Statistics of the Minimal Denominator Function and Short Holonomy Vectors of Translation Surfaces

Albert Alejandro Artiles Calix

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Reading Committee:

Jayadev Athreya, Chair

Douglas Lind

Sara Billey

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Abstract

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Chair of the Supervisory Committee:
Jayadev S. Athreya

This dissertation addresses topics in dynamical systems, number theory, and the geometry of surfaces. It contains an introduction and three subsequent chapters.

The introduction 1 contains information about how the work presented in chapters 3 and 4 fits in with the past and current literature in the fields of Diophantine approximations and translation surfaces.

Chapter 2 contains mostly classical results and information that allows the reader to understand the contents of the subsequent chapters. It contains information about ergodic theory, Diophantine approximation, unimodular lattices, and translation surfaces.

Chapter 3 discusses the minimal denominator function in detail as defined by Meiss-Sander in [39]. We compute the limiting distribution of the minimal denominator function and also generalize it to the contexts of higher dimensional Diophantine approximation, linear forms, holonomy vectors of translation surfaces, and Heegner fields through the usage of machinery from the theory of equivariant processes.

Chapter 4 focuses on making the distribution of holonomy vectors of translation surfaces easier to compute by deriving a probability density function. This is done by combining the growing literature on the statistics of saddle connections with the theory of void and gap distributions as described by Marklof in [28].

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Chapter 1

INTRODUCTION

Over the past few decades, we have seen an increased interest in the connections between dynamical systems, number theory, and geometry. One celebrated result due to Margulis is the proof of the Oppenheim Conjecture in [26] using methods from ergodic theory and homogeneous dynamics to answer a question about quadratic forms. In the 1990s, Ratner published a series of papers that dealt with the connection between the dynamics, topology, and geometry of unipotent flows. These theorems have become classical at this point. A good reference for these theorems is the book by Morris [32], which summarized the main results of Ratner's vast work. More recently, Athreya-Ghosh [4] used techniques from homogeneous dynamics and unipotent flows to answer a question of Erdös-Szüsz-Turán from the 1958 paper [15] about the distribution of the number of solutions to an inequality describing different approximations to real numbers.

While studying the dynamics of circle rotations in 2020, Meiss-Sander came across the following question: Given a real number x and a tolerance $\delta > 0$, how complicated must a rational number be in order to be a solution to the inequality $\left|x - \frac{p}{q}\right| < \delta$? The original question can be found in section 3 of [39]. This inquiry gave rise to the formulation of the minimal denominator function or MDF for short:

$$\tilde{q}_{\min}(x, \delta) = \left\{q \in \mathbb{N} : \text{ there exists } p \in \mathbb{Z} \text{ such that } \left|x - \frac{p}{q}\right| < \delta\right\}, \tag{1.0.1}$$

where $x \in [0, 1]$ and δ is a positive tolerance. This dissertation provides a probabilistic answer to the question asked by Meiss-Sander. We computed the distribution of the MDF in Theorem 3.1.5 in terms of subsets of the space of unimodular lattices. The theorem was proved by translating the Meiss-Sander number theory question to a question about equidistribution along long closed horocycles in the space of unimodular lattices. This connection then inspired the formulation of Theorem 3.2.6, which allows us to turn weak* convergence results into equidistribution results. In particular, given $t > 0$, Theorem 3.2.6

allows us to turn the counting problem: *how often does one expect to see t intersections between a lattice and a random subset of* $\mathbb{R}^n$, to a question about weak* convergence of measures. This theorem fits in with the recent work by Athreya-Ghosh [4] and Marklof-Strombergsson [30] of using equivariant processes to answer questions dealing with the statistics of discrete subsets of $\mathbb{R}^n$. Recently, Kim [22] showed that the probability of having k intersections between a centrally symmetric Borel set and a lattice in $\mathbb{R}^n$ can be approximated by a Poisson distribution when n is large. Theorem 3.2.6 does not require the test sets to be centrally symmetric and does not require the value of n to be large, however, due to its generality, we are unable to write the limiting distribution in terms of well-known distributions in contrast to Kim's work.

Theorem 3.1.5 also provides information about the distribution of the Farey Sequence. This theorem describes the hit time between a random interval of length 2δ in $[0, 1]$ and the Farey Sequence. More information about this relation can be found in §2.3.2. We are not the first to explore the statistics of the Farey Sequence. Hall [19] computed the distributions of gaps between consecutive pairs of elements in the Farey Sequence using analytic number theory techniques. More recently, Athreya-Cheung recovered Hall's theorem using techniques from homogeneous dynamics. Boca-Zaharescu have extensively studied different statistics of the Farey Sequence in [8].

Theorem 3.2.6 has applications outside of number theory. Theorem 3.3.21 provides a probabilistic description of the length of the shortest holonomy vector of a translation surface lying in a randomly selected thin cone. This theorem is directly connected to the work of Masur [31] and of Veech [43] from the 1990s. Their results on the asymptotics of the number of saddle connections of translation surfaces and $SL(2, \mathbb{R})$ ergodic measures on Borel spaces pave the way for a lot of the current work in the field today. This includes the work of Athreya-Chaika [1] on the distribution of slope gaps on translation surfaces, the computation and implementation of algorithms to provide density functions for the slope gaps of Veech surfaces ([42],[25]), and the equidistribution of saddle connections of translation surfaces to the Lebesgue measure on the surface by Dozier [12].

Chapter 4 of this dissertation provides a computational improvement to Theorem 3.3.21 by describing its limiting distribution. This is encapsulated in Theorem 4.2.5 which combines the work of Marklof, [27],[28],[29], and the work of many authors, such as Athreya-Chaika [1], Athreya-Chaika-Lelièvre [2], Uyanik-Work [42], and Kumanduri-Sanchez-Wang [25] to provide explicit density functions for the distribution of short holonomy vectors of Veech surfaces. This links the results from §3.3.4 to the question about finding the slope gap distribution of translation surfaces through the connection between the void and gap distributions of sequences in $[0, 1]$.

Chapter 2
BACKGROUND

2.1 Dynamical Systems

Dynamical systems model the evolution of physical phenomena over time. To this end, we provide the following very abstract definition:

Definition 2.1.1. *A* ***dynamical system*** *is an ordered triple* $(X, M, \cdot)$*, where* X *is a set,* M *is a monoid, and* $\cdot : M \times X \to X$ *is a monoid action.*

Dynamical systems come in many different flavors depending on the properties and structures one imposes on the set X and the monoid M. These branches usually go by names described by *adjective* dynamics. Some examples are:

1. Measurable Dynamics: Where X is a measure space and M acts via measurable maps.

2. Topological Dynamics: Where X is a topological space and M acts via continuous maps.

3. Holomorphic Dynamics: Where X is a complex manifold and M acts via holomorphic maps.

4. Smooth Dynamics: Where X is a smooth manifold and M acts via smooth maps.

5. Symbolic Dynamics: Where X is a space of sequences of symbols and M is generated by what we call a shift map.

This dissertation will be concerned mainly with measurable and topological dynamics.

2.1.1 Ergodic Theory

Ergodic theory is the branch of measurable dynamics that is concerned with *measure-preserving transformations*. This section will provide the basic definitions and main theorems of the subject.

Definition 2.1.2. *A **measure-preserving system** is a quadruple $(X, \mathcal{B}, \mu, M)$, where X is a set, $\mathcal{B}$ is a σ-algebra on X and μ is a measure on X. M is a monoid acting on X via measurable maps such that for each $T \in M$ and $A \in \mathcal{B}$,*

$$\mu(T^{-1}(A)) = \mu(A).$$

In the rest of this document, unless otherwise stated, μ will be assumed to be a probability measure.

We will usually drop the σ-algbera from the quadruple $(X, \mathcal{B}, \mu, M)$ and just write (X, μ, M) for convenience. If M is generated by a single element T, then we will write (X, μ, T) to represent (X, μ, M).

One technique used to study dynamical systems, and mathematics in general, is to break or decompose the object of study into easier-to-understand pieces. If one were to take a measure-preserving system (X, μ, M) and found a subset A of X with measure $0 < \mu(A) < 1$ such that $T(A) \subset A$ and $T(X \setminus A) \subset X \setminus A$ for each $T \in M$, then one could break the study of X by looking at the two measure-preserving systems (A, λ_1, M) and $(X \setminus A, \lambda_2, M)$ where $\lambda_1(B) = \frac{1}{\mu(A)}\mu(B)$ and $\lambda_2(C) = \frac{1}{\mu(X \setminus A)}\mu(C)$. To this end, we define an ergodic measure-preserving system as a measure-preserving system that cannot be decomposed further down in this manner. More explicitly, we have:

Definition 2.1.3. *Let (X, μ, M) be a measure-preserving system. We say (X, μ, M) is **ergodic** if whenever $A \subset X$ is such that for all $T \in M$, $T^{-1}A = A$, then $\mu(A) = 0$ or $\mu(A) = 1$.*

Ergodic measure-preserving systems are in a way the basic building blocks when studying measure-preserving systems, similar to simple groups in group theory or irreducible modules in the theory of modules.

Ergodic theory is concerned with the long-term behavior of dynamical systems. A foundational result of the subject is the Pointwise Ergodic Theorem, also known as Birkhoff's Ergodic Theorem, proved by Birkhoff in [7].

Theorem 2.1.4. ***Birkhoff (1931)*** *Suppose that (X, μ, T) is a measure-preserving system and $f : X \to \mathbb{R}$ is an $L^1(\mu)$ function. Then there exists $f^* \in L^1(\mu)$ such that for μ-a.e. $x \in X$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f \circ T^i(x) = f^*(x).$$

Moreover, if (X, μ, T) is ergodic, then $f^*(x)$ is constant for μ-a.e. x and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f \circ T^i(x) = \int_X f d\mu.$$

Many generalizations of Theorem 2.1.4 have been proved throughout the years. Generalizations have been proved for actions of groups such as $\mathbb{Z}^m$ and more generally by amenable groups [14]. As a side note, we also have the Mean Ergodic Theorem proved by Von Neumann before Birkhoff in [34] which works for L^2 functions and provides a geometric definition of the limiting function through projections into an orthogonal space [14].

Equidistribution: One key behavior of orbits in a dynamical system we will be particularly focused on will be equidistribution. The idea of equidistribution is that of a sequence becoming dense within a topological space in a uniform way. More precisely, we have the following definition:

Definition 2.1.5. *Let X be a topological space endowed with a Borel probability measure μ. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ in X* ***equidistributes*** *with respect to μ if for each open subset U of X,*

$$\lim_{N\to\infty} \frac{|\{n \le N : x_n \in U\}|}{N} = \mu(U). \tag{2.1.6}$$

In particular, the Pointwise Ergodic Theorem states that if (X, μ, T) is a measure-preserving system where μ is a Borel measure on X and $\mu(U) > 0$ for every open U, then for μ-almost every $x \in X$, the sequence $(T^n(x))_{n\in\mathbb{N}}$ equidistributes.

2.1.2 Homogeneous Dynamics

Homogeneous dynamics concerns itself with dynamics in homogeneous spaces. In this section, we will review some of the basic definitions and theorems as well as discuss some of the main ideas in the field with an emphasis on matrix groups.

Definition 2.1.7. *Let (X, μ, G) be a measure-preserving system where G is a group acting on X via homeomorphisms. If G acts transitively on X, we say (X, μ, G) is a* ***homogeneous dynamical system*** *and X is a* ***homogeneous (G-)space****.*

We now provide a few examples of homogeneous dynamical systems.

1. Take $X = \mathbb{R}/\mathbb{Z}$, which is topologically a circle, and μ be the normalized arclength on the circle X. Let $G = \mathbb{R}/\mathbb{Z}$ itself. Let G act on X via left multiplication. The triple (X, μ, G) is a homogeneous dynamical system.

2. Let S^2 be the unit sphere in $\mathbb{R}^3$. Let μ be the normalized surface area and $G = O(3)$, the group of orthogonal matrices. G acts on $\mathbb{R}^3$ and this induces a transitive action on S^2. $O(3)$ preserves the Euclidean metric on $\mathbb{R}^3$. This implies $O(3)$ acts via measure-preserving transformations on S^2 making the triple $(S^2, \mu, O(3))$ into a homogeneous dynamical system.

3. Let $X = \mathbb{Z}/2\mathbb{Z}$, the group with two elements. Define $Y = X^{\mathbb{N}}$ equipped with the product topology and the product group structure. By Tychonoff's theorem, we have that Y is a compact group. Define μ_∞ be the measure on Y for which $\mu_\infty(\{(x_n)_{n\in\mathbb{N}} : x_1 = a_1, x_2 = a_2, ..., x_n = a_n\}) = \frac{1}{2^n}$, where $a_i \in X$ for $i \in \{1, ..., n\}$. One can show, using Kolmogorov's Extension Theorem found in appendix A of [14], that μ_∞ is, in fact, a measure on the Borel σ-algebra of Y, μ_∞ is invariant under the action of Y on itself, and that it is a probability measure making (Y, μ_∞, Y) into a homogeneous dynamical system.

Recipe to make homogeneous spaces: Examples of homogeneous dynamical systems are abundant. We provide here an alternative description of homogeneous spaces that allows us to get many examples.

Theorem 2.1.8. *Let G be a locally compact group acting on itself via left multiplication. There exists a non-trivial Radon measure μ (not necessarily finite) such that (G, μ, G) is a measure-preserving system.*

The measure μ from Theorem 2.1.8 is called the *left Haar measure of G*. There is an analogous theorem for when G acts on itself via right multiplication.

Suppose G is a Lie group acting transitively on a set X and let $p \in X$. Let Γ_p be the stabilizer of p under this action. Then we have a bijection from the coset space G/Γ_p to X given by $g\Gamma_p \mapsto g \cdot p$. The left Haar measure, $\widetilde{\mu}$, on G can then be pushed forward to a G-invariant measure, μ, on G/Γ_p. By identifying X and G/Γ_p via our bijection, we can transfer the topology of G/Γ_p and measure μ to X. By abuse of notation, we write the new system as (X, μ, G).

$$\begin{tikzcd}
(G, \widetilde{\mu}, G) \arrow[d] \arrow[dr] & \\
(G/\Gamma_p, \mu, G) \arrow[r, "\simeq"'] & (X, \mu, G)
\end{tikzcd}$$

By another abuse of notation, the measure μ in the dynamical system (X, μ, G) is also called the (left) Haar measure even though X itself need not be a group.

2.2 *The Special Linear Group,* $SL(n, \mathbb{R})$

The group of determinant 1 of $n \times n$ matrices with real entries is known as the special linear group over $\mathbb{R}$ and is represented by $SL(n, \mathbb{R})$. $SL(n, \mathbb{R})$ is precisely the group of orientation-preserving linear transformations that preserves the Lebesgue measure on $\mathbb{R}^n$.

2.2.1 Unimodular Lattices

Definition 2.2.1. *A* ***unimodular lattice*** Λ *in* $\mathbb{R}^n$ *is a maximal discrete subgroup of* $\mathbb{R}^n$ *such that* $\mathrm{vol}(\mathbb{R}^n/\Lambda) = 1$. *We denote the set of unimodular lattices in* $\mathbb{R}^n$ *by* $\mathscr{X}_n$.

The main example we have of a unimodular lattice in $\mathbb{R}^n$ is $\mathbb{Z}^n$. One way to get more examples of unimodular lattices in $\mathbb{R}^n$ is by taking $g \in SL(n, \mathbb{R})$ and translating $\mathbb{Z}^n$ by g to form the unimodular lattice $g\mathbb{Z}^n$. The next theorem tells us that we get all unimodular lattices through this procedure.

Theorem 2.2.2. *The group $SL(n,\mathbb{R})$ acts transitively on $\mathscr{X}_n$. The stabilizer of $\mathbb{Z}^n$ is $SL(n,\mathbb{Z})$.*

Using Theorem 2.2.2 and the discussion from §2.1.2, we are able to identify $\mathscr{X}_n$ with $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$.

Theorem 2.2.3. *There exists a unique probability measure μ_n such that $(\mathscr{X}_n, \mu_n, SL(2,\mathbb{R}))$ is a measure-preserving system.*

A proof of Theorem 2.2.3 can be found in chapter 7 of [33]. This allows us to define the following measure-preserving system $(\mathscr{X}_n, \mu_n, SL(n,\mathbb{R}))$, where μ_n is a Borel probability measure and allows us to use tools from homogeneous dynamics, probability theory, and ergodic theory to study unimodular lattices.

2.2.2 Discrete Subgroups of $SL(2,\mathbb{R})$

Definition 2.2.4. *A **lattice** in $SL(n,\mathbb{R})$ is a discrete subgroup Γ of $SL(n,\mathbb{R})$ such that $SL(n,\mathbb{R})/\Gamma$ has finite Haar measure.*

Just as we saw in §2.2.1, $SL(2,\mathbb{Z})$ is a lattice of $SL(2,\mathbb{R})$. Other examples of lattices include the Farey group found in [35], and triangle Hecke groups to name a few.

