Deformations of Categories of Coherent Sheaves and Fourier-Mukai Transforms

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Abstract

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In modern algebraic geometry, an algebraic variety is often studied by way of its category of coherent sheaves or derived category. Recent work by Toda has shown that infinitesimal deformations of the category of coherent sheaves can be described as twisted sheaves on a noncommutative deformation of the variety. This thesis generalizes Toda's work by creating a chain of inclusions from deformations of schemes to commutative deformations to deformations of the category of coherent sheaves. We define projections from coherent deformations to commutative deformations to scheme deformations and show that the fiber of the projection from commutative deformations to schemes is a gerbe.

We also prove that for two derived equivalent K3 surfaces in characteristic p and any scheme deformation of one of these, there is a scheme deformation of the other so that the two deformations are also derived equivalent.

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1 Introduction

This thesis studies deformations of the category of coherent sheaves on a scheme and how derived equivalences lift to such deformations. This is an interesting area of study for many reasons. Deformation arguments are very powerful, so it is useful to better understand how the derived category of a scheme interacts with deformations. Deformations of the category of coherent sheaves can be seen as a generalization of deformations of the scheme, so it is interesting to study how these more general deformations behave. In particular, some of these deformations are noncommutative.

The basic question that motivates our study is the following: Let X_0 and Y_0 be two schemes, and Φ_0 be an equivalence between the bounded derived category of coherent sheaves of X_0 and Y_0. Let X be a deformation of X_0. Is there a deformation Y of Y_0 and an equivalence Φ between the derived categories of X and Y, with Φ reducing to Φ_0?

$$\begin{array}{c} D(X) \xrightarrow{\Phi} D(Y) \\ \Big\downarrow \\ D(X_0) \xrightarrow{\Phi_0} D(Y_0) \end{array}$$

The answer to this question is *not always*. But the problem can be remedied if we replace deformations of schemes with deformations of the categories of coherent sheaves.

Deformations of categories

Deformation theory of abelian categories is developed in [LVdB2] and [LVdB1] as a generalization of Gerstenhaber's deformation theory of algebras [GS]. We are particularly interested in deformations of the category $\mathrm{Coh}(X_0)$ of coherent sheaves on some scheme X_0, which can be seen as a generalization of the deformation theory of X_0 as a scheme.

The infinitesimal deformations of $\mathrm{Coh}(X_0)$ are parameterized by the second Hochschild cohomology of X_0, which decomposes via the Hochschild-Kostant-Rosenberg (HKR) isomorphism [Yek] into a direct sum of cohomology groups of wedge products of the tangent sheaf:

$$HH^2(X_0) \cong H^2(X_0, \mathcal{O}_{X_0}) \oplus H^1(X_0, T_{X_0}) \oplus H^0(X_0, \wedge^2 T_{X_0}).$$

In [Toda], Toda gives a description of an infinitesimal deformation of $\mathrm{Coh}(X_0)$

as a category of twisted sheaves on a deformation of X_0 with a noncommutative structure sheaf, with each part of this description corresponding to one summand of the HKR decomposition. Toda describes deformations of $\mathrm{Coh}(X_0)$ from the complex numbers $\mathbf{C}$ to the complex dual numbers $\mathbf{C}[\varepsilon]/(\varepsilon^2)$.

The main result of the first part of the thesis is a generalization of the HKR decomposition, which is manifested as inclusions of deformation pseudofunctors

$$\mathrm{Def}^{\,\mathrm{sch}} \to \mathrm{Def}^{\,\mathrm{comm}} \to \mathrm{Def}^{\,\mathrm{coh}} \tag{1}$$

and left inverses

$$\mathrm{Def}^{\,\mathrm{sch}} \leftarrow \mathrm{Def}^{\,\mathrm{comm}} \leftarrow \mathrm{Def}^{\,\mathrm{coh}}, \tag{2}$$

where the notation signifies deformations of schemes, commutative deformations of $\mathrm{Coh}(X_0)$, and deformations of $\mathrm{Coh}(X_0)$, respectively.

The map $\mathrm{Def}^{\,\mathrm{coh}} \to \mathrm{Def}^{\,\mathrm{comm}}$ of (2) requires a square-zero deformation, but all other maps exist in general. Furthermore, each fiber of the map $\mathrm{Def}^{\,\mathrm{comm}} \to \mathrm{Def}^{\,\mathrm{sch}}$ in (2) is a category of gerbes.

Hence (1) is the analog of the inclusions of summands of the Hochschild Cohomology

$$H^1(X_0, T_{X_0}) \to H^2(X_0, \mathcal{O}_{X_0}) \oplus H^1(X_0, T_{X_0}) \to HH^2(X_0).$$

Fourier-Mukai Transforms

A Fourier-Mukai transform is a type of functor between the derived categories of coherent sheaves of two schemes that can be written in terms of a single object in the derived category of the product scheme [Muk, Huy]. It is a theorem of Orlov that any derived equivalence of smooth projective varieties is a Fourier-Mukai transform [Orl1].

Toda also studied the relationship between Fourier-Mukai equivalences and deformations of $\mathrm{Coh}(X_0)$ and showed that given an equivalence $D^b(X_0) \to D^b(Y_0)$ and a first-order deformation of $\mathrm{Coh}(X_0)$, there is a corresponding deformation of $\mathrm{Coh}(Y_0)$ so that the equivalence lifts to the derived categories of the deformations [Toda].

These transforms enrich the study of deformations of $\mathrm{Coh}(X_0)$ because the induced map $HH^2(X_0) \to HH^2(Y_0)$ often mixes summands of the HKR decomposition, in many cases creating a duality between deformations of a scheme, which are easy to understand, and twisted or noncommutative deformations, which are not.

We prove that for two derived equivalent K3 surfaces X_0 and Y_0 in characteristic p and any infinitesimal scheme deformation X of X_0, there is a scheme deformation Y of Y_0 so that the Y is derived equivalent to X.

Overview

In Chapter 2, we review the theory of deformations of schemes and summarize the theory of deformations of abelian categories. Then in Chapter 3, we study $\mathrm{Def}^{\,\mathrm{sch}}$, $\mathrm{Def}^{\,\mathrm{comm}}$, and $\mathrm{Def}^{\,\mathrm{coh}}$. In Chapter 4, we discuss the theory of Fourier-Mukai transforms. In Chapter 5, we prove the results about lifting Fourier-Mukai partnerships.

2 Deformations of schemes and categories

For the convenience of the reader, we will review deformations of schemes and summarize Lowen and Vandenberg's theory of deformations of abelian categories.

2.1 Deformations of schemes

The notion of a deformation comes from the idea of "continuously deforming" one space into another. In algebraic geometry, a collection of schemes that "vary continuously" is called a *family*, and is usually defined as a space over a base $X \to B$. The fibers of this map give the individual schemes in the family. The condition of "varying continuously" is that the map be *flat*.

The terms *family* and *deformation* are often used interchangeably, but by *deformation theory*, we generally mean families over a nilpotent extension of a given base scheme. The example to have in mind is a scheme X over a field k, in which case deformations of X to $k[\epsilon]/\epsilon^2$ are simply families over the base $k[\epsilon]/\epsilon^2$.

Definition 2.1.1 (Deformation) Let $A \to A_0$ be a map of commutative rings with nilpotent ideal J. Let X_0 be a scheme over A_0. A *deformation* of X_0 to A is a scheme X, flat over A, together with an inclusion $\iota : X_0 \to X$ of A-schemes so that $\iota \otimes_A A_0$ is an isomorphism.

Definition 2.1.2 (Equivalence of deformations) Two deformations X and X' are considered *equivalent* if there is an isomorphism that is compatible with the given inclusions:

$$\begin{tikzcd} X \arrow[rr, "\cong"] & & X' \\ & X_0 \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

Remark 2.1.3 It is useful to note that the underlying topological spaces of X_0, X, and X' are all the same, so that we could really make the definition of a deformation in terms of a map of structure sheaves $\mathcal{O}_X \to \mathcal{O}_{X_0}$, and then the equivalence of deformations looks like this:

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr, "\cong"] \arrow[dr, two heads] & & \mathcal{O}_{X'} \arrow[dl, two heads] \\ & \mathcal{O}_{X_0} & \end{tikzcd}$$

2.1.4 Category of deformations One way to package this information together is to form a category of deformations of X_0 to A, where the objects are deformations and the morphisms are equivalences of deformations. Note that this category is in fact a *groupoid*, that is, all morphisms are isomorphisms.

For example, if X_0 is a scheme over k, then the set of deformations to $k[\epsilon]/\epsilon^2$, up to equivalence, is given by $H^1(X_0, T_{X_0})$. This set of isomorphism classes is called the *skeleton* of the deformation category.

2.1.5 Deformation functor It is also useful to study how deformations to different rings are related. We can quantify this by defining a deformation functor F from the category $\underline{\mathrm{Art}}_{A_0}$ of Artinian rings with surjection to A_0, to the category $\underline{\mathrm{Set}}$ of sets.

Given a map in $\underline{\mathrm{Art}}_{A_0}$, that is, a map $A' \to A$ of Artinian rings compatible with the maps to A_0, any deformation of X_0 to A' can be reduced to A by tensoring over A' with A. So we can define the deformation functor by

$$\begin{aligned} F(A) &= \{\text{deformations of } X_0 \text{ to } A\}/\text{equivalence} \\ F(A' \to A) &= \text{reduction} \end{aligned}$$

Using this notation, we can write $F(k[\epsilon]/\epsilon^2) = H^1(X_0, T_{X_0})$. We call this the tangent space of F. If a moduli space of schemes exists near the point representing X_0, this is the tangent space of the moduli space at that point.

2.1.6 Deformation pseudofunctor Finally, by combining these two ways of packaging deformation data, we can have the deformation "functor" take its values in the category of groupoids instead of sets. In other words,

$$F(A) = \text{the category of deformations}$$
$$F(A' \to A) = \text{reduction}$$

This is in fact not a functor, but a *pseudofunctor* (also known as a *lax 2-functor*). The difference in this case is subtle. For example, $F(A \xrightarrow{\text{id}} A)$ is the functor $- \otimes_A A$, which is not equal to the identity functor, but is naturally isomorphic.

2.1.7 Pseudofunctor terminology Two pseudofunctors are *equivalent* if there is a natural transformation between them such that for fixed A, the categories $F(A)$ and $G(A)$ are equivalent.

2.2 Deformations of categories

Lowen and Van den Bergh define deformations of linear and abelian categories in [LVdB2].

Let $A \to A_0$ be as above.

2.2.1 Base change for categories Let $\mathcal{D}$ be an A_0-linear category, and $\mathcal{C}$ be an A-linear category.

Define $\overline{\mathcal{D}}$ to be the A-linear category with the same objects and morphisms, with the morphism groups thought of as A-modules using the map $A \to A_0$.

Define $A_0 \otimes_A \mathcal{C}$ to be the A_0-linear category formed by tensoring the morphism groups by A_0. This category has the same objects as $\mathcal{C}$, with morphisms $(A_0 \otimes_A \mathcal{C})(A, B) = A_0 \otimes_A \mathcal{C}(A, B)$. Note $A_0 \otimes_A (-)$ is left adjoint to $\overline{(-)}$.

