Discrete Approximations to Time-changed Brownian Motions

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Abstract

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We give a general discrete approximation scheme of time-changed Brownian motions on $\mathbb{R}^d$. Our approximation scheme works for any smooth measure with full quasi-support on $\mathbb{R}^d$ with suitable initial distributions. Under some mild conditions on the smooth measure, the discrete approximation scheme works for every starting point.

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DEDICATION

To Mom,

for all her care, guidance, and support in my life.

Chapter 1

INTRODUCTION AND MAIN RESULTS

Time change is an important transformation for Markov processes and for Brownian motion in particular. Let $d \geq 1$ be a positive integer and W be the standard Brownian motion in $\mathbb{R}^d$. Given a positive Borel measurable function f on $\mathbb{R}^d$, one can define the integral of f along Brownian sample path up to time $t \geq 0$ by $A_t = \int_0^t f(W_s)ds$. Let A^{-1} be the (time) inverse function of A (note that the inverse exists as A_t is continuous and strictly increasing in t). It defines a time-changed $X_t = W_{A_t^{-1}}$. Then the process X is a random time-changed process of Brownian motion with respect to the function f. One can not only do time change of Brownian motion using a positive Borel measurable function f, but using a large class of measures called smooth measures ([CF12, Definition 2.3.13]). Let μ be a smooth measure with full quasi-support. One can construct the time-changed Brownian motion with respect to μ. By [CF12, Theorem 4.1.1] there exists a positive continuous additive functional (PCAF) A of the Brownian motion W whose Revuz measure is μ. Define $X_t = W_{A_t^{-1}}$ where $A_t^{-1} := \inf\{s \geq 0 : F_s > t\}$ is the generalized inverse function of A. Then X is called the time-changed Brownian motion with respect to μ. When $\mu(dx) = f(x)dx$ (dx is the Lebesgue measure) for some positive $f \in C(\mathbb{R}^d)$, $A_t = \int_0^t f(W_s)ds$ and the time-changed process coincide with the previous definition. See Section 2.1 below for a brief introduction of these concepts and refer to [FOT11, CF12] for detailed discussion.

We are interested in developing a scheme to simulate the time-changed Brownian motion X that only uses μ (no need for Brownian sample paths or inverses of PCAFs whatever) as input so that it is easy to implement and do simulation. It is well known that Brownian motion is the scaling limit of simple random walks. A natural idea is to change the holding time or waiting time of random walk according to the mass of μ at that place in a suitable way. We show such scheme indeed works. In fact, a more general scheme can be developed.

For this purpose, for any $r > 0$, we consider a continuous time simple random walk process $\widetilde{W}^r$ approximating the standard Brownian motion W, and measure μ_r which approximates μ vaguely. Then we do time change on $\widetilde{W}^r$ to get a time-changed random walk process X^r and show that X^r converges to the time-changed Brownian motion X as $r \downarrow 0$. We construct X^r as follows. Let ϱ be a probability density function (with respect to the Lebesgue measure) that has zero mean and covariance $M_\varrho \delta_{ij}$ for some constant $M_\varrho > 0$, where δ_{ij} is the Kronecker delta. Let ϕ be a probability density function (with respect to the Lebesgue measue). For each $r > 0$, set $\phi_r(x) := r^{-d}\phi(x/r)$, $\mu_r(dx) := (\phi_r * \mu)(dx) = \int \phi_r(y - x)\mu(dy)dx$, $m_r(dx) := \frac{M_\varrho^{-1}}{r^2}dx$, $\lambda_r(x) := \frac{dm_r}{d\mu_r}(x) = \left(M_\varrho r^2 \int \phi_r(y-x)\mu(dy)\right)^{-1}$, $\varrho_r(x) := r^{-d}\varrho(x/r)$ and $Q_r(x, dy) := \varrho_r(y-x)dy$. We construct a continuous-time random walk process X^r waits for an exponential distributed time with rate λ_r and then jumps with transition probability Q_r. For this purpose, let $\{\xi_i\}_{i=1}^\infty$ be i.i.d. random variables with probability density function ϱ. Let $\{\eta_i\}_{i=0}^\infty$ be i.i.d. exponential distributed with parameter 1 and independent of $\{\xi_i\}_{i=0}^\infty$. Let ξ_0 be a random variable in $\mathbb{R}^d$ independent of $\{\xi_i\}_{i=1}^\infty$ and $\{\eta_i\}_{i=0}^\infty$. For each $r > 0$ let $\xi_i^r := r\xi_i$ (hence it has distribution ϱ_r) and set $W_n^r := \xi_0 + \sum_{i=1}^n \xi_i^r$ for $n \in \mathbb{N}$ ($\sum_{i=1}^n \xi_i^r = 0$ if $n = 0$). Thus W_n^r is a random walk process with initial distribution ξ_0 and step distribution ϱ_r. Then define $\eta_i^r(x) := \eta_i/\lambda_r(x)$ (hence it is exponentially distributed with parameter λ_r) and $N_t^r := \inf\{n \in \mathbb{N} : \sum_{i=0}^n \eta_i^r(W_i^r) > t\}$. Then the process $X_t^r := W_{N_t^r}^r$ is the so-called regular step process with road map Q_r and speed function λ_r (see, e.g. [CF12, 2.2.1]). When $\mu(dx)$ is the Lebesgue measure we denote the corresponding X^r by $\widetilde{W}^r$. Note that $\widetilde{W}^r$ is simply the continuous time random walk with step distribution ϱ and jumping rate $M_\varrho^{-1}r^{-2}$ (or mean holding time $M_\varrho r^2$).

It is natural to expect that the time-changed continuous time random walk X^r converges to the time-changed Brownian motion X as $r \downarrow 0$. Under some mild assumptions on ϱ, μ and ϕ, we show that the answer is positive. When $\mu(dx)$ is the Lebesgue measure we have $\lambda_r(x) = \frac{M_\varrho^{-1}}{r^2}$ for any $x \in \mathbb{R}^d$ and $r > 0$, X^r converges to the standard Brownian motion starting from ξ_0. For general full quasi-support smooth measure μ, X^r will converge to the corresponding time-changed Brownian motion with respect to the measure μ. Here is a diagram that illustrate our approximation scheme.

$$\begin{tikzcd}
\widetilde{W}_t^r \arrow[r, "\text{Donsker's invarinace principle}"] \arrow[d, "\text{t.c. by } \mu_r"] & \text{BM} \arrow[d, "\text{t.c. by } \mu"] \\
X_t^r \arrow[r, "\text{Our approximation scheme}"] & \text{t.c.BM}
\end{tikzcd}$$

Throughout this paper we add some basic assumptions on the probability densities ϱ and the measure μ:

(a) $\int x\varrho(x)dx = 0, \int x_i x_j \varrho(x)dx = M_\varrho \delta_{ij}$ and ϱ is symmetric about the origin

(b) μ has full quasi-support and does not charge polar sets.

Note that Assumption (a) implies ϱ has zero mean and $\widetilde{W}^r$ converges to Brownain motion by Donsker's invarinace principle. The Assumption (b) allow us to contruct a *continuous* time change of Brownian motion according to the measure μ. (Without full quasi-support, time changes will not be continuous and the time-changed processes will have jumps.)

We now list some other assumptions on the probability densities ϱ, ϕ and measure μ:

(A.1) ϕ is bounded and has compact support and weak derivative (in the sense of Schwartz distributions) such that $|\nabla\phi| \le C\varrho$ on $\mathbb{R}^d$ for some constant $C > 0$.

(A.2) ϱ is bounded above by a constant $H_\varrho > 0$. Moreover there exists $\delta > 0$ such that ϱ has a finite $\max(2, d-2+\delta)$-th absolute moment.

(A.3) For any $R > 0$, there exists $C_R > 0$ and $\beta_\mu = \beta_\mu(R) > d-2$ such that for any $x \in B(0,R)$ and $r \in (0,1)$ we have $\mu(B(x,r)) \le C_R r^{\beta_\mu}$. In addition,

$$\lim_{r\downarrow 0} r^{\beta_\mu - d + 2} \log\left(\int_{B(0,2R)} \frac{dx}{\int \phi((x-y)/r)\mu(dy)}\right) = 0. \tag{1.1}$$

A sufficient condition for (A.1) is that there exists an open ball $B_\varrho \subseteq \mathbb{R}^d$ on which ϱ is bounded below by a constant $h_\varrho > 0$, in which case we can easily find ϕ supported in B_ϱ with bounded weak derivative. A typical example of ϱ is the uniform distribution of the unit ball $\varrho(x) = \frac{\mathbb{1}_{B(0,1)}(x)}{|B(0,1)|}$, where $B(x,r) := \{y \in \mathbb{R}^d : |y-x| < r\}$ and $|B(x,r)|$ is the volume. In this case, we can calculate $M_\varrho = \frac{1}{d+2}$. A sufficient condition for (1.1) is that for any $R > 0$, there exists $c_R > 0$ and $\beta'_\mu > 0$ such that for any $x \in B(0,R)$ and $r \in (0,1)$ we have $\mu(B(x,r)) \geq c_R r^{\beta'_\mu}$. A typical nontrivial example of μ that satisfies (A.3) is the Liouville measure from Liouville quantum gravity, which we will briefly introduce after Theorem 1.2.

We show that (ϱ, ϕ) satisfying (A.1) is enough to prove that X^r converges in Skorokhod topology to the time changed Brownian motion X with initial distribution absolutely continuous to the symmetrizing measure (Theorem 1.1). The additional conditions (A.2) and (A.3) will allow us to establish point-wise starting distribution convergence in Skorokhod topology (Theorem 1.2).

Remark. The moment condition in $(A.2)$ can be dropped when $d \leq 3$ as we always assume ϱ have a finite second moment. Under (A.3) we can in fact show that μ charge no polar sets and is a smooth measure in strict sense (Proposition 4.18). See [FOT11, CF12] for the definition of smooth measures in strict sense. This allows us to define time-changed Brownian motion starting point-wisely.

Let $(\Omega, \mathcal{A}, \{\mathbb{P}_x\}_{x\in\mathbb{R}^d})$ be the probability space that $\{\xi_i, \eta_i\}_{i=0}^\infty$ live on with $\mathbb{P}_x\{\xi_0 = x\} = 1$. Set $\mathbb{P}_\nu = \int \nu(dx)\mathbb{P}_x$ for any measure ν. Let $\mathbf{D}([0,\infty);\mathbb{R}^d)$ denote the space of right continuous paths with left limits.

The following two theorems are the main results of this paper. Note that both theorems are about time change of the standard Brownian motion, but the result can be easily extended to Brownian motion with constant diffusion matrix through a linear transformation.

Theorem 1.1 (Approximation under symmetrizing measures). *Under assumption (A.1), for any $f_0 \geq 0$ in $C_c(\mathbb{R}^d)$, set $\nu_r = f_0 \cdot \mu_r$ and $\nu = f_0 \cdot \mu$. Then the law of $\{X^r; t \geq 0\}$ under $\mathbb{P}_{\nu_r}$ converges weakly in $\mathbf{D}([0,\infty);\mathbb{R}^d)$ in Skorokhod topology to the time-changed Brownian* *motion X by μ with initial distribution ν.*

Theorem 1.2 (Approximation for every starting point)**.** *Under assumptions (A.1), (A.2) and (A.3), for any sequence $x_r \in \mathbb{R}^d$ converges to $x_0 \in \mathbb{R}^d$, the law of $\{X^r; t \geq 0\}$ under $\mathbb{P}_{x_r}$ converges weakly in $\mathbf{D}([0,\infty); \mathbb{R}^d)$ in Skorokhod topology to the law of $\{X; t \geq 0\}$ starting from x_0 as $r \downarrow 0$.*

Remark. We can also use continuous time random walk on $r\mathbb{Z}^d$ to approximate time-changed Brownian motion. Detailed discussion is given in Chapter 5.

Our results have some straightforward applications. For example, it can be applied to the Liouville Brownian motion constructed and defined in [Ber15, GRV16], which is the time-changed Brownian motion with respect to the Liouville meausure. For each $\gamma \in (0, 2)$ the Liouville measure M_γ is the weak limit $\lim_{\varepsilon \downarrow 0} e^{\gamma h_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}(h_\varepsilon^2(z))} dz$ where h_ε is the circle average of a Gaussian free field. It is shown in [AK16, Theorem 3.1] (see also[GRV16, Theorem 2.2]) that almost surely (in Gaussian free fields), for any $\varepsilon > 0$ and $R > 1$ the Liouville measure satisfies

$$C_1 r^{\alpha_1+\varepsilon} \leq M_\gamma(B(x,r)) \leq C_2 r^{\alpha_2-\varepsilon}, \qquad \forall x \in B(0,R), r \in (0,1)$$

where $\alpha_1 = \frac{1}{2}(\gamma+2)^2$ and $\alpha_2 = \frac{1}{2}(2-\gamma)^2$. This implies that the Liouville measure satisfies the condition (A.3) as long as ϕ is chosen to be bounded below away from 0 on an open ball. Furthermore, by [GRV16, Theorem 2.7] the PCAF with Revuz measure μ is strictly increasing for any starting point $x \in \mathbb{R}^d$. This implies that the quasi support of μ is $\mathbb{R}^d$ (i.e. has full quasi-support) by [CF12, Theorem 5.2.1 (i)]. Hence Theorem 1.2 gives the Liouville Brownian motion a time-changed random walk approximation scheme. This is one of the motivations of this paper.

Corollary 1.3. *Let $d = 2$ and fix $\gamma \in (0, 2)$. Let μ be the Liouville measure with parameter γ and X be the Liouville Brownian motion. Then for any sequence $x_r \in \mathbb{R}^d$ converges to $x_0 \in \mathbb{R}^d$, the time-changed random walk $\{X^r; t \geq 0\}$ starting from x_r converges in distribution in $\mathbf{D}([0,\infty); \mathbb{R}^d)$ in Skorokhod topology to the Liouville Brownian motion X starting from x_0.*

We use some notation convention throughout this paper. We will fix the dimension $d \geq 1$. We use $\mathbb{N}$ and $\mathbb{N}^*$ to denote nonegative integers and positive integers respectively. The symbols c, C with or without numerical subscripts stand for positive constants whose value may change from place to place, and they may only depend on the dimension d and values related to ϱ, μ or ϕ, unless the dependency of other variables is explicitly specified. By adding letter subscripts except numbers to the symbols c, C we indicate their dependence on those subscripts, and their values may also change from place to place. We use $r > 0$ as sup/sub-script on W, X, Q etc to denote the step size r in the approximation scheme. Sometimes we will fix a sequence $r_n \leq 1$ and $r_n \downarrow 0$ and with a slight abuse of notation, we will replace sup/sub-script on W, X, Q etc by n to denote the corresponding object when $r = r_n$. When r replaced by n, we assume a sequence $r_n \in (0, 1]$ tending to 0 (as $n \to \infty$) is fixed. We use $o_h(1)$ denotes values going to 0 as the subscript h approaches to some value (usually 0 or ∞). We use $L^2(E; \mu)$ denote L^2-space on E with measure μ; when $E = \mathbb{R}^d$ we simply write $L^2(\mu)$. Given a Dirichlet form $(\mathcal{E}, \mathcal{F})$, we denote $\mathcal{E}(u) = \mathcal{E}(u, u)$ for $u \in \mathcal{F}$.

Chapter 2

PRELIMINARIES

In this chapter we introduce some necesary concepts and propositions that will be used later.

2.1 Basics of Potential theory

We states basic concepts from potential theory. For detailed discussions, see [FOT11, CF12].

Let E be a locally compact separable metric space, m is a positive Radon measure on E. Let Ω be the sample space and $(X_t, \mathbb{P}_x)$ a m-symmetric Hunt process taking $(E, \mathcal{B}(E))$, whose Dirichlet form is $(\mathcal{E}, \mathcal{F})$. For a closed subset F of E, define

$$\mathcal{F}_F := \{f \in \mathcal{F} : f = 0 \ m\text{-a.e. on } E \setminus F\}.$$

Let $\mathcal{E}_1(\cdot) = \mathcal{E}(\cdot) + \|\cdot\|^2_{L^2(E;m)}$. Denote by $\mathcal{O}$ the family of all open subsets of E. For $A \in \mathcal{O}$ we define

$$\mathcal{L}_A = \{u \in \mathcal{F} : u \geq 1 \ m\text{-a.e. on } A\},$$

$$\mathrm{Cap}(A) = \begin{cases} \inf_{u \in \mathcal{L}_A} \mathcal{E}_1(u,u), & \mathcal{L}_A \neq \emptyset \\ \infty & \mathcal{L}_A = \emptyset, \end{cases}$$

and for any set $A \subseteq E$ we let

$$\mathrm{Cap}(A) = \inf_{B \in \mathcal{O}, A \subseteq B} \mathrm{Cap}(B).$$

We call this the capacity of A.

An increasing sequence $\{F_k, k \geq 1\}$ of closed sets of E is an $\mathcal{E}$-nest if $\cup_k \mathcal{F}_{F_k}$ is $\mathcal{E}_1$-dense in F.

A subset N of E is called $\mathcal{E}$-polar if there is an $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ such that $N \subseteq \cap_{k\geq 1}(E \setminus F_k)$.

A measure μ is called smooth if μ charges no $\mathcal{E}$-polar set and there exists a $\mathcal{E}$-nest $\{F_k, k \geq 1\}$ such that $\mu(F_k) < \infty$ for every $k \geq 1$.

Let $\mathcal{F}_t := \sigma\{X_s; s \leq t\}$ and θ be the shift operator of paths (i.e. $\theta_t(\omega) = \omega_{\cdot+t}$). A numerical function $A_t(\omega)$ of two variables $t \geq 0$, $\omega \in \Omega$ is called an additive functional of X if there exist $\Lambda \in \mathcal{F}_\infty$ and an m-inessential set $N \subset E$ with

$$\mathbb{P}_x(\Lambda) = 1 \text{ for } x \in E\backslash N \quad \text{ and } \quad \theta_t\Lambda \subset \Lambda \text{ for } t > 0,$$

and the following conditions are satisfied:

- For each $t \geq 0$, $A_t|_\Lambda$ is $\mathcal{F}_t|_\Lambda$-measurable, where $A_t|_\Lambda$ is the restriction of A_t to Λ and $\mathcal{F}_t|_\Lambda$ is the σ-algebra of $\mathcal{F}_t$ restricted to Λ.

- For any $\omega \in \Lambda$, $A_\cdot(\omega)$ is right continuous on $[0, \infty)$ has the left limits on $(0, \zeta(\omega))$, $A_0(\omega) = 0, |A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, and $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for $t \geq \zeta(\omega)$.

- The additivity

  $$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega) \qquad \text{for every } t, s \geq 0,$$

  is satisfied.

2.2 Convergences in different Hilbert spaces

Those concepts are mainly from Kuwae and Shioya's paper [KS03]. One can also refer to [Kol05] for a concise introduction about these concepts.

Definition 2.1. *We say that a sequence of Hilbert spaces* $\mathcal{H}_n$ *converges to a Hilbert space* $\mathcal{H}$ *if there exists a dense subspace* $\mathcal{C} \subseteq \mathcal{H}$ *and a sequence of linear operators* $\Phi_n : \mathcal{C} \to \mathcal{H}_n$ *such that* $\lim_{n\to\infty} \|\Phi_n u\|_{\mathcal{H}_n} = \|u\|_{\mathcal{H}}$ *for every* $u \in \mathcal{C}$.

For the following discussion, we assume that the Hilbert space $\mathcal{H}_n$ converges to the Hilbert space in the sense of Definition 2.1.

Definition 2.2. *(Strong convergence) We say that a sequence* $u_n \in \mathcal{H}_n$ *strongly converges to* $u \in \mathcal{H}$ *if there exists a sequence* $v_m \in \mathcal{C}$ *strongly converges to* u *in* $\mathcal{H}$ *such that*

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \|\Phi_n v_m - u_n\|_{\mathcal{H}_n} = 0.$$

Definition 2.3. *(Weak convergence) We say that a sequence of* $u_n \in \mathcal{H}_n$ *weakly converges to* $u \in \mathcal{H}$ *if every sequence* $v_n \in \mathcal{H}_n$ *strongly converges to* $v \in \mathcal{H}$*, we have*

$$(u_n, v_n)_{\mathcal{H}_n} \to (u, v)_{\mathcal{H}}.$$

Proposition 2.4. *For any sequence* $u_n \in \mathcal{H}_n$ *that converges to* $u \in \mathcal{H}$ *in the sense of Definition 2.3 we have*

$$\sup_n \|u_n\|_{\mathcal{H}_n} < \infty \qquad \text{and} \qquad \|u\|_H \leq \underline{\lim}_n \|u_n\|_{\mathcal{H}_n}.$$

Proof. See [KS03, Lemma 2.3]. □

Proposition 2.5. *Assume that* $u_n \in \mathcal{H}_n$ *converges to* $u \in \mathcal{H}$ *and* $v_n \in \mathcal{H}_n$ *converges to* $v \in \mathcal{H}$ *in the sense of Definition 2.2. Then* $u = v$ *if and only if* $\lim_{n\to\infty} \|u_n - v_n\|_{\mathcal{H}_n} = 0$.

Proof. See [Kol05, Lemma 7.1]. □

Proposition 2.6. *The convergence of* u_n *to* u *in the sense of Definition 2.3 is equivalent to the following: for every* $v \in \mathcal{C}$ *we have* $(u_n, \Phi_n v)_{\mathcal{H}_n} \to (u, v)_{\mathcal{H}}$.

Proof. Assume for every $v \in \mathcal{C}$ we have $(u_n, \Phi_n v)_{\mathcal{H}_n} \to (u, v)_{\mathcal{H}}$, then for any $w \in \mathcal{H}$, $w_n \in \mathcal{H}_n$ converges strongly to w, and any $\varepsilon > 0$ we can find $w_\varepsilon \in \mathcal{C}$ such that $\|w_\varepsilon - w\|_{\mathcal{H}} \leq \varepsilon$ and $\overline{\lim}_{n\to\infty} \|\Phi_n w_\varepsilon - w_n\|_{\mathcal{H}_n} \leq \varepsilon$. Then by Cauchy-Schwarz inequality

$$\overline{\lim_{n\to\infty}} |(u_n, w_n)_{\mathcal{H}_n} - (u_n, \Phi_n w_\varepsilon)_{\mathcal{H}_n}| \leq \varepsilon \sup_n \|u_n\|_{\mathcal{H}_n}$$

and

$$\overline{\lim_{n\to\infty}} |(u,w)_{\mathcal{H}} - (u,w_\varepsilon)_{\mathcal{H}}| \le \varepsilon \|u\|_{\mathcal{H}}.$$

By Proposition 2.4 we know $\sup_n \|u_n\|_{\mathcal{H}_n} < \infty$. As we have $(u_n, \Phi_n w_\varepsilon)_{\mathcal{H}_n} \to (u, w_\varepsilon)_{\mathcal{H}}$ and $\varepsilon > 0$ is arbitrary, we get

$$\overline{\lim_{n\to\infty}} |(u_n,w_n)_{\mathcal{H}_n} - (u,w)_{\mathcal{H}}| = 0$$

hence the strong convergence of u_n to u follows.

The other direction is obvious. □

Proposition 2.7. *If $u_n \in \mathcal{C} \subseteq \mathcal{H}$ converges to $u \in \mathcal{H}$ strongly in $\mathcal{H}$, then there exists a nondecreasing subsequence $k_n \uparrow \infty$ and $v_n := u_{k_n}$ such that $\Phi_n v_n \in \mathcal{H}_n$ converges strongly to $u \in \mathcal{H}$ in the sense of Definition 2.2.*

Proof. By definition we have for any $m, n \ge 0$

$$\lim_{k\to\infty} \|\Phi_k(u_m - u_n)\|_{H_k} = \|u_m - u_n\|_{\mathcal{H}}$$

and

$$\lim_{N\to\infty} \sup_{m,n\ge N} \|u_m - u_n\|_{\mathcal{H}} = 0.$$

Hence for any $n \in \mathbb{N}^*$, we can find $N_n \uparrow \infty$ such that for any $m \ge N_n$

$$\|u_m - u_{N_n}\|_{\mathcal{H}} \le 1/(2n)$$

and there exists $k_{n,m} \in \mathbb{N}$ such that

$$\sup_{k\ge k_{n,m}} \|\Phi_k(u_m - u_{N_n})\|_{H_k} \le \|u_m - u_{N_n}\|_{\mathcal{H}} + 1/(2n) \le 1/n.$$

Then we find a sequence $\{k_{i,N_i}\}_{i=1}^\infty$. Define $v_n = u_{N_n}$ if $k_{n,N_n} \le n < k_{n+1,N_{n+1}}$. Then we have for $m \in [k_{j,N_j}, k_{j+1,N_{j+1}}) \cap \mathbb{N}^*$

$$\overline{\lim_{n\to\infty}} \|\Phi_n(v_n - v_m)\|_{\mathcal{H}_n} \le 1/j$$

hence

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \|\Phi_n v_n - \Phi_n v_m\|_{\mathcal{H}_n} = 0.$$

As v_m converges to u strongly in $\mathcal{H}$, we have $\Phi_n v_n$ converges to u in the sense of Definition 2.2.

$\square$

Definition 2.8 (Operator strong convergence)**.** *We say that a sequence of bounded operators T_n on $\mathcal{H}_n$ strongly converges to an bounded operator T on $\mathcal{H}$ if for every sequence u_n strongly tending to $u \in \mathcal{H}$, the sequence $T_n(u_n)$ strongly tends to $T(u)$.*

2.3 Mosco convergence

Mosco convergence is a convergence between Dirichlet forms which was first introduced by Mosco in [Mos94]. It is extended to Dirichlet forms in varying spaces in [KS03]. See also [Kim06, CKK13]. Let $\mathcal{H}_n$ be Hilbert spaces which converges to the Hilbert space $\mathcal{H}$ in the sense of Definition 2.1.