2.2.3 Geodesic and Horocyclic Flows

In this section, we will introduce two flows on $\mathscr{X}_2$ and describe some of their properties.

Geodesic Flow: Let

$$A = \left\{ g_t = \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{bmatrix} : t \in \mathbb{R} \right\} \subset SL(2,\mathbb{R}). \tag{2.2.5}$$

A acts on $\mathscr{X}_2$ via left multiplication. This produces a flow on $\mathscr{X}_2$ known as the *geodesic flow.* Since $A \subset SL(2,\mathbb{R})$, its action preserves the measure μ_2. It turns out that the measure-preserving system $(\mathscr{X}_2, \mu_2, A)$ is an ergodic system. A proof of this can be found in [14]. In fact, if Γ is a lattice in $SL(2,\mathbb{R})$, the system $(SL(2,\mathbb{R})/\Gamma, \mu, A)$ is an ergodic measure-preserving system, where μ is the Haar measure on $SL(2,\mathbb{R})/\Gamma$.

Comment: The term geodesic flow comes from the fact that this flow can be interpreted as the unit speed flow on the unit tangent bundle of the modular curve. This correspondence is well-known and can be found in chapter 9 of [14], however, it is outside of the scope of our current work and we will therefore omit it.

Horocyclic Flow: Another important flow on $\mathscr{X}_2$ is given by left multiplication by the following subgroup of $SL(2,\mathbb{R})$:

$$H = \left\{ h_s = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} : s \in \mathbb{R} \right\} \subset SL(2,\mathbb{R}). \tag{2.2.6}$$

This flow is known as the *horocylic* or *horocycle* flow. Just as in the case of the geodesic flow, $(\mathscr{X}_2, \mu_2, H)$ is an ergodic measure-preserving system. In fact, $(SL(2,\mathbb{R})/\Gamma, \mu, H)$ is an ergodic measure-preserving system whenever Γ is a lattice and μ is the Haar measure on $SL(2,\mathbb{R})/\Gamma$. However, a lot more can be said about the horocyclic flow.

Definition 2.2.7. *A **horocycle** is the orbit of a point $\Lambda \in \mathscr{X}_2$ under the horocyclic flow.*

Theorem 2.2.8. ***(Dani-Smillie 1984)*** *Let l_n be a sequence of horocycles with periods p_n, respectively. Let λ_n be the uniform measure on l_n. If $p_n \to \infty$ then the sequence of measures λ_n converges weak* to μ_2.*

Theorem 2.2.8 is proved in [11]. It will be the main tool in §3.1 and generalizations of it will be used in §3.3.1, §3.3.2, §3.3.4. It also allows us to classify all ergodic measures for the horocyclic flow, see [20] for a full proof of this statement.

The horocyclic flow and geodesic flow are related by the following equation:

$$g_t h_s g_{-t} = h_{se^{-t}}, \tag{2.2.9}$$

which in particular tells us that conjugating the horocyclic flow by the geodesic flow returns a slowed-down version of the horocyclic flow.

2.3 Diophantine Approximation

At its root, Diophantine approximation is concerned with approximating real numbers via rational numbers. In particular, we are interested in quantitative and qualitative statements about how "well" we can approximate a real number x via "simple" rational numbers $\frac{p_n}{q_n}$.

2.3.1 Classical approximation theorems

Three very important theorems in this field are the following:

Theorem 2.3.1. ***(Density of Rational Numbers)*** *Let $\varepsilon > 0$ and $x \in \mathbb{R}$. Then there exists a rational number r such that $|x - r| < \varepsilon$.*

Theorem 2.3.1 provides us with the qualitative property that rational numbers are dense within the real numbers and it is one of the most important theorems in real analysis and analytic number theory. However, the theorem does not provide any information about how to find the rational number r.

The next theorem due to Dirichlet refines Theorem 2.3.1 by providing a bound on the complexity of the rational number needed to approximate the real number x.

Theorem 2.3.2. ***(Dirichlet)*** *Let $x \in [0, 1]$ and $Q > 0$. Then there exists $0 \le p \le q \le Q$ such that $\left|x - \frac{p}{q}\right| < \frac{1}{Q}$.*

Theorem 2.3.2 provides additional information about the complexity of the rational number r needed in Theorem 2.3.1 to approximate the real number x up to a certain tolerance ε.

By looking at different subsets of $\mathbb{R}$ or $\mathbb{C}$, one can get stronger results. For instance, Roth's theorem proved in [37], provides quantitative information about the approximation of algebraic numbers via rational numbers.

Theorem 2.3.3. ***(Roth 1955)*** *Let x be an irrational algebraic number. Then for every $\varepsilon > 0$, the collection of rational numbers $\frac{p}{q}$ satisfying the inequality $\left|x - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}$ is finite.*

The part where we demand $\varepsilon > 0$ in Theorem 2.3.3 is necessary for the conclusion in the sense that if we choose $\varepsilon = 0$, Theorem 2.3.2 tells us that we always have an infinite number of solutions for the inequality $\left|x - \frac{p}{q}\right| < \frac{1}{q^2}$.

2.3.2 *Farey Sequence*

The rational numbers are dense in the collection of real numbers. We focus our view here on the interval $[0,1]$ and provide an exhaustion of $\mathbb{Q} \cap [0,1]$ by finite sets.

Definition 2.3.4. *The* ***Farey Sequence****,* $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$*, is the sequence of subsets of* $\mathbb{Q} \cap [0,1]$ *where the* n*-th term is given by*

$$\mathcal{F}_n = \left\{ \frac{p}{q} : 0 \le p \le q \le n,\ 0 < q,\ \ \gcd(p,q) = 1 \right\}. \tag{2.3.5}$$

Here we provide the first few elements of the $\mathcal{F}$:

$$\mathcal{F}_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}, \tag{2.3.6}$$

$$\mathcal{F}_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}, \tag{2.3.7}$$

$$\mathcal{F}_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}, \tag{2.3.8}$$

$$\mathcal{F}_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}, \tag{2.3.9}$$

$$\mathcal{F}_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}. \tag{2.3.10}$$

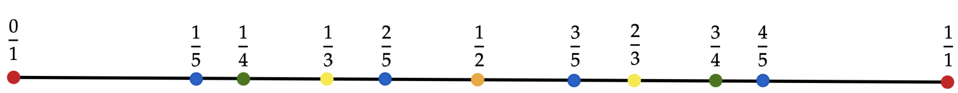

Figure 2.1: Depiction of $\mathcal{F}_5$ graded by colors. The colors represent the first elements of $\mathcal{F}$ in which the point appears.

Figure 2.2: Depiction of $\mathcal{F}_{15}$.

It is well known that the Farey Sequence equidistributes on $[0,1]$. A proof of this can be found in Chapter 5 of [24]. However, the Farey sequence does not behave like a sequence of i.i.d. random variables. Because of this semi-random behavior, we would like to explore some of the finer statistics of the distribution of the Farey Sequence. One way we can explore these finer statistics is by understanding the frequency of the gaps between consecutive terms in $\mathcal{F}_n$ where n is a very large number. The distribution of these gaps, when properly renormalized was computed first by Hall in [19] using techniques from analytic number theory and later by Athreya-Cheung using techniques from homogeneous dynamics [3].

A precise statement of the computation and description of the Hall distribution can be found [19] and [3]. Here we express only the formula for the distribution as it will be used and generalized in chapter 4.

Definition 2.3.11. *The* ***Hall distribution*** *is given by*

$$H(s) = \begin{cases} 1 & s \in \left(0, \frac{3}{\pi^2}\right], \\ -1 + \frac{6}{\pi^2 s} - \frac{6}{\pi^2 s}\log\left(\frac{3}{\pi^2 s}\right) & s \in \left[\frac{3}{\pi^2}, \frac{12}{\pi^2 s}\right], \\ -1 + \frac{6}{\pi^2 s} + 2\sqrt{\frac{1}{4} - \frac{3}{\pi^2 s}} - \frac{12}{\pi^2 s}\log\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{3}{\pi^2 s}}\right) & s \in \left[\frac{12}{\pi^2}, \infty\right). \end{cases} \tag{2.3.12}$$

$H(t)$ represents the probability that a gap between consecutive terms in $\mathcal{F}_n$ is larger than $\frac{3}{\pi^2 n^2}t$.

2.4 Translation Surfaces

Translation surfaces are a mathematical object that arises naturally in questions related to complex analysis, billiards on rational polygons, and statistical mechanics to name a few. In this section, we describe only the basic properties of translation surfaces we need. For a more comprehensive reference the reader may see [?AthMas],[44], and [45].

2.4.1 Definitions and basic examples of translation surfaces

We provide two equivalent definitions of a translation surface.

Definition 2.4.1. *A* ***translation surface*** *is a tuple (X, ω), where X is a compact Riemann surface and ω is a holomorphic 1-form on X.*

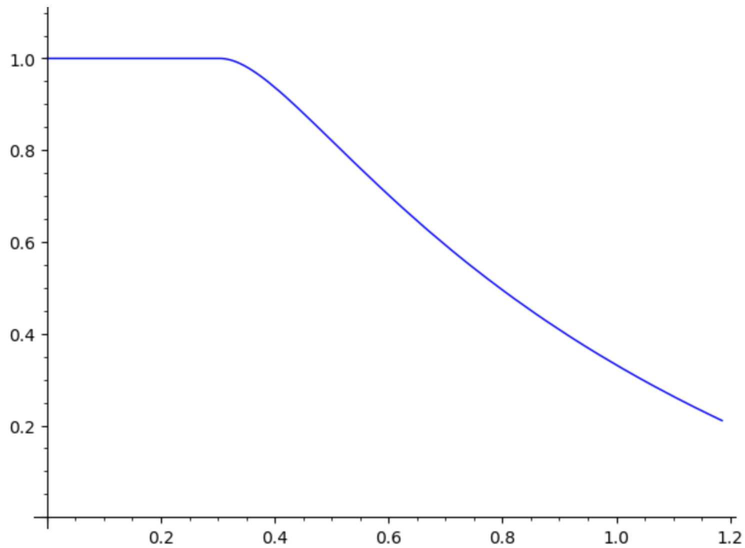

Figure 2.3: Graph of the Hall distribution

Definition 2.4.2. *A* ***translation surface*** *is a finite union of disjoint polygons* $P = \bigsqcup_{i=1}^{n} P_i \subset \mathbb{C}$ *such that the sides of* P *can be partitioned into parallel congruent pairs* $\{s_a, s_b\}$ *and each* s_a *and* s_b *in the same equivalence class are identified via Euclidean translation.*

Translation surfaces are also known as flat surfaces or abelian differentials in the literature. We usually drop the X from the (X, ω) notation for a translation surface and call ω the translation surface unless it causes confusion.

We now provide some examples of translation surfaces.

Square Flat Torus: Using definition 2.4.1, we may construct the square flat torus by looking at the Riemann Surface $\mathbb{C}/\mathbb{Z}[i]$. The holomorphic 1-form on the torus will be the push-forward of the form dz on $\mathbb{C}$.

Using definition 2.4.2, we may construct the square flat torus as follows: Let P be the unit square $[0, 1]^2 \subset \mathbb{C}$. We identify the left and right sides via the translation $T(z) = z + 1$ and the top and bottom sides via $T(z) = z + i$.

The flat torus can be visualized as the world from the video games *Asteroids* and *Pac-Man*. A world where exiting the screen from the right means you enter the screen from the

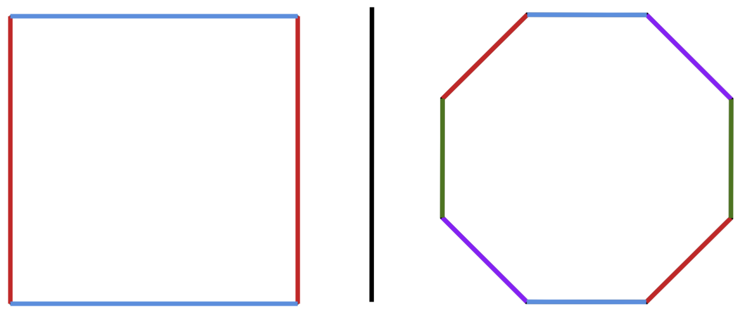

Figure 2.4: Left: Depiction of a square flat torus. Right: Depiction of the translation surface arising from a regular octagon. Two sides are identified via Euclidean translation if they have the same color.

left and exiting from the top means you enter from the bottom. The left side of Figure 2.4 provides a picture of a square flat torus.

Regular Polygons: If n is even, we can look at the regular n-gon and identify opposite sides via Euclidean translation. Figure 2.4 depicts the surface arising from the regular octagon on the right.

If n is odd, we can take the regular n-gon and reflect along one of its sides, we would then get a $2(n-1)$-gon. We can then proceed to identify parallel sides via Euclidean translation. This translation surface is known as the double $n-$gon. Figure 2.5 depicts the double 5-gon, also known as the double pentagon.

Two translation surfaces are equivalent if one can take one, do a finite number of cuts, and rearrange the pieces into the other by translating them. This procedure is sometimes called the *cut-translate-paste procedure* or *scissors congruence.*

Translation surfaces have the property that they possess a flat metric outside the zeros of ω. This geometry can be seen more easily as the geometry inherited from $\mathbb{C}$. The zeros of ω are called *cone points* or *saddle points*. The focus of §3.3.4 and chapter 4 will be on understanding the relative location of saddle points in relation to one another. To be

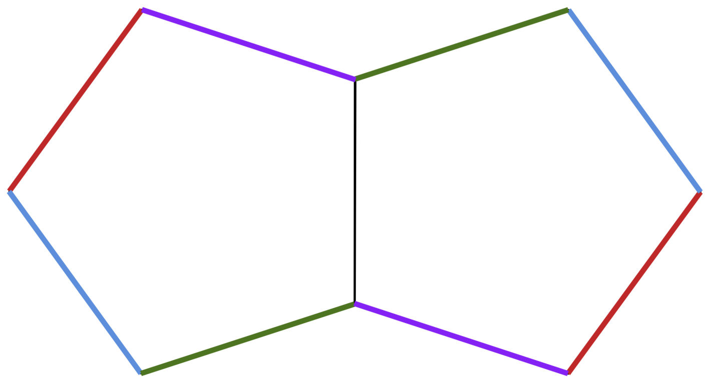

Figure 2.5: Depiction of the double pentagon. Two sides are identified via Euclidean translation if they have the same color. The black side in the middle is the axis of reflection but is not a side of the polygon.

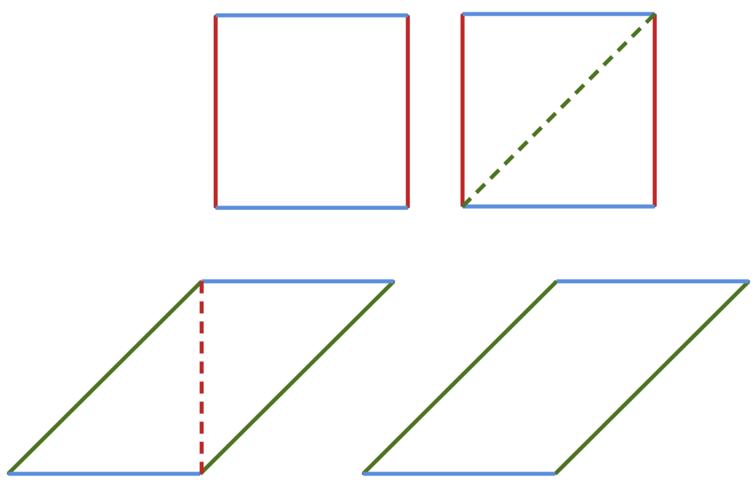

Figure 2.6: The cut translate and paste procedure on the square flat torus.

precise, we provide the following definition:

Definition 2.4.3. *Let ω be a translation surface. A **saddle connection** is a geodesic connecting two (not necessarily distinct) saddle points of ω and not passing through any other cone points.*

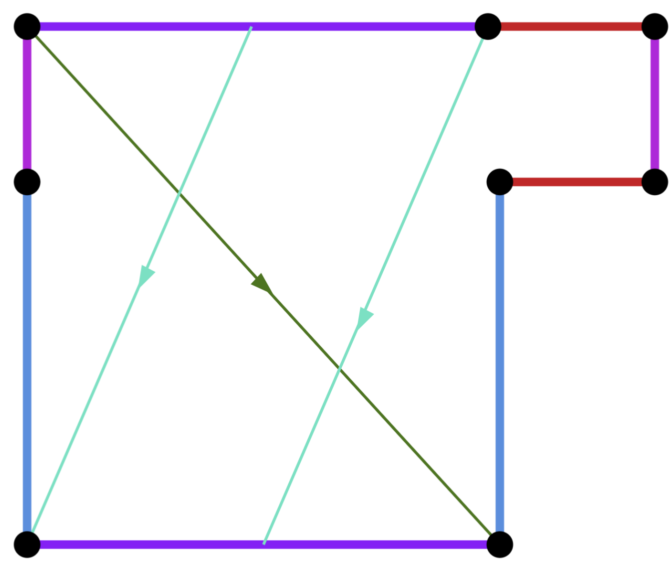

Figure 2.7: Depiction of two saddle connections, one in cyan and another in green, on a genus 2 translation surface.