The right adjoint to $\overline{(-)}$ is denoted by $(-)_{A_0}$. We call $\mathcal{C}_{A_0}$ the category of A_0-objects in $\mathcal{C}$. It has a concrete definition, but this will not be relevant here.

Definition 2.2.2 (Deformations of linear categories) For an A_0-linear category $\mathcal{D}$, an A-deformation of $\mathcal{D}$ is an A-linear category $\mathcal{C}$, flat over A, together with an A-linear functor $\mathcal{C} \to \overline{\mathcal{D}}$ that induces an equivalence $A_0 \otimes_A \mathcal{C} \to \mathcal{D}$.

Definition 2.2.3 (Equivalence of deformations) If $f : \mathcal{C}_1 \to \mathcal{D}$ and $g : \mathcal{C}_2 \to \mathcal{D}$ are two deformations of $\mathcal{D}$ to A, then an *equivalence of deformations* is an equivalence of A-linear categories $\Phi : C_1 \to C_2$ together with a natural isomorphism between $g \circ \Phi$ and f.

Definition 2.2.4 (Deformations of abelian categories) For an abelian A_0-linear category $\mathcal{D}$, an A-deformation of $\mathcal{D}$ is an A-linear abelian category $\mathcal{C}$, flat over A, together with an A-linear functor $\overline{\mathcal{D}} \to \mathcal{C}$ that induces an equivalence $\mathcal{D} \to \mathcal{C}_{A_0}$.

There is a similar definition of equivalence of deformations of abelian categories, but we will not need it.

Although we are studying deformations of the abelian category $\mathrm{Coh}(X)$, we will only need to work with deformations of linear categories, due to the following construction and theorem.

2.2.5 Category corresponding to a scheme Let X_0 be a scheme, and let $\mathcal{B}$ be a basis of open affine sets of X_0. Define an A_0-linear category $\mathfrak{u}$, whose objects are the elements of $\mathcal{B}$, with morphisms

$$\mathfrak{u}_{\mathcal{B}}(U,V) = \begin{cases} \mathcal{O}_{X_0}(U) & \text{if } U \subseteq V \\ 0 & \text{otherwise.} \end{cases}$$

This defines functors $\iota_{\mathcal{B}} : \underline{\mathrm{Sch}}/A_0 \to A_0\text{-}\underline{\mathrm{Cat}}$, which take $X_0 \mapsto \mathfrak{u}_{\mathcal{B}}$.

Theorem 2.2.6 (Lowen and Van den Berg) *If X_0 is a separated, quasi-compact, Noetherian scheme, and $\mathcal{B}$ is a basis of open affine sets, then the pseudofunctor of linear deformations of $\iota_{\mathcal{B}}(X_0)$ is equivalent to the pseudofunctor of abelian deformations of* $\mathrm{Coh}(X_0)$.

Proof. The pseudofunctor of deformations of $\iota(X_0)$ is equivalent to the pseudofunctor of deformations of $\mathrm{Mod}(\mathcal{O}_{X_0})$ [LVdB2]. If X_0 is separated and quasi-compact, then the pseudofunctor of deformations of $\mathrm{Mod}(\mathcal{O}_{X_0})$ is equivalent to the pseudofunctor of deformations of $\mathrm{Qch}(X_0)$ [Low]. Finally, for a locally coherent category such as $\mathrm{Qch}(X_0)$, the deformation pseudofunctor is equivalent to the deformation pseudofunctor of the finitely presented subcategory [LVdB2], which is $\mathrm{Coh}(X_0)$ if X_0 is Noetherian. $\square$

Corollary 2.2.7 *If $\mathcal{B}'$ and $\mathcal{B}$ are two bases of open affine sets, the pseudofunctor of deformations of $\iota_{\mathcal{B}}$ is equivalent to the pseudofunctor of deformations of $\iota_{\mathcal{B}'}$.*

Note 2.2.8 It is not true that $\iota_{\mathcal{B}}(X_0)$ and $\iota_{\mathcal{B}'}(X_0)$ are equivalent categories, only that they have equivalent deformation pseudofunctors.

Notation 2.2.9 For convenience, we will almost always consider the basis $\mathcal{B}$ defined by all open affine sets of X_0, and write ι instead of $\iota_{\mathcal{B}}$.

Remark 2.2.10 Clearly $\iota(X_0) \otimes A$ is equivalent to $\iota(X_0 \otimes A)$, which shows ι defines a natural transformation.

Notation 2.2.11 We will write $\mathrm{Def}_{X_0}^{\mathrm{coh}}$ for the pseudofunctor of deformations of the category $\iota(X_0)$, which is equivalent to the pseudofunctor of deformations of the abelian category of coherent sheaves. We will leave off the X_0 when it is understood.

Remark 2.2.12 As we discussed above, we will generally look at the deformation pseudofunctor from $\underline{\mathrm{Art}}_{A_0}$ to the category of groupoids. To simplify, we will often fix a ring A and look only at a single groupoid of deformations, e.g. $\mathrm{Def}^{\mathrm{coh}}(A)$.

Summary 2.2.13 Just to keep things straight, $\mathrm{Def}^{\mathrm{coh}}(A)$ is a groupoid whose objects are deformations and morphisms are equivalences of deformations. In turn, each deformation is itself an A-linear category, and each equivalence of deformations is an equivalence of categories. There are certainly category-theoretic issues to worry about in this, but they are carefully studied in [LVdB2] using universes.

3 Deformations of Coh(X)

As in the last chapter, let $\pi : A \to A_0$ be a map of commutative rings with nilpotent ideal J. Let X_0 be a scheme over A_0. We will study deformations of $\iota(X_0)$ to A.

3.1 Deformation subfunctors

3.1.1 Splitting via the HKR isomorphism In [Toda], Toda describes deformations of $\iota(X_0)$ for a scheme X_0 over $\mathbf{C}$ to $\mathbf{C}[\epsilon]/\epsilon^2$, which serves as our starting point.

A deformation of $\iota(X_0)$ in this case is given by an element of the Hochschild cohomology $HH^2(X_0)$, which splits via the HKR isomorphism into direct summands

$$HH^2(X_0) \cong H^2(X_0, \mathcal{O}_{X_0}) \oplus H^1(X_0, T_{X_0}) \oplus H^0(X_0, \wedge^2 T_{X_0}).$$

Given an affine open covering of X_0 and a Čech cocycle representing (α, β, γ), Toda constructs a deformation of X_0 using β and a sheaf $\mathcal{A}$ of (possibly noncommutative) rings using γ. Then he defines deformations of $\mathrm{Coh}(X_0)$ to be α-twisted

$\mathcal{A}$-modules, that is a collection of $\mathcal{A}$-modules, one on each of the covering sets, with isomorphisms on the double overlaps, so that the cocycle composition of these isomorphisms on the triple overlaps is exactly multiplication by α.

3.1.2 Tower of embeddings We will study a generalization of this decomposition, which we will describe as a tower of embeddings

$$\mathrm{Def}^{\,\mathrm{sch}} \to \mathrm{Def}^{\,\mathrm{comm}} \to \mathrm{Def}^{\,\mathrm{coh}},$$

which we will now define.

Definition 3.1.3 (Commutative deformations) We say that a category $\mathcal{C}$ is commutative if, for each object U, the ring of endomorphisms $\mathcal{C}(U,U)$ is a commutative. The category $\mathrm{Def}^{\,\mathrm{comm}}_{X_0}(A)$ is the full subcategory of $\mathrm{Def}^{\,\mathrm{coh}}_{X_0}(A)$ consisting of objects which are commutative categories.

Definition 3.1.4 (Scheme deformations) The category $\mathrm{Def}^{\,\mathrm{sch}}_{X_0}(A)$ is the full subcategory of $\mathrm{Def}^{\,\mathrm{coh}}_{X_0}(A)$ consisting of categories equivalent to $\iota(X)$ for some scheme X.

3.1.5 Scheme deformations are commutative Note that $\mathrm{Def}^{\,\mathrm{sch}}_{X_0}(A)$ is also a subcategory of $\mathrm{Def}^{\,\mathrm{comm}}_{X_0}(A)$.

Remark 3.1.6 We would like $\mathrm{Def}^{\,\mathrm{sch}}(A)$ to be the deformations of X_0 as a scheme, although this is not how we have defined it. We will prove later that these are equivalent, that is, that ι defines an equivalence between the category of deformations of X_0 to A and $\mathrm{Def}^{\,\mathrm{sch}}(A)$.

3.1.7 Subfunctors Finally, the reduction of a commutative deformation will remain commutative, and the reduction of a scheme deformation will remain a scheme deformation, so we can use these deformations to define sub-pseudofunctors $\mathrm{Def}^{\mathrm{comm}}$ and $\mathrm{Def}^{\mathrm{sch}}$.

3.2 Properties of deformations of linear categories

Although deformations of a category may seem to carry a lot of information, they have nice properties that make them less unwieldy.

Proposition 3.2.1 *Let $\mathcal{D}$ be an arbitrary A_0-linear category, and let $\Phi : \mathcal{C} \to \mathcal{D}$ be a deformation to A.*

1. *If $\mathcal{D}(U, V)$ is a rank 1 free right $\mathcal{D}(U, U)$-module for each U and V, then the same is true for $\mathcal{C}$.*

2. *If a right module generator for $\mathcal{D}(U, V)$ composed with a right module generator for $\mathcal{D}(V, W)$ is a generator for $\mathcal{D}(U, W)$, then the same is true for $\mathcal{C}$.*

Proof. We will write U for both an object in $\mathcal{C}$ and its image under the functor to $\mathcal{D}$.

Let π denote the map $M \to N$. We will first show that any preimage of a generator of N under π is a generator of M. By flatness, $J \otimes M \cong \ker(M \to N)$. Let $\theta \in M$ map to a generator of N. Suppose that θ generates $J^{n+1}M$ for some n. Then for $x \in M$, we have $\pi(x) = \pi(\theta)\pi(z)$ for some $z \in M$, which means

$x = \theta z + j'$ for $j' \in J$. Now for any $j \in J^n$, we have $jx = \theta jz + j'j = \theta jz + \theta z'$, meaning that θ generates $J^n M$. Since J is nilpotent, we use induction to deduce that θ generates M.

This proves that M is at least generated by a single element, that is, there is a surjective homomorphism $\phi : S \to M$. Let $K = \ker(\phi)$; since $\phi \otimes_A A_0$ is an isomorphism, $K \otimes_A A_0 = 0$, which means that $J \otimes K \to K$ is surjective. Hence we get a chain of surjections

$$0 \to \cdots \to J^2 \otimes K \to J \otimes K \to K.$$

So K is 0, which means that M is in fact free of rank 1.

For the second assertion, let f and g be morphisms in $\mathcal{C}$ that generate the appropriate modules. Because each map $\mathcal{C}(U,V) \to \mathcal{D}(U,V)$ is surjective, f and g also generate the appropriate modules in $\mathcal{D}$. By assumption, so does their product. Since the preimage of a generator is a generator, the result follows.

$\square$

Note that for any scheme X_0, the category $\iota(\mathcal{O}_{X_0})$ satisfies these two properties, so any deformation of $\iota(\mathcal{O}_{X_0})$ does also.

3.2.2 Objects of a deformation In the above proof, we used the same letter to denote both an object in $\mathcal{C}$ and an object in $\mathcal{D}$, even though Φ may send two different objects to the same one. The proof is still correct.

