# A Microlocal Analysis of the Lévy Generator with Conjugate Points

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**Abstract**

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We analyze the microlocal structure of the infinitesimal generator of a Lévy process on a compact, connected Riemannian manifold when conjugate points are allowed. We show that if there are no singular conjugate pairs, then the infinitesimal generator equals two pseudodifferential operators plus an infinite sum of Fourier integral operators. This extends known results for the flat torus, the round sphere, and Anosov manifolds.

# Contents

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# Dedication

To the humans, dogs, and cats who made this possible.

# Chapter 1

# Introduction

In this dissertation, we will study the infinitesimal generator of a Lévy process on a Riemannian manifold $(M, g)$. Loosely speaking, a Lévy process on a Riemannian manifold looks like a Brownian motion interlaced with jumps along geodesics at random times. The infinitesimal generator encodes certain information about the process and reflects the geometry of $(M, g)$. For example, the infinitesimal generator of the most famous Lévy process, Brownian motion, is the Laplace-Beltrami operator. Although Brownian motion on Riemannian manifolds has been well studied [23, 9], the theory for more general Lévy processes has received less attention. Hunt introduced Lévy processes on Lie groups [24], while Gangolli initiated the study of symmetric spaces [12, 13]. Applebaum and Estrade were the first to construct Lévy processes on more general Riemannian manifolds, under a natural assumption on the Lévy measure [4]. We will analyze the microlocal structure of the infinitesimal generator of this Lévy process.

One motivation for studying Lévy processes on Riemannian manifolds is to shed light on the Lévy flight foraging hypothesis, which is the basis of several biological models [48, 6] and search algorithms [52, 53, 15, 26]. This controversial hypothesis claims that Lévy processes are a better model of animal foraging behavior than Brownian motion, in the sense that they can optimize search efficiency [40, 49]. Although the underlying geometry is often curved, until recently this topic had only been studied in Euclidean space. The expected time for a pure-jump Lévy process to find a small target on a Riemannian manifold was first studied in [7], while [37, 36, 35] considered Brownian motion. A numerical comparison of [7] and [35] was first performed in [47], confirming that Brownian motion might be the faster search strategy for small targets on the 2-torus. This suggests that the underlying geometry could be an important factor in determining whether the Lévy flight foraging hypothesis is valid in a given context.

In [7], conjugate points are one of the main geometric influences on the expected stopping time. Specifically, the authors show that on the round sphere the expected stopping time exhibits singular behavior at antipodal points, but no such anomaly occurs on the flat torus or Anosov manifolds. (We say that $(M, g)$ is an Anosov manifold if its geodesic flow is an Anosov flow on its unit sphere bundle [1, 28], or equivalently, if $g$ lies in the $C^2$-interior of the set of metrics on $M$ without conjugate points [38].) Conjugate points influence the expected stopping time through the microlocal structure of the Lévy generator (i.e., the infinitesimal generator of a pure-jump Lévy process). On the round sphere the Lévy generator decomposes into a pseudodifferential operator and a Fourier integral operator, while on the flat torus and on Anosov manifolds it is simply a pseudodifferential operator. Inspired by these results, as well as the microlocal analysis of the geodesic X-ray transform performed in [42, 34, 17], we prove a similar theorem in a more general geometric context.

Our main result, Theorem 4.2.9, shows that if there are no singular conjugate pairs, then the Lévy generator equals two pseudodifferential operators plus an infinite sum of Fourier integral operators. Each Fourier integral operator is associated with conjugate pairs of a given order, and the order of the operator depends on the order of those conjugate pairs and the dimension of $M$. The canonical relation of each Fourier integral operator is related to the geometry of the set of conjugate pairs. When $(M, g)$ is the round sphere or an Anosov manifold, we recover the results of [7]. Our theorem also covers all possibilities in two dimensions. This microlocal approach means that we can describe the singularities of the Lévy generator.

Here is an outline of what lies ahead. In Chapter 2, we describe Lévy processes and their infinitesimal generators, providing the probabilistic context. In Chapter 3, we discuss the main concepts that we will need from microlocal analysis. In Chapter 4, we prove our main result: the Lévy generator equals two pseudodifferential operators plus an infinite sum of Fourier integral operators.

# Chapter 2

# Lévy Processes and Infinitesimal Generators

The purpose of this chapter is to define the Lévy generator and (hopefully) motivate why it has the form that it does. Strictly speaking, none of this material is necessary to understand the content of Chapters 3 and 4; the reader may take Definition 4.1.1 for granted and adopt a purely microlocal mindset. However, given that our main result is about the Lévy generator, we thought it appropriate to provide some of the probabilistic background.

Throughout this chapter, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and $n$ a positive integer. (So $\Omega$ is a set, $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.)

This chapter is organized as follows. A Lévy process is a certain kind of stochastic process, so Section 2.1 is a brief discussion of stochastic processes. Section 2.2 defines Lévy processes on $\mathbb{R}^n$ and discusses their structure in light of the Lévy-Khintchine formula. Section 2.3 defines the infinitesimal generator of a Feller process and describes some of its features in the case of a Lévy process on $\mathbb{R}^n$. Finally, Section 2.4 defines Lévy processes on a Riemannian manifold and describes their infinitesimal generators.

The material in Sections 2.1-2.3 is largely drawn from [3]. Section 2.4 summarizes the main results of [4] with some assistance from [39].

## 2.1 Stochastic Processes

A stochastic process is a collection of random variables indexed by time, so we begin with a quick discussion of random variables.

We always equip $\mathbb{R}^n$ with the ***Borel σ-algebra***, denoted by $\mathcal{B}(\mathbb{R}^n)$. A ***random variable on*** $\mathbb{R}^n$ is a measurable function $X : \Omega \to \mathbb{R}^n$. The ***distribution of X*** is the Borel probability measure $\mu_X$ on $\mathbb{R}^n$ defined by

$$\mu_X(B) := \mathbb{P}(X^{-1}(B)) \text{ for all } B \in \mathcal{B}(\mathbb{R}^n).$$

Two random variables $X$ and $Y$ are called ***independent*** if

$$\mathbb{P}(X \in B \text{ and } Y \in C) = \mathbb{P}(X \in B)\mathbb{P}(Y \in C) \text{ for all } B, C \in \mathcal{B}(\mathbb{R}^n).$$

The definition of independence for a finite number of random variables is analogous.

Given a Borel measurable function $f : \mathbb{R}^n \to \mathbb{C}$, the ***expected value of f(X)*** is

$$\mathbb{E}(f(X)) := \int_\Omega f(X(\omega))\, \mathbb{P}(d\omega) = \int_{\mathbb{R}^n} f(y)\, \mu_X(dy).$$

The ***characteristic function of X*** is the function $\phi_X : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\phi_X(x) := \mathbb{E}(e^{ix\cdot X}) = \int_\Omega e^{ix\cdot X(\omega)}\, \mathbb{P}(d\omega) = \int_{\mathbb{R}^n} e^{ix\cdot y}\, \mu_X(dy).$$

The characteristic function of $X$ uniquely determines the distribution of $X$ (i.e., if two random variables have the same characteristic function, then they must have the same distribution).

Now we turn our attention from random variables to stochastic processes, which model the evolution of chance in time.

**Definition 2.1.1.** *For each $t \geq 0$, let $X(t) : \Omega \to \mathbb{R}^n$ be a random variable on $\mathbb{R}^n$. Then the collection $X = (X(t))_{t\geq 0}$ is called a* ***stochastic process on*** $\mathbb{R}^n$.

As we shall see in Section 2.2, a Lévy process on $\mathbb{R}^n$ is a certain kind of stochastic process on $\mathbb{R}^n$. The rest of this section provides the ingredients for the definition of a Lévy process on $\mathbb{R}^n$ and introduces a few other concepts that will prove useful.

Let $X$ be a stochastic process. Given $x_0 \in \mathbb{R}^n$, we say that ***$X$ starts at $x_0$*** if $X(0) = x_0$ almost surely (i.e., $X(0)(\omega) = x_0$ for almost every $\omega \in \Omega$). If an object uniquely determines

$$\mathbb{P}\left(X(t_1) \in B_{t_1}, \dots, X(t_{k-1}) \in B_{t_{k-1}}, \text{ and } X(t_k) \in B_{t_k}\right) \tag{2.1}$$

for all $k \in \mathbb{N}$, times $t_1, \dots, t_k \geq 0$, and $B_{t_1}, \dots, B_{t_k} \in \mathcal{B}(\mathbb{R}^n)$, we say that this object ***uniquely identifies $X$***.

For each $\omega \in \Omega$, the function $t \mapsto X(t)(\omega)$ from $[0, \infty)$ to $\mathbb{R}^n$ is called a ***sample path of $X$***. We say that $X$ is ***continuous*** if almost all its sample paths are continuous. Similarly, we say that $X$ is ***càdlàg*** if almost all its sample paths are càdlàg (i.e., right-continuous with left limits). If $X$ is càdlàg, then for almost every $\omega \in \Omega$, the left-hand limit

$$X(t-) := \lim_{s \nearrow t} X(s)$$

exists for all $t \geq 0$. Hence, for such $\omega$, the ***jump size of $X$ at time $t$***,

$$\Delta X(t) := X(t) - X(t-), \tag{2.2}$$

exists for all $t \geq 0$. Moreover, for such $\omega$, the sample path $t \mapsto X(t)(\omega)$ can only have jump discontinuities (and thus has at most countably many discontinuities).

The ***increments of $X$*** are the random variables $X(t) - X(s)$ for $0 \leq s \leq t$. We say that $X$ has ***stationary increments*** if $X(t) - X(s)$ and $X(t-s) - X(0)$ have the same distribution whenever $0 \leq s \leq t$. Intuitively, this means that the change in $X$ over a given time interval depends only on the length of that interval, not on when the observation began. We say that $X$ has ***independent increments*** if $X(t_2) - X(t_1), \dots, X(t_{k+1}) - X(t_k)$ are independent for all $k \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_{k+1}$. Intuitively, this means that the change in $X$ over a given time interval does not depend on its changes during any non-overlapping time interval (past or future).

## 2.2 Lévy Processes on Euclidean Space

We open this section with the definition of a Lévy process on $\mathbb{R}^n$ and a few examples.

**Definition 2.2.1.** *We call a stochastic process $X$ on $\mathbb{R}^n$ a* ***Lévy process*** *if it starts at $0$, is càdlàg, and has stationary and independent increments.*

**Example 2.2.2.** *A stochastic process of the form $(bt)_{t\geq 0}$ for a fixed vector $b \in \mathbb{R}^n$ is called a* ***drift process****. Every drift process is a Lévy process on $\mathbb{R}^n$.*

A drift process is simply a deterministic motion in a fixed direction. A more interesting example of a Lévy process is a Brownian motion.

**Example 2.2.3.** *Let $A$ be a non-negative definite symmetric $n \times n$ matrix. We call a Lévy process $B_A = (B_A(t))_{t\geq 0}$ a* ***Brownian motion on $\mathbb{R}^n$*** *with covariance matrix $A$ if it is continuous and $B_A(t)$ is normally distributed with mean vector $0$ and covariance matrix $tA$ for each $t \geq 0$. A Brownian motion on $\mathbb{R}^n$ is a Lévy process on $\mathbb{R}^n$.*

**Remark 2.2.4.** *When $A$ is the $n\times n$ identity matrix (denoted by $I$), $B_A$ is called a* ***standard Brownian motion on $\mathbb{R}^n$*** *and is usually denoted by $B = (B(t))_{t\geq 0}$.*

The sum of a drift process and a Brownian motion on $\mathbb{R}^n$ is called a ***Brownian motion with drift***. Note that all Brownian motions with drift are continuous.

It turns out that every Lévy process on $\mathbb{R}^n$ is a sum of a Brownian motion with drift and a third kind of process called a jump process (the kind of Lévy process in which we are most interested). This is a consequence of the Lévy-Khintchine formula, which we discuss next.

To prepare for the Lévy-Khintchine formula, we need a few definitions. The ***characteristic function*** of a Lévy process $X$ is the function from $[0, \infty) \times \mathbb{R}^n$ to $\mathbb{C}$ defined by

$$(t, x) \mapsto \phi_{X(t)}(x),$$

which uniquely identifies $X$. Let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^n$, and let $\mathbf{1}_{\mathbb{B}}$ be its indicator function. We call a Borel measure $\nu$ on $\mathbb{R}^n \setminus \{0\}$ a ***Lévy measure*** if

$$\int_{\mathbb{R}^n\setminus\{0\}} \min(1, |y|^2)\, \nu(dy) < \infty. \tag{2.3}$$

Among other things, (2.3) ensures that the integral in (2.4) is finite. The significance of (2.3) will become clearer near the end of this section.

**Theorem 2.2.5** (Lévy-Khintchine Formula)**.** *If $X$ is a Lévy process on $\mathbb{R}^n$, then there is a vector $b \in \mathbb{R}^n$, a non-negative definite symmetric $n \times n$ matrix $A$, and a Lévy measure $\nu$ on $\mathbb{R}^n \setminus \{0\}$ such that the characteristic function of $X$ equals $(t, x) \mapsto e^{t\psi(x)}$, where*

$$\psi(x) := ib \cdot x - \frac{1}{2} x \cdot Ax + \int_{\mathbb{R}^n\setminus\{0\}} (e^{ix\cdot y} - 1 - ix \cdot y \mathbf{1}_{\mathbb{B}}(y))\, \nu(dy). \tag{2.4}$$

*Conversely, if $\psi : \mathbb{R}^n \to \mathbb{C}$ is of the form (2.4), then the map $(t, x) \mapsto e^{t\psi(x)}$ is the characteristic function of a Lévy process on $\mathbb{R}^n$.*

**Remark 2.2.6.** *We call the triplet $(b, A, \nu)$ the **characteristics of $X$**. We call the function $\psi : \mathbb{R}^n \to \mathbb{C}$ defined by (2.4) the **characteristic exponent of $X$**.*

The Lévy-Khintchine formula says that the characteristic function of a Lévy process is of a specific form that is uniquely determined by $(b, A, \nu)$. Since a Lévy process is uniquely identified by its characteristic function, this means that the characteristics $(b, A, \nu)$ uniquely identify a Lévy process.

Let $X$ be a Lévy process with characteristics $(b, A, \nu)$. Then its characteristic function is

$$(t, x) \mapsto e^{t\psi_b(x)} e^{t\psi_A(x)} e^{t\psi_\nu(x)},$$

where $\psi_b$, $\psi_A$, $\psi_\nu$ are the three respective terms on the right-hand side of (2.4). Our choice to use $b$ and $A$ in the Lévy-Khintchine formula is intentional: $\psi_b$ is the characteristic exponent of the drift process $(bt)_{t\geq 0}$, while $\psi_A$ is the characteristic exponent of a Brownian motion on $\mathbb{R}^n$ with covariance matrix $A$. So, if $\nu = 0$, then $X$ is a Brownian motion with drift of the form $(bt + B_A(t))_{t\geq 0}$. This implies that every Lévy process on $\mathbb{R}^n$ can be decomposed into the sum of a drift process, a Brownian motion, and a process regulated by the Lévy measure $\nu$. This last (and most mysterious) process is called a pure-jump process.

**Definition 2.2.7.** *Let $X$ be a Lévy process on $\mathbb{R}^n$. We call $X$ a* ***pure-jump process*** *if its characteristics are of the form $(0, 0, \nu)$ (and $\nu \neq 0$).*

Recall that a Brownian motion with drift is continuous. Since a Lévy process is the sum of a Brownian motion with drift and a pure-jump process, this implies that (if we are willing to ignore events of probability zero) any discontinuity in the sample paths of a Lévy process is due to its pure-jump term. And since a Lévy process is cádlág, these discontinuities must be jump discontinuities (hence the name "pure-jump process"). So, in some sense, the Lévy measure regulates the jumps of a Lévy process. To get a better idea of what this means, let us consider a few examples. (The examples and commentary below are largely based on the Overview in [3], to which we are indebted.)

In the following discussion, we suppose that $b$, $A$, and $\nu$ are all nonzero. For the moment, we also suppose that $\nu$ is a finite measure. This implies that $y \mapsto e^{ix\cdot y} - 1$ and $y \mapsto ix \cdot y \mathbf{1}_{\mathbb{B}}(y)$ are $\nu$-integrable for each $x \in \mathbb{R}^n$. Hence, we can safely absorb the third term of the integral in (2.4) into the drift term. That is,

$$\psi(x) = i\tilde{b} \cdot x - \frac{1}{2} x \cdot Ax + \int_{\mathbb{R}^n \setminus \{0\}} \left(e^{ix\cdot y} - 1\right) \nu(dy), \tag{2.5}$$

where $\tilde{b} = b - \int_{\mathbb{B}\setminus\{0\}} y \, \nu(dy)$. The advantage of this maneuver is that the integral in (2.5) is the characteristic exponent of a well known Lévy process.

As a first example, we consider when the Lévy measure is a multiple of a Dirac measure.

**Example 2.2.8.** *Let $\delta_h$ be the Dirac measure concentrated at a fixed $h \in \mathbb{R}^n \setminus \{0\}$. Suppose*

$$\nu = \lambda \delta_h \text{ for a fixed } \lambda > 0.$$

*Then the integral in (2.5) is the characteristic exponent of a Poisson process $N = (N(t))_{t\geq 0}$*

*of intensity $\lambda$ taking values in the set $\{kh : k \in \mathbb{N} \cup \{0\}\}$. This means that*

$$\mathbb{P}(N(t) = kh) = e^{-\lambda t}\frac{(\lambda t)^k}{k!} \text{ for all } t \geq 0 \text{ and } k \in \mathbb{N} \cup \{0\}.$$

*The sample paths of $N$ are piecewise constant with jumps of size $h$ at random times $(T_k)_{k\in\mathbb{N}}$. Each $T_k$ is a random variable on $\mathbb{R}^n$ defined by*

$$T_k := \inf\{t \geq 0 : N(t) = kh\}.$$

*Let $X$ be a Lévy process with characteristics $(b, A, \nu)$. Then*

$$X(t) = \tilde{b}t + B_A(t) + N(t).$$

*The sample paths of $X$ behave as follows. From $t = 0$ until time $T_1$, they follow the sample paths of a Brownian motion with drift. At time $T_1$, they have a jump of size $h$. Between times $T_1$ and $T_2$, they again follow the sample paths of a Brownian motion with drift, then have a jump of size $h$ at time $T_2$. This pattern of a Brownian motion with drift interlaced with jumps of size $h$ at random times continues indefinitely.*

Next, we consider when the Lévy measure is a linear combination of Dirac measures.

**Example 2.2.9.** *Fix $m \in \mathbb{N}$. For $j = 1, \dots, m$, fix $\lambda_j > 0$ and $h_j \in \mathbb{R}^n \setminus \{0\}$. Suppose*

$$\nu = \sum_{j=1}^{m} \lambda_j \delta_{h_j}.$$

*Let $X$ be a Lévy process with characteristics $(b, A, \nu)$. Then*

$$X(t) = \tilde{b}t + B_A(t) + \sum_{j=1}^{m} N_j(t),$$

*where $N_1, \dots, N_m$ are (independent) Poisson processes, the $j$th having intensity $\lambda_j$ and taking values in $\{kh_j : k \in \mathbb{N} \cup \{0\}\}$. Therefore, the sample paths of $X$ still resemble a Brownian motion with drift interlaced with jumps at random times, but now each jump size could be any of the numbers $h_1, \dots, h_m$.*

A similar picture emerges for any Lévy process whose Lévy measure is finite, except that now the jump sizes themselves may be random.

**Example 2.2.10.** *Let $X$ be a Lévy process with characteristics $(b, A, \nu)$, and suppose $\nu$ is finite. Then the sample paths of $X$ have the form*

$$X(t) = bt + B_A(t) + \sum_{0 \le s \le t} \Delta X(s),$$

*where $\Delta X(s)$ is the jump size of $X$ at time $s$, defined in* (2.2). *(Since $X$ is càdlàg, the sum on the right-hand side is countable.) Hence, in general, both the times and sizes of the jumps are random.*

Things become more subtle if $\nu$ is not finite. If

$$\int_{\mathbb{R}^n \setminus \{0\}} \min(1, |y|)\, \nu(dy) < \infty, \tag{2.6}$$

then one can use Poisson random measures to show that the sample paths of $X$ still resemble a Brownian motion with drift interlaced with random jumps at random times. (The only difference between (2.6) and (2.3) is that $|y|$ has replaced $|y|^2$.) The more challenging case is where $\nu$ satisfies (2.3) but not (2.6). Then $y \mapsto e^{ix \cdot y} - 1$ may no longer be $\nu$-integrable, but $y \mapsto e^{ix \cdot y} - 1 - ix \cdot y \mathbf{1}_{\mathbb{B}}(y)$ always is. Intuitively, $\nu$ can no longer distinguish small jumps from drift, so we must amalgamate both into the integral term. Despite this subtlety, we can still interpret such a Lévy process as a Brownian motion with drift interlaced with random "jumps" at random times.

To close this section, we highlight a collection of Lévy processes that will play a crucial role in Section 2.4. Here and elsewhere, $O(n)$ denotes the set of orthogonal $n \times n$ matrices.

**Definition 2.2.11.** *A Lévy measure $\nu$ is called **isotropic** if it is invariant under orthogonal transformations. In other words,*

$$\nu(OB) = \nu(B) \text{ for all } O \in O(n) \text{ and } B \in \mathcal{B}(\mathbb{R}^n \setminus \{0\}).$$

*We call a Lévy process on $\mathbb{R}^n$ **isotropic** if its Lévy measure is isotropic.*

**Remark 2.2.12.** *A Lévy process on $\mathbb{R}^n$ is isotropic if and only if its characteristics are of the form $(0, aI, \nu)$, where $a \geq 0$ and $\nu$ is isotropic. In particular, an isotropic Lévy process on $\mathbb{R}^n$ does not have a drift term.*

## 2.3 Infinitesimal Generators

Throughout this section, $E$ denotes either $\mathbb{R}^n$ or a compact, connected Riemannian manifold. Much of the terminology from Section 2.1 carries over naturally when $\mathbb{R}^n$ is replaced by an arbitrary Riemannian manifold. However, in general, we can no longer discuss the increments of a stochastic process on $E$, which were the focal point of Definition 2.2.1. Thus, to define Lévy processes on more general Riemannian manifolds, we need a different approach. Section 2.4 describes this new approach, while the present section lays the foundation.

