Brownian Motion on Spaces with Varying Dimension

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Abstract

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In this thesis we introduce and study Brownian motion with or without drift on state spaces with varying dimension.

- Starting with a concrete such state space that is the plane with an infinite pole on it, we construct a Brownian motion on it and derive sharp two-sided global estimates on its transition density functions (also called heat kernel). These two-sided estimates are of Guassian type. However, we show that the parabolic Harnack inequality fails for such process and the measure on the underlying state space does not satisfy volume doubling property.

- Brownian motion on some other state spaces with varying dimensions are also studied in this thesis. For instance, we study Brownian motion on a plane with multiple lines and Brownian motion on a plane with an arc.

- Similar to Brownian motion with varying dimension, drifted Brownian motion with varying dimension can be characterized by infinitesimal generators and by non-symmetric Dirichlet forms. By establishing and using Duhamel's formula, we show the transition density of Brownian motion with drift on spaces with varying dimension is comparable to that of Brownian motion with varying dimension without drift. We also derive the Green function estimates for this process on bounded smooth domains and establish Hölder regularity for its parabolic functions.

TABLE OF CONTENTS

Page

LIST OF FIGURES

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I would also like to express gratitude to all the financial, academic and technical support from University of Washington, Department of Mathematics, including all the library and computer facilities.

DEDICATION

I dedicate this thesis to my beloved family,

for their constant support and unconditional love.

Forever and always, I love you all dearly.

Chapter 1

INTRODUCTION

1.1 The Motivation of This Thesis

Brownian motion is a building block in modern probability theory. Named after the botanist Robert Brown in 1828, "Brownian motion" describes the irregular random movement of pollen grains suspended in fluid resulting from their collision with the quick atoms or molecules in the media. Serving great significance in industry as well as pure mathematics, a tremendous range of applications of Brownian motion has grown far beyond motions of microsopic particles and has been seen, for instance, in physics and in modelings of stock prices fluctuations.

As an important research object of pure mathematics, Brownian motion was first constructed on Euclidean spaces and later extended to Riemannian manifolds. There is also a lively research area concerning the construction of Brownian motion on unusual spaces including non-smooth media, disconnected subsets of Euclidean spaces or fractals. For instance, in [2], Bass and Burdzy studied "fiber Brownian motion" that moves like 2-dimensional Brownian motion in a part of its state space, but it evolves like a 1-dimensional Brownian motion if it happens to be on one of many "fibers" in its state space. The process is obtained as a weak limit of reflected Brownian motions in domains with very thin tubes. Although the dimension of the state space of the Brownian motion studied in [2] is also changing, the object that I study in my thesis is quite different. There have also been work done studying how to produce a new Markov process by "gluing" together several Markov processes living on different state spaces. For example, in [14], Evans and Sowers studied how to produce a new Markov process by partially "collapsing" a process under equivalence relation. However, our method is also very different from theirs.

In this thesis, I study Brownian motion on state spaces with varying dimension, the simpliest case of which is an infinite 1-dimensional pole installed on a 2-dimensional plane.

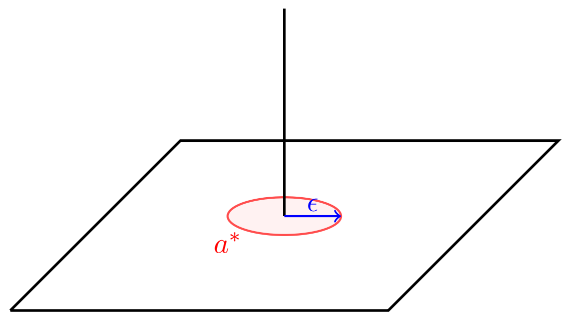

Figure 1.1: State space of BMVD

However, it is known that a singleton in the plane is polar (i.e. of zero capacity) with respect to Brownian motion, by which we mean 2-dimensional Brownian motion does not hit a singleton in finite time. Consequently, Brownian motion cannot be constructed in the usual sense on such a state space because once a Brownian motion particle is on the plane, it will never have the chance to climb up the pole. Therefore, we will need to define Brownian motion with varying dimension as a "darning" process. Roughly speaking, we "collapse" (or short) a small closed disk $B_\epsilon = B(0,\epsilon)$ on $\mathbb{R}^2$ into a singleton a^* which therefore has positive capacity, namely Brownian motion on the plane does hit a closed disk in finite time with probability one, and then we install a pole at a^*. See Figure 1.1.

1.2 Notations and Metrics on the State Space

To be more precise, the state space of Brownian motion with varying dimension is defined in the following way. Suppose for $\epsilon > 0$, B_ϵ is the closed disc on $\mathbb{R}^2$ centered at $(0,0)$ with radius ϵ. Let $D_0 := \mathbb{R}^2 \setminus B_\epsilon$. By identifying B_ϵ with a singleton denoted by a^*, we can introduce a topological space $E := D_0 \cup \{a^*\} \cup \mathbb{R}_+$, with a neighborhood of a^* defined as $\{a^*\} \cup (D_1 \cap \mathbb{R}_+) \cup (D_2 \cap D_0)$ for some neighborhood D_1 of 0 in $\mathbb{R}_+$ and D_2 of B_ϵ in D_0. Fix $p > 0$. Let m_p be the measure on E whose restriction on $\mathbb{R}_+$ and D_0 is the Lebesgue measure times p and 1, respectively. We set $m_p(\{a^*\}) = 0$.

Throughout this paper we will denote the geodesic metric on E by $\rho(\cdot,\cdot)$. Namely, $\forall x, y \in E$, $\rho(x,y)$ is the shortest distance between x and y. We also denote the Euclidean metric by $|\cdot|_e$. Namely, $\forall x, y \in D_0 = E \cap \mathbb{R}^2$, $|x-y|_e$ is simply the Euclidean distance on $\mathbb{R}^2$ between x and y. For $x \in D_0$, $|x|_\rho = \rho(a^*, x)$. Apparently,

$$\rho(x,y) = |x-y|_e \wedge (|x|_\rho + |y|_\rho), \quad x, y \in D_0.$$

1.3 Background of Studying Heat Kernel Estimates

A Markov process is characterized by its transition function. However, in many cases it is impossible to get the explicit formula for the transition density of a process, which motivates probabilists as well as analysts to study its "estimates". Therefore, obtaining sharp two-sided bounds of the transition density functions of strong Markov processes is an important question both in probability and in analysis. This field of research has been called "heat kernel estimates" because the densities of the heat semigroup, which can also be viewed as the fundamental solutions to the heat equations driven by the Laplacian operator, are exactly the transition densities of Brownian motion on Euclidean spaces. This indeed provides deep connection between probability, PDE and geometry through "heat kernel".

Another connection between probability and analysis is through infinitesimal generators. In particular, the infinitesimal generator of Brownian motion is the Laplacian operator, and Brownian motion can be constructed on Riemannian manifolds by relating its transition semigroup to the Laplace-Beltrami operator on such manifolds, which enables mathematicians to study geometry through probabilistic approaches.

A great amount of influential work in this rich and fruitful direction has been done at least in the past 50 years. It was first showed in 1968 by Aronson [1] that the fundamental solutions of $\mathcal{L}u = u_t$ have Gaussian-type upper and lower bounds, where the operator $\mathcal{L}u := \{A_{i,j}(x)u_{x_i}\}_{x_j}$ satisfies that there exist constants $c, M > 0$ such that for all $\xi \in \mathbb{R}^d$ and for almost all (x,t),

$$A_{i,j}(x)\xi_i\xi_j \geq c\|\xi\|^2 \text{ and } |A_{i,j}(x)| \leq M.$$

To be more precise, it has been shown in [1] that suppose $p(t,x,y)$ is the fundamental solution of $\mathcal{L}u = u_t$, where $\mathcal{L}$ satisfies the conditions described above, then there exist constants $C_i > 0$, $1 \le i \le 4$ depending only on T and the structure of $\mathcal{L}$ such that

$$\frac{C_1}{t^{-d/2}} \exp\{-C_2|x-y|^2/t\} \le p(t,x,y) \le \frac{C_3}{t^{-d/2}} \exp\{-C_4|x-y|^2/t\}.$$

On a weighted complete Riemannian manifold (M, g), we say that the two-sided Aronson-type Gaussian bounds for heat kernels hold if

$$p(t,x,y) \asymp \frac{C_1}{t^{d/2}} \exp\left(-\frac{C_2|x-y|^2}{t}\right), \quad \forall x, y \in \mathbb{R}^d, t > 0,$$

where $C_1, C_2 > 0$ are constants and the sign "$\asymp$" means that both $\ge$ and $\le$ are satisfied but possibly with different values of the constants C_1, C_2. There have been remarkable results in the field of global analysis on manifolds. Let Δ be the Laplace-Beltrami operator on a complete Riemannian manifold M with the Riemannian mteric d and Riemannian measure μ. Li-Yau proved in [21] that if M has non-negative Ricci curvature, then the heat kernel $p(t,x,y)$ satisfies

$$\frac{C_1}{\mu\left(B(x,t^{1/2})\right)} \exp\left(-\frac{d(x,y)}{C_1 t}\right) \le p(t,x,y) \le \frac{C_2}{\mu\left(B(x,t^{1/2})\right)} \exp\left(-\frac{d(x,y)}{C_2 t}\right).$$

It was proved a few years later by A. Grigor'yan and L. Saloff-Coste that on a weighted complete Riemannian manifold (M, d), the two-sided Aronson-type Gaussian estimate for heat kernels can be characterized by any of the following equivalent ways.

- The parabolic Harnack inequality: A positive solution u of the heat equation in a cylinder of the form $Q = (s, s+r^2) \times B(x,r)$ satisfies

$$\sup_{Q_-}\{u\} \le C \inf_{Q_+}\{u\},$$

  where $Q_- = (s + r^2/5, s + 2r^2/5) \times B(x, r/2)$, $Q_+ = (s + 3r^2/5, s + 4r^2/5) \times B(x, r/2)$.

- The conjunction of

  - The volume doubling property: $\mu(B(x,2r)) \le C\mu(B((x,r))$, $\forall x \in M, r > 0$.
  - The Poincaré inequality: $\forall x \in M, r > 0, B = B(x,r)$

$$\int_B |f - f_B|^2 \,\mu(dx) \le Cr^2 \int_B |\nabla f|^2 \,\mu(dx), \quad \forall f \in C_c^\infty(B),$$

where $f_B = \frac{1}{\mu(B)} \int_B f(x)\mu(dx)$.

The same characterization has been further extended to strongly local Dirichlet forms by Karl-Theodor Sturm in [20].

1.4 Case of Brownian Motion with Varying Dimension

The primary motivation of my research is to study the properties of Brownian motion with varying dimension (BMVD in abbreviation), in particular, the global sharp two-sided estimates of the transition density functions. We first characterize of the L^2-infinitesimal generator $\mathcal{L}$ of BMVD in Section 2.3. Recall the notations defined in Section 1.1, we show that $u \in L^2(E; m_p)$ is in the domain of the generator $\mathcal{L}$ if and only if Δu exists as an L^2-integrable function in the distributional sense when restricted to D_0 and $\mathbb{R}_+$, and u satisfies zero-flux condition at a^*; see Theorem 2.3.1 for details. It is not difficult to see that BMVD has a transition density function $p(t, x, y)$ with respect to the measure m_p, which is also called the fundamental solution (or heat kernel) for $\mathcal{L}$. Note that $p(t, x, y)$ is symmetric in x and y.

Although it can be shown that the BMVD studied in thesis thesis is also associated to the Laplacian operator on its state space, one cannot expect its transition density function to be in the same Gaussian type as Brownian motion on Euclidean spaces. The reason is that the volume-doubling property near the darning point a^* fails due to the varying dimension of the state space. (To see this, denote $B_\rho(x, r)$ the ball centered at $x \in E$ with radius r under ρ-metric. Then for small $r > 0$ and $x_0 \in D_0$ with $|x_0|_\rho = r$, $m_p(B_\rho(x_0, r)) = \pi r^2$ while $m_p(B_\rho(x_0, 2r)) = 2\epsilon r + r^2 + r$. Thus there does not exist a constant $C > 0$ so that $m_p(B_\rho(x, 2r)) \leq C m_p(B_\rho(x, r))$ for all $x \in E$ and every $r \in (0, 1]$.) So we can not employ the results of A. Grigor'yan and L. Saloff-Coste to obtain heat kernel estimates through volume doubling and Poincaré inequality or through parabolic Harnack inequality. In fact, we will show the parabolic Harnack inequality fails for BMVD. Therefore the behavior of its heat kernel depends on both time and the position of the points. Indeed it turns out its heat kernel behavior switches between 1-dimensional Gaussian type and 2-dimensional Gaussian type.

There have been other results on studying heat kernels on Riemannian manifolds whose geometric structure exhibits some behavior of "varying dimension". For instance, heat kernel behaviors on manifolds with ends have been studied in [16], which gives us some idea on how the keat kernel behavior changes in response to the curvature. However, the setting in [16] is still Laplacian operators defined on Riemannian manifolds.

Another goal of this thesis is to study Brownian motion with drift on spaces with varying dimension. Similar to BMVD, Brownian motion with drift on spaces with varying dimension is a diffusion process on the same state space such that its restriction on $\mathbb{R}_+$ is $1-$dimensional Brownian motion with drift, and its restriction on D_0 is 2-dimensional Brownian motion with drift on D_0. It admits no killing or sojourn at the darning point a^*. Such a process can be conveniently defined in terms of the Dirichlet form.

On $\mathbb{R}^d$, Brownian motion with drift b is characterized by its infinitesimal generator $\mathcal{L}^b = \Delta + b \cdot \nabla$, where the drift term b satisfies the Kato class condition that $b \in \mathbf{K}_{d+1}$ or $|b|^2 \in \mathbf{K}_d$. We say a function $b : \mathbb{R}^d \to \mathbb{R}$ is in Kato class $\mathbf{K}_\alpha$ if and only if

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} \frac{|b(y)|}{|x-y|^{\alpha-2}} dy = 0, \quad \text{for } \alpha \ge 3,$$

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} \log\left(|x-y|^{-1}\right) |b(y)|\, dy = 0, \quad \text{for } \alpha = 2,$$

and

$$\sup_{x} \int_{x\in\mathbb{R}^d} \int_{|x-y|\le 1} |b(y)|\, dy < \infty, \quad \text{for } \alpha = 1.$$

It was proved by Z.-Q. Chen and Z. Zhao in [11] that when $|b|^2 \in \mathbf{K}_d$, the bilinear form $\mathcal{L}^b = \Delta + b \cdot \nabla$ is lower semibounded, closable, Markovian and satisfies Silverstein's sector condition, thus there is a minimal diffusion process associated with it. Later it was proved by R. Bass and Z.-Q.Chen in [3] that when $b \in \mathbf{K}_{d+1}$, there is a unique weak solution to the SDE

$$dX_t = dB_t + b(X_t)dt, \qquad x = x_0$$

which is a strong Markov process associated to the infinitesimal generator $\mathcal{L}^b = \Delta + b \cdot \nabla$. We point out that on $\mathbb{R}^d$, $L^p \subset \mathbf{K}_{d+1} \cap \left\{b : |b|^2 \in \mathbf{K}_d\right\}$ for $p > d$.

The question of sharp two-sided bounds for transition densities of drifted Brownian motion on Euclidean spaces has been completely answered in the past few decades. It was first established by Aronson that the transition density $p(t,x,y)$ of drifted Brownian motion on $\mathbb{R}^d$ has the following two-sided Gaussian-type bounds provided that $b \in L^p(B(0,R))$ for some $p > d$ and $R > 0$ and bounded outside $B(0,R)$.

$$\frac{C_1}{t^{d/2}} \exp\left(-\frac{C_1|x-y|^2}{t}\right) \le p(t,x,y) \le \frac{C_2}{t^{d/2}} \exp\left(-\frac{C_2|x-y|^2}{t}\right).$$

Later it was proved by Q. S. Zhang in [27] that the above heat kernel estimate holds for $b \in \mathbf{K}_{d+1}$. Also readers may refer to [23] by Riahi and [18, Proposition 2.3] by P. Kim and R. Song.

In this thesis, we study Brownian motion with drift on spaces with varying dimension by obtaining its sharp two-sided heat kernel estimates. Recall the notations defined in Section 1.2. Let $b : E \to \mathbb{R}$ be a measurable function. In this thesis we assume b can be decomposed as $b = b_1 + b_2$, where $b_1 \in L^\infty(E)$ and b_2 satisfies that

1. $b_2|_{\mathbb{R}_+} \in L^p(\mathbb{R}_+)$ with $p \in (1,\infty]$
2. $b_2|_{D_0} \in L^p(D_0)$ with $p \in (2,\infty]$.

We denote such a family of b by $L^\infty(E) + L^{p_1,p_2}(E)$ for some $p_1 \in (1,\infty]$ and $p_2 \in (2,\infty]$, where $L^{p_1,p_2}(E) := \{f : f|_{\mathbb{R}_+} \in L^{p_1}(\mathbb{R}_+),\ f|_{D_0} \in L^{p_2}(D_0).\}$.

By establishing and using Duhamel's formula for BMVD, we show for drift functions in the class of $L^\infty(E) + L^{p_1,p_2}(E)$ with $p_1 \in (1,\infty]$ and $p_2 \in (2,\infty]$, the heat kernels of drifted BMVD have the same two-sided Gaussian-type bounds as BMVD without drift.

1.5 Key Ingredients

The main difficulty in the study of BMVD was that, since the dimension of the state space E is changing and the volume-doubling property fails, most of the typical approaches do not work for the BMVD. Despite of the fact that Nash's inequality is valid for BMVD, the upper bound given by Nash's inequality doesn't turn out to be sharp. To tackle such difficulities, one of the key methods is to "project" the state space of the BMVD to $\mathbb{R}$ by considering its radial process. Using Fukushima decomposition, one can get an stochastic differential equation characterizing the radial process of BMVD, from which we derive the small time heat kernel estimate for this radial process.

However, the stochastic differential equation characterizing the radial process does not directly yield the large time estimate, as it involves a non-constant drift term. By obtaining the two-sided estimate for the on-diagonal heat kernel at the darning point a^* through some probabilistic method, we therefore establish the global off-diagonal heat kernel estimates for large time.

In this theis, we define BMVD with drift as a strong Markov process in terms of Dirichlet form. Unlike BMVD without drift, BMVD with drift in general is non-symmetric. It also turns out that the method of radial process is no longer applicable in this case, because the process is no longer rotational-invariant on the plane. To attack the difficulty, we first realize that the process can be equivalently characterized by Girsanov transform. Thus by establishing and using the Duhamel's formula for BMVD, we show that the small time heat kernel for Brownian motion with drift on space with varying dimension has the same two-sided Gaussian-type bounds as BMVD for the class of drift functions in $L^\infty(E) + L^{p_1,p_2}(E)$.

Chapter 2
PRELIMINARY

2.1 Definition of Brownian motion with varying dimension

The motivation of this thesis is to study the properties of a symmetric Hunt process whose behavior is similar to Brownian motion but lives on a state space whose dimension is varying. Such a process is called Brownian motion with varying dimension (BMVD in abbreviation). The simplest state space with varying dimension is $\mathbb{R}^2 \cup \mathbb{R}^1$ which is embedded in $\mathbb{R}^3$ in the following way:

$$\mathbb{R}^2 \times \mathbb{R}_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0 \text{ or } x_1 = x_2 = 0 \text{ and } x_3 > 0\}.$$

One cannot construct a Brownian motion on such a space in the usual sense, because $2-$dimensional Brownian motion does not hit a singleton. Therefore, in order to construct a symmetric Hunt process whose behavior is similar to Brownian motion with a state space that has varying dimension, we collapse a closed disc on $\mathbb{R}^2$ to a singleton, and then define the desired Brownian motion with varying dimension as a "darning" process whose definition will be given later. See Figure 1.1.

As has been defined in Section 1.2, the state space of Brownian motion with varying dimension is $E = D_0 \cup \{a^*\} \cup \mathbb{R}_+$, where $D_0 = \mathbb{R}^2 \backslash B_\epsilon$. A neighborhood of a^* is defined as $\{a^*\} \cup (D_1 \cap \mathbb{R}_+) \cup (D_2 \cap D_0)$ for some neighborhood D_1 of 0 in $\mathbb{R}_+$ and D_2 of B_ϵ in D_0. For fixed $p > 0$, m_p is the measure on E whose restriction on $\mathbb{R}_+$ and D_0 is the Lebesgue measure times p and 1, respectively. $m_p(\{a^*\}) = 0$.

Definition 2.1.1 (BMVD)**.** *Fix $p > 0$. The BMVD studied in this paper, denoted by X, is an m_p-symmetric diffusion on E such that (see, for example, [9].)*

1. *its part process in $\mathbb{R}_+$ or D_0 has the same law as standard Brownian motion in $\mathbb{R}_+$ or D_0;*

2. it admits no killings on a^*;

It follows from the m_p-symmetry and the fact that $m_p(\{a^*\}) = 0$ that such a process spends zero Lebesgue amount of time (i.e. zero sojourn time) at a^*.

Here and in the rest of this thesis, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$. We also follow the convention that in the statements of the theorems or propositions $C, C_1, \cdots$ denote positive constants, whereas in their proofs $c, c_1, \cdots$ denote positive constants whose value might change along the lines of a proof.

2.2 Existence and Uniqueness of BMVD

The following theorem addresses the existence and uniquess of BMVD that we are studying.

Theorem 2.2.1. *For every $\epsilon > 0$ and $p > 0$, BMVD X on E with parameter (ϵ, p) exists and is unique. Its associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m_p)$ is given by*

$$\begin{cases} \mathcal{F} = \left\{f : f|_{\mathbb{R}^2} \in W^{1,2}(\mathbb{R}^2), f \text{ is constant } \mathcal{E}-q.e. \text{ on } B_\epsilon, f|_{\mathbb{R}_+} \in W^{1,2}(\mathbb{R}_+), f|_{\mathbb{R}}(0) = f|_{B_\epsilon}\right\}, \\ \mathcal{E}(f, g) = \sum_{i=1,2} \int_{\mathbb{R}^2 \setminus B_\epsilon} \frac{\partial f(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_i} dx + p \int_{\mathbb{R}_+} f'(x) g'(x) dx, \end{cases}$$

where "q.e." means "quasi-everywhere" with respect to Brownian motion on $\mathbb{R}^2$, whose definition can be found, for instance, in [9].

Proof. Existence: Let $u_0(x) = \mathbb{E}^x\left[e^{-T_{a^*}}\right]$. Also let

$$\mathcal{F} = \mathcal{E}_1\text{-closure of linear span of } W_0^{1,2}(E \setminus \{a^*\}) \cup \{u_0\},$$

where

$$\mathcal{E}(f, g) = \int_{D_0} \nabla f(x) \cdot \nabla g(x) dx + p \int_{\mathbb{R}_+} f'(x) g'(x) dx. \tag{2.2.1}$$

It is easy to check that $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on $L^2(E; m_p)$. So there is a "reversible" conservative diffusion process X on E associated with it. The process X is a BMVD on E.

Uniqueness: Conversely, if X is a BMVD, it suffices to check from definition that its associated Dirichlet form denoted by $(\mathcal{E}^*, \mathcal{F}^*)$ has to be the one given in (2.2.1). Indeed, since a^* is non-polar for X, for all $u \in \mathcal{F}^*$, $H^1_{a^*}u(x) := \mathbb{E}^x\left[e^{-T_{a^*}}u(X_{T_{a^*}})\right] = u(a^*)\mathbb{E}^x[T_{a^*}] \in \mathcal{F}$ and $u - H^1_{a^*}u(x) \in W^{1,2}_0(E\backslash\{a^*\})$. Thus $\mathcal{F}^* \subset \mathcal{F}$. The other direction of inclusion holds because X^D has the same distribution as subprocess of Brownian motion killed upon leaving D, which proves $\mathcal{F} = \mathcal{F}^*$. $(\mathcal{E}, \mathcal{F})$ is strongly local so for every bounded $u \in \mathcal{F}^* = \mathcal{F}$,

$$\mathcal{E}(u,u) = \mu^c_{\langle u\rangle}(E\backslash\{a^*\}) = \mu^c_{\langle u\rangle}(E\backslash\{a^*\}) + \mu^c_{\langle u\rangle}(a^*) = \mu^c_{\langle u\rangle}(E\backslash\{a^*\}) = \mu^c_{\langle u\rangle}(D_0) + \mu^c_{\langle u\rangle}(\mathbb{R}_+),$$

where the third equality is due to the fact that $m_p(\{a^*\}) = 0$. See, for example, [9, Proposition 4.3.1]. i.e.,

$$\mathcal{E}(u,u) = \int_{D_0} |u(x)|^2 dx + p\int_{\mathbb{R}_+} |u'(x)|^2 dx, \quad \forall u \in \mathcal{F}^*.$$

The proof is thus complete. □

2.3 Characterization of the Infinitesimal Generator associated with BMVD

To give the characterization of the infinitesimal generator $\mathcal{L}$ associated with BMVD, let us first define the "flux" $\mathcal{N}_p(u)(a^*)$ at the darning point a^* of a function $u \in \mathcal{D}(\mathcal{L})$ as follows.

$$\mathcal{N}_p(u)(a^*) := \int_E \nabla u(x)\cdot u_0(x) m_p(dx) + \int_E \Delta u(x) u_0(x) m_p(dx), \tag{2.3.1}$$

where $u_0(x) = \mathbb{E}^x\left[e^{-T_{a^*}}\right]$. The following theorem characterizes the infinitesimal generator associated with BMVD.

Theorem 2.3.1. *A function $u \in \mathcal{F}$ is in $\mathcal{D}(\mathcal{L})$ if and only if the distributional Laplacian Δu of u exists as an L^2-integrable function on $E\backslash\{a^*\}$ and u has zero flux at a^*. Moreover, for $u \in \mathcal{D}(\mathcal{L})$, $\mathcal{L}u = \Delta u$ on $E\backslash\{a^*\}$.*

Proof. Let $\mathcal{L}$ be the L^2-generator of BMVD X. $u \in \mathcal{D}(\mathcal{L})$ if and only if $u \in \mathcal{F}$ and there is some $f \in L^2(E; m)$ so that

$$\mathcal{E}(u,v) = -\int_E f(x)v(x)m_p(dx) \quad \text{for every } v \in \mathcal{F}.$$

Denote the above f by $\mathcal{L}u$. The above is equivalent to

$$\int_E \nabla u(x)\cdot\nabla v(x)m_p(dx) = -\int_E f(x)v(x)m_p(dx) \quad \text{for every } v\in C_c^\infty(E\backslash\{a^*\}) \qquad (2.3.2)$$

and

$$\int_E \nabla u(x)\cdot\nabla u_0(x)m_p(dx) = -\int_E f(x)u_0(x)m_p(dx). \qquad (2.3.3)$$

(2.3.2) says that $f=\Delta u\in L^2(E;dx)$, and (2.3.3) is equivalent to $\mathcal{N}_p(u)(a^*)=0$. □

2.4 Main Results: Sharp Two-sided Heat Kernel Bounds for BMVD

The main purpose of this thesis is to establish the sharp two-sided estimates on $p(t,x,y)$ in Theorem 2.4.1 and Theorem 2.4.2. To state the theorems, we first recall that one of the most interesting heat kernel estimates is the following Gaussian type estimate on a Riemannian manifold (M,ρ)

$$p(t,x,y)\asymp\frac{C_1}{t^{d/2}}\exp\left(-\frac{C_2\rho(x,y)^2}{t}\right),\quad \forall x,y\in(M,\rho), t>0,$$

where $C_1,C_2>0$ are constants and the sign $\asymp$ means that both $\geq$ and $\leq$ are satisfied but possibly with different values of the constants C_1,C_2. For example, the above estimate holds for the heat kernel of uniformly elliptic operators in divergence form in $\mathbb{R}^d$.

For notation convenience, in this paper we denote

$$\overline{p}_{_D}(t,x,y):=p(t,x,y)-p_{_D}(t,x,y), \qquad (2.4.1)$$

where D is a domain of E and $p_{_D}(t,x,y)$ is the transition density of the part process killed upon exiting D. The "intuitive" difference between $p_{_D}(t,x,y)$ and $\overline{p}_{_D}(t,x,y)$ is that for $p_{_D}(t,x,y)$, the trajectory started from x hits y at time t without exiting D, while for $\overline{p}_{_D}(t,x,y)$, the trajectory has to exit D once and then return to y.

The main results are as follows.

Theorem 2.4.1. *Let $T>0$ be fixed. Then when $t\in(0,T]$, there exist positive constants C_i, $1\leq i\leq 22$, such that the transition density $p(t,x,y)$ of BMVD satisfies the following estimates.*

1. *For* $x, y \in \mathbb{R}_+$,

$$\frac{C_1}{\sqrt{t}} e^{-\frac{C_2|x-y|^2}{t}} \le p(t,x,y) \le \frac{C_3}{\sqrt{t}} e^{-\frac{C_4|x-y|^2}{t}}. \tag{2.4.2}$$

2. *For* $x \in \mathbb{R}_+$, $y \in D_0 \cup \{a^*\}$,

$$\frac{C_5}{\sqrt{t}} e^{-\frac{C_6\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_7}{\sqrt{t}} e^{-\frac{C_8\rho(x,y)^2}{t}}, \quad \text{when } |y|_\rho \le 1; \tag{2.4.3}$$

whereas

$$\frac{C_9}{t} e^{-\frac{C_{10}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{11}}{t} e^{-\frac{C_{12}\rho(x,y)^2}{t}}, \quad \text{when } |y|_\rho > 1, \tag{2.4.4}$$

3. *For* $x, y \in D_0 \cup \{a^*\}$, *when* $|x|_\rho < 1$, $|y|_\rho < 1$,

$$\begin{aligned} \frac{C_{13}}{\sqrt{t}} e^{-\frac{C_{14}\rho(x,y)^2}{t}} + \frac{C_{13}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{15}|x-y|_e^2}{t}} &\le p(t,x,y) \\ \le \frac{C_{16}}{\sqrt{t}} e^{-\frac{C_{17}\rho(x,y)^2}{t}} + \frac{C_{16}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{18}|x-y|_e^2}{t}}; \end{aligned} \tag{2.4.5}$$

otherwise

$$\frac{C_{19}}{t} e^{-\frac{C_{20}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{21}}{t} e^{-\frac{C_{22}\rho(x,y)^2}{t}}, \tag{2.4.6}$$

It is easy to see that the above three cases cover all the possible locations of the points $x, y \in E$, up to switching x and y. The large time heat kernel estimates for BMVD are given by the next theorem, which is very different from the small time estimates.