For our purposes, we would actually like to identify the objects of the deformation with the objects of the original category. If the original category is of the form $\iota(X_0)$, note that no two distinct objects are isomorphic. For simplicity, we will require the deformation to have the same property, which means that Φ is a bijection on objects. To do this, we need to make sure that there are no non-isomorphic objects of $\mathcal{C}$ which become isomorphic in $\mathcal{D}$.

Proposition 3.2.3 *If $\Phi : \mathcal{C} \to \mathcal{D}$ is a deformation as above, with V and W objects of $\mathcal{C}$ and $U \coloneqq \Phi(V) = \Phi(W)$, then V and W are isomorphic objects of $\mathcal{C}$.*

Proof. Let $f \in \mathcal{C}(V, W)$ and $g \in \mathcal{C}(W, V)$ be preimages of $1 \in \mathcal{D}(U, U)$. Then $g \circ f \in \mathcal{C}(V, V)$ is also a preimage of $1 \in \mathcal{D}(U, U)$. For any map of rings with nilpotent kernel, every preimage of a unit is a unit, hence $h \circ g \circ f$ is the identity for some h, so $V \cong W$. □

3.3 Noncommutative deformations

We continue with $0 \to J \to A \to A_0 \to 0$ exact and J nilpotent. Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a deformation to A, with $\mathcal{D} = \iota(X_0)$.

In this section only, we will assume $J^2 = 0$. Our goal is to get a commutative deformation from an arbitrary deformation, that is, to remove the noncommutative part.

Definition 3.3.1 (Commutification) Let $\mathcal{C} \in \mathrm{Def}^{\mathrm{coh}}(A)$, and for each $U \subseteq V$, choose a $\mathcal{C}(U, U)$-module generator θ_{UV} of $\mathcal{C}(U, V)$, with $\theta_{UU} = 1$. Define $\mathrm{P}_A(\mathcal{C})$ as the category which has the same objects and A-modules of morphisms as $\mathcal{C}$, but with composition defined as

$$g * f = \tfrac{1}{2}\theta_{VW}(yx + xy)\theta_{UV}, \tag{3.3.1.1}$$

where $g = \theta_{VW}y$ and $f = x\theta_{UV}$. We are using implicit multiplication to denote composition in $\mathcal{C}$, and $*$ to denote composition in $\mathrm{P}_A(C)$.

3.3.2 The resulting category is commutative If f and g are elements of $\mathcal{C}(U, U)$, then $g * f = \frac{1}{2}(gf + fg) = f * g$.

3.3.3 Well defined operation Since $\mathcal{C}(V, W)$ is a free $\mathcal{C}(V, V)$-module, there is only one way to write $g = \theta_{VW}y$. However, it is possible that $f = x\theta_{UV} = x'\theta_{UV}$. In this case the difference between the definition using x and the definition using x' is half of

$$\theta_{VW}(x' - x)y\theta_{UV} = \theta_{VW}(x' - x)\theta_{UV}z = 0$$

for some $z \in \mathcal{C}(U, U)$. Therefore, the operation is well defined.

3.3.4 Independent of choice of generators If you replace each θ_{UV} with θ'_{UV}, there is some element $u \in \mathcal{C}(U, U)$ which reduces to 1 in $\mathcal{D}$ so that $\theta u = \theta'$. Since u is of the form $1 + j$ for some j in the commutative ring $J \otimes \mathcal{C}(U, U)$, u is central.

Let x' and y' be such that $g = \theta' y'$ and $f = \theta' x'$. Then $\theta y = \theta u y'$, which means

$y = uy'$. Also, $x\theta = x'\theta u$. This gives us

$$\begin{aligned}
\theta' x' y' \theta' &= \theta u x' y' \theta u \\
&= \theta u x' \theta z u \\
&= \theta u (x' \theta u) z \\
&= \theta u (x\theta) z \\
&= \theta u x y' \theta \\
&= \theta x (u y') \theta \\
&= \theta x y \theta.
\end{aligned}$$

Since (3.3.1.1) can be rewritten as

$$g * f = \frac{1}{2}\left(gf + \theta x y \theta\right),$$

this shows the functor does not depend on choice of θ.

3.3.5 Composition is associative The new composition is associative. For $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} X$, write $h\theta_{VW} = \theta_{WX}\theta_{VW}z$, $g = \theta_{VW}y$, and $f = x\theta_{UV}$. Then we have

$$\begin{aligned}
4(hg)f &= \theta_{WX}\theta_{VW}(zyx + yzx + xzy + xyz)\theta_{UV} \\
4h(gf) &= \theta_{WX}\theta_{VW}(zyx + zxy + yxz + xyz)\theta_{UV}
\end{aligned}$$

The difference is $(xz - zx)y - y(xz - zx) \in J^2 \otimes \mathcal{C}(U, U)$, which must be zero.

3.3.6 Value on maps of deformations If you have a map of deformations $\mathcal{C} \to \mathcal{C}'$ over $\mathcal{D}$, then you can define a functor $\mathrm{P}_A(C) \to \mathrm{P}_A(C')$, which is identical to the functor from $\mathcal{C} \to \mathcal{C}'$. The only thing to check is that this preserves the redefined composition, and the argument for this is the same one we used to show that the composition doesn't depend on choice of θ.

Theorem 3.3.7 *If $J^2 = 0$ then the functor*

$$\mathrm{P}_A : \mathrm{Def}^{\mathrm{coh}}(A) \to \mathrm{Def}^{\mathrm{comm}}(A)$$

is a left inverse to the inclusion $\mathrm{Def}^{\mathrm{comm}}(A) \to \mathrm{Def}^{\mathrm{coh}}(A)$.

Proof. We have shown that P_A defines a functor $\mathrm{Def}^{\mathrm{coh}}(A) \to \mathrm{Def}^{\mathrm{comm}}(A)$. It is clear that P_A is the identity on deformations that are already commutative. □

3.4 The sheaf corresponding to a commutative deformation

As before, $0 \to J \to A \to A_0$ is exact and J is nilpotent. Let $\mathcal{C} \in \mathrm{Def}^{\mathrm{comm}}(A)$ be a commutative deformation of $\iota(X_0)$.

3.4.1 Get a restriction map from the bimodule structure Whenever you have an S-R-bimodule M that is free of rank 1 as a right R-module, with S and R commutative, there is a map $S \to \mathrm{Hom}_R(M, M) \cong R$. In words, multiplication by s

on the left is an R-module homomorphism, and therefore is given by multiplication by an element r of R. You can calculate the image of s by choosing any R-module generator θ of M and finding the unique $r \in R$ such that $s\theta = \theta r$.

Remark 3.4.2 Although we have defined the map without choosing a generator of M, it is easy to see that two choices of generators differ by a unit in R, that is, $\theta' = \theta u$. If $s\theta = \theta r$, then $s\theta' = s\theta u = \theta r u = \theta' u^{-1} r u$. Because R is commutative, $u^{-1} r u = r$. If R and S are not commutative but $J^2 = 0$, then u will be central, and the construction that follows will still work. Of course, in that case we can first replace the deformation by a commutative one using the results of Section 3.3, which gives the same result.

Definition 3.4.3 Let $\mathcal{B}$ be the set of affine open sets of X_0. Define a presheaf $\chi(\mathcal{C})$ of rings on $\mathcal{B}$ by

$$\chi(\mathcal{C})(U) = \mathcal{C}(U, U)$$

with restriction maps defined as above. The requirement that the composition of two generators is also a generator ensures that a composition of two restriction maps is again a restriction map, because the image of s under the double restriction can be calculated by moving s past $\theta_1 \theta_2$, which is the same as moving it first past θ_1 and then past θ_2.

Remark 3.4.4 The restriction on $\chi(\mathcal{C})$ is the standard restriction of $\mathcal{O}_{X_0}$ after tensoring with A_0. Also, χ takes a functor $\mathcal{C} \to \mathcal{C}'$ to a map of presheaves $\chi(C) \to \chi(C')$. These are simple exercises left to the reader.

We need to prove one small algebraic result before proving that this presheaf is a sheaf.

Lemma 3.4.5 *Let $M^\bullet$ be a complex of flat A-modules, and $0 \to J \to A \to A_0 \to 0$ be a nilpotent extension. If $M^\bullet \otimes_A A_0$ is exact then $M^\bullet$ is exact.*

Proof. Consider the exact sequence $0 \to J^{n+1} \to J^n \to J^n/J^{n+1} \to 0$, and note that the A-module structure on the quotient factors through A_0. Tensoring $M^\bullet$ with this exact sequence gives

$$0 \to J^{n+1} \otimes M^\bullet \to J^n \otimes M^\bullet \to J^n/J^{n+1} \otimes M^\bullet \to 0.$$

The last term is isomorphic to $J^n/J^{n+1} \otimes_{A_0} (A_0 \otimes_A M^\bullet)$. The hypothesis, combined with the fact that $M^\bullet \otimes_A A_0$ is a complex of flat A_0-modules, ensures this is exact. The first two terms have the same cohomology. By induction, the cohomology of $M^\bullet$ is the same as that of $J^n \otimes M^\bullet$ for any n. Since $J^n = 0$ for large enough n, $M^\bullet$ itself is exact. □

Corollary 3.4.6 *The presheaf $\chi(\mathcal{C})$ is a $\mathcal{B}$-sheaf.*

Proof. Take $M^\bullet$ to be the sheaf complex $\mathcal{C}(U, U) \to \prod \mathcal{C}(U_i, U_i) \to \prod \mathcal{C}(U_{ij}, U_{ij})$, where the first map is restriction and the second is the difference of the two possible restrictions. Note that U is affine, hence quasi-compact, so I can assume the products are finite. This is a complex of flat A-modules which becomes exact when tensored with A_0. Therefore, $M^\bullet$ is exact. □

This defines a unique sheaf on the topological space $|X_0|$. Notice that $\chi(\mathcal{C})\otimes A_0 \cong \mathcal{O}_{X_0}$, so $\chi(\mathcal{C})$ is the structure sheaf of a deformation X of X_0. When convenient, we will write $\chi(\mathcal{C}) = X$.

3.5 The gerbe corresponding to a commutative deformation

A *gerbe* is a locally non-empty locally connected stack of groupoids. For more background, see [Vis].

Let $\mathcal{C}$ be a commutative deformation of $\iota(X_0)$. Then we can describe $\mathcal{C}$ as a gerbe over the scheme X that corresponds to $\chi(\mathcal{C})$.

Definition 3.5.1 To each Zariski open subset U of X, define $\mathcal{C}|_U$ as the full subcategory of $\mathcal{C}$ generated by objects corresponding to subsets of U. For each such U, define

$$\mathfrak{X}(U) = \mathrm{Func}^*_{X_0}(\mathcal{C}|_U, \iota\chi(\mathcal{C}|_U)).$$

By $\mathrm{Func}^*_{X_0}$, I mean the category (in fact, groupoid) whose objects are equivalences of categories that become the identity on X_0, and whose morphisms are natural isomorphisms that become the identity on X_0.

We have identified the objects of $\mathcal{C}$ and $\iota\chi(\mathcal{C})$ with the affine open sets of X_0. These equivalences preserve this identification. The value of the equivalences on morphisms is what distinguishes them.

