AN ANNIHILATING – BRANCHING PARTICLE MODEL FOR THE HEAT EQUATION WITH AVERAGE TEMPERATURE ZERO

Krzysztof Burdzy and **Jeremy Quastel**

Abstract. We consider two species of particles performing random walks in a domain in $\mathbb{R}^d$ with reflecting boundary conditions, which annihilate on contact. In addition there is a conservation law so that the total number of particles of each type is preserved: When the two particles of different species annihilate each other, particles of each species, chosen at random, give birth. We assume initially equal numbers of each species and show that the system has a diffusive scaling limit in which the densities of the two species are well approximated by the positive and negative parts of the solution of the heat equation normalized to have constant L^1 norm. In particular, the higher Neumann eigenfunctions appear as asymptotically stable states at the diffusive time scale.

1. Introduction.

A branching particle system representation for the heat equation solution with positive temperature was introduced in [BHIM] and later studied in [BHM] (see also [GK1] and [GK2]). Here is an informal description of that model and one of the main results, proved in [BHM]. Suppose that D is an open set in $\mathbb{R}^d$ and N Brownian particles move independently inside D. Whenever one of these particles hits the boundary of D, it is killed and one of the other particles, randomly chosen, splits into two particles, so that the number N of particles remains constant. When the number of particles goes to infinity and the initial (normalized) distribution of particles converges to a measure on D then the particle density

---

Research partially supported by NSF grant DMS-0303310 (KB) and NSERC (JQ).

converges to the solution of the heat equation for every time $t \geq 0$, with the appropriate initial condition, normalized so that it has a constant total mass for all times $t \geq 0$.

The main purpose of this article is to study a related model that involves two different types of particles. We call them $+$ and $-$ particles. The two types of particles perform independent symmetric random walks with reflection on the boundary, and annihilate each other on contact. When two particles of different signs annihilate each other then two other particles are chosen randomly, one $+$ particle and one $-$ particle, and each one of these particles splits into two particles of the same type as the parent particle, so that the numbers of $+$ and $-$ particles are conserved.

The reader might have noticed that we have changed the Brownian particles to symmetric random walks. This is because except in dimension 1, the Brownian particles would not meet, so one would have to have them annihilate when they came within some $\varepsilon > 0$ of each other. So we might as well just put the particles on a discrete lattice. Hence, particles in our new model will be represented by random walks on the lattice $\varepsilon\mathbb{Z}^d \cap D$ with ε^{-d} comparable to the number of particles. We will assume that particles reflect from the boundary of D in a way to be described precisely later.

Our main result says that when the numbers of $+$ and $-$ particles are equal, say N, and $N \to \infty$, then the densities of the two types of particles converge to the positive and negative parts of the solution of the heat equation normalized to have the correct total masses.

We consider here only the case when the numbers of $+$ and $-$ particles are equal. When the numbers of the two types of particles are not equal there should still be a hydrodynamic limit, but the expected limiting equation is far less simple. The same holds when there are more than two species of particles. See [BHIM], [CBH] and [CMBH] for details. The last two articles contain a large number of non-rigorous results based on simulations and informal calculations. They are a rich source of conjectures for mathematicians. These problems are quite difficult because unlike most problems in hydrodynamic limits, the invariant measures of these systems are unknown. Of course, well inside the regions where each individual species resides one expects a product of Poisson distributions. However, it is not apriori clear that such regions exist, and anyway, all the interesting behavior in the system is at the boundaries between the species.

2. Preliminaries.

We collect in this section a few results that may be known but we could not find a ready reference for them.

We start with a discussion of a single particle model—reflected random walk on a lattice region approximating a region $D \subset \mathbb{R}^d$. Let D be a bounded connected open set in $\mathbb{R}^d$, $d \geq 2$, and let $D_\varepsilon = \varepsilon\mathbb{Z}^d \cap D$ for $\varepsilon > 0$. We want to construct a nearest-neighbor continuous time random walk W_t^ε on D_ε. Moreover, we want W_t^ε to converge to reflecting Brownian motion in D as $\varepsilon \to 0$. A natural choice for W_t^ε would be a process that jumps to one of its neighbors with equal probabilities, after a suitable time delay. In the main part of our project, such a process is somewhat inconvenient to deal with. We will use a different model for a "reflecting random walk" on D_ε, described below.

The process W_t^ε is a finite state continuous time Markov process so it is fully described by the jump distribution and holding time at each state. We start by describing its jump distribution. Let ∂D_ε be the set of $x \in D_\varepsilon$ which have fewer than $2d$ neighbors in D_ε. If $x \in D_\varepsilon \setminus \partial D_\varepsilon$ then the random walk W_t^ε will jump from x to any of its neighbors with equal probabilities.

We will assume from now on that D has an analytic boundary. This is used later to get some easy estimates on the lattice Laplacian of eigenfunctions of the true Laplacian. One expects the results of this article to hold with much weaker assumptions on the boundary, but we have not pursued this here.

If ε is very small and $x \in \partial D_\varepsilon$ has fewer than d neighbors in D_ε then this implies that the normal vector to the boundary close to x is almost parallel to one of the axes. It is not hard to see that by removing from D_ε all $x \in \partial D_\varepsilon$ with fewer than d neighbors in D_ε, we obtain a set D'_ε with the property that all $x \in \partial D'_\varepsilon$ have at least d neighbors in D'_ε. By abuse of notation, we will refer to this set as D_ε. Note that every point in ∂D_ε lies at a distance from ∂D not exceeding 2ε

For $x \in \partial D_\varepsilon$ let $x' \in \partial D$ be the closest point to x on ∂D and let $\mathbf{n}(x')$ be the inward unit normal vector at x'. Let $\{\mathbf{e}_1(x') \stackrel{\text{df}}{=} \mathbf{n}(x'), \mathbf{e}_2(x'), \dots, \mathbf{e}_d(x')\}$ be an orthonormal basis, depending on x. Let p_{xy} denote the probability that W^ε makes the next jump from x to y (assuming it is at x now) and note that, by assumption,

$$\sum_{y \in D_\varepsilon, |x-y|=\varepsilon} p_{xy} = 1. \tag{2.1}$$

We want to find p_{xy} for $x \in \partial D_\varepsilon$ so that for some $c_1 > 0$,

$$\sum_{y \in D_\varepsilon, |x-y|=\varepsilon} p_{xy}(y - x) = c_1 \mathbf{n}(x'). \tag{2.2}$$

Note that equations (2.2) impose only $d-1$ constraints because $c_1 > 0$ is arbitrary. Hence, (2.1) and (2.2) effectively form a set of only d equations, and we have at least d unknowns p_{xy}. It is not hard to see that (2.1)-(2.2) have a solution.

Next we choose the holding times. For $x \in D_\varepsilon \setminus \partial D_\varepsilon$ we let the holding time at x have the mean $h_\varepsilon(x) = \varepsilon^2$. This corresponds to the usual space-time scaling for Brownian motion. For $x \in \partial D_\varepsilon$, the mean $h_\varepsilon(x)$ of the holding time at x is chosen so that

$$\lim_{s \downarrow 0} \frac{1}{s} \sum_{1 \le j \le d} E(((W^\varepsilon_{t+s} - W^\varepsilon_t) \cdot \mathbf{e}_j(x'))^2 \mid W^\varepsilon_t = x) = 1. \tag{2.3}$$

The discrete Laplacian Δ_ε, i.e., the generator of the process W^ε is given by

$$\Delta_\varepsilon f(x) = h_\varepsilon^{-1}(x) \sum_{y \in D_\varepsilon, |y-x| = \varepsilon} p_{xy}(f(y) - f(x)). \tag{2.4}$$

The following lemma is not used later in the paper but we state it with an informal proof to provide an intuitive basis for the interpretation of our main results.

Lemma 2.1. *Suppose that $x_\varepsilon \in D_\varepsilon$ and $x_\varepsilon \to x_0 \in \overline{D}$ as $\varepsilon \to 0$. If $W^\varepsilon_0 = x_\varepsilon$, then $\{W^\varepsilon_t, t \ge 0\}$ converge weakly to the reflecting Brownian motion on $\overline{D}$ starting from x_0, as $\varepsilon \to 0$.*

Proof (Sketch). Let $\{V^\varepsilon_n, n \ge 0\}$ be the skeletal discrete time random walk on D_ε, corresponding to W^ε_t. It is easy to see that it will suffice to show that $\widetilde{V}^\varepsilon_t \stackrel{\text{df}}{=} V^\varepsilon_{[t/\varepsilon^2]}$ converge to reflecting Brownian motion. Consider a small $c_1 > 0$ and choose $r > 0$ so small that for any $x', z' \in \partial D$ with $|x' - z'| < 2r$, we have $|\mathbf{n}(x') - \mathbf{n}(z')| < c_1$. Let τ_A denote the hitting time of a set A. If $B(x_0, r) \subset D$ then obviously $\{\widetilde{V}^\varepsilon_t, t \le \tau_{B(x_0,r)^c}\}$ converge weakly to Brownian motion starting from x_0, stopped at the hitting time $B(x_0, r)$.

