

**Configurational transition in a Fleming-Viot-type model
and probabilistic interpretation of Laplacian eigenfunctions**

by

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Abstract

We analyze and simulate a two dimensional Brownian multi-type particle system with death and branching (birth) depending on the position of particles of different types. The system is confined in the two dimensional box, whose boundaries act as the sink of Brownian particles. The branching rate matches the death rate so that the total number of particles is kept constant. In the case of m types of particles in the rectangular box of size a, b and elongated shape $a \gg b$ we observe that the stationary distribution of particles corresponds to the m -th Laplacian eigenfunction. For smaller elongations $a > b$ we find a configurational transition to a new limiting distribution. The ratio a/b for which the transition occurs is related to the value of the m -th eigenvalue of the Laplacian with rectangular boundaries.

I Introduction

It is remarkable how the simple systems with few deterministic rules (such as Life [1] or cellular automata [2]) can generate complex structures. In the population dynamics however one often uses stochastic models. For example as has been recently shown [3] the addition of stochastic factor into the Life game favors diversity of structures, contrary to the original model in which diversity is the decreasing function of time. Introduction of the probabilistic factor in the cellular automata in the description of the dynamics of the social impact in the population [4] leads to the complex spatial and temporal intermittent behavior. In the genome population dynamics [5-6] one uses stochastic processes such as super-Brownian motion or Fleming-Viot processes. The model presented in this paper is a special type of population dynamic stochastic model.

The dynamics of systems with two competing species has been studied with the emphasis on the spatial heterogeneity influence on the temporal evolution and spatial organization [7-9]. In the case of strong competition only one of the two survives, which means that the average lifetime of two species can be different. In our model we are concerned with the spatial distribution of three or more competing species. Our model differs from those mentioned above because we study the case when the total number of particles is constant and the average lifetime of different species is the same in the long time limit.

Here we describe a behavior of a population of m different types of particles with Brownian dynamics confined in the two dimensional box, whose walls act as sinks for the particles. Additionally we assume that if two particles of different type occupy the same lattice point, both are killed. The birth rules are chosen in such a way as to keep the number of particles constant at each time step and to ensure that the average lifetime of any type of particles is the same in the long time limit. As an

example we show (Fig.1) the stationary configurations for 3 types of particles in a rectangular box of size $a \times b$. For $a/b > 1.63$ the particles of different types occupy domains of rectangular shapes (Fig.1a). We call such configuration elementary and show (in section V) that it is related to the third Laplacian eigenfunction. When the size ratio decreases below 1.63 the configuration changes its character as shown in Figs.1bc. Here the domains have the shapes that are not related to the Laplacian eigenfunctions. We shall call this transition the configurational transition. In Section VII we show that the ratio of rectangle sides a/b at the transition can be obtained from the simple condition involving the third Laplacian eigenvalue. In a natural way our model provides the probabilistic interpretation of the Laplacian eigenfunctions. We would like to emphasize that our stochastic model leads to a deterministic limiting distribution, whereas for example in super-Brownian motion and Fleming-Viot processes the limiting distribution has the fractal nature.

The paper is organized as follows. In Section II we briefly discuss the discrete analogues of super-Brownian motion and Fleming-Viot processes. In Sections III and IV we describe in detail our model. The connection between the stationary state of the model and the Laplacian eigenfunctions is given in Section V and computer simulations are described in Section VI. The analysis of the configurational transition and the concluding remarks are contained in Section VII.

II Super-Brownian and Fleming-Viot processes:

particle systems with death rate independent of position

Super-Brownian motion and Fleming-Viot processes are usually discussed in the continuous time and space state setting. We will present their discrete analogues for the purpose of comparison with our own model introduced in Sections III and IV below. In the first model we consider particles on the two dimensional lattice. At every time step $t = 1, 2, 3 \dots$, each particle either dies or branches off into two