If $V \subseteq U$, then there is functor $\mathfrak{X}(U) \to \mathfrak{X}(V)$ which takes an equivalence of categories $\mathcal{C}|_U \to \iota\chi\mathcal{C}|_U$ and restricts it to the full subcategory $\mathcal{C}|_V$, and the image becomes the $\iota\chi\mathcal{C}|_V$. This makes $\mathfrak{X}$ into a presheaf of groupoids.

Lemma 3.5.2 (Locally non-empty) *If U is an affine open set, there is an equivalence of categories $\mathcal{C}|_U \to \iota\chi(\mathcal{C}|_U)$.*

Proof. Note that there is a special system of generators for $\iota\chi(C)(W,V)$ that has the special property that any composition of these generators gives another of these generators, viz., $1 \in \chi(C)$. An equivalence of categories exists if and only if there is a similar consistent set of generators for $\mathcal{C}(U,V)$. Because there is a final object in $\mathcal{C}|_U$, we can choose arbitrary $\theta_{WU} \in \mathcal{C}(W,U)$ and then demand that $\theta_{WV} \in \mathcal{C}(W,V)$ is such that $\theta_{VU}\theta_{WV} = \theta_{WU}$. Note that if $T \subseteq W \subseteq V \subseteq U$, then we can compare $\theta_{WV}\theta_{TW}$ and θ_{TV} by composing both with θ_{VU}, after which they become equal. If for some x we have $\theta_{VU}x = 0$, then $x = \theta_{TV}x'$, which implies $\theta_{VU}\theta_{TV}x' = 0$, which means $x' = 0$, which implies $x = 0$. $\square$

3.5.3 Natural isomorphisms Suppose $\mathcal{F} \in \mathrm{Func}^*_{X_0}(\mathcal{C}, \iota\chi\mathcal{C})$. A natural isomorphism from $\mathcal{F}$ to itself consists of a collection $\{s_V \mid V \subseteq U\}$ such that for any $x \in \mathcal{C}(W,V)$ the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "s_w"] \arrow[d, "F(x)"'] & W \arrow[d, "F(x)"] \\ V \arrow[r, "s_V"] & V \end{tikzcd}$$

The commutativity condition implies that there is an invertible section $s \in \chi(\mathcal{C})(U)^*$ so that each s_V is the restriction of s. The requirement that the natural isomorphism be the identity on X_0 means this section should map to 1, which means it is an element of $(1 + J\mathcal{O}_X)(U)$.

Lemma 3.5.4 (Locally connected) *For any U, with $F, G \in \mathfrak{X}(U)$, there is a natural isomorphism $F \to G$.*

Proof. Since our functors are equivalences of categories, this is the same as showing that any auto-equivalence of category H of $\iota\chi(\mathcal{C})$ is naturally isomorphic to the identity.

Note that any isomorphism of sheaves $f : \chi(\mathcal{C}) \to \chi(\mathcal{C})$ induces a functor on $\iota\chi(\mathcal{C})$ that is naturally isomorphic to the identity. The functor is given by $\Phi \mapsto f\Phi f^{-1}$, and the natural transformation is given by f, so we may assume that H is the identity on $\mathcal{C}(V, V)$.

Since each $\iota\chi(\mathcal{C})(W, V)$ is identified with $\iota\chi(\mathcal{C})(W, W)$, we have $H(1_{WV}) = u_{WV}$ (thought of as an element of $\iota\chi(\mathcal{C})(W, W)$). The natural isomorphism we are looking for is

$$\begin{tikzcd} W \arrow[r, "u_{WU}^{-1}"] \arrow[d, "1"'] & W \arrow[d, "u_{WV}"] \\ V \arrow[r, "u_{VU}^{-1}"] & V \end{tikzcd}$$

Commutativity is a result of $u_{VU}u_{WV} = u_{WU}$. $\square$

3.5.5 Descent We will now show that $\mathfrak{X}$ satisfies the descent conditions, making it into a stack and hence a $(1 + J\mathcal{O}_X)$-gerbe.

Lemma 3.5.6 (Gluing morphisms) *The functor*

$$\Phi : \mathfrak{X}(U) \to \mathfrak{X}(\{U_i\})$$

is fully faithful.

Proof. Given $F, G \in \mathfrak{X}(U)$, morphisms $F \to G$ are natural which reduce to the identity, and are a torsor under $1 + J\mathcal{O}_X|_U \cong \ker(\chi\mathcal{C} \to \mathcal{O}_{X_0})|_U$. If two of these natural isomorphisms are locally the same, they are in fact the same. Further, a collection of local isomorphisms which agree on the double overlaps glue together to give a natural isomorphism between F and G. □

This next lemma will be needed in order to glue objects. We will eventually use descent data to find a "locally defined" functor, and this lemma allows us to extend the local definition to a global one.

Lemma 3.5.7 *Suppose that there is a subcategory $\mathfrak{U}$ of $\mathcal{C}$ whose objects form a basis for X, and which has a single morphism for each $W \subseteq V$.*

For each W, let $\{W_i\}$ be the set of all elements of $\mathfrak{U}$ which are subsets of W. The morphisms of $\mathfrak{U}$ make this into a diagram in $\mathcal{C}$. There exist maps in $\mathcal{C}(W_i, W)$ for each W of $\mathfrak{U}$ inducing an isomorphism $\lim_i \operatorname{colim}_j \mathcal{C}(W_i, V_j) \cong \mathcal{C}(W, V)$.

The maps can be chosen as lifts of specified generators $\eta_i \in \mathcal{C}(W_i, W) \otimes A_0$.

Proof. Begin by fixing W and find maps $\theta_i \in \mathcal{C}(W_i, W)$ making $W \cong \operatorname{colim}_i W_i$. (The maps could alternatively be found by using Čech cocycles.)

The morphisms of the diagram (W_i) define maps $\mathcal{C}(W_j, W) \to \mathcal{C}(W_i, W)$ for $W_i \subseteq W_j$, defining a presheaf

$$\mathcal{F} : W_i \mapsto \{\theta \in \mathcal{C}(W_i, W) \colon \overline{\theta} = \eta\}$$

on W. The presheaf is defined only locally but defines a unique sheaf on W.

Note that $\mathcal{F}$ is a torsor under the action of $1 + J\mathcal{O}_X \cong \ker(\chi\mathcal{C}^* \to \chi(\mathcal{C} \otimes A_0)^*)$ given by composition. Since W is affine, $\mathcal{F}$ is trivial, that is, $\mathcal{F} \cong (1 + J\mathcal{O}_X)|_W$. A global section of $\mathcal{F}$ corresponds to an element of $\mathcal{C}(W, W)$, which restricts to give elements of $\mathcal{C}(W_i, W)$ consistent with the maps $\mathcal{C}(W_j, W_i)$.

Let V be an arbitrary object of $\mathcal{C}$. Choose a generator θ of $\mathcal{C}(W, V)$ as a $\mathcal{C}(W, W)$-module, and a cover $\{W_{i_j}\}$ of W consisting of elements of $\mathfrak{U}$. Let $\theta_{i_j} \in \mathcal{C}(W_{i_j}, W)$ be the map defined above.

This gives the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}(W,V) \arrow[r, "s \mapsto s\theta_{i_j}"] & \prod \mathcal{C}(W_{i_j}, V) \arrow[r, shift left] \arrow[r, shift right] & \prod \mathcal{C}(W_{i_j} \cap W_{i_k}, V) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{C}(W,W) \arrow[r] \arrow[u, "\wr"'] & \prod \chi\mathcal{C}(W_{i_j}) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, "\wr"'] & \prod \chi\mathcal{C}(W_{i_j} \cap W_{i_k}) \arrow[r] \arrow[u, "\wr"'] & 0
\end{tikzcd}$$

The diagram is commutative because we have $s \in \mathcal{C}(W, W)$ maps to $\{\theta \circ s \circ \theta_{i_j}\}$ going up first then right, and to $\{\theta \circ \theta_{i_j} \circ \rho_i(s)\}$ going right first then up, and $\theta_i \circ \rho_i(s) = s \circ \theta_i$ by definition.

The map $\mathcal{C}(W, V) \to \lim \mathcal{C}(W_i, V)$ is injective because if it is zero on each

$\mathcal{C}(W_i, V)$, then it is zero on each $\mathcal{C}(W_{i_j}, V)$, so it is zero to begin with.

Given a compatible collection of morphisms $s_i \in \mathcal{C}(W_i, V)$, we can look only at the s_{i_j}, which must glue to form $s \in \mathcal{C}(W, V)$ such that $s \circ \theta_{i_j} = s_{i_j}$. Finally, s_i and $s \circ \theta_i$ agree on each $W_i \cap W_{i_j}$, so by injectivity of the sheaf condition, they are the same. Therefore the map is surjective, which proves $\mathcal{C}(W, V) \cong \lim \mathcal{C}(W_i, V)$.

Lastly we must show that for fixed W_i, the map $\operatorname{colim}_j \mathcal{C}(W_i, V_j) \to \mathcal{C}(W_i, V)$ is an isomorphism.

Since $W_i = V_j$ for some j, for every nonzero group in the colimit, there is an isomorphism $\mathcal{C}(W_i, W_i) \to \mathcal{C}(W_i, V_j)$ because the latter is a rank 1 free module. This makes the colimit isomorphic to $\mathcal{C}(W_i, W_i)$. Similarly, there is an isomorphism $\mathcal{C}(W_i, W_i) \to \mathcal{C}(W_i, V)$. Under these identifications, the map $\operatorname{colim}_j \mathcal{C}(W_i, V_j) \to \mathcal{C}(W_i, V)$ is the identity.

We can arrange these last isomorphisms into a diagram indexed by W_i and take the limit to get an isomorphism $\lim_i \operatorname{colim}_j \mathcal{C}(W_i, V_j) \to \lim_i \mathcal{C}(W_i, V)$. We can combine this with the isomorphism $\mathcal{C}(W, V) \to \lim_i \mathcal{C}(W_i, V)$ to get the isomorphism we desire. $\square$

Lemma 3.5.8 (Gluing objects) *Let U be an open set of X and let $\{U_i \colon i \in \mathcal{I}\}$ be an open cover of U. Let $(\{F_i\}, \{\phi_{ij}\})$ be an object with descent data on $\{U_i\}$, which consists of a collection $\{F_i\}$ of objects of $\mathfrak{X}(U_i)$ and a set $\{\phi_{ij}\}$ of isomorphisms $F_j|_{U_{ij}} \to F_i|_{U_{ij}}$, with $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ when restricted to U_{ijk}. Then there exists an object G of $\mathfrak{X}(U)$ and natural isomorphisms $\alpha_i : G|_{U_i} \to F_i$ such that $\phi_{ij} \circ \alpha_j = \alpha_i$ when restricted to U_{ij}.*

Proof. The proof will proceed in two steps. We will first use the descent data to define the functor G on small open sets. Then we will use Lemma 3.5.7 to extend the functor to all open sets.

Let $\mathfrak{U}$ be the set of objects of $\mathcal{C}$ corresponding to open sets that lie inside some U_i. These are the small open sets I am talking about.