Definition 2.9 (Mosco Convergence)**.** *We say that a sequence of quadratic forms $(\mathcal{E}^n, \mathcal{F}^n)$ on $\mathcal{H}_n$ is Mosco convergent to a quadratic form $(\mathcal{E}, \mathcal{F})$ on $\mathcal{H}$ if the following conditions hold:*

- *For any sequence $u_n \in \mathcal{F}^n$ weakly converges to $u \in \mathcal{H}$ with $\sup_n \mathcal{E}^n(u_n) < \infty$, we have $\mathcal{E}(u) \le \varliminf_n \mathcal{E}^n(u_n)$.*
- *For every $u \in \mathcal{F}$ there exist a strong convergent sequence $u_n \to u$ with $u_n \in \mathcal{F}^n$ such that $\mathcal{E}(u) = \lim_n \mathcal{E}^n(u_n)$.*

Remark. For any Dirichlet form $(\mathcal{F}, \mathcal{E})$, if we make the convention that $\mathcal{E}(u) = \infty$ for $u \in H \setminus \mathcal{F}$. Then Definition 2.9 is equivalent to

- For any sequence $u_n \in \mathcal{H}_n$ weakly converges to $u \in \mathcal{H}$ we have $\mathcal{E}(u) \le \varliminf_n \mathcal{E}^n(u_n)$.
- For every $u \in \mathcal{H}$ there exist a strong convergent sequence $u_n \to u$ with $u_n \in \mathcal{H}_n$ such that $\mathcal{E}(u) = \lim_n \mathcal{E}^n(u_n)$.

We now give the Dirichlet forms of X^r and X. Because $\varrho(x) = \varrho(-x)$ for any $x \in \mathbb{R}^d$, it is easy to check that $m_r(dx) = \frac{M_\varrho^{-1}}{r^2} dx$ is a symmetrizing measure of $Q_r(x, dy) = \varrho_r(y - x)dy$, i.e.,

$$Q_r(x,dy)m_r(dx) = Q_r(y,dx)m_r(dy).$$

By [CF12, Theorem 2.2.2] (with the discussion on the cases of unbounded speed measure on page 55 of [CF12]), we know the Dirichlet form of X^r on $L^2(\mathbb{R}^d;\mu_r)$ is

$$\begin{aligned}\mathcal{F}^r &= \{f \in L^2_{loc}(\mathbb{R}^d) : \mathcal{E}^r(f) < \infty\} \cap L^2(\mathbb{R}^d;\mu_r)\\ \mathcal{E}^r(u) &= \frac{1}{2}\int\int (u(x)-u(y))^2 Q_r(x,dy)m_r(dx).\end{aligned}$$

By [CF12, Theorem 5.2.2] the Dirichlet form of the time-changed Brownian motion X on $L^2(\mathbb{R}^d;\mu)$ is

$$\begin{aligned}\mathcal{F}^\mu &= \widetilde{BL}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d;\mu)\\ \mathcal{E}(u) &= \frac{1}{2}\int |\nabla u(x)|^2 dx,\end{aligned}$$

where $\widetilde{BL}(D)$ denotes the set of quasi-continuous versions of functions in the Beppo-Levi space $BL(D) = W^{1,2}_{\text{loc}}(D)$ in a domain $D \subseteq \mathbb{R}^d$. This identifies any sub-sequential limit of X^r will have the same finite dimensional distributions as the time-changed Brownian motion X.

Besides the two Dirichlet forms $(\mathcal{F}^r, \mathcal{E}^r)$ on $L^2(\mathbb{R}^d;\mu_r)$ and $(\mathcal{F}^\mu, \mathcal{E})$ on $L^2(\mathbb{R}^d;\mu)$, we introduce some other Dirichlet forms. Let E be an open set of $\mathbb{R}^d$ and define Dirichlet forms $(\mathcal{F}^{r,E}, \mathcal{E}^{r,E})$ on $L^2(E;\mu_r|_E)$ by

$$\begin{cases}\mathcal{F}^{r,E} = & \{u \in L^2_{\text{loc}}(E) : \mathcal{E}^r(u) < \infty\} \cap L^2(E;\mu_r|_E)\\ \mathcal{E}^{r,E}(u) = & \frac{1}{2}\int_{E\times E}(u(x)-u(y))^2 Q_r(x,dy)m_r(dx)\\ & +\int_E u(x)^2(1-Q_r(x,E))m_r(dx)\end{cases} \tag{2.1}$$

and $(\mathcal{F}^{\mu,E}, \mathcal{E}^{\mu,E})$ on $L^2(E;\mu)$ by

$$\begin{cases}\mathcal{F}^{\mu,E} & = \overline{C_c^\infty(E)}^{\mathcal{E}_1^{\mu,E}} \cap L^2(E;\mu)\\ \mathcal{E}^{\mu,E}(u) & = \frac{1}{2}\int_E |\nabla u(x)| dx.\end{cases} \tag{2.2}$$

where $\overline{C_c^\infty(E)}^{\mathcal{E}_1^{\mu,E}}$ denotes the completement of $C_c^\infty(E)$ with respect to $\mathcal{E}_1^{\mu,E} = \mathcal{E}^{\mu,E}(\cdot) + \|\cdot\|^2_{L^2(E;\mu)}$. Note that $\mathcal{E}^{r,E}(u)$ (resp. $\mathcal{E}^{\mu,R}(u)$) is the same as $\mathcal{E}^r(u)$ (resp. $\mathcal{E}(u)$) if we extend the function $u \in L^2(E;\mu_r|_E)$ to be 0 outside E. We also define $\mathcal{E}_1^\mu(u) = \mathcal{E}(u) + (u,u)_\mu$ for $u \in \mathcal{F}^\mu$ where $(\cdot,\cdot)_\mu$ is the $L^2(\mu)$ inner product.

By [CF12, Theorem 3.3.8], the Dirichlet form $(\mathcal{F}^{r,E}, \mathcal{E}^{r,E})$ on $L^2(E;\mu_r|_E)$ is associated to $X^{r,E}$, the killed process of X^r upon leaving E. Similarly, the Dirichlet form $(\mathcal{F}^{\mu,E}, \mathcal{E}^{\mu,E})$ on $L^2(E;\mu)$ is associated to X^E, the killed process of X upon leaving E.

We will establish the Mosco convergence of $(\mathcal{F}^{r,E}, \mathcal{E}^{r,E})$ to $(\mathcal{F}^{\mu,E}, \mathcal{E}^{\mu,E})$ and the Mosco convergence of $(\mathcal{F}^r, \mathcal{E}^r)$ to $(\mathcal{F}^\mu, \mathcal{E})$ in Proposition 3.7. Then by [KS03, Theorem 2.4] we have Mosco convergence of Dirichlet forms in the sense of Definition 2.9 is equivalent to the L^2 convergence of the semigroups in the sense of Definition 2.8. It then implies the finite dimensional distribution vague convergence. For a proof of this fact, one can refer the discussion at the end of Chapter 6 of [Kol05].

Chapter 3

APPROXIMATION UNDER SYMMETRIZING MEASURES

In this chapter we provide a proof for Theorem 1.1. The main idea is to establish tightness and identify the limit using Dirichlet form theory. The convergence will be established first on a weaker topology called pseudo-path topology (Proposition 3.9), then we can strengthen the convergence to the Skorokhod topology.

3.1 Convergence in finite dimensional distributions

We start with some simple results.

Lemma 3.1. *For any unit vector $e \in \mathbb{R}^d$ we have $\int \langle e, x\rangle^2_{\mathbb{R}^d} \varrho(x)dx = M_\varrho < \infty$. Further more, we have for any $r > 0$, $x, z \in \mathbb{R}^d$*

$$\frac{{M_\varrho}^{-1}}{r^2} \int |\langle x, y-z\rangle|^2 \varrho_r(y-z)dy = |x|^2.$$

Proof. Let $e = \sum_{i=1}^d c_i e_i$, then $\sum_{i=1}^d c_i^2 = 1$, hence by orthogonality

$$\int \langle e, x\rangle^2_{\mathbb{R}^d} \varrho(x)dx = \sum_{i=1}^d c_i^2 \int \langle e_i, x\rangle^2_{\mathbb{R}^d} \varrho(x)dx = M_\varrho.$$

Using the identity above we get

$$\begin{aligned}\frac{M_\varrho^{-1}}{r^2} \int \langle x, y-z\rangle^2_{\mathbb{R}^d} \varrho_r(y-z)dy &= |x|^2 M_\varrho^{-1} \int \langle \frac{x}{|x|}, \frac{y-z}{r}\rangle^2_{\mathbb{R}^d} \varrho_r(y-z)dy \\ &= |x|^2 M_\varrho^{-1} \int \langle \frac{x}{|x|}, y\rangle^2_{\mathbb{R}^d} \varrho(y)dy \\ &= |x|^2,\end{aligned}$$

which prove the second statement.

□

Recall that $\mu_r(dx) = (\phi_r * \mu)(dx)$ and $Q_r(x, dy) = \varrho_r(y - x)dy$.

Lemma 3.2. *The following holds.*

(1) For $p \in [1, \infty)$ and $f \in L^p(\mu_r)$, $\phi_r * f \in L^p(\mu)$.

(2) For any function $f \in L^1(\mu_r)$, we have $\mu_r(f) := \int f d\mu_r = \int (\phi_r * f)(x)\mu(dx)$.

(3) For any continuous function f with compact support, $\lim_{r\to 0} \mu_r(f) = \mu(f)$.

(4) $f_r \in L^2(\mu_r)$ converge weakly to $f \in L^2(\mu)$ if and only if $\phi_r * f_r$ converges weakly to f in $L^2(\mu)$.

Proof. Notice that by Jensen's inequality

$$\int (\phi_r * f)^p d\mu \le \int \phi_r * (|f|^p) d\mu = \int |f|^p d\mu_r.$$

Hence (1) holds. By Fubini's Theorem

$$\begin{aligned}\int f(x)\mu_r(dx) &= \int f(x) \int \phi_r(y - x)\mu(dy)dx \\ &= \int\int f(x)\phi_r(y - x)dx\mu(dy) \\ &= \int (\phi_r * f)(y)\mu(dy).\end{aligned}$$

Then (2) and (3) holds. For (4), Let $g \in C_c^2(\mathbb{R}^d)$. Since g is continuous and has compact support, it is uniformly continuous, thus $\sup_{x,y:|x-y|\le r} |g(y) - g(x)| = o_r(1)$ which goes to 0 as $r \downarrow 0$. By Fubini theorem,

$$\begin{aligned}\int g f_r d\mu_r &= \int \phi_r * (g f_r) d\mu \\ &= \int (\phi_r * f_r)(x)(g(x) + o_r(1))\mu(dx) \\ &= \int (\phi_r * f_r) g d\mu + o_r(1) \int \phi_r * f_r d\mu\end{aligned}$$

Note that $\int \phi_r * f_r d\mu = \int f_r d\mu_r$ which is uniformly bounded in r under either weak convergence condition. Hence we see

$$\lim_{r\downarrow 0}\int g f_r d\mu_r = \lim_{r\downarrow 0}\int (\phi_r * f_r) g d\mu$$

provided that the limit exists, and the two weak convergences are equivalent by Proposition 2.6. $\square$

Next we derive some basic properties of bilinear forms $\mathcal{E}$ and $\mathcal{E}^r$.

Lemma 3.3. *For any probability density function ϕ (with respect to the Lebesgue measure on $\mathbb{R}^d$), we have $\mathcal{E}(\phi * u) \leq \mathcal{E}(u)$ for $u \in \mathcal{F}$ and $\mathcal{E}^r(\phi * u) \leq \mathcal{E}^r(u)$ for $u \in \mathcal{F}^r$.*

Proof. Since $\nabla(\phi * u) = \phi * \nabla u$ (by definition of weak derivative), by Jensen's inequality and shifting invariance of the Lebesgue measure we have

$$\begin{aligned}
\mathcal{E}(\phi * u) &= \frac{1}{2}\int |\nabla(\phi * u)(x)|^2 dx \\
&= \frac{1}{2}\int |(\phi * \nabla u)(x)|^2 dx \\
&\leq \frac{1}{2}\int\int \phi(z)|\nabla u(x-z)|^2 dz dx \\
&= \frac{1}{2}\int\int |\nabla u(x-z)|^2 dx \phi(z) dz \\
&= \frac{1}{2}\int |\nabla u(x)|^2 dx \\
&= \mathcal{E}(u).
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{E}^r(\phi * u) &= \frac{M_\varrho^{-1}}{2r^2}\int\int(\phi * u(x) - \phi * u(y))^2 \varrho_r(y-x)dydx \\
&= \frac{M_\varrho^{-1}}{2r^2}\int\int(\int(u(x-z) - u(y-z))\phi(z)dz)^2 \varrho_r(y-x)dydx \\
&\leq \frac{M_\varrho^{-1}}{2r^2}\int\int\int(u(x-z) - u(y-z))^2\phi(z)dz\varrho_r(y-x)dydx \\
&= \frac{M_\varrho^{-1}}{2r^2}\int\int\int(u(x-z) - u(y-z))^2\varrho_r(y-x)dydx\phi(z)dz \\
&= \frac{M_\varrho^{-1}}{2r^2}\int\int\int(u(x) - u(y))^2\varrho_r(y-x)dydx\phi(z)dz \\
&= \mathcal{E}^r(u)\int\phi(z)dz = \mathcal{E}^r(u).
\end{aligned}$$

where the forth equality uses the translation invariance of the Lebesgue measure. □

Proposition 3.4. *For any* $r > 0$*,* $u \in L^2_{loc}(\mathbb{R}^d)$ *such that* $\mathcal{E}^r(u)$ *is finite, we have* $\mathcal{E}^r(u) \leq \mathcal{E}^{r/k}(u)$ *for any* $k \in \mathbb{N}^*$*. Furthermore, if* $\mathcal{E}(u) < \infty$ *and* $u_s \in W^{1,2}_{loc}(\mathbb{R}^d)$ *such that* $\lim_{s\downarrow 0}\mathcal{E}(u_s - u) = 0$*, then* $\mathcal{E}^r(u) \leq \underline{\lim}_{s\downarrow 0}\mathcal{E}^s(u_s)$ *for any* $r > 0$*.*

Proof. Recall by definition

$$\mathcal{E}^r(u) = \frac{M_\varrho^{-1}}{2r^2}\int\int(u(x) - u(y))^2\varrho_r(x-y)dydx.$$

For any $x, y \in \mathbb{R}^d$ set $x_i = x + \frac{i}{k}(y-x)$ for $i = 0, 1, .., k$. Then

$$\begin{aligned}
\mathcal{E}^r(u) &= \frac{M_\varrho^{-1}}{2r^2}\int\int(u(x) - u(y))^2\varrho_r(y-x)dydx \\
&= \frac{M_\varrho^{-1}}{2r^2}\int\int(\sum_{i=0}^{k-1}u(x_i) - u(x_{i+1}))^2\varrho_r(y-x)dydx \\
&\leq \frac{M_\varrho^{-1}}{2r^2}\int\int k\sum_{i=0}^{k-1}(u(x_i) - u(x_{i+1}))^2\varrho_r(y-x)dydx.
\end{aligned}$$

Note that $dydx = k^d dx_i dx_{i+1}$ and $|y-x| = k|x_i - x_{i+1}|$ for any $i = 0, ..., k-1$. By change of variables and shifting invariance we have

$$
\begin{aligned}
\mathcal{E}^r(u) &\leq \frac{M_\varrho^{-1}}{2r^2}\int\int k^{d+1}\sum_{i=0}^{k-1}(u(x_i)-u(x_{i+1}))^2\varrho_r(k(x_i-y_{i+1}))dx_idx_{i+1}.\\
&= \frac{M_\varrho^{-1}}{2r^2}\int\int k^{d+2}(u(x_0)-u(x_1))^2\varrho_r(k(x_0-y_1))dx_0dx_1\\
&= \frac{M_\varrho^{-1}}{2(r/k)^2}\int\int(u(x)-u(y))^2\varrho_{r/k}(x-y)dxdy\\
&= \mathcal{E}^{r/k}(u).
\end{aligned}
$$

Finally, for any $r > s > 0$, we can find $k = k(s,r) \in \mathbb{N}^*$ such that $|r - ks| \leq s$. Let $c = c(s) = ks/r$. Then $|c-1| \leq s/r \to 0$ as $s \downarrow 0$. Denote $u^c(\cdot) = u(c\ \cdot)$, then it is straightforward to check $\mathcal{E}^{cr}(u) = c^{d-2}\mathcal{E}^r(u^c)$. We will show that $\lim_{s\downarrow 0}\mathcal{E}(u_s^c - u) = 0$. To see this, first note that $\mathcal{E}(u_s^c) = c^{d-2}\mathcal{E}(u_s) \to \mathcal{E}(u)$ and for any $v \in C_c^\infty(\mathbb{R}^d)$ we have

$$
\mathcal{E}(u_s^c, v) = c^{1-d}\int \nabla u_s(x)\nabla v(x/c)dx \to \mathcal{E}(u,v)
$$

as $s \downarrow 0$. A standard approximation argument of v to u in $\mathcal{E}$ shows $\mathcal{E}(u_s^c, u) \to \mathcal{E}(u,u)$ and hence

$$
\lim_{s\downarrow 0}\mathcal{E}(u_s^c - u) = \lim_{s\downarrow 0}[\mathcal{E}(u_s^c) - 2\mathcal{E}(u_s^c, u) + \mathcal{E}(u)] = 0.
$$

Now by a property of BL functions (cf. [DL45]) there exist constants C_s such that $u_s^c + C_s$ is L^2_{loc}-convergent to u. Taking a sequence of s such that $\mathcal{E}^s(u_s) \to \underline{\lim}_{s\downarrow 0}\mathcal{E}^s(u_s)$, and extracting a further subsequence (say along $I \subseteq (0,1)$) we have $u_s^c + C_s$ converges a.e. to u along I. By Fatou lemma,

$$
\begin{aligned}
\mathcal{E}^r(u) &\leq \underline{\lim}_{s\in I, s\downarrow 0}\ \mathcal{E}^r(u_s^c + C_s)\\
&= \underline{\lim}_{s\in I, s\downarrow 0}\ c^{d-2}\mathcal{E}^{cr}(u_s)\\
&= \underline{\lim}_{s\in I, s\downarrow 0}\ \mathcal{E}^{ks}(u_s)\\
&\leq \underline{\lim}_{s\in I, s\downarrow 0}\ \mathcal{E}^s(u_s)\\
&= \underline{\lim}_{s\downarrow 0}\mathcal{E}^s(u_s).
\end{aligned}
$$

This completes the proof.

$\square$

Proposition 3.5. *Suppose (A.1) holds. If* $\underline{\lim}_{r\downarrow 0}\mathcal{E}^r(u)<\infty$*, then we have* $u\in W^{1,2}_{loc}(\mathbb{R}^d)$*. In addition, for any* $u\in W^{1,2}_{loc}(\mathbb{R}^d)$*, we have* $\lim_{r\downarrow 0}\mathcal{E}^r(u)$ *exists and* $\mathcal{E}(u)=\lim_{r\downarrow 0}\mathcal{E}^r(u)$*.*

Proof. Since $\mathcal{E}^r(u)<\infty$ for some small $r>0$, we have $\int(u(x)-u(y))^2\varrho_r(y-x)dy<\infty$ for $x\in\mathbb{R}^d$ Lebesgue almost everywhere. By (A.1), u is integable on $x+B_\varrho$ for $x\in\mathbb{R}^d$ a.e. and hence $u\in L^2_{\text{loc}}(dx)$.

Now we show u has weak derivative $\nabla u\in(L^2(\mathbb{R}^d))^d$. For any $\varepsilon>0$ let $u_\varepsilon=\varphi_\varepsilon * u$ where φ_ε is any smooth mollifier with compact support. For each $R>0$, by Taylor approximation we know

$$\begin{aligned}&\frac{M_\varrho^{-1}}{2r^2}\int_{|x|<R}\int_{|y-x|<R}|u_\varepsilon(y)-u_\varepsilon(x)|^2\varrho_r(y-x)dydx\\ =&\frac{M_\varrho^{-1}}{2r^2}\int_{|x|<R}\int_{|y-x|<R}\langle\nabla u_\varepsilon(x),y-x\rangle^2_{\mathbb{R}^d}\varrho_r(y-x)dydx+o_r(1)\\ =&\sum_{i,j=1}^d\frac{1}{2}\int_{|x|<R}\partial_i u_\varepsilon(x)M_{ij}^{r,R}\partial_j u_\varepsilon(x)dx\end{aligned}$$

where $M_{ij}^{r,R}=\frac{M_\varrho^{-1}}{r^2}\int_{|y|<R}y_iy_j\varrho_r(y)dy$ and the Taylor error term is of $o_r(1)$ because $|x|<R$ and $|y|<2R$. We claim $M_{ij}^{r,R}\to M_\varrho\delta_{ij}$ as $r\downarrow 0$. This is because

$$\begin{aligned}\left|\frac{M_\varrho^{-1}}{r^2}\int_{|y|\geq R}y_iy_j\varrho_r(y)dy\right|&\leq\frac{M_\varrho^{-1}}{2r^2}\int_{|y|\geq R}(y_i^2+y_j^2)\varrho_r(y)dy\\ &\leq M_\varrho^{-1}\int_{|y|\geq R/r}|y|^2\varrho(y)dy.\end{aligned}$$

Together with Lemma 3.1 the claim is proved. Hence

$$\lim_{r\downarrow 0}\frac{M_\varrho^{-1}}{2r^2}\int_{|x|<R}\int_{|x-y|<R}|u_\varepsilon(y)-u_\varepsilon(x)|^2\varrho_r(y-x)dydx=\frac{1}{2}\int_{|x|<R}|\nabla u_\varepsilon(x)|^2dx.$$

On the other hand,

$$\lim_{r\downarrow 0}\frac{M_\varrho^{-1}}{2r^2}\int_{|x|<R}\int_{|y-x|<R}|u_\varepsilon(y)-u_\varepsilon(x)|^2\varrho_r(y-x)dydx\leq\underline{\lim}_{r\downarrow 0}\mathcal{E}^r(u_\varepsilon)\leq\underline{\lim}_{r\downarrow 0}\mathcal{E}^r(u)$$

where the last inequality we use Lemma 3.3. Hence we have

$$\mathcal{E}(u_\varepsilon) = \lim_{R\uparrow\infty} \frac{1}{2}\int_{|x|<R} |\nabla u_\varepsilon(x)|^2 dx \le \varliminf_{r\downarrow 0} \mathcal{E}^r(u)$$

holds for any $\varepsilon > 0$. Namely, $|\nabla u_\varepsilon|$ is bounded in $L^2(\mathbb{R}^d)$. Then we can find a subsequence $\varepsilon_n \downarrow 0$ such that for all coordinate $i \in \{1, 2, ..., d\}$ each derivate $\partial_i u_{\varepsilon_n}$ converges weakly in $L^2(\mathbb{R}^d)$ to some function $v_i \in L^2(\mathbb{R}^d)$. Then for any $f \in C_c^\infty(\mathbb{R}^d)$ we have

$$(f, v_i)_{L^2(\mathbb{R}^d)} = \lim_{n\to\infty} (f, \partial_i u_{\varepsilon_n})_{L^2(\mathbb{R}^d)} = \lim_{n\to\infty} (\partial_i f, u_{\varepsilon_n})_{L^2(\mathbb{R}^d)} = (\partial_i f, u)_{L^2(\mathbb{R}^d)}$$

which shows $\partial_i u = v_i \in L^2(\mathbb{R}^d)$. Then $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)$.

Finally, by Taylor approximation again

$$\frac{1}{2}\int_{|x|<R} |\nabla u(x)|^2 dx = \lim_{r\downarrow 0} \frac{M_\varrho^{-1}}{2r^2} \int_{|x|<R}\int_{|x-y|<R} |u(y) - u(x)|^2 \varrho_r(y-x) dy dx \le \varliminf_{r\downarrow 0} \mathcal{E}^r(u)$$

for any $R > 0$ hence $\mathcal{E}(u) \le \varliminf_{r\downarrow 0} \mathcal{E}^r(u)$. If furthermore u has compact support, then $u \in L^2(\mathbb{R}^d)$ and by Fourier transform we have

$$\begin{aligned}
\mathcal{E}^r(u) &= \frac{M_\varrho^{-1}}{r^2} \int\int |\hat{u}(z)|^2 (1 - \cos(\langle y, z\rangle_{\mathbb{R}^d})) \varrho_r(y) dy dz \\
&\le \frac{M_\varrho^{-1}}{r^2} \int\int |\hat{u}(z)|^2 \frac{1}{2} |y|^2 |z|^2 \varrho_r(y) dy dz \\
&= \frac{1}{2}\int |\hat{u}(z)|^2 |z|^2 dz = \mathcal{E}(u).
\end{aligned}$$

For general function $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)$, we may choose a sequence of functions $u_s \in C_c^\infty(\mathbb{R}^d)$ such that $\mathcal{E}(u_s - u) \to 0$. Then by Proposition 3.4 we have for any $r > 0$

$$\mathcal{E}^r(u) \le \varliminf_{s\downarrow 0} \mathcal{E}^s(u_s) \le \varliminf_{s\downarrow 0} \mathcal{E}(u_s) = \mathcal{E}(u).$$

Hence $\varlimsup_{r\downarrow 0} \mathcal{E}^r(u) \le \mathcal{E}(u) \le \varliminf_{r\downarrow 0} \mathcal{E}^r(u)$, which shows $\lim_{r\downarrow 0} \mathcal{E}(u) = \mathcal{E}(u)$ □

To prove Mosco convergence, we need the following lemma.