In order to keep track of saddle connections we introduce the following definition.

Definition 2.4.4. *Let ω be a translation surface and γ be a saddle connection of ω. The* ***holonomy vector*** *associated to γ is given by*

$$z_\gamma = \int_\gamma \omega \in \mathbb{C}. \tag{2.4.5}$$

If we write $z_\gamma = x_\gamma + iy_\gamma$, we see that x_γ describes the horizontal displacement and y_γ describes the vertical displacement of γ.

2.4.2 The $SL(2,\mathbb{R})$ action on translation surfaces

$SL(2,\mathbb{R})$ acts naturally on $\mathbb{R}^2 \simeq \mathbb{C}$ via the linear action. This linear action induces an action on the set of polygons on the plane, which in turn induces an action on the space of translation surfaces.

Definition 2.4.6. *Given a translation surface ω, the stabilizer of ω under the $SL(2,\mathbb{R})$ action is called the* ***Veech group*** *and is denoted by $SL(\omega)$.*

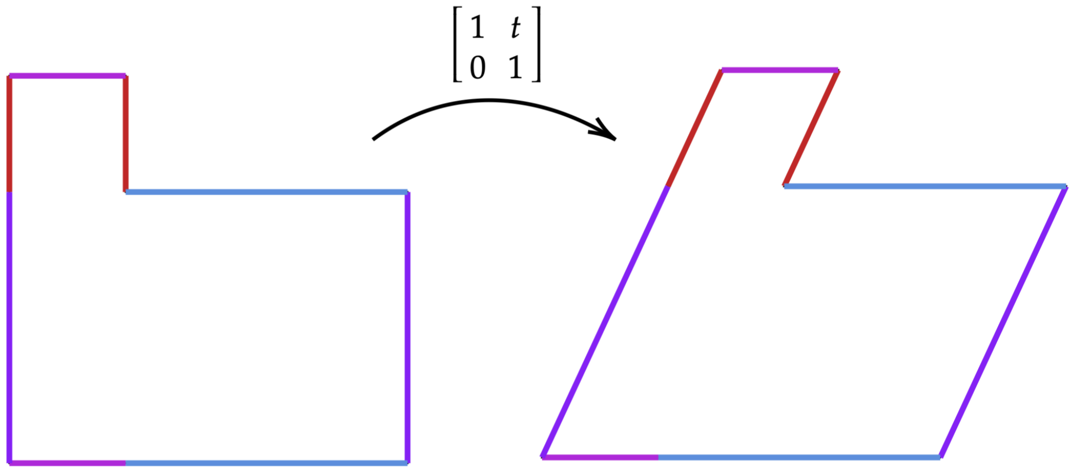

Figure 2.8: Horizontal shear of a translation surface.

The Veech group of most translation surfaces is trivial, however, some translation surfaces exhibit a large Veech group. They are called *Veech surfaces* or *lattice surfaces.*

Definition 2.4.7. *The translation surface ω is called a* ***Veech*** *or* ***lattice surface*** *if its Veech group has finite covolume in $SL(2, \mathbb{R})$.*

Some examples of Veech surfaces include the flat torus and the surfaces arising from regular polygons discussed above. An example of an explicit non-Veech surface can be found in [38]. Most translation surfaces are not Veech, but the collection of Veech surfaces is dense in the moduli space of translation surfaces [45].

The Veech group of the flat square torus is $SL(2, \mathbb{Z})$. Edwards-Sanderson-Schmidt [13] provide a general algorithm to compute Veech groups.

Chapter 3

THE MINIMAL DENOMINATOR FUNCTION AND GEOMETRIC GENERALIZATIONS

This chapter contains the main two theorems of this dissertation as well as many examples illustrating their application to different contexts in Diophantine approximation and the geometry of surfaces. The two main theorems are Theorem 3.1.5, describing the limiting distribution of the minimal denominator function, and Theorem 3.2.6, which allows us to generalize the techniques from Theorem 3.1.5 to higher dimensions, linear forms, saddle connections of translation surfaces, and Heegner fields.

3.1 Minimizing denominators

In this section, we will explore one of the main topics of this dissertation, *the minimal denominator function*, or *MDF* for short. This function comes out of a question posted by Meiss-Sander in section 3 of [39]. They asked the following: Given a real number x and a tolerance $\delta > 0$, what is the rational number $\frac{p}{q}$ in the interval $(x-\delta, x+\delta)$ with the smallest denominator q? Finding the explicit $\frac{p}{q}$ is usually a very hard question. To this end, Meiss-Sander defined the following function:

$$\widetilde{q}_{\min}(x,\delta) = \min\left\{q \in \mathbb{N} : \left|x - \frac{p}{q}\right| < \delta, p \in \mathbb{Z}\right\}. \tag{3.1.1}$$

Figure 3.1 depicts both the chaos and order found in the graph of $\widetilde{q}_{\min}(x,\delta)$ for a fixed and small δ.

The graph has mirror symmetry about the number $\frac{1}{2}$ and dips down near rational numbers with small denominators. This happens because we can view $\widetilde{q}_{\min}(x,\delta)$ in the following way: We place a small interval of radius δ centered at x. Then $\widetilde{q}_{\min}(x,\delta) = Q$ if and only if $\mathcal{F}_Q$ has a non-empty intersection with $(x-\delta, x+\delta)$ but $\mathcal{F}_{Q-1}$ does not. This realization will be important when looking at Theorem 3.1.7 and later on in chapter 4 when exploring the statistics of saddle connections on translation surfaces.

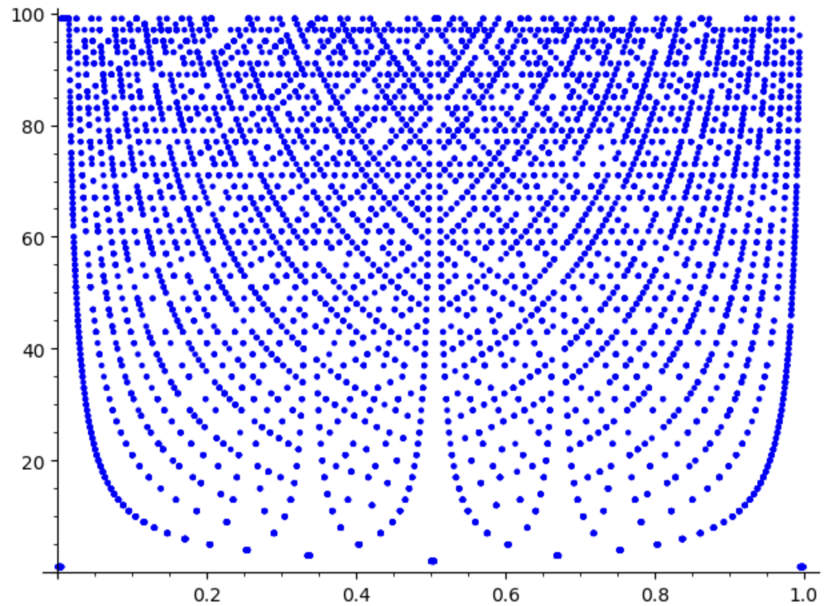

Figure 3.1: Graph of $\widetilde{q}_{\min}(x, 0.01)$.

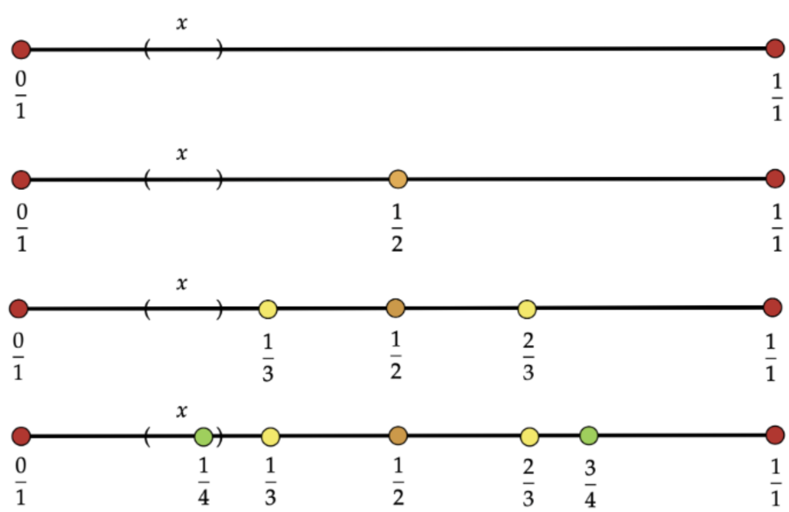

Figure 3.2: Using the Farey sequence to compute $\widetilde{q}_{\min}(x, \delta)$.

Chen-Haynes computed in [9] the asymptotics of the expected value of the slightly different function:

$$q_{\min}(x,\delta) = \min\left\{q \in \mathbb{N} : \left|x - \frac{p}{q}\right| < \frac{\delta}{2}, p \in \mathbb{Z}\right\}. \tag{3.1.2}$$

Theorem 3.1.3. ***(Chen-Haynes 2022)*** *With the notation decribed above,*

$$\int_0^1 q_{\min}(x,\delta)dx \sim \frac{16}{\pi^2}\frac{1}{\sqrt{\delta}} + O(\log^2(\delta)). \tag{3.1.4}$$

The notation $f(t) \sim g(t)$ found in Theorem 3.1.3 means that $\lim_{t\to 0}\frac{f(t)}{g(t)} = 1$.

Our contributions come from relating a normalized version of the MDF inspired by Theorem 3.1.3 and relating it to an equidistribution problem in the space of unimodular lattices.

Theorem 3.1.5. *Let P be the uniform measure on $[0,1]$, $T \in \mathbb{R}$, and μ_2 be the Haar measure on $\mathscr{X}_2$. Then*

$$\lim_{\delta\to 0} P\left(x \in [0,1] : \sqrt{\delta}\widetilde{q}_{\min}(x,\delta) \leq T\right) = \mu_2\left(\Lambda \in X_2 : F^1(\Lambda) \leq T\right). \tag{3.1.6}$$

The function F_1 is part of a one-parameter family of functions which will be described in Definition 3.1.9. The method of proof of Theorem 3.1.5 allows us to generalize to a broader set of examples, these are described in §3.3.

Marklof improved the statement of Theorem 3.1.5 by computing the probability density function of the limiting distribution in [28].

Theorem 3.1.7. ***(Marklof 2024)*** *Let P be the uniform measure on $[0,1]$, $T \in \mathbb{R}$, and $\mathcal{D} \subset [0,1]$ such that $P(\partial\mathcal{D}) = 0$. Then,*

$$\lim_{\delta\to 0} P\left(x \in \mathcal{D} : \sqrt{\delta}q_{\min}(x,\delta) \geq T\right) = P(\mathcal{D})\int_T^\infty \eta(s)ds, \tag{3.1.8}$$

where $\eta(s) = \frac{6}{\pi^2}sH\left(\frac{3}{\pi^2}s^2\right)$, and $H(t)$ is the Hall distribution discussed in §2.3.2.

The symbol ∂A denotes the boundary of the set A.

Marklof's approach is adapted in chapter 4 to the context of translation surfaces.

3.1.1 Proof of Theorem 3.1.5

We begin by creating a connection between $\widetilde{q}_{\min}(x,\delta)$ and the integer lattice in two dimensions.

Definition 3.1.9. *Let $\Lambda \in \mathscr{X}_2$ and let $\delta > 0$. We define $F_\delta^1 : \mathscr{X}_2 \to \mathbb{R}$ by*

$$F_\delta^1(\Lambda) = \min \left\{ u \in \mathbb{R} : \text{ there exists } v \in \mathbb{R} \text{ such that } \begin{bmatrix} u \\ v \end{bmatrix} \in \Lambda \cap C_\delta \right\}, \tag{3.1.10}$$

where $C_\delta = \left\{ [x, y]^T \in \mathbb{R}^2 : x > 0, |y| < \delta x \right\}$.

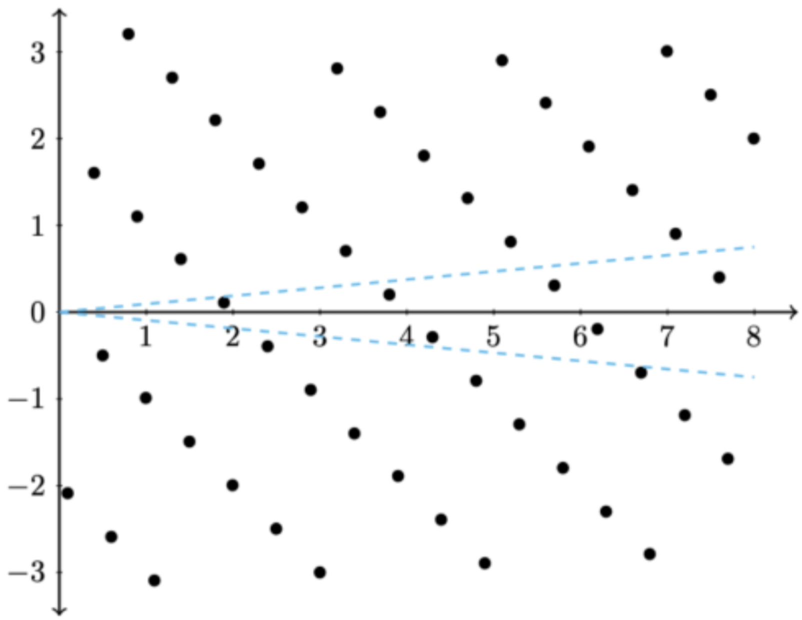

Figure 3.3: A depiction of C_δ intersecting a lattice Λ.

Figure 3.3 depicts a random lattice Λ and its intersection with the horizontal cone C_δ. The x-coordinate of the first point of Λ one sees inside the cone when moving from left to right is the value of $F_\delta(\Lambda)$. The parameter δ provides us with infinitely many functions that behave in predictable ways when precomposed with the geodesic and horocyclic flows. Lemma 3.1.11 provides us with explicit formulas for how F_δ changes with respect to these flows.

Lemma 3.1.11. *Let $\delta > 0$ and F_δ be defined as above. Then,*

1. *for all $t \in \mathbb{R}$, $F_\delta(g_t\Lambda) = e^{\frac{t}{2}} F_{\delta e^t}(\Lambda)$;*

2. for all $s \in \mathbb{R}$, $\tilde{q}_{\min}(s,\delta) = F_\delta(h_s\mathbb{Z}^2)$.

Proof. We will prove part 1 of Lemma 3.1.11 by doing a computation.

$$\begin{aligned}
F_\delta^1(g_t\Lambda) &= \min\left\{u : \begin{bmatrix} u \\ v \end{bmatrix} \in g_t\Lambda \cap C_\delta\right\} \\
&= \min\left\{u : g_{-t}\begin{bmatrix} u \\ v \end{bmatrix} \in \Lambda \cap g_{-t}C_\delta\right\} \\
&= \min\left\{u : \begin{bmatrix} e^{-\frac{t}{2}}u \\ e^{\frac{t}{2}}v \end{bmatrix} \in \Lambda \cap C_{\delta e^t}\right\} \\
&= e^{\frac{t}{2}}\min\left\{u' : \begin{bmatrix} u' \\ v' \end{bmatrix} \in \Lambda \cap C_{\delta e^t}\right\} \\
&= e^{\frac{t}{2}} F_{\delta e^t}^1(\Lambda).
\end{aligned}$$

This completes the proof of the first part of Lemma 3.1.11.

We now proceed to prove part 2 of 3.1.11. We begin by defining a correspondence between rational numbers and lattice points. If $\frac{p}{q}$ is a rational number written in simplest form, we identify it with the integer vector $[q,p]^T$. We claim that $\frac{p}{q} \in (x-\delta, x+\delta)$ precisely when $h_x[q,p]^T \in C_\delta$.

