# Dissipative mechanisms in SNS junctions and magnetic vortex lattices

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**Abstract**

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When a normal metal is subject to an external field and a transport current is induced, the dissipation rate is typically controlled by the elastic scattering time of the quasiparticles. In superconductors, the presence of the superfluid condensate complicates quasi-particle dynamics, and as a result the dissipation rate can depend on additional time scales. In particular, when the condensate is accelerated by the external field, a spectral flow of quasi-particle energy levels are induced, resulting in a mechanism of dissipation which is controlled by the inelastic relaxation time. In this work I theoretically investigate the role of this mechanism in two experimentally relevant superconducting systems: superconductor-normal metal-superconductor junctions and magnetic vortex lattices formed in type-II superconducting films. In the former, I develop a theory of current-voltage characteristics and show that there is a regime at small voltages/currents with features that are distinct from the large voltage/current regime, which is described by the conventional theory. In the latter, I calculate the microwave absorption coefficient and show that there is a broad range of parameters where the mechanism of dissipation related to spectral flow gives the dominant contribution.

# Dissipative mechanisms in SNS junctions and magnetic vortex lattices

## CONTENTS

## I. INTRODUCTION

Superconductivity, that is the complete disappearance of dc electric resistivity in a material, was first observed in the early 1910's following improvements in refrigeration techniques which

allowed various metals to be cooled to sufficiently low temperatures. In the decades the discovery of superconductivity, scientists struggled to understand the fundamental nature of the phenomenon. It was not until the 1950's and 1960's, following the work of Bardeen, Cooper, Shrieffer [1] (hereafter referred as BCS), that a satisfactory microscopic theory of conventional superconductivity was developed. Although BCS theory is essentially based on a self consistent mean field treatment, it captures the essential physics which gives rise the condensation of Cooper pairs into a coherent superconducting state and has been largely successful in describing a variety of superconducting phenomena.

While there are a vast amount of topics which fall under the umbrella of superconductivity, this dissertation will focus on the topic of dissipation in superconducting systems. Dissipation in superconductors is generally determined by quasi-particle scattering processes, which can be characterized by the two time scales: the elastic relaxation time $\tau_{el}$ and the inelastic relaxation time $\tau_{in}$. In most superconductors the momentum relaxation rate is much faster than the energy relaxation rate and $\tau_{in} \gg \tau_{el}$. In the conventional theory, dissipation rate is controlled by $\tau_{el}$ and the long inelastic time $\tau_{in}$ plays no role (See for example [2–4]). However, due to a mechanism of dissipation which is related to the spectral motion of quasi-particle energy levels, there is an additional contribution to the dissipation rate which is instead controlled by $\tau_{in}$. Because of the hierarchy of time scales $\tau_{in} \gg \tau_{el}$, the contribution to the dissipation of the latter type is distinct from the conventional result and can be much larger in some cases. To demonstrate this effect I will focus on the two particular superconducting systems, magnetic vortex lattices and superconductor-normal metal-superconductor (SNS) junctions.

The sections of this dissertation will be organized in the following way. In Sec. II I will outline the general theoretical tools which be used for calculations in subsequent sections. Here I will give a brief summary of conventional BCS theory and present a kinetic theory for quasi-particles in the presence of a spectral flow. Sec. III will focus on the problem of microwave absorption in the magnetic vortex lattices formed by type-II superconducting films in a perpendicular magnetic field. Using the tools developed in the previous chapter, the microwave absorption coefficient is calculated and compared to the conventional result. In Sec. IV, a theory of current-voltage (I-U) characteristics in SNS is developed and in Sec. V it is applied to the case of non-reciprocal junctions. In Sec. VI the related topic of negative critical currents in SNS junctions is discussed. Finally, in Sec. VII a rigorous derivation of the kinetic equation introduced in Sec. II using a method based on Green's functions is given.

## II. THEORETICAL DESCRIPTION OF THE SUPERCONDUCTING STATE

### A. The Cooper Instability

The conventional BCS theory of superconductivity is based on Cooper's theorem about the instability of the groundstate of an electron gas with an arbitrarily small attractive interaction. To demonstrate this effect, let us consider the quantum mechanical problem where two electrons in the groundstate of a normal metal interact via potential $V(\mathbf{r_1} - \mathbf{r_2})$, independent of their spins. I will assume that the presence of the other electrons, which occupy all single particle states below the Fermi level, manifests only through the Pauli exclusion principle. The wavefunction of the two electrons $\psi(\mathbf{r_1}, \mathbf{r_2})$ is determined by the Schrodinger equation,

$$\left[\frac{-1}{2m}\left({\nabla_\mathbf{1}}^2 + {\nabla_\mathbf{2}}^2\right) + V(\mathbf{r_1} - \mathbf{r_2})\right]\psi(\mathbf{r_1}, \mathbf{r_2}) = (\epsilon + 2\epsilon_F)\,\psi(\mathbf{r_1}, \mathbf{r_2}). \tag{1}$$

Here $\epsilon$ is the energy measured with respect to the groundstate energy of two non-interacting electrons, which is twice the Fermi energy $\epsilon_F$. It is convenient to can change coordinates from $\mathbf{r_1}$ and $\mathbf{r_2}$ to $\mathbf{r} = (\mathbf{r_1} + \mathbf{r_2})/2$ and $\tilde{\mathbf{r}} = (\mathbf{r_1} - \mathbf{r_2})/2$ . If the center of mass is assumed to be at rest, we can choose coordinates such that the wavefunction only depends on the relative coordinate $\tilde{\mathbf{r}}$,

$$\left[\frac{-1}{m}\frac{d^2}{d\mathbf{r}^2} + V(\tilde{\mathbf{r}})\right]\psi(\tilde{\mathbf{r}}) = (\epsilon + 2\epsilon_F)\,\psi(\tilde{\mathbf{r}}). \tag{2}$$

In the momentum representation,

$$\psi(\mathbf{k}) = \int d^3\tilde{\mathbf{r}} e^{-i\mathbf{k}\cdot\tilde{\mathbf{r}}}\psi(\tilde{\mathbf{r}}), \tag{3}$$

the Schrodinger equation is given by,

$$\left(2\xi(k) - \epsilon)\psi(\mathbf{k})\right) + \int \frac{d^3\mathbf{k}'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}')\psi(\mathbf{k}') = 0. \tag{4}$$

Here $\xi(k) = \frac{k^2}{2m} - \epsilon_F$ and the interaction potential $V(\mathbf{k} - \mathbf{k}')$ is given by,

$$V(\mathbf{k} - \mathbf{k}') = \int d^3\tilde{\mathbf{r}} e^{-i\mathbf{k}\cdot\tilde{\mathbf{r}}} V(\tilde{\mathbf{r}}). \tag{5}$$

The interaction potential and can be expanded in the spherical harmonics,

$$V(\mathbf{k} - \mathbf{k}') = \sum_{l=0}^{\infty} V_l(\mathbf{k}, \mathbf{k}') \sum_{m=-l}^{l} Y_{lm}(\mathbf{k}) Y^*_{lm}(\mathbf{k}'), \tag{6}$$

where $Y_{lm}$ are the spherical harmonics with orbital angular momentum $l$ and its z-projection $m$. Assuming that the interaction is attractive near the Fermi surface, the coefficients $V_l(\mathbf{k},\mathbf{k}')$ have the form

$$V_l(\mathbf{k},\mathbf{k}') = \begin{cases} -V_l, & \epsilon_F < \frac{k^2}{2m}, \frac{k'^2}{2m} < \epsilon_F + \Delta_l \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Here $V_l > 0$ and $\epsilon_F \ll \Delta_l$. Substituting Eq. (7) back into Eq. (4) we have

$$\Big(2\xi(k) - \epsilon\Big)\psi(\mathbf{k}) - N_0 \sum_{l=0}^{\infty} V_l \int_0^{\Delta_l} d\xi \int \frac{d\Omega}{4\pi} \sum_{m=-l}^{l} Y_{lm}(\mathbf{k}) Y^*_{lm}(\mathbf{k}')\psi(\mathbf{k}') = 0 \tag{8}$$

Here $N_0 = \frac{mk_F}{2\pi^2}$ is the density of states at the Fermi level, and we have expressed the integral over $\mathbf{k}'$ as an integral over the solid angle $\Omega$ and the energy $\xi(k) = \frac{k^2}{2m} - \epsilon_F$. By expanding the wavefunction $\psi(\mathbf{k})$ in the spherical harmonics,

$$\psi(\mathbf{k}) = \sum_{l}^{\infty} a_l \psi_l(\mathbf{k}) = \sum_{l}^{\infty} a_l \sum_{m=-l}^{l} b_m Y_{lm}(\mathbf{k}), \tag{9}$$

and using the orthonormality of the spherical functions,

$$\int \frac{d\Omega}{4\pi} Y_{lm}(\mathbf{k}) Y^*_{l'm'}(\mathbf{k}) = \delta_{ll'}\delta mm', \tag{10}$$

it can be shown that solutions of Eq. (8) are given by wavefunctions with definite angular momentum $\psi_l(\mathbf{k})$, and the corresponding energies $\epsilon_l$ are determined by the relation,

$$\psi_l(\mathbf{k}) = \frac{N_0 V_l \int_0^{\Delta_l} d\xi \psi_l(\mathbf{k})}{2\xi(k) - \epsilon_l} \tag{11}$$

Integrating the above equation with respect to energy gives,

$$\begin{aligned} \int_0^{\Delta_l} d\xi \psi_l(\mathbf{k}) &= \Big( \int d\xi \psi_l(\mathbf{k}) \Big) N_0 V_l \int_0^{\Delta_l} d\xi \frac{1}{2\xi(k) - \epsilon_l} \\ 1 &= \frac{N_0 V_l}{2} \log\left( \frac{\Delta_k - \epsilon_l/2}{-\epsilon_l/2} \right). \end{aligned} \tag{12}$$

Thus the energies are given by formula,

$$\epsilon_l = -2\Delta_l e^{-\frac{2}{N_0 V_l}}, \tag{13}$$

and are negative for all angular momenta $l$ as well as arbitrarily small interaction $V_l$. This means that in the presence of an arbitrarily weak attractive interaction, it becomes energetically favorable for electrons to pair together rather than to fill up a Fermi surface, and thus the groundstate of the electron gas becomes unstable. This gives us a hint that attractive interactions lead to the formation of cooper pairs in the groundstate, and are essential for superconductivity.

### B. The self consistent mean field method

The spectrum of elementary excitations in a superconductor can be obtained by diagonalizing an effective Hamiltonian $\hat{H}_{eff}$ which includes an attractive pair interaction between electrons,

$$\hat{H}_{eff} = \hat{H}_0 + \frac{1}{2}\sum_{\alpha\beta\lambda\mu}\int d^3\mathbf{r_1}\int d^3\mathbf{r_2}\hat{\psi}^+_\alpha(\mathbf{r_1})\hat{\psi}^+_\beta(\mathbf{r_2})V_{\alpha\beta,\lambda\kappa}(\mathbf{r_1}-\mathbf{r_2})\hat{\psi}_\lambda(\mathbf{r_1})\hat{\psi}_\kappa(\mathbf{r_2}). \tag{14}$$

Here $\psi^+_\alpha(\mathbf{r}), \psi_\alpha(\mathbf{r})$ are the electron field operators with spin index $\alpha$ and $V_{\alpha\beta,\lambda\kappa}(\mathbf{r_1}-\mathbf{r_2})$ describes the effective two particle interaction, and $\hat{H}_0$ is the single particle Hamiltonian. In the position representation $\hat{H}_0$ is given by,

$$\hat{H}_0 = \sum_\alpha \int d^3\mathbf{r}\hat{\psi}^+_\alpha(\mathbf{r})\left(\frac{1}{2m}\left(-i\boldsymbol{\nabla}_\mathbf{r} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi_0(\mathbf{r}) - \epsilon_F\right)\hat{\psi}_\alpha(\mathbf{r}), \tag{15}$$

where $\phi_0(\mathbf{r})$ and $\mathbf{A}$ are the scalar and vector potentials. In BCS theory, the Hamiltonian in Eq. (14) is treated in the mean field approximation. In particular, the following mean field quantities are introduced

$$\begin{aligned} F_{\alpha\beta}(\mathbf{r_1},\mathbf{r_2}) &= \langle\hat{\psi}_\alpha(\mathbf{r_1})\hat{\psi}_\beta(\mathbf{r_2})\rangle \\ F^+_{\alpha\beta}(\mathbf{r_1},\mathbf{r_2}) &= \langle\hat{\psi}^+_\alpha(\mathbf{r_1})\hat{\psi}^+_\beta(\mathbf{r_2})\rangle. \end{aligned} \tag{16}$$

Here $\langle ...\rangle$ denotes averaging over the groundstate [1]. By allowing the groundstate of a superconductor to be a superposition of states with different numbers of Cooper pairs, the quantities $F_{k\alpha\beta}$ and $F^+_{k\alpha\beta}$ are non-zero [2]. Expanding Eq. (14) in the mean field approximation we get the BCS Hamiltonian

$$\begin{aligned} \hat{H}_{BCS} = \hat{H}_0 + \frac{1}{2}\sum_{\alpha\beta\lambda\kappa}\int d^3\mathbf{r_1}\int d^3\mathbf{r_2}V_{\alpha\beta,\lambda\kappa}(\mathbf{r_1}-\mathbf{r_2})\Big(&F_{\lambda\kappa}(\mathbf{r_1},\mathbf{r_2})\hat{\psi}^+_\alpha(\mathbf{r_1})\hat{\psi}^+_\beta(\mathbf{r_2}) \\ &+ F^+_{\alpha\beta}(\mathbf{r_1},\mathbf{r_2})\hat{\psi}_\lambda(\mathbf{r_1})\hat{\psi}_\kappa(\mathbf{r_2}) - F^+_{\alpha\beta}(\mathbf{r_1},\mathbf{r_2})F_{\lambda\mu}(\mathbf{r_1},\mathbf{r_2})\Big). \end{aligned} \tag{17}$$

In most superconductors the effective attraction between electrons is mediated by electron-phonon interactions [3], and the interaction potential can be treated as point like

$$V_{\alpha\beta,\lambda\kappa}(\mathbf{r_1}-\mathbf{r_2}) = V_0\delta(\mathbf{r_1}-\mathbf{r_2})\delta_{\alpha(-\beta)}\delta_{\alpha\lambda}\delta_{\lambda(-\kappa)}. \tag{18}$$

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[1] At finite temperature, the average is taken with respect to the appropriate density matrix

[2] Strictly speaking the Hamiltonian of Eq. (14) conserves particle number and the true groundstate must have a well defined particle number $N$. However, relaxing this condition of definite particle number allows us to use the mean field method results in a groundstate which is close to the true groundstate in Fock space.

[3] The mean field method can still be applied in unconventional superconductors where the interaction potential is more complicated. However, since this work is focused on s-wave superconductors, I will only focus on the case where the interactions are point-like for simplicity.

In this case we can introduce the order parameter $\Delta(\mathbf{r})$ as well as its conjugate $\Delta^+(\mathbf{r})$,

$$\Delta(\mathbf{r}) = -V_0 \sum_{\lambda} \int d^3\mathbf{r} F_{\lambda(-\lambda)}(\mathbf{r},\mathbf{r}),$$
$$\Delta^+(\mathbf{r}) = -V_0 \sum_{\lambda} \int d^3\mathbf{r} F^+_{\lambda(-\lambda)}(\mathbf{r},\mathbf{r}), \tag{19}$$

and express the BCS Hamiltonian in the following form

$$\hat{H}_{BCS}(\mathbf{r}) = \sum_{\alpha} \int d^3\mathbf{r} \Big[ \hat{\psi}^+_{\alpha}(\mathbf{r}) \Big( \frac{1}{2m} \Big( -i\boldsymbol{\nabla}_{\mathbf{r}} - \frac{e}{c}\mathbf{A} \Big)^2 + e\phi_0(\mathbf{r}) - \epsilon_F \Big) \hat{\psi}_{\alpha}(\mathbf{r}) + \Delta(\mathbf{r}) \psi^+_{\alpha}(\mathbf{r}) \psi^+_{-\alpha}(\mathbf{r}) + \Delta^+(\mathbf{r}) \psi^+_{\alpha}(\mathbf{r}) \psi^+_{-\alpha}(\mathbf{r}) \Big]. \tag{20}$$

Since $\Delta(\mathbf{r})$ and $\Delta^+(\mathbf{r})$ are complex numbers rather than operators, the mean field Hamiltonian is quadratic in creation/annihilation operators and can be diagonalized by changing the basis. To do this, we introduce the quasiparticle operators $\hat{\gamma}^+_{n\alpha}, \hat{\gamma}_{n\beta}$, which are defined in terms of the electron operators by the following relations,

$$\hat{\psi}_{\alpha}(\mathbf{r}) = \sum_n \big( u_n(\mathbf{r}) \hat{\gamma}_{n\alpha} - v^*_n(\mathbf{r}) \hat{\gamma}^+_{n(-\alpha)} \big)$$
$$\hat{\psi}^+_{\alpha}(\mathbf{r}) = \sum_n \big( u^*_n(\mathbf{r}) \hat{\gamma}^+_{n\alpha} - v_n(\mathbf{r}) \hat{\gamma}_{n(-\alpha)} \big). \tag{21}$$

The Bogoliubov amplitudes $u_{n\beta}(\mathbf{r}), v_{n\beta}(\mathbf{r})$ satisfy the normalization condition

$$\int d^3\mathbf{r} \big( |u_n(\mathbf{r})|^2 + |v_n(\mathbf{r})|^2 \big) = 1, \tag{22}$$

and are chosen such that the quasiparticle operators $\hat{\gamma}^+_{n\alpha}, \hat{\gamma}_{n\beta}$ have Fermionic commutation relations,

$$\{ \gamma^+_{\mathbf{k}\alpha} \gamma_{\mathbf{k}'\beta} \} = \delta_{\alpha\beta} \delta_{\mathbf{k}\mathbf{k}'}$$
$$\{ \gamma_{\mathbf{k}\alpha} \gamma_{\mathbf{k}'\beta} \} = \{ \gamma^+_{\mathbf{k}\alpha} \gamma^+_{\mathbf{k}'\beta} \} = 0 \tag{23}$$

and diagonalize the BCS Hamiltonian,

$$\hat{H}_{BCS} = \epsilon_0 + \sum_{n\alpha} \epsilon_n \hat{\gamma}^+_{n\alpha} \hat{\gamma}_{n\alpha}. \tag{24}$$

Here $\epsilon_0$ is the groundstate energy and $\epsilon_n$ is the spectrum of the quasiparticle excitations. Using Eqs. (20)-(24), we arrive at the Bogoliubov-deGennes (BdG) equations

$$\begin{pmatrix} \frac{1}{2m} \left( -i\boldsymbol{\nabla}_{\mathbf{r}} - \frac{e}{c}\mathbf{A} \right)^2 + e\phi_0(\mathbf{r}) - \epsilon_F & |\Delta(\mathbf{r})| e^{i\chi(\mathbf{r})} \\ |\Delta(\mathbf{r})| e^{-i\chi(\mathbf{r})} & \frac{-1}{2m} \left( -i\boldsymbol{\nabla}_{\mathbf{r}} - \frac{e}{c}\mathbf{A} \right)^2 + e\phi_0(\mathbf{r}) - \epsilon_F \end{pmatrix} \begin{pmatrix} u_n(\mathbf{r}) \\ v_n(\mathbf{r}) \end{pmatrix} = \epsilon_n \begin{pmatrix} u_n(\mathbf{r}) \\ v_n(\mathbf{r}) \end{pmatrix}. \tag{25}$$

Here $|\Delta|$ and $\chi$ are the modulus and phase of the order parameter,

$$\Delta(\mathbf{r}) = |\Delta(\mathbf{r})|e^{i\chi(\mathbf{r})}. \tag{26}$$

Since this procedure is based on a mean field approach, Eq. (25) should be supplemented with a self consistency condition. Using Eqs. (16), (20), and (21), we get

$$\Delta(\mathbf{r}) = V_0 \sum_n v_n^*(\mathbf{r}) u_n(\mathbf{r}). \tag{27}$$

Eqs. (25) and (27) form a closed set of equations which can be solved to determine the energy spectrum $\epsilon_n$.

In the case of a uniform superconductor, the quasiparticle excitations are momentum eigenstates and the spectrum is given by,

$$\epsilon(\mathbf{p}, \mathbf{p_s}) = \sqrt{\xi^2(\mathbf{p}) + |\Delta|^2} + \frac{\mathbf{p_s} \cdot \mathbf{p}}{m}. \tag{28}$$

Here $\xi(\mathbf{p}) = \frac{p^2}{2m} - \epsilon_F$ and $\mathbf{p_s}$ is the superfluid momentum defined by the expression,

$$\mathbf{p_s}(\mathbf{r}) = \frac{1}{2}\left(\boldsymbol{\nabla}\chi(\mathbf{r}) - \frac{2e}{c}\mathbf{A}(\mathbf{r})\right). \tag{29}$$

Since the phase of the order parameter $\chi$ depends on the choice of gauge, the spectrum will in general depend on $\chi$ only through the superfluid momentum $\mathbf{p_s}$ [4].

### C. Clean and Dirty superconductors

In the presence of disorder, the BdG Hamiltonian contains a random impurity potential $U_{imp}(\mathbf{r})$ and the solutions of the BdG equations are altered. With this in mind, it is useful to make a distinction between clean and dirty superconductors.

In clean superconductors, the impurity potential sufficiently weak such that the momentum of the eigenstates remain well defined and the spectrum can be calculated using BdG equations without including $U_{imp}(\mathbf{r})$. In such systems it is possible to construct quasiparticle wavepackets out of the solutions of Eq. (25) such that the width of the packet in position space $\Delta r$ that is much larger than the mean free path $l = v_F\tau_{el}$, and a width in momentum space $\Delta p$ which is smaller than a characteristic momentum of the system $\delta p_c$. Since the width of the wavepackets

[4] It follows Eqs. (25) - (27) that that under a gauge transformation $\mathbf{A} \to \mathbf{A} + \frac{c}{e}\boldsymbol{\nabla} f$, the phase transforms as $\chi \to \chi + 2f$

are related by the uncertainty relation $\Delta p \Delta r \sim 1$, such wavepackets can be constructed only when

$$l\delta p_c \gg 1. \tag{30}$$

The characteristic momentum depends on the details of the system as well as the energy interval of quasiparticles one is interested in, and as a result there is some ambiguity in what is meant by "clean" in the general case. For the case of a uniform superconductor, where the quasparticles of interest are typically in an energy interval of order $|\Delta|$ and not too close to the gap edge $|\Delta| - v_F p_s$, we can use Eq. (28) to get $\delta p_c \sim \frac{|\Delta|}{v_F}$. Thus a uniform superconductor is said to be clean when $|\Delta|\tau_{el} \gg 1$.

In the opposite limit of dirty superconductors, where $\delta p_c |\Delta| \ll 1$, the quasiparticle momentum is no longer a good quantum number and the exact energy levels are highly sensitive to the configuration of the impurity potential. While the BdG equations can in principle be used to calculate $\epsilon_n$ for any particular impurity potential $U_{imp}(\mathbf{r})$, such calculations are often impractical. In such cases, it is more useful to calculate the density of states $\nu(\epsilon)$ averaged over all possible configurations of the impurity potential, which can be obtained by solving Usadel's equations[5] .

### D. Kinetics of quasiparticles in the presence of spectral flow

The kinetics of normal metals is well described by the Boltzmann kinetic equation, which is valid as long as the frequency and gradients of the perturbations are small compared to the characteristic energy and momentum of the quasiparticles. However in superconductors the dynamics of the superfluid condensate and the quasiparticles become coupled, which introduces additional energy and momentum scales associated with the condensate. As a result, the conditions necessary for a Boltzmann-equation type description become more stringent and is satisfied only in clean superconductors [2]. Because of this, the kinetics of superconductors is much more complicated compared to normal metals.

To account for these complications, the kinetic theory of superconductors can be rigorously formulated using a method involving Green's functions. Although the general equations obtained from such an approach are quite complicated, in some cases they can be reduced to a relatively simple form. In particular, we will take advantage of the fact that in most superconductors there is a hierarchy of time scales where the elastic relaxation time of quasiparticles

$\tau_{el}$ is much smaller than the inelastic relaxation time $\tau_{in}$. We will focus on the regime where frequency of the external fields $\omega$ is small compared the modulus of the order parameter $|\Delta|$ as well the elastic relaxation rate of quasiparticles $\tau_{el}^{-1}$.

The former condition that $\omega \ll |\Delta|$ allows us to treat $|\Delta(\mathbf{r})|$ as a constant. This means that the dynamics of the condensate manifests only in the time dependence of $\mathbf{p_s}(\mathbf{r}, t)$, which is controlled by the Josephson relation,

$$\dot{\mathbf{p}}_\mathbf{s}(\mathbf{r}, t) = \boldsymbol{\nabla}\Phi(\mathbf{r}, t) + e\mathbf{E}(\mathbf{r}, t). \tag{31}$$

Here $\mathbf{E} = -\boldsymbol{\nabla}\phi_0 - \partial_t\mathbf{A}$ is the electric field and $\Phi$ is the gauge invariant potential defined by the expression,

$$\Phi(\mathbf{r}, t) = \frac{1}{2}\frac{d\chi}{dt} + e\phi_0(\mathbf{r}, t). \tag{32}$$

The gauge invariant potential is related to asymmetric populations of electrons and holes generated by spatial variations of external fields.

The latter condition $\omega \ll \tau_{el}^{-1}$ implies that quasiparticles are able to fully relax within each energy manifold, allowing us to describe the population of quasiparticles using a distribution function $n(\epsilon, \mathbf{r}, t)$ which depends on energy.