Recall that we have a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A family of sub-$\sigma$-algebras $(\mathcal{F}_t)_{t\geq 0}$ of $\mathcal{F}$ is called a ***filtration*** if $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $0 \leq s \leq t$. Intuitively, a filtration models a flow of information for the ongoing experiment modeled by $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathcal{F}_t$ representing the information accumulated by time $t$. We say that a stochastic process $X$ on $E$ is ***adapted to $(\mathcal{F}_t)_{t\geq 0}$*** if $X(t)$ is $(\mathcal{F}_t, \mathcal{B}(E))$-measurable for every $t \geq 0$. Intuitively, this means that the information about $X$ keeps pace with time; we cannot "see the future." If $X$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$, then we can define conditional probabilities and expectations involving $X$ and the "information" $\mathcal{F}_t$. This allows us to introduce Markov and Feller processes, which will play a supporting role in our new approach to Lévy processes.

**Definition 2.3.1.** *Let $X$ be a stochastic process on $E$ adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. We say that $X$ is a* ***Markov process*** *if the following holds for all $B \in \mathcal{B}(E)$ and $0 \leq s \leq t$:*

$$\mathbb{P}(X(t) \in B \,|\, \mathcal{F}_s) = \mathbb{P}(X(t) \in B \,|\, X(s)). \tag{2.7}$$

**Remark 2.3.2.** *If the filtration is not particularly important to a given discussion, we will say statements like "Let $X$ be a Markov process" and leave the filtration unspecified.*

The Markov property (2.7) says that for any Borel set $B$ and times $s \le t$, the conditional probability of $X(t)$ being in $B$ given $\mathcal{F}_s$ is precisely the conditional probability of $X(t)$ being in $B$ given $X(s)$. Intuitively, since $\mathcal{F}_s$ includes all the information accumulated by time $s$, while $X(s)$ only indicates the present state of $X$, (2.7) means that future states of $X$ depend only on its present state, not on any of its past states.

We are interested in Markov processes because every Lévy process on $\mathbb{R}^n$ (when adapted to an appropriate filtration) is a Markov process.

**Example 2.3.3.** *Consider a Lévy process $X$ on $\mathbb{R}^n$. For each $t \ge 0$, let $\mathcal{F}_t$ be the $\sigma$-algebra generated by the random variable $X(t)$. Then $X$ is adapted to $(\mathcal{F}_t)_{t\ge 0}$ (true of any stochastic process) and a Markov process (due to $X$ having stationary and independent increments).*

In light of this example, our definition of Lévy process on a compact, connected Riemannian manifold should imply that such a process is a Markov process. As we shall see, this is the reason why we will be restricted to isotropic Lévy processes.

Our next goal is to define the infinitesimal generator of a Feller process (a kind of Markov process). We are motivated by the following facts. First, it turns out that every Lévy process on $\mathbb{R}^n$ is a Feller process, and the definition of the latter makes sense when $E$ is an arbitrary Riemannian manifold. Second, a Feller process starting at a given point is uniquely identified by its infinitesimal generator, similar to how a Lévy process on $\mathbb{R}^n$ is uniquely identified by its characteristic function. So, it is reasonable to conjecture that a Lévy process on a compact, connected Riemannian manifold can be seen as a Feller process whose infinitesimal generator has a specific form, and this will indeed be the case.

Before defining Feller processes and infinitesimal generators, we need to introduce the transition semigroup. And these definitions involve some new function spaces. Let $B_b(E, \mathbb{R})$ be the Banach space of bounded, Borel measurable real-valued functions on $E$ equipped with the supremum norm $\|\cdot\|_\infty$. Then let $C_0(E, \mathbb{R})$ be the closed subspace of $B_b(E, \mathbb{R})$ consisting of continuous functions that vanish at infinity. (Note that if $E$ is compact, then $C_0(E, \mathbb{R})$ is simply the space of continuous real-valued functions on $E$.)

**Definition 2.3.4.** *Let $X$ be a Markov process. The **transition semigroup of $X$** is the family $(T_t)_{t\geq 0}$ of bounded linear operators on $B_b(E, \mathbb{R})$ defined by*

$$(T_t f)(x) := \mathbb{E}\big(f(X(t)) \,|\, X(0) = x\big), \tag{2.8}$$

*where $t \geq 0$, $f \in B_b(E, \mathbb{R})$, and $x \in E$. (The right-hand side is the conditional expectation of the random variable $f(X(t))$ given $X(0) = x$.)*

To gain some insight into transition semigroups, let us examine what it represents in the case of a standard Brownian motion on $\mathbb{R}^n$ (cf. Remark 2.2.4).

**Example 2.3.5.** *Let $B = (B(t))_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}^n$. Given a bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$, define $u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ by*

$$u(t, x) := \mathbb{E}\big(f(x + B(t))\big).$$

*Then $u(0, x) = f(x)$ for all $x \in \mathbb{R}^n$, and $u$ solves the diffusion (or heat) equation*

$$\partial_t u = \frac{1}{2}\Delta u, \tag{2.9}$$

*where $\Delta$ is the Laplacian on $\mathbb{R}^n$. Moreover, letting $(T_t)_{t\geq 0}$ be the transition semigroup of $B$,*

$$u(t, x) = \mathbb{E}\big(f(B(t)) \,|\, B(0) = x\big) = (T_t f)(x).$$

*Therefore, the transition semigroup of $B$ is the source-to-solution operator for (2.9).*

With the transition semigroup defined, we can turn our attention to Feller processes.

**Definition 2.3.6.** *A **Feller process** is a Markov process $X$ that satisfies the following:*

*(i) $T_t$ maps $C_0(E, \mathbb{R})$ into itself for all $t \geq 0$.*

*(ii) $\lim_{t\searrow 0} \|T_t f - f\|_\infty = 0$ for all $f \in C_0(E, \mathbb{R})$.*

**Remark 2.3.7.** *A Feller process is uniquely identified by its starting point and transition semigroup. (For us, the starting point will always be known.)*

A Feller process $X$ is essentially a "well behaved" Markov process, in the sense that

$$(T_t \mathbf{1}_B)(x) = \mathbb{P}(X(t) \in B \,|X(0) = x)$$

depends continuously on the starting point $x$. In other words, small changes in the starting point lead to small changes in the distribution of $X(t)$.

**Example 2.3.8.** *Every Lévy process on $\mathbb{R}^n$ is a Feller process.*

With the notion of a Feller process in hand, we can introduce its infinitesimal generator, which will play a role similar to that of the characteristic function in Section 2.2.

**Definition 2.3.9.** *Let $X$ be a Feller process and $(T_t)_{t\geq 0}$ be its transition semigroup. Let*

$$D_L := \left\{ f \in C_0(E, \mathbb{R}) : \textit{there exists } g_f \in C_0(E, \mathbb{R}) \textit{ such that } \lim_{t \searrow 0} \left\| \frac{T_t f - f}{t} - g_f \right\|_\infty = 0 \right\}.$$

*Then the **infinitesimal generator of $X$** is the (typically unbounded) linear operator $L$ with domain $D_L$ defined by*

$$Lf := \lim_{t \searrow 0} \frac{T_t f - f}{t}.$$

**Remark 2.3.10.** *The infinitesimal generator determines the transition semigroup (in the sense that two Feller processes with the same infinitesimal generator have the same transition semigroup). Therefore, by Remark 2.3.7, a Feller process is uniquely identified by its starting point and infinitesimal generator.*

One way to understand the behavior of a function near a point is to consider its derivative. Unfortunately, this is not possible for many stochastic processes. (Even Brownian motion has sample paths that are non-differentiable almost everywhere.) But it turns out that we can differentiate the transition semigroup, and this is precisely what the infinitesimal generator does. Indeed, if $X$ is a Feller process with transition semigroup $(T_t)_{t\geq 0}$, then

$$L(T_t f) = \frac{d}{dt}(T_t f) \text{ for all } f \in D_L.$$

Since $T_0 f = f$ (just plug $t = 0$ into (2.8)), this implies that

$$(Lf)(x) = \frac{d}{dt}\bigg|_{t=0} (T_t f)(x) = \frac{d}{dt}\bigg|_{t=0} \mathbb{E}(f(X(t))|X(0) = x)$$

for all $f \in D_L$ and $x \in E$. In this way, the infinitesimal generator $L$ helps us to understand the behavior of $X$ near its starting point.

The remainder of this section examines the infinitesimal generator of a Lévy process on $\mathbb{R}^n$. Since the results below depend on the Fourier transform, it will be convenient to restrict our attention to functions in $C_c^\infty(\mathbb{R}^n, \mathbb{R})$ (the ***set of compactly supported smooth real-valued functions on*** $\mathbb{R}^n$). If $L$ is the infinitesimal generator of a Lévy process on $\mathbb{R}^n$, then $C_c^\infty(\mathbb{R}^n, \mathbb{R})$ is contained in $D_L$ and $L$ is determined by its restriction to $C_c^\infty(\mathbb{R}^n, \mathbb{R})$, so this is no real loss.

The following result shows that the infinitesimal generator of a Lévy process on $\mathbb{R}^n$ has a specific form that mirrors (and in fact arises from) the Lévy-Khintchine formula.

**Proposition 2.3.11.** *Let $X$ be a Lévy process on $\mathbb{R}^n$ with characteristics $(b, A, \nu)$, characteristic exponent $\psi$, and infinitesimal generator $L$. Write $b = (b^j)$ and $A = (a^{jk})$. Then*

$$(Lf)(x) = b^j \partial_j f(x) + \frac{1}{2} a^{jk} \partial_j \partial_k f(x) + \int_{\mathbb{R}^n \setminus \{0\}} \big(f(x+y) - f(x) - y^j \partial_j f(x) \mathbf{1}_{\mathbb{B}}(y)\big)\, \nu(dy) \tag{2.10}$$

*for all $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$.*

Notice that the structure of the infinitesimal generator is completely determined by the Lévy-Khintchine formula: the drift term corresponds to a first-order differential operator, the Brownian term corresponds to a second-order differential operator, and the pure-jump term corresponds to an integral operator. Hence, we can read off our intuitive description of the sample paths from the form of the infinitesimal generator, which is useful in situations where no Lévy-Khintchine formula is available (e.g., many Riemannian manifolds).

The infinitesimal generator of a standard Brownian motion on $\mathbb{R}^n$ appeared in (2.9).

**Example 2.3.12.** *The infinitesimal generator of a standard Brownian motion on* $\mathbb{R}^n$ *is* $\frac{1}{2}\Delta$. *(Take* $b = 0$, $A = I$, *and* $\nu = 0$ *in Proposition 2.10.)*

Another example of interest is an isotropic Lévy process on $\mathbb{R}^n$.

**Example 2.3.13.** *Let* $X$ *be an isotropic Lévy process on* $\mathbb{R}^n$. *Then (by Remark 2.2.12) its characteristics are of the form* $(0, aI, \nu)$, *where* $a \geq 0$ *and* $\nu$ *is isotropic, so* (2.10) *becomes*

$$(Lf)(x) = \frac{1}{2}a\Delta f(x) + \int_{\mathbb{R}^n\setminus\{0\}} \left(f(x+y) - f(x) - y^j\partial_j f(x)\mathbf{1}_{\mathbb{B}}(y)\right)\nu(dy). \tag{2.11}$$

The infinitesimal generator of an isotropic Lévy process on a compact, connected Riemannian manifold will resemble (2.11), so this is a helpful example to keep in mind.

Returning to the case of an arbitrary Lévy process on $\mathbb{R}^n$, it is worth noting that Proposition 2.3.11 is a consequence of the following result.

**Proposition 2.3.14.** *The infinitesimal generator* $L$ *of a Lévy process on* $\mathbb{R}^n$ *with characteristic exponent* $\psi$ *is given by*

$$(Lf)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\psi(\xi)f(y)\,dy\,d\xi$$

*for all* $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ *and* $x \in \mathbb{R}^n$.

In the terminology of Section 3.3, Proposition 2.3.14 essentially means that the infinitesimal generator of a Lévy process on $\mathbb{R}^n$ is a pseudodifferential operator on $\mathbb{R}^n$ with amplitude $\psi$. (We say "essentially" because in general $\psi$ is not even differentiable, let alone smooth, and hence is not an amplitude in the sense of Definition 3.3.2.) Our main theorem (Theorem 4.2.9) may be viewed as a generalization of this result for the infinitesimal generator of an isotropic Lévy process on a compact, connected Riemannian manifold, a process that we will define in the next section.

## 2.4 Lévy Processes on a Riemannian Manifold

Throughout this section, we let $(M, g)$ denote a compact, connected Riemannian manifold of dimension $n$ and $\Delta_g$ its Laplace-Beltrami operator. Our aim in this section is to motivate and summarize the construction of isotropic Lévy processes on $M$ carried out in [4]. Their approach is motivated by the Eells-Elworthy construction of Brownian motion on $M$, which in turn is motivated by the Euclidean case, so we back up and begin there.

There are several ways to define a standard Brownian motion $B = (B(t))_{t \geq 0}$ on $\mathbb{R}^n$. In Remark 2.2.4, we defined $B$ to be a continuous Lévy process on $\mathbb{R}^n$ in which $B(t)$ is normally distributed with mean vector 0 and covariance matrix $tI$ for each $t \geq 0$. Another option is to define such a process through its infinitesimal generator. Indeed, since a Feller process is uniquely identified by its starting point and infinitesimal generator (Remark 2.3.10), we could define a Lévy process on $\mathbb{R}^n$ to be a Feller process starting at 0 with infinitesimal generator $\frac{1}{2}\Delta$ (cf. Example 2.3.12). Both definitions have strengths and weaknesses. A strength of the second definition is that it naturally extends if $\mathbb{R}^n$ is replaced by $(M, g)$. (Replace 0 with some $x \in M$ and $\frac{1}{2}\Delta$ with $\frac{1}{2}\Delta_g$.) This is how we will define Brownian motion on $M$.

**Definition 2.4.1.** *A **Brownian motion on M** starting at $x \in M$ is a Feller process starting at $x$ whose infinitesimal generator is $\frac{1}{2}\Delta_g$.*

One topic we have not addressed (even in $\mathbb{R}^n$) is how one might construct a Brownian motion. Norbert Wiener was the first to do so in $\mathbb{R}^n$ [51]. On $M$, the idea is to somehow "transfer" a Brownian motion on $\mathbb{R}^n$ to one on $M$. One way to do this is to solve a certain stochastic differential equation (SDE) in each local chart of $M$ and show the solutions patch together to produce a globally defined Feller process on $M$ whose infinitesimal generator is $\frac{1}{2}\Delta_g$. Although possible [25], this approach is difficult to execute (and would be even more so for processes with jumps). Another approach is to seek to construct Brownian motion as the global solution of an SDE on $M$. Unfortunately, this is only possible if $M$ is parallelizable. But Eells and Elworthy showed that Brownian motion on $M$ can always be obtained as the projection of the solution of an SDE on the orthonormal frame bundle of $M$ (see [9, 10, 32]). This is the approach used in [4] to define and construct more general Lévy processes on $M$.

In the next three paragraphs, we list some background material to prepare for the relevant results of [4]. To avoid getting off track, we provide minimal commentary along the way.

Let $T_pM$ denote the tangent space of $M$ at $p \in M$, and let $OM_p$ be the ***set of linear isometries from $\mathbb{R}^n$ to $T_pM$***. Each $r \in OM_p$ can be viewed as a choice of orthonormal basis for $T_pM$ (e.g., the image under $r$ of the standard basis vectors $e_1, \dots, e_n$ for $\mathbb{R}^n$). We call the disjoint union

$$OM := \coprod_{p \in M} OM_p$$

the ***orthonormal frame bundle of M***. Since $M$ and $O(n)$ (the orthogonal group) are compact, $OM$ is compact. The Riemannian metric of $M$ can be used to define a Riemannian metric on $OM$. In particular, we may consider the orthogonal complement of a subspace of the tangent space of $OM$.

Let $\pi_{OM} : OM \to M$ be the projection map, and let $d\pi_{OM}|_r$ be its differential at $r \in OM_p$. The ***vertical subspace at r*** $\in OM_p$ is the kernel of $d\pi_{OM}|_r$. The ***horizontal subspace at r*** $\in OM_p$ (denoted by $H_rOM$) is the orthogonal complement of the vertical subspace at $r$. For each $r \in OM_p$, the restriction of $d\pi_{OM}|_r$ to $H_rOM$ is a linear isomorphism from $H_rOM$ to $T_pM$. Thus, for each $v \in T_pM$ and $r \in OM_p$, there is a unique vector $v^H \in H_rOM$ such that $d\pi_{OM}|_r(v^H) = v$. We call $v^H$ the ***horizontal lift of v***. The horizontal lift of a vector field on $M$ is defined similarly.

The ***canonical horizontal vector fields*** $\{H_y : y \in \mathbb{R}^n\}$ of $OM$ are defined by

$$H_y(r) = r(y)^H \text{ for each } y \in \mathbb{R}^n \text{ and } r \in OM.$$

When $y = e_j$ (the $j$th standard basis vector for $\mathbb{R}^n$), we instead write

$$H_j := H_{e_j} \text{ for } j = 1, \dots, n.$$

Since $OM$ is compact, the canonical horizontal vector fields are complete. In particular, we can define a map $\mathrm{Exp}(H_y) : OM \to OM$ by sending each $r \in OM$ to the orthonormal frame obtained by following for time 1 the integral curve of $H_y$ starting at $r$.

With this setup, we can turn our attention to the relevant results of [4]. In what follows, we assume that $R$ is a càdlàg stochastic process on $OM$. (Technically, we should assume that $R$ is a semimartigale defined up to an explosion time, but we ignore this for simplicity.) Recall that $R(t-)$ denotes the left-hand limit of $R$ at $t$, while $\Delta R(t)$ is the jump of $R$ at $t$.

**Definition 2.4.2.** *Let $Y = (Y(t))_{t\geq 0}$ be a Lévy process on $\mathbb{R}^n$ with characteristics $(b, A, \nu)$. Write $Y(t) = (Y^j(t))$, $b = (b^j)$, and $A = (a^{jk})$. Suppose*

*(i) $R$ starts at $r_0 \in OM$.*

*(ii) $R$ solves the following SDE on $OM$ for all $u \in C^\infty(OM, \mathbb{R})$ and $t \geq 0$:*

$$
\begin{aligned}
u(R(t)) = u(r) &+ \int_0^t H_j u(R(s-))\, dY^j(s) + \frac{1}{2}\int_0^t a^{jk} H_j H_k u(R(s))\, ds \\
&+ \sum_{0\leq s\leq t} \left[ u\left(\mathrm{Exp}\left(H_{\Delta Y(s)}\right)(R(s-))\right) - u(R(s-)) - \Delta Y^j(s) H_j u(R(s-))\right].
\end{aligned}
\tag{2.12}
$$

*Then we say that $R$ is a **horizontal Lévy process on $OM$** with initial frame $r_0$.*

**Remark 2.4.3.** *The first integral in* (2.12) *is an Itô integral (see, for instance, Chapter 4 of [3]). Since the Itô integral is with respect to $Y$, we say that $Y$ drives the SDE. For a treatment of SDEs on manifolds driven by processes with jumps, see Chapter 7 of [29].*

By [2, 4], (2.12) indeed has a unique càdlàg solution $R = (R(t))_{t\geq 0}$ starting at any given $r_0 \in OM$. Moreover, $R$ is a Feller process, and its infinitesimal generator $\tilde{L}$ is defined by

$$
\begin{aligned}
(\tilde{L}u)(r) := {}& b^j H_j u(r) + \frac{1}{2} a^{jk} H_j H_k u(r) \\
&+ \int_{\mathbb{R}^n\setminus\{0\}} \left[u(\,\mathrm{Exp}(H_y)(r)) - u(r) - y^j H_j u(r) \mathbf{1}_{\mathbb{B}}(y)\right] \nu(dy)
\end{aligned}
\tag{2.13}
$$

for all $u \in C^\infty(OM, \mathbb{R})$ and $r \in OM$. Note the similarity between (2.13) and (2.10). This is an indication that we are on the right track.

So far, we have worked on $OM$. Now we project. Consider the càdlàg stochastic process on $M$ defined by $X := \pi_{OM}(R)$. Then $X$ is our candidate to be a "Lévy process on $M$." Unfortunately, $X$ may not be a Markov process. Since every Lévy process on $\mathbb{R}^n$ is a Markov process (Example 2.3.3), it is unreasonable to call every such process $X$ "a Lévy process on $M$." The issue is that the finite-dimensional distributions of $X$ (defined in (2.1)) may depend on the choice of initial frame $r_0$ (see [2]). This violates the fact that the future states of a Markov process can only depend on its present state, not on any of its past states. The way around this is to assume that the Lévy process $Y$ on $\mathbb{R}^n$ that drives (2.12) is isotropic. Then $X$ is a Feller process and thus a reasonable candidate to be a "Lévy process on $M$" (see [4]).

**Definition 2.4.4.** *Suppose $Y$ is an isotropic Lévy process on $\mathbb{R}^n$. Let $R$ be the unique cádlág solution of* (2.12) *starting at $r_0 \in OM$. Then we call $X := \pi_{OM}(R)$ an **isotropic Lévy process on M**. If $Y$ is also a pure-jump process, then we call $X$ an **isotropic pure-jump Lévy process on M**.*

**Remark 2.4.5.** *An isotropic Lévy process on $M$ is a Feller process (by Theorem 3.1 in [4]), so it is uniquely identified by its initial frame and infinitesimal generator (by Remark 2.3.10).*

As one would hope, a Brownian motion on $M$ is an isotropic Lévy process on $M$.

**Example 2.4.6.** *Let $Y$ be a standard Brownian motion on $\mathbb{R}^n$. Then the solution $R$ of* (2.12) *is called a **horizontal Brownian motion on OM**. Its infinitesimal generator is $\frac{1}{2}\Delta_H$, where $\Delta_H$ is **Bochner's horizontal Laplacian**:*

$$\Delta_H := \sum_{j=1}^{n} H_j^2.$$

*The Lévy process $X := \pi_{OM}(R)$ is a Brownian motion on $M$ as defined by Eells and Elworthy (see [9, 10, 32]). Therefore, [4] extends the well known Eells-Elworthy construction.*

Let $L$ be the infinitesimal generator of an isotropic Lévy process on $M$. As one might expect, $L$ is the projection of $\tilde{L}$ (the infinitesimal generator of $R$) onto $M$. Specifically,

$$(Lf)(p) := (\tilde{L}(f \circ \pi_{OM}))(r) \tag{2.14}$$

for all $f \in C^\infty(M, \mathbb{R})$, $p \in M$, and $r \in OM_p$. Since $\tilde{L}$ is explicitly given in (2.13), we would expect that $L$ can be written quite explicitly, and indeed it can.