Theorem 2.4.2. *Let* $T > 0$ *be fixed. Then when* $t \in [T, \infty)$, *there exist positive constants* C_i, $23 \le i \le 40$, *such that the transition density* $p(t,x,y)$ *of BMVD satisfies the following estimates:*

1. *For* $x, y \in D_0$,

$$\frac{C_{23}}{t} e^{-\frac{C_{24}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{25}}{t} e^{-\frac{C_{26}\rho(x,y)^2}{t}}.$$

2. *For* $x \in \mathbb{R}_+$, $y \in D_0 \cup \{a^*\}$,

   when $|y|_\rho \le 1$,

$$\frac{C_{27}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\cdot\log t\right]e^{-\frac{C_{28}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{29}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\cdot\log t\right]e^{-\frac{C_{30}\rho(x,y)^2}{t}};$$

   when $|y|_\rho > 1$,

$$\frac{C_{31}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1+\log\left(1+\frac{\sqrt{t}}{|y|_\rho}\right)\right)\right]e^{-\frac{C_{32}\rho(x,y)^2}{t}} \le p(t,x,y)$$
$$\le \frac{C_{33}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1+\log\left(1+\frac{\sqrt{t}}{|y|_\rho}\right)\right)\right]e^{-\frac{C_{34}\rho(x,y)^2}{t}},$$

3. *For* $x, y \in \mathbb{R}_+$,

$$\frac{C_{35}}{\sqrt{t}}\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1\wedge\frac{|y|}{\sqrt{t}}\right)e^{-\frac{C_{36}|x-y|^2}{t}} + C_{35}\left[\frac{1}{t}+\frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right]e^{-\frac{C_{37}(|x|^2+|y|^2)}{t}}$$
$$\le p(t,x,y)$$
$$\le \frac{C_{38}}{\sqrt{t}}\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1\wedge\frac{|y|}{\sqrt{t}}\right)e^{-\frac{C_{39}|x-y|^2}{t}} + C_{38}\left[\frac{1}{t}+\frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right]e^{-\frac{C_{40}(|x|^2+|y|^2)}{t}}.$$

Remark 2.4.3. *In the above two theorems, T can be chosen as an arbitrary positive constant. Changing the T value will only result in possible different values of C_i's . The forms of the heat kernel bounds will remain the same.*

Since the dimension of the state space E is varying, many of the typical approaches do not work for BMVD. One of the key methods we use to get the small time heat kernel estimates is the method of radial process. By applying Fukushima decomposition we get the SDE characterization for the radial process of BMVD. From there by invoking the explicit transition density of 2−Bessel process, we compute its hitting time distribution to get the desired heat kernel bounds for BMVD when at least one of x and y is on $\mathbb{R}^2$.

However, the SDE characterizing the radial process does not directly yield the large time estimate, as it involves a non-constant drift term. The key step is to get the two-sided estimate of $p(t, a^*, a^*)$ for $t > T$, where T is some fixed time. To do this, we first show $p(t, a^*, a^*) \le C/t$ when $t > T$ by applying the hitting time estimates of 2-dimensional

Bessel process obtained in [17]. Since this is equivalent to the fact that for the radial process $p^{(Y)}(t,0,0) \le C/t$. We thus compute the symmetrizing measure as well as the intrinsic metric of Y as a 1-dimensional diffusion, from which one can see that the volume of $B(0,r)$ under the symmetrizing measure is comparable to $r+r^2$. We then apply [12, Theorem 7.2] to conclude that $p(t,a^*,a^*) \asymp 1/t$. Once the two-sided bound for $p(t,a^*,a^*)$ is established, $p(t,x,y)$ for arbitrary pairs of $(x,y)\in E\times E$ can be bounded sharply again by invoking the Brownian hitting time distribution as well as the heat kernel estimates for part Brownian motion.

We also establish the Hölder-continuity for the parabolic functions of BMVD in Theorem 5.0.4. However, parabolic Harnack inequality does not hold for BMVD. An counterexample is given in Remark 3.4.5.

2.5 BMVD with Drift: Definition and Main Results

2.5.1 Definition of BMVD with Drift

In this subsection, we introduce Brownian motion with drift on spaces with varying dimension. Let $b: E\to\mathbb{R}$ be in the family of $L^\infty(E)+L^{p_1,p_2}(E)$ for some $p_1\in(1,\infty]$ and $p_2\in(2,\infty]$, where $L^{p_1,p_2}(E) := \left\{f: f|_{\mathbb{R}_+}\in L^{p_1}(\mathbb{R}_+),\, f|_{D_0}\in L^{p_2}(D_0)\right\}$. We first need to define BMVD with drift b. We call the Hunt process associated with the following Dirichlet form "drifted Brownian motion with varying dimension" denoted by X^b.

$$\mathcal{E}^b\left(f,\ g\right) = \mathcal{E}^0\left(f,\ g\right) - \left(b\cdot\nabla f,\ g\right), \quad \mathcal{D}(\mathcal{E}^b) = \mathcal{D}(\mathcal{E}^0),$$

where $\mathcal{D}(\mathcal{E}^0) = \big\{f: f|_{\mathbb{R}^2}\in W^{1,2}(\mathbb{R}^2), f$ is constant $\mathcal{E}-$q.e. on $B_\epsilon, f|_{\mathbb{R}_+}\in W^{1,2}(\mathbb{R}_+), f|_{\mathbb{R}_+}(0) = f|_{B_\epsilon}\big\}$. One can check directly that this is a regular non-symmetric Dirichlet space on $L^2(E,m_p)$, thus there is a continuous Hunt process on E associated with it.

We show in Section 7.1 that such a process can also be characterized by Girsanov transform as follows.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_t := \exp\left(\int_0^t b(X_s)dX_s - \int_0^t |b(X_s)|^2 ds\right), \tag{2.5.1}$$

We also show in Section 7.1 that the infinitesimal generator of drifted BMVD as an L^2-generator denoted by $\mathcal{L}^b$ is the following.

$$\mathcal{L}^b u = \mathcal{L}u + b(x)\nabla u(x) \cdot \mathbf{1}_{D_0} + b(x)\frac{du}{dx} \cdot \mathbf{1}_{\mathbb{R}_+}$$

with its domain being

$$\mathcal{D}(\mathcal{L}^b) = \left\{ u \in \mathcal{D}(\mathcal{E}) : \mathcal{L}u \in L^2(E), \text{u satisfies the same "zero flux" condition as BMVD} \right\},$$

where $\mathcal{L}$ is the infinitesimal generator of non-drifted BMVD. The definition of "flux" is given by (2.3.1).

It has also been shown in Section 7.1 that such a process can be characterized through Girsanov transform. Unlike BMVD without drift, such a process in general is non-symmetric when the drift term is non-trivial. The main result we have for drifted BMVD is that, by establishing and using the Duhamel's formula for BMVD, we show that the small time heat kernel for Brownian motion with drift on space with varying dimension has the same two-sided Gaussian-type bounds as BMVD for the class of drift functions in $L^\infty(E) + L^{p_1,p_2}(E)$.

2.5.2 Main results about BMVD with drift

Theorem 2.5.1. *Let $T > 0$ be fixd. Let $b : E \to \mathbb{R}$ be in the family of $L^\infty(E) + L^{p_1,p_2}(E)$ for some $p_1 \in (1, \infty]$ and $p_2 \in (2, \infty]$. Let $p^b(t, x, y)$ be the transition density of BMVD with drift b. There exists constants C_1, C_2, β_1, $\beta_2 > 0$ such that it holds*

$$C_1 p^0_{\beta_2}(t, x, y) \le p^b(t, x, y) \le C_2 p^0_{\beta_1}(t, x, y), \qquad (t, x, y) \in (0, T] \times E \times E.$$

Let $X^{b,D}$ be the part process of drifted BMVD killed upon exiting a domian D of E. Let $p^b_D(t, x, y)$ be the transition density of $X^{b,D}$. As a corollary, by making use of the explicit boundary decay rate of killed Brownian motion with drift as well as the boundary Harnack principle for Brownian motion, the estimates on Green function for the part process of drifted BMVD defined as

$$g^b_D(x, y) := \int_0^\infty p^b_D(t, x, y)dt$$

are not hard to obtain. Before giving the statement of the results, we first recall that an open set $D \subset \mathbb{R}^d$ is called to be $C^{1,1}$ if there exist a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}-$function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0, \dots, 0)$, $\|\nabla\phi\|_\infty \le \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(z)| \le \Lambda|x - z|$ and an orthonormal coordinate system CS_z : $y = (y_1, \dots, y_d) := (\tilde{y}, y_d)$ with its origin at z such that

$$B(z, R_0) \cap D_0 = \{y \in B(0, R_0) \text{ in } CS_z : y_d > \phi(\tilde{y})\}.$$

For the state space E, an open set $D \subset E$ will be called $C^{1,1}$ in E, if $D \cap \mathbb{R}_+$ is a $C^{1,1}$ open set in $\mathbb{R}$, and $D \cap D_0$ is also a $C^{1,1}$ open set in $\mathbb{R}^2$.

Theorem 2.5.2. *Let $g^b_D(x, y)$ be the Green function of drifted BMVD killed upon exiting D, where D is a bounded $C^{1,1}$ domain of E. It is also assumed that $a^* \in D$.*

$$g^b_D(x, y) \asymp \begin{cases} \delta_D(x) \wedge \delta_D(y), & x \in \mathbb{R}_+,\ y \in \mathbb{R}_+; \\ \\ (\delta_D(y) \wedge 1)\,(\delta_D \wedge 1) + \ln\left(1 + \frac{\delta_{D,\epsilon}(x)\delta_{D,\epsilon}(y)}{|x-y|^2_\epsilon}\right), & x \in D_0,\ y \in D_0; \\ \\ (\delta_D(y) \wedge 1) \cdot \delta_D(x), & x \in \mathbb{R}_+,\ y \in D_0, \end{cases}$$

where $\delta_D(\cdot) = dist(\cdot, \partial D)$, $\delta_{D,\epsilon}(\cdot) = dist\Big(\cdot, \partial\big(D \cap D_0\big)\Big)$.

Chapter 3

SMALL TIME HEAT KERNEL ESTIMATE

3.1 Nash-type Inequality adapted to BMVD

This lemma is a version of Nash-type inequality adapted to our case which yields both on-diagonal and off-diagonal heat kernel upper bound estimates. Recall it has been set that $D_0 = \mathbb{R}^2 \backslash B_e(0,\epsilon)$.

Lemma 3.1.1. *There exists $C_1 > 0$ such that*

$$\|f\|^2_{L^2(E)} \le C_1 \left(\mathcal{E}(f,f)^{\frac{1}{2}} \|f\|_{L^1(E)} + \mathcal{E}(f,f)^{\ \frac{1}{3}} \|f\|^{\frac{4}{3}}_{L^1(E)} \right), \qquad \forall f \in \mathcal{F}.$$

Proof. We consider $D_0 = \mathbb{R}^2 \backslash B_e(0,\epsilon)$ which is an inner-uniform domain of $\mathbb{R}^2$. Therefore for the reflecting Brownian motion on D_0 it holds

$$p^{RBM}(t,x,y) \asymp \frac{1}{V(x,\sqrt{t})} \le ct^{-1},$$

which is equivalent to the following Nash's inequality

$$\|f\|^2_{L^2(D_0)} \le c \|\nabla f\|_{L^2(D_0)} \cdot \|f\|_{L^1(D_0)}, \quad \forall f \in W^{1,2}(D_0) \cap L^2(D_0) \cap L^1(D_0).$$

On the half real line, by considering the 1-dimensional reflecting Brownian motion on $\mathbb{R}_+$ one sees that $\|f\|^3_{L^2(\mathbb{R}_+)} \le C \|f\|^2_{L^1(\mathbb{R}_+)} \|f'\|_{L^2(\mathbb{R}_+)}$, $\forall f \in W^{1,2}(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$. The desired inequality now follows by combining these two inequalities. □

Proposition 3.1.2. *There exists $C_2 > 0$ such that*

$$\|\overline{P_t}\|_{1\to\infty} \le C_2 \left(\frac{1}{t} + \frac{1}{t^{1/2}} \right), \qquad \forall t \in (0, +\infty].$$

Proof. This follows immediately from Lemma 3.1.1 by [8, Corollary 2.12] □

3.2 Upper Bound Estimate Using Davis Method

Proposition 3.2.1. *There exist* $C_3, C_4 > 0$ *such that*

$$p(t,x,y) \le C_3 \left(\frac{1}{t} + \frac{1}{t^{1/2}} \right) e^{-C_4 \frac{\rho(x,y)^2}{t}}, \quad \forall x,y \in E, \quad \forall t > 0.$$

Proof. Fix $x_0, y_0 \in E$, $t_0 > 0$. Set $\alpha := \rho(y_0, x_0)/4t_0$ and $\psi(x) := \alpha \cdot \rho(x, a^*)$. Then we define $\psi_n(x) = \psi(x) \wedge n$. Note

$$e^{-2\psi_n} |\nabla e^{\psi_n}|^2 = |\nabla \psi_n|^2 = |\alpha|^2 \cdot \mathbf{1}_{\left\{\rho(x,a^*) \le \frac{n}{|\alpha|}\right\}}(x)$$

as well as

$$e^{2\psi_n} |\nabla e^{-\psi_n}|^2 = |\nabla \psi_n|^2 = |\alpha|^2 \cdot \mathbf{1}_{\left\{\rho(x,a^*) \le \frac{n}{|\alpha|}\right\}}(x).$$

It thus follows by [8, Corollary 3.28],

$$p(t,x,y) \le c \left(\frac{1}{t} + \frac{1}{t^{1/2}} \right) \exp\left(-\rho\left(\psi(y), \psi(x)\right) + 2t|\alpha|^2 \right), \quad \text{a.e. } y, t > 0, \tag{3.2.1}$$

By the smoothness of p, we may drop the a.e. Taking $t = t_0, x = x_0$ and $y = y_0$ in (3.2.1) completes the proof. $\square$

Proposition 3.2.2. *There exist constants* $C_5, C_6 > 0$ *such that*

$$\mathbb{P}^x \left(\rho\left(X_t, x\right) \ge \lambda \right) \le \frac{C_5}{t} e^{-\frac{C_6 \lambda^2}{t}}, \qquad \textit{for all } t \le 1, \quad x \in E.$$

Proof. The idea of the proof is straightforward. It is simply the integration of the two sides of Proposition 3.2.1. Here we omit the details. $\square$

Proposition 3.2.3. *There exist constants* $C_7, C_8 > 0$ *such that*

$$\mathbb{P}^x \left(\sup_{s \le t} \rho\left(X_s, x\right) \ge \lambda \right) \le \frac{C_7}{t} e^{-\frac{C_8 \lambda^2}{t}}, \qquad \textit{for all } t \le 1, \quad x \in E.$$

Proof. Let $\tau_\lambda = \inf\{t > 0 : \rho\left(X_t, X_0\right) \ge \lambda\}$. By the strong Markov property, we have

$$\begin{aligned}
\mathbb{P}^x \left(\sup_{0 \le s \le t} \rho\left(X_s, x\right) \ge \lambda \right) &\le \mathbb{P}^x \big(\rho\left(X_t, x\right) \ge \lambda/2 \big) + \mathbb{P}^x \big(\tau_\lambda < t, \rho\left(X_t, X_{\tau_\lambda}\right) \ge \lambda/2 \big) \\
&\le \mathbb{P}^x \big(\rho\left(X_t, x\right) \ge \lambda/2 \big) + \int_0^t \mathbb{E}^x \Big(\mathbb{P}^{X_s} \big(\rho\left(X_{t-s}, X_0\right) \ge \lambda/2 \big); \tau \in ds \Big) \\
&\le \mathbb{P}^x \big(\rho\left(X_t, x\right) \ge \lambda \big) + \sup_{y \in E, 0 \le u \le t} \mathbb{P}^y \big(\rho\left(X_u, y\right) \ge \lambda/2 \big) \\
&\le 2 \sup_{y \in E, 0 \le u \le t} \mathbb{P}^y \big(\rho\left(X_u, y\right) \ge \lambda/2 \big).
\end{aligned}$$

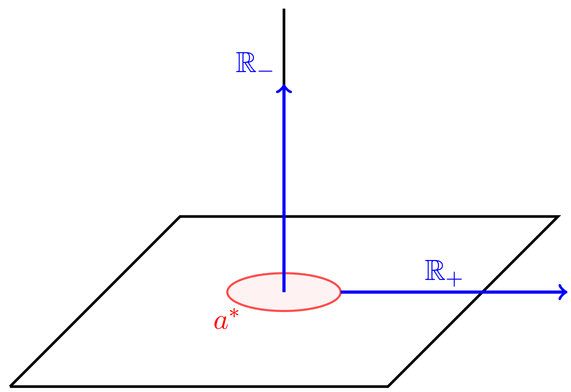

Figure 3.1: Radial Process

The conclusion now follows from the above inequality and Proposition 3.2.2. □

3.3 Fukushima Decomposition and Radial Process

In order to get the sharp two-sided heat kernel estimates, we consider the radial process of X. Namely, we project X to $\mathbb{R}$ by applying the following mapping from E to $\mathbb{R}$: (See Figure 3.3)

$$u(x) = \begin{cases} -|x|, & x \in \mathbb{R}_+; \\ \\ |x|_\rho, & x \in D_0. \end{cases} \tag{3.3.1}$$

It thus holds $u \in \mathcal{F}_{loc}$, where $\mathcal{F}_{loc}$ denotes the local Dirichlet space, whose definition can be found, for instance, in [9]. Fukushima decomposition (see, for example, [15, Chapter 5]) tells us that $u(X)$ can be uniquely decomposed as the sum of a martingale additive functional and a continuous additive functional of zero energy. i.e., there uniquely exist $M^{[u]}$ and $N^{[u]}$ such that

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad \mathbb{P}^x\text{-a.s. for q.e. } x \in E,$$

where $M_t^{[u]}$ is a martingale additive functional of X, and $N_t^{[u]}$ is a continuous additive functional of zero energy of X. Now we may explicitly compute $M^{[u]}$ and $N^{[u]}$. First we compute the Revuz measure of $N^{[u]}$ denoted by ν (To see the relationship between a Markov process local time and its characterizing Revuz measure, one might refer to, for instance,

[9]). For any $\psi \in C_c^\infty(E)$,

$$
\begin{aligned}
\int_E \psi d\nu = \mathcal{E}(u,\psi) &= \int_{D_0} \nabla|x| \cdot \nabla\psi dx + p\int_{\mathbb{R}_+} (-1)\psi' dx \\
&= \int_{D_0} \frac{x}{|x|} \cdot \nabla\psi dx - p\int_{\mathbb{R}_+} \psi' dx \\
&= -\int_{D_0} \operatorname{div}\left(\frac{x}{|x|}\right)\psi dx - \int_{\partial B_e(0,\epsilon)} \psi(0)\frac{\partial}{\partial\vec{n}}\left(\frac{x}{|x|}\right) d\tau + p\psi(0) \\
&= -\int_{D_0} \frac{1}{|x|}\psi dx + (2\pi\epsilon - p)\psi(0).
\end{aligned}
$$

where $D_0 = \mathbb{R}^2 \backslash B_e(0,\epsilon)$, $\vec{n}$ is the outward pointing unit vector normal of the surface $\partial B_e(0,\epsilon)$, and $d\tau$ is the surface measure on S^1. The computation above shows

$$
\nu(dx) = -\frac{1}{|x|}\mathbf{1}_{\{D_0\}}dx + (2\pi\epsilon - p)\delta_{\{0\}},
$$

which in turn shows

$$
dN_t^{[u]} = \frac{1}{u(X_t)+\epsilon}\mathbf{1}_{\{X_t \in D_0\}}dt + (2\pi\epsilon - p)dL_t^0,
$$

where L_t^0 is the local time at 0 in the sense of Revuz measure. The correspondence between a local time and its Revuz measure can be found, for instance, [9, Theorem A.3.5, Appendix A]. Now in order to get $M^{[u]}$, we need to find $\mu_{\langle M^{[u]}\rangle}$, which is the Revuz measure of the martingale additive functional $M_t^{[u]}$. Let $u_n = (-n) \vee u \wedge n$, and it immediately follows $u_n \in \mathcal{F}$. Let $\mathcal{F}_b := \{f \in \mathcal{F} : f$ has a q.e. continuous version that is bounded.$\}$. By [15, Theorem 5.5.2], for any $f \in \mathcal{F}_b \cap C_c(E)$,

$$
\begin{aligned}
\mu_{\langle M^{[u_n]}\rangle}(f) &= 2\mathcal{E}(u_n \cdot f, u_n) - \mathcal{E}(u_n^2, f) = 2\int \nabla(u_n \cdot f)\nabla u_n - \int (\nabla u_n)^2 \cdot \nabla f \\
&= 2\int |\nabla u_n|^2 \cdot f + 2\int u_n \nabla f \nabla u_n - 2\int u_n \nabla u_n \nabla f \\
&= 2\int |\nabla u_n|^2 f,
\end{aligned}
$$

which shows

$$
\mu_{\langle M^{[u_n]}\rangle}(dx) = 2|\nabla u_n|^2(dx) = 2 \cdot \mathbf{1}_{B_\rho(a^*,n)}(dx).
$$

By [9, Proposition 4.1.9]

$$
M_t^{[u_n]} = B_{t \wedge \tau_{B_\rho(a^*,n)}},
$$

where B_t is the standard 1-dimensional Brownian motion, and $\tau_{B_\rho(a^*,n)}$ is the exit time of $B_\rho(a^*, n)$. Now it follows from Lemma 5.5.1 in [15],

$$M_t^{[u]} = M_t^{[u_n]} \quad \text{on each } B_\rho(a^*, n),$$

which means $M_t^{[u]} = B_t$, since n is arbitrary. Combining this with what we have got for $N_t^{[u]}$, we are ready to conclude

$$du(X_t) = dB_t + \frac{1}{u(X_t) + \epsilon}\mathbf{1}_{\{X_t \in D_0\}}dt + (2\pi\epsilon - p)dL_t^0(X), \tag{3.3.2}$$

where $L_t^0(X)$ is the local time with respect to X in sense of Revuz measure. For the rest of this article, we set $Y := u(X)$. Local time can also be defined in terms of semi-martingale local time with respect to the 1-dimensional diffusion process Y. For the definition of semi-martingale local time, one may refer to, for example, [18]. To find the relationship between $L_t^0(X)$ in sense of Revuz measure and $L_t^0(Y)$ as a semi-martingale local time, we have the following proposition.

Proposition 3.3.1.

$$4\pi\epsilon L_t^0(X) = L_t^0(Y), \tag{3.3.3}$$

where the left hand side is defined as the local time corresponding to the Dirac measure $\delta_{\{0\}}$*, while the right hand side is a non-symmetric semi-martingale local time.*

Proof. It has been established that the radial process Y can be characterized by the following SDE:

$$dY_t = dB_t + \frac{1}{Y_t + \epsilon}\mathbf{1}_{\{Y_t > 0\}} + (2\pi\epsilon - p)dL_t^0(X).$$

For the original process X, consider $v(X_t) := |X_t|_\rho$, $v \in \mathcal{F}_{loc}$. Therefore by Fukushima decomposition, similar to what we've done above, it holds

$$dv(X_t) = d\widetilde{B}_t + \frac{1}{|X_t| + \epsilon}\mathbf{1}_{\{X_t \in D\}}dt + (2\pi\epsilon + p)dL_t^0(X),$$

where the notation $\widetilde{B}$ means that this Brownian motion might not be the same Brownian motion as the one appearing in the SDE that characterizes Y. i.e.,

$$d|X_t| = d\widetilde{B}_t + \frac{1}{|X_t| + \epsilon}\mathbf{1}_{\{X_t \in D\}}dt + (2\pi\epsilon + p)dL_t^0(X).$$

i.e.,

$$d|Y_t| = d\widetilde{B}_t + \frac{1}{Y_t + \epsilon}\mathbf{1}_{\{Y_t>0\}}dt + (2\pi\epsilon + p)dL_t^0(X).$$

On the other hand, by Tanaka's formula (see, for example, [18]), we have

$$\begin{aligned}
d|Y_t| &= \operatorname{sgn}(Y_t)dY_t + dL_t^0(Y) \\
&= \operatorname{sgn}(Y_t)dB_t + \operatorname{sgn}(Y_t)\frac{1}{Y_t+\epsilon}\mathbf{1}_{\{Y_t>0\}}dt + \operatorname{sgn}(Y_t)(2\pi\epsilon - p)dL_t^0(X) + dL_t^0(Y) \\
&= \operatorname{sgn}(Y_t)dB_t + \frac{1}{Y_t+\epsilon}\mathbf{1}_{\{Y_t>0\}}dt + (2\pi\epsilon - p)\operatorname{sgn}(Y_t)dL_t^0(X) + dL_t^0(Y),
\end{aligned}$$

where $\operatorname{sgn}(\cdot)$ takes value 1 on $(0, +\infty)$, and -1 on $(-\infty, 0]$. Since the decomposition of a semi-martingale as the sum of a local martingale and an increasing process with finite variation is unique, it has to hold

$$(2\pi\epsilon + p)L_t^0(X) = \operatorname{sgn}(Y_t)(2\pi\epsilon - p)L_t^0(X) + L_t^0(Y).$$

The local time $L_t^0(Y)$ has increments only on $\{Y_t = 0\}$, therefore

$$(2\pi\epsilon + p)L_t^0(X) = (p - 2\pi\epsilon)L_t^0(X) + L_t^0(Y).$$

i.e.,

$$4\pi\epsilon L_t^0(X) = L_t^0(Y).$$

□

The semi-martingale local time in (3.3.3) is non-symmetric in the following sense:

$$L_t^0(Y) = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t 1_{[0,\epsilon[}(Y_s)d\langle Y\rangle_s,$$

where it only takes the increments when Y is in $[0, \epsilon[$ instead of $]-\epsilon, \epsilon[$. Indeed, in order to study the excursion of Y at 0, one needs to convert the non-symmetric semi-martingale local time to the symmetric one which is defined as follows:

$$\widehat{L}_t^0(Y) := \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t 1_{]-\epsilon,\epsilon[}(Y_s)d\langle Y\rangle_s,$$

Actually, if we define $\tilde{u}(x) := -u(x)$ and then apply all the exactly same argument as above to $\tilde{u}$, it will follow that

$$2pL_t^0(X) = L_t^0(-Y),$$

which implies that for the symmetric local time $\widehat{L}_t^0$ it holds

$$\begin{aligned}\widehat{L}_t^0(Y) &= \frac{1}{2}\left(L_t^0(Y) + L_t^0(-Y)\right)\\ &= \frac{1}{2}\left(4\pi\epsilon L_t^0(X) + 2pL_t^0(X)\right)\\ &= (2\pi\epsilon + p)L_t^0(X),\end{aligned}$$

which together with (3.3.2) gives the following proposition that characterizes the excursion of the process X at a^*.

Proposition 3.3.2.

$$dY_t = dB_t + \frac{1}{Y_t+\epsilon}\mathbf{1}_{\{Y_t>0\}}dt + \frac{2\pi\epsilon - p}{2\pi\epsilon + p}d\widehat{L}_t^0(Y). \tag{3.3.4}$$

3.4 Small Time Heat Kernel Estimates

To study the heat kernel esimates of X, let us first prove the following proposition regarding the radial process Y which will be used later in the context.

Proposition 3.4.1. *There exist $C_i > 0$, $1 \le i \le 4$, such that the following estimate holds:*

$$\frac{C_1}{\sqrt{t}}e^{-\frac{C_2|x-y|^2}{t}} \le p^{(Y)}(t,x,y) \le \frac{C_3}{\sqrt{t}}e^{-\frac{C_4|x-y|^2}{t}},\ (t,x,y) \in (0,1] \times \mathbb{R} \times \mathbb{R}. \tag{3.4.1}$$

Proof. Let $\beta := \frac{2\pi\varepsilon - p}{2\pi\varepsilon + p}$ and Z be the skew Brownian motion

$$dZ_t = dB_t + \beta\widehat{L}_t^0(Z),$$

The diffusion process Y can be obtained from Z through drift perturbation. The transition density function $p_0(t,x,y)$ of Z is explicitly known and enjoys the two-sided Aronson-type Gaussian estimates (3.4.1); see, e.g., [24]. One can check that

$$|\nabla_x p_0(t,x,y)| \le c_1 t^{-1}\exp(-c_2|x-y|^2/\sqrt{t}).$$

From which one can deduce (3.4.1) by using the same argument as that for Theorem A in Zhang [27, §4]. □

The following corollary is an immediate consequence of the above proposition.

Corollary 3.4.2. *There exist* $C_i > 0$, $5 \le i \le 8$, *such that the following estimate holds:*

$$\frac{C_5}{\sqrt{t}} e^{-\frac{C_6|x-y|^2}{t}} \le p^{(X)}(t,x,y) \le \frac{C_7}{\sqrt{t}} e^{-\frac{C_8|x-y|^2}{t}},\ (t,x,y) \in (0,1] \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Now we return to the estimate of $p(t,x,y)$ when $x \in \mathbb{R}$, and $y \in D_0$, for which we have the following proposition:

Proposition 3.4.3. *There exist constants* $C_i > 0$, $9 \le i \le 16$, *such that for all* $x \in \mathbb{R}_+$, $y \in D_0$, $t \in [0,1]$ *the following estimates hold:*
When $|y|_\rho \le 1$,

$$\frac{C_9}{\sqrt{t}} e^{-\frac{C_{10}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{11}}{\sqrt{t}} e^{-\frac{C_{12}\rho(x,y)^2}{t}};$$

when $|y|_\rho > 1$,

$$\frac{C_{13}}{t} e^{-\frac{C_{14}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{15}}{t} e^{-\frac{C_{16}\rho(x,y)^2}{t}}.$$

Proof. We first note that in this case

$$p^{(X)}(t,x,y) = \int_{s=0}^{t} \mathbb{P}^y(T_{a^*} \in ds) p(t-s,0,x).$$

By the rotational-invariance of 2−dimensional Brownian motion, $\mathbb{P}^y(T_{a^*} \in ds)$ as a function in y only depends on $|y|_\rho$, therefore so is $p^{(X)}(t,x,y)$ as a function in y. Now let $r = |y|_\rho$ and set $q(t,x,r) := p(t,x,y)$ for $r = |y|_\rho$. For all $a > b > 0$, $x \in \mathbb{R}_+$,

$$\int_a^b p^{(Y)}(t,-|x|,y)dy = \int_{y \in D_0, a<|y|_\rho<b} p^{(X)}(t,x,y) m_p(dy) = \int_{y \in \mathbb{R}^2, a+\epsilon<|y|<b+\epsilon} p^{(X)}(t,x,y) m_p(dy)$$
$$= \int_a^b (r+\epsilon) q(t,x,r) dr.$$

By the uniqueness of probability density functions, this implies when $x \in \mathbb{R}_+$, $y \in D_0$,

$$p^{(Y)}(t,-|x|,|y|_\rho) = (r+\epsilon) q(t,x,r) = (r+\epsilon) p^{(X)}(t,x,y), \tag{3.4.2}$$

where $r = |y|_\rho$. i.e.,

$$p^{(X)}(t,x,y) = \frac{1}{|y|_\rho + \epsilon} p^{(Y)}(t,-|x|,|y|_\rho),$$

where

$$\frac{c_1}{\sqrt{t}} e^{-\frac{c_2\rho(x,y)^2}{t}} \le p^{(Y)}(t,-|x|,|y|_\rho) \le \frac{c_3}{\sqrt{t}} e^{-\frac{c_4\rho(x,y)^2}{t}}$$

When $|y|_\rho < 1$, the right hand side above is comparable to $\dfrac{1}{\sqrt{t}} e^{-\frac{c\rho(x,y)^2}{t}}$.
When $|y|_\rho > 1$, we have

$$\frac{1}{(|y|_\rho + \epsilon)\sqrt{t}} e^{-\frac{c\rho(x,y)^2}{t}} \asymp \frac{1}{|y|_\rho \sqrt{t}} e^{-\frac{c\rho(x,y)^2}{t}} = \frac{\sqrt{t}}{|y|_\rho t} e^{-\frac{c\rho(x,y)^2}{t}},$$

Notice the fact that when $\frac{\sqrt{t}}{|y|_\rho} < 1$, for any $c > 0$,

$$\frac{\sqrt{t}}{|y|_\rho} \geq e^{-\frac{c|y|_\rho^2}{t}}.$$

It thus follows

$$\frac{c_6}{t} e^{-\frac{c_6\rho(x,y)^2}{t}} \leq \frac{1}{(|y|_\rho + \epsilon)\sqrt{t}} e^{-\frac{c\rho(x,y)^2}{t}} \leq \frac{c_7}{t} e^{-\frac{c_8\rho(x,y)^2}{t}}.$$

i.e.,

$$\frac{c_6}{t} e^{-\frac{c_6\rho(x,y)^2}{t}} \leq p^{(X)}(t,x,y) \leq \frac{c_7}{t} e^{-\frac{c_8\rho(x,y)^2}{t}}.$$

□

Now let us consider the third case of Theorem 2.4.1 which is the case that both x and y are in D_0.