offsprings with probability $1/2$. If the particle branches, both offsprings occupy the same lattice site as the parent particle and then each goes to one of the four neighbour lattice sites with probability $1/4$. The events are independent for all the particles in the population. Suppose that at time $t = 1$ the particle system consists of j particles and every particle is located at $(0,0)$. Let X_s^j be a measure-valued process whose value at time s is defined as follows. The measure $X_s^j(A)$ of an open subset A of \mathbf{R}^2 is equal to the number of particles at time $t = [s]$ which lie in $\sqrt{j}A$ ($[s]$ is the integer part of s). Consider the sequence of processes $\{X_{ju}^j/j, u \geq 0\}_{j \geq 1}$ where $s = ju$ and u plays the role of the rescaled time. This sequence of processes converges as $j \rightarrow \infty$ to a measure-valued diffusion called super-Brownian motion with the initial state $\delta_{(0,0)}$ (mass 1 concentrated at $(0,0)$). The limiting distribution of the process has the fractal nature in dimensions $d \geq 2$. The volume occupied by the particles with the linear size of the system, L , scales as L^2 irrespective of d ($L \rightarrow 0$). For $d = 1$, the limiting distribution of the process at a fixed time has a continuous density.

The second model differs from the first one in that the population size is fixed and equal to j . The dynamics are now the following. First suppose that $k = 1, 2, \dots, j-1$ and $n \geq 1$. In order to obtain the state of the process at time $t = nj + k + 1$ from that at $t = nj + k$, we choose randomly one particle and kill it. Next, another particle is chosen from the surviving ones and it branches into two offspring which occupy the same lattice site as the parent particle. If $t = nj$ then we obtain the new configuration at time $t = nj + 1$ by letting each of the particles move to one of the 4 nearest sites on the lattice, with probability $1/4$, independent of all other particles. We renormalize the system in order to obtain a continuous limit. Suppose that at time $t = 1$ the system consists of j particles located at $(0,0)$. Let X_s^j be a measure defined as in the first model, i.e., the measure

$X_s^j(A)$ of an open subset A of \mathbf{R}^2 is equal to the number of particles at time $t = [s]$ which lie in $\sqrt{j}A$. Then the sequence of processes $\{X_{j^{2u}}^j/j, u \geq 0\}_{j \geq 1}$ ($s = j^{2u}$) converges as $j \rightarrow \infty$ to a measure-valued diffusion called the Fleming-Viot process [6, 10] with the initial state $\delta_{(0,0)}$. This process has the same fractal nature as the super-Brownian motion. The Fleming-Viot process is the super-Brownian motion when the latter is conditioned to have a constant total number of particles.

Recently there has been growing interest in models incorporating dependence of the motion of individual particles on the current configuration [10,11]. We propose to study a model with a constant population size in which particles can die and branch. In our model, the death of a particle will depend on its location and thus it differs from the two aforementioned models. The simplest case is when a particle dies if and only if it moves to a set of the designated sites on the lattice.

III Particle system with death depending on position

We fix a connected subset D_ε of the square lattice with the mesh size ε , denoted $(\varepsilon\mathbf{Z})^2$. The particles in our model die if they move outside D_ε , so D_ε plays the role of the state space. The number of particles is fixed and equal to j . Transitions from the state of the system at time $t = k$ to that at time $t = k + 1$ may be described as follows. First each of the particles goes to one of the 4 nearest sites on the lattice $(\varepsilon\mathbf{Z})^2$, with probability 1/4, independent of all other particles. Then all particles which are outside D_ε die. An equal number of particles is chosen uniformly from among the surviving particles. Each of the chosen particles splits into two offspring which occupy the same site as the parent particle. Hence, the number of particles in our model is constant between generations.

Fix some open connected set $D \subset \mathbf{R}^2$ and let $D_\varepsilon = D \cap (\varepsilon\mathbf{Z})^2$. Suppose that at time $t = 1$ each of the j particles occupy sites in D_ε . Let $X_s^{j,\varepsilon}$ be the measure valued process whose value at time s is defined as follows. The measure $X_s^{j,\varepsilon}(A)$ of

an open subset A of \mathbf{R}^2 is equal to the number of particles which are in A at time $[s]$.

The qualitative long time behavior of our system is much different from that in the case of the super-Brownian motion or Fleming-Viot process. A typical particle configuration in both models discussed in the previous section has a fractal nature. Rigorously speaking, the limiting continuous models are measure-valued diffusions whose states are measures supported on fractal sets [10]. In our model, increasing the number of particles j and decreasing the mesh ε of the lattice results, in the long run, in a stable distribution which is a suitably normalized first eigenfunction of the Laplacian on D with zero boundary values. In other words, if $f(x, y)$ denotes the first eigenfunction of the Laplacian with zero boundary values in D then $\lim X_{s/\varepsilon^2}^{j,\varepsilon}(dx, dy)/j = cf(x, y)dxdy$ where $0 < c < \infty$ and the limit is taken as $\varepsilon \rightarrow 0$, $j \rightarrow \infty$ and $s \rightarrow \infty$. Here $c = 1/\int_D f(x, y)dxdy$.