Suppose $Y$ is an isotropic Lévy process on $\mathbb{R}^n$. By Remark 2.2.12, the characteristics of $Y$ are of the form $(0, aI, \nu)$, where $a \geq 0$ and $\nu$ is isotropic. Then (2.13) becomes

$$\tilde{L} = \frac{1}{2} a \Delta_H + \tilde{\mathcal{A}}, \tag{2.15}$$

where $\Delta_H$ is Bochner's horizontal Laplacian and $\tilde{\mathcal{A}}$ is defined by

$$(\tilde{\mathcal{A}}u)(r) := \int_{\mathbb{R}^n \setminus \{0\}} \left[ u(\operatorname{Exp}(H_y)(r)) - u(r) - y^j H_j u(r) \mathbf{1}_{\mathbb{B}}(y) \right] \nu(dy).$$

for all $u \in C^\infty(OM, \mathbb{R})$ and $r \in OM$. The terms of $\tilde{L}$ correspond to the characteristics of $Y$. Hence, the sample paths of $R$ resemble a horizontal Brownian motion on $OM$ interlaced with random jumps at random times, and $\tilde{\mathcal{A}}$ is the infinitesimal generator of the jump part (cf. the paragraph after Proposition 2.3.11).

With this understanding of $\tilde{L}$, we turn our attention to $L$, the infinitesimal generator of $X := \pi_{OM}(R)$. For each $p \in M$, define a Lévy measure $\nu_p$ on $T_pM$ by

$$\nu_p(B) := \nu(r^{-1}(B)) \text{ for some } r \in OM_p \text{ and all } B \in \mathcal{B}(T_pM). \tag{2.16}$$

Since $\nu$ is isotropic, $\nu_p$ is independent of the choice of $r$ (by Lemma 2.1 in [4]) and isotropic. For each $p \in M$ and $v \in T_pM$, let $\gamma_v$ be the geodesic of $(M, g)$ through $p$ with initial velocity $v$, and let $\exp_p(v) = \gamma_v(1) \in M$. Then (2.14) and (2.15) imply that

$$L = \frac{1}{2} a \Delta_g + \mathcal{A},$$

where $\mathcal{A}$ is defined by

$$(\mathcal{A}f)(p) := \int_{T_pM \setminus \{0\}} \left[ f(\exp_p(v)) - f(p) - vf(p) \mathbf{1}_{\mathbb{B}}(v) \right] \nu_p(dv) \tag{2.17}$$

for all $f \in C^\infty(M, \mathbb{R})$ and $p \in M$. (Here, $\mathbf{1}_{\mathbb{B}}$ is the indicator function of the open unit ball in $T_pM$.) The terms of $L$ also correspond to the characteristics of the driving Lévy process

$Y$. Hence, the sample paths of $X$ resemble a Brownian motion on $M$ interlaced with random jumps at random times, and $\mathcal{A}$ is the infinitesimal generator of the jump part.

We will analyze the microlocal structure of $\mathcal{A}$ in Chapter 4. To close the present chapter, we connect (2.17) to Definition 4.1.1. The first step is to rewrite (2.17) as a principal value integral. Fix $f \in C^\infty(M, \mathbb{R})$ and $p \in M$. Since $\nu_p$ is isotropic, it is invariant under the change of variables $v \mapsto -v$. Thus, (2.17) is equivalent to

$$(\mathcal{A}f)(p) = \int_{T_pM\setminus\{0\}} \left[f(\exp_p(-v)) - f(p) + vf(p)\mathbf{1}_{\mathbb{B}}(v)\right] \nu_p(dv). \tag{2.18}$$

Adding (2.17) and (2.18) and dividing by two, we get

$$\begin{aligned}(\mathcal{A}f)(p) &= \frac{1}{2}\int_{T_pM\setminus\{0\}} \left[f(\exp_p(v)) - 2f(p) + f(\exp_p(-v))\right] \nu_p(dv)\\ &= \frac{1}{2}\lim_{\varepsilon\searrow 0}\int_{T_pM\setminus\mathbb{B}_\varepsilon} \left[f(\exp_p(v)) - 2f(p) + f(\exp_p(-v))\right] \nu_p(dv),\end{aligned} \tag{2.19}$$

where $\mathbb{B}_\varepsilon$ is the open unit ball of radius $\varepsilon$ in $T_pM$. Since $f$ is bounded and $\nu_p(T_pM \setminus \mathbb{B}_\varepsilon)$ is finite for every $\varepsilon > 0$ (cf. (2.3)), we can divide (2.19) into two integrals, make the change of variables $v \mapsto -v$ in the second, and recombine. We obtain

$$(\mathcal{A}f)(p) = \text{p.v.}\int_{T_pM\setminus\{0\}} \left[f(\exp_p(v)) - f(p)\right] \nu_p(dv). \tag{2.20}$$

The second step is to specify the Lévy measure $\nu$. Following [7], we choose $\nu$ so that $\mathcal{A}$ coincides with the fractional Laplace-Beltrami operator when $(M, g)$ is the flat torus. Fix $\alpha \in (0, 1)$, and let

$$C_{n,\alpha} := \frac{4^\alpha\Gamma(n/2 + \alpha)}{\pi^{n/2}|\Gamma(-\alpha)|},$$

where $\Gamma$ is the Gamma function. (This is the constant that appears in the singular integral operator definition of the fractional Laplacian on $\mathbb{R}^n$.) Define a Lévy measure $\nu$ on $\mathbb{R}^n$ by

$$\nu(B) := C_{n,\alpha}\int_B \frac{1}{|y|^{n+2\alpha}}\,dy \text{ for all } B \in \mathcal{B}(\mathbb{R}^n).$$

We take $\nu$ to be the Lévy measure of the isotropic Lévy process $Y$ on $\mathbb{R}^n$ that drives (2.12).

For each $p \in M$, define $\nu_p$ as in (2.16). Then

$$\nu_p(B) = C_{n,\alpha} \int_B \frac{1}{|v|_g^{n+2\alpha}}\, dT_p(v) \text{ for all } B \in \mathcal{B}(T_pM), \tag{2.21}$$

where $dT_p$ is the Riemannian density of $T_pM$ induced by the metric $g|_{T_pM}$. Using (2.21) to change measures in (2.20), we obtain

$$(\mathcal{A}f)(p) = C_{n,\alpha}\, \text{p.v.} \int_{T_pM\setminus\{0\}} \frac{f(\exp_p(v)) - f(p)}{|v|_g^{n+2\alpha}}\, dT_p(v) \tag{2.22}$$

for all $f \in C^\infty(M, \mathbb{R})$ and $p \in M$, which agrees with Definition 4.1.1.

**Definition 2.4.7.** *We call the operator $\mathcal{A}$ defined by* (2.22) *the* ***Lévy generator****.*

**Remark 2.4.8.** *In the realm of probability theory, it makes sense for $\mathcal{A}$ to act on real-valued functions. However, microlocal analysis is built on the Fourier transform, which in general turns a real-valued function into a complex-valued function. So, in Chapter 4, we will allow $\mathcal{A}$ to act on complex-valued functions (see Definition 4.1.1).*

In summary, the Lévy generator is the infinitesimal generator of an isotropic pure-jump Lévy process on $M$ whose family of Lévy measures is defined by (2.21). Since we will only consider the infinitesimal generators of such Lévy processes in Chapter 4, there is no harm in referring to (2.22) by the shorter and less precise name “Lévy generator.”

# Chapter 3

# Microlocal Preliminaries

This chapter provides some background on microlocal analysis to prepare for Chapter 4. The most comprehensive introduction to microlocal analysis is Hörmander's four-volume series [19, 20, 21, 22]. For a more accessible introduction to the subject, the reader could consult [16] or [44, 45]. To avoid going too far afield, we assume that the reader is comfortable working with smooth manifolds. See [31] for an introduction to smooth manifolds.

Throughout this chapter, $X$ and $Y$ are smooth manifolds, $n$ is a positive integer, and $\Omega$ is an open subset of $\mathbb{R}^n$.

This chapter is organized as follows. Section 3.1 quickly runs through our notation for some relevant concepts related to smooth manifolds. Section 3.2 defines distributions and the wave front set, allowing us to discuss singularities. Section 3.3 introduces pseudodifferential operators as generalizations of differential operators. Section 3.4 states the Schwartz kernel theorem and hints at the connection between pseudodifferential operators and Fourier integral operators. Section 3.5 takes up the laborious task of defining Fourier integral operators. Section 3.6 builds up the statement of the clean intersection calculus, which provides some conditions for when a composition of two Fourier integral operators is guaranteed to be a Fourier integral operator.

## 3.1 Function, Tangent, and Cotangent Spaces

This brief section fixes our notation for some function spaces, the tangent and cotangent spaces, and the differential of a function. Some of these concepts were introduced in Chapter 2, but we reintroduce them here to keep our promise that Chapter 2 is optional.

We reserve the term "function" for a map whose codomain is the complex plane $\mathbb{C}$. We denote the ***set of continuous functions on $X$*** by $C(X)$ and the ***set of smooth maps from $X$ to $Y$*** by $C^\infty(X,Y)$. If $Y = \mathbb{C}$, we denote $C^\infty(X,Y)$ more simply by $C^\infty(X)$. We call $C^\infty(X)$ the ***space of smooth functions on $X$*** and equip it with the compact-open topology. (So, convergence in $C^\infty(X)$ is the uniform convergence of the functions in a sequence and all their derivatives on every compact subset of $X$.) Giving $C^\infty(X)$ a topology allows us to say statements like "$u : C^\infty(X) \to \mathbb{C}$ is continuous." We denote the ***support of $\psi$*** $\in C^\infty(X)$ by $\operatorname{supp}(\psi)$. We say that $\psi$ is ***compactly supported*** if $\operatorname{supp}(\psi)$ is compact. The ***space of compactly supported smooth functions on $X$*** is denoted by $C_c^\infty(X)$, and we equip it with the inductive limit topology. (This is the topology induced by the space of smooth functions on $X$ supported in $K$ as $K$ ranges over all compact subsets of $X$.)

The ***tangent space of $X$ at $x$*** $\in X$ is denoted by $T_xX$. The disjoint union

$$TX := \coprod_{x\in X} T_xX$$

is called the ***tangent bundle of $X$***. Similarly, the ***cotangent space of $X$ at $x$*** $\in X$ is denoted by $T_x^*X$, and the ***cotangent bundle of $X$*** is

$$T^*X := \coprod_{x\in X} T_x^*X.$$

The cotangent bundle with the zero section removed is denoted by

$$T^*X \setminus \{0\} := \{(x,\xi) \in T^*X : \xi \neq 0\}.$$

The ***diagonal of $T^*X \times T^*X$*** is the set

$$\Delta(T^*X) := \left\{((x,\xi),(x,\xi)) : (x,\xi) \in T^*X\right\}. \tag{3.1}$$

We say that a subset $\Gamma$ of $T^*X$ is ***conic*** if

$$(x,\xi) \in \Gamma \implies (x,t\xi) \in \Gamma \text{ for all } t > 0. \tag{3.2}$$

Similarly, we say that a subset $\Gamma$ of $X \times \mathbb{R}^n$ is ***conic*** if (3.2) holds. (These two definitions coincide when $X$ is an open subset of $\mathbb{R}^n$ because $T^*X = X \times \mathbb{R}^n$.)

Given $F \in C^\infty(X,Y)$, we denote the ***differential of $F$ at $x$*** $\in X$ by

$$dF|_x : T_xX \to T_{F(x)}Y.$$

The corresponding dual linear map is denoted by

$$dF|_x^t : T^*_{F(x)}Y \to T^*_xX.$$

The ***global differential of $F$***, denoted by

$$dF : TX \to TY,$$

is the map whose restriction to $T_xX$ is $dF|_x$ for each $x \in X$. Given another smooth manifold $Z$, a smooth map $G \in C^\infty(X \times Y, Z)$, and a point $y \in Y$, we let

$$d_xG|_{(x,y)} : T_xX \to T_{G(x,y)}Z$$

denote the differential of $G(\cdot,y) \in C^\infty(X,Z)$ at $x$. The corresponding dual linear map is

$$d_xG|^t_{(x,y)} : T_{G(x,y)}Z \to T_xX.$$

## 3.2 Distributions and the Wave Front Set

This section introduces "generalized functions" (called distributions) and their singularities (elements of the wave front set). For a more thorough introduction to distributions and the wave front set, see [11].

Observe that each $u \in C_c^\infty(\mathbb{R}^n)$ defines a continuous linear map $C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ by

$$\langle u, \varphi \rangle := \int_{\mathbb{R}^n} u\varphi \, dx, \quad \varphi \in C_c^\infty(\mathbb{R}^n). \tag{3.3}$$

In fact, this map uniquely determines $u$, so $C_c^\infty(\mathbb{R}^n)$ can be seen as a subset of the collection of continuous linear maps $C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$. Such maps are called distributions (on $\mathbb{R}^n$).

**Definition 3.2.1.** *A* ***distribution on $X$*** *is a continuous linear map $C_c^\infty(X) \to \mathbb{C}$. We denote the space of all such maps by $\mathcal{D}'(X)$.*

**Definition 3.2.2.** *A* ***compactly supported distribution on $X$*** *is a continuous linear map $C^\infty(X) \to \mathbb{C}$. We denote the space of all such maps by $\mathcal{E}'(X)$.*

**Remark 3.2.3.** *The action of $u \in \mathcal{D}'(X)$ on $\varphi \in C_c^\infty(X)$ is denoted by $\langle u, \varphi \rangle$, and similarly if $u \in \mathcal{E}'(X)$ on $\varphi \in C^\infty(X)$.*

We saw in (3.3) that each $u \in C_c^\infty(\mathbb{R}^n)$ defines a distribution given by integration against $u$. Since $u$ is a smooth function, it is reasonable to think of (3.3) as a "smooth" distribution. The following definition makes this idea precise.

**Definition 3.2.4.** *Let $u \in \mathcal{D}'(X)$ and $x_0 \in X$. We say that $u$ is* ***smooth near $x_0$*** *if there exists $\chi \in C_c^\infty(X)$ with $\chi = 1$ near $x_0$ such that $\chi u \in C^\infty(X)$. Given an open set $U \subset X$, we say that $u$ is* ***smooth on $U$*** *if $u$ is smooth near each point of $U$.*

In other words, a distribution is smooth near a point if and only if it is equal to a smooth function in a neighborhood of that point. (For us, neighborhoods are always open sets.) This leads us to the singular support, which consists of all the points where a distribution fails to be smooth.

**Definition 3.2.5.** *Let $u \in \mathcal{D}'(X)$. The **singular support of $u$**, denoted by* $\operatorname{singsupp}(u)$*, is the complement of the largest open subset of $X$ on which $u$ is smooth.*

**Example 3.2.6.** *Let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^n$, and let $\mathbf{1}_{\mathbb{B}}$ be its indicator function. Then* $\operatorname{singsupp}(\mathbf{1}_{\mathbb{B}})$ *is $\partial\mathbb{B}$, the boundary of $\mathbb{B}$ (i.e., the $(n-1)$-dimensional sphere).*

The rest of this section builds up to the definition of the wave front set, which gives a more refined notion of a singularity than the singular support. We begin in $\mathbb{R}^n$.

In $\mathbb{R}^n$, the definition of the wave front set is based on the Fourier transform. The Fourier transform of $u \in \mathcal{E}'(\mathbb{R}^n)$ is denoted by $\hat{u}$. When $u \in C_c^\infty(\mathbb{R}^n)$, its Fourier transform is the smooth function $\hat{u} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\hat{u}(\xi) := \int_{\mathbb{R}^n} e^{-iy\cdot\xi} u(y)\, dy. \tag{3.4}$$

It turns out that smoothness of a distribution corresponds to decay in its Fourier transform:

**Lemma 3.2.7.** *Let $u \in \mathcal{E}'(\mathbb{R}^n)$. Then $u$ is smooth on $\mathbb{R}^n$ if and only if the following holds:*

$$\textit{for all } j \in \mathbb{N} \textit{ there exists } C_j \geq 0 \textit{ such that } |\hat{u}(\xi)| \leq C_j(1 + |\xi|)^{-j} \textit{ for all } \xi \in \mathbb{R}^n. \tag{3.5}$$

If we localize (3.5) to an open cone in $\mathbb{R}^n$, we can think of this condition as smoothness in that cone. (A ***cone in $\mathbb{R}^n$*** is a set $V \subset \mathbb{R}^n$ such that $\xi \in V$ implies $t\xi \in V$ for all $t > 0$.) To also localize in the base variable $x$, we can multiply $u$ by a smooth bump function before taking the Fourier transform. This leads us to the following definition, which can be seen as a counterpart to Definition 3.2.4. (Recall that $\Omega$ denotes an open subset of $\mathbb{R}^n$.)

**Definition 3.2.8.** *Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$. We say that $u$ is **smooth near $(x_0, \xi_0)$** if there exists $\chi \in C_c^\infty(\Omega)$ with $\chi = 1$ near $x_0$ and an open cone $V$ in $\mathbb{R}^n$ containing $\xi_0$ such that the following holds:*

$$\textit{for all } j \in \mathbb{N} \textit{ there exists } C_j \geq 0 \textit{ such that } |\widehat{\chi u}(\xi)| \leq C_j(1 + |\xi|)^{-j} \textit{ for all } \xi \in V.$$

*Given a conic open set $\Gamma \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$, we say that $u$ is **smooth in $\Gamma$** if $u$ is smooth* *near each point of* $\Gamma$*.*

The next lemma connects Definitions 3.2.4 and 3.2.8.

**Lemma 3.2.9.** *Let* $u \in \mathcal{D}'(\Omega)$ *and* $x_0 \in \Omega$*. Then* $u$ *is smooth near* $x_0$ *if and only if* $u$ *is smooth near* $(x_0, \xi)$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$*.*

The forward direction of Lemma 3.2.9 follows from the forward direction of Lemma 3.2.7. The backward direction is more challenging. It follows from the backward direction of Lemma 3.2.7, the compactness of the closed unit ball $\overline{\mathbb{B}}$, and the fact that $\chi u$ is "at least as smooth as $u$" for all $\chi \in C_c^\infty(\Omega)$ (see Lemma 11.1.2 and Proposition 11.1.1 in [11]).

This brings us to the definition of the wave front set. While the singular support consists of all the points where a distribution fails to be smooth, the wave front set consists of all the points *and directions* where a distribution fails to be smooth.

**Definition 3.2.10.** *Let* $u \in \mathcal{D}'(\Omega)$*. The* ***wave front set of u****, denoted by* $\mathrm{WF}(u)$*, is the complement of the largest conic open subset of* $\Omega \times (\mathbb{R}^n \setminus \{0\})$ *in which* $u$ *is smooth. The* ***twisted wave front set of u*** *is*

$$\mathrm{WF}'(u) := \{(x, -\xi) : (x, \xi) \in \mathrm{WF}(u)\}. \tag{3.6}$$

The next proposition (a corollary of Lemma 3.2.9) connects Definitions 3.2.5 and 3.2.10.

**Proposition 3.2.11.** *Let* $u \in \mathcal{D}'(\Omega)$*. Then*

$$\mathrm{singsupp}(u) = \{x \in \Omega : \textit{there exists } \xi \neq 0 \textit{ such that } (x, \xi) \in \mathrm{WF}(u)\}.$$

In other words, the projection of the wave front set onto $\Omega$ is precisely the singular support. Hence, the wave front set refines the singular support by also describing the direction of each singularity.

**Example 3.2.12.** *Let* $\mathbb{B}$ *be the open unit ball in* $\mathbb{R}^n$*, and let* $\mathbf{1}_{\mathbb{B}}$ *be its indicator function. By Example 3.2.6 and Proposition 3.2.11,* $\mathrm{WF}(\mathbf{1}_{\mathbb{B}})$ *is contained in* $\partial\mathbb{B} \times (\mathbb{R}^n \setminus \{0\})$*. Determining*

$\mathrm{WF}(\mathbf{1}_{\mathbb{B}})$ *exactly requires more effort. Intuitively,* $\mathbf{1}_{\mathbb{B}}$ *is smooth in directions that are tangent to* $\partial\mathbb{B}$ *and not smooth in directions that are orthogonal to* $\partial\mathbb{B}$*. And in fact*

$$\mathrm{WF}(\mathbf{1}_{\mathbb{B}}) = \{(x, \xi) : |x| = 1 \text{ and } \xi \neq 0 \text{ is orthogonal to } \partial\mathbb{B} \text{ at } x\}.$$

So far, we have only defined the wave front set of a distribution on an open subset of $\mathbb{R}^n$. The key to extending the definition to distributions on an arbitrary smooth manifold is that the transformation law of the wave front set under coordinate changes is that of covectors. To state this precisely, we need to introduce the pullback of a distribution.

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open sets, $f : \Omega_1 \to \Omega_2$ be a diffeomorphism, and $u \in \mathcal{D}'(\Omega_2)$. Set $h := f^{-1}$. The ***pullback of $u$ by $f$*** is the distribution $f^*u$ on $\Omega_1$ defined by

$$\langle f^*u, \varphi\rangle := \left\langle u, \left|\det(dh|_y)\right| \varphi(h(y))\right\rangle \text{ for all } \varphi \in C_c^\infty(\Omega_1).$$

The next lemma expresses that wave front set of $f^*u$ in terms of the wave front set of $u$.