Proof of claim: Notice that $h_x[q,p]^T = [q, p-qx]^T$. We then have that $[q,p-qx]^T \in C_\delta$ precisely when $\delta > \left|\frac{p-qx}{q}\right| = \left|x - \frac{p}{q}\right|$, which is equivalent to $\frac{p}{q} \in (x-\delta, x+\delta)$. This completes the proof of our claim.

Our claim implies that the set of denominators of fractions in $(x-\delta, x+\delta)$ is precisely the set of x-coordinates of the lattice points in $h_x\mathbb{Z}^2 \cap C_\delta$. In particular, they have the same minimum, which means $\widetilde{q}_{\min}(x,\delta) = F_\delta^1(h_x\mathbb{Z}^2)$ as desired.

□

Lemma 3.1.11 allows us to understand how the quantity $F_\delta^1(\Lambda)$ changes as we change Λ in relation to the geodesic flow. Notice that if $t = -\log\delta$, we have that

$$F_\delta^1(g_{-\log\delta}\Lambda) = \delta^{-\frac{1}{2}} F_1^1(\Lambda) \tag{3.1.12}$$

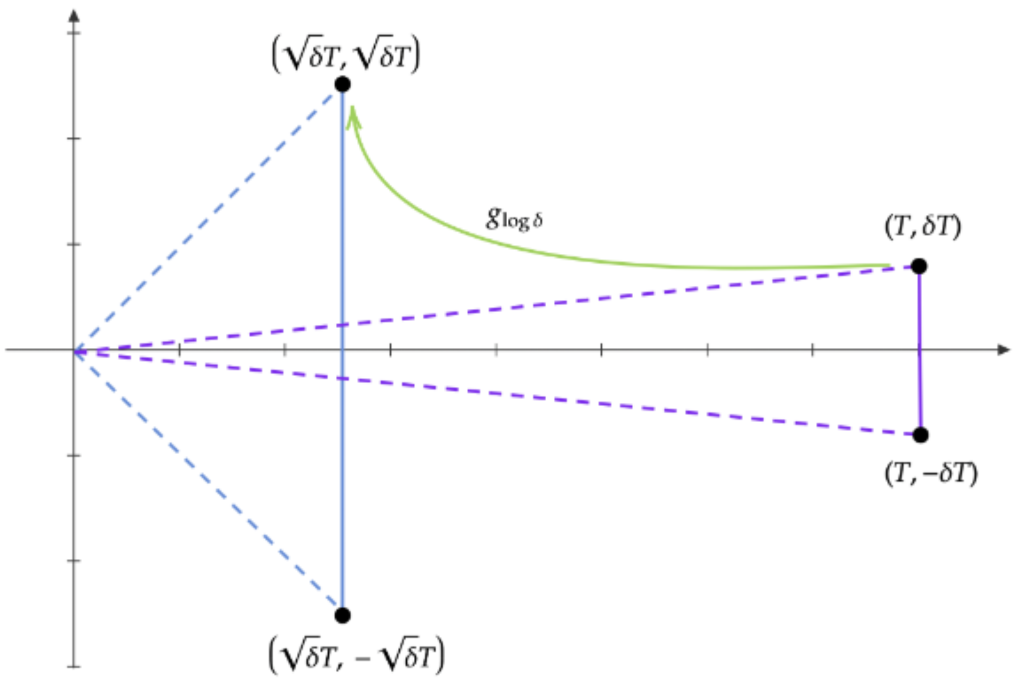

Figure 3.4: The action of $g_{\log \delta}$ on the cone C_δ.

for each $\Lambda \in X_2$. The identity 3.1.12 allows us to exchange the problem of understanding $q_{\min}$ as a problem with a fixed lattice and changing region of intersection with a randomly selected lattice intersecting a fixed region of $\mathbb{R}^2$. Figure 3.4 contains a visual representation of the effect of $g_{\log \delta}$ on the cone C_δ.

Theorem 3.1.13. *Let P denote the uniform probability measure on $[0,1]$. Then for every $T \in \mathbb{R}$, as $\delta \to 0$,*

$$P\left(\left\{x : \sqrt{\delta}\widetilde{q}_{\min}(x,\delta) \leq T\right\}\right) \to \mu_2\left(\left\{\Lambda \in X_2 : F_1^1(\Lambda) \leq T\right\}\right). \tag{3.1.14}$$

Proof. By the second equation of Lemma 3.1.11, $\sqrt{\delta}\widetilde{q}_{\min}(x,\delta) \leq T$ precisely when $\sqrt{\delta}F_\delta^1(h_x\mathbb{Z}^2) \leq T$. This means that

$$P\left(\left\{x : \sqrt{\delta}\widetilde{q}_{\min}(x,\delta) \leq T\right\}\right) = P\left(\left\{x : \sqrt{\delta}F_\delta^1(h_x\mathbb{Z}^2) \leq T\right\}\right).$$

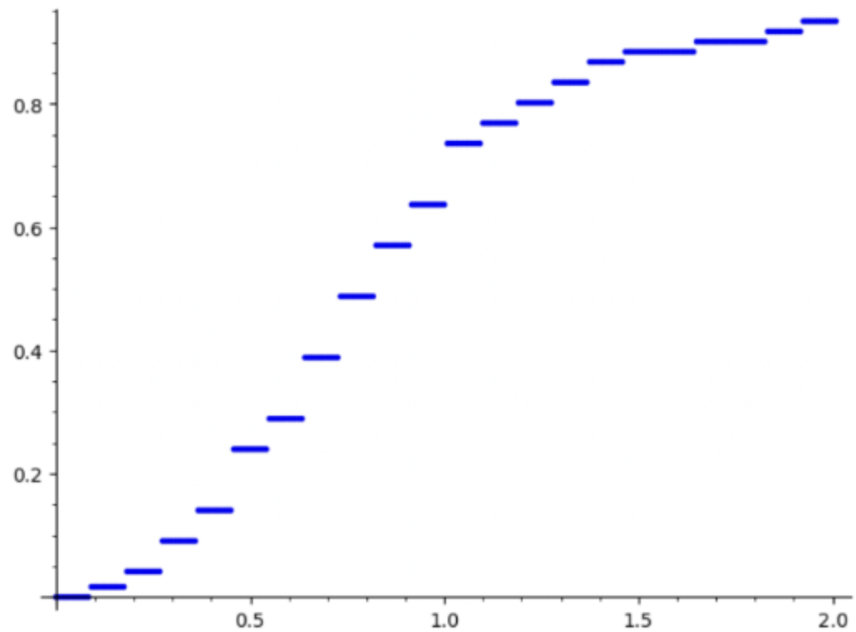

Figure 3.5: Cumulative distribution of $\widetilde{q}_{\min}(x, 0.01)$

Using equation 3.1.12, we get that

$$\begin{aligned} P\left(\left\{x : \sqrt{\delta}F_{\delta}^{1}(h_x\mathbb{Z}^2) \leq T\right\}\right) &= P\left(\left\{x : \sqrt{\delta}F_{\delta}^{1}(g_{-\log\delta}g_{\log\delta}h_x\mathbb{Z}^2) \leq T\right\}\right) \\ &= P\left(\left\{x : F_1^1(g_{\log\delta}h_x\mathbb{Z}^2) \leq T\right\}\right). \end{aligned}$$

Notice that $g_{\log\delta}h_x = h_{\frac{x}{\delta}}g_{\log\delta}$. This means that the lattice $g_{\log\delta}\mathbb{Z}^2$ has period δ^{-1} under the horocyclic flow. Hence, by Theorem 2.2.8, we have that as $\delta \to 0$,

$$P\left(\left\{x : F_1^1(g_{\log\delta}h_x\mathbb{Z}^2) \leq T\right\}\right) \to \mu_2\left(\left\{\Lambda : F_1^1(\Lambda) \leq T\right\}\right),$$

as desired.

□

3.2 Equivariant Processes

This section will introduce the concept of *equivariant processes*, also known as *Siegel-Veech measures*. We will use the main theorem of this section, Theorem 3.2.6, to provide a general framework for the examples described in §3.1 and §3.3. It will be beneficial to keep in mind the proof and setup of Theorem 3.1.5 as we discuss the contents of this section.

The ideas in this section come from the work of many people. Some references include Veech's work [43] where he generalizes the work of Siegel [40]. These ideas were then extended by Marklof-Strombergsson [30] and Athreya-Ghosh [4].

We will describe three things prior to defining equivariant processes, an action on a space, an action on measure in $\mathbb{R}^n$ and a map connecting both actions:

Action on Space: Let G be a subgroup of $GL(d, \mathbb{R})$ for $d \geq 2$. Let G act on the standard Borel space (X, λ) via measure-preserving transformations. An example of this is when G is either H or A, $X = \mathscr{X}_2$, and $\lambda = \mu_2$ from §2.2.

Actions on Measures: Let $\mathscr{M}(\mathbb{R}^d)$ be the space of σ-finite Radon Borel measure on $\mathbb{R}^d$. The linear action of G on $\mathbb{R}^d$ induces an action of G on $\mathscr{M}(\mathbb{R}^d)$ as follows: If $A \subset \mathbb{R}^d$ is a Borel set, $g \in G$, and $\nu \in \mathscr{M}(\mathbb{R}^d)$, then

$$(g \cdot \nu)(A) = \nu(g^{-1}(A)). \tag{3.2.1}$$

Definition 3.2.2. *With the setup above, let $\phi : X \to \mathscr{M}(\mathbb{R}^d)$ be a measurable map. We call ϕ an **equivariant process map** if for all $g \in G$, $x \in X$, and $A \subset \mathbb{R}^d$ Borel,*

$$\phi(gx)(A) = (g \cdot \phi(x))(A). \tag{3.2.3}$$

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "x \mapsto gx"'] & \mathscr{M}(\mathbb{R}^d) \arrow[d, "\nu \mapsto g \cdot \nu"] \\ X \arrow[r, "\phi"'] & \mathscr{M}(\mathbb{R}^d) \end{tikzcd}$$

Definition 3.2.4. *Let $d \geq 2$, $G \subset GL(d, \mathbb{R})$ be a subgroup, and (X, λ) be a standard Borel space equipped with a G-action that preserves the measure μ. The triple (X, μ, ϕ) is called an **equivariant process** if $\phi : X \to \mathscr{M}(\mathbb{R}^d)$ is an equivariant process map.*

3.2.1 Chen-Haynes Distributions

We prove a general result that will allow us to frame the MDF question from §3.1 as an equivariant process and generalize it to different contexts.

Let $\mathcal{S} = \{S_T\}_{T>0}$ be a family of Borel subsets of $\mathbb{R}^m$ with the property that if $T_1 \leq T_2$, then $S_{T_1} \subset S_{T_2}$. Let (X_n, λ_n, ν_n) be a sequence of equivariant processes. We define the *t-th Chen-Haynes distribution* associated to $\mathcal{S}$ and (X_n, λ_n, ν_n) by

$$\xi(X_n, \lambda_n, \nu_n, \mathcal{S}, t)(T) = \lim_{n\to\infty} \lambda_n \left(\{x \in X_n : \nu_n(x)(S_T) \geq t\}\right). \tag{3.2.5}$$

We specialize the setup above by looking at sequences of G-equivariant processes over a fixed space and a fixed equivariant process map. Let (X, λ_n, ν) be such a sequence of G-equivariant processes. Then we have the following result which allows us to exchange weak* convergence results for equidistribution results.

Theorem 3.2.6. *Suppose that $\lambda(\partial\{x \in X : \nu(x)(S_T) \geq t\}) = 0$. If $\lambda_n \xrightarrow{*} \lambda$, then*

$$\xi(X, \lambda_n, \nu, \mathcal{S}, t)(T) = \lambda\left(\{x \in X : \nu(x)(S_T) \geq t\}\right). \tag{3.2.7}$$

Proof. Since all of our measures are Radon Borel, we have that weak* convergence of measures implies that for all bounded continuous functions $f : X \to \mathbb{R}$, $\lambda_n(f) \to \lambda(f)$. This means that if A is a set subset of X with $\lambda(\partial A) = 0$, then we may approximate χ_A from above and below via bounded continuous functions. This means that $\lambda_n(A) \to \lambda(A)$. We complete the proof by setting $A = \{x \in X : \nu(x)(S_T) \geq t\}$.

□

Following our example, if $A \subset \mathbb{R}^2$, $\Lambda \in \mathscr{X}_2$ and

$$\phi(\Lambda) = \sum_{x\in\Lambda} \delta_x, \tag{3.2.8}$$

where δ_x is the Dirac-delta measure with support $\{x\}$, then $\nu(\Lambda)(A)$ is the cardinality of the intersection of A and Λ. If $k \in \mathbb{N}$, we can see that the k-th Chen-Haynes distribution at time T for the equivariant process (X, μ, ϕ) is the measure of unimodular lattices that intersect S_T in at least k points.

Theorem 3.1.13 revisited: In the case of Theorem 3.1.13, we see that the sets $S_T = C_1 \cap \{[x,y]^T \in \mathbb{R}^2 : x \leq T\}$ and the measures λ_n given by the uniform measure on the h_s-orbit of $g_{\log 1/n}\mathbb{Z}^2$ in X_2 provides us with a sequence of standard Borel spaces. Let $G = \{h_s : s \in \mathbb{R}\} \subset SL(2,\mathbb{R})$ and $\nu : \mathscr{X}_2 \to \mathcal{M}(\mathbb{R}^2)$ be given by

$$\nu(\Lambda) = \sum_{x\in\Lambda} \delta_x, \tag{3.2.9}$$

where δ_x is the Dirac-delta measure with support $\{x\}$.

This construction provides us with a sequence of equivariant processes $(\mathscr{X}_2, \lambda_n, \nu)$

Notice that $F_1^1(\Lambda) \leq T$ precisely when $\nu(\Lambda)(S_T) \geq 1$. This rephrases Theorem 3.1.13 as the computation of the Chen-Haynes distribution associated with the sequence $(\mathscr{X}_2, \lambda_n, \nu)$ and the family of cones $\mathcal{S} = \{S_T : T > 0\}$.

$$\begin{aligned}
\xi(X_2, \lambda_n, \nu, \mathcal{S}, 1)(T) &= \lim_{n\to\infty} \lambda_n \left(\{\Lambda \in X_2 : \nu(\Lambda)(S_T) \geq 1\}\right) \\
&= \lim_{n\to\infty} \lambda_n \left(\left\{\Lambda \in X_2 : F_1^1(\Lambda) \leq T\right\}\right) \\
&= \mu_2 \left(\left\{\Lambda \in X_2 : F_1^1(\Lambda) \leq T\right\}\right).
\end{aligned}$$

In what follows, we will compute the Chen-Haynes distributions of equivariant processes associated to higher dimensional Diophantine approximations, the holonomy vectors of translation surfaces, and Diophantine approximation over Heegner fields. We will proceed similarly, providing appropriate families of sets $\mathcal{S}$, sequences of measure $\{\lambda_n\}$ in the pertinent space, and weak* convergence results which satisfy the hypotheses of Theorem 3.2.6. The equivariant process map will always be of the form seen in equation 3.2.9.

3.3 Examples

3.3.1 Joint Diophantine Approximations

In this section, we ask the following question. Given real numbers $x_1, x_2, ..., x_m$, what is the best way to approximate them via rational numbers $r_1, r_2..., r_m$ all at the same time? More precisely:

Motivating question: Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_m \end{bmatrix}^T \in [0,1]^m$, let $\|\cdot\|$ denote the max norm in $\mathbb{R}^m$ and let $\delta > 0$. What is the smallest $q \in \mathbb{N}$ such that there exists an integer point $\mathbf{p} = \begin{bmatrix} p_1 & p_2 & \dots & p_m \end{bmatrix}^T \in \mathbb{Z}^m$ such that $\left\|\mathbf{x} - \frac{1}{q}\mathbf{p}\right\| < \delta$?

An equivalent statement is: When looking at the rational points in $B_\delta(\mathbf{x})$, we can compute the least common multiple of the denominators of the coordinates of each of these points. What is the smallest number one can get? More precisely: $\mathbf{r} = \begin{bmatrix} \frac{p_1}{q_1} & \frac{p_2}{q_2} & \dots & \frac{p_m}{q_m} \end{bmatrix}$, where the entries are written in simplest form. Let $\operatorname{Denom}(\mathbf{r}) = \operatorname{lcm}(q_1, q_2, ..., q_m)$. We ask what is $\min\{\operatorname{Denom}(\mathbf{r}) : \|\mathbf{x} - \mathbf{r}\| < \delta\}$?