To see this, notice first that because of the diffusive scaling $x \rightarrow \varepsilon x$ and $s \rightarrow \varepsilon^{-2}s$, each particle, in the limit $\varepsilon \rightarrow 0$, executes a Brownian motion in D with a jump, upon exiting D , to a point occupied by a fellow particle chosen uniformly at random. Second, since particles interact only through the boundary of D by a random choice from the remaining particles, the equal time pair correlations are inversely proportional to the total particle number j , and therefore the particles are uncorrelated in the limit $j \rightarrow \infty$. Thus the limiting measure $X_s = \lim_{\varepsilon, j} X_{s/\varepsilon^2}^{\varepsilon, j}/j$ exists and is deterministic. Let us express this limit via its density $X_s(A) = \int_A \xi(s; x, y)dxdy$. Since all particles reside in D it follows that $\xi(s; x, y) \geq 0$ and it vanishes for points on the boundary of D . Now the average exit time of a typical particle from D is the reciprocal of λ_1 , the first eigenvalue of the Laplacian in D with zero boundary values, which shows that the per particle rate at which jumps take place is exactly λ_1 . Thus the density $\xi(s; x, y)$ is the solution of

a heat flow problem in D with a heat source of strength $\lambda_1 \xi(s; x, y)$ and absorption at the boundary, i.e.,

$$\partial \xi / \partial s + \Delta \xi = \lambda_1 \xi. \quad (3.1)$$

Here, $\Delta = -1/2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ is the Laplacian. As $s \rightarrow \infty$, the density converges to a solution of the stationary problem for which the normalized first eigenfunction is the unique non-negative solution having total integral 1.

IV Multi-type particle system

The first eigenfunction of the Laplacian with zero boundary values already has a natural probabilistic interpretation [12] and the model described in the previous section provides a new one. It seems that so far the higher eigenfunctions do not have a natural probabilistic interpretation. A model described in this section may be a first step towards such an interpretation.

Fix a connected subset D_ε of the square lattice $(\varepsilon \mathbf{Z})^2$ with the mesh size ε . In this model, each particle will reside in D_ε and have one of m possible types \mathcal{L}_k , $k = 1, 2, \dots, m$. Typically, at each time $t = l$ some particles will be chosen to split into two offspring. In such a case we will say that a new offspring was born at time $t = l$ and if this particle is killed at some later time $t = n$ then we will say that the lifetime of this particle was $T = n - l$. The transition mechanism of the system, which depends on the positions, types, and lifetimes of the particles, is the following one. First each of the particles goes to one of the 4 nearest sites on the lattice $(\varepsilon \mathbf{Z})^2$, with probability 1/4, independent of all other particles. Then all particles which moved outside D_ε are killed. If a site in D_ε is occupied by particles of several types, then two particles of different types are chosen randomly and also are killed. We repeat the procedure, killing pairs of particles of different type occupying the same site until there are no sites in D_ε with more than one type of particle. Killed

particles will be replaced with new offspring as follows. For every k , we choose n_k (to be defined below) particles of type \mathcal{L}_k randomly from among the surviving ones and each of these particles splits into two offspring of the same type which then occupy the same site as the parent particle. Now we define n_k . Let n_k^1 be the number of particles of type \mathcal{L}_k which died because they moved outside D_ε . Let n_k^2 be the number of the *pairs* of particles which were killed inside D_ε such that the types and lifetimes of the particles involved were (\mathcal{L}_i, T_1) and (\mathcal{L}_k, T_2) and $T_1 > T_2$ (i.e., the particle with type \mathcal{L}_k had a shorter lifetime). Let n_k^3 be defined just as n_k^2 except that we replace the condition $T_1 > T_2$ with the condition $T_1 = T_2$. Then we set $n_k = n_k^1 + 2n_k^2 + n_k^3$. Note that the total number of particles in our model is constant between generations but the number of particles of type \mathcal{L}_k can vary, for each k .