**Lemma 3.2.13.** *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *be open sets,* $f : \Omega_1 \to \Omega_2$ *be a diffeomorphism, and* $u \in \mathcal{D}'(\Omega_2)$*. Then*

$$\mathrm{WF}(f^*u) = \{(x, df|_x^t\, \eta) \in \Omega_1 \times (\mathbb{R}^n \setminus \{0\}) : (f(x), \eta) \in \mathrm{WF}(u)\}.$$

The diffeomorphism $f : \Omega_1 \to \Omega_2$ induces a diffeomorphism $F : T^*\Omega_2 \to T^*\Omega_1$ defined by

$$F(y, \eta) := \left(f^{-1}(y), df|_{f^{-1}(y)}^t\, \eta\right).$$

Then Lemma 3.2.13 says precisely that

$$\mathrm{WF}(f^*u) = F(\mathrm{WF}(u)).$$

In other words, the wave front set transforms under coordinate changes as a subset of the cotangent bundle. Therefore, we can define the wave front set of a distribution on a smooth manifold $X$ as a conic subset of $T^*X \setminus \{0\}$.

Here is the procedure. Suppose $u \in \mathcal{D}'(X)$, and let $(U, \kappa)$ be a local coordinate chart for

$X$. Consider the distribution $(\kappa^{-1})^*(u|_U)$ on $\kappa(U) \subset \mathbb{R}^n$, where $u|_U \in \mathcal{D}'(U)$ is the restriction of $u$ to $U$. Let $K : T^*(\kappa(U)) \to T^*U$ be the diffeomorphism induced by $\kappa : U \to \kappa(U)$. Define

$$\mathrm{WF}(u|_U) := K\left(\mathrm{WF}\left((\kappa^{-1})^*(u|_U)\right)\right).$$

Lemma 3.2.13 implies that $\mathrm{WF}(u|_U)$ is independent of the choice of coordinates on $U$, so we can cover $X$ with local coordinate charts to obtain a well-defined subset of $T^*X \setminus \{0\}$.

**Definition 3.2.14.** *Let $u \in \mathcal{D}'(X)$. The* ***wave front set of u****, denoted by* $\mathrm{WF}(u)$*, is the union of all* $\mathrm{WF}(u|_U)$ *as* $(U, \kappa)$ *ranges over all local coordinate charts for $X$. The* ***twisted wave front set of u****, denoted by* $\mathrm{WF}'(u)$*, is defined as in* (3.6)*.*

Similar to the Euclidean case, we can think of $\mathrm{WF}(u)$ as the complement of the largest conic open subset of $T^*X \setminus \{0\}$ in which $u$ is smooth.

## 3.3 Pseudodifferential Operators

This section introduces pseudodifferential operators, one of the main kinds of operators that we will encounter in Chapter 4. Similar to our approach to the wave front set in Section 3.2, we will first work on an open subset $\Omega$ of $\mathbb{R}^n$.

To motivate the definition of a pseudodifferential operator, we start with an example.

**Example 3.3.1.** *Suppose $m \in \mathbb{N}$ and $A$ is a differential operator of order $m$ with coefficients in $C_c^\infty(\Omega)$. The symbol of $A$ is the function $a \in C^\infty(\Omega \times \mathbb{R}^n)$ defined by*

$$a(x,\xi) := \sum_{|\gamma| \le m} a_\gamma(x)\xi^\gamma,$$

*where $a_\gamma$ are the coefficients of $A$. Using the Fourier inversion theorem, one can show that*

$$(Af)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi) f(y)\, dy\, d\xi \tag{3.7}$$

*for all $f \in C_c^\infty(\Omega)$. Moreover, for every pair of multi-indices $\alpha, \beta$ and every compact subset*

*$K$ of $\Omega$, there is a constant $C_{\alpha,\beta,K} \geq 0$ such that for all $(x,\xi) \in K \times \mathbb{R}^n$,*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq C_{\alpha,\beta,K}(1+|\xi|)^{m-|\beta|}.$$

The last sentence in Example 3.3.1 is the definition of a symbol of order $m$, which can be used to define pseudodifferential operators (as any operator of the form (3.7)). However, it turns out to be more convenient to allow the symbol $a(x,\xi)$ to depend on $y$, and this is the approach we will take.

In the remainder of this section, $m$ denotes an arbitrary real number.

**Definition 3.3.2.** *We say that a smooth function $a(x,y,\xi)$ on $\Omega \times \Omega \times \mathbb{R}^n$ is an **amplitude of order m**, written $a \in S^m(\Omega,\Omega)$, if for all multi-indices $\alpha, \beta, \gamma$ and every compact subset $K$ of $\Omega$, there is a constant $C_{\alpha,\beta,\gamma,K} \geq 0$ such that*

$$\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x,y,\xi)\right| \leq C_{\alpha,\beta,\gamma,K}(1+|\xi|)^{m-|\gamma|} \tag{3.8}$$

*for all $(x,y,\xi) \in \Omega \times \Omega \times \mathbb{R}^n$.*

One reason for requiring amplitudes to satisfy (3.8) is that it ensures

$$x \mapsto (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y)\, dy\, d\xi$$

is a smooth function of $x \in \mathbb{R}^n$. Every differential operator maps smooth functions to smooth functions, and this ensures that pseudodifferential operators also possess this property.

**Definition 3.3.3.** *Consider a continuous linear operator*

$$A : C_c^\infty(\Omega) \to C^\infty(\Omega).$$

*We say that $A$ is a **pseudodifferential operator on Ω of order m**, written $A \in \Psi^m(\Omega)$, if there exists $a \in S^m(\Omega,\Omega)$ such that*

$$(Af)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi) f(y)\, dy\, d\xi$$

*for all $f \in C_c^\infty(\Omega)$ and $x \in \mathbb{R}^n$.*

**Example 3.3.4.** *By Example 3.3.1, every differential operator on $\Omega$ of order $m$ (with coefficients in $C_c^\infty(\Omega)$) is a pseudodifferential operator on $\Omega$ of order $m$.*

To extend Definition 3.3.3 to an arbitrary smooth manifold $X$, we will use the restriction and transfer of an operator. Let $U$ be an open subset of $X$, and consider an operator

$$A : C_c^\infty(X) \to C^\infty(X).$$

The ***restriction of $A$ to $U$***, denoted by denoted by $A_U$, is the composition

$$C_c^\infty(U) \to C_c^\infty(X) \xrightarrow{A} C^\infty(X) \to C^\infty(U),$$

where the first arrow is extension by zero to $X \setminus U$ and the last is the natural restriction map. Given a local coordinate chart $(U, \kappa)$ for $X$, the ***transfer of $A_U$ via $\kappa$*** is the operator

$$A_{U,\kappa} : C_c^\infty(\kappa(U)) \to C^\infty(\kappa(U))$$

defined by

$$A_{U,\kappa} f := (A_U(f \circ \kappa)) \circ \kappa^{-1}.$$

**Definition 3.3.5.** *Consider a continuous linear operator*

$$A : C_c^\infty(X) \to C^\infty(X).$$

*We say that $A$ is a* ***pseudodifferential operator on $X$ of order $m$****, written $A \in \Psi^m(X)$, if, for every local coordinate chart $(U, \kappa)$ for $X$, the transfer of $A_U$ via $\kappa$ is in $\Psi^m(\kappa(U))$.*

We can extend any pseudodifferential operator $A$ on $X$ to a continuous linear operator $\mathcal{E}'(X) \to \mathcal{D}'(X)$. To see how, consider the ***transpose of $A$***,

$$A^t : C_c^\infty(X) \to C^\infty(X),$$

which is a continuous linear operator defined by

$$\langle A^t \psi, \varphi \rangle := \langle \psi, A\varphi \rangle \text{ for all } \varphi, \psi \in C_c^\infty(X). \tag{3.9}$$

Then we can extend $A$ to a continuous linear operator $A : \mathcal{E}'(X) \to \mathcal{D}'(X)$ by letting

$$\langle Au, \varphi \rangle := \langle u, A^t \varphi \rangle \text{ for all } u \in \mathcal{E}'(X) \text{ and } \varphi \in C_c^\infty(X) \tag{3.10}$$

It is natural to wonder if there is a relationship between the singularities of $u$ and those of $Au$, and the next proposition shows that there is one.

**Proposition 3.3.6.** *If $A$ is a pseudodifferential operator on $X$ and $u \in \mathcal{E}'(X)$, then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u).$$

In other words, pseudodifferential operators do not introduce new singularities.

## 3.4 The Schwartz Kernel Theorem

The previous section introduced pseudodifferential operators as generalizations of differential operators. Many operators that we will encounter in Chapter 4 are Fourier integral operators, which are generalizations of pseudodifferential operators. In the next section, we will define a Fourier integral operator to be an operator whose Schwartz kernel is of a specific form (see Definition 3.5.15). In preparation, the present section discusses the Schwartz kernel theorem and examines the Schwartz kernel of a pseudodifferential operator.

Recall that $X$ and $Y$ denote smooth manifolds. The ***tensor product*** of $\varphi \in C_c^\infty(X)$ and $\psi \in C_c^\infty(Y)$ is the function $\varphi \otimes \psi \in C_c^\infty(X \times Y)$ defined by

$$(\varphi \otimes \psi)(x, y) = \varphi(x)\psi(y) \text{ for all } x \in X \text{ and } y \in Y.$$

As we shall see, the Schwartz kernel is determined by its action on such functions.

Before stating the Schwartz kernel theorem, we motivate it with an example.

**Example 3.4.1.** *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open sets. Each $K \in C_c^\infty(\Omega_1 \times \Omega_2)$ defines a continuous* *linear operator* $A : C_c^\infty(\Omega_2) \to C_c^\infty(\Omega_1)$ *by*

$$(A\psi)(x) := \int_{\Omega_2} K(x,y)\psi(y)\,dy.$$

*Observe that for all* $\varphi \in C_c^\infty(\Omega_1)$ *and* $\psi \in C_c^\infty(\Omega_2)$*,*

$$\langle A\psi, \varphi\rangle = \int_{\Omega_1} (A\psi)(x)\varphi(x)\,dx = \int_{\Omega_1}\int_{\Omega_2} K(x,y)\varphi(x)\psi(y)\,dy\,dx.$$

*Hence,*

$$\langle A\psi, \varphi\rangle = \langle K, \varphi \otimes \psi\rangle \text{ for all } \varphi \in C_c^\infty(\Omega_1) \text{ and } \psi \in C_c^\infty(\Omega_2). \tag{3.11}$$

*Conversely, given any continuous linear operator* $A : C_c^\infty(\Omega_2) \to C_c^\infty(\Omega_1)$*, there is a unique* $K \in C_c^\infty(\Omega_1 \times \Omega_2)$ *such that* (3.11) *holds.*

Example 3.4.1 is a special case of the Schwartz kernel theorem.

**Theorem 3.4.2** (Schwartz Kernel Theorem)**.** *A linear operator* $A : C_c^\infty(Y) \to \mathcal{D}'(X)$ *is continuous if and only if there exists* $K_A \in \mathcal{D}'(X \times Y)$ *such that*

$$\langle A\psi, \varphi\rangle = \langle K_A, \varphi \otimes \psi\rangle \text{ for all } \varphi \in C_c^\infty(X) \text{ and } \psi \in C_c^\infty(Y).$$

*Moreover, the distribution* $K_A$ *is uniquely determined by* $A$*.*

**Definition 3.4.3.** *Let* $A : C_c^\infty(Y) \to \mathcal{D}'(X)$ *be a continuous linear operator. The distribution* $K_A \in \mathcal{D}'(X \times Y)$ *given by Proposition 3.4.2 is called the* ***Schwartz kernel of A****.*

Among other things, the Schwartz kernel theorem allows us to extend the notion of the wave front set from distributions to continuous linear operators.

**Definition 3.4.4.** *Let* $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ *be a continuous linear operator. The* ***wave front set of A****, denoted by* $\mathrm{WF}(A)$*, is the wave front set of the Schwartz kernel of* $A$*.*

Definition 3.4.4 used the fact that a continuous linear operator defines a unique Schwartz kernel. We can go the other way, too: we can *define* a continuous linear operator by specifying its Schwartz kernel. This is the approach that we will use to define Fourier integral operators in Section 3.5. For now, let us determine the Schwartz kernel of a pseudodifferential operator on an open set $\Omega \subset \mathbb{R}^n$ and consider in what ways we might be able to generalize.

**Example 3.4.5.** *Let $m \in \mathbb{R}$. Suppose $A \in \Psi^m(\Omega)$ with amplitude $a \in S^m(\Omega, \Omega)$. Then*

$$(A\psi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi)\psi(y)\, dy\, d\xi$$

*for all $\psi \in C_c^\infty(\Omega)$ and $x \in \Omega$. Since $A\psi$ is a smooth function,*

$$\langle A\psi, \varphi \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi)\varphi(x)\psi(y)\, dy\, d\xi\, dx$$

*for all $\varphi \in C_c^\infty(\Omega)$. Thus, intuitively,*

$$\langle A\psi, \varphi \rangle = \langle K_A, \varphi \otimes \psi \rangle \text{ for all } \varphi, \psi \in C_c^\infty(\Omega),$$

*where $K_A \in \mathcal{D}'(\Omega \times \Omega)$ is oscillatory integral defined by*

$$K_A := (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi)\, d\xi. \tag{3.12}$$

**Remark 3.4.6.** *To indicate that an integral should be interpreted as an oscillatory integral, we omit the domain of integration as we did in* (3.12)*. Since $a$ satisfies* (3.8)*,* (3.12) *is absolutely convergent (and thus an ordinary integral) whenever $m < -n$. When $m \geq -n$, an integration by parts argument relying on* (3.8) *can be used to rigorously define* (3.12)*. For a detailed discussion of oscillatory integrals, see [43].*

**Remark 3.4.7.** *Using* (3.12)*, one can show that*

$$\mathrm{WF}(A) \subset \left\{ ((x,\xi),(x,-\xi)) : (x,\xi) \in T^*\Omega \setminus \{0\} \right\}.$$

Now that we know the Schwartz kernel of a pseudodifferential operator, let us consider how we might generalize. The key observation is that (3.12) still defines a distribution if

$$\Omega \times \Omega \times \mathbb{R}^n \ni (x,y,\xi) \mapsto (x-y)\cdot\xi$$

is replaced by an arbitrary phase function.

**Definition 3.4.8.** *Fix $n_1, n_2, N \in \mathbb{N}$ and let $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$ be open sets. A* ***phase function on $\Omega_1 \times \Omega_2 \times (\mathbb{R}^N \setminus \{0\})$*** *is a smooth real-valued function $\phi(x, y, \theta)$ on $\Omega_1 \times \Omega_2 \times (\mathbb{R}^N \setminus \{0\})$ that is positively homogeneous of degree 1 in $\theta$ and satisfies*

$$\left(d_x\phi|_{(x,y,\theta)}, d_\theta\phi|_{(x,y,\theta)}\right) \neq (0,0) \text{ and } \left(d_y\phi|_{(x,y,\theta)}, d_\theta\phi|_{(x,y,\theta)}\right) \neq (0,0)$$

*for all $(x, y, \theta) \in \Omega_1 \times \Omega_2 \times (\mathbb{R}^N \setminus \{0\})$.*

In the setting of Definition 3.4.8, we say that a smooth function $a(x, y, \theta)$ on $\Omega_1 \times \Omega_2 \times \mathbb{R}^N$ is an ***amplitude*** if it satisfies the analogous condition to the one given in Definition 3.3.2.

**Lemma 3.4.9.** *Fix $n_1, n_2, N \in \mathbb{N}$ and let $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$ be open sets. If $a$ is an amplitude and $\phi$ is a phase function on $\Omega_1 \times \Omega_2 \times (\mathbb{R}^N \setminus \{0\})$, then*

$$I_{a,\phi} := (2\pi)^{-\frac{n_1+n_2}{4}-\frac{N}{2}} \int e^{i\phi(x,y,\theta)} a(x, y, \theta)\, d\theta \tag{3.13}$$

*is a distribution on $\Omega_1 \times \Omega_2$.*

**Remark 3.4.10.** *One can show that*

$$\mathrm{WF}(I_{a,\phi}) \subset \left\{\left(\left(x, d_x\phi|_{(x,y,\theta)}\right), \left(y, d_y\phi|_{(x,y,\theta)}\right)\right) : d_\theta\phi|_{(x,y,\theta)} = 0\right\}.$$

*Note that this agrees with Remark 3.4.7 in the specific case of a pseudodifferential operator.*

In Example 3.4.5, we saw that the Schwartz kernel of a pseudodifferential operator on $\Omega \subset \mathbb{R}^n$ is given by (3.13) in the specific case where $n_1 = n_2 = N = n$, $\Omega_1 = \Omega_2 = \Omega$, and $\phi(x, y, \theta) = (x - y) \cdot \theta$. (One of the reasons for the factor in front of the integral in (3.13) is to conform with the factor in front of the integral in (3.12).) Lemma 3.4.9 provides a wider class of distributions that all resemble (3.12). By the Schwartz kernel theorem, these distributions $I_{a,\phi}$ correspond to a class of continuous linear operators $C_c^\infty(\Omega_2) \to \mathcal{D}'(\Omega_1)$ (a class that includes all pseudodifferential operators when $\Omega_1 = \Omega_2$). As we shall see in the next section, Fourier integral operators belong to this wider class of operators.

## 3.5 Fourier Integral Operators

Given how we defined pseudodifferential operators on a smooth manifold in Section 3.3, the reader might guess that a Fourier integral operator from one smooth manifold to another is an operator whose Schwartz kernel is locally of the form (3.13). This is essentially correct, but the fact that phase functions can often only be defined locally presents some difficulties on the road to globalization. Fortunately, it turns out that every *clean* phase function locally defines a Lagrangian submanifold that could exist globally. This observation points to the "correct" approach to Fourier integral operators: start with a Lagrangian submanifold, associate it locally with clean phase functions, and then work with oscillatory integrals of the form (3.13). This is the rough approach we will follow in this section.

With this in mind, we first discuss Lagrangian submanifolds. A ***symplectic manifold*** is a smooth manifold equipped with a closed nondegenerate 2-form, called the ***symplectic form***. For example, the cotangent bundle is a symplectic manifold when equipped with its canonical symplectic form (see Proposition 22.11 in [31]).

A Lagrangian submanifold is a certain kind of submanifold of symplectic manifold.

**Definition 3.5.1.** *A submanifold $S$ of a symplectic manifold is called a **Lagrangian submanifold** if $T_xS$ equals its symplectic complement for all $x \in S$.*

**Remark 3.5.2.** *For us, a "submanifold" without further qualification means an "immersed or embedded submanifold" (in the sense of Chapter 5 in [31]). In particular, Definition 3.5.1 implies that a Lagrangian submanifold may be immersed or embedded.*

We are ultimately interested in Lagrangian submanifolds of $T^*(X \times Y) \setminus \{0\}$, where $X$ and $Y$ are smooth manifolds. But for the next few pages (through Definition 3.5.10), it will be convenient to replace $X \times Y$ with an arbitrary smooth manifold $Z$ of dimension $n_Z$. We let $\omega_Z$ denote the canonical symplectic form on $T^*Z$, and we also fix a positive integer $N$.

With that said, we state the manifold version of Definition 3.4.8.

**Definition 3.5.3.** *Let $\Gamma$ be a conic open subset of $Z \times (\mathbb{R}^N \setminus \{0\})$. A* ***phase function on $\Gamma$*** *is a smooth real-valued function $\phi(z, \theta)$ that is positively homogeneous of degree 1 in $\theta$ and satisfies the following for all $(z, \theta) \in \Gamma$:*

$$d\phi|_{(z,\theta)} = \left(d_z\phi|_{(z,\theta)}, d_\theta\phi|_{(z,\theta)}\right) \neq (0, 0).$$

Having introduced Lagrangian submanifolds and phase functions, we now want to describe which phase functions are associated with Lagrangian submanifolds. The next definition provides the right phase functions to work with.

**Definition 3.5.4.** *Let $\Gamma$ be a conic open subset of $Z \times (\mathbb{R}^N \setminus \{0\})$ and let $\phi$ be a phase function on $\Gamma$. Consider the set*

$$\Sigma_\phi := \left\{(z, \theta) \in \Gamma : d_\theta\phi|_{(z,\theta)} = 0\right\}$$

*and the map $F : \Sigma_\phi \to T^*Z \setminus \{0\}$ defined by*

$$F(z, \theta) := \left(z, d_z\phi|_{(z,\theta)}\right). \tag{3.14}$$

*We say that $\phi$ is a* ***clean phase function*** *if $\Sigma_\phi$ is a submanifold of $\Gamma$ and $F$ has rank $n_Z$ at each point of $\Sigma_\phi$.*

**Remark 3.5.5.** *We call $e := \dim\left(\Sigma_\phi\right) - n_Z$ the* ***excess of $\phi$****. It will come into play when we discuss the clean intersection calculus in Section 3.6.*

Suppose $\phi$ is a clean phase function on $\Gamma$. Then (3.14) has constant rank, so the rank theorem (Theorem 4.12 in [31]) implies that each point $(z, \theta) \in \Sigma_\phi$ has a neighborhood $\Gamma_{z,\theta}$ in $\Sigma_\phi$ such that $F\left(\Gamma_{z,\theta}\right)$ is a submanifold of $T^*Z \setminus \{0\}$. Hence

$$F\left(\Sigma_\phi\right) = \bigcup_{(z,\theta)\in\Sigma_\phi} F\left(\Gamma_{z,\theta}\right).$$

Since the submanifolds $F\left(\Gamma_{z,\theta}\right)$ might intersect, it is not clear if $F\left(\Sigma_\phi\right)$ is a smooth manifold. But the next lemma shows that $F\left(\Sigma_\phi\right)$ is indeed a smooth manifold and more.

**Lemma 3.5.6.** *Let $\Gamma$ be a conic open subset of $Z \times (\mathbb{R}^N \setminus \{0\})$. Suppose $\phi$ is a clean phase function on $\Gamma$. Then*

$$\Lambda_\phi := F(\Sigma_\phi) = \left\{\left(z, d_z\phi|_{(z,\theta)}\right) : (z,\theta) \in \Gamma \text{ and } d_\theta\phi|_{(z,\theta)} = 0\right\} \tag{3.15}$$

*is a conic immersed Lagrangian submanifold of $T^*Z \setminus \{0\}$.*

**Definition 3.5.7.** *If $\Gamma$ is a conic open subset of $Z \times (\mathbb{R}^N \setminus \{0\})$ and $\phi$ is a clean phase function on $\Gamma$, then we say that* ***$\phi$ microlocally parametrizes $\Lambda_\phi$****.*

By Lemma 3.5.6, a clean phase function $\phi$ defines a conic immersed Lagrangian submanifold $\Lambda_\phi$. Since $\phi$ is generally only defined locally, the same is true of $\Lambda_\phi$. A key step toward the global definition of Fourier integral operators is the observation that we can start with a globally defined Lagrangian submanifold and locally associate it with clean phase functions. Specifically, given any conic immersed Lagrangian submanifold $\Lambda$ of $T^*Z \setminus \{0\}$ and any point $(z,\zeta) \in \Lambda$, we can find a conic open set $\Gamma \subset Z \times (\mathbb{R}^N \setminus \{0\})$ and a clean phase function $\phi$ on $\Gamma$ such that $\Lambda_\phi = \Lambda$ near $(z,\zeta)$. The next piece of the puzzle is to find a suitable definition of an amplitude on $\Gamma$.

