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Some Theorems on
the Resolution Property and
the Brauer map

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Abstract

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Using formal-local methods, we prove that a separated and normal Deligne-Mumford surface must satisfy the resolution property, this includes the first class of separated algebraic spaces which are not schemes. Our analysis passes through the case of gerbes and an arbitrarily singular Deligne-Mumford curve, each of which we establish independently. Our methods can be extended to give new results on the surjectivity of the Brauer map. For example, we show that on a generically reduced variety, any cohomological Brauer class is represented by an Azumaya algebra away from a closed subset of codimension ≥ 3 . We also investigate generically trivial Brauer classes in high codimension which arise from singularities.

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DEDICATION

to my father

Chapter 1

INTRODUCTION

An algebraic stack has the resolution property if every coherent sheaf is the quotient of a vector bundle. Knowing a space has the resolution property is a fundamental question and can be incredibly useful, yet, in most cases, it is surprisingly hard to verify. When X is a scheme we have an affirmative answer when there exists an ample line bundle, or more generally, an ample family of line bundles. Examples of such X are \mathbf{Q} -factorial varieties with affine diagonal or quasiprojective schemes with arbitrary singularities. Beyond this, affirmative answers become very difficult to come by. Indeed, since there exist normal schemes of dimension 2 which do not admit nontrivial line bundles, a fundamentally different approach must be considered. In [54] Schröer and Vezzosi prove that two-dimensional, normal schemes which are finite type and separated over a field satisfy the resolution property. Their methods were refined and extended by Gross in [23] who proved the same for proper schemes of dimension 2, of finite type over a noetherian ring. Curiously, both these proofs use the existence of affine neighborhoods and as such do not apply to the case of algebraic spaces. The case of algebraic stacks is very important but much more difficult: often, one has to simultaneously prove that the stack is a quotient in the sense of [17] and that its moduli space (if one exists at all!) satisfies the resolution property.

Theorem 1.0.1. *Let X be a separated, tame Deligne-Mumford stack of finite type over a field k .*

1. *If X is 1-dimensional then it satisfies the resolution property.*
2. *If X is 2-dimensional and normal then it satisfies the resolution property.*

In particular, they are global quotient stacks.

Our strategy is to use the process of rigidification to reduce to the case of a stack with generically trivial stabilizers. However, this process forces us to consider the case of gerbes. To deal with this difficulty we prove new results on the surjectivity of the Brauer map:

Theorem 1.0.2. *Let X be a tame, separated Deligne-Mumford stack,*

1. *Suppose that X is a noetherian algebraic space with a regular locus which is a dense open subset (e.g. a generically reduced variety). Then for every class $\alpha \in \text{Br}'(X)$ there is a Zariski open neighborhood $U \subset X$ with $\text{codim}(X \setminus U) \geq 3$ such that $\alpha|_U \in \text{Br}(U)$.*
2. *Suppose X is of finite type over a field k , 2-dimensional, and normal with generically trivial stabilizers. Then the geometric and cohomological Brauer groups coincide.*

The most important characterization of the resolution property is that of Totaro and Gross [57]: a quasi-compact, quasi-separated (qcqs) algebraic stack with affine diagonal satisfies the resolution property if and only if it can be expressed as a quotient stack $[W/GL_n]$ where W is quasiaffine. Totaro proved this in the normal and noetherian setting but Gross [22] extended this to qcqs algebraic stacks and even relativized the problem. To define what it means for a morphism to have the resolution property, Gross employs the notion of generating sheaves first described by Olsson and Starr in [51]. Equipped with this formalism one quickly realizes that a host of problems can be rephrased by asking whether or not a particular morphism has the resolution property. For example, if X is a tame Artin stack with moduli space M , then the morphism $X \rightarrow M$ has the resolution property if and only if X is a quotient stack. This is a fundamental question and in the instance of μ_n or \mathbf{G}_m -gerbes, it is closely related to an important problem regarding the Brauer group.

In his original treatises on the Brauer group [26], Grothendieck posed the question: does every cohomological Brauer class arise from a PGL_n -torsor? In other words, is the Brauer map $\text{Br}(X) \rightarrow \text{Br}'(X)$ surjective? An affirmative answer for any particular scheme would be useful. For example, a cohomological interpretation of the group of Azumaya algebras (under the Brauer equivalence) would greatly assist in calculations. These sort of issues concern

those who study the existence of rational points on varieties or, for that matter, anyone who wishes to give geometric meaning to a cohomology class. Viewed in this way, we may reinterpret Gabber’s result on the surjectivity of the Brauer map: it states that every μ_n gerbe morphism $X \rightarrow M$ satisfies the resolution property whenever M admits an ample line bundle (see [17], [19], [16]). Extending Gabber’s result is an important problem and a general solution has remained out of reach for almost 50 years. Note that this is unknown even when M is a smooth scheme of dimension 3, finite type and separated over the complex numbers.

It is a novelty that the surjectivity of the Brauer map is equivalent to the resolution property of a particular gerbe morphism but the story gets much better! Gabber’s result can be used to verify the resolution property for a large class of Deligne-Mumford stacks that are *not* gerbes! Kresch and Vistoli [37] proved that if a (generically tame) smooth Deligne-Mumford stack which is separated and finite type over a field has quasiprojective coarse moduli space then it satisfies the resolution property. The proof of this fact is marvelous but quite difficult: they give a method to reduce the general situation to the case of μ_n gerbes, whence Gabber’s result instantly applies. Their techniques are a mix of Giraud’s nonabelian theory [20] and projective methods a la Bertini.

Our results have been known in some very special cases: that normal separated *schemes* of dimension 2 satisfy the resolution property was first proven in [54], but the authors did not address the case of algebraic spaces. Thus, even for algebraic spaces Theorem 1.0.1(2) is already new. Moreover, Kresch, Vistoli and Gabber’s results apply when X is smooth and/or admits an ample line bundle. Thus the theorems in this paper are really about the non-smooth, non-quasiprojective setting. Here, the problem is much more difficult because we cannot leverage the existence of *any* natural vector bundles (e.g. polarizing, cotangent or jet bundles) and in fact one must actively *construct* vector bundles instead. Another special but important case is the following: μ_n -gerbe morphisms over geometrically normal, separated 2-dimensional algebraic spaces over a field satisfy the resolution property (Schröer [53]). We improve Schröer’s result in two ways: first we allow the stacky structure to vary and we show the resolution property holds absolutely (Theorem 1.0.2(2)). Second, Theorem 1.0.2(1) implies

that over generically reduced surfaces the geometric and cohomological Brauer groups coincide. This result can also be contrasted with Grothendieck's early result [26]: all cohomological Brauer classes on regular noetherian schemes are represented by Azumaya algebras away from a subset of codimension ≥ 3 . As long as the spaces are separated our contribution shows we can replace the hypothesis of regularity with regularity on a dense open subset.

A fundamental tool we exploit is closely related to recent work on Tannakian duality as in [42], [29] or [12]. Using this theory, we can prove the following

Lemma 1.0.3. *Let X be a noetherian algebraic space with affine diagonal. If X admits a stratification:*

$$\begin{array}{ccc} \emptyset & \longrightarrow & \text{Spec } B \\ \downarrow & & \downarrow \textit{open} \\ \text{Spec } A = (X \setminus \text{Spec } B)_{\textit{red}} & \xrightarrow{\textit{closed}} & X \end{array}$$

then there is a noetherian I -adic ring \hat{A} with $\hat{A}/I \cong A$, a morphism $\text{Spec } \hat{A} \rightarrow X$ which is a flat neighborhood of $\text{Spec } A \subset X$ that induces a cartesian pushout square as below.

$$\begin{array}{ccc} \text{Spec } C & \longrightarrow & \text{Spec } B \\ \downarrow & & \downarrow \textit{open} \\ \text{Spec } \hat{A} & \xrightarrow{\textit{flat}} & X \end{array}$$

Chapter 2

BACKGROUND

2.1 The Resolution Property and Quotient Stacks

In this section we recall precise definitions of the (relative) resolution property (following Gross [22] and [21]) and quotient stacks before explaining some basic consequences of the definitions. At the end we relate the two concepts using results of Totaro and Gross.

Definition 2.1.1. Let X be a quasi-compact and quasi-separated (qcqs) algebraic stack and $\mathcal{G}\{F_i\}_{i \in I}$ a set of finitely-presented quasicoherent sheaves. The objects in \mathcal{G} is said to be a generating family of \mathcal{O}_X -modules if for any quasicoherent sheaf M there exists a surjection $\bigoplus F_i \rightarrow M$ where $F_i \in \mathcal{G}$.

A relative version of the above is

Definition 2.1.2. Let $f : X \rightarrow Y$ be a qcqs morphism of algebraic stacks. We say a family of finitely-presented quasicoherent \mathcal{O}_X modules $\mathcal{G} = \{F_i\}$ is f -generating if every quasicoherent \mathcal{O}_X -module admits a surjection

$$\bigoplus_i F_i \otimes f^* N_i \rightarrow M$$

for some family of quasicoherent \mathcal{O}_Y -modules $\{N_i\}_{i \in I}$ and $F_i \in \mathcal{G}$.

The family is said to be universally f -generating if $\{F_i|_{X \times_Y T}\}_{i \in I}$ is $f_T : X \times_Y T \rightarrow Y$ -generating for all morphisms $T \rightarrow Y$.

Now we can define the resolution property

Definition 2.1.3. Let $f : X \rightarrow Y$ be a qcqs morphism of algebraic stacks. We say f has the resolution property if there exists a universally f -generating family of finitely-presented locally

free \mathcal{O}_X -modules $\{V_i\}_{i \in I}$. Let X be a qcqs algebraic stack, we say that X has the resolution property if there exists a generating family of finitely-presented locally free \mathcal{O}_X -modules $\{V_i\}_{i \in I}$.

Remark 2.1.4. For a qcqs algebraic stack X , the resolution property of the canonical morphism $f : X \rightarrow \text{Spec } \mathbf{Z}$ is equivalent to the resolution property of X . Indeed, if a universally f -generating family of locally free sheaves on X exists then every quasicohherent module F can be surjected onto by a direct sum $\bigoplus V_i^{\oplus m_i} \rightarrow F$. Conversely if X satisfies the resolution property then take a generating family of locally free sheaves $\{V_i\}$, it is certainly f -generating hence it is *universally* f -generating by Gross [22] Corollary 1.10 and the fact that $\text{Spec } \mathbf{Z}$ has affine diagonal.

The relativization will come in handy because if a qcqs algebraic stack X maps to Y , to show that X satisfies the resolution property it suffices to show that $X \rightarrow Y$ and Y both satisfy the resolution property. We give an accessible proof below for the sake of completeness. A more general result is given in Gross 1.8(v) [22].

Lemma 2.1.5. *Suppose $f : X \rightarrow Y$ is a qcqs morphism between qcqs algebraic stacks. Then X satisfies the resolution property if f and Y both satisfy the resolution property.*

Proof. Suppose that f and Y satisfy the resolution property. Let $\{V_i\}_{i \in I}$ be a generating family of vector bundles on Y and $\{W_j\}_{j \in J}$ a universally f -generating family of vector bundles on X . Fix a quasicohherent \mathcal{O}_X -module M , we know there exists a surjection

$$\bigoplus_{j \in J} W_j^{\oplus n_j} \otimes f^* N_j \rightarrow M$$

for some family of quasicohherent \mathcal{O}_Y -modules $\{N_j\}_{j \in J}$. Since $\{V_i\}_{i \in I}$ is a generating family of vector bundles on Y , there exists surjections $\bigoplus_{i \in I} V_i^{\oplus m_{ij}} \rightarrow N_j$. We may pull these maps back to X to obtain surjections

$$\bigoplus_{i \in I} f^* V_i^{\oplus m_{ij}} \rightarrow f^* N_j$$

After tensoring with $W_j^{\oplus n_j}$ for each j we obtain the following composition of surjections:

$$\bigoplus_{j \in J} [W_j^{\oplus n_j} \otimes (\bigoplus_{i \in I} f^* V_i^{m_{ij}})] \rightarrow \bigoplus_{j \in J} W_j^{\oplus n_j} \otimes f^* N_j \rightarrow M$$

since tensor products commute with direct sums it follows that the family of vector bundles

$$\{W_i^{\oplus n} \otimes f^* V_j^{\oplus m}\}_{(i,j,n,m) \in I \times J \times \mathbb{Z}^2}$$

is generating on X .

□

One of the difficulties of the resolution property is that we do not know when it descends. However, under additional hypothesis one can argue the following.

Lemma 2.1.6. *Suppose $f : X \rightarrow Y$ is a finite, faithfully flat and finitely-presented morphism between qcqs S -stacks. Then if $X \rightarrow S$ satisfies the resolution property, $Y \rightarrow S$ does as well.*

Proof. This is proven by Gross in [22] 4.3(vii). □

Now we introduce the notion of a quotient stack following [17].

Definition 2.1.7. Let X be a qcqs algebraic stack, we say that X is a quotient stack if it is isomorphic to $[Z/GL_n]$ where Z is an algebraic space.

In general, it is very difficult to verify if an arbitrary algebraic stack is a quotient stack but we give a criterion below. Recall that a morphism of algebraic stacks $X \rightarrow Y$ is said to be projective if there exists a finite type quasicoherent sheaf E on Y and a factorization $X \rightarrow \mathbf{P}(E)_Y \rightarrow Y$ where $\mathbf{P}(E)_Y$ is the projectivization of E over Y and $X \rightarrow \mathbf{P}(E)_Y$ is a closed immersion.

Proposition 2.1.8. *Let X be an algebraic stack finite type over a noetherian base scheme S , then the following are equivalent*

1. X is a quotient stack

2. *There exists a locally free sheaf of finite rank V on X so that for every geometric point $x : \text{Spec } k \rightarrow X$, the stabilizer action I_x on the vector space $V_{\text{Spec } k, x}$ is faithful i.e. the induced morphism of group schemes $I_x \rightarrow GL(V_{\text{Spec } k, x})$ is injective.*
3. *There exists a faithfully flat, projective morphism $Y \rightarrow X$ where Y is a quotient stack.*

Proof. See [17] Lemmas 2.12 and 2.13. □

Remark 2.1.9. At first glance, it may seem very restrictive to only consider stacks of the form $[Z/GL_n]$ but in fact this class of stacks includes all those of the form $[Y/G]$ where Y is an algebraic space and G a flat linear algebraic group (i.e those embeddable into GL_n). Indeed, because G is linear, there is a closed embedding $G \subset GL_n$ and we may consider the contracted product $Y \times^G GL_n = [Y \times GL_n/G]$ and view it as a GL_n -space via right translation. Moreover $[Y/G] \cong [Y \times^G GL_n/GL_n]$.

Definition 2.1.10. Let \mathcal{X} be a quasicompact and separated algebraic stack and V a vector bundle. If for every geometric point $x : \text{Spec } k \rightarrow X$ the stabilizer action I_x on the vector space $V_{\text{Spec } k, x}$ is faithful then we say V is a faithful vector bundle.

Proposition 2.1.11. *Let \mathcal{X} be a qcqs algebraic stack with affine diagonal and let V be a finite rank vector bundle. Then V is a faithful vector bundle if and only if $\text{Frame}(V)$ is an algebraic space.*

Proof. See the proof of Lemma 2.12 in [17]. □

This brings us to a result that relates these two seemingly disparate notions: a qcqs algebraic stack satisfies the resolution property if and only if it admits a very special quotient stack presentation.

Theorem 2.1.12. *(Totaro, Gross) Let X be a qcqs algebraic stack with affine inertia. Then X satisfies the resolution property if and only if $X = [W/GL_n]$ where W is a quasiprojective scheme.*

Proof. This is Theorem 5.10 of [22] where we take $Y = \text{Spec } \mathbf{Z}$. □

Another relationship between these two notions is a criterion for when an algebraic stack with finite inertia is a quotient stack.

Theorem 2.1.13. *Let X be an algebraic stack of finite type, over a noetherian base scheme S . Assume moreover that X has finite diagonal. If $\pi : X \rightarrow M$ denotes the coarse moduli space map, then X is a quotient stack if and only if π has the resolution property.*

Proof. Suppose that $X = [Z/GL_n]$ for an algebraic space Z . We will show that $Z \rightarrow M$ is an affine morphism then use Theorem 5.10 of [22] to conclude that π has the resolution property. Note that by [17] Theorem 2.7 there exists a finite surjective morphism $f : Y \rightarrow X$ where Y is a scheme. It follows that $\pi \circ f : Y \rightarrow M$ is also finite (hence affine). Suppose that M is affine, then Y is affine and because $Z \times_X Y$ is a GL_n -torsor over Y it is also affine. However, $Z \times_X Y \rightarrow Z$ is a finite surjective morphism so by Chevalley's theorem (Tag 07VP, [56]) Z must be affine as well.

Conversely, if π has the resolution property Theorem 5.10 of [22] implies there is a quasiaffine classifying map $X \rightarrow BGL_n$ over M . Therefore, pulling back the universal frame bundle yields a GL_n -torsor $Z \rightarrow X$ where Z is quasiaffine over M . In particular, Z is an algebraic space and therefore $X = [Z/GL_n]$ is a quotient stack. □

2.2 Deformation Theory of Vector Bundles on Stacks

First, we recall the deformation theory of vector bundles on Artin stacks and then we discuss the rigidity of representation types on tame Artin stacks as in [1]. Fix a square zero $f : \mathcal{X}_0 \rightarrow \mathcal{X}$ thickening of algebraic stacks, that is, a closed immersion whose defining ideal is square zero. Since we will work over the lisse-étale site of an Artin stack, in general there does not exist a final object. However, if e denotes the singleton sheaf then recall that if \mathcal{Y} is a stack over the site $\mathcal{X}_{\text{lis-ét}}$, then a section of $\mathcal{Y} \rightarrow \mathcal{X}_{\text{lis-ét}}$ corresponds to a morphism of stacks $F : e \rightarrow \mathcal{Y}$ over $\mathcal{X}_{\text{lis-ét}}$. If we replace \mathcal{Y} with an equivalent split fibered category, then for

every smooth morphism $t : T \rightarrow \mathcal{X}$, F yields an object $F(T, t)$ of $Y(T)$ which is compatible with restrictions. Moreover, a natural transformation between two functors $F, G : e \rightarrow \mathcal{Y}$ is necessarily an isomorphism since \mathcal{Y} is a groupoid.

Definition 2.2.1. Let V_0 be a vector bundle on \mathcal{X}_0 , a deformation of V_0 to \mathcal{X} is a pair (V, ϕ) where V vector bundle on \mathcal{X} and $\phi : V|_{\mathcal{X}_0} \rightarrow V_0$ is an isomorphism.

Proposition 2.2.2. Let $\mathcal{X}_0 \rightarrow \mathcal{X}$ be a square zero thickening of algebraic stacks with defining ideal $I \subset \mathcal{O}_{\mathcal{X}}$ and V_0 a vector bundle on \mathcal{X}_0 .

1. There exists an obstruction $o \in \text{Ext}_{\mathcal{O}_{\mathcal{X}_0}}^2(V_0, I \otimes V_0)$ whose vanishing is equivalent to the existence of a deformation of V_0 .
2. If a deformation of V_0 exists, the set of isomorphism classes of pairs (V, ϕ) lifting V_0 is a (trivial) torsor under $\text{Ext}_{\mathcal{O}_{\mathcal{X}_0}}^1(V_0, I \otimes V_0)$.

Proof. We will show that the stack of deformations of V_0 , \mathcal{D} , is a gerbe over $\mathcal{X}_{\text{lis-ét}}$ banded by the quasicoherent sheaf $f_*\text{Hom}_{\mathcal{O}_{\mathcal{X}_0}}(V_0, V_0 \otimes I)$. To see why this suffices first recall that equivalence classes of gerbes on $\mathcal{X}_{\text{lis-ét}}$ banded by an abelian sheaf A correspond to cohomology classes in $H^2(\mathcal{X}_{\text{lis-ét}}, A)$ (see [50] Theorem 12.2.8 or [20] Chapter 4, Theorem 3.4.2) and

$$H^2(\mathcal{X}_{\text{lis-ét}}, f_*\underline{\text{Hom}}_{\mathcal{O}_{\mathcal{X}_0}}(V_0, V_0 \otimes I)) \cong H^2(\mathcal{X}_0, \underline{\text{Hom}}_{\mathcal{O}_{\mathcal{X}_0}}(V_0, V_0 \otimes I)) \cong \text{Ext}_{\mathcal{O}_{\mathcal{X}_0}}^2(V_0, V_0 \otimes I)$$

Moreover, under this correspondence, a gerbe is trivial over $\mathcal{X}_{\text{lis-ét}}$ iff it corresponds to the trivial cohomology class. The first result follows since a gerbe is trivial if and only if there is an equivalence $\mathcal{D} \cong BA_{\mathcal{X}_{\text{lis-ét}}}$ and such equivalences correspond to sections of \mathcal{D} over $\mathcal{X}_{\text{lis-ét}}$ i.e. deformations of V_0 . The second result follows because isomorphism classes of sections of \mathcal{D} correspond to $f_*\text{Hom}_{\mathcal{O}_{\mathcal{X}_0}}(V_0, V_0 \otimes I)$ -torsors which in turn correspond to cohomology classes in

$$H^1(\mathcal{X}_{\text{lis-ét}}, f_*\underline{\text{Hom}}_{\mathcal{O}_{\mathcal{X}_0}}(V_0, V_0 \otimes I)) \cong H^1(\mathcal{X}_0, \underline{\text{Hom}}_{\mathcal{O}_{\mathcal{X}_0}}(V_0, V_0 \otimes I)) \cong \text{Ext}_{\mathcal{O}_{\mathcal{X}_0}}^1(V_0, V_0 \otimes I)$$

Define a category fibered in groupoids \mathcal{D} over $\mathcal{X}_{\text{lis-ét}}$ whose $T \in \mathcal{X}_{\text{lis-ét}}$ points consist of deformations (V, ϕ) of $V_0|_{T_0}$ where $T_0 = T \times_{\mathcal{X}} \mathcal{X}_0$. An arrow $a : (V, \phi) \rightarrow (V', \phi')$ over T is an isomorphism $a : V \cong V'$ which is compatible with the isomorphisms ϕ, ϕ' . The pullback of an object $(V, \phi) \in \mathcal{D}(T)$ along a morphism $t : T' \rightarrow T$ in $\mathcal{X}_{\text{lis-ét}}$ is simply $(t^*V, t^*\phi)$. The proof that \mathcal{D} is a stack on $\mathcal{X}_{\text{lis-ét}}$ is standard and we omit the details: the key points are that sheaves on a site form a stack and a sheaf on an algebraic space being a vector bundle is local in the étale topology. Our goal is to show that the stack \mathcal{D} is a gerbe banded by $f_*\text{Hom}_{\mathcal{O}_{\mathcal{X}_0}}(V_0, V_0 \otimes I)$ i.e. étale locally on an object T of $\mathcal{X}_{\text{lis-ét}}$, the stack \mathcal{D} admits a section, any two sections in $\mathcal{D}(T)$ are étale locally on T isomorphic and any section in $\mathcal{D}(T)$ has an automorphism group that is functorially identified with $(f_T)_*\underline{\text{Hom}}_{\mathcal{O}_{\mathcal{X}}}(V_0, V_0 \otimes I)$.

1. In fact, local sections exist: let $U \rightarrow T$ be a étale cover by a scheme, then $U \times_T \mathcal{X}_0 = U_0 \rightarrow U$ is a thickening (and hence induces a topological isomorphism) and $V_0|_{U_0}$ a vector bundle on U_0 . Therefore we may refine U with a trivializing Zariski cover of $V_0|_{U_0}$, call it U' , but because the trivial vector bundle always deforms this yields a section in $\mathcal{D}(U')$.
2. Next consider two sections $s, s' \in \mathcal{D}(T)$, by taking a further cover of T , $T' \rightarrow T$ where $T' = \text{Spec } A$ is affine we may suppose they are $s = (\mathcal{O}_{T'}^n, \phi)$ and $s' = (\mathcal{O}_{T'}^n, \phi')$. We need an isomorphism compatible with ϕ and ϕ' , but $\phi^{-1} \circ \phi' \in GL_n(T'_0)$ can be lifted to a section $\sigma \in GL_n(T')$, and $\sigma : \mathcal{O}_{T'}^n \rightarrow \mathcal{O}_{T'}^n$ yields an isomorphism of deformations between s and s' .
3. Fix an object s in \mathcal{D} over $T \in \mathcal{X}_{\text{lis-ét}}$, this is a smooth morphism $t : T \rightarrow \mathcal{X}$ and we write $f_T : T \times_{\mathcal{X}} \mathcal{X}_0 = T_0 \rightarrow T$ for the induced thickening. The object s corresponds to a deformation (V, ϕ) of $V_0|_{T_0}$ on T . Since t is flat we obtain an exact sequence of sheaves on $T_{\text{ét}}$

$$0 \rightarrow I|_{T_0} \rightarrow \mathcal{O}_T \rightarrow \mathcal{O}_{T_0} \rightarrow 0$$

We tensor this sequence with V and use the identification $\phi : V|_{T_0} \cong V_0|_{T_0}$ to obtain

$$0 \rightarrow I|_{T_0} \otimes_{\mathcal{O}_{T_0}} V_0|_{T_0} \rightarrow V \rightarrow V_0|_{T_0} \rightarrow 0$$

One sees that an automorphism of V respecting ϕ must induce the identity modulo $I|_{T_0} \otimes_{\mathcal{O}_{T_0}} V_0|_{T_0}$. It follows that automorphisms correspond to \mathcal{O}_T -module morphisms $V \rightarrow I|_{T_0} \otimes_{\mathcal{O}_{T_0}} V_0|_{T_0}$ i.e.

$$\begin{aligned} \text{Aut}_T(s) &= \text{Hom}_{\mathcal{O}_T}(V, I|_{T_0} \otimes_{\mathcal{O}_{T_0}} V_0|_{T_0}) \\ &= \text{Hom}_{\mathcal{O}_{T_0}}(V_0|_{T_0}, I|_{T_0} \otimes_{\mathcal{O}_{T_0}} V_0|_{T_0}) \\ &= \Gamma(T, (f_T)_* \underline{\text{Hom}}_{\mathcal{O}_{T_0}}(V_0|_{T_0}, I|_{T_0} \otimes_{\mathcal{O}_{T_0}} V_0|_{T_0})) \\ &= \Gamma(T, f_* \underline{\text{Hom}}_{\mathcal{O}_{\mathcal{X}_0}}(V_0, V_0 \otimes I)) \end{aligned}$$

The result follows. □

Below we begin a discussion of deformation-theoretic phenomena of vector bundles on tame Artin gerbes.