Suppose that $B(x_0, r) \cap \partial D \ne \emptyset$. Let

$$\begin{aligned} V^{\varepsilon,1}_n &= \sum_{j \le n-1} (V^\varepsilon_{j+1} - V^\varepsilon_j) \mathbf{1}_{V^\varepsilon_j \notin \partial D_\varepsilon}, \\ V^{\varepsilon,2}_n &= \sum_{j \le n-1} (V^\varepsilon_{j+1} - V^\varepsilon_j) \mathbf{1}_{V^\varepsilon_j \in \partial D_\varepsilon}. \end{aligned}$$

The process $V^{\varepsilon,1}_j$ will make on the order of ε^{-2} steps before exiting $B(x_0, r)$. Since $V^{\varepsilon,2}_n$ has a non-vanishing drift in one direction (when its increment is not identically zero), it can make only on the order of ε^{-1} steps before exiting $B(x_0, r)$. Hence, $\widetilde{V}^{\varepsilon,1}_t \stackrel{\text{df}}{=} V^{\varepsilon,1}_{[t/\varepsilon^2]}$ converges to a Brownian motion and $\widetilde{V}^{\varepsilon,2}_t \stackrel{\text{df}}{=} V^{\varepsilon,2}_{[t/\varepsilon^2]}$ converges to a process with finite variation. The direction of increase of the limit of $\widetilde{V}_t^{\varepsilon,2}$ is within c_1 of $\mathbf{n}(x_0')$, as long as the process stays within $B(x_0, r)$. We can cover ∂D with a finite number of balls such that the normal vectors to ∂D are within arbitrarily small constant $c_1 > 0$ inside each of the balls. Splitting the random walk path into pieces so that every piece is contained in one of these balls, we can prove that the limit of $\widetilde{V}_t^{\varepsilon}$ is the sum of a Brownian motion and a process with finite variation which increases only when the process is on the boundary of D, and the direction of increase is always perpendicular to ∂D. This characterizes the reflecting Brownian motion in $\overline{D}$. $\square$

We have assumed that D has has an analytic boundary. This implies that D has a discrete spectrum for the Laplacian with Neumann boundary conditions. In the Dirichlet case, the spectrum is discrete for any bounded open connected set D. In the Neumann case, the spectrum is not necessarily discrete if D is bounded but has a non-smooth boundary, see, e.g., [HSS]. It is quite easy to see that the spectrum is discrete for the Neumann Laplacian when ∂D can be represented locally by a Lipschitz function (see, e.g., Section 2 of [BB]). This condition on the boundary can be substantially relaxed—it is enough to suppose that the boundary is locally represented by a function which is Hölder continuous with sufficiently small exponent, see, e.g., [NS]. As we said, our assumptions on the smoothness of ∂D are much stronger than that, for technical reasons.

Let $-\lambda_n$, $n = 0, 1, 2, \dots$ be the eigenvalues of the Laplacian on D with the Neumann boundary conditions, and let ϕ_n be the corresponding eigenfunctions. We list the same eigenvalue more than once, if it has a multiplicity greater than 1. For a measure μ on $\overline{D}$, we let $\hat{\mu}_n = \int_{\overline{D}} \phi_n d\mu$.

The following review of random measures is based on Section 3 of [D]. A sequence of finite measures μ^n on a space Λ converges weakly to μ if for any bounded continuous function f we have $\int_\Lambda f(x) d\mu^n(x) \to \int_\Lambda f(x) d\mu(x)$ as $n \to \infty$. Let $M_F(\Lambda)$ denote the space of all finite measures on Λ equipped with the topology of weak convergence, let $M_1(\Lambda)$ be the subspace of $M_F(\Lambda)$ consisting of probability measures, and let $M_{F,c}(\Lambda)$ denote the subspace of $M_F(\Lambda)$ consisting of measures with total variation less than or equal to c. Let $\mathcal{B}(\overline{D})$ denote the family of Borel subsets of $\overline{D}$. Suppose that for some probability space $(\Omega, \mathcal{F}, P)$, the function $\mu : \mathcal{B}(\overline{D}) \times \Omega \to \mathbb{R}$ has the following two properties, (i) for a fixed ω, $\mu(\,\cdot\,, \omega)$ is a finite measure on $\overline{D}$, and (ii) for a fixed $A \in \mathcal{B}(\overline{D})$, $\mu(A, \,\cdot\,)$ is a random variable. Then the distribution of μ is an element of $M_1(M_F(\overline{D}))$. We will refer to μ as a "random measure." The space of right-continuous functions with left limits, f :

$(0,\infty) \to \Lambda$, equipped with the Skorohod topology (see Section 3.6 of [D]) will be denoted by $S((0,\infty),\Lambda)$. In this article, we will be concerned with the convergence of processes on the open half-line $(0,\infty)$, not the usual semi-closed half-line $[0,\infty)$. The convergence in the Skorohod topology on $(0,\infty)$ is defined as the convergence in the Skorohod topology on every compact subset of $(0,\infty)$.

Lemma 2.2. *Suppose that $f : \overline{D} \to \mathbb{R}$ is a continuous (and hence bounded) function and let $a_n = \int_{\overline{D}} f(x)\phi_n(x)dx$. Suppose that $c_1 < \infty$ and μ^k, $k \geq 1$, are random measures (possibly defined on different probability spaces), with distributions in $M_1(M_{F,c_1}(\overline{D}))$. Assume that for every fixed n, $\lim_{k\to\infty} \hat{\mu}_n^k = a_n$ in distribution. Then the distributions of μ^k converge weakly in $M_1(M_F(\overline{D}))$ to δ_μ, where $\mu(dx) = f(x)dx$.*

Proof. Let $d\nu^k(x) = d\mu^k(x) - f(x)dx$ and note that $\hat{\nu}_n^k \to 0$ in distribution as $k \to \infty$, for every n. It will suffice to show that the distributions of ν^k converge weakly in $M_1(M_F(\overline{D}))$ to $\delta_{\mathbf{0}}$, where $\mathbf{0}$ is the measure identically equal to 0. Since $\overline{D}$ is compact, so is $M_{F,c_1}(\overline{D})$ and it follows that the sequence ν^k is tight and contains a convergent subsequence. Let ν be the weak limit of a subsequence of ν^{k_j}. It will be enough to show that $\nu = \delta_{\mathbf{0}}$.

Let $G_n : M_F(\overline{D}) \to \mathbb{R}$ be defined by $G_n(\sigma) = \left|\int_{\overline{D}} \phi_n(x)d\sigma(x)\right| \wedge 1$. The functionals G_n are continuous and bounded, and distributions of ν^{k_j} converge weakly to ν in $M_1(M_F(\overline{D}))$, so for every fixed n, $EG_n(\nu^{k_j}) \to EG_n(\nu)$ as $j \to \infty$. By assumption, $\hat{\nu}_n^k \to 0$ in distribution for every n, so $EG_n(\nu) = \lim_{j\to\infty} EG_n(\nu^{k_j}) = 0$ for every n. Hence, ν is supported on measures $\sigma \in M_F(\overline{D})$ with the property that $\hat{\sigma}_n = 0$ for all n. It will suffice to show that every measure with this property and finite total variation is identically equal to 0.

Fix a non-random measure σ on $\overline{D}$ with a finite total variation and such that $\hat{\sigma}_n = 0$ for all n. Fix any Borel set $A \subset \overline{D}$. It will be enough to show that $\int_{\overline{D}} \mathbf{1}_A(x)d\sigma(x) = 0$.

According to the Weyl formula, $\lambda_n \sim n^{2/d}$ (see [NS] for a recent strong version of this theorem). By Theorem 1 of [G], $\|\phi_n\|_\infty \leq c_2\lambda_n^{(d-1)/4}$, so $\|\phi_n\|_\infty \leq c_3 n^{(d-1)/2d}$.