Fix $m \in \mathbb{R}$. Mimicking Definition 3.3.2, we denote by $S^m(Z \times \mathbb{R}^N)$ the set of functions $a \in C^\infty(Z \times \mathbb{R}^N)$ that satisfy the following condition: for every differential operator $P$ on $Z$ (with smooth coefficients), every multi-index $\alpha$, and every compact subset $K$ of $Z$, there is a constant $C_{P,\alpha,K} \geq 0$ such that

$$|P\partial_\theta^\alpha a(z,\theta)| \leq C_{P,\alpha,K}(1 + |\theta|)^{m-|\alpha|}$$

for all $(z,\theta) \in K \times \mathbb{R}^N$. We say that a subset $S$ of $Z \times (\mathbb{R}^N \setminus \{0\})$ is ***conically compact*** if it is closed, conic, and its projection on $Z$ is compact.

**Definition 3.5.8.** *Let $m \in \mathbb{R}$ and let $\Gamma$ be a conic open subset of $Z \times (\mathbb{R}^N \setminus \{0\})$. We say that a smooth function $a : \Gamma \to \mathbb{C}$ is an* ***amplitude on $\Gamma$ of order $m$****, written $a \in S^m(\Gamma)$, if, in each conically compact subset of $\Gamma$, $a$ equals an element of $S^m(Z \times \mathbb{R}^N)$.*

Now we can start putting everything together.

**Lemma 3.5.9.** *Let $\Gamma$ be a conic open subset of $Z \times (\mathbb{R}^N \setminus \{0\})$. Suppose $\phi$ is a clean phase function on $\Gamma$ and $a$ is an amplitude on $\Gamma$. Then the oscillatory integral*

$$\int e^{i\phi(z,\theta)} a(z,\theta)\, d\theta$$

*is a distribution on $Z$. Moreover,*

$$\mathrm{WF}(I_{a,\phi}) \subset \Lambda_\phi,$$

*where $\Lambda_\phi$ is the set defined in* (3.15)*.*

This leads us to the definition of a Lagrangian distribution.

**Definition 3.5.10.** *Let $m \in \mathbb{R}$ and let $\Lambda$ be a Lagrangian submanifold of $T^*Z \setminus \{0\}$. A* ***Lagrangian distribution of order $m$ associated with $\Lambda$*** *is a locally finite sum of distributions of the form*

$$(2\pi)^{-\frac{n_Z}{4}-\frac{N}{2}} \int e^{i\phi(z,\theta)} a(z,\theta)\, d\theta,$$

*where $\phi$ is a clean phase function on a conic open set $\Gamma \subset Z \times (\mathbb{R}^N \setminus \{0\})$ that microlocally parametrizes $\Lambda_\phi$ and $a \in S^{m+\frac{n_Z}{4}-\frac{N}{2}-\frac{e}{2}}(\Gamma)$ ($e$ being the excess of $\phi$).*

**Remark 3.5.11.** *The seemingly arbitrary power of $2\pi$ and order of the amplitude $a$ ensure that this terminology and notation are consistent with those for pseudodifferential operators.*

Now that we have defined Lagrangian distributions, we can return to our original goal of defining Fourier integral operators. We resume working with smooth manifolds $X$ and $Y$. (The role of $Z$ in the preceding discussion will be played by $X \times Y$.) We denote the canonical symplectic forms on $T^*X$ and $T^*Y$ by $\omega_X$ and $\omega_Y$, respectively. Unless stated otherwise, we equip $T^*(X \times Y) \setminus \{0\}$ with the symplectic form $\omega_X - \omega_Y$.

The role of the Lagrangian submanifold $\Lambda$ in the preceding discussion will be played by a canonical relation.

**Definition 3.5.12.** *A* ***canonical relation from $T^*Y$ to $T^*X$*** *is a closed conic Lagrangian submanifold of $T^*(X \times Y) \setminus \{0\}$ contained in $(T^*X \setminus \{0\}) \times (T^*Y \setminus \{0\})$. If $C$ is a canonical relation, the corresponding* ***twisted canonical relation*** *is*

$$C' := \{((x,\xi),(y,-\eta)) : ((x,\xi),(y,\eta)) \in C\}.$$

**Remark 3.5.13.** *Unlike [22, 45], we allow canonical relations to be merely immersed. When a canonical relation is immersed but not embedded, we call it a* ***local canonical relation****.*

**Remark 3.5.14.** *The twisted canonical relation is a Lagrangian submanifold of $T^*(X \times Y) \setminus \{0\}$ when the latter is equipped with the standard symplectic form $\omega_X + \omega_Y$.*

At long last, we have reached the definition of a Fourier integral operator.

**Definition 3.5.15.** *Let $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ be a continuous linear operator, $m \in \mathbb{R}$, and $C$ be a canonical relation from $T^*Y$ to $T^*X$. We say that $A$ is a* ***Fourier integral operator from $Y$ to $X$ of order $m$ associated with $C$****, written $A \in \mathcal{I}^m(X \times Y, C')$, if the Schwartz kernel of $A$ is a Lagrangian distribution of order $m$ associated with $C'$.*

As promised, every pseudodifferential operator is a Fourier integral operator.

**Example 3.5.16.** *If $X = Y$ and $C$ is the diagonal of $(T^*X \setminus \{0\}) \times (T^*X \setminus \{0\})$, then*

$$\mathcal{I}^m(X \times X, C') = \Psi^m(X).$$

*In other words, the pseudodifferential operators on $X$ are precisely the Fourier integral operators from $X$ to itself whose canonical relation is the diagonal of $(T^*X \setminus \{0\}) \times (T^*X \setminus \{0\})$.*

A crucial property of Fourier integral operators is that wave front sets are mapped by the canonical relation.

**Proposition 3.5.17.** *If $A \in \mathcal{I}^m(X \times Y, C')$ and $u \in \mathcal{E}'(Y)$, then*

$$\mathrm{WF}(Au) \subset C(\mathrm{WF}(u)),$$

*where*

$$C(\mathrm{WF}(u)) := \{(x, \xi) \in T^*X : \textit{there exists } (y, \eta) \in \mathrm{WF}(u) \textit{ such that } (x, \xi, y, \eta) \in C\}.$$

Proposition 3.5.17 (like Proposition 3.3.6) helps us track where singularities go. We can reformulate Proposition 3.5.17 as a statement about the wave front set of an operator (rather than a distribution), and this version will be more convenient to apply in Chapter 4.

**Proposition 3.5.18.** *If* $A \in \mathcal{I}^m(X \times Y, C')$*, then* $\mathrm{WF}'(A) \subset C$*.*

## 3.6 The Clean Intersection Calculus

When is the composition of two Fourier integral operators a Fourier integral operator? That is the question that we would like to address in this section. Since Fourier integral operators generalize pseudodifferential operators, which generalize differential operators, let's first consider the composition of two differential operators and of two pseudodifferential operators. For simplicity, we initially work on an open set $\Omega \subset \mathbb{R}^n$ before moving to smooth manifolds.

**Example 3.6.1.** *For* $j = 1, 2$*, suppose*

$$A_j : C_c^\infty(\Omega) \to C_c^\infty(\Omega)$$

*is a differential operator of order* $m_j$*. Since the range of* $A_2$ *is contained in the domain of* $A_1$*, we can compose* $A_1$ *and* $A_2$ *to obtain a new operator*

$$A_1 \circ A_2 : C_c^\infty(\Omega) \to C_c^\infty(\Omega).$$

*And by the produce rule,* $A_1 \circ A_2$ *is a differential operator on* $\Omega$ *of order* $m_1 + m_2$*.*

In short, a composition of differential operators is a differential operator. What about the composition of two pseudodifferential operators?

**Example 3.6.2.** *For $j = 1, 2$, suppose*

$$A_j : C_c^\infty(\Omega) \to C^\infty(\Omega)$$

*is a pseudodifferential operator of order $m_j$. Since $A_2 f$ may not be compactly supported even though $f$ is, in general we cannot compose $A_1$ and $A_2$. But if we know that $A_2$ maps $C_c^\infty(\Omega)$ into itself, then we can define the composition*

$$A_1 \circ A_2 : C_c^\infty(\Omega) \to C^\infty(\Omega).$$

*Moreover, one can show that $A_1 \circ A_2$ is a pseudodifferential operator on $\Omega$ of order $m_1 + m_2$.*

**Remark 3.6.3.** *Since $A_1$ and $A_2$ extend to continuous linear operators $\mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ (see* (3.10)*), we could have used $\mathcal{E}'(\Omega)$ instead of $C_c^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ instead of $C^\infty(\Omega)$.*

Example 3.6.2 shows that a composition of two pseudodifferential operators is a pseudodifferential operator as long as the operator acting first preserves compact support. Motivated by this, we make the following definition.

**Definition 3.6.4.** *A pseudodifferential operator $A$ on a smooth manifold $X$ is called **properly supported** if both $A$ and its transpose $A^t$ (see* (3.9)*) map $\mathcal{E}'(X)$ into itself.*

Among other things, requiring the transpose to also map $\mathcal{E}'(X)$ into itself ensures that familiar properties such as $(A_1 \circ A_2)^t = A_2^t \circ A_1^t$ carry over to pseudodifferential operators.

By Example 3.6.2 and Definition 3.6.4, we can conclude the following.

**Proposition 3.6.5.** *Suppose $A_1 \in \Psi^{m_1}(X)$ and $A_2 \in \Psi^{m_2}(X)$ are properly supported. Then $A_1 \circ A_2 \in \Psi^{m_1+m_2}(X)$ and $A_1 \circ A_2$ is properly supported.*

To address the composition of non-properly supported pseudodifferential operators, we need to introduce smoothing operators.

**Definition 3.6.6.** *A continuous linear operator $\mathcal{E}'(X) \to \mathcal{D}'(X)$ is called a **smoothing operator** if it maps $\mathcal{E}'(X)$ into $C^\infty(X)$.*

It turns out that for any pseudodifferential operator $A$ on $X$, there is a properly supported pseudodifferential operator $\tilde{A}$ on $X$ such that $A - \tilde{A}$ is a smoothing operator. Hence, if we are willing to work modulo smoothing operators, we can always compose two pseudodifferential operators, and their composition is a pseudodifferential operator of the expected order.

Now we turn our attention toward the task of composing two Fourier integral operators. Since Fourier integral operators (like pseudodifferential operators) do not preserve compact support in general, we will need to work with those that are properly supported. (Note that the transpose $A^t$ of a Fourier integral operator $A$ is defined analogously to (3.9).)

**Definition 3.6.7.** *A Fourier integral operator $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ is called* ***properly supported*** *if $A$ maps $\mathcal{E}'(Y)$ into $\mathcal{E}'(X)$ and $A^t$ maps $\mathcal{E}'(X)$ into $\mathcal{E}'(Y)$.*

While there is a clean-cut answer for when we can compose two pseudodifferential operators, the situation for Fourier integral operators is murkier. Indeed, there are no known necessary and sufficient conditions for when a composition of two Fourier integral operators is again a Fourier integral operator. But some sufficient conditions are known, and we will focus on one called the clean intersection calculus.

To get a sense for why composing two Fourier integral operators is a delicate task, recall that a Fourier integral operator is associated with a closed conic Lagrangian submanifold called its canonical relation (see Definitions 3.5.12 and 3.5.15). If we compose two Fourier integral operators, the "canonical relation" of their composition should be the composition of their canonical relations. Although we can always compose two canonical relations (as we would any other relations), the composition may not be Lagrangian, smooth, or even closed. In this way, the composition of two Fourier integral operators might be defined but fail to be a Fourier integral operator. To guarantee that it is one, we need to set some parameters on how the original canonical relations interact. Let us look for such conditions.

In what follows, $X$, $Y$, and $Z$ denote smooth manifolds.

**Definition 3.6.8.** *Let $C_1$ be a canonical relation from $T^*Y$ to $T^*X$, and let $C_2$ be a canonical* *relation from $T^*Z$ to $T^*Y$. The **composition of $C_1$ and $C_2$** is the relation*

$$C_1 \circ C_2 := \Big\{((x,\xi),(z,\zeta)) \in T^*X \times T^*Z : \exists\, (y,\eta) \in T^*Y \textit{ such that } ((x,\xi),(y,\eta)) \in C_1 \\ \textit{and } ((y,\eta),(z,\zeta)) \in C_2\Big\}.$$

When can we be certain that $C_1 \circ C_2$ is a canonical relation from $T^*Z$ to $T^*X$? Two parts of Definition 3.5.12 are automatically satisfied. Specifically, $C_1 \circ C_2$ is conic because $C_1$ and $C_2$ are conic, and $C_1 \circ C_2$ is contained in $(T^*X \setminus \{0\}) \times (T^*Z \setminus \{0\})$ because $C_1$ is contained in $(T^*X \setminus \{0\}) \times (T^*Y \setminus \{0\})$ and $C_2$ is contained in $(T^*Y \setminus \{0\}) \times (T^*Z \setminus \{0\})$. Therefore, we only need to find conditions that ensure that $C_1 \circ C_2$ is a closed Lagrangian submanifold of $T^*(X \times Z) \setminus \{0\}$. To find such conditions, we will shift our perspective and view $C_1 \circ C_2$ as the projection of the intersection of two smooth manifolds. Let

$$C := (C_1 \times C_2) \cap (T^*X \times \Delta(T^*Y) \times T^*Z), \tag{3.16}$$

where $\Delta(T^*Y)$ is the diagonal of $T^*Y \times T^*Y$ (see (3.1)). Let

$$\pi_C : C \to T^*(X \times Z) \setminus \{0\}$$

be the projection map. Then

$$C_1 \circ C_2 = \pi_C(C).$$

So, the question is, when is $\pi_C(C)$ a closed Lagrangian submanifold of $T^*(X \times Z) \setminus \{0\}$?

Let us first focus on ensuring that $\pi_C(C)$ is closed. Since $C$ is closed in itself, we could assume that $\pi_C$ is a closed map (i.e., it takes closed subsets to closed subsets). However, in the proof of the clean intersection calculus, it turns out to be crucial to work with amplitudes whose supports are conically compact. This is not always possible if $\pi_C$ is merely a closed map, but it is always possible if $\pi_C$ is a proper map.

**Definition 3.6.9.** *Let $N_1$ and $N_2$ be topological spaces. We say that $f : N_1 \to N_2$ is a **proper map** if the preimage of each compact subset of $N_2$ is compact.*

If $\pi_C$ is a proper map, one can show that $\pi_C(C)$ is a closed subset of $T^*(X \times Z) \setminus \{0\}$ and that we may suppose the amplitudes of the two Fourier integral operators we would like to compose have conically compact support.

The next task is to ensure that $\pi_C(C)$ is a submanifold of $T^*(X \times Z) \setminus \{0\}$. For this to be possible, $C$ must be a smooth manifold. One way to guarantee this is to assume that the intersection (3.16) is transverse.

**Definition 3.6.10.** *Let $S$ and $\tilde{S}$ be embedded submanifolds of a smooth manifold $N$. We say that the intersection $S \cap \tilde{S}$ is **transverse** if for each $x \in S \cap \tilde{S}$, the tangent spaces $T_xS$ and $T_x\tilde{S}$ together span $T_xN$.*

Assuming (3.16) is a transverse intersection leads to the transverse intersection calculus of Hörmander [18]. But $C$ is also a smooth manifold under the weaker assumption that the intersection (3.16) is merely clean.

**Definition 3.6.11.** *Let $S$ and $\tilde{S}$ be embedded submanifolds of a smooth manifold $N$. We say that the intersection $S \cap \tilde{S}$ is **clean** if $S \cap \tilde{S}$ is an embedded submanifold of $N$ and*

$$T_x(S \cap \tilde{S}) = T_xS \cap T_x\tilde{S} \text{ for all } x \in S \cap \tilde{S}.$$

*In this case, the **excess of the intersection** is*

$$e := \operatorname{codim} S + \operatorname{codim} \tilde{S} - \operatorname{codim}(S \cap \tilde{S}) \geq 0.$$

**Remark 3.6.12.** *If $S \cap \tilde{S}$ is a transverse intersection, then the excess is zero. Hence, the excess measures "how much a clean intersection fails to be transverse."*

**Remark 3.6.13.** *The excess of* (3.16) *equals the dimension of the fibers of $\pi_C$. In particular, if $\pi_C$ is injective, then the excess is zero (and the intersection is transverse).*

Assuming (3.16) is clean leads to the clean intersection calculus of Duistermaat-Guillemin [8]. This is the approach that we will use in Chapter 4.

So far, we have seen that if $\pi_C$ is a proper map and the intersection (3.16) is clean, then $\pi_C(C)$ is a submanifold of and closed in $T^*(X \times Z) \setminus \{0\}$. Without additional assumptions, one can show that $\pi_C(C)$ is a Lagrangian submanifold of $T^*(X \times Z) \setminus \{0\}$. This means that $C_1 \circ C_2 = \pi_C(C)$ is a local canonical relation (cf. Remark 3.5.13). If we want to avoid self-intersections of $C_1 \circ C_2$ (and thus ensure that it is an embedded submanifold), we may additionally assume that the fibers of $\pi_C$ are connected.

The preceding discussion is summarized in the clean intersection calculus.

**Theorem 3.6.14** (Clean Intersection Calculus)**.** *Let $X$, $Y$, and $Z$ be smooth manifolds, let $C_1$ be a canonical relation from $T^*Y$ to $T^*X$, and let $C_2$ be a canonical relation from $T^*Z$ to $T^*Y$. Let $m_1, m_2 \in \mathbb{R}$, and suppose that*

$$A_1 \in \mathcal{I}^{m_1}(X \times Y, C_1') \text{ and } A_2 \in \mathcal{I}^{m_2}(Y \times Z, C_2')$$

*are properly supported. In addition, assume that the following holds:*

*(i) The intersection*

$$C := (C_1 \times C_2) \cap \big(T^*X \times \Delta(T^*Y) \times T^*Z\big) \tag{3.17}$$

*is clean with excess $e$.*

*(ii) The projection map $\pi_C : C \to T^*(X \times Z) \setminus \{0\}$ is proper.*

*(iii) The fibers of $\pi_C$ are connected.*

*Then*

$$A_1 \circ A_2 \in \mathcal{I}^{m_1+m_2+\frac{e}{2}}(X \times Z, C').$$

**Remark 3.6.15.** *The excess of the intersection* (3.17) *is precisely the excess of a phase function in a local representation of $A_1 \circ A_2$ (cf. Remark 3.5.5).*

**Remark 3.6.16.** *If we are willing to allow local canonical relations, we can omit point (iii).*

In summary, a composition of Fourier integral operators may not be a Fourier integral operator, but it is if the conditions of the clean intersection calculus are met. The proofs of our results in Chapter 4 will largely consist of checking these conditions.

We close this section with an example of the clean intersection calculus that will appear in Chapter 4.

**Example 3.6.17.** *Suppose $A_1$ is a properly supported pseudodifferential operator on $Y$, and let $A_2$ be a Fourier integral operator from $Z$ to $Y$. Since $A_1$ is properly supported, it extends to a continuous linear operator $\mathcal{D}'(Y) \to \mathcal{D}'(Y)$. Hence, the composition $A_1 \circ A_2$ is defined. Since $A_1$ is a pseudodifferential operator, its canonical relation is*

$$C_1 = \left\{ ((y, \eta), (y, \eta)) : (y, \eta) \in T^*Y \setminus \{0\} \right\}.$$

*It follows that (i)-(iii) are satisfied (with $X = Y$). Therefore, composing a properly supported pseudodifferential operator and a Fourier integral operator yields a Fourier integral operator.*

# Chapter 4

# Analysis of the Lévy Generator

This chapter (published in [46]) brings us to our main result. In Section 4.1, we discuss the necessary background, including an initial decomposition of the Lévy generator into pseudodifferential operators and a remainder term. In Section 4.2, we prove that the remainder term is an infinite sum of Fourier integral operators.

Throughout this chapter, we make the following assumption:

**Assumption 4.0.1.** $(M, g)$ *is a compact, connected Riemannian manifold of dimension* $n \geq 2$.

## 4.1 Preliminaries

This section introduces our notation, provides the main definitions, and recalls a result from [7] that will guide our analysis of the Lévy generator.

### 4.1.1 The Lévy Generator

Before defining the Lévy generator, let us specify our notation for vector bundles and introduce the exponential map.

Unless stated otherwise, we will denote the projection map associated with a vector bundle by $\pi$ with a subscript indicating the total space of the bundle (e.g., $\pi_{TM}$ is the projection map from the tangent bundle $TM$ to $M$). When referring to a point in a vector bundle, one must decide whether to specify the base point. For example, should a point in $TM$ be denoted by $(x, v) \in TM$ or simply $v \in TM$? We will often omit the base point, but we will include it when there is potential for confusion. We do follow the convention that $(x, v) \in TM$ means $v \in TM$ and $x = \pi_{TM}(v)$, and similarly for points in other vector bundles.

For each $v \in TM$, let $\gamma_v$ be the ***maximal geodesic with initial data*** $\dot{\boldsymbol{\gamma}}_{\boldsymbol{v}}(\mathbf{0}) = \boldsymbol{v}$. Since we view $\dot{\gamma}_v(0)$ as a point in $TM$, this initial data includes (loosely speaking) both the initial position and velocity of the geodesic. Each geodesic $\gamma_v$ is defined on all of $\mathbb{R}$ because $M$ is compact. Thus we can define the ***exponential map*** $\exp : TM \to M$ by

$$\exp(v) := \gamma_v(1).$$

For each $x \in M$, the ***restriction of the exponential map to*** $\boldsymbol{T_x M}$ will be denoted by $\exp_x$.