Theorem 3.4.4. *There exist constants $C_i > 0$, $17 \leq i \leq 26$, such that for all $t \in [0,1]$, $x, y \in D_0$ the following estimates hold:*
When $|x|_\rho < 1$, $|y|_\rho < 1$,

$$\begin{aligned} &\frac{C_{17}}{\sqrt{t}} e^{-\frac{C_{18}\rho(x,y)^2}{t}} + \frac{C_{17}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{19}|x-y|_e^2}{t}} \leq p(t,x,y) \\ &\leq \frac{C_{20}}{\sqrt{t}} e^{-\frac{C_{21}\rho(x,y)^2}{t}} + \frac{C_{20}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{22}|x-y|_e^2}{t}}; \end{aligned} \tag{3.4.3}$$

otherwise

$$\frac{C_{23}}{t} e^{-\frac{C_{24}\rho(x,y)^2}{t}} \leq p(t,x,y) \leq \frac{C_{25}}{t} e^{-\frac{C_{26}\rho(x,y)^2}{t}}, \tag{3.4.4}$$

where $|\cdot|_e$ and $|\cdot|_\rho$ denote the Euclidean metric and the geodesic metric respectively.

Proof. For $x \in D_0$, $t \leq 1$, we notice

$$p(t,x,y) = \overline{p}_{D_0}(t,x,y) + p_{D_0}(t,x,y),$$

Thus by Markov property,

$$\int_{s=0}^{t} \mathbb{P}^x(T_{\{a^*\}} \in ds) p(t-s, a^*, y) + \frac{c_1}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{c_2|x-y|_e^2}{t}} \le p(t,x,y)$$
$$\le \int_{s=0}^{t} \mathbb{P}^x(T_{\{a^*\}} \in ds) p(t-s, a^*, y) + \frac{c_3}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{c_4|x-y|_e^2}{t}}$$

where the second term of the upper and lower bounds is due to the heat kernel estimate of 2−dimensional killed Brownian motion. As we've mentioned in the proof to Proposition 3.4.3, $p(t-s, a^*, y)$ as a function in y only depends on $|y|_\rho$, therefore so is $\overline{p}_{D_0}(t,x,y)$ as a function in y. We thus again set $q(t,x,r) := \overline{p}_{D_0}(t,x,y)$ for $r = |y|_\rho$. Now for any interval $(a,b) \subset \mathbb{R}_+$, we have

$$\int_a^b \overline{p}^{(Y)}_{\mathbb{R}_+}(t, |x|_\rho, r) dr = \int_{a \le |y|_\rho \le b} \overline{p}^{(X)}_{D_0}(t,x,y) m_p(dy)$$
$$= \int_{a+\epsilon \le |y| \le b+\epsilon} \overline{p}^{(X)}_{D_0}(t,x,y) m_p(dy) = \int_a^b (r+\epsilon) q(t,x,r) dr.$$

It follows by Monotone Class Theorem

$$\overline{p}^{(Y)}_{\mathbb{R}_+}(t, |x|_\rho, r) \asymp (r+\epsilon) q(t,x,r), \quad r = |y|_\rho,\ y \in D_0.$$

Further we have

$$q(t,x,r) \asymp \frac{1}{|y|_\rho + \epsilon} \overline{p}^{(Y)}_{\mathbb{R}_+}(t, |x|_\rho, |y|_\rho) \tag{3.4.5}$$
$$= \frac{1}{|y|_\rho + \epsilon} \int_0^t \mathbb{P}^{(Y),|x|_\rho}(T_{\{0\}} \in ds) p^{(Y)}(t-s, 0, |y|_\rho) \tag{3.4.6}$$
$$\asymp \frac{1}{|y|_\rho + \epsilon} \int_0^t \mathbb{P}^{(Y),|x|_\rho}(T_{\{0\}} \in ds) p^{(Y)}(t-s, 0, -|y|_\rho) \tag{3.4.7}$$
$$= \frac{1}{|y|_\rho + \epsilon} \cdot p^{(Y)}(t, |x|_\rho, -|y|_\rho),$$

where (3.4.5) is due to (3.4.2), (3.4.7) is due to Proposition 3.4.1. To see (3.4.6) we have for any measurable set $A \subset \mathbb{R}_+$,

$$\int_A \int_0^t \mathbb{P}^{(Y),|x|_\rho}(T_{\{0\}} \in ds) p^{(Y)}(t-s, 0, |y|_\rho) d|y|_\rho = \int_0^t \mathbb{P}^{(Y),|x|_\rho}(T_{\{0\}} \in ds) \int_A p^{(Y)}(t-s, 0, |y|_\rho) d|y|_\rho$$
$$= \int_0^t P^{(Y)}_{t-s}(0, A)\, \mathbb{P}^{(Y),|x|_\rho}\left(T_{\{0\}} \in ds\right)$$
$$= \mathbb{P}^{(Y),|x|_\rho}\left(X_t \in A; T_{\{0\}} \le t\right)$$
$$= \int_A \overline{p}^{(Y)}_{\mathbb{R}_+}(t, |x|_\rho, |y|_\rho)\, d|y|_\rho.$$

Therefore by the uniqueness of the density function, (3.4.6) holds. Thus again by Proposition 3.4.1 it holds

$$\frac{c_1}{(|y|_\rho+\epsilon)\sqrt{t}}e^{-\frac{c_2(|x|_\rho^2+|y|_\rho^2)}{t}} \le q(t,x,r) \le \frac{c_3}{(|y|_\rho+\epsilon)\sqrt{t}}e^{-\frac{c_4(|x|_\rho^2+|y|_\rho^2)}{t}}, \tag{3.4.8}$$

Now we consider two different cases.

Case 1. $|x|_\rho<1, |y|_\rho<1$. Letting $r=|y|_\rho$ it holds

$$p(t,x,y)=\overline{p}_{D_0}(t,x,y)+p_{D_0}(t,x,y)=q(t,x,r)+p_{D_0}(t,x,y),$$

which yields

$$\begin{aligned}\frac{c_5}{\sqrt{t}}e^{-\frac{c_6(|x|_\rho+|y|_\rho)^2}{t}}+\frac{c_5}{t}\left(1\wedge\frac{|x|_\rho}{\sqrt{t}}\right)\left(1\wedge\frac{|y|_\rho}{\sqrt{t}}\right)e^{-\frac{c_7|x-y|_e^2}{t}} &\le p(t,x,y)\\ \le \frac{c_8}{\sqrt{t}}e^{-\frac{c_9(|x|_\rho+|y|_\rho)^2}{t}}+\frac{c_8}{t}\left(1\wedge\frac{|x|_\rho}{\sqrt{t}}\right)\left(1\wedge\frac{|y|_\rho}{\sqrt{t}}\right)e^{-\frac{c_{10}|x-y|_e^2}{t}}&.\end{aligned}\tag{3.4.9}$$

We now claim that the right hand side of (3.4.9) can be written as

$$\frac{c_{11}}{\sqrt{t}}e^{-\frac{c_{12}\rho(x,y)^2}{t}}+\frac{c_{11}}{t}\left(1\wedge\frac{|x|_\rho}{\sqrt{t}}\right)\left(1\wedge\frac{|y|_\rho}{\sqrt{t}}\right)e^{-\frac{c_{13}|x-y|_e^2}{t}}. \tag{3.4.10}$$

Indeed, when at least one of $\frac{|x|_\rho}{\sqrt{t}}$ and $\frac{|y|_\rho}{\sqrt{t}}$ is less than 1, it holds that $e^{-\frac{c_{14}(|x|_\rho+|y|_\rho)^2}{t}}\asymp e^{-\frac{c_{15}\rho(x,y)^2}{t}}$. Therefore (3.4.9) becomes

$$\begin{aligned}\frac{c_5}{\sqrt{t}}e^{-\frac{c_6\rho(x,y)^2}{t}}+\frac{c_5}{t}\left(1\wedge\frac{|x|_\rho}{\sqrt{t}}\right)\left(1\wedge\frac{|y|_\rho}{\sqrt{t}}\right)e^{-\frac{c_7|x-y|_e^2}{t}} &\le p(t,x,y)\\ \le \frac{c_8}{\sqrt{t}}e^{-\frac{c_9\rho(x,y)^2}{t}}+\frac{c_8}{t}\left(1\wedge\frac{|x|_\rho}{\sqrt{t}}\right)\left(1\wedge\frac{|y|_\rho}{\sqrt{t}}\right)e^{-\frac{c_{10}|x-y|_e^2}{t}}&.\end{aligned}$$

However, when both $\frac{|x|_\rho}{\sqrt{t}}$ and $\frac{|y|_\rho}{\sqrt{t}}$ are greater than 1, for the right hand side of (3.4.9) it holds

$$\begin{aligned}\frac{1}{\sqrt{t}}e^{-\frac{c_{12}(|x|_\rho+|y|_\rho)^2}{t}}+\frac{1}{t}e^{-\frac{c_{13}|x-y|_e^2}{t}} &\asymp \frac{1}{\sqrt{t}}e^{-\frac{c_{12}(|x|_\rho+|y|_\rho)^2}{t}}+\left(\frac{1}{\sqrt{t}}+\frac{1}{t}\right)e^{-\frac{c_{13}|x-y|_e^2}{t}}\\ &\asymp \frac{1}{\sqrt{t}}e^{-\frac{c_{16}\rho(x,y)^2}{t}}+\frac{1}{t}e^{-\frac{c_{13}|x-y|_e^2}{t}}\\ &= \frac{1}{\sqrt{t}}e^{-\frac{c_{17}\rho(x,y)^2}{t}}+\frac{1}{t}\left(1\wedge\frac{|x|_\rho}{\sqrt{t}}\right)\left(1\wedge\frac{|y|_\rho}{\sqrt{t}}\right)e^{-\frac{c_{13}|x-y|_e^2}{t}},\end{aligned}$$

where one can see the second " $\asymp$ " by considering the two cases $|x|_\rho+|y|_\rho\le 2\rho(x,y)$ and $|x|_\rho+|y|_\rho\ge 2\rho(x,y)=2|x-y|_e$ separately. This proves the desired result.

Case 2. At least one of $|x|_\rho$ and $|y|_\rho$ is greater than 1. Without loss of generality, we assume $|y|_\rho > 1 > \sqrt{t}$. Same to the previous case we have

$$\begin{aligned} p(t,x,y) &= \overline{p}_{D_0}(t,x,y) + p_{D_0}(t,x,y) \\ &= q(t,x,r) + p_{D_0}(t,x,y), \end{aligned} \tag{3.4.11}$$

where $r = |y|_\rho$. By (3.4.8), we have

$$\frac{c_{18}}{|y|_\rho \sqrt{t}} e^{-\frac{c_{19}(|x|_\rho + |y|_\rho)^2}{t}} \le q(t,x,r) \le \frac{c_{20}}{|y|_\rho \sqrt{t}} e^{-\frac{c_{21}(|x|_\rho + |y|_\rho)^2}{t}},$$

which is equivalent to the fact that

$$\frac{c_{18}}{t} e^{-\frac{c_{19}(|x|_\rho + |y|_\rho)^2}{t}} \le q(t,x,r) \le \frac{c_{20}}{t} e^{-\frac{c_{21}(|x|_\rho + |y|_\rho)^2}{t}} \tag{3.4.12}$$

since $|y|_\rho > 1 > \sqrt{t}$. It thus follows from (3.4.11) and (3.4.12) that

$$\begin{aligned} \frac{c_{22}}{t} e^{-\frac{c_{23}(|x|_\rho + |y|_\rho)^2}{t}} + \frac{c_{22}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{c_{24}|x-y|_e^2}{t}} &\le p(t,x,y) \\ \le \frac{c_{25}}{t} e^{-\frac{c_{26}(|x|_\rho + |y|_\rho)^2}{t}} + \frac{c_{25}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{c_{27}|x-y|_e^2}{t}} & \end{aligned} \tag{3.4.13}$$

We again consider (3.4.13) in two subcases. When one of $\frac{|x|_\rho}{\sqrt{t}}$ and $\frac{|y|_\rho}{\sqrt{t}}$ is less than 1, it holds that $e^{-\frac{c(|x|_\rho + |y|_\rho)^2}{t}} \asymp e^{-\frac{c\rho(x,y)^2}{t}}$. The desired conclusion thus readily follows from the upper bound estimate Proposition 3.2.1 or the fact that $|x-y|_e \ge \rho(x,y)$. Whereas when both $\frac{|x|_\rho}{\sqrt{t}}$ and $\frac{|y|_\rho}{\sqrt{t}}$ are greater than 1, the second term of (3.4.13) becomes $\frac{1}{t} e^{-\frac{c|x-y|_e^2}{t}}$. Therefore (3.4.13) is reduced to

$$\begin{aligned} \frac{c_{28}}{t} e^{-\frac{c_{29}\rho(x,y)^2}{t}} \le \frac{c_{30}}{t} e^{-\frac{c_{31}(|x|_\rho + |y|_\rho)^2}{t}} + \frac{c_{30}}{t} e^{-\frac{c_{32}|x-y|_e^2}{t}} &\le p(t,x,y) \\ \le \frac{c_{33}}{t} e^{-\frac{c_{34}(|x|_\rho + |y|_\rho)^2}{t}} + \frac{c_{33}}{t} e^{-\frac{c_{35}|x-y|_e^2}{t}} &\le \frac{c_{36}}{t} e^{-\frac{c_{37}\rho(x,y)^2}{t}}, \end{aligned}$$

since $\rho(x,y) = |x-y|_e \wedge (|x|_\rho + |y|_\rho)$. The proof is thus complete. □

Remark 3.4.5. *We have the following remark regarding the small time heat kernel estimates.*

1. *One cannot expect to rewrite the estimate of* (3.4.3) *as* $\frac{1}{t}\exp(-\frac{c\rho(x,y)^2}{t})$. *The counterexample is that* $x = y = a^*$, *in which case* x *and* y *can be viewed as either on* $\mathbb{R}$ *or*

on D_0*, therefore both Proposition 3.4.1 and Proposition 3.4.3 have already confirmed that* $p(t,x,y) \asymp \frac{1}{\sqrt{t}}$*, which is consistent with the* (3.4.3)*.*

2. *The Euclidean distance appearing in* (3.4.3) *cannot be replaced with geodesic distance. To find an counteon* D_0 *whose Euclidean coordinates are rexample, one may consider two points* x *and* y *both* $\left(\epsilon + \frac{1}{\sqrt{t}}, 0\right)$ *and* $\left(-\epsilon - \frac{1}{\sqrt{t}}, 0\right)$ *respectively, in which case the estimate of* (3.4.3) *is comparable with* $\frac{1}{\sqrt{t}} + \frac{1}{t}\exp(-\frac{\epsilon^2}{t})$*, but if we replaced* $|x-y|_e$ *with* $\rho(x,y)$*, it would be comparable with* $\frac{1}{\sqrt{t}} + \frac{1}{t}$*. For fixed* ϵ*, as* t *gets close to* 0*,* $\frac{1}{\sqrt{t}} + \frac{1}{t}\exp(-\frac{\epsilon^2}{t}) \asymp \frac{1}{\sqrt{t}}$*, but* $\frac{1}{\sqrt{t}} + \frac{1}{t} \asymp \frac{1}{t}$*.*

Chapter 4

LARGE TIME HEAT KERNEL ESTIMATE

4.1 On-diagonal Large Time Estimate at the Darning Point a^*

We still denote the BMVD process by X, and denote the radial process of X by Y which has been defined by (3.3.2). Unless otherwise stated, it is always assumed in this section that $t > 1$.

Proposition 4.1.1. *$p(t, a^*, a^*)$ is decreasing in t, for all $t > 0$.*

Proof.

$$\begin{aligned}\frac{d}{dt}p(t,a^*,a^*) &= \frac{d}{dt}\int_x p(t/2,a^*,x)^2 m_p(dx)\\ &= \int \left(\frac{d}{dt}p(t/2,a^*,x)\right) p(t/2,a^*,x)m_p(dx)\\ &= \int \mathcal{L}_x p(t/2,a^*,x)p(t/2,a^*,x)m_p(dx)\\ &= -\mathcal{E}\left(p(t/2,a^*,x),p(t/2,a^*,x)\right) \le 0,\end{aligned}$$

which shows the monitonicity of $p(t, a^*, a^*)$. □

Proposition 4.1.2.

$$p(t,a^*,a^*) \le C_1\frac{\log t}{t} \quad \textit{for } t \in [4,\infty).$$

Proof. When $x \in D_0$, $1 < |x|_\rho < \sqrt{t}$,

$$p(t,x,a^*) = \int_{s=0}^t \mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,a^*) \ge p(t,a^*,a^*)\mathbb{P}^x(T_{a^*}\le t) \asymp p(t,a^*,a^*)\left(1-\frac{\log|x|_e}{\log\sqrt{t}}\right),$$

where the second inequality above is due to Proposition 4.1.1, and the " $\asymp$ " is due to the hitting time estimate in [17, p.2-p.3]. Therefore

$$
\begin{aligned}
1 \geq \mathbb{P}^{a^*}\left(X_t \in D_0 \cap \left\{1 < |x|_\rho < \sqrt{t}\right\}\right) &= \int_{D_0 \cap \{1<|x|_\rho<\sqrt{t}\}} p(t, a^*, x) m_p(dx) \\
&\geq \int_{1+\epsilon}^{\sqrt{t}+\epsilon} p(t, a^*, a^*) \left(1 - \frac{\log r}{\log \sqrt{t}}\right) prdr \\
&\asymp \int_1^{\sqrt{t}} p(t, a^*, a^*) \left(1 - \frac{\log r}{\log \sqrt{t}}\right) rdr.
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
p(t, a^*, a^*) &\leq c \cdot \left[\int_1^{\sqrt{t}} \left(1 - \frac{\log r}{\log \sqrt{t}}\right) rdr\right]^{-1} \\
&= c \cdot \left[\int_1^{\sqrt{t}} rdr - \frac{1}{\log \sqrt{t}} \int_1^{\sqrt{t}} r \log r dr\right]^{-1} \\
&= c \cdot \left(\frac{1}{2}(t-1) - \frac{2}{\log t}\left(\frac{1}{4} t \log t - \frac{1}{4} t + \frac{1}{4}\right)\right)^{-1} \\
&\asymp \left(\frac{t}{\log t}\right)^{-1} = \frac{\log t}{t}.
\end{aligned}
$$

□

Proposition 4.1.3. *when $t > 4$, there exists some $C_2 > 0$ such that the following inequality holds:*

$$
\frac{1}{\log t} \int_0^t p(s, a^*, a^*) ds \leq C_2.
$$

Proof. Again by the hitting time estimate due to [17, p.2-p.3], when $x \in \left\{D_0 \cap 1 < |x|_\rho < \sqrt{t}\right\}$,

$$
p(t, x, a^*) \geq \int_{t/2}^t \mathbb{P}^x(T_{a^*} \in ds) p(t-s, a^*, a^*) \asymp \frac{\log |x|_\rho}{t(\log t)^2} \int_0^{t/2} p(s, a^*, a^*) ds.
$$

Therefore

$$
\begin{aligned}
1 \geq \mathbb{P}^{a^*}\left(X_t \in D_0 \cap \left\{1 < |x|_\rho < \sqrt{t}\right\}\right) &= \int_{D_0 \cap \left\{1<|x|_\rho<\sqrt{t}\right\}} p(t, a^*, x) m_p(dx) \\
&\geq p \int_1^{\sqrt{t}} \frac{(r+\epsilon)\log r}{t(\log t)^2} dr \cdot \int_0^{t/2} p(s, a^*, a^*) ds \\
&\asymp \int_1^{\sqrt{t}} \frac{r \log r}{t(\log t)^2} dr \cdot \int_0^{t/2} p(s, a^*, a^*) ds \\
&\asymp \frac{t \log t}{t(\log t)^2} \int_0^{t/2} p(s, a^*, a^*) ds \\
&= \frac{1}{\log t} \int_0^{t/2} p(s, a^*, a^*) ds.
\end{aligned}
$$

Equivalently, for some $c > 0$,

$$
\int_0^t p(s, a^*, a^*) \leq c \log t, \quad t > 4.
$$

□

Proposition 4.1.4. *Assume $t > 4$. There exists $C_3 > 0$ such that when $x \in D_0 \cap \left\{1 < |x|_\rho < \sqrt{t}\right\}$, it holds*

$$
p(t, a^*, x) \leq C_3 \left[\frac{1}{t} \log\left(\frac{\sqrt{t}}{|x|_e}\right) + \frac{1}{t}\right].
$$

Proof. By the hitting time estimate in [17, p.2-p.3],

$$
\begin{aligned}
p(t, x, a^*) &= \int_0^{t/2} \mathbb{P}^x(T_{a^*} \in ds) p(t-s, a^*, a^*) + \int_{t/2}^t \mathbb{P}^x(T_{a^*} \in ds) p(t-s, a^*, a^*) \\
&\leq \frac{\log t}{t} \mathbb{P}^x(T_{a^*} \leq t/2) + \frac{\log |x|_\rho}{t(\log t)^2} \int_0^{t/2} p(s, a^*, a^*) ds \\
&\leq \frac{\log t}{t}\left(1 - \frac{\log |x|_e}{\log \sqrt{t}}\right) + \frac{\log |x|_\rho}{t \log t} \\
&\leq \frac{2}{t} \log\left(\frac{\sqrt{t}}{|x|_e}\right) + \frac{1}{t}.
\end{aligned}
$$

Proposition 4.1.3 is used in the second inequality. □

Proposition 4.1.5. *There exist $C_4, C_5 > 0$ such that when $x \in D_0$, $|x|_\rho > \sqrt{t} > 1$, it holds*

$$
p(t, a^*, x) \leq C_4 \left(\frac{1}{t} e^{-\frac{C_5 |x|_\rho^2}{t}} + \frac{\left(\log\left(|x|_\rho^2/t\right)\right)^2}{|x|_\rho^2 (\log t)^2}\right).
$$

Proof.

$$p(t,a^*,x) = \int_0^t \mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,a^*)$$
$$= \int_0^{t/2} \mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,a^*) + \int_{t/2}^t \mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,a^*). \quad (4.1.1)$$

For the first term of (4.1.1) by Proposition 4.1.1 we have

$$\begin{aligned}\int_0^{t/2} \mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,a^*) &\le p(t/2,a^*,a^*)\mathbb{P}^x(T_{a^*}\le t/2)\\ &\le p(t/2,a^*,a^*)\frac{1}{\log|x|_\rho}e^{-\frac{c_1|x|_\rho^2}{t}}\\ &\le c_2\cdot\frac{\log t}{t\log|x|_\rho}e^{-\frac{c_1|x|_\rho^2}{t}}\\ &\le \frac{c_2}{t}e^{-\frac{c_1|x|_\rho^2}{t}}, \end{aligned} \quad (4.1.2)$$

where the second " $\le$ " is due to the hitting time estimate in [17, p.2-p.3]. The second last inequality above is due to Proposition 4.1.2. For the second term of (4.1.1), it holds

$$\begin{aligned}\int_{t/2}^t \mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,a^*)ds &\le \sup_{s\in[t/2,t]}\left(\frac{d\,\mathbb{P}^x(T_{a^*}\le\xi)}{d\xi}\bigg|_{\xi=s}\right)\cdot\int_{t/2}^t p(t-s,a^*,a^*)ds\\ &\le \sup_{s\in[t/2,t]}\left(\frac{d\,\mathbb{P}^x(T_{a^*}\le\xi)}{d\xi}\bigg|_{\xi=s}\right)\log t\\ &\le \log t\left(\frac{\log|x|_e}{t(\log t)^2}e^{-\frac{c_3|x|_e^2}{t}} + \frac{\left(\log\left(|x|_e^2/t\right)\right)^2}{|x|_e^2(\log t)^3}\right) \quad (4.1.3)\\ &\asymp \frac{\log|x|_\rho}{t\log t}e^{-\frac{c_3|x|_\rho^2}{t}} + \frac{\left(\log\left(|x|_\rho^2/t\right)\right)^2}{|x|_\rho^2(\log t)^2} \quad (4.1.4)\\ &\asymp \frac{1}{t}e^{-\frac{c|x|_\rho^2}{t}} + \frac{\left(\log\left(|x|_\rho^2/t\right)\right)^2}{|x|_\rho^2(\log t)^2}, \quad (4.1.5)\end{aligned}$$

where the second " $\le$ " is due to Proposition 4.1.3. (4.1.3) is due to [26, Theorem 2], and (4.1.4) is due to the fact that $|x|_\rho \asymp |x|_e$ since it is assumed $|x|_\rho > \sqrt{t} > 1$. It is used in (4.1.5) that when $|x|_\rho > \sqrt{t}$,

$$\frac{\log|x|_\rho}{\log t}e^{-\frac{c|x|_\rho^2}{t}} \le \text{ const.}$$

(4.1.5) together with (4.1.2) completes the proof. □

Proposition 4.1.6. *There exist $C_6, C_7 > 0$ such that when $x \in \mathbb{R}_+$, it holds*

$$p(t, a^*, x) \leq C_6 \left(\frac{\log t}{t} e^{-\frac{C_7 x^2}{t}} \right), \quad t > 4.$$

Proof. By Proposition 4.1.2 and hitting time distribution of 1-dimensional Brownian motion,

$$\begin{aligned} p(t, a^*, x) &= \int_0^t \mathbb{P}^x(T_{a^*} \in ds) p(t-s, a^*, a^*) \\ &= \int_0^1 \mathbb{P}^x(T_{a^*} \in ds) p(t-s, a^*, a^*) + \int_1^t \frac{x}{\sqrt{s^3}} e^{-\frac{c_1 x^2}{s}} p(t-s, a^*, a^*) ds \\ &\leq \frac{\log t}{t} e^{-c_2 x^2} + \int_1^{t/2} \frac{x}{\sqrt{s^3}} e^{-\frac{c_3 x^2}{s}} \cdot \frac{\log(t-s)}{t-s} ds + \int_{t/2}^t \frac{x}{\sqrt{s^3}} e^{-\frac{c_4 x^2}{s}} p(t-s, a^*, a^*) ds \\ &\leq \frac{\log t}{t} e^{-c_2 x^2} + \frac{\log t}{t} \int_1^{t/2} \frac{x}{\sqrt{s^3}} e^{-\frac{c_3 x^2}{s}} ds + \int_{t/2}^{t-1} \frac{x}{\sqrt{s^3}} e^{-\frac{c_4 x^2}{s}} p(t-s, a^*, a^*) ds \\ &+ \int_{t-1}^t \frac{x}{\sqrt{s^3}} e^{-\frac{c_4 x^2}{s}} \frac{1}{\sqrt{t-s}} ds. \end{aligned} \tag{4.1.6}$$

For the second term of (4.1.6) we have

$$\frac{\log t}{t} \int_1^{t/2} \frac{x}{\sqrt{s^3}} e^{-\frac{c_3 x^2}{t}} ds \asymp \frac{\log t}{t} \int_{s=1}^{t/2} e^{-\frac{c_3 x^2}{s}} d\left(\frac{x}{\sqrt{s}}\right) \leq c_5 \cdot \frac{\log t}{t} e^{-\frac{c_3 x^2}{t}}. \tag{4.1.7}$$

For the third term of (4.1.6) it holds

$$\begin{aligned} \int_{t/2}^{t-1} \frac{x}{\sqrt{s^3}} e^{-\frac{c_4 x^2}{s}} p(t-s, a^*, a^*) ds &\leq \frac{x}{\sqrt{t^3}} e^{-\frac{c_4 x^2}{t}} \cdot \int_1^{t/2} p(s, a^*, a^*) ds \\ &\leq \frac{x}{\sqrt{t^3}} e^{-\frac{c_4 x^2}{t}} \cdot \log t \leq \frac{c_6}{t} e^{-\frac{c_4 x^2}{t}} \cdot \log t, \end{aligned} \tag{4.1.8}$$

where the first " $\leq$ " above is due to Proposition 4.1.3. The last term of (4.1.6) is bounded by

$$\frac{x}{\sqrt{t^3}} e^{-\frac{c_4 x^2}{t}} \int_{t-1}^t \frac{1}{\sqrt{t-s}} ds \leq \frac{x}{\sqrt{t^3}} e^{-\frac{c_4 x^2}{t}} \leq \frac{c_7}{t} e^{-\frac{c_4 x^2}{t}}. \tag{4.1.9}$$

(4.1.7) together with (4.1.8) and (4.1.9) shows

$$p(t, a^*, x) \leq c_8 \left(\frac{\log t}{t} e^{-c_2 x^2} + \frac{\log t}{t} e^{-\frac{c_4 x^2}{t}} + \frac{1}{t} e^{-\frac{c_4 x^2}{t}} \right) \leq c_8 \cdot \frac{\log t}{t} e^{-\frac{c_9 x^2}{t}}.$$

□

Lemma 4.1.7. *There exists $C_8 > 0$ such that*

$$p(t, a^*, a^*) \leq \frac{C_8}{t}, \qquad t > 1.$$

Proof. By semi-group property

$$\begin{aligned} p(t,a^*,a^*) &= \int_E p(t/2,a^*,x)^2 m_p(dx) \\ &= \left(\int_{\mathbb{R}_+} + \int_{D_0\cap\{0<|x|_\rho<1\}} + \int_{D_0\cap\{1<|x|_\rho<\sqrt{t}\}} + \int_{D_0\cap\{|x|_\rho>\sqrt{t}\}} \right) p(t/2,a^*,x)^2 m_p(dx). \end{aligned} \tag{4.1.10}$$

Thus we consider the right hand side above in several cases.