To answer this question, we define the *m-dimensional minimal denominator function.*

Definition 3.3.1. *Let $\delta > 0$ and $\mathbf{x} \in [0,1]^m$.*

$$Q^m(\mathbf{x}, \delta) = \min\left\{q \in \mathbb{N} : \text{ there exists } \mathbf{p} \in \mathbb{Z}^m \text{ such that } \left\|\mathbf{x} - \frac{1}{q}\mathbf{p}\right\| < \delta\right\}. \tag{3.3.2}$$

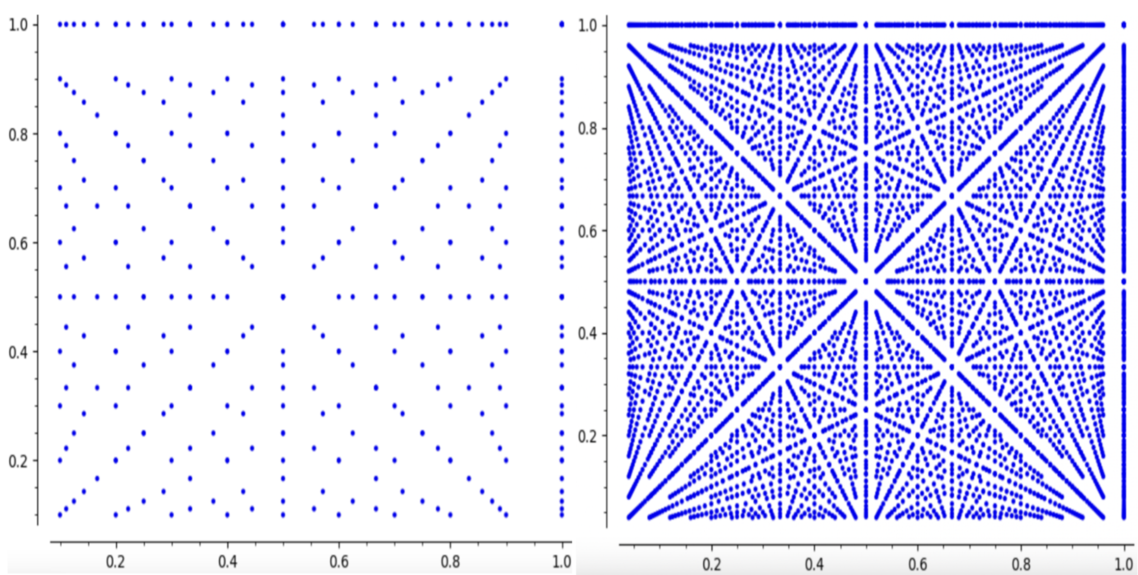

Figure 3.6: Left: Points in $[0,1]^2$ where Denom is at most 10. Right: Points in $[0,1]^2$ where Denom is at most 25.

Figure 3.6 provides a picture of the rational points with the smallest least common denominators in two dimensions. This is a generalization of the Farey sequence seen in

§2.3. In particular, we are able to see the Farey sequence appear on the sides of the unit square in Figure 3.6.

Motivated by the work in §3.1, we study the statistics of $Q^m(\mathbf{x},\delta)$ by relating it to the theory of lattices.

Define the following function $F_\delta^m : \mathscr{X}_{m+1} \to \mathbb{R}$ by

$$F_\delta^m(\Lambda) = \min\left\{ u : \text{there exists } \mathbf{v} \in \mathbb{R}^m \text{ such that } \begin{bmatrix} u \\ \mathbf{v} \end{bmatrix} \in \Lambda \cap C_\delta^m \right\},$$

where $C_\delta^m = \left\{ \begin{bmatrix} t \\ t\mathbf{x} \end{bmatrix} : \|\mathbf{x}\| < \delta, t > 0 \right\}$. With these new definitions, we are now able to compute the limiting cumulative distribution for a properly normalized version of $Q^m(\mathbf{x},\delta)$, that being $\delta^{\frac{m}{m+1}} Q^m(x,\delta)$.

Theorem 3.3.3. *Let P denote the uniform distribution on $[0,1]^m$ and $T \in \mathbb{R}$. Then, as $\delta \to 0$,*

$$P\left(\left\{\mathbf{x} \in [0,1]^m : \delta^{\frac{m}{m+1}} Q^m(\mathbf{x},\delta) \le T\right\}\right) \to \mu_{m+1}\left(\{\Lambda \in X_{m+1} : F_1^m(\Lambda) \le T\}\right).$$

Corollary 3.3.4. *As $\delta \to 0$,*

$$\mathbb{E}_{\mathbf{x}}\left[\delta^{\frac{m}{m+1}} Q^m(\mathbf{x},\delta)\right] = \delta^{\frac{m}{m+1}} \int_{[0,1]^m} Q^m(\mathbf{x},\delta)\,dP(\mathbf{x}) \to \int_{X_{m+1}} F_1^m(\Lambda)\,d\mu_{m+1}(\Lambda).$$

We omit the proof of Theorem 3.3.3 as it will be a consequence of Theorem 3.3.14.

3.3.2 Approximating Linear Transformations

Let m and n be positive integers. Endow $\mathbb{R}^m$ and $\mathbb{R}^n$ with their max norms $\|\cdot\|_m$ and $\|\cdot\|_n$, respectively. The norms $\|\cdot\|_m$ and $\|\cdot\|_n$ induce a norm, $\|\cdot\|$, on $\mathbb{R}^m \oplus \mathbb{R}^n = \mathbb{R}^{m+n}$ given by $\left\|[\mathbf{u},\mathbf{v}]^T\right\| = \|\mathbf{u}\|_n + \|\mathbf{v}\|_m$. Let X be an $n \times m$ matrix with entries in $[0,1]$. For $\delta > 0$, define

$$Q^{m,n}(X,\delta) = \min_{(\mathbf{q},\mathbf{p}) \in \mathbb{Z}^n \oplus \mathbb{Z}^m} \left\{ \|\mathbf{q}\|_n : \|X\mathbf{q} - \mathbf{p}\|_m < \delta \|\mathbf{q}\|_n \right\}.$$

Notice that the case when $n = 1$, $Q^{m,1}(\mathbf{x},\delta) = Q^m(\mathbf{x},\delta)$ and when m and n are both 1, $Q^{1,1}(x,\delta) = q_{\min}(x,\delta)$.

We may identify the collection of $m \times n$ matrices with entries in $[0,1]$ with $[0,1]^{mn}$. $Q^{m,n}(X,\delta)$ then models the question: Given a randomly selected $X \in [0,1]^{mn}$, what is the shortest integer vector $\mathbf{q} \in \mathbb{Z}^n$ that lands within an appropriately sized neighborhood of $\mathbb{Z}^m$ under X. The size of this neighborhood depends on the size of $\mathbf{q}$.

To study the statistics of $Q^{m,n}$ we will relate it to the theory of lattices just as we did in §3.1 and give appropriate generalizations of the geodesic and horocyclic flows.

3.3.3 Lattice Interpretation

Define $F_\delta^{m,n} : X_{m+n} \to \mathbb{R}$ by

$$F_\delta^{m,n}(\Lambda) = \min \left\{ \|\mathbf{u}\|_n : \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \in \Lambda \cap C_\delta^{m,n} \right\},$$

where

$$C_\delta^{m,n} = \left\{ r \begin{bmatrix} \mathbf{s} \\ \mathbf{t} \end{bmatrix} : r > 0, \|\mathbf{s}\|_n = 1, \|\mathbf{t}\|_m < \delta \right\}.$$

The higher-dimensional versions of the horocyclic and geodesic flows that we will use are the following: Let $m,n \geq 1$ and let $X \in [0,1]^{mn}$, define

$$h_X^{m,n} = \begin{bmatrix} \mathrm{Id}_n & 0 \\ -X & \mathrm{Id}_m \end{bmatrix} \qquad \text{and} \qquad g_t^{m,n} = \begin{bmatrix} e^{\frac{t}{m+n}}\mathrm{Id}_n & 0 \\ 0 & e^{\frac{-nt}{m(m+n)}}\mathrm{Id}_m \end{bmatrix}.$$

The next lemma is a higher dimensional analog of Lemma 3.1.11.

Lemma 3.3.5. *With the notation as above,*

1. *for all* $t \in \mathbb{R}$, $F_\delta^{m,n}(g_t^{m,n}\Lambda) = e^{\frac{t}{m+n}} F_{\delta e^{\frac{t}{m}}}(\Lambda)$;

2. *for all* $X \in [0,1]^{mn}$, $Q^{m,n}(X,\delta) = F_\delta^{m,n}(h_X^{m,n}\mathbb{Z}^{m+n})$.

Proof. We begin with the proof of the first equation above. We first explore the effects of the geodesic flow on the cone $C_\delta^{m,n}$. Just as in Lemma 3.1.11, we have that $g_t^{m,n}$ expands our cone in the direction of $\mathbb{R}^m$ by $e^{\frac{t}{m}}$. More precisely, we have

$$g_t^{m,n} C_\delta^{m,n} = C_{\delta e^{-\frac{t}{m}}}^{m,n}. \tag{3.3.6}$$

Hence, we have the following computation:

$$\begin{aligned}
F_\delta^{m,n}(g_t^{m,n}\Lambda) &= \min\left\{\|\mathbf{u}\|_n : \begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} \in g_t^{m,n}\Lambda \cap C_\delta^{m,n}\right\}\\
&= \min\left\{\|\mathbf{u}\|_n : g_{-t}^{m,n}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} \in \Lambda \cap g_{-t}^{m,n} C_\delta^{m,n}\right\}\\
&= \min\left\{\|\mathbf{u}\|_n : \begin{bmatrix}e^{-\frac{t}{m+n}}\mathbf{u}\\ e^{-\frac{nt}{m(m+n)}}\mathbf{v}\end{bmatrix} \in \Lambda \cap C_{\delta e^{\frac{t}{m}}}^{m,n}\right\}\\
&= e^{\frac{t}{m+n}}\min\left\{\left\|\mathbf{u}'\right\|_n : \begin{bmatrix}\mathbf{u}'\\ \mathbf{v}'\end{bmatrix} \in \Lambda \cap C_{\delta e^{\frac{t}{m}}}^{m,n}\right\}\\
&= e^{\frac{t}{m+n}} F_{\delta e^{\frac{t}{m}}}^{m,n}(\Lambda),
\end{aligned}$$

as desired.

We now proceed to prove the second equation above. We first show that $Q^{m,n}(X,\delta) \geq F_\delta^{m,n}(h_X^{m,n}\mathbb{Z}^{m+n})$. Let $(\mathbf{q},\mathbf{p}) \in \mathbb{Z}^n \oplus \mathbb{Z}^m$ such that $\|X\mathbf{q}-\mathbf{p}\|_m < \delta\|\mathbf{q}\|_n$ and $\|\mathbf{q}\|_n$ is minimized. Then by definition, $Q^{m,n}(X,\delta) = \|\mathbf{q}\|_n$. We then have that

$$h_X\begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix} = \begin{bmatrix}\mathbf{q}\\ \mathbf{p}-X\mathbf{q}\end{bmatrix} \in h_X\mathbb{Z}^{m+n} \cap C_\delta^{m,n}.$$

Hence, we have that

$$F_\delta^{m,n}(h_X\mathbb{Z}^{m+}) \leq \|\mathbf{q}\|_n = Q_\delta^{m,n}(X,\delta).$$

Next, we show that $F_\delta^{m,n}(h_X\mathbb{Z}^{m+n}) \geq Q^{m,n}(X,\delta)$. Let $(\mathbf{a},\mathbf{b}) \in h_X\mathbb{Z}^{m+n} \cap C_\delta^{m,n}$ such that $\|\mathbf{a}\|_n$ is minimized. This means $F_\delta^{m,n}(h_X\mathbb{Z}^{m+n}) = \|\mathbf{a}\|_n$. This means that there exists $(\mathbf{q},\mathbf{p}) \in \mathbb{Z}^n \oplus \mathbb{Z}^m$ such that

$$h_X\begin{bmatrix}\mathbf{q}\\ \mathbf{p}\end{bmatrix} = \begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}.$$

That is $\mathbf{a} = \mathbf{q}$ and $\mathbf{p} - X\mathbf{q} = \mathbf{b}$. Since

$$\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = \|\mathbf{a}\|_n\begin{bmatrix}\frac{\mathbf{a}}{\|\mathbf{a}\|_n}\\ \frac{\mathbf{b}}{\|\mathbf{a}\|_n}\end{bmatrix} \in C_\delta^{m,n},$$

it follows that $\left\|\frac{\mathbf{b}}{\|\mathbf{a}\|_n}\right\|_m < \delta$ which means $\|\mathbf{b}\|_m < \delta\|\mathbf{a}\|_n$. This can be rewritten as $\|\mathbf{p} - X\mathbf{q}\|_m < \delta\|\mathbf{q}\|_n$. That is

$$Q^{m,n}(X,\delta) \le \|\mathbf{q}\|_n = \|\mathbf{a}\|_n = F_\delta^{m,n}(h_X\mathbb{Z}^{m+n}).$$

This completes the proof of Lemma 3.3.5. □

Notice that when $t = -\log(\delta^m)$, the first equation from Lemma 3.3.5 says that

$$F_\delta^{m,n}(g^{m,n}_{-\log(\delta^m)}\Lambda) = \delta^{-\frac{m}{m+n}} F_1^{m,n}(\Lambda). \tag{3.3.7}$$

Before proceeding to state the main theorem of this section, we state an equidistribution theorem due to Kleinbock and Margulis [23] which will play the role of Theorem 2.2.8 in the proof of Theorem 3.1.13.

Almost uniformly continuous functions: Let (Y,μ) be a topological space equipped with its Borel σ-algebra and a measure μ. We say a function $\Psi : Y \to \mathbb{R}$ is *almost uniformly continuous* if there exist sequences $\{\Psi_i\}_{i\in\mathbb{N}}$ and $\{\Psi^j\}_{j\in\mathbb{N}}$ of uniformly continuous functions on Y such that Ψ_i increases almost surely to Ψ and Ψ_j decreases almost surely to Ψ. In particular, with the setup above, if μ is a regular probability measure and A is a measurable subset of Y with $\mu(\partial A) = 0$, then the indicator function of A, χ_A, is almost uniformly continuous.

Lemma 3.3.8. ***(Kleinbock-Margulis 1996)*** *Let $f \in L^2(M_{m\times n}(\mathbb{R}))$ with compact support. Then for any almost uniformly continuous $\Psi \in L^2(X_{m+n})$, any compact subset L of X_{m+n}, and any $\varepsilon > 0$, the exists $T > 0$ such that*

$$\left| \int_{M_{m\times n}(\mathbb{R})} f(h_X)\Psi(g_t h_X\Lambda)dX - \int_{M_{m\times n}(\mathbb{R})} f(h_X)dX \int_{X_{m+n}} \Psi(\Lambda)d\mu_{m+n}(\Lambda) \right| < \varepsilon \tag{3.3.9}$$

for any $\Lambda \in L$ and all $t > T$.

Before proving Theorem 3.3.14, we give some notation and a lemma which will be useful in the proof. For $R, T \in \mathbb{R}$, we define the following sets:

1. $A_T = \{\Lambda \in X_{m+n} : F_1^{m,n}(\Lambda) \le T\}$,

2. $B_T = X_{m+n} \setminus A_T$,

3. $C^R = \overline{C_1^{m,n}} \cap \{[\mathbf{u}, \mathbf{v}]^T \in \mathbb{R}^n \oplus \mathbb{R}^m : \|\mathbf{u}\|_n \leq R\}$,

4. For each $\Lambda \in X_{m+n}$, $\Lambda^* = \Lambda \setminus \{\mathbf{0}\}$.

Lemma 3.3.10. *With the notation as above, we have the following:*

1. $\overline{A_T} \subset A_T \cup \{\Lambda \in X_{m+n} : \Lambda^* \cap \partial C_1^{m,n} \neq \emptyset\}$,

2. $\partial A_T \subset \{\Lambda \in X_{m+n} : F_1^{m,n} = T\} \cup \{\Lambda \in X_{m+n} : \Lambda^* \cap \partial C_1^{m,n} \neq \emptyset\}$,

3. $\mu_{m+n}(\partial A_T) = 0$.