Lemma 3.6. *Let ϕ be a probability density function as in (A.1). Set $\phi_r(x) = \phi(x/r)/r^d$. Let $u_r \in \mathcal{F}^r$ and $\sup_r \mathcal{E}^r(u_r) < \infty$. Then $\sup_r \mathcal{E}(\phi_r * u_r) < \infty$.*

Proof. For any $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we have

$$\begin{aligned}\nabla(\phi_r * u)(x) &= \nabla \int \phi_r(z)u(x-z)dz \\ &= \nabla \int \phi_r(z)(u(x-z)-u(x))dz \\ &= \int \frac{1}{r^{d+1}}(\nabla\phi)(z/r)(u(x-z)-u(x))dz \\ &= \int \frac{1}{r}(\nabla\phi)(z)(u(x-rz)-u(x))dz\end{aligned}$$

hence by (A.1)

$$\begin{aligned}|\nabla(\phi_r * u)(x)| &\le \frac{1}{r}\int |\nabla\phi(z)||u(x-rz)-u(x)|dz \\ &\le \frac{C}{r}\int |u(x-rz)-u(x)|\varrho(z)dz \\ &= \frac{C}{r}Q_r|u-u(x)|(x)\end{aligned}$$

Then we have by Jensen inequality

$$\begin{aligned}\sup_r \mathcal{E}(\phi_r * u_r) &\le \sup_r \frac{C}{r^2}\int (Q_r|u_r-u_r(x)|)^2(x)dx \\ &\le \sup_r \frac{C}{r^2}\int Q_r(u_r-u_r(x))^2(x)dx \\ &= C\sup_r \mathcal{E}^r(u_r) < \infty.\end{aligned}$$

This finishes the proof. □

For domain $E \subseteq \mathbb{R}^d$, we set set $\mathcal{H}_n = L^2(E;\mu_n)$, $H = L^2(E;\mu)$, $\mathcal{C} = C_c^\infty(E)$ and Φ_n be the embedding map from $\mathcal{C}$ to $\mathcal{H}_n$. It is clear that $\mathcal{H}_n$ converges to $\mathcal{H}$ in the sense of Definition 2.1. Recall that $(\mathcal{F}^{r,E}, \mathcal{E}^{r,E})$ and $(\mathcal{F}^{\mu,E}, \mathcal{E}^{\mu,E})$ are defined in (2.1) and (2.2) respectively in Section 2.3. The processes $X^{r,E}$ and X^E are the killed processes associated to the Dirichlet forms. We are now ready to prove the Mosco convergence of the Dirichlet forms.

Proposition 3.7. *The Dirichlet form* $(\mathcal{F}^{r,E},\mathcal{E}^{r,E})$ *on* $L^2(E;\mu_r|_E)$ *(resp.* $(\mathcal{F}^r,\mathcal{E}^r)$ *on* $L^2(\mathbb{R}^d;\mu_r)$*) is Mosco convergent to* $(\mathcal{F}^{\mu,E},\mathcal{E}^{\mu,E})$ *on* $L^2(E;\mu|_E)$ *(resp.* $(\mathcal{F}^\mu,\mathcal{E})$ *on* $L^2(\mathbb{R}^d;\mu)$*). In particular,* $X^{r,E}$ *(resp.* X^r*) converges in finite dimensional distribution in vague topology to* X^E *(resp.* X*).*

Proof. The relation between Mosco convergence and finite dimensional distribution convergence has been discussed at the end of Section 2.3. We will focus on establishing Mosco convergences.

Take any sequence $r_n \downarrow 0$. We only need to show the convergence holds along any such sequence. With slight abuse of notation, replace the sub/sup-script r in $\mathcal{F}^r,\mathcal{E}^r,Q_r,\mu_r$, etc by n to denote the corresponding things when $r=r_n$. Let $u_n\in L^2(\mu_n)$ converge weakly to $u\in L^2(\mu)$ in the sense of Definition 2.3 and $\sup_n\mathcal{E}^n(u_n)<\infty$. Set $v_n=\phi_n * u_n$. By Lemma 3.2 we get $v_n\to u$ weakly in $L^2(\mu)$. By Lemma 3.3, Lemma 3.6 and choosing a subsequence (still denote as u_n, v_n), we may assume that $\overline{\lim}_n\,\mathcal{E}^n(v_n)\le\underline{\lim}_n\,\mathcal{E}^n(u_n)<\infty$ and $\sup_n\mathcal{E}(v_n)<\infty$. By weak convergence in $L^2(\mu)$ we know $\sup_n\|v_n\|_{L^2(\mu)}<\infty$ hence v_n is a bounded sequence in $\mathcal{F}^\mu$ with respect to $\mathcal{E}_1^\mu$. Then by the Banach-Saks theorem, we can find a subsequence of v_n (still denote as v_n) such that $w_n:=(\sum_{i=1}^n v_i)/n$ converge strongly to a quasi-continuous function $\widetilde{u}\in\mathcal{F}^\mu$ in $\mathcal{E}_1^\mu$. As w_n converges to u weakly in $L^2(\mu)$, we know $\widetilde{u}$ is a quasi-continuous μ-version of u by the uniqueness of weak limits. By Proposition 3.4, triangle inequalities and Lemma 3.3 we have for any $k\in\mathbb{N}$

$$\mathcal{E}^k(\widetilde{u})\le\underline{\lim}_{n\to\infty}\mathcal{E}^n(w_n)\le\underline{\lim}_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^n\sqrt{\mathcal{E}^n(v_i)}\right)^2\le\overline{\lim}_{n\to\infty}\mathcal{E}^n(v_n)\le\underline{\lim}_{n\to\infty}\mathcal{E}^n(u_n).$$

hence by Proposition 3.5 we know $\mathcal{E}(\widetilde{u})\le\underline{\lim}_{k\to\infty}\mathcal{E}^k(\widetilde{u})\le\underline{\lim}_{n\to\infty}\mathcal{E}^n(u_n)$. This establishes the first condition of Mosco convergence.

For the second condition of Mosco convergence, by [CF12, Theorem 5.1.6] we know $C_c^\infty(\mathbb{R}^d)$ is $\mathcal{E}_1^\mu$-dense in $\mathcal{F}^\mu$. For $u\in\mathcal{F}^\mu$ we can choose $u_n\in C_c^\infty(\mathbb{R}^d)$ $\mathcal{E}_1^\mu$-converges to u. By Proposition 2.7 we can find subsequence v_n of u_n that converges in the sense of Definition 2.2. Note that v_n also $\mathcal{E}$-converges to u. By Proposition 3.4 and Proposition 3.5 we have $\overline{\lim}_{n\to\infty}\mathcal{E}^{r_n}(v_n)\le\overline{\lim}_{n\to\infty}\lim_{k\to\infty}\mathcal{E}^{r_n/k}(v_n)=\overline{\lim}_{n\to\infty}\mathcal{E}(v_n)=\mathcal{E}(u)$.

For $(\mathcal{F}^{r,E}, \mathcal{E}^{r,E})$ the proof is similar. For the first condition, just note that if $u_n \in L^2(E;\mu_n|_E)$ converge weakly to $u \in L^2(E;\mu|_E)$, then by extending u_n, u to be 0 outside E we still have $v_n := \phi_n * u_n \in L^2(\mu)$ converge weakly to u in $L^2(\mu)$ and in the same way as the proof above we will be able to find the quasi-continuous version $\widetilde{u}$ of u in $\mathcal{F}^\mu$. Then

$$\mathcal{E}^{\mu,E}(\widetilde{u}) = \mathcal{E}(\widetilde{u}) \le \varliminf_{n\to\infty} \mathcal{E}^n(v_n) \le \varliminf_{n\to\infty} \mathcal{E}^n(u_n) = \varliminf_{n\to\infty} \mathcal{E}^{n,E}(u_n).$$

For the second condition, $C_c^\infty(E)$ is still $\mathcal{E}_1^\mu$-dense in $\mathcal{F}^\mu$ and by Proposition 2.7 we can find a sequence $u_n \in C_c^\infty(E) \subseteq L^2(E;\mu_n)$ converges in the sense of Definition 2.2 to $u \in L^2(E;\mu)$ and in $\mathcal{E}$. Hence

$$\varlimsup_{n\to\infty} \mathcal{E}^{n,E}(u_n) = \varlimsup_{n\to\infty} \mathcal{E}^n(u_n) \le \varlimsup_{n\to\infty} \lim_{k\to\infty} \mathcal{E}^k(u_n) = \varlimsup_{n\to\infty} \mathcal{E}(u_n) = \mathcal{E}(u) = \mathcal{E}^{\mu,E}(u).$$

The proof is completed. □

3.2 Convergence in Skorokhod topology

In this chapter, we will improve convergence in finite dimensional distribution to convergence in Skorokhod topology. We will use a localization argument and apply a result by Aldous [Ald89, Proposition 1.2].

For a path $\Gamma_\cdot$ taking values in $\mathbb{R}^d$ timed by $[0,\infty)$ or $\mathbb{N}$, denote $\tau_R(\Gamma) := \inf\{s \ge 0 : |\Gamma_s| \ge R\}$. We first prepare a lemma that will be used in Proposition 3.10. Recall that $\nu = f_0 \cdot \mu$ where $f_0 \in C_c(\mathbb{R}^d)$ is given in Theorem 1.1.

Lemma 3.8. *There exists a sequence of $R_k \uparrow \infty$ as $k \to \infty$ such that $\mathbb{P}_\nu\{\tau_{R_k}(X) = t\} = 0$ for any $t \in \mathbb{R}_+$.*

Proof. Let P_t be the transition semigroup of X. Then P_t is contractive on $L^1(\mathbb{R}^d;\mu)$ and

$$\mathbb{P}_\nu\{\tau_R(X) = t\} \le \mathbb{P}_\nu\{X_t \in \partial B(0,R)\} = \int P_t \mathbb{1}_{\partial B(0,R)} f_0 d\mu \le \|f_0\|_\infty \mu(\partial B(0,R)).$$

The existence of a sequence $R_k \uparrow \infty$ that $\mu(\partial B(0, R_k))$ follows from the countable additivity of a sigma finite measure. □

Take a sequence $R_k \uparrow \infty$ from Lemma 3.8. Let $R \in \{R_k\}_k$ be large enough such that $f_0 = 0$ outside $B(0, R)$. Throughout the rest of this chapter, we fix this R and define notations without emphasising the dependency on R. In particular, we denote $\tau(\Gamma) := \tau_R(\Gamma)$ for a path Γ.

We next show the tightness of the processes on a weak topology called pseudo-path topology. The topology is also called Meyer-Zheng topology as it was first introduced in the paper [MZ84]. Pseudo-path topology is the topology that paths are convergent in measure. Namely, a path ω^n converges in pseudo-path topology to ω if $\int_0^\infty 1 \wedge |\omega_t^n - \omega_t| e^{-t} dt \to 0$ as $n \to \infty$.

Let $\check{X}^r$ (resp. $\check{X}$) denote the killed process of X^r (resp. $\check{X}$) upon leaving $B(0, R)$.

Proposition 3.9. *The law of X^r (resp. $\check{X}^r$) with initial distribution $f_0 \cdot \mu_r$ converges in pseudo-path topology to the law of X (resp. $\check{X}$) with initial distribution $f_0 \cdot \mu$.*

Proof. Take any sequence $r_n \downarrow 0$, let $\mathbb{P}^n$ be the law of X^{r_n} with initial distribution $f_0 \cdot \mu_{r_n}$. First we show tightness of $\{\mathbb{P}^n\}$ in $\mathbf{D}([0, \infty), \mathbb{R}^d_\partial)$ in pseudo-path topology, where $\mathbb{R}^d_\partial = \mathbb{R}^d \cup \{\partial\}$, ∂ being the cemetery point. By a corollary in page 358 of [MZ84], $\{\mathbb{P}^n\}$ is tight if and only if for any relatively compact disjoint open sets E and F, $\sup_n \mathbb{E}^n[N_T^{E,F}] < \infty$, where $N^{E,F}(\omega)$ is the number of crossings from E to F by a path ω over $[0, T]$ and $\mathbb{E}^n$ is taking expectation using $\mathbb{P}^n$. By the main theorem of [CFS01] we have estimate

$$\mathbb{E}^n[N_T^{E,F}] \leq 2T \inf\{\mathcal{E}^n(u, u) : u \in \mathcal{F}^n, u = 0 \text{ on } F, u = 1 \text{ on } E\} \|f_0\|_\infty$$

Since $C_c^\infty \subseteq \mathcal{F}^n \cap \mathcal{F}$ and by Proposition 3.4 $\mathcal{E}^r(u) \leq \mathcal{E}^{r/2}(u) \to \mathcal{E}(u)$ as $r \downarrow 0$ for any $u \in C_c^\infty$, we have $\sup_n \mathbb{E}^n[N_T^{E,F}] < \infty$ and hence $\{\mathbb{P}^n\}$ is tight in $\mathbf{D}([0, \infty), \mathbb{R}^d_\partial)$ in pseudo-path topology.

By Proposition 3.7 we have finite dmiensional distribution vague convergence of the processes. The tightness in pseudo-path topology together with the identification of limit gives us the convergence in pseudo-path topology. Indeed, By tightness, for any subsequence $r_n \to 0$, there is a sub-subsequence r_{n_k} that $X^{r_{n_k}}$ converge in distribution to $\widetilde{X}$ (may depend on the sequence) in pseudo-path topology. By [MZ84, Theorem 5] there exists a subsequence of r_{n_k} and a Lebesgue null set $\mathcal{N} \subseteq [0, \infty)$ such that finite dimensional distribution of $X^{r_{n_k}}$ converges to $\widetilde{X}$ for time outside $\mathcal{N}$. By Mosco convergence we know $\widetilde{X}$ and X have the same finite dimensional distribution outside $\mathcal{N}$. Since finite dimensional distribution on a dense set of time indices completely determines the law of a càdlàg process, Hence X^r converges in law in pseudo-path topology to the time-changed Brownian motion X.

The proof for $\check{X}$ is simliar. ☐

Now we improve the convergence with respect to a stronger topology. For given $f_0 \in C_c(\mathbb{R}^d)$, recall that $\nu_r = f_0 \cdot \mu_r$ and $\nu = f_0 \cdot \mu$. Let Y^r (resp. Y) be the stopped processes of X^r (resp. X) upon leaving $B(0, R)$. More precisely, define $Y^r_\cdot := X^r_{\cdot \wedge \tau(X^r)}$ and $Y_\cdot := X_{\cdot \wedge \tau(X)}$.

Proposition 3.10. *The stopping time $\tau(X^r)$ under $\mathbb{P}_{\nu_r}$ converges in law to $\tau(X)$ under $\mathbb{P}_\nu$. The stopped process Y^r under $\mathbb{P}_{\nu_r}$ converges in law to Y under $\mathbb{P}_\nu$ on $\mathbf{D}([0, \infty); \mathbb{R}^d)$ equiped with Skorokhod topology.*

Proof. By Mosco convergence Proposition 3.7 for any bounded function g on $\mathbb{R}^d_\partial := \mathbb{R}^d \cup \{\partial\}$ we have

$$\lim_{r \downarrow 0} \mathbb{E}_{\nu_r}(g(\check{X}^r_t)) = \mathbb{E}_\nu(g(\check{X}_t)).$$

Taking $g = f\mathbb{1}_E$ for any $f \in C_b(\mathbb{R}^d)$ we get

$$\lim_{r \downarrow 0} \mathbb{E}_{\nu_r}[f(X^r_t), \tau(X^r) > t] = \mathbb{E}_\nu[f(X_t), \tau(X) > t].$$

In addition, we can take $f = 1_{\mathbb{R}^d}$ to see that $\mathbb{P}_{\nu_r}(\tau(X^r) > t) \to \mathbb{P}_\nu(\tau(X) > t)$. Hence $\tau(X^n)$ under $\mathbb{P}_{\nu_r}$ converges in distribution to $\tau(X)$ under $\mathbb{P}_\nu$.

The main difficulty is to show Y^r converges to Y in finite dimensional distribution. Once proved, noting that Y^r is an uniformly integrable martingale as $\sup_t |Y^r_t|$ is stochastically dominated by $R + r|\xi_1|$, by [Ald89, Proposition 1.2] we know the convergence holds in Skorokhod topology. Let r_n be any sequence of real numbers that goes down to 0. For the remaining of the proof, we denote any sub/sup script r_n by n for simplicity of notation (e.g. denote X^{r_n} by X^n etc.) and let $\tau(X^n)$ (resp. $\tau(X)$) be the exit time of X^n (resp. X) upon leaving $B(0,R)$. To show finite dimensional distribution of Y^n converging to that of Y, given positive integer k, for $0 < t_1 < t_2 < \cdots < t_k$, we need to show that

$$\lim_{n\to\infty} \mathbb{E}_{\nu_n}[\prod_{i=1}^{k} f_i(Y^n_{t_i})] = \mathbb{E}_{\nu}[\prod_{i=1}^{k} f_i(Y_{t_i})].$$

Notice that

$$\begin{aligned}\mathbb{E}_{\nu_n}[\prod_{i=1}^{k} f_i(Y^n_{t_i})] &= \mathbb{E}_{\nu_n}[\prod_{i=1}^{k} f_i(X^n_{\tau(X^n)}); \tau(X^n) \le t_1] \\ &\quad + \sum_{j=1}^{k-1} \mathbb{E}_{\nu_n}[\prod_{i=1}^{j} f_i(X^n_{t_i}) \prod_{i=j+1}^{k} f_i(X^n_{\tau(X^n)}); t_j < \tau(X^n) \le t_{j+1}] \qquad (3.1) \\ &\quad + \mathbb{E}_{\nu_n}[\prod_{i=1}^{k} f_i(X^n_{t_i}); t_k < \tau(X^n)].\end{aligned}$$

The last term $\mathbb{E}_{\nu_n}[\prod_{i=1}^{k} f_i(X^n_{t_i}); t_k < \tau(X^n)]$ converges to $\mathbb{E}_{\nu}[\prod_{i=1}^{k} f_i(X^n_{t_i}); t_k < \tau(X)]$ because of the finite dimensional distribution convergence of $\check{X}^r$ to $\check{X}$. To prove the first two terms converges to

$$\mathbb{E}_{\nu}[\prod_{i=1}^{k} f_i(X_{\tau(X)}); \tau(X) \le t_1] + \sum_{j=1}^{k-1} \mathbb{E}_{\nu}[\prod_{i=1}^{j} f_i(X_{t_i}) \prod_{i=j+1}^{k} f_i(X_{\tau(X)}); t_j < \tau(X) \le t_{j+1}]$$

we construct a probability space such that $(X, X^n_{\tau(X^n)}, \tau(X^n)) \to (\widetilde{X}, \widetilde{\chi}, \widetilde{\tau})$ almost surely and show that $(\widetilde{X}, \widetilde{\chi}, \widetilde{\tau})$ has the same distribution as $(X, X_{\tau(X)}, \tau(X))$. We divide the proof into several steps.

Step 1: Construct probability spaces. Because $X^n, \check{X}^n, \tau(X^n)$ converge in distribution to $X, \check{X}, \tau(X)$ respectively and $X^n_{\tau(X^n)}$ is stochastically domainated by $R + |\xi_1|$ (if $r_n \le 1$), the law of $(X^n, \check{X}^n, \tau(X^n), X^n_{\tau(X^n)})$ is tight. Let $(Z, \check{Z}, \widetilde{\tau}, \widetilde{\chi})$ be any of its subsequencial limit. We want to show that $(Z, \check{Z}, \widetilde{\tau}, \widetilde{\chi})$ has the same law as $(X, \check{X}, \tau, X_\tau)$. By applying

Dudley's extension of Skorokhod's representation theorem, we may assume that along that subsequence $(X^n, \check{X}^n, \tau(X^n), X^n_{\tau(X^n)})$ converges to $(Z, \check{Z}, \widetilde{\tau}, \widetilde{\chi})$ almost surely, where the first two components are in pseudo-path topology whereas the last two components are in $\mathbb{R}$ and $\mathbb{R}^d$ respectively. Let $\mathbb{P}$ denote the probability measure for this almost-surely convergence.

Step 2: Identify $\check{Z}$. Note that

$$\int_0^\infty (1 \wedge |\check{X}^n_t - \check{Z}_t|)e^{-t}dt = \int_0^{\tau(X^n)-} (1 \wedge |X^n_t - \check{Z}_t|)e^{-t}dt + \int_{\tau(X^n)}^\infty (1 \wedge |\partial - \check{Z}_t|)e^{-t}dt$$

where we extend the Euclidian distance to the cemetery point by $|\partial - x| := \infty$ if $x \neq \partial$ else 0. Because almost surely the left hand side converges to 0 and $\tau(X^n) \to \widetilde{\tau}$, $X^n \to Z$ in pseudo-path, $\check{Z}$ is right continuous, we get $\check{Z}$ has the following representation

$$\check{Z}_t = \begin{cases} Z_t & \text{if } t < \widetilde{\tau} \\ \partial & \text{if } t \geq \widetilde{\tau} \end{cases}.$$

Step 3: Identify $\widetilde{\tau} = \tau(Z)$. To prove that, we need to show $Z_t \in B(0,R)$ for any $t < \widetilde{\tau}$ and $Z_{\widetilde{\tau}} \in \partial B(0,R)$. Because $\check{Z}$ being the limit of $\check{X}^n$ has the same distribution as $\check{X}$, we know $\check{Z} \in B(0,R) \cup \{\partial\}$. Also note that $Z_t \in \mathbb{R}^d$ for all t and $Z_t = \check{Z}_t$ for $t < \widetilde{\tau}$, hence we get $Z_t \in B(0,R)$ for any $t < \widetilde{\tau}$. Again from the representation of $\check{Z}$ we see $\widetilde{\tau}$ is exactly the exit time of $\check{Z}$ upon leaving $B(0,R)$, i.e., $\widetilde{\tau} = \tau(\check{Z})$. Together with the fact that $Z_t = \check{Z}_t$ for $t < \widetilde{\tau}$ we have

$$\mathbb{P}\{Z_{\widetilde{\tau}-} \in \partial B(0,R)\} = \mathbb{P}\{\check{Z}_{\tau(\check{Z})-} \in \partial B(0,R)\} = \mathbb{P}\{\check{X}_{\tau(\check{X})-} \in \partial B(0,R)\} = 1.$$

where the last equality follows from that fact that $\widetilde{\tau} = \sup\{t \in \mathbb{Q}_+ : \check{Z}_t \in B(0,R)\}$ is a measurable function of the path $\check{Z}$ and $\check{Z}$ has the same distribution as the killed time-changed Brownian motion $\check{X}$. Then we get $Z_{\widetilde{\tau}-} \in \partial B(0,R)$ almost surely. By right-continuity of Z we get $Z_{\widetilde{\tau}} = Z_{\widetilde{\tau}-} \in \partial B(0,R)$. We complete the proof of $\widetilde{\tau} = \tau(Z)$. This also identifies that $\check{Z}$ is indeed the killed process of Z upon leaving $B(0,R)$.

Step 4: Identify $\widetilde{\chi} = Z_{\tau(Z)}$. For any $\sigma > 0$ we introduce the weighted moving average processes

$$\bar{X}^{n,\sigma}_t := \frac{1}{\Sigma} \int_{(t-\sigma)\vee 0}^t X^n_s e^{-s} ds, \qquad \bar{X}^{\infty,\sigma}_t := \frac{1}{\Sigma} \int_{(t-\sigma)\vee 0}^t Z_s e^{-s} ds$$

where $\Sigma = \int_{(t-\sigma)\vee 0}^{t} e^{-s} ds$. Note that by the pseudo-path convergence

$$|\bar{X}_t^{n,\sigma} - \bar{X}_t^{\infty,\sigma}| \leq \frac{1}{\Sigma} \int_{(t-\sigma)\vee 0}^{t} |X_s^n - Z_s| e^{-s} ds \xrightarrow{n\to\infty} 0$$

uniformly in t for which $\{X_s^n, Z_s\}_{s<t}$ are bounded. Hence we see for any $\varepsilon, \sigma > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\{\sup_{0<t<\tau(X^n)} |\bar{X}_t^{n,\sigma} - \bar{X}_t^{\infty,\sigma}| \leq \varepsilon, \tau(X^n) < \tau_{2R}(Z)\} = 1$$

where $\tau_{2R}(Z)$ is the first exit time of Z of $B(0, 2R)$. Note that $\lim_{n\to\infty} \tau(X^n) = \tau(Z) < \tau_{2R}(Z)$ so the probability can be made convergent to 1. Next we set

$$T_{\varepsilon,\sigma}^n := \inf\{t < \tau(X^n) : |\bar{X}_t^{n,\sigma}| > R - \varepsilon, |X_t^n - \bar{X}_t^{n,\sigma}| < \varepsilon\}.$$

Clearly $T_{\varepsilon,\sigma}^n$ are stopping times for the right continuous filtration generated by X^n and we claim for any $\varepsilon > 0$

$$\varliminf_{L\uparrow\infty} \varliminf_{\sigma\downarrow 0} \varliminf_{n\to\infty} \mathbb{P}\{T_{\varepsilon,\sigma}^n < L\} = 1. \tag{3.2}$$

To see that, first notice that $\lim_{\sigma\downarrow 0} \lim_{n\to\infty} \bar{X}_t^{n,\sigma} = \lim_{\sigma\downarrow 0} \bar{X}_t^{\infty,\sigma} = Z_t$ for any $t < \tau(Z) = \lim_{n\to\infty} \tau(X^n)$ and there exists $t < \tau(Z)$ such that $|Z_t| > R - \varepsilon$, hence

$$\varliminf_{\sigma\downarrow 0} \varliminf_{n\to\infty} \mathbb{P}\{\sup_{t<\tau(X^n)} |\bar{X}_t^{n,\sigma}| > R - \varepsilon\} = 1. \tag{3.3}$$

Next by contradiction assume

$$\varlimsup_{L\uparrow\infty} \varlimsup_{\sigma\downarrow 0} \varlimsup_{n\to\infty} \mathbb{P}\{\inf_{t<\tau(X^n)\wedge L} |X_t^n - \bar{X}_t^{n,\sigma}| \geq \varepsilon\} > 0, \tag{3.4}$$

then with an uniform strict positive probability that for any sufficient large $L > 0$ there exists a sequence of $\sigma = \sigma_k \downarrow 0$, $\varlimsup_{n\to\infty} \inf_{t<\tau(X^n)\wedge L} |X_t^n - \bar{X}_t^{n,\sigma}| \geq \varepsilon$. Then together on the event

$$A(n, \sigma, L, \varepsilon) := \{\sup_{s,s'\in[0,L], |s-s'|\leq\sigma} |Z_s - Z_{s'}| \leq \varepsilon/3\} \cap \varliminf_{n\to\infty} \{\sup_{0<t<\tau(X^n)} |\bar{X}_t^{\infty,\sigma} - \bar{X}_t^{n,\sigma}| \leq \varepsilon/3\}$$

we have $|Z_t - \bar{X}_t^{\infty,\sigma}| \leq \varepsilon/3$ and

$$|X_t^n - Z_t| \geq |X_t^n - \bar{X}_t^{n,\sigma}| - |\bar{X}_t^{\infty,\sigma} - \bar{X}_t^{n,\sigma}| - |Z_t - \bar{X}_t^{\infty,\sigma}| \geq \varepsilon/3$$

for any $t < \tau(X^n) \wedge L$, hence

$$\overline{\lim_{n\to\infty}} \int_0^{\tau(X^n)\wedge L} |X_t^n - Z_t| e^{-t} dt \geq (1 - e^{-\tau(X)\wedge L})\varepsilon/3 > 0. \tag{3.5}$$

But for any $L_k > 0$, we may choose $\sigma_k > 0$ small enough such that $\mathbb{P}[A(n, \sigma_k, L_k, \varepsilon)] \uparrow 1$. Then we get with positive probability (3.5) holds, which contradicts to the fact that X^n pseudo-path converges to Z almost surely. Hence assumption (3.4) is false, i.e.

$$\overline{\lim_{\sigma\downarrow 0}}\, \overline{\lim_{n\to\infty}}\, \mathbb{P}\{\inf_{t<\tau(X^n)} |X_t^n - \bar{X}_t^{n,\sigma}| < \varepsilon\} = 1.$$

This together with (3.3) proves (3.2).