Case 1. $x \in \mathbb{R}_+$. By Proposition 4.1.6

$$\int_{\mathbb{R}_+} p(t/2,a^*,x)^2 m_p(dx) \le c_1 \left(\frac{\log t}{t}\right)^2 \int_0^\infty e^{-\frac{c_2 x^2}{t}} m_p(dx) \asymp \frac{(\log t)^2}{t^{3/2}}. \tag{4.1.11}$$

Case 2. $x \in D_0 \cap \{0 < |x|_\rho < 1\}$. Recall that we have the following upper bound estimate due to Nash's inequality:

$$p(t,x,y) \le \frac{c_3}{\sqrt{t}} e^{-\frac{c_4 \rho(x,y)^2}{t}}, \quad t > 1.$$

It thus follows

$$\begin{aligned} \int_{D_0\cap\{0<|x|_\rho<1\}} p(t/2,a^*,x)^2 m_p(dx) &\le \left(\sup_{x\in D_0\cap\{0<|x|_\rho<1\}} p(t/2,a^*,x)^2 \right) \left| D_0 \cap \{0<|x|_\rho<1\} \right| \\ &\le \left(\frac{c_5}{\sqrt{t}}\right)^2 \asymp \frac{1}{t}. \end{aligned} \tag{4.1.12}$$

Case 3. $x \in D_0 \cap \{|x|_\rho > \sqrt{t}\}$. It has been showed in Proposition 4.1.5 that

$$p(t,a^*,x) \le c_6 \left(\frac{1}{t} e^{-\frac{c_7 |x|_\rho^2}{t}} + \frac{\left(\log\left(|x|_\rho^2/t\right)\right)^2}{|x|_\rho^2 (\log t)^2} \right), \quad x \in D_0 \cap \{|x|_\rho > \sqrt{t}\}.$$

It thus follows

$$\begin{aligned} \int_{D_0\cap\{|x|_\rho>\sqrt{t}\}} p(t/2,a^*,x)^2 m_p(dx) &\le \int_{D_0\cap\{|x|_\rho>\sqrt{t}\}} \left(\frac{c_6}{t} e^{-\frac{c_7|x|_\rho^2}{t}} \right)^2 m_p(dx) \\ &\quad + \int_{D_0\cap\{|x|_\rho>\sqrt{t}\}} \left(c_6 \frac{\left(\log\left(|x|_\rho^2/t\right)\right)^2}{|x|_\rho^2 (\log t)^2} \right)^2 m_p(dx). \end{aligned} \tag{4.1.13}$$

For the first term on the right hand side of (4.1.13) we have

$$\int_{D_0\cap\{|x|_\rho>\sqrt{t}\}} \left(\frac{1}{t} e^{-\frac{c_7|x|_\rho^2}{t}} \right)^2 m_p(dx) \asymp \int_{\sqrt{t}}^\infty \frac{r}{t^2} e^{-\frac{c_7 r^2}{t}} dr \asymp \int_{r=\sqrt{t}}^\infty \frac{1}{t} e^{-\frac{c_7 r^2}{t}} d\left(\frac{r^2}{t}\right) \asymp \frac{1}{t}. \tag{4.1.14}$$

For the second term on the right side of (4.1.13) we have

$$\int_{D_0\cap\{|x|_\rho>\sqrt{t}\}}\left(\frac{\left(\log\left(x^2/t\right)\right)^2}{x^2\left(\log t\right)^2}\right)^2 m_p(dx) = p\int_{\sqrt{t}}^{\infty}\frac{r\left(\log(r^2/t)\right)^4}{r^4(\log t)^4}dr = p\int_{\sqrt{t}}^{\infty}\frac{\left(\log(r^2/t)\right)^4}{r^3(\log t)^4}dr$$
$$= p\int_1^{\infty}\frac{(\log u)^4}{u^3t^{3/2}(\log t)^4}\sqrt{t}\,du = p\int_1^{\infty}\frac{(\log u)^4}{u^3t(\log t)^4}\,du$$
$$\asymp \frac{1}{t(\log t)^4}. \tag{4.1.15}$$

(4.1.14) together with (4.1.15) shows that

$$\int_{D_0\cap\{|x|_\rho>\sqrt{t}\}} p(t/2,a^*,x)^2 m_p(dx) \le \frac{c_8}{t}. \tag{4.1.16}$$

Case 4. $x\in D_0\cap\{1<|x|_\rho<\sqrt{t}\}$. It is showed in Proposition 4.1.4 that when $x\in D_0\cap\{1<|x|_\rho<\sqrt{t}\}$,

$$p(t,x,a^*)\le c_9\left(\frac{1}{t}\log\left(\frac{\sqrt{t}}{|x|_e}\right)+\frac{1}{t}\right).$$

It thus follows

$$\int_{D_0\cap\{1<|x|_\rho<\sqrt{t}\}} p(t/2,a^*,x)^2 m_p(dx)$$
$$\asymp \frac{1}{t^2}\left|D_0\cap\left\{1<|x|_\rho<\sqrt{t}\right\}\right| + \frac{1}{t^2}\int_{x\in D_0\cap\{1<|x|_\rho<\sqrt{t}\}}\left(\log\left(\frac{\sqrt{t}}{|x|_e}\right)\right)^2 m_p(dx)$$
$$\asymp \frac{1}{t}+\frac{1}{t^2}\int_{1+\epsilon}^{\sqrt{t}+\epsilon} r\left(\log\left(\frac{\sqrt{t}}{r}\right)\right)^2 dr$$
$$\asymp \frac{1}{t}+\frac{1}{t^2}\int_{1}^{\sqrt{t}} r\left(\log\left(\frac{\sqrt{t}}{r}\right)\right)^2 dr$$
$$\asymp \frac{1}{t}+\frac{1}{t^2}\int_{u=1}^{\sqrt{t}} \frac{\sqrt{t}}{u}\left(\log u\right)^2\frac{\sqrt{t}}{u^2}du$$
$$\asymp \frac{1}{t}+\frac{1}{t}\int_{u=1}^{\sqrt{t}} \frac{(\log u)^2}{u^3}du \asymp \frac{1}{t}, \tag{4.1.17}$$

where in the 4th " $\asymp$ " the change of variable $u=\sqrt{t}/r$ is made. Combining (4.1.11),(4.1.12), (4.1.16) and (4.1.17) one has

$$p(t,a^*,a^*)\le \frac{c_{10}}{t}.$$

□

Proposition 4.1.8. *Let Y be the radial process of X which is $1-$dimensional. The symmetrizing measure of Y is*

$$\mu(dx) \asymp \mathbf{1}_{(-\infty,0)}dx + (x+\epsilon)\mathbf{1}_{[0,\infty)}dx,$$

The intrinsic metric is thus the Euclidean metric.

Proof. Recall that the radial process Y can be characterized as follows.

$$dY_t = dB_t + \frac{1}{Y_t+\epsilon}\mathbf{1}_{\{Y_t>0\}}dt + \frac{2\pi\epsilon - p}{2\pi\epsilon + p}d\widehat{L}_t^0(Y).$$

Without loss of generality we first assume $p = 2\pi\epsilon$, because a different p value only affects the metric on $\mathbb{R}_+$. It thus follows that

$$\left(u', v'\right)_\mu = -\left(\mathcal{L}u, v\right)_\mu, \qquad u, v \in \mathcal{D}(\mathcal{L}),$$

where

$$\mathcal{L}u = u'' + \frac{1}{x+\epsilon}\mathbf{1}_{\{x>0\}}u'.$$

Now let μ be the measure in the statement of the proposition. Let $u, v \in \mathcal{D} \cap C_c^\infty(\mathbb{R})$. By doing integration by parts on $\mathbb{R}_+$ one has

$$\begin{aligned}
-\left(\mathcal{L}u, v\right)_\mu &= -\int_{\mathbb{R}_+}\left(\frac{1}{x+\epsilon}u' + u''\right)v\cdot(x+\epsilon)dx + \int_{\mathbb{R}_-}u'v'dx \\
&= -\int_{\mathbb{R}_+}\left(u'v + u''v\cdot(x+\epsilon)\right)dx + \int_{\mathbb{R}_-}u'v'dx \\
&= -\int_{\mathbb{R}_+}vd\left(u'(x+\epsilon)\right) + \int_{\mathbb{R}_-}u'v'dx \\
&= \int_{\mathbb{R}_+}u'(x+\epsilon)dv + \int_{\mathbb{R}_-}u'v'dx \\
&= \int_{\mathbb{R}_+}u'v'\,(x+\epsilon)dx + \int_{\mathbb{R}_-}u'v'dx = \int_{\mathbb{R}}u'v'\mu(dx),
\end{aligned}$$

which proves that $\mu(dx) = \mathbf{1}_{(-\infty,0)}dx + (x+\epsilon)\mathbf{1}_{[0,\infty)}dx$ is the symmetrizing measure of the radial process Y under the assumption that $p = 2\pi\epsilon$. Let $\rho_{\mathcal{E}}$ be the intrinsic metric associated with $\mathcal{E}(u,v) := (u',v')_\mu$. By the definition of intrinsic metric,

$$\rho_{\mathcal{E}}(x,y) = \sup\left\{u(x) - u(y) : u \in \mathcal{D}(\mathcal{E}) \cap C_c(\mathbb{R}), d\Gamma(u,u) \le d\mu\right\},$$

where $d\Gamma(u,u)$ is defined as follows:

$$\int_A d\Gamma(u,u) := \int_A (u')^2 d\mu.$$

Since μ has the same support as Lebesgue measure, $u' = 1$ a.e. with respect to μ is equivalent to the fact that $u' = 1$ a.e. with respect to Lebesgue measure. It thus follows that $\rho_{\mathcal{E}}$ is the same as $1-$dimensional Euclidean metric, which proves that for $p = 2\pi\epsilon$, $\mu(dx) = \mathbf{1}_{(-\infty,0)}dx + (x+\epsilon)\mathbf{1}_{[0,\infty)}dx$. The conclusion for the general case follows immediately. □

Theorem 4.1.9.

$$p(t, a^*, a^*) \asymp \frac{1}{t}, \qquad t > 1.$$

Proof. It is proved in Lemma 4.1.7

$$p^{m,Y}(t,0,0) \asymp p^X(t,a^*,a^*) \le \frac{c_1}{t},$$

where the superscript m means this is the probability density of Y with respect to the $1-$dimensional Lebesgue measure m. Since locally near the point 0, the symmetrizing measure μ of Y differs from m only up to a constant, it actually holds

$$p^{\mu,Y}(t,0,0) \asymp p^{m,Y}(t,0,0) \asymp p^X(t,a^*,a^*) \le \frac{c_2}{t}.$$

Now we want to claim the statement of the theorem holds by applying [12, Theorem 7.2] to the radial process Y. $p^{\mu,Y}$ is the Dirichlet heat kernel with respect to the measure μ on $\mathbb{R}$. Indeed in view of Proposition 4.1.8, the condition of [12, Thereom 7.2] is satisfied by choosing $z = 0$ and

$$v(r) := Kr\mathbf{1}_{\{0<r<1\}} + Kr^2\mathbf{1}_{\{1\le r<\infty\}},$$

where K is some sufficiently large constant. It is remarked after [12, Theorem 7.2] that the T in that theorem can be equal to ∞. We already have

$$p^{\mu,Y}(t,0,0) \le \frac{c_3}{t}, \qquad t > 1.$$

Thus by applying [12, Theorem 7.2], it holds

$$p^X(t,a^*,a^*) \asymp p^{\mu,Y}(t,0,0) \asymp \frac{1}{t}, \qquad t > 1.$$

□

4.2 Off-diagonal Large Time Heat Kernel Estimate

Corollary 4.2.1. *There exist* $C_i > 0$, $9 \le i \le 12$ *such that*

$$\frac{C_9}{t} e^{-\frac{C_{10}|x|_\rho^2}{t}} \le p(t, a^*, x) \le \frac{C_{11}}{t} e^{-\frac{C_{12}|x|_\rho^2}{t}}, \qquad t > 1,\, x \in D_0.$$

Proof. We prove this for several different cases.

Case 1. $1 < |x|_\rho < 2\sqrt{t}$. In this case, by the hitting time estimate in [17],

$$\begin{aligned} p(t,x,a^*) &= \int_0^{t/2} \mathbb{P}^x(T_{a^*} \in ds) p(t-s, a^*, a^*) + \int_{t/2}^{t} \mathbb{P}^x(T_{a^*} \in ds) p(t-s, a^*, a^*) \\ &\asymp p(t, a^*, a^*) \mathbb{P}^x(T_{a^*} \le t/2) + \frac{\log |x|_\rho}{t(\log t)^2} \int_0^{t/2} p(s, a^*, a^*) ds \\ &\asymp p(t, a^*, a^*) \left(1 - \frac{\log |x|_\rho}{\log \sqrt{t}}\right) + \frac{\log |x|_\rho}{t \log t} \\ &\asymp \frac{1}{t} \left(1 - \frac{\log |x|_\rho}{\log \sqrt{t}}\right) + \frac{\log |x|_\rho}{t \log t} \asymp \frac{1}{t}. \end{aligned}$$

Case 2. $|x|_\rho > 2\sqrt{t}$. We denote a standard 2−dimensional Brownian motion by B.

$$\begin{aligned} p(t,x,a^*) &= \mathbb{E}^x \int_0^t \mathbf{1}_{\{T_{B_\rho(a^*,1)} \in ds\}} p\left(t-s, X_{T_{B_\rho(a^*,1)}}, a^*\right) \\ &\le \mathbb{E}^x \int_0^t \mathbf{1}_{\{T_{B_\rho(a^*,1)} \in ds\}} \cdot \frac{c_1}{t-s} e^{-\frac{c_2}{t-s}} \qquad (4.2.1) \\ &\le \mathbb{E}^{2-BM,x} \int_0^t \mathbf{1}_{\{T_{B_e(0,1+\epsilon)} \in ds\}} p^{2-BM}\left(c_2(t-s), B_{T_{B_e(0,1+\epsilon)}}, 0\right) \qquad (4.2.2) \\ &\le p^{2-BM}(c_2 t, x, 0) \le \frac{c_1}{t} e^{-\frac{c_2 |x|_e^2}{t}}. \end{aligned}$$

Also the other direction of the inequalities hold as follows.

$$\begin{aligned} p(t,x,a^*) &= \mathbb{E}^x \int_0^t \mathbf{1}_{\{T_{B_\rho(a^*,1)} \in ds\}} p\left(t-s, X_{T_{B_\rho(a^*,1)}}, a^*\right) \\ &\ge \mathbb{E}^x \int_0^t \mathbf{1}_{\{T_{B_\rho(a^*,1)} \in ds\}} \cdot \frac{c_1}{t-s} e^{-\frac{c_2}{t-s}} \qquad (4.2.3) \\ &\ge \mathbb{E}^{2-BM,x} \int_0^t \mathbf{1}_{\{T_{B_e(0,1+\epsilon)} \in ds\}} p^{2-BM}\left(c_2(t-s), B_{T_{B_e(0,1+\epsilon)}}, 0\right) \qquad (4.2.4) \\ &\ge p^{2-BM}(c_2 t, x, 0) \ge \frac{c_1}{t} e^{-\frac{c_2 |x|_e^2}{t}}. \end{aligned}$$

(4.2.1) and (4.2.3) are due to the result of the first case as well as the small time estimate. (4.2.2) and (4.2.4) are due to the fact that $\mathbb{P}^x(T_{B_\rho(a^*,1)} \in ds)$ has the same distribution as

$\mathbb{P}^{2-BM,x}(T_{B_e(0,1+\epsilon)} \in ds)$ as well as the following:

$$\frac{c_3}{t-s}e^{-\frac{c_4}{t-s}} \leq \frac{c_3}{t-s}e^{-\frac{c_5(1+\epsilon^2)}{t-s}} \leq p^{2-BM}\left(t-s, B_{T_{B_e(0,1+\epsilon)}}, 0\right) \leq \frac{c_6}{t-s}e^{-\frac{c_7(1+\epsilon^2)}{t-s}} \leq \frac{c_6}{t-s}e^{-\frac{c_7}{t-s}},$$

since it is assumed $t > 1$. $|x|_\rho > 2\sqrt{t} > 2$ implies that $|x|_\rho \asymp |x|_e$. It thus follows

$$\frac{c_8}{t}e^{-\frac{c_9|x|_\rho^2}{t}} \leq p(t,x,a^*) \leq \frac{c_{10}}{t}e^{-\frac{c_{11}|x|_\rho^2}{t}}.$$

Case 3. $0 < |x|_\rho < 1$.

$$\begin{aligned} p(t,a^*,x) = &\int_{D_0\cap B(a^*,2)} p(t/2,a^*,y)p(t/2,y,x)m_p(dy) \\ &+ \int_{D_0\cap B^c(a^*,2)} p(t/2,a^*,y)p(t/2,y,x)m_p(dy) + \int_{\mathbb{R}_+} p(t/2,a^*,y)p(t/2,y,x)m_p(dy). \end{aligned} \tag{4.2.5}$$

By the upper bound estimate due to Nash's inequality, we have for the first term on the right hand side of (4.2.5)

$$\int_{y\in D_0\cap B(a^*,2)} p(t/2,x,y)p(t/2,y,z)m_p(dy) \leq \int_{D_0\cap B(a^*,2)} \left(\frac{c_{12}}{\sqrt{t}}\right)^2 m_p(dy) \asymp \frac{1}{t}. \tag{4.2.6}$$

For the second term on the right hand side of (4.2.5), it holds

$$\begin{aligned} &\int_{D_0\cap B^c(a^*,2)} p(t/2,x,y)p(t/2,y,z)m_p(dy) \\ \leq &\int_{x\in E} \mathbb{P}^x\left(X_{t/2} \in D_0\cap B^c(a^*,2)\right) \sup_{y\in D_0\cap B^c(a^*,2)} p(t/2,a^*,y)m_p(dz) \\ \leq &\sup_{y\in D_0\cap B^c(a^*,2)} p(t/2,a^*,y) \leq \frac{c_{13}}{t}, \end{aligned} \tag{4.2.7}$$

where the last inequality is due to the results of the previous two cases. Now we consider the third term on the right hand side of (4.2.5).

$$\begin{aligned}
\int_{\mathbb{R}_+} p(t/2,a^*,y)p(t/2,y,x)m_p(dy) &= \int_{\mathbb{R}_+} p(t/2,a^*,y)p(t/2,x,y)m_p(dy)\\
&= \int_{\mathbb{R}_+} p(t/2,a^*,y)\int_{s=0}^{t/2}\mathbb{P}^x\left(T_{a^*}\in ds\right)p(t/2-s,a^*,y)m_p(dy)\\
&= \int_{\mathbb{R}_+} p(t/2,a^*,y)\int_{s=0}^{t/2}\mathbb{P}^x\left(T_{a^*}\in ds\right)p(t/2-s,y,a^*)m_p(dy)\\
&= \int_{s=0}^{t/2}\mathbb{P}^x\left(T_{a^*}\in ds\right)\int_{\mathbb{R}_+} p(t/2,a^*,y)p(t/2-s,y,a^*)m_p(dy)\\
&= \int_{s=0}^{t/2}\mathbb{P}^x\left(T_{a^*}\in ds\right)p(t-s,a^*,a^*)\\
&\asymp \frac{1}{t}\cdot\mathbb{P}^x\left(T_{a^*}\le t/2\right)\asymp\frac{1}{t}, \qquad (4.2.8)
\end{aligned}$$

where the last " $\asymp$ " is due to the fact that $|x|_\rho<1$ and $t>1$. (4.2.8) together with (4.2.6) and (4.2.7) shows that

$$p(t,a^*,x)\asymp\frac{1}{t},\quad t>1, x\in D_0, 0<|x|_\rho<1.$$

The proof of Case 3 is thus complete. □

The next corollary regards the estimate of $p(t,a^*,x)$ when $x\in\mathbb{R}_+$.

Corollary 4.2.2. *There exist constants $C_i>0$, $13\le i\le 16$ such that*

$$C_{13}\left[\frac{1}{t}+\frac{\log t}{t}\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\right]e^{-\frac{C_{14}|x|^2}{t}}\le p(t,a^*,x)\le C_{15}\left[\frac{1}{t}+\frac{\log t}{t}\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\right]e^{-\frac{C_{16}|x|^2}{t}},$$

for $x\in\mathbb{R}_+, t>1$.

Proof. By Markov property we have

$$\begin{aligned}
p(t,a^*,x) &= \int_0^t\mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,a^*)\\
&= \int_0^1\mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,a^*)+\int_1^t\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}p(t-s,a^*,a^*)ds\\
&\asymp\frac{1}{t}e^{-c_2x^2}+\int_1^{t/2}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\cdot\frac{1}{t-s}ds+\int_{t/2}^t\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}p(t-s,a^*,a^*)ds.
\end{aligned}\qquad (4.2.9)$$

For the second term of (4.2.9) it holds

$$\begin{aligned}\int_1^{t/2} \frac{x}{\sqrt{s^3}} e^{-\frac{c_3 x^2}{s}} \cdot \frac{1}{t-s} ds &\asymp \frac{1}{t} \int_1^{t/2} \frac{x}{\sqrt{s^3}} e^{-\frac{c_3 x^2}{s}} ds \\ &\asymp \frac{1}{t} \int_1^{t/2} \frac{x}{\sqrt{s^3}} e^{-\frac{c_3 x^2}{s}} ds \\ &= \frac{1}{t} \int_{s=1}^{t/2} e^{-\frac{c_3 x^2}{s}} d\left(\frac{x}{\sqrt{s}}\right) \asymp \frac{1}{t} e^{-\frac{c_3 x^2}{t}}. \end{aligned} \tag{4.2.10}$$

For the third term of (4.2.9) it holds

$$c_4 \cdot \frac{x}{\sqrt{t^3}} e^{-\frac{c_1 x^2}{t}} \cdot \log t \le \int_{t/2}^{t} \frac{x}{\sqrt{s^3}} e^{-\frac{c_1 x^2}{s}} p(t-s, a^*, a^*) ds \le c_5 \cdot \frac{x}{\sqrt{t^3}} e^{-\frac{c_1 x^2}{t}} \cdot \log t.$$

i.e.,

$$c_4 \cdot \frac{\log t}{t} \left(1 \wedge \frac{x}{\sqrt{t}}\right) e^{-\frac{c_1 x^2}{t}} \le \int_{t/2}^{t} \frac{x}{\sqrt{s^3}} e^{-\frac{c_1 x^2}{s}} p(t-s, a^*, a^*) ds \le c_5 \cdot \frac{\log t}{t} \left(1 \wedge \frac{x}{\sqrt{t}}\right) e^{-\frac{c_1 x^2}{t}}. \tag{4.2.11}$$

The desired conclusion follows by combining (4.2.9), (4.2.11) and (4.2.10). □

Now we try to estimate $p(t, x, y)$ for arbitrary pairs of $(x, y) \in D_0 \times D$ when $t > 1$.

Theorem 4.2.3. *There exist constants $C_i > 0$, $17 \le i \le 20$, such that the following estimate holds:*

$$\frac{C_{17}}{t} e^{-\frac{C_{18}\rho(x,y)^2}{t}} \le p(t, x, y) \le \frac{C_{19}}{t} e^{-\frac{C_{20}\rho(x,y)^2}{t}}, \quad (t, x, y) \in (1, \infty) \times D_0 \times D_0.$$

Proof. We prove this for two different cases.

Case 1. Either $|x|_\rho > 1$ or $|y|_\rho > 1$ holds. Without loss of generality, we assume $|y|_\rho > 1$. We notice in this case, by elementary geometry, it actually holds

$$\rho(x, y) \asymp |x - y|_e. \tag{4.2.12}$$

Denote a standard 2−dimensional Brownian motion by B. We have

$$\begin{aligned}
\overline{p}_{D_0}(t,x,y) &= \int_0^t \mathbb{P}^x\left(T_{a^*}\in ds\right)p(t-s,a^*,y) \\
&\le \int_0^t \mathbb{P}^x\left(T_{a^*}\in ds\right)\frac{c_1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}} && (4.2.13)\\
&= \mathbb{E}^{2-BM,x}\int_0^t \mathbf{1}_{\{T_{B_e(0,\epsilon)}\in ds\}}\cdot\frac{c_1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}} && (4.2.14)\\
&\le c_1\mathbb{E}^{2-BM,x}\int_0^t \mathbf{1}_{\{T_{B_e(0,\epsilon)}\in ds\}}\cdot p^{2-BM}\left(c_2(t-s),B_{T_{B_e(0,\epsilon)}},y\right) && (4.2.15)\\
&\le c_1\overline{p}^{2-BM}_{\mathbb{R}^2\setminus B_e(0,\epsilon)}(c_2t,x,y).
\end{aligned}$$

We also have the other direction of the inequalities as follows.

$$\begin{aligned}
\overline{p}_{D_0}(t,x,y) &= \int_0^t \mathbb{P}^x\left(T_{a^*}\in ds\right)p(t-s,a^*,y) \\
&\ge \int_0^t \mathbb{P}^x\left(T_{a^*}\in ds\right)\frac{c_4}{t-s}e^{-\frac{c_3|y|_\rho^2}{t-s}} && (4.2.16)\\
&= \mathbb{E}^{2-BM,x}\int_0^t \mathbf{1}_{\{T_{B_e(0,\epsilon)}\in ds\}}\cdot\frac{c_4}{t-s}e^{-\frac{c_3|y|_\rho^2}{t-s}} && (4.2.17)\\
&\ge c_4\mathbb{E}^{2-BM,x}\int_0^t \mathbf{1}_{\{T_{B_e(0,\epsilon)}\in ds\}}\cdot p^{2-BM}\left(c_3(t-s),B_{T_{B_e(0,\epsilon)}},y\right) && (4.2.18)\\
&\ge c_4\overline{p}^{2-BM}_{\mathbb{R}^2\setminus B_e(0,\epsilon)}(c_3t,x,y).
\end{aligned}$$

(4.2.13) and (4.2.16) are due to Corollary 4.2.1. (4.2.14) and (4.2.17) because $\mathbb{P}^x(T_{a^*}\in ds)$ has the same distribution as $\mathbb{P}^{2-BM,x}(T_{B_e(0,\epsilon)}\in ds)$. (4.2.15) and (4.2.18) are due to the assumption that $|y|_\rho>1$, which implies $|y|_\rho\asymp|y|_\rho-\epsilon$. It thus follows

$$p(t,x,y)=\overline{p}_{D_0}(t,x,y)+p_{D_0}(t,x,y).$$

i.e.,

$$c_4\overline{p}^{2-BM}_{\mathbb{R}^2\setminus B_e(0,\epsilon)}(c_3t,x,y)+p^{2-BM}_{\mathbb{R}^2\setminus B_e(0,\epsilon)}(t,x,y)\le p(t,x,y)\le c_1\overline{p}^{2-BM}_{\mathbb{R}^2\setminus B_e(0,\epsilon)}(c_2t,x,y)+p^{2-BM}_{\mathbb{R}^2\setminus B_e(0,\epsilon)}(t,x,y).$$

Hence

$$\frac{c_5}{t}e^{-\frac{c_6|x-y|_e^2}{t}}\le p(t,x,y)\le\frac{c_7}{t}e^{-\frac{c_8|x-y|_e^2}{t}}.$$

Case 2. Both $|x|_\rho$ and $|y|_\rho$ are less than 1. The proof of this case is similar to Case 3 of

Corollary 4.2.1.

$$p(t,x,y) = \int_{D_0\cap B(a^*,2)} p(t/2,x,z)p(t/2,z,y)m_p(dz)$$
$$+\int_{D_0\cap B^c(a^*,2)} p(t/2,x,z)p(t/2,z,y)m_p(dz) + \int_{\mathbb{R}_+} p(t/2,x,z)p(t/2,z,y)m_p(dz). \tag{4.2.19}$$

By the upper bound estimate due to Nash's inequality, we have for the first term on the right hand side of (4.2.19)

$$\int_{D_0\cap B(a^*,2)} p(t/2,x,z)p(t/2,z,y)m_p(dz) \le \int_{D_0\cap B(a^*,2)} \left(\frac{c_9}{\sqrt{t}}\right)^2 m_p(dz) \asymp \frac{1}{t}. \tag{4.2.20}$$

For the second term on the right hand side of (4.2.19), it holds

$$\begin{aligned}
&\int_{D_0\cap B^c(a^*,2)} p(t/2,x,z)p(t/2,z,y)m_p(dz) \\
&\le \mathbb{P}^x\left(X_{t/2}\in D_0\cap B^c(a^*,2)\right) \sup_{z\in D_0\cap B^c(a^*,2),|y|_\rho<1} p(t/2,y,z) \\
&\le \sup_{y,z\in D_0,|y|_\rho<1,|z|_\rho>2} p(t/2,y,z) \le \frac{c_{10}}{t},
\end{aligned} \tag{4.2.21}$$

where the last inequality is due to the result of the first case. Now we consider the third term on the right hand side of (4.2.19).

$$\begin{aligned}
\int_{\mathbb{R}_+} p(t/2,x,z)p(t/2,z,y)m_p(dz) &= \int_{\mathbb{R}_+} p(t/2,x,z)p(t/2,y,z)m_p(dz) \\
&= \int_{\mathbb{R}_+} p(t/2,x,z)\int_{s=0}^{t/2} \mathbb{P}^y\left(T_{a^*}\in ds\right)p(t/2-s,a^*,z)m_p(dz) \\
&= \int_{s=0}^{t/2} \mathbb{P}^y\left(T_{a^*}\in ds\right)\int_{\mathbb{R}_+} p(t/2,x,z)p(t/2-s,a^*,z)m_p(dz) \\
&= \int_{s=0}^{t/2} \mathbb{P}^y\left(T_{a^*}\in ds\right)p(t-s,a^*,x) \\
&\asymp \frac{1}{t}\cdot\mathbb{P}^y\left(T_{a^*}\le t/2\right) \asymp \frac{1}{t},
\end{aligned} \tag{4.2.22}$$

where the last " $\asymp$ " is due to the fact that $|y|_\rho < 1$ and $t>1$. The second last " $\asymp$ " is due to Corollary 4.2.1. Therefore again replacing the three terms on the right hand side of (4.2.19) with (4.2.20), (4.2.21) and (4.2.22) yields

$$p(t,x,y)\asymp\frac{1}{t}, \quad (x,y)\in\left(D_0\cap B(a^*,1)\right)\times\left(D_0\cap B(a^*,1)\right).$$

The proof is thus complete. □

Next theorem regards the case that $x \in \mathbb{R}$, $y \in D_0$.

