \\ N_{\mathfrak{g}}(L) &= \langle h_1 + 2h_2, x_\beta \mid \beta \in \Phi^+ \setminus \{\alpha_1\}\rangle, \end{aligned}$$

have dimensions 7, 9, and 6 respectively. Conjugate subalgebras have conjugate, hence equidimensional, normalizers therefore the subalgebras $\mathrm{Lie}(C_3)$, $\mathrm{Lie}(C_5)$, and L are non-conjugate. □

5.2.10 Types E_6, E_8, and F_4

For these types there are too many cases for us to reasonably tackle them by hand. Instead we have written Magma code that attempts to confirm that $\mathbb{E}(\mathfrak{u}) \subseteq U \cdot \mathrm{Lie}(\max(\Phi))$ holds for Lie algebras of a given type. This code is available online [54], is successful for types

E_6, E_8, and F_4, and moreover confirms that if $\mathcal{E} \in \mathbb{E}(\mathfrak{u})$ is defined over a subfield $F \subseteq k$ then the conjugating element may be taken from the F-points of U. We warn that while the computations for E_6 and F_4 are very quick the computation for E_8 takes several hours and it is helpful to import the list of maximal sets of commuting roots from code written in Sage [52], where that aspect of the computation is much faster.

Summarizing the above discussion, we now have the following result.

Theorem 5.2.8. *Let G be simple algebraic group with root system Φ, not of type A_2 or G_2. Then*

$$\mathbb{E}(\mathfrak{u}) = \bigcup_{I \in \max(\Phi)} U \cdot \mathrm{Lie}(I).$$

Moreover, for any subfield $F \subseteq k$ the F-points of $\mathbb{E}(\mathfrak{u})$ are contained in $U(F) \cdot \mathrm{Lie}(I)$ for some $I \in \max(\Phi)$.

If $\dot{w} \in N_G(T)$ is a representative of the Weyl group element $w \in W$ then $\dot{w} \cdot x_\alpha$ is proportional to $x_{w \cdot \alpha}$. Thus if both I and wI are contained in $\max(\Phi)$ then $\dot{w} \cdot \mathrm{Lie}(I) = \mathrm{Lie}(wI)$. As Φ is not of type G_2 we know that every set in $\max(\Phi)$ is W-conjugate to an ideal. This gives the following corollary.

Corollary 5.2.9. *Let G be simple algebraic group with root system Φ, not of type A_2 or G_2. Then*

$$\mathbb{E}(\mathfrak{u}) \subseteq \bigcup_{\substack{I \in \max(\Phi) \\ I \text{ an ideal}}} G \cdot \mathrm{Lie}(I).$$

Moreover, for any subfield $F \subseteq k$ the F-points of $\mathbb{E}(\mathfrak{u})$ are contained in $G(F) \cdot \mathrm{Lie}(I)$ for some ideal $I \in \max(\Phi)$.

5.2.11 Type G_2 when $p = 3$

We now break with the assumption, made at the start of the current section, that p is equal or greater than the length of the longest root string in Φ. In type G_2 there are root strings of length 4 and we now consider the case $p = 3$.

The results of this section are based on the property that if $\alpha \neq \beta$ are positive roots then $[x_\alpha, x_\beta] = 0$ if and only if α and β commute. The techniques that follow from that assumption are still valid in the current case if we replace "commuting" with a different combinatorial property that holds if and only if the corresponding x_α and x_β commute.

Definition 5.2.10. Let $\alpha \neq \beta$ be positive roots and p a prime. We say that α and β *p-commute* if they commute or if the α-string through β begins at $(-p+1)\alpha + \beta$.

The above notion is a direct translation of the condition that the structure constant $N_{\alpha,\beta}$ is either 0 or p (it cannot be a larger multiple of p) therefore the product $[x_\alpha, x_\beta] = 0$ if and only if α and β p-commute. There are 3 sets of maximal p-commuting roots in Φ^+, they are

$$\begin{aligned} R_1 &= \{\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}, \\ R_2 &= \{\alpha_1, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}, \\ R_3 &= \{\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}. \end{aligned}$$

Geometrically each set consists of 4 rotationally consecutive roots. As long and short roots alternate and the Weyl group is the Dihedral group on the long roots we see that these three sets are conjugate.

The three cases of elementary abelian Lie subalgebras that come from these roots are as follows. If $\mathrm{LT}(\mathcal{E}) = R_1$ then $\mathcal{E} = \mathrm{Lie}(R_1)$ because R_1 is minimal in the ordering. If $\mathrm{LT}(\mathcal{E}) = R_2$ then

$$\mathcal{E} = \langle x_{\alpha_1} + c x_{\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2} \rangle.$$

Conjugating by $\exp(\mathrm{ad}(-c x_{\alpha_2}))$ yields $\mathrm{Lie}(R_2)$. Lastly if $\mathrm{LT}(\mathcal{E}) = R_2$ then

$$\mathcal{E} = \langle x_{\alpha_2} + c x_{3\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2} \rangle.$$

Conjugating by $\exp(\mathrm{ad}(\sqrt[3]{c} x_{\alpha_2}))$ yields $\mathrm{Lie}(R_3)$. Thus we have $\mathbb{E}(\mathfrak{u}) \subseteq G \cdot \mathrm{Lie}(R_1)$.

5.3 *Calculation of* $\mathbb{E}(\mathfrak{g})$ *for* $\mathfrak{g} = \operatorname{Lie} G$

We now calculate, when p is separably good, the variety $\mathbb{E}(\mathfrak{g})$ when $\mathfrak{g}$ is the Lie algebra of a reductive group G. We start by reducing to the case of a simple algebraic group.