Now we can choose a sequence $\varepsilon_k \downarrow 0, L_k \uparrow \infty, \sigma_k \downarrow 0, n_k \uparrow \infty$ such that $\mathbb{P}\{T_k > L_k\} = o_k(1)$ where $T_k := T^{n_k}_{\varepsilon_k,\sigma_k}$ and $\mathbb{P}[A_k] = 1 - o_k(1)$ where $A_k = A(n_k, \sigma_k, L_k, \varepsilon_k)$. Let $\delta_k = \delta_k(\varepsilon_k) \downarrow 0$ be such that

$$\inf_{R-\varepsilon_k<|x|<R} H_{n_k}(x, B(x, \delta_k)) = 1 - o_k(1) \tag{3.6}$$

where $H_n(x, dy) = \mathbb{P}_x\{W^{r_n}_{\tau(W^{r_n})} \in dy\}$ is the hitting distribution of the random walk W^{r_n} starting from x upon leaving $B(0, R)$. The proof of (3.6) is postponed to Proposition 3.11. Then by the strong Markov property

$$\begin{aligned}
\mathbb{P}\{|X^{n_k}_{\tau(X^{n_k})} - X^{n_k}_{T_k}| < \delta_k,\ T_k < L_k\} &= \mathbb{E}[H_{n_k}(X^{n_k}_{T_k}, B(X^{n_k}_{T_k}, \delta_k)), T_k < L_k] \\
&\geq \inf_{R-\varepsilon_k<|x|<R} H_{n_k}(x, B(x, \delta_k)) \cdot \mathbb{P}\{T_k < L_k\} \\
&= 1 - o_k(1).
\end{aligned}$$

Moreover, on the event $A_k \cap \{T_k < L_k\}$ we have

$$|X^{n_k}_{T_k} - Z_{T_k}| \leq |Z_{T_k} - \bar{X}^{\infty,\sigma_k}_{T_k}| + |\bar{X}^{\infty,\sigma_k}_{T_k} - \bar{X}^{n_k,\sigma_k}_{T_k}| + |\bar{X}^{n_k,\sigma_k}_{T_k} - X^{n_k}_{T_k}| \leq \varepsilon_k/3 + \varepsilon_k/3 + \varepsilon_k$$

hence $|X^{n_k}_{\tau(X^{n_k})} - Z_{T_k}| \leq 4\varepsilon_k/3 + \delta_k$ with probability $1 - o_k(1)$. We finally claim Z_{T_k} converges to $Z_{\tau(Z)}$ in probability, then we can conclude that $\widetilde{\chi} = \lim_{k\to\infty} X^{n_k}_{\tau(X^{n_k})} = Z_{\tau(Z)}$. To see it, just notice that on $A_k \cap \{T_k < L_k\}$ we have $|\bar{X}^{n_k,\sigma_k}_t - Z_t| \leq 2\varepsilon_k/3$ hence $T_k > \tau_{R-2\varepsilon_k}(Z)$.

Since $T_k < \tau(X^n) \to \tau(Z)$ and $\tau_{R-2\varepsilon_k}(Z) \uparrow \tau(Z)$, we see $T_k \to \tau(Z)$ and hence $Z_{T_k} \to Z_{\tau(Z)}$ by continuity.

Step 5: Identify the law. Combining that $\widetilde{\tau} = \tau(Z)$, $\widetilde{\chi} = Z_{\tau(Z)}$ and $\check{Z}$ is the killed process of Z upon leaving $B(0, R)$, we see that along a subsequence

$$(X^n, \check{X}^n, \tau(X^n), X^n_{\tau(X^n)}) \to (Z, \check{Z}, \tau(Z), Z_{\tau(Z)})$$

almost surely. Since $\check{Z}, \tau(Z), Z_{\tau(Z)}$ are measurable functions of the path Z and Z has the same distribution as X, we see that along any subsequence the tuple $\mathbb{X}^n := (X^n, \tau(X^n), X^n_{\tau(X^n)})$ converges in law to the tuple $\mathbb{X} := (X, \tau(X), X_{\tau(X)})$, where the first two components are in pseudo-path topology whereas the last two components are in $\mathbb{R}$ and $\mathbb{R}^d$ respectively. The proof for $\mathbb{X}^n$ converges in law to $\mathbb{X}$ is completed.

Finally, for any subsequence $I \subseteq \mathbb{N}$, by [MZ84, Theorem 5] there exists a further subsequence $I' \subseteq I$ and a dense subset $T = T(I) \subseteq \mathbb{R}_+$ such that $\mathbb{X}^n_T := (\{X^n_t\}_{t \in T}, \tau(X^n), X^n_{\tau(X^n)})$ converges in law to $\mathbb{X}_T := (\{X_t\}_{t \in T}, \tau(X), X_{\tau(X^n)})$. By Lemma 3.8 the discontinuity of function $(x_1, ..., x_k, s) \mapsto \mathbb{1}_{(a,b]}(s) \prod_{i=1}^k f_i(x_i)$ for any $b > a > 0$ has zero probability under $\mathbb{P}_\nu$, hence the last two terms in Equation (3.1) converges and we get $\mathbb{E}_{\nu_n}[\prod_{i=1}^k f_i(Y^n_{t_i})]$ converges to $\mathbb{E}_\nu[\prod_{i=1}^k f_i(Y_{t_i})]$ along I'. Because finite dimensional distribution on a dense set of time indices completely determines the law of a càdlàg process, by [Ald89, Proposition 1.2] we know Y^n converges in law in Skorokhod topology to Y along I'. Because I and r_n are arbitray, we get the desired convergence of Y^r to Y. □

We now give the proof of (3.6) used in the proof of Proposition 3.10. Recall that W^r are discrete-time simple random walks defined in Chapter 1 and $\tau(\Gamma)$ is the exit time of path Γ leaving $B(0, R)$.

Proposition 3.11. *The hitting distribution* $H_n(x, dy) = \mathbb{P}_x\{W^{r_n}_{\tau(W^{r_n})} \in dy\}$ *of random walk* W^{r_n} *converges weakly to the hitting distribution of the Brownian motion* $H(x, dy) = \mathbb{P}_x\{W_{\tau(W)} \in dy\}$. *Furthermore, for any* $\varepsilon_k \downarrow 0$ *we can find* $\delta_k \downarrow 0$ *and* $n_k \uparrow \infty$ *such that* $\sup_{R-\varepsilon_k<|x|<R} H_{n_k}(x, B(x, \delta_k)) = 1 - o_k(1)$.

Proof. We first show that for any starting point the random walk W^r converges in distribution in Skorokhod topology (as the limit W is continuous, the topology is the same as the uniform topology on any finite time interval) to the Brownian motion W. This is the classical Donsker's invariance principle in high dimensions, but we did not find existing references, so we give the proof here. Without loss of generality we may assume the starting point is the origin. Let $W^{r,i}_t$ be the projection of W^r to the first coordinate. Namely $W^{r,i}_n = \langle e_i, W^r_n\rangle_{\mathbb{R}^d}$. By Lemma 3.1 we see $\mathrm{Var}(W^{r,i}_1) = r^2 M_\varrho$. Hence by the classical Donsker's invariance principle $S^{r,i}_t : t \mapsto W^{r,i}_{\lfloor t(M_\varrho^{-1})/r^2\rfloor}$ converges in distribution (in uniformly topology on any finite time interval) to the standard Brownian motion B^i. Hence the law of $(S^{r,1}, ..., S^{r,d})$ is tight. Assume $(B^1, ..., B^d)$ is any of its sub-sequential limit. By Skorokhod's representation theorem we may put $\{S^{r,i}\}_{i=1}^d$ and $\{B^i\}_{i=1}^d$ in the same probability space such that $\{S^{r,i}\}_{i=1}^d$ converges to $\{B^i\}_{i=1}^d$ in probability. We want to show that $\{B^i\}_{i=1}^d$ are independent. For two square-integrable martingales M^1 and M^2, let $\langle M^1, M^2\rangle$ be the covariation process of M^1, M^2 and $\langle M^1\rangle := \langle M^1, M^1\rangle$ be the predictable quadratic variation of M. First notice that $S^{r,i} + S^{r,j}$ is the inner product of W^r and $e_i + e_j$. Then it is straightforward to calculate $\langle S^{r,i} + S^{r,j}\rangle_t = \|e_i + e_j\|^2 r^2 M_\varrho \lfloor \frac{t}{r^2 M_\varrho}\rfloor$ (which increases to $2t$ as $r \downarrow 0$). Then note that for any $t > 0$

$$\lim_{r\downarrow 0} \mathbb{E}[\langle (B^i + B^j) - (S^{r,i} + S^{r,j})\rangle_t] = \lim_{r\downarrow 0} \mathbb{E}[((B^i_t + B^j_t) - (S^{r,i}_t + S^{r,j}_t))^2] = 0.$$

This implies $\langle S^{r,i} + S^{r,j}\rangle_t$ converges to $\langle B^i + B^j\rangle_t$ in L^1 (hence in probability) because of the following inequality (see, e.g., [Kal97, Proposition 23.1])

$$\mathbb{E}[|\langle M^1\rangle_t - \langle M^2\rangle_t|] = \mathbb{E}[|\langle M^1 + M^2, M^1 - M^2\rangle_t|] \le \sqrt{\mathbb{E}\langle M^1 + M^2\rangle_t \cdot \mathbb{E}\langle M^1 - M^2\rangle_t}.$$

Passing to the limit as $r \downarrow 0$ we have $\langle B^i + B^j\rangle_t = 2t$. Similarly we can get $\langle B^i - B^j\rangle_t = 2t$. Then the covariation process $\langle B^i, B^j\rangle_t = 0$ by the polarization identity. Hence by the characterization of d-dimensional Brownian motion (see, e.g., [Kal97, Theorem 16.3]) we get $\{B^i\}_{i=1}^d$ is the d-dimensional Brownian motion and $\{S^{r,i}\}_{i=1}^d$ converges in probability (uniformly on any finite time interval) to the d-dimensional Brownian motion.

As the hitting distributions of X^r and $S^r := (S^{r,1}, ..., S^{r,d})$ are exactly the same by their definition, we have $H_n(x, dy) = \mathbb{P}_x\{S^{r_n}_{\tau(S^{r_n})} \in dy\}$. We now show that $\mathbb{P}_x\{S^r_{\tau(S^r)} \in dy\}$ converges weakly to $H(x, dy)$ as $r \downarrow 0$ for each $x \in B(0, R)$. By shifting we only need to show $\mathbb{P}_0\{S^r_{\tau_x(S^r)} \in dy\}$ converges weakly to $H(x, dy)$ where τ_x is the hitting time of $B(-x, R)$. First note that because almost surely S^r converges uniformly on any bounded time interval, $\tau_x(S^r)$ converges to $\tau_x(W)$ on

$$A := \{W_s \in B(-x, R)\ \forall s < \tau_x\} = \{\forall \varepsilon > 0,\ \exists s \in [\tau_x, \tau_x + \varepsilon),\ |W_s + x| > R\}$$

the event that W never touches $\partial B(-x, R)$ before exiting $B(-x, R)$. Since every point on $\partial B(-x, R)$ is regular for the standard Brownian motion, the set A has probability 1 thus almost surely $\tau_x(S^r)$ converges to $\tau_x(W)$. This together with the fact that S^r converges to W in Skorokhod topology in probability gives us that $S^r_{\tau_x(S^r)} \to W_{\tau_x(W)}$ in probability by the continuous mapping theorem. Hence $H_n(x, dy)$ converges weakly to $H(x, dy)$ for each $x \in \mathbb{R}^d$. In particular, for $f \in C_c(\mathbb{R}^d)$ we have $H_n f \to Hf$ point-wisely. In addition $H_n f$ are harmonic (with respect to X^n) inside $B(0, R)$ hence by Proposition 4.7 proved in the next chapter, for all n sufficient small we have

$$\sup_{x \in B(0,R)} |H_n f(x) - Hf(x)| \le \sup_{y \in E_k} |H_n f(y) - Hf(y)| + o_k(1)$$

where E_k is a (finite) $1/k$-net of $B(0, R)$. Letting $n \to \infty$ and then $k \to \infty$ we see $H_n f$ uniformly converges to Hf inside $B(0, R)$.

Finally, as $H(r \cdot e_1, dy)$ converges weakly to Dirac measure at $R \cdot e_1$ as $r \to R$, for each $\varepsilon_k > 0$ we can choose $l_k \downarrow 0$ and nonnegative functions $f_k \in C_c(\mathbb{R}^d)$ with $f_k = 1$ in $B(R \cdot e_1, l_k)$ and $f_k = 0$ outside $B(R \cdot e_1, l_k)$ such that $Hf_k((R - \varepsilon_k) \cdot e_1) \uparrow 1$ as $k \uparrow \infty$. Then for $\varepsilon_k > 0$ we can take $\delta_k = 2l_{R-\varepsilon_k} + \varepsilon_k$, and using rotation invariance to get

$$\inf_{R-\varepsilon_k < |x| < R} H(x, B(x, \delta_k)) \ge Hf_k((R - \varepsilon_k) \cdot e_1) = 1 - o_k(1)$$

In addition for each k we may choose n_k such that $\sup_{R-\varepsilon_k<|x|<R} |Hf_k(x) - H_{n_k} f_k(x)| = o_k(1)$. Hence

$$\inf_{R-\varepsilon_k < |x| < R} H_{n_k}(x, B(x, \delta_k)) \ge \inf_{R-\varepsilon_k < |x| < R} H(x, B(x, \delta_k)) - o_k(1) = 1 - o_k(1).$$

We complete the proof. $\square$

Finally we prove Theorem 1.1.

Proof of Theorem 1.1. We only need to show that for each fixed $T > 0$ and each closed set F in $\mathbf{D}[0,T]$ (with Skorokhod topology), we have

$$\varlimsup_{r \downarrow 0} \mathbb{P}\left(X^r_{[0,T]} \in F\right) \leq \mathbb{P}\left(X_{[0,T]} \in F\right).$$

For this propose, for any $\varepsilon > 0$ choose $R > 0$ large enough that $\mathbb{P}\left(\tau_R(X) \leq T\right) \leq \varepsilon$. Let Y^r be the stopped process of X^r upon leaving $B(0,R)$ and $\tau_R(X^r)$ be the stopping time. Then

$$\begin{aligned}\mathbb{P}\left(X^r_{[0,T]} \in F\right) &\leq \mathbb{P}\left(Y^r_{[0,T]} \in F, \tau_R(X^r) > T\right) + \mathbb{P}\left(\tau_R(X^r) \leq T\right) \\ &\leq \mathbb{P}\left(Y^r_{[0,T]} \in F \cap \{\omega \in \mathbf{D}[0,T] : \sup_{t \in [0,T]} |\omega_t| \leq R\}\right) + \mathbb{P}\left(\tau_R(X^r) \leq T\right)\end{aligned}$$

By Proposition 3.10 we know $\tau_R(X^r)$ converges in distribution to $\tau_R(X)$. Hence

$$\begin{aligned}\varlimsup_{r \downarrow 0} \mathbb{P}\left(X^r_{[0,T]} \in F\right) &\leq \mathbb{P}\left(Y_{[0,T]} \in F \cap \{\omega \in \mathbf{D}[0,T] : \sup_{t \in [0,T]} |\omega_t| \leq R\}\right) + \mathbb{P}\left(\tau_R(X) \leq T\right) \\ &= \mathbb{P}\left(X_{[0,T]} \in F \cap \{\omega \in \mathbf{D}[0,T] : \sup_{t \in [0,T]} |\omega_t| \leq R\}\right) + \varepsilon\end{aligned}$$

Finally let $R \to \infty$ and then $\varepsilon \to 0$ we get

$$\varlimsup_{r \downarrow 0} \mathbb{P}\left(X^r_{[0,T]} \in F\right) \leq \mathbb{P}\left(X_{[0,T]} \in F\right).$$

This shows that X^r converge to X in law on $\mathbf{D}([0,\infty); \mathbb{R}^d)$ equiped with Skorokhod topology.

$\square$

Chapter 4

APPROXIMATION UNDER INDIVIDUAL STARTING POINTS

In this chapter we assume the measure μ has full quasi-support on $\mathbb{R}^d$ and satisfies the condition (A.3). Namely, for any $R > 0$, there exists $\beta_\mu = \beta_\mu(R) > d-2$ such that for any $x \in B(0,R)$ and $r \in (0,1)$ we have $\mu(B(x,r)) \le C_R r^{\beta_\mu}$. This condition ensures that the measure μ is a smooth measure (Proposition 4.18). In addition,

$$\lim_{r \downarrow 0} r^{\beta_\mu - d + 2} \log\left(\int_{B(0,2R)} \frac{dx}{\int \phi((x-y)/r)\mu(dy)} \right) = 0. \tag{4.1}$$

This condition is used in the proof of Proposition 4.16. We also assume ϱ satisfies (A.1) and (A.2).

For pointwise starting convergence, we need some regularities or estimates of the analytic counterparts (Green's function, harmonic functions, transition semigroup) of the continuous-time random walk processes with or without time change.

Recall that $\widetilde{W}^n$ is the continuous time simple random walk (without time change) with scaling $r = r_n$. As under $\mathbb{P}_x$ the process $\widetilde{W}^n$ starting at the single point will have positive probability staying at the starting point for some time, the transition group of $\widetilde{W}^n$ is not absolutely continuous with respect to the Lebesgue measure, we need to work around this diffiucty.

Let $\widetilde{W}^{n,D}$ be the process $\widetilde{W}^n$ killed upon leaving D, and $q_k^{n,D}(x,y)$ is the transition density of W_k^n killed upon leaving D. Note that $q_k^{n,D}(x,y)$ is symmetric in x, y. When one of x is the origin, we denote the density by $q_k^{n,D}(y)$

Let $g^{n,D}(x,y)$ be the occupation time density of $\widetilde{W}^n$ from the second jump till leaving the domain $D \subseteq \mathbb{R}^d$, i.e., for any $f \in C_b(\mathbb{R}^d)$ we have

$$\begin{aligned}\int f(y) g^{n,D}(x,y)dy &:= \mathbb{E}_x\left[\int_{\eta_0^n+\eta_1^n}^{\tau_D(\widetilde{W}^n)\vee(\eta_0^n+\eta_1^n)} f(\widetilde{W}_t^n)dt; W_1^n \in D\right] \\ &= \sum_{k=2}^{\infty}\int_0^{\infty} e^{-M_\varrho^{-1}r_n^{-2}t}\frac{(M_\varrho^{-1}r_n^{-2}t)^{k-2}}{(k-2)!}dt\ \mathbb{E}_x[f(W_k^n); k<\tau_D(W_n)] \\ &= \sum_{k=2}^{\infty} M_\varrho r_n^2 \int f(y) q_k^{n,D}(y)dy.\end{aligned}$$

Hence

$$g^{n,D}(x,y) = M_\varrho r_n^2 \sum_{k=2}^{\infty} q_k^{n,D}(x,y)$$

Note that $g^{n,D}(x,y)$ is symmetric in x,y. When x is the origin, we denote the density by $g^{n,D}(y)$. In addition, it enjoys shifting property, i.e., for any $x \in \mathbb{R}^d$ we have $g^{n,D}(x,\cdot) = g^{n,D-x}(\cdot - x)$. When the underlying scaling $r_n = 1$ we denote $g^{n,D}$ by g^D. When $D = \mathbb{R}^d$ we denote $g^{n,D}$ by g^n.

Proposition 4.1 and Proposition 4.7 at the beginnings of Section 4.1 and Section 4.2 are the keys to prove the Hölder regularity of the semigroup of the process $X^{n,D}$, the process killed X^n upon leaving the domain $D \subseteq \mathbb{R}^d$.

4.1 Green's function estimates

In this chapter we prove the following crude bound for Green's function estimates.

Proposition 4.1. *For $R > 0$ and $D = B(0,R)$.*

$$g^{n,D}(z) \le g(z)\mathbb{1}_{B(0,R)}(z)$$

for all $z \in \mathbb{R}^d, 0 < |z| < Rn \in \mathbb{N}$, where $g(z) := \begin{cases} C_1 & \text{if } d=1 \\ C_2\log(|z|^{-1}) + C_2 & \text{if } d=2 \\ C_d|z|^{2-d} & \text{if } d\ge 3\end{cases}$

Recall that $\widetilde{W}^n$ is the continuous time random walk when $\mu = m_0$ the Lebesgue measure and W^n is the discrete time random walk (without time change). Let p^n_t denotes the density of $\widetilde{W}^n_t \circ \theta_{\eta^n_0}$ where θ is the shift operator. For $k \geq 1$ let q^n_k be the transition density of W^n_k. When $r_n = 1$ we omit the supscript of q. then we can see that

$$\begin{aligned} p^n_t(x) &= \sum_{k=1}^{\infty} \mathbb{P}\{N^n_t = k-1\} q^n_k(x) \\ &= \sum_{k=1}^{\infty} e^{-M_\varrho^{-1} r_n^{-2} t} \frac{(M_\varrho^{-1} r_n^{-2} t)^{k-1}}{(k-1)!} q^n_k(x). \end{aligned}$$

Recall that $\int |x|^2 \varrho(x) dx = M_\varrho d$. Let $\bar{p}(z) = \frac{1}{(2\pi M_\varrho d)^{d/2}} \exp\{-|z|^2/(2M_\varrho d)\}$ be the normal density with zero mean and variance $M_\varrho d$ and $\bar{p}_k(z) = k^{-d/2}\bar{p}(z/\sqrt{k})$

Proposition 4.2. *Under the condition* $\|\varrho\|_\infty < \infty$ *we have*

$$\sup_{z \in \mathbb{R}^d} |q_k(z) - \bar{p}_k(z)| = o(k^{-d/2})$$

as $k \to \infty$.

Proof. This is essentially due to the local central limit theorem. The result in [IL71, Theorem 4.3.1] can be generalized to d dimension. We have

$$\sup_{z \in \mathbb{R}^d} |k^{d/2} q_k(\sqrt{k} z) - \bar{p}(z)| = o_k(1).$$

Plugging in $\bar{p}(z) = k^{d/2}\bar{p}_k(\sqrt{k}z)$ we get the desired result. $\square$

Corollary 4.3. *There exists* $C > 0$ *such that for* $k \geq 1$ *we have*

$$\|q_k\|_\infty \leq C k^{-d/2}$$

and for any $L > 0$ *there exists* $c_L > 0$ *and* $K_L > 0$ *such that*

$$q_k(z) \geq c_L k^{-d/2}$$

for any $k \geq K_L$ *and all* $|z| \leq L\sqrt{k}$.

Proof. From Proposition 4.2 we see that

$$\|q_k\|_\infty \le \|\bar{p}_k\|_\infty + o(k^{-d/2}) \le C'k^{-d/2}$$

for any k sufficient large. Hence we may take $C = C' \vee H_\varrho$. On the other hand, we can find $K_L > 0$ such that

$$\inf_{|z|\le L\sqrt{k}} q_k(z) \ge \inf_{|z|\le L\sqrt{k}} \bar{p}_k(z) - o(k^{-d/2}) \ge c_L k^{-d/2}$$

for any $k \ge K_L$. We complete the proof. □

Proposition 4.4. *There exists $C > 0$ such that*

$$\|p_t^n\|_\infty \le Ct^{-d/2}$$

for all $t > 0$, $n \in \mathbb{N}$; and for any $L > 0$ there exist $c'_L, K'_L > 0$ such that

$$p_t^n(x) \ge c'_L t^{-d/2}$$

for all $n \in \mathbb{N}$ and $t > 0$ with $t/(M_\varrho r_n^2) > K'_L$ and for all $|x| \le L(2M_\varrho)^{-1/2}\sqrt{t}$.