Theorem 4.2.4. *There exist constants $C_i > 0$, $21 \le i \le 28$, such that the following estimates hold for $(t,x,y) \in (1,\infty) \times \mathbb{R}_+ \times D_0$:*
When $|y|_\rho < 1$,

$$\frac{C_{21}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\cdot\log t\right]e^{-\frac{C_{22}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{23}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\cdot\log t\right]e^{-\frac{C_{24}\rho(x,y)^2}{t}}; \tag{4.2.23}$$

when $|y|_\rho \ge 1$,

$$\begin{aligned}&\frac{C_{25}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1+\log\left(1+\frac{\sqrt{t}}{|y|_\rho}\right)\right)\right]e^{-\frac{C_{26}\rho(x,y)^2}{t}} \le p(t,x,y)\\ &\le \frac{C_{27}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1+\log\left(1+\frac{\sqrt{t}}{|y|_\rho}\right)\right)\right]e^{-\frac{C_{28}\rho(x,y)^2}{t}}.\end{aligned} \tag{4.2.24}$$

Proof. We prove this for different cases.
Case 1. $|y|_\rho < 1$. In this case

$$\frac{c_1}{\sqrt{t}}e^{-\frac{c_2|y|_\rho^2}{t}} \le p(t,a^*,y) \le \frac{c_3}{\sqrt{t}}e^{-\frac{c_4|y|_\rho^2}{t}}, \quad t<1; \tag{4.2.25}$$

and

$$\frac{c_5}{t}e^{-\frac{c_6|y|_\rho^2}{t}} \le p(t,a^*,y) \le \frac{c_7}{t}e^{-\frac{c_8|y|_\rho^2}{t}}, \quad t>1, \tag{4.2.26}$$

Thus by Markov property, we have

$$\begin{aligned}p(t,x,y) &\asymp \int_0^t \mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,y)\\ &\asymp \int_0^{t/2}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds + \int_{t/2}^{t-1}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds\\ &+\int_{t-1}^t\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{\sqrt{t-s}}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds\end{aligned} \tag{4.2.27}$$

For the first term on the right hand side of (4.2.27) we have

$$\begin{aligned}\frac{c_9}{t}e^{-\frac{c_{10}|y|_\rho^2}{t}}\int_{s=0}^{t/2}e^{-\frac{c_{11}|x|^2}{s}}d\left(\frac{|x|}{\sqrt{s}}\right) &\le \int_0^{t/2}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds\\ &\le \frac{c_{12}}{t}e^{-\frac{c_{13}|y|_\rho^2}{t}}\int_{s=0}^{t/2}e^{-\frac{c_{14}|x|^2}{s}}d\left(\frac{|x|}{\sqrt{s}}\right).\end{aligned}$$

Thus

$$\frac{c_{15}}{t}e^{-\frac{c_{16}\rho(x,y)^2}{t}} \le \int_0^{t/2}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds \le \frac{c_{17}}{t}e^{-\frac{c_{18}\rho(x,y)^2}{t}}. \tag{4.2.28}$$

For the second term on the right hand side of (4.2.27) we have

$$\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{19}}{t}e^{-\frac{c_{20}x^2}{t}}\int_1^{t/2}\frac{1}{s}e^{-\frac{c_{21}|y|_\rho^2}{s}}ds\le\int_{t/2}^{t-1}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds$$
$$\le\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{22}}{t}e^{-\frac{c_{23}x^2}{t}}\int_1^{t/2}\frac{1}{s}e^{-\frac{c_{24}|y|_\rho^2}{s}}ds.$$

By making a change of variable $u=|y|/\sqrt{s}$ we have

$$\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{25}}{t}e^{-\frac{c_{26}x^2}{t}}\int_{u=\frac{|y|}{\sqrt{t/2}}}^{|y|}\frac{1}{u}e^{-c_{27}u^2}du\le\int_{t/2}^{t-1}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds$$
$$\le\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{28}}{t}e^{-\frac{c_{29}x^2}{t}}\int_{u=\frac{|y|}{\sqrt{t/2}}}^{|y|}\frac{1}{u}e^{-c_{30}u^2}du.$$

Noticing $|y|_\rho<1$ one has

$$\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{25}}{t}e^{-\frac{c_{26}x^2}{t}}\log\sqrt{t}\le\int_{t/2}^{t-1}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{t-s}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds\le\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{28}}{t}e^{-\frac{c_{29}x^2}{t}}\log\sqrt{t}.\tag{4.2.29}$$

For the third term on the right hand side of (4.2.27) we have

$$\int_{t-1}^{t}\frac{x}{\sqrt{s^3}}e^{-\frac{c_1x^2}{s}}\frac{1}{\sqrt{t-s}}e^{-\frac{c_2|y|_\rho^2}{t-s}}ds\le\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{30}}{t}e^{-\frac{c_{31}x^2}{t}}\int_0^1\frac{1}{\sqrt{s}}e^{-\frac{c_{32}|y|_\rho^2}{s}}ds$$
$$=\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{30}}{t}e^{-\frac{c_{31}x^2}{t}}.\tag{4.2.30}$$

It thus follows from (4.2.28), (4.2.29) and (4.2.30) that

$$\frac{c_{32}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\log t\right]e^{-\frac{c_{33}\rho(x,y)^2}{t}}\le p(t,x,y)\le\frac{c_{27}}{t}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\log t\right]e^{-\frac{c_{28}\rho(x,y)^2}{t}},$$
$$x\in\mathbb{R}_+,y\in D_0,|y|_\rho<1.$$

Case 2. $|y|_\rho>1$, in which case

$$\frac{c_1}{t}e^{-\frac{c_2|y|_\rho^2}{t}}\le p(t,a^*,y)\le\frac{c_3}{t}e^{-\frac{c_4|y|_\rho^2}{t}},\quad t>0.$$

Again by Markov property it holds

$$p(t,x,y)\asymp\int_0^t\mathbb{P}^x(T_{a^*}\in ds)p(t-s,a^*,y)$$
$$\asymp\int_0^{t/2}\frac{x}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}\frac{1}{t-s}e^{-\frac{c_6|y|_\rho^2}{t-s}}ds+\int_{t/2}^{t}\frac{x}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}\frac{1}{t-s}e^{-\frac{c_6|y|_\rho^2}{t-s}}ds.\tag{4.2.31}$$

For the first term on the right hand side of (4.2.31) we have

$$\frac{c_7}{t}e^{-\frac{c_8|y|_\rho^2}{t}}\int_0^{t/2}\frac{|x|}{\sqrt{s^3}}e^{-\frac{c_9x^2}{s}}ds \le \int_0^{t/2}\frac{x}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}\frac{1}{t-s}e^{-\frac{c_6|y|_\rho^2}{t-s}}ds$$
$$\le \frac{c_{10}}{t}e^{-\frac{c_{11}|y|_\rho^2}{t}}\int_0^{t/2}\frac{|x|}{\sqrt{s^3}}e^{-\frac{c_{12}x^2}{s}}ds.$$

i.e.,

$$\frac{c_7}{t}e^{-\frac{c_8|y|_\rho^2}{t}}\int_{s=0}^{t/2}e^{-\frac{c_9x^2}{s}}d\left(\frac{x}{\sqrt{s}}\right) \le \int_0^{t/2}\frac{x}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}\frac{1}{t-s}e^{-\frac{c_6|y|_\rho^2}{t-s}}ds$$
$$\le \frac{c_{10}}{t}e^{-\frac{c_{11}|y|_\rho^2}{t}}\int_{s=0}^{t/2}e^{-\frac{c_{12}x^2}{s}}d\left(\frac{x}{\sqrt{s}}\right).$$

i.e.,

$$\frac{c_7}{t}e^{-\frac{c_8\rho(x,y)^2}{t}} \le \int_0^{t/2}\frac{x}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}\frac{1}{t-s}e^{-\frac{c_6|y|_\rho^2}{t-s}}ds \le \frac{c_9}{t}e^{-\frac{c_{10}\rho(x,y)^2}{t}}. \tag{4.2.32}$$

For the second term on the right hand side of (4.2.31) it holds

$$\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{11}}{t}e^{-\frac{c_{12}x^2}{t}}\int_0^{t/2}\frac{1}{s}e^{-\frac{c_{13}|y|_\rho^2}{s}}ds \le \int_{t/2}^{t}\frac{x}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}\frac{1}{t-s}e^{-\frac{c_6|y|_\rho^2}{t-s}}ds$$
$$\le \left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{14}}{t}e^{-\frac{c_{15}x^2}{t}}\int_0^{t/2}\frac{1}{s}e^{-\frac{c_{16}|y|_\rho^2}{s}}ds.$$

By making a change of variable that $u = |y|/\sqrt{s}$ we have

$$\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{11}}{t}e^{-\frac{c_{12}x^2}{t}}\int_{u=\frac{|y|_\rho}{\sqrt{t/2}}}^{\infty}\frac{1}{u}e^{-c_{13}u^2}du \le \int_{t/2}^{t}\frac{x}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}\frac{1}{t-s}e^{-\frac{c_6|y|_\rho^2}{t-s}}ds$$
$$\le \left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{14}}{t}e^{-\frac{c_{15}x^2}{t}}\int_{u=\frac{|y|_\rho}{\sqrt{t/2}}}^{\infty}\frac{1}{u}e^{-c_{16}u^2}du.$$

By considering $|y|_\rho < \sqrt{t}$ and $|y|_\rho \ge \sqrt{t}$ separately one can see

$$\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{11}}{t}e^{-\frac{c_{12}x^2}{t}}\left(1+\log\left(1+\frac{\sqrt{t}}{|y|_\rho}\right)\right)e^{-\frac{c_{13}|y|_\rho^2}{t}} \le \int_{t/2}^{t}\frac{x}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}\frac{1}{t-s}e^{-\frac{c_6|y|_\rho^2}{t-s}}ds$$
$$\le \left(1\wedge\frac{|x|}{\sqrt{t}}\right)\frac{c_{14}}{t}e^{-\frac{c_{15}x^2}{t}}\left(1+\log\left(1+\frac{\sqrt{t}}{|y|_\rho}\right)\right)e^{-\frac{c_{16}|y|_\rho^2}{t}} \tag{4.2.33}$$

By replacing the right hand side of (4.2.31) with (4.2.32) and (4.2.33) we have shown that when $y \in D_0,\ |y|_\rho > 1$, it holds

$$\frac{c_{17}}{t}e^{-\frac{c_{18}\rho(x,y)^2}{t}}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1+\log\left(1+\frac{\sqrt{t}}{|y|_\rho}\right)\right)\right] \le p(t,x,y)$$
$$\le \frac{c_{19}}{t}e^{-\frac{c_{20}\rho(x,y)^2}{t}}\left[1+\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1+\log\left(1+\frac{\sqrt{t}}{|y|_\rho}\right)\right)\right].$$

□

To show the two-sided estimate for arbitrary $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$, we first prove the following lemma.

Lemma 4.2.5. *For any constant $c > 0$, it holds*

$$\int_1^t \frac{1}{s} + \frac{\log s}{s}\left(1 \wedge \frac{|y|}{\sqrt{s}}\right) e^{-\frac{c|y|^2}{s}} ds \asymp \log t, \quad y \in \mathbb{R}_+,\ |y| < \sqrt{t}.$$

Proof. It is obvious that

$$\int_1^t \frac{1}{s} + \frac{\log s}{s}\left(1 \wedge \frac{|y|}{\sqrt{s}}\right) e^{-\frac{c|y|^2}{s}} ds \geq \log t.$$

To see the other direction, we have

$$\begin{aligned}
\int_1^t \frac{1}{s} ds + \int_1^t \frac{\log s}{s}\left(1 \wedge \frac{|y|}{\sqrt{s}}\right) e^{-\frac{c|y|^2}{s}} ds &\leq \log t + \int_1^t \frac{\log s}{s}\frac{|y|}{\sqrt{s}} e^{-\frac{c|y|^2}{s}} ds \\
&\leq \log t + \log t \int_1^t \frac{|y|}{\sqrt{s^3}} e^{-\frac{c|y|^2}{s}} ds \\
&\asymp \log t + \log t \int_{s=1}^t e^{-\frac{c|y|^2}{s}} d\left(\frac{|y|}{\sqrt{s}}\right) \\
&\leq \log t + \log t \cdot e^{-\frac{c|y|^2}{t}} \\
&\asymp \log t.
\end{aligned}$$

The proof is thus complete since we've showed the inequality in both directions. □

Theorem 4.2.6. *There exist constants $C_i > 0$, $29 \leq i \leq 34$, such that the following estimate holds for $(t, x, y) \in (1, \infty) \times \mathbb{R}_+ \times \mathbb{R}_+$*

$$\begin{aligned}
&\frac{C_{29}}{\sqrt{t}}\left(1 \wedge \frac{|x|}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|}{\sqrt{t}}\right) e^{-\frac{C_{30}|x-y|^2}{t}} + C_{29}\left[\frac{1}{t} + \frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right] e^{-\frac{C_{31}(|x|^2+|y|^2)}{t}} \leq p(t, x, y) \\
&\leq \frac{C_{32}}{\sqrt{t}}\left(1 \wedge \frac{|x|}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|}{\sqrt{t}}\right) e^{-\frac{C_{33}|x-y|^2}{t}} + C_{32}\left[\frac{1}{t} + \frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right] e^{-\frac{C_{34}(|x|^2+|y|^2)}{t}},
\end{aligned} \tag{4.2.34}$$

Proof. We first notice that

$$p(t, x, y) = p_{\mathbb{R}_+}(t, x, y) + \int_0^t \mathbb{P}^x(T_{a^*} \in ds) p(t-s, a^*, y). \tag{4.2.35}$$

Thus again we prove this theorem by considering several different cases.

Case 1. $|x| > \sqrt{t}$, $|y| > 2\sqrt{t}$. In this case we actually have

$$p(t, x, y) \geq p_{\mathbb{R}_+}(t, x, y) \geq \frac{c_1}{\sqrt{t}} e^{-\frac{c_2|x-y|^2}{t}}.$$

Therefore by the upper bound estimate due to Nash's inequality, it holds

$$\frac{c_1}{\sqrt{t}}e^{-\frac{c_2|x-y|^2}{t}} \le p(t,x,y) \le \frac{c_3}{\sqrt{t}}e^{-\frac{c_4|x-y|^2}{t}}.$$

Case 2. $|x| < \sqrt{t}$ or $|y| < \sqrt{t}$ holds. Without loss of generality we assume $|y| < 2\sqrt{t}$. By Corollary 4.2.2 we have

$$\begin{aligned} p(t,x,y) &= \int_0^t \frac{|x|}{\sqrt{s^3}}e^{-\frac{c_5x^2}{s}}p(t-s,a^*,y)dy + p_{\mathbb{R}_+}(t,x,y) \\ &\asymp p_{\mathbb{R}_+}(t,x,y) + \int_0^{t/2} \frac{|x|}{\sqrt{s^3}}e^{-\frac{c_6x^2}{s}}\left[\frac{1}{t-s} + \frac{\log(t-s)}{t-s}\left(1\wedge\frac{|y|}{\sqrt{t-s}}\right)\right]e^{-\frac{c_7y^2}{t-s}}ds \\ &+ \int_{t/2}^{t-1} \frac{|x|}{\sqrt{s^3}}e^{-\frac{c_6x^2}{s}}\left[\frac{1}{t-s} + \frac{\log(t-s)}{t-s}\left(1\wedge\frac{|y|}{\sqrt{t-s}}\right)\right]e^{-\frac{c_7y^2}{t-s}}ds \\ &+ \int_{t-1}^{t} \frac{|x|}{\sqrt{s^3}}e^{-\frac{c_8x^2}{s}}\frac{1}{\sqrt{t-s}}e^{-\frac{c_9y^2}{t-s}}ds \end{aligned} \tag{4.2.36}$$

For the second term on the right hand side of (4.2.36) we have

$$\begin{aligned} &\left[\frac{1}{t} + \frac{\log t}{t}\left(1\wedge\frac{|y|}{\sqrt{t}}\right)\right]e^{-\frac{c_{10}y^2}{t}}\int_{s=0}^{t/2} e^{-\frac{c_{11}|x|^2}{s}}d\left(\frac{|x|}{\sqrt{s}}\right) \\ &\qquad \le \int_0^{t/2} \frac{|x|}{\sqrt{s^3}}e^{-\frac{c_6x^2}{s}}\left[\frac{1}{t-s} + \frac{\log(t-s)}{t-s}\left(1\wedge\frac{|y|}{\sqrt{t-s}}\right)\right]e^{-\frac{c_7y^2}{t-s}}ds \\ &\qquad \le \left[\frac{1}{t} + \frac{\log t}{t}\left(1\wedge\frac{|y|}{\sqrt{t}}\right)\right]e^{-\frac{c_{12}y^2}{t}}\int_{s=0}^{t/2} e^{-\frac{c_{13}|x|^2}{s}}d\left(\frac{|x|}{\sqrt{s}}\right), \end{aligned}$$

i.e.,

$$\begin{aligned} \left[\frac{1}{t} + \frac{|y|\log t}{\sqrt{t^3}}\right]e^{-\frac{c_{14}(|x|^2+|y|^2)}{t}} &\le \int_0^{t/2} \frac{|x|}{\sqrt{s^3}}e^{-\frac{c_6x^2}{s}}\left[\frac{1}{t-s} + \frac{\log(t-s)}{t-s}\left(1\wedge\frac{|y|}{\sqrt{t-s}}\right)\right]e^{-\frac{c_7y^2}{t-s}}ds \\ &\le \left[\frac{1}{t} + \frac{|y|\log t}{\sqrt{t^3}}\right]e^{-\frac{c_{15}(|x|^2+|y|^2)}{t}}. \end{aligned} \tag{4.2.37}$$

For the third term on the right hand side of (4.2.36) we have

$$\begin{aligned} &\frac{|x|}{\sqrt{t^3}}e^{-\frac{c_{16}|x|^2}{t}}\int_1^{t/2}\left[\frac{1}{s} + \frac{\log s}{s}\left(1\wedge\frac{|y|}{\sqrt{s}}\right)\right]e^{-\frac{c_{17}y^2}{s}}ds \\ &\le \int_{t/2}^{t-1} \frac{|x|}{\sqrt{s^3}}e^{-\frac{c_6x^2}{s}}\left[\frac{1}{t-s} + \frac{\log(t-s)}{t-s}\left(1\wedge\frac{|y|}{\sqrt{t-s}}\right)\right]e^{-\frac{c_7y^2}{t-s}}ds \\ &\le \frac{|x|}{\sqrt{t^3}}e^{-\frac{c_{18}|x|^2}{t}}\int_1^{t/2}\left[\frac{1}{s} + \frac{\log s}{s}\left(1\wedge\frac{|y|}{\sqrt{s}}\right)\right]e^{-\frac{c_{19}y^2}{s}}ds. \end{aligned}$$

It thus follows from Lemma 4.2.5 that

$$c_{20}\frac{|x|\log t}{\sqrt{t^3}}e^{-\frac{c_{21}(|x|^2+|y|^2)}{t}} \le \int_{t/2}^{t-1}\frac{|x|}{\sqrt{s^3}}e^{-\frac{c_6x^2}{s}}\left[\frac{1}{t-s}+\frac{\log(t-s)}{t-s}\left(1\wedge\frac{|y|}{\sqrt{t-s}}\right)\right]e^{-\frac{c_7y^2}{t-s}}ds$$
$$\le c_{22}\frac{|x|\log t}{\sqrt{t^3}}e^{-\frac{c_{23}(|x|^2+|y|^2)}{t}}. \tag{4.2.38}$$

For the fourth term on the right hand side of (4.2.36) we have

$$\int_{t-1}^{t}\frac{|x|}{\sqrt{s^3}}e^{-\frac{c_8x^2}{s}}\frac{1}{\sqrt{t-s}}e^{-\frac{c_9y^2}{t-s}}ds \le \frac{|x|}{\sqrt{t^3}}e^{-\frac{c_{10}|x|^2}{t}}\int_{t-1}^{t}\frac{1}{\sqrt{t-s}}ds$$
$$\asymp \frac{|x|}{\sqrt{t^3}}e^{-\frac{c_{10}|x|^2}{t}} \le \frac{c_{11}}{t}e^{-\frac{c_{12}|x|^2}{t}} \asymp \frac{1}{t}e^{-\frac{c_{12}(|x|^2+|y|^2)}{t}}, \tag{4.2.39}$$

where the last " $\asymp$ " is due to the assumption that $|y|_\rho < 2\sqrt{t}$. Combining (4.2.37), (4.2.38) and (4.2.39) we have for the right hand side of (4.2.36) it holds for Case 2,

$$p(t,x,y) \asymp p_{\mathbb{R}_+}(t,x,y) + \left[\frac{1}{t}+\frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right]e^{-\frac{c_{13}(|x|^2+|y|^2)}{t}}, \text{ when } |x|<\sqrt{t} \text{ or } |y|<\sqrt{t}.$$

Now recall

$$\frac{c_{14}}{\sqrt{t}}\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1\wedge\frac{|y|}{\sqrt{t}}\right)e^{-\frac{c_{15}|x-y|^2}{t}} \le p_{\mathbb{R}_+}(t,x,y) \le \frac{c_{16}}{\sqrt{t}}\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1\wedge\frac{|y|}{\sqrt{t}}\right)e^{-\frac{c_{17}|x-y|^2}{t}},$$

we can summarize both cases as

$$\frac{c_{18}}{\sqrt{t}}\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1\wedge\frac{|y|}{\sqrt{t}}\right)e^{-\frac{c_{19}|x-y|^2}{t}} + c_{18}\left[\frac{1}{t}+\frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right]e^{-\frac{c_{20}(|x|^2+|y|^2)}{t}} \le p(t,x,y)$$
$$\le \frac{c_{21}}{\sqrt{t}}\left(1\wedge\frac{|x|}{\sqrt{t}}\right)\left(1\wedge\frac{|y|}{\sqrt{t}}\right)e^{-\frac{c_{22}|x-y|^2}{t}} + c_{21}\left[\frac{1}{t}+\frac{\log t}{t}\left(\frac{|x|+|y|}{\sqrt{t}}\right)\right]e^{-\frac{c_{23}(|x|^2+|y|^2)}{t}},$$

for all $(t,x,y)\in(1,\infty)\times\mathbb{R}_+\times\mathbb{R}_+$. □

Chapter 5

HÖLDER-CONTINUITY OF PARABOLIC FUNCTIONS

To show the Hölder-continuity of parabolic functions with respect to X, we begin with proving the following lemma.

Lemma 5.0.7. *For any ball $B_\rho(x,R) \subset E$, there exists some $C_1 > 0$ such that for any $y, z \in B(x, R/2)$, it holds*

$$p_{B(x,R)}(t,y,z) \geq C_1 p(t,y,z), \quad t < 1.$$

Proof. For notation convenience, we denote $B(x,R)$ by B. It suffices to show $\overline{p}_B(t,y,z)$ is sufficiently small in comparison with $p(t,y,z)$. Indeed it follows from the small time heat kernel estimate that

$$\begin{aligned}
\overline{p}_B(t,y,z) &= \mathbb{E}^y \int_{s=0}^t \mathbf{1}_{\{\tau_B \in ds\}} p(t-s, X_{\tau_B}, z) \\
&\leq \mathbb{E}^y \int_{s=0}^t \mathbf{1}_{\{\tau_B \in ds\}} \frac{c_1}{t-s} e^{-\frac{c_2 R^2}{t-s}} ds \\
&\leq \frac{c_1}{t} e^{-\frac{c_2 R^2}{t}} \int_{s=0}^t \mathbb{P}^y(\tau_B \in ds) \\
&= \frac{c_1}{t} e^{-\frac{c_2 R^2}{t}} \mathbb{P}^y(\tau_B < t) \leq c_1 e^{-\frac{c_2 R^2}{t}}.
\end{aligned}$$

Since it is proved when t is sufficiently small, $p(t,y,z) \geq \frac{c_3}{\sqrt{t}} e^{-\frac{c_4 \rho(y,z)^2}{t}} \geq \frac{c_5}{\sqrt{t}} e^{-\frac{c_6 R^2}{t}}$, it has to hold for some $c_7 > 0$

$$p_{B(x,R)}(t,y,z) \geq c_7 p(t,y,z), \quad t < 1.$$

$\square$

Let $Z_s = (V_s, X_s)$ be the space-time process of X where $V_s = V_0 + s$. We start with the following lemma which will be later used. In the remaining context of this section, $Q(t,x,R) := [t, t+R^2] \times B_\rho(x,R)$.

Lemma 5.0.8. *Fix $R_0 > 0$. There exists some constant $C_4 > 0$ such that for all $0 < R < R_0$, any $x_0 \in E$, any $v \in B_\rho(x_0, R/2)$ and any $A \subset Q(0, x_0, R/2)$ such that $\frac{|A|}{|Q(0,x_0,R/2)|} \geq \frac{1}{3}$,*

$$\mathbb{P}^{(0,v)}(T_A < \tau_R) \geq C_3, \tag{5.0.1}$$

where $\tau_R = \tau_{Q(0,x_0,R)}$.

Proof. We are going to estimate the expected time that the space-time process Z spends in A before exiting $Q(0, x_0, R)$. Let $X^{B(x_0,R)}$ denote the process X killed upon exiting the ball $B(x_0, R)$ and let $p^{B(x_0,R)}$ be its transition density. Let $A_s := \{x \in E : (s, x) \in A\}$. Then

$$\begin{aligned}\mathbb{E}^{(0,v)} \int_0^{\tau_R} \mathbf{1}_A (s, X_s)\, ds &= \mathbb{E}^{(0,v)} \int_0^{\tau_R} \mathbf{1}_A \left(s, X^{B(x_0,R)}\right) ds = \int_0^{R^2} \mathbb{P}^{(0,v)} \left(\left(s, X^{B(x_0,R)}\right) \in A\right) ds \\ &= \int_0^{R^2} \mathbb{P}^v \left(X_s^{B(x_0,R)} \in A_s\right) ds = \int_0^{R^2} \int_{A_s} p^{B(x_0,R)}(s, v, y) m_p(dy) ds.\end{aligned} \tag{5.0.2}$$

To bounded the right hand side of (5.0.2) from below, we consider two cases.

Case 1. $B(x_0, R) > R/6$, in which case $|A| > \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{6} R^3$. Since there exists some large positive integer K such that

$$\int_0^{R^2/K} \int_{A_s} 1 \cdot m_p(dy) ds \leq \int_{R^2/K}^{R^2/4} \int_{A_s} 1 \cdot m_p(dy) ds, \tag{5.0.3}$$

for the right hand side of (5.0.2) we have

$$\begin{aligned}\int_0^{R^2} \int_{A_s} p^{B(x_0,R)}(s, v, y) m_p(dy) ds &\geq \int_{R^2/K}^{R^2/4} \frac{c_1}{\sqrt{s}} e^{-\frac{c_2 \rho(y,v)^2}{s}} m_p(dy) ds \\ &\asymp \int_{R^2/K}^{R^2/4} \frac{c_1}{\sqrt{s}} m_p(dy) ds \asymp \frac{1}{R} \int_{R^2/K}^{R^2/4} 1 \cdot m_p(dy) ds \geq \frac{c_3}{R} |A|,\end{aligned}$$

where the last inequality is due to (5.0.3), and the first inequality is due to small time heat kernel estimate as well as Lemma 5.0.7.

Case 2. $B(x_0, R) \leq R/6$, in which case $x_0 \in D_0$, $\rho(x_0, a^*) \geq \frac{5R}{6}$; $v \in D_0$, $\rho(v, a^*) \geq R/3$; $\forall (s, y) \in A$, $y \in D_0$, $\rho(y, a^*) \geq R/3$. Thus there exists some $c > 0$ such that $|A| \geq cR^4$. Again there exists some large positive integer K such that

$$\int_0^{R^2/K} \int_{A_s} 1 \cdot m_p(dy) ds \leq \int_{R^2/K}^{R^2/4} \int_{A_s} 1 \cdot m_p(dy) ds.$$

Now for the right hand side of (5.0.2) we have

$$\int_0^{R^2}\int_{A_s} p^{B(x_0,R)}(s,v,y)m_p(dy)ds \geq \int_{R^2/K}^{R^2} \frac{c_5}{s} e^{-\frac{c_6\rho(y,v)^2}{s}} m_p(dz)ds$$
$$\geq \frac{c_5}{R}\int_{R^2/K}^{R^2}\int_{A_s} 1\cdot m_p(dy)ds \geq \frac{c_5|A|}{R^2},$$

where the first inequality is due to the fact that

$$p^B(s,y,v) \asymp p(s,y,v) \geq \frac{c_5}{s} e^{\frac{c_6\rho(y,z)^2}{s}}, (s,y,v) \in \left(\frac{R^2}{2}, R^2\right) \times D_0 \times D_0, |y|_\rho > \frac{R}{3}, |v|_\rho > \frac{R}{2}.$$

On the other hand,

$$\mathbb{E}^{(0,v)} \int_0^{\tau_R} \mathbf{1}_A(s,X_s)ds = \int_0^{\infty} \mathbb{P}^{(0,v)} \left(\int_0^{\tau_R} \mathbf{1}_A(s,X_s)ds > u\right) du$$
$$= \int_0^{R^2} \mathbb{P}^{(0,v)} \left(\int_0^{\tau_R} \mathbf{1}_A(s,X_s)ds > u\right) du$$
$$\leq \int_0^{R^2} \mathbb{P}^{(0,v)} \left(\int_0^{\tau_R} \mathbf{1}_A(s,X_s)ds > 0\right) du$$
$$\leq R^2\mathbb{P}^{(0,v)}(T_A < \tau_R).$$

The lemma has been proved by combining the results above. □

Theorem 5.0.9. *Fix $R_0 > 0$, there is a constant $C_5 = C(R_0) > 0$ such that for every $0 < R \leq R_0$, $x_0 \in E$, and every bounded parabolic function q in $Q(0, x_0, 2R)$, it holds that*

$$|q(s,x) - q(t,y)| \leq C\|q\|_{\infty,R} R^{-\beta} \left(|t-s|^{1/2} + \rho(x,y)\right)^{\beta}, \tag{5.0.4}$$

$\forall (s,x),(t,y) \in Q(0,x_0,R)$, where $\|q\|_{\infty,R} := \sup_{(t,y)\in 4R^2\times E} |q(t,y)|$.