Lemma 2.2.3. *Consider a tame Artin gerbe $\mathcal{X}_A \rightarrow \text{Spec } A$ over an Artinian ring with residue field k . Let V and W be vector bundles on \mathcal{X} . If $V|_{\mathcal{X} \times_{\text{Spec } A} \text{Spec } K} \cong W|_{\mathcal{X} \times_{\text{Spec } A} \text{Spec } K}$ for some field extension $k \subset K$ then there exists an isomorphism $V \cong W$ on \mathcal{X} .*

Proof. Consider the sheaves $\underline{\text{Isom}}_{\mathcal{X}}(V, W) = I \subset \underline{\text{Hom}}_{\mathcal{X}}(V, W) = H$ over $\text{Spec } A$. Since gerbes are always flat and locally of finite presentation over their coarse space (see [56] Tag 06QI) it follows that H is representable by a finitely presented abelian cone over $\text{Spec } A$, (see Theorem D of [28]). Our hypothesis is that the open subfunctor, I , admits a $\text{Spec } K$ -point.

Since $H \times_{\text{Spec } A} \text{Spec } k$ is a abelian cone of finite type over a field, it must be isomorphic to \mathbf{A}_k^n . By hypothesis the open subfunctor $I \times_{\text{Spec } A} \text{Spec } k \subset H \times_{\text{Spec } A} \text{Spec } k \cong \mathbf{A}_k^n$ is nonempty after extending coefficients to the larger field K which implies $I \times_{\text{Spec } A} \text{Spec } k$ is nonempty.

However, nonempty open subschemes of \mathbf{A}_k^n always have a k -rational point when k is infinite i.e. there exists an isomorphism $\phi : V|_{\mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} k} \cong W|_{\mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} k}$. If k is finite then we exploit Lang's theorem: no smooth connected algebraic group over a finite field admits nontrivial torsors. Indeed, I is a (right) torsor under $\mathrm{Aut}(W)$ which itself is open in the *affine* space $\underline{\mathrm{Hom}}_{\mathcal{X}}(W, W)$ and hence smooth and connected. It follows that I admits a k -point.

It follows that (W, ϕ) is a deformation of $V|_{\mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} k}$ to \mathcal{X}_A . Since (V, can) is another deformation, the lemma will follow once we show deformations are unique.

By factoring the map $A \rightarrow k$ into square zero extensions we may consider a deformation situation as above. Let J be an ideal sheaf defining a square-zero thickening. To show

$$H^1(\mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} k, J \otimes \mathrm{End}(V)) = 0$$

i.e. that the deformation spaces vanish, observe that the Leray spectral sequence for the coarse space map $\mathcal{X} \times_{\mathrm{Spec} A} \mathrm{Spec} k \rightarrow \mathrm{Spec} k$ degenerates by tameness. Thus, the deformation spaces vanish identically since they correspond to the cohomology of a coherent sheaf over a point. The result follows. \square

Remark 2.2.4. We learned the technique above from our advisor, Max Lieblich. One can find a similar argument in Lemma 7.6 of [40]. The following lemma is also similar.

Lemma 2.2.5. *Let $\pi : \mathcal{X} \rightarrow \mathrm{Spec} R$ be a coarse space morphism from a tame Artin stack to a local noetherian ring with residue field k . Fix two vector bundles on V and W on \mathcal{X} so that $V_k \cong W_k$ become isomorphic on the closed fiber $\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k$, then $V \cong W$.*

Proof. Denote the closed fiber by $\mathcal{X} \times_X \mathrm{Spec} k = \mathcal{X}_k$ and the closed immersion by $\iota : \mathcal{X}_k \rightarrow \mathcal{X}$. We will show that the natural adjunction morphism of coherent sheaves

$$\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}}(V, W) \rightarrow \iota_* \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}_k}}(V_k, W_k)$$

is surjective on global sections. It is surjective as a morphism of sheaves. Note that there is an equality of functors $\Gamma(\mathcal{X}, -) = \Gamma(\mathrm{Spec} R, -) \circ \pi_*(-)$, and both π_* and $\Gamma(\mathrm{Spec} R, -)$ are

exact functors on $\mathrm{Qcoh}(\mathcal{X})$ and $\mathrm{Qcoh}(\mathrm{Spec} R)$. It follows that $\Gamma(\mathcal{X}, -)$ sends a surjection to a surjection, so that

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(V, W) \rightarrow \iota_* \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}_k}}(V_k, W_k)$$

is surjective. Thus, we may lift the isomorphism $V_k \cong W_k$ to a homomorphism $\phi : V \rightarrow W$. Since it is an isomorphism on the closed fiber, Nakayama's lemma implies it is surjective. However, since they are vector bundles of the same rank, the surjection ϕ must be an isomorphism. □

Lemma 2.2.6. *Let V be a vector bundle on a tame Artin gerbe $\mathcal{X} \rightarrow \mathrm{Spec} R$ over a strictly henselian local ring. Assume moreover that the local ring contains a copy of its residue field $k \subset R$. Then there is an isomorphism*

$$(\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k) \times_{\mathrm{Spec} k} \mathrm{Spec} R \cong \mathcal{X}$$

and a commutative diagram all of whose squares are cartesian.

$$\begin{array}{ccccc} \mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec} k & \longrightarrow & \mathrm{Spec} R & \longrightarrow & \mathrm{Spec} k \end{array}$$

Via this identification, we have an isomorphism

$$(V|_{\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k})|_{(\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k) \times_{\mathrm{Spec} k} \mathrm{Spec} R} \cong V$$

In particular, if a vector bundle V on $\mathrm{Spec} R$ is faithful at a point $\mathrm{Spec} K = \eta \in \mathrm{Spec} R$ then it is faithful at the closed point $\mathrm{Spec} k$.

Proof. First observe that $(\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k) \times_{\mathrm{Spec} k} \mathrm{Spec} R$ and \mathcal{X} are two tame Artin gerbes over $\mathrm{Spec} R$ which are isomorphic over the closed point. Moreover, they are both trivial gerbes over $\mathrm{Spec} R$ since any smooth surjective morphism to a strictly Henselian local ring admits a

section and the coarse space morphism of a gerbe is always smooth! Therefore we can write $(\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k) \times_{\mathrm{Spec} k} \mathrm{Spec} R \cong BG'$ and $\mathcal{X} \cong BG$ for finite flat linearly reductive group schemes G and G' over $\mathrm{Spec} R$. By the first method of proof in Lemma 2.17 of [1] it follows that there exists an isomorphism $BG \cong BG'$ over $\mathrm{Spec} R$. The first claim follows. Let V_k denote the restriction of V to the closed fiber i.e. $V_k = V|_{\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k}$. It follows that

$$W = V_k|_{(\mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} k) \times_{\mathrm{Spec} k} \mathrm{Spec} R} \cong V$$

since they are both vector bundles whose restrictions to the closed fiber are isomorphic. Then Lemma 2.2.5 shows that $V \cong W$.

We now explain how to deduce the final claim of the lemma. Note that a vector bundle V is faithful at a point $p : \mathrm{Spec} L \rightarrow \mathrm{Spec} R$ exactly when the fiber of the total space of the frame bundle $\mathrm{Frame}(V_p) \rightarrow \mathcal{X} \times_{\mathrm{Spec} R} \mathrm{Spec} L \rightarrow \mathrm{Spec} L$ is an algebraic space. However, the composed morphism $\mathrm{Frame}(V_K) \rightarrow \mathrm{Frame}(V) \rightarrow \mathrm{Frame}(V_k)$ is stabilizer preserving since it is pulled back along the morphisms of schemes $\mathrm{Spec} K \rightarrow \mathrm{Spec} R \rightarrow \mathrm{Spec} k$ (see for example Lemma 7.6 of [48]). Therefore because $\mathrm{Frame}(V_K) \rightarrow \mathrm{Frame}(V_k)$ is fpqc and stabilizer preserving $I_{\mathrm{Frame}(V_K)} \rightarrow \mathrm{Frame}(V_K)$ being an isomorphism implies $I_{\mathrm{Frame}(V_k)} \rightarrow \mathrm{Frame}(V_k)$ is as well i.e. if $\mathrm{Frame}(V_K)$ is an algebraic space then so is $\mathrm{Frame}(V_k)$, as desired.

□

We should interpret this as saying that on a tame Artin gerbe the representation type of the fibers of a vector bundle is rigid. Using this, we prove a very useful result: the locus where a vector bundle on a tame gerbe is faithful is open and closed.

Proposition 2.2.7. *Let V be a vector bundle on a tame Artin gerbe $\mathcal{X} \rightarrow X$ which is of finite type and separated over a field k . Then the locus where V is faithful is open and closed.*

Proof. First we show the faithful locus is open. Let K denote the kernel of the natural morphism of group sheaves over \mathcal{X} :

$$0 \rightarrow K \rightarrow I_{\mathcal{X}} \rightarrow \underline{\mathrm{Aut}}(V)$$

Note that K is finite over \mathcal{X} and hence corresponds to a coherent algebra \mathcal{A} . The locus where V is faithful is precisely the locus where \mathcal{A} has fiber rank ≤ 1 . Since fiber rank is an upper semicontinuous function it follows that the faithful locus is open, call it U . To show that the faithful locus is closed it suffices to show that if $\eta \in U$ and if $\eta \rightsquigarrow p$ is a closed point that η specializes to, then $p \in U$. First, we reduce to the case when k is algebraically closed and p is a k -point.

Our goal is to show that $\text{Frame}(V_p)$ is an algebraic space if $\text{Frame}(V_\eta)$ is. So if we base change along the morphism $\text{Spec } \bar{k} \rightarrow \text{Spec } k$ we obtain stabilizer preserving morphisms $f : \mathcal{X}_{\bar{k}} = \mathcal{X} \times_{\text{Spec } k} \text{Spec } \bar{k} \rightarrow \mathcal{X}$, $\text{Frame}(V_{f^{-1}(p)}) \rightarrow \text{Frame}(V_p)$, and $\text{Frame}(V_{f^{-1}(\eta)}) \rightarrow \text{Frame}(V_\eta)$. Thus it suffices to show $\text{Frame}(V_{f^{-1}(p)})$ is an algebraic space. But $f^{-1}(p)$ is a finite Artinian \bar{k} -scheme so by Tag 0BPW of [56] it suffices to assume $f^{-1}(p)$ is a finite union of \bar{k} -points p' . Since f is a flat morphism it satisfies the going-down property and hence for any $p' \in f^{-1}(p) \subset \mathcal{X}_{\bar{k}}$ there is a $\eta' \in f^{-1}(\eta) \subset f^{-1}(U) \subset \mathcal{X}_{\bar{k}}$ with $\eta' \rightsquigarrow p'$. It remains to show that $\text{Frame}(V_{p'})$ is an algebraic space when $\text{Frame}(V_{\eta'})$ is.

Consider the strictly local ring about p' and pullback along $g : \text{Spec}(\mathcal{O}_{p'}^{sh}) \rightarrow X$ to obtain a map of algebraic spaces $\text{Frame}(V_{g^{-1}(\eta')}) \rightarrow \text{Frame}(V_{\eta'})$. Since p' is a \bar{k} -point, the previous lemma applies to the gerbe $\mathcal{X} \times_X \mathcal{O}_{p'}^{sh} \rightarrow \mathcal{O}_{p'}^{sh}$ and the vector bundle $V|_{\mathcal{X} \times_X \mathcal{O}_{p'}^{sh}}$. Since this vector bundle is faithful at a point in $g^{-1}(\eta')$ it shows that $\text{Frame}(V_{p'})$ has no nontrivial stabilizers. The result follows. □

In the case when \mathcal{X} is a gerbe banded by μ_n we can say more

Proposition 2.2.8. *Let $\mathcal{X} \rightarrow X$ be a gerbe banded by μ_n and V a vector bundle on \mathcal{X} . The locus where V is a twisted vector bundle is open and closed.*

Proof. There is a canonical decomposition of $V = \bigoplus_{\chi} V_{\chi}$ where χ runs through the characters of μ_n . In fact V is twisted precisely when $V_{\iota} = V$ where $\iota : \mu_n \rightarrow \mathbf{G}_m$ is the standard character, see 3.1.1.4 of [39]. Let m denote the rank of V on a connected component of $\mathcal{X}_0 \subset \mathcal{X}$, then the locus where V is twisted on \mathcal{X}_0 is precisely the locus $\{x \in \mathcal{X}_0 | \text{rank}((V_{\iota})_x) = m\}$. Since V_{ι}

is a direct summand of a vector bundle, its rank is a locally constant function on \mathcal{X}_0 and in particular is constant on components. Thus the locus where V is twisted is open and closed. \square

2.3 Mayer Vietoris squares and Flat Neighborhoods via Tannakian Duality

Definition 2.3.1. Consider the cartesian square of algebraic stacks, where i is an open immersion.

$$\begin{array}{ccc} U \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \xrightarrow{i} & X \end{array}$$

In addition, suppose that $W \times_X Y \rightarrow Y$ is an isomorphism for every $W \rightarrow X$ whose scheme-theoretic image is disjoint from U . Following [30] such a square is called a weak Mayer-Vietoris square. If in addition the scheme-theoretic image of any such W is tor-independent of f , the square is called a tor-independent Mayer-Vietoris square. If the square is weak and in addition f is flat then it is called a flat Mayer-Vietoris square.

Remark 2.3.2. For our purposes we will not need tor-independent Mayer-Vietoris square which aren't already flat. The reason we introduce the concept is to ensure a smooth transition between our exposition and that of [30]. In general it may seem difficult to check if a cartesian square is of this form, so we begin by giving sufficient conditions for a square to be Mayer-Vietoris of some type.

Lemma 2.3.3. *Suppose we have a flat/weak Mayer-Vietoris square as above as well as a morphism of algebraic stacks $X' \rightarrow X$. We may base change the square to X' to obtain*

$$\begin{array}{ccc} U'_Y & \longrightarrow & Y' \\ \downarrow & & \downarrow f' \\ U' & \xrightarrow{i'} & X' \end{array}$$

then this new square is also a flat/weak Mayer-Vietoris square. Moreover, the property of being a flat/tor/weak Mayer-Vietoris square is flat-local on X .

Proof. See Lemma 3.1 of [30] □

Lemma 2.3.4. *Let X be a quasicompact algebraic stack and consider the cartesian square*

$$\begin{array}{ccc} U \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ U & \xrightarrow{i} & X \end{array}$$

where f is flat. Let Z denote the reduced induced structure on $X \setminus U$ and suppose it yields a finitely presented closed immersion $Z \rightarrow X$. Then the diagram is a flat Mayer-Vietoris square if and only if the morphism $Y \times_X Z \rightarrow Z$ is an isomorphism.

Proof. The only if part is immediate from the definition. Now suppose that $Y \times_X Z \rightarrow Z$ is an isomorphism. Since $f : Y \rightarrow X$ is flat it is certainly tor-independent relative to Z . Lemma 3.2 (2) of [30] then implies that the square is tor-independent, in particular it is a weak Mayer-Vietoris square where f is flat. The result follows. □

Theorem 2.3.5. *Suppose we have a flat Mayer-Vietoris square such that the open immersion is quasicompact. Then the natural functor*

$$\mathrm{Qcoh}(X) \rightarrow \mathrm{Qcoh}(U) \times_{\mathrm{Qcoh}(U_Y)} \mathrm{Qcoh}(Y)$$

is an equivalence of categories. Consequently, to obtain a finite rank vector bundle on X it suffices to find one on Y , another on U along with a specified isomorphism between their restrictions to U_Y . More precisely, the natural functor

$$\mathrm{Vect}(X) \rightarrow \mathrm{Vect}(U) \times_{\mathrm{Vect}(U_Y)} \mathrm{Vect}(Y)$$

is an equivalence of categories.

Proof. The first statement is Theorem B (1) of [30]. We explain the second statement: fully faithfulness follows from the quasicohherent case since the natural functor $\text{Vect}(X) \rightarrow \text{Qcoh}(X)$ is fully faithful. To show essential surjectivity we choose an triple (V, W, ϕ) and observe that it must be isomorphic to the restriction $(F|_U, F|_Y, \text{can})$ where F is a quasicohherent sheaf on X . It remains to show that F must be a finite rank vector bundle. After passing to a smooth cover we may assume X is a scheme and then this follows because (1) a quasicohherent sheaf which is finitely generated after a faithfully flat base change must be finitely generated and (2) a finitely presented sheaf on X which is locally free after a faithfully flat cover must be locally free. Indeed, it must be flat because flatness is reflected by faithfully flat maps and now the local freeness follows since flat finitely-presented modules are automatically free over local rings and again the finite presentation allows us to conclude the module is free in a neighborhood. \square

The next result tells us that the schematic locus of a separated locally noetherian algebraic space always contains the codimension 1 points. A very special case of this fact is that separated locally noetherian algebraic spaces of dimension 1 are always schemes.

Proposition 2.3.6. *Suppose X is a locally noetherian separated algebraic space. If $x \in X$ is a point with $\dim \mathcal{O}_p^{sh} \leq 1$, then there is an open subscheme $U \subset X$.*

Proof. This is Tag 0ADD in [56]. \square

The next theorem guarantees that given any finite set of codimension ≤ 1 points in a separated locally noetherian algebraic space, there exists a affine open subscheme containing this finite set.

Proposition 2.3.7. *Suppose X is a separated, locally noetherian algebraic space and x_1, \dots, x_n is a finite set of points all having codimension ≤ 1 . Then there is an affine open $\text{Spec } B \subset X$ containing all the x_i .*

Proof. By the previous proposition we may find a open subscheme $U \subset X$ containing all the

x_i . Now, since any finite set of points in a scheme admits a quasicompact open neighborhood, we may replace U with a noetherian scheme. To finish, apply Tag 09NN in [56]. \square

Remark 2.3.8. One says an algebraic space satisfies the Chevellay-Kleimann property if every finite set of points admits a common affine open neighborhood. The previous two propositions may be interpreted as saying every separated, locally noetherian algebraic space satisfies the Chevellay-Kleimann property in codimension ≤ 1 .

A novelty in dealing with algebraic spaces and stacks is that points and closed subsets need not admit affine open neighborhoods, for this reason we consider a more flexible concept: flat neighborhoods. Let Z denote a closed substack of an algebraic stack X and I_Z the corresponding quasicoherent ideal sheaf. Then $Z^{[n]}$ is defined to be the closed substack defined by the quasicoherent ideal sheaf $I_Z^n \subset \mathcal{O}_X$.

Definition 2.3.9. Let X be a locally noetherian algebraic stack and $Z \rightarrow X$ a closed substack. We say a morphism $V \rightarrow X$ is a flat neighborhood of Z if it is flat and the pullback $V \times_X Z \rightarrow Z$ is an isomorphism for every n .

We begin with a preparatory lemma which can be proven without the Tannakian machine. Roughly, it says that closed points on decent algebraic spaces admit local flat neighborhoods.

Lemma 2.3.10. *Let X be a quasiseparated noetherian algebraic space. Let $z \in X$ be a closed point, then there is a noetherian local ring (R, m, k) and a morphism $\text{Spec } R \rightarrow X$ (sending m to z) which is a flat neighborhood of z . Moreover, the dimension of R is equal to the codimension of the point $z \in X$.*

Proof. Since X is quasiseparated it is decent (Tag 03JX in [56]) and therefore by Tag 0BBP in [56] there is a étale morphism $(U, u) \rightarrow (X, z)$ where U is a affine scheme, u is the only point in U lying over z , and $u : \text{Spec } k(u) \rightarrow X$ is a monomorphism. The closed point z also admits the monomorphism given by the closed immersion $\text{Spec } \mathcal{O}_X/I_z = \text{Spec } k(z) \rightarrow X$, see Tag 0AHB in [56]. When we base change $\text{Spec } k(z) \rightarrow X$ along $\text{Spec } k(u) \rightarrow U \rightarrow X$ we obtain monomorphisms $\text{Spec } k(z) \times_X \text{Spec } k(u) \rightarrow \text{Spec } k(u)$ and $\text{Spec } k(z) \times_X \text{Spec } k(u) \rightarrow \text{Spec } k(z)$.

By Tag 03DP in [56] these must both be isomorphisms and it follows that $k(z) \cong k(u)$ over X . Since we chose U so that the fiber over z was a singleton this implies $U \times_X \text{Spec } k(z) \rightarrow k(z)$ is an isomorphism. Therefore, $\text{Spec } \mathcal{O}_{U,u} \rightarrow X$ is a flat neighborhood of $z \in X$ as desired. \square

Remark 2.3.11. In order to exhibit flat neighborhoods of closed substacks which have nontrivial geometry we require the notion of Coherent Tannakian duality as discussed in [29]. To motivate this concept, fix a noetherian algebraic stack X and observe that a morphism $f : T \rightarrow X$ induces a functor of abelian categories $f^* : \text{Coh}(X) \rightarrow \text{Coh}(T)$. This functor is right exact, respects tensor products and sends $\mathcal{O}_X \mapsto \mathcal{O}_T$. Let $\text{Hom}_{r\otimes, \cong}(\text{Coh}(X), \text{Coh}(T))$ denote the category of functors satisfying these conditions with the morphisms being natural isomorphisms of functors. Coarsely, the following version of Tannkian duality asserts that every functor as described above comes from a morphism $X \rightarrow T$. More precisely, we have the following

Theorem 2.3.12. (*Hall-Rydh*) *Let X be a noetherian algebraic stack with quasi-affine diagonal. If T is a locally noetherian algebraic stack, the functor*

$$\text{Hom}(T, X) \rightarrow \text{Hom}_{r\otimes, \cong}(\text{Coh}(X), \text{Coh}(T))$$

is an equivalence of categories.

Proof. Let $\text{Hom}_{c\otimes, \cong}^{ft}(\text{Qcoh}(X), \text{Qcoh}(T))$ denote the category whose objects are right-exact functors $F : \text{Qcoh}(X) \rightarrow \text{Qcoh}(T)$ which satisfy $F(\mathcal{O}_X) = \mathcal{O}_T$, preserve small direct sums and finite type sheaves, and respect tensor products. The morphisms of this category are the natural isomorphisms of functors. Then, when X and T are locally noetherian the natural functor

$$\text{Hom}_{r\otimes, \cong}(\text{Coh}(X), \text{Coh}(T)) \rightarrow \text{Hom}_{c\otimes, \cong}^{ft}(\text{Qcoh}(X), \text{Qcoh}(T))$$

is an equivalence. The result now follows from Theorem 8.4(i) in [29]. \square

Corollary 2.3.13. *Let A be a noetherian I -adically complete ring and suppose X is a noetherian algebraic stack with quasi-affine diagonal. Then $X(\mathrm{Spec} A) \cong \lim X(\mathrm{Spec} A/I^n)$.*

Proof. This is Corollary 1.5 of [29]. □

Theorem 2.3.14. *Let X be a noetherian, separated, algebraic space. Suppose $Z = Z^{[0]} = \mathrm{Spec} A_0 \rightarrow X$ is a closed immersion, then there is a flat neighborhood of Z given by a noetherian affine scheme $Y = \mathrm{Spec} A \rightarrow X$ of dimension $\leq \dim(X)$.*

Proof. Since $Z \rightarrow X$ is a closed immersion, it corresponds to a quasicoherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Let $Z^{[i]}$ denote the closed subscheme corresponding to $\mathcal{I}^{i+1} \subset \mathcal{O}_X$. Because these are affine over X and there are surjective morphisms of \mathcal{O}_X -algebras $\mathcal{O}_X/\mathcal{I}^{j+1} \rightarrow \mathcal{O}_X/\mathcal{I}^j$ we obtain closed immersions $Z^{[j-1]} \rightarrow Z^{[j]}$ for every $j \geq 1$. Since these are finite surjective morphisms and Z is affine, Tag 07VP in [56] implies every $Z^{[i]}$ is an affine scheme. Since the global sections functor is exact on affine schemes we obtain for each $i \geq j$ a surjective ring map

$$\phi_{ij} : \Gamma(Z^{[i]}, \mathcal{O}_X/\mathcal{I}^{i+1}) = A_i \rightarrow \Gamma(Z^{[j]}, \mathcal{O}_X/\mathcal{I}^{j+1}) = A_j$$

If I_i denotes the kernel of $A_i \rightarrow A_0$ then one sees that the kernel of ϕ_{ij} is I_i^{j+1} . Moreover, $I_1/I_1^2 = I_1$ is a finitely generated A_0 -module since it is an ideal of the noetherian ring A_1 . It follows from 0.7.2.7 and 0.7.2.8 in [25] that the limit $A = \lim A_i$ is an adic noetherian ring with respect to the kernel of the natural map $A \rightarrow A_0$, call it I .