To obtain a relatively simple equation for the distribution function $n(\epsilon, \mathbf{r}, t)$, we will make the additional assumption that there is a local relationship between the density of states $\nu(\epsilon, t)$ and the superfluid momentum $\mathbf{p_s}(\mathbf{r}, t)$,

$$\nu(\epsilon, t) = \int d^3\mathbf{r}\tilde{\nu}\left(\epsilon, \mathbf{p}_\mathbf{s}(\mathbf{r}, t)\right). \tag{33}$$

Although the specific conditions necessary for this local approximation depend on the details of the system, they are valid when the external fields vary slowly compared to the characteristic time and length scales of the system. Within this regime, the kinetic equation which controls the time evolution of $n(\epsilon, \mathbf{r}, t)$ is given by,

$$\partial_t n(\epsilon, \mathbf{r}, t) + \partial_\epsilon n(\epsilon, \mathbf{r}, t)\dot{\mathbf{p}}_\mathbf{s}(\mathbf{r}, t) \cdot \tilde{\mathbf{V}}_\nu(\epsilon, \mathbf{r}) - \boldsymbol{\nabla}_j\left(D^*_{ij}(\epsilon, \mathbf{r})\boldsymbol{\nabla}_i n(\epsilon, \mathbf{r}, t)\right) = I_{in}\{n(\epsilon, \mathbf{r}, t)\}. \tag{34}$$

Here $D^*_{ij}$ is the generalized diffusion coefficient, $I_{in}$ is the scattering integral, which we will treat in the relaxation time approximation

$$I_{in}\{n(\epsilon, \mathbf{r}, t)\} = \frac{n_F(\epsilon) - n(\epsilon, t)}{\tau_{in}}, \tag{35}$$

and $\tilde{\mathbf{V}}_\nu(\epsilon, \mathbf{p_s})$ is the "level sensitivity" defined in terms of the local density of states $\tilde{\nu}(\epsilon, \mathbf{r}, t)$.

$$\tilde{\mathbf{V}}_\nu(\epsilon, \mathbf{p_s}) = \frac{-1}{\tilde{\nu}(\epsilon, \mathbf{p_s})} \int_0^\epsilon d\tilde{\epsilon} \frac{d\tilde{\nu}(\epsilon, \mathbf{p_s}(t))}{d\mathbf{p_s}}. \tag{36}$$

While the form of the kinetic equation is fairly general, the parameters $D^*_{ij}$ and $V_\nu$ depend on the details of the system.

The second term on the LHS of Eq. (34) is a source term which is related to the motion of energy levels that is induced by a time dependent superfluid momentum $\mathbf{p_s}$, and the third term is describes describes diffusion of quasi-particles. Since the distribution can relax via both diffusion as well as inelastic processes, the entropy production rate $\dot{S}$ associated with the distribution function $n(\epsilon, \mathbf{r}, t)$ is given by the formula

$$\dot{S} = \frac{-1}{T} \int d\epsilon (\partial_\epsilon n_F(\epsilon))^{-1} \int d\mathbf{r} \tilde{\nu}(\epsilon, \mathbf{p_s}(\mathbf{r})) \left( \frac{\delta n^2(\epsilon, \mathbf{r}, t)}{\tau_{in}} - D^*_{ij}(\epsilon, \mathbf{p_s}(\mathbf{r}, t)) \boldsymbol{\nabla}_i n(\epsilon, \mathbf{r}, t) \boldsymbol{\nabla}_k n(\epsilon, \mathbf{r}, t) \right), \tag{37}$$

where $T$ is the temperature and the brackets $\langle ... \rangle$ denote averaging over time.

While the parameters $D^*_{ij}$, $V_\nu$, and $\nu$ depend on the details of the system, the form of the kinetic equation is fairly general and can be applied to a variety of superconducting systems. Equation (34) as been used to study dissipation in bulk superconductors [6–8], SNS junctions [9, 10], and magnetic vortex lattices [11, 12]. In the case of clean superconductors, Eq. (34) can be easily derived from the Boltzmann equation (See Ref. [8]). In the diffusive limit, the kinetic equation was derived in Ref. [9] using a method based on Green's functions. This derivation will be presented in Sec. VII.

*Phenomenological derivation of kinetic equation in the uniform case*

A phenomenological derivation of Eq. (34) can be obtained by first considering the uniform case where $\mathbf{p_s}(t)$ and $n(\epsilon, t)$ are constant in space. At sufficiently small frequencies where the system is in the adiabatic regime, we can introduce an instantaneous energy spectrum which depends on the value of $\mathbf{p_s}(t)$ . Note that the total number of levels in the system must be conserved, and therefore the density of states associated with the instantaneous spectrum $\nu(\epsilon, \mathbf{p_s}(t))$ must obey a continuity equation,

$$\partial_t \nu(\epsilon, \mathbf{p_s}(t)) + \partial_\epsilon \left[ \nu(\epsilon, \mathbf{p_s}(t)) v_\nu(\epsilon, \mathbf{p_s}(t)) \right] = 0, \tag{38}$$

where $v_\nu(\epsilon, \mathbf{p_s}(t))$ is the "level velocity" in energy space. Using Eq. (31), the level velocity can be expressed in the form

$$v_\nu(\epsilon, \mathbf{p_s}(t)) = \dot{\mathbf{p}}_\mathbf{s}(\mathbf{r}, t) \cdot \left( \frac{-1}{\nu(\epsilon, t)} \int_0^\epsilon d\tilde{\epsilon} \frac{d\nu(\tilde{\epsilon}, \mathbf{p_s})}{d\mathbf{p_s}} \right) \tag{39}$$

In the absence of inelastic scattering, the total number of quasi-particles is also conserved, which gives us a second continuity equation

$$\partial_t \left[ n(\epsilon, t) \nu(\epsilon, \mathbf{p_s}(t)) \right] + \partial_\epsilon \left[ n(\epsilon, t) \nu(\epsilon, \mathbf{p_s}(t)) v_\nu(\epsilon, \mathbf{p_s}(t)) \right] = 0, \tag{40}$$

Substituting Eq. (38) into Eq. (40), using the expression for the level velocity (39), and adding the $I_{in}$ to the RHS to account for scattering, we get

$$\partial_t n(\epsilon, t) + \partial_\epsilon n(\epsilon, t) \dot{\mathbf{p}}_\mathbf{s}(\mathbf{r}, t) \cdot \left( \frac{-1}{\nu(\epsilon, t)} \int_0^\epsilon d\tilde{\epsilon} \frac{d\nu(\tilde{\epsilon}, \mathbf{p_s})}{d\mathbf{p_s}} \right) = I_{in}\{n(\epsilon, t)\}. \tag{41}$$

By restoring the position dependence in $\mathbf{p_s}(\mathbf{r}, t)$ and adding a diffusion term, we arrive at Eq. (34).

With this in mind, the source term in Eq. (34) should be interpreted as the influence of the motion of energy levels on the quasiparticle population. This source term is absent in the Boltzmann equation for normal metals and is capable of generating a non-equilibrium quasi-particle population in energy space. In some cases, the non-equilibrium distribution that is generated can relax only via inelastic processes, resulting in a dissipation rate which is controlled by the large inelastic relaxation time $\tau_{in}$. As a result, there is a mechanism of dissipation which exists only in superconductors, and can be much larger than mechanisms of dissipation in the normal state which are controlled by the relatively smaller elastic relaxation time $\tau_{el}$.

## III. MICROWAVE ABSORPTION OF DISORDERED MAGNETIC VORTEX LATTICES

Films of type-II superconductors placed in a magnetic field normal to the plane, which is weaker than the upper critical field $H_{c2}$, host Abrikosov vortices whose density is set by the condition that the average flux of the magnetic field per vortex is equal to the flux quantum $\Phi_0 = \frac{\pi\hbar c}{e}$ [13]. In the absence of disorder, the vortices are arranged in a periodic manner and a magnetic vortex lattice is formed. The dissipation in such systems has been studied extensively since the work of Bardeen and Stephen [14–20]. Most articles on the topic of microwave

absorption in these vortex lattices are based on the Bardeen-Stephen theory, where the mechanism of dissipation is due to the motion of vortex cores and the microwave conductivity is proportional to the elastic relaxation time $\tau_{el}$.

In this chapter I describe a new mechanism of microwave absorption in disordered vortex lattices in the presence of an *ac*-electric filed $\mathbf{E}(t) = \mathbf{E}\cos(\omega t)$, which exists even the absence of vortex motion and is larger than the conventional result in a broad range of physical parameters. I will begin by summarizing the conventional result based on the Bardeen-Stephen picture, followed by a qualitative description of a new mechanism of dissipation which is not captured by the Bardeen-Stephen theory. Next I will use the methods developed in the previous chapter to calculate the microwave absorption coefficient $\sigma(\omega)$ and compare it with the convention result.

### A. Bardeen-Stephen theory

In the Bardeen-Stephen theory [14], dissipation in the system arises as a result of the friction forced caused by the motion of the vortex cores. The friction force $\boldsymbol{F} = -\eta_{BS}\boldsymbol{v}_v$ can be expressed in terms of the vortex velocity $\boldsymbol{v}_v$ and the Bardeen-Stephen vortex viscosity $\eta_{BS} = \Phi_0 H_{c2}\sigma_n/c^2$. Equating the rate of viscous energy dissipation to Joule heat, $\frac{H}{\Phi_0}\eta_{BS}\boldsymbol{v}_v^2 = \sigma_{BS}E^2$, one obtains

$$\sigma_{BS} \approx \frac{H\eta_{BS}\mathbf{v}_v^2}{\Phi_0 E^2}. \tag{42}$$

In the presence of disorder the vortex lattice is pinned, and is capable of supporting a dissipationless *dc*-current density which is smaller than the critical current $J_c$. In the flux flow regime, where $J \gg J_c$ and the pinning forces can be neglected, the friction force must be equal to the Magnus force induced by the transport current. In this case the vortex velocity is given by,

$$\boldsymbol{v}_v \approx c\left(\frac{\mathbf{E}\times\mathbf{H}}{H^2}\right), \tag{43}$$

and the conductivity is given by the conventional Bardeen-Stephen result,

$$\sigma_{BS} \sim \sigma_n \frac{H_{c2}}{H}. \tag{44}$$

It should be noted that Eq. (44) is relevant only in the non-linear regime. On the other hand, *ac*-electric fields $\mathbf{E}(t) = \mathbf{E}_0\cos(\omega t)$ induce dissipation in superconductors even in the linear

regime. In most articles on microwave absorption, the dissipative conductivity $\sigma'_{BS}(\omega)$ in the linear regime is evaluated phenomenologically using the Bardeen-Stephen expression for the vortex viscosity, $\eta_{BS}$, and the vortex velocity $\mathbf{v}'_v(t)$ which is modified by the pinning forces (see, for example, [19, 21, 22]). The modified vortex velocity is $\mathbf{v}'_v(t)$ can be expressed in the form,

$$\boldsymbol{v}'_v(t) \approx c\left(\frac{\boldsymbol{E}(t)\times\boldsymbol{H}}{H^2}\right)\left(1-\frac{\lambda_L^2}{\lambda_{eff}^2(H)}\right). \tag{45}$$

Here $\lambda_L$ is the London length, and $\lambda_{eff}(H)$ is the Campbell [23] length. The latter depends on the pinning strength and characterizes the macroscopic superfluid density of the system. Assuming that the pinning strength is determined by the vortex cores and is independent of the magnetic field, the ratio of the Campbell and London length may be expressed in the form [23, 24]

$$\frac{\lambda_{eff}^2(H)}{\lambda_L^2} = 1+\frac{\Phi_0 H d}{8\pi\lambda_L^2 k}. \tag{46}$$

Here $k$ is the average "spring constant", which relates the average pinning force on the vortex $\boldsymbol{F}=-k\delta\boldsymbol{r}$, to the average vortex displacement $\delta\boldsymbol{r}$.

Based on the Bardeen-Stephen picture, the microwave conductivity $\sigma_{BS}(\omega)$ given by,

$$\sigma_{BS}(\omega) \approx \sigma_n\frac{H_{c2}}{H}\left(1-\frac{\lambda_L^2}{\lambda_{eff}^2(H)}\right)^2. \tag{47}$$

In the limit of strong pinning, $k\to\infty$, we have $\lambda_{eff}^2(H)/\lambda_L^2\to 1$, and the microwave absorption vanishes.

### B. Mechanisms of dissipation in a rigid vortex lattice

Let us consider the limit of a perfectly rigid vortex lattice, where the Bardeen-Stephen contribution to the microwave conductivity vanishes $\sigma_{BS}(\omega)=0$. Although the vortex cores do not move, the microwave electric field will still accelerate the condensate and deform the vortex plaquettes. This deformation is felt by quasiparticles which have energies below the threshold of percolation between different vortex plaquettes, which are trapped inside the vortex plaquettes. The percolation threshold $\epsilon^*$ arises because in the presence of a superfluid momentum $\mathbf{p}_s^{(0)}(\boldsymbol{r})$ the energy gap of the quasiparticle states is shifted down from $\Delta$ by $\delta\epsilon(\mathbf{p}_s^{(0)}(\boldsymbol{r}))$, as illustrated in Fig. 1. This deformation will modify the quasi-particle energy

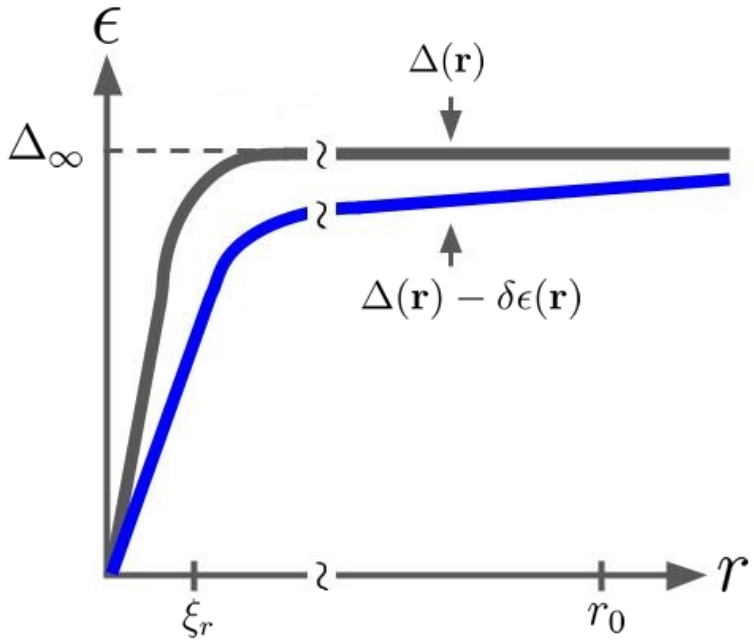



Figure 1: a) The black line denotes the dependence of the modulus of the order parameter $\Delta(r)$ on the distance from the vortex core $r$, while the blue line denotes the $r$-dependence of the spectral edge $\Delta(r) - \epsilon_{min}(r)$.

levels within a vortex plaquette and generate a non-equilibrium distribution, which then relaxes and generate dissipation.

This effect can be described in terms of the time dependence of local density of state $\tilde{\nu}\left(\epsilon, \mathbf{p_s}(\mathbf{r}, t)\right)$. Since the density of states $\nu(\epsilon)$ is a scalar, and the acceleration of the condensate is proportional to the electric field $\dot{\boldsymbol{p}}_s \propto \boldsymbol{E}$, its time derivative can have a linear dependence on $\dot{\boldsymbol{p}}_s$ only in the presence of a *dc*-superfluid momentum $\mathbf{p_s^{(0)}}(\mathbf{r})$. As a result, the level velocity which is induced must be proportional to the *dc*-superfluid momentum, $v_\nu \sim \mathbf{p}_s^{(0)}(\mathbf{r})$. Due to the non-uniformity of $\mathbf{p_s}^{(0)}(\mathbf{r})$ within a plaquette of the vortex lattice, the non-equilibrium distribution generated by spectral flow of energy levels is non-uniform and can relax via both inelastic processes as well as diffusion. As a result, the dissipation rate has 2 contributions corresponding to the 2 channels of relaxation, The relative importance of these two channels depends on frequency of the microwave field $\omega$.

Because of pinning the vortex lattice in distorted, and the spatial distribution of the superfluid momentum $\mathbf{p}_s^{(0)}(\mathbf{r})$ is not symmetric. Therefore, the non-equilibrium part of the of the distribution function cannot relax completely by diffusion across the plaquettes, and ultimately its relaxation is achieved by inelastic scattering processes. Since the inelastic relaxation

time, $\tau_{in}$, is typically much larger than the diffusion time across a vortex plaquette, $\tau_D$, the microwave absorption at small frequencies is controlled by $\tau_{in}$. Thus, the low frequency conductivity $\sigma(\omega) \sim K\tau_{in}$ is proportional to the inelastic relaxation time and a parameter $K$ characterizing the degree of lattice distortion. As the frequency increases to the regime where $\omega \gg \tau_{in}^{-1}$, the period of oscillations becomes small compared to $\tau_{in}$ and the dissipation rate via inelastic processes is suppressed. At higher frequencies, the dissipation is dominated by diffusive of quasiparticles and the conductivity is controlled by, $\tau_D$, the characteristic diffusion time across a vortex plaquette.

In the limit of a rigid lattice both of of these contributions dominate the Bardeen-Stephen conductivity $\sigma_{BS}$. However, because both $\tau_D$ and $\tau_{in}$ are typically much larger than $\tau_{el}$, the contribution to the conductivity from these effects will dominate the conventional Bardeen-Stephen conductivity in a broad range of parameters.

### C. Calculating the microwave conductivity

To illustrate the effects described in the previous section, I will calculate the microwave conductivity of a diffusive vortex lattice using the methods described in Sec. II. The microwave absorption coefficient is related to the entropy production rate by the relation,

$$T\langle \dot{S} \rangle = \frac{\sigma(\omega)}{2} \mathbf{E}_0^2. \tag{48}$$

Using Eq. (37), the entropy production rate can be expressed in terms of the distribution function $n(\epsilon, \mathbf{r}, t)$, which can be obtained by solving the kinetic equation (34). To calculate $\sigma(\omega)$, I will need expressions for the parameters $D^*(\epsilon, \mathbf{r})$, $\nu(\epsilon, \mathbf{r})$, and $\tilde{V}_\nu(\epsilon, \mathbf{p_s})$, which can be obtained from Usadel's equation. In the local approximation [5], the solutions to Usadel's equations are essentially the same as in those calculated in Ref. [7], for the case of uniform $\mathbf{p_s}^{(0)}$. In the energy interval $\Delta - \delta\epsilon(\mathbf{p}_s^{(0)}(\boldsymbol{r})) < \epsilon < \Delta$, which gives the dominant contribution to the entropy production, the expressions for the relevant parameters are given by

$$\frac{\nu(\epsilon, \mathbf{p_s})}{\nu_n} \approx \eta^{-1/3}(\mathbf{p_s}), \tag{49}$$

$$\mathbf{V}_\nu(\epsilon, \mathbf{p_s}) \approx D p_s \eta^{-1/3}(\mathbf{p_s}), \tag{50}$$

$$D^*_{ij}(\mathbf{p_s}) \approx D\eta^{1/3}(\mathbf{p_s})\delta_{ij}, \tag{51}$$

[5] This is applicable to the vortex lattice everywhere which is not too close to the vortex cores. As we shall see, the dominant contribution comes from regions which are far from the vortex cores, so this approximation is not problematic.

where we have defined

$$\eta(\mathbf{p_s}) = \frac{(v_F p_s)^2 \tau_{\rm el}}{\Delta}. \tag{52}$$

In addition to these parameters, an expression for the condensate acceleration $\dot{\mathbf{p}}_\mathbf{s}(\mathbf{r}, t)$ is required. This will be the focus of the next section.

*Spatial distribution of the condensate acceleration*

At low frequencies in the London regime, where the modulus of the order parameter $|\Delta|$ outside the vortex cores is approximately uniform, the quasiparticle density of states depends on the local instantaneous condensate momentum $\boldsymbol{p}_s(\boldsymbol{r}, t)$, and its rate of change is proportional to the local condensate acceleration, $\dot{\boldsymbol{p}}_s(\boldsymbol{r}, t)$. In thin films, the Pearl length [**?** ], which characterizes the screening of the magnetic field, practically always exceeds the inter-vortex distance, $\sim l_H$. In this regime we can neglect the small in-homogeneity of the magnetic field $\boldsymbol{H}(\boldsymbol{r})$ caused by the diamagnetic currents.

In the presence of a microwave field $\boldsymbol{E}(t)$ the vortex positions $\boldsymbol{r}_j(t)$ become time-dependent, and the condensate acceleration is given by,

$$\dot{\boldsymbol{p}}_s(\boldsymbol{r}, t) = \hbar\hat{z} \times \sum_j \frac{2(\boldsymbol{r} - \boldsymbol{r}_j(t))(\boldsymbol{r} - \boldsymbol{r}_j(t)) \cdot \dot{\boldsymbol{r}}_j(t) - |\boldsymbol{r} - \boldsymbol{r}_j(t)|^2 \dot{\boldsymbol{r}}_j(t)}{|\boldsymbol{r} - \boldsymbol{r}_j(t)|^4} + e\boldsymbol{E}(t). \tag{53}$$

In a perfectly rigid lattice, the first term vanishes and the condensate acceleration is given by the second term, $e\boldsymbol{E}(t)$. The first term describes the modification caused by the motion of the vortices.

Let us consider the spatial distribution of $\dot{\boldsymbol{p}}_s(\boldsymbol{r}, t)$ inside the plaquette of a given vortex $j$, that is at $|\boldsymbol{r} - \boldsymbol{r}_j| \lesssim l_H$. The term $j$ in the sum in Eq. (53), which corresponds to the motion of the native core, $\dot{\boldsymbol{r}}_j$, produces a contribution $\dot{\boldsymbol{p}}_s^{(1)}(\boldsymbol{r}) \sim \hbar\dot{\boldsymbol{r}}_j/|\boldsymbol{r} - \boldsymbol{r}_j(t)|^2$, which decays rapidly with the distance from the core. The remaining contribution, $\dot{\boldsymbol{p}}_s^{(2)}(\boldsymbol{r})$, which is produced by the motion of the other vortices together with the second term, does not fall off with the distance and may be approximated by a constant for $|\boldsymbol{r} - \boldsymbol{r}_j(t)| \lesssim l_H$.

By order of magnitude, $\dot{\boldsymbol{p}}_s^{(2)}$ coincides with the spatial average of the condensate acceleration in the system, $\langle\dot{\boldsymbol{p}}_s(t)\rangle$. In the linear approximation and at sufficiently low frequencies, where the viscous forces are negligible in comparison to pinning, $\langle\dot{\boldsymbol{p}}_s(t)\rangle$ is related to the electric field

by the Campbell formula,

$$\dot{\boldsymbol{p}}_s^{(2)} \sim \langle \dot{\boldsymbol{p}}_s(t) \rangle = \frac{\lambda_L^2}{\lambda_{eff}^2(H)} e\boldsymbol{E}(t). \tag{54}$$

Similarly, the typical vortex velocity $\dot{\boldsymbol{r}}_j$, which determines the magnitude of $\dot{\boldsymbol{p}}_s^{(1)}(\boldsymbol{r})$, is on the order of the vortex velocity $\boldsymbol{v}_v'$.

In the next section, we evaluate the dissipation arising from the two contributions to the condensate acceleration. The dissipation caused by $\dot{\boldsymbol{p}}_s^{(1)}$ is dominated by quasiparticles at short distances from the core, and corresponds to the Bardeen-Stephen contribution to the conductivity. The dissipation caused by $\dot{\boldsymbol{p}}_s^{(2)}(\boldsymbol{r})$ is dominated by quasiparticles at distances on the order of $l_H$ from the vortex core. We will show that this term produces the dominant contribution in a wide interval of physical parameters.

*Inelastic relaxation contribution*

Let us first consider the contribution to the conductivity due to inelastic processes. At frequencies $\omega \ll \tau_D^{-1}$, the non-equilibrium part of the distribution function $\delta n$ can be treated as spatially uniform. Introducing $\langle ... \rangle_k$ as the average over the kth plaquette, the kinetic equation is given

$$\partial_t \langle \delta n \rangle_k(\epsilon, t) + \langle \dot{\mathbf{p}}_s \cdot \boldsymbol{V}(\epsilon, \boldsymbol{p}_\mathrm{s}(\mathbf{r})) \rangle_k \partial_\epsilon n_F(\epsilon) = \frac{-\delta \langle n \rangle_k}{\tau_{in}} \tag{55}$$

Note that in the case of perfectly symmetric lattice the second term in Eq. (55) vanishes. This is indicative of the fact that in a perfectly symmetric lattice the distribution function can relax entirely via diffusion. However in the presence of disorder, the superfluid momentum around vortices is asymmetric and this term is nonzero. Solving Eq. (55) with respect to $\delta \langle n \rangle_k$, substituting the result into Eq. (37) followed by Eq. (48), and finally averaging over plaquettes, at $T \sim> \Delta$ we get

$$\frac{\sigma}{\sigma_\mathrm{n}} \approx \frac{\tau_\mathrm{in}}{\tau_\mathrm{el}} \frac{\lambda_L^4}{\lambda_{eff}^4(H)} \frac{1}{\left[1 + (\omega \tau_\mathrm{in})^2\right]} \int_0^{\epsilon^*} d\epsilon \frac{\langle \langle \nu(\epsilon, \mathbf{p}_\mathrm{s}(\mathbf{r})) \rangle_k \langle \mathbf{n} \cdot \mathbf{V}(\epsilon, \mathbf{r}) \rangle_k^2 \rangle \rangle}{T \nu_n v_\mathrm{F}^2} \tag{56}$$

Here $\sigma_n = e^2 \tilde{\nu}_N$, $e$ is the electron charge, $D = \frac{v_F^2 \tau_{el}}{2}$ is the electron diffusion coefficient in normal metal, $v_F$ is the Fermi velocity in the normal metal, $\tilde{\nu}_N$ is the normal metal density of states at the Fermi surface, $\mathbf{n}$ is a unit vector in the direction of $\mathbf{E}$, and outer brackets $\langle \langle ... \rangle \rangle$ indicate averaging over plaquettes.