Fix $\alpha \in (0, 1)$. It is well known that the fractional Laplacian of order $\alpha$ is the infinitesimal generator of a $2\alpha$-stable jump process on Euclidean space [30, 33]. To ensure that the Lévy generator is consistent with the fractional Laplacian, we will include the constant

$$C_{n,\alpha} := \frac{4^\alpha \Gamma(n/2 + \alpha)}{\pi^{n/2} |\Gamma(-\alpha)|},$$

where $\Gamma$ is the Gamma function.

**Definition 4.1.1.** *The* ***Lévy generator****, denoted by $\mathcal{A}$, is defined for $f \in C^\infty(M)$ and $x \in M$ by*

$$(\mathcal{A}f)(x) := C_{n,\alpha} \text{ p.v.} \int_{T_x M \setminus \{0\}} \frac{f(\exp_x(v)) - f(x)}{|v|_g^{n+2\alpha}} \, dT_x(v),$$

*where $dT_x$ is the Riemannian density of $T_x M$ induced by the metric $g|_{T_x M}$ and $|v|_g$ is the length of the vector $v$.*

### 4.1.2 Averaging Along the Geodesic Flow

This subsection introduces one of the operators that will appear in our initial decomposition of the Lévy generator.

Let $SM$ be the ***unit sphere bundle over M***. Its fiber over a point $x \in M$ is

$$S_x M := \{v \in T_x M : |v|_g = 1\}.$$

Since $M$ is compact and $n$-dimensional, $SM$ is a compact manifold of dimension $2n-1$, and it is an embedded submanifold of $TM$. We will write $\pi$ rather than $\pi_{SM}$ for the projection map $SM \to M$, and we will use $\iota_{SM}$ for the inclusion map $\iota_{SM} : SM \hookrightarrow TM$.

The ***geodesic flow on SM*** is the smooth map $\Phi : SM \times \mathbb{R} \to SM$ defined by

$$\Phi(v, s) := \dot{\gamma}_v(s).$$

It is a smooth submersion because $\Phi(\cdot, s) : SM \to SM$ is a diffeomorphism for each $s \in \mathbb{R}$. We will denote the differential of the map $\Phi(\cdot, s)$ at a vector $v \in SM$ by $d_v \Phi|_{(v,s)}$. Also, the ***geodesic flow on TM*** will be denoted by $\tilde{\Phi}$.

Since $(M, g)$ is a compact Riemannian manifold, its ***injectivity radius*** (denoted by $r_{\text{inj}}$) is positive and finite. Choose a bump function $\chi \in C_c^\infty(\mathbb{R})$ satisfying $\chi(s) = 1$ for $|s| < r_{\text{inj}}^2/4$ and $\chi(s) = 0$ for $|s| > r_{\text{inj}}^2/2$. Let

$$a(s) := \begin{cases} (1 - \chi(s^2))s^{-1-2\alpha} & \text{if } s \geq 0, \\ 0 & \text{if } s \leq 0. \end{cases}$$

Then we can define an operator $R_a : C^\infty(SM) \to C(SM)$ by

$$(R_a u)(v) := C_{n,\alpha} \int_{\mathbb{R}} a(s) u(\Phi(v, s))\, ds. \tag{4.1}$$

We may think of $R_a u$ as a certain average of $u$ along the geodesic flow. This operator will play a crucial role in our analysis of the Lévy generator.

### 4.1.3 Pushforward and Pullback by a Smooth Submersion

Next we will define the pushforward and pullback and set our notation for Fourier integral operators.

Recall that if $F : X \to Y$ is a smooth map between smooth manifolds, then for each $x \in X$ the differential $dF|_x : T_x X \to T_{F(x)} Y$ yields a dual linear map

$$dF|_x^t : T^*_{F(x)} Y \to T^*_x X.$$

Since $dF|_x^t$ is given in coordinates by the transpose of the matrix of $dF|_x$, if $F$ is a smooth submersion then $dF|_x^t$ is injective for all $x \in X$.

**Definition 4.1.2.** *Let $X$ and $Y$ be smooth manifolds of dimension $n_X$ and $n_Y$, respectively, with smooth positive densities $dx$ and $dy$. Let $F : X \to Y$ be a smooth submersion. Then the* ***pushforward*** *$F_* : C_c^\infty(X) \to C_c^\infty(Y)$ and the* ***pullback*** *$F^* : C_c^\infty(Y) \to C^\infty(X)$ are defined by the requirement that*

$$\int_Y (F_*\varphi)(y)\psi(y)\, dy = \int_X \varphi(x)\psi(F(x))\, dx = \int_X \varphi(x)(F^*\psi)(x)\, dx$$

*for all $\varphi \in C_c^\infty(X)$ and $\psi \in C_c^\infty(Y)$.*

Explicitly, the pushforward by $F$ integrates over the level sets of $F$, while the pullback precomposes with $F$.

We would like to use Definition 4.1.2 to express $R_a$ as a composition of simpler operators. Let $\tilde{a}^m$ be the operator that multiplies by the smooth function

$$\tilde{a}(v, s) := C_{n,\alpha}\, a(s).$$

Then $\tilde{a}^m$ is a properly supported pseudodifferential operator on $SM \times \mathbb{R}$ of order 0. Let

$$p : SM \times \mathbb{R} \to SM$$

be the projection map. Since $p_*$ integrates a function over $\mathbb{R}$, (4.1) suggests that $R_a$ equals the composition $p_* \circ \tilde{a}^m \circ \Phi^*$. This is not quite correct since $\tilde{a}^m \circ \Phi^* u$ may not have compact support, even if $u \in C^\infty(SM)$. But we can use a partition of unity to define the composition of $p_*$ and $\tilde{a}^m \circ \Phi^*$, and this is how we will view the operator $R_a$.

It is known, going back at least to [14], that the pushforward and pullback by a smooth submersion are both Fourier integral operators. We will state the version from Lemma 1 of [17] to have this result in the precise format we need.

**Lemma 4.1.3.** *Suppose we are in the setting of Definition 4.1.2. Then the pushforward $F_*$ and the pullback $F^*$ are both Fourier integral operators of order $(n_Y - n_X)/4$. The canonical relation of $F_*$ is*

$$C_{F_*} := \left\{ (\eta, dF|_x^t \,\eta) : x \in X,\ \eta \in T^*_{F(x)} Y \setminus \{0\} \right\},$$

*while the canonical relation of $F^*$ is*

$$C_{F^*} := \left\{ (dF|_x^t \,\eta, \eta) : x \in X,\ \eta \in T^*_{F(x)} Y \setminus \{0\} \right\}.$$

**Remark 4.1.4.** *Lemma 4.1.3 implies that*

$$\begin{aligned}
\Phi^* &\in \mathcal{I}^{-1/4}((SM \times \mathbb{R}) \times SM, C'_{\Phi^*}), \\
p_* &\in \mathcal{I}^{-1/4}(SM \times (SM \times \mathbb{R}), C'_{p_*}), \\
\pi^* &\in \mathcal{I}^{(1-n)/4}(SM \times M, C'_{\pi^*}), \\
\pi_* &\in \mathcal{I}^{(1-n)/4}(M \times SM, C'_{\pi_*}).
\end{aligned}$$

In the next subsection, we will see that the key to determining the microlocal structure of $\mathcal{A}$ is to show that $\pi_* \circ R_a \circ \pi^*$ is an infinite sum of Fourier integral operators. Our main tool will be the clean intersection calculus (see Theorem 3.6.14).

### 4.1.4 Initial Decomposition of the Lévy Generator

The next result, which is a consequence of Theorem 1.6 in [7], is the starting point for our microlocal analysis of $\mathcal{A}$. The idea of the proof is to split $\mathcal{A}$ into two parts: one where

$\exp_x$ is injective, and a remainder term involving $\pi_*$, $R_a$, and $\pi^*$. This decomposition is unnecessary if $(M, g)$ is the flat torus, since $-\mathcal{A}$ is the fractional Laplace-Beltrami operator in that case (see Theorem 1.4 in [7]). Thus we may safely make the following assumption throughout the paper:

**Assumption 4.1.5.** *$(M, g)$ is not the flat torus.*

**Theorem 4.1.6.** *Let $\alpha \in (0, 1)$. Then there exist $\mathcal{A}_{2\alpha} \in \Psi^{2\alpha}(M)$ and $\mathcal{A}_0 \in \Psi^0(M)$ such that*

$$\mathcal{A} = \mathcal{A}_{2\alpha} + \mathcal{A}_0 + \pi_* \circ R_a \circ \pi^*.$$

*Proof.* Fix $f \in C^\infty(M)$ and $x \in M$. Write $\mathcal{A} = A_1 + A_2$, where

$$(A_1 f)(x) := C_{n,\alpha}\, \text{p.v.} \int_{T_x M \setminus \{0\}} \chi\left(|v|_g^2\right) \frac{f(\exp_x(v)) - f(x)}{|v|_g^{n+2\alpha}}\, dT_x(v),$$

$$(A_2 f)(x) := C_{n,\alpha} \int_{T_x M \setminus \{0\}} \left(1 - \chi\left(|v|_g^2\right)\right) \frac{f(\exp_x(v)) - f(x)}{|v|_g^{n+2\alpha}}\, dT_x(v).$$

By our choice of $\chi$, the integrand of $A_1$ vanishes whenever $|v|_g^2 > r_{\text{inj}}^2/2$, so we can make the change of variables $y = \exp_x(v)$. Then

$$(A_1 f)(x) = C_{n,\alpha}\, \text{p.v.} \int_M \chi(d_g(x,y)^2) \frac{f(y) - f(x)}{d_g(x,y)^{n+2\alpha}} J(x,y)\, dV_g(y),$$

where $d_g(x, y)$ is the Riemannian distance from $x$ to $y$, $J(x, y)$ is the Jacobian determinant of the map $y \mapsto \exp_x^{-1}(y)$, and $dV_g$ is the Riemannian density of $(M, g)$. Hence $\mathcal{A}_{2\alpha} := A_1$ is a pseudodifferential operator on $M$ of order $2\alpha$. (See Theorem 1.6 in [7]. A similar argument may be found in Section 5 of [41], where the authors show that the normal operator of the geodesic X-ray transform on a simple manifold is a pseudodifferential operator of order $-1$.)

For $A_2$, use polar coordinates to write

$$(A_2 f)(x) = C_{n,\alpha} \int_{S_x M} \int_0^\infty \left(1 - \chi\left(s^2\right)\right) \frac{f(\exp_x(sv)) - f(x)}{s^{1+2\alpha}}\, ds\, dS_x(v),$$

where $dS_x$ is the Riemannian density of $S_x M$ induced by the metric $g|_{S_x M}$. Since $\pi_*$ integrates over the fibers of $\pi$, we can write

$$(A_2 f)(x) = (\pi_* \circ R_a \circ \pi^* f)(x) - (\pi_* \circ R_a \circ \pi^* 1)(x) f(x).$$

Let $\mathcal{A}_0$ be the operator that multiplies by the constant function $-(\pi_* \circ R_a \circ \pi^* 1)$. Since $\mathcal{A}_0$ is a pseudodifferential operator on $M$ of order 0, this completes the proof. □

Due to Theorem 4.1.6, the microlocal analysis of $\mathcal{A}$ boils down to showing that $\pi_* \circ R_a \circ \pi^*$ is an infinite sum of Fourier integral operators, a task we will now take up.

## 4.2 Microlocal Structure of the Lévy Generator

This section contains our main result: a microlocal decomposition of $\mathcal{A}$ into pseudodifferential operators and Fourier integral operators. In light of Theorem 4.1.6, a natural first step would be to show that $R_a$ is a Fourier integral operator, but this is not true in general (see the paragraph following Remark 4.2.3). Fortunately, since we will eventually split $\pi_* \circ R_a \circ \pi^*$ into several pieces using a partition of unity on $SM \times \mathbb{R}$, it is sufficient to show that

$$R_{a,\psi} := p_* \circ (\psi \tilde{a})^m \circ \Phi^*$$

is a Fourier integral operator, where $\psi \in C_c^\infty(SM \times \mathbb{R})$ has sufficiently small support.

### 4.2.1 $R_{a,\psi}$ Is a Fourier Integral Operator

Since the multiplication operator $(\psi \tilde{a})^m$ is in $\Psi^0(SM \times \mathbb{R})$ and properly supported, we know $(\psi \tilde{a})^m \circ \Phi^*$ is in $\mathcal{I}^{-1/4}((SM \times \mathbb{R}) \times SM, C'_{\Phi^*})$. Therefore, to prove that $R_{a,\psi}$ is a Fourier integral operator, it suffices to show that the clean intersection calculus applies to $p_*$ and $(\psi \tilde{a})^m \circ \Phi^*$. This is the content of the next theorem. Recall that $r_{\text{inj}} > 0$ denotes the injectivity radius of the compact Riemannian manifold $(M, g)$.

**Theorem 4.2.1.** *Suppose $(M, g)$ satisfies Assumptions 4.0.1 and 4.1.5. Fix an open interval*

*$I$ with length less than $r_{\text{inj}}$ and take any function $\psi \in C_c^\infty(SM \times I)$. Let*

$$C_{R_{a,\psi}} := \Big\{(\xi, \tilde{\xi}) \in T^*SM \times T^*SM : \exists\, s \in I \text{ such that } dp|^t_{(\pi_{T^*SM}(\xi),s)}\, \xi = d\Phi|^t_{(\pi_{T^*SM}(\xi),s)}\, \tilde{\xi}\Big\}.$$

*Then $R_{a,\psi} \in \mathcal{I}^{-1/2}(SM \times SM, C'_{R_{a,\psi}})$.*

*Proof.* We will use the clean intersection calculus for Fourier integral operators. As noted above, we can use a partition of unity to define the composition of $p_*$ and $(\psi\tilde{a})^m \circ \Phi^*$. Then by localizing and reducing to the case of two properly supported Fourier integral operators, the proof boils down to an analysis of the canonical relations.

The twisted wave front set of $(\psi\tilde{a})^m \circ \Phi^*$ is contained in $\mathrm{WF}'((\psi\tilde{a})^m) \circ C_{\Phi^*}$, and Lemma 4.1.3 implies that $\mathrm{WF}'((\psi\tilde{a})^m) \circ C_{\Phi^*}$ is contained in

$$\widetilde{C}_{\Phi^*} := \left\{ \left( d\Phi|^t_{(v,s)}\, \tilde{\xi}, \tilde{\xi} \right) : (v,s) \in SM \times I,\ \tilde{\xi} \in T^*_{\Phi(v,s)}SM \right\}.$$

Hence, when we microlocalize in the proof of the clean intersection calculus (see [22]), we obtain smoothing operators except near points in $C_{p_*} \circ \widetilde{C}_{\Phi^*}$. Using Lemma 4.1.3, we find

$$C_{R_{a,\psi}} = C_{p_*} \circ \widetilde{C}_{\Phi^*}.$$

Therefore, by the clean intersection calculus, it is enough to prove the following:

(i) The intersection

$$C := (C_{p_*} \times \widetilde{C}_{\Phi^*}) \cap \big(T^*SM \times \Delta(T^*(SM \times \mathbb{R})) \times T^*SM\big) \tag{4.2}$$

is clean in the sense that $C$ is an embedded submanifold, and at every point $c \in C$ the tangent space $T_cC$ equals the intersection of the tangent spaces of the two manifolds being intersected.

(ii) The projection map $\pi_C : C \to T^*(SM \times SM) \setminus \{0\}$ is proper.

(iii) For every $(\xi, \tilde{\xi}) \in T^*(SM \times SM) \setminus \{0\}$, the fiber $\pi_C^{-1}(\xi, \tilde{\xi})$ is connected.

If we are willing to allow local canonical relations, then point (iii) can be omitted. Since the length of $I$ is less than $r_{\text{inj}}$, which is a lower bound for the length of any periodic geodesic, $\pi_C$ is injective. So point (iii) holds in this case, and assuming points (i) and (ii) are true the excess of the intersection (4.2) is zero.

To begin proving point (i), consider the map

$$G : T^*SM \times I \to T^*(SM \times I)$$

defined by

$$G(\xi, s) := dp|^t_{(\pi_{T^*SM}(\xi),s)}\, \xi.$$

If we include the base points, then

$$G(\pi_{T^*SM}(\xi), \xi, s) = ((\pi_{T^*SM}(\xi), s), (\xi, 0)), \tag{4.3}$$

so $G$ is a smooth embedding. Let $Z$ be the smooth rank-$(2n-1)$ subbundle of $T^*(SM \times I)$ whose fiber over each point $(v, s) \in SM \times I$ is

$$Z_{(v,s)} := \text{Range}\left(d\Phi|^t_{(v,s)}\right).$$

In other words, $Z_{(v,s)}$ is the subspace of $T^*_{(v,s)}(SM \times I)$ that is conormal to the kernel of $d\Phi|_{(v,s)}$. To see that $Z$ is a smooth subbundle, just use the rank theorem to locally write

$$\Phi\left(y^1, \dots, y^{2n}\right) = \left(y^1, \dots, y^{2n-1}\right),$$

and note that $\left(dy^1, \dots, dy^{2n-1}\right)$ is a smooth local frame for $Z$. Because $d\Phi|^t_{(v,s)}$ is a linear isomorphism from $T^*_{\Phi(v,s)}SM$ to $Z_{(v,s)}$, we can define a smooth map

$$d\Phi^{-t} : Z \to T^*SM \tag{4.4}$$

whose restriction to each fiber $Z_{(v,s)}$ is the inverse $(d\Phi|^t_{(v,s)})^{-1} : Z_{(v,s)} \to T^*_{\Phi(v,s)}SM$.

The domain of our smooth parametrization of $C$ will be the set

$$\mathcal{O} := \mathrm{Range}(G) \cap Z.$$

We claim that $\mathcal{O}$ is an embedded submanifold of dimension $4n-2$. Let

$$q : Z \to \mathbb{R} \tag{4.5}$$

be the restriction to $Z$ of the projection map

$$T^*(SM \times I) \ni \big((v,s),(\xi,\sigma)\big) \mapsto \sigma.$$

Since $\mathcal{O}$ equals the level set $q^{-1}(0)$, the claim is true if $dq|_\zeta$ is nonzero for all $\zeta \in Z$. Let $\pi_Z(\zeta) = (v,s)$. Then $\zeta = d\Phi|^t_{(v,s)}\,\theta$ for some $\theta \in T^*_{\Phi(v,s)}SM$. Choose coordinates for $SM$ near $v$ and near $\Phi(v,s)$. Then locally we can write $\Phi = (\Phi^1, \ldots, \Phi^{2n-1})$. Fix natural coordinates for $T^*SM$ associated with the coordinates near $\Phi(v,s)$. Let $(\theta_1, \ldots, \theta_{2n-1})$ be the coordinate representation of the covector $\theta \in T^*_{\Phi(v,s)}SM$. Since $\Phi(v,\cdot)$ is a unit-speed geodesic, we may suppose without loss of generality that

$$\frac{d}{d\tilde{s}}\bigg|_{\tilde{s}=s} \Phi^1(v,\tilde{s}) \neq 0.$$

Define a curve $\beta = (\beta^1, \ldots, \beta^{2n})$ in $T^*_{(v,s)}(SM \times I)$ as the composition of $d\Phi|^t_{(v,s)}$ and the curve

$$\mathbb{R} \ni \tau \mapsto (\theta_1 + \tau, \theta_2, \ldots, \theta_{2n-1}) \in T^*_{\Phi(v,s)}SM.$$

Then $\beta$ is a smooth curve in $Z_{(v,s)}$ such that $\beta(0) = \zeta$. Moreover,

$$\frac{d}{d\tau}\bigg|_{\tau=0} (q \circ \beta)(\tau) = \frac{d}{d\tau}\bigg|_{\tau=0} \beta^{2n}(\tau) = \frac{d}{d\tilde{s}}\bigg|_{\tilde{s}=s} \Phi^1(v,\tilde{s}) \neq 0. \tag{4.6}$$

Therefore, as claimed, $\mathcal{O}$ is an embedded submanifold of dimension $4n-2$.

Now we can use $\mathcal{O}$ to parametrize $C$. Indeed, if $G_\xi^{-1}$ is the inverse of (4.3) composed with the projection onto the $\xi$ component, then the map

$$P_C : \mathcal{O} \to T^*SM \times \mathcal{O} \times \mathcal{O} \times T^*SM$$

defined by

$$P_C(\zeta) := \left(G_\xi^{-1}(\zeta), \zeta, \zeta, d\Phi^{-t}\zeta\right)$$

is a smooth embedding, so $C$ is an embedded submanifold of dimension $4n-2$.

To finish proving point (i), fix $c \in C$. Since $T_cC$ is necessarily contained in the intersection of the tangent spaces of the manifolds on the right-hand side of (4.2), it is enough to show the reverse containment. Observe that $C_{p_*} \times \widetilde{C}_{\Phi^*}$ can be parametrized by the map

$$P_{C_{p_*} \times \widetilde{C}_{\Phi^*}} : T^*SM \times \mathbb{R} \times Z \to T^*SM \times T^*(SM \times \mathbb{R}) \times Z \times T^*SM$$

defined by

$$P_{C_{p_*} \times \widetilde{C}_{\Phi^*}}(\xi, s, \zeta) := (\xi, G(\xi, s), \zeta, d\Phi^{-t}\zeta)\,.$$

(Here $G$ is the natural extension of (4.3) to $T^*SM \times \mathbb{R}$.) Hence any vector $X \in T_c(C_{p_*} \times \widetilde{C}_{\Phi^*})$ is the velocity of some smooth curve

$$\mathbb{R} \ni \tau \mapsto (\xi(\tau), s(\tau), \zeta(\tau)) \in T^*SM \times \mathbb{R} \times Z,$$

meaning $P_{C_{p_*} \times \widetilde{C}_{\Phi^*}}(\xi(0), s(0), \zeta(0)) = c$ and

$$X = \frac{d}{d\tau}\bigg|_{\tau=0} P_{C_{p_*} \times \widetilde{C}_{\Phi^*}}(\xi(\tau), s(\tau), \zeta(\tau)).$$

The vector $X$ is also in $T_c\big(T^*SM \times \Delta\big(T^*(SM \times I)\big) \times T^*SM\big)$ if and only if

$$\frac{d}{d\tau}\bigg|_{\tau=0} G(\xi(\tau), s(\tau)) = \frac{d}{d\tau}\bigg|_{\tau=0} \zeta(\tau).$$

Thus, in any local coordinates, $G(\xi, s)$ and $\zeta$ agree to first order at $\tau = 0$. Then (4.3) implies that the same is true of $\xi$ and $G_\xi^{-1}(\zeta)$. Hence

$$\begin{aligned} X &= \frac{d}{d\tau}\Big|_{\tau=0} P_{C_{p_*}\times \widetilde{C}_{\Phi^*}}(\xi(\tau), s(\tau), \zeta(\tau)) \\ &= \frac{d}{d\tau}\Big|_{\tau=0} P_C(\zeta(\tau)), \end{aligned}$$

which means $X \in T_c C$. This completes the proof of point (i).