Theorem 5.3.1. *Let $G_1, \ldots, G_n$ be the simple algebraic subgroups of the derived group $[G, G]$ and let $\mathfrak{g}_i = \operatorname{Lie}(G_i)$. Then*

$$\mathbb{E}(\mathfrak{g}) = \prod_{i=1}^{n} \mathbb{E}(\mathfrak{g}_i).$$

Proof. Follows from theorem 5.1.12. □

Henceforth we assume that G is a simple algebraic group. In all but the A_2 and G_2 cases we find that $\mathbb{E}(\mathfrak{g})$ is a disjoint union of flag varieties corresponding to ideals of maximal commuting roots. In the A_2 and G_2 cases we find that the $\mathbb{E}(\mathfrak{g})$ are irreducible varieties of dimensions 5 and 8 respectively, each given as the union of three G-orbits.

5.3.1 Reduction to the unipotent case

Our first step is to show that every maximal elementary abelian may be conjugated into $\mathfrak{u}$, the Lie algebra of the unipotent radical of the Borel. This is a result of Levy et. al. based on the following theorem.

Theorem 5.3.2 ([35, 2.2])**.** *Let G be a semisimple group and p a very non-torsion prime for G. Let U be a unipotent subgroup scheme of G. Then U is contained in a Borel subgroup of G.*

Remark 5.3.3. Reading the proof in Levy et al., one finds that the theorem above holds for any prime when G is SL_n or Sp_n, and consequently it holds for any group which is separably isogeneous to SL_n or Sp_n.

Corollary 5.3.4. *Let G be semisimple and p either separably good or very non-torsion for G. Then any elementary subalgebra $\mathcal{E}$ of $\mathfrak{g}$ may be conjugated into $\mathfrak{u}$.*

Proof. Per remark (a) following the above theorem (*loc. cit.*) observe that the first Frobenius kernel $G_1 \leq G$ is a closed normal subgroup isomorphic to the infinitesimal group scheme $\underline{\mathfrak{g}}$ that one obtains from the restricted Lie algebra $\mathfrak{g}$ [31, I.9.6] and the conjugation action $G \curvearrowright G_1$ corresponds to the adjoint action $G \curvearrowright \mathfrak{g}$. Let $\underline{\mathcal{E}} \subseteq \underline{\mathfrak{g}}$ be the subgroup scheme corresponding to an elementary subalgebra. As $\mathcal{E}$ consists entirely of nilpotent elements of $\mathfrak{g}$ we have, by Engel's theorem, that $\underline{\mathcal{E}}$ is a unipotent subgroup scheme of G. The above theorem then gives that $\underline{\mathcal{E}}$ may be conjugated into our choice of Borel B and therefore $\mathcal{E}$ may be conjugated into the Lie algebra of B. As $\mathcal{E}$ is nilpotent its conjugate must then lie in the nilpotent radical $\mathfrak{u}$ of this Lie algebra. □

Remark 5.3.5. Explicitly, the condition on p in corollary 5.3.4 amounts to p separably good or G of type G_2 and $p = 3$. The latter produces an interesting counter-example to the separability theorem lemma 5.3.8 which we explain in example 5.3.10.

Example 5.3.6. This conjugation property is not true for a general prime. One can check that when $p = 2$ the nilpotent cone of $\mathfrak{pgl}_2$ is a linear subspace of dimension 2. This is necessarily maximal among elementary subalgebras and is not contained in the Lie algebra of any Borel. Examples of maximal elementary subalgebras which are not contained in a Borel exist for $\mathfrak{pgl}_3$ when $p = 3$ (see Levy et al. [35]) and $\mathfrak{pgl}_4$ when $p = 2$. While it is true that for all n there exists an elementary subalgebra of $\mathfrak{pgl}_n$ which is not contained in a Borel, it is not known for larger n whether such an elementary subalgebra can be maximal.

Corollary 5.3.7. *Let G be semisimple and p separably good for G. Then*

$$\mathbb{E}(\mathfrak{g}) = \bigcup_{\substack{R \in \max(\Phi) \\ R \text{ an ideal}}} G \cdot \mathrm{Lie}(R).$$

Proof. Follows immediately from corollary 5.2.9 and the corollary above. □

5.3.2 Ideal orbits

Now we geometrically identify orbits of the form $G \cdot \mathrm{Lie}(R)$ where $R \in \max(\Phi)$. Observe that such $\mathcal{E} = \mathrm{Lie}(R)$ are fixed by B under the adjoint action (see Malle and Testerman [37, 15.4]). The stabilizer $P = \mathrm{Stab}_G(\mathcal{E})$ is then a standard parabolic and is generated by B and representatives $\dot{s}_i \in N_G(T)$ of some collection of simple roots s_i in the Weyl group $W = N_G(T)/C_G(T)$. As $\dot{s}_i \,\mathrm{Lie}(R) = \mathrm{Lie}(s_i R)$ we have $\dot{s}_i \in P$ if and only if $s_i \in \mathrm{Stab}_W(R)$. Thus P is identified by the information in Table 5.3 above.

The orbit map $\pi\colon G \to G \cdot \mathcal{E}$ factors to a bijective morphism $G/P \to G \cdot \mathcal{E}$ and we show below that this morphism is in fact an isomorphism. Note that the fact that the orbit map π induces an isomorphism $G/P \cong G \cdot \mathcal{E}$ is equivalent to π being separable. This, in turn, is equivalent to the condition that the kernel of the tangent map at the identity $\ker d\pi_1$ is contained in $\mathrm{Lie}(P)$ (see, for example, [10, 6.7]). We show that the latter holds in theorem 5.3.9.

Lemma 5.3.8. *If* $\pi\colon G \to \mathbb{E}(r, \mathfrak{g})$ *is the orbit map* $g \mapsto g \cdot \mathcal{E}$ *for some elementary subalgebra* $\mathcal{E} \in \mathbb{E}(r, \mathfrak{g})$ *then* $\ker \mathrm{d}\pi_1 = N_{\mathfrak{g}}(\mathcal{E})$.