Proof. Choosing $\varepsilon = 1/2$, standard exponential estimates shows that there exist $c_1, c_2 > 0$ such that $\mathbb{P}\{|N_s - s| \ge \varepsilon s\} \le c_1 e^{-c_2 s}$ for any $s > 0$ where N is a Poisson process. In particular we can choose $c_1 < 1$ if we require s be sufficiently large, say $s > C_0$. Setting $s = \frac{t}{M_\varrho r_n^2}$

$$\begin{aligned} p_t^n(x) &= \sum_{k=1}^\infty \mathbb{P}\{N_t^n = k-1\} q_k^n(x) \\ &\le c_1 e^{-c_2 t/(M_\varrho r_n^2)} H_\varrho r_n^{-d} + \sum \mathbb{P}\{N_t^n = k-1\} q_k^n(x) \end{aligned}$$

where here and for the remainder of this proof, we write just $\sum$ to denote the sum over all intergers k with $|k - \frac{t}{M_\varrho r_n^2}| < \varepsilon \frac{t}{M_\varrho r_n^2}$. Hence for these k we have $(1-\varepsilon)t < M_\varrho r_n^2 k < (1+\varepsilon)t$. Then

$$\begin{aligned} \sum \mathbb{P}\{N_t^n = k-1\} q_k^n(x) &\le \sum \mathbb{P}\{N_t^n = k-1\} C r_n^{-d} k^{-d/2} \\ &\le \sum \mathbb{P}\{N_t^n = k-1\} C (t/M_\varrho)^{-d/2} \\ &\le C(t/M_\varrho)^{-d/2}. \end{aligned}$$

For the first part we have

$$c_1 e^{-c_2 t/(M_\varrho r_n^2)} H_\varrho r_n^{-d} = f\left(\frac{t}{M_\varrho r_n^2}\right) H_\varrho (t/M_\varrho)^{-d/2}$$

where $f(z) = c_1 e^{-c_2 z} z^d$ is bounded for $z > 0$. Hence together we have $p_t^n \leq C(t/M_\varrho)^{-d/2}$.

For the lower bound, when $\frac{t}{M_\varrho r_n^2} > C_0$ we may choosing $c_1 < 1$. For any $L > 0$, let c_L, K_L be as in Corollary 4.3 and let $\frac{(1-\varepsilon)t}{M_\varrho r_n^2} \geq K_L$. Then $k \geq K_L$ under the summation set and

$$\begin{aligned} p_t^n(x) &\geq \sum \mathbb{P}\{N_t^n = k-1\} q_k^n(x) \\ &\geq \sum \mathbb{P}\{N_t^n = k-1\} r_n^{-d} c_L k^{-d/2} \\ &\geq (1 - c_1 e^{-c_2 t}) c_L \left(\frac{(1+\varepsilon)t}{M_\varrho}\right)^{-d/2} \\ &\geq (1-c_1) c_L (M_\varrho/2)^{d/2} t^{-d/2} \end{aligned}$$

for all $|x| \leq L\sqrt{\frac{(1-\varepsilon)t}{M_\varrho r_n^2}} r_n = L(2M_\varrho)^{-1/2}\sqrt{t}$. Hence we can set $c_L' = (1-c_1)c_L(M_\varrho/2)^{d/2}$. Combining the restriction $\frac{t}{M_\varrho r_n^2} > C_0$ and $\frac{(1-\varepsilon)t}{M_\varrho r_n^2} \geq K_L$ we get $\frac{t}{M_\varrho r_n^2} \geq C_0 \vee (2K_L)$. Hence we can set $K_L' = C_0 \vee (2K_L)$. □

Proposition 4.5. *If* $\int |z|^m \varrho(z) dz < \infty$ *for some* $m \geq 2$*, then we have*

$$q_k(x) \leq C H_\varrho \left(\sqrt{k}/|x|\right)^m.$$

for any $k \geq 2$*.*

Proof. The proof is inspired by [LL10, Proposition 2.4.6]. For $k \geq 2$, let $\bar{k} = k/2$ if k is even and $k = (k+1)/2$ otherwise. We have

$$\begin{aligned} q_k(x) &= q_{k-\bar{k}} * q_{\bar{k}}(x) \\ &= \int_{|z|>|x|/2} q_{k-\bar{k}}(x-z) q_{\bar{k}}(z) dz + \int_{|z|\leq|x|/2} q_{k-\bar{k}}(x-z) q_{\bar{k}}(z) dz \\ &\leq H_\varrho \mathbb{P}_0(|W_{\bar{k}}| \geq |x|/2) + H_\varrho \mathbb{P}_0(|W_{k-\bar{k}}| \geq |x|/2) \end{aligned}$$

We claim $\mathbb{P}_0(|W_{\bar{k}}| \geq |x|/2) \leq C(\sqrt{k}/|x|)^m$. To see this, we can apply BDG inequality (see, e.g. [Kal97, Theorem 23.12]) to get $\mathbb{E}_0[|W_{\bar{k}}|^m] \leq C k^{m/2}$, and then by Chebyshev inequality

$$\mathbb{P}_0(|W_{\bar{k}}| > |x|/2) \leq C \frac{\mathbb{E}_0[|W_{\bar{k}}|^m]}{|x|^m} \leq C(\sqrt{k}/|x|)^m.$$

We have a similar bound for $\mathbb{P}_0(|W_{k-\bar{k}}| \geq |x|/2)$. Hence we finish the proof. □

Proposition 4.6. *Let $R > 0$ and $D \subseteq B(0,R)$. For $d = 1,2$, there exists $c(R) > 0$ such that for all $k \geq 2$ and r_n sufficiently small that for any $x \in D/r_n$*

$$q_k^{D/r_n}(x) \leq Ck^{-d/2}e^{-cr_n^2 k}.$$

Proof. Let $\bar{k} = k/2$ if k is even and $k = (k+1)/2$ otherwise. Then by the Chapman-Kolmogorov equation and Corollary 4.3

$$q_k^{D/r_n}(x) \leq Ck^{-d/2} \int q_{\bar{k}}^{D/r_n}(y)dy.$$

On the other hand,

$$\int q_{\bar{k}}^{D/r_n}(y)dy \leq \mathbb{P}_0 \left(\max_{i \leq \bar{k}} |W_i^1| \leq R/r_n \right).$$

where W^1 denotes W^r for $r = 1$ (i.e. the discrete time random walk without scaling). Let j_n be the smallest positive integer such that $j_n \geq \bar{k}/\lfloor r_n^{-2} \rfloor$. Then by the strong Markov property,

$$\begin{aligned}
\mathbb{P}_0(\max_{i \leq \bar{k}} |W_i^1| \leq R/r_n) &\leq \sup_{|x| \leq R/r_n} \mathbb{P}_x(\max_{i \leq 1/r_n^2} |W_i^1| \leq R/r_n)^{j_n} \\
&\leq \mathbb{P}_0(\sup_{i \leq \lfloor r_n^{-2} \rfloor} |r_n W_i^1| \leq 3R)^{j_n} \\
&\leq e^{-cr_n^2 k}.
\end{aligned}$$

The last inequality holds for sufficient small r_n because by Donsker's theorem

$$\sup_{i \leq \lfloor r_n^{-2} \rfloor} |r_n W_i^1| \to \sup_{0 \leq t \leq M_\varrho d} |W_t|$$

in distribution, where W is the stand Brownian motion. □

Recall that $q_k^{n,D}$ is the transition density of W_k^n killed upon leaving domain $D \subseteq \mathbb{R}^d$ and $g^{n,D}(y) = M_\varrho r_n^2 \sum_{k=2}^{\infty} q_k^{n,D}(y)$. When the underlying scaling $r_n = 1$ we omit the first supscript. When $D = \mathbb{R}^d$ we omit the second supscript.

Proof of Proposition 4.1. By scaling we can see that

$$q_k^{n,D}(z) = r_n^{-d} q_k^{D/r_n}(z/r_n)$$

This implies

$$g^{n,D}(z) = M_\varrho \sum_{k=2}^{\infty} r_n^{2-d} q_k^{D/r_n}(z/r_n). \tag{4.2}$$

First we show the case when $d \geq 3$. Using Equation (4.2) we write

$$g^{n,D}(z) = M_\varrho |z|^{2-d} \sum_{k=2}^{\infty} (|z|/r_n)^{d-2} q_k^{D/r_n}(z/r_n).$$

As $q_k^{D/r_n} \leq q_k$, we only need to show that

$$\sum_{k=2}^{\infty} |z|^{2-d} q_k(z) < \infty.$$

For $k \leq |z|^2$ we apply Proposition 4.5 to get

$$\begin{aligned} \sum_{k \leq |z|^2} q_k(z) &\leq \sum_{k \leq |z|^2} C k^{-d/2} (\sqrt{k}/|z|)^m \\ &= C|z|^{-m} \sum_{k \leq |z|^2} k^{(m-d)/2}. \end{aligned}$$

When $m > d-2$, the sum is integrable and bounded by $C|z|^{m-d+2}$, hence

$$\sum_{k \leq |z|^2} q_k(z) \leq C|z|^{2-d}.$$

For $k \geq |z|^2$ we apply Corollary 4.3 to see that

$$\sum_{k \geq |z|^2} q_k(z) \leq C \sum_{k \geq |z|^2} k^{-d/2} = O(|z|^{2-d}).$$

For $d = 2$, by Proposition 4.5

$$\sum_{k \leq |z/r_n|^2} q_k^D(z/r_n) \leq \sum_{k \leq |z/r_n|^2} C k^{-1} (\sqrt{k}/|z/r_n|)^2 = C.$$

On the other hand, by Proposition 4.6

$$\begin{aligned}\sum_{k>|z/r_n|^2} q_k^D(z/r_n) &\leq \sum_{k>|z/r_n|^2} \frac{1}{kr_n^2} e^{-cr_n^2 k} r_n^2 \\ &\leq \sum_{r_n^2 k>|z|^2} \int_{kr_n^2}^{(k+1)r_n^2} \frac{1}{y} e^{-cy} dy \\ &\leq \int_{|z|^2}^{\infty} \frac{1}{y} e^{-cy} dy \\ &\leq C \log(|z|^{-1}).\end{aligned}$$

For $d = 1$, by Proposition 4.6

$$\begin{aligned}\sum_{k=1}^{\infty} r_n q_k^{D/r_n}(z/r_n) &\leq \sum_{k=1}^{\infty} r_n C k^{-1/2} e^{-cr_n^2 k} \\ &= \sum_{k=1}^{\infty} C (r_n^2 k)^{-1/2} e^{-cr_n^2 k} r_n^2 \\ &\leq C \int_0^{\infty} y^{-1/2} e^{-cy} dy < \infty\end{aligned}$$

This completes the proof of the proposition. □

4.2 Hölder regularity of harmonic functions

We say a function h is harmonic in an bounded open set $D \subseteq \mathbb{R}^d$ with respect to a process Z if for any relatively compact subset D_1 of D, $h(x) = \mathbb{E}_x[h(Z_{\tau_{D_1}}); \tau_{D_1} < \infty]$ for all $x \in D_1$.

We prove the following Hölder regularity of harmonic functions.

Proposition 4.7. *There is $\gamma' > 0, c_0 > 0$ and $C > 0$ such that if h is bounded and harmonic with respect to X^n in a ball $B(x_0, 2r)$ with $r \geq c_0 r_n$, then*

$$|h(x) - h(y)| \leq C \left(\frac{|x-y|}{r} \right)^{\gamma'} \|h\|_\infty \qquad \text{for } x, y \in B(x_0, r).$$

We use the standard approach to prove the Hölder regularity of harmonic functions. But since X^n has positive probability to stay at the starting point, the transition semigroup is not absolutely continuous with respect to the Lebesgue measure, we need to get around with it.

Lemma 4.8. *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\mathbb{P}_x\{\tau_{B(x,R)}(\widetilde{W}^n) \leq \delta R^2\} \leq \varepsilon$$

for all $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $R > 0$.

Proof. Let φ be a rotation invariant $C^2(\mathbb{R}^d)$ function such that $\varphi(0) = 0$ and $\varphi(r) = 1$ for $r \geq 1$. Define $\varphi_R(x) = \varphi(x/R)$. Then $\varphi_R(\widetilde{W}^n_t) - \int_0^t \mathcal{L}^n \varphi_R(\widetilde{W}^n_s)ds$ is a martingale, where $\mathcal{L}^n = \lambda_n(Q_n(\cdot, dx) - id)$ is the infinitismal generator of $\widetilde{W}^n$. By Taylor approximation error bound we have for some $L > 0$ (depending only on the choice of φ)

$$|\varphi(y + x) - \varphi(x) - \nabla\varphi(x) \cdot y| \leq \frac{L}{2}|y|^2$$

Using the fact that ϱ has zero mean, we have

$$\begin{aligned}
|\mathcal{L}^n \varphi_R(x)| &= \left| \frac{M_\varrho^{-1}}{r_n^2} \int (\varphi(\frac{r_n y + x}{R}) - \varphi(\frac{x}{R}))\varrho_n(y)dy \right| \\
&= \left| \frac{M_\varrho^{-1}}{r_n^2} \int (\varphi(\frac{r_n y + x}{R}) - \varphi(\frac{x}{R}) - \nabla\varphi(\frac{x}{R}) \cdot \frac{r_n y}{R})\varrho_n(y)dy \right| \\
&\leq \frac{M_\varrho^{-1}}{r_n^2} \int \frac{L r_n^2 |y|^2}{2R^2} \varrho_n(y)dy \\
&= \frac{L}{2R^2}
\end{aligned}$$

for any $x \in \mathbb{R}^d$ Hence $\varphi_R(\widetilde{W}^n_t) - \frac{L}{2R^2}t$ is a supmartingale. Then

$$\mathbb{P}_x\{\tau_{B(x,R)}(\widetilde{W}^n) \leq \delta R^2\} \leq \mathbb{E}_x[\varphi_R(\widetilde{W}^n_{\tau_{B(x,R)} \wedge \delta R^2})] \leq \frac{L}{2R^2}\mathbb{E}_x[\tau_{B(x,R)}(\widetilde{W}^n) \wedge \delta R^2] \leq \frac{L\delta}{2}.$$

Choosing $\delta = 2\varepsilon/L$ finishes the proof. □

Lemma 4.9. *There exist $C_1, c_1 > 0$ such that for any $x_0, x \in \mathbb{R}^d, n \in \mathbb{N}, r > c_1 r_n$, we have $\mathbb{E}_x[\tau_{B(x_0,r)}(\widetilde{W}^n)] \leq C_1 r^2$.*

Proof. By Proposition 4.4, taking $t = c_2 r^2$ for c_2 to be determined later, we have

$$\begin{aligned}
\mathbb{P}_x\{\tau_{B(x_0,r)}(\widetilde{W}^n) > t\} &\le \mathbb{E}_x[\mathbb{P}_{W_1^n}\{\widetilde{W}^n_{t-\eta_0^n} \in B(x_0,r)\}; \eta_0^n \le t/2] + \mathbb{P}_x\{\eta_0^n > t/2\} \\
&\le \sup_{s\in[t/2,t]} \int_{B(x_0,r)} p_s^n(y-x)dy + \mathbb{P}_x\{\eta_0^n > t/2\} \\
&\le Ct^{-d/2}|B(x,r)| + e^{-\frac{t/2}{M_\varrho r_n^2}} \\
&\le Cc_2^{-d/2} + e^{-c(c_2/c_1)^2}.
\end{aligned}$$

By choosing c_2 large enough and then c_1 small enough, we can make the above less than $1/2$. By the Markov property, for $m \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}_x\left(\tau_{B(x_0,r)}(\widetilde{W}^n) > (m+1)t\right) &\le \mathbb{E}_x\left[\mathbb{P}_{\widetilde{W}^n_{mt}}\left(\tau_{B(x_0,r)}(\widetilde{W}^n) > t\right); \tau_{B(x_0,r)}(\widetilde{W}^n) > mt\right] \\
&\le \frac{1}{2}\mathbb{P}_x\left(\tau_{B(x_0,r)}(\widetilde{W}^n) > mt\right).
\end{aligned}$$

By induction,

$$\mathbb{P}_x\left(\tau_{B(x_0,r)}(\widetilde{W}^n) > mt\right) \le 2^{-m} \text{ for all } m \in \mathbb{N}.$$

which proves the lemma. $\square$

Proposition 4.10. *There exist constants $c_1, c_2 > 0$ and $\Theta > 1$ such that for any $x_0 \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $t > 0$ with $t \ge c_1 M_\varrho r_n^2$, $D = B(x_0, r)$ with $r \ge \Theta t^{1/2}$, $x \in B(x_0, c_1 t^{1/2})$, and any bounded integrable function $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with support in $B(x_0, c_1 t^{1/2}/2)$ we have*

$$\mathbb{E}_x[f(\widetilde{W}_t^{n,D})] \ge c_2 t^{-d/2}\|f\|_1 - 3e^{-\frac{t/4}{M_\varrho r_n^2}}\|f\|_\infty.$$

Proof. For simplicity of notation we set $Y = \widetilde{W}^n$ and $T = \tau_D(\widetilde{W}^n)$. By Proposition 4.4 there exists $c_1 > 0$ such that $p_s^n(z) \ge cs^{-d/2}$ for any $n \in \mathbb{N}$, $s > 0$ and $|z| \le 2c_1 s^{1/2}$. For bounded function $f \in L^1(\mathbb{R}^d)$ with support in $B(x_0, c_1 t^{1/2}/2)$, we have

$$\mathbb{E}_x[f(\widetilde{W}_t^{n,D})] = \mathbb{E}_x[f(Y_t)] - \mathbb{E}_x[f(Y_t); T \le t].$$

For $\mathbb{E}_x[f(Y_t)]$ we have

$$\begin{aligned}\mathbb{E}_x[f(Y_t)] &\geq \mathbb{E}_x[\mathbb{E}_{W_1^n}[f(Y_{t-\eta_0^n})]; \eta_0^n \leq t/2] - \|f\|_\infty \mathbb{P}_x\{\eta_0^n > t/2\} \\ &\geq \inf_{x \in B(x_0, c_1 t^{1/2})} \inf_{s \in [t/2, t]} \int_{B(x_0, c_1 t^{1/2}/2)} f(y) p_s^n(y-x) dy - \|f\|_\infty e^{-\frac{t/2}{M_\varrho r_n^2}} \\ &\geq c t^{-d/2} \|f\|_1 - \|f\|_\infty e^{-\frac{t/2}{M_\varrho r_n^2}}.\end{aligned}$$

For $\mathbb{E}_x[f(Y_t); T \leq t]$ we first estimate the exit probability $\mathbb{P}_x\{T \leq t/2\}$. For any $\varepsilon \in (0,1)$, by Lemma 4.8 there exists $\delta = \delta(\varepsilon)$ such that $\mathbb{P}_x\{\tau_{B(x,s)}(\widetilde{W}^n) \leq s^2\delta\} \leq \varepsilon$ for any $s > 0$. If $t/2 \leq (r - |x - x_0|)^2\delta$, then we have

$$\begin{aligned}\mathbb{P}_x\{T \leq t/2\} &\leq \mathbb{P}_x\{\tau_{B(x, r-|x-x_0|)}(\widetilde{W}^n) \leq t/2\} \\ &\leq \mathbb{P}_x\{\tau_{B(x, r-|x-x_0|)}(\widetilde{W}^n) \leq (r - |x-x_0|)^2\delta\} \\ &\leq \varepsilon.\end{aligned}$$

The above holds when $r \geq (|x - x_0| + (2\delta)^{-1/2})t^{1/2}$. So we set $\Theta = c_1/2 + (2\delta)^{-1/2}$. Backing to $\mathbb{E}_x[f(Y_t); T \leq t]$ we have

$$\begin{aligned}\mathbb{E}_x[f(Y_t); T \leq t] &= \mathbb{E}_x[f(Y_t); T \leq t/2] + \mathbb{E}_x[f(Y_t); t/2 < T \leq t] \\ &=: S_1(x) + S_2(x)\end{aligned}$$

For $S_1(x)$, using the strong Markov property, Lemma 4.8 and Proposition 4.4 we have

$$\begin{aligned}S_1(x) = \mathbb{E}_x[f(Y_t); T \leq t/2] &= \mathbb{E}_x[\mathbb{E}_{Y_T} f(Y_{t-T}); T \leq t/2] \\ &\leq \mathbb{P}_x\{T \leq t/2\} \sup_{z \in D^c, s \in [t/2, t]} \mathbb{E}_z f(Y_s) \\ &\leq \varepsilon \sup_{z \in D^c, s \in [t/2, t]} (\mathbb{E}_z[f(Y_s); \eta_0^n \leq t/4] + \mathbb{E}_z[f(Y_s); \eta_0^n > t/4]) \\ &\leq \varepsilon \left(C t^{-d/2} \|f\|_1 + \|f\|_\infty e^{-\frac{t/4}{M_\varrho r_n^2}} \right).\end{aligned}$$

For $S_2(x)$, using the strong Markov property again we get

$$\begin{aligned}S_2(x) &\le \mathbb{E}_x[\mathbb{E}_{W_1^n}[f(Y_{t-\eta_0^n}); t/2 < T + \eta_0^n \le t]; \eta_0^n \le t/4] + \|f\|_\infty \mathbb{P}_x\{\eta_0^n > t/4\} \\ &\le \sup_{s\in[3t/4,t]} \mathbb{E}_x \mathbb{E}_{W_1^n}[f(Y_s); t/4 < T \le s] + \|f\|_\infty \mathbb{P}_x\{\eta_0^n > t/4\} \\ &\le \sup_{s\in[3t/4,t]} \int \mathbb{E}_y[f(Y_s); t/4 < T \le s] q_1^n(y)dy + \|f\|_\infty e^{-\frac{t/4}{M_\varrho r_n^2}}\end{aligned}$$

For any $s \in [3t/4, t]$ let $T'_s = \sup\{l \le s : \widetilde{W}_l^n \in D^c\}$ be the last hitting time of D^c before time s. Then we have $T'_s \ge T$ and

$$\mathbb{E}_y[f(Y_s); t/4 < T \le s] \le \mathbb{E}_y[f(Y_s); t/4 \le T'_s \le s]$$

Let $g \in L^1(\mathbb{R}^d)$ and $t/4 = t_0 < t_1 < ... < t_k = s$ with each $t_i = t/4 + (i/k)(s - t/4)$. For any $0 \le j \le k$

$$\int g(y)\mathbb{E}_y[f(Y_s); t/4 \le T'_s \le s]dy \le \mathbb{E}_{m_0}[g(Y_0)\mathbb{1}_{D^c}(Y_{t_j}) \prod_{i=j}^{k} \mathbb{1}_D(Y_{t_i}) f(Y_s)].$$

where m_0 is the Lebesgue measure. By time reversal (see e.g. [FOT11, Lemma 4.1.2]) we have

$$\mathbb{E}_{m_0}[g(Y_0)\mathbb{1}_{D^c}(Y_{t_j}) \prod_{i=j}^{k} \mathbb{1}_D(Y_{t_i}) f(Y_s)] = \mathbb{E}_{m_0}[g(Y_s)\mathbb{1}_{D^c}(Y_{s-t_j}) \prod_{i=j}^{k} \mathbb{1}_D(Y_{s-t_i}) f(Y_0)].$$

Summing over $j = 0, 1, ...,, k$ and letting $k \to \infty$, the right hand side above tends to $\mathbb{E}_{m_0}[g(Y_s)f(Y_0); T \le s - t/4]$. Hence

$$\begin{aligned}\int \mathbb{E}_y[f(Y_s); t/4 < T \le s]g(x)dx &\le \mathbb{E}_{m_0}[g(Y_s)f(Y_0); T \le s - t/4] \\ &\le \mathbb{E}_{m_0}[g(Y_s)f(Y_0); T \le 3t/4].\end{aligned}$$

By applying the same estimate for S_1 we can get

$$\mathbb{E}_{m_0}[g(Y_s)f(Y_0); T \le 3t/4] \le \varepsilon(Ct^{-d/2}\|g\|_1 + \|g\|_\infty e^{-\frac{t/4}{M_\varrho r_n^2}})\|f\|_1$$

Note that the above inequality holds for any bounded $g \in L^1(\mathbb{R}^d)$, hence we choose $g = q_1^n$ to get

$$\begin{aligned}
S_2(x) &\le \varepsilon(Ct^{-d/2}\|q_1^n\|_1 + \|q_1^n\|_\infty e^{-\frac{t/4}{M_\varrho r_n^2}})\|f\|_1 + \|f\|_\infty e^{-\frac{t/4}{M_\varrho r_n^2}} \\
&\le \varepsilon\left(C + H_\varrho\left(\frac{t}{r_n^2}\right)^{d/2} e^{-\frac{t/4}{M_\varrho r_n^2}}\right) t^{-d/2}\|f\|_1 + \|f\|_\infty e^{-\frac{t/4}{M_\varrho r_n^2}} \\
&\le \varepsilon C t^{-d/2}\|f\|_1 + \|f\|_\infty e^{-\frac{t/4}{M_\varrho r_n^2}}
\end{aligned}$$

where the last inequality we use that the function $z \to z^{d/2}e^{-z/(4M_\varrho)}$ is bounded for $z > 0$. Finally we have

$$\begin{aligned}
\mathbb{E}_x[f(\widetilde{W}_t^{n,D})] &= \mathbb{E}_x[f(Y_t)] - S_1(x) - S_2(x) \\
&\ge (c - \varepsilon C)t^{-d/2}\|f\|_1 - 3\|f\|_\infty e^{-\frac{t/4}{M_\varrho r_n^2}}.
\end{aligned}$$

Letting ε be small enough such that $c - \varepsilon C > 0$ yields the result. □

Corollary 4.11. *There exists $c > 0$ and $\Xi \in (0,1)$ that can be chosen arbitrarily close to 1 such that for any $\kappa \in (0,1]$, there exists $c_\kappa > 0$ such that for any $n \in \mathbb{N}, r \ge c_\kappa r_n$, $x \in \mathbb{R}^d$, and $A \subseteq B(x, \Xi r)$ with $|A|/|B(x,\Xi r)| \ge \kappa$ and $y \in B(x, 2\Xi r)$ we have*

$$\mathbb{P}_y\{\sigma_A(\widetilde{W}^n) < \tau_{B(x,r)}(\widetilde{W}^n)\} \ge c\kappa.$$