For each $V \in \mathfrak{U}$, choose $i \in \mathcal{I}$ such that $V \subseteq U_i$, and denote it by $m(V)$. For $x \in \mathcal{C}(W, V)$, let $j = m(W)$ and $i = m(V)$, and define

$$G(x) = F_i(x) \circ \phi_{ij}(W),$$

using the convention $\phi_{ii}(W) = \mathrm{id} \in \iota\chi(\mathcal{C})(W, W)$. Given $y \in \mathcal{C}(W', W)$ and $x \in \mathcal{C}(W, V)$, with $m(W') = k$, $m(W) = j$, and $m(V) = i$, we have

$$\begin{aligned} G(x \circ y) &= F_i(x \circ y) \circ \phi_{ik}(W') \\ &= F_i(x) \circ F_i(y) \circ \phi_{ij}(W') \circ \phi_{jk}(W') \\ &= F_i(x) \circ \phi_{ij}(W) \circ F_j(y) \circ \phi_{jk}(W') \\ &= G(x) \circ G(y)\,. \end{aligned}$$

Note that we have defined enough of G to determine $G|_{U_i}$. For any $W \subseteq U_i$, define $\alpha_i(W) = \phi_{ik}(W)$, where $k = m(W)$. Clearly $\phi_{ij} \circ \alpha_j = \alpha_i$. You can confirm that α_i is an isomorphism between $G|_{U_i}$ and F_i by checking that for any $x \in \mathcal{C}(W, V)$ with $m(W) = k$ and $m(V) = j$, the outer rectangle of the following diagram commutes.

$$\begin{tikzcd}
W \arrow[r, "\phi_{jk}"'] \arrow[rr, bend left, "G(x)"] \arrow[d, "\alpha_i = \phi_{ik}"'] & W \arrow[r, "F_j(x)"'] \arrow[d, "\phi_{ij}"] & V \arrow[d, "\phi_{ij} = \alpha_i"] \\
W \arrow[r, equal] & W \arrow[r, "F_i(x)"] & V
\end{tikzcd}$$

The two inner squares clearly commute.

Note that each $\iota\chi\mathcal{C}(W, V)$ has a special morphism 1_{WV} which corresponds to $1 \in \chi\mathcal{C}(W)$. Then the morphisms $\{G^{-1}(1_{WV})\}$ make $\mathfrak{U}$ into a subcategory of $\mathcal{C}$ as required for Lemma 3.5.7. Also, $\mathcal{C}(W, V) \otimes A_0$ has this special 1_{WV}, so we can use the lemma to define maps

$$\mathcal{C}(W, V) \to \lim \operatorname{colim} \mathcal{C}(W_i, V_j) \xrightarrow{G} \lim \operatorname{colim} \iota\chi\mathcal{C}(W_i, V_j) \to \iota\chi\mathcal{C}(W, V)$$

which commute with the maps to $\mathcal{C}(W, V) \otimes A_0$. Since G commutes with limits and colimits, this composition agrees with G where both are defined.

The composition is therefore an equivalence of categories that reduces to the identity and with the correct descent data, that is, a gluing of the descent data we started with.

$\square$

3.5.9 Comparing to scheme deformations Now we are ready to compare deformations of categories and deformations of schemes.

Theorem 3.5.10 *The functor χ induces a natural transformation*

$$\mathrm{Def}^{\mathrm{comm}} \to \mathrm{Def}^{\mathrm{sch}}$$

which is a left inverse to the inclusion $\mathrm{Def}^{\mathrm{sch}} \to \mathrm{Def}^{\mathrm{comm}}$.

Proof. For fixed A, the functor is defined by $\iota \circ \chi$, which is natural in A. For a scheme X, you can use $1_U \in \mathcal{O}(U)$ as the generator for $\iota(X)(U, V)$. Then for $s \in \mathcal{O}(V)$, note that $s(1_U) = (1_U)\rho(s)$, where $\rho : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is the restriction map. Therefore, $\chi(\iota(X))$ gives back X. $\square$

Now we will prove that $\mathrm{Def}^{\mathrm{sch}}_{X_0}(A)$ is in fact equivalent to the category of scheme deformations of X_0 to A.

Proposition 3.5.11 *The categorification functor ι induces a weak equivalence between the pseudofunctor of deformations of X_0 as a scheme, and the pseudofunctor* $\mathrm{Def}^{\mathrm{sch}}$.

Proof. By definition, ι is essentially surjective, so we only need to show that it is fully faithful.

Since χ gives a left inverse of ι, we know that ι is injective on morphisms.

In the other direction, things are a little more complicated. Recall that in $\mathrm{Def}^{\mathrm{sch}}(A)$, equivalences of categories are the morphisms, and natural isomorphisms of these equivalences are the 2-morphisms. We must show that any equivalence $\Phi : \iota(X) \to \iota(X')$ is naturally isomorphic to $\iota(f)$ for some $f : X \to X'$. We can form $\Psi = \iota(f)^{-1} \circ \Phi : \iota(X) \to \iota(X)$. Since $\iota(X) = \iota\chi\iota(X)$, both Ψ and the identity are in $\mathfrak{X}(X)$, hence are naturally isomorphic. Thus $\Phi \cong \iota(f)$. $\square$

This justifies the notation $\mathrm{Def}^{\mathrm{sch}}$.

4 Fourier-Mukai Transforms

In this section we will discuss some basic properties of Fourier-Mukai transforms. For this chapter I will work in the bounded derived category of coherent sheaves $D^b(X)$. I will denote derived pushforward, pullback and tensor by f_*, f^*, and $\otimes$, respectively (instead of the often used Rf_*, Lf^*, and $\otimes^L$). I will use a bullet to indicate elements of the derived category, except in the case of kernels of Fourier-Mukai transforms, where I omit the bullet as is customary. My main reference for this section is [Huy].

4.1 Definition of a Fourier-Mukai transform

Definition 4.1.1 Let X and Y be smooth projective varieties. Let $\mathcal{P} \in D^b(X \times Y)$. The *Fourier-Mukai transform* $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is

$$\Phi_{\mathcal{P}}(\mathcal{F}^\bullet) := q_*(p^*(\mathcal{F}^\bullet) \otimes \mathcal{P}).$$

We call $\mathcal{P}$ the *kernel* of $\Phi_{\mathcal{P}}$.

Note that the definition of transform is not symmetric in X and Y. In particular, $\mathcal{P}$ could also be considered as the kernel of a transform from Y to X. Also, let me again emphasize that $\mathcal{P}$ may be a true complex, and does not have to be a sheaf.

4.2 Examples of Fourier-Mukai transforms

- The functor f_* is a Fourier-Mukai transform with kernel $\mathcal{O}_\Gamma$, where $\Gamma \subseteq X \times Y$ is the graph of f.

- The functor f^* is a Fourier-Mukai transform with kernel $\mathcal{O}_\Gamma$. Note that this time, $\mathcal{O}_\Gamma$ is used to define a transform $D^b(Y) \to D^b(X)$. In this case, the two functors defined by $\mathcal{O}_\Gamma$ are adjoints, but this is usually not true.

- The identity functor is a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta$.

- The shift functor $[N]$ is a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta[N]$.

4.2.1 Transform induced by a family One fundamental example is when $\mathcal{P}$ is a sheaf on $S \times X$, flat over S. In this case we often think of $\mathcal{P}$ as a family of sheaves on X, parameterized by S. The Fourier-Mukai transform induced by $\mathcal{P}$ encodes information about this family. As an illustration of one way this happens, let $s_0 \in S$ and suppose $k(s_0) = k$. A tangent vector at s_0 is determined by a length-two subscheme concentrated at s_0:

$$0 \to k \to \mathcal{O}_Z \to k \to 0.$$

Pulling back to $S \times X$ and tensoring with the flat sheaf $\mathcal{P}$, we get

$$0 \to P_{s_0} \to \mathcal{P}|_{Z\times X} \to \mathcal{P}_{s_0} \to 0.$$

Thus we have a map $T_{s_0}S \to \mathrm{Ext}^1_X(\mathcal{P}_{X_0}, \mathcal{P}_{X_0})$, which is called the Kodaira-Spencer map. This map can be described in terms of a Fourier-Mukai transform as follows (See [Huy, Section 5.1]):

$$\begin{tikzcd} T_{s_0}(S) \cong \mathrm{Ext}^1_S(k,k) \arrow[r, "\text{Kodaira-Spencer}"] \arrow[d, "\cong"] & \mathrm{Ext}^1(\mathcal{P}_{s_0}, \mathcal{P}_{s_0}) \arrow[d, "\cong"] \\ \mathrm{Hom}_{D^b(S)}(k, k[1]) \arrow[r, "\Phi_{\mathcal{P}}"] & \mathrm{Hom}_{D^b(X)}(\mathcal{P}_{s_0}, \mathcal{P}_{s_0}[1]). \end{tikzcd}$$

4.2.2 Moduli space of sheaves on a K3 surface Let v be a Mukai vector and h the first Chern class of an ample line bundle on a K3 surface X. We denote by $M_v(h)$ the coarse moduli space of semi-stable sheaves (with respect to h) with Mukai vector v. If v and v' are integral (1,1) Mukai vectors with $\langle v, v'\rangle$ in the Mukai pairing, then there exists an h such that $M_v(h)$ is fine and parametrizes only stable sheaves. In addition, $M_v(h)$ is a smooth algebraic symplectic variety and $\omega_{M_v(h)} \cong \mathcal{O}$. If $\langle v, v\rangle = 0$ (in which case we say that v is isotropic), then $\dim M_v(h) = 2$. If we choose a connected component of $M_v(h)$, we have a K3 surface parametrizing stable sheaves with Mukai vector v on X (See [Huy, Proposition 10.24]).

The restriction of the universal sheaf of $M_v(h)$ to a connected component M gives us an M-flat sheaf, and Mukai proved that this sheaf induces an equivalence

$D^b(X) \to D^b(M)$.

4.3 Computations with Fourier-Mukai transforms

First, here are a couple of tools that help us with computations.

Proposition 4.3.1 (Flat base change) *Given a Cartesian diagram*

$$\begin{tikzcd} Y' \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{q}"] & X' \arrow[d, "q"] \\ Y \arrow[r, "p"] & X \end{tikzcd}$$

with p flat and q proper, and $\mathcal{F}^\bullet \in D(X')$, we have a functorial isomorphism

$$\tilde{q}_*\tilde{p}^*(\mathcal{F}^\bullet) \xrightarrow{\sim} p^*q_*(\mathcal{F}^\bullet).$$

Remember that the pushforwards and pullbacks are derived pushforwards and pullbacks, although because p and $\tilde{p}$ are flat, the derived pullbacks can be computed by pulling back each sheaf in the complex.

Proof. See [Hart, III.8.3]. $\square$

Proposition 4.3.2 (Projection formula) *Let $f : X \to Y$ be a proper morphism of projective schemes over a field. For $\mathcal{F}^\bullet \in D^b(X)$ and $\mathcal{E}^\bullet \in D^b(Y)$, we have*

$$f_*(\mathcal{F}^\bullet) \otimes \mathcal{E}^\bullet \xrightarrow{\sim} f_*(\mathcal{F}^\bullet \otimes f^*(\mathcal{E}^\bullet)).$$

Proof. This proposition is a consequence of the standard projection formula. See [Huy, Chapter 3]. □

Now we will look at what happens when you compose Fourier-Mukai transforms, look at some exterior tensor product results that will be useful later on, and discuss Orlov's result and some of its consequences.