Proof. Let $\underline{e} = (e_1, \ldots, e_r)$ be a basis of $\mathcal{E}$. As $\mathbb{E}(r, \mathfrak{g})$ is a closed subvariety in $\mathrm{Grass}(r, \mathfrak{g})$ we may consider the latter to be the codomain of π. This gives the following diagram

$$\begin{tikzcd}
 & (\mathfrak{g}^{\times r})^\circ \arrow[d, "\phi"] \\
G \arrow[ur, "\widetilde{\pi}", "g\mapsto g\cdot\underline{e}"'] \arrow[r, "\pi", "g\mapsto g\cdot\mathcal{E}"'] & \mathrm{Grass}(r, \mathfrak{g})
\end{tikzcd} \tag{5.3.8.1}$$

where $(\mathfrak{g}^{\times r})^\circ$ is the open subset of $\mathfrak{g}^{\times r}$ consisting of linearly independent r-tuples and $\phi\colon (\mathfrak{g}^{\times r})^\circ \to \mathrm{Grass}(r, \mathfrak{g})$ is the canonical projection. The map ϕ is, in particular, a GL_r-torsor and hence is locally trivial. Its tangent map at $\underline{e}$ can be identified as the linear map

$$\mathrm{d}\phi_{\underline{e}} : \mathrm{Hom}(\mathcal{E}, \mathfrak{g}) \to \mathrm{Hom}(\mathcal{E}, \mathfrak{g}/\mathcal{E})$$

induced by the projection $\mathfrak{g} \to \mathfrak{g}/\mathcal{E}$. Indeed, locally on the affine neighborhood defined by the non-vanishing of the Plucker coordinate associated with $\mathcal{E}$, the torsor trivializes as follows:

$$\begin{tikzcd}
\mathbb{A}^{r(n-r)} \times \mathrm{GL}_r \arrow[r, hook] \arrow[d, "\phi"] & \mathbb{A}^{rn} = \mathbb{M}_{r\times n} \simeq \mathrm{Hom}_k(\mathcal{E}, \mathfrak{g}) \arrow[d] \\
\mathbb{A}^{r(n-r)} \arrow[r, equal] & \mathrm{Hom}(\mathcal{E}, \mathfrak{g}/\mathcal{E}).
\end{tikzcd}$$

Replace diagram 5.3.8.1 with the corresponding diagram for tangent spaces:

$$\begin{tikzcd}
 & \mathfrak{g}^{\times r} \cong \mathrm{Hom}(\mathcal{E}, \mathfrak{g}) \arrow[d, "\mathrm{d}\phi_{\underline{e}}"] \\
\mathfrak{g} \arrow[ur, "g\mapsto([g,\underline{e}])", "\mathrm{d}\widetilde{\pi}_1"'] \arrow[r, "\mathrm{d}\pi_1"'] & \mathrm{Hom}(\mathcal{E}, \mathfrak{g}/\mathcal{E}).
\end{tikzcd}$$

We are interested in $\ker \mathrm{d}\pi_1 = \ker(\mathrm{d}\phi_1 \circ \mathrm{d}\widetilde{\pi}_1)$. Note that in diagram 5.3.8.1 the identification $\mathfrak{g}^{\times r} \cong \mathrm{Hom}(\mathcal{E}, \mathfrak{g})$ is given by sending an r-tuple $(g_1, \ldots, g_r)$ to the map $\mathcal{E} \to \mathfrak{g}$ defined by $e_i \mapsto g_i$. Hence, the kernel of the vertical map is $\mathcal{E}^{\times r} \subset \mathfrak{g}^{\times r}$. The map $\mathrm{d}\widetilde{\pi}_1$ is given by $g \mapsto ([g, e_1], \ldots, [g, e_r])$. To land in $\ker \mathrm{d}\phi_1 = \mathcal{E}^{\times r}$ we must have $[g, e_i] \in \mathcal{E}$ for any $1 \leq i \leq r$. Hence, $\mathrm{d}\phi_1 \circ \mathrm{d}\widetilde{\pi}_1(g) = 0$ if and only if $g \in N_{\mathfrak{g}}(\mathcal{E})$. □

Theorem 5.3.9. *If* $P = \mathrm{Stab}_G(\mathcal{E})$ *is parabolic then the orbit* $G \cdot \mathcal{E} \subseteq \mathbb{E}(\mathfrak{g})$ *is isomorphic to the flag variety* G/P.

Proof. By Borel [10, 6.7] and the above lemma this is equivalent to the statement that $N_{\mathfrak{g}}(\mathcal{E}) \subseteq \mathrm{Lie}(P)$. Without loss of generality we assume P is a standard parabolic, then $\mathcal{E}$ is fixed by the torus and therefore is a Chevalley subalgebra. The normalizer $N_{\mathfrak{g}}(\mathcal{E})$ is then a Chevalley subalgebra as well so it suffices to choose $\alpha \in \Phi$ and show that $x_\alpha \in N_{\mathfrak{g}}(\mathcal{E})$ implies $x_\alpha \in \mathrm{Lie}(P)$.

We assume that p is separably good for G so, in particular, p is greater or equal to the length of the longest root string in Φ. This implies that any structure constants which are zero in k are zero in $\mathbb{Z}$. We get then that the $\mathbb{Z}$-form of x_α normalizes the $\mathbb{Z}$-form of $\mathcal{E}$. The action of the root space U_α on $\mathfrak{g}$ is given by exponentiating the adjoint action of the $\mathbb{Z}$-form of x_α and then base changing to k, thus U_α stabilizes $\mathcal{E}$. As x_α spans the Lie algebra of U_α we have $x_\alpha \in \mathrm{Lie}(P)$ as desired. □