By the previous theorem we obtain a map $Y = \mathrm{Spec} A \rightarrow X$. It also has the property that the projection map $Y \times_X Z \rightarrow Z$ is an isomorphism. Indeed, $Y \times_X Z \rightarrow Y$ is a closed immersion and hence corresponds to a A -algebra in $\mathrm{Coh}(\mathrm{Spec} A)$, call it \mathcal{B} . However, since $\mathrm{Coh}(\mathrm{Spec} A) \rightarrow \lim \mathrm{Coh}(\mathrm{Spec} A/I^i)$ is an equivalence of categories it suffices to determine the coherent algebra structure of \mathcal{B} modulo I^j for every $j > 0$. However, pulling back this coherent algebra to $\mathrm{Spec} A/I^i = Z^{[i-1]}$ yields the sheaf of algebras corresponding to the affine morphism $Z \times_X Z^{[i-1]} \rightarrow Z^{[i-1]}$ and $Z \times_X Z^{[i-1]} \cong Z$ over $Z^{[i-1]}$. It follows that $Y \times_X Z \cong Z$

over $\text{Spec } A$ because their sheaves of coherent algebras are isomorphic over $\text{Spec } A/I^{j+1} = Z^{[j]}$ for all $j \geq 0$. For each i a similar argument shows that the projection map $Y \times_X Z^{[i]} \rightarrow Z^{[i]}$ is an isomorphism since $Z^{[j]} \times_X Z^{[i]} \cong Z^{[i]}$ over X for all $j \geq i$. Since both X and Y are noetherian a variant of the local criterion for flatness (Tag 0523 in [56]) implies that $Y \rightarrow X$ is flat over the points of $Y \times_X Z = V(I) \subset \text{Spec } A$. However, since A is I -adic all maximal ideals must contain I (Theorem 8.2(i), pg. 57 in [43]) and therefore the morphism $\text{Spec } A \rightarrow X$ must be flat.

To see why $\dim(A) \leq \dim(X)$ observe that at any closed point x of $\text{Spec } A$ the fiber $f^{-1}(f(x))$ is a singleton. Now apply [43] Theorem 15.1 (i) to conclude that the height of x is no larger than the height of $f(x)$.

□

2.4 Classical K-Theory

The goal of this section is to describe fundamental theorems on the K-theory of noetherian rings. Fix a commutative ring and consider the free abelian group on all isomorphism classes of finitely generated projective modules of constant rank over R , this forms a commutative ring if we define multiplication by using the tensor product. We form $K_0(\text{Spec } R)$ (or $K(R)$ for short) as the quotient of this ring by the subgroup generated by elements of the form $[P_1] - [P_2] - [P_3]$ when $0 \rightarrow P_2 \rightarrow P_1 \rightarrow P_3 \rightarrow 0$ is an exact sequence. The resulting subgroup is closed under $-\otimes_R P$. Thus, the subgroup is an ideal and the resulting quotient is the ring $K(R)$. By the hypothesis that each such module has constant rank, one can deduce that the rank function $P \mapsto \dim_{k(\eta)} P_\eta$ is well behaved and descends to $K(R)$, more precisely, there exists a ring homomorphism $\text{rk} : K(R) \rightarrow \mathbf{Z}$. We say that a module M is stably-isomorphic to M' if there exists $m \in \mathbf{Z}$ with $M \oplus R^m \cong M' \oplus R^m$, it is not difficult to see that two projective modules are stably isomorphic if and only if they belong to the same class in $K(R)$. We begin by listing two fundamental theorems below.

Theorem 2.4.1. *Let R be a noetherian ring with Krull dimension d and $K(R)$ the associated*

Grothendieck ring described above. Then

1. (Serre) *If P is a projective module such that $\text{rk}(P) = n > d$ then $P \cong P_0 \oplus R^{n-d}$*
2. (Bass) *If P and P' are stably isomorphic projective modules with rank larger than d then $P \cong P'$.*

Proof. For Serre's theorem see Theorem 1 in [55], for Bass' see Chapter IX, 4.1 in [10]. \square

Using these results and elementary methods one can say more:

Proposition 2.4.2. *Let R and $K(R)$ be as above. Then*

1. *The kernel of the rank function is a nil ideal.*
2. *If two projective modules P and Q belong to the same class in $K(R)$ then there is a positive integer N such that $P^{\oplus N} \cong Q^{\oplus M}$.*
3. *Any class in $K(R)$ with positive rank has an integer multiple which is represented by a projective module P .*
4. *If P has $P^{\otimes n} \cong R^{\oplus m}$ then there is a integer N such that $P^{\oplus N}$ is also free.*

Proof. This is exactly Proposition 3.1.4.3 in [41], there one can find a sketch of the proof. For the details see [19] (Lemma K, pg. 188) for the first statement, Proposition 4.2. Chapter IX in [10] for the second and third. These statements are closely related to the theorems of Bass and Serre above. We present a proof of 4. since we do not understand the corresponding proof in [41].

In $K(R)$ one can write $[P] = a + \beta$ where $a = [R^{\oplus \text{rk}(P)}]$ and $\beta = [P] - [R^{\oplus \text{rk}(P)}]$ is a class with zero rank. By hypothesis $[P]^n = (a + \beta)^n = [R^{\oplus c}]$ and multiplying this expression out in $K(R)$ we deduce that $a^n = [R^{\oplus c}]$ since a^n is free and is the only summand in $a^n + na^{n-1}\beta + \dots \beta^n$ with nonzero rank. It follows that

$$na^{n-1}\beta + \dots \beta^n = 0$$

in $K(R)$. Let k be the smallest positive integer with $\beta^k = 0$ (which exists by 1.), then multiplying the equation above by β^{k-2} yields

$$n \operatorname{rk}(P)^{n-1} \beta^{k-1} = 0$$

Thus some multiple of β^{k-1} vanishes in K-theory. Multiplying the equation above by β^{k-3} yields

$$n \operatorname{rk}(P)^{n-1} \beta^{k-2} + l \beta^{k-1} = 0$$

so multiplying this by $n \operatorname{rk}(P)^{n-1}$ shows $(n \operatorname{rk}(P)^{n-1})^2 \beta^{k-2}$ must be zero. By induction it follows that

$$(n \operatorname{rk}(P)^{n-1})^{k-1} \beta = 0$$

This implies $i[P] = j[R]$ for integers i and j . The second part of the proposition now shows $P^{\oplus N}$ is free for large N . \square

Corollary 2.4.3. *Given a projective module P and a fixed positive integer n , there are nonzero free modules F_0 and F_1 and a projective P' such that $P \otimes P'^{\otimes n} \otimes F_0 \cong F_1$.*

Proof. This is Corollary 3.1.4.4 of [41]. \square

2.5 Rigidification

The purpose of this section is to prove the following

Theorem 2.5.1. *Let \mathcal{X} be a separated, normal and noetherian Deligne-Mumford stack, then there exists a flat subgroup stack $G \subset I_{\mathcal{X}}$ such that the associated rigidification \mathcal{X}^{rig} has generically trivial stabilizers.*

Remark 2.5.2. This was used in [37] when \mathcal{X} is regular without proof. Moreover, in a preprint of [1] the case when \mathcal{X} is regular appeared in the appendix as Remark A.3. A complete argument in the normal setting can be found within the proof of 2.1 in [49]. The proof below has a slightly different flavor, so we include it here.

Proof. Since \mathcal{X} is noetherian and normal we may assume \mathcal{X} is integral. By hypothesis the morphism $f : I_{\mathcal{X}} \rightarrow \mathcal{X}$ is finite and unramified. Let $G \subset I_{\mathcal{X}}$ denote the scheme-theoretic closure of the generic fiber of f , we will show that G is a flat subgroup stack of $I_{\mathcal{X}}$. Note that $g : G \rightarrow \mathcal{X}$ is finite, unramified and has the property that every component dominates \mathcal{X} . Since it is also representable we may appeal to Tag 04HJ to find a étale neighborhood $(U, u) \rightarrow (\mathcal{X}, x)$ which yields a cartesian square:

$$\begin{array}{ccc} V = \bigsqcup_{i=1}^n V_i & \longrightarrow & G \\ \downarrow (f_i) & & \downarrow \\ U & \xrightarrow{i} & \mathcal{X} \end{array}$$

where $f_i : V_i \rightarrow U$ is a closed immersion and V_i connected. Since being a flat morphism is étale local, to show that g is flat it suffices to show that each f_i is an open immersion. Since i is étale it follows that U is normal and therefore we may write $U = \bigsqcup U_i$ where the U_i are irreducible components of U . It follows f_i factors as a closed immersion $f'_i : V_i \rightarrow U_{i'}$ where $U_{i'}$ is a normal (integral) scheme. If we show that f'_i is surjective then since $U_{i'}$ is reduced it must be an isomorphism so $V_i \rightarrow U$ is a open immersion. Since the morphism is finite we only need to show that a generic point of V_i maps to the generic point of $U_{i'}$. But every generic point of V_i is a generic point of V and so it must map to a generic point of G (since $V \rightarrow G$ is étale, see Tag 0ABV) which in turn maps to a generic point of \mathcal{X} . Since $i : U \rightarrow \mathcal{X}$ is étale the generic fiber consists precisely of the generic points of U i.e. every generic point of V_i maps to the generic point of $U_{i'}$. It follows that f_i is an open immersion and that f is flat. In fact, because f is also finite and unramified, it follows that f is finite étale.

Next we show that $G \subset I_{\mathcal{X}}$ is a subgroup-stack. Because the inversion map is an isomorphism over \mathcal{X} it must preserve the generic fiber of $I_{\mathcal{X}} \rightarrow \mathcal{X}$, so it sends the closure of the generic fiber, G , to itself. Consider the induced multiplication map $m : G \times_{\mathcal{X}} G \rightarrow I_{\mathcal{X}} \times_{\mathcal{X}} I_{\mathcal{X}} \rightarrow I_{\mathcal{X}}$, we need to show that it factors through the closed substack $G \subset I_{\mathcal{X}}$. Observe that $G \times_{\mathcal{X}} G$ is finite étale over \mathcal{X} (since G is), hence every generic point of $G \times_{\mathcal{X}} G$

maps to the generic point of \mathcal{X} . But since m is a map over \mathcal{X} every generic point of $G \times_{\mathcal{X}} G$ maps to a point in the generic fiber of $I_{\mathcal{X}} \rightarrow \mathcal{X}$. Consider the cartesian diagram

$$\begin{array}{ccc} K & \xrightarrow{i'} & G \times_{\mathcal{X}} G \\ \downarrow & & \downarrow \\ G & \xrightarrow{i} & I_{\mathcal{X}} \end{array}$$

By the previous discussion the closed substack $i' : K \rightarrow G \times_{\mathcal{X}} G$ must contain all the generic points of the (reduced) stack $G \times_{\mathcal{X}} G$, thus i' is an isomorphism. This implies G is a flat subgroup stack of $I_{\mathcal{X}}$. The result now follows from Theorem A.1 of [1] applied to the subgroup stack $G \subset I_{\mathcal{X}}$. □

2.6 Artin Contractions

Here we describe a method of constructing Algebraic spaces which are not, in general, schemes. All algebraic spaces will be separated and finite-type over a fixed field k .

Theorem 2.6.1. *Let \mathfrak{X}' be a formal algebraic space, let $Y' = V(I')$ be the closed subspace defined by a defining ideal I' of \mathfrak{X}' , and let $f : Y' \rightarrow Y$ be a proper map of algebraic spaces. Suppose the following conditions are satisfied:*

1. *For every coherent sheaf F on \mathfrak{X}' , we have*

$$R^1 f_*(I^n F / I^{n+1} F) = 0 \text{ if } n \gg 0$$

2. *For every n , the map of sheaves on Y*

$$f_*(\mathcal{O}_{\mathfrak{X}'}/I^n) \times_{f_*\mathcal{O}_{Y'}} \mathcal{O}_Y \rightarrow \mathcal{O}_Y$$

is surjective

Then there is a formal modification $\mathfrak{f} : \mathfrak{X}' \rightarrow \mathfrak{X}$ and a defining ideal $I \subset \mathcal{O}_{\mathfrak{X}}$ such that $V(I) = Y$ and that \mathfrak{f} induces f on Y' .

Proof. This is precisely [6][Theorem 6.2] □

Remark 2.6.2. Suppose we are working with k -varieties and that Y is a k -point, then the second condition is vacuous. Formal modifications constructed when Y is a single point should be thought of as formal contractions.

Remark 2.6.3. The above theorem can be understood as giving sufficient cohomological criteria for determining if a contraction can exist formally locally. The next result guarantees that a contraction exists in the category of algebraic spaces as long as it exists formally locally.

Theorem 2.6.4. *Let Y' be a closed subset of an algebraic space X' of finite type over k , and let \mathfrak{X}' be the formal completion of X' along Y' . Given a formal modification $\mathfrak{f}_1 : \mathfrak{X}' \rightarrow \mathfrak{X}$, there is a modification*

$$F : X' \rightarrow X, Y \subset X$$

and an isomorphism ϕ of its completion \mathfrak{f} with \mathfrak{f}_1 . The pair (f, ϕ) is determined up to unique isomorphism.

Proof. This is precisely [6][Theorem 3.1]. □

Remark 2.6.5. If we seek to contract a integral subvariety $Y' \subset X'$ where X' is normal (i.e. $Y = \text{Spec } k$ and f is the structure morphism) then the resulting space can be taken to be normal. Indeed, replace X in the theorem above by its normalization. By the universal property of normalization and the normality of X' we obtain a proper map $X' \rightarrow X$ which contracts Y' .

Joseph Mazur has given a simpler cohomological condition in [44]. Indeed, he shows that to check a subvariety is contractible, it is enough to verify that it has ample conormal bundle. Since we will only contract curves, we give a simple proof below.

Theorem 2.6.6. *Let $Y' \subset X'$ be a closed immersion of smooth k -varieties with Y' a proper curve. Moreover, assume that Y' has ample conormal bundle, i.e. $I'_{Y'}/I'^2_{Y'}$ is ample. Then there*

is a morphism $f : X' \rightarrow X$ of algebraic spaces which maps Y' to a closed point $p \in X$ and f is an isomorphism along $X \setminus \{p\}$, i.e. Y' can be contracted to a point.

Proof. Copying the notation of the previous theorems, we set \mathfrak{X} to be the completion of X' along Y' , I' the ideal defining Y' in X' , $Y = \text{Spec } k$, and $Y' \rightarrow \text{Spec } k = Y$ the structure map. Fix a coherent sheaf F on \mathfrak{X}' , to apply 2.6.1 and 2.6.4 we need to show

$$H^1(Y', I'^n F / I'^{n+1} F) = 0, n \gg 0$$

However, the sheaf $I'^n F / I'^{n+1} F$ fits naturally into an exact sequence

$$0 \rightarrow K \rightarrow \text{Sym}^n I' / I'^2 \otimes F \rightarrow I'^n F / I'^{n+1} F \rightarrow 0$$

and by the ampleness of the conormal bundle I' / I'^2 the cohomology of $\text{Sym}^n I' / I'^2 \otimes F$ vanishes for $n \gg 0$ (see Proposition 3.3 in [31]). Moreover, since K is topologically supported on the curve Y' it has no second cohomology. Therefore, taking the long exact sequence associated to the above sequence shows that $I'^n F / I'^{n+1} F$ has vanishing first cohomology for all large n . Theorem 2.6.1 gives us a formal contraction and Theorem 2.6.4 allows us to realize this formal contraction in the category of algebraic spaces.

□

When studying the surjectivity of the Brauer map we will turn to Artin contractions as a source of interesting examples. For the moment, we shall revisit a famous example of Nagata: an algebraic space surface which is not a scheme.

Theorem 2.6.7. (Nagata) *Let k be an uncountable algebraically closed field. There is a normal algebraic space surface which is finite-type and proper over k which is not a scheme.*

Proof. Fix an elliptic curve $E \subset \mathbf{P}_k^2$ with base point P . Choose 10 points P_1, \dots, P_{10} in such a way that the corresponding degree zero line bundles have no relations. More precisely, the map

$$\mathbf{Z}^{10} \rightarrow \langle \mathcal{O}(P_1 - P), \dots, \mathcal{O}(P_{10} - P) \rangle \subset \text{Pic}_{E/k}^\circ(k)$$

sending $e_i \mapsto \mathcal{O}(P_i - P)$ is an isomorphism. To show this is possible we will choose these points inductively. First, note that the torsion of $\text{Pic}_{E/k}^\circ(k)$ is countable and since $\text{Pic}_{E/k}^\circ(k) = E(k)$ is uncountable there exists a point P_1 corresponding to a non-torsion class. If we have chosen P_1, \dots, P_i then observe that $\langle \mathcal{O}(P_1 - P), \dots, \mathcal{O}(P_i - P) \rangle + \text{Pic}_{E/k}^\circ(k)_{\text{tors}}$ is still countable. Thus we may choose P_{i+1} with the desired property.

Blow up \mathbf{P}_k^2 along $\{P_1, \dots, P_{10}\}$ and denote the result as X and let the strict transform of the elliptic curve be denoted by $E \subset X$. Observe that $E^2 = -1$ in X i.e. the ideal sheaf of E in X restricted to E has degree -1 . This implies the conormal bundle of E is ample and the theorem above implies we may contract E in X . Let $f : X \rightarrow Y$ denote this contraction where Y is normal. Since the map is surjective, Y is proper over k because X is. It remains to check that Y cannot be a scheme.

If it were, there would exist a affine open neighborhood of the point $f(E) = Q : \text{Spec } A \subset Y$. It is well-known that complements of affine open neighborhoods over separated normal schemes have pure codimension 1. Thus there is a curve, C , in Y not meeting Q . It follows that C (viewed as a divisor in X) cannot meet E . Moreover, in \mathbf{P}_k^2 the closure of the generic points of C can only meet E at $\{P_1, \dots, P_{10}\}$. But since C is linearly equivalent to nL for a line $L \subset H$ it follows that the P_i have a nontrivial relation in $\text{Pic}_{E/k}^\circ(k)$, a contradiction. \square

Remark 2.6.8. In 2.11 of [7] Artin shows that all normal surfaces over a prime field of characteristic $p \neq 0$ are projective. Although the proof is stated for schemes, the same proof works verbatim as Chow's lemma holds for such algebraic spaces. The example given above obviously fails in this case as there exists no non-torsion points on finite-type group schemes over such fields.

Chapter 3

THE RESOLUTION PROPERTY: PART ONE

3.1 Separated Algebraic Space Surfaces satisfy the Resolution Property

In this section we discuss the question of the resolution property for algebraic spaces. In [23] Gross shows that when X is a 2-dimensional *scheme*, finite type and proper over a noetherian ring, it must satisfy the resolution property. Here we give a limited extension to the case when X is an algebraic space.

Theorem 3.1.1. *Let X be a 2-dimensional, integral algebraic space, which is finite type and separated over a field. If X is regular in codimension 1, then it must satisfy the resolution property.*

The novelty of Theorem 3.1.1 is that it addresses the first class of algebraic spaces which admit non-schematic examples. Prior to this, there was no class of non-schematic algebraic spaces for which the resolution property was known. There are two reasons Schröer-Vezzosi [54] and Gross' [23] method does not immediately extend to the algebraic space setting. First, they make an initial reduction which requires the existence of an affine neighborhood about arbitrary points of X . In particular, they show that there is a almost-ample family of rank 1 sheaves which are locally free away from finitely many closed points and moreover that there are enough of these so that every coherent sheaf is a quotient of a polynomial combination of such rank one sheaves. Second, Gross believed that his local analysis (see Problem 2.8 in [23]) required the existence of Zariski neighborhoods. However, using the formal flat descent as in [46] or [30], we argue that étale (or even flat) neighborhoods suffice.

The strategy we employ here is that of Gross in [23]. We begin with an arbitrary coherent sheaf F on X and aim to show there is vector bundle V along with a surjection $V \rightarrow F \rightarrow 0$.

A vague description of the argument goes like this: locally free surjections arise locally via extension classes in certain local Ext groups and we glue these together to obtain a global Ext class with desirable properties. The proof is broken up into pieces: at each stage we replace F with a sheaf which is closer to being locally free.

We begin with an observation of Gross:

Lemma 3.1.2. *Let Y be a noetherian, quasicompact, and integral algebraic stack with affine diagonal. Then every coherent sheaf is the quotient of a torsion-free coherent sheaf.*

Proof. There exists a smooth cover $\pi : U \rightarrow Y$ from an affine scheme, fix a coherent sheaf F on Y . Since U is affine there exists a surjection $V \rightarrow \pi^*F$ from a free module V . When we pushforward this morphism it remains a surjection as the morphism is affine, moreover since π is faithfully flat we see that π_*V is faithfully flat over Y . Consider the fiber diagram of sheaves below:

$$\begin{array}{ccc} K & \longrightarrow & F \\ \downarrow f' & & \downarrow i \\ \pi_*V & \xrightarrow{i'} & \pi_*\pi^*F \end{array}$$

Observe that $K \rightarrow F$ is surjective, K is quasicoherent, and that $K \rightarrow \pi_*V$ is injective. Since K is the union of its coherent subsheaves (Proposition 15.4 in [38]) and F is coherent we may find a coherent subsheaf $J \subset K$ which surjects onto F . Since $J \subset \pi_*V$ and the latter is torsion-free, J must be as well.

□

Remark 3.1.3. When X is a normal noetherian algebraic space of dimension 2 of finite type and separated over a field k of characteristic 0 we can replace an arbitrary coherent sheaf on X with one which is reflexive using the following trick. By (see Corollaire 16.6.2 in [38]) there is a normal scheme Y along with a action by a finite constant group G so that $X \cong Y/G$. In

other words there is a coarse space morphism $\pi : [Y/G] \rightarrow X$. Since $Y \rightarrow [Y/G]$ is finite étale and Y is a normal algebraic surface which is a scheme we may use Schröer-Vezzosi's or Gross' result to deduce that Y satisfies the resolution property. By Lemma 2.1.6 it follows that $[Y/G]$ satisfies the resolution property. As such there exists a $V \rightarrow \pi^*F \rightarrow 0$ which remains surjective upon pushforward since $[Y/G]$ is tame. So we have $\pi_*V \rightarrow \pi_*\pi^*F = F \rightarrow 0$ where π_*V is a reflexive coherent sheaf.

To see why it is reflexive first note that away from finitely many closed points $S \subset X$, X and $[Y/G]$ are both regular, let $U = X - S$. Then $[Y/G] \times_X U \rightarrow U$ is a coarse space map and it is flat here since it is a quasi-finite morphism between smooth stacks of the same dimension. Since $V|_{[Y/G] \times_X U}$ is flat over U , when we pushforward $(\pi_U)_*V|_{[Y/G] \times_X U}$ (see Corollary 1.3 (3) in [47]), this is locally free on U and therefore its pushforward to X is reflexive since $X - U$ has codimension ≥ 2 . Since V is reflexive on $[Y/G]$ this reflexive module is naturally isomorphic to π_*V .

Definition 3.1.4. Let X be a locally noetherian algebraic space, we say that a coherent sheaf F is F_k if $F|_{\text{Spec } \mathcal{O}_{X,x}}$ is free for all $x \in X$ with $\dim \mathcal{O}_{X,x}^{sh} \leq k$ and for all other x we have $\text{pd}_{\mathcal{O}_{X,x}^{sh}}(F|_{\text{Spec } \mathcal{O}_{X,x}}) \leq 1$.

Roughly a coherent sheaf is F_k if is locally free on an open subset containing all the codimension $\leq k$ points and on its singular locus it is locally of projective dimension 1.

Proposition 3.1.5. (Gross) *Let R be a noetherian local ring of dimension 2 and U its punctured spectrum. For any finitely generated R -module M which is locally free of rank ≥ 1 on U there exists an exact sequence of finitely generated R -modules which are locally free of constant rank on U*

$$0 \rightarrow L \rightarrow N \rightarrow M \rightarrow 0$$

such that N has projective dimension ≤ 1 and $L|_U \cong \det(M|_U)^\vee$.

Proof. This is Proposition 2.3 of [23]. When R is normal, this follows from a weaker result of Bourbaki (see Theorem 2.14 of [18]). □

Now we extend Gross' Proposition 2.6 in [23] to the setting of algebraic spaces.

Proposition 3.1.6. *Let X be a dimension 2, quasiseparated, noetherian, algebraic space and F a coherent sheaf which is locally free of constant rank away from a closed set $Z \subset X$ of codimension 2. There is a coherent sheaf L on X such that $L|_{X-Z} \cong \det(F|_{X-Z})^\vee$. Moreover, there exists an obstruction $o \in H^2(X, \underline{\text{Hom}}(F, L))$ whose vanishing is sufficient to guarantee the existence of an exact sequence of coherent sheaves in X*

$$0 \rightarrow L \rightarrow N \rightarrow F \rightarrow 0$$

where N is F_1 .