Proof. With loss of generality, assume $0 \leq q(s) \leq \|q\|_{\infty,R} = 1$. First assume $x_0 = a^*$. Let $\eta = 1 - c_1/4$, $\rho = \frac{1}{2} < \eta$. Note that for every $(t,x) \in Q(0,a^*,R)$, q is parabolic in $Q(t,x,R) \subset Q(0,a^*,2R)$. We will show that $\sup_{Q(t,x,\rho^k R)} |q| - \inf_{Q(t,x,\rho^k R)} |q| \leq \eta^k$, all k. For notation convenience, we denote $Q(t,x,\rho^k R)$ to be Q_k. Define $a_i = \inf_{Q_i} q$, $b_i = \sup_{Q_i} q$. Clearly, $b_i - a_i \leq 1 \leq \eta$, for all $i \leq 0$. Now suppose $b_i - a_i \leq \eta^i$ for all $i \leq k$ and we are going to show that $b_{k+1} - a_{k+1} \leq \eta^{k+1}$. Observe that $Q_{k+1} \subset Q_k$ and so $a_k \leq q \leq b_k$ on Q_{k+1}. Define $A' := \{z \in Q_{k+1}, q(z) \leq (a_k + b_k)/2\}$. We may suppose $\frac{|A'|}{|Q_{k+1}|} \geq 1/2$, for if not, we use $1-q$ instead of q. Let A be a compact subset of A' such that $\frac{|A|}{|Q_{k+1}|} \geq 1/3$. For any given $\epsilon > 0$, pick $z_1, z_2 \in Q_{k+1}$ such that $q(z_1) \geq b_{k+1} - \epsilon$ and $q(z_1) \leq a_{k+1} + \epsilon$.

$$\begin{aligned}
b_{k+1} - a_{k+1} - 2\epsilon &\leq q(z_1) - q(z_2) \\
&= \mathbb{E}^{z_1}\left[q(Z_{T_A \wedge \tau_{k+1}}) - q(z_2)\right] \\
&= \mathbb{E}^{z_1}\left[q(Z_{T_A}) - q(z_2); T_A < \tau_{k+1}\right] + \mathbb{E}^{z_1}\left[q(Z_{\tau_{k+1}}) - q(z_2); T_A > \tau_{k+1}\right] \\
&\leq \left(\frac{a_k + b_k}{2} - a_k\right)\mathbb{P}^{z_1}(T_A < \tau_{k+1}) + (b_k - a_k)\mathbb{P}^{z_1}(T_A > \tau_{k+1}) \\
&= (b_k - a_k)\left[1 - \mathbb{P}^{z_1}(T_A < \tau_{k+1})/2\right] \\
&\leq \eta^k(1 - c_1/2) \\
&\leq \eta^{k+1}.
\end{aligned}$$

Since ϵ is arbitrary, the claim is proved. For $z = (s,x)$ and $w = (t,y)$ in $Q(0, a^*, R)$ with $s \leq t$, let k be the smallest integer such that $|z - w| := \left(|t-s|^{1/2} + \rho(x,y)\right) \leq \rho^k R$. Thus $\log(|z-w|/R) \geq (k+1)\log\rho$, for $w \in Q(s, x, \rho^k R)$, and

$$|q(z) - q(w)| \leq \eta^k = e^{k\log\eta} \leq c_2\left(\frac{|z-w|}{R}\right)^{\log\eta/\log\rho},$$

which proves (5.0.4) for the case that $x_0 = a^*$.

In general, for any $x_0 \in E^*$, there are two cases:

Case 1. $|x|_\rho < R/4$. For this case, q is parabolic in $Q(0, a^*, 3R/4) \subset Q(0, x_0, R)$, therefore (5.0.4) holds on $Q(0, a^*, 3R/8)$which contains $Q(0, x_0, R/8)$, i.e.,

$$|q(t,x) - q(s,y)| \leq C(R_0) \sup_{Q(0,a^*,3R/4)} |q| R^{-\beta}\left(|t-s|^{1/2} - \rho(x,y)\right)^\beta, \ \forall (t,x), (s,y) \in Q(0, a^*, 3R/8).$$

Therefore

$$|q(t,x) - q(s,y)| \leq C(R_0) \sup_{Q(0,x_0,R)} |q| R^{-\beta}\left(|t-s|^{1/2} - \rho(x,y)\right)^\beta, \quad \forall (t,x), (s,y) \in Q(0, x_0, R/8).$$

Case 2. $|x|_\rho \geq R/4$. Since $Q(0, x_0, R/4)$ is disjoint from the origin, it holds that $\forall (t,x), (s,y) \in Q(0, x_0, R/8)$,

$$\begin{aligned}
|q(t,x) - q(s,y)| &\leq C(R_0) \sup_{Q(0,x_0,R/4)} |q| R^{-\beta}\left(|t-s|^{1/2} - \rho(x,y)\right)^\beta \\
&\leq C(R_0) \sup_{Q(0,x_0,R)} |q| R^{-\beta}\left(|t-s|^{1/2} - \rho(x,y)\right)^\beta.
\end{aligned}$$

The proof is complete. $\square$

Remark 5.0.10. *Parabolic Harnack inequality does not hold for the process X. An counterexample is as follows: For s arbitrarily small, fix $y \in \mathbb{R}^2$ such that $|y|_\rho = \sqrt{s}$. Set $Q_+ := (3s/2, 2s) \times B_\rho(y, 2\sqrt{s})$ and $Q_- := (s/2, s) \times B(y, 2\sqrt{s})$. Pick $u(t,x) := p(t,x,y)$. Since we are assuming s to be small, it holds that $\sup_{Q_+} u = \frac{1}{\sqrt{s}} + \frac{1}{s}$, but $\inf_{Q_-} u = \frac{1}{\sqrt{s}}$.*

Chapter 6

OTHER EXAMPLES OF BMVD

6.1 Planary BMVD with Multiple Straight Lines

In this section we study the BMVD denoted by X constructed on a state space which can be viewed as a plane $\mathbb{R}^2$ with two straight half lines L_1 and L_2 vertically attached on it. To be more precise, let B_1 and B_2 be two disjoint closed discs on $\mathbb{R}^2$. Let a_1^* and a_2^* be the two darning points on $\mathbb{R}^2$, where a_i^* is obtained by collapsing $B_i \subset \mathbb{R}^2$ to a singleton, $i = 1, 2$. We set $a_i^* = L_i \cap \mathbb{R}^2, i = 1, 2$. Let $D_0 := \mathbb{R}^2 \backslash (B_1 \sqcup B_2)$, then the state space of X is thus $E := L_1 \sqcup L_2 \sqcup D_0 \sqcup \{a_1^*, a_2^*\}$. See Figure 6.1. Fix $p > 0$. Let m_p denote the measure on E whose restriction on $L_i, i = 1, 2$ or D_0 is the Lebesgue measure times p and 1, respectively, and $m_p(\{a_i^*\}) = 0$, $i = 1, 2$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X on $L^2(E, m_p)$ is given by

$$\begin{cases} \mathcal{F} = \left\{f : f|_{\mathbb{R}^2} \in W^{1,2}(\mathbb{R}^2), f \text{ is const. } \mathcal{E}\text{-q.e. on } B_i, f|_{L_i} \in W^{1,2}(\mathbb{R}), \ f|_{B_i} = f|_{L_i}(a_i^*), i = 1, 2. \right\} \\ \mathcal{E}(f, g) = \mathbf{D}(f, g) := \dfrac{1}{2} \displaystyle\int_{D_0} \frac{\partial f(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_i} m_p(dx) + \sum_{i=1,2} \frac{1}{2} \int_{L_i} f'(x) g'(x) m_p(dx). \end{cases}$$

For the remaining of this section, without loss of generality we assume that $|a_1^* - a_2^*|_\rho = 4$.

The goal of this section is to give a two-sided small time heat kernel estimate. Unless otherwise stated, it is always assumed in this section that $t < 1$. First, we notice that by exactly the same argument in Section 3, the same results as from Proposition 3.1.1 to Proposition 3.2.1 still hold for this case. Thus we state the off-diagonal upper bound estimate as follows without repeating the proof.

Proposition 6.1.1. *There exist $C_1, C_2 > 0$ such that*

$$p(t, x, y) \le C_1 \left(\frac{1}{t} + \frac{1}{t^{1/2}} \right) e^{-\frac{C_2 \rho(x,y)^2}{t}}, \qquad \forall x, y \in E, t \in (0, 1].$$

The Proposition 6.1.2 stated as follows will be used repeatedly in the remaining context.

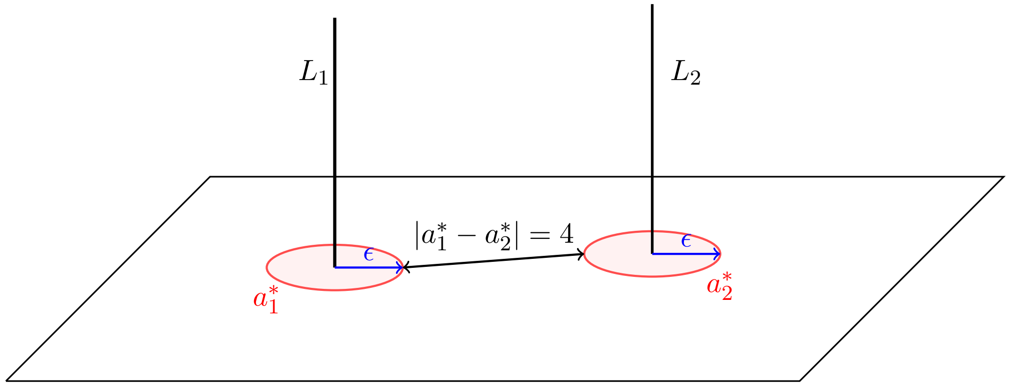

Figure 6.1: BMVD with multiple lines

Proposition 6.1.2. *There exist constant $C_3, C_4 > 0$ such that for any $(t,x,y) \in (0,1] \times L_i \times L_i$, $i = 1$ or 2, it holds*

$$\overline{p}^{(X)}_{L_i \cup B_\rho(a_i^*,2)}(t,x,y) \le e^{-\frac{C_3}{t}} \cdot e^{-\frac{C_4 \rho(x,y)^2}{t}}, \quad t \in (0,1].$$

Proof.

$$\begin{aligned}
\overline{p}^{(X)}_{L_i \cup B_\rho(a_i^*,2)}(t,x,y) &\le \int_{s=0}^t \mathbb{P}^x \left(\tau_{L_i \sqcup \left(D_0 \cap B_\rho(a_i^*,2)\right)} \in ds\right) \sup_{z \in \partial(D_0 \cap B_\rho(a_i^*,2))} p^{(X)}(t-s,z,y) \\
&\le \int_{s=0}^t \mathbb{P}^x \left(\tau_{L_i \cup B_\rho(a_i^*,2)} \in ds\right) \frac{1}{t-s} e^{-\frac{c_1 |y|_\rho^2 + c_2}{t-s}} \\
&\asymp \int_{s=0}^t \mathbb{P}^x \left(\tau_{L_i \cup B_\rho(a_i^*,2)} \in ds\right) e^{-\frac{c_1 |y|_\rho^2 + c_2}{t-s}} \\
&\le e^{-\frac{c_1 |y|_\rho^2 + c_2}{t}} \int_{s=0}^t \mathbb{P}^x (\tau_{L_i \cup B_\rho(a_i^*,2)} \in ds) \\
&= e^{-\frac{c_1 |y|_\rho^2 + c_2}{t}} \cdot \mathbb{P}^x (\tau_{L_i \cup B_\rho(a_i^*,2)} \le t) \\
&\le e^{-\frac{c_2}{t}} \cdot e^{-\frac{c_1 (|x|^2 + |y|^2)}{t}} \\
&\le e^{-\frac{c_2}{t}} \cdot e^{-\frac{c_1 \rho(x,y)^2}{t}},
\end{aligned}$$

∎

The main result of this section is the following.

Theorem 6.1.3. *There exist constants $C_i > 0$, $5 \le i \le 18$, such that for all $t \in [0,1]$, the following estimates hold:*

When $\rho(x, a_i^*) < 1$, $\rho(y, a_i^*) < 1$, $x, y \in D_0, i = 1$ or 2,

$$\frac{C_5}{\sqrt{t}} e^{-\frac{C_6 \rho(x,y)^2}{t}} + \frac{C_5}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_7|x-y|_e^2}{t}} \le p(t,x,y)$$
$$\le \frac{C_8}{\sqrt{t}} e^{-\frac{C_9 \rho(x,y)^2}{t}} + \frac{C_8}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{10}|x-y|_e^2}{t}};$$

when $x \in L_j$, $y \in D_0$, $\left|y - a_j^*\right|_\rho < 1$, $j = 1$ or 2,

$$\frac{C_{11}}{\sqrt{t}} e^{-\frac{C_{12}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{13}}{\sqrt{t}} e^{-\frac{C_{12}\rho(x,y)^2}{t}};$$

otherwise

$$\frac{C_{15}}{t} e^{-\frac{C_{16}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{17}}{t} e^{-\frac{C_{18}\rho(x,y)^2}{t}},$$

where $|\cdot|_e$ or $|\cdot|_\rho$ denote the Euclidean metric or the geodesic metric respectively.

Proof. We prove the theorem by considering the different cases depending on the locations of x and y. In the following context we use $d(\cdot,\cdot)$ to denote the geodesic distance from a point to a susbet of E.

Case 1. $x, y \in D_0$, $\rho(x, \{a_1^*, a_2^*\}) > 1$, $\rho(y, \{a_1^*, a_2^*\}) > 1$. In this case, by considering the killed process upon hitting $B_\rho(a_1^*, 1) \sqcup B_\rho(a_2^*, 1)$ and the upper bound estimate, it holds

$$\begin{aligned} p(t,x,y) &\asymp \overline{p}_{E\setminus(B_\rho(a_1^*,1)\sqcup B_\rho(a_2^*,1))}(t,x,y) + p_{E\setminus(B_\rho(a_1^*,1)\sqcup B_\rho(a_2^*,1))}(t,x,y) \\ &\ge p_{E\setminus(B_\rho(a_1^*,1)\sqcup B_\rho(a_2^*,1))}(t,x,y) \\ &= p^{2-BM}_{\mathbb{R}^2\setminus(B_1\cup B_2)}(t,x,y) \\ &\ge \frac{c_1}{t} e^{-\frac{c_2|x-y|_e^2}{t}} \ge \frac{c_1}{t} e^{-\frac{c_3\rho(x,y)^2}{t}}. \end{aligned}$$

The last "$\ge$" is due to the fact that when $\rho(x, a_1^* \sqcup a_2^*) > 1$, $\rho(y, a_1^* \sqcup a_2^*) > 1$, $|x-y|_e \asymp \rho(x,y)$ by elementary geometry. On the other hand by Nash's inequality we have

$$p(t,x,y) \le \frac{c_4}{t} e^{-\frac{c_5\rho(x,y)^2}{t}}, \quad t < 1.$$

It thus readily follows in this case

$$\frac{c_1}{t} e^{-\frac{c_3\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{c_4}{t} e^{-\frac{c_5\rho(x,y)^2}{t}}.$$

Case 2. $x, y \in D_0$, $\rho(x, a_i^*) < 1$, $\rho\left(y, \{a_1^*, a_2^*\}\right) > 1$. Without loss of generality, we assume $\rho(x, a_1^*) < 1$. By considering the hitting time of $\{a_2^*\}$ we have

$$p(t,x,y) = \overline{p}_{E\setminus\{a_2^*\}}(t,x,y) + p_{E\setminus\{a_2^*\}}(t,x,y) \geq p_{E\setminus\{a_2^*\}}(t,x,y), \tag{6.1.1}$$

To get an estimate of (6.1.1), we replace the darning point a_2^* with the regular Euclidean disc B_2, which is actually what we have studied in Section 4. we let $p^{(s)}(t,x,y)$ be the transition density of BMVD whose state space is $\mathbb{R}_+ \cup \{a_1^*\} \cup (\mathbb{R}^2\setminus B_1)$, which is just the process we have studied in the previous few sections. It has already been showed in Section 4 that

$$\frac{c_6}{t}e^{-\frac{c_7\rho(x,y)^2}{t}} \leq p^{(s)}(t,x,y) \leq \frac{c_8}{t}e^{-\frac{c_9\rho(x,y)^2}{t}}, \; x,y \in D_0, \rho(x,a_1^*) < 1, \rho(y,a_1^*) > 1. \tag{6.1.2}$$

Since

$$p^{(s)}(t,x,y) = p^{(s)}_{E\setminus B_2}(t,x,y) + \overline{p}^{(s)}_{E\setminus B_2}(t,x,y), \tag{6.1.3}$$

we now need to show the second term on the right hand side of (6.1.3) is sufficiently small in comparison with the first term. Let the superscript "(s)" denote the probability or the density function of the process whose state space only involves a single darning point on $\mathbb{R}^2$. It holds

$$\begin{aligned}
\overline{p}^{(s)}_{E\setminus B_2}(t,x,y) &= \int_{s=0}^{t} \mathbb{P}^{(s),y}(T_{B_2} \in ds) p^{(s)}(t, X^{(s)}_{T_{B_2}}, x) \\
&\leq \sup_{z\in\partial B_2} \int_{s=0}^{t} \mathbb{P}^{(s),y}(T_{B_2} \in ds) p^{(s)}(t,z,x) \\
&\leq \int_{s=0}^{t} \mathbb{P}^{(s),y}(T_{B_2} \in ds) \frac{1}{t-s} e^{-\frac{c_{10}\rho(x,\partial B_2)}{t-s}} \\
&\leq e^{-\frac{c_{10}\rho(x,\partial B_2)}{t}} \cdot \mathbb{P}^{(s),y}(T_{B_2} < t) \\
&\leq e^{-\frac{c_{10}\rho(x,\partial B_2)}{t}} \cdot \mathbb{P}^{(s),y}(T_{B_2\cup\{a_1^*\}} < t) \\
&\leq \exp\left(-\frac{c_{10}\rho(x,\partial B_2)}{t}\right) \cdot \exp\left(-\frac{c_{11}\rho(y,\partial B_2 \cup \{a_1^*\})}{t}\right) \\
&\leq e^{-c_{12}/t} \cdot e^{-\frac{c_{13}\rho(x,y)^2}{t}},
\end{aligned}$$

where the second last inequality is due to the condition that $\rho(y, \{a_1^*, a_2^*\}) > 1$, and the last inequality is due to the fact that $\rho(x, a_1^*) < 1$. Now that we've showed

$$\frac{c_{14}}{t}e^{-\frac{c_{15}\rho(x,y)^2}{t}} \leq p^{(s)}_{E\setminus B_2}(t,x,y) + \overline{p}^{(s)}_{E\setminus B_2}(t,x,y) \leq \frac{c_{16}}{t}e^{-\frac{c_{17}\rho(x,y)^2}{t}}$$

as well as that

$$\overline{p}^{(s)}_{E\setminus B_2}(t,x,y) \le e^{-c_{12}/t} \cdot e^{-\frac{c_{13}\rho(x,y)^2}{t}},$$

it follows for the right hand side of (6.1.1) it holds

$$p_{E\setminus\{a_2^*\}}(t,x,y) = p^{(s)}_{E\setminus B_2}(t,x,y) \ge \frac{c_{18}}{t}e^{-\frac{c_{19}\rho(x,y)^2}{t}},\ t<1, \tag{6.1.4}$$

where the equality above is due to the fact that before the process hits B_2, it doesn't matter if B_2 is a regular disc or a collapsed darning point. Combining (6.1.4) and (6.1.1) we have

$$p(t,x,y) \ge \frac{c_{20}}{t}e^{-\frac{c_{21}\rho(x,y)^2}{t}}.$$

Invoking the upper bound estimate that $p(t,x,y) \le \frac{c_{22}}{t}e^{-\frac{c_{23}\rho(x,y)^2}{t}}$, we readily conclude

$$\frac{c_{24}}{t}e^{-\frac{c_{25}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{c_{26}}{t}e^{-\frac{c_{27}\rho(x,y)^2}{t}}.$$

Case 3. $x,y\in D_0$, and both x and y are close to the same darning point a_i^*, i.e., $\rho(x,a_i^*)<1$, $\rho(y,a_i^*)<1$. For this case, we consider the killed process upon exiting $B_\rho(a_i^*,3)$. The transition density $p(t,x,y)$ can be decomposed as follows

$$p(t,x,y) = p_{L_i\cup B_\rho(a_i^*,3)}(t,x,y) + \overline{p}_{L_i\cup B_\rho(a_i^*,3)}(t,x,y),$$

Similar to Proposition 6.1.2, we first show

$$p(t,x,y) \asymp p_{L_i\cup B_\rho(a_i^*,3)}(t,x,y). \tag{6.1.5}$$

Indeed since $\rho(x,a_i^*)<1$, $\rho(y,a_i^*)<1$, $x,y\in D_0$, it holds

$$
\begin{aligned}
\overline{p}_{L_i\cup B_\rho(a_i^*,3)}(t,x,y) &\le \int_{s=0}^t \mathbb{P}^x\left(\tau_{L_i\cup B_\rho(a_i^*,3)}\in ds\right)\sup_{z\in\partial(D_0\cap B_\rho(a_i^*,3))} p^{(X)}(t-s,z,y)\\
&\le \int_{s=0}^t \mathbb{P}^x\left(\tau_{L_i\cup B_\rho(a_i^*,3)}\in ds\right)\sup_{z\in\partial(D_0\cap B_\rho(z_i^*,3))} \frac{c_{28}}{t-s}e^{-\frac{c_{29}\rho(y,z)^2}{t-s}}\\
&\le \int_{s=0}^t \mathbb{P}^x\left(\tau_{L_i\cup B_\rho(a_i^*,3)}\in ds\right)\frac{c_{28}}{t-s}e^{-\frac{2c_{29}}{t-s}}\\
&\le \int_{s=0}^t \mathbb{P}^x\left(\tau_{L_i\cup B_\rho(a_i^*,3)}\in ds\right)\frac{c_{28}}{t-s}e^{-\frac{c_{29}\rho(y,a_i^*)^2}{t-s}}\cdot e^{-\frac{c_{30}}{t-s}}\\
&\le \int_{s=0}^t \mathbb{P}^x\left(\tau_{L_i\cup B_\rho(a_i^*,3)}\in ds\right)e^{-\frac{c_{29}\rho(y,a_i^*)^2}{t-s}}\cdot e^{-\frac{c_{30}}{t-s}}\\
&\le e^{-\frac{c_{29}\left(\rho(y,a_i^*)^2+1\right)}{t}}\int_{s=0}^t \mathbb{P}^x(\tau_{L_i\cup B_\rho(a_i^*,3)}\in ds)\\
&= e^{-\frac{c_{29}\left(\rho(y,a_i^*)^2+1\right)}{t}}\cdot \mathbb{P}^x(\tau_{L_i\cup B_\rho(a_i^*,3)}\le t)\\
&\le e^{-\frac{30c}{t}}\cdot e^{-\frac{c_{29}\left(\rho(x,a_i^*)^2+\rho(y,a_i^*)^2\right)}{t}}\\
&\le e^{-\frac{c_{30}}{t}}\cdot e^{-\frac{c_{29}\rho(x,y)^2}{t}},
\end{aligned}
$$

where the second last "$\asymp$" above is due to Proposition 3.2.3 as well as the assumption that $\rho(x,a_i^*)<1$, which proves (6.1.5). By considering the radial process which corresponds to the killed process of X upon exiting $B_\rho(a_i^*,3)$, we can apply the similar argument as Case 2 and conclude

$$
p(t,x,y)\asymp p_{L_i\cup B_\rho(a_i^*,3)}(t,x,y)\ge \frac{c_{31}}{\sqrt{t}}e^{-\frac{c_{32}\rho(x,y)^2}{t}}+\frac{c_{31}}{t}\left(1\wedge\frac{|x|_\rho}{\sqrt{t}}\right)\left(1\wedge\frac{|y|_\rho}{\sqrt{t}}\right)e^{-\frac{c_{33}|x-y|_e^2}{t}}.
$$

and that

$$
p(t,x,y)\asymp p_{L_i\cup B_\rho(a_i^*,3)}(t,x,y)\le \frac{c_{34}}{\sqrt{t}}e^{-\frac{c_{35}\rho(x,y)^2}{t}}+\frac{c_{34}}{t}\left(1\wedge\frac{|x|_\rho}{\sqrt{t}}\right)\left(1\wedge\frac{|y|_\rho}{\sqrt{t}}\right)e^{-\frac{c_{36}|x-y|_e^2}{t}}.
$$

Case 4. $x,y\in D_0$, $\rho(x,a_1^*)<1$, $\rho(y,a_2^*)<1$. Note that in this case actually $2\le\rho(x,y)\le 6$. Let $\widetilde{D}:=D_0\cap\left(B_\rho(a_1^*,3/2)\cup B_\rho(a_2^*,3/2)\right)^c\cap\{z:|y-z|_\rho\le 8\le 4\rho(x,y)\}$. Intuitively, $\widetilde{D}$ is the region that lies away from a_1^* and a_2^* but "centered" at y. By the result of Case 2

$$
\frac{c_{37}}{t}e^{-\frac{c_{38}\rho(y,z)^2}{t}}\le p(t,z,y)\le\frac{c_{39}}{t}e^{-\frac{c_{40}\rho(y,z)^2}{t}},\ z\in\widetilde{D},\ \rho(y,a_2^*)<1.
$$

It thus follows

$$\begin{aligned}
p(t,x,y) &\geq \int_{\widetilde{D}} p(t/2,x,z)p(t/2,z,y)m_p(dz) \geq \inf_{z\in\widetilde{D}}\left(\frac{c_{41}}{t}e^{-\frac{c_{42}|z-y|_\rho^2}{t}}\right)\int_{\widetilde{D}} p(t/2,x,z)m_p(dz) \\
&\geq \frac{c_{41}}{t}e^{-\frac{c_{42}\rho(x,y)^2}{t}}\int_{\widetilde{D}} p(t/2,x,z)m_p(dz) \\
&\asymp \frac{c_{41}}{t}e^{-\frac{c_{42}\rho(x,y)^2}{t}}\int_{\widetilde{D}} \frac{c}{t}e^{-\frac{c_{43}|x-z|_\rho^2}{t}}m_p(dz) \\
&\asymp \frac{c_{41}}{t}e^{-\frac{c_{42}\rho(x,y)^2}{t}}\cdot e^{-c_{43}/t} \asymp \frac{c_{41}}{t}e^{-\frac{c_{42}\rho(x,y)^2}{t}},
\end{aligned}$$

where the last three inequalities above are due to the fact that when $z\in\widetilde{D}$, $\rho(y,z)\leq 8\leq 4\rho(x,y)$. The last two "$\asymp$"'s are due to the fact that $1/2\leq\rho(x,z)\leq\rho(x,y)+\rho(y,z)\leq 10$, and thus

$$\int_{\widetilde{D}} \frac{c_{43}}{t}e^{-\frac{c_{44}\rho(x,z)^2}{t}}m_p(dz) = \mathbb{P}^x(X_{t/2}\in\widetilde{D}) \asymp e^{-c_{45}/t} \geq e^{-\frac{c_{46}\rho(x,y)^2}{t}},$$

since in the current case, $2<\rho(x,y)<6$.

Case 5. $x\in L_i$, $y\in D_0$. Again for this case, we need to consider several different subcases. Without loss of generality, assume $i=1$.

Subcase 1. $\rho(y,a_1^*)<1$. By exactly the same argument as the proof of Proposition 6.1.2, we have

$$p(t,x,y)\asymp p_{B_\rho(a_1^*,2)\cup L_i}(t,x,y),$$

which implies

$$\frac{c_{47}}{\sqrt{t}}e^{-\frac{c_{48}\rho(x,y)^2}{t}} \leq p(t,x,y) \leq \frac{c_{49}}{\sqrt{t}}e^{-\frac{c_{50}\rho(x,y)^2}{t}}.$$

Subcase 2. $\rho(y,a_1^*)\geq 1$. By the Markov property of X and the result of Case 4,

$$p(t,x,y)=\int_{s=0}^t \mathbb{P}^x(T_{\{a_1^*\}}\in ds)\cdot p(t-s,a_1^*,y) \asymp \int_0^t \frac{|x-a_1^*|}{s^{3/2}}e^{-\frac{c_{51}\rho(x,a_1^*)^2}{s}}\cdot\frac{c_{52}}{t-s}e^{-\frac{c_{53}\rho(y,a_1^*)^2}{t-s}}ds.$$

In order to estimate the right hand side, one may consider the following two situations.

If $\rho(x,a_1^*)>\sqrt{t}$, then

$$\begin{aligned}
\int_0^t \frac{\rho(x,a_1^*)}{s^{3/2}}e^{-\frac{c_{51}\rho(x,a_1^*)^2}{s}}\frac{c_{52}}{t-s}e^{-\frac{c_{53}\rho(y,a_1^*)^2}{t-s}}ds &\geq \int_0^t \frac{c_{52}}{s}e^{-\frac{c_{51}\rho(x,a_1^*)^2}{s}}\cdot\frac{1}{t-s}e^{-\frac{c_{53}\rho(y,a_1^*)^2}{t-s}}ds \\
&\geq \int_{t/3}^{2t/3}\frac{c_{52}}{t-s}e^{-\frac{c_{51}\rho(y,a_1^*)^2}{t-s}}\cdot\frac{1}{s}e^{-\frac{c_{53}\rho(x,a_1^*)^2}{s}}ds \\
&\geq \frac{c_{52}}{t}e^{-\frac{c_{51}\left(\rho(x,a_1^*)^2+\rho(y,a_1^*)^2\right)}{t}} \geq \frac{c_{52}}{t}e^{-\frac{c_{51}\rho(x,y)^2}{t}}.
\end{aligned}$$

On the other hand, if $\rho(x, a_1^*) < \sqrt{t} < 1$. Again, one may assume $\rho(x, a_1^*) < \sqrt{t}/2$, for otherwise it would be the same as the previous case. Now that $t > 4\rho(x, a_1^*)^2$,

$$\begin{aligned}
&\int_0^t \frac{\rho(x, a_1^*)}{s^{3/2}} e^{-\frac{c_{51}\rho(x,a_1^*)^2}{s}} \frac{c_{52}}{t-s} e^{-\frac{c_{53}\rho(y,a_1^*)^2}{t-s}} ds \\
\geq &\int_{\rho(x,a_1^*)^2}^{2\rho(x,a_1^*)^2} \frac{\rho(x, a_1^*)}{s^{3/2}} e^{-\frac{c_{51}\rho(x,a_1^*)^2}{s}} \frac{c_{52}}{t-s} e^{-\frac{c_{53}\rho(y,a_1^*)^2}{t-s}} ds \\
\geq &\int_{\rho(x,a_1^*)^2}^{2\rho(x,a_1^*)^2} \frac{c_{52}}{s} e^{-\frac{c_{51}\rho(x,a_1^*)^2}{s}} \frac{1}{t} e^{-\frac{c_{53}\rho(y,a_1^*)^2}{t}} ds \\
\geq &\int_{\rho(x,a_1^*)^2}^{2\rho(x,a_1^*)^2} \frac{c_{52}}{s} \cdot \frac{1}{t} e^{-\frac{c_{51}\rho(x-y)^2}{t}} ds \\
\geq &\frac{c_{52}}{t} e^{-\frac{c_{51}\rho(x-y)^2}{t}} \frac{1}{\rho(x, a_1^*)^2} \cdot \rho(x, a_1^*)^2 \asymp \frac{1}{t} e^{-\frac{c_{51}\rho(x,y)^2}{t}},
\end{aligned}$$

where the second inequality is due to the relationship that $t > \rho(x, a_1^*)^2$, and the third inequality is due to the fact that $\rho(x, a_1^*) + \rho(y, a_1^*) \leq 1 + 5 \leq 3\rho(x, y)$, since it is assumed that $\rho(a_1^*, a_2^*) = 4$ and that $\rho(x, a_1^*) < 1$, $\rho(y, a_2^*) < 1$. Due to the heat kernel upper bound estimate of that $p(t, x, y) \leq \frac{c_{54}}{t} e^{-\frac{c_{55}\rho(x,y)^2}{t}}$, it also holds for this case

$$\frac{c_{56}}{t} e^{-\frac{c_{57}\rho(x,y)^2}{t}} \leq p(t, x, y) \leq \frac{c_{58}}{t} e^{-\frac{c_{59}\rho(x,y)^2}{t}}.$$

Case 6. The remaining case is that $x \in L_1$, $y \in L_2$. Let $d_1 := \rho(x, a_1^*)$, $d_2 := \rho(y, a_2^*) + \rho(a_1^*, a_2^*)$. As in case 5,

$$\begin{aligned}
p(t, x, y) &= \int_0^t \mathbb{P}^x(T_{\{a_1^*\}} \in ds) \cdot p(t-s, a_1^*, y) ds \\
&\geq \int_0^t \frac{d_1}{s^{3/2}} e^{-\frac{c_{60} d_1^2}{s}} \frac{c_{61}}{t-s} e^{-\frac{c_{62} d_2^2}{t-s}} ds \geq \frac{c_{63}}{t} e^{-\frac{c_{64}\rho(x,y)^2}{t}}.
\end{aligned}$$

Also the other direction of the inequalities holds.

$$\begin{aligned}
p(t, x, y) &= \int_0^t \mathbb{P}^x(T_{\{a_1^*\}} \in ds) \cdot p(t-s, a_1^*, y) ds \\
&\leq \int_0^t \frac{d_1}{s^{3/2}} e^{-\frac{c_{65} d_1^2}{s}} \frac{c_{66}}{t-s} e^{-\frac{c_{67} d_2^2}{t-s}} ds \leq \frac{c_{68}}{t} e^{-\frac{c_{69}\rho(x,y)^2}{t}}.
\end{aligned}$$

It thus follows that in this case

$$\frac{c_{63}}{t} e^{-\frac{c_{64}\rho(x,y)^2}{t}} \leq p(t, x, y) \leq \frac{c_{68}}{t} e^{-\frac{c_{69}\rho(x,y)^2}{t}}.$$

The above cases have covered all the possible locations of x and y. $\square$

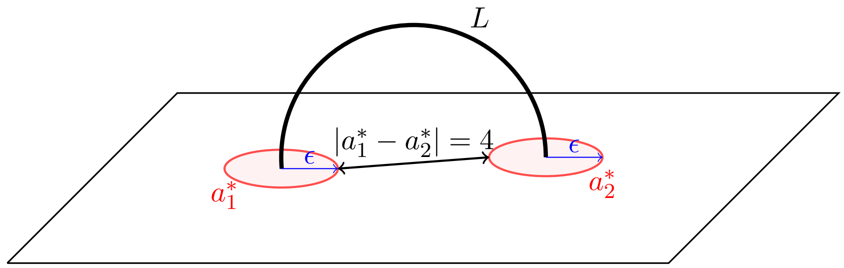

Figure 6.2: BMVD with an arc

6.2 Planary BMVD with an Arch

In this section we study another example of multi-dimensional Brownian motion. The BMVD denoted by X is now constructed on a state space which can be intuitively viewed as a plane $\mathbb{R}^2$ with an arch of a real line segment denoted by L attached to it.