Proof. We begin by proving the containment in 1. Fix $T > 0$. Let $\Lambda \in \overline{A_T}$. Suppose $\Lambda \in \overline{A_T}$ and

$\Lambda \notin \{\Lambda \in X_{m+n} : \Lambda^* \cap \partial C_1^{m,n} \neq \emptyset\}$. If $F_1^{m,n}(\Lambda) \leq T$, we are done, so suppose $F_1^{m,n}(\Lambda) = T_0 > T$. Let B be the closed ball in $\mathbb{R}^{m+n}$ centered at the origin with radius R, where R is chosen such that $C^{T_0} \subset \mathrm{Int}(B)$ and $\partial B \cap \Lambda = \emptyset$. Since B is compact, Λ is discrete, and $F_1^{m,n}(\Lambda) = T_0 < R$, it follows that $B \cap \Lambda$ is finite and non-empty. This implies that there exists $\delta > 0$ such that if $[\mathbf{u}, \mathbf{v}]^T \in B \cap \Lambda$ with $\|\mathbf{u}\|_n \in (T_0 - \delta, T_0 + \delta)$, then $\|\mathbf{u}\|_n = T_0$. Let $d = \mathrm{dist}(\partial B, \Lambda)$. By our choice of R we have that $d > 0$. Define $\eta = \frac{1}{2}\min\{d, \delta, 1, T_0 - T\}$. Consider B' to be the ball centered at the origin with radius $2R$. Since $\Lambda \cap B'$ is finite and the action of $SL(m+n, \mathbb{R})$ on $\mathbb{R}^{m+n}$ is continuous, there exists a neighborhood U of $\mathrm{Id} \in SL(m+n, \mathbb{R})$ such that if $\mathbf{w} \in B' \cap \Lambda$ and $g \in U$, $\|g\mathbf{w} - \mathbf{w}\| < \eta$. Since $\Lambda \in \overline{A_T}$, there exists a sequence $\Lambda_i \in A_T$ converging to Λ. Since Λ_i is converging to Λ, we can write $\Lambda_i = g_i\Lambda$ where $g_i \in U$ for i large enough. Then this implies that

$$|T_0 - T| \leq |F_1^{m,n}(\Lambda) - F_1^{m,n}(\Lambda_i)| < \eta < |T_0 - T|.$$

This is a contradiction. This means our assumption that $F_1^{m,n}(\Lambda) = T_0 > T$ is false. Hence, $F_1^{m,n}(\Lambda) \leq T$, so $\Lambda \in A_T$. This completes the proof of part 1 from the Lemma 3.3.10.

We now proceed to prove the containment in 2. One can use a similar argument as the one used in the proof of the first containment from Lemma 3.3.10 to show that

$$\overline{B_T} \subset B_T \cup \{\Lambda \in X_{m+n} : \Lambda^* \cap \partial C_1^{m,n} \neq \emptyset\}.$$

This then implies

$$\partial A_T = \overline{A_T} \cap \overline{B_T} \subset \{\Lambda \in X_{m+n} : F_1^{m,n}(\Lambda) = T\} \cup \{\Lambda \in X_{m+n} : \Lambda^* \cap \partial C_1^{m,n} \neq \emptyset\}. \quad (3.3.11)$$

Finally, we prove equation 3. Let $D = \left\{[\mathbf{u}, \mathbf{v}]^T \in \mathbb{R}^{m+n} : \|\mathbf{u}\|_n = T\right\} \cup \partial C_1^{m,n}$. Notice that D has Lebesgue measure 0 on $\mathbb{R}^{m+n}$, this then implies that $\{\Lambda \in X_{m+n} : \Lambda^* \cap D \neq \emptyset\}$ has measure 0 by the Siegel formula. Since

$$\begin{aligned} \partial A_T = \overline{A_T} \cap \overline{B_T} &\subset \{\Lambda \in X_{m+n} : F_1^{m,n}(\Lambda) = T\} \cup \{\Lambda \in X_{m+n} : \Lambda^* \cap \partial C_1^{m,n} \neq \emptyset\} && (3.3.12) \\ &= \{\Lambda \in X_{m+n} : \Lambda^* \cap D \neq \emptyset\}, && (3.3.13) \end{aligned}$$

We conclude that ∂A_T has measure zero.

□

The theorem due to Siegel used in Lemma 3.3.10 can be found as theorem 2 in [40]. This theorem was later generalized by Veech for a more general collection of discrete subsets of $\mathbb{R}^2$ in [43].

We are now ready to state and prove Theorem 3.3.14

Theorem 3.3.14. *Let P be the uniform probability measure on $[0,1]^{mn}$. Then as $\delta \to 0$,*

$$P\left(\left\{X \in M_{m\times n}([0,1]) : \delta^{\frac{m}{m+n}} Q^{m,n}(X,\delta) \leq T\right\}\right) \to \mu_{m+n}\left(\{\Lambda \in X_{m+n} : F_1^{m,n}(\Lambda) \leq T\}\right). \quad (3.3.15)$$

Proof. Lemma 3.3.5, allows us to interchange
$P\left(\left\{X \in M_{m\times n}([0,1]) : \delta^{\frac{m}{m+n}} Q^{m,n}(X,\delta) \leq T\right\}\right)$
for $P\left(\left\{X \in M_{m\times n}([0,1]) : \delta^{\frac{m}{m+n}} F_\delta^{m,n}(h_X \mathbb{Z}^{m+n}) \leq T\right\}\right)$.
By the third equation in Lemma 3.3.10, we get that χ_{A_T} is almost uniformly continuous.

Using equation 3.3.7, we get that

$$\begin{aligned}
&P\left(\left\{X \in M_{m\times n}([0,1]) : \delta^{\frac{m}{m+n}} F_\delta^{m,n}(h_X\mathbb{Z}^{m+n}) \leq T\right\}\right) \\
&\quad = P\left(\left\{X \in M_{m\times n}([0,1]) : F_\delta^{m,n}(g_{-\log(\delta^m)}h_X\mathbb{Z}^{m+n}) \leq T\right\}\right) \\
&\quad = \int_{[0,1]^{mn}} \chi_{A_T}\left(g_{-\log(\delta^m)}h_X\mathbb{Z}^{m+n}\right) dP(X).
\end{aligned}$$

We can then apply Lemma 3.3.8 by setting $f = \chi_{[0,1]^{mn}}$ and $\Psi = \chi_{A_T}$.

$$\lim_{\delta\to 0}\int_{[0,1]^{mn}} \chi_{[0,1]^{mn}}(X)\chi_A\left(g_{-\log(\delta^m)}h_X\mathbb{Z}^{m+n}\right) dP(X) = \int_{X_{m+n}} \chi_{A_T}(\Lambda)d\mu_{m+n}(\Lambda) = \mu(A_T).$$

as claimed. □

Corollary 3.3.16.

$$\lim_{\delta\to 0}\mathbb{E}_X\left[\delta^{\frac{m}{m+n}}Q^{m,n}(X,\delta)\right] = \int_{X_{m+n}} F_1^{m,n}(\Lambda)d\mu_{m+n}(\Lambda). \tag{3.3.17}$$

3.3.4 Existence of limiting distribution for Veech surfaces

Given $g > 0$ and an integer partition α of $2g-2$, we define $\mathcal{H}(\alpha)$ to be the moduli space of translation surfaces ω with genus g, area 1, and zeros with orders given by α. The space $\mathcal{H}(\alpha)$ has a natural topology and $SL(2,\mathbb{R})$ invariant Borel probability measure, which we call the Masur-Smillie-Veehc measure and denote by μ_{MSV}. Zorich [45] contains a detailed discussion of these structures on $\mathcal{H}(\alpha)$.

Of particular interest to us is the fact that the action of $SL(2,\mathbb{R})$ on each connected component of $\mathcal{H}(\alpha)$ is continuous and ergodic.

We focus our view first on Veech surfaces. Let ω be a Veech surface with Veech group Γ_ω. The $SL(2,\mathbb{R})$ orbit of ω is parameterized by $SL(2,\mathbb{R})/\Gamma_\omega$.

For simplicity, we will denote $SL(2,\mathbb{R})/\Gamma_\omega$ by Y_ω. It turns out the quotient space Y_ω is never compact and, therefore, it must have a positive finite number of cusps. This non-compactness guarantees the existence of closed horocyles [10]. With this in mind, we state an important theorem due to Dani-Smillie [11].

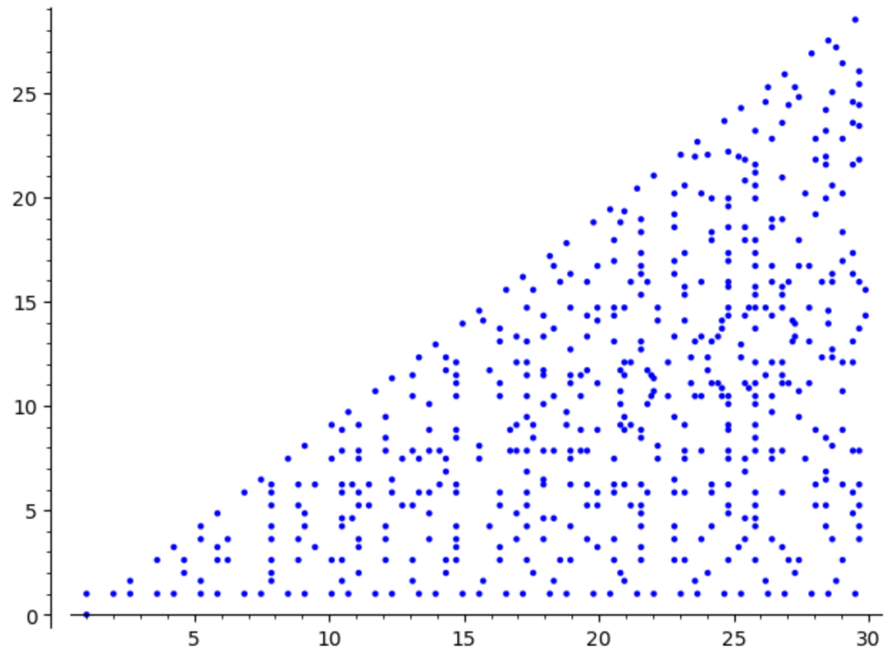

Figure 3.7: Depiction of the holonomy vectors with slope between 0 and 1 of a genus 2 translation surface.

Theorem 3.3.18. ***(Dani-Smillie 1984)*** *Let ω be a Veech surface and $(\omega_i)_{i\in\mathbb{N}}$ be a sequence of points in Y_ω. Suppose ω_i has period s_i under the action of the horocyclic flow. Let ν_i be the uniform measure on the orbit of ω_i. If $s_i \to \infty$, then*

$$\nu_i \overset{*}{\rightharpoonup} \frac{1}{\mu_\omega(Y_\omega)}\mu_\omega,$$

where μ_ω is the Haar measure on Y_ω.

A natural extension of the function $\tilde{q}_{\min}$ in the context of translation surfaces is the following:

Question: What is the point with the smallest x-coordinate in $\Lambda_\omega \cap h_{-s}C_\delta$ as s ranges over $\mathbb{R}$?

Due to the Veech dichotomy [21], we know that in the context of Veech surfaces, we can always find (up to rotating the surface if necessary) an $\alpha > 0$ such that h_α belongs to the

Veech group of our surface.

Let $\delta > 0$ and consider the following function:

$$\Psi(\omega,\delta) = \min\left\{\mathrm{Re}(z_\gamma) : \gamma \text{ is a saddle connection of } \omega \text{ and } z_\gamma \in \Lambda(\omega)\cap C_\delta\right\}. \qquad (3.3.19)$$

Lemma 3.3.20. *Let ω be a translation surface, then*

1. *for all $t \in \mathbb{R}$, $\Psi(g_t\omega,\delta) = e^{\frac{t}{2}}\Psi(\omega, e^t\delta)$;*

2. *For all $\delta > 0$, $\Psi(g_{-\log\delta}\omega,\delta) = \sqrt{\delta}\Psi(\omega,1)$.*

Theorem 3.3.21. *Let ω be a Veech surface and suppose that $h_\alpha \in \Gamma_\omega$ for some $\alpha > 0$. Let P be the uniform probability measure on $[0,\alpha]$. Then as $\delta \to 0$,*

$$P\left(\left\{s\in[0,\alpha] : \sqrt{\delta}\Psi(h_s\omega,\delta)\le T\right\}\right) \to \frac{\mu_\omega\left(\{g\Gamma_\omega \in Y_\omega : \Psi(g\Gamma_\omega,1)\le T\}\right)}{\mu_\omega\left(Y_\omega\right)}.$$

Proof. This proof has the same strategy as that of Theorem 3.1.13, we change to the appropriate equidistribution theorem to pass to the limit.

Let

$$A = \{k\Gamma_\omega \in Y_\omega : \Psi(k\omega,1)\le T\}.$$

We now proceed to the computation

$$\begin{aligned}
P\left(\left\{s\in[0,\alpha] : \sqrt{\delta}\Psi(h_s\omega,\delta)\le T\right\}\right) &= P\left(\left\{s\in[0,\alpha] : \sqrt{\delta}\Psi(g_{-\log\delta}g_{\log\delta}h_s\omega,\delta)\le T\right\}\right)\\
&= P\left(\{s\in[0,\alpha] : \Psi(g_{\log\delta}h_s\omega,1)\le T\}\right)\\
&= \frac{1}{\alpha}\int_0^\alpha \chi_A(g_{\log\delta}h_s\omega)ds.
\end{aligned}$$

Since $g_{\log\delta}h_s = h_{\frac{s}{\delta}}g_{\log\delta}$, we have that the period of $g_{\log\delta}\omega$ under the horocyclic flow is $\frac{\alpha}{\delta}$ and hence by Theorem 3.3.18, the orbit $g_{\log\delta}\omega$ under the horocyclic flow is becoming equidistributed as $\delta\to 0$. This means that

$$\lim_{\delta\to 0}\frac{1}{\alpha}\int_0^\alpha \chi_A(g_{\log\delta}h_s\omega)ds = \lim_{\delta\to 0}\frac{\delta}{\alpha}\int_0^{\frac{\alpha}{\delta}}\chi_A(h_s g_{\log\delta}\omega)ds = \frac{1}{\mu_\omega(Y_\omega)}\int_{Y_\omega}\chi_A(g\Gamma_\omega)d\mu_\omega.$$

And we have that

$$\frac{1}{\mu_\omega(Y_\omega)}\int_{Y_\omega}\chi_A(g\Gamma_\omega)d\mu_\omega = \frac{\mu_\omega\left(\{g\Gamma_\omega \in Y_\omega : \Psi(g\Gamma_\omega, 1) \le T\}\right)}{\mu_\omega\left(Y_\omega\right)},$$

which is what we wanted to show. $\square$

Corollary 3.3.22. *Let ω be a Veech surface and suppose that $h_\alpha \in SL(\omega)$ for some $\alpha > 0$, then as $\delta \to 0$,*

$$\mathbb{E}_x\left[\sqrt{\delta}\Psi(h_s\omega, \delta)\right] = \frac{\sqrt{\delta}}{\alpha}\int_0^\alpha \Psi(h_s\omega, \delta)ds \to \frac{1}{\mu_\omega(X_\omega)}\int_{X_\omega}\Psi(g\Gamma_\omega, 1)d\mu_\omega.$$

3.3.5 Ratios of Gaussian Integers

Let $\mathbb{C}$ denote the field of complex numbers endowed with its Euclidean norm. In this section, we look at a complex version of the MDF from §3.1. Let $|x+iy| = \sqrt{x^2+y^2}$, the usual norm on complex numbers. Define the *complex minimal denominator function* $q_{\min}^{\mathbb{C}} : [0,1]^2 \to \mathbb{R}$ by:

$$q_{\min}^{\mathbb{C}}(z,\delta) = \min\left\{|c+di| > 0 : c+di \in \mathbb{Z}[i] \text{ and there exists } a+bi \in \mathbb{Z}[i] \text{ such that } \left|z - \frac{a+bi}{c+di}\right| < \delta\right\}. \tag{3.3.23}$$

Here we view $[0,1]^2$ as a subset of $\mathbb{C}$.

The elements of $\mathbb{Z}[i]$ are called *Gaussian integers* and they form a Euclidean domain, which allows us to make the following definition:

Definition 3.3.24. *Let $a+bi$, $c+di \in \mathbb{Z}[i]$ such that $c+di \neq 0$. We say the ratio $\frac{a+bi}{c+di}$ is in **simplest form** if whenever $w \in \mathbb{Z}[i]$ such that w divides $a+bi$ and $c+di$, it follows that w is a unit.*

Note: The set of units in $\mathbb{Z}[i]$ is the set $M = \{1, -1, i, -i\}$.

$\mathbb{Z}[i]$ is the ring of integers in $\mathbb{Q}[i]$. In particular, this means that any elements in $\mathbb{Q}[i]$ can be written as a ratio of elements in $\mathbb{Z}[i]$.