Proof. Let $Y = \bigsqcup_{z \in Z} \text{Spec } R_z$ be a disjoint union formed by the local noetherian flat neighborhoods of each of the closed points $z \in Z$ just as in Lemma 2.3.10. Observe that the natural map $Y \rightarrow X$ is a flat neighborhood of Z . Therefore if we set $U = X \setminus Z$ then we obtain a flat Mayer-Vietoris square.

$$\begin{array}{ccc} \bigsqcup_{z \in Z} (\text{Spec } R_z \setminus \{m_z\}) & \longrightarrow & Y = \bigsqcup_{z \in Z} \text{Spec } R_z \\ \downarrow & & \downarrow f \\ U & \xrightarrow{i} & X \end{array}$$

Restrict F to Y and on each noetherian local ring R_z apply Proposition 3.6 to $F|_{\text{Spec } R_z}$. This yields an exact sequence, or an extension class, call it σ_z :

$$0 \rightarrow L_z \rightarrow N_z \rightarrow F|_{\text{Spec } R_z} \rightarrow 0$$

of R_z modules where N_z and L_z are locally free away from m_z , N_z has projective dimension ≤ 1 and isomorphisms

$$L_z|_{(\text{Spec } R_z \setminus \{m_z\})} \cong \det(F|_{(\text{Spec } R_z \setminus \{m_z\})})^\vee \cong \det(F|_U)^\vee|_{(\text{Spec } R_z \setminus \{m_z\})}$$

These isomorphisms and the formal gluing theorem 2.3.5 imply that there exists a coherent sheaf $L \in \text{Qcoh}(X)$ which restricts to $\det(F|_U)^\vee$ over U , $\bigsqcup L_z$ over Y , and induces the isomorphisms $\det(F|_U)^\vee|_{(\text{Spec } R_z \setminus \{m_z\})} \cong L_z|_{(\text{Spec } R_z \setminus \{m_z\})}$ on the overlap $Y \times_X U = \bigsqcup_{z \in Z} (\text{Spec } R_z \setminus \{m_z\})$.

As such, we may view the sequences above as extension classes $\sigma_z \in \text{Ext}^1(F|_{\text{Spec } R_z}, L_z)$ and we claim these glue to yield a global section of $E = \underline{\text{Ext}}^1(F, L)$. A global section of E naturally corresponds to a morphism of \mathcal{O}_X -modules: $\mathcal{O}_X \rightarrow E$. However, since $E|_U = 0$ (because F is locally free on U) and we have $(\sigma_z)_{z \in Z} : \mathcal{O}_Y \rightarrow E|_Y = \text{Ext}^1(F|_Y, L|_Y)$, put together we view them as a morphism of triples

$$(0, (\sigma_z)_{z \in Z}) : (\mathcal{O}_U, \mathcal{O}_Y, \text{can}) \rightarrow (0, \underline{\text{Ext}}^1(F|_Y, L|_Y), \text{can})$$

In other words we obtain a morphism in the category $\text{Qcoh}(U) \times_{\text{Qcoh}(U_Y)} \text{Qcoh}(Y)$. By Theorem 2.3.5 this must come from a morphism $\mathcal{O}_X \rightarrow E$ over X i.e. there is a section $\sigma \in H^0(X, \underline{\text{Ext}}^1(F, L))$ that extends $(\sigma_z)_{z \in Z}$. Finally, by considering the low degree terms of the local-global Ext spectral sequence

$$H^p(X, \underline{\text{Ext}}^q(F, L)) \Rightarrow \text{Ext}^{p+q}(F, L)$$

we obtain the exact sequence

$$\text{Ext}^1(F, L) \rightarrow H^0(X, \underline{\text{Ext}}^1(F, L)) \rightarrow H^2(X, \underline{\text{Hom}}(F, L))$$

Therefore, for $\sigma \in H^0(X, \underline{\text{Ext}}^1(F, L))$ to come from $\text{Ext}^1(F, L)$ we need $\sigma \mapsto 0$ under the second map. The image of σ in $H^2(X, \underline{\text{Hom}}(F, L))$ is the obstruction whose vanishing is equivalent to the existence of a global extension

$$0 \rightarrow L \rightarrow N \rightarrow F \rightarrow 0$$

extending the local extensions $(\sigma_z)_{z \in Z}$.

□

Proposition 3.1.7. (Gross) *Let X be a dimension 2, noetherian, algebraic space and F a coherent sheaf which is F_1 . Let Z denote the (finite) locus where F is not locally free. Assume, moreover that F has constant rank at all generic points of X . Then, there is a vector bundle V of constant rank on X and an obstruction $o \in H^2(X, \text{Hom}(F, V^{\oplus m}))$ whose vanishing guarantees the existence of a locally free resolution*

$$0 \rightarrow V^{\oplus m} \rightarrow E \rightarrow F \rightarrow 0$$

Proof. The proof is very similar to the previous proposition, so we just give a sketch here. Let $Y = \bigsqcup_{z \in Z} \text{Spec } R_z \rightarrow X$ be a flat neighborhood of Z which consists of local noetherian rings.

Since F is F_1 we observe that there is a exact sequence of sheaves on $\text{Spec } R_{z_i}$

$$0 \rightarrow W_{z_i} \rightarrow E_{z_i} \rightarrow F|_{\text{Spec } R_{z_i}} \rightarrow 0$$

where W_{z_i} and E_{z_i} are free. Since the R_{z_i} is local we may identify $E_{z_i} \cong \mathcal{O}_{\text{Spec } R_{z_i}}^{\oplus n_i}$. However, as the z_i varies the E_{z_i} may have different ranks, call them n_1, \dots, n_k . Next, replace $\phi_i : \mathcal{O}_{\text{Spec } R_{z_i}}^{\oplus n_i} \rightarrow F|_{\text{Spec } R_{z_i}}$ with $\bigoplus_{j=1}^k \mathcal{O}_{\text{Spec } R_{z_i}}^{\oplus n_j} \rightarrow \mathcal{O}_{\text{Spec } R_{z_i}}^{\oplus n_i} \rightarrow F|_{\text{Spec } R_{z_i}}$ where the first map is projection onto the i th coordinate and the second is ϕ_i . In this way we obtain an exact sequence of sheaves of constant rank on Y

$$0 \rightarrow \mathcal{O}_Y^m \rightarrow \mathcal{O}_Y^n \rightarrow F|_Y \rightarrow 0$$

where $n = \sum_{j=0}^k n_j$ and $m = n - \text{rank}(F)$.

Thus we obtain a section $\sigma_Y \in \underline{\text{Ext}}^1(F|_Y, \mathcal{O}_Y^m)$. Just as in the proof of the previous proposition we can use Theorem 2.3.5 to infer the existence of a global section σ of $\underline{\text{Ext}}^1(F, \mathcal{O}_X^m)$ which extends the σ_Y . Using the low degree terms of the local-global Ext spectral sequence the obstruction to lifting $\sigma \in H^0(X, \underline{\text{Ext}}^1(F, \mathcal{O}_X^m))$ to an element in $\text{Ext}^1(F, \mathcal{O}_X^m)$ lies in $H^2(X, \text{Hom}(F, \mathcal{O}_X^m))$. The result follows.

□

Thus far we have identified cohomological obstructions in degree two whose vanishing would yield a locally free resolution. The lemma below explains why we may assume that these always vanish.

Lemma 3.1.8. *Suppose X is a integral algebraic space of dimension 2 which is finite type and separated over a field. Moreover, assume X is regular in codimension 1. Let $Z \subset X$ be a closed subset of codimension 2 and suppose F a coherent sheaf which is locally free away from Z . To show F admits a surjection by a vector bundle, it suffices to show $F|_{X \setminus \{q\}}$ and $F|_{X \setminus \{p\}}$ admits a surjection by a vector bundle on the punctured spaces $X \setminus \{q\}$, $X \setminus \{p\}$ where $q, p \in X$ are distinct regular closed points where F is already locally free. In particular, to show F can be resolved by locally frees it suffices to assume X is nonproper.*

Proof. Choose two regular closed points $p, q \in X$ where F is locally free. This can be done since X is regular away from finitely many closed points and so we can certainly find two which are disjoint from Z . Denote the open immersions by $j_p : X \setminus \{p\} \rightarrow X$ and $j_q : X \setminus \{q\} \rightarrow X$ and observe that since F is locally free (and therefore, reflexive) at p and X is regular near p the natural map $F \rightarrow (j_p)_*(j_p)^*F$ is an isomorphism. The same argument shows that $F \rightarrow (j_q)_*(j_q)^*F$ is an isomorphism.

Suppose we find surjections $V_q \rightarrow j_q^*F$ and $V_p \rightarrow j_p^*F$, pushing them forward and taking their sum yields a surjection $(j_q)_*V_q \oplus (j_p)_*V_p \rightarrow F$. Since the closed points have codimension 2 the pushforward of the vector bundles along j_p or j_q is reflexive and now Tag 0B3N in [56] applied to the 2-dimensional regular local rings $\mathcal{O}_{X,p}^{sh}$, $\mathcal{O}_{X,q}^{sh}$ and the respective modules $(j_q)_*V_q$, $(j_p)_*V_p$ shows that they are locally free on all of X . The result follows. □

Remark 3.1.9. The trick of puncturing X to eliminate cohomological obstructions was independently known to Gross. Although it does not appear in his published papers, [23], [22], one can find a similar argument in his thesis (see 2.1.13 of [21]).

Now we are ready to prove the following

Theorem 3.1.10. *Let X be a 2-dimensional, integral algebraic space, which is finite type and separated over a field. If X is regular in codimension 1, then it must satisfy the resolution property.*

Proof. Begin with a coherent sheaf F . By applying lemma 3.1.2 to X there is a surjection $F' \rightarrow F$ where F' is a coherent torsion-free sheaf. Therefore it suffices to show torsion free coherent sheaves admit surjections by vector bundles, i.e. we may suppose that F is torsion free. Since X is regular in codimension one this implies F is locally free in codimension one and hence the locus where F isn't locally free is a finite set of closed points $Z = \{p_1, \dots, p_n\} \subset X$. Using Lemma 3.1.8 we may further assume that X is nonproper by puncturing X . Since X is 2-dimensional the main theorem of [34] implies X has no coherent cohomology in degree 2!

Since all top coherent cohomology vanishes, Proposition 3.1.6 implies there exists an exact sequence

$$0 \rightarrow L \rightarrow N \rightarrow F \rightarrow 0$$

where N is F_1 . Thus it suffices to assume F is F_1 . To conclude we apply Proposition 3.1.7 and cohomological vanishing in degree 2 to deduce that F appears in an extension:

$$0 \rightarrow V \rightarrow E \rightarrow F \rightarrow 0$$

where E is locally free.

□

Remark 3.1.11. The result of [34] is actually stated for schemes. In particular, it argues that a schematic variety X with no proper components must have cohomological dimension $\leq \dim X - 1$. However, the same proof provided there works for *algebraic space* varieties as well.

3.2 Tame Deligne-Mumford Curves are Quotient Stacks

In this section we show that Deligne-Mumford curves are always global quotient stacks and satisfy the resolution property. This has been known in the smooth case by (2.17 in [17])

or [11] in the analytic or topological category) but there is no proof in the general setting.

Lemma 3.2.1. *Let \mathcal{X} be a separated DM stack whose coarse space X is a scheme. Then there is a Zariski-open covering $U_i \subset X$ and finite locally free morphisms $\mathcal{Y}_i \rightarrow \mathcal{X}_i = \mathcal{X} \times_X U_i$ where \mathcal{Y}_i is a DM stack which is Zariski locally a quotient stack by finite groups.*

Proof. Let $\mathcal{X} \rightarrow X$ be the coarse space morphism. The statement is Zariski local on X so we may assume that X is affine. By Lemma 2.2.3 of [2] there is a étale covering $U_i \rightarrow X$ so that $\mathcal{X} \times_X U_i \cong [Z/G]$ where G is a finite constant group and Z is a G -scheme. Since X is quasicompact we may refine and replace $U_i \rightarrow X$ with an affine étale surjection $U = \text{Spec } A \rightarrow X$. By Tag 02LH in [56] there is a surjective finite locally free morphism $Y'_i \rightarrow X$ and an open covering $\bigcup_j Y'_{ij} = Y'_i$ so that $Y'_{ij} \rightarrow X$ factors through $Y'_{ij} \rightarrow U \rightarrow X$. Set $\mathcal{Y}_i = Y'_i \times_X \mathcal{X}$ to conclude. □

Remark 3.2.2. The previous Lemma will allow us to assume many properties which hold étale locally, actually hold Zariski locally. For instance, when trying to show that X is a quotient stack the previous lemma will allow us to assume that the Y_i above and X are Zariski-locally quotient stacks. Indeed, suppose we know that Y_i and X being Zariski locally quotient stacks implies they are global quotient stacks, then since the property of being a quotient stack descends along finite flat morphisms it follows that X is Zariski locally a quotient stack and again that implies X itself is a global quotient stack. The same trick was used in Gabber's thesis to prove that the Brauer map is surjective over affine schemes.

Lemma 3.2.3. *(Kresch) Suppose that \mathcal{X} is a DM stack which is separated and finite type over a field. Suppose moreover that the coarse space, X , is a scheme. Then \mathcal{X} is a quotient stack Zariski locally if and only if \mathcal{X} is Zariski locally a quotient stack by finite groups.*

Proof. See Proposition 5.2 in [36]. □

Example 3.2.4. We will show that the hypothesis of X being a scheme is necessary. Consider Nagata's example of an algebraic space surface which isn't a scheme: blow up 10 very general points on an elliptic curve $E \subset \mathbf{P}^2$ and contract E to obtain an algebraic space \overline{X} . The resulting space has the curious property that every curve in \overline{X} goes through the singular point. Thus, any Zariski open neighborhood of the singular point contains all the codimension ≤ 1 points of \overline{X} . We claim that there exists a μ_n gerbe $\mathcal{X} \rightarrow X$ which remains nontrivial in every Zariski neighborhood of the singular point. Indeed, by considering the low-degree terms of the Leray spectral sequence along $X \rightarrow \overline{X}$ for the sheaf \mathbf{G}_m we see that there are torsion classes in $\text{Pic}(E)$ which give rise to nontrivial torsion classes in $H^2(X, \mathbf{G}_m)$. Let $U \subset \overline{X}$ be a Zariski open neighborhood of p . By using the functoriality of the Leray spectral sequence one can check that this torsion element of $H^2(\overline{X}, \mathbf{G}_m)$ remains nontrivial when restricted to U .

Suppose that there is a Zariski neighborhood $U \subset \overline{X}$ of p where \mathcal{X} becomes a quotient stack by a finite group $[Y/G]$. Then $Y \rightarrow [Y/G] \rightarrow U$ is étale and by normalizing \overline{X} in Y we obtain a finite étale cover of \overline{X} . Indeed, the normalization yields a finite morphism $Z \rightarrow \overline{X}$ which is étale over U . It remains to check that the morphism is étale at finitely many codimension 2 points in the inverse image of $X \setminus U$. Since the points outside U are regular and Z is normal we use purity of the branch locus (Tag 0BMB in [56]) to conclude that the map is étale. However, since \overline{X} is birational to the projective plane it admits no nontrivial finite étale covers. It follows that $Y \rightarrow U$ admits a section and that $[Y/G] = \mathcal{X} \times_X U$ is a trivial μ_n gerbe. Thus, \mathcal{X} is not Zariski-locally a quotient stack by finite groups. To conclude our example we need to show that this μ_n gerbe is in fact a global quotient stack. In fact we will see later that all μ_n -gerbes on this algebraic space are global quotient stacks (see theorem 4.3.2).

Lemma 3.2.5. *Suppose G is a finite constant group scheme, Z a separated noetherian algebraic space with a right G -action and let $\mathcal{X} = [Z/G]$ be the corresponding stack quotient. If $f : Z \rightarrow \mathcal{X}$ is the quotient map then $f_*\mathcal{O}_Z$ is a vector bundle of rank $|G|$ and the representation type of $f_*\mathcal{O}_Z$ for any geometric point $x : \text{Spec } L \rightarrow \mathcal{X}$ is $L[H]^{\oplus |G:H|}$ where*

$H = \text{Stab}(x)$ is the stabilizer of the point x in $\mathcal{X}(L)$.

Proof. Since $f : Z \rightarrow [Z/G]$ is a left G -torsor, f is a finite étale cover of degree $|G|$. It follows that $f_*\mathcal{O}_Z$ is a vector bundle of rank $|G|$.

Next, we explain what we mean by the second statement of the lemma. Let $\text{Spec } L \rightarrow \mathcal{G}_x \rightarrow \mathcal{X}$ be the associated residual gerbe (see Tags 06MU, 06UI, 06QK). That is, $\mathcal{G}_x \rightarrow \mathcal{X}$ is a monomorphism, and \mathcal{G}_x is a reduced gerbe whose underlying topological space is a singleton. Thus the sheafification of \mathcal{G}_x is represented by a point $\text{Spec } k$, let $\pi : \mathcal{G}_x \rightarrow \text{Spec } k$ be the associated map. Then the pullback $\mathcal{G}_x \times_{\text{Spec } k} \text{Spec } L$ is trivial gerbe isomorphic to $BH_{\text{Spec } L}$ where H is a finite constant group scheme. Since $BH_{\text{Spec } L} \rightarrow \mathcal{G}_x \rightarrow [Z/G]$ is the composition of stabilizer-preserving morphisms we may identify H with the stabilizer of x in $\mathcal{X}(L)$. The second statement claims that the $H_{\text{Spec } L}$ -representation corresponding to the vector bundle $f_*\mathcal{O}_Z|_{BH_{\text{Spec } L}}$ on $BH_{\text{Spec } L}$ is $L[H]^{\oplus |G:H|}$. The integer $|G : H|$ will be explained below.

We obtain a morphism $BH_{\text{Spec } L} \rightarrow \mathcal{G}_x \rightarrow [Z/G] \rightarrow BG_{\mathbf{Z}}$ which is representable since it is the composition of representable morphisms. Moreover if $\pi : \text{Spec } \mathbf{Z} \rightarrow BG_{\mathbf{Z}}$ is the canonical morphism, then

$$f_*\mathcal{O}_Z|_{BH_{\text{Spec } L}} \cong \pi_*\mathcal{O}_{\text{Spec } \mathbf{Z}}|_{BH_{\text{Spec } L}}$$

This is because $Z \cong [Z/G] \times_{BG_{\text{Spec } \mathbf{Z}}} \text{Spec } \mathbf{Z}$ over $[Z/G]$ and π is an affine morphism so cohomology commutes with base change. Thus, we study the induced morphism $h : BH_{\text{Spec } L} \rightarrow BG_{\text{Spec } L}$ because if $\pi_L : \text{Spec } L \rightarrow BG_{\text{Spec } L}$ is the canonical map, then

$$f_*\mathcal{O}_Z|_{BH_{\text{Spec } L}} \cong h^*(\pi_L)_*\mathcal{O}_{\text{Spec } L}$$

However, h is representable so we know it comes from an injective homomorphism of groups $H_L \rightarrow G_L$. Indeed, the proof of Lemma 3.8 in [1] gives an explicit description of the representable morphisms $BH_{\text{Spec } L} \rightarrow BG_{\text{Spec } L}$: they correspond to inner-isomorphism classes of injective morphisms $H \rightarrow G$. Thus it makes sense to consider the integer $|G : H|$. Next, we identify the category of quasicoherent sheaves on BH_L and BG_L with the categories of H and G representations over L respectively. Under this identification the pullback functor

$h^* : \mathrm{QCoh}(BG) \rightarrow \mathrm{QCoh}(BH)$ corresponds to the restriction functor on representations and the pushforward functor corresponds to the induction functor on representations. Thus, pushing forward the structure sheaf along $\pi_L : B\{e\}_L \cong L \rightarrow BG_{\mathrm{Spec} L}$ corresponds to inducing the trivial representation along the inclusion $\{e\} \rightarrow G$. Therefore $(\pi_L)_* \mathcal{O}_{\mathrm{Spec} L}$ corresponds to the regular representation of G . Moreover, if we choose a set of left coset representatives of H in G the restriction of the regular representation of G to H can be described by an isomorphism

$$\mathrm{Res}_H^G L[G] \cong L[H]^{\oplus |G:H|}$$

□

Theorem 3.2.6. *Let \mathcal{X} be a tame DM stack which is separated and finite type over a field k and suppose that all of the components of X are of dimension ≤ 1 , then X is a global quotient stack.*

Proof. First we explain why it suffices to show X is Zariski-locally a quotient stack by finite groups implies it is a quotient stack. Assume any \mathcal{X} as in the statement which is also Zariski-locally a quotient stack by finite groups is a global quotient stack. Apply lemma 3.2.1 and note that if $\mathcal{Y}_i \rightarrow \mathcal{X}_i$ is a finite faithfully flat covering where \mathcal{Y}_i is Zariski locally a quotient stack by finite groups then \mathcal{Y}_i is a quotient stack. Indeed, \mathcal{Y}_i satisfies all the hypothesis as in the theorem so our assumption implies \mathcal{Y}_i is a global quotient stack. This in turn implies that \mathcal{X}_i is a quotient stack by Lemma 2.1.8. Now we apply Kresch's result (Lemma 3.2.3) above to deduce that \mathcal{X} is Zariski locally a quotient by finite groups and again our assumption implies \mathcal{X} is a global quotient stack. It follows that we may assume \mathcal{X} is Zariski locally a quotient stack by finite groups.

The next reduction we make is to assume that \mathcal{X} is reduced. Indeed, if $\mathcal{X}_{\mathrm{red}}$ is a quotient stack, then it admits a faithful vector bundle V . We will deform along the nilpotent closed immersion $\mathcal{X}_{\mathrm{red}} \rightarrow \mathcal{X}$ by factoring the extension into square zero extensions. Then the obstruction to deforming V at each stage lies in $H^2(\mathcal{X}_{\mathrm{red}}, \mathrm{End}(V) \otimes I)$ where I is an ideal sheaf. Since \mathcal{X} is tame the Leray spectral sequence for the coarse space map $\mathcal{X} \rightarrow X$:

$$H^p(X, R^q \pi_*(\text{End}(V) \otimes I)) \Rightarrow H^{p+q}(\mathcal{X}, \text{End}(V) \otimes I)$$

degenerates. In particular we obtain isomorphisms $H^2(\mathcal{X}, \text{End}(V) \otimes I) \cong H^2(X, \pi_*(\text{End}(V) \otimes I))$ and the latter vanishes by Grothendieck vanishing because X is a scheme of dimension 1.

Suppose \mathcal{X} is Zariski locally of the form $[Z_i/G_i]$ for a reduced scheme Z_i and a finite constant group scheme G_i . We will exhibit faithful vector bundles on an open cover and show that they can be glued. Note that $I_{\mathcal{X}} \rightarrow \mathcal{X}$ is flat on a dense open subset of \mathcal{X} because \mathcal{X} is reduced and hence generically smooth. Let q_1, \dots, q_n denote the closed points where $I_{\mathcal{X}} \rightarrow \mathcal{X}$ is nonflat or where \mathcal{X} is singular. Also, include in this finite list at least one point from each irreducible component of \mathcal{X} . Let $\mathcal{U}_1, \dots, \mathcal{U}_n$ be open neighborhoods of these points that do not contain more than one of the q_i , such that \mathcal{U}_i are each quotient stacks by finite groups.

Next, we construct a vector bundle on each of these \mathcal{U}_i . Suppose $\mathcal{U}_i \cong [Z_i/G_i]$. The standard quotient map is a G_i -torsor $f_i : Z_i \rightarrow [Z_i/G_i] \cong \mathcal{U}_i$. Let V_i denote $(f_i)_*(\mathcal{O}_{Z_i})^{\oplus(\prod_{i \neq j} |G_j|)}$. A few observations are in order. First, V_i is vector bundle of constant rank $\prod_{k=1}^n |G_k|$. Second, the geometric fiber at any point $p \in \mathcal{U}_i$ of V_i always induces some power of the regular representation of the stabilizer at p by Lemma 3.2.5. In other words, at each geometric residual gerbe where the V_i are defined, they are pairwise isomorphic. It remains to show that these vector bundles can be patched together on an open subset of the q_1, \dots, q_n .

By induction assume there is a vector bundle on a neighborhood $\mathcal{U}'_{\leq j} = \mathcal{U}'_1 \cup \dots \cup \mathcal{U}'_j$, call it $V_{\leq j}$, where $q_i \in \mathcal{U}'_i \subset \mathcal{U}_i$ and which satisfies $V_{\leq j}|_{\mathcal{U}'_k} \cong V_k|_{\mathcal{U}'_k}$ for $k \leq j$. We claim that we may extend this vector bundle over a open neighborhood of q_{j+1} contained in \mathcal{U}_{j+1} . Consider $\mathcal{U}_{j+1} \cap \mathcal{U}'_{\leq j}$, since it contains none of the q_i , it is a (possibly disconnected) smooth gerbe. If we can show that there are isomorphisms $V_{\leq j} \cong V_{j+1}$ over the generic points (or more precisely, the generic residual gerbes) of the components of $\mathcal{U}_{j+1} \cap \mathcal{U}'_{\leq j}$ then this would complete the induction step. Indeed, given such an isomorphism we may spread it out over a dense open neighborhood \mathcal{W} of $\mathcal{U}_{j+1} \cap \mathcal{U}'_{\leq j}$ and then shrinking \mathcal{U}_{j+1} by throwing out a few closed points of $\mathcal{U}_{j+1} \cap \mathcal{U}'_{\leq j}$ does the job.