Substituting the expressions in Eq. (49) into Eq. (56), we get

$$\frac{\sigma}{\sigma_{\rm n}} \approx K \frac{\lambda_L^4}{\lambda_{eff}^4(H)} \frac{\tau_{in}\Delta}{\left[1+(\omega\tau_{\rm in})^2\right]} \left(\frac{H}{H_{c2}}\right)^{2/3}, \tag{57}$$

where the parameter,

$$K(\mathbf{H}) = \frac{\left\langle \left\langle \mathbf{n}\cdot\mathbf{V}(\epsilon,\mathbf{r})\right\rangle_k^2\right\rangle_{(\epsilon\sim\Delta-\delta\epsilon)}}{\frac{D_n}{l_H\eta^{1/3}}}, \tag{58}$$

characterizes the degree of non-symmetricity of the vortex. The value of $K(\mathbf{H})$ is not universal and depends on the details of the pinning potential and the asymmetric part of the spatial distribution of the superfluid momentum inside packets of the vortex lattice $\mathbf{p}_s^{(0)}(\mathbf{r})$. If both the relative amplitude fluctuation of $\mathbf{V}(\mathbf{r})$, and their correlation radius are of order one, then $K \approx 1$. Since $\tau_{in}$ is typically, much larger than $\tau_D$ and $\tau_{in}$, Eq. (57) gives the main contribution to $\sigma$ in the low frequency regime.

*Diffusion Contribution*

At $\omega\tau_{in} \gg 1$ the discussed above contribution decreases with the frequency, and eventually the microwave conductivity is controlled by spatial diffusion on distances of order, or smaller than $l_H$. In this limit an neglect the effects of lattice distortion, and the kinetic equation can be rewritten in the axially symmetric approximation where $p_s^{(0)} = \hbar/r$. We can also neglect the term in Eq. (34) responsible for the inelastic relaxation. In polar coordinates the kinetic equation is given by,

$$\begin{aligned} -i\omega\delta n(\epsilon,\theta,r) - \frac{D\left(p_s^{(0)}(r)\right)}{r^2}\partial_\theta^2\delta n(\epsilon,\theta,r) - \frac{1}{r}\partial_r\left[rD\left(p_s^{(0)}(r)\right)\partial_r\delta n(\epsilon,\theta,r)\right] \\ = \langle\dot{p}_s(t)\rangle v_F\partial_\epsilon n_F(\epsilon)\left(\frac{r}{v_F\tau_{el}\Delta}\right)^{-1/3}\cos\theta, \end{aligned} \tag{59}$$

where $\delta n(\epsilon,r,\theta,t) = \delta n(\epsilon,\theta,r)e^{i\omega t}$. Substituting the long-distance part of the condensate acceleration given by Eq (54) into the equation, the solution is given by

$$\begin{aligned} \delta n(\epsilon,\theta,r)\Big|_{r\gg\xi} =& \langle\dot{p}_s(t)\rangle v_F\partial_\epsilon n_F(\epsilon) \\ &\times\left(\frac{L_\omega}{v_F\tau_{el}\Delta}\right)^{-1/3}\cos\theta\begin{cases}\frac{1}{D_n}\left(\frac{\xi}{L_\omega}\right)^{2/3}\left(\frac{r}{L_\epsilon}\right)^{-7/3}, & r/L_\omega \ll 1,\\ \frac{i}{\omega}\left(\frac{r}{L_\omega}\right)^{-1/3}, & r/L_\omega \gg 1.\end{cases} \end{aligned} \tag{60}$$

Here $L_\omega$ is defined by $\omega = D\left(p_s^{(0)}(L_\omega)\right)/L_\omega^2$.

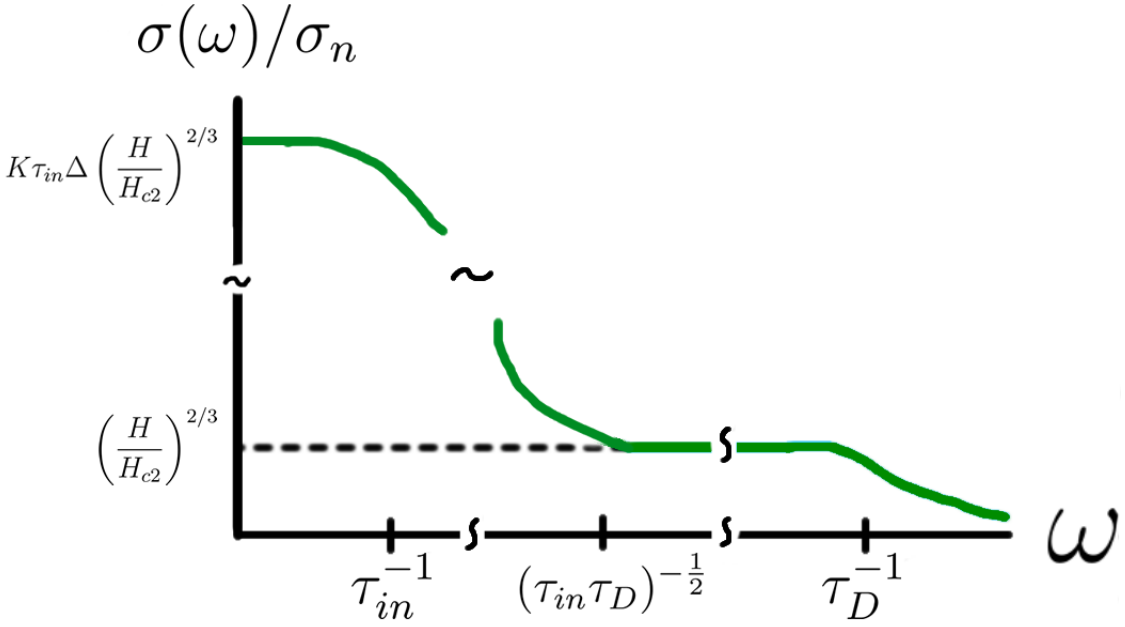



Figure 2: Qualitative sketch of the frequency dependence of $\sigma(\omega)$.

Substituting Eq. (60) into Eq. (37) and recalling that $l_H^2 \frac{\Delta}{D_n} \sim H_{c2}/H$ we get

$$\frac{\sigma}{\sigma_n} \approx \left(\frac{\lambda_L^4}{\lambda_{eff}^4(H)}\right)\left(\frac{H_{c2}}{H}\right)^{2/3} \begin{cases} 1, & \omega \ll \tau_D^{-1}, \\ (\omega\tau_D)^{-5/4}, & \omega \gg \tau_D^{-1}. \end{cases} \tag{61}$$

In the diffusive regime, the diffusion time is given by $\tau_D^{-1} = D\left(\frac{v_F^2\tau_{el}}{\Delta}\right)^{1/3} l_H^{-8/3}$.

Comparing Eq. (57) and Eq. (61) we conclude that , for $\omega < \tau_D^{-1}$, the latter gives the main contribution to the conductivity. if $\omega > \omega^* = \sqrt{\frac{K\Delta}{\tau_{in}}}(H/H_{c2})^{2/3}$. We present qualitative picture of the frequency dependence of the conductivity $\sigma(\omega)$ in Fig. 2. Finally, we note that replacing $\langle \dot{\boldsymbol{p}}_s(\boldsymbol{r})\rangle$ in Eq. (59) by $\dot{\boldsymbol{p}}_s^{(1)}(\boldsymbol{r})$, gives a contribution to dissipation, which is dominated by short distances from the core, and reproduces the Bardeen-Stephen result $\sigma_{BS}$.

*Comparison with Bardeen-Stephen result*

Finally, lets us compare our results for $\sigma$ at $\omega^* < \omega < 1/\tau_D$, with the Bardeen-Stephen conductivity in the presence of pinning $\sigma_{BS}$. Thus, using Eqs. (47) and (61) we get

$$\frac{\sigma}{\sigma'_{BS}} \sim \left(\frac{\lambda_L^2}{\lambda_{eff}^2 - \lambda_L^2}\right)^2 \left(\frac{\xi_s}{r_0}\right)^{2/3}. \tag{62}$$

Consider for example a strong pinning case where the condensation energy of the vortex core changes by a factor of the order unity when the vortex is displaced by a distance $\xi_s$ from

the equilibrium position, then $k(0) \sim d\Delta^2\nu_n$. Recalling that $\lambda_L^2 = mc^2/4\pi n_s e^2$, where $n_s$ is the superfluid density, for $\omega^* < \omega < \tau_D^{-1}$, we get

$$\frac{\sigma}{\sigma'_{BS}} \sim \left(\frac{L_H^2}{\xi_s^2}\right)^{5/3} > 1. \tag{63}$$

In this case the high frequency conductivity is given by Eq. (61) in arbitrary magnetic field smaller than $H_{c2}$. In the case where pinning is weak, Eq. (61) describes the conductivity only in sufficiently small magnetic field.

## IV. I-U CHARACTERISTICS OF SNS JUNCTIONS

The theory of current-voltage (I-U) characteristics of superconducting weak links at relatively large voltages has been developed in many articles (see for example [25–28], and references therein). However, at small voltages the I-U characteristics exhibit interesting features which are quite different from those at large voltages, and has attracted much less attention. The origin of this small voltage regime can be understood as follows.

Due to Andreev reflection at the normal metal-superconductor boundaries, low energy quasiparticles ($\epsilon < |\Delta|$) are trapped in the normal region of the SNS junction. Because quasiparticles pick up the phase of the superconductor as they reflect off the boundaries, the spectrum of these bound states depend on the phase difference across the junction $\phi$ . When a voltage is applied across the junction, the phase changes in time via the Josephson relation,

$$\dot{\phi}(t) = 2eU(t). \tag{64}$$

As a result, the energy levels change in time and a "spectral flow" is induced in energy space. At small voltages quasiparticles move adiabatically with their energy levels, generating a non-equilibrium distribution of quasiparticles which can only relax via inelastic processes. As a result, the dissipation rate generated by this mechanism is proportional to the inelastic relaxation time $\tau_{in}$, which in most cases is much larger than the elastic relaxation time $\tau_{el}$ which controls the dissipation in the large voltage regime.

In this chapter we use the methods developed in the previous section to develop a theory of I-U characteristics in this low voltage regime. I will begin by presenting the general equations which describe the kinetics of SNS junctions in the adiabatic regime. Next I will use these equations to derive general characteristics features in the I-U characteristics of voltage biased

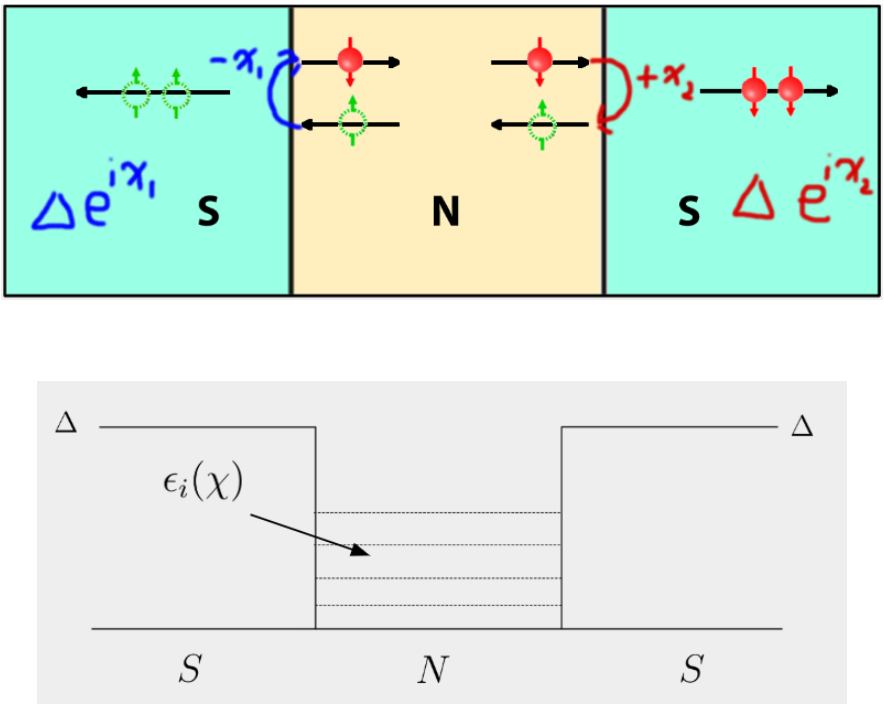



Figure 3: A schematic picture of a) Andreev reflection in an SNS junction b) bound state energy levels in an SNS junction

and current biased SNS junctions. At the end of the chapter we will apply these results to the case of ballistic single channel junctions and diffusive multi-channel junctions .

### A. Kinetics of SNS junctions

While Eq. (34) can be used to describe SNS junctions, we will primarily be interested in the regime where the voltage across the junction $U$ is small compared to the inverse time of quasiparticle propagation across the junction. In this approximation, the distribution function as spatially uniform within the normal region and the diffusion term in the kinetic equation can be dropped. Furthermore, the density of states in the junction $\nu(\epsilon, \phi)$ depends only on the total phase difference across the junction $\phi$, which evolves in time via the Josephson relation Eq. (64). As a result, Eq. (34) can be reduced to the form,

$$\partial_t n(\epsilon, t) + \partial_\epsilon n(\epsilon, t)\,(eU(t))\,V_\nu(\epsilon, \phi) = I_{in}\{n(\epsilon, t)\}, \tag{65}$$

where $V_\nu(\epsilon,t)$ is defined in terms of the density of states,

$$V_\nu\left(\epsilon,\phi\right)=\frac{-1}{\nu(\epsilon,\phi)}\int_0^\epsilon d\tilde{\epsilon}\frac{d\tilde{\nu}(\epsilon,\phi)}{d\phi}. \tag{66}$$

The current through the junction can be expressed in terms of $n(\epsilon,t)$ using the formula,

$$J(t)=J_s\left(\phi(t),T\right)+2e\int_0^\infty d\epsilon\left(n(\epsilon,t)-n_F(\epsilon)\right)\nu\left(\epsilon,\phi(t)\right)V_\nu\left(\epsilon,\phi(t)\right). \tag{67}$$

Here $J_s\left(\phi(t),T\right)$ is the supercurrent, which is given by

$$J_s\left(\phi(t),0\right)=2e\int_0^\infty \nu\left(\epsilon,\phi(t)\right)V_\nu\left(\epsilon,\phi(t)\right)\left(1+n_F(\epsilon)\right) \tag{68}$$

In Sec. VII we show that Eq. (65) can be obtained by integrating Eq. (34) over the junction, and derive Eq. (67) using identities involving Green's functions.

*Lagrangian and Eularian coordinates*

In some cases the exact energy levels $\epsilon_i(t)$ in the junction can be calculated and it is more convenient to write Eq. (65) in terms of the occupation number of a particular level $n_i(t)$ rather than the population of quasiparticles at a particular energy $n(\epsilon,t)$. This is analogous to changing from Euler to Lagrange coordinates in hydrodynamics, where the former describes the fluid at a particular fixed position while the latter follows an infinitesimal fluid elemental as it moves. To do change to "Lagrange" coordinates, we use the identities

$$\nu(\epsilon,t)=\sum_i\delta\left(\epsilon-\epsilon_i(t)\right), \tag{69}$$

$$n(\epsilon,t)=\sum_i n_i(t)\delta\left(\epsilon-\epsilon_i(t)\right). \tag{70}$$

Substituting these expressions into Eqs. (65)-(68) gives,

$$\partial_t n_i(t)=\frac{n_F\left[\epsilon_i(t)\right]-n_i}{\tau_{in}} \tag{71}$$

$$J(t)=J_s\left(\phi(t),T\right)+2e\sum_i\frac{\partial\epsilon_i\left(\phi(t)\right)}{\partial\phi}\left(n_i-n_F(\epsilon_i)\right) \tag{72}$$

$$J_s\left(\phi(t),T\right)=2e\sum_i\frac{d\epsilon_i(\phi(t))}{d\phi}(1+n_F(\epsilon_i)) \tag{73}$$

In the following sections, we will choose to work in Eularian coordinates unless otherwise stated.

### B. Voltage biased junctions

To calculate the I-V characteristics at fixed voltage we need to calculate the average current $\langle J(U)\rangle$, which can be obtained by substituting the solution to Eq. (65) into Eq. (67). To solve the kinetic equation, it is convenient to introduce the following change in variables $(\epsilon, t) \to (N, t)$, where $N$ is given by,

$$N\left(\epsilon, t\right) = \int_0^{\epsilon} d\tilde{\epsilon} \nu\left(\tilde{\epsilon}, \phi(t)\right). \tag{74}$$

Here $\epsilon(N, \phi)$ should be interpreted as the energy of the Nth energy level counting from $\epsilon = 0$. The advantage of working in these variables is that Eq. (65) simplifies to the form,

$$\partial_t n\left(N, t\right) = \frac{n_F\left[\epsilon\left(N, \phi(t)\right)\right] - n\left(N, t\right)}{\tau_{in}}. \tag{75}$$

At time scales larger than $\tau_{in}$, solutions to (75) are insensitive to initial conditions and converge to the following expression,

$$n\left(N, t\right) = \int_0^{\infty} \frac{d\tau}{\tau_{in}} e^{-\frac{\tau}{\tau_{in}}} n_F\left[\epsilon\left(N, \phi(t-\tau)\right)\right]. \tag{76}$$

Here we note that if the voltage across the junction is fixed, the phase difference $\phi$ simply winds at a constant rate, which means that

$$\phi(t-\tau) = \phi(t) - 2eU\tau. \tag{77}$$

Changing the integration over time $\tau$ to an integration over phase $\gamma = 2eU\tau$, we get

$$n(N, t) = \frac{1}{2e|U|\tau_{in}} \int_0^{\infty} d\gamma e^{-\gamma/(2e|U|\tau_{in})} n_F\left[\epsilon\big(N, \phi(t) - \mathrm{sgn}(U)\gamma\big)\right], \tag{78}$$

where $\mathrm{sgn}(U) = \frac{U}{|U|}$. Substituting this into the expression for the current (67) and averaging over time, we get

$$\langle J(U)\rangle = \frac{e}{2e|U|\pi} \int_0^{\infty} dN \int_{-\pi}^{\pi} \partial_\phi \epsilon(N, \phi) d\gamma \int_0^{\infty} d\gamma e^{-\gamma/(2e|U|\tau_{in})} n_F\left[\epsilon\big(N, \phi - \mathrm{sgn}(U)\gamma\big)\right]. \tag{79}$$

Here we have used the fact that averaging over $t$ is equivalent to averaging over $\phi$, as well as the identity

$$V_\nu(N, \phi) = -\partial_\phi \epsilon(N, \phi) \tag{80}$$

When the temperature is small compared to the gap in the electrodes $T \ll |\Delta|$ and but still large compared to the characteristic energy scales of the junction, the expression for the

current can be expanded in powers of $1/T$. To lowest order in $1/T$, the average current is given by

$$\langle J(U)\rangle = \frac{e}{2e|U|\pi}\int_0^\infty dN\int_{-\pi}^{\pi} d\phi\partial_\phi\epsilon(N,\phi)\partial_\epsilon n_F\left[\epsilon(N,\phi)\right] \times\int_0^\infty d\gamma e^{-\gamma/(2e|U|\tau_{in})}\Big(\epsilon\left(N,\phi-\mathrm{sgn}(U)\gamma\right)-\epsilon(N,\phi)\Big) \tag{81}$$

*Small voltages*

At small voltages $e|U|\ll\tau_{in}^{-1}$, we can expand the expression for the current up to linear order in $eU\tau_{in}$. Note that the dominant contribution to the integral in second line of Eq. (81) is within the width of the exponential where $\gamma\lesssim e|U|\tau_{in}\ll 1$, allowing us to expand the in the following way

$$\begin{aligned}\int_0^\infty d\gamma e^{-\gamma/(2e|U|\tau_{in})}\Big(\epsilon\left(N,\phi-\mathrm{sgn}(U)\gamma\right)-\epsilon(N,\phi)\Big) &= -\mathrm{sgn}(U)\int_0^\infty d\gamma e^{-\gamma/(2e|U|\tau_{in})}\partial_\phi\epsilon(N,\phi)\gamma\\ &= \mathrm{sgn}(U)(2eU)^2\partial_\phi\epsilon(N,\phi)\end{aligned} \tag{82}$$

Substituting this into Eq. (81) and performing the integral over $\gamma$, we get

$$\langle J(U)\rangle = \frac{e}{2e|U|\pi}\int_0^\infty dN\int_{-\pi}^{\pi} d\phi\partial_\epsilon n_F\left[\epsilon(N,\phi)\right]\Big(\partial_\phi\epsilon(N,\phi)\epsilon(N,\phi)+2\mathrm{sgn}(U)(2eU)^2\left(\partial_\phi\epsilon(N,\phi)\right)^2\Big) \tag{83}$$

Changing variables back to $(\epsilon,t)$, we arrive at the following expression for the average current

$$\langle J(U)\rangle = \langle g_1(\phi)\rangle_\phi(2eU\tau_{in}), \tag{84}$$

where $\langle ...\rangle_\phi$ denotes averaging over $\phi$ and $g_1(\phi)$ is given by

$$g_1(\phi) = -2e\int_0^\infty d\epsilon\partial_\epsilon n_F(\epsilon)\nu(\epsilon,\phi)V_\nu^2(\epsilon,\phi). \tag{85}$$

In Lagrange coordinates, the expression for $g_1(\chi)$ is given by,

$$g_1(\phi) = -2e\sum_i\partial_\epsilon n_F\left[\epsilon_1(\chi)\right]\left(\partial_\phi\epsilon_i(\phi)\right)^2 \tag{86}$$

The linear conductance of the junction is thus given by $G = 2e\tau_{in}\langle g_1(\phi)\rangle_\phi$, and is proportional to the inelastic relaxation time.

*Large voltages*

In the case of large voltage $eU \gg \tau_{in}^{-1}$, the width of the exponential in Eq. (81) is large compared to the period of oscillations of $\epsilon(N, \phi - \mathrm{sgn}(U)\gamma)$. As a result, we expect the average current to be small. To estimate the average current in this limit, we can expand $\epsilon(N, \phi)$ in a Fourier series

$$\epsilon(N, \phi) = \langle \epsilon(N) \rangle_\phi + \sum_k C_k(N) e^{ikx}. \tag{87}$$

For typical phase dependence of the quasiparticle spectrum, the Fourier sums are dominated by $k$ of order unity. Substituting this into Eq. (81) gives,

$$\begin{aligned} \langle J(U) \rangle &= -2e(2eU\tau_{in}) \int_0^\infty dN \partial_\epsilon n_F \left[ \langle \epsilon(N) \rangle_\phi \right] \sum_{k \neq 0} \frac{k^2}{1 + (2keU\tau_{in})^2} |C_k(N)|^2 \\ &\approx \left( \frac{1}{1 + (eU\tau_{in})^2} \right) 2e(2eU\tau_{in}) \int_0^\infty dN \partial_\epsilon n_F \left[ \langle \epsilon(N) \rangle_\phi \right] \sum_{k \neq 0} (-k^2) |C_k(N)|^2. \end{aligned} \tag{88}$$

Comparing this to the expression for the current in the small voltage limit, we get the following estimate

$$\langle J(U) \rangle \approx \frac{\langle g_1(\phi) \rangle_\phi}{eU\tau_{in}}. \tag{89}$$

Thus when the voltage is large compared to $\tau_{in}^{-1}$, the average current decreases with increasing voltage.

*General features in the voltage viased I-U characteristics*

Based on Eqs. (84) and (89), we see that the I-U curve is non-monotonic. At small voltages the current increases with the voltage up until a threshold $U_{max} \sim (e\tau_{in})^{-1}$ where the current reaches a maximum current $J_{max} \approx \langle g_1(\phi) \rangle_\phi$. Beyond this maximum, the current decreases with increasing voltage $\langle J \rangle \sim 1/U$.

It should be noted that the expressions for the current in Eqs. (84) and (89) is calculated in the approximation where the distribution function is uniform and isotropic. As a result the conventional contribution to the dissipative current $J_N \approx G_N U$, where $G_N$ is the conductance of the normal region of the junction, is neglected. At small voltages $eU \ll \tau_{in}^{-1}$ where $\langle J(U) \rangle \sim \tau_{in}$, the $J_N \sim \tau_{el}$ gives only a small contribution to the linear conductance and can be dropped. However, at large voltages where the contribution to the current controlled by $\tau_{in}$ decreases

with $U$, the $J_N$ eventually becomes the dominant contribution to the current. With this in mind, the current decreases with the voltage until a threshold $U_{min}$, where the current reaches a minimum. We can estimate $U_{min}$ by demanding that $G_N U_{min}$ and $GU/(2eU\tau_{in})^2$ are of the same order, which gives

$$U_{min} \approx \frac{1}{\tau_{in}} \sqrt{\frac{G}{G_N}}. \tag{90}$$

For voltage $U \gg U_{min}$ the current increases monotonically with the voltage and the I-U characteristics approaches the curves described by the conventional theory.

Thus the I-U curves at fixed voltage have an N type shape and their features are qualitatively summarized by Fig. 4.

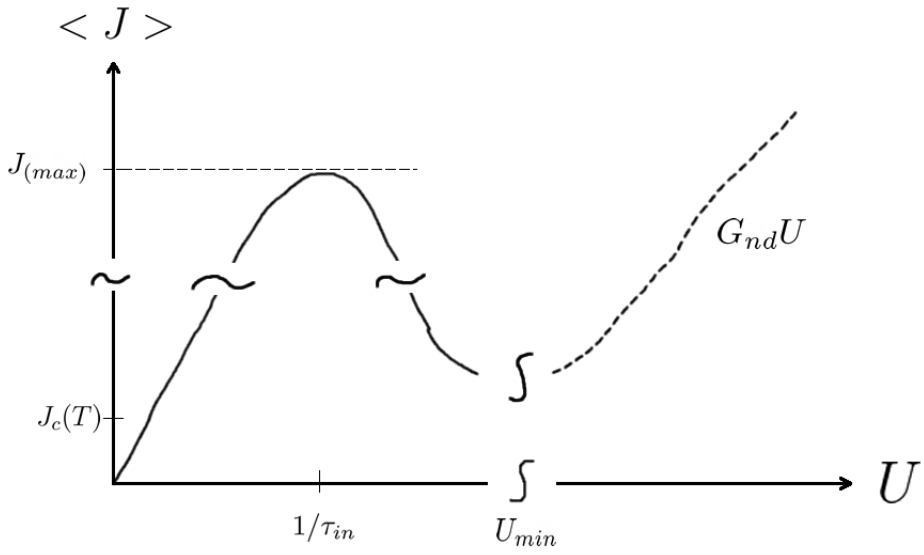



Figure 4: A schematic picture of an I-U characteristics of a voltage-biased SNS junction.