It remains to prove point (ii). Since $P_C$ is a diffeomorphism onto $C$, it suffices to show

$$\pi_C \circ P_C : \mathcal{O} \to C_{R_{a,\psi}} \tag{4.7}$$

has a continuous inverse. Suppose $((\xi_j, \tilde{\xi}_j))_{j=1}^\infty$ is a sequence in $C_{R_{a,\psi}}$ converging to some point $(\xi, \tilde{\xi}) \in C_{R_{a,\psi}}$. Since $\pi_C \circ P_C$ is injective, there exist unique $\zeta_j, \zeta \in \mathcal{O}$ such that

$$(\xi_j, \tilde{\xi}_j) = \left(G_\xi^{-1}(\zeta_j), d\Phi^{-t}\zeta_j\right) \quad \text{and} \quad (\xi, \tilde{\xi}) = \left(G_\xi^{-1}(\zeta), d\Phi^{-t}\zeta\right).$$

Let $\pi_Z(\zeta_j) = (v_j, s_j)$ and $\pi_Z(\zeta) = (v, s)$. Then by the definition of $d\Phi^{-t}$,

$$\pi_{T^*SM}(\tilde{\xi}_j) = \Phi(v_j, s_j) \quad \text{and} \quad \pi_{T^*SM}(\tilde{\xi}) = \Phi(v, s).$$

Since $(\xi_j, \tilde{\xi}_j) \to (\xi, \tilde{\xi})$, we know $v_j \to v$ and $\Phi(v_j, s_j) \to \Phi(v, s)$. Because the length of $I$ is less than $r_{\text{inj}}$, every subsequence of $(s_j)$ has a subsequence that converges to $s$. Thus $s_j \to s$, which means $\pi_Z(\zeta_j) \to \pi_Z(\zeta)$. Hence

$$(\pi_C \circ P_C)^{-1}(\xi_j, \tilde{\xi}_j) = \zeta_j \to \zeta = (\pi_C \circ P_C)^{-1}(\xi, \tilde{\xi}).$$

Therefore $(\pi_C \circ P_C)^{-1}$ is continuous, so point (ii) holds.

Since points (i)-(iii) hold, the excess is zero, and $p_*$ and $(\psi\tilde{a})^m \circ \Phi^*$ are Fourier integral operators of order $-1/4$, we conclude that $R_{a,\psi} \in \mathcal{I}^{-1/2}(SM \times SM, C'_{R_{a,\psi}})$. $\square$

**Remark 4.2.2.** *Since $a \in C^\infty(\mathbb{R})$ is supported away from 0, a similar argument as the* *second paragraph of the proof of Theorem 4.2.1 shows that* $\mathrm{WF}'(R_{a,\psi})$ *is contained in*

$$\Big\{(\xi,\tilde{\xi}) \in T^*SM \times T^*SM : \exists\, s \in I \setminus \{0\} \text{ such that } (\pi_{T^*SM}(\xi), s) \in \operatorname{supp}(\psi) \\ \text{and } dp|^t_{(\pi_{T^*SM}(\xi),s)}\, \xi = d\Phi|^t_{(\pi_{T^*SM}(\xi),s)}\, \tilde{\xi}\Big\}.$$

*This observation will be useful in the proof of Theorem 4.2.9.*

**Remark 4.2.3.** *It follows from points (i)-(iii) that* (4.7) *is a smooth embedding.*

It is worthwhile to mention why, in general, we cannot modify our proof of Theorem 4.2.1 to show that $R_a$ is a Fourier integral operator. The key issue is that point (ii) may not hold, i.e., the projection map $\pi_C : C \to T^*SM \times T^*SM$ may not be proper. In our proof of point (ii), it is crucial that the sequence $(s_j)$ is contained an interval of length at most $r_{\mathrm{inj}}$. If $\psi \in C^\infty(SM \times \mathbb{R})$ does not have compact support, then we cannot guarantee this.

### 4.2.2 Composition with $\pi^*$

Recall that our goal is to understand the microlocal structure of $\pi_* \circ R_a \circ \pi^*$. By a partition of unity, this boils down to understanding the microlocal structure of $\pi_* \circ R_{a,\psi} \circ \pi^*$. Theorem 4.2.1 demonstrated that $R_{a,\psi}$ is a Fourier integral operator if the support of $\psi$ is sufficiently small. The next theorem shows the same is true of

$$L_\psi := R_{a,\psi} \circ \pi^*.$$

**Theorem 4.2.4.** *Suppose* $(M,g)$ *satisfies Assumptions 4.0.1 and 4.1.5. Fix an open interval* $I$ *with length less than* $r_{\mathrm{inj}}$ *and take any function* $\psi \in C_c^\infty(SM \times I)$. *Let*

$$C_{L_\psi} := \big\{(\xi,\tilde{\eta}) \in T^*SM \times T^*M : \exists\, s \in I \text{ such that} \\ dp|^t_{(\pi_{T^*SM}(\xi),s)}\, \xi = d\Phi|^t_{(\pi_{T^*SM}(\xi),s)} \circ d\pi|^t_{\Phi(\pi_{T^*SM}(\xi),s)}\, \tilde{\eta}\big\}.$$

*Then* $L_\psi \in \mathcal{I}^{-(n+1)/4}(SM \times M, C'_{L_\psi})$.

*Proof.* Using Theorem 4.2.1 and Lemma 4.1.3, one can check that $C_{L_\psi}$ equals $C_{R_{a,\psi}} \circ C_{\pi^*}$.

Hence it suffices to show that points (i)-(iii) in the proof of Theorem 4.2.1 hold with (4.2) replaced by

$$C := (C_{R_{a,\psi}} \times C_{\pi^*}) \cap (T^*SM \times \Delta(T^*SM) \times T^*M). \tag{4.8}$$

Since $SM$ is compact, point (ii) holds by p. 18 of [22]. As in the proof of Theorem 4.2.1, $\pi_C$ is injective because the length of $I$ is less than $r_{\text{inj}}$. So point (iii) holds, and assuming point (i) is true the excess of the intersection (4.8) is zero.

To begin proving point (i), let $V$ be the smooth rank-$n$ subbundle of $T^*SM$ whose fiber over each vector $v \in SM$ is

$$V_v := \text{Range}\,(d\pi|_v^t)\,.$$

Similar to (4.4), we can define a smooth map

$$d\pi^{-t} : V \to T^*M$$

whose restriction to each fiber $V_v$ is the inverse $(d\pi|_v^t)^{-1} : V_v \to T^*_{\pi(v)}M$. Let

$$\widetilde{Z} := \left(d\Phi^{-t}\right)^{-1}(V).$$

Take $Z$ and $\mathcal{O}$ the same as in the proof of Theorem 4.2.1. Since $d\Phi^{-t} : Z \to T^*SM$ is a smooth submersion and $V$ is an embedded codimension-$(n-1)$ submanifold of $T^*SM$, we know $\widetilde{Z}$ is an embedded codimension-$(n-1)$ submanifold of $Z$.

Using a similar argument as the proof of Theorem 4.2.1, we will show that

$$\widetilde{\mathcal{O}} := \mathcal{O} \cap \widetilde{Z}$$

is an embedded submanifold of dimension $3n-1$. Let $\tilde{q} : \widetilde{Z} \to \mathbb{R}$ be the restriction to $\widetilde{Z}$ of the function $q$ defined in (4.5). Then $\widetilde{\mathcal{O}}$ equals the level set $\tilde{q}^{-1}(0)$, so it suffices to show that $d\tilde{q}|_{\tilde{\zeta}}$ is nonzero for all $\tilde{\zeta} \in \widetilde{Z}$. Let $\pi_{\widetilde{Z}}(\tilde{\zeta}) = (v, s)$. Then for some $\tilde{\eta} \in T^*_{\gamma_v(s)}M$, we have

$$\tilde{\zeta} = d\Phi|_{(v,s)}^t \circ d\pi|_{\Phi(v,s)}^t\, \tilde{\eta}.$$

Choose coordinates for $SM$ near $\Phi(v, s)$, which yield corresponding coordinates for $M$ near

$\gamma_v(s)$. Fix natural coordinates on $T^*M$. Then locally we can write $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_n)$. Define a curve $\beta$ in $T^*SM$ as the composition of $d\Phi|^t_{(v,s)} \circ d\pi|^t_{\Phi(v,s)}$ and the curve

$$\mathbb{R} \ni \tau \mapsto (\tilde{\eta}_1 + \tau, \tilde{\eta}_2, \dots, \tilde{\eta}_n) \in T^*_{\gamma_v(s)}M.$$

Then $\beta$ is a smooth curve in $\widetilde{Z}$ such that $\beta(0) = \tilde{\zeta}$. Moreover, similar to (4.6),

$$\frac{d}{d\tau}\bigg|_{\tau=0} (\tilde{q} \circ \beta)(\tau) \neq 0.$$

This proves that $\widetilde{\mathcal{O}}$ is an embedded submanifold of dimension $3n-1$.

Now we can use $\widetilde{\mathcal{O}}$ to parametrize $C$ via the map

$$P_C : \widetilde{\mathcal{O}} \to T^*SM \times T^*SM \times T^*SM \times T^*M$$

defined by

$$P_C(\tilde{\zeta}) := \left(G_\xi^{-1}(\tilde{\zeta}), d\Phi^{-t}\tilde{\zeta}, d\Phi^{-t}\tilde{\zeta}, d\pi^{-t} \circ d\Phi^{-t}\tilde{\zeta}\right).$$

By Remark 4.2.3, the map

$$\mathcal{O} \ni \zeta \mapsto \left(G_\xi^{-1}(\zeta), d\Phi^{-t}\zeta\right) \in T^*SM \times T^*SM$$

is a smooth embedding, and hence so is its restriction to $\widetilde{\mathcal{O}}$. Thus $P_C$ is a smooth embedding, so $C$ is an embedded submanifold of dimension $3n-1$.

To complete the proof of (i), note that $C_{R_{a,\psi}} \times C_{\pi^*}$ is parametrized by the map

$$P_{C_{R_{a,\psi}} \times C_{\pi^*}} : \mathcal{O} \times V \to T^*SM \times T^*SM \times T^*SM \times T^*M$$

given by

$$P_{C_{R_{a,\psi}} \times C_{\pi^*}}(\zeta, \theta) := \left(G_\xi^{-1}(\zeta), d\Phi^{-t}\zeta, \theta, d\pi^{-t}\theta\right).$$

Fix $c \in C$ and suppose $X \in T_c(C_{R_{a,\psi}} \times C_{\pi^*})$. Then there is a smooth curve

$$\mathbb{R} \ni \tau \mapsto (\zeta(\tau), \theta(\tau)) \in \mathcal{O} \times V$$

such that $P_{C_{R_{a,\psi}} \times C_{\pi^*}}(\zeta(0), \theta(0)) = c$ and the velocity of this curve at zero is $X$. As in the proof of Theorem 4.2.1, if $X \in T_c(T^*SM \times \Delta(T^*SM) \times T^*M)$ as well, then the derivatives of $d\Phi^{-t}\zeta$ and $\theta$ agree at $\tau = 0$ in any local coordinates. It follows that

$$\begin{aligned} X &= \frac{d}{d\tau}\Big|_{\tau=0} P_{C_{R_{a,\psi}} \times C_{\pi^*}}(\zeta(\tau), \theta(\tau)) \\ &= \frac{d}{d\tau}\Big|_{\tau=0} P_C(\zeta(\tau)), \end{aligned}$$

which means $X \in T_c C$. Therefore the intersection (4.8) is clean.

Since points (i)-(iii) hold, the excess is zero, and $R_{a,\psi}$ and $\pi^*$ are Fourier integral operators of order $-1/2$ and $(1-n)/4$, respectively, it follows that $L_\psi \in \mathcal{I}^{-(n+1)/4}(SM \times M, C'_{L_\psi})$. $\square$

It remains to consider the composition of $\pi_*$ and $R_a \circ \pi^*$. The main difficulty in this case is that the composition of canonical relations may have multiple connected components. One component corresponds to a smoothing operator, and the others appear only when there are conjugate points. If we rule out certain types of conjugate points, then these additional components give rise to Fourier integral operators whose canonical relations and orders can be determined. This is the content of Theorem 4.2.9, our main result. In the next subsection, we will provide the additional definitions and lemmas needed to state and prove it.

### 4.2.3 Conjugate Pairs

Though conjugate points along a geodesic are often defined in terms of vanishing Jacobi fields, it will be more convenient to work with the corresponding velocity vectors of the geodesic instead. This leads us to the notion of a conjugate pair.

**Definition 4.2.5.** *We call $(v, s) \in SM \times (\mathbb{R} \setminus \{0\})$ a* ***conjugate pair*** *if*

$$K_{(v,s)} := \ker\left(d\pi|_{\Phi(v,s)} \circ d_v\Phi|_{(v,s)}\right) \cap \ker(d\pi|_v) \neq \{0\}.$$

*If the dimension of $K_{(v,s)}$ is $1 \leq k \leq n-1$, then $(v, s)$ is a* ***conjugate pair of order*** *$k$. The* ***set of regular conjugate pairs of order*** *$k$, denoted by $\mathcal{C}_{R,k}$, is the set of conjugate* *pairs that have a neighborhood $U$ in $SM \times \mathbb{R}$ such that all other conjugate pairs in $U$ have order $k$. The* ***set of singular conjugate pairs****, denoted by $\mathcal{C}_S$, is the set of conjugate pairs that are not in $\mathcal{C}_{R,k}$ for any $k$.*

Our definition of a conjugate pair agrees with the one given in [17] and provides a different way to parametrize the same set in the case of a compact, non-trapping Riemannian manifold with strictly convex boundary. We choose to view conjugate pairs as points in $SM \times (\mathbb{R}\setminus\{0\})$ rather than in $SM \times SM$ to allow the possibility of conjugate points along periodic geodesics. By Lemma 3 in [17], Definition 4.2.5 is equivalent to the traditional definition of conjugate points in terms of vanishing Jacobi fields along a geodesic.

Fix an interval $I$ with length less than $r_{\text{inj}}$. For our purposes, it will suffice to work with the following subset of $\mathcal{C}_{R,k}$:

$$\mathcal{C}_{R,k,I} := \{(v,s) \in \mathcal{C}_{R,k} : s \in I\}.$$

However, all statements in this subsection remain true with $\mathcal{C}_{R,k,I}$ replaced by $\mathcal{C}_{R,k}$.

The crucial assumption in Theorem 4.2.9 is that there are no singular conjugate pairs. This matters because, as the next lemma shows, the set of regular conjugate pairs in $\mathcal{C}_{R,k,I}$ is a smooth manifold, which may not be true of the set of all conjugate pairs.

**Proposition 4.2.6.** *For each integer $1 \le k \le n-1$, the set $\mathcal{C}_{R,k,I}$ is an embedded $(2n-1)$-dimensional submanifold of $SM \times I$, and the set*

$$E_{R,k,I} := \left\{((v,s),X) \in \mathcal{C}_{R,k,I} \times TSM : X \in K_{(v,s)}\right\}$$

*is a smooth vector bundle of rank $k$ over $\mathcal{C}_{R,k,I}$.*

*Proof.* For the first point, it is enough to show that each point in $\mathcal{C}_{R,k,I}$ has a neighborhood $U$ in $SM \times I$ such that $\mathcal{C}_{R,k,I} \cap U$ is an embedded submanifold of dimension $2n-1$. To prove this local statement, we will extend the methods of [50].

Fix $(v,s) \in \mathcal{C}_{R,k,I}$. Let $d_F\exp$ be the differential in the fiber variables of the exponential map $\exp : TM \to M$. By [50], we can find coordinate neighborhoods $W_1$ of $sv$ in $TM$ and $W_2$

of $\exp(sv)$ in $M$ such that the $(k-1)$st elementary symmetric polynomial in the eigenvalues of $d_F \exp$ (denoted by $\sigma_{k-1}$) has nonzero derivative in the radial direction. Then $\sigma_{k-1}^{-1}(0)$ is an embedded $(2n-1)$-dimensional submanifold of $TM \setminus \{0\}$, and it equals the set of vectors in $W_1$ with conjugate points of order $k$ or higher in $W_2$.

Choose a neighborhood $U$ of $(v, s)$ in $SM \times I$ such that all other conjugate pairs in $U$ have order $k$. Supposing without loss of generality that $s > 0$, we may assume $U \subset SM \times (0, \infty)$. Consider the smooth map $f : TM \setminus \{0\} \to SM \times \mathbb{R}$ defined by

$$f(w) := \left( \frac{w}{|w|_g}, |w|_g \right).$$

Then $f$ is a smooth immersion and satisfies

$$f\left(\sigma_{k-1}^{-1}(0) \cap f^{-1}(U)\right) = \mathcal{C}_{R,k,I} \cap U.$$

Since $f|_{\sigma_{k-1}^{-1}(0) \cap f^{-1}(U)}$ has a continuous inverse defined on its image by

$$\mathcal{C}_{R,k,I} \cap U \ni (\tilde{v}, \tilde{s}) \mapsto \tilde{s}\tilde{v} \in TM \setminus \{0\},$$

it follows that $\mathcal{C}_{R,k,I} \cap U$ is an embedded submanifold of dimension $2n-1$.

To prove the second point, let $\mathcal{V}_{R,k,I}$ be the pullback of $\ker(d\pi)$ by the map

$$\mathcal{C}_{R,k,I} \ni (v, s) \mapsto v \in SM.$$

Then $E_{R,k,I}$ is the kernel of the smooth bundle homomorphism

$$\mathcal{V}_{R,k,I} \ni ((v, s), X) \mapsto \left(\Phi(v, s),\, d\pi|_{\Phi(v,s)} \circ d_v\Phi|_{(v,s)} X\right) \in TSM.$$

By the definition of $\mathcal{C}_{R,k,I}$, the restriction of this map to each fiber has constant rank $n-1-k$. Since we can view this map as a bundle homomorphism over $\mathcal{C}_{R,k,I}$ by pulling back $TSM$ by $\Phi$, it follows that $E_{R,k,I}$ is a smooth rank-$k$ subbundle of $\mathcal{V}_{R,k,I}$. $\square$

Next we will turn $TM$ into a symplectic manifold and make some remarks. Let $\omega$ be the ***canonical symplectic form on $T^*M$***, and let $\flat_g : TM \to T^*M$ be the ***musical***

***isomorphism induced by the metric g***. Then we can define a symplectic form $\omega_g$ on $\boldsymbol{TM}$ by

$$\omega_g(X, Y) := \omega(d\flat_g X, d\flat_g Y), \quad X, Y \in T(TM).$$

Then $\tilde{\Phi}(\cdot, s)$ is a symplectomorphism for each $s \in \mathbb{R}$, the kernel of $d\pi_{TM}|_v$ is a Lagrangian subspace of $T_v(TM)$ for each $v \in TM$, and in natural coordinates $(x^i, v^i)$ on $TM$ we have

$$\omega_g = \xi^\ell \frac{\partial g_{i\ell}}{\partial x^j}\, dx^j \wedge dx^i + g_{ij}\, dv^j \wedge dx^i. \tag{4.9}$$

In turn, $\omega_g$ induces a smooth bundle isomorphism $\flat_{\omega_g} : T(TM) \to T^*(TM)$ defined by

$$[\flat_{\omega_g}(X)](Y) := \omega_g(X, Y) = (X \lrcorner\, \omega_g)(Y),$$

where $X \lrcorner\, \omega_g$ is ***interior multiplication by X***. We will denote the inverse of $\flat_{\omega_g}$ by $\sharp_{\omega_g}$.

The next lemma defines a smooth bundle homomorphism that will help us describe the canonical relations of the various pieces of $\pi_* \circ R_a \circ \pi^*$.

**Lemma 4.2.7.** *For each integer* $1 \leq k \leq n-1$*, there is a smooth bundle homomorphism*

$$F_{k,I} : E_{R,k,I} \to T^*(M \times M) = T^*M \times T^*M$$

*defined by the requirement that*

$$\Big(d\iota_{SM}|_v X \lrcorner\, \omega_g, \Big(d\iota_{SM}|_{\Phi(v,s)} \circ d_v\Phi|_{(v,s)} X\Big) \lrcorner\, \omega_g\Big) = d\pi_{T(M\times M)}|^t_{(v,\Phi(v,s))}\, F_{k,I}\big((v,s), X\big). \tag{4.10}$$

The proof of this result is essentially the same as that of Lemma 4 in [17], so we do not repeat the details here. Observe that the image of $F_{k,I}$ is given by pairs of conjugate points in $M \times M$ together with the covariant derivatives of Jacobi fields that vanish at both of those points. See [17] for further details on the geometric interpretation of $F_{k,I}$.

Our final lemma is the key geometric tool in the proof of Theorem 4.2.9.