Again, we consider the high density limit distribution for the system. Fix some open connected set $D \subset \mathbf{R}^2$, let $D_\varepsilon = D \cap (\varepsilon\mathbf{Z})^2$ and assume that at time $t = 1$ all particles occupy sites in D_ε . Recall that we have the total of j particles which belong to m different types \mathcal{L}_k . Let the measure $X_s^{k,j,\varepsilon}$ of an open subset A of \mathbf{R}^2 be equal to the number of particles of type \mathcal{L}_k which are in A at time $[s]$.

Fix $m \geq 2$ and $D \subset \mathbf{R}^2$ and let $j \rightarrow \infty$, $\varepsilon \rightarrow 0$ and $s \rightarrow \infty$. In the limit, for every k , the measure $X_{s/\varepsilon^2}^{k,j,\varepsilon}(dx, dy)/j$ will converge to $c_k f_k(x, y) dx dy$ where $0 < c_k < \infty$ and f_k is the first eigenfunction of the Laplacian with zero boundary values on a subdomain D_k of D . Because of the dynamics, particles of different types become segregated so the subdomains D_k are disjoint and their union is D .

Our transformation rules have been chosen so that the average lifetimes of particles of different types are equal in the limit. For if at a certain time the average lifetime of particles of type \mathcal{L}_k is smaller than that for type \mathcal{L}_n , the collisions of the particles of these two types will result in an increase of the number of particles of

type \mathcal{L}_k . This will imply the growth of the subregion D_k occupied by particles of type \mathcal{L}_k and hence their average lifetime will increase. The opposite will be true for the particles of type \mathcal{L}_n and so in the limit the average lifetimes of all types of particles will be the same.

The average lifetime of a particle of type \mathcal{L}_k is equal to the inverse of the first eigenvalue in D_k . Hence, the first eigenvalue for the Laplacian with zero boundary conditions in D_k is the same for every k , in the limit.

Let (x, y) be a point on the boundary between two subregions D_k and D_n and let N be the normal unit vector to the boundary at (x, y) pointing inside D_n . Note that the normal unit vector \hat{N} at (x, y) pointing inside D_n is the same as $-N$. Then we must have $\partial c_k f_k / \partial N = -\partial c_n f_n / \partial(\hat{N})$ because the particles of both types are killed on the boundary at the same rate.

V Limit distribution and Laplacian eigenfunctions

Let $F(x, y)dxdy = F_m(x, y)dxdy$ be equal to the limit of $X_{s/\varepsilon^2}^{k,j,\varepsilon}(dx, dy)/j$ on D_k . In other words, $F(x, y) = c_k f_k(x, y)$ on D_k and the constants c_k are such that $\partial c_k f_k / \partial N = -\partial c_n f_n / \partial(\hat{N})$ on the boundary between D_k and D_n , where N is the inward normal vector on the boundary of D_k and $\hat{N} = -N$. Hence, $\partial F / \partial N = -\partial F / \partial(\hat{N})$ on the boundary between D_k and D_n .

Suppose that g is an eigenfunction for the Laplacian in D with zero boundary values. The lines where g is equal to zero are called the “nodal lines” and they divide D into a number of subregions \tilde{D}_k . The function g is differentiable, so we must have $\partial|g|/\partial N = -\partial|g|/\partial(\hat{N})$ on the boundary between \tilde{D}_k and \tilde{D}_n . Moreover, $|g|$ is the first eigenfunction for the Laplacian on every subregion \tilde{D}_k . This suggests that F_m may be equal to $|g|$ for some eigenfunction g of the Laplacian in D .

A simple example shows that for some D and m , the limit distribution F_m cannot be equal to $|g|$ for any eigenfunction g in D . This is the case when an odd

number of “nodal lines” for F_m meet at a single point. The number of nodal lines meeting at one point must be even for an eigenfunction. Fig. 1b illustrates a limit distribution for a system with 3 particle types. In this case, there are three nodal lines for F_m which meet at one point and consequently F_m cannot be equal to $|g|$ in this case.

One may ask, then, when the limit distribution F_m for a multi-type particle system corresponds to a higher eigenfunction. We concentrated our efforts on one particular class of domains, namely rectangles D because in this case, the eigenvalues and the corresponding eigenfunctions can be explicitly calculated.