Similar to the notations in the previous sections, let B_1 and B_2 be two disjointed Euclidean discs on $\mathbb{R}^2$, and let a_1^* and a_2^* be the two points obtained by darning $B_i, i = 1, 2$ respectively. Also let a_i^*, $i = 1, 2$ be the two endpoints of L. Set $D_0 := \mathbb{R}^2 \backslash (B_1 \cap B_2)$. The state space of X is $E := D_0 \cup L \cup \{a_1^*, a_2^*\}$ which is embedded in $\mathbb{R}^3$. See Figure 6.2.

Fix $p > 0$. Let m_p denote the measure on E whose restriction on L or D_0 is the Lebesgue measure times 1 and p, respectively, and $m_p(\{a_i^*\}) = 0$, $i = 1, 2$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X on $L^2(E, m_p)$ is thus given by

$$\begin{cases} \mathcal{F} = \left\{ f : f|_{\mathbb{R}^2} \in W^{1,2}(\mathbb{R}^2),\ f \text{ is constant } \mathcal{E}\text{-q.e. on } B_i\,, f|_L \in W^{1,2}(L),\ f|_{B_i} = f|_L(a_i^*)\,, i = 1, 2. \right\} \\ \mathcal{E}(f, g) = \mathbf{D}(f, g) := \displaystyle\int_{D_0} \frac{\partial f(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_i} m_p(dx) + \sum_{i=1,2} \int_L f'(x) g'(x) m_p(dx). \end{cases}$$

As in the previous section, we are interested in obtaining a two-sided small time heat kernel estimate for X, therefore unless otherwise stated, it is always assumed in this section that $t < 1$. Most of the proofs in this section are similar to those in the previous section. Again we assume without loss of generality that $\rho(a_1^*, a_2^*) = 4 = |L|_\rho$. To be more precise, we assume that the geodesic distance between a_1^* and a_2^* is 4, which is the same as the length of L.

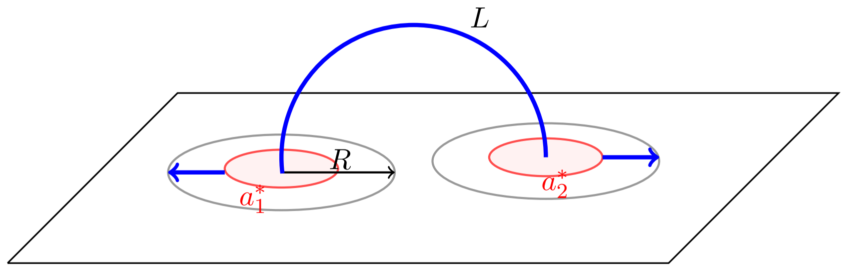

Figure 6.3: The radial process of BMVD with an arc

In order to give the heat kernel estimate results, we start with considering the case that both x and y are on L and have the following proposition.

Proposition 6.2.1. *There exists $C_i > 0$, $1 \le i \le 4$ such that*

$$\frac{C_1}{\sqrt{t}} e^{-\frac{C_2 \rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_3}{\sqrt{t}} e^{-\frac{C_4 \rho(x,y)^2}{t}}, \quad t \in [0,1],\ x, y \in L.$$

Proof. We choose $R > 0$ so small that $B_\rho(a_1^*, R) \cap B_\rho(a_2^*, R) = \varnothing$. Since Proposition 6.1.2 remains true for the BMVD in this section, we have

$$p(t,x,y) \asymp p_{L \sqcup B_\rho(a_1^*,R) \sqcup B_\rho(a_2^*,R)}(t,x,y). \tag{6.2.1}$$

We consider the part process killed upon exiting $L \sqcup B_\rho(a_1^*, R) \sqcup B_\rho(a_2^*, R)$ and denote such a part process by X^k. Without loss of generality, we assume the radius of the two collapsed discs is ϵ. We can thus define a "radial" process of X^k in the following way (see Figure 6.2):

$$u(x) = \begin{cases} -\rho(x, a_1^*), & x \in B_\rho(a_1^*, R); \\ \frac{2\pi\epsilon}{p} \cdot \rho(x, a_1^*), & x \in L; \\ 4 + \rho(x, a_2^*), & x \in B_\rho(a_2^*, R), \end{cases}$$

where 4 is the length of L. Let $Y_t := u(X_t^k)$ be the "radial" process of the part process X^k.

Similar to the computation having been done in Section 4, For any $\psi \in C_c^\infty(E)$,

$$\begin{aligned}
\int_E \psi d\nu = \mathcal{E}(u,\psi) &= \int_{B_1} \nabla(-\rho(x,a_1^*))\cdot\nabla\psi dx + \int_{B_2}\nabla\rho(x,a_2^*)\cdot\nabla\psi dx + 2\pi\epsilon\int_0^4 \psi' dx \\
&= -\int_{B_1}\frac{x}{\rho(x,a_1^*)}\cdot\nabla\psi dx + \int_{B_2}\frac{x}{\rho(x,a_2^*)}\cdot\nabla\psi dx + 2\pi\epsilon\int_0^4\psi' dx \\
&= \int_{B_1}\mathrm{div}\left(\frac{x}{\rho(x,a_1^*)}\right)\psi dx + \int_{\partial B_1}\psi(a_1^*)\frac{\partial}{\partial\vec{n}}\left(\frac{x}{\rho(x,a_1^*)}\right)d\tau - \int_{B_2}\mathrm{div}\left(\frac{x}{\rho(x,a_2^*)}\right)\psi dx \\
&\quad - \int_{\partial B_2}\psi(a_2^*)\frac{\partial}{\partial\vec{n}}\left(\frac{x}{\rho(x,a_2^*)}\right)d\tau + 2\pi\epsilon\big(\psi(a_2^*)-\psi(a_1^*)\big) \\
&= \int_{B_1}\frac{1}{\rho(x,a_1^*)}\psi dx - \int_{B_2}\frac{1}{\rho(x,a_2^*)}\psi dx,
\end{aligned}$$

where $D=\mathbb{R}^2\backslash B_e(0,\epsilon)$, $\vec{n}$ is the outward pointing unit vector normal of the surface $\partial B_e(0,\epsilon)$, and $d\tau$ is the surface measure on S^1. Therefore Y can be characterized by the following SDE:

$$dY_t = \frac{2\pi\epsilon}{p}\cdot\mathbf{1}_{[-R,4+R]}dB_t - \left(\frac{1}{Y_t-4+\epsilon}\mathbf{1}_{\{4<Y_t<4+R\}} + \frac{1}{Y_t-\epsilon}\mathbf{1}_{\{-R<Y_t<0\}}\right)dt.$$

According to Proposition 3.4.1,

$$\frac{c_1}{\sqrt{t}}e^{-\frac{c_2\rho(x,y)^2}{t}} \le p^Y(t,x,y) \asymp \frac{c_3}{\sqrt{t}}e^{-\frac{c_4\rho(x,y)^2}{t}}, \qquad 0<x,y<4,\ t<1.$$

It thus follows from (6.2.1)

$$p(t,x,y)\asymp p_{L\sqcup B_\rho(a_1^*,R)\sqcup B_\rho(a_2^*,R)}(t,x,y) = p^Y(t,x,y),\ x,y\in L, t\in(0,1].$$

i.e.,

$$\frac{c_5}{\sqrt{t}}e^{-\frac{c_6\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{c_7}{\sqrt{t}}e^{-\frac{c_8\rho(x,y)^2}{t}}, \quad x,y\in L, t\in(0,1].$$

□

The following proposition regards the case that $x\in L$ and $y\in D_0$.

Proposition 6.2.2. *For $x\in L$, $y\in D_0$, there exist constants $C_i>0$, $5\le i\le 12$ such that for all $t\in[0,1]$ the following estimates hold:*
When $\rho(y,a_i^*)<1$, $i=1$ or 2,

$$\frac{C_5}{\sqrt{t}}e^{-\frac{C_6\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_7}{\sqrt{t}}e^{-\frac{C_8\rho(x,y)^2}{t}},$$

otherwise

$$\frac{C_9}{t}e^{-\frac{C_{10}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{11}}{t}e^{-\frac{C_{12}\rho(x,y)^2}{t}},$$

Proof. For the case that $\rho(y, a_i^*) < 1$ where $i = 1$ or 2, we consider the part process killed upon exiting $L \sqcup B_\rho(a_1^*, 3/2) \sqcup B_\rho(a_2^*, 3/2)$. By the same argument as Proposition 6.2.1 we know

$$p(t,x,y) \asymp p_{L\sqcup B_\rho(a_1^*,3/2)\sqcup B_\rho(a_2^*,3/2)}(t,x,y)$$

i.e.,

$$\frac{c_1}{\sqrt{t}}e^{-\frac{c_2\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{c_3}{\sqrt{t}}e^{-\frac{c_4\rho(x,y)^2}{t}}.$$

On the other hand, if $\rho(y, a_i^*) > 1$, $i = 1$ and 2. Since $x \in L$ and we assume in this section that $|L| = 4$, we may without loss of generality assume that $\rho(x, a_1^*) < 2$. By elementary geometry one sees

$$\rho(x,a_1^*) + \rho(y,a_1^*) \asymp \rho(x,y) \asymp \rho(y,a_1^*) \ge 1. \tag{6.2.2}$$

Now we consider the part process killed upon hitting $B_\rho(a_2^*, 1/2)$. It follows

$$\begin{aligned} p(t,x,y) &\ge \int_{s=0}^{t} \mathbb{P}^y_{E\setminus B_\rho(a_2^*,1/2)}(T_{a_1^*} \in ds)p(t-s,a_1^*,x) \\ &\ge \int_{s=0}^{t/2} \mathbb{P}^y_{E\setminus B_\rho(a_2^*,1/2)}(T_{a_1^*} \in ds)p(t-s,a_1^*,x) \\ &\ge \frac{c_5}{\sqrt{t}}e^{-\frac{c_6\rho(x,a_1^*)^2}{t}} \cdot \mathbb{P}^y_{E\setminus B_\rho(a_2^*,1/2)}(T_{a_1^*} < t/2) \quad (6.2.3) \\ &\ge \frac{c_5}{\sqrt{t}}e^{-\frac{c_6\rho(x,a_1^*)^2}{t}} \cdot e^{-\frac{c_7\rho(y,a_1^*)^2}{t}} \quad (6.2.4) \\ &\ge \frac{c_5}{t}e^{-\frac{c_6\rho(x,y)^2}{t}}. \quad (6.2.5) \end{aligned}$$

In the above displays, (6.2.3) is due to Proposition 6.2.1; (6.2.4) is due to the hitting time estimate of standard 2−dimensional Brownian motion, and (6.2.5) is due to (6.2.2). □

Now the remaining case is that both x and y are on D_0.

Theorem 6.2.3. *For $x, y \in D_0$, there exist constants $C_i > 0$, $13 \le i \le 18$, such that for all $t \in [0, 1]$ the following estimates hold:*

When $x, y \in B_\rho(a_i^*, 1)$, $i = 1$ *or* 2,

$$\frac{C_{13}}{\sqrt{t}} e^{-\frac{C_{14}\rho(x,y)^2}{t}} + \frac{C_{13}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{15}|x-y|_e^2}{t}} \le p(t,x,y)$$
$$\le \frac{C_{16}}{\sqrt{t}} e^{-\frac{C_{17}\rho(x,y)^2}{t}} + \frac{C_{16}}{t}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t}}\right)\left(1 \wedge \frac{|y|_\rho}{\sqrt{t}}\right) e^{-\frac{C_{18}|x-y|_e^2}{t}},$$

otherwise,

$$\frac{C_{19}}{t} e^{-\frac{C_{20}\rho(x,y)^2}{t}} \le p(t,x,y) \le \frac{C_{21}}{t} e^{-\frac{C_{22}\rho(x,y)^2}{t}},$$

Proof. Actually when $x, y \in D_0,\ t < 1$, we can adapt the results as well as their proofs of cases $1-4$ in Theorem 6.1.3. To see this, we first show for the process of the current section, the following statement holds. It roughly says the starting from D_0, the process does not leave $D_0 \cup B_\rho(a_1^*, 3/2) \cup B_\rho(a_2^*, 3/2)$ before $t < 1$.

$$\begin{aligned}
&\overline{p}_{D_0\cup B_\rho(a_1^*,3/2)\cup B_\rho(a_2^*,3/2)}(t,x,y) \\
&\le \int_{s=0}^t \mathbb{P}^x\left(\tau_{D_0\cup B_\rho(a_1^*,3/2)\cup B_\rho(a_2^*,3/2)} \in ds\right) \sup_{z\in\partial\left(D_0\cup B(a_1^*,3/2)\cup B(a_2^*,3/2)\right)} p(t-s,z,y) \\
&\le \int_{s=0}^t \mathbb{P}^x\left(\tau_{D_0\cup B_\rho(a_1^*,3/2)\cup B_\rho(a_2^*,3/2)} \in ds\right) \frac{c_1}{t-s} \exp\left(-\frac{c_2\left(\rho(y,a_1^*)^2 \wedge \rho(y,a_2^*)^2\right)}{t-s}\right) \cdot e^{-\frac{c_3}{t-s}} \\
&\asymp \int_{s=0}^t \mathbb{P}^x\left(\tau_{D_0\cup B_\rho(a_1^*,3/2)\cup B_\rho(a_2^*,3/2)} \in ds\right) \exp\left(-\frac{c_2\left(\rho(y,a_1^*)^2 \wedge \rho(y,a_2^*)^2\right)}{t-s}\right) \cdot e^{-\frac{c_3}{t-s}} \\
&\le \exp\left(-\frac{c_2\left(\rho(y,a_1^*)^2 \wedge \rho(y,a_2^*)^2\right)}{t}\right) \cdot e^{-\frac{c_3}{t}} \int_{s=0}^t \mathbb{P}^x(\tau_{D_0\cup B_\rho(a_1^*,3/2)\cup B_\rho(a_2^*,3/2)} \in ds) \\
&= \exp\left(-\frac{c_2\left(\rho(y,a_1^*)^2 \wedge \rho(y,a_2^*)^2\right)}{t}\right) \cdot e^{-\frac{c_3}{t}} \cdot \mathbb{P}^x(\tau_{D_0\cup B_\rho(a_1^*,3/2)\cup B_\rho(a_2^*,3/2)} \le t) \\
&\le \exp\left(-\frac{c_2\left(\rho(y,a_1^*)^2 \wedge \rho(y,a_2^*)^2\right)}{t}\right) \cdot e^{-\frac{c_3}{t}} \cdot \exp\left(-\frac{c\left(\rho(x,a_1^*)^2 \wedge \rho(x,a_2^*)^2\right)}{t}\right) \\
&\le e^{-\frac{c_3}{t}} \cdot \exp\left(-\frac{c_2\rho(x,y)^2}{t}\right).
\end{aligned}$$

Actually one immediately sees that exactly the same argument as above holds for the process discussed in Section 7 which is planary BMVD with multiple straight lines, because it only depends on the trajectory up to exiting $(D_0 \cup B_\rho(a_1^*, 3/2) \cup B_\rho(a_2^*, 3/2))$. Therefore, by denoting the transition density of planary BMVD with multiple straight lines by $p^{(m)}$, it holds

$$p^{(m)}(t,x,y) \asymp p^{(m)}_{D_0\cup B(a_1^*,3/2)\cup B(a_2^*,3/2)}(t,x,y), \quad x,y\in D_0, t<1. \tag{6.2.6}$$

The idea of the proof to the above relationship is similar to the proof to Case 2 of Theorem 6.1.3. Indeed for planary BMVD starting from D_0, the distribution of BMVD with an arc is the same as that of BMVD with two straight lines. i.e.,

$$p^{(m)}_{D_0\cup B(a_1^*,3/2)\cup B(a_2^*,3/2)}(t,x,y) = p_{D_0\cup B(a_1^*,3/2)\cup B(a_2^*,3/2)}(t,x,y), \quad x,y\in D, t<1.$$

Therefore recalling (6.2.6), we conclude when $x,y\in D_0$,

$$\begin{aligned}
p(t,x,y) &= p_{D_0\cup B(a_1^*,3/2)\cup B(a_2^*,3/2)}(t,x,y) + \overline{p}_{D_0\cup B_\rho(a_1^*,3/2)\cup B_\rho(a_2^*,3/2)}(t,x,y)\\
&\asymp p^{(m)}_{D_0\cup B(a_1^*,3/2)\cup B(a_2^*,3/2)}(t,x,y)\\
&\asymp p^{(m)}(t,x,y),
\end{aligned}$$

which means the results as well as their proofs of cases $1-4$ in Theorem 6.1.3 apply to this current Theorem. □

Chapter 7

BMVD WITH DRIFT

7.1 Girsanov Transform of BMVD and Its Resolvent Kernel

Let $b : E \to \mathbb{R}$ be in the family of $L^{\infty}(E) + L^{p_1,p_2}(E)$ for some $p_1 \in (1,\infty]$ and $p_2 \in (2,\infty]$, where $L^{p_1,p_2}(E) := \left\{f : f|_{\mathbb{R}_+} \in L^{p_1}(\mathbb{R}_+),\ f|_{D_0} \in L^{p_2}(D_0)\right\}$. We have defined BMVD with drift in terms of Dirichlet forms in Section 2.5.1 as the Hunt process associated with the following Dirichlet form:

$$\mathcal{E}^b\,(f,\ g) = \mathcal{E}^0\,(f,\ g) - (b \cdot \nabla f,\ g)\,, \quad \mathcal{D}(\mathcal{E}^b) = \mathcal{D}(\mathcal{E}^0),$$

where $\mathcal{D}(\mathcal{E}^0) = \big\{f : f|_{\mathbb{R}^2} \in W^{1,2}(\mathbb{R}^2), f$ is constant $\mathcal{E}-$q.e. on $B_\epsilon, f|_{\mathbb{R}_+} \in W^{1,2}(\mathbb{R}_+), f|_{\mathbb{R}}(0) = f|_{B_\epsilon}\big\}$.

In this section, we characterize such a process by Girsanov transform and identify its resolvent kernels.

First we define a family of probability measures $\mathbb{Q}$ in terms of Girsanov transform as follows. We set $M^1_t := \displaystyle\int_0^t \mathbf{1}_{\mathbb{R}_+}(X_s)dX_s$ and $M^2_t := \displaystyle\int_0^t \mathbf{1}_{D_0}(X_s)dX_s$. To see how these stochastic intergrals are rigorously defined, we set $\rho(x) := \rho(x, a^*)$ for $x \in E$, and $A_\delta := \{x \in \mathbb{R}_+, \rho(x) > \delta\}$, $B_\delta := \{x \in D_0, \rho(x) > \delta\}$. We define a sequence of stopping times as follows.

$$\begin{aligned}
&S^\delta_0 = \inf\{t > 0 : X_t \in A_\delta\};\\
&T^\delta_1 = \sigma_{\{a^*\}};\\
&S^\delta_1 = \sigma_{A_\delta} \circ \theta_{T^\delta_1} + T^\delta_1;\\
&T^\delta_2 = \sigma_{\{a^*\}} \circ \theta_{S^\delta_1} + S^\delta_1;\\
&S^\delta_2 = \sigma_{A_\delta} \circ \theta_{T^\delta_2} + T^\delta_2;\\
&\cdots
\end{aligned}$$

For each $t > 0$, we define

$$M_t^{1,\delta} := \sum_{n\geq 1} \int_{S_n^\delta \wedge t}^{T_n^\delta \wedge t} 1 \cdot dX_s = \int_0^t \sum_n \mathbf{1}_{\{s\in[S_n^\delta,\, T_n^\delta]\}} dX_s.$$

The above stochastic integral is well-defined because the restriction of X on $\mathbb{R}_+$ has the same distribution as 1-dimensional Brownian motion on $\mathbb{R}_+$. Thus due to the strong Markov property of X as well as the fact that the summation is finite, the above 1-dimensional stochastic integral is well-defined. Since we have for each fixed $t > 0$,

$$\sum_n \mathbf{1}_{\{s\in[S_n^\delta,\, T_n^\delta]\}} \uparrow \mathbf{1}_{\{X_s\in\mathbb{R}_+\}} \text{ a.s. }, \quad \text{as } \delta \to 0.$$

It follows that there is a unique square-integrable martingale

$$M_t^1 = \int_0^t \mathbf{1}_{\{X_s\in\mathbb{R}_+\}} dX_s.$$

Similarly, to define M_t^2, we define

$$\begin{aligned}
\widehat{S}_0^\delta &= \inf\{t > 0 : X_t \in B_\delta\}; \\
\widehat{T}_1^\delta &= \sigma_{\{a^*\}}; \\
\widehat{S}_1^\delta &= \sigma_{B_\delta} \circ \theta_{\widehat{T}_1^\delta} + \widehat{T}_1^\delta; \\
\widehat{T}_2^\delta &= \sigma_{\{a^*\}} \circ \theta_{\widehat{S}_1^\delta} + \widehat{S}_1^\delta; \\
\widehat{S}_2^\delta &= \sigma_{B_\delta} \circ \theta_{\widehat{T}_2^\delta} + \widehat{T}_2^\delta; \\
&\cdots
\end{aligned}$$

Similarly, for each $t > 0$, we define

$$M_t^{2,\delta} := \sum_{n\geq 1} \int_{\widehat{S}_{n-1}^\delta \wedge t}^{\widehat{T}_n^\delta \wedge t} 1 \cdot dX_s = \int_0^t \sum_n \mathbf{1}_{\{s\in[\widehat{S}_{n-1}^\delta,\, \widehat{T}_n^\delta]\}} dX_s.$$

Similarly, the above stochastic integral is well-defined because the restriction of X on D_0 has the same distribution as 2-dimensional Brownian motion on D_0. Thus due to the strong Markov property of X as well as the fact that the summation is finite, the above 2-dimensional stochastic integral is well-defined. Also we have

$$\sum_n \mathbf{1}_{\{s\in[\widehat{S}_{n-1}^\delta,\, \widehat{T}_n^\delta]\}} \uparrow \mathbf{1}_{\{X_s\in D_0\}} \text{ a.s. }, \quad \text{as } \delta \to 0.$$

Therefore there is a unique square-integrable martingale

$$M_t^2 = \int_0^t \mathbf{1}_{\{X_s \in D_0\}} dX_s.$$

Now we define

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_t := \exp\bigg(\int_0^t b(M_s^1) dM_s^1 + \int_0^t b(M_s^2) dM_s^2 - \int_0^t |b(X_s)|^2 ds \bigg), \tag{7.1.1}$$

where $(X_t)_{t>0}$ is BMVD. Let G_α^0 be the resolvents of BMVD. We first need to prove the relationship between the process defined by $\mathbb{Q}$, namely, by the Girsanov transform of BMVD, and BMVD with drift defined in terms of Dirichlet forms. That is, we need to show the following theorem which consists of two statements.

In the remaining context, we just write $(M_t)_{t>0}$ and suppress the superscripts to avoid confusion caused by notation.

Theorem 7.1.1. *The following two statements hold.*

1. The family of probability measures $\mathbb{Q}$ *defined in* (7.1.1) *determines a continuous Hunt process. If we denote the resolvents of this process by* G_α^b, *it holds* $G_\alpha^b := \sum_{n=0}^{\infty} G_\alpha^0 (b \cdot \nabla G_\alpha^0)^n$, *where the infinite sum converges.*

2. The process determined by $\mathbb{Q}$ *which is defined in* (7.1.1) *has a Dirichlet form expression*

$$\mathcal{E}^b(f,g) := \mathcal{E}^0(f,g) - (b \cdot \nabla f, g), \quad \mathcal{D}(\mathcal{E}^b) = \mathcal{D}(\mathcal{E}). \tag{7.1.2}$$

Recall that $\mathcal{D}(\mathcal{E}) = \big\{ f : f|_{\mathbb{R}^2} \in W^{1,2}(\mathbb{R}^2), f$ is constant $\mathcal{E}-$q.e. on $B_\epsilon, f|_{\mathbb{R}_+} \in W^{1,2}(\mathbb{R}_+), f|_{\mathbb{R}}(0) = f|_{B_\epsilon} \big\}$. The argument of the proof is almost the same as in [13]. Before proving Theorem 7.1.1, let us first prove several lemmas. We start with observing that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^t b(X_s) dX_s - \int_0^t |b(X_s)|^2 ds \right) = M_t$$

uniquely defines a family of probability measures. To prove the first claim, we need to find the relationship between the following two families of operators.

$$G_\alpha^b f(x) = \mathbb{E}_{\mathbb{Q}}^x \left(\int_0^\infty e^{-\alpha t} f(X_t) dt \right), \quad f \in b\mathcal{B}(E);$$
$$G_\alpha^0 f(x) = \mathbb{E}_0^x \left(\int_0^\infty e^{-\alpha t} f(X_t) dt \right), \quad f \in b\mathcal{B}(E),$$

where $\mathbb{E}_0^x$ is the expectation corresponds to the case of $b=0$.

As is showed in Theorem 3.1 of [13], $\mathbb{E}_0^x M_t^2 < \infty$. Since

$$\left|P_t^b f(x)\right| = \left|\mathbb{E}_0^x(M_t f(X_t))\right| \le \left(\mathbb{E}_0^x M_t^2\right)^{1/2} \left(\mathbb{E}_0^x f^2(X_t)\right)^{1/2},$$

by setting $\alpha_0 := \sup_{x\in E} \mathbb{E}_0^x \left(\int_0^t |b(X_s)|^2 ds\right)$ we have

$$\|P_t^b f\|_2^2 \le e^{\frac{1}{2}\alpha_0 t} \|P_t^b f^2\|_1 \le e^{\frac{1}{2}\alpha_0 t} \|f\|_2^2.$$

Thus $\|P_t^b f\|_2^2 \le e^{\alpha_0 t}$, which implies

$$\|G_\alpha^b\|_{2,2} \le \frac{1}{\alpha - \alpha_0},$$

where $\|\cdot\|_{2,2}$ is the operator norm from L^2 to L^2.

Recall that $\mathcal{E}^0$ is the Dirichlet form of BMVD therefore it is symmetric. We first have the following lemma which is in the same form as Lemma 3.3 in [13].