### C. Current Biased

In a current biased setup, the SNS junction undergoes a transition into a resistive state when the bias current $J$ exceeds the critical current $J_c(T)$. In this case the phase different $\phi(t)$ increase monotonically, while the voltage $U(t)$ changes periodically in time, as illustrated in Fig. 5. In the following we will be interested in the dependence of the voltage averaged over the period of oscillations, $\langle U(t) \rangle$, on the bias current $J$.

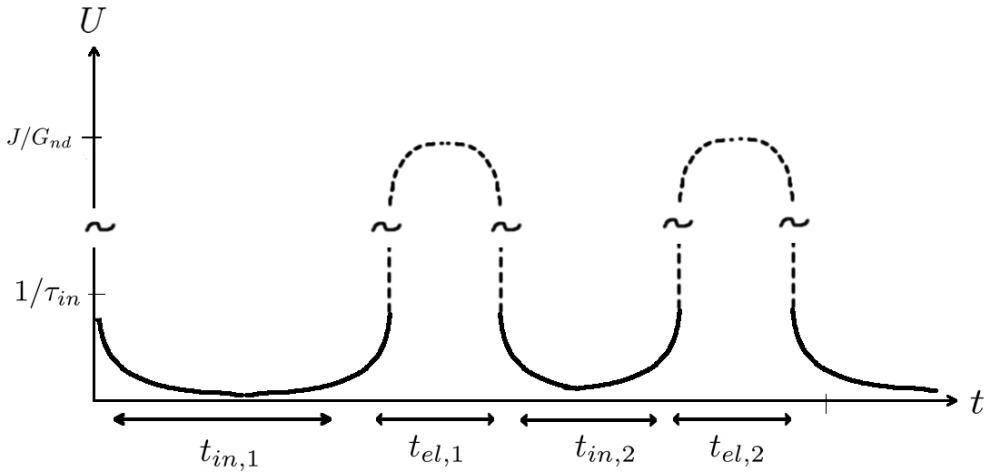



Figure 5: Time dependence of voltage at a current-based SNS junction when $J > J_c(T)$.

When the voltage is small compared to $\tau_{in}^{-1}$ the expression for the instananeous current is given by,

$$J(t) = J_c(T)Y(\chi(t), T) + g_1(\phi)\dot{\phi}\tau_{in}, \tag{91}$$

where $J_s(\chi(t), T) = J_c(T)Y(\chi(t), T)$. It is important to note however that the level sensitivity $V_\nu(\epsilon, \phi)$ vanishes at the time-reversal invariant points $\phi = \pi n$, where $n$ is an integer. As a result, according to Eqs. (68) and (85), both $J_s$ and $g_1$ vanishes, and in some intervals near these points the bias current must be carried by the normal contribution $J_N$. Thus, the phase and time periods of the oscillations can be separated into two "elastic" and two "inelastic" intervals,

$$t_p = (t_{in,1} + t_{in,2} + t_{el,1} + t_{el,2}) \qquad 2\pi = \phi_{in,1} + \phi_{in,2} + \phi_{el,1} + \phi_{el,2} \tag{92}$$

in which the bias current is dominated by the contributions which are proportional to $\tau_{el}$ and $\tau_{in}$ respectively.

The boundaries of the elastic intervals $\chi_{el}$ can be determined from the condition that, at $\dot{\phi} \sim \tau_{in}^{-1}$ the bias current can be carried by the maximal inelastic contribution, $J = g_1(\phi)$. In the vicinity of the time-reversal invariant points, $\phi = \pi n + \delta\phi$, we have

$$g_1(\phi) \approx \frac{\delta\chi^2}{2} \left.\frac{d^2 g_1(\phi)}{d\phi^2}\right|_{\phi=\pi n} \approx J_{max}\frac{\delta\phi^2}{2}. \tag{93}$$

As a result, we get the following estimate for the width of the elastic phase intervals

$$\phi_{el,1}, \phi_{el,2} \sim \sqrt{\frac{J}{J_{max}}} \tag{94}$$

Inside the inelastic intervals the phase winds at a rate of order of $\dot{\chi}_{in} = (J/G)$ , whereas inside the elastic interval it winds at a rate $\dot{\chi}_{el} = eJ/G_N$. Therefore we can estimate,

$$\frac{t_{el}}{t_{in}} \sim \frac{G_N}{G}\sqrt{\frac{J}{J_{max}}} \sim \frac{\tau_{el}}{\tau_{in}}\sqrt{\frac{J}{J_{max}}} \ll 1. \tag{95}$$

In the regime where $J \ll \langle g_1(\phi)\rangle_\phi$, the elastic intervals are small compared to the inelastic intervals. Thus, using the Josephson relation (64), the average voltage can be expressed in terms of the duration of the diagonal time intervals only,

$$\langle U\rangle = \frac{\pi}{et_p} \approx \frac{\pi}{e\left(t_{d,1}+t_{d,2}\right)}. \tag{96}$$

If the instantaneous phase is not to close to the time-reversal invariant points, $\phi = n\pi$, the time evolution of $\phi(t)$ is described by Eq. (91) Using this relation, the duration of the inelastic time intervals may be expressed as

$$t_{d,i} = \tau_{in}\int_{\phi_{in,i}} \frac{g_1(\phi)d\phi}{J - J_c(T)Y(\phi,T)}, \tag{97}$$

where $i = 1, 2$, and the integration is taken over the phase interval $\chi_{d,i}$

At small excess current, $J - J_c(T) \ll J_c(T)$, we can expand $Y(\phi, T)$ near its maximum at $\phi = \phi_m$, while at $J \gg J_c(T)$ we can neglect the second term in the denominator in Eq. (97). Then, using Eq. (96) we get

$$\langle U(J)\rangle = (2\tau_{in})^{-1}\begin{cases} \frac{\sqrt{2J_c(T)(J-J_c(T))}}{g_1(\phi_m)} & J - J_c(T) \ll J_c(T) \\ \sim \frac{J}{2\langle g_1(\phi)\rangle_\phi} & J_c(T) \ll J \ll J_{max}. \end{cases} \tag{98}$$

When the bias current, $J$ reaches a threshold $J_{jump} \sim J_{max}$, the widths of the phase intervals $\chi_{in,i}$ shrinks to zero, and the voltage-current dependence $\langle U(J)\rangle$ jumps to the branch dominated by $J_N$. The qualitative features of the I-U characteristics at fixed current is summarized by Fig. 6

### D. Application to clean and diffusive junctions

In this section we apply the results from the previous sections to the case ballistic junctions formed by a 1D wire and diffusive planar junctions (See Fig. 7). We will focus on the case where the length of the normal metal region $L$ is much larger than the superconducting coherence

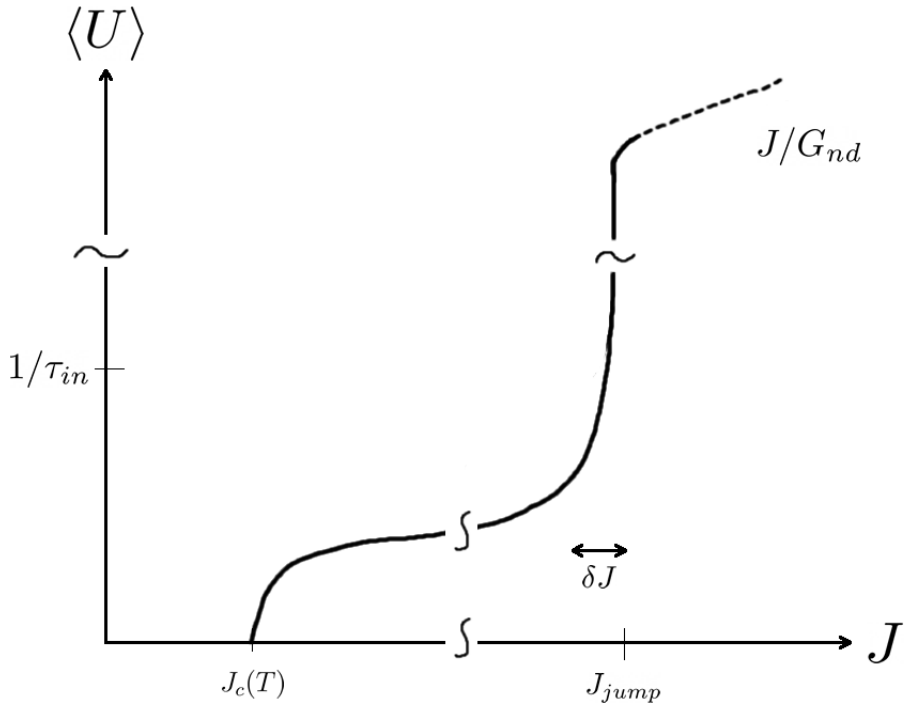



Figure 6: Schematic picture of the I-U characteristic of the current-based SNS junction.

length in the electrodes $\xi = \frac{v_F}{|\Delta|}$. In these cases the superconducting proximity effect can be neglected and the order parameter has the form

$$\Delta(\mathbf{r}) = \begin{cases} |\Delta| e^{i\chi_2} & x < -\frac{L}{2} \\ 0 & -\frac{L}{2} < x < \frac{L}{2} \\ |\Delta| e^{i\chi_1} & x > \frac{L}{2}, \end{cases} \tag{99}$$

where the phase difference across the junction is given by $\phi = \chi_2 - \chi_1$. To obtain an explicit expression for the parameters which characterize the I-U characteristics, we need to calculate $g_1(\phi)$, which is expressed in terms of the spectrum of the junction.

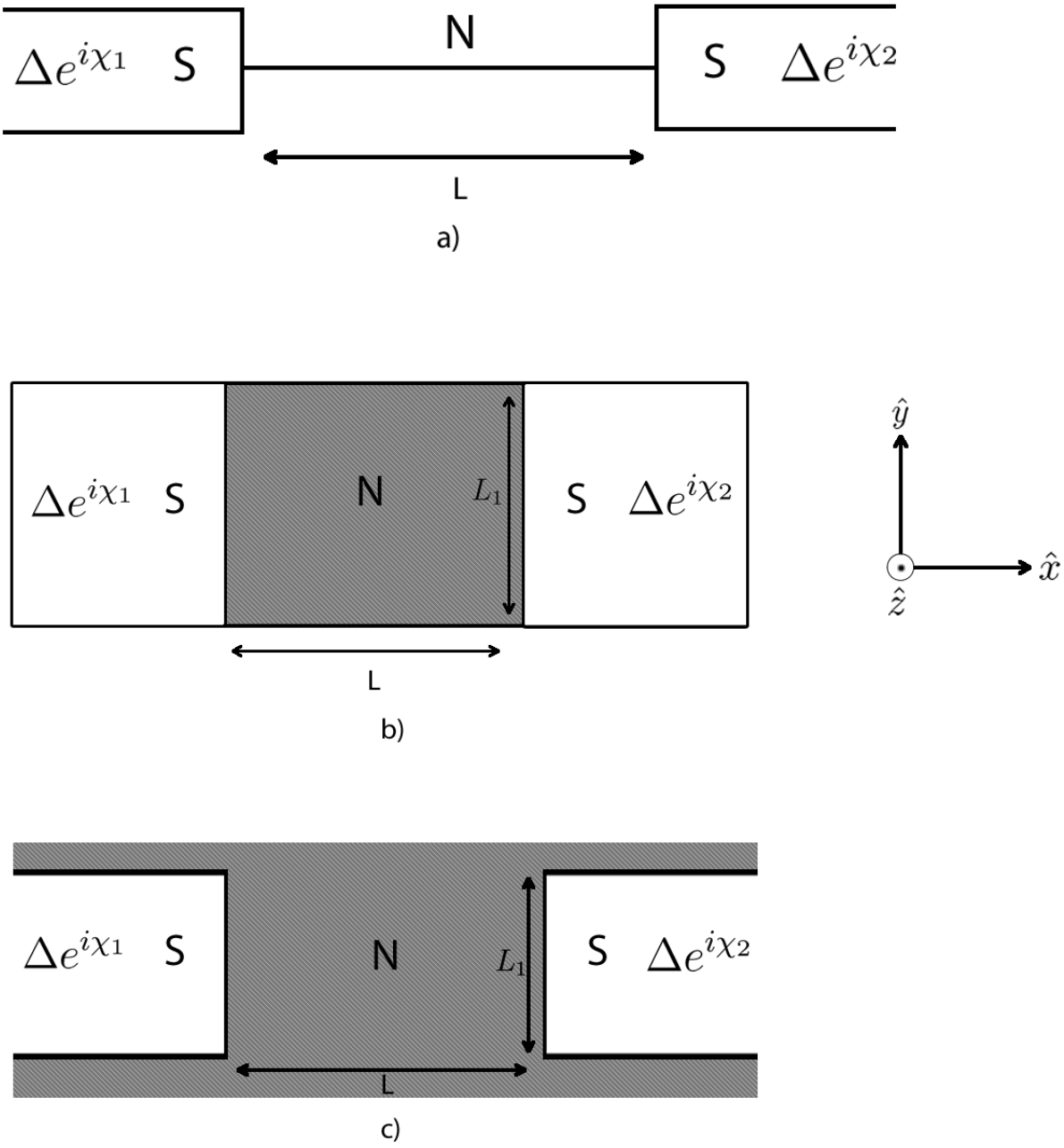



Figure 7: Qualitative representation of a) a ballistic 1D SNS junction where $l \gg L$ b) a diffusive SNS junction where $l \ll L$, with closed boundaries c) a diffusive junction with open boundaries

*Ballistic junctions*

The spectrum in ballistic be calculated using the BdG equations (25), which in this case is given by

$$\begin{pmatrix} \frac{\hat{p}^2}{2m} - \mu & \Delta(x) \\ \Delta*(x) & -\frac{\hat{p}^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} \psi_e(x) \\ \psi_h(x) \end{pmatrix} = \epsilon_n \begin{pmatrix} \psi_e(x) \\ \psi_h(x) \end{pmatrix}. \tag{100}$$

For energies $\epsilon < |\Delta|$, the general solution is given by

$$\begin{bmatrix} \psi_e \\ \psi_h \end{bmatrix}(x) = \begin{cases} Ae^{k_s^-(x+\frac{L}{2})} \begin{bmatrix} 1 \\ \gamma \end{bmatrix} + Be^{-k_s^+(x+\frac{L}{2})} \begin{bmatrix} 1 \\ \gamma^* \end{bmatrix} & x < -\frac{L}{2} \\ Ce^{ik^+x} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + De^{-ik^+} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + Ee^{ik^-} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + Fe^{-ik^-} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & -\frac{L}{2} < x < \frac{L}{2}, \\ Ge^{-k_s^-(x-\frac{L}{2})} \begin{bmatrix} 1 \\ \gamma e^{i\chi} \end{bmatrix} + He^{k_s^+(x-\frac{L}{2})} \begin{bmatrix} 1 \\ \gamma^* e^{i\chi} \end{bmatrix} & x > \frac{L}{2} \end{cases} \tag{101}$$

where we have introduced

$$\zeta = \sqrt{|\Delta|^2 - \epsilon^2}$$
$$\gamma = \frac{\epsilon + i\zeta}{|\Delta|}$$
$$k^{\pm} = k_F \pm \frac{\epsilon}{v_F}, k_s^{\pm} = k_F \pm \frac{\zeta}{v_F}.$$

The choice of coefficients depend on the boundary conditions at the SN interfaces. For the case of perfect transmission at the boundaries, the wavefunctions should be continuous at the SN interfaces. This yields the well known spectrum of Andreev boundstates derived by many decades earlier [29]

$$\epsilon_n^{\pm}(\phi) = \frac{v_F}{L}\left(\frac{\pi}{2} + n\pi \pm \frac{\phi}{2}\right). \tag{102}$$

The spectrum has two branches, corresponding quasiparticles carrying current towards the $+x$ and $-x$ directions. In the opposite limit where the transmission coefficient at the boundaries is small $r \ll 1$, the spectrum is given by

$$\epsilon_n^{\pm}(\phi) = \frac{n\pi v_F}{L} \pm 2\frac{v_F}{L} r\sqrt{2(r^2 + \cos\phi)} \tag{103}$$

The characteristic energy scale is set by the typical spacing between adjacent energy levels $\frac{v_F}{L}$, so we will focus on the case where $\frac{v_F}{L} \ll T \ll |\Delta|$. Substituting Eqs. (102) and (103) into

Eq. (86) we get the following estimates for the linear conductance,

$$G = \frac{e^2}{\pi}\frac{v_F}{L}\tau_{in}A(r), \quad T \gg v_F/L, \tag{104}$$

where we have defined

$$A(r) = \begin{cases} 1 & r = 1, \\ 8r^2 & r \ll 1. \end{cases} \tag{105}$$

We note that the conductance of a pure single channel SNS junction, Eq. (104), exceeds the normal state conductance $Ae^2/\hbar$, by a large factor $\frac{v_F}{L}\tau_{in} \gg 1$. Substituting Eq. (104) into the expressions for $J_{jump}$ and $J_{max}$, we get

$$J_{jump} \sim J_{max} \sim \frac{ev_F}{L}A(r). \tag{106}$$

The maximal current turns out to be temperature independent. The reason for this is that at low energies, $\epsilon_n \ll \Delta$, the sensitivity of the levels to a change in $\chi$ is independent of the energy.

It is instructive to compare value of $J_{max}$ and $J_{jump}$ with the critical current $J_c(T)$, which can be obtained by substituting Eqs. (102) and (102) into Eq. (68).

$$J_c(T) = B(r)\frac{ev_F}{2L}\begin{cases} 1 & , \quad T \ll \frac{v_F}{L}, \\ \exp(-\frac{2\pi TL}{v_F}) & , \quad T \gg \frac{v_F}{L}. \end{cases} \tag{107}$$

where the dimensionless coefficient $B(r)$ has the following limiting values at high and low contact transparencies,

$$B(r) = \begin{cases} 1 & r = 1, \\ r^2/2\pi & r \ll 1. \end{cases} \tag{108}$$

Comparing Eqs. (106) and (107) we arrive to a somewhat surprising conclusion that at high temperatures, $T \gg v_F/L$, the values of $J_{max}$ and $J_{jump}$ are of order of the critical current at zero temperature,

$$J_{max} \sim J_{jump} \sim J_c(0) \gg J_c(T). \tag{109}$$

*Diffusive planar junctions*

The spectrum of diffusive junctions can be obtained by solving Usadel's equations with appropriate boundary conditions. In the normal region where $\Delta = 0$, Eqs. (185)-(187) is

given by

$$\frac{D}{2}\left(\partial_x^2\theta - \frac{1}{2}\sin(2\theta)\left(\partial_x\tilde{\chi}\right)^2\right) = -\,\epsilon\sin\theta, \tag{110}$$

$$\partial_x\left(\sin^2\theta\partial_x\tilde{\chi}\right) = 0, \tag{111}$$

$$\nu(\epsilon,\mathbf{r}) = \tilde{\nu}_N\int d^3\mathbf{r}\cos\theta_1\cosh\theta_2 \tag{112}$$

In the case of perfectly transmitting interfaces $r = 1$, the boundary conditions are given by (see Ref. [30])

$$\theta(\epsilon, x = \pm L/2) = \frac{\pi}{2} \qquad\qquad \tilde{\chi}(\epsilon, x = \pm L/2) = \pm\frac{\chi}{2}. \tag{113}$$

Solutions of Eqs. (110),(111) were investigated in several articles (see for example, Refs. [31, 32]). The density of states in the normal region of SNS junctions differs from that in the normal metal only at small energies of the order of mini-gap $E_g \sim E_T = D/L^2$. For our purposes we need only rough features of $\epsilon$ and $\chi$ dependencies of the density of states,

$$\nu(\epsilon,\chi) = 2LL_1\tilde{\nu}_N\begin{cases} 0 & ,\epsilon < E_g(\chi), \\ h(\epsilon,\chi) & ,\epsilon - E_g(\chi) \sim E_g(\chi), \\ 1 & ,\epsilon - E_g(\chi) >> E_g(\chi). \end{cases} \tag{114}$$

where $h(\epsilon,\chi)$ is of order unity. Substituting Eq. (114) into Eq. (85) we get

$$G \approx G_N\tau_{in}\frac{E_g^2(0)}{T}, \tag{115}$$

where $G_N = e^2E_T\tilde{\nu}_N v$ is the conductance of the normal metal. The magnitude of $J_{max}$ and $J_{Jump}$ can be estimated to be

$$J_{max} \sim J_c(0)\frac{E_g(0)}{T} \tag{116}$$

Here $J_c(0) = \frac{1}{e}G_N E_g(0)$ is the critical current of a diffusive SNS junction at $T = 0$. We note that the value of $J_{max}$ can be significantly larger than $J_c(T)$.

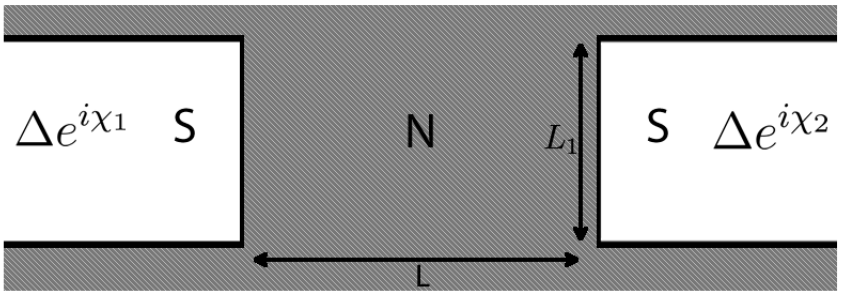
*Diffusive junctions with open boundary*

It should be noted that results presented above are valid in situations where the low energy quasi-particles are trapped inside the normal region of the junction, and the only channel of

the quasi-particle relaxation is the inelastic energy relaxation. In a different geometry, where the normal region of the junction is open to the bulk normal metal, as shown in Fig. ??, there is another channel of the relaxation via diffusion of quasi-particles into the bulk of the normal metal. In this case one can obtain an estimate for the conductance of the system substituting in Eq. (115)

$$\tau_{in} \to min[\tau^{(*)}, \tau_{in}] \tag{117}$$

where $\tau^* \sim L_1^2/D$ is the time of diffusion on the length $L_1$.




c)

## V. I-U CHARACTERISTICS OF NON-CENTROSYMMETRIC SNS JUNCTION

In this section we apply the results of the previous chapter to the case of non-centrosymmetric junctions to study the non-reciprocity in the I-U characteristics of the small voltage regime. We will show that similar to the linear conductance, the non-reciprocal part of the current can be expressed entirely in terms of the $\phi$-dependent density of states. Furthermore, the magnitude of the non-reciprocity is controlled by the inelastic relaxation time $\tau_{in}$ and can be much larger than non-reciprocity of non-centrosymmetric metals which is controlled by $\tau_{el}$. At the end of this chapter we consider the specific case of a diffusive junction with spin orbit coupling as well as an in plane magnetic field in the normal region, and estimate the magnitude of the non-reciprocity in the I-U characteristics.

### A. Symmetry restrictions

In non-centrosymmetric junctions, the current can be even in the voltage $U$ or odd in the magnetic field $\mathbf{H}$

$$J(U,\mathbf{H}) + J(-U,\mathbf{H}) \neq 0 \tag{118}$$

$$J(U,\mathbf{H}) - J(U,-\mathbf{H}) \neq 0. \tag{119}$$

This means that in the case of a voltage biased junction, the average current $\langle J(U,\mathbf{H})\rangle$ can have terms which are even in $U$ and odd in $\mathbf{H}$. However, due to Onsagers theorem [33, 34], the linear conductance of time reversal systems must be reciprocal

$$G_{Linear}(\mathbf{H}) = G_{Linear}(-\mathbf{H}). \tag{120}$$

This means that terms of the form $U\mathbf{H}$ are not allowed, and the non-reciprocity in the I-U characteristics can manifest only the the non-linear regime.

We showed in the previous chapter that the I-U characteristics in the small voltage regime is determined entirely by the density of states $\nu(\epsilon,\phi,\mathbf{H})$. From Eq. (79) that if the density of states is symmetric with respect to $\phi$, then $\langle J(U,\mathbf{H})\rangle = -\langle J(-U,\mathbf{H})\rangle$ and the I-U characteristics is reciprocal. Thus non-reciprocity in the small voltage regime is allowed only if the density of states which is odd in $\phi$. Since $\nu(\epsilon,\phi,\mathbf{H})$ is a scalar, it must be invariant upon time reversal,

$$\nu(\epsilon,\chi,\mathbf{H}) = \nu(\epsilon,-\chi,-\mathbf{H}). \tag{121}$$

In the absence of centrosymmetry, this is the only symmetry restriction on $\nu(\epsilon,\phi,\mathbf{H})$, and terms which are odd in both $\mathbf{H}$ and $\chi$ are allowed. Thus in the presence of a magnetic field (or some other mechanism which breaks time reversal symmetry) , the I-U characteristics of non-centrosymmetric SNS junctions can be non-reciprocal. This is distinct from non-reciprocal effects in normal metals, which require only broken centrosymmetry and can persist in the absence of magnetic fields [6].

It is important to note that in the special case where the influence of $\mathbf{H}$ on $\nu(\epsilon,\chi,\mathbf{H})$ reduces a constant phase shift $\phi(\mathbf{H})$,

$$\nu(\epsilon,\chi,\mathbf{H}) = \nu_0(\epsilon,\chi+\phi(\mathbf{H})), \tag{122}$$

[6] The reason for this is that dissipative currents necessarily produce entropy, and the direction of increasing entropy introduces a direction in time

the I-U characteristics remain reciprocal. We note however, that if different junctions obeying Eq. (122) are connected in parallel [35, 36], the critical current and resistance at current bias are nonreciprocal.