Let P_t be the transition semigroup for the reflected Brownian motion in $\overline{D}$ and for some fixed $t > 0$, let $g(x) = P_t\mathbf{1}_A(x)$. Then $g(x) = \sum_n (\hat{\mathbf{1}}_A)_n \phi_n(x) e^{-\lambda_n t}$, where $(\hat{\mathbf{1}}_A)_n = \int_{\overline{D}} \mathbf{1}_A(x)\phi_n(x)dx$. If we write $g(x) = \sum_n \hat{g}_n\phi_n(x)$, then

$$|\hat{g}_n| \leq |D|\|\phi_n\|_\infty e^{-\lambda_n t} \leq c_4 n^{(d-1)/2d} e^{-c_5 n^{2/d} t}.$$

Without loss of generality assume that the total variation of σ is not greater than 1 and note that,

$$\int_{\overline{D}} |\phi_n(x)|d|\sigma(x)| \leq |D|\|\phi_n\|_\infty \leq c_6 n^{(d-1)/2d}.$$

This and other estimates obtained so far imply that,

$$\sum_n \int_{\overline{D}} |\hat{g}_n||\phi_n(x)|d|\sigma(x)| \leq \sum_n c_4 n^{(d-1)/2d} e^{-c_5 n^{2/d} t} c_6 n^{(d-1)/2d} < \infty.$$

Hence, we can change the order of integration and summation in the following formula,

$$\int_{\overline{D}} g(x) d\sigma(x) = \int_{\overline{D}} \sum_n \hat{g}_n \phi_n(x) d\sigma(x) = \sum_n \hat{g}_n \int_{\overline{D}} \phi_n(x) d\sigma(x) = 0.$$

We have proved that $\int_{\overline{D}} P_t \mathbf{1}_A(x) d\sigma(x) = 0$ for every $t > 0$. Clearly, $|P_t \mathbf{1}_A(x)| \leq 1$ for all x, and $P_t \mathbf{1}_A(x) \to \mathbf{1}_A(x)$ for almost every $x \in \overline{D}$, so by the dominated convergence,

$$\int_{\overline{D}} \mathbf{1}_A(x) d\sigma(x) = \lim_{t \to 0} \int_{\overline{D}} P_t \mathbf{1}_A(x) d\sigma(x) = 0. \qquad \square$$

The following result is completely elementary so we leave its proof to the reader.

Lemma 2.3. *Suppose that for every $\varepsilon > 0$, we have a real valued process $\{R^\varepsilon(t), t > 0\}$ which is equal to $R_1^\varepsilon(t) + R_2^\varepsilon(t) + R_3^\varepsilon(t)$, and these processes satisfy the following conditions.*

(i) For every $\varepsilon > 0$, $t \to R_1^\varepsilon(t)$ is right-continuous and non-decreasing a.s.

(ii) For every fixed $t > 0$, the family $\{R_1^\varepsilon(t), \varepsilon > 0\}$ is tight.

(iii) For every fixed $t > 0$, $\sup_{0 \leq s \leq t} |R_2^\varepsilon(s)|$ converges to 0 in distribution, as $\varepsilon \to 0$.

(iv) For every $t > 0$,

$$\limsup_{\delta_1, \delta_2 \to 0} \limsup_{\varepsilon \to 0} P\left(\sup_{0 \leq t_1, t_2 \leq t, |t_1 - t_2| \leq \delta_1} |R_3^\varepsilon(t_2) - R_3^\varepsilon(t_1)| \geq \delta_2 \right) = 0.$$

Then the family $\{R^\varepsilon, \varepsilon > 0\}$ is tight in $M_1(S((0,\infty), \mathbb{R}))$.

3. Main results.

We will now describe the main object of our study, a particle system. Our description will be partly informal because this will make it more accessible without loss of rigor.

The state of the particle system at time $t \geq 0$ will be encoded as an integer-valued random function $\eta(t) = \eta_x(t)$ on D_ε. We will often suppress t in the notation and write η_x, with the convention that if $\eta(x) = \eta_x > 0$ then there are η_x particles of type $+$ at $x \in D_\varepsilon$, and $\eta_x < 0$ signifies the presence of η_x particles of type $-$ at x. Obviously, $\eta_x = 0$ means that there are no particles at x. Since $+$ and $-$ particles annihilate each other, we do not have to have a notation representing both types of particles at the same site of D_ε. We assume that there are N particles of type $+$ and N particles of type $-$ for every t.

The easiest way to describe the evolution of η is to use the particle picture. Each particle performs continuous time symmetric simple random walk (defined in Section 2), independent of other particles, until one of the particles hits a particle of the other type. When a particle jumps to a site occupied by at least one particle of the different sign, two particles of opposite signs annihilate each other. At the same time, one particle is chosen uniformly from the family of + particles and one − particle is chosen uniformly as well. Each of the two particles splits into two offspring of the same type as the parent, so that the number of particles of each type remains constant. The particles move independently until the next annihilation and birth event, and the evolution continues in the same manner. From the point of view of the mathematical description of the model, it is more convenient to represent the "annihilation and birth" event as a jump of the annihilated + particle to a randomly chosen + particle, and the same for the annihilated − particle.

In the above description of the dynamics of the system, when we say that a particle is chosen "uniformly," it means that we choose one of the N particles with the same probability $1/N$; in other words, a particle that is annihilated may be the one that splits into two offspring. In such a case, the offspring are born at the site where the annihilated particle resided just before the jump that lead to its annihilation. This is a different convention than in [BHM] but this convention will simplify some formulas and, of course, it makes little difference when N is large.

The informal description given above can be rigorously expressed in terms of the generator $L = L_\varepsilon$ for the process. Let $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$. The configuration η such that $\eta_x = 1$ and $\eta_y = 0$ for $y \neq x$ will be denoted δ_x. The formula for the generator L of the process η is

$$\begin{aligned} Lf(\eta) = \sum_{x,y\in D_\varepsilon, |x-y|=\varepsilon} & h_\varepsilon^{-1}(x) p_{xy} \Big\{ \eta_x^+ \mathbf{1}_{\{\eta_y \geq 0\}} (f(\eta - \delta_x + \delta_y) - f(\eta)) \\ & + \eta_x^- \mathbf{1}_{\{\eta_y \leq 0\}} (f(\eta + \delta_x - \delta_y) - f(\eta)) \\ & + \eta_x^+ \mathbf{1}_{\{\eta_y < 0\}} N^{-2} \sum_{u,v\in D_\varepsilon} \eta_u^+ \eta_v^- (f(\eta - \delta_x + \delta_y + \delta_u - \delta_v) - f(\eta)) \\ & + \eta_x^- \mathbf{1}_{\{\eta_y > 0\}} N^{-2} \sum_{u,v\in D_\varepsilon} \eta_u^- \eta_v^+ (f(\eta + \delta_x - \delta_y - \delta_u + \delta_v) - f(\eta)) \Big\}. \end{aligned}$$

The normalized particle density $u(x,t) = u^{N,\varepsilon}(x,t)$ is defined by $u(x,t) = N^{-1}\varepsilon^{-d}\eta_x$ for $x \in D_\varepsilon$. Typically, we will be interested in the population size of order $N \sim \varepsilon^d$. Let P_t denote the heat semigroup on $\overline{D}$, corresponding to the reflecting Brownian motion, and for a measure μ on $\overline{D}$, let $P_t\mu$ denote the measure with the density $\int_{\overline{D}} P_t(x, \cdot\,) d\mu(x)$. When

$\mu = \sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x,0)\delta_x$ and δ_x denotes the probability measure with the unit mass at x (by abuse of notation), then we will write $P_t u^{N,\varepsilon}(0)$ to denote $P_t\mu$. In other words, $P_t u^{N,\varepsilon}(0)dy = \sum_{x \in D_\varepsilon} u^{N,\varepsilon}(x,0)P_t(x,y)dy$.

For a measure μ on $\overline{D}$, we define $\overline{\mu}$ to be $c\mu$ where $c = c(\mu)$ is a normalizing constant chosen so that the total variation of $\overline{\mu}$ is equal to 2. For definiteness, we let $\overline{\mu} \equiv 0$ if $\mu \equiv 0$.

For the meaning of $M_1(S((0,\infty), M_F(\overline{D})))$, see Section 2.

Theorem 3.1. *Suppose that $\varepsilon \to 0$ and $N \to \infty$ in such a way that $c_1\varepsilon^{-d} \le N \le c_2\varepsilon^{-d}$ for some constants $0 < c_1 < c_2 < \infty$. Assume that D has an analytic boundary, $u^{N,\varepsilon}(x,0)$ are non-random, and for some signed measure μ on $\overline{D}$ which does not vanish identically, $\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x,0)\delta_x \to \mu$ in $M_F(\overline{D})$ as $\varepsilon \to 0$. Then, as $\varepsilon \to 0$,*

$$\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x,\,\cdot\,)\delta_x - \overline{P_\cdot\mu} \to \delta_{\mathbf{0}}, \qquad \text{in } M_1(S((0,\infty), M_F(\overline{D}))).$$

The symbol $\delta_{\mathbf{0}}$ in the last formula refers to the process identically equal to the null measure.