Proposition 4.3.3 (Mukai) *Let $\mathcal{P} \in D^b(X \times Y)$. The left adjoint to $\Phi_{\mathcal{P}}$ is a Fourier-Mukai transform with kernel*

$$\mathcal{P}_L := \mathcal{P}^\vee \otimes p^*\omega_Y[\dim(Y)].$$

The right adjoint to $\Phi_{\mathcal{P}}$ is a Fourier-Mukai transform with kernel

$$\mathcal{P}_R := \mathcal{P}^\vee \otimes q^*\omega_X[\dim(X)].$$

Here $^\vee$ denotes the derived dual.

Proof. See [Huy, p. 117]. The proof is a consequence of Grothendieck-Verdier duality. □

Proposition 4.3.4 ([Huy, Ex. 5.13]) *Let $\mathcal{P}_1 \in D^b(X_1 \times Y_3)$ and $\mathcal{P}_2 \in D^b(X_2 \times Y_4)$. Then $\mathcal{P}_1 \boxtimes \mathcal{P}_2$ induces a Fourier-Mukai transform $D^b(X_1 \times X_2) \to D^b(Y_3 \times Y_4)$. Let $\mathcal{R} \in D^b(X_1 \times X_2)$, and let $\mathcal{S} = \Phi_{\mathcal{P}_1 \boxtimes \mathcal{P}_2}(\mathcal{R}) \in D^b(Y_3 \times Y_4)$. The following diagram* *commutes.*

$$\begin{tikzcd} D^b(X_1) \arrow[d, "\Phi_{\mathcal{R}}"] & D^b(Y_3) \arrow[l, "\Phi_{\mathcal{P}_1}"'] \arrow[d, "\Phi_{\mathcal{S}}"] \\ D^b(X_2) \arrow[r, "\Phi_{\mathcal{P}_2}"] & D^b(Y_4) \end{tikzcd}$$

Proof. I will denote all projections by π, with superscripts indicating the source and subscripts indicating the target. I will omit the superscripts when the source is the full quadruple product.

For any $\mathcal{F} \in D^b(Y_3)$ we have

$$\begin{aligned}
\Phi_{\mathcal{P}_2}\Phi_{\mathcal{R}}\Phi_{\mathcal{P}_1}(\mathcal{F}) &= \pi^{24}_{4*}(\pi^{24*}_{2}\pi^{12}_{2*}(\pi^{12*}_{1}\pi^{13}_{1*}(\pi^{13*}_{3}\mathcal{F} \otimes \mathcal{P}_1) \otimes \mathcal{R}) \otimes \mathcal{P}_2) \\
&= \pi^{24}_{4*}(\pi^{124}_{24*}\pi^{124*}_{12}(\pi^{123}_{12*}\pi^{123*}_{13}(\pi^{13*}_{3}\mathcal{F} \otimes \mathcal{P}_1) \otimes \mathcal{R}) \otimes \mathcal{P}_2) \\
&= \pi^{24}_{4*}(\pi^{124}_{24*}(\pi^{124*}_{12}\pi^{123}_{12*}(\pi^{123*}_{3}\mathcal{F} \otimes \pi^{123*}_{13}\mathcal{P}_1) \otimes \pi^{124*}_{12}\mathcal{R}) \otimes \mathcal{P}_2) \\
&= \pi^{24}_{4*}(\pi^{124}_{24*}(\pi_{124*}\pi^{*}_{123}(\pi^{123*}_{3}\mathcal{F} \otimes \pi^{123*}_{13}\mathcal{P}_1) \otimes \pi^{124*}_{12}\mathcal{R}) \otimes \mathcal{P}_2) \\
&= \pi^{24}_{4*}(\pi^{124}_{24*}(\pi_{124*}(\pi^{*}_{3}\mathcal{F} \otimes \pi^{*}_{13}\mathcal{P}_1 \otimes \pi^{*}_{12}\mathcal{R})) \otimes \mathcal{P}_2) \\
&= \pi^{24}_{4*}(\pi_{24*}(\pi^{*}_{3}\mathcal{F} \otimes \pi^{*}_{13}\mathcal{P}_1 \otimes \pi^{*}_{12}\mathcal{R} \otimes \pi^{*}_{24}\mathcal{P}_2)) \\
&= \pi_{4*}(\pi^{*}_{3}\mathcal{F} \otimes \pi^{*}_{13}\mathcal{P}_1 \otimes \pi^{*}_{12}\mathcal{R} \otimes \pi^{*}_{24}\mathcal{P}_2) \\
&= \pi^{34}_{4*}(\pi^{34*}_{3}\mathcal{F} \otimes \pi_{34*}(\pi^{*}_{13}\mathcal{P}_1 \otimes \pi^{*}_{12}\mathcal{R} \otimes \pi^{*}_{24}\mathcal{P}_2)) \\
&= \pi^{34}_{4*}(\pi^{34*}_{3}\mathcal{F} \otimes \mathcal{S}) \\
&= \Phi_{\mathcal{S}}(\mathcal{F}).
\end{aligned}$$

Going from the first line to the second, we use flat base change, then simplification, flat base change, simplification, projection formula, projection formula, projection formula. □

Proposition 4.3.5 (Mukai) *If $\mathcal{P} \in D^b(X \times Y)$ and $\mathcal{Q} \in D^b(Y \times Z)$, define*

$$\mathcal{R} := \pi_{XY*}(\pi_{XY}^*\mathcal{P} \otimes \pi_{YZ}^*\mathcal{Q}).$$

Then $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} = \Phi_{\mathcal{R}}$.

Proof. The proof is similar in character to the proof of Proposition 4.3.4, and is given in [Huy, p. 118]. □

Theorem 4.3.6 (Orlov) *Let $\Phi : D^b(X) \to D^b(Y)$ be a fully faithful exact functor with left and right adjoints. Then up to isomorphism there is a unique $\mathcal{P} \in D^b(X \times Y)$ such that $\Phi \cong \Phi_{\mathcal{P}}$.*

Proof. The proof is found in [Orl2]. The proof constructs the object $\mathcal{P}$ and shows that $\Phi_{\mathcal{P}} \cong \Phi$ when restricted to an ample sequence. A discussion of ample sequences can be found in [Huy, Chapter 4]. □

I will use this theorem exclusively in the case that Φ is an equivalence of categories.

Proposition 4.3.7 ([Huy, Ex. 5.18]) *The Fourier-Mukai transform $\Phi_{\mathcal{P}}$ is an equivalence if and only if the following two conditions are satisfied:*

$$i)\ \pi_{13*}(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P}_L) \cong \mathcal{O}_{\Delta_X} \qquad ii)\ \pi_{13*}(\pi_{12}^*\mathcal{P}_L \otimes \pi_{23}^*\mathcal{P}) \cong \mathcal{O}_{\Delta_Y},$$

using the same notation as above on $X \times Y \times X$ and $Y \times X \times Y$, respectively.

Proof. If the conditions are satisfied, then by the composition formula in Proposition 4.3.5, $\Phi_{\mathcal{P}_L} \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{O}_{\Delta_X}} \cong \mathrm{id}_{D(X)}$, and composing in the reverse order gives $\mathrm{id}_{D(Y)}$. Therefore $\Phi_{\mathcal{P}}$ is an equivalence.

Conversely, if $\Phi_{\mathcal{P}}$ is an equivalence, then there is a natural isomorphism $\Phi_{\mathcal{P}_L} \circ \Phi_{\mathcal{P}} \xrightarrow{\sim} \mathrm{id}_{D(X)}$. By Orlov, the two functors have isomorphic kernels. The kernel of the first is $\pi_{13*}(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{P}_L)$ and the identity has kernel $\mathcal{O}_{\Delta_X}$, so these are isomorphic and the first condition is satisfied. The second is by similar reasoning. $\square$

Proposition 4.3.8 *If the Fourier-Mukai transforms with kernels $\mathcal{P}_i \in D^b(X_i \times Y_i)$ for $i = 1, 2$ are equivalences, then the Fourier-Mukai transform with kernel $\mathcal{P}_1 \boxtimes \mathcal{P}_2$ is also.*

Proof. The proof uses the criterion of Proposition 4.3.7. It is straightforward, using only the projection formula and flat base change, but it is notationally difficult. I will use the same notation for projections as in Proposition 4.3.4, this time on the six-fold product $X \times X \times Y \times Y \times X \times X$.

First, note that $(\mathcal{P}_1 \boxtimes \mathcal{P}_2)_L \cong \mathcal{P}_{1L} \boxtimes \mathcal{P}_{2L}$ because the derived dual complex distributes across the tensor product and the dualizing sheaf of a product is the tensor product of the dualizing sheaves of the factors. Using this isomorphism, we prove the first condition:

$$
\begin{aligned}
&\pi_{1256*}(\pi_{13}^{*}\mathcal{P}_1 \otimes \pi_{24}^{*}\mathcal{P}_2 \otimes \pi_{35}^{*}\mathcal{P}_{1L} \otimes \pi_{46}^{*}\mathcal{P}_{2L}) \cong \\
&\qquad \cong \pi_{1256*}^{12356}\pi_{12356*}(\pi_{12356}^{*}\pi_{135}^{12356*}(\pi_{13}^{135*}\mathcal{P}_1 \otimes \pi_{35}^{135*}\mathcal{P}_{1L}) \otimes \pi_{24}^{*}\mathcal{P}_2 \otimes \pi_{46}^{*}\mathcal{P}_{2L}) \\
&\qquad \cong \pi_{1256*}^{12356}(\pi_{135}^{12356*}(\pi_{13}^{135*}\mathcal{P}_1 \otimes \pi_{35}^{135*}\mathcal{P}_{1L}) \otimes \pi_{12356*}\pi_{246}^{*}(\pi_{24}^{246*}\mathcal{P}_2 \otimes \pi_{46}^{246*}\mathcal{P}_{2L})) \\
&\qquad \cong \pi_{1256*}^{12356}(\pi_{135}^{12356*}(\pi_{13}^{135*}\mathcal{P}_1 \otimes \pi_{35}^{135*}\mathcal{P}_{1L}) \otimes \pi_{26}^{12356*}\pi_{26*}^{246}(\pi_{24}^{246*}\mathcal{P}_2 \otimes \pi_{46}^{246*}\mathcal{P}_{2L})) \\
&\qquad \cong \pi_{1256*}^{12356}(\pi_{135}^{12356*}(\pi_{13}^{135*}\mathcal{P}_1 \otimes \pi_{35}^{135*}\mathcal{P}_{1L}) \otimes \pi_{1256}^{12356*}\pi_{26}^{1256*}\mathcal{O}_{\Delta_X}) \\
&\qquad \cong \pi_{1256*}^{12356}\pi_{135}^{12356*}(\pi_{13}^{135*}\mathcal{P}_1 \otimes \pi_{35}^{135*}\mathcal{P}_{1L}) \otimes \pi_{26}^{1256*}\mathcal{O}_{\Delta_X} \\
&\qquad \cong \pi_{15}^{1256*}\pi_{15*}^{135}(\pi_{13}^{135*}\mathcal{P}_1 \otimes \pi_{35}^{135*}\mathcal{P}_{1L}) \otimes \pi_{26}^{1256*}\mathcal{O}_{\Delta_X} \\
&\qquad \cong \pi_{15}^{1256*}\mathcal{O}_{\Delta_X} \otimes \pi_{26}^{1256*}\mathcal{O}_{\Delta_X} \\
&\qquad \cong \mathcal{O}_{\Delta_{X\times X}}.
\end{aligned}
$$

Most lines here are just rearranging, but from lines 2 to 3 we use the projection formula and from lines 3 to 4 and 6 to 7 we use the flat base change theorem. The second condition is proved in the same manner. □

4.4 The cohomological Fourier-Mukai transform

Let us first discuss the Mukai vector, which will allow us to consider the cohomological Fourier-Mukai transform and also be very useful later on.