### B. Non-reciprocity in voltage biased junctions

Let us first consider the case of low voltages where $eU \ll \tau_{in}^{-1}$. Since non-reciprocal features only manifest in the non-linear I-U characteristics, we must expand the expression for $\langle J(U)\rangle$ obtained in Eq. (79) to quadratic order in $eU\tau_{in}$. Furthermore, to obtain a non-vanishing result we must also expand to quadratic order in $1/T$. Performing this expansion, we arrive at the following expression

$$\begin{aligned}\langle J(U,\mathbf{H})\rangle = &-\frac{e}{\pi}(2eU\tau_{in})\int_0^\infty dN\int_{-\pi}^{\pi}d\phi\partial_\epsilon n_F(\epsilon\,(N,\phi,\mathbf{H}))\,(\partial_\phi\epsilon(N,\phi,\mathbf{H}))^2\\ &+\frac{e}{2\pi}(2eU\tau_{in})^2\int_0^\infty dN\int_{-\pi}^{\pi}d\phi\partial_\phi\Big(\partial_\epsilon n_F(\epsilon(N,\phi,\mathbf{H}))\,(\partial_\phi\epsilon(N,\phi,\mathbf{H}))^2\Big)\\ &+\frac{e}{2\pi}(2eU\tau_{in})^2\int_0^\infty dN\int_{-\pi}^{\pi}d\phi\partial_\epsilon^2 n_F(\epsilon(N,\phi,\mathbf{H}))\,(\partial_\phi\epsilon(N,\phi,\mathbf{H}))^3\end{aligned}\tag{123}$$

Changing variables back to $(\epsilon,t)$, we get the following expression for $\langle J\rangle$,

$$\langle J(U,\mathbf{H})\rangle = \langle g_1(\phi,\mathbf{H})\rangle_\phi(2eU\tau_{in}) + \langle g_2(\phi,\mathbf{H})\rangle_\phi(2eU\tau_{in})^2,\tag{124}$$

Here $g_1(\phi,\mathbf{H})$ is given by Eq. (85) and we have defined,

$$\begin{aligned}g_2(\phi,\mathbf{H}) = 2e\int_0^\infty d\epsilon\int_{-\pi}^{\pi}d\phi\Big(&\partial_\epsilon n_F(\epsilon)\partial_\phi\left[\nu(\epsilon,\phi,\mathbf{H})V_\nu^2(\epsilon,\phi,\mathbf{H})\right]\\ &+\partial_\epsilon^2 n_F(\epsilon)\nu(\epsilon,\phi,\mathbf{H})\Big(V_\nu(\epsilon,\phi,\mathbf{H})\Big)^3\Big).\end{aligned}\tag{125}$$

Note that the linear current is proportional to $\tau_{in}$ while the non-reciprocal current is proportional to $\tau_{in}^2$. The first term in Eq. (125) vanishes upon integration over $\phi$, since it is a total derivative of a periodic function, which means that only the second term contributes to $\langle g_2(\phi,\mathbf{H})\rangle_\phi$. In a centrosymmetric junction where $\nu$ is even in $\chi$, $V_\nu$ is odd $\chi$ and thus $g_2(\phi,\mathbf{H})$ vanishes upon integration of $\phi$, consistent with the symmetry principles discussed in the previous section.

In the limit of large voltages $eU \gg \tau_{in}^{-1}$, we can once again substitute the Fourier expansion (87) into the expression for the non-reciprocal part of the current $\delta J(U,\mathbf{H}) = \frac{1}{2}(\langle J(U,\mathbf{H})\rangle +$

$\langle J(-U,\mathbf{H})\rangle$),

$$\delta J(U,\mathbf{H}) = \frac{e}{2\pi}\int_{-\pi}^{\pi} d\phi \partial_\phi \epsilon(N,\phi,\mathbf{H}) \int_0^\infty \partial_\epsilon^2 n_F(\epsilon) \times \sum_{k_1,k_2} \frac{C_{k_1}(N)C_{k_2}(N)}{1+(k_1+k_2)^2(2eU\tau_{in})^2} e^{i(k_1+k_2)\phi} \approx \frac{1}{(eU\tau_{in})^2}\langle g_2(\phi,\mathbf{H})\rangle_\phi. \quad (126)$$

In this regime the total current is given by,

$$\langle J(U,\mathbf{H})\rangle \approx \frac{\langle g_1(\phi,\mathbf{H})\rangle_\phi}{eU\tau_{in}} + \frac{\langle g_2(\phi,\mathbf{H})\rangle_\phi}{(2eU\tau_{in})^2}, \quad (127)$$

From Eqs. (124) and (127), we estimate that the non-reciprocity in $J_{max}(\mathbf{H})$ and $U_{min}(\mathbf{H})$ is given by,

$$\delta J_{max}(\mathbf{H}) \approx \langle g_2(\chi,\mathbf{H})\rangle_\phi \quad (128)$$

$$\delta U_{min}(\mathbf{H}) \approx (e\tau_{in})^{-1}\left(\frac{\langle g_2(\phi,\mathbf{H})\rangle_\phi}{\langle g_1(\phi,\mathbf{H})\rangle_\phi}\right)\left(\frac{G(\mathbf{H})}{G_N}\right) \quad (129)$$

### C. Non-reciprocity in current biased junctions

Let us now turn to the consideration of nonreciprocity in the current-bias setup. The nonreciprocity of the critical current $J_c$ of SNS junctions has been studied in several articles [37–45] .

Here we study the I-U characteristics $\bar{U}(J)$ at currents larger than the critical current. In the interval where $J - J_c(\mathbf{H},T) \ll J_c(\mathbf{H},T)$, the dominant contribution to the period $t_p$ comes from the interval of phase near $\chi_m(\mathbf{H})$, where the supercurrent reaches its maximum value. In this interval the voltage is small compared $\tau_{in}^{-1}$ and the current is given by [7],

$$J = J_s(\phi(t),\mathbf{H},T) + g_1(\phi,\mathbf{H})\dot{\chi}\tau_{in}. \quad (130)$$

Integrating the equation gives the following expression for the period $t_p$,

$$t_p \approx \tau_{in}\int_{-\pi}^{\pi} \frac{g_1(\phi_m(\mathbf{H}),\mathbf{H})d\phi}{J - J_s(\phi,\mathbf{H},T)}. \quad (131)$$

[7] Including the term which is proportional to $g_2(\phi,\mathbf{H})$ results in corrections which are higher order in$(J - J_c(\mathbf{H},T))/J_c(\mathbf{H},T)$

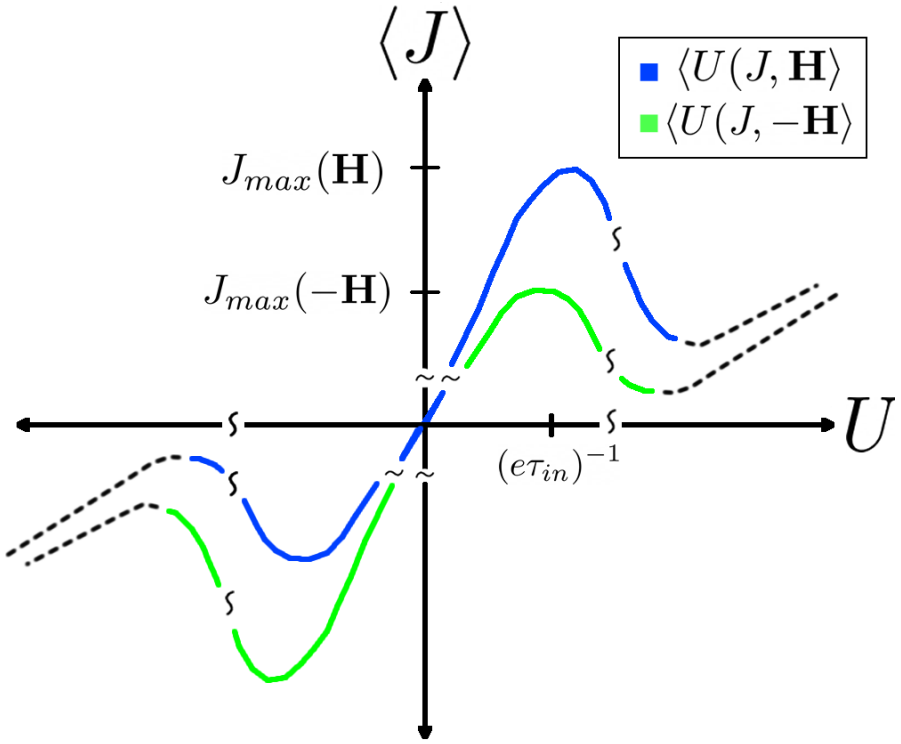



Figure 8: The I-U characteristics of a nonreciprocal SNS junction at low voltage bias for opposite signs of the magnetic field are sketched in blue and green. The dashed lines correspond to the high voltage regime.

Expanding the super-current near $\phi = \phi_m(\mathbf{H})$,

$$J_s(\phi, \mathbf{H}, T) \approx J_c(\mathbf{H}, T) + \frac{(\phi - \phi_m(\mathbf{H}))^2}{2} \partial_\phi^2 J_s(\phi, \mathbf{H}, T)\big|_{\phi=\phi_m}, \tag{132}$$

keeping terms which are lowest order in $\mathbf{H}$ as well as $(J - J_c(\mathbf{H}, T))/J_c(\mathbf{H}, T)$, we using Eq. (96), we get the following expression for the average voltage,

$$\langle U(J, \mathbf{H}) \rangle \approx A(\mathbf{H}) \sqrt{(J - J_c(\mathbf{H}, T))}, \tag{133}$$

where

$$A(\mathbf{H}) = \frac{\sqrt{-\partial_\phi^2 J_s(\phi, \mathbf{H}, T)|_{\phi=\phi_m(\mathbf{H})}}}{2^{5/2} e\tau_{in} g_1(\phi_m(\mathbf{H}), \mathbf{H})}. \tag{134}$$

The voltage in Eq. (133) has the standard square root dependence on the excess current $J - J_c(\mathbf{H}, T)$. The nonreciprocity in this regime is characterized by the nonreciprocity of the critical current $J_c(\mathbf{H}, T)$ and the coefficient $A(\mathbf{H})$.

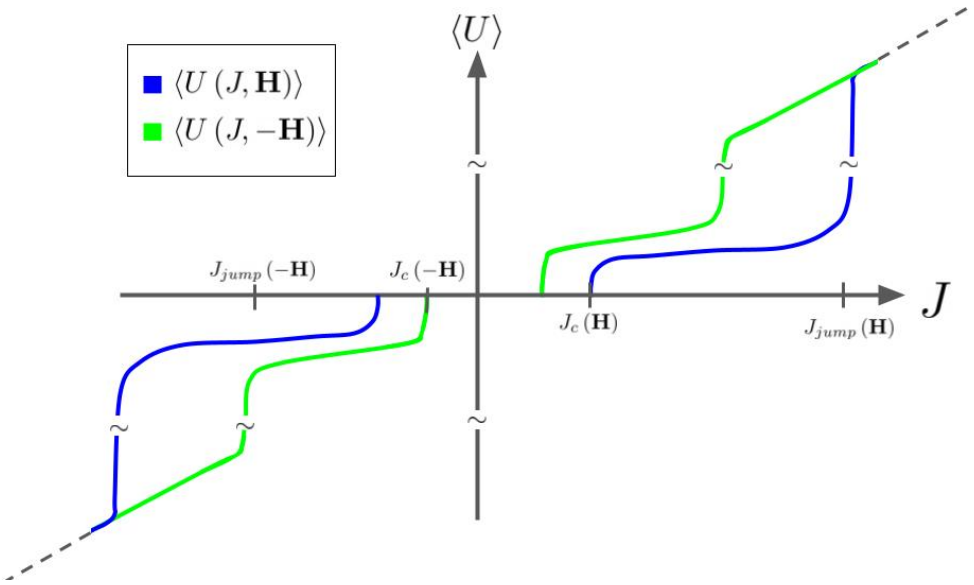



Figure 9: A qualitative picture of the I-U characteristics of a nonreciprocal SNS junction at fixed current. The blue and green curves correspond to the low voltage regime regime of the I-U characteristics for opposite signs of magnetic field. The dashed lines correspond to the high voltage regime.

### D. Application to diffusive planar junction with spin-orbit coupling

To estimate the magnitude of the effect, below we apply the general results obtained above to a planar junction of length $L$ and width $L_1$(shown in Fig. 10), in which the normal region is described by the following Hamiltonian,

$$H = \mathbf{p}^2/2m - E_F + \beta^{\alpha i} p^i \sigma^\alpha + V_{imp}(\mathbf{r}) + g\mu_0 \mathbf{H}\cdot\boldsymbol{\sigma}. \tag{135}$$

Here $E_F$ is the Fermi energy, $m$ is the electron mass, $\sigma_i$ are the Pauli matrices in spin space, $g$ is the g-factor, $\mu_0$ is the Bohr magneton, and $V_{imp}(\mathbf{r})$ is the random impurity potential. For Rashba spin-orbit coupling $\beta^{\alpha i} = \alpha_R \epsilon^{\alpha i j}\hat{n}^j$, where $\hat{n}$ is a unit polar vector, and for Dresselhaus spin-orbit coupling $\beta^{\alpha i} = \alpha_D \delta^{\alpha i}$.

The direction of the magnetic field is chosen to be parallel to the film, as depicted in Fig. 10. Therefore, it enters the Hamiltonian, Eq. (135), only via the Zeeman term.

Below we will focus on linear in $\mathbf{H}$ contribution to the nonreciprocity of the I-U characteristics. We consider a case of weak spin-orbit coupling $\beta p_F \ll \tau_{el}^{-1}$ and focus on the diffusive regime, $L \gg \sqrt{D\tau_{so}}$. Here $D = v_F^2\tau_{el}/2$ is the typical value of the electron diffusion coefficient in the normal region, $\tau_{so}$ is the spin relaxation time, and $v_F$ is the Fermi-velocity. We also

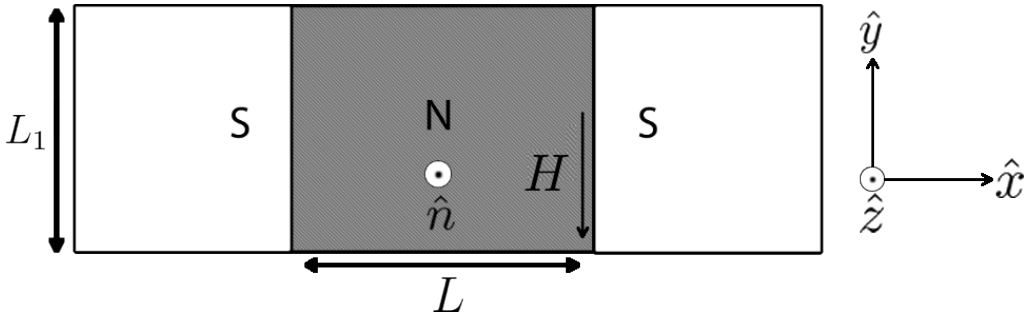



Figure 10: Top down view of a planar SNS junction. The junction is aligned along the $\hat{\mathbf{x}}$ direction, there is a parallel magnetic field $\mathbf{H}$ directed in the $\hat{\mathbf{y}}$ direction, and there is an out of plane vector $\hat{\mathbf{n}}$ pointing in the $\hat{\mathbf{z}}$ direction which breaks inversion symmetry.

assume that the distance between the superconductors, $L$, is much larger than the coherence length in the superconductors, and therefore the order parameter $\Delta(\mathbf{r})$ has a constant modulus $\Delta$ in the superconducting leads, and vanishes in the normal region, see Fig. 10. After averaging over the random impurity potential, the density of states in the SNS junction $\nu(\epsilon, \chi, \mathbf{H})$ is obtained by solving Usadel's equation in the presence of spin-orbit coupling and a magnetic field [46–48]. Here we present the main results, leaving the details of the calculation to VII D.

The main feature of the density of states of a diffusive SNS junction is the existence of a mini-gap at $\chi = 0$ of order $E_T \sim D/L^2$ [32, 49–51] (for simplicity we restrict ourselves to the case where the S-N boundaries of the junction are transparent $r = 1$). The density of states $\nu(\epsilon, \chi, \mathbf{H})$ exhibits a significant $\chi$-dependence only for energies of the order of the mini-gap. This means that the level sensitivity, $V_\nu(\epsilon, \phi, \mathbf{H})$, is peaked in the energy interval $\epsilon \gtrsim E_T$.

It should be noted that if the diffusion coefficient $D(x)$ and the strength of spin-orbit coupling $\beta^{\alpha x}(x)$ depend only on the $x$ coordinate, the density of states at $\mathbf{H} \neq 0$ can be written in the form Eq. (122) with

$$\phi(\mathbf{H}) = \int_{-L/2}^{L/2} dx \frac{2\tau_{so} g\mu_0 \beta^{\alpha x}(x) \mathbf{H}^\alpha}{D(x)}. \tag{136}$$

Therefore, in this idealized 1D model the I-U characteristics are reciprocal. However, in the general case where $D(\mathbf{r})$ and $\beta(\mathbf{r})$ are functions of the two coordinates, $x$ and $y$, or the shape of the normal metal part of the junction is not rectangular, the I-U characteristics of the junction are non-reciprocal.

Below we estimate the degree of non-reciprocity of the I-U characteristics in the case where $L \lesssim L_1$, the amplitude of fluctuations the diffusion coefficient in the y-direction is of order $\delta D$,

and the correlation length of such fluctuations is of order $L_1$. In this case we get the following estimates for $\langle g_1\rangle$ and $\langle g_2\rangle$,

$$\langle g_1\rangle \sim e\nu_N \frac{E_T^3}{T}, \quad \langle g_2\rangle \sim \langle g_1\rangle \frac{\beta(g\mu_0 H)\tau_{so}E_T}{LT^2}\left(\frac{\delta D}{D}\right)^2, \tag{137}$$

where $\tau_{so}^{-1} = 4p_F^2\beta^2\tau_{el}$. For the nonreciprocal part $\delta A \equiv [A(\mathbf{H}) - A(-\mathbf{H})]$ of the coefficient $A(\mathbf{H})$ in Eq. (134) we find

$$\delta A \sim \frac{\sqrt{J_c(0,T)}}{e\tau_{in}\langle g_1\rangle}\frac{\tau_{so}\beta(g\mu_0 H)}{LE_T}\left(\frac{\delta D}{D}\right)^2. \tag{138}$$

The parameters in Eqs. (137)–(138) characterize the degree of nonreciprocity of I-U characteristics of SNS junctions with weak spin-orbit coupling in both the voltage and current bias cases.

We note that transport in diffusive junctions in which the normal region is formed by a surface of a topological insulator can also be analysed using Usadel's equation. This is done in Appendix B. The magnitude of nonreciprocity in this case is obtained by setting in Eqs. (137) and (138) $\tau_{so} \to \tau_{el}$ and $\beta \to v$, where $v$ is the velocity of the relativistic dispersion.

## VI. NEGATIVE CRITICAL CURRENTS IN 1-D SNS JUNCTIONS*

Although not directly related to the topic of dissipation, the problem of negative critical currents in SNS junctions is another interesting problem. The supercurrent given by Eq. (73) can be expressed in terms of the Josephson energy of the junction $E_J(\phi)$,

$$J_s(\phi) = 2e\frac{dE(\phi)}{d\phi}$$
$$E_J(\phi) = \int d\epsilon n_F(\epsilon)\epsilon\nu(\epsilon,\phi) \tag{139}$$

In general $E(\phi)$ is a $2\pi$ periodic function of $\phi$ and can be expanded in a Fourier series. However, in most cases the first harmonic gives the dominant contribution and contributions from harmonics can be dropped [8]. In such cases the energy and the supercurrent can be expressed in the form,

$$E_J(\phi) = \frac{J_c}{2e}(1-\cos(\phi)) \tag{140}$$
$$J_s = J_c\sin(\phi). \tag{141}$$

[8] This is valid for for junctions which are weakly coupled, or at temperatures which are large compared to the energy scale of the Andreev bound states

In most junctions, the sign of $J_c$ is positive and the energy is minimized at $\phi = 0$. It is possible to prove that in the single particle approximation the critical current $J_c$ is always positive [52]. However, beyond the non-interacting electron approximation, there are no general principles that determine the sign of $J_c$. If the sign of $J_c$ is negative, the groundstate lies at $\phi = \pi$ and the junction is said to have "negatve critical current".

Several physical mechanisms of negative currents have been proposed. The sign of the critical current of a superconductor-ferromagnet superconductor junction is an oscillating function of the magnetization and the length of the ferromagnet (see Refs. [53, 54]). Even in the absence of macroscopic magnetization, the critical current can be negative if it is mediated by tunneling through a magnetic impurity [55], a resonant state [56–58] or a quantum dot in the Coulomb blockade regime [59].

In this section, I will consider a mechanism of negative critical current in a 1-D SNS junction in the Coulomb blockade regime which can be traced back to the node theorem for electron wave functions in 1-D. I will show that the sign of the critical current alternates as a function of the number of electrons in the normal region. Namely, the critical current is positive when the number of electrons is even, and negative if it is odd, even when the number of electrons in the junction is large.

### A. Tunnel Hamiltonian model of a SNS junction

To demonstrate this effect, we consider a junction described by the follow Hamiltonian,

$$\hat{H} = E_C(\hat{N} - \mathcal{N}_0)^2 + \hat{H}^{1\mathrm{D}} + \sum_{i=1,2} \left[\hat{H}_t^{(i)} + \hat{H}_{\mathrm{SC}}^{(i)}\right]. \tag{142}$$

In this expression, $\hat{H}_{SC}^{(}i)$ is the Hamiltonian of the i-th superconducting lead and $\hat{H}^{(}1D)$ is the Hamiltonian of the normal metal region,

$$\hat{H}^{1\mathrm{D}} = \sum_{m,\sigma} \xi_m c_{\sigma m}^\dagger c_{\sigma m}. \tag{143}$$

Here $c_{\sigma m}$ is the annihilation operator of an electron with spin $\sigma =\uparrow,\downarrow$ and the $m$-th single-electron state, whose energy $\xi_m$ is measured relative to the chemical potential (we will assume that $\xi_m$ is a monotonic function of index $m$, and $\xi_0 = 0$). The mean level spacing for the conductor is $\delta \sim v_{\mathrm{F}}/L$, where $L$ is the conductor's length, and $v_{\mathrm{F}}$ is the Fermi velocity.

The tunneling Hamiltonian may be expressed in the form,

$$\hat{H}_t^{(i)} = \sum_{m,k,\sigma} t_{mk}^{(i)} c_{\sigma m}^{\dagger} a_{\sigma k}^{(i)} + \text{H.c.}, \tag{144}$$

where $a_{\sigma k}^{(i)}$ denotes the electron annihilation operator in state $k$ in the superconducting lead $i$, and $t_{mk}^{(i)}$ denotes the tunneling matrix element between a state $k$ in the lead and a state $m$ in the normal region. The lowest order of the perturbation theory with respect to $H_t^{(i)}$, which yields the dependence of the junction energy on $(\chi_1 - \chi_2)$, is fourth order. For simplicity we assume that $\Delta$, the quasiparticle energy gap in superconductors, is larger than the Coulomb energy, and mean level spacing: $\Delta \gg E_C, \delta$. In this regime quasiparticles can tunnel from superconductors to the normal metal wire only by pairs, and the part of the pair-tunneling Hamiltonian between superconductor $i$ and the metallic wire may be written in the form

$$\hat{H}_{\text{T}}^{(i)} = e^{i\chi_i} \sum_{mn} T_{mn}^{(i)} c_{\uparrow m}^{\dagger} c_{\downarrow n}^{\dagger} + \text{H.c.}, \tag{145}$$

where $\chi_i$ is the order parameter phase for the $i$-th superconducting lead, $T_{mn}^{(i)}$ denotes the tunneling amplitude of a Cooper pair from lead $i$ into the states $m$ and $n$ in the normal region. For low-lying excited states $m$ and $n$ in the wire, satisfying $\Delta \gg |\xi_n|, |\xi_m|$, amplitude $T_{mn}^{(i)}$ can be expressed in terms of the single-particle tunneling amplitudes in Eq. (147) in the form

$$T_{mn}^{(i)} = -\sum_k \frac{t_{km}^{(i)} t_{kn}^{(i)} \left|\langle a_{\uparrow k} a_{\downarrow k} \rangle\right|}{\epsilon_k}. \tag{146}$$

The single particle amplitudes can be expressed in terms of the wavefunctions of the single particle states in the normal region $\psi_n(x)$ and the leads $\phi_k^{(i)}(x)$,

$$t_{mk}^{(i)} = \frac{1}{2m^*} \left( \phi_k^{(i)}(x) \partial_x \psi_m(x) - \psi_m(x) \partial_x \phi_k^{(i)}(x) \right) \Big|_{x=x_i}. \tag{147}$$

Here $m^*$ is the electron mass, x is the axis along the junction, and $x_i$ is located inside the tunneling barrier between the wire and the lead.

In this approximation expression, $\epsilon_k$ is the quasiparticle energy in state $k$ of the superconductor, and $\langle a_{\uparrow k} a_{\downarrow k} \rangle$ denotes the Cooper pair condensation amplitude. Once higher-energy degrees of freedom are "integrated out", the effective model Hamiltonian reads

$$\hat{H}_{\text{eff}} = E_C (\hat{N} - \mathcal{N}_0)^2 + \hat{H}^{\text{1D}} + \hat{H}_{\text{T}}^{(1)} + \hat{H}_{\text{T}}^{(2)}. \tag{148}$$

Note that it contains degrees of freedom in the normal metal only.