With this in mind, we can write a complex version of the Farey sequence. We define $\mathcal{F}^{\mathbb{C}} = (\mathcal{F}_Q^{\mathbb{C}})_{Q\ge 1}$, where

$$\mathcal{F}_Q^{\mathbb{C}} = \left\{\frac{a+bi}{c+di} \in [0,1]^2 : c^2+d^2 \le 2Q^2\right\}. \tag{3.3.25}$$

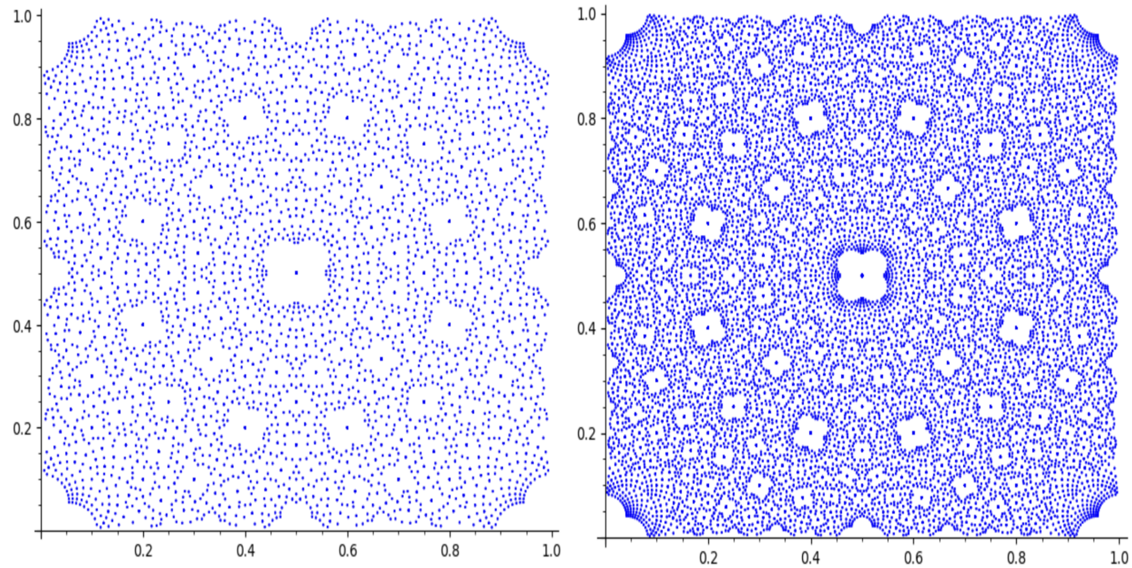

Figure 3.8: Left: Depiction of $\mathcal{F}_7^{\mathbb{C}}$. Right: Depiction of $\mathcal{F}_{12}^{\mathbb{C}}$.

Reduction to homogeneous dynamics: Denote by $SL(2,\mathbb{C})$ the group of 2×2 matrices with complex coefficients and determinant 1. We are interested in the following subgroup

$$SL(2,\mathbb{Z}[i]) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}[i], ad-bc = 1 \right\}. \tag{3.3.26}$$

Definition 3.3.27. *A* ***complex 2-dimensional unimodular lattice*** *is a subgroup $\Lambda \subset \mathbb{C}^2$ such that $\Lambda = g\mathbb{Z}[i]^2$ for some $g \in SL(2,\mathbb{C})$. We denote the set of complex 2-dimensional unimodular lattices by $\mathscr{X}_2^{\mathbb{C}}$.*

Just like with (real) unimodular lattices, $\mathscr{X}_2^{\mathbb{C}}$ is in bijective correspondence with the quotient space $SL(2,\mathbb{C})/SL(2,\mathbb{Z}[i])$. In particular, we can transplant the topology and Haar measure from $SL(2,\mathbb{C})/SL(2,\mathbb{Z}[i])$ to $\mathscr{X}_2^{\mathbb{C}}$. We denote the Haar measure on $\mathscr{X}_2^{\mathbb{C}}$ by $\mu_2^{\mathbb{C}}$. Humbert proved that $\mathscr{X}_2^{\mathbb{C}}$ has finite measure. [18] contains a proof of this statement in more generality. We will discuss this further in §3.3.5.

With this in mind, we are able to define the following function: $F_\delta^{\mathbb{C}} : \mathscr{X}_2^{\mathbb{C}} \to \mathbb{R}$ by

$$F_\delta^{\mathbb{C}}(\Lambda) = \min \left\{ |z| : \begin{bmatrix} z \\ w \end{bmatrix} \in \Lambda \cap C_\delta^{\mathbb{C}} \right\}, \tag{3.3.28}$$

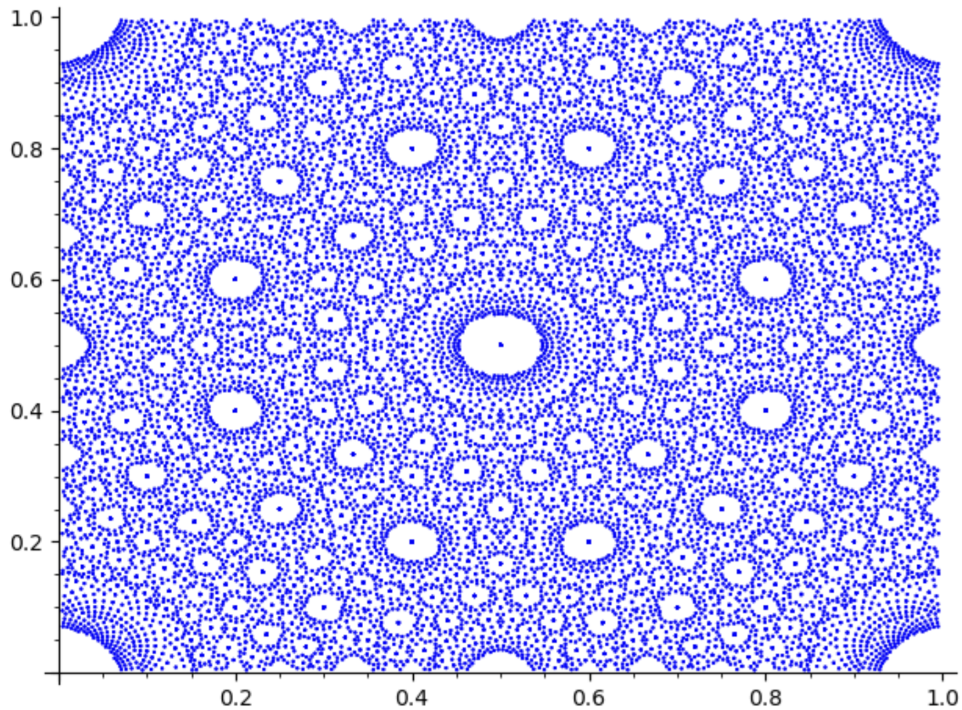

Figure 3.9: Depiction of $\mathcal{F}_{15}^{\mathbb{C}}$

where $C_\delta^{\mathbb{C}} = \left\{ r \begin{bmatrix} s \\ t \end{bmatrix} : r \in \mathbb{C}^*, |s| = 1, |t| < \delta \right\}$.

Our versions of the horocyclic and geodesic flows in the complex world will be given by $H = \left\{ \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} : s \in \mathbb{C} \right\}$ and $A = \left\{ \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{bmatrix} : t \in \mathbb{R} \right\}$.

This provides us with the following lemma that mirrors Lemma 3.1.11.

Lemma 3.3.29. *For all $z \in [0,1]^2$, $t \in \mathbb{R}$, we have that*

1. *for all $t \in \mathbb{R}$, $F_\delta^{\mathbb{C}}(g_t\Lambda) = e^{\frac{t}{2}} F_{\delta e^t}^{\mathbb{C}}(\Lambda)$;*

2. *for all $z \in [0,1]^2$, $q_{\min}^{\mathbb{C}}(z,\delta) = F_\delta^{\mathbb{C}}(h_z \mathbb{Z}[i]^2)$.*

The proof of Lemma 3.3.29 is similar to the proof of Lemma 3.1.11, so we omit it.

The weak* convergence theorem that we will use in order to apply Theorem 3.2.6 will be Theorem 0.1 from [36].

We first observe that the orbit of $\mathbb{Z}[i]^2$ under H is an embedded torus in $\mathscr{X}_2^{\mathbb{C}}$. It follows that the orbit of $g_t\mathbb{Z}[i]$ under H is also an embedded torus in $\mathscr{X}_2^{\mathbb{C}}$. Theorem 0.1 from [36] tells us the uniform probability measures on these tori converge weak* to the Haar measure $\mu_2^{\mathbb{C}}$. We are able to state the following theorem:

Theorem 3.3.30. *Let P denote the uniform probability measure on $[0,1]^2 \subset \mathbb{C}$ and $T \in \mathbb{R}$. Then*

$$\lim_{\delta \to 0} P(\{z \in [0,1]^2 : \sqrt{\delta} q_{\min}^{\mathbb{C}}(x,\delta) \leq T\}) = \mu_2^{\mathbb{C}}(\{\Lambda \in \mathscr{X}_2^{\mathbb{C}} : F_1^{\mathbb{C}}(\Lambda) \leq T\}). \tag{3.3.31}$$

The proof of Theorem 3.3.30 is similar to the proof of Theorem 3.1.5 so we omit it.

3.3.6 Heegner Fields

We can extend the arguments about approximating complex numbers via ratios of Gaussian integers to the family of number fields called Heegner Fields. First, we will provide some preliminary definitions.

Definition 3.3.32. *An* ***algebraic number field*** *is a finite extension of $\mathbb{Q}$.*

Definition 3.3.33. *Let K be an algebraic number field which is also a subset of $\mathbb{C}$. The* ***ring of integers*** *of K, denoted by $\mathcal{O}_K$, is the collection of roots of monic polynomials with integer coefficients that belong to K.*

Theorem 3.3.34. *Let $D \geq 1$ be a square-free integer. If $K = \mathbb{Q}[\sqrt{-d}]$, then*

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}\left[\sqrt{-D}\right] & -D \equiv 2,3 \mod 4, \\ \\ \mathbb{Z}\left[\frac{1+\sqrt{-D}}{2}\right] & -D \equiv 1 \mod 4. \end{cases} \tag{3.3.35}$$

We are particularly interested in the case where $\mathcal{O}_K$ is a *unique factorization domain* or *UFD*, for short. The following result was conjectured by Gauss. The first proof came from Heegner in 1952 with some minor errors. Baker [6] and Stark [41] provided the first full proof in 1966 and 1967, respectively.

Theorem 3.3.36. ***(Heegner 1952, Baker 1966, Stark 1967)*** *Let $D \in \mathbb{N}$ be a square-free integer. The ring of integers of $\mathbb{Q}[\sqrt{-D}]$ is a UFD if and only if $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$.*

If D is one of the elements of the list in Theorem 3.3.36, we refer to $K = \mathbb{Q}[\sqrt{-D}]$ as a *Heegner field*.

Given a complex number $z \in K$, we may write it as a fraction $z = \frac{a+b\sqrt{-D}}{c+d\sqrt{-D}}$, where $a + b\sqrt{-D}, c + d\sqrt{-D} \in \mathcal{O}_K$. If this fraction has the property that if $w \in \mathcal{O}_K$ and w divides $a + b\sqrt{-D}$ and $c + d\sqrt{-D}$, then w is a unit, we call $z = \frac{a+b\sqrt{-D}}{c+d\sqrt{-D}}$ a *simplest form* (just like in the case of $\mathbb{Q}[i]$ and $\mathbb{Z}[i]$).

We can then define a version of the MDF on Heegner fields. For $z \in [0, 1]^2 \subset \mathbb{C}$ and $\delta > 0$,

$$q^K_{\min}(z, \delta) = \min \left\{ |q| : q \in \mathcal{O}_K, \text{ and there exists } p \in \mathcal{O}_K \text{ such that } \left| z - \frac{p}{q} \right| < \delta \right\}. \quad (3.3.37)$$

The homogeneous space that we will use to compute the distribution of $\sqrt{\delta} q^K_{\min}$ will be similar to the one for the Gaussian integers.

Definition 3.3.38. *The special linear group of 2 by 2 matrices with entries in $\mathcal{O}_K$ is given by*

$$SL(2, \mathcal{O}_K) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathcal{O}_K, ad - bc = 1 \right\}.$$

We also define the projective special linear group by

$$PSL(2, \mathcal{O}_K) = SL(2, \mathcal{O}_K)/\{\pm \mathrm{Id}\}.$$

Definition 3.3.39. *Let $K = \mathbb{Q}[\sqrt{-D}]$ where D is a square free-positive integer. The* ***Bianchi group*** *of K is $PSL(2, \mathcal{O}_K)$. The topological space $PSL(2, \mathbb{C})/PSL(2, \mathcal{O}_K)$ is called the* ***Bianchi Orbifold*** *of K.*

For the remainder of this section, we will assume K is always a Heegner field.

It was first proved by Humbert that $PSL(2, \mathbb{C})/PSL(2, \mathcal{O}_K)$ has a finite Haar measure. [18] contains an explicit description of the volume in terms of the Dedekind-Zeta function of the field K.

Definition 3.3.40. *A **Bianchi** K**-lattice**, or just **Bianchi lattice**, is a subgroup $\Lambda \subset \mathbb{C}^2$ such that there exists $g \in SL(2,\mathbb{C})$ such that $\Lambda = g\mathcal{O}_K^2$. We denote the set of Bianchi K-lattices by $\mathscr{X}_2^K$.*

Notice also that the homogeneous $SL(2,\mathbb{C})/SL(2,\mathcal{O}_K)$ and $PSL(2,\mathbb{C})/PSL(2,\mathcal{O}_K)$ are isomorphic and so we identify them both with $\mathscr{X}_2^K$.

With this in mind, we define $F_\delta^K : \mathscr{X}_2^K \to \mathbb{R}$ by

$$F_\delta^K(\Lambda) = \min\left\{ |z| : \begin{bmatrix} z \\ w \end{bmatrix} \in \Lambda \cap C_\delta^{\mathbb{C}} \right\}, \tag{3.3.41}$$

Since Theorem 0.1 from [36] deals precisely with Bianchi orbifolds of Heegner fields, we are able to extend Theorem 3.3.30 to the following theorem

Theorem 3.3.42. *Let K be a Heegner field. Then*

$$\lim_{\delta\to 0} P\left(\{z \in [0,1]^2 : \sqrt{\delta} q_{\min}^K(x,\delta) \le T\}\right) = \mu_2^K\left(\{\Lambda \in \mathscr{X}_2^K : F_1^K(\Lambda) \le T\}\right), \tag{3.3.43}$$

where μ_2^K is the Haar probability measure of $\mathscr{X}_2^K$.

Note on Non-Heegner Fields: One can ask about approximating complex numbers via quadratic extensions of $\mathbb{Q}[i]$ that are not Heegner fields, however, we no longer have a well-defined notion of minimal denominator since ratios of algebraic integers. For instance, in the field $\mathbb{Q}[\sqrt{-5}]$ we have the same complex number $\frac{2}{1-\sqrt{-5}} = \frac{1+\sqrt{-5}}{3}$ written in two different ways. One can still write functions to describe approximations of complex numbers in these fields, but the notion of best approximation will depend on the way we decide to represent the elements of $\mathbb{Q}[\sqrt{D}]$.

Chapter 4

DENSITY FUNCTIONS OF SHORT HOLONOMY VECTORS

The previous chapter was devoted to proving the existence of the limiting distributions of $q_{\min}$ and its generalizations to different contexts. This was done by computing the size of different subsets of homogeneous spaces. To do this, we begin by providing some background on the relation between the distribution of short holonomy vectors and the slope gap distributions of translation surfaces

4.1 Void and Gap Distributions

In this section, we summarize some results from [27] and [28] which we will be needing in chapter 4.

Let $(\xi_j)_{j=1}^{\infty}$ be a sequence of real numbers in $[0,1]$. For $N \in \mathbb{N}$, we let $\Xi_N = \{\zeta_1 \leq \zeta_2 \leq ... \leq \zeta_N\}$ be the reordering of $\{\xi_j : 1 \leq j \leq N\}$ by the natural order of $[0,1]$. We assume that the sequence Ξ_N equidistributes in $[0,1]$ with respect to the Lebesgue measure.

We may think of equidistribution as a first test for the randomness of the sequence ξ_j since the law of large numbers states that a sequence of i.i.d. random variables will equidistribute in [0,1] almost surely. However, there are deterministic sequences of variables that equidistribute in $[0,1]$ that do not arise from i.i.d. random variables. An example of such a sequence is the Farey sequence [24] or the sequence $\{n\alpha \mod 1 : n \in \mathbb{N}\}$ where $\alpha \notin \mathbb{Q}$ [14].

A finer test of the randomness of sequences in $[0,1]$ is the distribution of the gaps between consecutive elements in Ξ_N as N tends to infinity.

Gap Distributions: Suppose that sequence of sequences Ξ_N equidistributes in [0,1]. We would like to know the statistics of the *gaps* between consecutive terms. Since adjacent elements in Ξ_N become closer and closer as $N \to \infty$, it would be easier to renormalize the size of the gaps between consecutive elements so that we can think about gap sizes in $[0,\infty)$ instead. Since $|\Xi_N| = N$, it seems reasonable to normalize the gaps by looking at $N(\zeta_{j+1} - \zeta_j)$. Then we can ask, what proportion of renormalized gaps in Ξ_N are less than a given value L. We refer to this distribution as the gap distribution of Ξ_N. More formally, we have the following definition. Let $\Xi = (\Xi_N)_{N=1}^{\infty}$.