Note that we have not assumed that the total variation of μ is equal to 2. The total variation of μ cannot be larger than 2 but it can be smaller than 2 when the particles are tightly interspersed at time 0. If the total variation of μ is less than 2, then for small ε, the particle configuration has an almost instantaneous jump at time 0 to a configuration that approximates $\overline{\mu}$. For this reason, we obtain convergence only in $M_1(S((0,\infty), M_F(\overline{D})))$, not in $M_1(S([0,\infty), M_F(\overline{D})))$.

The assumption that $u^{N,\varepsilon}(x,0)$ are non-random measures is easy to remove—we added it for technical convenience only.

Theorem 3.1 is a special case of Theorem 3.2 below. We need some more notation to present this more general result.

Recall from Section 2 that $-\lambda_n$, $n = 0,1,2,\dots$ are the eigenvalues of the Laplacian on D with the Neumann boundary conditions, and ϕ_n are the corresponding eigenfunctions. For a measure μ on $\overline{D}$, we write $\hat{\mu}_n = \int_{\overline{D}} \phi_n d\mu$. For a function $f : D_\varepsilon \to \mathbb{R}$ we let $\langle f, g\rangle = \varepsilon^d \sum_{x \in D_\varepsilon} f(x)g(x)$. The Fourier coefficients for the "density" $u(x,t)$ will be denoted $\hat{u}_n = \hat{u}_n(t) = \langle u(t), \phi_n\rangle$. In other words, $\hat{u}_n$ is the n-th Fourier coefficient for the measure $\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x,t)\delta_x$.

Note that if $\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x,0)\delta_x \to \mu$ in $M_F(\overline{D})$ as $\varepsilon \to 0$, where μ is a signed measure that is not identically equal to 0, then for some n there exists $a > 0$ such that $|\hat{u}_n(0)| \ge a$ for sufficiently small $\varepsilon > 0$ (see Lemma 2.2 and its proof).

Theorem 3.2. *Suppose that $\varepsilon \to 0$ and $N \to \infty$ in such a way that $c_1\varepsilon^{-d} \le N \le c_2\varepsilon^{-d}$ for some constants $0 < c_1 < c_2 < \infty$. Assume that D has an analytic boundary and for some n there exists $a > 0$ such that $\inf_{N,\varepsilon} |\hat{u}_n(0)| = a$. Then, as $\varepsilon \to 0$,*

$$\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x,\,\cdot\,)\delta_x - \overline{P.u^{N,\varepsilon}(0)} \to \delta_{\mathbf{0}}, \qquad \textit{in } M_1(S((0,\infty), M_F(\overline{D}))).$$

Proof. The proof will be divided into several steps.

Step 1. In this step, we will show that for $z \in D_\varepsilon$ and $f(\eta) = \eta_z$,

$$L_\varepsilon f(\eta) = \Delta_\varepsilon^* \eta_z + V\eta_z, \tag{3.1}$$

where $V = V_\varepsilon(\eta)$ is the (normalized, instantaneous) jump intensity for the particle system in state η, defined by

$$V = 2N^{-1} \sum_{x,y \in D_\varepsilon, |x-y|=\varepsilon} h_\varepsilon^{-1}(x) p_{xy}(\eta_x^+ \mathbf{1}_{\{\eta_y<0\}} + \eta_x^- \mathbf{1}_{\{\eta_y>0\}}),$$

and

$$\Delta_\varepsilon^* f(x) = \sum_{y \in D_\varepsilon, |y-x|=\varepsilon} (h_\varepsilon^{-1}(y) p_{yx} f(y) - h_\varepsilon^{-1}(x) p_{xy} f(x)).$$

This operator is the adjoint of the discrete Laplacian Δ_ε given by

$$\Delta_\varepsilon f(x) = h_\varepsilon^{-1}(x) \sum_{y \in D_\varepsilon, |y-x|=\varepsilon} p_{xy}(f(y) - f(x)).$$

Fix some $z \in D_\varepsilon$ and let $f(\eta) = \eta_z$. Then

$$\begin{aligned} Lf(\eta) = &\sum_{x,y \in D_\varepsilon, |x-y|=\varepsilon} h_\varepsilon^{-1}(x) p_{xy} \Big\{ \eta_x^+ \mathbf{1}_{\{\eta_y \ge 0\}} (\mathbf{1}_{\{y=z\}} - \mathbf{1}_{\{x=z\}}) \\ &+ \eta_x^- \mathbf{1}_{\{\eta_y \le 0\}} (\mathbf{1}_{\{x=z\}} - \mathbf{1}_{\{y=z\}}) \\ &+ \eta_x^+ \mathbf{1}_{\{\eta_y<0\}} N^{-2} \sum_{u,v \in D_\varepsilon} \eta_u^+ \eta_v^- (\mathbf{1}_{\{y=z\}} + \mathbf{1}_{\{u=z\}} - \mathbf{1}_{\{x=z\}} - \mathbf{1}_{\{v=z\}}) \\ &+ \eta_x^- \mathbf{1}_{\{\eta_y>0\}} N^{-2} \sum_{u,v \in D_\varepsilon} \eta_u^- \eta_v^+ (\mathbf{1}_{\{x=z\}} + \mathbf{1}_{\{v=z\}} - \mathbf{1}_{\{y=z\}} - \mathbf{1}_{\{u=z\}}) \Big\}. \end{aligned} \tag{3.2}$$

The sum of the terms in (3.2) with the indicators $\mathbf{1}_{\{x=z\}}$ is equal to

$$\begin{aligned}
&\sum_{y\in D_\varepsilon, |z-y|=\varepsilon} h_\varepsilon^{-1}(z) p_{zy}\Big[-\eta_z^+ \mathbf{1}_{\{\eta_y\geq 0\}} + \eta_z^- \mathbf{1}_{\{\eta_y\leq 0\}} \\
&\quad - \eta_z^+ \mathbf{1}_{\{\eta_y<0\}} N^{-2} \sum_{u,v\in D_\varepsilon} \eta_u^+ \eta_v^- + \eta_z^- \mathbf{1}_{\{\eta_y>0\}} N^{-2} \sum_{u,v\in D_\varepsilon} \eta_u^- \eta_v^+ \Big] \\
&= \sum_{y\in D_\varepsilon, |z-y|=\varepsilon} h_\varepsilon^{-1}(z) p_{zy}\Big[-\eta_z^+ \mathbf{1}_{\{\eta_y\geq 0\}} + \eta_z^- \mathbf{1}_{\{\eta_y\leq 0\}} \\
&\quad - \eta_z^+ \mathbf{1}_{\{\eta_y<0\}} N^{-2} N^2 \quad + \eta_z^- \mathbf{1}_{\{\eta_y>0\}} N^{-2} N^2 \Big] \\
&= - \sum_{y\in D_\varepsilon, |z-y|=\varepsilon} h_\varepsilon^{-1}(z) p_{zy} \eta_z .
\end{aligned} \tag{3.3}$$

A similar calculation shows that the sum of the terms in (3.2) with the indicators $\mathbf{1}_{\{y=z\}}$ is equal to

$$\sum_{x\in D_\varepsilon, |z-x|=\varepsilon} h_\varepsilon^{-1}(x) p_{xz} \eta_x . \tag{3.4}$$

The sum of the terms in (3.2) with the indicators $\mathbf{1}_{\{u=z\}}$ is equal to

$$\begin{aligned}
&\sum_{x,y\in D_\varepsilon, |x-y|=\varepsilon} h_\varepsilon^{-1}(x) p_{xy}\Big[\eta_x^+ \mathbf{1}_{\{\eta_y<0\}} N^{-2} \sum_{v\in D_\varepsilon} \eta_z^+ \eta_v^- - \eta_x^- \mathbf{1}_{\{\eta_y>0\}} N^{-2} \sum_{v\in D_\varepsilon} \eta_z^- \eta_v^+ \Big] \\
&= \sum_{x,y\in D_\varepsilon, |x-y|=\varepsilon} h_\varepsilon^{-1}(x) p_{xy}\Big[\eta_x^+ \mathbf{1}_{\{\eta_y<0\}} N^{-2} \eta_z^+ N - \eta_x^- \mathbf{1}_{\{\eta_y>0\}} N^{-2} \eta_z^- N \Big] \\
&= (1/2) V_\varepsilon \eta_z .
\end{aligned} \tag{3.5}$$

Similarly, the sum of the terms in (3.2) with the indicators $\mathbf{1}_{\{v=z\}}$ is equal to $(1/2)V_\varepsilon\eta_z$. Combining this and (3.2)-(3.5), we obtain (3.1).