### B. Perturbative calculation of Josephson energy

Evaluating the Josephson coupling energy $E_J$ of the system using second-order perturbation theory in powers of $\hat{H}_{\mathrm{T1,2}}$

$$E_J = \sum_{mn} \frac{\langle 0|\hat{H}_T^{(1)}|\uparrow m,\downarrow n\rangle\langle\uparrow m,\downarrow n|\hat{H}_T^{(2)}|0\rangle}{E_0 - E_{mn}} + \text{c.c.}, \tag{149}$$

where $|0\rangle$ is the ground state of the wire, while $|\uparrow m,\downarrow n\rangle$ is the two-particle excited state. The excited state is characterized by presence (or absence) of two electrons with opposite spins, one on level $n$, another on level $m$. The structure of the ground state $|0\rangle$ depends on the parity of $N_0$. When $N_0$ is even, $|0\rangle$ is spin singlet, and any $m \geq 0$ level is empty, any $m < 0$ level is doubly occupied. This ground state remains stable as long as the gate potential $\mathcal{N}_0$ satisfies the following inequalities

$$-E_C < 2E_C(N_0 - \mathcal{N}_0) < E_C + |\xi_{-1}|. \tag{150}$$

When either of these strict inequalities become equality, the ground state becomes charge-degenerate. For example, if $2E_C(N_0 - \mathcal{N}_0) = -E_C$, the state with $N_0$ and a state with $N_0 + 1$ become degenerate. The latter state has one extra electron occupying $m = 0$ level, making the total number of electrons in the normal region odd. The ground state with odd $N_0$ is stable when $2|N_0 - \mathcal{N}_0| < 1$. For odd $N_0$, the ground state is spin doublet. In the following, the single-electron index $m$ is redefined as follows: $m \to m - M$. This way, all levels with $m < 0$ are empty. A single electron resides on the $m = 0$ level if $N_0$ is odd, otherwise it is empty. We cam also assume without loss of generality that $\xi_0 = 0$.

The perturbative expression for the Josephson energy Eq. (149) can be reduced to the following expression,

$$E_J(\phi) = -(E_+ + E_-)\cos(\phi), \tag{151}$$

where the energies $E_\pm$ represent contributions corresponding to two-electron and two-hole tunneling processes,

$$E_\pm = 2\sum_{mn} \frac{T_{mn}^{(1)} T_{mn}^{(2)} \Theta(\pm m \pm 1/2)\Theta(\pm n \pm 1/2)}{4E_C \pm [4E_C(N_0 - \mathcal{N}_0) + \xi_m + \xi_n]}. \tag{152}$$

The choice of sign sign on RHS is dictated by the sign on the LHS and the $1/2$ shifts in the Heviside functions $\Theta(x)$ are introduced to avoid the ambiguity at $\Theta(0)$. Note that the signs of the terms in the sum are determined by the products $T_{mn}^{(1)} T_{mn}^{(2)}$.

By substituting the expression for the single particle amplitudes Eq. (147) into Eq. (146), it can be shown that the sign of the pair tunneling amplitudes is determined by the derivatives of the single particle wavefunctions at the barriers $T^{(i)}_{mn} \propto \partial_x\psi_m\partial_x\psi_n\big|_{x=x_i}$. The sign $T^{(1)}_{mn}T^{(2)}_{mn}$ is then given by,

$$T^{(1)}_{mn}T^{(2)}_{mn} \propto \partial_x\psi_m(0)\partial_x\psi_n(0)\partial_x\psi_m(L)\partial_x\psi_n(L) \tag{153}$$

The 1-D node theorem guarantees that $\partial_x\psi_m(0)\partial_x\psi_m(L) \propto (-1)^m$. As a result, the product $T^{(1)}_{mn}T^{(2)}_{mn}$ alternates sign as a function of $m$ and $n$,

$$T^{(1)}_{mn}T^{(2)}_{mn} = (-1)^{m+n}|T^{(1)}_{mn}T^{(2)}_{mn}|/ \tag{154}$$

The RHS in Eq. (154) is the sum of an alternating series, and the sign is determined by the sign of the first term in the series, which is different for the case where $N_0$ is even and odd. As a result, the Josephson coupling oscillates as a function of $N_0$. To demonstrate this explicitly, we consider the case of a clean 1-D junction where the single particle wavefunctions are known. *clean 1-D junction* For the case of a clean wire where there length of the junction is much larger than the mean free path $L \gg l$, the wavefunctions are sinusoidal

$$\begin{aligned} \psi_m &\propto \sin(k_F x) \\ k_m &= \frac{\pi N_0}{2L} + \frac{\pi m}{L}, \end{aligned} \tag{155}$$

and the single particle energies are given by $\xi_m = m\delta$. In this regime, $|T^{(1)}_{mn}T^{(2)}_{mn}|$ may be considered independent of $m$ and $n$, while the sign of the product $T^{(1)}_{mn}T^{(2)}_{mn}$ satisfies Eq. (154). Thus, Eq. (154) for even number of electrons can be expressed as

$$E^{(\text{even})}_{\pm} = E_0 \sum_{\substack{m\geq 0\\ n\geq 0}} \frac{(-1)^{m+n}}{\kappa_\pm + m + n}. \tag{156}$$

Here $E_0 \propto g^{(1)}g^{(2)}\delta$, and the dimensionless offset parameters are

$$\kappa_\pm = \frac{4E_C}{\delta}\left[1 + (N_0 - \mathcal{N}_0) + \frac{(1\mp 1)\delta}{4E_C}\right]. \tag{157}$$

Note that $\kappa_\pm > 0$, as ensured by inequalities (150), and $\kappa_+ + \kappa_- \approx 2$. The latter relation means that at least one of $\kappa$'s is of order unity, and neither of them exceed 2.

Therefore, we reduce the issue of finding the Josephson coupling to the task of evaluating

the sum in Eq. (156). To proceed, we rewrite this double sum in the form

$$\mathcal{S}_\pm = \sum_{n=0}^{+\infty} (-1)^n f(\kappa_\pm + n), \tag{158}$$

$$\text{where} \quad f(y) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{y+m}. \tag{159}$$

Since the right-hand side of Eq. (159) is a sign-alternating series satisfying the Leibniz criterion [sequence $(y+m)^{-1}$ monotonically decreases to zero for growing index $m$], we conclude that, for positive $y$, the series is convergent, and function $f(y)$ is finite. Moreover, the Leibniz theorem guarantees that, for positive $y$, $f(y) > 0$ since the first term in the sum (159) is positive. Additionally, it is easy to prove that $f(y)$ decreases monotonically when $y \to +\infty$. Indeed, the derivative of $f$

$$f'(y) = \sum_{\ell=0}^{+\infty} \left[ \frac{1}{(y+2\ell+1)^2} - \frac{1}{(y+2\ell)^2} \right] \tag{160}$$

is negative since it is a convergent series of strictly negative terms. Therefore, the series in Eq. (158) also passes the Leibniz test. Furthermore, $\mathcal{S}_\pm > 0$ since $f(\kappa_\pm)$ are both positive. Thus for an even number of electrons in the wire the critical current $J_c$ is positive.

Let us now consider the situation with an odd number of electrons in the wire. In this case from Eq. (149) we obtain the following expression for the counterpart of $E_\pm^{(\text{even})}$ in Eq. (156),

$$E_\pm^{(\text{odd})} = E_0 \sum_{\substack{m\geq 0 \\ n \geq 1}} \frac{(-1)^{m+n}}{\chi_\pm + m + n}. \tag{161}$$

Here the offset parameters $\chi_\pm > 0$ are defined similar to Eq. (157) . The crucial difference between Eqs. (161) and (156) is that the summation range for $n$ starts from 0 in Eq. (156), while in Eq. (161) index $n$ runs from 1. This follows from the fact that the state $n = 0$ is singly occupied.. Repeating the consideration between Eqs. (158) and (160) we find that $E_\pm^{(\text{odd})} < 0$. Thus, for an odd number of electrons in the wire the critical current $J_c$ is negative. Thus, we conclude that an addition or subtraction of a single electron from the conductor changes the sign of the critical current even in the case where the number of electrons is large.

The reason for the opposite sign of the critical current $J_c$ in the cases of even and odd number of electrons in the wire is related to the fact that the sign of the corresponding alternating series, Eqs. (156) and (161), are determined by the sign of the terms with the smallest energy denominator.

## VII. DERIVATION OF KINETIC EQUATIONS

In this chapter I will present a derivation of the kinetic equations for the case of diffusive superconductors. Our starting point will be the Gorkov equations for the impurity averaged Green's functions in Keldysh representation [60]. We will then write these equations in the quasi-classical approximation, followed by the diffusion approximation. Next we will introduce two generalized distribution functions $f$ and $f_1$ to parameterize the Keldysh component of the Green's functions. Using this parameterization together with the Gorkov equations in the diffusion approximation, we will derive the Larkin-Ovchinnikov equations that describe the kinetics of dirty superconductors. Finally we will show that in cases where the retarded and advanced component of the Green's functions can be calculated in the local approximation, and when effects associated with charge imbalance can be neglected, the Larkin-Ovchinnikov can be reduced to Eqs. (34).

### A. The Gorkov equations for the Keldysh Green's functions

For this chapter I will denote matrices in Nambu space with a hat, $\hat{A}$, matrices in both Nambu and Keldysh space with a check, $\check{A}$, and I will choose units such that $\hbar = c = 1$. The Gorkov equations for the impurity averaged Green's functions in Keldysh representation is given by,

$$\left(i\hat{\tau}_3\partial_{t_1} + \frac{1}{2m}\left(\nabla_{\mathbf{r_1}} - ie\hat{\tau}_3\mathbf{A}(\mathbf{r_1},t_1)\right)^2 + \mu + \check{\Delta}(\mathbf{r_1},t_1) - e\phi_0(\mathbf{r_1},t_1)\right)\check{G}(\mathbf{r_1},\mathbf{r_2},t_1,t_2) \\ -(\check{\Sigma}\otimes\check{G})(\mathbf{r_1},\mathbf{r_2},t_1,t_2) = \delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\delta(t_1-t_2). \quad (162)$$

Here $\check{\Sigma}$ is the self energy, $\hat{\tau}_i$ are Pauli matrices in Nambu space, $\mathbf{A}(\mathbf{r},t)$ is the vector potential, $\hat{\Delta}(\mathbf{r},t)$ is the superconducting order parameter, $\mu$ is the chemical potential [9], $\phi_0(\mathbf{r})$ is the scalar potential, and cross operator represents a convolution,

$$(O_1\otimes O_2)(\mathbf{r_1},\mathbf{r_2},t_1,t_2) = \int d\mathbf{r}\int dt O_1(\mathbf{r_1},\mathbf{r},t_1,t)O_2(\mathbf{r},\mathbf{r_2},t,t_2). \quad (163)$$

The Keldysh space matrices have the form

$$\check{G} = \begin{pmatrix}\hat{G}^R & \hat{G}^K\\ 0 & \hat{G}^A\end{pmatrix}, \qquad \check{\Sigma} = \begin{pmatrix}\hat{\Sigma}^R & \hat{\Sigma}^K\\ 0 & \hat{\Sigma}^A\end{pmatrix}, \qquad \check{\Delta} = \begin{pmatrix}\hat{\Delta} & 0\\ 0 & \hat{\Delta}\end{pmatrix}, \quad (164)$$

[9] Here we have also assumed that the renormalization of the chemical potential is already absorbed in $\mu$

The matrix structure of $\hat{G}^{R,A}$ and $\hat{\Delta}$ in Nambu space has the form,

$$\hat{G}^{R,A} = \begin{pmatrix} G^{R,A} & F^{R,A} \\ -F^{R,A+} & -G^{R,A} \end{pmatrix} \qquad \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^{+} & 0 \end{pmatrix} \tag{165}$$

where $G^{R,A}$ and $F^{R,A}$ are retarded and advanced components of the normal and anomalous Green's functions.

Equation (162) should be supplemented with its conjugate equation,

$$\check{G}(\mathbf{r_1},\mathbf{r_2},t_1,t_2)\Big( -i\hat{\tau}_3\partial_{t_2} + \frac{1}{2m}\Big(\nabla_{\mathbf{r_2}} + ie\hat{\tau}_3\mathbf{A}(\mathbf{r_2},t_2)\Big)^2 + \mu + \hat{\Delta}(\mathbf{r_2},t_2) - e\phi_0(\mathbf{r_2},t_2)\Big) \\ -(\check{G}\otimes\check{\Sigma})(\mathbf{r_1},\mathbf{r_2},t_1,t_2) = \delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\delta(t_1-t_2), \tag{166}$$

where the derivatives are understood to be acting towards the left, as well as the self-consistency condition for the order parameter $\hat{\Delta}(\mathbf{r},t)$ and the gauge invariant potential $\Phi(\mathbf{r},t)$,

$$\hat{G}^{K}(\mathbf{r},\mathbf{r},t,t) = \begin{pmatrix} \frac{\tilde{\nu}_N}{2}\Phi(\mathbf{r},t) & \frac{1}{\lambda}\Delta(\mathbf{r},t) \\ -\frac{1}{\lambda}\Delta^{+}(\mathbf{r},t) & \frac{\tilde{\nu}_N}{2}\Phi(\mathbf{r},t) \end{pmatrix} \tag{167}$$

Here $\lambda$ is the electron interaction constant and $\tilde{\nu}_N$ is the density of states per unit volume in the normal metal.

*Quasi-classical approximation for the Gorkov equations*

In cases where the external fields $\phi_0(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ are slowly varying in space and time, the Gorkov equations can be significantly simplified. In particular, when the characteristic frequency of the fields $\omega$ is less than the Fermi energy $\omega \ll \epsilon_F$ and the characteristic length scale of spatial variations $L_c$ is longer than the Fermi wavelength $L_c \gg p_F^{-1}$, the quasi-classical approximation used to convert the Gorkov equations to a set of first order differential equations for the Greens functions. To do this, we first subtract Eq. (166) from Eq. (162)

$$i\hat{\tau}_3\partial_{t_1}\check{G}(\mathbf{r_1},\mathbf{r_2},t_1,t_2) + i\hat{\partial}_{t_2}\check{G}(\mathbf{r_1},\mathbf{r_2},t_1,t_2)\hat{\tau}_3 \\ +\Big(\frac{1}{2m}\Big(\nabla_{\mathbf{r_1}} - ie\mathbf{A}(\mathbf{r_1},t_1)\Big)^2 + \mu + \hat{\Delta}(\mathbf{r_1},t_1) - e\phi_0(\mathbf{r_1},t_1)\Big)\check{G}(\mathbf{r_1},\mathbf{r_2},t_1,t_2) \\ -\check{G}(\mathbf{r_1},\mathbf{r_2},t_1,t_2)\Big(\frac{1}{2m}(\nabla_{\mathbf{r_2}} + ie\hat{\tau}_3\mathbf{A}(\mathbf{r_2},t_2))^2 + \mu + \hat{\Delta}(\mathbf{r_2},t_2) - e\phi_0(\mathbf{r_2},t_2)\Big) \\ = (\check{\Sigma}\otimes\check{G})(\mathbf{r_1},\mathbf{r_2},t_1,t_2) - (\check{G}\otimes\check{\Sigma})(\mathbf{r_1},\mathbf{r_2},t_1,t_2). \tag{168}$$

Next we introduce the Wigner coordinates,

$$\begin{aligned} \boldsymbol{r} &= \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2), \quad \tilde{\boldsymbol{r}} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \\ t &= \frac{1}{2}(t_1 + t_2), \quad \tilde{t} = t_1 - t_2, \end{aligned} \tag{169}$$

and Fourier transform Eq. (168) over the relative position $\tilde{\mathbf{r}}$ as well as the relative time $\tilde{t}$. Dropping terms which are second order in derivatives, we arrive at the following equation

$$\begin{aligned} \frac{1}{2}\partial_t\{\hat{\tau}_3, \check{G}(\epsilon,\mathbf{r},t,\mathbf{p})\} - i\epsilon[\hat{\tau}_3, \check{G}(\epsilon,\mathbf{r},t,\mathbf{p})] + \frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\check{G}(\epsilon,\mathbf{r},\mathbf{p}) \\ +[\hat{H}(\mathbf{r},t,\mathbf{p}), \check{G}(\epsilon,\mathbf{r},t,\mathbf{p})] - \frac{i}{2}\{\partial_t\hat{H}(\mathbf{r},t,\mathbf{p}), \partial_\epsilon\check{G}(\epsilon,\mathbf{r},t,\mathbf{p})\} \\ -\frac{e}{2m}\mathbf{A}(\mathbf{r},t)\cdot\boldsymbol{\nabla}\{\hat{\tau}_3, \check{G}(\epsilon,\mathbf{r},t,\mathbf{p})\} + \frac{i}{2}\{\boldsymbol{\nabla}\hat{H}(\mathbf{r},t,\mathbf{p}), \nabla_{\mathbf{p}}\check{G}(\epsilon,\mathbf{r},t,\mathbf{p})\} \\ = -i[\check{\Sigma}(\epsilon,\mathbf{r},t,\mathbf{p}), \check{G}(\epsilon,\mathbf{r},t,\mathbf{p})] \\ +\frac{1}{2}\{\boldsymbol{\nabla}\check{\Sigma}(\epsilon,\mathbf{r},t,\mathbf{p}), \nabla_{\mathbf{p}}\check{G}(\epsilon,\mathbf{r},t,\mathbf{p})\} - \frac{1}{2}\{\nabla_{\mathbf{p}}\check{\Sigma}(\epsilon,\mathbf{r},t,\mathbf{p}), \boldsymbol{\nabla}\check{G}(\epsilon,\mathbf{r},t,\mathbf{p})\} \\ -\frac{1}{2}\{\partial_t\check{\Sigma}(\epsilon,\mathbf{r},t,\mathbf{p}), \partial_\epsilon\check{G}(\epsilon,\mathbf{r},t,\mathbf{p})\} + \frac{1}{2}\{\partial_\epsilon\check{\Sigma}(\epsilon,\mathbf{r},t,\mathbf{p}), \partial_t\check{G}(\epsilon,\mathbf{r},t,\mathbf{p})\}. \end{aligned} \tag{170}$$

Here the brackets $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ stand for commutators and anti-commutators, and we have defined,

$$\check{G}(\epsilon,\mathbf{r},t,\mathbf{p}) = \int dt \int d^3\mathbf{r}\check{G}(\mathbf{r_1},\mathbf{r_2},t_1,t_2)e^{-i\mathbf{p}\cdot\tilde{\mathbf{r}}+i\epsilon\tilde{t}}, \tag{171}$$

$$\hat{H}(\mathbf{r},t,\mathbf{p}) = \frac{-ie}{m}\mathbf{A}(\mathbf{r},t)\cdot\mathbf{p}\hat{\tau}_3 - i\hat{\Delta}(\mathbf{r},t) + \frac{ie^2}{m}\mathbf{A}^2(\mathbf{r},t) + ie\phi_0(\mathbf{r},t). \tag{172}$$

*The diffusion approximation for Gorkov equations*

The self-energy $\check{\Sigma} = \check{\Sigma}_{el} + \check{\Sigma}_{in}$ is a sum of two contributions corresponding to elastic and inelastic scattering respectively. In the case when the total scattering rate $\check{\Sigma}$ is smaller than the characteristic quasi-particle energy $\epsilon_c$, it can be dropped from the equation for the retarded Green's function. In this case the quasi-particle momentum is a good quantum number, and one can use a conventional Boltzmann kinetic equation for quasi-particle distribution function to describe slow superconducting dynamics [2]. In this case, Eq. (34) can be derived straightforwardly by integrating the Boltzmann equation over the direction of the momentum. We will be interested in the opposite limit, where the quasi-particle momentum is not a good quantum number, and

$$\tau_{el}^{-1} > \epsilon_c > \tau_{in}^{-1}. \tag{173}$$

In this case $\check{\Sigma}_{in}$ still can be dropped from the equation for the retarded and advanced Green's functions, however $\check{\Sigma}_{el}$ is the largest term in Eq. (170), and can not be neglected.

An effective approach to describe the quasi-particle dynamics in this limit was developed in Ref. [3]. This method is based on the fact that the elastic part of the self-energy can be expressed in terms of the Green's functions,

$$\check{\Sigma}_{el}(\epsilon, \mathbf{r}, t) = \frac{-1}{2\pi\tau_{el}} \int d^3\mathbf{p} \check{G}(\epsilon, \mathbf{r}, t, \mathbf{p}), \tag{174}$$

and thus $\check{\Sigma}_{el}$ does not depend on $\mathbf{p}$. With this in mind, we can integrate Eq. (170) over $\xi_p = \frac{\mathbf{p}^2}{2m} - \mu$, keeping the direction of the momentum $\mathbf{n} = \mathbf{p}/\mathbf{p}$ fixed. Keeping terms which are leading order in spacial gradients we get,

$$\begin{aligned} &\frac{1}{2}\partial_t\{\hat{\tau}_3, \check{g}(\epsilon, \mathbf{r}, t, \mathbf{n})\} - i\epsilon[\hat{\tau}_3, \check{g}(\epsilon, \mathbf{r}, t, \mathbf{n})] + v_F\mathbf{n}\cdot\boldsymbol{\nabla}\check{g}(\epsilon, \mathbf{r}, \mathbf{n}) \\ &+[\hat{H}(\mathbf{r}, t, p_F\mathbf{n}), \check{g}(\epsilon, \mathbf{r}, t, \mathbf{n})] - \frac{i}{2}\{\partial_t\hat{H}(\mathbf{r}, t, p_F\mathbf{n}), \partial_\epsilon\check{g}(\epsilon, \mathbf{r}, t, \mathbf{n})\} \\ &= -i[\check{\Sigma}_{el}(\epsilon, \mathbf{r}, t), \check{g}(\epsilon, \mathbf{r}, t, \mathbf{n})] - i[\check{\Sigma}_{in}(\epsilon, \mathbf{r}, t), \check{g}(\epsilon, \mathbf{r}, t, \mathbf{n})]. \end{aligned} \tag{175}$$

Here we have defined the quasiclassical Green's function,

$$\check{g}(\epsilon, \mathbf{r}, t, \mathbf{n}) = \frac{i}{\pi}\int d\xi_p \check{G}(\epsilon, \mathbf{r}, t, \mathbf{p}). \tag{176}$$

The factor of $i/\pi$ is included to have the same notation as in Ref. [3].

In the diffusive limit the Greens functions are almost isotropic and we can expand $\check{g}(\epsilon, \mathbf{r}, t, \mathbf{n})$ in the spherical harmonics,

$$\check{g}(\epsilon, \mathbf{r}, t, \mathbf{n}) = \check{g}_0(\epsilon, \mathbf{r}, t) + \check{\mathbf{g}}_\mathbf{1}(\epsilon, \mathbf{r}, t)\cdot\mathbf{n}, \qquad \check{g}_0(\boldsymbol{r}, t_1, t_2) \gg \check{\mathbf{g}}_1(\boldsymbol{r}, t_1, t_2)\cdot\boldsymbol{n}. \tag{177}$$

Enforcing the normalization conditions for $(\check{g}\otimes\check{g})(\mathbf{r}, t_1, t_2, \boldsymbol{n})$ (see for example [61]),

$$(\check{g}\otimes\check{g})(\mathbf{r}, t_1, t_2, \boldsymbol{n}) = \delta(t_1 - t_2), \tag{178}$$

we get the following equations for $\check{g}_0$ and $\check{g}_1$,

$$\check{g}_0(\epsilon, \mathbf{r}, t)\check{g}_0(\epsilon, \mathbf{r}, t) = 1, \qquad \check{\mathbf{g}}_\mathbf{1}(\epsilon, \mathbf{r}, t)\check{g}_0(\epsilon, \mathbf{r}, t) = -\check{g}_0(\epsilon, \mathbf{r}, t)\check{\mathbf{g}}_\mathbf{1}(\epsilon, \mathbf{r}, t). \tag{179}$$

Substituting Eq. (177) into (175), using Eq. (A8a) and the fact that $\check{\Sigma}_{el} = \frac{-i}{2\tau_{el}}\check{g}_0$, in the linear in spacial gradients approximation we get

$$\check{\mathbf{g}}_\mathbf{1}(\epsilon, \mathbf{r}, t) = -\frac{3D}{v_F}\left(\check{g}_0(\epsilon, \mathbf{r}, t)\boldsymbol{\partial}_\mathbf{r}\check{g}_0(\epsilon, \mathbf{r}, t) + \frac{e}{2}\partial_t\mathbf{A}(\mathbf{r}, t)\check{g}_0(\epsilon, \mathbf{r}, t)\{\hat{\tau}_3, \partial_\epsilon\check{g}_0(\epsilon, \mathbf{r}, t)\}\right). \tag{180}$$

Here $\boldsymbol{\partial}_{\mathbf{r}} = \boldsymbol{\nabla} - ie\mathbf{A}(\mathbf{r},t)[\hat{\tau}_3,\cdot]$ is the covariant derivative. Substituting Eqs. (177), (180) into (175), and averaging the result over direction of $\mathbf{n}$, we get an equation for the isotropic part of the Green's functions $\check{g}_0$,

$$\begin{aligned}\frac{1}{2}\partial_t\{\hat{\tau}_3,\check{g}_0(\epsilon,\mathbf{r},t)\} - i\epsilon[\hat{\tau}_3,\check{g}_0(\epsilon,\mathbf{r},t)] - D\boldsymbol{\partial}_{\mathbf{r}}\cdot\Big(\check{g}_0(\epsilon,\mathbf{r},t)\boldsymbol{\partial}_{\mathbf{r}}\check{g}_0(\epsilon,\mathbf{r},t)\Big)\\ +\frac{eD}{2}\partial_t\mathbf{A}(\mathbf{r},t)\partial_\epsilon\{\hat{\tau}_3,\check{g}_0(\epsilon,\mathbf{r},t)\boldsymbol{\partial}_{\mathbf{r}}\check{g}_0(\epsilon,\mathbf{r},t)\} - i[\hat{\Delta}(\mathbf{r},t),\check{g}_0(\epsilon,\mathbf{r},t)]\\ -\frac{1}{2}\{\partial_t\hat{\Delta}(\mathbf{r},t),\partial_\epsilon\check{g}_0(\epsilon,\mathbf{r},t)\} + e\partial_t\phi_0(\mathbf{r},t)\partial_\epsilon\check{g}_0(\epsilon,\mathbf{r},t)\\ = -i[\check{\Sigma}_{in}(\epsilon,\mathbf{r},t),\check{g}_0(\epsilon,\mathbf{r},t)].\end{aligned} \tag{181}$$