Proof. Applying Proposition 4.10 with $r = \Theta t^{1/2}$ and $f = \mathbb{1}_A$ to get

$$\begin{aligned}
\mathbb{P}_y\{\sigma_A(\widetilde{W}^n) < \tau_{B(x,r)}(\widetilde{W}^n)\} &\ge \mathbb{P}_y\{\widetilde{W}_t^n \in A; t < \tau_{B(x,r)}(\widetilde{W}^n)\} \\
&\ge c_2 t^{-d/2}|A| - 3e^{-\frac{t/4}{M_\varrho r_n^2}} \\
&= c_2\Theta^d r^{-d}|A| - 3e^{-\frac{r^2}{4\Theta^2 M_\varrho r_n^2}}
\end{aligned}$$

provided $t \ge c_1 M_\varrho r_n^2$, $y \in B(x, c_1 t^{1/2})$ and $A \subseteq B(x, c_1 t^{1/2}/2)$, where c_1, c_2 are from Proposition 4.10. Hence $r = \Theta t^{1/2} \ge \Theta(c_1 M_\varrho r_n^2)^{1/2}$ and $A \subseteq B(x, c_1 r/(2\Theta))$. Set $\Xi = \frac{c_1}{2\Theta} = \frac{c_1}{c_1 + 2(2\delta)^{-1/2}}$ where δ is from the proof of Proposition 4.10 and $\Theta = c_1/2 + (2\delta)^{-1/2}$, then $y \in B(x, 2\Xi r)$ and $A \subseteq B(x, \Xi r)$. We next apply $|A|/|B(x,\Xi r)| \ge \kappa$ to get

$$c_2\Theta^d r^{-d}|A| \ge c_3(\Xi\Theta)^d\kappa \ge c_4\kappa.$$

Finally setting $c_\kappa = (\Theta^2 c_1 M_\varrho)^{1/2} \vee (-4\Theta^2 M_\varrho \ln(c_4\kappa/6))^{1/2}$ we get

$$c_2\Theta^d r^{-d}|A| - 3e^{-\frac{r^2}{4\Theta^2 M_\varrho r_n^2}} \geq c_4\kappa - c_4\kappa/2 = c_4\kappa/2 = c\kappa.$$

Finally, we note that c_1 from Proposition 4.10 is the coefficient $L(2M_\varrho)^{-1/2}$ in Proposition 4.4, which can be chosen to be arbitrarily large. Hence $\Xi = \frac{c_1}{c_1+2(2\delta)^{-1/2}}$ can be chosen arbitrarily close to 1. We complete the proof. □

Lemma 4.12. *For any $\kappa \in (0,1]$, there exists $c_\kappa, c > 0$ such that for any $x \in \mathbb{R}^d$, $r \geq c_\kappa r_n$, any compact set $A \subseteq B(x,r)$ with $|A|/|B(x,r)| \geq \kappa$ and any $y \in B(x,r)$ we have*

$$\mathbb{P}_y\{\sigma_A(\widetilde{W}^n) < \tau_{B(x,r)}(\widetilde{W}^n)\} \geq c\kappa.$$

Proof. Given $\kappa \in (0,1]$, there exists $\Xi' \in (1/2,1)$ such that $\frac{|B(x,r)\backslash B(x,\Xi' r)|}{|B(x,r)|} < \kappa/2$. Hence $A' = A \cap B(x,\Xi' r)$ satisfies $|A'|/|B(x,r)| \geq \kappa/2$. We choose $\Xi > \Xi' > 1/2$ so that $A' \subseteq B(x,\Xi r)$ and $y \in B(x,r) \subseteq B(x,2\Xi r)$, then apply Corollary 4.11 to get that there exists $c > 0$ such that for any $r \geq c_\kappa r_n$

$$\mathbb{P}_y\{\sigma_{A'}(\widetilde{W}^n) < \tau_{B(x,r)}(\widetilde{W}^n)\} \geq c\kappa.$$

Note that $A' \subseteq A$ hence $\sigma_A(\widetilde{W}^n) \leq \sigma_{A'}(\widetilde{W}^n)$, and then

$$\mathbb{P}_y\{\sigma_A(\widetilde{W}^n) < \tau_{B(x,r)}(\widetilde{W}^n)\} \geq \mathbb{P}_y\{\sigma_{A'}(\widetilde{W}^n) < \tau_{B(x,r)}(\widetilde{W}^n)\} \geq c\kappa.$$

We complete the proof. □

Proof of Proposition 4.7. By Lemma 4.12, there exists $c_0', c_1 > 0$ such that for any $x \in \mathbb{R}^d, r \geq c_0' r_n$, $A \subseteq B(x,r)$ with $|A| > |B(x,r)|/3$ and $y \in B(x,r)$ we have

$$P_y(\sigma_A(\widetilde{W}^n) < \tau_{B(x,r)}(\widetilde{W}^n)) \geq c_1.$$

By Lévy system formula, we have for any bounded function φ and region $D \subseteq \mathbb{R}^d$

$$\mathbb{E}_x[\varphi(\widetilde{W}^n_{\tau_D}); \widetilde{W}^n_{\tau_D} \neq \widetilde{W}^n_{\tau_D-}] = \mathbb{E}_x[\int_0^{\tau_D} \int \frac{M_\varrho^{-1}}{r_n^2} Q_n(\widetilde{W}^n_s, dz)\varphi(z)ds].$$

Taking $\varphi(z) = \mathbb{1}_{B(x,r')^c}$ and $D = B(x,r)$ for $r' \geq 2r$ and noticing that $\widetilde{W}^n \in D$ before time τ_D we get

$$\begin{aligned}
\mathbb{P}_x\left(\widetilde{W}^n_{\tau_{B(x,r)}} \notin B(x,r')\right) &= \mathbb{E}_x[\int_0^{\tau_{B(x,r)}} \int_{B(x,r')^c} \frac{M_\varrho^{-1}}{r_n^2} \varrho_n(z - \widetilde{W}^n_s) dz ds] \\
&\leq \mathbb{E}_x[\tau_{B(x,r)}(\widetilde{W}^n)] \int_{B(0,r'-r)^c} \frac{M_\varrho^{-1}}{r_n^2} \varrho_n(z) dz \\
&\leq \mathbb{E}_x[\tau_{B(x,r)}(\widetilde{W}^n)] \frac{1}{(r'-r)^2} \frac{M_\varrho^{-1}(r'-r)^2}{r_n^2} \int_{B(0,\frac{r'-r}{r_n})^c} \varrho(z) dz.
\end{aligned}$$

By Lemma 4.9 there exists $c_0'' > 0$ such that $\mathbb{E}_x[\tau_{B(x,r)}(\widetilde{W}^n)] \leq cr^2$ for any $r \geq c_0'' r_n$. Because ϱ has finite second moment, we have

$$\frac{M_\varrho^{-1}(r'-r)^2}{r_n^2} \int_{B(0,\frac{r'-r}{r_n})^c} \varrho(z) dz \leq M_\varrho^{-1} \int_{B(0,\frac{r'-r}{r_n})^c} |z|^2 \varrho(z) dz \leq d.$$

Also note that $\frac{1}{(r'-r)^2} \leq \frac{4}{(r')^2}$ as $r' \geq 2r$. Hence we conclude there exists $c_2 > 0$ such that

$$\mathbb{P}_x\left(\widetilde{W}^n_{\tau_{B(x,r)}} \notin B(x,r')\right) \leq c_2 (\frac{r}{r'})^2.$$

Let $\zeta = 1 - \frac{c_1}{4}$, $\rho = \frac{1}{2} \wedge (\frac{\zeta}{2})^{1/2} \wedge (\frac{c_1 \zeta}{8 c_2})^{1/2}$ and $c_0 = c_0' \wedge c_0''$. Let h be $\mathcal{E}^n$-harmonic in $B(x_0, 2r)$ for $x_0 \in \mathbb{R}^d$. By scaling without loss of generality, assume $0 \leq h \leq 1$. For any $x \in B(x_0, r)$, $k \geq 0$, set $B_k = B(x, \rho^k r)$, $a_k = \inf_{B_k} h$, $b_k = \inf_{B_k} h$ and $\tau_k = \tau_{B_k}(\widetilde{W}^n)$. We use induction to show $b_k - a_k \leq \zeta^k$ for $k \geq 0$ and $\rho^k r \geq c_0 r_n$. Clearly $b_i - a_i \leq 1 \leq \zeta^i$ for all $i \leq 0$. Now suppose $b_i - a_i \leq \zeta^i$ for all $i \leq k$, we are going to show that $b_{k+1} - a_{k+1} \leq \zeta^{k+1}$. Set $A' = \{z \in B_{k+1} : h(z) \leq (a_k + b_k)/2\}$. We may assume $|A'| \geq |B_{k+1}|/2$ otherwise we can replace h by $1 - h$. Choose compact set $A \subseteq A'$ such that $|A|/|B_k| > 1/3$. For $\varepsilon > 0$ choose

$z_1,\, z_2 \in B_{k+1}$ such that $h(z_1) \geq b_{k+1} - \varepsilon$ and $h(z_2) \leq a_{k+1} - \varepsilon$. Then

$$\begin{aligned}
b_{k+1} - a_{k+1} - 2\varepsilon &\leq h\left(z_1\right) - h\left(z_2\right) \\
&= \mathbb{E}_{z_1}\left[h\left(\widetilde{W}^n_{\sigma_A \wedge \tau_{k+1}}\right) - h\left(z_2\right)\right] \\
&= \mathbb{E}_{z_1}\left[h\left(\widetilde{W}^n_{\sigma_A}\right) - h\left(z_2\right); \sigma_A < \tau_{k+1}\right] \\
&\quad + \mathbb{E}_{z_1}\left[h\left(\widetilde{W}^n_{\tau_{k+1}}\right) - h\left(z_2\right); \sigma_A > \tau_{k+1},\ \widetilde{W}^n_{\tau_{k+1}} \in B_k\right] \\
&\quad + \sum_{i=1}^{\infty} \mathbb{E}_{z_1}\left[h\left(Z_{\tau_{k+1}}\right) - h\left(z_2\right); \sigma_A > \tau_{k+1},\ \widetilde{W}^n_{\tau_{k+1}} \in B_{k-i} \backslash B_{k+1-i}\right] \\
&\leq \left(\frac{a_k + b_k}{2} - a_k\right) \mathbb{P}_{z_1}\left(\sigma_A < \tau_{k+1}\right) \quad + \left(b_k - a_k\right) \mathbb{P}_{z_1}\left(\sigma_A > \tau_{k+1}\right) \\
&\qquad + \sum_{i=1}^{\infty} \left(b_{k-i} - a_{k-i}\right) \mathbb{P}_{z_1}\left(\widetilde{W}^n_{\tau_{k+1}} \notin B_{k+1-i}\right) \\
&\leq \left(b_k - a_k\right)\left(1 - \frac{\mathbb{P}_{z_1}\left(\sigma_A < \tau_{k+1}\right)}{2}\right) + \sum_{i=1}^{\infty} c_2 \zeta^k \left(\rho^2/\zeta\right)^i \\
&\leq \zeta^k \left(1 - \frac{c_1}{2}\right) + 2c_2 \zeta^{k-1} \rho^2 \\
&\leq \zeta^k \left(1 - \frac{c_1}{2}\right) + \frac{c_1}{4} \zeta^{k-1} \zeta^2 \\
&= \zeta^{k+1}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $b_k - a_k \leq \zeta^k$ for all $k \geq 0$.

Now for $x, y \in B(x_0, r)$ choose k be the largest integer such that $|x-y| \vee c_0 r_n \leq \rho^k r$ Then $\log(\frac{|x-y| \vee c_0 r_n}{r}) \geq (k+1)\log\rho$ and

$$|h(x) - h(y)| \leq \zeta^k \leq c_3 (\frac{|x-y| \vee c_0 r_n}{r})^{\log\zeta/\log\rho}.$$

□

4.3 Hölder regularity of time-changed semigroups

We first prepare several lemmas. The main proposition is given in Proposition 4.16 and Proposition 4.17.

Lemma 4.13. *Let g be the function given in Proposition 4.1. For $R > 2$, there exist $\beta > 0$ and $C_R > 0$ such that for any $x \in B(0,R)$ and $r \in (0,1)$*

$$(g\mathbb{1}_{B(0,r)} * \mu)(x) \equiv \int_{B(x,r)} g(x-y)\mu(dy) \leq C_R r^{\beta}.$$

Proof. The case when $d = 1$ is obvious as by Proposition 4.1 and assumption (A.3)

$$\int_{B(x,r)} g(x-y)\mu(dy) \leq C\mu(B(x,r)) \leq C_R r^{\beta_\mu}.$$

For $d \geq 2$ we can divide $B(x,r)$ into a sequence of annulus $A_k = \{y \in \mathbb{R}^d : 2^{-k-1}r < |y-x| < 2^{-k}r\}$, then

$$\begin{aligned}\int_{B(x,r)} g(x-y)\mu(dy) \leq B(0,r) &\leq \sum_{k=0}^{\infty} g(2^{-k}r)\mu(A_k \setminus A_{k+1}) \\ &\leq \sum_{k=0}^{\infty} g(2^{-k}r) C_R (2^{-k}r)^{\beta_\mu}.\end{aligned}$$

When $d = 2$, take $R > 2$ such that $K \subseteq B(0, R-1)$,

$$\sum_{k=0}^{\infty} g(2^{-k}r) C_R (2^{-k}r)^{\beta_\mu} \leq \sum_{k=0}^{\infty} C_d(-\log(2^{-k}r) + \log R) C_R (2^{-k}r)^{\beta_\mu} \leq C r^{\beta'}$$

for some $\beta' \in (0, \beta_\mu)$. When $d \geq 3$,

$$\sum_{k=0}^{\infty} g(2^{-k}r) C_R (2^{-k}r)^{\beta_\mu} \leq \sum_{k=0}^{\infty} C|2^{-k}r|^{2-d} C_R (2^{-k}r)^{\beta_\mu} \leq C_R r^{\beta''}$$

for some $\beta'' = \beta_\mu - (d-2) > 0$. Hence the result is proved. □

Lemma 4.14. *Given $R > 2$ and any bounded domain $D \subseteq B(0,R) \subseteq \mathbb{R}^d$, there exist $c_1(R) > 0, \beta > 0$ such that for any $r \in (0,1)$, any $x_0 \in B(0,R/2)$ and $x \in B(x_0,r)$ and $n \in \mathbb{N}$ we have*

$$\mathbb{E}_x[\tau_{B(x_0,r)}(X^n)] \leq c_1 r^{\beta} + o_{r_n}(1).$$

In addition, for any $r \in (0,1)$

$$\mathbb{E}_x[\tau_{B(x_0,r)}(X)] \leq c_1 r^{\beta}.$$

Proof. Let $B = B(x_0, r) \subseteq D$. Note that by strong Markov property we have

$$\begin{aligned}
\mathbb{E}_x[\tau_B(X^n)] &= \mathbb{E}_x[\tau_B(X^n); \eta_0^n \geq \tau_B(X^n)] + \mathbb{E}_x[\tau_B(X^n); \eta_0^n < \tau_B(X^n)] \\
&\leq \mathbb{E}_x[\eta_0^n] + \mathbb{E}_x[\tau_B(X^n); \eta_0^n + \eta_1^n \leq \tau_B(X^n)] \\
&\leq \mathbb{E}_x[\eta_0^n] + \mathbb{E}_x[\eta_0^n + \eta_1^n + \tau_B(X^n \circ \theta_{\eta_0^n+\eta_1^n}); W_1^n \in B] \\
&= \mathbb{E}_x[2\eta_0^n(W_0^n) + \eta_1^n(W_1^n); W_1^n \in B] + \mathbb{E}_x[\tau_B(X^n \circ \theta_{\eta_0^n+\eta_1^n}); W_1^n \in B].
\end{aligned}$$

For the first part

$$\begin{aligned}
\mathbb{E}_x[2\eta_0^n(W_0^n) + \eta_1^n(W_1^n); W_1^n \in B] &\leq 3 \sup_{x \in B} \frac{1}{\lambda_n(x)} \\
&= 3 \sup_{x \in B} r_n^2 M_\varrho \int \phi_n(y - x)\mu(dy) \\
&\leq 3\|\phi\|_\infty \sup_{x \in D} r_n^{2-d} M_\varrho \mu(B(x, r_n)) \\
&\leq 3C_R \|\phi\|_\infty M_\varrho r_n^{2-d+\beta_\mu} = o_{r_n}(1).
\end{aligned}$$

The last equation is because $\beta_\mu > d - 2$, the above goes to 0 as $r_n \downarrow 0$. For the second part,

$$\begin{aligned}
\mathbb{E}_x[\tau_B(X^n \circ \theta_{\eta_0^n+\eta_1^n}); W_1^n \in B] &= \int g^{n,B}(x, y)\mu_n(dy) \\
&\leq \int g^{n,B(x,2r)}(x, y)\mu_n(dy) \\
&\leq \int g(x - y)\mathbb{1}_{B(x,2r)}(y)\mu_n(dy) \\
&= (g\mathbb{1}_{B(0,2r)} * (\phi_n * \mu))(x) \\
&= (\phi_n * (g\mathbb{1}_{B(0,2r)} * \mu))(x) \\
&\leq (\phi_n * (C_R r^\beta))(x) = C_R r^\beta
\end{aligned}$$

where we use Lemma 4.13 and Proposition 4.1. For the process X we have

$$\begin{aligned}
\mathbb{E}_x[\tau_B(X)] &\leq \int g(x - y)\mathbb{1}_{B(x,2r)}(y)\mu_n(dy) \\
&= (g\mathbb{1}_{B(0,2r)} * (\phi_n * \mu))(x) \\
&= (\phi_n * (g\mathbb{1}_{B(0,2r)} * \mu))(x) \\
&\leq (\phi_n * (C_R r^\beta))(x) = C_R r^\beta.
\end{aligned}$$

The proof is completed. □

Given bounded domain $D \subseteq B(0,R) \subseteq \mathbb{R}^d$, with the convention that functions take value 0 at the cemetery point, define the resolvent $U^{n,D}_\alpha f(x) := \mathbb{E}_x \int_0^\infty e^{-\alpha t} f(X^{n,D}_t)dt$ and the transition semigroup of $P^{n,D}_t f(x) := \mathbb{E}_x f(X^{n,D}_t)$.

Lemma 4.15. *There exist* $\kappa > 0, n_0(R) > 0, C_{R,\kappa} > 0$ *such that for all* $n \geq n_0$ *and any nonegative* $h \in L^2(D;\mu_n)$ *with* $\|h\|_{1,n} \leq 1$

$$\|P^{n,D}_t h\|^2_{2,n} \leq C_{R,\kappa}\left((t - O_n \log \|h\|_{2,n}) \vee 0\right)^{-1/\kappa}$$

where $O_n = O(r_n^{\beta_\mu - d+2})$.

Proof. Let $G^{n,D} f(x) := \mathbb{E}_x \int_0^\infty f(X^{n,D}_t)dt$ be the Green's operator. Then

$$G^{n,D}1(x) = \mathbb{E}_x[\tau_D(X^n)].$$

Like in the proof of Lemma 4.14, we can show that for any open set $U \subseteq D \subseteq B(0,R)$ and $p > 1$

$$\begin{aligned}
\|G^{n,U}1\|_\infty &\leq O(r_n^{\beta_\mu-d+2}) + \sup_{x\in U}\int g^{n,U}(x,y)\mu_n(dy) \\
&\leq O(r_n^{\beta_\mu-d+2}) + \sup_{x\in U}\int_U g(x-y)\mu_n(dy) \\
&\leq O(r_n^{\beta_\mu-d+2}) + \mu_n(U)^{\frac{p}{p-1}} \sup_{x\in B(0,R)}\left(\int_{B(x,2R)} g(x-y)^p \mu_n(dy)\right)^{1/p} \\
&= O(r_n^{\beta_\mu-d+2}) + \mu_n(U)^{\frac{p}{p-1}} \sup_{x\in B(0,R)}\left(\phi_n * (g^p \mathbb{1}_{B(0,2R)} * \mu)(x)\right)^{1/p}.
\end{aligned}$$

By the same method as in Lemma 4.13 we can show $(g^p\mathbb{1}_{B(0,2R)} * \mu)(x) \leq C_R R^\beta$ for some $\beta \in (0,\beta_\mu)$ if $d = 2$. In the case $d \geq 3$ we can choose $p > 1$ such that $\beta = \beta_\mu - p(d-2) > 0$ and get $(g^p\mathbb{1}_{B(0,2R)} * \mu)(x) \leq C_R R^\beta$. Hence there is a constant C_R such that

$$\|G^{n,U}1\|_\infty \leq O(r_n^{\beta_\mu-d+2}) + C_R \mu_n(U)^{\frac{p}{p-1}}.$$

By [GT12, Lemma 3.2] we know the smallest eigenvalue $\Lambda_{\min}$ of the generator $\mathcal{A}^{n,U}$ of $\mathcal{E}^{n,U}$ on $L^2(B(0,R);\mu_n)$ satisfies

$$\Lambda_{\min}(U) \geq \|G^{n,U}1\|_\infty^{-1} \geq (C_R\mu_n(U)^\kappa + O_n)^{-1}. \tag{4.3}$$

where $O_n = O(r_n^{\beta_\mu - d+2})$ and $\kappa = \frac{p}{p-1}$. We then follow the proof of [GH14, Lemma 5.4, 5.5]. For any $u \in \mathcal{F}^{n,D} \cap C_0(U)$ and $u \geq 0$, The set $U_s := \{x \in U : u > s\}$ is open for every $s > 0$. By Markov property we have for any $t \geq 0$,

$$\mathcal{E}(u) \geq \mathcal{E}((u-t)_+).$$

When $t > s$, $(u-t)_+$ vanish outside U_t, hence $\mathcal{E}((u-t)_+) \geq \Lambda_{\min}(U_s)\int_{U_s}(u-t)_+^2\mu_n(dx)$. Let $A = \|u\|_{n,1}$ and $B = \|u\|_{n,2}^2$. Since $u \geq 0$, we can use the inequality $(u-t)_+^2 \geq u^2 - 2tu$ to get

$$\mathcal{E}((u-t)_+) \geq \Lambda_{\min}(U_s)(B - 2tA). \tag{4.4}$$

On the other hand, we have

$$\mu\left(U_s\right) \leqslant \frac{1}{s}\int_{U_s} u d\mu \leqslant \frac{A}{s}. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we get

$$\mathcal{E}((u-t)_+) \geq \frac{1}{C_R(A/s)^\kappa + O_n}\left(B - 2tA\right). \tag{4.6}$$

and letting $t \downarrow s = \frac{B}{4A}$ we get

$$\mathcal{E}^{n,U}(u) \geq \frac{1}{2}\frac{\|u\|_{2,n}^2}{C_R(\|u\|_{1,n}^2/\|u\|_{2,n}^2)^\kappa + O_n} \tag{4.7}$$

Set $u_t = P_t^{n,D}h$ for $h \geq 0$ with $\|h\|_{1,n} \leq 1$. and denote $J(t) = \|u_t\|_{2,n}^2$. Using the fact that $P_t^{n,D}$ is contractive in $L^1(D;\mu_n)$ (because $P_t^{n,D}$ is symmetric in $L^2(D;\mu_n)$ and $P_t^{n,D}1 \leq 1$) we have

$$\frac{dJ}{dt} = -2\mathcal{E}^{n,D}(u_t) \leq -\frac{J}{C_R J^{-\kappa} + O_n}$$

or

$$C_R\frac{dJ}{J^{\kappa+1}} + O_n\frac{dJ}{J} \leq -dt.$$

Integrating from 0 to t yields

$$-\frac{C_R}{\kappa}(J^{-\kappa} - \|h\|_{2,n}^{-2\kappa}) + O_n \log \frac{J}{\|h\|_{2,n}^2} \leq -t.$$

Rearranging to get

$$t - O_n \log \|h\|_{2,n}^2 \leq \frac{C_R}{\kappa}(J^{-\kappa} - \|h\|_{2,n}^{-2\kappa}) + O_n \log \frac{1}{J} \leq C_{R,\kappa} J^{-\kappa}$$

where we use the fact that $\log z \leq c_\kappa z^\kappa$ for $z > 0$ in the last inequality and it holds for all sufficient large n as $O_n \downarrow 0$. The proof is completed. □

Proposition 4.16. *There is $\gamma > 0$, $c_0 > 0$ and $n_0 > 0$ that for any $R > 2, t > 0, \alpha > 0$, $D \subseteq B(0,R)$, there are $C_1(R,\alpha) > 0$, $C_2(R,t) > 0$ such that for any $n > n_0$ and bounded function f supported on D and $x, y \in B(0,R)$ with $|x-y| \geq c_0 r_n$,*

$$|U_\alpha^{n,D} f(x) - U_\alpha^{n,D} f(y)| \leq C_1(|x-y|^\gamma + o_n(1))\|f\|_\infty$$
$$|P_t^{n,D} f(x) - P_t^{n,D} f(y)| \leq C_2(|x-y|^\gamma + o_n(1))\|f\|_\infty$$

where $\|f\|_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|$ and $\|f\|_{2,n}^2 = \int f^2(x)\mu_n(dx)$.

Proof. To show the first inequality, we use a similar proof in [CCK15, Proposition 2.4]. We need to check the following conditions: for $R > 2$ there exist $c_1(R), c_2(R), \beta, \gamma' \in (0,\infty)$, $n_0 \in \mathbb{N}$ such that for any $x_0 \in B(0,R)$, and $r \in [c_0 r_n, 1)$, the following two hold.

- For all $x \in B(x_0, r/2)$,

$$\mathbb{E}_x[\tau_{B(x_0,r)}(X^n)] \leq c_1 r^\beta + o_{r_n}(1).$$

- There is $\gamma' > 0$ such that if h is bounded and harmonic with respect to X^n in a ball $B(x_0, 2r)$, then

$$|h(x) - h(y)| \leq c_2 \left(\frac{|x-y|}{r}\right)^{\gamma'} \|h\|_\infty \qquad \text{for } x, y \in B(x_0, r).$$

The first condition is satisfied by Lemma 4.14. Since continuous time change does not change the harmonicity, harmonic functions with respect to X^n are the same as that with respect to the continuous-time random walk $\widetilde{W}$ (without time change). Thus the second condition is satisfied by Proposition 4.7.