Step 2. We will now derive estimates for the Fourier coefficients of u. Note that η is a finite state continuous time Markov process. Hence, it is elementary to check that the Fourier coefficient $\hat{u}_n$ satisfies the following stochastic differential equation

$$d\hat{u}_n = L\hat{u}_n dt + dM_n, \tag{3.6}$$

where $M_n(t)$ is a martingale. Since ϕ_n is the n-th eigenfunction, $\Delta\phi_n = -\lambda_n\phi_n$, where Δ is the Laplacian on D with the Neumann boundary conditions. Recall that we have assumed that D has an analytic boundary. It was pointed out in [BP] that in view of Theorem 5.7.1' on page 169 of [M], if the coefficients of a second order elliptic equation are real analytic on a bounded analytic domain D up to the boundary, then for every point z

on ∂D there exists a ball B centered at z such that solutions of the elliptic equation can be extended to be real analytic functions on B. This and (2.1)-(2.4) imply that there exists a series expansion for ϕ_n that yields

$$\Delta_\varepsilon \phi_n(x) = \Delta \phi_n(x) + \psi_\varepsilon(x) = -\lambda_n \phi_n(x) + \psi_\varepsilon(x),$$

where $|\psi_\varepsilon(x)| \le c(n)\varepsilon$. We have $L\eta_x = \Delta_\varepsilon^* \eta_x + V\eta_x$ and

$$\hat{u}_n = \langle u, \phi_n \rangle = \varepsilon^d \sum_{x \in D_\varepsilon} N^{-1} \varepsilon^{-d} \eta_x \phi_n(x) = N^{-1} \sum_{x \in D_\varepsilon} \eta_x \phi_n(x),$$

so

$$\begin{aligned}
L\hat{u}_n &= N^{-1} \sum_{x \in D_\varepsilon} (L\eta_x)\phi_n(x) \qquad (3.7)\\
&= N^{-1} \sum_{x \in D_\varepsilon} (\Delta_\varepsilon^* \eta_x)\phi_n(x) + N^{-1} \sum_{x \in D_\varepsilon} (V\eta_x)\phi_n(x) \\
&= N^{-1} \sum_{x \in D_\varepsilon} \left(\sum_{y \in D_\varepsilon, |y-x| = \varepsilon} (h_\varepsilon^{-1}(y) p_{yx} \eta_y - h_\varepsilon^{-1}(x) p_{xy} \eta_x) \right) \phi_n(x) \\
&\quad + N^{-1} V \sum_{x \in D_\varepsilon} \eta_x \phi_n(x) \\
&= N^{-1} \sum_{x \in D_\varepsilon} \left(\sum_{y \in D_\varepsilon, |y-x| = \varepsilon} (h_\varepsilon^{-1}(x) p_{xy} \phi_n(y) - h_\varepsilon^{-1}(x) p_{xy} \phi_n(x)) \right) \eta_x \\
&\quad + N^{-1} V \sum_{x \in D_\varepsilon} \eta_x \phi_n(x) \\
&= N^{-1} \sum_{x \in D_\varepsilon} \eta_x (\Delta_\varepsilon \phi_n(x)) + V N^{-1} \sum_{x \in D_\varepsilon} \eta_x \phi_n(x) \\
&= N^{-1} \sum_{x \in D_\varepsilon} \eta_x (-\lambda_n \phi_n(x) + \psi_\varepsilon(x)) + V N^{-1} \sum_{x \in D_\varepsilon} \eta_x \phi_n(x) \\
&= (V - \lambda_n)\hat{u}_n + \Psi_{\varepsilon,n},
\end{aligned}$$

where $|\Psi_{\varepsilon,n}| \le c(n)\varepsilon$. Note that

$$\lim_{s \to 0} \frac{E(M_n(t+s) - M_n(t))^2}{E[(\hat{u}_n(t+s) - \hat{u}_n(t))^2]} = 1,$$

because

$$\lim_{s \to 0} \frac{[E(\hat{u}_n(t+s) - \hat{u}_n(t))]^2}{E[(\hat{u}_n(t+s) - \hat{u}_n(t))^2]} = 0.$$

It follows that

$$E[M_n^2(t)] = \int_0^t E[A_n(s)]ds, \tag{3.8}$$

where $A_n = A_n(t,\eta)$ is given by the following formula,

$$\begin{aligned}
A_n &= \lim_{s\to 0} E[(\hat{u}_n(t+s) - \hat{u}_n(t))^2] \\
&= N^{-2} \sum_{x,y\in D_\varepsilon, |x-y|=\varepsilon} h_\varepsilon^{-1}(x) p_{xy} \Big\{ \eta_x^+ \mathbf{1}_{\{\eta_y \geq 0\}} (\phi_n(y) - \phi_n(x))^2 \\
&+ \eta_x^- \mathbf{1}_{\{\eta_y \leq 0\}} (\phi_n(x) - \phi_n(y))^2 \\
&+ \eta_x^+ \mathbf{1}_{\{\eta_y < 0\}} N^{-2} \sum_{u,v\in D_\varepsilon} \eta_u^+ \eta_v^- (\phi_n(y) - \phi_n(x) + \phi_n(u) - \phi_n(v))^2 \\
&+ \eta_x^- \mathbf{1}_{\{\eta_y > 0\}} N^{-2} \sum_{u,v\in D_\varepsilon} \eta_u^- \eta_v^+ (\phi_n(x) - \phi_n(y) - \phi_n(u) + \phi_n(v))^2 \Big\}.
\end{aligned}$$

This implies the following bound for A_n, for small ε,

$$A_n \leq c_1 N^{-1} V (\varepsilon^2 \|\nabla \phi_n\|_\infty^2 + \|\phi_n\|_\infty^2) \leq \beta_n N^{-1} V, \tag{3.9}$$

where $\beta_n < \infty$ depends on ϕ_n, and ∇ stands for the usual gradient acting on functions defined on $\mathbb{R}^d$. It follows from (3.6) and (3.7) that

$$\begin{aligned}
\hat{u}_n(t) &= e^{\int_0^t (V(r)-\lambda_n)dr} \left(\hat{u}_n(0) + \int_0^t e^{-\int_0^s (V(r)-\lambda_n)dr} dM_s + \int_0^t e^{-\int_0^s (V(r)-\lambda_n)dr} \Psi_{\varepsilon,n}(s) ds \right) \\
&= \hat{u}_n(0) e^{\int_0^t (V(r)-\lambda_n)dr} + \int_0^t e^{\int_s^t (V(r)-\lambda_n)dr} dM_s + \int_0^t e^{\int_s^t (V(r)-\lambda_n)dr} \Psi_{\varepsilon,n}(s) ds \\
&\stackrel{\text{df}}{=} \hat{u}_n(0) e^{\int_0^t (V(r)-\lambda_n)dr} + R_{n,1}(t) + R_{n,2}(t).
\end{aligned} \tag{3.10}$$

In view of (3.8) and (3.9), we have

$$\begin{aligned}
E[R_{n,1}^2(t)] &= E\left[\int_0^t e^{2\int_s^t (V(r)-\lambda_n)dr} A_n(s) ds \right] \\
&\leq \beta_n N^{-1} E\left[\int_0^t V(s) e^{2\int_s^t (V(r)-\lambda_n)dr} ds \right].
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^t V(s) e^{2\int_s^t (V(r)-\lambda_n)dr} ds \\
&\qquad = \lambda_n \int_0^t e^{2\int_s^t (V(r)-\lambda_n)dr} ds + (1/2) \left(e^{2\int_0^t (V(r)-\lambda_n)dr} - 1 \right) \\
&\qquad \leq c_2 e^{2\int_0^t V(r)dr},
\end{aligned}$$

where c_2 depends on t and n. Thus, for some c_3 that depends on t and n,

$$E[R_{n,1}^2(t)] \le c_3 \beta_n N^{-1} E\left[\int_0^t e^{2\int_s^t V(r)dr} ds\right].$$

For the second remainder, we have the following estimate,

$$\begin{aligned} E[R_{n,2}^2(t)] &\le (\sup_{s \le t} \Psi_{\varepsilon,n}^2(s)) E\left[\int_0^t e^{2\int_s^t (V(r)-\lambda_n)dr} ds\right] \\ &\le c(n)^2 \varepsilon^2 E\left[\int_0^t e^{2\int_s^t (V(r)-\lambda_n)dr} ds\right]. \end{aligned}$$

Hence,

$$E[(R_{n,1}(t) + R_{n,2}(t))^2] \le (c_3 \beta_n N^{-1} + c(n)^2 \varepsilon^2) E\left[e^{2\int_0^t V(r)dr}\right]. \tag{3.11}$$