*Usadel's equations*

In most cases $\hat{g}_0^{R,A}$ relaxes on a time scale which is on the order of $|\Delta|^{-1}$. When the characteristic frequency of the fields is small compared to $|\Delta|$ and $\hat{g}_0^{R,A}$ can be calculated in the adiabatic approximation. In this approximation, time derivatives are dropped in the diagonal components of Eq. (181) and we get Usadel's equations [5] in matrix form

$$i\epsilon[\hat{\tau}_3,\hat{g}_0^R(\epsilon,\mathbf{r},t)] + D\boldsymbol{\partial}_{\mathbf{r}}\cdot\Big(\hat{g}_0^R(\epsilon,\mathbf{r},t)\boldsymbol{\partial}_{\mathbf{r}}\hat{g}_0^R(\epsilon,\mathbf{r},t)\Big) + i[\hat{\Delta}(\mathbf{r},t),\hat{g}_0^R(\epsilon,\mathbf{r},t)] = 0. \tag{182}$$

The local density can be expressed in terms of $\hat{g}_0^R(\epsilon,\mathbf{r},t)$ using the identity,

$$\tilde{\nu}(\epsilon,\mathbf{r},t) = \frac{\tilde{\nu}_N}{2}\mathrm{Re}\bigg(\int d^3\mathbf{r}\mathrm{Tr}\big(\hat{\tau}_3\hat{g}_0^R(\epsilon,\mathbf{r},t)\big)\bigg). \tag{183}$$

It is often convenient to parameterize the $\hat{g}_0^R$ in the following way,

$$\hat{g}_0^R = \begin{pmatrix}\cos\theta & \sin\theta e^{i\chi}\\ \sin\theta e^{-i\chi} & -\cos\theta\end{pmatrix}, \tag{184}$$

where $\theta = \theta_1 + i\theta_2$ and $\chi = \chi_1 + i\chi_2$ are complex numbers. Substituting this parameterization into Eqs. (182) and (183) yields

$$\boldsymbol{\nabla}\cdot\big(D\boldsymbol{\nabla}\chi\sin^2\theta\big) = 0, \tag{185}$$

$$\frac{1}{2}\boldsymbol{\nabla}\cdot\big(D\boldsymbol{\nabla}\theta\big) + i\epsilon\sin\theta + |\Delta|\cos\theta - \frac{D}{4}\sin 2\theta\big(\boldsymbol{\nabla}\chi\big)^2 = 0, \tag{186}$$

$$\nu(\epsilon,\mathbf{r}) = \tilde{\nu}_N\int d^3\mathbf{r}\Re\cos\theta. \tag{187}$$

### B. The Larkin-Ovchinnikov equations

While $\hat{g}_0^R$ and $\hat{g}_0^A$ can be obtain from Eq. (182), information about the quasiparticle dynamics is stored in the Keldysh components of $\check{g}_0$. In clean superconductors where $\tau_{el}^{-1} \ll \epsilon_c$, the Keldysh Green's functions can be parameterized in terms of the conventional distribution function $n(\mathbf{r}, \mathbf{p}, t)$ and the Keldysh component of the Gorkov equations can be reduced to the Boltzmann equation. While this parameterization is not possible in the diffusive case, we can introduce the matrix $\hat{f}(\mathbf{r}, t_1, t_2, \mathbf{n})$ and parameterize $\hat{g}^K$ in the following way,

$$\hat{g}^K(t_1, t_2, \boldsymbol{n}) = (\hat{g}^R \otimes \hat{f})(\mathbf{r}, t_1, t_2, \boldsymbol{n}) - (\hat{f} \otimes \hat{g}^A)(\mathbf{r}, t_1, t_2, \boldsymbol{n}). \tag{188}$$

This is possible because the normalization condition Eq. (178) is satisfied for any choice of $\hat{f}$. Substituting Eq. (188) into the off diagonal component of Eq. (175) yields only two linearly independent equations for $\hat{f}$. This means that $\hat{f}$ characterized by two parameters, and we can choose it to be diagonal. Introducing the two generalized distribution functions $f$ and $f_1$, we can express $\hat{f}$ in the form

$$\hat{f}(\mathbf{r}, t_1, t_2, \boldsymbol{n}) = f(\mathbf{r}, t_1, t_2, \boldsymbol{n}) + \hat{\tau}_3 f_1(\mathbf{r}, t_1, t_2, \boldsymbol{n}). \tag{189}$$

To obtain $\hat{g}_0^K(\epsilon, \mathbf{r}, t)$ in the diffusive limit, we substitute Eq. (189) into Eq. (188) and Fourier transform with respect to the relative time difference. Dropping the dependence on $\mathbf{n}$ and as well as terms which are first order in time derivatives[10], we get

$$\hat{g}_0^K(\epsilon, \mathbf{r}, t) = 2f(\epsilon, \mathbf{r}, t)\hat{\delta}(\epsilon, \mathbf{r}, t) + 2f_1(\epsilon, \mathbf{r}, t)\hat{\alpha}(\epsilon, \mathbf{r}, t), \tag{190}$$

where we have defined,

$$2\hat{\alpha}(\epsilon, \mathbf{r}, t) = \hat{g}_0^R(\epsilon, \mathbf{r}, t)\hat{\tau}_3 - \hat{\tau}_3\hat{g}_0^A(\epsilon, \mathbf{r}, t), \tag{191}$$

$$2\hat{\delta}(\epsilon, \mathbf{r}, t) = \hat{g}_0^R(\epsilon, \mathbf{r}, t) - \hat{g}_0^A(\epsilon, \mathbf{r}, t). \tag{192}$$

To obtain the kinetic equations for $f$ and $f_1$, we substitute Eq. (190) into the Keldysh component of Eq. (181). The first equation is obtained by taking the trace of the resulting equation in Nambu space, and the second equation is obtained by multiplying by $\hat{\tau}_3$ before taking the trace. Although it is not necessary, it is convenient at this point to pick a gauge

[10] Keeping higher order terms will only result in terms which are 2nd order in electric field

where $\phi(\mathbf{r},t)=0$ [11]. Keeping terms which are first order in derivatives of $\mathbf{A}$, dropping terms which contain time derivatives of $|\Delta|$, and using Eq. (182) to simplify expressions involving $\hat{g}^{R,A}$ when possible, we arrive at the Larkin-Ovchinnikov equations

$$\tilde{\nu}(\epsilon,\mathbf{r},t)\partial_t f(\epsilon,\mathbf{r},t)-\frac{1}{4}eD\tilde{\nu}_N\partial_\epsilon f(\epsilon,\mathbf{r},t)\Big(\mathbf{E}\cdot\mathbf{j}_\epsilon(\mathbf{r},t)+4i\Phi(\mathbf{r},t)\mathrm{Tr}\big\{\hat{\tau}_3\hat{\Delta}(\mathbf{r},t)\hat{\delta}(\epsilon,\mathbf{r},t)\big\}\Big)$$
$$-\frac{1}{4}D\tilde{\nu}_N\boldsymbol{\nabla}\cdot\big(\Pi_1(\epsilon,\mathbf{r},t)\boldsymbol{\nabla}f(\epsilon,\mathbf{r},t)\big)-\frac{1}{4}D\tilde{\nu}_N\mathbf{j}_\epsilon(\mathbf{r},t)\cdot\boldsymbol{\nabla}f_1(\epsilon,\mathbf{r},t)=\tilde{I}_1\{f\},\quad(193)$$

$$\partial_t\big(f_1(\epsilon,\mathbf{r},t)\tilde{\nu}(\epsilon,\mathbf{r},t)\big)-\frac{1}{4}D\tilde{\nu}_N\boldsymbol{\nabla}\cdot\big(\Pi_2(\epsilon,\mathbf{r},t)\boldsymbol{\nabla}f_1(\epsilon,\mathbf{r},t)\big)-D\tilde{\nu}_N\mathbf{j}_\epsilon(\mathbf{r},t)\cdot\boldsymbol{\nabla}f(\epsilon,\mathbf{r},t)$$
$$-\frac{i\tilde{\nu}_N}{2}f_1(\epsilon,\mathbf{r},t)\mathrm{Tr}\big\{\hat{\gamma}\hat{\Delta}(\epsilon,\mathbf{r},t)\big\}-i\tilde{\nu}_N\Phi(\mathbf{r},t)\partial_\epsilon f(\epsilon,\mathbf{r},t)\mathrm{Tr}\big\{\hat{\tau}_3\hat{\Delta}(\epsilon,\mathbf{r},t)\hat{\gamma}\big\}=\tilde{I}_2\{f_1\}.\quad(194)$$

Here $D=\frac{v_F^2\tau_{el}}{3}$ is the diffusion coefficient of the normal metal, Tr denotes a trace in Nambu space, and we have defined

$$\mathbf{j}_\epsilon=\mathrm{Tr}\big\{\hat{\tau}_3\big(\hat{g}_0^R\boldsymbol{\partial}_\mathbf{r}g_0^R-\hat{g}_0^A\boldsymbol{\partial}_\mathbf{r}g_0^A\big)\big\},\quad(195)$$

$$\Pi_1=\mathrm{Tr}\{1-\hat{g}_0^A\hat{g}_0^R\}=2+2|g_0^R|^2-2|F_0^R|^2,\quad(196)$$

$$\Pi_2=\mathrm{Tr}\{1-\hat{\tau}_3\hat{g}_0^A\hat{\tau}_3\hat{g}_0^R\}=2+2|g_0^R|^2+2|F_0^R|^2,\quad(197)$$

$$\hat{\gamma}=\frac{1}{2}(\hat{g}_0^R+\hat{g}_0^A)=\frac{1}{2}\begin{pmatrix}g_0^R-(g_0^R)^* & F_0^R-(F^{R+})^*\\ -F_0^{R+}+(F_0^R)^* & (g_0^R)^*-g_0^R,\end{pmatrix}\quad(198)$$

The inelastic scattering integrals $\tilde{I}_1\{f\}$ and $\tilde{I}_2\{f_1\}$ vanishes in equillibrium, when $f=f^{(eq)}=\tanh(\epsilon/2T)$ and $f_1=0$. The characteristic time scale associated with both $I_1\{f\}$ and $I_2\{f_1\}$ is of the order of $\tau_{in}$.

By substituting Eq. (190) into Eq. (167) , the self consistency conditions can be expressed in the form

$$\hat{\Delta}(\mathbf{r},t)=\lambda\int_{-\infty}^{\infty}d\epsilon\big[f(\epsilon,\mathbf{r},t)(F(\epsilon,\mathbf{r},t)-(F^+)^*(\epsilon,\mathbf{r},t))-f_1(\epsilon,\mathbf{r},t)(F(\epsilon,\mathbf{r},t)+(F^+)^*(\epsilon,\mathbf{r},t))\big]\quad(199)$$

$$\Phi(\mathbf{r},t)=\frac{1}{\tilde{\nu}_N}\int_{-\infty}^{\infty}d\epsilon\nu(\epsilon,\mathbf{r},t)f_1(\epsilon,\mathbf{r},t).\quad(200)$$

The distribution function $f_1$ is related to charge imbalance generated by an unequal population of electron and hole like quasiparticles. Note that if $f$ is uniform, the solution to Eq. (194) is given by $f_1=0$, which reflects the fact that the charge valance vanishes in the uniform case.

[11] In Ref. [3] the authors choose to write the equations for $f$ and $f_1$ without choosing a particular gauge. However, the resulting equations were quite complicated and they were not written in terms of gauge invariant quantities. For the sake of simplicity I will work in the gauge $\phi(\mathbf{r},t)=0$ in this section

*The quasi-uniform approximation for the LO equations*

Although Eqs. (193) and (194) are still quite complicated, in many cases several terms in the equations can be dropped. In particular, in problems where the charge imbalance generated by the external fields is small, $f_1$ relaxes on a time scale which is much smaller than the relaxation time of $f$. In such cases, Eq. (193) can be treated in the quasi-uniform approximation, where terms which contain $f_1$ are dropped while terms containing gradients of $f$ are retained,

$$\tilde{\nu}(\epsilon,\mathbf{r},t)\partial_t f(\epsilon,\mathbf{r},t) - \frac{1}{4}eD\tilde{\nu}_N\partial_\epsilon f(\epsilon,\mathbf{r},t)\mathbf{E}\cdot\mathbf{j}_\epsilon(\mathbf{r},t) \\ -\frac{1}{4}D\tilde{\nu}_N\boldsymbol{\nabla}\cdot\big(\Pi_1(\epsilon,\mathbf{r},t)\boldsymbol{\nabla}f(\epsilon,\mathbf{r},t)\big) = \tilde{I}_1\{f\}. \tag{201}$$

Dividing the equation by $\tilde{\nu}(\epsilon,\mathbf{r},t)$ we have,

$$\partial_t f(\epsilon,\mathbf{r},t) - \frac{eD\tilde{\nu}_N}{4\tilde{\nu}(\epsilon,\mathbf{r},t)}\partial_\epsilon f(\epsilon,\mathbf{r},t)\mathbf{E}\cdot\mathbf{j}_\epsilon(\mathbf{r},t) - \boldsymbol{\nabla}\cdot\big(D^*(\epsilon,\mathbf{r})\boldsymbol{\nabla}f(\epsilon,\mathbf{r},t)\big) = I_1\{f\}, \tag{202}$$

where we have defined $I_1 = \tilde{I}_1/\tilde{\nu}$ and,

$$D^*(\epsilon,\mathbf{r},t) = \frac{D}{4}\left(\frac{\tilde{\nu}_N}{\tilde{\nu}(\epsilon,\mathbf{r},t)}\right)\Pi_1(\epsilon,\mathbf{r},t) \\ = \frac{D}{4}\left(\frac{\cos\theta_1}{\cosh\theta_2}\right), \tag{203}$$

where $\theta_{1,2}$ is given by the solution to Eqs. (185).

Here we note that Eq. (202) resembles Eq. (34), the kinetic equation presented in Sec. II. Although the second term in Eq. (202) resembles the spectral flow term, the parameter $\mathbf{j}_\epsilon$ is expressed in terms of the $\hat{g}_0^{R,A}$ and cannot be generally expressed in terms of the local density of states $\tilde{\nu}$. In the next section we will show that this is possible in the local approximation, where Eq. (202) can be reduced to Eq. (34).

### C. Expressing spectral flow in terms of the density of states

Although the derivation of the Larkin-Ovchinnikov equations is quite technically challenging, Eqs. (193) and (194) were derived several decades ago and the calculations in the previous sections of this chapter are not original. However to complete the derivation of Eq.(34), we will show that in the local approximation $\mathbf{j}_\epsilon$ can be expressed in terms of the density of states,

which is a new result. To do this, we use Eq. (180) and (195) to express $\mathbf{j}_\epsilon$ in the following form,

$$\mathbf{j}_\epsilon = \frac{2}{m\pi D}\mathrm{Im}\bigg(\int d^3\mathbf{p}\mathrm{Tr}\Big\{\hat{\tau}_3\hat{G}^R(\epsilon,\mathbf{r},t,\mathbf{p})\Big\}\mathbf{p}\bigg), \tag{204}$$

where $\hat{G}^R$ is the retarded Green's function (not the quasi-classical Green's function). Differentiating of both sides of Eq. (204) with respect to $\epsilon$ we get,

$$\partial_\epsilon\mathbf{j}_\epsilon(t) = \frac{2}{\pi D}\partial_\epsilon\mathrm{Im}\bigg(\int d^3\mathbf{p}\mathrm{Tr}\Big\{\hat{G}^R(\epsilon,\mathbf{r},t,\mathbf{p})\frac{\mathbf{p}}{m}\hat{\tau}_3\Big\}\bigg). \tag{205}$$

In the local approximation the Hamiltonian is a local function of the $\mathbf{A}$, and we can use the fact that,

$$\frac{d\hat{H}}{d\mathbf{A}} = -\frac{ie\mathbf{p}}{m}\hat{\tau}_3 + \frac{2ie^2}{m}\mathbf{A}. \tag{206}$$

Using Eq. (206) as well as the fact that that $\mathbf{p} \ll e\mathbf{A}$ in the quasi-classical approximation we can write Eq. (205) in the form

$$\partial_\epsilon\mathbf{j}_\epsilon = \frac{2i}{e\pi D}\partial_\epsilon\mathrm{Im}\bigg(\int d^3\mathbf{p}\mathrm{Tr}\Big\{\hat{G}^R(\epsilon,\mathbf{r},t,\mathbf{p})\frac{d\hat{H}}{d\mathbf{A}}\Big\}\bigg). \tag{207}$$

To proceed further, we need to derive the following identity relating derivatives of the Green's functions.

$$\int d^3\mathbf{p}\mathrm{Tr}\Big\{\hat{\tau}_3\frac{d\hat{G}}{d\lambda}\Big\} = i\partial_\epsilon\int d^3\mathbf{p}\mathrm{Tr}\Big\{\hat{G}\frac{d\hat{H}}{d\lambda}\Big\}. \tag{208}$$

In order to derive this identity, first consider a Hamiltonian and corresponding Green's function with some parametric dependence on $\lambda$,

$$\hat{\mathcal{G}}(\epsilon,\lambda) = \frac{1}{i\epsilon\hat{\tau}_3 - \hat{\mathcal{H}}(\lambda)}. \tag{209}$$

Here $\hat{\mathcal{H}}$ is the Hamiltonian with a particular impurity potential, and $\hat{\mathcal{G}}$ is the exact Green's function of this Hamiltonian. Calculating the mixed derivatives of the spectral determinant by performing the derivatives $\partial_\epsilon$ and $\partial_\lambda$ in opposite orders, we have the following relations,

$$\partial_\lambda\partial_\epsilon\int d^3\mathbf{p}\mathrm{Tr}\Big(\ln\big(\hat{\mathcal{G}}^{-1}\big)\Big) = \partial_\epsilon\partial_\lambda\int d^3\mathbf{p}\mathrm{Tr}\Big(\ln\big(\hat{\mathcal{G}}^{-1}\big)\Big), \tag{210}$$

$$\partial_\lambda\int d^3\mathbf{p}\mathrm{Tr}\big(\hat{\mathcal{G}}\partial_\epsilon\hat{\mathcal{G}}^{-1}\big) = \partial_\epsilon\int d^3\mathbf{p}\mathrm{Tr}\Big(\hat{\mathcal{G}}\partial_\lambda\hat{\mathcal{G}}^{-1}\Big), \tag{211}$$

$$\partial_\lambda\int d^3\mathbf{p}\mathrm{Tr}\big(\hat{\tau}_3\hat{\mathcal{G}}\big) = i\partial_\epsilon\int d^3\mathbf{p}\mathrm{Tr}\big(\hat{\mathcal{G}}\partial_\lambda\hat{\mathcal{H}}\big). \tag{212}$$

Next we average Eq. (212) over impurity configurations. In the case where $\partial_\lambda \hat{H}$ is independent of the impurity potential, we have equation (208). Using the Eqs. (207) and (208), in the case of $\lambda \equiv \mathbf{A}$, we have,

$$\partial_\epsilon \mathbf{j}_\epsilon = \frac{2}{e\pi D}\mathrm{Im}\left( \int d^3\mathbf{p}\mathrm{Tr}\left\{ \hat{\tau}_3 \frac{d\hat{G}}{d\mathbf{A}} \right\} \right) = \frac{4}{eD\tilde{\nu}_N}\frac{d\tilde{\nu}}{d\mathbf{A}}. \tag{213}$$

Integrating Eq. (213) with respect to $\epsilon$ and using the fact that $\mathbf{p_s} = e\mathbf{A}$ in our chosen gauge, we arrive at the following gauge invariant expression for $\mathbf{j}_\epsilon$,

$$\mathbf{j}_\epsilon(\mathbf{r},t) = \frac{-4}{D\tilde{\nu}_N}\int_0^\epsilon d\tilde{\epsilon}\frac{d\tilde{\nu}(\tilde{\epsilon},\mathbf{p_s}(\mathbf{r}),t)}{d\mathbf{p_s}} = \frac{4\tilde{\nu}(\epsilon,\mathbf{r},t)}{D\tilde{\nu}_N}\tilde{\mathbf{V}}_\nu(\epsilon,\mathbf{r},t), \tag{214}$$

where $\tilde{\mathbf{V}}_\nu(\epsilon,\mathbf{r},t)$ is defined by Eq. (36). Substituting this into Eqs. (202) we obtain an equation for $f$ which is expressed entirely in terms of gauge invariant parameters.

$$\partial_t f(\epsilon,\mathbf{r},t) + \dot{\mathbf{p}}_\mathbf{s}(\mathbf{r},t)\cdot\tilde{\mathbf{V}}_\nu(\epsilon,\mathbf{r},t)\partial_\epsilon f(\epsilon,\mathbf{r},t) - \boldsymbol{\nabla}\left( D^*(\epsilon,\mathbf{r},t)\boldsymbol{\nabla} f(\epsilon,\mathbf{r},t) \right) = I_1\{f\} \tag{215}$$

Finally, we can express $f(\epsilon,\mathbf{r},t)$ in terms of the conventional distribution function $n(\epsilon,\mathbf{r},t)$ ,

$$f(\epsilon,\mathbf{r},t) = 1 - 2n(\epsilon,\mathbf{r},t). \tag{216}$$

Substituting this into Eq. (215), we obtain Eq. (34).

### D. Kinetic equation in diffusive SNS junctions

In the case of diffusive SNS junctions, the characteristic relaxation rate associated with spatial relaxation of $f$ is given by the Thouless energy $\tau_D^{-1} = E_T$. If the voltage across the junction is small compared to the Thouless energy $eU \ll E_T$, we can work in the approximation where distribution function is uniform within the junction. With this in mind, we multiply Eq. (215) by the local density of states $\tilde{\nu}(\epsilon,\mathbf{r},t)$ and integrate over the junction to get,

$$\nu(\epsilon,t)\partial_t f(\epsilon,t) + \frac{e\tilde{\nu}_N D}{4}\partial_\epsilon f(\epsilon,t)\int d^3\mathbf{r}\mathbf{E}(\mathbf{r},t)\cdot\mathbf{j}_\epsilon(\mathbf{r},t) = \nu(\epsilon,t)\tilde{I}_1\{f\}. \tag{217}$$

Here $\nu(\epsilon,t) = \int d^3\mathbf{r}\tilde{\nu}(\epsilon,\mathbf{r},t)$ is the total density of states in the junction.

For the diffusive junctions considered in this work (See Fig. 7), the width of the junction is much larger than the length of the junction and the dependence of the Green's functions on the directions perpendicular to the junction can be neglected. In this case $\mathbf{j}_\epsilon(x)$ points in the x-direction (which is chosen to be parallel to the junction) and depends only on the

x coordinate. Choosing a gauge where $\mathbf{E}(x) = -\boldsymbol{\nabla}\phi_0(x,t)$ and integrating the 2nd term by parts, we get

$$\nu(\epsilon,t)\partial_t f(\epsilon,t) + \frac{e\tilde{\nu}_N SD}{4}\partial_\epsilon f(\epsilon,t)\Big( -\phi_0(x,t)j_\epsilon(x,t)\Big|_{-L/2}^{L/2} + \int dx\phi_0(x,t)\partial_x j_\epsilon(x,t)\Big) = \nu(\epsilon,t)\tilde{I}_1\{f\}, \quad (218)$$

where $S$ is the cross sectional area of the junction.

To proceed we must return to Usadel's equations for $\hat{g}_0^R$ and use the fact that these equations have a first integral. In the normal region of the junction where $|\Delta| = 0$, Usadel's equation is given by

$$i\epsilon\big[\hat{\tau}_3, \hat{g}_0^R(\epsilon,\mathbf{r},t)\big] + D\boldsymbol{\partial}_\mathbf{r}\cdot\Big(\hat{g}_0^R(\epsilon,\mathbf{r},t)\boldsymbol{\partial}_\mathbf{r}\hat{g}_0^R(\epsilon,\mathbf{r},t)\Big) = 0. \quad (219)$$

Multiplying this equation by $\hat{\tau}_3$ , taking the trace in Nambu space, and using the definition for $\mathbf{j}_\epsilon$ given in Eq. (195) , we arrive at the following identity,

$$\boldsymbol{\nabla}\cdot\mathbf{j}_\epsilon(\mathbf{r}) = 0, \quad (220)$$

which in this case means that $j_\epsilon$ is constant along the junction. With this in mind Eq. (221) can be reduced to the form,

$$\partial_t f(\epsilon,t) - \frac{e\tilde{\nu}_N DS}{4\nu(\epsilon,t)}\partial_\epsilon f(\epsilon,t)U(t)j_\epsilon(t) = \tilde{I}_1\{f\}. \quad (221)$$

Here $U(t) = \phi_0\left(\frac{L}{2},t\right) - \phi_0\left(\frac{-L}{2},t\right)$ is the voltage across the junction.