Example 5.3.10. To see how the above argument can fail when p is less than the maximal length of a root string in Φ consider G of type G_2 and $p = 3$. We saw in subsection 5.2.11 that $\mathcal{E} = \mathrm{Lie}(R_1)$ is a maximal elementary abelian. The stabilizer of $\mathcal{E}$ is the Borel B but one can check that in $\mathfrak{g}_{\mathbb{Z}}$ we have $[x_{-\alpha_1}, \mathcal{E}] = 3\mathbb{Z}x_{x_{\alpha_2}} + \mathcal{E}$, thus $x_{-\alpha_1}$ normalizes $\mathcal{E}$ in $\mathfrak{g}_{\mathbb{F}_3}$. If $M_0 \in \mathbb{M}_{14}(\mathbb{Z})$ is the matrix of $\operatorname{ad}(x_{-\alpha_1})$ in $\mathfrak{g}_{\mathbb{Z}}$ and $M_3 \in \mathbb{M}_{14}(\mathbb{F}_3)$ is its mod 3 image, i.e., the matrix of $\operatorname{ad}(x_{-\alpha_1})$ in $\mathfrak{g}_{\mathbb{F}_3}$, then $M_0^3 \neq 0$ but $M_3^3 = 0$. Thus even though $\exp(M_0)$ and $\exp(M_3)$ are both well defined and $\exp(M_3)$ stabilizes $\mathcal{E}$, the action of $U_{-\alpha_1}$ is given by the mod 3 image of $\exp(M_0)$ and this does not equal $\exp(M_3)$.

5.3.3 Majority case

Retaining the notation from the previous section we now consider G which is *not* of type A_2 or G_2.

Theorem 5.3.11. *Let G be a simple algebraic group, not of type A_2 or G_2. Assume that p is separably good for G. Then*

$$\mathbb{E}(\mathfrak{g}) = \coprod_{\substack{R\in\max(\Phi)\\ R \text{ an ideal}}} G/P_R,$$

where $P_R = \operatorname{Stab}_G(\operatorname{Lie}(R))$.

Proof. From corollary 5.3.7 the variety $\mathbb{E}(\mathfrak{g})$ is a union of orbits $G\cdot\operatorname{Lie}(R)$ where R ranges over the ideals in $\max(\Phi)$ and theorem 5.3.9 gives that each orbit $G\cdot\operatorname{Lie}(R)$ is isomorphic to the flag variety G/P_R. These orbits are therefore closed and we need only prove that they are distinct.

By the Bruhat decomposition two such $\operatorname{Lie}(R)$ are conjugate if and only if the corresponding R are conjugate via the Weyl group. At most one ideal in $\max(\Phi)$ is not of the form Φ_i^{rad} for some i so lemma 5.1.6 gives that distinct maximal commuting ideals are non-conjugate as desired. $\square$

Combining theorem 5.3.11 with Table 5.2 we get, except for types A_2 and G_2, the explicit type-by-type calculation of $\mathbb{E}(\mathfrak{g})$ found in Table 5.4.

Table 5.4: $\mathbb{E}(\mathfrak{g})$ for p separably good.

Type	Restrictions on rank	$\mathbb{E}(\mathfrak{g})$
A_{2n}	$n = 1$	Irreducible, 5-dimensional
	$n \geq 2$	$G/P_{\Delta\setminus\{\alpha_n\}} \coprod G/P_{\Delta\setminus\{\alpha_{n+1}\}}$
A_{2n+1}	$n \geq 0$	$G/P_{\Delta\setminus\{\alpha_{n+1}\}}$
B_n	$n = 2, 3$	$G/P_{\Delta\setminus\{\alpha_1\}}$
	$n = 4$	$G/P_{\Delta\setminus\{\alpha_1\}} \coprod G/P_{\{\alpha_2,\alpha_3\}}$
	$n \geq 5$	$G/P_{\Delta\setminus\{\alpha_1,\alpha_n\}}$
C_n	$n \geq 3$	$G/P_{\Delta\setminus\{\alpha_n\}}$
D_n	$n = 4$	$G/P_{\Delta\setminus\{\alpha_1\}} \coprod G/P_{\Delta\setminus\{\alpha_3\}} \coprod G/P_{\Delta\setminus\{\alpha_4\}}$
	$n \geq 5$	$G/P_{\Delta\setminus\{\alpha_{n-1}\}} \coprod G/P_{\Delta\setminus\{\alpha_n\}}$
E_6		$G/P_{\Delta\setminus\{\alpha_1\}} \coprod G/P_{\Delta\setminus\{\alpha_6\}}$
E_7		$G/P_{\Delta\setminus\{\alpha_7\}}$
E_8		$G/P_{\Delta\setminus\{\alpha_2\}}$
F_4		$G/P_{\{\alpha_1,\alpha_3\}}$
G_2		Irreducible, 8-dimensional

5.3.4 Type A_2

In type A_2 there are 3 disjoint orbits, two of which contain Chevalley subalgebras corresponding to ideals and one which does not contain a Chevalley subalgebra. The following are representatives of the three orbits

$$L_1 = \langle x_{\alpha_2}, x_{\alpha_1+\alpha_2} \rangle,$$

$$L_2 = \langle x_{\alpha_1}, x_{\alpha_1+\alpha_2} \rangle,$$

$$L_3 = \langle x_{\alpha_1} + x_{\alpha_2}, x_{\alpha_1+\alpha_2} \rangle.$$

The stabilizers of these subalgebras are

$$\begin{aligned}
\mathrm{Stab}_G(L_1) &= P_1,\\
\mathrm{Stab}_G(L_2) &= P_2,\\
\mathrm{Stab}_G(L_3) &= \langle \alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_2), U \mid \lambda_1^3 = \lambda_2^3\rangle,
\end{aligned}$$

and so the orbits have dimensions

$$\begin{aligned}
\dim(G\cdot L_1) &= 2,\\
\dim(G\cdot L_2) &= 2,\\
\dim(G\cdot L_3) &= 5.
\end{aligned}$$

As $G\cdot L_1 \simeq G/P_1$ and $G\cdot L_2 \simeq G/P_2$ they are closed orbits and $G\cdot L_3$ is open. The map $\mathbb{P}^1 \to \mathbb{E}(\mathfrak{g})$ given by $[a:b] \mapsto \langle ax_{\alpha_1} + bx_{\alpha_2}, x_{\alpha_1+\alpha_2}\rangle$ then yields that the closure of $G\cdot L_3$ contains the other two orbits. So $G\cdot L_3$ is dense, hence $\mathbb{E}(\mathfrak{g})$ is irreducible of dimension 5.