Recall that for some n there exists $a > 0$ such that $\inf_{N,\varepsilon} |\hat{u}_n(0)| = a$. Let n_0 be the smallest n satisfying this condition. It follows from (3.10) and (3.11) that

$$E\left[\left(\hat{u}_{n_0}(t) - \hat{u}_{n_0}(0) e^{\int_0^t (V(r)-\lambda_{n_0})dr}\right)^2\right] \le (c_3 \beta_{n_0} N^{-1} + c(n_0)^2 \varepsilon^2) E\left[e^{2\int_0^t V(r)dr}\right]. \tag{3.12}$$

Suppose that N is large enough (and, consequently, ε is small) so that

$$2(c_3 \beta_{n_0} N^{-1} + c(n_0)^2 \varepsilon^2) \le (1/2) a^2 e^{-2\lambda_{n_0} t}. \tag{3.13}$$

Since $\|\phi_{n_0}\|_\infty < \infty$ and $\varepsilon^d \sum_x |u(x,t)| = 2$, we obtain

$$|\hat{u}_{n_0}(t)| = \left| \varepsilon^d \sum_{x \in D_\varepsilon} u(x,t) \phi_{n_0}(x) \right| \le c_4. \tag{3.14}$$

By (3.12), (3.13) and (3.14),

$$\begin{aligned} a^2 e^{-2\lambda_{n_0} t} E\left[e^{2\int_0^t V(r)dr}\right] &\le E\left[\left(\hat{u}_{n_0}(0) e^{\int_0^t (V(r)-\lambda_{n_0})dr}\right)^2\right] \\ &\le 2E\left[\left(\hat{u}_{n_0}(t) - \hat{u}_{n_0}(0) e^{\int_0^t (V(r)-\lambda_{n_0})dr}\right)^2\right] + 2E[(\hat{u}_{n_0}(t))^2] \\ &\le 2(c_3 \beta_{n_0} N^{-1} + c(n_0)^2 \varepsilon^2) E\left[e^{2\int_0^t V(r)dr}\right] + 2c_4^2 \\ &\le (1/2) a^2 e^{-2\lambda_{n_0} t} E\left[e^{2\int_0^t V(r)dr}\right] + 2c_4^2, \end{aligned}$$

so

$$E\left[e^{2\int_0^t V(r)dr}\right] \le 4c_4^2 a^{-2} e^{2\lambda_{n_0} t}. \tag{3.15}$$

Note that $\widehat{P_t u(0)}_n = e^{-\lambda_n t}\hat{u}_n(0)$ and let $v(x,t) = e^{\int_0^t V^{N,\varepsilon}(r)dr}(P_t u^{N,\varepsilon}(0))(x)$. We combine (3.10), (3.11) and (3.15) to see that if $N \to \infty$ (and, therefore, $\varepsilon \to 0$), then for every fixed n and $t > 0$, $\hat{u}_n(t) - \hat{v}_n(t) \to 0$ in distribution. Then Lemma 2.2 shows that

$$\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x,t)\delta_x - e^{\int_0^t V^{N,\varepsilon}(r)dr} P_t u^{N,\varepsilon}(0) \to \delta_{\mathbf{0}}, \qquad \text{in } M_1(M_F(\overline{D})). \tag{3.16}$$

Step 3. It is not obvious that the normalization of $P_t u^{N,\varepsilon}(0)$ in (3.16) is the same as in the statement of Theorem 3.2. It is conceivable that a sizeable proportion of positive and negative particles are tightly interspersed so that their masses cancel each other in the limit. We will show that this is not the case—intuitively, the two populations occupy disjoint parts of D.

Let $B_\delta(x)$ denote a hypercube in D_ε, centered at x, with side length δ. We will consider only $\delta > \varepsilon$. We set

$$\Lambda_\delta(t,x) = \min\left(\sum_{y \in B_\delta(x)} \eta_y^+, \sum_{y \in B_\delta(x)} \eta_y^-\right).$$

Note that if $\varepsilon < \delta/2$ and $x \in D_\delta$ then $B_\delta(x)$ contains at least $(\delta/2\varepsilon)^d$ sites $y \in D_\varepsilon$. Fix an arbitrarily small $c_0 > 0$. Suppose that for some $B_\delta(x)$, we have $\Lambda_\delta(t,x) \ge c_0(\delta/2\varepsilon)^d$. Given this assumption, we will show that the number of ($+$ and $-$) particles that are located in $B_\delta(x)$ at time t and collide with a particle of the opposite sign before time $t+\delta^2$ has expectation greater than $c_1\Lambda_\delta(t,x)$. Suppose without loss of generality that there are fewer $+$ than $-$ particles in $B_\delta(x)$ at time t, i.e., $\Lambda_\delta(t,x) = \sum_{y \in B_\delta(x)} \eta_x^+$. Consider independent continuous time reflecting random walks Y_k, $1 \le k \le \Lambda_\delta(t,x)$, starting from the same points as the locations of the $+$ particles at time t in $B_\delta(x)$. The distribution of a process $\{Y_k(s), s \ge t\}$ is the same as for a single particle in our process η, except that Y_k's do not interact with other particles. For $k < (1/2)(\delta/2\varepsilon)^d \wedge \Lambda_\delta(t,x)$,

$$P(Y_k(t+\delta^2) \ne Y_j(t+\delta^2), 1 \le j \le k-1) > p_1 > 0,$$

where p_1 depends only on the dimension d. This implies that with probability p_2, the number of different sites occupied by $Y_k(t+\delta^2)$, $1 \le k \le \Lambda_\delta(t,x)$, is greater than $c_2(\delta/\varepsilon)^d$, where $p_2, c_2 > 0$ depend only on d and c_0.

Choose $\Lambda_\delta(t,x)$ locations of the $-$ particles at time t in $B_\delta(x)$. From each of these points, start a continuous time reflecting random walk Z_k. Assume that Z_k's, $1 \le k \le \Lambda_\delta(t,x)$, are independent, and they are independent of Y_k's. Suppose that the number of different sites occupied by $Y_k(t+\delta^2)$, $1 \le k \le \Lambda_\delta(t,x)$, is greater than $c_2(\delta/\varepsilon)^d$ and call the set of these sites Γ. Then it is easy to see that with probability greater than $p_3 > 0$, the number of distinct sites in Γ occupied by $Z_k(t+\delta^2)$, $1 \le k \le (c_2/2)(\delta/\varepsilon)^d$, is greater than $c_3(\delta/\varepsilon)^d$.

Suppose that there are at least $c_2(\delta/\varepsilon)^d$ sites in Γ and the number of distinct sites in Γ occupied by $Z_k(t+\delta^2)$, $1 \le k \le (c_2/2)(\delta/\varepsilon)^d$, is greater than $c_3(\delta/\varepsilon)^d$. Find the first time $t_1 > t$ when some Y_k and Z_j occupy the same site and call these particles "eliminated." Then, by induction, find the smallest $t_m > t_{m-1}$ when some non-eliminated Y_k and Z_j occupy the same site and eliminate this pair of particles. Note that the total number of eliminated pairs by the time $t+\delta^2$ cannot be smaller than $(c_3/2)(\delta/\varepsilon)^d$.

Now we return to our original model, with interactions between particles. Consider the set of $+$ and $-$ particles in η that reside at the same locations as Y_k's and Z_k's at time t in $B_\delta(x)$. Choose from this set a pair (Q_+, Q_-) consisting of a $+$ and a $-$ particle and suppose that it would have been "eliminated" in the scheme described above, i.e., if these two particles had been Y_k and Z_j for some k and j. If Q_+ and Q_- do not meet before time $t+\delta^2$, it means that one of these particles must have met a particle of the opposite sign (different from Q_+ and Q_-) before time $t+\delta^2$, and hence at least one of particles Q_+ and Q_- has a jump before time $t+\delta^2$. Thus, with probability greater than $p_2p_3 > 0$, there will be at least $(c_3/4)(\delta/\varepsilon)^d$ jumps between times t and $t+\delta^2$, by particles that are located in $B_\delta(x)$ at time t, assuming that $\Lambda_\delta(t,x) \ge c_0(\delta/2\varepsilon)^d$.