Let $\text{Spec } K \rightarrow X$ be the inclusion of any generic point of the coarse space of $\mathcal{U}_{j+1} \cap \mathcal{U}'_{\leq j}$. If \bar{K} is the algebraic closure then $\mathcal{X} \times_X \text{Spec } \bar{K} \cong BH_{\bar{K}}$ because any étale gerbe must admit a section over an algebraically closed field. It follows that the vector bundles V_{j+1} and $V_{\leq j}$ are isomorphic over $\mathcal{X} \times_X \text{Spec } \bar{K}$ since they correspond to isomorphic representations of $H = \text{Stab}(\text{Spec } \bar{K} \rightarrow X)$, namely, the same power of the regular representation of H . By Lemma 2.2.3 we may conclude that the vector bundles V_{j+1} and $V_{\leq j}$ agree on $\mathcal{X} \times_X \text{Spec } K$. This completes the induction, thereby implying the existence of a vector bundle V on an open neighborhood $\mathcal{U} \subset \mathcal{X}$ which contains all the q_i . Taking a reflexive hull of a coherent extension yields a vector bundle, V' on all of \mathcal{X} . Indeed, the points not in \mathcal{U} are smooth 1-dimensional points and hence being torsion-free forces local-freeness. It remains to see why V' is faithful. This follows because the faithful locus is dense and only remains to be checked at the locus where \mathcal{X} is a gerbe, thus Proposition 2.2.7 applies.

Observe that zero dimensional Deligne-Mumford stacks are always quotient stacks. As above, we immediately reduce to the case when the Deligne-Mumford stack is reduced and étale locally on the coarse space, we know it is a quotient. However, every étale morphism whose target is a point is finite, hence a dimension 0 Deligne-Mumford stack is always a quotient. Thus we know that any finite type, tame, separated DM stack whose components have dimension ≤ 1 is a quotient stack. Indeed, use the argument above to produce a faithful vector bundle over each component. Then by taking appropriate sums of each of the vector bundles we can arrange for them to have constant rank.

□

Corollary 3.2.7. *Let \mathcal{X} be a tame Deligne-Mumford stack which is separated and finite type over a field k and suppose that all of the components of X are of dimension ≤ 1 , then \mathcal{X} admits a faithful vector bundle of constant rank with trivial determinant.*

Proof. By the previous theorem there exists a faithful vector bundle of constant rank on \mathcal{X} , call it V . Set $W = V \oplus \det(V)^\vee$, it is still faithful since V is but it also has trivial determinant

since the determinant is additive. \square

Corollary 3.2.8. *Let \mathcal{X} be a tame Deligne-Mumford stack which is separated and finite type over a field k and suppose that all of the components of X are of dimension ≤ 1 , then \mathcal{X} satisfies the resolution property.*

Proof. Let $\mathcal{X} \rightarrow X$ denote the coarse space map. By the previous corollary, this morphism satisfies the resolution property. Since X is 1-dimensional, it is quasiprojective and thus satisfies the resolution property. By 2.1.13 and 2.1.5, it follows that \mathcal{X} satisfies the resolution property. \square

Chapter 4

THE SURJECTIVITY OF THE BRAUER MAP

4.1 Grothendieck's Question

We begin by briefly reviewing the theory of the Brauer group with a focus on its relationship to gerbes and twisted sheaves. Let X denote an algebraic stack for what follows.

Definition 4.1.1. The cohomological Brauer group of X is $H^2(X, \mathbf{G}_m)_{\text{tors}}$, we denote it by $\text{Br}'(X)$.

Proposition 4.1.2. *The following sequence is exact in the category of sheaves groups on X*

$$1 \rightarrow \mathbf{G}_m \rightarrow \mathbf{GL}_n \rightarrow \mathbf{PGL}_n \rightarrow 1$$

Proof. See [20] pg. 341. □

Following Giraud's nonabelian formalism [20], we obtain a long exact sequence (up to degree 2) of pointed sets for any n :

$$\dots \rightarrow H^1(X, \mathbf{G}_m) \rightarrow H^1(X, \mathbf{GL}_n) \rightarrow H^1(X, \mathbf{PGL}_n) \rightarrow H^2(X, \mathbf{G}_m) \rightarrow \dots$$

Here, $H^1(X, G)$ denotes isomorphism classes of left G -torsors and $H^2(X, \mathbf{G}_m)$ denotes isomorphism classes of \mathbf{G}_m -gerbes. Let δ_n denote the connecting map from degree one to two. Noting that \mathbf{PGL}_n -torsors correspond to Azumaya algebras of degree n we obtain

Theorem 4.1.3. *Let X be an algebraic stack, then there is a natural injection $\delta : \text{Br}(X) \rightarrow \text{Br}'(X)$, we call it the Brauer map. The map sends the class of an Azumaya Algebra A to the \mathbf{G}_m -gerbe of trivializations of A , \mathcal{X}_A . More precisely, $\mathcal{X}_A(T)$ is the category of pairs (V, ϕ) where V is a vector bundle on T and a trivialization $\phi : \text{End}(V) \cong A|_T$. A morphism from (V, ϕ) to (W, ψ) is an isomorphism from $V \rightarrow W$ compatible with the trivializations ϕ and ψ .*

Proof. See [20] Chapter V.4.4. □

Now we can formulate a problem posed by Grothendieck (See [26])

Question 4.1.4. (*Grothendieck*) Is $\delta : \text{Br}(X) \rightarrow \text{Br}'(X)$ surjective?

In other words, does every torsion cohomology class come from an Azumaya algebra? In what follows, we define twisted sheaves, at the expense of introducing stacky complexity, they will clarify the question above. Suppose that $D \subset \mathbf{G}_m$ is a subgroup scheme.

Definition 4.1.5. Let \mathcal{X} be a D -gerbe on X . Any sheaf F on \mathcal{X} has a right action it inherits from the left action of $\text{Aut}(x) \cong D$ on $x \in \mathcal{X}$: if $\phi : x \rightarrow x$ is an automorphism then so is $F(\phi) : F(x) \rightarrow F(x)$. If we assume that F is also quasicoherent then it also has a left action of D it obtains via the \mathcal{O} -module structure. We say a quasicoherent sheaf F on \mathcal{X} is twisted if this left action is the one associated to the right action above.

The following theorem explains the connection between the Brauer map and twisted sheaves.

Theorem 4.1.6. *Let X be an quasicompact algebraic stack and $\mathcal{X} \in \text{H}^2(X, \mathbf{G}_m)$ a \mathbf{G}_m -gerbe. Then the following statements are equivalent.*

1. \mathcal{X} is isomorphic to \mathcal{X}_A for some Azumaya algebra A on X .
2. There exists a twisted vector bundle of nonzero constant rank on \mathcal{X} .
3. There exists a μ_n gerbe \mathcal{Y} admitting a twisted vector bundle and a map of stacks $\mathcal{Y} \rightarrow \mathcal{X}$ equivariant for the inclusion $\mu_n \subset \mathbf{G}_m$. Moreover, any μ_n -gerbe \mathcal{Y} with a map $\mathcal{Y} \rightarrow \mathcal{X}$ equivariant for $\mu_n \subset \mathbf{G}_m$ admits a twisted vector bundle.
4. There exists a μ_n gerbe \mathcal{Y} so that the structure map $\mathcal{Y} \rightarrow X$ has the resolution property and a map of stacks $\mathcal{Y} \rightarrow \mathcal{X}$ equivariant for the inclusion $\mu_n \subset \mathbf{G}_m$.

5. *There is a finite, finitely presented flat map $Y \rightarrow X$ so that $\mathcal{X}|_Y$ is isomorphic to \mathcal{X}_A for an Azumaya algebra A on Y .*

Proof. 1. \Rightarrow 2.

Let T be any algebraic space over X , then $\mathcal{X}(T)$ is the groupoid of pairs (V, ϕ) where V is a vector bundle on T and ϕ is an isomorphism $\text{End}(V) \rightarrow A|_T$. There is a tautological vector bundle on this stack: it is the sheaf $(V, \phi) \mapsto \Gamma(T, V)$ where the restriction maps are those induced by morphisms in \mathcal{X}_A . Noting that the automorphisms of (V, ϕ) are precisely left multiplication on V by an element in $\mathbf{G}_m(T)$, 2. follows.

2. \Rightarrow 1.

Let V be a twisted vector bundle of nonzero constant rank and consider $\text{End}(V)$. Since the relative stabilizers of $\mathcal{X} \rightarrow X$ act trivially on this algebra, $\text{End}(V)$ is actually an algebra on X . In fact, it is an Azumaya algebra on X . Indeed, étale locally on X , the gerbe map $\mathcal{X} \rightarrow X$ admits a section which, in turn, yields an equivalence $\mathcal{X} \rightarrow B\mathbf{G}_m$. This equivalence endows \mathcal{X} with a twisted line bundle, L , which gives rise to a trivialization of $\text{End}(V) \cong \text{End}(V \otimes L^\vee)$. In fact, given any T point of \mathcal{X} , we obtain a trivialization of $\text{End}(V)$. In other words we obtain a map $\mathcal{X} \rightarrow \mathcal{X}_{\text{End}(V)}$ which respects the \mathbf{G}_m gerbe structure. Since this must be an isomorphism, the first statement now follows.

2. \Rightarrow 3.

Observe that the previous argument shows $\mathcal{X}_A \cong \mathcal{X}$ for some Azumaya algebra A on X . Next, note that the corresponding cohomology class is torsion and hence admits a lift to a cohomological class in $H^2(X, \mu_n)$ via the cohomology of the Kummer sequence. By the nonabelian formalism of Giraud, this implies there exists a μ_n gerbe \mathcal{Y} and a map $\mathcal{Y} \rightarrow \mathcal{X}$ equivariant for $\mu_n \rightarrow \mathbf{G}_m$. If \mathcal{Y} is any μ_n -gerbe admitting such a morphism, we may pull back a given twisted vector bundle on \mathcal{X} to obtain one over \mathcal{Y} .

3. \Rightarrow 1.

Suppose 3. holds and let V denote a twisted vector bundle on the μ_n gerbe \mathcal{Y} . Set $A = \text{End}(V)$ and observe that this is the pullback of an Azumaya algebra on X . Indeed, μ_n

acts trivially on $\text{End}(V)$ and étale locally on X , it is isomorphic to $\text{End}(V \otimes L^\vee)$ where L is some fixed twisted line bundle on $B\mu_n$. Let A denote this Azumaya algebra and consider the \mathbf{G}_m -gerbe associated to it, \mathcal{X}_A whose T -points are $(W, \phi : \text{End}(W) \cong A|_T)$. There is a natural map of gerbes $\mathcal{Y} \rightarrow \mathcal{X}_A$ equivariant for $\mu_n \subset \mathbf{G}_m$. Indeed, a T -point of \mathcal{Y} induces a trivialization $\mathcal{Y}_T \cong B\mu_n$ i.e. a T -point of \mathcal{Y} yields a twisted line bundle L on \mathcal{Y}_T . But then $V|_{\mathcal{Y}_T} \otimes L^\vee$ is a vector bundle on X and there is a natural isomorphism $\phi : A|_T = \text{End}(V)|_T \cong \text{End}(V \otimes L^\vee)$, i.e. a trivialization of the Azumaya algebra A . In summary, a T -point of \mathcal{Y} naturally induces a T -point of \mathcal{X}_A . One can check the maps on stabilizer groups respect the inclusion $\mu_n \subset \mathbf{G}_m$. However, since there is also a map $\mathcal{Y} \rightarrow \mathcal{X}$ equivariant for $\mu_n \subset \mathbf{G}_m$ this means $\mathcal{X}_A \cong \mathcal{X}$. This shows that 3. implies 1.

3. \Rightarrow 4.

Suppose 3. holds. Fix a twisted vector bundle V on \mathcal{Y} , we will show that its frame bundle is quasiaffine over X . This shows $\mathcal{Y} \rightarrow X$ factors through a quasiaffine morphism $\mathcal{Y} \rightarrow BGL_{n,X}$ so by Theorem 5.10 in [22], $\mathcal{Y} \rightarrow X$ satisfies the resolution property. Let F denote the total space of the frame bundle of V and let $F \rightarrow \mathcal{Y} \rightarrow X$ be the associated maps. By passing to a smooth cover of X we may suppose that it is a scheme and that $\mathcal{Y} \cong B\mu_n$. Now F is an algebraic space (because on a μ_n gerbe over a scheme the action of the inertia stack on a twisted sheaf is faithful). Moreover, $F \rightarrow \mathcal{Y}$ is a GL_n -torsor and $\mathcal{Y} = B\mu_n \rightarrow X$ is cohomologically affine hence the morphism $F \rightarrow X$ is cohomologically affine. By Proposition 3.3 in [3], the morphism is affine. This completes the proof that 3. implies 4.

4. \Rightarrow 3.

Now suppose 5. holds. Since $\mathcal{Y} \rightarrow X$ is a μ_n gerbe the relative inertia $I_f \rightarrow \mathcal{Y}$ has geometric fibers isomorphic to μ_n . Thus the assumption that f has the resolution property and Theorem 5.10 of [22] implies there is a vector bundle V on \mathcal{Y} whose frame bundle is quasiaffine over X . We may decompose $V = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} V_i$ into its eigensheaves with respect to the action of μ_n . Let $S \subset \{1, \dots, n-1\}$ be the subset consisting of integers where $V_i \neq 0$. If there was a $i \in S$ relatively prime to n then there is a k such that $V_i^{\otimes k}$ which is 1-twisted vector bundle. Suppose every i is not relatively prime to n . There cannot be a prime p dividing

every integer in S and n , otherwise $\mu_p \subset \mu_n$ would act trivially on the fibers of V . This is contrary to the fact that μ_n acts faithfully on the geometric fibers of V . It follows that there exists a polynomial combination of the V_i 's that is 1-twisted. This shows 4. implies 3.

1. \Leftrightarrow 5.

For the proof that 5. implies 1, we refer the reader to Chapter 2, Lemma 4 in [19]. The converse is trivial.

□

4.2 Gluing Twisted Vector Bundles along flat neighborhoods

We begin with a series of lemmas which extend the following result from Gabber's thesis: if $X = U \cup V$ is a scheme with U, V and $U \cap V$ affine then $\text{Br}(X) = \text{Br}'(X)$. The idea of the original proof is his but what follows is inspired by an adapted form from Lieblich [39].

Lemma 4.2.1. *Suppose an algebraic stack \mathcal{X} appears in a pushout square*

$$\begin{array}{ccc} \text{Spec } C & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X} \end{array}$$

such that every μ_n -gerbe \mathcal{G} on \mathcal{X} satisfies the following two properties:

1. *The natural functor*

$$\text{Vect}(\mathcal{G}) \rightarrow \text{Vect}(\mathcal{G}_{\mathcal{X}'}) \times_{\text{Vect}(\mathcal{G}_{\text{Spec}(C)})} \text{Vect}(\mathcal{G}_{\mathcal{Y}})$$

is an equivalence of categories.

2. *There exists twisted vector bundles of nonzero constant rank V and W on $\mathcal{G}_{\mathcal{X}'}$ and $\mathcal{G}_{\mathcal{Y}}$ respectively which satisfy $V|_{\mathcal{G}_{\text{Spec } C}}^{\otimes n} \cong W|_{\mathcal{G}_{\text{Spec } C}}^{\otimes n} \cong \mathcal{O}_{\mathcal{G}_{\text{Spec } C}}^{\oplus m}$.*

Then every μ_n -gerbe on \mathcal{X} admits a twisted vector bundle of nonzero constant rank, and in particular $\text{Br}(\mathcal{X}) = \text{Br}'(\mathcal{X})$.

Proof. Let \mathcal{G} denote a μ_n -gerbe on \mathcal{X} , it suffices to exhibit a nonzero twisted vector bundle of constant rank. Let V and W be twisted vector bundles as in the second hypothesis of the lemma. If we show that some direct sum of V agrees with a direct sum of W on $\mathcal{G}_{\text{Spec } C}$. The first hypothesis of the lemma allows us to conclude that there is a twisted vector bundle on \mathcal{G} .

Observe that on $\mathcal{G}_{\text{Spec } C}$ both the untwisted modules $W|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee}$, $V|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee}$ correspond to (the pullback of) projective modules on $\text{Spec } C$. Moreover, their n th-tensor powers are both free! It follows from K-theory, (2.4.2 (4)) that there is a N such that $(W|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee})^{\oplus N} \cong (V|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee})^{\oplus N} \cong F$ where F is a free C -module. Also note that there is an obvious isomorphism

$$V|_{\mathcal{G}_{\text{Spec } C}} \otimes (W|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee}) \cong W|_{\mathcal{G}_{\text{Spec } C}} \otimes (V|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee})$$

Thus, we obtain

$$\begin{aligned} V|_{\mathcal{G}_{\text{Spec } C}}^{\oplus M} &\cong V|_{\mathcal{G}_{\text{Spec } C}} \otimes F \\ &\cong V|_{\mathcal{G}_{\text{Spec } C}} \otimes (W|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee})^{\oplus N} \\ &\cong [V|_{\mathcal{G}_{\text{Spec } C}} \otimes (W|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee})]^{\oplus N} \\ &\cong [W|_{\mathcal{G}_{\text{Spec } C}} \otimes (V|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee})]^{\oplus N} \\ &\cong W|_{\mathcal{G}_{\text{Spec } C}} \otimes (V|_{\mathcal{G}_{\text{Spec } C}} \otimes V|_{\mathcal{G}_{\text{Spec } C}}^{\vee})^{\oplus N} \\ &\cong W|_{\mathcal{G}_{\text{Spec } C}} \otimes F \\ &\cong W|_{\mathcal{G}_{\text{Spec } C}}^{\oplus M} \end{aligned}$$

The result follows. □

Lemma 4.2.2. *Let X be a noetherian algebraic space which appears in a flat Mayer-Vietoris square*

$$\begin{array}{ccc}
\mathrm{Spec} C & \longrightarrow & \mathrm{Spec} B \\
\downarrow & & \downarrow f \\
\mathrm{Spec} A & \xrightarrow{i} & X
\end{array}$$

where i is an open immersion and f is a flat neighborhood of $(X \setminus \mathrm{Spec} A)_{red}$. Then $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Proof. Consider a Brauer class $\alpha \in \mathrm{Br}'(X)$, it suffices to find a twisted vector bundle of nonzero constant rank on a μ_n -gerbe \mathcal{X} mapping to the class α . Observe that by base-changing $\mathcal{X} \rightarrow X$ along the underlying square we obtain gerbes $\mathcal{X}_{\mathrm{Spec} A}$, $\mathcal{X}_{\mathrm{Spec} B}$, $\mathcal{X}_{\mathrm{Spec} C}$ representing the restricted Brauer classes and together they form a flat Mayer-Vietoris square by Lemma 2.3.3.

$$\begin{array}{ccccc}
& & \mathcal{X}_{\mathrm{Spec} C} & \longrightarrow & \mathcal{X}_{\mathrm{Spec} B} \\
& \swarrow & \downarrow & & \swarrow \\
\mathcal{X}_{\mathrm{Spec} A} & \longrightarrow & \mathcal{X} & & \\
\downarrow & & \downarrow & & \downarrow \\
& & \mathrm{Spec} C & \longrightarrow & \mathrm{Spec} B \\
\downarrow & \swarrow & \downarrow & & \swarrow \\
\mathrm{Spec} A & \longrightarrow & X & &
\end{array}$$

By Gabber's thesis [19] we know that every cohomology class in the cohomological Brauer group of an affine scheme is represented by an Azumaya algebra. It follows that there exists a nonzero twisted vector bundle of constant rank on $\mathcal{X}_{\mathrm{Spec} A}$ and $\mathcal{X}_{\mathrm{Spec} B}$.

On $\mathcal{X}_{\mathrm{Spec} A}$, let V denote a nonzero twisted vector bundle of constant rank, and on $\mathcal{X}_{\mathrm{Spec} B}$ let W denote another twisted bundle. Since these are 1-twisted sheaves on a μ_n -gerbe their n th tensor power has trivial stabilizer action so, in particular, $V^{\otimes n}$ is the pullback of a vector bundle on $\mathrm{Spec}(A)$ and analogously for $W^{\otimes n}$. By classical K-theory (Lemma 2.4.3) we learn that there exists a projective module P on $\mathrm{Spec}(A)$ such that $V^{\otimes n} \otimes P^{\otimes n} \otimes F_0$ is a free A -module for some free F_0 . Thus, by replacing V with $V \otimes P \otimes F_0$ we may assume that

$V^{\otimes n}$ is a free A -module. We do the same with W so that $W^{\otimes n}$ is free. Finally, by taking appropriate direct sums of V and W on $\mathcal{X}_{\text{Spec } A}$ and $\mathcal{X}_{\text{Spec } B}$ respectively, we may assume that they have the same constant rank! Note that both $V^{\oplus m}$ and $W^{\oplus l}$ still have the property that their n th tensor power is free for any $l, m > 0$. Now Theorem 2.3.5 allows us to invoke Lemma 4.2.1 and we conclude that $\text{Br}(X) = \text{Br}(X')$. □

Moving forward, we try to isolate some algebraic spaces X which can be realized inside a diagram as above. More precisely, we isolate X which can be presented as the pushout of affine schemes in a flat Mayer-Vietoris square.

4.3 There are enough Azumaya Algebras in codimension 2

Proposition 4.3.1. *Let X be a separated, noetherian algebraic space which admits an affine open subscheme $\text{Spec}(A) \rightarrow X$ whose complement $Z = (X \setminus \text{Spec}(A))_{\text{red}}$ is a closed subscheme factoring through a closed affine subscheme $Z \rightarrow \text{Spec}(R) \rightarrow X$. Then X appears in a flat Mayer-Vietoris square with affine schemes. Moreover, every μ_n -gerbe over X admits a nonzero twisted vector bundle of constant rank.*

Proof. By hypothesis we have a closed immersion $\text{Spec } R \rightarrow X$ which contains Z . By Theorem 2.3.14 we may assert the existence of a flat affine neighborhood of $\text{Spec } R$, call it $\text{Spec } B \rightarrow X$. Since it is a flat neighborhood of $\text{Spec } R$ it is also a flat neighborhood for the closed subscheme $Z \subset \text{Spec } R$. This yields the following cartesian square

$$\begin{array}{ccc} \text{Spec } C & \longrightarrow & \text{Spec } B \\ \downarrow & & \downarrow f \\ \text{Spec } A & \xrightarrow{i} & X \end{array}$$

where f is a flat neighborhood of Z . By Theorem 4.2.2, it follows that every μ_n gerbe admits a nonzero twisted vector bundle of constant rank. □

Theorem 4.3.2. *Let X be a separated, noetherian algebraic space whose regular locus contains a dense open subset. Then for any $\alpha \in \text{Br}'(X)$ there exists a open $U \subset X$ with $\text{codim}(X \setminus U) \geq 3$ such that $\alpha|_U \in \text{Br}(U)$.*

Proof. Consider an arbitrary μ_n -gerbe \mathcal{X} on X . Using the results above we will show there exists a nonzero twisted vector bundle on $\mathcal{X}|_U$ where $U \subset X$ is a dense open subset containing all codimension ≤ 2 points. Our strategy will be to find a nonzero twisted vector bundle V on a dense open subspace containing all the *singular* codimension ≤ 2 points. Given such a V , any reflexive extension V' of this twisted vector bundle will be locally free at *all* codimension ≤ 2 points. Indeed, it will be locally free at all singular codimension ≤ 2 points by construction and on the remaining smooth points of codimension ≤ 2 the Auslander-Buchsbaum formula implies V' is locally free. Passing to the open locus where V is locally free yields the result.

Since the regular locus of X is dense and open, the complement, $\text{Sing}(X)$ is a closed subspace of codimension ≥ 1 and therefore we may write

$$\text{Sing}(X) = C_1 \cup \dots \cup C_n \cup B_1 \cup \dots \cup B_m$$

where the C_i are codimension 1 components and the B_i are components of higher codimension. Using Proposition 2.3.7 there exists a dense open affine subscheme $\text{Spec}(A) \subset X$ containing the generic points of each of the C_i . This means all but finitely many codimension 2 points of X lying along $\text{Sing}(X)$ are contained in $\text{Spec}(A)$. Indeed, since $\text{Spec } A \cap C_i$ contains the generic point of the noetherian space C_i , there are finitely many components in $C_i \setminus (\text{Spec } A \cap C_i)$ all of whose generic points have codimension ≥ 2 in X . Therefore, the union of the generic points of the finitely many components of $C_1 \setminus (\text{Spec } A \cap C_1), \dots, C_n \setminus (\text{Spec } A \cap C_n)$ is finite and since the B_i are already codimension ≥ 2 they contribute at most one codimension 2 point each.

Now, since $\text{Spec } A$ is dense the (finitely many) codimension 2 points of X which lie on $\text{Sing}(X) \cap (X \setminus \text{Spec } A)$ must all be codimension ≤ 1 points of $(X \setminus \text{Spec } A)_{\text{red}}$. As such, this finite set of points admits a common affine open neighborhood of $(X \setminus \text{Spec } A)_{\text{red}}$ by Proposition 2.3.7, denote it by $\text{Spec } B \subset (X \setminus \text{Spec } A)_{\text{red}}$. However, because $(X \setminus \text{Spec } A)_{\text{red}}$

has the subspace topology (see Tag 04CE in [56]), there is a open subspace $W \subset X$ so that $W \cap (X \setminus \text{Spec } A)_{red} = \text{Spec } B$.