**Lemma 4.2.8.** *Let $(v,s) \in SM \times I$, $\tilde{v} = \Phi(v,s)$, $\eta \in T^*_{\pi(v)}M$, and $\tilde{\eta} \in T^*_{\pi(\tilde{v})}M$. Then*

$$dp|^t_{(v,s)} \circ d\pi|^t_v \, \eta = d\Phi|^t_{(v,s)} \circ d\pi|^t_{\tilde{v}} \, \tilde{\eta} \tag{4.11}$$

*if and only if*

$$d\pi|^t_v \, \eta = d_v\Phi|^t_{(v,s)} \circ d\pi|^t_{\tilde{v}} \, \tilde{\eta} \quad \textit{and} \quad \eta(v) = \tilde{\eta}(\tilde{v}) = 0. \tag{4.12}$$

*If (4.12) holds and $s \neq 0$ then $(v,s)$ is a conjugate pair, and if $(v,s) \in \mathcal{C}_{R,k,I}$ then $(\eta, \tilde{\eta}) \in F_{k,I}(E_{R,k,I})$. Conversely, if $(\eta,\tilde{\eta}) \in F_{k,I}(E_{R,k,I})$ then (4.12) holds for some $(v,s) \in \mathcal{C}_{R,k,I}$.*

*Proof.* To see that (4.11) and (4.12) are equivalent, first note that

$$dp|^t_{(v,s)} \circ d\pi|^t_v \, \eta = (d\pi|^t_v \, \eta, 0)\,,$$
$$d\Phi|^t_{(v,s)} \circ d\pi|^t_{\tilde{v}} \, \tilde{\eta} = \left(d_v\Phi|^t_{(v,s)} \circ d\pi|^t_{\tilde{v}} \, \tilde{\eta}, \tilde{\eta}(\tilde{v})\right),$$

by (4.3) and the fact that $d\pi|^t_{\tilde{v}} \, \tilde{\eta}(\dot{\Phi}(v,s)) = \tilde{\eta}(\tilde{v})$. Hence (4.11) holds if and only if

$$d\pi|^t_v \, \eta = d_v\Phi|^t_{(v,s)} \circ d\pi|^t_{\tilde{v}} \, \tilde{\eta} \quad \text{and} \quad \tilde{\eta}(\tilde{v}) = 0.$$

Thus (4.12) implies (4.11). For the converse, just apply both sides of (4.11) to the vector $(\dot{\Phi}(v,0), 0) \in T_{(v,s)}(SM \times I)$ and deduce that $\eta(v) = \tilde{\eta}(\tilde{v})$.

Before addressing the claims about conjugate pairs, let us make a few observations. It will be useful to work with $TM$ rather than $SM$. To connect the two, observe that

$$d\pi|^t_v = d\iota_{SM}|^t_v \circ d\pi_{TM}|^t_v, \tag{4.13}$$
$$d_v\Phi|^t_{(v,s)} \circ d\iota_{SM}|^t_{\tilde{v}} = d\iota_{SM}|^t_v \circ d_v\tilde{\Phi}|^t_{(v,s)}, \tag{4.14}$$

due to the identities $\pi = \pi_{TM} \circ \iota_{SM}$ and $\iota_{SM} \circ \Phi = \tilde{\Phi}(\iota_{SM}(\cdot), \cdot)$. Let

$$X := (d\pi_{TM}|^t_v \, \eta)^{\sharp_{\omega_g}} \in T_v(TM). \tag{4.15}$$

Equivalently, applying $\flat_{\omega_g}$, we have

$$X \lrcorner \, \omega_g = d\pi_{TM}|^t_v \, \eta. \tag{4.16}$$

In the next paragraph, we will prove the following analogue of the first condition in (4.12):

$$d\pi_{TM}|_v^t\,\eta = d_v\tilde{\Phi}|_{(v,s)}^t \circ d\pi_{TM}|_{\tilde{v}}^t\,\tilde{\eta}. \tag{4.17}$$

Now suppose (4.12) holds and $s \neq 0$. We will divide the proof that $(v, s)$ is a conjugate pair into three steps. The first one is to prove (4.17). By (4.13), (4.14), and (4.12),

$$d\iota_{SM}|_v^t \circ d\pi_{TM}|_v^t\,\eta = d\iota_{SM}|_v^t \circ d_v\tilde{\Phi}|_{(v,s)}^t \circ d\pi_{TM}|_{\tilde{v}}^t\,\tilde{\eta}.$$

Since $\ker(d\iota_{SM}|_v^t)$ is the span of the differential of $TM \ni w \mapsto |w|_g^2$, this means

$$d\pi_{TM}|_v^t\,\eta = d_v\tilde{\Phi}|_{(v,s)}^t \circ d\pi_{TM}|_{\tilde{v}}^t\,\tilde{\eta} + \frac{\tau}{2}\,d(|w|_g^2)\big|_v$$

for some $\tau \in \mathbb{R}$. Applying both sides to a radial vector $r \in T_v(TM)$, we find

$$0 = \tilde{\eta}\left(d\pi_{TM}|_{\tilde{v}} \circ d_v\tilde{\Phi}|_{(v,s)}\,r\right) + \tilde{\tau}$$

for some $\tilde{\tau} \in \mathbb{R}$. Since $\tilde{\eta}(\tilde{v}) = 0$ by assumption and the vector $d\pi_{TM}|_{\tilde{v}} \circ d_v\tilde{\Phi}|_{(v,s)}\,r$ is parallel to $\tilde{v}$, this implies that $\tau = 0$ and hence completes the proof of (4.17). As a corollary, observe that since $\tilde{\Phi}(\cdot, s)$ is a symplectomorphism, (4.17) implies that

$$d\pi_{TM}|_{\tilde{v}}^t\,\tilde{\eta} = d_v\tilde{\Phi}|_{(v,s)}X \lrcorner\, \omega_g. \tag{4.18}$$

The second step is to show that (4.15) is in $\mathrm{Range}(d\iota_{SM}|_v)$. Since (4.16) implies that $X \lrcorner\, \omega_g$ vanishes on $\ker(d\pi_{TM}|_v)$, which is a Lagrangian subspace, $X$ must be in $\ker(d\pi_{TM}|_v)$. Choose a natural coordinate system $(x^i, v^i)$ on $TM$ near $v$ such that $g_{ij} = \delta_{ij}$ at $\pi(v)$, and

$$v = d\pi_{TM}|_v\,\frac{\partial}{\partial x^1}. \tag{4.19}$$

Then $\mathrm{Range}(d\iota_{SM}|_v)$ is the span of the vectors $\partial/\partial x^1, \dots, \partial/\partial x^n, \partial/\partial v^2, \dots, \partial/\partial v^n$. Since $X$ is in $\ker(d\pi_{TM}|_v)$, we can write

$$X = a^j\,\frac{\partial}{\partial v^\ell}$$

for some $a^j \in \mathbb{R}$. Using (4.16), (4.19), and the assumption $\eta(v) = 0$, we find

$$\omega_g \left( X, \frac{\partial}{\partial x^1} \right) = 0.$$

Using (4.9) and the fact that $g_{ij} = \delta_{ij}$ at $\pi(v)$, this implies $a^1 = 0$. Hence $X$ is in $\mathrm{Range}(d\iota_{SM}|_v)$, so we can define the vector $d\iota_{SM}|_v^{-1} X \in T_v SM$.

The third step is to show that $d\iota_{SM}|_v^{-1} X$ is in $K_{(v,s)}$, meaning

$$d\iota_{SM}|_v^{-1} X \in \ker \left( d\pi|_{\tilde{v}} \circ d_v \Phi|_{(v,s)} \right) \cap \ker(d\pi|_v).$$

Since $X$ is in $\ker(d\pi_{TM}|_v)$ and $d\pi|_v = d\pi_{TM}|_v \circ d\iota_{SM}|_v$, we know $d\iota_{SM}|_v^{-1} X$ is in $\ker(d\pi|_v)$. Hence it is enough to prove that

$$d\pi|_{\tilde{v}} \circ d_v \Phi|_{(v,s)} \circ d\iota_{SM}|_v^{-1} X = 0.$$

By the transposes of (4.13) and (4.14), this is equivalent to showing

$$d\pi_{TM}|_{\tilde{v}} \circ d_v \tilde{\Phi}|_{(v,s)} X = 0.$$

But (4.18) implies that $d_v \tilde{\Phi}|_{(v,s)} X \lrcorner\, \omega_g$ vanishes on the Lagrangian subspace $\ker(d\pi_{TM}|_{\tilde{v}})$, so $d_v \tilde{\Phi}|_{(v,s)} X$ is indeed in $\ker(d\pi_{TM}|_{\tilde{v}})$. This proves that $(v, s)$ is a conjugate pair.

Now suppose $(v, s) \in \mathcal{C}_{R,k,I}$. Since $d\iota_{SM}|_v^{-1} X$ is in $K_{(v,s)}$, this means

$$\big((v, s), d\iota_{SM}|_v^{-1} X\big) \in E_{R,k,I}.$$

Using (4.16) in the first line below and (4.14) (transposed) and (4.18) in the second, we find

$$\begin{aligned} (d\iota_{SM}|_v \circ d\iota_{SM}|_v^{-1} X) \lrcorner\, \omega_g &= X \lrcorner\, \omega_g = d\pi_{TM}|_v^t \,\eta, \\ \left( d\iota_{SM}|_{\tilde{v}} \circ d_v \Phi|_{(v,s)} \circ d\iota_{SM}|_v^{-1} X \right) \lrcorner\, \omega_g &= d_v \tilde{\Phi}|_{(v,s)} X \lrcorner\, \omega_g = d\pi_{TM}|_{\tilde{v}}^t \,\tilde{\eta}. \end{aligned}$$

Hence $(\eta, \tilde{\eta}) = F_{k,I}\big((v, s), d\iota_{SM}|_v^{-1} X\big)$, which proves that $(\eta, \tilde{\eta}) \in F_{k,I}(E_{R,k,I})$.

Conversely, suppose $(\eta, \tilde{\eta}) = F_{k,I}((v,s), X)$. Unpacking (4.10), this means

$$d\iota_{SM}|_v X \lrcorner \, \omega_g = d\pi_{TM}|_v^t \, \eta, \tag{4.20}$$

$$\left(d\iota_{SM}|_{\tilde{v}} \circ d_v \Phi|_{(v,s)} X\right) \lrcorner \, \omega_g = d\pi_{TM}|_{\tilde{v}}^t \, \tilde{\eta}. \tag{4.21}$$

Let $Y \in T_{\tilde{v}}(TM)$. Then (4.21) and the transpose of (4.14) imply that

$$\omega_g \left(d_v \tilde{\Phi}|_{(v,s)} \circ d\iota_{SM}|_v X, Y\right) = d\pi_{TM}|_{\tilde{v}}^t \, \tilde{\eta}(Y).$$

Using that $\tilde{\Phi}(\cdot, s)$ is a symplectomorphism together with (4.20), we find

$$\left(d_v \tilde{\Phi}|_{(v,s)}^t\right)^{-1} \circ d\pi_{TM}|_v^t \, \eta(Y) = d\pi_{TM}|_{\tilde{v}}^t \, \tilde{\eta}(Y).$$

Hence (4.17) holds in this direction of the proof as well. Applying $d\iota_{SM}|_v^t$ to both sides of that equation and rewriting with (4.13) and (4.14) establishes the first condition in (4.12).

Now take the same natural coordinates on $TM$ described above (4.19). Then

$$d\iota_{SM}|_v X = \sum_{j=2}^{n} a^j \frac{\partial}{\partial v^j}$$

for some $a^j \in \mathbb{R}$. By (4.19), (4.20), and (4.9), we get

$$\eta(v) = d\pi_{TM}|_v^t \, \eta \left(\frac{\partial}{\partial x^1}\right) = (d\iota_{SM}|_v X \lrcorner \, \omega_g) \left(\frac{\partial}{\partial x^1}\right) = \omega_g \left(d\iota_{SM}|_v X, \frac{\partial}{\partial x^1}\right) = 0,$$

and $\tilde{\eta}(\tilde{v}) = 0$ by a similar argument. This establishes the second condition in (4.12). □

### 4.2.4 Final Decomposition of the Lévy Generator

Now we can state our main result. It refines Theorem 4.1.6 by showing that $\pi_* \circ R_a \circ \pi^*$ is an infinite sum of Fourier integral operators when there are no singular conjugate pairs.

**Theorem 4.2.9.** *Suppose $\mathcal{C}_S = \varnothing$ and $(M, g)$ satisfies Assumptions 4.0.1 and 4.1.5. Let $(I_j)_{j=1}^{\infty}$ be an open cover of $\mathbb{R}$ by intervals of length less than $r_{\mathrm{inj}}$. Then for each $j \geq 1$ and* *for* $k = 1$ *to* $n-1$*, the sets*

$$C_{A_{k,I_j}} := F_{k,I_j}(E_{R,k,I_j}) \subset T^*(M \times M)$$

*are either local canonical relations or empty. Let* $C_{A_{k,1,I_j}}, \dots, C_{A_{k,M_{k,j},I_j}}$ *be the connected components of* $C_{A_{k,I_j}}$*. Let* $\mathcal{A}_{2\alpha}$ *and* $\mathcal{A}_0$ *be the pseudodifferential operators from Theorem 4.1.6. Then* $\mathcal{A}$ *is of the form*

$$\mathcal{A}_{2\alpha} + \mathcal{A}_0 + \sum_{j=1}^{\infty} \left( \mathcal{A}_{-\infty,I_j} + \sum_{k=1}^{n-1} \sum_{m=1}^{M_{k,j}} A_{k,m,I_j} \right),$$

*where* $\mathcal{A}_{-\infty,I_j}$ *is a smoothing operator, and either*

$$A_{k,m,I_j} \in \mathcal{I}^{-(n-k+1)/2}(M \times M, C'_{A_{k,m,I_j}}),$$

*or* $M_{k,j} = 1$ *and* $A_{k,1,I_j} = 0$ *if* $C_{A_{k,I_j}} = \varnothing$.

*Proof.* We must show that $\pi_* \circ R_a \circ \pi^*$ is an infinite sum of Fourier integral operators. The clean intersection calculus does not directly apply in this case, but we will cut up $\pi_* \circ R_a \circ \pi^*$ so that it applies to each separate piece.

Since $\mathcal{C}_S = \varnothing$, for each $j \geq 1$ and for $k = 1$ to $n-1$ we can find bounded open subsets $U_{k,j}$ of $SM \times I_j$ with disjoint closures such that $\mathcal{C}_{R,k,I_j} \subset U_{k,j}$. Write

$$U_{k,j} = \bigcup_{m=1}^{M_{k,j}} U_{k,m,j},$$

where each open set $U_{k,m,j}$ contains exactly one of the connected components of $\mathcal{C}_{R,k,I_j}$. Then we can construct a partition of unity $\{\psi_{k,m,j}\}$ on $SM \times \mathbb{R}$ such that

$$\mathrm{supp}(\psi_{0,1,j}) \subset (SM \times I_j) \setminus \left( \bigcup_{k=1}^{n-1} \mathcal{C}_{R,k,I_j} \right)$$

for each $j \geq 1$ and $\mathrm{supp}(\psi_{k,m,j}) \subset U_{k,m,j}$ for $k = 1$ to $n-1$ and $m = 1$ to $M_{k,j}$. Setting

$M_{0,j} = 1$, we see that $\pi_* \circ R_a \circ \pi^*$ is of the form

$$\sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \sum_{m=1}^{M_{k,j}} \pi_* \circ L_{\psi_{k,m,j}}.$$

By Remark 4.2.2, the twisted wave front set of $L_{\psi_{k,m,j}}$ is contained in

$$C_{L,k,m,j} := \{(\xi, \tilde{\eta}) \in T^*SM \times T^*M : \exists\, s \in I_j \setminus \{0\} \text{ such that } (\pi_{T^*SM}(\xi), s) \in \operatorname{supp}(\psi_{k,m,j}) \\ \text{and } dp|^t_{(\pi_{T^*SM}(\xi),s)}\, \xi = d\Phi|^t_{(\pi_{T^*SM}(\xi),s)} \circ d\pi|^t_{\Phi(\pi_{T^*SM}(\xi),s)}\, \tilde{\eta}\}.$$

Therefore we obtain smoothing operators except near points in

$$C_{\pi_*} \circ C_{L,k,m,j} = \Big\{(\eta, \tilde{\eta}) \in T^*M \times T^*M : \exists\, (v,s) \in SM \times (I_j \setminus \{0\}) \text{ such that} \\ (v,s) \in \operatorname{supp}(\psi_{k,m,j}) \text{ and } dp|^t_{(v,s)} \circ d\pi|^t_v\, \eta = d\Phi|^t_{(v,s)} \circ d\pi|^t_{\Phi(v,s)}\, \tilde{\eta}\Big\}.$$

Lemma 4.2.8 implies that $C_{\pi_*} \circ C_{L,0,1,j}$ is empty, so

$$\mathcal{A}_{-\infty,I_j} := \pi_* \circ L_{\psi_{0,1,j}}$$

is a smoothing operator. For $k \geq 1$, Lemma 4.2.8 implies that

$$C_{\pi_*} \circ C_{L,k,m,j} = C_{A_{k,m,I_j}}.$$

If any of these compositions are empty, we can absorb the corresponding operator into $\mathcal{A}_{-\infty,I_j}$ and set $A_{k,m,I_j} = 0$.

It remains to show that if $C_{\pi_*} \circ C_{L,k,m,j}$ is nonempty, then the clean intersection calculus applies to $\pi_* \circ L_{\psi_{k,m,j}}$. We refer to points (i)-(ii) in the proof of Theorem 4.2.1. (Since we omit point (iii), we only obtain local canonical relations in general.) Define

$$C_{k,m,j} := (C_{\pi_*} \times C_{L,k,m,j}) \cap (T^*M \times \Delta(T^*SM) \times T^*M).$$

Let $E_{R,k,m,j}$ be the restriction of $E_{R,k,j}$ to the $m$th connected component of $\mathcal{C}_{R,k,I_j}$, and let $F^{\ell}_{k,I_j}$ be the $\ell$th component function of $F_{k,I_j}$ for $\ell = 1, 2$. Then Lemma 4.2.8 implies that

$C_{k,m,j}$ is a connected component of

$$\Big\{ \Big( F^1_{k,I_j}((v,s),X), d\iota_{SM}|_v^t(d\iota_{SM}|_v X \lrcorner\, \omega_g), d\iota_{SM}|_v^t(d\iota_{SM}|_v X \lrcorner\, \omega_g), F^2_{k,I_j}((v,s),X) \Big) : \\ ((v,s),X) \in E_{R,k,m,j} \Big\}.$$

Thus, to show that $C_{k,m,j}$ is an embedded submanifold, it is enough to prove that

$$E_{R,k,m,j} \ni ((v,s),X) \mapsto \big(v, d\iota_{SM}|_v^t(d\iota_{SM}|_v X \lrcorner\, \omega_g)\big) \in T^*SM \tag{4.22}$$

is a smooth embedding. Let $p_{k,m,j}$ be the projection of the $m$th connected component of $\mathcal{C}_{R,k,I_j}$ onto $SM$. Let $\mathcal{V}_{R,k,m,j}$ be the pullback of $\ker(d\pi)$ by $p_{k,m,j}$. Then $E_{R,k,m,j}$ is a smooth subbundle of $\mathcal{V}_{R,k,m,j}$ (as in the proof of Proposition 4.2.6), so it suffices to show that the extension of (4.22) to $\mathcal{V}_{R,k,m,j}$ is a smooth embedding. Observe that the bundle homomorphism

$$\mathcal{V}_{R,k,m,j} \ni ((v,s),X) \mapsto (v,X) \in TSM$$

covers $p_{k,m,j}$ and is a smooth embedding (because it is a proper injective immersion). This implies that $p_{k,m,j}$ is a smooth embedding, so $\mathcal{V}_{R,k,m,j}$ is smoothly isomorphic to the restriction of $\ker(d\pi)$ to $\mathrm{Range}(p_{k,m,j})$. Hence it suffices to show that

$$\ker(d\pi)|_{\mathrm{Range}(p_{k,m,j})} \ni (v,X) \mapsto \big(v, d\iota_{SM}|_v^t(d\iota_{SM}|_v X \lrcorner\, \omega_g)\big) \in T^*SM$$

is a smooth embedding. But (4.9) implies that this map is injective in each fiber, and hence a smooth bundle isomorphism onto its image. Therefore (4.22) is a smooth embedding, which proves that $C_{k,m,j}$ is a connected embedded submanifold of dimension $2n+k-1$.

Now let $c = (c_1, c_2, c_3, c_4) \in C_{k,m,j}$ be arbitrary, and consider the set

$$D_{k,m,j} := T_c(C_{\pi_*} \times C_{L,k,m,j}) \cap T_c(T^*M \times \Delta(T^*SM) \times T^*M) \\ \subset T_{c_1}(T^*M) \times T_{c_2}(T^*SM) \times T_{c_3}(T^*SM) \times T_{c_4}(T^*M).$$

Since $T_c C_{k,m,j}$ has dimension $2n+k-1$ and is contained in $D_{k,m,j}$, the intersection is clean if the dimension of $D_{k,m,j}$ is at most $2n+k-1$. Suppose $(Y_1, Y_2, Y_3, Y_4) \in D_{k,m,j}$. Then

$Y_2 = Y_3$, and an examination of $C_{L,k,m,j}$ shows that $Y_3$ determines $Y_4$. Hence the dimension of $D_{k,m,j}$ is at most $3n-1$ (the dimension of $C_{\pi_*}$). But for fixed $(v,s)$, the set

$$\left\{ d\iota_{SM}|_v^t (d\iota_{SM}|_v X \lrcorner\, \omega_g) : ((v,s),X) \in E_{R,k,m,j} \right\}$$

is a $k$-dimensional vector space contained in

$$\left\{ d\pi|_v^t \, \eta : \eta \in T^*_{\pi(v)}M \right\}.$$

Therefore the dimension of $D_{k,m,j}$ is at most $3n-1-(n-k) = 2n+k-1$, so the intersection is clean with excess $k-1$. Because $SM$ is compact, the projection map

$$\pi_{k,m,j} : C_{k,m,j} \to T^*M \times T^*M$$

is proper by p. 18 of [22]. Since $\pi_*$ and $L_{\psi_{k,m,j}}$ are Fourier integral operators of order $(1-n)/4$ and $-(n+1)/4$, respectively, we conclude that

$$A_{k,m,I_j} := \pi_* \circ L_{\psi_{k,m,j}}$$

is in $\mathcal{I}^{-(n-k+1)/2}(M \times M, C'_{A_{k,m,I_j}})$ whenever the set $C_{A_{k,m,I_j}}$ is nonempty. $\square$

Two special cases of Theorem 4.2.9 are worth mentioning. First, if $(M,g)$ is Anosov, then it has no conjugate points [38], so each Fourier integral operator $A_{k,m,I_j}$ is zero and we recover Theorem 1.6 in [7]. Second, Theorem 4.2.9 covers all possibilities in two dimensions because singular conjugate pairs cannot exist (since conjugate pairs can only have order 1). In higher dimensions, the generic case includes singular conjugate pairs [5, 27].

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