Let $D = \{(x, y) \in \mathbf{R}^2 : 0 < x < a, 0 < y < b\}$. Then all eigenvalues of the Laplacian in D with zero boundary values are given by $\lambda_{j,k} = \pi^2[(j/a)^2 + (k/b)^2]$, where j and k are arbitrary integers greater than 0 [13]. The eigenfunction corresponding to $\lambda_{j,k}$ has the form $f_{j,k}(x, y) = \sin((j\pi/a)x) \sin((k\pi/b)y)$. It may happen that $\lambda_{j_1,k_1} = \lambda_{j_2,k_2}$ even though $j_1 \neq j_2$ and $k_1 \neq k_2$ but this is possible for only a countable number of side ratios $r = b/a$. We can also write $\lambda_{j,k}$ as $(\pi/a)^2(j^2 + (k/r)^2)$.

It is intuitively clear that when the number of types of particles m is constant but the side ratio of D is very large then particles of different types will occupy m rectangles arranged in a linear order (see for example Fig.1a). We will call this arrangement “elementary.” It corresponds to an eigenfunction $f_{j,k}$ of the Laplacian with either $j = 1$ or $k = 1$. On the other hand, when m is fixed, say $m = 3$, and the side ratio is close to 1, we obtain in computer simulations a configuration illustrated in Fig.1b which does not correspond to any eigenfunction. We determined by simulation the critical side ratio at which we observe the transition between the elementary configuration and a configuration which does not correspond to any eigenfunction.

VI Computer simulations

Further discussion of the limiting distributions and eigenvalues will be illustrated by computer simulations so we make a digression to explain our figures. In all simulations we took D to be a rectangle. The figures show the regions D and the boundaries between the subregions occupied by different particle types. All simulations were done for rectangles D with sides $b = 100$ and $100 < a < 300$. Because of memory constraints, the results of the simulations were compressed in the following way. Every region D was divided into a number of small identical rectangles, usually with side lengths between 5 and 10. The numbers of particles of different type were found in every small rectangle and the rectangle was declared of type \mathcal{L}_k if the number of particles of this type was the greatest of all particle types. Only rectangles close to the boundaries between D_k 's contained different particles types. In our simulations, almost all other rectangles contained only one type of particles.

We have simulated long time behavior of the system in rectangles of different side ratios with 100,000 particles. Most simulations ran for 150,000 or 200,000 timesteps. The starting configurations included "elementary configurations," other configurations with polygonal separating lines and totally random configurations. We used various initial proportions of different particle types. We did simulations with $m = 3, 4$ and 5 particle types. In each case we determined the critical side ratio $r_m = a/b$ at which we observed a transformation of the stationary configuration from the elementary configuration to a configuration which did not correspond to an eigenfunction.

When the number of particle types is $m = 3$, the critical side ratio is 1.64 ± 0.01 (Fig. 1). The simulations starting from various initial distributions show that the limit distribution is elementary for the ratio 1.65 and it is not for the ratio 1.62.

In the case of side length ratios 1.63 and 1.64, the particle configuration had a tendency to preserve its initial shape. However, several independent simulations done for the side ratios 1.63 and 1.64 suggest that the elementary configuration is unstable in the long run and that once the configuration changes somewhat the shape, it moves quickly towards the limit distribution illustrated in Fig.1b. These remarks are based on computer animation and cannot be effectively illustrated by figures.

The results of the simulations are most clear in the case of 4 particle types. Each of the simulations was started from an asymmetric configuration. The critical ratio is 2.26 ± 0.01 . The particle distributions are given in Fig. 2.

Simulations with 5 particle types (Fig.3) were also started from asymmetric distributions. In this case, the critical side ratio is 2.85 ± 0.01 .

VII Configurational transition and Laplacian eigenvalues

We will argue that the critical side ratios obtained from the computer simulations match exceptionally well the critical rectangle side ratios for the following problem. *When is it true that the elementary configuration with m subregions corresponds to the m -th eigenfunction?* We order the eigenfunctions according to their eigenvalues, i.e., the m -th eigenfunction corresponds to m -th smallest eigenvalue.