Let $K(t)$ be the number of collisions before t. Then $EK(t) = (N/2)E\int_0^t V(s)ds$. Let $H_\delta(t) = \{x \in D_\delta : \Lambda_\delta(t,x) \ge c_0(\delta/2\varepsilon)^d\}$. We see that for some c_1 depending on c_0,

$$EK(t) \ge \sum_{k=0}^{[t/\delta^2]-1} E \sum_{x \in H_\delta(k\delta^2)} c_1[\Lambda_\delta(k\delta^2, x)].$$

For the same reason, for $s \in [0, \delta^2]$,

$$EK(t) \ge \sum_{k=0}^{[(t-s)/\delta^2]-1} E \sum_{x \in H_\delta(k\delta^2)} c_1[\Lambda_\delta(k\delta^2 + s, x)].$$

Hence,

$$EK(t) \ge \int_{\delta^2}^{t-2\delta^2} \delta^{-2} E \sum_{x \in H_\delta(s)} c_1[\Lambda_\delta(s,x)]ds. \tag{3.17}$$

For a function f on D_ε let $\Xi(x,\delta,f) = \sum_{y\in B_\delta(x)} f(y)$. Recall that $u_t = u(x,t) = N^{-1}\varepsilon^{-d}\eta_x$. We have

$$\Xi(x,\delta,|u_t|) - |\Xi(x,\delta,u_t)| = 2N^{-1}\varepsilon^{-d}\Lambda_\delta(t,x). \tag{3.18}$$

If $x \notin H_\delta(s)$ then either

$$\sum_{y\in B_\delta(x)} \eta_y^+ \le c_0(\delta/2\varepsilon)^d \qquad \text{or} \qquad \sum_{y\in B_\delta(x)} \eta_y^- \le c_0(\delta/2\varepsilon)^d.$$

Hence,

$$\sum_{x\notin H_\delta(s)} (\Xi(x,\delta,|u_s|)-|\Xi(x,\delta,u_s)|) \le \sum_{x\notin H_\delta(s)} c_0(\delta/2\varepsilon)^d \le \sum_{x\in D_\delta} c_0(\delta/2\varepsilon)^d \le c_0c_4\varepsilon^{-d}, \tag{3.19}$$

where c_4 is a constant depending only on D. Recall that $N \ge c_5\varepsilon^{-d}$. In view of (3.15), (3.17), (3.18) and (3.19),

$$\begin{aligned}
&E\left[\int_{\delta^2}^{t-2\delta^2}\left(2-\sum_{x\in D_\delta}|\Xi(x,\delta,\varepsilon^d u_s)|\right)ds\right]\\
&= E\left[\int_{\delta^2}^{t-2\delta^2}\sum_{x\in D_\delta}(\Xi(x,\delta,|\varepsilon^d u_s|)-|\Xi(x,\delta,\varepsilon^d u_s)|)ds\right]\\
&= \varepsilon^d E\left[\int_{\delta^2}^{t-2\delta^2}\sum_{x\in D_\delta}(\Xi(x,\delta,|u_s|)-|\Xi(x,\delta,u_s)|)ds\right]\\
&= \varepsilon^d E\left[\int_{\delta^2}^{t-2\delta^2}\sum_{x\notin H_\delta(s)}(\Xi(x,\delta,|u_s|)-|\Xi(x,\delta,u_s)|)ds\right]\\
&\qquad + \varepsilon^d E\left[\int_{\delta^2}^{t-2\delta^2}\sum_{x\in H_\delta(s)}(\Xi(x,\delta,|u_s|)-|\Xi(x,\delta,u_s)|)ds\right]\\
&\le c_0c_4t + 2\varepsilon^d E\left[\int_{\delta^2}^{t-2\delta^2}\sum_{x\in H_\delta(s)} N^{-1}\varepsilon^{-d}\Lambda_\delta(s,x)ds\right]\\
&\le c_0c_4t + 2c_6N^{-1}\delta^2 EK(t)\\
&= c_0c_4t + 2c_6\delta^2 E\left[\int_0^t V(s)ds\right]\\
&\le c_0c_4t + c_7\delta^2,
\end{aligned} \tag{3.20}$$

where c_7 depends on c_0. For a fixed $t>0$ and $n>0$, we can find $c_0>0$ and $\delta_n \in (0,2^{-n})$ so small that the right hand side of (3.20) is less than 2^{-n}. Let $\mathcal{T}_n$ be the set of $s\in[0,t]$

such that for $\varepsilon < \delta_n$,

$$E\left[2 - \sum_{x \in D_{\delta_n}} |\Xi(x, \delta, \varepsilon^d u_s)|\right] \geq n^2 2^{-n}.$$

Then, by (3.20), $|\mathcal{T}_n| \leq n^{-2}$. Choose arbitrarily small $c_* > 0$ and let n_0 be so large that $\sum_{n \geq n_0} |\mathcal{T}_n| < c_*$. Let $\mathcal{T}_* = (0, t] \setminus \bigcup_{n \geq n_0} \mathcal{T}_n$. For $s \in \mathcal{T}_*$, and $\varepsilon < \delta_n$,

$$P\left(2 - \sum_{x \in D_{\delta_n}} |\Xi(x, \delta, \varepsilon^d u_s)| \geq n^3 2^{-n}\right) \leq n^{-1}. \tag{3.21}$$

By passing to a subsequence, if necessary, we may assume that $\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x, s)\delta_x$ converge in $M_1(M_F(\overline{D}))$. It follows from (3.21) that any limit of $\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x, s)\delta_x$ is supported on measures with the total variation 2, for every $s \in \mathcal{T}_*$. Since $c_* > 0$ in the definition of $\mathcal{T}_*$ is arbitrarily small, we obtain the same conclusion for almost every $s \in [0, t]$. This and (3.16) imply that for almost every $s > 0$,

$$\sum_{x \in D_\varepsilon} \varepsilon^d u^{N,\varepsilon}(x, s)\delta_x - \overline{P_s u^{N,\varepsilon}(0)} \to \delta_{\mathbf{0}}, \qquad \text{in } M_1(M_F(\overline{D})). \tag{3.22}$$

Step 4. Next we will show that the convergence holds not only for every fixed $t > 0$ but also in the Skorohod topology on $M_1(S((0, \infty), M_F(\overline{D})))$. Fix a smooth function φ on $\overline{D}$ and let $w_t^\varepsilon = w_t = \langle u_t, \varphi \rangle$. We will show that the family of processes $\{w^\varepsilon, \varepsilon > 0\}$ is tight in $M_1(S((0, \infty), \mathbb{R}))$. In order to prove that, we will first derive some estimates for w_t similar to the estimates for $\hat{u}_n$. We have

$$dw_t = Lw_t dt + dM_\varphi, \tag{3.23}$$

where $M_\varphi(t)$ is a martingale. Since φ is smooth, there exists a series expansion for φ that yields

$$\Delta_\varepsilon \varphi(x) = \Delta\varphi + \psi_\varepsilon(x),$$

where $|\psi_\varepsilon(x)| \leq c_\varphi \varepsilon$. We have $L\eta_x = \Delta_\varepsilon^* \eta_x + V\eta_x$ and

$$w_t = \langle u, \varphi \rangle = \varepsilon^d \sum_{x \in D_\varepsilon} N^{-1} \varepsilon^{-d} \eta_x \varphi(x) = N^{-1} \sum_{x \in D_\varepsilon} \eta_x \varphi(x),$$

so

$$Lw = N^{-1} \sum_{x \in D_\varepsilon} (L\eta_x)\varphi(x) \tag{3.24}$$

$$
\begin{aligned}
&= N^{-1} \sum_{x\in D_\varepsilon} (\Delta_\varepsilon^* \eta_x)\varphi(x) + N^{-1} \sum_{x\in D_\varepsilon} (V\eta_x)\varphi(x) \\
&= N^{-1} \sum_{x\in D_\varepsilon} \left(\sum_{y\in D_\varepsilon, |y-x|=\varepsilon} (h_\varepsilon^{-1}(y) p_{yx}\eta_y - h_\varepsilon^{-1}(x) p_{xy}\eta_x) \right) \varphi(x) \\
&\quad + N^{-1} V \sum_{x\in D_\varepsilon} \eta_x \varphi(x) \\
&= N^{-1} \sum_{x\in D_\varepsilon} \left(\sum_{y\in D_\varepsilon, |y-x|=\varepsilon} (h_\varepsilon^{-1}(x) p_{xy}\varphi(y) - h_\varepsilon^{-1}(x) p_{xy}\varphi(x)) \right) \eta_x \\
&\quad + N^{-1} V \sum_{x\in D_\varepsilon} \eta_x \varphi(x) \\
&= N^{-1} \sum_{x\in D_\varepsilon} \eta_x (\Delta_\varepsilon \varphi(x)) + V N^{-1} \sum_{x\in D_\varepsilon} \eta_x \varphi(x) \\
&= N^{-1} \sum_{x\in D_\varepsilon} \eta_x (\Delta\varphi(x) + \psi_\varepsilon(x)) + V N^{-1} \sum_{x\in D_\varepsilon} \eta_x \varphi(x) \\
&= \langle u, \Delta\varphi \rangle + \Psi_\varepsilon + V w,
\end{aligned}
$$

where $|\Psi_\varepsilon| \le c_\varphi \varepsilon$. Note that

$$
E[M_\varphi^2(t)] = \int_0^t E[A_\varphi(s)]ds, \tag{3.25}
$$

where $A_\varphi = A_\varphi(t, \eta)$ is given by the following formula,

$$
\begin{aligned}
A_\varphi &= \lim_{s\to 0} E[(w(t+s) - w(t))^2] \\
&= N^{-2} \sum_{x,y\in D_\varepsilon, |x-y|=\varepsilon} h_\varepsilon^{-1}(x) p_{xy} \Big\{ \eta_x^+ \mathbf{1}_{\{\eta_y \ge 0\}} (\varphi(y) - \varphi(x))^2 \\
&+ \eta_x^- \mathbf{1}_{\{\eta_y \le 0\}} (\varphi(x) - \varphi(y))^2 \\
&+ \eta_x^+ \mathbf{1}_{\{\eta_y < 0\}} N^{-2} \sum_{u,v\in D_\varepsilon} \eta_u^+ \eta_v^- (\varphi(y) - \varphi(x) + \varphi(u) - \varphi(v))^2 \\
&+ \eta_x^- \mathbf{1}_{\{\eta_y > 0\}} N^{-2} \sum_{u,v\in D_\varepsilon} \eta_u^- \eta_v^+ (\varphi(x) - \varphi(y) - \varphi(u) + \varphi(v))^2 \Big\}.
\end{aligned}
$$