5.3.5 Type G_2

In type G_2 there are again 3 disjoint orbits, two of which contain Chevalley subalgebras and one which does not, but now only one orbit contains a Chevalley subalgebra corresponding to an ideal. The following are representatives of the three orbits

$$\begin{aligned}
L &= \langle x_{\alpha_2} + x_{3\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2}\rangle\\
\mathrm{Lie}(C_3) &= \langle x_{\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2}\rangle,\\
\mathrm{Lie}(C_5) &= \langle x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2}\rangle,
\end{aligned}$$

and C_5 is the ideal. The stabilizers of these subalgebras are

$$\begin{aligned}
\mathrm{Stab}_G(L) &= \langle \alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_2), U_\alpha \mid \lambda_1^2 = \lambda_2^3,\ \alpha \in \Phi^+ \setminus \{\alpha_1\}\rangle,\\
\mathrm{Stab}_G(\mathrm{Lie}(C_3)) &= \langle T, U_\alpha \mid \alpha \in \Phi^+ \setminus \{\alpha_1\}\rangle,\\
\mathrm{Stab}_G(\mathrm{Lie}(C_5)) &= P_2,
\end{aligned}$$

so the orbits have dimensions

$$\begin{aligned}\dim(G \cdot L) &= 8,\\ \dim(G \cdot \mathrm{Lie}(C_3)) &= 7,\\ \dim(G \cdot \mathrm{Lie}(C_5)) &= 5.\end{aligned}$$

As $\mathbb{E}(\mathfrak{g})$ is a closed subvariety of the Grassmannian it is complete. The Borel fixed point theorem then gives that $G \cdot \mathrm{Lie}(C_5)$ is the only closed orbit. Boundaries of orbits are unions of orbits of smaller dimension therefore the closure of $G \cdot \mathrm{Lie}(C_3)$ is its union with $G \cdot \mathrm{Lie}(C_5)$ and the orbit $G \cdot L$ is open. To see that $G \cdot \mathrm{Lie}(C_3)$ is not open we need that the closure of $G \cdot L$ contains $G \cdot \mathrm{Lie}(C_3)$; one sees this from the map $\mathbb{A}^1 \to \mathbb{E}(\mathfrak{g})$ defined by $a \mapsto \langle x_{\alpha_2} + a x_{3\alpha_1+\alpha_2}, x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2}\rangle$ which sends $\mathbb{A}^1 \setminus \{0\}$ into $G \cdot L$ and 0 to $\mathrm{Lie}(C_3)$. In particular, we have now shown that $G \cdot L$ is dense and so $\mathbb{E}(\mathfrak{g})$ is irreducible of dimension 8.

5.4 Applications to Chevalley groups

Let $k = \overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ and let G be a reductive k-group, defined and split over $\mathbb{Z}$ as above. Our calculation of $\mathbb{E}(\mathfrak{g})$ yields information about the maximal elementary abelian p-subgroups of the Chevalley groups $G(\mathbb{F}_q)$ for $q = p^r$ and their conjugacy classes. Since any such subgroup can be conjugated into the Sylow p-subgroup $U(\mathbb{F}_q)$, we can assume that G is semi-simple simply connected. Assume p is good for G, and let $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$ be the varieties of p-unipotent and p-nilpotent elements in G and $\mathfrak{g}$ respectively.

To translate between the Lie algebra and Chevalley group we use the theorem below.

Theorem 5.4.1 ([49, 4.3])**.** *There exists a unique isomorphism* $\phi\colon \mathcal{N}_1(\mathfrak{g}) \to \mathcal{U}_1(G)$ *satisfying*

1. ϕ *is* G*-equivariant,*
2. $x, y \in \mathfrak{g}$ *commute if and only if* $\phi(x), \phi(y) \in G$ *commute,*
3. $x \in \mathfrak{g}_{\mathbb{F}_q}$ *if and only if* $\phi(x) \in G(\mathbb{F}_q)$*.*

Such a ϕ gives an inclusion preserving bijection between p-nilpotent commutative subsets of $\mathfrak{g}_{\mathbb{F}_q}$ and p-unipotent commutative subsets of $G(\mathbb{F}_q)$. A maximal commuting set of p-unipotent elements in $G(\mathbb{F}_q)$ is necessarily a maximal elementary abelian subgroup. Similarly a maximal set of commuting p-nilpotent elements in $\mathfrak{g}_{\mathbb{F}_q}$ is necessarily a maximal elementary subalgebra, and therefore corresponds to a maximal elementary subalgebra of $\mathfrak{g}$ that is defined over $\mathbb{F}_q$. As the $\mathbb{F}_q$-rational points of the Grassmannian are exactly the subspaces defined over $\mathbb{F}_q$ we now have the following.