The instantaneous values of $\hat{g}_0^R$ can be obtained by solving Usadel's equations Eq. (182) with appropriate boundary conditions, and it can be shown that the density of states in the junction $\nu\left(\epsilon,\phi(t)\right)$ depends only on the phase difference across the junction,

$$\phi(t) = \int_{-L/2}^{L/2} dx p_s(x,t), \quad (222)$$

which evolves in time via Eq. (64). Finally we can use Eq. (214), the expression for $\mathbf{j}_\epsilon(t)$ derived in the previous section (214), to write $j_\epsilon(t)$ in terms of $\nu(\epsilon,\phi(t))$.

$$\begin{aligned} j_\epsilon(t) &= \frac{1}{SL}\int d^3\mathbf{r}\mathbf{j}_\epsilon(t)\cdot\mathbf{x} \\ &= \frac{-4}{SD\tilde{\nu}_N}\int_0^\epsilon d\tilde{\epsilon}\frac{1}{L}\int d^3\mathbf{r}\frac{d\tilde{\nu}(\tilde{\epsilon},p_s(x),t)}{dp_s(x)} \\ &= \frac{-4}{SD\tilde{\nu}_N}\int_0^\epsilon d\tilde{\epsilon}\frac{d\nu(\epsilon,\phi(t))}{d\phi} \\ &= \frac{4\nu(\epsilon,t)}{SD\tilde{\nu}_N}V_\nu(\epsilon,t) \end{aligned} \quad (223)$$

Substituting this back into Eq. (221), we get

$$\nu(\epsilon,t)\partial_t f(\epsilon,t)+\partial_\epsilon f(\epsilon,t)\left(2eU(t)\right)V_\nu(\epsilon,t)=\nu(\epsilon,t)\tilde{I}_1\{f\}. \tag{224}$$

To get the expression for the current, we express density $\mathbf{j}$ in terms of the Keldysh Green's function,

$$\mathbf{j}(\mathbf{r},\mathbf{t})=-\frac{e\tilde{\nu}_N v_F}{4}\int_{-\infty}^{\infty}d\epsilon\int\frac{d\Omega_n}{4\pi}\mathrm{Tr}\{\hat{\tau}_3\hat{g}^K(\epsilon,\mathbf{r},t,\mathbf{n})\}\mathbf{n}. \tag{225}$$

Here $\int\frac{d\Omega_n}{4\pi}$ indicates an integration over the direction of the momentum. Substituting the Keldysh component of Eq. (180) into Eq. (225) and keeping terms which are leading order in the derivatives of the fields, we get

$$\mathbf{j}(\mathbf{r},t)=\frac{eD\tilde{\nu}_N}{4}\int_{-\infty}^{\infty}d\epsilon\mathbf{j}_\epsilon(\mathbf{r},t)f(\epsilon,\mathbf{r},t), \tag{226}$$

Using the expression for $\mathbf{j}_\epsilon$ (223) in Eq. (223), the total current through the junction $J(t)$ is given by,

$$J(t)=2e\int_{-\infty}^{\infty}d\epsilon\nu(\epsilon,t)V_\nu(\epsilon,t)f(\epsilon,t), \tag{227}$$

Substituting Eq. (216) into the kinetic equation and the expressionf or the current, we arrive at Eqs. (65) and (67).

## APPENDIX A: DERIVATION OF USADEL'S EQUATIONS IN DIFFUSIVE JUNCTIONS WITH SPIN-ORBIT COUPLING AND A ZEEMAN FIELD

In this section we derive Usadel's equations for the electron Green's functions in the diffusive approximation in the presence of spin-orbit coupling and a Zeeman field. We then we show that in the special case of quasi-one dimensional geometry the I-U characteristics of the junctions turn out to be reciprocal. Thus, the non-reciprocity of the junctions is related to either a general character of position-dependence of the microscopic parameters describing the normal metal, such as $D(\mathbf{r})$ and $\beta(\mathbf{r})$, or geometry of the normal region.

### Weak spin-orbit coupling

We will first consider the case of weak spin-orbit coupling, where $\beta p_F \ll \tau_{el}^{-1}$ . The Hamiltonian describing a 2D SNS junction in the presence of weak spin-orbit coupling and an in plane magnetic field has the form

$$H=\left(\xi_p+\beta^{ij}\sigma^i p^j+V_{imp}(\mathbf{r})\right)\tau_3+g\mu_0\mathbf{H}\cdot\boldsymbol{\sigma}+\Delta(\mathbf{r})\tau_1. \tag{A1}$$

Here $\xi_p = p^2/2m - E_F$, $\tau_i$ are the Pauli matrices in Nambu space, and $\Delta(\mathbf{r})$ is the superconducting order parameter. We will focus on the case where the length of the normal metal region is much larger than the superconducting coherence length of the nodes $L \gg \xi$, and $\Delta(\mathbf{r})$ has the form

$$\Delta(\mathbf{r}) = \begin{cases} \Delta, & x < -\frac{L}{2}, \\ 0, & -\frac{L}{2} < x < \frac{L}{2}, \\ \Delta e^{-i\chi}, & x > \frac{L}{2}. \end{cases} \tag{A2}$$

We start with the Eilenberger equation [62] corresponding to the Hamiltonian in Eq. (A1)

$$\left[\epsilon \underline{\tau}_3 - \Delta(\mathbf{r})\tau_1 + A_0 \underline{\tau}_3, \underline{g}\right] + i v_F \hat{p}^k \tilde{\nabla}^k_{\mathbf{r}} \underline{g} - \frac{i}{2m}\left\{A_k, \nabla^k_{\mathbf{r}} \underline{g}\right\} = \frac{i}{2\tau_{el}}\left[\underline{g}_0, \underline{g}\right], \qquad \underline{g}^2 = 1. \tag{A3}$$

Here $\underline{g}(\mathbf{r}, \epsilon, \hat{\mathbf{p}})$ is the quasi-classical Green's function, which is a matrix in Nambu-spin space, $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$ denote the anti-commutator and commutator respectively, $\hat{p}$ is the unit vector pointing in the direction of $\vec{p}$, and $g_0$ is the Green's function averaged over the direction of $\hat{p}$. Finally, $\mathbf{h} = g\mu_0 \mathbf{H}$ is the Zeeman energy,

$$A_0 = -h^\alpha \sigma^\alpha, \qquad A_k = -m\beta^{\alpha k}\sigma^\alpha, \tag{A4}$$

and $\tilde{\nabla}^k_{\mathbf{r}} = \nabla^k_{\mathbf{r}} - i[A_k, \cdot]$.

We note that the third Eq. (A3) differs from the Eilenberger equation used in Ref. [63]. The reason for this is because in Ref. [63] the author chooses to perform a unitary transformation of the semiclassical Green's functions which eliminates terms which are linear in momentum and spin-orbit coupling, we choose not to do this. The density of sates in the normal region can be written in terms of the Green's function in the form,

$$\nu(\epsilon) = \frac{\nu_N}{4L_1 L}\int d\mathbf{r} \Re\left[\mathrm{Tr}_{\tau,\sigma}\left(\tau_3 \underline{g}_0(\epsilon, \mathbf{r})\right)\right]. \tag{A5}$$

Here $\nu_N$ is the total density of states of the normal metal part of the junction in the absence of the superconductors, $\mathrm{Tr}_{\tau,\sigma}$ denotes the trace over Nambu-spin space, and the integral is taken over the area of the normal region.

In the diffusive regime where the Green's functions are nearly isotropic, we can expand the Green's function into its zeroth and first angular harmonics,

$$\underline{g} = \underline{g}_0 + \underline{g}^k_1 \hat{p}^k, \quad |\underline{\boldsymbol{g}}_1| \ll |\underline{g}_0|. \tag{A6}$$

With these assumptions, the normalization conditions are then

$$\underline{g}_0^2 = 1, \quad \left\{\underline{g}_0, \underline{g}_1^k\right\} = 0. \tag{A7}$$

Substituting Eq. (A6) into Eq. (A3), and using Eq. (A7) we get the equations for $g_0$ and $g_1^k$

$$\left[\epsilon\underline{\tau}_3 - \Delta(\mathbf{r})\tau_1 + A_0\underline{\tau}_3, \underline{g}_0\right] + \frac{iv_F}{2}\tilde{\nabla}_{\mathbf{r}}^k\underline{g}_1^k - \frac{i}{2m}\left\{A_k, \nabla_{\mathbf{r}}^k\underline{g}_0\right\} = 0, \tag{A8a}$$

$$v_F\tilde{\nabla}_{\mathbf{r}}^k\underline{g}_0 - \frac{1}{2\tau_{\mathrm{el}}}\left[g_0, \underline{g}_1^k\right] = 0. \tag{A8b}$$

Multiplying both sides of Eq. (A8b) by $\underline{g}_0$ and using Eq. (A7), we get

$$\underline{g}_1^k = -v_F\tau_{\mathrm{el}}\underline{g}_0\tilde{\nabla}_{\mathbf{r}}^k\underline{g}_0. \tag{A9}$$

Substituting Eq. (A9) back into Eq. (A8a), we arrive at Usadel's equation,

$$\left[\epsilon\underline{\tau}_3 - \Delta(\mathbf{r})\tau_1 + A_0\underline{\tau}_3, \underline{g}_0\right] - i\tilde{\nabla}_{\mathbf{r}}^k\left(D(\mathbf{r})\underline{g}_0\tilde{\nabla}_{\mathbf{r}}^k\underline{g}_0\right) - \frac{i}{2m}\left\{A_k(\mathbf{r}), \nabla_{\mathbf{r}}^k\underline{g}_0\right\} = 0. \tag{A10}$$

To account for long range variations of the microscopic parameters, we allow $D(\mathbf{r})$ and $A_k(\mathbf{r})$ to depend on position. We note that Eq. (A10) is valid provided that $D(\mathbf{r})$ and $A_k(\mathbf{r})$ change slowly on the scale of the spin relaxation length.

Equation (A10) is written in Nambu-spin space. If $h \ll \tau_{so}^{-1}$ , it is useful to decompose to the Green's functions into a singlet and triplet components,

$$\underline{g}_0 = \underline{g}_s + \underline{g}_t^\alpha\sigma^\alpha, \quad |\underline{g}_t^\alpha| \ll |\underline{g}_s|. \tag{A11}$$

Here $\underline{g}_s$, $\underline{g}_t^\alpha$ are matrices in Nambu space. In this approximation, the normalization conditions Eq. A7 yield

$$\underline{g}_s^2 = 1, \quad \left\{\underline{g}_s, \underline{g}_t^\alpha\right\} = 0. \tag{A12}$$

Substituting Eq. (A11) into (A10) and using Eq. (A12), we get

$$\left[\epsilon\underline{\tau}_3 - \Delta(\mathbf{r})\tau_1, \underline{g}_s\right] - i\nabla_{\mathbf{r}}^k\left(D(\mathbf{r})\underline{g}_s\nabla_{\mathbf{r}}^k\underline{g}_s\right) - h^\alpha\left[\underline{\tau}_3, \underline{g}_t^\alpha\right] + i\beta^{\alpha k}(\mathbf{r})\nabla_{\mathbf{r}}^k\underline{g}_t^\alpha = 0, \tag{A13a}$$

$$\begin{aligned}\frac{i}{\tau_{so}}\underline{g}_s\underline{g}_t^\alpha + \left[\epsilon\underline{\tau}_3 - \Delta(\mathbf{r})\tau_1, \underline{g}_t^i\right] - i\nabla_{\mathbf{r}}^k\left(D(\mathbf{r})\underline{g}_s\nabla_{\mathbf{r}}^k\underline{g}_t^i + \underline{g}_t^i\nabla_{\mathbf{r}}^k(D(\mathbf{r})\underline{g}_s)\right)\\ +2imD(\mathbf{r})\epsilon^{i\alpha\beta}\beta^{\alpha k}(\mathbf{r})\left(\underline{g}_s\nabla_{\mathbf{r}}^k\underline{g}_t^\beta + \underline{g}_t^\beta\nabla_{\mathbf{r}}^k\underline{g}_s + \nabla_{\mathbf{r}}^k\left(\underline{g}_s\underline{g}_t^\beta\right)\right) = h^i\left[\underline{\tau}_3, \underline{g}_s\right] - i\beta^{ik}(\mathbf{r})\nabla_{\mathbf{r}}^k\underline{g}_s.\end{aligned} \tag{A13b}$$

Note that we have dropped terms $\mathcal{O}\left(\underline{g}_t^2\right)$, and that only ordinary derivatives are left in the equations. The terms on the RHS of Eq. (A13b) act as source terms for the triplet Green's

functions. Thus, the triplet Green's function is generated either by the Zeeman field or the linear in gradient terms arising from spin orbit coupling.

Since we are interested in solutions for $\epsilon \sim E_T$ and $\Delta(\mathbf{r}) = 0$ in the normal metal, the second term on the LHS of Eqs. (A13a) is on the order of $E_T$. The typical length scale on which the Green's functions change is on the order of $L$ (see Ref. [**?** ]), so the gradients in Eqs. (A13a) and (A13b) are of order $\frac{1}{L}$. With this in mind, we note that the third term on the LHS of Eq. (A13b) is also on the order of $E_T$. In the case $E_T \ll \tau_{so}^{-1}$ the first term in Eq. (A13b) is the largest term on the LHS, and we get

$$\underline{g}_t^{\alpha} = -i\tau_{so}\left(\underline{g}_s\left[h^{\alpha}\underline{\tau}_3, \underline{g}_s\right] - i\beta^{\alpha k}(\mathbf{r})\underline{g}_s\nabla_{\mathbf{r}}^k\underline{g}_s\right). \tag{A14}$$

Finally, substituting Eq. (A14) into Eq. (A13a), we get the Usadel equation for $\underline{g}_s$

$$\begin{aligned}\left[\epsilon\underline{\tau}_3, \underline{g}_s\right] - i\nabla_{\mathbf{r}}^k\left(D(\mathbf{r})\underline{g}_s\nabla_{\mathbf{r}}^k\underline{g}_s\right) + i\tau_{so}h^2\left[\underline{\tau}_3, \underline{g}_s[\underline{\tau}_3, \underline{g}_s]\right] \\ +\tau_{so}\beta^{\alpha k}(\mathbf{r})h^{\alpha}\left(\left[\underline{\tau}_3, \underline{g}_s\nabla_{\mathbf{r}}^k\underline{g}_s\right] + \nabla_{\mathbf{r}}^k\left(\underline{g}_s\left[\underline{\tau}_3, \underline{g}_s\right]\right)\right) = 0.\end{aligned} \tag{A15}$$

In the subsequent equations we only keep terms which are linear in $h$, as we will be interested only in the part of the density of states which is linear in $\mathbf{H}$.

Let us first consider junctions with a 1D geometry, where $D(\mathbf{r})$ as well as $\beta(\mathbf{r})$ depend only on the $x$ coordinate. In this case, Eq. (A15) and Eq. (A5) are given by

$$[\epsilon\tau_3, g_s] - i\partial_x\left(D(x)g_s\partial_x g_s\right) + \tau_{so}\beta^{\alpha x}(xh^{\alpha}\left([\tau_3, g_s\partial_x g_s] + \partial_x\left(g_s\left[\tau_3, g_s\right]\right)\right) = 0, \tag{A16}$$

$$\nu(\epsilon) = \frac{\nu_N}{2L}\int_{-L/2}^{L/2} dx\Re\left[\mathrm{Tr}_{\tau}\left(\tau_3\underline{g}_s(\epsilon, x)\right)\right], \tag{A17}$$

where and $\mathrm{Tr}_{\tau}$ is the trace over Nambu space.

It is convenient to parameterize the Green's functions in the following form,

$$g_s(\epsilon, x) = \begin{pmatrix} \cos\theta(\epsilon, x) & \sin\theta(\epsilon, x)e^{i\tilde{\chi}(\epsilon,x)} \\ \sin\theta(\epsilon, x)e^{-i\tilde{\chi}(\epsilon,x)} & -\cos\theta(\epsilon, x)\end{pmatrix}, \tag{A18}$$

where $\theta(\epsilon, x)$ and $\tilde{\chi}(\epsilon, x)$ are complex variables. In this parametrization the Usadel equation reduces to the following two equations,

$$\partial_x\left[\left(D(x)\partial_x\tilde{\chi} + 2\tau_{so}\beta^{\alpha x}(x)h^{\alpha}\right)\sin^2\theta\right] = 0, \tag{A19}$$

$$\frac{1}{2}\partial_x\left(D(x)\partial_x\theta\right) + i\epsilon\sin\theta - \frac{D(x)}{4}\sin 2\theta\left(\partial_x\tilde{\chi} + \frac{2\tau_{so}\beta^{\alpha x}(x)h^{\alpha}}{D(x)}\right)^2 = 0, \tag{A20}$$

and the expression for the density of states becomes

$$\nu(\epsilon,\chi)=\frac{\nu_N}{L}\int_{-L/2}^{L/2}dx\Re\cos\theta(\epsilon,x). \tag{A21}$$

In the case of perfectly transmitting NS interfaces, the boundary conditions for Eqs. (A19), (A20) are given by

$$\theta(\epsilon,\pm L/2)=\frac{\pi}{2},\quad \tilde{\chi}(\epsilon,\pm L/2)=\pm\frac{\chi}{2}. \tag{A22}$$

We can further simplify these equations by making the change of variables $(\theta,\tilde{\chi})\rightarrow(\theta,\hat{\chi})$, where the shifted phase $\hat{\chi}$ is given by

$$\hat{\chi}=\tilde{\chi}+\int_{-L/2}^{x}dq\frac{2\tau_{so}\beta^{\alpha x}(q)h^{\alpha}}{D(q)}. \tag{A23}$$

In the new variables Eqs. (A19), (A20), and (A22) become

$$\partial_x\left[D(x)\partial_x\hat{\chi}\sin^2\theta\right]=0, \tag{A24}$$

$$\frac{1}{2}\partial_x\Big(D(x)\partial_x\theta\Big)+i\epsilon\sin\theta-\frac{D(x)}{4}\sin 2\theta\big(\partial_x\hat{\chi}\big)^2=0, \tag{A25}$$

$$\theta(\epsilon,\pm L/2)=\frac{\pi}{2},\qquad \tilde{\chi}(\epsilon,\pm L/2)=\pm\Big(\frac{\chi+\phi}{2}\Big). \tag{A26}$$

We note that Eqs. (A24)–(A26) are identical to Usadel's equations and their boundary conditions at $h=0$, with a phase difference of $\chi+\phi$ across the junction. Thus, the solution for $\theta$ has the form

$$\theta(\epsilon,x,\chi)=\theta_0(\epsilon,x,\chi+\phi) \tag{A27}$$

Here $\theta_0(\epsilon,x,\chi)$ is the solution for zero magnetic field and phase difference $\chi$, and $\phi$ is given by Eq. (27). Thus, we see that the phase dependence of the density of states has the form of Eq. (10). As discussed in the main text, in this case the I-U characteristics are reciprocal.

We would like to stress however that this result arose from the 1D character of the idealized model of the the SNS junction. In the general situation, where the parameters $D(\mathbf{r})$ and $\beta(\mathbf{r})$ are functions of $x$ and $y$, or the shape on the N-region is non-rectangular, the density of states $\nu(\chi)$ can not be expressed in the form of Eq. (25), and the I-U characteristics are nonreciprocal.

Consider for example a simple model where the diffusion coefficient is a function of $y$, a coordinate parallel to the SN interface,

$$D(x,y)=\begin{cases}D+\frac{\delta D}{2}, & 0<y<L_1/2,\\ D-\frac{\delta D}{2}, & -L_1/2<y<0,\end{cases} \tag{A28}$$

where $\delta D \ll D$. If $L_1 \gg L$ then the local density of states in the regions with $y > 0$ and $y < 0$ can be approximated $\mathbf{H} = 0$ solutions shifted by phases $\phi_\pm = \frac{2\tau_{so}\beta^{\alpha x}h^\alpha L}{D \pm \delta D/2}$ respectively. Then the total density of states can be written as

$$\nu(\epsilon, \chi, D) \approx \frac{1}{2}\Big(\nu_0(\epsilon, \chi + \phi_+, D + \delta D/2) + \nu_0(\epsilon, \chi + \phi_-, D - \delta D/2)\Big), \tag{A29}$$

where $\nu_0(\epsilon, \chi, D)$ is density of states of a junction with diffusion coefficient $D$ and dimensions $L_1 \times L$, at $\mathbf{H} = 0$. In this case the total density of states cannot be expressed in the form of equation of Eq. (25), and the I-U characteristics are non-reciprocal. Substituting Eq. (A29) into Eq. (17b) in the main text and keeping terms linear in $\mathbf{H}$, we get the following estimate for $\langle g_2 \rangle$,

$$\langle g_2(\chi, \mathbf{H})\rangle \approx \left(\frac{\tau_{so}\beta^{\alpha x}(g\mu_0\mathbf{H}^\alpha)}{LE_T}\right)\left(\frac{\delta D}{D}\right)^2 \frac{E_T^2}{T^2}\langle g_1(\chi, \mathbf{H})\rangle. \tag{A30}$$

Similarly, we substitute Eq. (A29) into Eq. (22) to obtain the following estimate for $\delta A$,

$$\delta A \approx \frac{\sqrt{J_c(0,T)}}{e\tau_{in}\langle g_1\rangle}\frac{\beta(\tau_{so}g\mu_0 H)}{LE_T}\left(\frac{\delta D}{D}\right)^2. \tag{A31}$$

Here we have used the fact that both the critical current and the phase $\chi_m(\mathbf{H})$, at which the supercurrent reaches its maximum, are functions of $\nu(\epsilon, \chi, D)$.

In the more general case where $L_1 \gtrsim L$, the scale of spacial fluctuations of $D(\mathbf{r})$ and $\beta(\mathbf{r})$ are of order $L$, and $\delta D \lesssim D$, we expect these estimates to be accurate to order unity.

### Superconductor-topological insulator-superconductor junction

In this section we consider a model in which the normal region of the junction is comprised of a conducting surface of a topological insulator described by the Hamiltonian

$$\mathcal{H}(k) = \left(\beta^{\alpha k}k^i\sigma^\alpha - \mu + V_{dis}\right)\tau_3 + h^\alpha\sigma^\alpha + \Delta\tau_1. \tag{A32}$$

This model can be viewed a limiting case of strong spin-orbit coupling in which the conduction band has a definite helicity.

The derivation of the Usadel equation in S-TI-S junction was done by Ref. [46]; here we will sketch the main steps. The Eilenberger equation for the Hamiltonian in Eq. (A32) is given by,

$$\begin{aligned}\left[\epsilon\underline{\tau}_3 - \Delta(\mathbf{r})\tau_1 - h^\alpha\sigma^\alpha\underline{\tau}_3, \underline{g}(\boldsymbol{n}, \epsilon, \mathbf{r})\right] + p_F\beta^{\alpha k}\hat{p}^k\left[\sigma^\alpha\underline{g}(\hat{\mathbf{p}}, \epsilon, \boldsymbol{r})\right] \\ + \frac{i}{2}\beta^{\alpha k}\left\{\sigma^\alpha, \nabla^k_{\mathbf{r}}\underline{g}(\hat{\mathbf{p}}, \epsilon, \boldsymbol{R})\right\} = -\frac{i}{2\tau_{\rm el}}\left[\langle\underline{g}\rangle, \underline{g}(\hat{\mathbf{p}}, \epsilon, \boldsymbol{R})\right],\end{aligned} \tag{A33}$$

In the regime of strong spin orbit coupling where the system has complete spin-momentum locking, the Green's function has the following helical spin structure,

$$\underline{g} = \frac{1}{2}\underline{g}' \left(1 + \hat{\beta}^{\alpha k}\hat{p}^k\sigma^\alpha\right), \tag{A34}$$

where $\underline{g}'$ is a matrix only in Nambu space. We then insert Eq. (A34) into Eq. (A33),

$$\left[\epsilon\underline{\tau}_3, -\Delta(\mathbf{r})\tau_1 - h^\alpha\sigma^\alpha, \frac{1}{2}\underline{g}'\left(1 + \hat{\beta}^{\gamma k}\sigma^\gamma\hat{p}^k\right)\right] + ip_F\epsilon^{\alpha\beta\gamma}\beta^{\alpha k}\hat{\beta}^{\beta i}\hat{p}^k\hat{p}^i g'\sigma^\gamma$$
$$+\frac{i\beta^{\alpha k}}{2}\nabla^k_{\mathbf{r}} g'\left(\sigma^\alpha + \hat{\beta}^{\alpha i}\hat{p}^i\right) + \frac{i}{2\tau_{\rm el}}\left[\left\langle\frac{1}{2}\underline{g}'\left(1 + \hat{\beta}^{\alpha k}\hat{p}^k\right)\right\rangle, \frac{1}{2}\underline{g}'\left(1 + \hat{\beta}^{\alpha k}\hat{p}^k\right)\right] = 0. \tag{A35}$$

In the diffusive limit where $v_F\tau_{el} \ll L$, we expand the Green's function into its zeroth and first moment $\underline{g}' = \underline{g}_0 + \underline{g}^k_1\hat{p}^k$, insert the expansion into Eq. (A35), project the equation onto its zeroth and first angular harmonics, and obtain two equations for $\underline{g}_0, \underline{g}_1$. After taking the trace of the resulting equations over spin indices, we arrive at the following equations,

$$\left[\epsilon\underline{\tau}_3 - \Delta(\mathbf{r})\tau_1, \underline{g}_0\right] + i\frac{\beta}{2}\hat{\nabla}^i_{\mathbf{r}}\underline{g}^i_1 = 0, \tag{A36}$$

$$\underline{g}^i_1 = -2\beta\tau_{\rm el}\underline{g}_0\hat{\nabla}^i_{\mathbf{r}}\underline{g}_0, \tag{A37}$$

where we have defined,

$$\hat{\nabla}^i_{\mathbf{r}}\cdot = \nabla^i_{\mathbf{r}}\cdot + i\frac{\hat{\beta}^{\alpha i}h^\alpha}{\beta}\left[\underline{\tau}_3, \cdot\right]. \tag{A38}$$

Substituting Eq. (A37) in (A36), we have Usadel's equation for the strong spin orbit case,

$$\left[\epsilon\underline{\tau}_3 - \Delta(\mathbf{r})\tau_1, \underline{g}_0\right] - i\nabla^i_{\mathbf{r}}\left(D(\mathbf{r})\underline{g}_0\nabla^i_{\mathbf{r}}\underline{g}_0\right)$$
$$+\tau_{el}\beta^{\alpha i}(\mathbf{r})h^\alpha\left(\nabla^i_{\mathbf{r}}\left(\underline{g}_0\left[\underline{\tau}_3, \underline{g}_0\right]\right) + \left[\underline{\tau}_3, \underline{g}_0\nabla^i_{\mathbf{r}}\underline{g}_0\right]\right) + i\tau_{el}h^2\left[\underline{\tau}_3, \underline{g}_0\left[\underline{\tau}_3, \underline{g}_0\right]\right] = 0, \tag{A39}$$

which is the same as Eq. (A15), the equation for $g_0$ in the case of weak spin-orbit coupling, with the replacement of $\tau_{so}$ with $\tau_{el}$ . This can be understood by the fact that in the case where spin-momentum locking is strong, spin-relaxation is limited only by the rate of elastic scattering. This suggests that the form of Eq. (A39) is universal within the diffusive regime, where the parameters in the equation depend on the strength of the spin-orbit coupling. As a result, we can apply the estimates for $\langle g_2\rangle$ and $\delta A$ given in Eqs. (A30) and (A31) to the case of a S-TI-S junction with the appropriate substitution of $\tau_{so} = \tau_{\rm el}$.

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