Observe that the open subspace $Y = \text{Spec } A \cup W \subset X$ contains all the singular codimension ≤ 2 points and satisfies the hypothesis of 4.2.2. Therefore we have $\text{Br}(Y) = \text{Br}'(Y)$ and in particular $\mathcal{X}|_Y$ admits a nonzero twisted vector bundle of finite rank. After taking a coherent extension of this vector bundle to all of X (see 15.5 in [38]) we may take a reflexive hull. Since the resulting coherent sheaf is already locally free on the singular codimension 2 points of \mathcal{X} it remains to check it at the smooth points codimension 2 points. This follows by the Auslander-Buchsbaum formula. That it is twisted follows from Proposition 2.2.8. □

Remark 4.3.3. When working with a μ_{pk} -gerbe \mathcal{X} in characteristic p note that there exists no étale atlas for \mathcal{X} . Thus to check the local freeness of a reflexive module at a regular codimension 2 point p of \mathcal{X} one has to be careful. In particular, if $U \rightarrow \mathcal{X}$ is a smooth atlas then a point lying above p may not be a codimension 2 point! That being said, there does exist a codimension 2 point lying above p , thus we can apply the Auslander-Buchsbaum formula at such a point.

4.4 A Stacky generalization of Schröer's Theorem

Lemma 4.4.1. *Let $\mathcal{X}_{\text{Spec } A} \rightarrow \text{Spec } A$ be a coarse space morphism of a tame Artin stack where A is noetherian and I -adic for some ideal I . Suppose moreover that there exists a vector bundle on $\mathcal{X}_{\text{Spec } A/I}$ with trivial determinant. Then \mathcal{X} admits a vector bundle V with trivial determinant lifting V_0 .*

Proof. To obtain such a vector bundle on $\mathcal{X}_{\text{Spec } A} \rightarrow \text{Spec } A$ we use deformation theory to get a formal system of vector bundles over $\{\mathcal{X}_{\text{Spec } A/I^n}\}$ and then algebraize this system using a generalization of Grothendieck's existence theorem. We deform the vector bundle V_0 along the sequence of thickenings

$$\mathcal{X}_{\text{Spec } A_0} \rightarrow \mathcal{X}_{\text{Spec } A/I^2} \rightarrow \mathcal{X}_{\text{Spec } A/I^3} \rightarrow \dots$$

At each stage a deformation (V_i, ϕ) on $\mathcal{X}_{\text{Spec } A/I^{i+1}}$ exists and is unique upto isomorphism by Proposition 2.2.2, the Leray spectral sequence for $\pi : \mathcal{X}_{\text{Spec } A_0} \rightarrow \text{Spec } A_0$, and tameness. Indeed the deformation and obstruction spaces are

$$H^i(\mathcal{X}_{\text{Spec } A_0}, \text{End}(V_{i-1}) \otimes I^{i+1}/I^{i+2}) \cong H^i(\text{Spec } A_0, \pi_*(\text{End}(V_{i-1}) \otimes I^{i+1}/I^{i+2}))$$

for $i = 1, 2$ respectively and these all vanish because an affine scheme has no quasicoherent cohomology! Note that a deformation of a vector bundle V_i also induces a deformation of its determinant, but since the deformation spaces are trivial and $\det(V_0) \cong \mathcal{O}_{\mathcal{X}_{\text{Spec } A_0}}$ this implies $\det(V_1) \cong \mathcal{O}_{\mathcal{X}_{\text{Spec } A/I^2}}$. By induction it follows that the V_i all have trivial determinants. Thus we have a compatible system $\{V_i\}$ of vector bundles all of whose determinants are trivial. By a generalization of Grothendieck's Existence theorem (see Theorem 1.4 of [52]) applied to $\{V_i\}$ and $\mathcal{X}_{\text{Spec } A} \rightarrow \text{Spec } A$ there exists a coherent sheaf on $\mathcal{X}_{\text{Spec } A}$ restricting to $\{V_i\}$. It is a vector bundle since it is a flat coherent sheaf on the noetherian stack $\mathcal{X}_{\text{Spec } A}$ (see Tag 0523 in [56]). Moreover, its determinant must be trivial by Grothendieck's existence theorem again since the corresponding formal determinant is trivial! This completes the proof of the lemma. \square

Theorem 4.4.2. *Let \mathcal{X} be a 2-dimensional tame, normal Deligne-Mumford stack which is finite type, separated over a field with generically trivial stabilizers, then $\text{Br}(\mathcal{X}) = \text{Br}'(\mathcal{X})$.*

Proof. Let \mathcal{G} be a μ_n -gerbe on \mathcal{X} and $\mathcal{X} \rightarrow X$ the coarse space map. It suffices to find an open substack $\mathcal{U} \subset \mathcal{X}$ containing the singular locus such that $\mathcal{G}|_{\mathcal{U}}$ admits a twisted vector bundle. Indeed, any reflexive coherent extension over $\mathcal{G}|_{\mathcal{U}} \subset \mathcal{G}$ will yield a twisted vector bundle on \mathcal{G} .

Let $\text{Spec } A \subset \mathcal{X}$ be a smooth affine open substack and denote by \mathcal{C} the complement with its reduced induced structure and $\mathcal{C} \rightarrow C$ the associated coarse space map. Since the singular locus of \mathcal{X} lies on \mathcal{C} we can denote by q_1, \dots, q_n the image of the singular points of \mathcal{X} in C . Let $\mathcal{G}_{\mathcal{C}} \rightarrow \mathcal{C}$ be the restricted μ_n -gerbe, we begin by showing there exists a twisted vector bundle on $\mathcal{G}_{\mathcal{C}}$. Since \mathcal{C} is 1-dimensional, tame, finite type and separated over a field it

is a quotient stack with quasiprojective coarse space (see theorem 4.6 above) and therefore by theorem 2.1 of [37] there exists a finite flat cover $Z \rightarrow \mathcal{C}$ where Z is a quasiprojective scheme. Now because \mathcal{G}_Z admits a nonzero twisted vector bundle, Theorem 4.1.6 implies $\mathcal{G}_{\mathcal{C}}$ admits one as well. It follows that $\mathcal{G}_{\mathcal{C}}$ admits a twisted vector bundle of constant (nonzero) rank $V_{\mathcal{C}}$. By replacing $V_{\mathcal{C}}$ with direct sums of itself we may suppose that it has a determinant with trivial stabilizer action and therefore is the pullback of a line bundle on C (see, for example, Lemma 3.2 of [37]). Moreover, by tensoring $V_{\mathcal{C}}$ with a large power of a ample line bundle on C we may also suppose that its determinant is very ample on C . Thus, by taking the complement of an appropriate section of the determinant we may find a open affine neighborhood $\text{Spec } B_0 \subset C$ containing the q_1, \dots, q_n and which trivializes $\det(V|_{\mathcal{C}})$. By Tag 04CE in [56] there exists a open subspace $W \subset X$ such that $W \cap C = \text{Spec } B_0$ and therefore a closed immersion $\text{Spec } B_0 \rightarrow W$. Therefore we may apply Theorem 2.3.14 to obtain a flat neighborhood $\text{Spec } B \rightarrow W$ of $\text{Spec } B_0$ which is affine, adic and 2-dimensional.

Applying the Lemma 4.4.1 to the coarse space morphism $\mathcal{G}_{\text{Spec } B} \rightarrow \text{Spec } B$ and the twisted vector bundle $V_{\mathcal{C}}|_{\text{Spec } B_0}$ we obtain a twisted vector bundle (see Lemma 2.2.8) $V_{\mathcal{G}_{\text{Spec } B}}$ on $\mathcal{G}_{\text{Spec } B}$ which has trivial determinant. Now observe that we obtain the following diagram whose bottom (and top) faces are flat Mayer-Vietoris squares.

$$\begin{array}{ccccc}
 & & \text{Spec } A \times_X \text{Spec } B & \longrightarrow & \mathcal{X}_{\text{Spec } B} \\
 & \swarrow & \downarrow & & \swarrow \\
 \text{Spec } A & \longrightarrow & \mathcal{X}_{\text{Spec } A \cup W} & & \downarrow \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \text{Spec } A \times_X \text{Spec } B & \longrightarrow & \text{Spec } B \\
 & \swarrow & \downarrow & & \swarrow \\
 \text{Spec } A & \longrightarrow & \text{Spec } A \cup W & & \downarrow
 \end{array}$$

As in the proof of Lemma 4.2.2, we may find a constant nonzero rank twisted vector bundle on W on $\mathcal{G}_{\text{Spec } A}$ whose n th tensor power is free. By taking direct sums of W and $V_{\mathcal{G}_{\text{Spec } B}}$ with themselves we may assume they have the same rank and note that $V_{\mathcal{G}_{\text{Spec } B}}$ still has trivial determinant. To apply Lemma 4.2.1 it remains to see why $V_{\mathcal{G}_{\text{Spec } B \times_X \text{Spec } A}}^{\otimes n}$ is trivial on

$\text{Spec } B \times_X \text{Spec } A$. But this follows because $\text{Spec } B \times_X \text{Spec } A$ is a noetherian 1-dimensional affine scheme and we may use Serre's splitting theorem (Theorem 2.4.1(1)), and the triviality of its determinant to deduce that it is isomorphic to a free module. Lemma 4.2.1 allows us to conclude.

□

4.5 Examples in codimension ≥ 3 arising from Artin contractions

The work above yields a positive answer to Grothendieck's question in codimension 2. A natural next step is to investigate the question in codimension 3. Indeed, Grothendieck's question is wide open even for smooth threefolds over the complex numbers. Towards understanding the existence of Azumaya algebras in high codimension, we initiate a study of generically trivial cohomological Brauer classes which arise from singularities in codimension ≥ 3 . We will consider a family of examples in high dimensions and we later reinterpret membership in the geometric Brauer group for the generically trivial Brauer classes.

Construction: Let Y be a smooth, separated, finite-type scheme of dimension ≥ 2 over an algebraically closed field k containing a smooth proper curve E of genus 1. Moreover, suppose that $N_{E/Y}$ is antiample i.e. I/I^2 is ample on E where I is the ideal sheaf of E in Y . Thus, Artin's constructions (see 2.6.5 and 2.6.6) explains that there is a proper birational morphism of normal algebraic spaces $\pi : Y \rightarrow X$ which contracts precisely E .

Proposition 4.5.1. *If $E \subset Y$ and $\pi : Y \rightarrow X$ are as above, then using the Leray spectral sequence for π we obtain an exact sequence*

$$0 \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}(Y) \longrightarrow \text{Pic}(E) \longrightarrow H^2(X, \mathbf{G}_m) \longrightarrow H^2(\text{Spec } K, \mathbf{G}_m)$$

where $\eta : \text{Spec } K \rightarrow X$ is the inclusion of the generic point.

Proof. Consider the exact sequence of low degree terms associated to the Leray spectral sequence for π and \mathbf{G}_m :

$$\begin{aligned} 0 \rightarrow \mathrm{Pic}(X) \rightarrow \mathrm{Pic}(Y) \rightarrow \mathrm{H}^0(X, \mathbf{R}^1 \pi_* \mathbf{G}_m) \rightarrow \mathrm{H}^2(X, \mathbf{G}_m) \rightarrow \\ \rightarrow \mathrm{E}_1^2 \rightarrow \mathrm{H}^1(X, \mathbf{R}^1 \pi_* \mathbf{G}_m) \rightarrow \mathrm{H}^3(X, \mathbf{G}_m) \end{aligned}$$

Indeed, since the map of sheaves $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Y$ is an isomorphism, we have $\pi_* \mathbf{G}_m \cong \mathbf{G}_m$. Here $E_1^2 = \ker(\mathrm{H}^2(Y, \mathbf{G}_m) \rightarrow \mathrm{H}^0(X, \mathbf{R}^2 \pi_* \mathbf{G}_m))$. However, because Y is smooth, the map $\mathrm{H}^2(Y, \mathbf{G}_m) \rightarrow \mathrm{H}^2(\mathrm{Spec} K, \mathbf{G}_m)$ is injective. Thus we can rewrite the sequence as

$$0 \rightarrow \mathrm{Pic}(X) \rightarrow \mathrm{Pic}(Y) \rightarrow \mathrm{H}^0(X, \mathbf{R}^1 \pi_* \mathbf{G}_m) \rightarrow \mathrm{H}^2(X, \mathbf{G}_m) \rightarrow \mathrm{H}^2(\mathrm{Spec} K, \mathbf{G}_m)$$

It is well known that $\mathbf{R}^1 \pi_* \mathbf{G}_m$ is the sheafification of the presheaf on X_{et}

$$U \mapsto \mathrm{H}^1(\pi^{-1}(U), \mathbf{G}_m) = \mathrm{Pic}(\pi^{-1}(U))$$

and the stalks are the same as that of this presheaf. Moreover, because π is an isomorphism away from $p \in X$ we find that $\mathbf{R}^1 \pi_* \mathbf{G}_m$ is supported only at p and so

$$\mathrm{H}^0(X, \mathbf{R}^1 \pi_* \mathbf{G}_m) = (R^1 \pi_* \mathbf{G}_m)_p = \lim \mathrm{H}^1(\pi^{-1}(U), \mathbf{G}_m) = \lim \mathrm{Pic}(\pi^{-1}(U))$$

where the limit ranges over all étale neighborhoods of p . By Lemma 3.1.16 of [45] or Tag 03Q9 in [56] this group is isomorphic to $\mathrm{Pic}(X_p^{sh})$ where $Y_p^{sh} = Y \times_X \mathrm{Spec}(\mathcal{O}_{X,p}^{sh})$. Consider the natural map $\mathrm{Pic}(Y_p^{sh}) \rightarrow \mathrm{Pic}(E)$ given by restricting line bundles to the closed fiber.

Lemma 4.5.2. *The natural map $\mathrm{Pic}(Y_p^{sh}) \rightarrow \mathrm{Pic}(E)$ is an isomorphism.*

Proof. Note that we have the diagram filled with fibered squares

$$\begin{array}{ccccccccc} E & \longrightarrow & \mathfrak{Y} & \longrightarrow & \widehat{E} & \longrightarrow & Y_p^{sh} & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \pi_c & & \downarrow \pi & & \downarrow \\ \mathrm{Spec} k(p) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_{X,p}^{sh}) & \longrightarrow & \mathrm{Spec} \widehat{\mathcal{O}_{X,p}^{sh}} & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{X,p}^{sh}) & \longrightarrow & X \end{array}$$

Fix a $L \in \text{Pic}(Y_p^{sh})$ and suppose $L|_{\widehat{E}} \cong \mathcal{O}_{\widehat{E}}$ then by flat base change and fpqc descent we know that the pushforward $\pi_* L$ is a line bundle on $\text{Spec}(\mathcal{O}_{X,p}^{sh})$ since $\pi_c \mathcal{O}_{\widehat{E}} \cong \mathcal{O}_{\text{Spec}(\widehat{\mathcal{O}_{X,p}^{sh}})}$. Since the base ring is local it must be trivial. Thus the adjunction map $\pi^* \pi_* L \rightarrow L$ is equivalent to $\mathcal{O}_{Y_p^{sh}} \rightarrow L$. This morphism is formation compatible with the flat base change and therefore restricts to the map $\pi_c^*(\pi_c)_* L|_{\widehat{E}} \rightarrow L|_{\widehat{E}}$. That this is an isomorphism implies L must have been the trivial bundle on Y_p^{sh} . It remains to check injectivity of the final two maps $\text{Pic}(\widehat{E}) \rightarrow \text{Pic}(\mathfrak{Y}) \rightarrow \text{Pic}(E)$. This amounts to showing two things: (1) every formal deformation of a fixed line bundle L on E is unique and (2) algebraizations of a line bundle from the formal scheme \mathfrak{Y} to \widehat{E} are unique. (1) follows because the deformations at the n th step are parametrized by $H^1(E, I^n/I^{n+1})$. Observe that $\text{Sym}^n(I/I^2) \cong I^n/I^{n+1}$ and therefore one can show that I^n/I^{n+1} is ample. Indeed, the n th tensor power $(I/I^2)^{\otimes n}$ is ample by 3.3 in [9], 5.2 in [31] and the hypothesis that I/I^2 is ample. Moreover, the n th symmetric power I^n/I^{n+1} is a quotient of the n th tensor power $(I/I^2)^{\otimes n}$ and ampleness descends along quotient maps by 2.2 in [31]. It follows by 1.1, 1.2 and 1.3 in [32] that $H^1(E, I^n/I^{n+1})$ vanishes for any $n \geq 1$. (2) follows from the Grothendieck Existence theorem as in 5.1.4 in [24].

To show surjectivity one produces a formal deformation of a fixed line bundle on E , then using Grothendieck's existence theorem we may algebraize this bundle to produce one on the fiber of $\text{Spec}(\widehat{\mathcal{O}_{X,p}^{sh}})$. Finally, one uses Popescu's theorem to show that we can further lift this bundle to the "henselized" fiber Y_p^{sh} . We may lift line bundles to arbitrary order because E is 1-dimensional and hence the obstruction spaces (which all lie in a degree 2 cohomology group) all vanish. Grothendieck's existence theorem now yields a lift of L to \widehat{E} . It remains to show that a line bundle, L on \widehat{E} lifts to one on Y_p^{sh} . The theory of Artin approximation is well known so we only give a sketch of the proof here. Define the functor of deformations of L :

$$F : \mathcal{O}_{X,p}^{sh}\text{-Algebras} \rightarrow \text{Set}$$

$$A \mapsto \{(L', \phi) \mid L' \in \text{Pic}(Y_A) \text{ with } \phi : L'|_{Y \times_X \text{Spec}(A \otimes_{\mathcal{O}_{X,p}^{sh}} k(p))} \cong L|_{A \otimes_{\mathcal{O}_{X,p}^{sh}} k(p)}\} / \cong$$

Where two pairs (L', ϕ') and (L'', ϕ'') are equivalent if there is an isomorphism between L' and L'' which is compatible with ϕ' and ϕ'' . We are looking for étale $\mathcal{O}_{X,p}^{sh}$ -algebra B such that $\text{Spec } B \rightarrow \mathcal{O}_{X,p}^{sh}$ is surjective and $F(B)$ is nonempty, for then we would obtain a $\mathcal{O}_{X,p}^{sh}$ -valued point of F . To show this we use Popescu's theorem [13] and the fact that $F(\text{colim } A_i) = \text{colim } F(A_i)$ to produce a smooth $\mathcal{O}_{X,p}^{sh}$ -algebra A_i for which $F(A_i)$ is nonempty. Indeed, Popescu's theorem tells us that $\widehat{\mathcal{O}_{X,p}^{sh}}$ is the colimit of smooth $\mathcal{O}_{X,p}^{sh}$ -algebras as desired. From here, it is a formal to find a étale such A_i : roughly, étale locally there are sections for a smooth morphism, restricting to this section yields the desired étale algebra. The interested reader is invited to find a more general argument in [13] or [5]. \square

The above lemma concludes the proof of 4.5.1. \square

Remark 4.5.3. Since this map comes from a spectral sequence, it isn't obvious how to describe it explicitly. In what follows we give a concrete description of the map. The idea is to consider sheaves on X that are "almost" locally free instead of considering locally free sheaves on Y .

Theorem 4.5.4. *The following sequences are exact in the category of abelian sheaves on the étale site of X .*

$$0 \rightarrow \text{Div}_X \rightarrow Z_X^1 \rightarrow \mathcal{P} \rightarrow 0$$

$$0 \rightarrow \mathbf{G}_m \rightarrow \iota_* \mathbf{G}_{m,K} \rightarrow \text{Div}_X \rightarrow 0$$

Here Div_X and Z_X^1 denotes the sheaf of Cartier and Weil divisors on X respectively. Moreover, $\iota : \text{Spec } K \rightarrow X$ is the inclusion of the generic point.

Proof. The first is exact on the left because any Cartier divisor can be realized as a line bundle with an embedding into the function field of X . In particular, given such a invertible sub sheaf $L \subset K$, we can take $\text{div}(1/f)$ for a local generator f and patch to form a global Weil divisor. If we get the zero Weil divisor then all local generators of L are units i.e. $L = \mathcal{O}_X$.

We define \mathcal{P} to be the cokernel of the inclusion $\text{Div}_X \rightarrow Z_X^1$ so exactness in the middle and the right follow by definition.

Exactness of the second sequence follows from two facts: exactness on the Zariski site and that the Zariski sequence is formation compatible with an étale base change. \square

Remark 4.5.5. Observe that \mathcal{P} is supported only at the point p since X is smooth elsewhere. Thus $\mathcal{P}(X)$ equals the stalk at p of the presheaf $U \mapsto Z_X^1(U)/\text{Div}_X(U)$ i.e.

$$\mathcal{P}(X) = \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh}))$$

Taking the long exact sequence in cohomology we obtain:

$$0 \rightarrow \text{Div}(X) \rightarrow Z_X^1(X) \rightarrow \mathcal{P}(X) \rightarrow H^1(X, \text{Div}_X) \rightarrow 0$$

To see this note that the sheaf of Weil divisors has vanishing first cohomology because we can write $Z_X^1 \cong \bigoplus_{x \in X^{(1)}} (i_x)_* \mathbf{Z}$ and each of these terms has trivial first cohomology. Indeed, $H^1(\text{Spec } k(x), \mathbf{Z}) = \text{Hom}_{\text{cont}}(G_{k(x)}, \mathbf{Z}) = 0$ since the Galois group of a field is compact whereas no nonzero subgroup of \mathbf{Z} is. For the second sequence in Theorem 4.5.4 we obtain:

$$H^1(X, \iota_* \mathbf{G}_{m,K}) \rightarrow H^1(X, \text{Div}_X) \rightarrow H^2(X, \mathbf{G}_m) \rightarrow H^2(X, \iota_* \mathbf{G}_{m,K})$$

However, by considering the Leray spectral sequence for $\iota : \text{Spec}(K) \rightarrow X$ we may refine this sequence. Note that $H^1(\text{Spec } K, \mathbf{G}_{m,K}) = \mathbf{R}^1 \iota_* \mathbf{G}_m = 0$ by Hilbert's theorem 90 which, in turn, implies $H^2(X, \iota_* \mathbf{G}_{m,K}) \subset H^2(\text{Spec } K, \mathbf{G}_m)$ and $H^1(X, \iota_* \mathbf{G}_{m,K}) = 0$ so:

$$0 \rightarrow H^1(X, \text{Div}_X) \rightarrow H^2(X, \mathbf{G}_m) \rightarrow H^2(\text{Spec } K, \mathbb{G}_m)$$

Thus we may realize the generically trivial cohomology classes as those which appear in the image of the following iterated coboundary map:

$$\mathcal{P}(X) \rightarrow H^1(X, \text{Div}_X) \rightarrow H^2(X, \mathbf{G}_m)$$

We describe this map below. Consider a generically trivial element $\alpha \in H^2(X, \mathbf{G}_m)$ which is represented by an Azumaya algebra A . By the generic triviality, $\alpha|_Y$ must be trivial. This

implies $A|_{X \setminus \{p\}} \cong \text{End}(E')$ for some locally free E' on $X \setminus \{p\}$. Moreover, because A is reflexive we have $A \cong \text{End}(E)$ with $E = i_* E'$ where E is also reflexive. Since A is Azumaya we must have

$$\text{End}(E|_{\text{Spec}(\mathcal{O}_{X,p}^{sh})}) \cong A|_{\text{Spec}(\mathcal{O}_{X,p}^{sh})} \cong \text{End}(\mathcal{O}^n)$$

This means implies $E|_{\text{Spec}(\mathcal{O}_{X,p}^{sh})} \cong \bigoplus^n L$ where L is a rank one reflexive sheaf. Indeed, by a theorem of Noether-Skolem, E and \mathcal{O}^n must differ by a line bundle on the punctured spectrum $\text{Spec}(\mathcal{O}_{X,p}^{sh}) \setminus \{p\}$. Pushing forward over the closed point yields the desired identification. Since X is normal L defines an element of $\mathcal{P}(X) = \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh}))$, call it s_A .

The hope is that the image of s_A under the coboundary map is $[A]$, and in Proposition 1.5 of [53] one can find an argument due to Gabber which (almost) realizes this dream. Since the exposition given there is not stated in this generality and is rather terse, we explain the idea here.

Proposition 4.5.6. *If X , A and s_A are as above then the class of A is the inverse of the image of s_A under the iterated coboundary map*

$$\mathcal{P}(X) = \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh})) \rightarrow \text{H}^1(X, \text{Div}_X) \rightarrow \text{H}^2(X, \mathbf{G}_m)$$

Proof. If $p : Z_X^1(X) \rightarrow \text{Cl}(\mathcal{O}_{X,p}^{sh}) \rightarrow 0$ is the surjection then under the first coboundary map the class of s_A gets mapped to the Div_X -torsor $p^{-1}(s_A) \in \text{H}^1(X, \text{Div}_X)$. This is the sheaf of liftings of the section $s_A \in \text{Cl}(\mathcal{O}_{X,p}^{sh})$ and under the second coboundary map this gets sent to the \mathbf{G}_m -gerbe \mathcal{Y} of liftings along the morphism $\pi : \mathbf{G}_{m,K} \rightarrow \text{Div}_X \rightarrow 0$. In particular, for an étale $T \rightarrow X$ the sections of $\mathcal{Y}(T)$ are pairs (M, ϕ) where M is a $\mathbf{G}_{m,K}$ -torsor and $\phi : M \rightarrow p^{-1}(s_A)|_T$ is a π -equivariant morphism. A morphism between such a pair in \mathcal{Y} is an isomorphism of $\mathbf{G}_{m,K}$ -torsors compatible with the associated π -equivariant maps. Note that the \mathbf{G}_m -action is given by left translation on M .