Definition 4.1.1. *Let Ξ_N be as defined above. The **gap distribution** of Ξ_N is given by*

$$\Phi_N(L) = \frac{|\{j < N : N(\zeta_{j+1} - \zeta_j) \le L\}|}{N}. \tag{4.1.2}$$

*We define the **limiting gap distribution** of $\Xi = (\Xi_N)_{N=1}^{\infty}$ by*

$$\Phi(L) = \lim_{N\to\infty} \Phi_N(L), \tag{4.1.3}$$

whenever it exists.

In the case in which Ξ arises from i.i.d. random, we know that the limiting gap distribution exists almost surely and is given by $\Phi(L) = Le^{-L}$. However, when Ξ is chosen to be the Farey sequence instead, $\Phi(L)$ is given by the Hall distribution 3.3.35. This tells us that the behavior of the Farey sequence although looking random from the perspective of equidistribution, its gaps do not mimic those of a sequence of i.i.d. random variables.

Void Distributions: The next question we will ask is the following: Let $L > 0$, what is the probability that an interval of length $\ell = \frac{L}{N}$ does *not* intersect Ξ_N? We can model this question by choosing a point $x \in [0,1]$ uniformly randomly and looking at the interval $[x, x+\ell)$ to see if it intersects Ξ_N. This can be generalized to higher dimensions, but we will stick to the one-dimensional case. For a higher dimensional description, one may see [28].

Definition 4.1.4. *Let P be the uniform measure on $[0,1]$. Define the **void distribution** of Ξ_N to be*

$$E_N(L) = P\left(\{x \in [0,1] : [x, x+\ell) \cap \Xi_N = \emptyset\}\right), \tag{4.1.5}$$

*and the **void distribution** of Ξ to be*

$$E(L) = \lim_{N\to\infty} E_N(L), \tag{4.1.6}$$

whenever it exists.

It turns out that both the limiting gap distribution and limiting void distributions of the sequence Ξ are related. The derivative of the void distribution is the gap distribution.

Theorem 4.1.7. *With the notation above,*

$$-\frac{d}{dL}E(L) = \Phi(L). \tag{4.1.8}$$

4.2 The density function of short holonomy vectors

Let ω be a translation surface. Define

$$\mathbb{P}\Lambda_\omega^R = \left\{ \frac{b}{a} : a + bi \in \Lambda_\omega, 0 \le \frac{b}{a} \le 1, 0 < a \le R \right\}. \tag{4.2.1}$$

Athreya-Chaika showed in [1] the following:

Let ω be a translation surface with genus $g > 0$. It follows that ω has $2g-2$ zeros, counting multiplicity. Let α denote the partition of $2g-2$ given in this way by the zeros of ω. $\mathcal{H}(\alpha)$ denotes the space of translation surfaces with zeros of orders given by α. We denote by $\mathcal{H}(\omega)$ the connected component of $\mathcal{H}(\alpha)$ that contains ω.

As described in §2.4, $SL(2,\mathbb{R})$ acts on $\mathcal{H}(\alpha)$ continuously. In fact, there exists an $SL(2,\mathbb{R})$ ergodic probability measure on $\mathcal{H}(\alpha)$. This measure is called the Masur-Smillie-Veech and we denote it by μ_{MSV}.

Athreya-Chaika proved in [1] the following theorem:

Theorem 4.2.2. ***(Athreya-Chaika 2012)*** *Let $g > 0$ and $\mathcal{H}$ be a connected component of the strata of translation surfaces of genus g. Let $\mu = \mu_{MSV}$ be the $SL(2,\mathbb{R})$-invariant probability measure on $\mathcal{H}$. For μ a.e. ω, the limiting gap distribution for $\mathbb{P}\Lambda_\omega^R$ exists.*

Not much is known about the limiting gap distribution of a generic translation surface, however, a lot more is known in the context when ω is chosen to be a Veech surface.[1], [2], [3], [42], and [25] contain computations for the gap distributions of different translation surfaces. [38] has computed the limiting gap distribution for the slopes of the holonomy vectors of double slit tori, which is an example of a non-Veech surface. We use their results together with the results of Markloff [27] and [28] to get explicit formulas for the distribution of short holonomy vectors of translation surfaces.

Lemma 4.2.3. *Let ω_1 and ω_2 be two translation surfaces such that $\omega_2 = g_{t_0}\omega_1$ for some $t \in \mathbb{R}$. Then*

$$P\left(\left\{s \in [0,1] : \sqrt{\delta}\Psi(h_s\omega_1, \delta) \leq T\right\}\right) = P\left(\left\{s \in [0, e^{-t_0}] : \sqrt{\delta}\Psi(h_s\omega_2, \delta) \leq T\right\}\right). \quad (4.2.4)$$

Proof. This proof is via computation. We rely on Lemma 3.3.20 and equation 2.2.9.

$$\begin{aligned}
P\left(\left\{s \in [0, e^{-t_0}] : \sqrt{\delta}\Psi(h_s\omega_2, \delta) \leq T\right\}\right) &= P\left(\left\{s \in [0, e^{-t_0}] : \sqrt{\delta}\Psi(g_{t_0}g_{-t_0}h_s\omega_2, \delta) \leq T\right\}\right) \\
&= P\left(\left\{s \in [0, e^{-t_0}] : \sqrt{\delta}\Psi(g_{t_0}g_{-t_0}h_s g_{t_0}\omega_1, \delta) \leq T\right\}\right) \\
&= P\left(\left\{s \in [0, e^{-t_0}] : \sqrt{\delta}\Psi(g_{t_0}h_{se^{t_0}}\omega_1, \delta) \leq T\right\}\right) \\
&= P\left(\left\{s \in [0, 1] : \sqrt{\delta}\Psi(g_{t_0}h_s\omega_1, \delta) \leq T\right\}\right) \\
&= P\left(\left\{s \in [0, e^{-t_0}] : \sqrt{\delta}e^{\frac{t_0}{2}}\Psi(h_s\omega_2, \delta e^{t_0}) \leq T\right\}\right) \\
&= P\left(\left\{s \in [0, e^{-t_0}] : \sqrt{\delta}\Psi(h_s\omega_1, \delta) \leq T\right\}\right).
\end{aligned}$$

This completes the proof. □

In the context when ω is a Veech surface, Lemma 4.2.3 allows us to assume that ω has period 1 under the horocyclic flow. For the remainder of this section, we will make this assumption.

Using the work of Veech [43] we know there exists $c > 0$ such that $\left|\mathbb{P}\Lambda_\omega^R\right| \sim cR^2$ as R grows to infinity. We will refer to c as the *projected Veech constant.*

Theorem 4.2.5. *Let ω be Veech surface such that $h_1\omega = \omega$ with projected Veech constant c. Let Φ be the limiting slope gap distribution of $\mathbb{P}\Lambda_\omega^R$. Then*

$$\lim_{\delta \to 0} P\left(s \in [0,1] : \sqrt{\delta}\Psi\left(h_s\omega\right) \geq T\right) = 2c\int_T^\infty t\Phi\left(ct^2\right)dt. \quad (4.2.6)$$

Proof. Notice that

$$\begin{aligned}
\Psi(h_s\omega) \geq T\delta^{-\frac{1}{2}} &\Leftrightarrow \left\{q + pi \in \Lambda : 0 < q \leq T\delta^{-\frac{1}{2}}, 0 \leq p \leq q, \frac{p}{q} \in \left(s - \frac{\delta}{2}, s + \frac{\delta}{2}\right)\right\} = \emptyset \\
&\Leftrightarrow \left(s - \frac{\delta}{2}, s + \frac{\delta}{2}\right) \cap \mathbb{P}\Lambda_\omega^{T\delta^{-\frac{1}{2}}} = \emptyset.
\end{aligned}$$

Let $Q = T\delta^{-\frac{1}{2}}$ and $x = cT^2$. Then $\delta = \frac{x}{cQ^2}$. Hence,

$$\Psi(h_s\omega) \geq T\delta^{-\frac{1}{2}} \Leftrightarrow \left(s - \frac{x}{2cQ^2}, s + \frac{x}{2cQ^2}\right) \cap \mathbb{P}\Lambda_\omega^Q = \emptyset. \tag{4.2.7}$$

Then,

$$\begin{aligned}\lim_{\delta\to 0} P\left(s \in [0,1] : \sqrt{\delta}\Psi\left(h_s\omega\right) \geq T\right) &= \lim_{Q\to\infty} P\left(\left\{s \in [0,1] : \left(s - \frac{x}{2cQ^2}, s + \frac{x}{2cQ^2}\right) \cap \mathbb{P}\Lambda_\omega^Q = \emptyset\right\}\right) \\ &= E(x).\end{aligned}$$

Theorem 3.3.21 allows us to know that the function $E(x)$ exists. We will now use Theorem 4.1.7 to compute explicit formulas for $E(x)$. If ω is a Veech surface, we know that the limiting gap distribution for ω exists [25] and is given by a continuous piece-wise real analytic function Φ with finitely many points of non-analyticity. More generally, given any connected stratum $\mathcal{H}$, with its Masur-Smillie-Veech invariant probability measure, μ_{MSV}, almost every flat surface has a limiting slope gap distribution. Let Φ be the slope gap distribution of ω. Then,

$$\begin{aligned}\frac{d}{dx}E(x) &= -\Phi(x) \\ E(x) &= \int_x^\infty \Phi(t)dt \\ &= \int_{cT^2}^\infty \Phi(t)dt \\ &= 2c\int_T^\infty u\Phi(cu^2)du.\end{aligned}$$

Hence,

$$\lim_{\delta\to 0} P\left(s \in [0,1] : \sqrt{\delta}\Psi\left(h_s\omega\right) \geq T\right) = 2c\int_T^\infty u\Phi(cu^2)du, \tag{4.2.8}$$

as desired. □

Next, we provide some explicit examples of surfaces for which we can compute explicitly the limiting short vector distribution.

4.2.1 Torus

The projective Veech constant for the torus is $c = \frac{3}{\pi^2}$, which is also the growth rate of the Farey sequence. The limiting gap distribution for the Farey sequence is given by Hall's distribution 3.3.35, and is denoted by $H(L)$. Hence, we recover the Theorem 3.1.7 by Marklof.

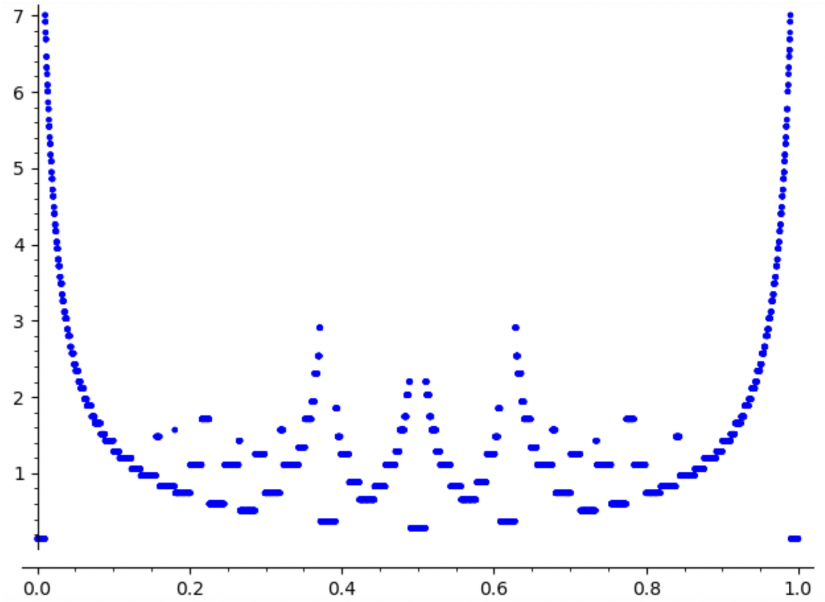

Figure 4.1: Graph of $\frac{1}{10}\Psi(h_s\omega, 0.01)$ where ω is the golden L and s ranges between 0 and 1.

4.2.2 Golden L

Let ω be the Golden L as described in [2].

$$\Lambda_\omega = SL(\omega)\begin{bmatrix}1\\0\end{bmatrix} \sqcup SL(\omega)\begin{bmatrix}\frac{1}{\varphi}\\0\end{bmatrix},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Since $[1,0]^T$ and $[\varphi^{-1},0]^T$ are colinear, it suffices to only look at the $SL(\omega)$ orbit of $[\varphi^{-1},0]^T$.

In particular

$$\left|\mathbb{P}\Lambda_\omega^R\right| = \left|SL(\omega)[\varphi^{-1},0]^T \cap T_R\right|,$$

where $T_R = \{(x,y) \in \mathbb{R}^2 : 0 \leq \frac{y}{x} \leq 1, 0 < x \leq R)\}$.

Using Theorem 16.1 and 16.4 from [43], we are able to compute

$$\left|\mathbb{P}\Lambda_\omega^R\right| = \left|SL(\omega)[\varphi^{-1},0]^T \cap T_R\right| \sim \left(\frac{10\varphi^3}{3\pi}\right)\left(\frac{1}{2}R^2\right) = \frac{5\varphi^3}{3\pi}R^2.$$

Therefore, we have that the distribution of short vectors for the golden L is given by

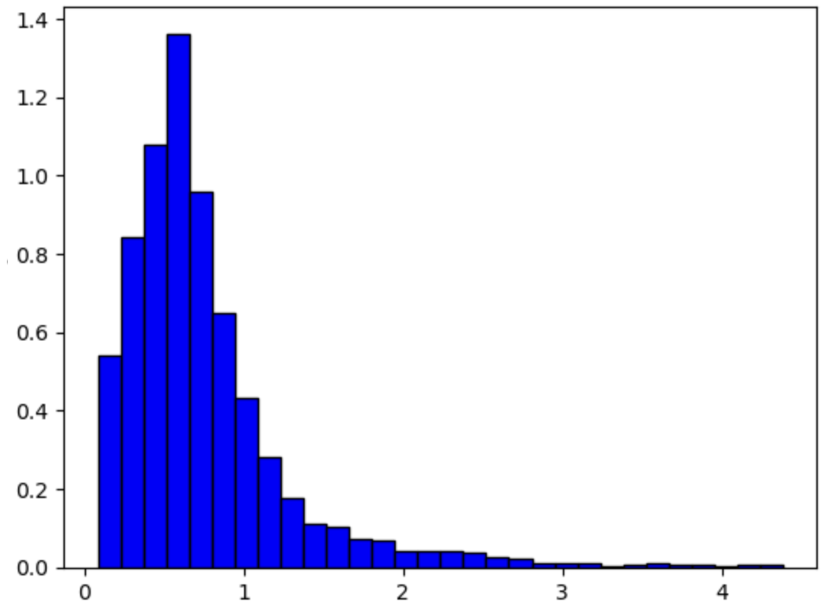

Figure 4.2: Probability Density function of the short holonomy vectors for the Golden L with $\delta = 0.01$

Corollary 4.2.9. *Let ω be the golden L and let Φ be the limiting gap distribution of ω. Then*

$$\lim_{\delta \to 0} P\left(s \in [0,1] : \sqrt{\delta}\Psi\left(h_s\omega\right) > T\right) = \frac{10\varphi^3}{3\pi}\int_T^\infty t\Phi\left(\frac{5\varphi^3}{3\pi}t^2\right) dt. \tag{4.2.10}$$

The limiting slope gap distribution of the golden L was computed explicitly in [2]. In particular, they provide explicit formulas for Φ in terms of elementary functions in Appendix A. Figure 4.2 depicts a histogram for the distribution of the short holonomy vectors of the golden L with $\delta = 0.01$.

4.2.3 Generic Translation Surfaces

Let α be an integer partition of $g > 1$. Let $\mathcal{H}$ be a connected component of the strata of genus g translation surfaces with zeros of order α. We endow $\mathcal{H}$ with the Smillie-Masur-Veech probability measure μ_{SMV}. We know the following data about the surfaces in $\mathcal{H}$.

1. μ_{SMV} almost every $\omega \in \mathcal{H}$ has the same limiting slope gap distribution. This can be found in [1].

2. μ_{SMV} almost every $\omega \in \mathcal{H}$ has a Veech constant, which in particular means it has a projected Veech constant. This was proved by Eskin-Masur in [16].

The distribution from part 1 is not explicitly known, but we know it has support at zero. This differs from the Veech case, where we know that the distribution does not have support at zero.

The computation of Veech constants for generic translation surfaces was done in [17] by Eskin-Masur-Zorich.

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