Theorem 5.4.2. *The map ϕ induces a $G(\mathbb{F}_q)$-equivariant bijection between the $\mathbb{F}_q$-rational points of $\mathbb{E}(\mathfrak{g})$ and the maximal elementary abelian subgroups of $G(\mathbb{F}_q)$.*

Even though we don't need this observation for our application to Chevalley groups, the following enhancement of corollary 5.3.4 is worthy of pointing out:

Corollary 5.4.3. *Let $\mathcal{E}$ be an $\mathbb{F}_q$-rational point of $\mathbb{E}(\mathfrak{g})$. Then $\mathcal{E}$ is $G(\mathbb{F}_q)$-conjugate to a subalgebra of $\mathfrak{u}$.*

Proof. The subalgebra $\mathcal{E}$ corresponds, using ϕ, to a maximal elementary abelian subgroup of $G(\mathbb{F}_q)$. As $U(\mathbb{F}_q)$ is a p-Sylow subgroup of $G(\mathbb{F}_q)$ an element $g \in G(\mathbb{F}_q)$ conjugates this elementary abelian into $U(\mathbb{F}_q)$. The equivariance of ϕ then gives that g conjugates $\mathcal{E}$ into $\mathfrak{u}$. ☐

Theorem 5.2.9 together with the calculation for A_2 in section 5.2 show that in all types except for G_2 if two elementary subalgebras in $\mathbb{E}(\mathfrak{u})$ are defined over $\mathbb{F}_q$ and conjugate by an element in $G(k)$ then they are already conjugate by an element in $G(\mathbb{F}_q)$. This observation, combined with theorem 5.4.2, allows us to translate the results for $\mathbb{E}(\mathfrak{u})$ from section 5.2 to the classification of conjugacy classes of maximal elementary abelian p-subgroups of $G(\mathbb{F}_q)$. Consequently, we recover the results of Barry [3] on maximal elementary abelian p-subgroups of Chevalley groups of classical type, and supplement Barry's results with similar information for the exceptional types.

Theorem 5.4.4. *Let G be a simple algebraic group defined and split over $\mathbb{Z}$. Assume p is good for G. Then the conjugacy classes and ranks of the elementary abelian p-subgroups of $G(\mathbb{F}_q)$ of maximal rank are given in Table 5.5, where in types E_8 and F_4 we take R to be the unique ideal in* $\max(\Phi)$.

Note that for G reductive, the representatives of conjugacy classes of maximal elementary abelian p-subgroups of $G(\mathbb{F}_q)$ are given by products of representatives for each simple entry in the direct product decomposition of the derived group $[G, G]$.

Example 5.4.5. In type G_2 we used the fact that k is algebraically closed in a nontrivial way in subsection 5.2.9 to conclude that $\mathbb{E}(\mathfrak{g})$ had three G-orbits. Thus we can only conclude that there are *at least* three conjugacy classes of maximal elementary abelian subgroups in $G(\mathbb{F}_q)$. In fact the number of conjugacy classes depends on q. For example, if $q = 5^r$ then $G(\mathbb{F}_q)$ has three conjugacy classes of elementary abelian subgroups when r is odd but has six such conjugacy classes when r is even.

As implied by the Quillen stratification theorem [43] the above calculation gives the number of irreducible components of maximal dimension for $\operatorname{Spec} H^*(G(\mathbb{F}_q), k)$.

Corollary 5.4.6. *Let G be a simple algebraic group defined and split over $\mathbb{Z}$. Assume p is good for G. Then the dimension of* $\operatorname{Spec} H^*(G(\mathbb{F}_q), k)$ *and the number of irreducible components of maximal dimension are given in Table 5.6.*

Proof. By Quillen [43] the dimension of $\operatorname{Spec} H^*(G(\mathbb{F}_q), k)$ equals the maximal rank of an elementary abelian p-subgroup of $G(\mathbb{F}_q)$. The maximal rank is the maximal order divided by p and, hence, can be read from Table 5.5. The Quillen stratification theorem implies that the number of irreducible components equals the number of conjugacy classes of maximal elementary abelian p-subgroups which once again can be read from Table 5.5. □

Table 5.5: Maximal elementary abelian subgroups of $G(\mathbb{F}_q)$, $q = p^r$.

Type	Restrictions on rank	# conjugacy classes	rank	Representatives
A_{2n}	$n=1$	3	$2r$	$\langle U_{\alpha_1}, U_{\alpha_1+\alpha_2}\rangle$ $\langle U_{\alpha_2}, U_{\alpha_1+\alpha_2}\rangle$ $\langle \phi(\mathbb{F}_q(x_{\alpha_1}+x_{\alpha_2})), U_{\alpha_1+\alpha_2}\rangle$
	$n \geq 2$	2	$n(n+1)r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{n+1}\rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{n}\rangle$
A_{2n+1}	$n \geq 0$	1	$(n+1)^2 r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{n+1}\rangle$
B_n	$n = 2, 3$	1	$(2n-1)r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{1}\rangle$
	$n = 4$	2	$7r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{1}\rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in S_1\rangle$
	$n \geq 5$	1	$\frac{1}{2}n(n-1)r + r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in S_1\rangle$
C_n	$n \geq 2$	1	$\frac{1}{2}n(n+1)r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{n}\rangle$
D_n	$n = 4$	3	$6r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{1}\rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{3}\rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{4}\rangle$
	$n \geq 5$	2	$\frac{1}{2}n(n-1)r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{n-1}\rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{n}\rangle$
E_6		2	$16r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{1}\rangle$ $\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{6}\rangle$
E_7		1	$27r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in \Phi^{\text{rad}}_{7}\rangle$
E_8		1	$36r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in R\rangle$
F_4		1	$9r$	$\langle U_\alpha(\mathbb{F}_q) \mid \alpha \in R\rangle$
G_2		≥ 3	$3r$	

Table 5.6: $\operatorname{Spec} H^*(G(\mathbb{F}_{p^r}), k)$.

Type	Restrictions on rank	# of irreducible components of max dimension	dimension
A_{2n}	$n = 1$	3	$2r$
	$n \geq 2$	2	$rn(n+1)$
A_{2n+1}	$n \geq 0$	1	$r(n+1)^2$
B_n	$n = 2, 3$	1	$r(2n-1)$
	$n = 4$	1	$7r - 1$
	$n \geq 4$	1	$\frac{1}{2}rn(n-1) + r$
C_n	$n \geq 2$	1	$\frac{1}{2}n(n+1)r$
D_n	$n = 4$	3	$6r$
	$n \geq 5$	2	$\frac{1}{2}n(n-1)r$
E_6		2	$16r$
E_7		1	$27r$
E_8		1	$36r$
F_4		1	$9r$
G_2		≥ 3	$3r$

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Appendix A

MAXIMAL SETS OF COMMUTING ROOTS

In the Appendix we give some details for the description of the maximal subsets of commuting roots found in Table 5.2 and the stabilizers of certain ideals in Table 5.3. Note that the maximal subsets of commuting roots are computed in Malcev [36] except that he skips the proof for E_8 and the paper is in Russian.