The \mathbf{G}_m -gerbe representing the class of A , \mathcal{X} , can be described similarly: the sections over T consists of pairs (V, ψ) where V is a vector bundle on T and $\psi : \text{End}(V) \rightarrow A|_T$ is an isomorphism of Azumaya algebras. Morphisms between two pairs is an isomorphism of vector

bundles respecting the associated trivializations of $A|_T$. Recall that A can be written as $\text{End}(F)$ with F reflexive and $F|_{\mathcal{O}_{\bar{X},p}^{sh}} = \bigoplus L$. Thus, by Noether-Skolem, $\psi : \text{End}(V) \rightarrow \text{End}(F|_T)$, induces an isomorphism $\bar{\psi} : V \otimes L' \cong F|_T$ where L' is reflexive of rank 1 on T . This hints at an alternate description of the gerbe \mathcal{X} . Indeed, define \mathcal{X}' as the stack whose T -points are triples $(V, L', \bar{\psi})$ where V is a vector bundle on T , L' a rank one reflexive sheaf, and $\bar{\psi} : V \otimes L' \cong F|_T$ an isomorphism. The morphisms between two triples are isomorphisms between the sheaves that respect the isomorphism with $F|_T$. One can check that this stack is a \mathbf{G}_m -gerbe where the \mathbf{G}_m action on $(V, L', \bar{\psi})$ is given by multiplication on V and inverse multiplication on L' . Finally, note that the construction above describes an equivariant map $\mathcal{X} \rightarrow \mathcal{X}'$.

Next we will describe an equivalence of stacks \mathcal{X}' and \mathcal{Y} that is equivariant for the inverse map $-1 : \mathbf{G}_m \rightarrow \mathbf{G}_m$. From this it follows that $[\mathcal{X}'] = -[\mathcal{Y}]$ as desired. Fix a T -section of \mathcal{X}' , $(V, L', \bar{\psi})$ and restrict the isomorphism to $\mathcal{O}_{\bar{X},p}^{sh}$: we obtain $\bigoplus L' \cong \bigoplus L$. From this we can deduce that L and L' yield the same class in $\text{Cl}(\mathcal{O}_{\bar{X},p}^{sh})$. Indeed, by tensoring with L^\vee on both sides and taking reflexive hulls we deduce that $\bigoplus (L' \otimes L^\vee)^{\vee\vee} \cong \bigoplus \mathcal{O}$. This implies $(L' \otimes L^\vee)^{\vee\vee}$ is a line bundle and hence L and L' must be isomorphic. Indeed, the Picard group of a local scheme is trivial so because the difference of the two Weil divisor classes is the class of a Cartier divisor, the difference must be null.

Let $\mathcal{M}^*(L')$ be the sheaf of invertible rational sections of L' on T . More precisely, if $T' \rightarrow T$ is étale then set $\mathcal{M}^*(L')(T' \rightarrow T) = \{(s, V)\} / \sim$ where V is a dense open of T and $s \in L'(V)$ doesn't vanish at any of the generic points of V . The equivalence relation is given by declaring (s, V) and (s', V') equal if the two sections agree on a dense open of $V \cap V'$. It is not difficult to see that $\mathcal{M}^*(L')$ is a $\iota_* \mathbf{G}_{m,K}$ -torsor. There is a natural map $f : \mathcal{M}^*(L') \rightarrow p^{-1}(s_A)$: send a section (s, V) over $T' \rightarrow T$ to $\text{div}(s)$, a Weil divisor on T' . Certainly this Weil divisor is equivalent to any other representing $L'|_{T'}$ but the previous paragraph implies $\text{div}(s) \in p^{-1}(s_A)(T')$. One can check that the map f is equivariant for the map $\pi : \iota_* \mathbf{G}_{m,K}|_T \rightarrow \text{Div}_X|_T$. In fact, we have constructed a map of stacks $\mathcal{X}' \rightarrow \mathcal{Y}$. However, since the action of \mathbf{G}_m on \mathcal{X} induces multiplication on L' by inverses we find that

the map of gerbes is antiequivariant. \square

Using this we rephrase a question about Brauer groups into a K-theoretic question along the resolution $Y \rightarrow X$, in particular:

Theorem 4.5.7. *Let $Y \rightarrow X$ be as above, then the following are equivalent*

- (a) *Every generically trivial Brauer class $\alpha \in \text{Br}'(X)$ is in the geometric Brauer group.*
- (b) *Suppose that $[L] \in \text{Pic}(E)/\text{Pic} Y$ with $[L^{\otimes n}] = [\mathcal{O}]$. Then it has a sum which lifts to a vector bundle on Y . More precisely, for any such L there is a vector bundle V on Y with $V|_E \cong \bigoplus^r L$ for some r .*
- (c) *Suppose that $[\mathcal{O}(D)] \in \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh}))/\text{Cl}(X)$ has $[\mathcal{O}(nD)] \cong [\mathcal{O}]$. Then there is a reflexive F on X which is locally free on $X \setminus \{p\}$ and $F|_{\text{Spec}(\mathcal{O}_{X,p}^{sh})} \cong \bigoplus^r \mathcal{O}(D)$ for some r .*

Proof. First observe that we have the following maps

$$\text{Pic}(E) \leftarrow \text{Pic}(Y_p^{sh}) \rightarrow \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh}))$$

the first is an isomorphism as described in 4.5.2 and the second is described via the following

$$\text{Pic}(Y_p^{sh}) \cong \text{Cl}(Y_p^{sh}) \rightarrow \text{Cl}(Y^{sh} \setminus E) \cong \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh}) \setminus \{p\}) \cong \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh}))$$

Note that the second map is an isomorphism when Y has dimension ≥ 3 and hence, in this case we obtain isomorphisms

$$\text{Pic}(E) \cong \text{Pic}(Y_p^{sh}) \cong \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh}))$$

However, in the 2-dimensional setting we have

$$\text{Pic}(Y_p^{sh})/\mathbf{ZO}(E)|_{Y_p^{sh}} \cong \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh}))$$

Suppose (b) holds and fix a representative in the class $[\mathcal{O}(D)]$ with $[\mathcal{O}(nD)] = [\mathcal{O}]$ as in (c). Then consider the corresponding line bundle $\mathcal{O}(D)|_{\text{Spec}(\mathcal{O}_{X,p}^{sh} \setminus \{p\})} = L' \in \text{Pic}(Y_p^{sh} \setminus E)$ and

let L be a line bundle on $\text{Pic}(Y_p^{sh}) = \text{Pic}(E)$ restricting to L' . We know that $L^{\otimes n} \mapsto \mathcal{O}(nD)$ and that there is a line bundle $M \in \text{Pic}(Y \setminus E) \cong \text{Cl}(X \setminus \{p\})$ with $M \mapsto \mathcal{O}(nD)$ as well. Thus, by formal glueing the pair $(L^{\otimes n}, M)$ is the restriction of a line bundle in $\text{Pic}(Y)$: apply 2.3.5 to the flat Mayer-Vietoris square:

$$\begin{array}{ccc} Y_p^{sh} \setminus E & \longrightarrow & Y \setminus E \\ \downarrow & & \downarrow \\ Y_p^{sh} & \longrightarrow & Y \end{array}$$

Thus $[L^{\otimes n}] = [\mathcal{O}]$ as in (b). So there is a vector bundle V with the property that $V|_E \cong L^{\oplus m}$. Note that

$$V|_{Y_p^{sh}} \cong L^{\oplus m}$$

as well since the former vector bundle is a deformation of $L^{\oplus m}$ and the deformation spaces of this vector bundle are identically zero. This implies $V|_{Y_p^{sh} \setminus E} \cong L'^{\oplus m} \cong \mathcal{O}(D)^{\oplus m}$ so set $F = i_*(V|_{X \setminus \{p\}})$ where $i : X \setminus \{p\} \rightarrow X$ to conclude that (c) holds. To see that (c) implies (b) choose a $[L] \in \text{Pic}(Y_p^{sh})/\text{Pic}(Y) \cong \text{Pic}(E)/\text{Pic}(Y)$ with the property that $[L^{\otimes m}] = [\mathcal{O}]$ and set $L' = L|_{Y_p^{sh} \setminus E}$. If we view this as a class $\mathcal{O}(D) \in \text{Cl}(\text{Spec } \mathcal{O}_{X,p}^{sh})$ one sees that $[\mathcal{O}(mD)] = [\mathcal{O}]$ in $\text{Cl}(\text{Spec } \mathcal{O}_{X,p}^{sh})/\text{Cl}(X)$. This follows from the commutativity of the following diagram of abelian groups.

$$\begin{array}{ccc} \text{Pic}(Y) & \longrightarrow & \text{Pic}(Y \setminus E) \cong \text{Cl}(X) \\ \downarrow & & \downarrow \\ \text{Pic}(Y_p^{sh}) & \xrightarrow{i} & \text{Pic}(Y_p^{sh} \setminus E) \cong \text{Cl}(\text{Spec } \mathcal{O}_p^{sh}) \end{array}$$

By (c) we see that there exists a reflexive module F on X , locally free on $X \setminus \{p\}$, with the property that $F|_{\text{Spec } \mathcal{O}_{X,p}^{sh}} \cong \mathcal{O}(D)^{\oplus n}$. Observe that $V' = (F|_Y)^{\vee\vee}$ is a vector bundle because

$$(F|_Y)^{\vee\vee}|_{Y_p^{sh}} \cong (\mathcal{O}(D)^{\oplus n}|_{Y_p^{sh}})^{\vee\vee} \cong L'^{\oplus n}$$

where $L'' \in \text{Pic}(Y_p^{sh})$. The first isomorphism follows from the fact that dualizing commutes with flat pullback, the second follows because a rank 1 reflexive sheaf on a regular scheme is always invertible and so V is locally free. We also have the isomorphism $L'^{\oplus n}|_{Y^{sh} \setminus E} \cong L'^{\oplus n}$. Indeed, the restriction of the natural map

$$F|_Y \rightarrow (F|_Y)^{\vee\vee}$$

to $Y_p^{sh} \setminus E$ is an isomorphism since $F|_{Y_p^{sh}}$ is already reflexive. But $F|_{Y_p^{sh} \setminus E} \cong L'^{\oplus n}$.

Thus, by Theorem 2.3.5 we see that the pair $(L^{\oplus n}, V'|_{Y \setminus E})$ can be glued along the following flat Mayer-Vietoris square

$$\begin{array}{ccc} Y_p^{sh} \setminus E & \longrightarrow & Y \setminus E \\ \downarrow & & \downarrow \\ Y_p^{sh} & \longrightarrow & Y \end{array}$$

to obtain a vector bundle V on Y with $V|_E = L^{\oplus n}$. This concludes the equivalence of (b) and (c).

Next we show that (a) implies (c). Recall that the image of $Z^1(X)/\text{Div}(X)$ in $\mathcal{P}(X)$ is precisely the kernel of the iterated coboundary map $\mathcal{P}(X) \rightarrow H^2(X, \mathbf{G}_m)$ i.e.

$$0 \rightarrow \text{Cl}(\text{Spec } \mathcal{O}_{X,p}^{sh})/\text{Cl}(X) \rightarrow H^2(X, \mathbf{G}_m) \rightarrow H^2(\text{Spec } K, \mathbf{G}_m)$$

is an exact sequence. Fix a class $a = [\mathcal{O}(D)] \in \text{Cl}(\text{Spec } \mathcal{O}_{X,p}^{sh})/\text{Cl}(X)$ with the property that $[\mathcal{O}(mD)] = [\mathcal{O}]$. This means $a \mapsto \alpha$ where $\alpha \in \text{Br}'(X)$. Since (a) implies that α is a geometric class there exists an Azumaya algebra A representing $-\alpha$. Moreover, by the discussion in the preceding remark we know that $A = \text{End}(F')$ for a reflexive sheaf F' which is locally free on $X \setminus \{p\}$ and by the preceding proposition it has the property that $F'|_{\text{Spec } \mathcal{O}_{X,p}^{sh}} \cong \mathcal{O}(D')^{\oplus n}$ where $[\mathcal{O}(D')] = [\mathcal{O}(D)]$. Thus, the difference $(\mathcal{O}(D) \otimes \mathcal{O}(D')^{\vee})^{\vee\vee}$ is the restriction of a globally defined reflexive rank one sheaf $\mathcal{O}(D'') \in \text{Cl}(X)$. Setting $F = (F' \otimes \mathcal{O}(D''))^{\vee\vee}$ yields a reflexive

sheaf which is locally free away from p and with the property that $F|_{\text{Spec } \mathcal{O}_{X,p}^{sh}} \cong \mathcal{O}(D)^{\oplus n}$. Thus, (c) follows.

Now suppose that (c) holds. Let $\mathcal{O}(D) \in \text{Cl}(\text{Spec}(\mathcal{O}_{X,p}^{sh})) = \mathcal{P}(X)$ map to $\text{Br}'(X)$, this implies there is a positive number n so that $\mathcal{O}(nD)$ the restriction of a reflexive rank one sheaf on X . By hypothesis there is a reflexive sheaf F on X , locally free on $X \setminus \{p\}$, with $F|_{\text{Spec}(\mathcal{O}_{X,p}^{sh})} = \mathcal{O}(D)^{\oplus m}$. It follows that $\text{End}(F)$ is a Azumaya algebra and the previous theorem says its class in $\text{Br}'(X)$ is the image of $\mathcal{O}(-D)$ under the map $\mathcal{P}(X) \rightarrow \text{H}^2(X, \mathbf{G}_m)$. \square

Remark 4.5.8. The previous proposition furnishes a concrete criterion for when generically trivial classes in $\text{Br}'(X)$ are geometric: indeed, one needs to check that torsion classes in $\text{Pic}(E)/\text{Pic}(Y)$ have a special lifting property. Moreover, the proposition allows us to check the geometricity of Brauer classes on these algebraic spaces X without reference to X itself. Instead, we may work on the smooth and projective resolution Y .

As an immediate application, we show that when Y as above is 2-dimensional then every generically trivial cohomological Brauer class on X is geometric.

Proposition 4.5.9. *Let Y be a smooth, separated, finite-type, k -scheme of dimension 2. Suppose Y contains a proper genus 1 curve, $E \subset Y$, with $N_{E/Y}$ antiample. If X denotes the algebraic space arising from the contraction then every generically trivial Brauer class on X is geometric.*

Proof. Fix a line bundle $[L] \in \text{Pic}(E)/\text{Pic}(Y)$ with $[L^{\otimes m}] = [\mathcal{O}]$. By the above theorem we need to show that there is a vector bundle V on Y and a positive integer n so that $V|_E \cong L^{\oplus n}$. We invoke the following: if a vector bundle W on E has a determinant which is the restriction of a line bundle on Y , then W is the restriction of a vector bundle on Y . See, for instance, 5.2 of [15] for a proof of this fact. Apply this to our setting by taking $W = L^{\oplus m}$: the vector bundle W has determinant $L^{\otimes m}$ which, by hypothesis is the restriction of a line bundle on Y . The result follows. \square

Remark 4.5.10. It follows that Nagata's 2-dimensional normal algebraic-space surface (see 2.6.7), X , satisfies $\text{Br}(X) = \text{Br}'(X)$. Indeed X is constructed by contracting a rigid elliptic

curve in a smooth rational surface. Thus, all Brauer classes are generically trivial as $\mathrm{Br}'(\mathbf{P}_k^n) = 0$. However, since X is generically smooth, 2-dimensional and separated, 4.3.2 applies. Indeed, every cohomological Brauer class on this algebraic space is represented by an Azumaya algebra away from a closed subset of codimension ≥ 3 . Even in the 3-dimensional setting this result doesn't tell us whether cohomology classes can be represented by Azumaya algebras near closed points. Thus, we will investigate a construction similar to Nagata's in dimension 3: we employ Artin's theory of formal modifications to contract a non-rational curve in a smooth threefold. A novelty here is that, unlike Nagata's example, the construction we describe gives rise to non-schematic algebraic spaces over all algebraically closed fields. In what follows we describe an explicit example away from characteristic 3.

Example 4.5.11. Consider \mathbf{P}_k^4 defined over some algebraically closed field k with $\mathrm{char} k \neq 3$. Let E denote the genus 1 curve cut out by the graded ideal $(x_3, x_4, \sum_{i=0}^4 x_i^3)$. From now onwards we will denote by $h(x_0, \dots, x_4)$ the polynomial $\sum_{i=0}^4 x_i^3$. We will show that there is a smooth degree 10 hypersurface $Y \subset \mathbf{P}_k^4$ containing E with the property that $N_{E/Y}$ is antiample.

Suppose such a Y exists. First, observe that the natural map

$$\mathrm{Pic}(\mathbf{P}_k^4) \rightarrow \mathrm{Pic}(Y)$$

is an isomorphism by the Grothendieck-Lefschetz theorem (see Corollary 3.7 of [27]). Thus every nontrivial effective divisor on Y is ample. Now, by 2.6.5 and 2.6.6 there exists a separated, normal algebraic space X along with a proper and birational morphism $\pi : Y \rightarrow X$ which contracts precisely E . We claim that X is not a scheme. Suppose it were, then there would exist an affine neighborhood $\mathrm{Spec} A \subset X$ about the singular point. Since this neighborhood is dense the complement $X \setminus \mathrm{Spec} A$ has pure codimension 1 by [56] Tag 0BCV. It follows that there is a codimension 1 closed subvariety $X \setminus \mathrm{Spec} A = D \subset Y$ which does not pass through E . Consider the line bundle

$$\mathcal{O}(D) \in \mathrm{Pic}(Y) \cong \mathbf{Z}$$

and note that it has a section which vanishes along a non-empty locus. It follows that $\mathcal{O}(D)$ must be ample on Y , but this contradicts the fact that it doesn't pass through E . It follows

that no affine open neighborhood can exist about the singular point in X , i.e. X is an algebraic space which is not a scheme. Thus, it remains to show that there is a smooth hypersurface $Y \subset \mathbf{P}_k^4$ containing E with $N_{E/Y}$ antiample.

Our argument will require a Bertini-type result which we were unable to find in the literature. We thank Lucas Braune for making us aware of the result. Although the proof is standard, we record it below.

Proposition 4.5.12. *Let Y be a scheme which is finite-type and smooth over an algebraically closed field k . Fix a $L \in \text{Pic}(Y)$, then let $m_p \subset \text{Spec } \mathcal{O}_{Y,p}$ and L_p denote the maximal ideal and $L|_{\text{Spec } \mathcal{O}_{Y,p}}$ respectively. Suppose that $V \subset H^0(Y, L)$ is a vector space which has the property that the natural map*

$$V \rightarrow L_p/m_p^2 L_p$$

is surjective for every $p \in Y$. Suppose moreover that $f \in H^0(Y, L)$ is a section. Then the general member of $f + V$ has a vanishing locus which is smooth over k .

Proof. Consider the incidence correspondence

$$I_{bad} = \{(p, s) | p \in V(s) \text{ and is singular at } p\} \subset I = \{(p, s) | p \in V(s)\} \subset Y \times V$$

and note that I and I_{bad} are closed subvarieties of $Y \times V$. Indeed, I is the vanishing of the universal section s_u over V and I_{bad} can be described locally as the vanishing of the partial derivatives of s_u (with respect to local coordinates of Y) and s_u . Next, observe that the fiber of $I_{bad} \rightarrow Y$ has dimension $\dim V - \dim Y - 1$. Indeed, the fiber over $p \in Y$ can be described as the $\ker[\pi : V \rightarrow L_p/m_p^2 L_p]$. Since the target vector space has dimension $1 + \dim Y$ (by smoothness of Y) and the map is surjective, we obtain the desired dimension count. It follows that $\dim I_{bad} \leq \dim V - 1$ which in turn implies that $I_{bad} \rightarrow V$ cannot be surjective. This implies the general member of V is smooth.

This completes the proof when $f = 0$, for the general case observe that $f + V$ can also be viewed as an affine space. Indeed, we do this via the identification $V \rightarrow f + V$ sending

$s \mapsto f + s$. Now we have

$$I_{bad} = \{(p, f + s) | f_p + s_p \in m_p^2\} \subset I = \{(p, f + s) | f_p + s_p = 0\} \subset Y \times (f + V)$$

where $I = V(f + \sum x_i s_i)$ where $\{s_i\}$ is a basis for V and x_i are the coordinates for V viewed as an affine space. Similarly, I_{bad} is locally described as the vanishing of the partials of $f + \sum x_i s_i$ with respect to local coordinates of Y . To finish the proof, repeat the same argument. Indeed, note that the natural map $\pi_f : f + V \rightarrow L_p/m_p^2 L_p$ is still surjective for all $p \in Y$. Moreover, $\pi_f^{-1}(0) = f + \pi^{-1}(-\pi(f)) \subset f + V$ and the latter is a translate of the kernel of π and therefore has the same dimension. \square

Consider $L = \mathcal{O}_{\mathbf{P}_k^4}(10)$ and set

$$f = x_3(x_1^9) + x_4(-x_0^4 x_1^5) + h(x_0, \dots, x_4)(x_0^4 x_2^3)$$

and

$$V = (h, x_3, x_4)^2 \cap H^0(\mathbf{P}_k^4, \mathcal{O}_{\mathbf{P}_k^4}(10)) \subset H^0(\mathbf{P}^4, L)$$

If we restrict L and V to the complement of $V(h, x_3, x_4) = E \subset \mathbf{P}_k^4$ we see that the hypothesis of the previous proposition is satisfied. Indeed, one of x_3, x_4, h will be a unit at a fixed point $p = [a_0; \dots; a_4]$ with $a_i = 1$. Suppose for example that $h(p) \neq 0$ then one can check $h^2(x_0 - a_0 x_i), \dots, h^2(x_4 - a_4 x_i)$ maps to a basis of L_p/m_p^2 . A similar argument works when either x_3 or x_4 is not zero. Thus, it follows that there is a section $f + s \in f + V$ which is smooth away from E . In fact, $Y = V(f + s)$ is smooth along E as well.

Observe that for a point $p \in E$ we have

$$(\partial_{x_2} f)(p) = ((3x_0^4 x_2^2)h + (\partial_{x_2} h)(x_0^4 x_2^3))(p) = 3x_0^4 x_2^5(p)$$

and

$$(\partial_{x_3} f)(p) = (x_1^9 + 3x_3^2(x_0^4 x_2^3))(p) = (x_1^9)(p)$$

If $x_1(p) = 0$ then $x_0^3 = -x_2^3$ which implies $x_0 x_2 \neq 0$. This tells us that $(\partial_{x_2} f)(p) \neq 0$ which implies smoothness of $Y = V(f + s)$ along E because $s \in (h, x_3, x_4)^2$. It remains to show

why the normal bundle $N_{E/Y}$ is antiample. By restricting the cotangent exact sequences corresponding to the embeddings $E \subset Y \subset \mathbf{P}_k^4$ to E and using the snake lemma we obtain the following commutative diagram

$$\begin{array}{ccccccc}
& & 0 & & & & 0 \\
& & \uparrow & & & & \uparrow \\
& & N_{E/Y}^\vee & & & & \\
& & \uparrow & & & & \uparrow \\
0 & \longrightarrow & I_E|_E & \longrightarrow & \Omega_{\mathbf{P}_k^4/k}|_E & \longrightarrow & \Omega_{E/k} \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & I_Y|_E & \longrightarrow & \Omega_{\mathbf{P}_k^4/k}|_E & \longrightarrow & \Omega_{Y/k}|_E \longrightarrow 0 \\
& & \uparrow & & & & \uparrow \\
& & 0 & & & & N_{E/Y}^\vee \\
& & & & & & \uparrow \\
& & & & & & 0
\end{array}$$

We will give an explicit description of the leftmost vertical exact sequence. Note that $\mathcal{O}(-10)|_E \cong I_Y = (f + s)$ and $\mathcal{O}(-1)|_E^{\oplus 2} \oplus \mathcal{O}(-3)|_E \cong I_E|_E$. Indeed, the natural map $\phi : \mathcal{O}(-1)|_E^{\oplus 2} \oplus \mathcal{O}(-3)|_E \rightarrow I_E$ induced by the inclusions $\mathcal{O}(-1) \cong (x_3) \subset I_E$, $\mathcal{O}(-1) \cong (x_4) \subset I_E$, and $\mathcal{O}(-3) \cong (h) \subset I_E$ is surjective. Upon restriction to E , the ideal sheaf I_E becomes a rank 3 vector bundle and hence the restriction $\phi|_E$ becomes an isomorphism because surjective maps between vector bundles of the same rank are necessarily isomorphisms. Via these identifications, one can see that the inclusion $I_Y|_E \rightarrow I_E|_E$ is described by the map $\mathcal{O}(-10)|_E \rightarrow \mathcal{O}(-1)|_E^{\oplus 2} \oplus \mathcal{O}(-3)|_E$ which, when twisted by $\mathcal{O}(10)|_E$, sends $1 \mapsto (x_1^9, -x_0^4 x_1^5, x_0^4 x_2^3)$. Dualizing this sequence, we obtain the following

$$0 \longrightarrow \mathcal{O}(-3)|_E \oplus \mathcal{O}(-2)|_E \xrightarrow{\begin{pmatrix} x_0^4 & 0 \\ x_1^4 & x_2^3 \\ 0 & x_1^5 \end{pmatrix}} \mathcal{O}(1)|_E^{\oplus 2} \oplus \mathcal{O}(3)|_E \xrightarrow{\begin{pmatrix} x_1^9 \\ -x_0^4 x_1^5 \\ x_0^4 x_2^3 \end{pmatrix}} \mathcal{O}(10)|_E \longrightarrow 0$$

By inspection, this sequence is a complex. To check the first map is injective note that the minors don't simultaneously vanish on E . Exactness in the middle follows by dimension reasons because for a complex of vector bundles one can check exactness on the fibers. It follows that $N_{E/Y} \cong \mathcal{O}(-3)|_E \oplus \mathcal{O}(-2)|_E$ and it follows that $N_{E/Y}$ is antiample.