4.4.1 Todd class The Todd class can be written in terms of the Chern roots α_i of a sheaf $\mathcal{E}$ and is defined as

$$\mathrm{Td}(\mathcal{E}) = \prod Q(\alpha_i), \text{ where } Q(x) = \frac{x}{1-e^{-x}}.$$

For surfaces, we only need to compute the first few terms, which are

$$1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2).$$

An important case is the Todd class of the tangent bundle, which is sometimes just referred to as the Todd class of a scheme and denoted Td_X. For a K3 surface, this is $1 + 2[X]$.

4.4.2 Square root of the Todd class Often, we need the square root of the Todd class, for example, for the Grothendieck-Riemann-Roch theorem. We can compute this as the square root of the power series, and as long as the first term is nonzero and has a square root, you can find a square root of the class. For a surface, it is easy to find the square root of a class (a_0, a_1, a_2) one step at a time: $b_0 = \sqrt{a_0}$, $b_1 = \frac{a_1}{2b_0}$, and $b_2 = \frac{a_2 - b_1^2}{2b_0}$. For a K3 surface X, this means we have $\sqrt{\mathrm{Td}_X} = 1 + [X]$.

4.4.3 The Mukai vector The Mukai vector of a coherent sheaf is an adjustment of the Chern character by the square root of the Todd class of the scheme.

$$v(\mathcal{F}) = \mathrm{ch}(\mathcal{F}).\sqrt{\mathrm{Td}_X}$$

4.4.4 Mukai vector of a K3 surface For a K3 surface, the Mukai vector of a coherent sheaf $\mathcal{F}$ is

$$v(\mathcal{F}) = (r, c_1, \frac{1}{2}(c_1^2 - 2c_2) + r).$$

When we look at a moduli space of coherent sheaves on a K3 surface X, we will fix the Mukai vector. This is the same as fixing the rank and Chern classes. The advantage of the Mukai vector is that it works nicely with Fourier-Mukai transforms. Note that we can also talk about the Mukai vector of an element of $K(X)$ or of $D^b(X)$.

4.4.5 Mukai pairing For a K3 surface, we define a pairing on $H^*(X, \mathcal{Z})$ called the Mukai pairing, which is defined as

$$\langle u, u' \rangle = r.s' + s.r' - \alpha.\alpha' \in H^4(X, \mathcal{Z}) \cong \mathcal{Z},$$

where $u = (r, \alpha, s)$ and $u' = (r', \alpha', s')$ [Orl2]. We call $H^*(X, \mathcal{Z})$ with this pairing the Mukai lattice. The lattice has a Hodge structure, and the (1,1) part consists of $H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X)$.

We can also define a Fourier-Mukai transform on cohomology as follows: for a class $\alpha \in H^*(X \times Y, \mathbf{Q})$, we define $\Phi_\alpha^H : H^*(X, \mathbf{Q}) \to H^*(Y, \mathbf{Q})$ by

$$\beta \mapsto p_*(\alpha.q^*(\beta)).$$

The Chern character maps $D^b(X) \to H^*(X, \mathbf{Q})$, but under this map the two

Fourier-Mukai transforms are not compatible. This is the reason we look at the Mukai vector instead of the Chern character.

Proposition 4.4.6 *If* $\mathcal{P} \in D^b(X \times Y)$ *and* $\mathcal{F}^\bullet \in D^b(X)$ *then*

$$\Phi^H_{v(\mathcal{P})}(v(\mathcal{F})) = v(\Phi_{\mathcal{P}}(\mathcal{F}^\bullet)).$$

In other words, the following digram commutes:

$$\begin{tikzcd} D^b(X) \arrow[r, "\Phi_{\mathcal{P}}"] \arrow[d, "v"] & D^b(Y) \arrow[d, "v"] \\ H^*(X, \mathbf{Q}) \arrow[r, "\Phi^H_{v(\mathcal{P})}"] & H^*(Y, \mathbf{Q}). \end{tikzcd}$$

Proof. See [Huy, Section 5.2]. This is a result of Grothendieck-Riemann-Roch. $\square$

5 Lifting Fourier-Mukai Kernels

For a Fourier-Mukai kernel Φ_0 on $X_0 \times Y_0$, and for any infinitesimal deformation of $\mathrm{Coh}(X_0)$, there is a unique deformation of $\mathrm{Coh}(Y_0)$ so that the derived equivalence lifts.

The infinitesimal deformations of $\mathrm{Coh}(X_0)$ are described by the Hochschild cohomology $HH^2(X_0)$. Thus there is a map $HH^2(X_0) \to HH^2(Y_0)$ induced by Φ_0. In this chapter, we study that map when the Fourier-Mukai transform is an autoequivalence given by tensoring with a line bundle. We also study the case where X_0 and Y_0 are K3 surfaces and the kernel of the Fourier-Mukai transform is the universal sheaf for a moduli space of sheaves on X_0.

5.1 Hochschild cohomology

Let X be a scheme.

Definition 5.1.1 The *Hochschild cohomology* of X is

$$HH^N(X) := \mathrm{Hom}_{D^b(X\times X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta[N]).$$

5.1.2 Hochschild class as a natural transformation The sheaves $\mathcal{O}_\Delta$ and $\mathcal{O}_\Delta[N]$ are the kernels of the identity functor and the shift-by-N functor, respectively. Therefore, an element of the Hochschild cohomology gives a natural transformation between these two functors. We should take care however, because two different elements could induce the same natural transformation, as shown in [Huy, p. 120].

Theorem 5.1.3 ([CW, Toda]) *Let X and Y be smooth projective varieties such that there exists an equivalence $\Phi : D^b(X) \to D^b(Y)$. Then Φ induces an isomorphism $\Phi^{HH} : HH^*(X) \to HH^*(Y)$.*

Proof. Since Φ is an equivalence of categories, it is induced by a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ for some $\mathcal{P} \in D^b(X \times Y)$ by Orlov's result. Then by Proposition 4.3.7, $\mathcal{P}_L$ also induces an equivalence $D^b(X) \to D^b(Y)$. Then by Proposition 4.3.8, $\mathcal{P} \boxtimes \mathcal{P}_L$ induces an equivalence $D^b(X \times X) \to D^b(Y \times Y)$. By Proposition 4.3.4, the Fourier-Mukai transform induced by $\Phi_{\mathcal{P}\boxtimes\mathcal{P}_L}(\mathcal{O}_{\Delta_X})$ is

$$\Phi_{\mathcal{P}_L} \circ \Phi_{\mathcal{O}_{\Delta_X}} \circ \Phi_{\mathcal{P}} \cong \mathrm{id}_{D^b(Y)},$$

so by Orlov's result, is isomorphic to $\mathcal{O}_{\Delta_Y}$. Thus $\Phi_{\mathcal{P}\boxtimes\mathcal{P}_L}$ gives a morphism

$$\mathrm{Hom}_{D^b(X\times X)}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}[N]) \to \mathrm{Hom}_{D^b(Y\times Y)}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y}[N]).$$

The group on the left is $HH^N(X)$ and on the right is $HH^N(Y)$.

$\square$

Since $HH^*(X)$ is somewhat mysterious, we relate it to the following, more straightforward ring.

Definition 5.1.4

$$HT^N(X) := \bigoplus_{p+q=N} H^p(X, \bigwedge^q T_X)$$

Theorem 5.1.5 ([Toda][Ginz, 9.4.1][Huy, p. 140]) *There exists an isomorphism of graded vector spaces*

$$I_{HKR} : HT^*(X) \to HH^*(X).$$

This is called the HKR isomorphism, and is named after Hochschild, Kostant, and Rosenberg. The idea is that for a regular k-algebra R, there is an isomorphism $T_R \to \mathrm{Hom}_R(\Omega_R, R)$. Using that $\mathrm{Ext}^1_{R\otimes_k R}(R,R) \cong T_R$ and $\mathrm{Tor}_1^{R\otimes_k R}(R,R) \cong \Omega_R$, we can extend this isomorphism to the higher cohomologies.

5.1.6 HKR is not a ring isomorphism The HKR isomorphism does not respect the ring structures of HH^* and HT^*. See [Huy, p. 140].

5.1.7 Induced map on HT Via the HKR isomorphisms, a derived equivalence Φ between schemes X and Y induces an isomorphism $\Phi^{HT} : HT^*(X) \to HT^*(Y)$ as follows:

$$\Phi^{HT} := I_Y^{-1} \circ \Phi^{HH} \circ I_X.$$

Lemma 5.1.8 ([Toda, Proposition 5.6]) *Let $\mathcal{P} \in D^b(X \times Y)$ be the kernel of a Fourier-Mukai equivalence $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$. Then* $\mathrm{RHom}(\mathcal{P}, \mathcal{P}) \cong HH^*(X)$.

Proof. It is enough to prove that the equivalence $\mathrm{id} \times \Phi_{\mathcal{P}} : D^b(X \times X) \to D^b(X \times Y)$

takes $\mathcal{O}_\Delta$ to $\mathcal{P}$.

Recall that $\Phi_{\mathcal{O}_\Delta}$ is the identity functor. The kernel of $\mathrm{id} \times \Phi_{\mathcal{P}}$ is $\mathcal{Q} := p_{13}^*\mathcal{O}_\Delta \otimes p_{24}^*\mathcal{P} \in D^b(X \times X \times X \times Y)$. By Proposition 4.3.4, the Fourier-Mukai transform defined by the kernel $\Phi_{\mathcal{Q}}(\mathcal{O}_\Delta)$ is given by composing $\Phi_{\mathcal{O}_\Delta}$, $\Phi_{\mathcal{O}_\Delta}$, and $\Phi_{\mathcal{P}}$. This is $\Phi_{\mathcal{P}}$, and since the transform is an equivalence, the kernel must be $\mathcal{P}$. $\square$

Proposition 5.1.9 *Let X and Y be K3 surfaces over k, with $\mathcal{P} \in D^b(X \times Y)$, flat over Y. Let X' and Y' be deformations of X and Y to A. Then there is at most one $\mathcal{P}' \in D^b(X' \times_A Y')$ such that $\mathcal{P}' \overset{\mathbf{L}}{\underset{A}{\otimes}} k \cong \mathcal{P}$.*

Proof. The set of such sheaves is a torsor under $\mathrm{Ext}^1(\mathcal{P}, \mathcal{P})$, which is isomorphic to $\mathrm{HH}^1(Y)$ by Lemma 5.1.8. This in turn is isomorphic to $H^1(\mathcal{O}_Y) \oplus H^0(\mathcal{T}_Y)$ by the HKR isomorphism, which is zero because Y is a K3 surface. $\square$

Lemma 5.1.10 *The Fourier-Mukai kernel $\mathcal{P}$ lifts to $\mathcal{P}'$ on $X' \times Y'$ if and only if Y' is the deformation corresponding to $\Phi(\sigma)$.*