We begin with the computation of maximal subsets of commuting roots. For type E, D_n when $n < 7$, and B_n when $n < 5$ we use a computer program which we have made available online [52]. For the remaining types we provide the following arguments.

A.0.1 Type A_n

One can check, as in Grantcharov and Serganova [27], that sending $J \subseteq \{1, 2, \ldots, n+1\}$ to the set of roots $\{\epsilon_i - \epsilon_j \mid i \in J, j \notin J\}$ yields a bijection between proper nontrivial subsets of $\{1, 2, \ldots, n+1\}$ and sets of inclusion maximal commutative subsets of Φ. As J gets sent to a set of size $|J|(n+1-|J|)$ we see that this set is of maximal order when $n = 2m$ and $|J| = m, m+1$ or when $n = 2m+1$ and $|J| = m+1$. It is a set of positive roots if and only if $J < \{1, \ldots, n+1\} \setminus J$, thus in type A_{2m} we have $J = \{1, \ldots, m\}$ or $\{1, \ldots, m+1\}$ yielding Φ_m^{rad} and $\Phi_{m+1}^{\mathrm{rad}}$, respectively, and in type A_{2m+1} we have $\{1, \ldots, m+1\}$ yielding $\Phi_{m+1}^{\mathrm{rad}}$.

A.0.2 Type B_n

We assume $n \geq 5$. The set $R = \{\epsilon_i \mid 1 \leq i \leq n\}$ is an inclusion maximal set of non-commuting roots so any maximal set of commuting roots in Φ^+ consists of a maximal set of commuting roots in $\Phi^+ \setminus R$ together with at most one element from R. Observe that $\Psi = \Phi \setminus \pm R$ is a root system of type D_n with simple roots $\{\alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\}$. The maximal set Ψ_n^{rad} can commute with any ϵ_i and yields S_i. The maximal set $\Psi_{n-1}^{\mathrm{rad}}$ commutes with ϵ_i when $i < n$ and yields S_i^*.

A.0.3 Type C_n

Sending $J \subseteq \{1, \ldots, n\}$ to the set

$$\phi(J) = \{\epsilon_i + \epsilon_{i'}, \epsilon_i - \epsilon_j, -\epsilon_j - \epsilon_{j'} \mid i, i' \in J \text{ and } j, j' \notin J\}$$

gives a bijection ϕ between the power set of $\{1, \ldots, n\}$ and inclusion maximal unipotent commuting subsets of Φ. Among those subsets J satisfying $|J| = m$, the number of positive roots in $\phi(J)$ attains a maximum of $\frac{1}{2}m(m+1) + m(n-m)$ when $J < \{1, \ldots, n\} \setminus J$ and this maximum value for a given m attains a maximum of $\frac{1}{2}n(n+1)$ when $m = n$. Thus we take the positive roots of $\phi(\{1, \ldots, n\})$ and get Φ_n^{rad}.

A.0.4 Type D_n

We assume $n \geq 7$.

Lemma A.0.7. *Let Φ be type D_n. If $R \subseteq \Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}}$ is an inclusion maximal set of commuting roots which contains $\epsilon_1 - \epsilon_2$ then $R = \Phi_1^{\mathrm{rad}}$ has order $2n-2$. If it does not contain $\epsilon_1 - \epsilon_2$ then it consists of the root $\epsilon_1 + \epsilon_2$ together with one choice of root from each of the sets $\{\epsilon_1 + \epsilon_r, \epsilon_2 - \epsilon_r\}_{2<r\leq n}$ and $\{\epsilon_1 - \epsilon_r, \epsilon_2 + \epsilon_r\}_{2<r\leq n}$, and hence has order $2n-3$.*

Proof. We have $\Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}} = \{\epsilon_1 \pm \epsilon_i, \epsilon_2 \pm \epsilon_j \mid 2 \leq i \leq n \text{ and } 3 \leq j \leq n\}$. If $\epsilon_1 - \epsilon_2$ is contained in our maximal set then the roots $\epsilon_2 \pm \epsilon_j$ are not, so the set contains at most the roots $\epsilon_1 \pm \epsilon_i$, i.e., the roots of Φ_1^{rad}. These indeed commute and there are $2n-2$ of them. If $\epsilon_1 - \epsilon_2$ is not contained in our maximal set then note that $\epsilon_1 + \epsilon_2$ is the longest root and therefore is contained in any inclusion maximal set of commuting roots. The remaining roots form the sets of non-commuting pairs given in the statement. One sees that roots from distinct pairs commute and there are $2n-4$ such pairs. $\square$

Observe that $m(\Phi) \geq |\Phi_n^{\mathrm{rad}}| = \frac{1}{2}n(n-1)$. Also Φ_1^{rad} is inclusion maximal and of smaller order so no element of $\max(\Phi)$ contains Φ_1^{rad}. Now $\Psi = \Phi \setminus \pm\Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}}$ is a root system of type D_{n-2} with simple roots $\{\alpha_3, \ldots, \alpha_n\}$. Every set of commuting roots in Φ^+ is the union of sets of commuting roots from Ψ^+ and $\Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}}$. By the lemma above the set of commuting roots from $\Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}}$ can have at most $2n-3$ elements and by induction the set from Ψ^+ can have at most $\frac{1}{2}(n-2)(n-3)$. These sum to the order of Φ_n^{rad} so a maximal set of commuting roots must be the union of a maximal set from Ψ^+ and a set of order $2n-3$ from $\Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}}$.