We obtain the following bound for A_φ, for small ε,

$$
A_\varphi \le c_1 N^{-1} V (\varepsilon^2 \|\nabla\varphi\|_\infty^2 + \|\varphi\|_\infty^2) \le \beta N^{-1} V, \tag{3.26}
$$

where $\beta < \infty$ depends on φ.

It follows from (3.23) and (3.24) that

$$
\begin{aligned}
w(t) &= e^{\int_0^t V(r))dr}\left(w(0) + \int_0^t e^{-\int_0^s V(r)dr} dM_s + \int_0^t e^{-\int_0^s V(r)dr}(\langle u, \Delta\varphi\rangle(s) + \Psi_\varepsilon(s))ds\right) \\
&= w(0)e^{\int_0^t V(r)dr} + \int_0^t e^{\int_s^t V(r)dr} dM_s + \int_0^t e^{\int_s^t V(r)dr}\Psi_\varepsilon(s)ds \\
&\quad + \int_0^t e^{\int_s^t V(r)dr}\langle u, \Delta\varphi\rangle(s)ds \\
&\stackrel{\text{df}}{=} w(0)e^{\int_0^t V(r)dr} + R_1(t) + R_2(t) + R_3(t). \qquad (3.27)
\end{aligned}
$$

In view of (3.25), (3.26) and (3.15), we have

$$
\begin{aligned}
E[R_1^2(t)] &= E\left[\int_0^t e^{2\int_s^t V(r)dr} A_n(s)ds\right] \\
&\leq c_2\beta N^{-1} E\left[\int_0^t V(s)e^{2\int_s^t V(r)dr} ds\right] \\
&= c_2\beta N^{-1} E\left[(1/2)\left(e^{2\int_0^t V(r)dr} - 1\right)\right] \\
&\leq c_3\beta N^{-1} E\left[e^{2\int_0^t V(r)dr}\right] \\
&\leq c_4 e^{2\lambda_{n_0} t} N^{-1}.
\end{aligned}
$$

This implies that

$$
E[\sup_{0\leq s\leq t} R_1^2(s)] \leq 4E[R_1^2(t)] \leq 4c_4 e^{2\lambda_{n_0} t} N^{-1}.
$$

A similar calculation and (3.15) show that

$$
E[\sup_{0\leq s\leq t} R_2^2(s)] \leq (\sup_{0\leq s\leq t} \Psi_\varepsilon^2(s)) E\left[\int_0^t e^{2\int_s^t V(r)dr} ds\right] \leq c_5\varepsilon^2.
$$

Hence,

$$
E[\sup_{0\leq s\leq t} (R_1(s) + R_2(s))^2] \leq c_6(N^{-1} + \varepsilon^2), \qquad (3.28)
$$

where c_6 depends on t and φ.

For the last term on the right hand side of (3.27), we observe that for $0 \leq t_1 < t_2 \leq t$,

$$
\begin{aligned}
|R_3(t_2) - R_3(t_1)| &= \left|\int_{t_1}^{t_2} e^{\int_s^t V(r)dr}\langle u, \Delta\varphi\rangle(s)ds\right| \\
&\leq \sup_{0\leq s\leq t} e^{\int_s^t V(r)dr} \int_{t_1}^{t_2} |\langle u, \Delta\varphi\rangle(s)|ds \\
&\leq c_7 e^{\int_0^t V(r)dr}\|\Delta\varphi\|_\infty |t_2 - t_1|.
\end{aligned}
$$

In view of (3.15),

$$\limsup_{\delta_1,\delta_2\to 0}\limsup_{\varepsilon\to 0} P\left(\sup_{0\le t_1,t_2\le t, |t_1-t_2|\le\delta_1} |R_3(t_2)-R_3(t_1)|\ge\delta_2\right)=0.$$

This, (3.27), (3.28) and Lemma 2.3 show that $\{w^\varepsilon, \varepsilon>0\}$ is a tight family of processes in $M_1(S((0,\infty),\mathbb{R}))$. Since smooth functions are dense in the set of continuous functions on $\overline{D}$ and the sum of two smooth functions is smooth, Theorem 3.7.1 of [D] shows that the family of processes $\left\{\sum_{x\in D_\varepsilon}\varepsilon^d u^{N,\varepsilon}(x,\,\cdot\,)\delta_x, \varepsilon>0\right\}$ is tight in $M_1(S((0,\infty),M_F(\overline{D})))$. Since $\{P_0 u^{N,\varepsilon}(0),\varepsilon>0\}$ are tight and the process $t\to\overline{P_t u^{N,\varepsilon}(0)}$ is a continuous function with values in $M_F(\overline{D})$, completely determined by $P_0 u^{N,\varepsilon}(0)$, we conclude that

$$\left\{\sum_{x\in D_\varepsilon}\varepsilon^d u^{N,\varepsilon}(x,\,\cdot\,)\delta_x-\overline{P_\cdot u^{N,\varepsilon}(0)},\varepsilon>0\right\}$$

is tight in $M_1(S((0,\infty),M_F(\overline{D})))$. By (2.13), any convergent subsequence of this family must be identically equal to $\delta_{\mathbf{0}}$, by the right continuity. $\square$

Acknowledgments. We are grateful to Rodrigo Bañuelos, Victor Ivrii, John Sylvester, John Toth and Steve Zelditch for very helpful advice.

REFERENCES

[BB] R. Bañuelos and K. Burdzy (1999) On the "hot spots" conjecture of J. Rauch *J. Func. Anal.* **164**, 1–33.

[BP] R. Bañuelos and M. Pang (2005) Level sets of the first nonconstant Neumann eigenfunction. (preprint)

[BHIM] K. Burdzy, R. Hołyst, D. Ingerman and P. March (1996) Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of Laplacian eigenfunctions *J. Phys. A* **29**, 2633–2642.

[BHM] K. Burdzy, R. Hołyst and P. March (2000) A Fleming-Viot particle representation of Dirichlet Laplacian *Comm. Math. Phys.* **214**, 679–703.

[CBH] O. Cybulski, V. Babin and R. Hołyst (2005) Minimization of the Renyi entropy production in the space partitioning process (preprint)

[CMBH] O. Cybulski, D. Matysiak, V. Babin and R. Hołyst (2005) Pattern formation in nonextensive thermodynamics: selection criterion based on the Renyi entropy production (preprint)

[D] D.A. Dawson (1993) Measure-valued Markov processes. *École d'Été de Probabilités de Saint-Flour XXI—1991*, 1–260, Lecture Notes in Math., 1541, Springer, Berlin.

[G] D. Grieser (2002) Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary *Comm. Partial Diff. Eq.* **27**, 1283–1299.

[GK1] I. Grigorescu and M. Kang (2004) Hydrodynamic limit for a Fleming-Viot type system *Stoch. Proc. Appl.* **110**, 111-143.

[GK2] I. Grigorescu and M. Kang (2004) Tagged particle limit for a Fleming-Viot type system (preprint)

[HSS] R. Hempel, L.A. Seco and B. Simon (1991) The essential spectrum of Neumann Laplacians on some bounded singular domains. *J. Func. Anal.* **102**, 448–483.

[M] C. Morrey (1966) *Multiple Integrals in the Calculus of Variations*, Springer Verlag, New York.

[NS] Yu. Netrusov and Yu. Safarov (2003) Weyl asymptotic formula for the Laplacian on domains with rough boundaries (preprint)

K.B.: Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98115-4350, USA, `burdzy@math.washington.edu`

J.Q.: Department of Mathematics, University of Toronto, 100 St. George St., Toronto, ON M5S 3G3, Canada, `quastel@math.toronto.edu`