Proof. Although it is true in more generality (which generality can be deduced from Toda's proof), let's assume that X' and Y' are schemes. In this case, the universal obstruction of $X' \times Y'$ applied to $\mathcal{P}$ gives $\Phi(\tau) - \sigma$. Huybrecht proved that this is zero exactly when $\mathcal{P}$ lifts. $\square$

5.1.11 Obstruction to lifting a kernel If $\mathcal{P}_0$ is a kernel on $X_0 \times Y_0$, the obstruction to lifting $\mathcal{P}_0$ to $X \times Y$ for some deformation X of X_0 and Y of Y_0 is given by

$\Phi(\sigma) - \tau$. In diagram form:

$$
\begin{tikzcd}
HH^2(X_0) \times HH^2(Y_0) \arrow[r] \arrow[rr, bend right, "\mathrm{id} - \Phi^{HH}"'] & \mathrm{Ext}^2(\mathcal{P}_0, \mathcal{P}_0) \arrow[r, "\sim"] & HH^2(X_0)
\end{tikzcd} .
$$

Corollary 5.1.12 *If X' and X'' are two deformations of X, and there is a derived equivalence between X' and X'' which is the identity on X, then in fact X' and X'' are isomorphic as deformations of X.*

Proof. The derived equivalence is a Fourier-Mukai transform, induced by a sheaf $\mathcal{P}$ on $X' \times X''$, which reduces to $\mathcal{O}_{\Delta_X}$ on $X \times X$, because $\mathcal{O}_{\Delta_X}$ induces the identity. Here Φ is also the identity, so the existence of $\mathcal{P}$ requires the obstruction class $\Phi(\tau) - \sigma = \tau - \sigma = 0$. $\square$

5.2 Lifting a sheaf to a deformation of Coh(X)

Let X_0 be a smooth projective scheme over an algebraically closed field k. The deformations of $\mathrm{Coh}(X_0)$ to $k[\epsilon]/(\epsilon^2)$ are given by $HH^2(X_0)$. We want to determine the obstruction to lifting a given sheaf or complex of sheaves to one of these deformations.

5.2.1 Toda's construction The deformation of $\mathrm{Coh}(X_0)$ corresponding to

$$(\alpha, \beta, \gamma) \in H^2(X_0, \mathcal{O}_{X_0}) \oplus H^1(X_0, T_{X_0}) \oplus H^0(X_0, \wedge^2 T_{X_0})$$

is the category of $(1 + \alpha\epsilon)$-twisted $\mathcal{A}$-modules. The sheaf $\mathcal{A}$ is the sheaf of rings which is equal to the structure sheaf $\mathcal{O}_X$ as sheaf of groups, where X is the deformation of X_0 corresponding to β. The multiplication of $\mathcal{A}$ is defined by $(a + b\epsilon) * (c + d\epsilon) = ac + (ad + bd + \gamma(a, c))\epsilon$.

More explicitly, an object in $\mathrm{Coh}(X_0, (\alpha, \beta, \gamma))$ is given by a collection $\{\mathcal{F}_i, \phi_{ij}\}$, where $\mathcal{F}_i$ is a sheaf of $\mathcal{A}|_{U_i}$-modules, and $\phi_{ij} : \mathcal{F}_i \to \mathcal{F}_j$ is an isomorphism on $U_i \cap U_j$. Furthermore, $\phi_{ki}\phi_{jk}\phi_{ij} = \mathrm{id}\,\alpha_{ijk}$, where α_{ijk} is a Čech representative of α.

5.2.2 Exponential Atiyah class In [Toda], the exponential Atiyah class is defined as a map in the derived category $\mathcal{P} \to \bigoplus \mathcal{P} \otimes \Omega^i[i]$. It is the identity in degree 0, the standard Atiyah class in degree 1, and the n-fold composition of the Atiyah class with itself in degree n.

Given a sheaf of X_0 with exponential Atiyah class (id, a_1, a_2) and a deformation of X to $k[\epsilon]/\epsilon^2$ given by $(\alpha, \beta, \gamma) \in HT^2(X)$, the obstruction to lifting the sheaf to a complex of sheaves on X is $\alpha\,\mathrm{id} + \beta a_1 + \gamma a_2 \in \mathrm{Ext}^2(\mathcal{P}, \mathcal{P})$.

5.3 Tensoring with a line bundle

Let L be a line bundle on X. Tensoring with L gives an autoequivalence of $D^b(X)$. The equivalence is the Fourier-Mukai transform with kernel $\Delta_*(L)$, but we will write the transform as Φ_L.

5.3.1 Picard action on Hochschild cohomology As with any Fourier-Mukai equivalence, Φ_L induces an isomorphism of $HH^2(X)$. That is, there is a map $\mathrm{Pic}(X) \to GL(HH^2(X))$. It is clearly a group homomorphism.

5.3.2 Exponential Atiyah class of a line bundle For a line bundle, the Atiyah class $L \to L \otimes \Omega[1]$ can be thought of as an element of $H^1(X, \Omega)$. The map $\mathrm{Pic}(X) \to H^1(X, \Omega)$ is given by the dlog map, which maps $a \mapsto \dfrac{da}{a}$. Over the complex numbers, the composition with $H^1(X, \Omega) \to H^2(X, \mathbf{Z}) \otimes \mathbf{C}$ gives the first Chern class.

Thus the exponential Atiyah class of L is $(1, a_1(L), a_2(L))$, with $a_1(L) = \operatorname{dlog} L$, and $a_2(L) = \operatorname{dlog} L \cup \operatorname{dlog} L$. Using Čech cohomology, you can compute $\operatorname{dlog} L$ by taking a representative τ_{ij} for L and using $\dfrac{d\tau_{ij}}{\tau_{ij}}$. Then $a_2(L)$ is computed by $\dfrac{d\tau_{ij} \wedge d\tau_{jk}}{\tau_{ij}\tau_{jk}} = \dfrac{d\tau_{ij} \wedge d\tau_{jk}}{\tau_{ik}}$.

The exponential Atiyah class is then an element of

$$H^0(X, \mathcal{O}) \oplus H^1(X, \Omega) \oplus H^2(X, \Omega^2).$$

The obstruction to lifting a line bundle is given by composing with an element of

$$HT^2(X) = H^2(X, \mathcal{O}) \oplus H^1(X, T_X) \oplus H^0(X, \wedge^2 T_X)$$

to get a class in $H^2(X, \mathcal{O})$.

5.3.3 The dlog map for pth powers Since dlog takes multiplication to addition, we have $\operatorname{dlog}(L^p) = 0$ if k has characteristic $p > 0$.

Proposition 5.3.4 ([vdGK]) *For a K3 surface X,* dlog *is an injection in characteristic 0, and an injection from* $\operatorname{Pic}(X)/p\operatorname{Pic}(X)$ *in characteristic p.*

Proposition 5.3.5 *For an element $(\alpha, \beta, \gamma) \in HT^2(X)$, we have*

$$\Phi_L(\alpha, \beta, \gamma) = (\alpha - \beta \operatorname{dlog} L - \gamma \operatorname{dlog}^2 L, \beta, \gamma)$$

Proof. Using Toda's notation, L lifts locally to $\mathcal{A}$, but glues on the overlap to $1 - (\gamma a_2(L) + \beta a_1(L))\epsilon$.

Tensoring a complex $\mathcal{F}$ by L will give a complex $\mathcal{F}_i \otimes L_i$ on each open set, and the gluings will now be $1 + \alpha - \beta a_1(L) - \gamma a_2(L)^2$.

This functor is clearly an equivalence, since its inverse can be given by tensoring with L^{-1}. It is also exact. So it is induced by a Fourier Mukai kernel, which must be a lift of $\Delta_*(L)$.

By Lemma 5.1.10, we are done. □

Proposition 5.3.6 *Let $L \in \operatorname{Pic} X$ and Φ_L be the Fourier-Mukai transform induced by the autoequivalence of $D^b(X)$ given by $(-) \otimes L$. Then Φ_L^{HH} is the identity if and only if L is a pth power.*

Proof. For a K3 surface, dlog is injective from $\operatorname{Pic}(X)/p\operatorname{Pic}(X)$. (In characteristic 0, it is injective.)

If $\operatorname{dlog} L$ is nonzero, then there is some β so that $\beta \operatorname{dlog} L \neq 0$. In this case, $\Phi_L(0, \beta, 0) = (-\beta \operatorname{dlog} L, \beta, 0)$. □

5.4 Lifting the equivalence relation

Theorem 5.4.1 *Suppose that X and Y are derived equivalent K3 surfaces over a field of characteristic $p > 0$. Then for any scheme deformation of X, there is a scheme deformation of Y so that the two deformations are derived equivalent.*

Proof. If there is a derived equivalence $\Phi : D^b(Y_0) \to D^b(X_0)$, and on Mukai vectors Φ maps $(0, 0, 1)$ to $v := (r, L, s)$, by composing with various automorphisms we can assume that r is prime to p and $Y_0 \cong M_{X_0}(v)$, the fine moduli space of stable sheaves of X_0 with Mukai vector v [LO, Proposition 6.2], with the derived equivalence given by the Fourier-Mukai transform with kernel $\mathcal{E}$.

Let a be such that $ar \equiv 1 \pmod{p}$, and let $\Psi : D^b(X_0) \to D^b(X_0)$ be the autoequivalence $(-) \otimes L^{-a}$. The composition $\Psi \circ \Phi$ maps the Mukai vector $(0, 0, 1)$ to $(r, (1 - ar)L, *)$. We can thus assume that L is a pth power.

Now for any first-order deformation X of X_0, the sheaf L lifts to X, and we can form the moduli space Y of sheaves on X with the same r and s, but with determinant the lift of L.

A universal sheaf on $X \times Y$ exists if and only if there is a line bundle b on X_0 such that $\mathrm{GCD}(r, bL, s) = 1$ [LO, Theorem 3.14]. So such a b must exist on X_0, but may not lift to X. We may replace b with pb, which does lift to X. Now $\mathrm{GCD}(p, r) = 1$ implies $\mathrm{GCD}(r, pbL, s) = 1$, so there is a universal sheaf on $X \times Y$, which must be the kernel of a Fourier-Mukai equivalence. □

Remark 5.4.2 It is tempting to say that there is a derived equivalence $D^b(X_0) \to D^b(Y_0)$ that induces a map on HH^2 that maps $H^1(T_{X_0}) \to H^1(T_{Y_0})$, but this is not what we have proven. The universal sheaf that we get using our argument may not reduce to $\mathcal{E}$. This could be overcome if not for the fact that for different deformations of X_0, we may get different universal sheaves.

Corollary 5.4.3 *If X_0 and Y_0 are derived equivalent K3 surfaces in characteristic 0, and there exists an equivalence $\Phi : D^b(X_0) \to D^b(Y_0)$ such that $\Phi((0,0,1)) = (r, rL, s)$ for some L, then any deformation of X_0 has a corresponding Y_0 that is a Fourier-Mukai partner.*

Proof. As in Theorem 5.4.1, we have $Y_0 \cong M_{X_0}(v)$, except this time after tensoring with L^{-1} we can assume that v is of the form $(r, 0, s)$. This implies that $\mathrm{GCD}(r, s) = 1$, which means that the universal sheaf exists on $X \times Y$. (In this case, we are using $b = 0$, which has no trouble lifting. □

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Vita

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