Now it suffices to take $R \in \max(\Psi)$ and check which roots in $\Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}}$ it commutes with. If $R = \Psi_n^{\mathrm{rad}}$ then $\epsilon_i + \epsilon_n \in R$ for all $2 < i < n$ so $\epsilon_1 - \epsilon_j, \epsilon_2 - \epsilon_j \notin R$ for all $2 < j \leq n$. This identifies a unique inclusion maximal set of commuting roots in $\Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}}$ and it's union with Ψ_n^{rad} is Φ_n^{rad}. If $R = \Psi_{n-1}^{\mathrm{rad}}$ then $\epsilon_i - \epsilon_n \in R$ for all $2 < i < n$ so $\epsilon_1 - \epsilon_i, \epsilon_2 - \epsilon_i, \epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_n \notin R$. Again this identifies the inclusion maximal set in $\Phi_{\alpha_1,\alpha_2}^{\mathrm{rad}}$ and it's union with $\Psi_{n-1}^{\mathrm{rad}}$ is $\Phi_{n-1}^{\mathrm{rad}}$.

A.0.5 Type F_4

Recall that the roots of F_4 are

$$\pm\epsilon_i, \pm\epsilon_i \pm \epsilon_j, \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4).$$

We will denote positive roots of the last type by $\epsilon_{ijk} = \frac{1}{2}(\epsilon_1 + i\epsilon_2 + j\epsilon_3 + k\epsilon_4)$ where $i, j, k \in \{\pm 1\}$ and will write, for example, ϵ_{+-+} instead of $\epsilon_{1,-1,1}$.

Lemma A.0.8. *If $\epsilon_{ijk} \neq \epsilon_{i'j'k'}$ then ϵ_{ijk} and $\epsilon_{i'j'k'}$ commute if and only if there is exactly one sign change between (i, j, k) and (i', j', k'). In particular, a commuting set of roots can have at most 2 roots of the form ϵ_{ijk}.*

Proof. Observe that the terms in $\epsilon_{ijk} + \epsilon_{i'j'k'}$ are exactly those e_t for which the sign did not change (including ϵ_1). If there are 1 or 2 such terms then $\epsilon_{ijk} + \epsilon_{i'j'k'}$ is a root. As the roots are distinct there cannot be 4 such terms, therefore for ϵ_{ijk} and $\epsilon_{i'j'k'}$ to commute there must be 3 such terms, hence exactly one sign change.

If (i,j,k), (i',j',k'), and (i'',j'',k'') are mutually distinct and there is exactly one sign change from (i,j,k) to (i',j',k') and (i'',j'',k'') then there are 2 sign changes from (i',j',k') to (i'',j'',k''). This proves that 3 roots of the form (i,j,k) cannot pairwise commute. $\square$

Now observe that the roots $\pm\epsilon_i$ and $\pm\epsilon_i \pm \epsilon_j$ give $B_4 \subseteq F_4$. As the commuting property is preserved when intersecting with a subroot system the lemma above gives that a maximal set of commuting roots in F_4 can have at most 9 roots: 7 from a maximal set in B_4 plus 2 additional roots of the form ϵ_{ijk}. This maximum is indeed attained so every maximal set of commuting positive roots in F_4 is identified by a tripel $(C, ijk, i'j'k')$, where $C \subseteq B_4$ is a maximal set of commuting roots and ϵ_{ijk} and $\epsilon i'j'k'$ are the two additional roots. We now compute all the possibilities.

$C = \Phi_1^{\text{rad}}$

Every ϵ_{ijk} commutes with $C = \{\epsilon_1, \epsilon_1 \pm \epsilon_i \mid i = 2,3,4\}$. Once (i,j,k) are chosen there are three tripels (i',j',k') which differ by a single sign change. This gives 24 ordered pairs of roots $(\epsilon_{ijk}, \epsilon_{i'j'k'})$ that can be added, thus 12 possible sets of roots for this case.

$C = S_t$

We have $C = \{\epsilon_t, \epsilon_i + \epsilon_j \mid 1 \leq i < j \leq 4\}$ and for ϵ_{ijk} to commute with ϵ_t the sign on ϵ_t must be positive. We cannot have more than one negative sign in (i,j,k) otherwise ϵ_{ijk} would not commute with some root of the form $\epsilon_i + \epsilon_j$. Thus the two roots of the form ϵ_{ijk} must be ϵ_{+++} and ϵ_{ijk} where there is exactly one negative sign in (i,j,k) and this negative sign is not on the ϵ_t term.

Thus for $C = S_1$ we get three maximal sets corresponding to the three choices for a negative sign and for $C = S_2, S_3, S_4$ we get two maximal sets each. This gives 9 possible sets of roots for this case.

$C = S_t^*$

We have $C = \{\epsilon_t, \epsilon_i + \epsilon_j, \epsilon_{i'} - \epsilon_4 \mid 1 \leq i < j < 4, 1 \leq i' < 4\}$. Because of the $\epsilon_{i'} - \epsilon_4$ terms the only ϵ_{ijk} with a single negative that commutes with C is ϵ_{++-}. As ϵ_{--+} and ϵ_{---} don't commute with $\epsilon_2 + \epsilon_3$ we find that the two additional elements must be ϵ_{++-} and ϵ_{+++} or ϵ_{++-} and $\epsilon ij-$ where exactly one of i, j is negative and the negative is not on the ϵ_t term.

The first choice is valid for any t. For the second when $C = S_1$ there are two choices for the additional negative and when $C = S_2, S_3$ there is one choice for the additional negative. This gives 7 possible sets of roots for this case.

A.0.6 Type G_2

There are 3 short and 3 long positive roots. No pair of short positive roots commute so there can be at most 1 short root in a maximal commuting set. The pair of long roots $(\alpha_2, 3\alpha_1 + \alpha_2)$ does not commute so a maximal set contains the highest root $3\alpha_1 + 2\alpha_2$ together with at most 1 other long and 1 short root. With this one can check that the maximal sets are

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},$$
$$\{\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},$$
$$\{\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},$$
$$\{\alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\},$$
$$\{2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$