Now we apply the above two conditions to show the resolvents are equi-Hölder. For $x_0 \in B(0,R)$ and $r \in (0,1)$, set $\tau_r^n := \tau_{B(x_0,r)}(X^n) \leq \tau_D(X^n)$. By the strong Markov property, for $x \in B(x_0, r/2)$

$$\begin{aligned} U_\alpha^{n,D} f(x) &= \mathbb{E}_x \int_0^{\tau_r^n} e^{-\alpha t} f(X_t^{n,D}) dt + \mathbb{E}_x^n[(e^{-\alpha\tau_r^n} - 1) U_\alpha^{n,D} f(X_{\tau_r^n}^n)] + \mathbb{E}_x^n[U_\alpha^{n,D} f(X_{\tau_r^n}^n)] \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We have

$$I_1 \leq \|f\|_\infty \mathbb{E}_x^n \tau_r^n \leq (c_1 r^\beta + o_{r_n}(1)) \|f\|_\infty$$

and by $\|U_\alpha^{n,D} f\|_\infty \leq \alpha^{-1} \|f\|_\infty$ and $|e^{-z} - 1| \leq z$ for $z \geq 0$ we have

$$I_2 \leq \alpha \mathbb{E}_x \tau_r^n \|U_\alpha^{n,D} f\|_\infty \leq (c_1 r^\beta + o_{r_n}(1)) \|f\|_\infty.$$

Note that I_3 as a function of x is bounded in $\mathbb{R}^d$ and harmonic in $B(x_0, r)$, hence for $x, y \in B(x_0, r)$ with $|x - y| \geq c_0 r_n$ our second condition applies. Combining all parts together and applying $\|U_\alpha^{n,D} f\|_\infty \leq \alpha^{-1} \|f\|_\infty$ again we get

$$|U_\alpha^{n,D} f(x) - U_\alpha^{n,D} f(y)| \leq c_3 \left(r^\beta + o_{r_n}(1) + \alpha^{-1} \left(\frac{|x-y|}{r} \right)^{\gamma'} \right) \|f\|_\infty.$$

For any distinct $x, y \in B(0,R)$ and $|x - y| < 1/4$, let $x_0 = x$ and $r = |x-y|^{1/2} (\geq c_0 r_n)$, then $y \in B(x_0, r)$ and hence

$$\begin{aligned} |U_\alpha^{n,D} f(x) - U_\alpha^{n,D} f(y)| &\leq c_3 \left(|x-y|^{\beta/2} + o_{r_n}(1) + \alpha^{-1} |x-y|^{\gamma'/2} \right) \|f\|_\infty \\ &= (C_1 |x-y|^\gamma + o_{r_n}(1)) \|f\|_\infty \end{aligned}$$

for $C_1 = c_3(1 + \alpha^{-1})$ and $\gamma = (\beta \wedge \gamma')/2$.

To show the equi-Hölder continuity of the semigroup, we follow the idea in [BKK10, Proposition 3.4]. But since we do not have ultracontractivity of the semigroup, extra work is needed. We note that the generator of the process $X^{n,D}$ is $\mathcal{A}^{n,D} = \lambda_n(Q_{n,D} - I)$, where I is the identity map and $Q_{n,D}f = Q_n(f\mathbb{1}_D)\mathbb{1}_D$. From now, we may let $\alpha = 1$ and won't emphrasis the dependency of constants on α. For any $s > 0$ denote $F_s := (\alpha I - \mathcal{A}^{n,D})P_s^{n,D}f$. Because the well-known relation between the resolvent and the generator that $U_\alpha^{n,D}(\alpha I - \mathcal{A}^{n,D}) = I$, and using the fact that $\mathcal{A}^{n,D}$ and $P_s^{n,D}$ are commutable, we have

$$\begin{aligned} P_t^{n,D} f &= U_\alpha^{n,D}(\alpha I - \mathcal{A}^{n,D})P_t^{n,D} f \\ &= U_\alpha^{n,D} P_{t/2}^{n,D}(\alpha I - \mathcal{A}^{n,D})P_{t/2}^{n,D} f \\ &= U_\alpha^{n,D} P_{t/2}^{n,D} F_{t/2}. \end{aligned}$$

We want to show $\|P_{t/2}^{n,D}F_{t/2}\|_\infty$ is bounded in n, then we can apply Hölder regularity of $U_\alpha^{n,D}$ to get the desired result. For $x \in D$, we have

$$\begin{aligned} P_{t/2}^{n,D}F_{t/2}(x) &= \mathbb{E}_x[F_{t/2}(X_{t/2}^{n,D}); t/2 \geq \eta_0^n] + \mathbb{E}_x[F_{t/2}(X_{t/2}^{n,D}); t/2 < \eta_0^n] \\ &= \mathbb{E}_x[\mathbb{E}_{W_1^n}[F_{t/2}(X_{t/2-\eta_0^n}^{n,D})]; W_1^n \in D, t/2 \geq \eta_0^n] + F_{t/2}(x)\mathbb{P}_x[t/2 < \eta_0^n] \\ &=: I_1' + I_2'. \end{aligned}$$

For I_2' we have

$$\begin{aligned} |I_2'| &= |F_{t/2}(x)|\mathbb{P}_x[t/2 < \eta_0^n] \\ &= e^{-\lambda_n(x)t/2}|\alpha I - \lambda_n(Q_{n,D} - I)P_{t/2}^{n,D} f|(x) \\ &\leq e^{-\lambda_n(x)t/2}(\alpha + 2\|f\|_\infty \lambda_n(x)) = o_n(1)\|f\|_\infty. \end{aligned}$$

For I_1' we have

$$\begin{aligned} I_1' &= \mathbb{E}_x[\mathbb{E}_{W_1^n}[F_{t/2}(X_{t/2-\eta_0^n}^{n,D})]; W_1^n \in D, t/2 \geq \eta_0^n] \\ &= \int_0^{t/2} Q_{n,D}P_{t/2-s}^{n,D}F_{t/2}(x)e^{-\lambda_n(x)s}\lambda_n(x)ds. \end{aligned}$$

We claim $\sup_{s\in(0,t/2)}\|Q_{n,D}P^{n,D}_{t/2-s}F_{t/2}\|_\infty$ is bounded in n, then I_1' is also bounded in n. Using spectrum representation theorem for self-adjoint operators, there exist projection operators $E_l = E^{n,D}_l$ on the space $L^2(D;\mu_n)$ such that

$$F_s = \int_0^\infty (\alpha+l)e^{-ls}dE_l(f).$$

Given $t_0>0$, for any $s\ge t_0$ we have $(\alpha+l)e^{-ls}\le C_{t_0}$, we have

$$\sup_{s\ge t_0}\|F_s\|_{2,n} = \int_0^\infty (\alpha+l)^2e^{-2ls}d\langle E_l(f),E_l(f)\rangle_n \le C_{t_0}\|f\|_{2,n}$$

where $\langle\cdot,\cdot\rangle_n$ denote the inner product in $L^2(D;\mu_n)$. We see $F_s\in L^2(D;\mu_n)$. For a function $h\in L^2(D;\mu_n)$, we have

$$\begin{aligned}
|\langle F_s,h\rangle_n| &= \left|\int_0^\infty (\alpha+l)e^{-lt}d\langle E_l(f),h\rangle_n\right| \\
&\le \left(\int_0^\infty (\alpha+l)e^{-ls}d\langle E_l(f),f\rangle_n\right)^{1/2}\left(\int_0^\infty (\alpha+l)e^{-ls}d\langle E_l(h),h\rangle_n\right)^{1/2} \\
&\le C_{t_0}\left(\int_0^\infty d\langle E_l(f),f\rangle_n\right)^{1/2}\left(\int_0^\infty e^{-ls/2}d\langle E_l(h),h\rangle_n\right)^{1/2} \\
&= C_{t_0}\|f\|_{2,n}\|P^{n,D}_{s/2}h\|_{2,n}.
\end{aligned}$$

For $x\in D$, by applying Lemma 4.15 with $h_{n,x}(y)=\varrho_n(y-x)\frac{dm_0}{d\mu_n}(y)\mathbb{1}_D(y)$, we get

$$\begin{aligned}
|(Q_{n,D}F_s)(x)| = |\langle F_s,h_{n,x}\rangle_n| &\le C_{t_0}\|f\|_{2,n}\|P^{n,D}_{s/2}h_{n,x}\|_{2,n} \\
&\le C_{t_0}\|f\|_{2,n}\left((C_Rs/2-O_n\log\|h_{n,x}\|_{2,n})\vee 0\right)^{-1/\kappa}
\end{aligned}$$

Note that

$$\begin{aligned}
\|h_{n,x}\|^2_{2,n} &= \int_D \varrho_n(y-x)^2\frac{dm_0}{d\mu_n}(y)^2\mu_n(dy) \\
&= \int_D \frac{\varrho_n(y-x)^2}{\int\phi_n(y-z)\mu(dz)}dy \\
&\le H_\varrho^2 r_n^{-d}\int_D \frac{1}{\int\phi((y-z)/r_n)\mu(dz)}dy.
\end{aligned}$$

By (A.3) we have $O_n \log \|h_{n,x}\|_2$ converges to 0 unifromly in $x \in D$. This shows there exists $n_0 = n_0(t_0, R)$ such that for any $n > n_0$

$$\|Q_{n,D}F_s\|_{\infty,D} \leq C_{t_0,R}\|f\|_{2,n}.$$

Now take $t_0 = t/2$, we have

$$\sup_{s\in(0,t/2)} \|Q_{n,D}P^{n,D}_{t/2-s}F_{t/2}\|_\infty = \sup_{s\in(0,t/2)} \|Q_{n,D}F_{t-s}\|_\infty \leq C_{t,R}\|f\|_{2,n}.$$

Hence

$$\|P^{n,D}_{t/2}F_{t/2}\|_\infty \leq C_{t,R}\|f\|_{2,n} + o_n(1)\|f\|_\infty \leq C_{t,R}\|f\|_\infty$$

where in the second inequality we use $\|f\|_{2,n} \leq \mu_n(D)\|f\|_\infty \leq C_R\|f\|_\infty$. Finally, applying Hölder regularity of $U^{n,D}_\alpha$ to get

$$\begin{aligned}
|P^{n,D}_t f(x) - P^{n,D}_t f(y)| &= |U^{n,D}_\alpha P^{n,D}_{t/2}F_{t/2}(x) - U^{n,D}_\alpha P^{n,D}_{t/2}F_{t/2}(y)| \\
&\leq C_1(|x-y|^\gamma + o_n(1))\|P^{n,D}_{t/2}F_{t/2}\|_\infty \\
&\leq C_2(|x-y|^\gamma + o_n(1))\|f\|_\infty.
\end{aligned}$$

The proof is completed. □

We can similarly show locally Hölder continuity of the resolvent $\{U^B_\alpha; \alpha > 0\}$ and semigroup $\{P^B_t; t \geq 0\}$ associated to the killed process X^B, where B is any ball in $\mathbb{R}^d$. We state the result and sketch the proof here. It will be used in Proposition 4.20.

Proposition 4.17. *The resolvent $\{U^B_\alpha; \alpha > 0\}$ and semigroup $\{P^B_t; t \geq 0\}$ are locally Hölder in B. More precisely, given any compact set $K \subseteq B$, $\alpha > 0$ and $t > 0$, there exists C_1 (depending on K, α) and C_2 (depending on K, t) such that for any $x, y \in K$*

$$|U^B_\alpha f(x) - U^B_\alpha f(y)| \leq C_1|x-y|^\gamma\|f\|_\infty$$

$$|P^B_t f(x) - P^B_t f(y)| \leq C_2|x-y|^\gamma\|f\|_2$$

where $\|f\|_\infty = \sup_{x\in\mathbb{R}^d}|f(x)|$ and $\|f\|_2^2 = \int f^2(x)\mu(dx)$.

Proof. For a stopping time τ that is less than τ_B, by the strong Markov property we have for any $f \in C_0(B)$

$$\begin{aligned}
U_\alpha^B f(x) &= \mathbb{E}_x \int_0^{\tau_B} e^{-\alpha t} f(X_t) dt \\
&= \mathbb{E}_x \int_0^{\tau} e^{-\alpha t} f(X_t) dt + \mathbb{E}_x \left[\mathbb{E}_{X_\tau} \int_0^{\tau_B} e^{-\alpha(t+\tau)} f(X_t) dt \right] \\
&= \mathbb{E}_x \int_0^{\tau} e^{-\alpha t} f(X_t) dt + \mathbb{E}_x \left[(e^{-\alpha\tau} - 1) R_\alpha^B f(X_\tau) \right] + \mathbb{E}_x \left[R_\alpha^B f(X_\tau) \right].
\end{aligned}$$

Let δ_K be the distance between B^c and K. For any distinct $x, y \in K$ with $|x-y| < \delta_K^2 \wedge (1/4)$ We set $\tau = \tau_{B(x,r)}$ where $r = |x-y|^{1/2}$ (this guarantees that $\tau < \tau_B$). Then using Lemma 4.14 we can apply the same estimate as in Proposition 4.16 to get the locally Hölder property of R_α^B. To prove Hölder property of the semigroup P_t^B, Let $G^B f(x) := \mathbb{E}_x \int_0^\infty f(X_t^B) dt$ be the Green's operator.

$$\begin{aligned}
\|G^U 1\|_\infty &\le \sup_{x \in U} \int_D g(x-y)\mu(dy) \\
&\le \mu(U)^{\frac{p}{p-1}} \sup_{x \in B(0,R)} \left(\int_{B(x,2R)} g(x,y)^p \mu(dy) \right)^{1/p} \\
&= \mu(U)^{\frac{p}{p-1}} \sup_{x \in B(0,R)} \left(g^p \mathbb{1}_{B(0,2R)} * \mu(x) \right)^{1/p}.
\end{aligned}$$

By the same estimate method as in Lemma 4.13 we can show $(g^p \mathbb{1}_{B(0,2R)} * \mu)(x) \le CR^\beta$ for some $\beta \in (0, \alpha)$ if $d = 2$. In the case $d > 3$ we can choose $p > 1$ such that $\beta = \alpha - p(d-2) > 0$ and get $(g^p \mathbb{1}_{B(0,2R)} * \mu)(x) \le CR^\beta$. Hence there is a constant $C = C(R)$ such that $\|G^{n,U} 1\|_\infty \le C\mu_n(U)^{\frac{p}{p-1}}$. By [GT12, Lemma 3.2] we know the smallest eigenvalue $\Lambda_{\min}$ of the generator $\mathcal{A}^{n,U}$ of $\mathcal{E}^{n,U}$ on $L^2(B(0,R), \mu_n)$ satisfies

$$\Lambda_{\min}(U) \ge \|G^{n,U} 1\|_\infty^{-1} \ge C\mu_n(U)^{-\frac{p}{p-1}}.$$

Then by [GH14, Lemma 5.5] we have $\|P_t^B\|_{L^1 \to L^\infty} < C_{B,t}$ hence the classical method in [BKK10, Proposition 3.4] applies and we get the Hölder continuity of P_t^B. $\square$

4.4 Convergence in Skorokhod topology under individual starting points

We may first use Green's function estimate Proposition 4.1 to prove that μ is a smooth measure.

Proposition 4.18. *The measure μ under the condition (A.3) is a smooth measure in strict sense.*

Proof. One can mimic the proof of Lemma 4.13 to show that for any $R > 0$

$$G_R\mu(x) := \int g_R(x, y)\mu(dx) < \infty$$

where $g_R(x, y)$ is the green kernel for the standard Brownian motion killed upon leaving $B(0, R)$. By [FOT11, Exercise 4.2.2] it shows for any compact set $K \subseteq \mathbb{R}^d$ the measure $1_K \cdot \mu$ is of finite energy integrals. hence by [CF12, Theorem 2.3.7] charge no polar set. Thus μ is a smooth measure in strict sense. □

With the above proposition, by [CF12, Theorem A.3.9] one may construct a strong Markov process $(X, \mathbb{P}_x)_{x\in\mathbb{R}^d}$ called the time-changed Brownian motion by μ.

We first establish the finite dimensional distribution convergence, given below.

Proposition 4.19. *For any sequence $x_n \in \mathbb{R}^d$ converges to $x_0 \in \mathbb{R}^d$, the law of $\{X^n; t \geq 0\}$ under $\mathbb{P}_{x_n}$ converges vaguely in finite dimensional distribution to the law of $\{X; t \geq 0\}$ under $\mathbb{P}_{x_0}$ as $n \to \infty$. In addition, for any $R > 2$, we have the law of $\tau_R(X^n)$ under $\mathbb{P}_{x_n}$ converges weakly to the law of $\tau_R(X)$ under $\mathbb{P}_{x_0}$, where $\tau_R(\Gamma)$ denotes the exit time of the path Γ from $B(0, R)$.*

Proof. Take $R > 2$ large enough such that $x_n \in B(0, R/2)$ and let $D = B(0, R)$. Let $f_0, f_1, ..., f_m$ be bounded functions with compact support and $0 = t_0 < t_1 < ... < t_m$, denote

$$\begin{aligned} G_n(x) &= f_0(x)P^{n,D}_{t_1-t_0}(f_1P^{n,D}_{t_2-t_1}(f_2 \cdots (f_{m-1}P^{n,D}_{t_m-t_{m-1}}f_m)\cdots))(x) \\ &= \mathbb{E}_x[f_0(X^{n,D}_{t_0})f_1(X^{n,D}_{t_1}) \cdots f_m(X^{n,D}_{t_m})] \end{aligned}$$

and $G(x) \equiv G_\infty := \mathbb{E}_x[f_0(X^D_{t_0})f_1(X^D_{t_1})\cdots f_m(X^D_{t_m})]$. We will show

$$\lim_{n\to\infty} |G_n(x_n) - G(x_0)| = 0.$$

Without loss of generality we assume $\|f_i\|_\infty \leq 1$ for $i = 0, ..., m$. Note that $\|G_k\|_\infty \leq \prod_{i=0}^m \|f_i\|_\infty \leq 1$ for all $k \in \mathbb{N} \cup \{\infty\}$. And by Mosco convergence Proposition 3.7 we know for any $f \in C_c(\mathbb{R}^d)$

$$\langle f, G_n\rangle_{\mu_n} \to \langle f, G\rangle_\mu.$$

For $x_0 \in \mathbb{R}^d$ and $\varepsilon > 0$ let $g_\varepsilon \in C_c(\mathbb{R}^d)$ be compactly supported in $B(x_0, \varepsilon)$ and $\int g_\varepsilon d\mu = 1$. Then

$$|G_n(x_n) - G(x_0)| \leq I_1 + I_2 + I_3 + I_4$$

where

$$\begin{aligned}
I_1 &= \left|G_n(x_n) - \int g_\varepsilon(x)G_n(x_n)d\mu_n\right|, \\
I_2 &= \left|\int g_\varepsilon(x)G_n(x_n)d\mu_n - \int g_\varepsilon(x)G_n(x)d\mu_n\right|, \\
I_3 &= \left|\int g_\varepsilon(x)G_n(x)d\mu_n - \int g_\varepsilon(x)G(x)d\mu\right|, \\
I_4 &= \left|\int g_\varepsilon(x)G(x)d\mu - G(x_0)\right|.
\end{aligned}$$

We have

$$\varlimsup_{n\to\infty} I_1 \leq \varlimsup_{n\to\infty}\left|1 - \int g_\varepsilon d\mu_n\right| = \varlimsup_{n\to\infty}\left|1 - \int g_\varepsilon d\mu\right| = 0.$$

For I_3 by Proposition 3.7 we have $\overline{\lim}_{n\to\infty} I_3 = 0$. For I_2 we use Proposition 4.16 to get

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\, I_2 &\le \overline{\lim_{n\to\infty}} \int g_\varepsilon(x)\, |G_n(x) - G_n(x_n)|\, \mu_n(dx) \\
&\le \overline{\lim_{n\to\infty}}\, 2\|g_\varepsilon\|_\infty \mu_n\{x : |x - x_n| \le c_0 r_n\} \\
&\quad + \overline{\lim_{n\to\infty}} \int_{|x-x_n|>c_0 r_n} g_\varepsilon(x)(C_2|x-x_n|^\gamma + o_{r_n}(1))\mu_n(dx) \\
&\le \overline{\lim_{n\to\infty}}\, 2\|g_\varepsilon\|_\infty \mu\{x : |x - x_n| \le (c_0 + 2r_\phi) r_n\} + C_2 \varepsilon^\gamma \int g_\varepsilon d\mu \\
&= o_\varepsilon(1)
\end{aligned}$$

where r_ϕ is the support radius of ϕ (i.e. $\operatorname{supp}(\phi) \subseteq B(0, r_\phi)$) and C_2 is from Proposition 4.16. Similarly for I_4 we have

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\, I_4 &\le \overline{\lim_{n\to\infty}}\, 2\|g_\varepsilon\|_\infty \mu\{x : |x - x_n| \le c_0 r_n\} + \int_{|x-x_n|>c_0 r_n} g_\varepsilon(x)|G(x) - G(x_0)|d\mu \\
&= o_\varepsilon(1).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\lim_{n\to\infty} G_n(x_n) = G(x_0)$$

or equivalently,

$$\lim_{n\to\infty} \mathbb{E}_{x_n}[f_0(X_{t_0}^{n,D}) f_1(X_{t_1}^{n,D}) \cdots f_m(X_{t_m}^{n,D})] = \mathbb{E}_{x_0}[f_0(X_{t_0}^{D}) f_1(X_{t_1}^{D}) \cdots f_m(X_{t_m}^{D})].$$

Taking $m = 1$ and $f_0 = \mathbb{1}_D$, we get

$$\lim_{n\to\infty} \mathbb{E}_{x_n} f_0(X_t^{n,D}) = \mathbb{E}_{x_0} f_0(X_t^D)$$

which is equivalent to $\lim_{n\to\infty} \mathbb{P}_{x_n}\{\tau_R(X^n) > t\} = \mathbb{P}_{x_0}\{\tau_R(X) > t\}$. Hence the law of $\tau_R(X^n)$ under $\mathbb{P}_{x_n}$ converges weakly to the law of $\tau_R(X)$ under $\mathbb{P}_{x_0}$. Next notice that for $n = \mathbb{N}^* \cup \{\infty\}$ we have

$$\begin{aligned}
\mathbb{E}_{x_n}[f_0(X_{t_0}^n) f_1(X_{t_1}^n) \cdots f_m(X_{t_m}^n)] &= \mathbb{E}_{x_n}[f_0(X_{t_0}^{n,D}) f_1(X_{t_1}^{n,D}) \cdots f_m(X_{t_m}^{n,D})] \\
&\quad + \mathbb{E}_{x_n}[f_0(X_{t_0}^{n,D}) f_1(X_{t_1}^{n,D}) \cdots f_m(X_{t_m}^{n,D}); t_m \ge \tau_R(X^n)].
\end{aligned}$$

Then

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \left|\mathbb{E}_{x_n}[f_0(X^n_{t_0})f_1(X^n_{t_1})\cdots f_m(X^n_{t_m})] - \mathbb{E}_{x_0}[f_0(X_{t_0})f_1(X_{t_1})\cdots f_m(X_{t_m})]\right| \\
\le& \overline{\lim_{n\to\infty}} \left|\mathbb{E}_{x_n}[f_0(X^{n,D}_{t_0})f_1(X^{n,D}_{t_1})\cdots f_m(X^{n,D}_{t_m})] - \mathbb{E}_{x_0}[f_0(X^D_{t_0})f_1(X^D_{t_1})\cdots f_m(X^D_{t_m})]\right| \\
&+ \overline{\lim_{n\to\infty}} \mathbb{P}_{x_n}\{t_m \ge \tau_R(X^n)\} + \mathbb{P}_{x_0}\{t_m \ge \tau_R(X)\} \\
=& 2\mathbb{P}_{x_0}\{t_m \ge \tau_R(X)\}.
\end{aligned}$$

Because R can be taken arbitrarily large, we see

$$\overline{\lim_{n\to\infty}} \mathbb{E}_{x_n}[f_0(X^n_{t_0})f_1(X^n_{t_1})\cdots f_m(X^n_{t_m})] = \mathbb{E}_{x_0}[f_0(X_{t_0})f_1(X_{t_1})\cdots f_m(X_{t_m})],$$

which shows the vague convergence of finite dimensional distributions of X^n under P_{x_n} to X under P_{x_0}.

□

Finally we establish the tightness of the law X^n in Skorokhod topology. The following proposition will be used to verify Aldous' tightness condition in Skorokhod topology and used in the proof of Proposition 4.21.