Recall the formulae for the eigenvalues of the Laplacian given in Sect. 3. We have $\lambda_{j,k} = (\pi/a)^2(j^2 + (k/r)^2)$ for a rectangle with sides equal to a and b and side ratio $r = b/a$. The elementary configuration is defined by the eigenfunction corresponding to $\lambda_{1,m}$. Whether $\lambda_{1,m}$ is the m -th eigenvalue depends only on r (it does not otherwise depend on the values of a and b). Note that $\lambda_{1,k} < \lambda_{1,m}$ for

$k < m$ so $\lambda_{1,m}$ is the m -th eigenvalue if and only if

$$\lambda_{1,m} < \lambda_{2,1}. \quad (7.1)$$

This is equivalent to (section V)

$$1^2 + (m/r)^2 < 2^2 + (1/r)^2. \quad (7.2)$$

We take $m = 3, 4, 5$ and solve this equation for r to obtain the following critical side ratios r_m : $r_3 = \sqrt{8/3} \approx 1.63$, $r_4 = \sqrt{5} \approx 2.24$, $r_5 = 2^{3/2} \approx 2.83$. Since our simulations were done on a discrete lattice, the critical side ratio values calculated for the rectangle D in \mathbf{R}^2 are only approximate. Eigenfunctions for the discrete Laplacian on a rectangle $D = \{(x, y) \in \mathbf{Z}^2 : 1 \leq x \leq a, 1 \leq y \leq b\}$ are given by $\tilde{f}(x, y) = g(x)h(y)$ where g and h satisfy $g(0) = g(a+1) = 0$, $h(0) = h(b+1) = 0$, and

$$\begin{aligned} g(x-1) - 2g(x) + g(x+1) &= \lambda_x g(x), & 1 \leq x \leq a, \\ h(y-1) - 2h(y) + h(y+1) &= \lambda_y h(y), & 1 \leq y \leq b. \end{aligned} \quad (7.3)$$

Then $\tilde{\lambda} = \lambda_x + \lambda_y$ is the eigenvalue corresponding to the eigenfunction $\tilde{f}(x, y) = g(x)h(y)$. If g changes sign $j-1$ times and h changes the sign $k-1$ times then $\tilde{\lambda} = \tilde{\lambda}_{j,k}$ is a discrete analog of $\lambda_{j,k}$. We solved the system of equations (7.3) numerically and determined $\tilde{\lambda}_{1,m}$ and $\tilde{\lambda}_{2,1}$. The critical side ratios in the discrete case are as follows:

$$\begin{aligned} r_3 &= 1.645 \pm 0.005, \\ r_4 &= 2.255 \pm 0.005, \\ r_5 &= 2.855 \pm 0.005. \end{aligned} \quad (7.4)$$

These values match very well the critical side ratios discussed in the previous section.

It is quite intriguing that the configurational transition takes place for side ratio related to the eigenvalue of the Laplacian (Eq(7.1-2)). It would be interesting to

find an explanation for this phenomenon. We hope that our results will be useful in the future studies of the population dynamics.

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Figure captions

Figure 1. Nodal lines for stationary distribution of particles with 3 particle types. Each region, separated by solid lines is occupied by only one type of particles. (a) The side ratio $r_3 = a/b = 1.64$; elementary configuration corresponding to the third Laplacian eigenfunction (b) $r_3 = 1.63$; configuration close to the transition point (non-elementary configuration) (c) $r_3 = 1$; configuration far from the transition point (non-elementary configuration).

Figure 2. Nodal lines for stationary distribution of particles with 4 particle types. Each region, separated by solid lines is occupied by only one type of particles. (a) The side ratio $r_4 = a/b = 2.27$; elementary configuration corresponding to the fourth Laplacian eigenfunction (b) $r_4 = 2.24$; configuration close to the transition point (non-elementary configuration) (c) $r_4 = 1$; configuration far from the transition point (non-elementary configuration).

Figure 3. Nodal lines for stationary distribution of particles with 5 particle types. Each region, separated by solid lines is occupied by only one type of particles. (a) The side ratio $r_5 = a/b = 2.88$; elementary configuration corresponding to the fifth Laplacian eigenfunction (b) $r_5 = 2.84$; configuration close to the transition point (non-elementary configuration) (c) $r_5 = 1$; configuration far from the transition point (non-elementary configuration).