4.6 Progress over finite fields

We have shown that Grothendieck's question has an affirmative answer for reduced, noetherian, separated algebraic spaces away from a subset of codimension ≥ 3 . The next natural question is: does it hold for threefolds? If X is a scheme, any point admits an affine neighborhood and hence Gabber's theorem tells us every cohomology class is Zariski-locally represented by an Azumaya algebra. Even for threefolds which are not schemes, it is not clear that such a result should hold about a fixed closed point. Here we give an affirmative answer when X is a 3-dimensional separated algebraic space, finite-type and separated over a finite field k . An essential ingredient is the following result of Artin: 2-dimensional algebraic space surfaces which are normal and defined over the algebraic closure of a finite field are always quasiprojective. This yields a partial solution Grothendieck's question applied to example 4.5.11 and indeed, it implies that Zariski locally on such contracted threefolds the equivalent conditions of 4.5.7 all hold.

Theorem 4.6.1. *Let X be a 3-dimensional separated algebraic space, finite-type and separated over $\overline{\mathbf{F}}_p$. Then there is a Zariski open covering $\bigcup U_i = X$ with the property that $\mathrm{Br}(U_i) = \mathrm{Br}'(U_i)$. Moreover, for any finite set of points $x_1, \dots, x_n \in X$ we can find a open set $U \subset X$ containing the x_i so that $\mathrm{Br}(U) = \mathrm{Br}'(U)$.*

Proof. Fix a dense open affine neighborhood $\mathrm{Spec} A \subset X$ and consider the complement $X \setminus U = S$. This S is a 2-dimensional algebraic space which is finite-type and separated over a finite field. Consider the normalization $S' \rightarrow S$ and note that because S is excellent this morphism is finite. We may compactify S' (see 1.2.1 in [14]) and then by applying 4.6 in [8] it follows that S' is a projective scheme. Then by 2.3 in [22] or Corollary 48 in [35] it follows

that S is a scheme with the Chevalley Kleiman property. In particular, there is a open affine subscheme $\text{Spec } B \subset S$ containing any finite set of points $x_1, \dots, x_n \in X$. Since $S \subset X$ has the subspace topology there is a open subset $W \subset X$ with the property that $W \times_X S = \text{Spec } B$. It follows that there is a cartesian square:

$$\begin{array}{ccc} \text{Spec } C & \longrightarrow & \text{Spec } \hat{B} \\ \downarrow & & \downarrow \text{flat} \\ \text{Spec } A & \xrightarrow{\text{open}} & W \cup \text{Spec } A \end{array}$$

The right-hand vertical map is a flat neighborhood of $\text{Spec } B \subset W \cup \text{Spec } A$ and we obtain it by Theorem 2.3.14. Note that the pullback of this flat neighborhood to $\text{Spec } A$ is affine since $W \cup \text{Spec } A$ is a separated algebraic space. Now, Lemma 2.3.4 implies that the diagram above is a flat Mayer-Vietoris square. Applying Lemma 4.2.2 shows that $\text{Br}(W \cup \text{Spec } A) = \text{Br}'(W \cup \text{Spec } A)$ as desired. □

4.7 Non-reduced phenomena

Here we address how the geometricity of a Brauer class defined over X can be deduced from the geometricity of its reduction. The first result we have is an easy consequence of the existence of the Frobenius in positive characteristic. Curiously, it only concerns Brauer classes whose period is prime to the base characteristic. The second is an application of a result of de Jong regarding modifications of Azumaya algebras.

Proposition 4.7.1. *Let $X \rightarrow X'$ be nilpotent thickening of algebraic spaces defined over \mathbb{F}_p . If every prime to p cohomology class in $\text{Br}'(X)$ is in $\text{Br}(X)$ then every prime to p cohomology class in $\text{Br}'(X')$ is in $\text{Br}(X')$.*

Proof. Note that the ideal sheaf in $\mathcal{O}_{X'}$ defining X is nilpotent, i.e. $I^m = 0$ for some $m > 0$. This means that if $x \in I$ then $x^{p^m} = 0$. So if we denote by F^m the iterated absolute Frobenius on X' , we have the following factorization

$$\begin{array}{ccc}
X & \longrightarrow & X' \\
& \swarrow f & \uparrow F^m \\
& & X'
\end{array}$$

The Frobenius pullback of a cohomology class $\alpha \in H^2(X', \mu_n)$ is $p\alpha$, so the pullback of the cohomology class α under the iterated Frobenius map is $p^m\alpha$. Thus, if \mathcal{X}' is the gerbe over X' corresponding to α then the gerbe $(F^m)^*\mathcal{X}'$ is the Brauer class corresponding to $p^m\alpha$. Moreover, by the commutative diagram above we have

$$(F^m)^*\mathcal{X}' \cong f^*(\mathcal{X}'|_X)$$

where $\mathcal{X}'|_X$ is the gerbe corresponding to $\alpha|_X$. Since this is a geometric Brauer class, $\mathcal{X}'|_X$ admits a twisted vector bundle V and pulling it back via f yields a vector bundle W . Since f induces a representable morphism $\mathcal{X}' \rightarrow \mathcal{X}'|_X$, it follows that $\underline{\text{Spec}}(\text{Sym}^* W)$ is an algebraic space because $\text{Spec} \text{Sym}^* V$ is an algebraic space. It follows that the μ_n gerbe corresponding to $p^m\alpha$ is a quotient stack. However, note that there is a morphism $\mathcal{X}' \rightarrow (F^m)^*\mathcal{X}'$ of stacks which induces the map $(-)^{p^m} : \mu_n \rightarrow \mu_n$ on their respective bandings. Indeed, the p^m th power map $\mu_n \rightarrow \mu_n$ induces the map on cohomology $H^2(X', \mu_n) \rightarrow H^2(X', \mu_n)$ which sends $\alpha \mapsto p^m\alpha$. Since n is prime to the characteristic, the p^m th power map induces an isomorphism of bands and thus the two stacks \mathcal{X}' and $(F^m)^*\mathcal{X}'$ are equivalent. It follows that \mathcal{X}' is also a quotient stack. \square

The second result is an application of de Jong's use of elementary transformations to unobstruct Azumaya algebras on surfaces. This was described in [15] via the deformation theory of Azumaya algebras and later in [33] using the deformation theory of sheaves on stacks.

Corollary 4.7.2. *Suppose $X \rightarrow \text{Spec } B$ is a proper, flat morphism of algebraic spaces with B a noetherian complete local ring where $X_0 = X \times_{\text{Spec } B} \text{Spec } B/m_B \rightarrow \text{Spec } B/m_B$ is a projective, reduced, 2-dimensional scheme which is equidimensional. Then every cohomological Brauer class whose order is prime to the characteristic of the residue field $B/m = k$ is geometric.*

Proof. Fix a cohomological Brauer class $\alpha \in \text{Br}'(X)$ of order n and consider its restriction $\alpha|_{X_0} \in \text{Br}'(X_0)$. Since X_0 is projective, there is an Azumaya algebra A with Brauer class $\alpha|_{X_0}$ with degree nk with a positive integer k so that the product is prime to the base characteristic. Indeed, the main result of [16] shows that the restriction $\alpha|_{X_0}$ is in the geometric Brauer group. Moreover, theorem 1 in [4] implies there is a Azumaya algebra representing $\alpha|_{X_0}$ of degree $\text{order}(\alpha|_{X_0})k$ with k prime to the characteristic. Since $\text{order}(\alpha|_{X_0})$ divides n we may tensor A with a suitable $\text{End}(\mathcal{O}_{X_0}^{\oplus r})$ so that it has the desired degree without changing its Brauer class. Now, there is a μ_{nk} -gerbe $\mathcal{Y} \rightarrow X$ with the property that $[\mathcal{Y}] \mapsto \alpha$ under the map $H^2(X, \mu_{nk}) \rightarrow H^2(X, \mathbf{G}_m)$. Consider the restriction of the μ_{nk} gerbe $\mathcal{Y}_0 = \mathcal{Y} \times_X X_0$. We know that $[A]$ and $[\mathcal{Y}_0]$ are equal in $H^2(X_0, \mathbf{G}_m)$. It follows by 2.2 of [15] that there is an elementary transformation A' of A so that $[A'] = [\mathcal{Y}_0]$.

There is a modification of the Azumaya algebra A' which is unobstructed and can therefore be deformed to the formal algebraic space $\widehat{X} = X \times_{\text{Spec } B} \text{Spf}(B)$. In fact, the deformation of an Azumaya algebra A' at the k th stage is obstructed by a class in $H^2(X_0, (A'/\mathcal{O}_X)|_k) \otimes m^k/m^{k+1}$ (see Lemma 3.1 of [15] and the following discussion). If sA' denotes the traceless morphisms then we only need to show that $H^0(X_0, sA' \otimes \omega_{X_0/k}) = 0$ as this group is dual to $H^2(X_0, (A'/\mathcal{O}_X)|_k)$. Apply Proposition 3.2 of [15] to conclude that we may find an unobstructed Azumaya algebra A'' on X_0 whose class in $H^2(X_0, \mu_{nk})$ is still $[\mathcal{Y}_0]$. It follows by the Grothendieck existence theorem (see, for example, 1.4 of [52]) that A'' is the restriction of an Azumaya algebra \widehat{A}'' on X . It remains to show that the Brauer class of \widehat{A}'' is α and for this it suffices to show that $H^2(X, \mu_{nk}) \rightarrow H^2(X_0, \mu_{nk})$ is injective. This follows from Tag 095T in [56].

□

Chapter 5

THE RESOLUTION PROPERTY: PART TWO

5.1 Tame Normal Orbifold-Surfaces are Quotient Stacks

Theorem 5.1.1. *Let \mathcal{X} be a normal tame Deligne-Mumford stack of dimension 2 which is finite type, separated over a field k and has generically trivial stabilizers. Then \mathcal{X} is a quotient stack.*

Proof. Step 1: It suffices to find a faithful vector bundle on a particular open neighborhood of \mathcal{X} .

Our goal is to exhibit a faithful vector bundle on \mathcal{X} . Since \mathcal{X} has generically trivial stabilizers we may find a smooth open affine subscheme $\text{Spec } B \subset \mathcal{X}$. Moreover, if we give the (topological) complement the reduced induced structure we obtain a Deligne-Mumford stack of dimension ≤ 1 , call it \mathcal{C} . Since \mathcal{C} is reduced, the inertia stack $I_{\mathcal{C}} \rightarrow \mathcal{C}$ is flat away from finitely many closed points and because \mathcal{X} is normal and 2-dimensional the singular locus of \mathcal{X} consists of finitely many closed points which all lie on $\mathcal{C} \subset \mathcal{X}$. Thus the union of the singular points of \mathcal{X} , the nonflat locus of $I_{\mathcal{C}} \rightarrow \mathcal{C}$, and a point from each irreducible component of \mathcal{C} is a finite set of points which we will denote by q_1, \dots, q_n . If we can find a faithful vector bundle in a neighborhood of this finite set then we can find a faithful vector bundle on all of \mathcal{X} . Indeed, by taking the reflexive hull of a coherent extension we obtain a vector bundle V on \mathcal{X} . To check this vector bundle is faithful it suffices to show it is faithful when we restrict it to \mathcal{C} . But we already know $V|_{\mathcal{C}}$ is faithful in a dense neighborhood about the points where \mathcal{C} is not a gerbe! Therefore, we may invoke Proposition 2.2.7 to deduce that it is faithful on all of \mathcal{C} .

Step 2: The construction of a special open neighborhood, $W \cup \text{Spec } B \subset \mathcal{X}$, about

$\{q_1, \dots, q_n\}$.

Let $\mathcal{C} \rightarrow C$ be the coarse space morphism for \mathcal{C} . Observe that the natural morphism $C \rightarrow X$ is a closed immersion: topologically it is true because $\mathcal{C} \rightarrow \mathcal{X}$ is a closed immersion and moreover the induced morphism of structure sheaves $\mathcal{O}_{\mathcal{X}} \rightarrow i_*\mathcal{O}_{\mathcal{C}}$ remains surjective when we push-forward along the tame coarse space map $\mathcal{X} \rightarrow X$. Let q'_1, \dots, q'_n denote the closed points of C corresponding to the closed points $q_1, \dots, q_n \in \mathcal{C}$. Since C is 1-dimensional, there exists an affine neighborhood $\text{Spec } A_0 \subset C$ containing $\{q'_1, \dots, q'_n\}$. By Tag 04CE in [56] there exists a open algebraic subspace $W \rightarrow X$ along with a closed immersion $\text{Spec } A_0 \rightarrow W$ which factors $\text{Spec } A_0 \rightarrow W \rightarrow X$ or topologically speaking: $W \cap C = \text{Spec } A_0$. Note that $W \cup \text{Spec } B$ is a neighborhood of q_1, \dots, q_n and therefore it suffices to find a faithful vector bundle on $W \cup \text{Spec } B$. By Theorem 2.3.14 we see that there exists a noetherian, adic, affine scheme of dimension 2 along with a map $\text{Spec } A \rightarrow W$ which is a flat neighborhood of $\text{Spec } A_0$, i.e. there exists a flat map $\text{Spec } A \rightarrow W$ so that the following square is cartesian

$$\begin{array}{ccc} \text{Spec } A_0 & \longrightarrow & \text{Spec } A \\ \downarrow & & \downarrow f \\ \text{Spec } A_0 & \xrightarrow{i} & W \end{array}$$

Because $W \cap (\text{Spec } B)^c = W \cap C = \text{Spec } A_0$ it follows that $W \cup \text{Spec } B \subset X$ fits into a flat Mayer-Vietoris square and we may base change along $\mathcal{X}_{W \cup \text{Spec } B} \rightarrow W \cup \text{Spec } B$ to obtain

$$\begin{array}{ccccc} & & \text{Spec } B \times_X \text{Spec } A & \longrightarrow & \text{Spec } B \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ \mathcal{X}_A & \longrightarrow & \mathcal{X}_{W \cup \text{Spec } B} & & \\ \downarrow & & \downarrow & & \downarrow \\ & \swarrow & \text{Spec } B \times_X \text{Spec } A & \longrightarrow & \text{Spec } B \\ \text{Spec } A & \longrightarrow & W \cup \text{Spec } B & & \end{array}$$

The right and back faces are cartesian because $\text{Spec } B \subset X$ is inside the locus where the coarse space map $\mathcal{X} \rightarrow X$ is an isomorphism. Since the bottom face is a flat Mayer-Vietoris square Lemma 2.3.3 implies the the top face is also a flat Mayer-Vietoris square. In what follows we invoke Theorem 2.3.5: if we can glue a vector bundle on $\text{Spec } B$ and a faithful one on \mathcal{X}_A on the fiber product $\text{Spec } B \times_X \text{Spec } A$ we obtain a faithful vector bundle on $\mathcal{X}_{W \cup \text{Spec } B}$.

Step 3: Constructing a faithful vector bundle on $W \cup \text{Spec } B$.

In fact, it suffices to find a vector bundle V on $\mathcal{X}_{\text{Spec } A}$ of constant nonzero rank which has trivial determinant and which is faithful on $\mathcal{X}_{\text{Spec } A_0}$. In this situation the restriction of V to $\text{Spec } A \times_X \text{Spec } B$ is trivial and therefore can be extended to $\text{Spec } B$. To see why the restriction of V to $\text{Spec } A \times_X \text{Spec } B$ is trivial note that since A is I -adic, I lies inside the Jacobson radical of A and therefore every maximal ideal of A . Thus $\text{Spec } B \times_X \text{Spec } A$ is an open subscheme of $\text{Spec } A$ which doesn't contain $V(I) = \text{Spec } A_0$ it follows that the open affine subscheme $\text{Spec } B \times_X \text{Spec } A$ has dimension $\leq \dim(\text{Spec } A) - 1 \leq 1$. Therefore by Serre's splitting theorem (see 2.4.1(1) above) there exists a decomposition

$$V|_{\text{Spec } B \times_X \text{Spec } A} \cong \mathcal{O}_{\text{Spec } B \times_X \text{Spec } A}^{\oplus m} \oplus L$$

where L is some line bundle. Taking determinants of both sides shows that $\det(V|_{\text{Spec } B \times_X \text{Spec } A}) \cong L$ but since $\det(V)$ is trivial this implies L is trivial. Thus we obtain a trivialization $V|_{\text{Spec } B \times_X \text{Spec } A} \cong \mathcal{O}_{\text{Spec } B \times_X \text{Spec } A}^{\oplus m}$ and we may extend this vector bundle over $\text{Spec } B$. Theorem 2.3.5 now yields a vector bundle on $\mathcal{X}_{W \cup \text{Spec } B}$ which restricts to V on $\mathcal{X}_{\text{Spec } A}$ and is trivial on $\text{Spec } B$, in particular it is faithful in a neighborhood of \mathcal{X} containing the points q_1, \dots, q_n . Indeed, it is faithful because it is so on the stacky locus of $\mathcal{X}_{W \cup \text{Spec } B}$. To finish, observe that $\mathcal{X}_{\text{Spec } A_0}$ is at most 1-dimensional. Thus, we may use Corollary 3.2.7 to obtain a faithful vector bundle V_0 on $\mathcal{X}_{\text{Spec } A_0}$ which has trivial determinant, now apply Lemma 4.4.1 to conclude.

□

5.2 Tame Deligne-Mumford Surfaces satisfy the Resolution Property

Now we turn to the proof that tame normal surfaces satisfy the resolution property, we follow the strategy of [37] in their proof of Theorem 2.2.

Lemma 5.2.1. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of separated Deligne-Mumford stacks which endows \mathcal{X} with the structure of an étale gerbe over \mathcal{Y} . Assume that \mathcal{Y} is connected. Then there is a finite étale cover $\mathcal{Y}' \rightarrow \mathcal{Y}$ and a constant group G so that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \rightarrow \mathcal{Y}'$ is a gerbe banded by G .*

Proof. Observe that the automorphism group of any two geometric points of \mathcal{X} over \mathcal{Y} are isomorphic to the same constant group G . Indeed, it suffices to compare the automorphism group of any point $p \in |\mathcal{Y}|$ with that of a generic point which specializes to it, thus we may pass to a connected étale neighborhood of our point: $U \rightarrow \mathcal{Y}$ so that $\mathcal{X} \times_{\mathcal{Y}} U \rightarrow U$ admits a section σ . This induces an isomorphism $\mathcal{X} \times_{\mathcal{Y}} U \cong B\text{Aut}(\sigma)$ where $\text{Aut}(\sigma)$ is a finite étale group scheme over U . By taking a further connected étale neighborhood $U' \rightarrow U$ of a lift of p we obtain an identification $\mathcal{X} \times_{\mathcal{Y}} U' \cong BG_{U'}$ where G is a constant group scheme over U' . It follows that every geometric point of $BG_{U'}$ has the same automorphism group. To see why, note that any geometric point $\text{Spec } K \rightarrow BG_U$ has an automorphism group $G' = \text{Aut}(T)$ where T is a G -torsor over $\text{Spec } K$. However, since K is algebraically closed it follows that T is trivial and that $G' = \text{Aut}_{G\text{-set}}(G) \cong G$. In fact, this shows that the band of the gerbe $\mathcal{X} \rightarrow \mathcal{Y}$ is étale locally isomorphic to the finite constant group G . This means that in the stack of bands over \mathcal{Y} the sheaf

$$\mathcal{Y}' = \text{Isom}_{\text{Band}(\mathcal{Y})}(\text{Band}(\mathcal{X}), G)$$

on \mathcal{Y} is a left $\text{Isom}_{\text{Band}(\mathcal{Y})}(G, G) = \text{Out}(G)_{\mathcal{Y}}$ -torsor. Since $G_{\mathcal{Y}}$ is a finite constant group its outer automorphism group is also finite constant. It follows that $\mathcal{Y}' \rightarrow \mathcal{Y}$ is a finite étale morphism and $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \rightarrow \mathcal{Y}'$ is a gerbe banded by G .

□

Theorem 5.2.2. *If \mathcal{X} is a tame, separated, normal Deligne-Mumford stack which is finite type over a field and dimension 2, then \mathcal{X} satisfies the resolution property.*

Proof. Let $\pi : \mathcal{X} \rightarrow X$ be the coarse space morphism. We may assume that \mathcal{X} is connected but because it is normal and noetherian this is the same as assuming it is integral. We may also assume that k is separably closed. Indeed, if $\mathcal{X} \times_k k^{\text{sep}}$ has the resolution property then there exists a vector bundle with quasiaffine frame bundle on $\mathcal{X} \times_{\text{Spec } k} \text{Spec } k'$ for a finite separable extension $k \subset k'$ and by Lemma 2.1.6 it follows that \mathcal{X} also satisfies the resolution property.

Our goal will be to dominate \mathcal{X} by a finite étale morphism from a stack which has the resolution property. By Theorem 2.5.1 we may rigidify $f : \mathcal{X} \rightarrow \mathcal{X}^{\text{rig}}$ so that \mathcal{X}^{rig} has generically trivial stabilizers and where f gives \mathcal{X} the structure of a étale gerbe over \mathcal{X}^{rig} . Using the previous lemma there is a finite étale cover $g : \mathcal{Y} \rightarrow \mathcal{X}^{\text{rig}}$ so that $\mathcal{Y} \times_{\mathcal{X}^{\text{rig}}} \mathcal{X} \rightarrow \mathcal{Y}$ is a gerbe banded by a finite constant group G . Since g is finite étale, \mathcal{Y} is a tame normal DM stack of dimension 2 with generically trivial stabilizers which is also finite type and separated over a field. Moreover since $\mathcal{Y} \times_{\mathcal{X}^{\text{rig}}} \mathcal{X} \rightarrow \mathcal{X}$ is finite étale it suffices to show $\mathcal{Y} \times_{\mathcal{X}^{\text{rig}}} \mathcal{X}$ satisfies the resolution property. Consider the stack of banded equivalences

$$I = \text{Isom}_{\text{id}_G}(BG_{\mathcal{Y}}, \mathcal{X} \times_{\mathcal{X}^{\text{rig}}} \mathcal{Y})$$

over \mathcal{Y} . It is a gerbe over \mathcal{Y} banded by the abelian constant group scheme $Z(G)_{\mathcal{Y}}$ (see Theorem 2.3.2 (iii) of [20]) and admits a finite flat surjective morphism to $\mathcal{X} \times_{\mathcal{X}^{\text{rig}}} \mathcal{Y}$ (see Remark 3.4 of [17]). Thus, it actually suffices to show that I satisfies the resolution property. Decomposing $Z(G) = \bigoplus \mathbf{Z}/n_i\mathbf{Z}$ yields projections $p_i : Z(G) \rightarrow \mathbf{Z}/n_i\mathbf{Z}$ and these induce maps $q_i : I \rightarrow \mathcal{R}_i$ where \mathcal{R}_i is a gerbe over \mathcal{Y} banded by $\mathbf{Z}/n_i\mathbf{Z}$. Moreover the maps q_i induce p_i on bands, it follows that we obtain a morphism of gerbes $q : I \rightarrow \mathcal{R}_1 \times_{\mathcal{Y}} \cdots \times_{\mathcal{Y}} \mathcal{R}_r$ inducing an equivalence on bands (see Corollary 2.2.8 in [20]). Thus, by [20] Proposition 2.2.6 (iii) it follows that q is an equivalence. If we show that each of the gerbe maps $\mathcal{R}_i \rightarrow \mathcal{Y}$ satisfy the resolution property, then so will their product $\mathcal{R}_1 \times_{\mathcal{Y}} \cdots \times_{\mathcal{Y}} \mathcal{R}_r$. Indeed, the resolution property of morphisms is stable under base change and composition (see [22] 1.8 (iv) and (v)).

But this will imply the morphism $I \rightarrow \mathcal{Y}$ has the resolution property, and by theorems 2.1.13 and 5.1.1 we know the coarse space map $\mathcal{Y} \rightarrow Y$ satisfies the resolution property. However, by theorem 3.1.1 the algebraic space Y satisfies the resolution property as well. Thus to complete the proof it suffices to show that a $\mathbf{Z}/n\mathbf{Z}$ -gerbe morphism, $\mathcal{R} \rightarrow \mathcal{Y}$ satisfies the resolution property.

Observe that because \mathcal{X} is tame and k is separably closed $\mathbf{Z}/n_i\mathbf{Z} \cong \mu_{n_i}$, so we may assume \mathcal{R} is a μ_n gerbe over \mathcal{Y} . By theorem 4.4.2 the cohomological and geometric Brauer group coincide for \mathcal{Y} so there exists a 1-twisted vector bundle V on \mathcal{R} . Therefore, by the proof of 3. implies 4. in theorem 4.1.6 the morphism $\mathcal{R} \rightarrow \mathcal{Y}$ satisfies the resolution property.

□

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