Proposition 4.20. *For any ball $B \subseteq \mathbb{R}^d$, the killed process X^B of the time-changed Brownian motion upon leaving B is a Feller process on the space $C_0(B)$ of continuous functions on $\bar{B}$ vanishing on the boundary ∂B. Hence its associated semigroup $\{P^B_t, t \ge 0\}$ satisfies $\lim_{t\to 0}\sup_{x\in B}|P^B_t f(x) - f(x)| = 0$ for any $f \in C_0(B)$.*

Proof. To show X^B is a Feller process, we want to show that for any $f \in C_0(B)$ we have $P^B_t f \in C_0(B)$ and $\lim_{t\downarrow 0} P^B_t f(x) = f(x)$ for any $x \in B$. Then the uniform convergence follows from [RY99, Proposition 3.2.3]. The continuity of $P^B_t f$ on B follows directly from Proposition 4.17. For any $a \in \partial B$, a is a regular point for X (i.e. $\mathbb{P}_a\{\sigma_B = 0\} = 1$ where $\sigma_B = \inf\{s > 0 : X_s \notin B\}$). This is because a is regular for the standard Brownian motion and X is the strictly increasing and continuous time-changed process of the standard Brownian motion. Then we claim $\lim_{x\to a}\mathbb{P}_x\{\tau_B > t\} = 0$ for any $t > 0$. Indeed, for $x \in \mathbb{R}^d$

$$\mathbb{P}_x\{\sigma_B > t\} = \lim_{\delta\downarrow 0}\mathbb{P}_x\{X_s \in B; \delta \le s \le t\} = \inf_{\delta>0}\mathbb{E}_x[\mathbb{P}_{X_\delta}\{X_s \in B; 0 \le s \le t-\delta\}].$$

Set $g_\delta(x) = \mathbb{P}_x\{X_s \in B; 0 \le s \le t-\delta\}$, then g_δ is bounded function vanishing outside B. Then $P_\delta g_\delta$ is continuous by Hölder continuity of P_δ (Proposition 4.16). In addition $g(x) := \mathbb{P}_x\{\tau_B > t\} = \inf_{\delta>0} P_\delta g_\delta(x)$ hence g is upper semicontinuous and $\overline{\lim}_{x\to a} \mathbb{P}_x\{\tau_B > t\} \le g(a) = 0$. Then we have $P_t^B f$ continuously vanish on the boundary as

$$\varlimsup_{\substack{x\in B\\x\to a}} P_t^B f(x) = \varlimsup_{\substack{x\in B\\x\to a}} \mathbb{E}_x[f(X_t); \tau_B > t] \le \|f\|_\infty \varlimsup_{\substack{x\in B\\x\to a}} \mathbb{P}_x\{\tau_B > t\} = 0.$$

Finally, for any $x \in B$, $\lim_{t\downarrow 0} P_t^B f(x) = f(x)$ by continuity of sample paths of X and the bounded convergence theorem. Then $\{P_t^B; t \ge 0\}$ is a Feller semigroup on $C_0(B)$. □

Proposition 4.21. *For any bounded sequence $x_n \in \mathbb{R}^d$, the law of $\{X^n; t \ge 0\}$ under $\mathbb{P}_{x_n}$ are tight in the Skorokhod topology.*

Proof. We apply the result of Aldous [Ald78] to prove tightness. For convenience of readers we state Aldous's result here. Let $T > 0$. If a sequence of stochastic process X^n taking values in the Skorokhod space $\mathbf{D}[0,T]$ is tight if it satisfies

- The sequence X_0^n is tight on the real line.
- The sequence $J(X^n) := \max_{t\in[0,T]} |X_t^n - X_{t-}^n|$ is tight on the real line
- For any stopping time $\tau_n \le T$ with respect to the natural filtration of X^n, $\delta_n \downarrow 0$, we have $X^n_{(\tau_n+\delta_n)\wedge T} - X^n_{\tau_n}$ converges in probability.

In our case, the first two conditions are clearly satisfied as the sequence $\{x_n\}$ is bounded and the jump $J(X^n)$ is stochastically dominated by $r_n|\xi_1|$. We now show the third condition is also true. For any $\varepsilon > 0$, we need to show that

$$\lim_{n\to\infty} \mathbb{P}_{x_n}\{|X^n_{\tau_n+\delta_n} - X^n_{\tau_n}| > \varepsilon\} = 0.$$

Let $\tau(X^n)$ be the exit time of X^n upon leaving $B(0,R)$. Choose $R > 0$ large enough such that $x_n \in B(0, R/2)$ for any n and $\mathbb{P}_{x_0}\{\tau(X) \le 2T\} < \varepsilon$. Then by Proposition 4.19 we have

$\mathbb{P}_{x_n}\{\tau(X^n) \le 2T\} < \varepsilon$ for any large enough n. By the strong Markov property

$$\begin{aligned}\mathbb{P}_{x_n}\{|X^n_{\tau_n+\delta_n} - X^n_{\tau_n}| > \varepsilon\} &= \mathbb{E}_{x_n}[\mathbb{P}_{X^n_{\tau_n}}\{|X^n_{\delta_n} - X^n_0| > \varepsilon\}; \tau(X^n) > 2T] + \mathbb{P}_{x_n}\{\tau(X^n) \le 2T\} \\ &\le \sup_{y \in B(0,R)} \mathbb{P}_y\{|X^n_{\delta_n} - y| > \varepsilon\} + 2\varepsilon.\end{aligned}$$

Let $y_1, ..., y_N$ be a $\varepsilon/3$-net of $B(0,R)$, and for each $k = 1, ..., N$ choose $f_k \in C_c^\infty(B(y_k, 2\varepsilon/3))$ such that $f_k(x) = 1$ for $|x - y_k| \le \varepsilon/3$. Then for $y \in B(y_k, \varepsilon/3)$

$$\begin{aligned}\mathbb{P}_y\{|X^n_{\delta_n} - y| > \varepsilon\} &\le \mathbb{P}_y\{|X^n_{\delta_n} - y_k| > 2\varepsilon/3\} \\ &\le \mathbb{P}_y\{\tau^n_{B(y_k,2\varepsilon/3)} \le \delta_n\} \\ &\le 1 - P^{n,B(y_k,2\varepsilon/3)}_{\delta_n} f_k(y)\end{aligned}$$

where τ^n_B is the exit time of X^n of B and $P^{n,B}_t$ is the transition semigroup of X^n killed upon leaving B.

Note that for any ball B, the transition semigroup P^B_t of killed process X upon leaving B is a Feller semigroup by Proposition 4.20. Hence $P^B_t f$ converges uniformly to f as $t \downarrow 0$ for any $f \in C_c(B)$. Since $f_k(x) = 1$ for any $x \in B(y_k, \varepsilon/3)$, we can choose $\delta_0 > 0$ such that $P^{B(y_k,2\varepsilon/3)}_{\delta_0} f_k(y) > 1 - \varepsilon$ for any $k = 1, ..., N$ and $y \in B(y_k, \varepsilon/3)$.

We claim $P^{n,B(y_k,2\varepsilon/3)}_{\delta} f_k$ converges uniformly to $P^{B(y_k,2\varepsilon/3)}_{\delta} f_k$ as $n \to \infty$ for any $k = 1, ..., N$ and $\delta > 0$. This is because $P^{n,B(y_k,2\varepsilon/3)}_{\delta} f_k$ converges pointwisely and $P^{n,B(y_k,2\varepsilon/3)}_{\delta} f_k$ is uniformly Hölder in n (Proposition 4.16). Thus using a finite ε-net trick as in the proof of Proposition 3.11 we can show there exists n_0 such that $P^{n,B(y_k,2\varepsilon/3)}_{\delta} f_k > 1 - 2\varepsilon$ for any $n > n_0$. Then for $y \in B(0,R)$ such that $y \in B(y_k, \varepsilon/3)$ and for any $\delta_n \in (0, \delta_0)$,

$$\begin{aligned}\mathbb{P}_y\{|X^n_{\delta_n} - y| > \varepsilon\} &\le \mathbb{P}_y\{\tau^n_{B(y_k,2\varepsilon/3)} \le \delta_n\} \\ &\le \mathbb{P}_y\{\tau^n_{B(y_k,2\varepsilon/3)} \le \delta_0\} \\ &\le 1 - P^{n,B(y_k,2\varepsilon/3)}_{\delta_0} f_k(y) \le 2\varepsilon\end{aligned}$$

hence $\mathbb{P}_{x_n}\{|X^n_{\tau_n+\delta_n} - X^n_{\tau_n}| > \varepsilon\} \le 2\varepsilon + 2\varepsilon = 4\varepsilon$ for any large enough n. This shows the third condition of Aldous's result holds, thus we have tightness on $\mathbf{D}[0,T]$. As $T > 0$ is arbitrary, tightness holds on $\mathbf{D}[0,\infty)$. □

Proof of Theorem 1.2. By Proposition 4.21 we know for any subsequence of n there is a further subsequence n_k such that the law of X^{n_k} under $\mathbb{P}_{x_{n_k}}$ converges in Shorohod topology to some process Z. By Proposition 4.19 we know Z must have the same finite dimensional distribution as the time changed Brownian motion X starting from x_0. This shows X^n under $\mathbb{P}_{x_n}$ converges to X under $\mathbb{P}_{x_0}$. □

Remark. As it is pointed out in [Ald89, Proposition 3.2], convergence in finite dimensional distributions in fact implies convergence in pseudo path topology. Hence the convergence of distribution in Skorokhod topology of the stopped processes of X^n under $\mathbb{P}_{x_n}$ can be established in the same way as in Proposition 3.10 and then the convergence can be extended to the non-stopped processes X^n by applying the same proof for Theorem 1.1 using the fact that the law of $\tau(X^n)$ under $\mathbb{P}^{(n)}_{x_n}$ converges weakly to the law of $\tau(X)$ under $\mathbb{P}_{x_0}$ given in Proposition 4.19, and the Theorem 1.2 follows. By proving the tightness of the law of X^n in Skorokhod topology under $\mathbb{P}_{x_n}$ using the result from [Ald78, Theorem 1] and Feller properties, we provide an alternative proof of Theorem 1.2.

Chapter 5

APPROXIMATION ON GRIDS

We can generalize the approximation results on grids. As the main idea is similar to the case on $\mathbb{R}^d$, we will only sketch the proofs and point out the differences.

5.1 *Setups and notations*

Let $\bar{\varrho}$ be a probability distribution on $\mathbb{Z}^d$ (a nonnegative function that $\sum_{z\in\mathbb{Z}^d}\bar{\varrho}(z)=1$) satisfying $\bar{\varrho}(z)=\bar{\varrho}(-z)$ for each $z\in\mathbb{Z}^d$, $\sum_{z\in\mathbb{Z}^d} z\bar{\varrho}(z)=0$ and $\sum_{z\in\mathbb{Z}^d} z_iz_j\bar{\varrho}(z)=M_{\bar{\varrho}}\delta_{ij}$ where $M_{\bar{\varrho}}>0$ and z_i,z_j are components of $z\in\mathbb{Z}^d$.

Let $\mathfrak{C}_0:=[-1/2,1/2]^d$. For each $z\in(r\mathbb{Z})^d$ and $r>0$ denote $\mathfrak{C}_{z,r}=z+r\mathfrak{C}_0$. Let $\bar{n}_r$ be the counting measure on $(r\mathbb{Z})^d$ (i.e. $\bar{n}_r(\{z\})=1$ for every $z\in(r\mathbb{Z})^d$), $\bar{m}_r(dz)=\frac{M_{\bar{\varrho}}^{-1}}{r^{2-d}}\bar{n}_r(dz)$ and $\bar{\mu}_r(dx)=\mu_r(\mathfrak{C}_{x,r})\bar{n}_r(dx)$. For each function $\bar{f}:(r\mathbb{Z})^d\to\mathbb{R}$, define $f:\mathbb{R}^d\to\mathbb{R}$ by

$$f=\sum_{z\in(r\mathbb{Z})^d}\bar{f}(z)\mathbb{1}_{\mathrm{Int}(\mathfrak{C}_{z,r})} \tag{5.1}$$

where $\mathrm{Int}(\mathfrak{C}_{z,r})$ is the interior of $\mathfrak{C}_{z,r}$. We call f the lift-up of $\bar{f}$ (from $(r\mathbb{Z})^d$). In particular, we may map $\bar{\varrho}$ to its lift-up ϱ (note that $\int_{\mathbb{R}^d}\varrho(x)dx=1$ as $|\mathfrak{C}_0|=1$) and choose a probability density function ϕ such that (ϱ,ϕ) satisfies (A.1). It is straightforward to check that this construction of ϱ satisfies the basic assumptions of ϱ in Chapter 1: $\varrho(x)=\varrho(-x)$ for every $x\in\mathbb{R}^d$, $\int_{\mathbb{R}^d}x\varrho(x)dx=0$ and $\int_{\mathbb{R}^d}x_ix_j\varrho(x)dx=M_\varrho\delta_{ij}$ where $M_\varrho=c_1M_{\bar{\varrho}}+c_2$ for some constants $c_1,c_2>0$.

We construction $\bar{X}^r$ like in Chapter 1 with jump distribution $\bar{\varrho}(z/r)\bar{n}_r(dz)$ instead of ϱ_r and jumping rate being $\frac{d\bar{m}_r}{d\bar{\mu}_r}=(M_{\bar{\varrho}}r^{2-d}\mu_r(\mathfrak{C}_{x,r}))^{-1}$ instead of $\frac{dm_r}{d\mu_r}(x)$. Then $\bar{X}^r$ is a random walk living on the grid $(r\mathbb{Z})^d$. We state similar results as Theorem 1.1 and Theorem 1.2.

Theorem 5.1. *For any $f_0 \geq 0$ in $C_c(\mathbb{R}^d)$, set $\nu_r = f_0 \cdot \bar{\mu}_r$ and $\nu = f_0 \cdot \mu$. Then the law of $\{\bar{X}^r; t \geq 0\}$ under $\mathbb{P}_{\nu_r}$ converges weakly in $\mathbf{D}([0,\infty);\mathbb{R}^d)$ in Skorokhod topology to the time-changed Brownian motion X by μ with initial distribution ν.*

Theorem 5.2. *Assume (A.3) and there exist $\delta > 0$ and $m = max(2, d-2+\delta)$ such that $\sum_{z\in\mathbb{Z}^d} |z|^m \bar{\varrho}(z) < \infty$. Then for any sequence $x_r \in (r\mathbb{Z})^d \subseteq \mathbb{R}^d$ converges to $x_0 \in \mathbb{R}^d$, the law of $\{\bar{X}^r; t \geq 0\}$ under $\mathbb{P}_{x_r}$ converges weakly in $\mathbf{D}([0,\infty);\mathbb{R}^d)$ in Skorokhod topology to the law of $\{X; t \geq 0\}$ starting from x_0 as $r \downarrow 0$.*

Remark. In the grid setting, conditions on $\bar{\varrho}$ are weakened, because (A.1) will be only applied through ϱ in the proof and $\bar{\varrho}$ is automatically bounded ((A.2) holds for $\bar{\varrho}$).

To prepare for the proofs, we introduce symbols on $(r\mathbb{Z})^d$. The general principle is adding bars to turn objects on $\mathbb{R}^d$ to $(r\mathbb{Z})^d$. Let $\bar{Q}_r(x,dy) := \bar{\varrho}(\frac{y-x}{r})\bar{n}_r(dy)$ for $x, y \in (r\mathbb{Z})^d$. It is clear that

$$\bar{Q}_r(x,dy)\bar{m}_r(dx) = \frac{M_{\bar{\varrho}}^{-1}}{r^{2-d}}\bar{\varrho}(\frac{y-x}{r})\bar{n}_r(dx)\bar{n}_r(dy) = \bar{Q}_r(y,dx)\bar{m}_r(dy).$$

Note that for any bounded function $\bar{f}$ on $(r\mathbb{Z})^d$, we have for $x \in \mathbb{Z}^d$

$$\bar{Q}_r\bar{f}(rx) = \int \bar{f}(y)\bar{\varrho}(\frac{y-x}{r})\bar{n}_r(dy) = \int f(ry)\varrho(y-rx)m_0(dy)$$

and

$$\int_{(r\mathbb{Z})^d} |\bar{f}(z)|^p \bar{\mu}_r(dz) = \sum_{z\in(r\mathbb{Z})^d} |\bar{f}(z)|^p \mu_r(\mathfrak{C}_{z,r}) = \int_{\mathbb{R}^d} |f(z)|^p \mu_r(dz).$$

We write down Dirichlet forms assoicated to $\bar{X}^r$:

$$\begin{aligned} \bar{\mathcal{F}}^r &= \{\bar{u} \in L^2_{\text{loc}}((r\mathbb{Z})^d) : \bar{\mathcal{E}}^r(\bar{u}) < \infty\} \cap L^2((r\mathbb{Z})^d; \bar{\mu}_r) \\ \bar{\mathcal{E}}^r(\bar{u}) &= \frac{1}{2}\int\int (\bar{u}(x) - \bar{u}(y))^2 \bar{Q}_r(x,dy)\bar{m}_r(dx). \end{aligned}$$

For $R > 2$ let $\bar{E}_r = B(0,R) \cap (r\mathbb{Z})^d$. Define $(\bar{\mathcal{F}}^{r,R}, \bar{\mathcal{E}}^{r,R})$ by

$$\begin{aligned} \bar{\mathcal{F}}^{r,R} =& \{\bar{u} \in L^2_{\text{loc}}((r\mathbb{Z})^d) : \mathcal{E}^r(\bar{u}) < \infty\} \cap L^2(\bar{E}_r; \bar{\mu}_r|_{\bar{E}_r}) \\ \bar{\mathcal{E}}^{r,R}(\bar{u}) =& \frac{1}{2}\int_{\bar{E}_r\times\bar{E}_r} (\bar{u}(x) - \bar{u}(y))^2 \bar{Q}_r(x,dy)\bar{m}_r(dx) + \int_{\bar{E}_r} \bar{u}(x)^2(1 - \bar{Q}_r(x,\bar{E}))\bar{m}_r(dx). \end{aligned}$$

The above Dirichlet form is associated to $\bar{X}^r$ killed upon exiting $B(0,R)$. Note that for any $\bar{u} \in \bar{\mathcal{F}}^r$ we have $u \in \mathcal{F}^r$. Moreover, for $\bar{u} \in \bar{\mathcal{F}}^{r,R}$ we have $u \in \mathcal{F}^{r,E_r}$ and $\bar{\mathcal{E}}^{r,R}(\bar{u}) = \mathcal{E}^{r,E_r}(u)$, where $E_r = \mathrm{Int}(\cup_{z\in\bar{E}_r}\mathfrak{C}_{z,r})$ and $(\mathcal{F}^{r,E_r}, \mathcal{E}^{r,E_r})$ are defined in (2.1).

5.2 Approximation under symmetrizing measures with settings on grids

Let $\bar{\mathcal{H}}_r := L^2((r\mathbb{Z})^d;\bar{\mu}_r)$, $\mathcal{H} := L^2(\mathbb{R}^d;\mu)$ and $\mathcal{C} := C_c^\infty(\mathbb{R}^d)$. Define $\bar{\Phi}_r : \mathcal{C} \to \mathcal{H}_r$ by $\bar{\Phi}(u)(z) = u(z)$ for $z \in (r\mathbb{Z})^d$. We can check

$$\|\Phi_r(u)\|_{\bar{\mathcal{H}}_r} = \sum_{z\in(r\mathbb{Z})^d} u(z)^2\mu_r(\mathfrak{C}_{z,r}) = \sum_{z\in(r\mathbb{Z})^d} \int_{\mathfrak{C}_{z,r}} u(x)^2\mu_r(dx) + o_r(1) \to \|u\|_{\mathcal{H}}$$

which shows $\mathcal{H}_r$ converges to $\mathcal{H}$ in the sense of Definition 2.1. In addition, $\bar{u}_r \in \bar{\mathcal{H}}_r$ converges to $u \in \mathcal{H}$ in the sense of Definition 2.3 if and only if $\phi_r * u_r$ converges to u in $\mathcal{H}$. Indeed, for any function $f \in \mathcal{C}$ we have

$$\begin{aligned}
\int_{(r\mathbb{Z})^d} \bar{u}_r\Phi_r(f)d\bar{\mu}_r &= \sum_{z\in(r\mathbb{Z})^d} \int_{\mathfrak{C}_{z,r}} u_r(x)f(z)\mu_r(dx) \\
&= \int_{\mathbb{R}^d} \phi_r * (u_r f)(x)\mu(dx) + o_r(1) \\
&= \int_{\mathbb{R}^d} \phi_r * u_r(x)f(x)\mu(dx) + o_r(1)
\end{aligned}$$

which shows

$$\lim_{r\downarrow 0}\langle \bar{u}_r, \Phi_r(f)\rangle_{\bar{\mathcal{H}}_r} = \lim_{r\downarrow 0}\langle \phi_r * u_r, f\rangle_{\mathcal{H}}$$

provided the limit of either side equal to $\langle u, f\rangle_{\mathcal{H}}$.

We now show Mosco convergence of the form $(\bar{\mathcal{F}}^r, \bar{\mathcal{E}}^r)$ to $(\mathcal{E}, \mathcal{F}^\mu)$. For any functions $\bar{u}_r \in \bar{\mathcal{H}}_r$ with $\sup_r \bar{\mathcal{E}}^r(\bar{u}_r) < \infty$ and converges to $u \in \mathcal{H}$ in the sense of Definition 2.3, we want to show that there exists a μ-version $\widetilde{u}$ of u such that $\mathcal{E}(\widetilde{u}) \leq \underline{\lim}_r \bar{\mathcal{E}}^r(\bar{u}_r)$. The proof in Proposition 3.7 still applies. On the other hand, for the second condition of Mosco convergence, by [CF12, Theorem 5.1.6] we know $C_c^\infty(\mathbb{R}^d)$ is $\mathcal{E}_1^\mu$-dense in $\mathcal{F}^\mu$. For $u \in \mathcal{F}^\mu$ we can choose $u_n \in C_c^\infty(\mathbb{R}^d)$ $\mathcal{E}_1^\mu$-converges to u. By Proposition 2.7 we can find subsequence v_n of u_n that converges in the sense of Definition 2.2. Note that v_n also $\mathcal{E}$-converges to u. By

Proposition 3.4 and Proposition 3.5 we have $\overline{\lim}_{n\to\infty}\bar{\mathcal{E}}^{r_n}(v_n) \le \overline{\lim}_{n\to\infty}\lim_{k\to\infty}\bar{\mathcal{E}}^{r_n/k}(v_n) = \overline{\lim}_{n\to\infty}\mathcal{E}(v_n) = \mathcal{E}(u)$. Hence we find $\bar{v}_n := v_n|_{(r_n\mathbb{Z})^d}$ such that $\overline{\lim}_{n\to\infty}\bar{\mathcal{E}}^{r_n}(\bar{v}_n) \le \mathcal{E}(u)$. For $(\bar{\mathcal{F}}^{r,R}, \bar{\mathcal{E}}^{r,R})$ the proof is similar.

Next we can follow the proof of Proposition 3.10 to show the convergence of stopped processes in Skorokhod topology. Then Theorem 1.1 can be extended to the approximation on grids(Theorem 5.1).

5.3 Approximation under individual starting points and settings on grids

We outline the proof of Theorem 5.2.

For the results starting from individual points, we need the Hölder regularity of the semigroups after time change. For this purpose, we need the uniform Green's function estimates like Proposition 4.1 and the Hölder regularity like Proposition 4.7 with setting on $\mathbb{Z}^d$. For Proposition 4.1 the main ingredient is the local central limit theorem and BDG inequality. These two are known to still hold in the setting on $\mathbb{Z}^d$ (see, e.g. [LL10, Theorem 2.3.9]). For Proposition 4.7, the Hölder regularity of the semigroups (thus the harmonic functions) of random walks without time change has been shown in [BK08, Theorem 4.9].

With Green's function estimates, we can get similar exit time estimates as in Lemma 4.14 and semigroup contractive properties as in Lemma 4.15. Then together with the Hölder regularity of the harmonic functions, we can follow the proof of Proposition 4.16 to show Hölder regularity of the semigroups after time change. Then Theorem 1.2 can be extended to the approximation on grids (Theorem 5.2).

BIBLIOGRAPHY

[AK16] Sebastian Andres and Naotaka Kajino. Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions. *Probab. Theory Related Fields*, 166(3-4):713–752, 2016.

[Ald78] David Aldous. Stopping times and tightness. *Ann. Probab.*, 6(2):335–340, 1978.

[Ald89] David Aldous. Stopping times and tightness. II. *Ann. Probab.*, 17(2):586–595, 1989.

[Ber15] Nathanaël Berestycki. Diffusion in planar Liouville quantum gravity. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(3):947–964, 2015.

[BK08] Richard F. Bass and Takashi Kumagai. Symmetric Markov chains on $\mathbb{Z}^d$ with unbounded range. *Trans. Amer. Math. Soc.*, 360(4):2041–2075, 2008.

[BKK10] Richard F. Bass, Moritz Kassmann, and Takashi Kumagai. Symmetric jump processes: localization, heat kernels and convergence. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(1):59–71, 2010.

[CCK15] Zhen-Qing Chen, David A. Croydon, and Takashi Kumagai. Quenched invariance principles for random walks and elliptic diffusions in random media with boundary. *Ann. Probab.*, 43(4):1594–1642, 2015.

[CF12] Zhen-Qing Chen and Masatoshi Fukushima. *Symmetric Markov processes, time change, and boundary theory*, volume 35 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2012.

[CFS01] Z.-Q. Chen, P. J. Fitzsimmons, and R. Song. Crossing estimates for symmetric Markov processes. *Probab. Theory Related Fields*, 120(1):68–84, 2001.

[CKK13] Zhen-Qing Chen, Panki Kim, and Takashi Kumagai. Discrete approximation of symmetric jump processes on metric measure spaces. *Probab. Theory Related Fields*, 155(3-4):703–749, 2013.

[DL45] J. Deny and J. L. Lions. Les espaces du type de Beppo Levi. *Ann. Inst. Fourier (Grenoble)*, 21:305–370, 1945.

[FOT11] Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2011.

[GH14] Alexander Grigor'yan and Jiaxin Hu. Upper bounds of heat kernels on doubling spaces. *Mosc. Math. J.*, 14(3):505–563, 641–642, 2014.

[GRV16] Christophe Garban, Rémi Rhodes, and Vincent Vargas. Liouville Brownian motion. *Ann. Probab.*, 44(4):3076–3110, 2016.

[GT12] Alexander Grigor'yan and Andras Telcs. Two-sided estimates of heat kernels on metric measure spaces. *Ann. Probab.*, 40(3):1212–1284, 2012.

[IL71] I. A. Ibragimov and Yu. V. Linnik. *Independent and stationary sequences of random variables.* Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov.

[Kal97] Olav Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, 1997.

[Kim06] Panki Kim. Weak convergence of censored and reflected stable processes. *Stochastic Process. Appl.*, 116(12):1792–1814, 2006.

[Kol05] Alexander V. Kolesnikov. Convergence of Dirichlet forms with changing speed measures on $\mathbb{R}^d$. *Forum Math.*, 17(2):225–259, 2005.

[KS03] Kazuhiro Kuwae and Takashi Shioya. Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry. *Comm. Anal. Geom.*, 11(4):599–673, 2003.

[LL10] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2010.

[Mos94] Umberto Mosco. Composite media and asymptotic Dirichlet forms. *J. Funct. Anal.*, 123(2):368–421, 1994.

[MZ84] P.-A. Meyer and W. A. Zheng. Tightness criteria for laws of semimartingales. *Ann. Inst. H. Poincaré Probab. Statist.*, 20(4):353–372, 1984.

[RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, third edition, 1999.