Lemma 7.1.2. *There exists a constant $\alpha_1 > \alpha_0$ such that for $\alpha > \alpha_1$, $f \in L^2(E)$*

$$\psi_n := G_\alpha^0 (b\cdot \nabla G_\alpha^0)^n f, \quad n = 0, 1, 2, \cdots$$

is in $\mathcal{D}(\mathcal{E})$. The infinite sum $\sum_{n=0}^\infty \psi_n$ converges in $\mathcal{D}(\mathcal{E})$ with respect to the norm $\|\cdot\|_{1,2}$, and for each $n \ge 1$,

$$\mathcal{E}_\alpha^0(\psi_n, g) = (b\cdot\nabla\psi_n, g), \quad g \in \mathcal{D}(\mathcal{E}). \tag{7.1.3}$$

Proof. Since b is assumed to be in the family of $L^\infty(E) + L^{p_1,p_2}(E)$ for some $p_1 \in (1,\infty]$ and $p_2 \in (2,\infty]$, where $L^{p_1,p_2}(E) := \left\{f : f|_{\mathbb{R}_+} \in L^{p_1}(\mathbb{R}_+),\ f|_{D_0} \in L^{p_2}(D_0)\right\}$, it follows that

$$\sup_{x\in E} \mathbb{E}_0^x \left(\int_0^\infty e^{-\alpha t} |b(X_t)|^2 dt\right) < \infty.$$

For $f \in L^2(E)$, $G_\alpha^0 f$ is in $\mathcal{D}(\mathcal{E})$ and $G_\alpha^0(b\cdot\nabla G_\alpha^0 f)$ is well-defined for $\alpha > \alpha_0$, because it holds for some constant $c > 0$ that

$$\begin{aligned}
&\mathbb{E}_0^x \left(\int_0^\infty e^{-\alpha s} |b(X_s)\cdot\nabla G_\alpha^0 f(X_s)| ds\right) \\
\le& \mathbb{E}_0^x \left(\int_0^\infty e^{-\alpha s} |b(X_s)|^2 ds\right)^{1/2} \mathbb{E}_0^x \left(\int_0^\infty e^{-\alpha s} |\nabla G_\alpha^0 f|^2(X_s) ds\right)^{1/2} \\
\le& \sqrt{c} \left(\mathbb{E}_0^x \int_0^\infty e^{-\alpha s} |\nabla G_\alpha^0 f|^2 (X_s) ds\right)^{1/2} < \infty.
\end{aligned}$$

Furthermore,

$$\|\psi_1\|_2^2 = \|G_\alpha^0(b\cdot\nabla G_\alpha^0 f)\|_2^2 \le c\int_0^\infty e^{-\alpha s}\|P_s^0(|\nabla G_\alpha^0 f|^2)\|_1 ds \le \frac{c}{\alpha}\|\nabla G_\alpha^0 f\|_2^2.$$

For $\beta > 0$, we define $\mathcal{E}^{(\beta)}(f,g) := \beta(f - \beta G_\beta^0 f, g)$, $\forall f, g \in L^2(E)$. It is known that

$$\mathcal{E}^{(\beta)}(f,f) \ge \mathcal{E}^{(0)}(\beta G_\beta^0 f, \beta G_\beta^0 f), \quad \forall f \in L^2(E)$$

and $f \in \mathcal{D}(\mathcal{E})$ if and only if $\sup_{\beta>0}\mathcal{E}^{(\beta)}(f,f) < \infty$, in which case

$$\mathcal{E}^{(0)}(f,g) = \lim_{\beta\to\infty}\mathcal{E}^{(\beta)}(f,g), \quad \text{for } g \in \mathcal{D}(\mathcal{E}).$$

By the resolvent identity

$$G_\alpha^0 - \beta G_\beta^0 G_\alpha^0 = \frac{\beta}{\beta-\alpha}G_\beta^0 - \frac{\alpha}{\beta-\alpha}G_\alpha^0,$$

we have

$$\mathcal{E}^{(\beta)}(\psi_1,\psi_1) = \frac{\beta}{\beta-\alpha}\left(b\cdot\nabla G_\alpha^0 f, \beta G_\beta^0\psi_1\right) - \frac{\alpha\beta}{\beta-\alpha}\|\psi_1\|_2^2.$$

Since b is in the family of $L^\infty(E) + L^{p_1,p_2}(E)$ for some $p_1 \in (1,\infty]$ and $p_2 \in (2,\infty]$, for some constant $c > 0$ it holds

$$\begin{aligned}\left|(\beta\nabla G_\alpha^0 f, \beta G_\beta^0\psi_1)\right| &\le \frac{1}{2}\|\nabla G_\alpha^0 f\|_2^2 + \frac{1}{2}\int |b|^2(\beta G_\beta^0\psi_1)^2 dx \\ &\le \frac{1}{2}\|\nabla G_\alpha^0 f\|_2^2 + c\|\beta G_\beta^0\psi_1\|_2^2 \le \frac{1}{2}\|\nabla G_\alpha^0 f\|_2^2 + c\|\psi_1\|_2^2.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{E}^{(\beta)}(\psi_1,\psi_1) &= \frac{\beta}{\beta-\alpha}\left(b\cdot\nabla G_\alpha^0 f, \beta G_\beta^0\psi_1\right) - \frac{\alpha\beta}{\beta-\alpha}\|\psi_1\|_2^2 \\ &\le \frac{\beta}{\beta-\alpha}\left(\frac{1}{2}\|\nabla G_\alpha^0 f\|_2^2 + c\|\psi_1\|_2^2\right) - \frac{\alpha\beta}{\beta-\alpha}\|\psi_1\|_2^2\end{aligned}$$

It follows that

$$\sup_{\beta>0}\mathcal{E}^{(\beta)}(\psi_1,\psi_1) = \lim_{\beta\to\infty}\mathcal{E}^{(\beta)}(\psi_1,\psi_1) = \|\nabla G_\alpha^0 f\|_2^2 + (c-\alpha)\|\psi_1\|_2^2 < \infty.$$

Thus $\psi_1 = G_\alpha^0(b\cdot\nabla G_\alpha^0 f) \in \mathcal{D}(\mathcal{E})$. Since $\beta G_\beta^0\psi_1$ converges to ψ_1 in $\mathcal{E}_1^0-$norm as $\beta\to\infty$ and $(f,g) \mapsto \int (b\cdot\nabla f)\cdot g dm$ is a continuous bilinear form in $\mathcal{D}(\mathcal{E})$ with respect to $\mathcal{E}_1^0-$norm,

$$\mathcal{E}^0(\psi_1,\psi_1) = \lim_{\beta\to\infty}\mathcal{E}^{(\beta)}(\psi_1,\psi_1) = (b\cdot\nabla G_\alpha^0 f, \psi_1) - \alpha\|\psi_1\|_2^2.$$

Thus

$$\mathcal{E}_\alpha^0(\psi_1,\psi_1) = (b\cdot\nabla G_\alpha^0 f,\psi_1).$$

A similar argument shows that for $\alpha > \alpha_0$,

$$\mathcal{E}_\alpha^0(\psi_1,g) = (b\cdot\nabla G_\alpha^0 f,g), \quad \forall g\in\mathcal{D}(\mathcal{E}).$$

For $\alpha > \alpha_1$, since b is bounded, there exists some $\lambda > 0$ such that

$$\mathcal{E}_\alpha^0(\psi_1,\psi_1) \le \|\nabla G_\alpha^0 f\|_2 \left(\int_E |b|^2\psi_1^2 dx\right)^{1/2} \le \left(\lambda\mathcal{E}_\alpha^0(G_\alpha^0 f, G_\alpha^0 f)\right)^{1/2}\left(\frac{1}{2\lambda}\mathcal{E}_\alpha^0(\psi_1,\psi_1)\right)^{1/2}.$$

Hence

$$\mathcal{E}_\alpha^0(\psi_1,\psi_1) \le \frac{1}{2}E_\alpha^0(G_\alpha^0 f, G_\alpha^0 f) = \frac{1}{2}\mathcal{E}_\alpha^0(\psi_0,\psi_0).$$

by induction we have $\psi_n\in\mathcal{D}(\mathcal{E})$ for $n\ge 2$,

$$\mathcal{E}_\alpha^0(\psi_n,\psi_n) \le \frac{1}{2}\mathcal{E}_\alpha^0(\psi_{n-1},\psi_{n-1})$$

and $\mathcal{E}_\alpha^0(\psi_n,g) = (b\cdot\nabla\psi_{n-1},g)$, $\forall g\in\mathcal{D}(\mathcal{E})$. It follows immediately that $\sum_{n=0}^{\infty}\psi_n$ converges in $(\mathcal{D}(\mathcal{E}),\|\cdot\|_{1,2})$. □

The next theorem characterizes the relationship between G_α^0 and G_α^b.

Theorem 7.1.3. *Let α_1 be the same constant as in Lemma 7.1.2. For $\alpha > \alpha_1$, and $f\in L^2(E)$, we have $G_\alpha^b f\in\mathcal{D}(\mathcal{E})$, and*

$$\begin{cases} G_\alpha^b f = G_\alpha^0 f + G_\alpha^b(b\cdot\nabla G_\alpha^0 f) \\ G_\alpha^b f = \sum_{n=0}^{\infty} G_\alpha^0(b\cdot\nabla G_\alpha^0)^n f \\ G_\alpha^b f = G_\alpha^0 f + G_\alpha^0(b\cdot\nabla G_\alpha^b f), \end{cases} \tag{7.1.4}$$

and

$$\mathcal{E}_\alpha^0\left(G_\alpha^0(b\cdot\nabla G_\alpha^b f), g\right) = \left(b\cdot\nabla G_\alpha^b f, g\right), \quad \forall g\in\mathcal{D}(\mathcal{E}). \tag{7.1.5}$$

Proof. Since $G^0_\alpha f \in \mathcal{D}(\mathcal{E})$, for $M_t = \exp\left(\int_0^t b(X_s)dX_s - \frac{1}{2}\int_0^t |b(X_s)|^2 ds\right)$, $t \geq 0$,

$$M_t = 1 + \int_0^t M_s b(X_s) dX_s,$$

where X is the BMVD. It follows that

$$G^0_\alpha f(X_t) = G^0_\alpha f(X_0) + \int_0^t (\nabla G^0_\alpha f)(X_s)dX_s + \int_0^t \big(\alpha G^0_\alpha f(X_s) - f(X_s)\big)ds.$$

Using integration by parts, one has

$$e^{-\alpha t}G^0_\alpha f(X_t) = G^0_\alpha f(X_0) + \int_0^t e^{-\alpha s}(\nabla G^0_\alpha f)(X_s)dX_s - \int_0^t e^{-\alpha s} f(X_s)ds.$$

Since $M_t = 1 + \int_0^t M_s \cdot b(X_s)dX_s$,

$$\mathbb{E}^x_0\left[M_t\left(e^{-\alpha t}G^0_\alpha f(X_t) - G^0_\alpha f(X_0) + \int_0^t e^{-\alpha s} f(X_s)ds\right)\right] = \mathbb{E}^x_0\left[\int_0^t e^{-\alpha s} M_s \left(b \cdot \nabla G^0_\alpha f\right)(X_s)ds\right].$$

Hence

$$\mathbb{E}^x_0\left[M_t e^{-\alpha t} G^0_\alpha f(X_t)\right] + \mathbb{E}^x_0\left[\int_0^t e^{-\alpha s} f(X_s)ds\right] = G^0_\alpha f(x) + \mathbb{E}^x_0 \int_0^t e^{-\alpha s} M_s \left(b \cdot \nabla G^0_\alpha f\right)(X_s)ds. \tag{7.1.6}$$

Again by integration by parts,

$$e^{-\frac{1}{2}\alpha t} M_t = 1 + \int_0^t e^{-\frac{1}{2}\alpha s} M_s b(X_s) dWs - \frac{\alpha}{2}\int_0^t e^{-\frac{1}{2}\alpha s} M_s ds.$$

Thus

$$\begin{aligned}\mathbb{E}^x_0\left(\int_0^t e^{-\alpha s} M_s^2 |b(X_s)|^2 ds\right) &= \mathbb{E}^x_0\left[\left(e^{-\frac{1}{2}\alpha t} M_t - 1 + \frac{\alpha}{2}\int_0^t e^{-\frac{1}{2}\alpha s} M_s ds\right)^2\right] \\ &\leq 3\mathbb{E}^x_0\left[e^{-\alpha t} M_t^2 + 1 + \frac{\alpha}{2}\int_0^\infty e^{-\frac{1}{2}\alpha s} M_s^2 ds\right] < \infty;\end{aligned}$$

and also for some constant $c > 0$ it holds

$$\begin{aligned}&\mathbb{E}^x_0\left[\int_0^\infty e^{-\alpha s} M_s |b \cdot \nabla G^0_\alpha f(X_s)| ds\right] \\ &\leq \left(\mathbb{E}^x_0 \int_0^\infty e^{-\alpha s} M_s^2 |b(X_s)|^2 ds\right)^{1/2} \left(\mathbb{E}^x_0 \int_0^\infty e^{-\alpha s} |\nabla G^0_\alpha f|^2 (X_s) ds\right)^{1/2} \\ &\leq c \cdot \left(\mathbb{E}^x_0 \int_0^\infty e^{-\alpha s} |\nabla G^0_\alpha f|^2 (X_s) ds\right)^{1/2} < \infty.\end{aligned}$$

$G^b_\alpha(b\cdot\nabla G^0_\alpha f)$ is therefore well-defined and

$$\|G^b_\alpha(b\cdot\nabla G^0_\alpha f)\|_2^2 \le c\cdot\int_0^\infty e^{-\alpha s}\|\nabla G^0_\alpha f\|_2^2 ds \le \frac{c}{\alpha}\|\nabla G^0_\alpha f\|_2^2.$$

By Cauchy-Schwarz inequality as well as the fact that $\sup_{x\in E}\mathbb{E}^x_0 M_t^2 \le e^{\frac{1}{2}\alpha_0 t}$ we conclude that

$$\mathbb{E}^x_0\left(\int_0^\infty e^{-\alpha s}M_s|f(X_s)|ds\right)<\infty \quad \text{and} \quad \lim_{t\to\infty}\mathbb{E}^x_0\left[M_t e^{-\alpha t}G^0_\alpha f(X_t)\right]=0.$$

Thus by letting $t\to\infty$ in (7.1.6), one gets

$$\mathbb{E}^x_0\int_0^\infty M_s e^{-\alpha s}f(X_s)ds = G^0_\alpha f(x)+\mathbb{E}^x_0\int_0^\infty e^{-\alpha s}M_s(b\cdot\nabla G^0_\alpha f)(X_s)ds.$$

That is

$$G^b_\alpha f(x)=G^0_\alpha f(x)+G^b_\alpha(b\cdot\nabla G^0_\alpha f)(x).$$

By the same argument

$$G^b_\alpha(b\cdot\nabla G^0_\alpha f)(x)=\sum_{k=1}^n G^0_\alpha(b\cdot\nabla G^0_\alpha)^k f(x)+G^b_\alpha(b\cdot\nabla G^0_\alpha)^{n+1}f(x).$$

If we set $\psi_k=G^0_\alpha(b\cdot\nabla G^0_\alpha)^k f$ which is known to be in $\mathcal{D}(\mathcal{E})$ for $\alpha>\alpha_1$, then

$$G^b_\alpha(b\cdot\nabla G^0_\alpha f)(x)=\sum_{k=1}^n\psi_k(x)+G^b_\alpha(b\cdot\nabla\psi_n)(x), \quad n\ge 1.$$

By the same reasoning as that for (3.37) in [13], there exists some constant $c>0$ such that

$$\|G^b_\alpha(b\cdot\nabla\psi_n)\|_2^2\le\frac{c}{\alpha}\|\nabla\psi_n\|_2^2$$

which converges to zero as $n\to\infty$ by lemma 7.1.2. It follows

$$G^b_\alpha(b\cdot\nabla G^0_\alpha f)(x)=\sum_{k=1}^\infty\psi_k(x).$$

The last " = " of (7.1.4) follows immediately from the second identity. (7.1.5) follows from (7.1.3) and the third identity of (7.1.4). □

Recall in (7.1.2) it is defined that $\mathcal{E}^b(f,g)=\frac{1}{2}\int\nabla f\cdot\nabla g dx-\int_E(b\cdot\nabla f)\cdot g\, m_p(dx)$, which is a non-symmetric Dirichlet form. We are now ready to prove Theorem 7.1.1.

Proof of Theorem 7.1.1: The first statement has already been proved. We only need to check the second statement.

For $f \in L^2(E)$, it has been proved that $G^b_\alpha f \in \mathcal{D}(\mathcal{E})$. For $g \in \mathcal{D}(\mathcal{E})$,

$$\begin{aligned}\mathcal{E}^b_\alpha(G^b_\alpha f, g) &= \mathcal{E}^0_\alpha\left(G^0_\alpha f + G^0_\alpha(b\cdot\nabla G^b_\alpha f), g\right) - (b\cdot\nabla G^b_\alpha f, g)\\ &= (f,g) + (b\cdot\nabla G^b_\alpha f, g) - (b\cdot\nabla G^b_\alpha f, g) = (f,g),\end{aligned}$$

which implies that the resolvents of $\mathcal{E}^b$ is $\{G^b_\alpha\}$, where by definition $G^b_\alpha f(x) = \mathbb{E}^x_{\mathbb{Q}} \int_0^\infty e^{-\alpha t} f(X_t)dt$.

Theorem 7.1.1 is therefore proved in view of the result of Theorem 7.1.3.

The followting theorem characterizes the infinitesimal generator associated with $(X^b)_{t>0}$.

Theorem 7.1.4. *A function $u \in \mathcal{D}(\mathcal{E}^b)$ is in $\mathcal{D}(\mathcal{L}^b)$ if and only if the distributional Laplacian Δu of u exists as an L^2-integrable function on $E\backslash\{a^*\}$ and u has the same zero flux property as BMVD at a^*. Moreover, for $u \in \mathcal{D}(\mathcal{L}^b)$, $\mathcal{L}^b u = \Delta u + b\nabla u$ on $E\backslash\{a^*\}$.*

Proof. Let $\mathcal{L}^b$ be the L^2-generator of BMVD X^b. $u \in \mathcal{D}(\mathcal{L}^b)$ if and only if $u \in \mathcal{D}(\mathcal{E}^b)$ and there is some $f \in L^2(E; m_p)$ so that

$$\mathcal{E}^b(u,v) = -\int_E f(x)v(x)m_p(dx) \quad \text{for every } v \in \mathcal{D}(\mathcal{E}^b).$$

Denote the above f by $\mathcal{L}^b u$. The above is equivalent to

$$\int_E \nabla u(x)\cdot\nabla v(x) m_p(dx) + \int_E b(x)\nabla u(x) v(x) m_p(dx) = -\int_E f(x)v(x)m_p(dx) \quad \text{for every } v \in C_c^\infty(E\backslash\{a^*\}) \tag{7.1.7}$$

and

$$\int_E \nabla u(x)\cdot\nabla u_0(x) m_p(dx) + \int_E b(x)\nabla u(x) u_0(x) m_p(dx) = -\int_E f(x)u_0(x)m_p(dx). \tag{7.1.8}$$

(7.1.7) says that $f = \Delta u + b\nabla u \in L^2(E; dx)$, and (7.1.8) is equivalent to $\mathcal{N}_p(u)(a^*) = 0$. □

7.2 Smallness of Perturbation

Unless otherwise stated, it is always assumed in this section that $t \in (0,1]$. The first claim we need to make is related to the smallness of $b(x)\cdot\nabla$ as a perturbation of Δ, which will thus lead to the small time heat kernel estimates of BMVD with drift. Before doing this, We first define $p^0_\alpha(t,x,y)$ as the canonical form of the heat kernel of BMVD in the following way.

$$p^0_\alpha(t,x,y)=\begin{cases}\frac{1}{\sqrt{t}}e^{-\frac{\alpha|x-y|^2}{t}}, & x\in\mathbb{R}_+,\ y\in\mathbb{R}_+, \text{ or } x\in\mathbb{R}_+,\ y\in D_0,\ |y|_\rho\le 1;\\ \frac{1}{t}e^{-\frac{\alpha\rho(x,y)^2}{t}}, & x\in D_0,\ y\in D_0,\ |x|_\rho+|y|_\rho\ge 1;\\ \frac{1}{t}e^{-\frac{\alpha\rho(x,y)^2}{t}}, & x\in\mathbb{R}_+,\ y\in D_0,\ |y|_\rho>1\ ;\\ \frac{1}{\sqrt{t}}e^{-\frac{\alpha_1\rho(x,y)^2}{t}}+\frac{1}{t}\left(1\wedge\frac{|x|^2_\rho}{t}\right)\left(1\wedge\frac{|y|^2_\rho}{t}\right)e^{-\frac{\alpha_2|x-y|^2_e}{t}}, & x\in D_0,\ y\in D,\ |x|_\rho+|y|_\rho\ge 1,\end{cases}$$

where $0<\alpha=\alpha_1\le\alpha_2$.

Proposition 7.2.1. *There exist constants α, β and γ satisfying $0<\alpha<\beta<\gamma$ such that it holds*

$$C_1p^0_\gamma(t,x,y)\le p(t,x,y)\le C_2p^0_\alpha(t,x,y),\quad (t,x,y)\in(0,1]\times E\times E, \tag{7.2.1}$$

as well as

$$|\nabla_x p(t,x,y)|\le\frac{C_3}{\sqrt{t}}p^0_\beta(t,x,y), \tag{7.2.2}$$

where C_i, $i=1,2,3$, are some positive constants.

Proof. (7.2.1) is the heat kernel estimates for BMVD which have already been shown. Since α can always be chosen to be larger and γ can always be chosen to be smaller, it suffices to show (7.2.2) for some $\beta>0$. We prove (7.2.2) by considering different regions of x and y. In the following proof, the values of the constants c_i and β_i may change from line to line for $i=1,2,\cdots$.

Case 1. $x\in\mathbb{R}_+$, $y\in\mathbb{R}_+$. In this case

$$\begin{aligned}\nabla_x p(t,x,y)&=\nabla_x p_{\mathbb{R}_+}(t,x,y)+\nabla_x\overline{p}_{\mathbb{R}_+}(t,x,y)\\&=\nabla_x p^{1-BM}_{\mathbb{R}_+}(t,x,y)+\nabla_x\int_0^t\mathbb{P}^x\left(T_{a^*}\in ds\right)p(t-s,a^*,y)\\&=\nabla_x p^{1-BM}_{\mathbb{R}_+}(t,x,y)+\int_0^t\nabla_x\mathbb{P}^x\left(T_{a^*}\in ds\right)p(t-s,a^*,y).\end{aligned} \tag{7.2.3}$$

We now notice for some constants $c > 0$ and $\beta > 0$ it holds

$$\left|\nabla_x p_{\mathbb{R}_+}^{1-BM}(t,x,y)\right| \leq \frac{c_1}{t} e^{-\frac{\beta|x-y|^2}{t}}; \tag{7.2.4}$$

and that

$$\left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*} \in ds\right) p(t-s,a^*,y)\right| \leq \left|\int_0^t \frac{\partial}{\partial x}\left(\frac{|x|}{s^{3/2}} e^{-\frac{\beta|x|^2}{s}}\right) \frac{c_2}{\sqrt{t-s}} e^{-\frac{\beta y^2}{t-s}} ds\right| \leq \frac{c_3}{t} e^{-\frac{\beta|x-y|^2}{t}}. \tag{7.2.5}$$

By replacing the two terms on the right hand side of (7.2.3) with (7.2.4) and (7.2.5) respectively, it has been proved that for Case 1,

$$|\nabla_x p(t,x,y)| \leq \frac{c_4}{\sqrt{t}} p_b^0(t,x,y).$$

Case 2. $x \in \mathbb{R}_+$, $y \in D_0$. We denote $|y|_\rho$ which is the geometric distance from y to a^* by r. Due to the rotational invariance of $p(t,x,y)$ in y for this case, we may consider $\nabla_x p(t,x,y)$ as a function in r.

When $|y|_\rho < 1$, it holds

$$\begin{aligned}
|\nabla_x p(t,x,y)| &= \left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*} \in ds\right) p(t-s,a^*,y)\right| \\
&\leq \left|\int_0^t \frac{\partial}{\partial x}\left(\frac{|x|}{s^{3/2}} e^{-\frac{\beta x^2}{s}}\right) \frac{c_1}{\sqrt{t-s}} e^{-\frac{\beta y^2}{t-s}} ds\right| \leq \frac{c_2}{t} e^{-\frac{\beta\rho(x,y)^2}{t}}.
\end{aligned} \tag{7.2.6}$$

However, when $|y|_\rho \geq 1$, it holds for some constants $c, \beta > 0$ that

$$\begin{aligned}
\left|\nabla_x p(t,x,y)\right| &= \left|\nabla_x \int_0^x \mathbb{P}^x(T_{\{0\}} \in ds) p(t-s,0,y)\right| \\
&= \left|\int_0^t \nabla_x \mathbb{P}^x(T_{\{0\}} \in ds) p(t-s,0,y)\right| \\
&\leq \frac{c_3|x|}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{s^3(t-s)^2}} e^{-\beta\left(\frac{x^2}{2s}+\frac{|y|^2}{2(t-s)}\right)} ds \\
&= \frac{c_3|x|}{\sqrt{2\pi}} \int_0^{t/2} \frac{1}{t\sqrt{s^3}} e^{-\frac{\beta x^2}{2s}-\frac{\beta|y|^2}{2t}} ds + \int_{t/2}^t \frac{1}{t^{3/2}(t-s)} e^{-\frac{\beta x^2}{2t}-\frac{\beta|y|^2}{2(t-s)}} ds \\
&\leq \frac{c_3|x|}{\sqrt{2\pi}} \left(\frac{1}{t} e^{-\frac{\beta|y|^2}{2t}} \int_0^{t/2} \frac{1}{\sqrt{s^3}} e^{-\frac{\beta x^2}{2s}} ds + \frac{1}{t^{3/2}} e^{-\frac{\beta x^2}{2t}} \int_{t/2}^t \frac{1}{t-s} e^{-\frac{\beta|y|^2}{2(t-s)}} ds\right) \\
&\leq \frac{c_3|x|}{\sqrt{2\pi}} \left(\frac{1}{t} e^{-\frac{\beta\rho(x,y)^2}{t}} + \frac{1}{t^{3/2}} e^{-\frac{\beta\rho(x,y)^2}{t}}\right) \leq \frac{c_4|x|}{t^2} e^{-\frac{\beta\rho(x,y)^2}{t}} \leq \frac{c_5}{t^{3/2}} e^{-\frac{\beta\rho(x,y)^2}{t}}.
\end{aligned}$$

Case 3. Still we consider the case that $x \in \mathbb{R}_+$, $y \in D_0$, but now we consider $\nabla_y p(t,x,y)$. We denote $|y|_\rho$ by r. Now in order to partial differentiate $p(t,x,y)$ in y, we first show that the radial process Y is a 1-dimensional diffusion process with drift term in Kato class $K_{1,1}$. It has been shown that Y has an stochastic differential equation characterization as follows.

$$dY_t = dB_t + \frac{1}{Y_t+\epsilon}\mathbf{1}_{\{Y_t>0\}}dt + (2\pi\epsilon - p)dL_t^0,$$

where the local time term can be neglected for the purpose of heat kernel estimates by taking $p = 2\pi\epsilon$. We set $f(x) = \frac{1}{x+\epsilon}\mathbf{1}_{\{x>0\}}$, and one can conveniently check that $f \in K_{1,1}$:

$$\lim_{r\downarrow 0}\sup_{x\in\mathbb{R}_+, x\geq r}\int_{x-r}^{x+r}\frac{1}{y+\epsilon}dy = \lim_{r\downarrow 0}\sup_{x\geq r}\log\frac{x+r+\epsilon}{x-r+\epsilon} = \lim_{r\downarrow 0}\log\frac{2r+\epsilon}{\epsilon} = 0,$$

which shows that Y can be charicterized by

$$dY_t = dB_t + f(Y_t)dt, \qquad \text{where } f(x) = \frac{1}{x+\epsilon}\mathbf{1}_{\{x>0\}} \in K_{1,1}.$$

Therefore by Theorem 2.4 of [18],

$$\frac{\partial}{\partial r}p^Y(t,x,r) \leq \frac{c_1}{t}e^{-\frac{c_2(x^2+r^2)}{t}}.$$

Since when $x \in \mathbb{R}_+$, $y \in D_0$, $p^X(t,x,y) = \frac{1}{r}p^Y(t,-x,r)$, it holds

$$\nabla_y p^X(t,x,y) = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}p^Y(t,-x,r)\frac{\partial r}{\partial x_1} - p^Y(t,-x,r)\frac{\partial}{\partial x_1}\right),\right.$$
$$\left.\left(r\frac{\partial}{\partial r}p^Y(t,-x,r)\frac{\partial r}{\partial x_2} - p^Y(t,-x,r)\frac{\partial r}{\partial x_2}\right)\right].$$

Recall $p^Y(t,-x,r) \leq \frac{c_1}{\sqrt{t}}e^{-\frac{c_2(x^2+r^2)}{t}}$. When $x \in \mathbb{R}_+$, we have

$$\left|\nabla_y p^X(t,x,y)\right| \leq \frac{c_1}{t}e^{-\frac{c_2(x^2+|y|_\rho^2)}{t}}.$$

i.e.

$$|\nabla_y p(t,x,y)| \leq \frac{c_3}{\sqrt{t}}\,p_\beta^0(t,x,y) \quad \text{when } x \in \mathbb{R}_+, y \in D_0. \tag{7.2.7}$$

Case 4. $x, y \in D_0$. In this case we have

$$\nabla_x p(t,x,y) = \nabla_x p_{D_0}(t,x,y) + \nabla_x \overline{p}_{D_0}(t,x,y). \tag{7.2.8}$$

By the result shown in [18],

$$|\nabla_x p(t,x,y)| \le \frac{c_1}{\sqrt{t}} p_{D_0}(t,x,y). \tag{7.2.9}$$

For the second term on the right hand side of (7.2.8), we have

$$\nabla_x \overline{p}_{D_0}(t,x,y) = \int_0^t \mathbb{P}^y\left(T_{a^*} \in ds\right) \nabla_x p(t-s,a^*,x) = \int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*} \in ds\right) p(t-s,a^*,y). \tag{7.2.10}$$

Now we consider (7.2.10) under different cases.

Subcase 1. $|x|_\rho > 1$. In this case it holds

$$\begin{aligned} \left|\nabla_x \overline{p}_{D_0}(t,x,y)\right| &\le \int_0^t \mathbb{P}^y\left(T_{a^*} \in ds\right) \left|\nabla_x p(t-s,a^*,x)\right| \\ &\le \int_0^t \mathbb{P}^y\left(T_{a^*} \in ds\right) \frac{1}{\sqrt{t-s}} \frac{c_2}{t-s} e^{-\frac{\beta|x|_\rho^2}{t-s}} \\ &\le \int_0^t \mathbb{P}^y\left(T_{a^*} \in ds\right) c_3 \cdot e^{-\frac{\beta|x|_\rho^2}{t-s}} \le c_4 \cdot e^{-\frac{\beta(|x|_\rho^2+|y|_\rho^2)}{t}} \le \frac{c_5}{\sqrt{t}} p_\beta^0(t,x,y), \end{aligned} \tag{7.2.11}$$

where the second " $\le$ " is due to the result of Case 3.

Subcase 2. $|x|_\rho < 1$, $|y|_\rho < 1$. For this subcase it holds

$$\begin{aligned} \left|\nabla_x \overline{p}_{D_0}(t,x,y)\right| &= \left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*} \in ds\right) p(t-s,a^*,y)\right| \\ &\le \left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*} \in ds\right) \frac{c_6}{\sqrt{t-s}} e^{-\frac{\beta|y|_\rho^2}{t-s}}\right| \\ &\le \left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*} \in ds\right) c_7 \cdot p(t-s,a^*,\beta|y|_\rho)\right| \\ &= \left|\nabla_x \int_0^t \mathbb{P}^x\left(T_{a^*} \in ds\right) c_7 \cdot p(t-s,a^*,\beta|y|_\rho)\right| \\ &= c_7 \left|\nabla_x p(t,x,\beta|y|_\rho)\right| \le \frac{c_8}{t} e^{-\frac{\beta(|x|_\rho^2+|y|_\rho^2)}{t}} \le \frac{c_9}{\sqrt{t}} p_\beta^0(t,x,y). \end{aligned} \tag{7.2.12}$$

Subcase 3. $|x|_\rho < 1$, $|y|_\rho > 1$. In this case we have

$$\begin{aligned}
\left|\nabla_x \overline{p}_{D_0}(t,x,y)\right| &= \left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*}\in ds\right) p(t-s,a^*,y)\right| \\
&\le \left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*}\in ds\right) \frac{c_{10}}{t-s} e^{-\frac{\beta|y|_\rho^2}{t-s}}\right| \\
&\le \left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*}\in ds\right) \frac{c_{11}}{\sqrt{t-s}} e^{-\frac{\beta|y|_\rho^2}{t-s}}\right| \\
&\le \left|\int_0^t \nabla_x \mathbb{P}^x\left(T_{a^*}\in ds\right) c_{12}\cdot p(t-s,a^*,\beta|y|_\rho)\right| \\
&= c_{12}\left|\nabla_x p(t,x,\beta|y|_\rho)\right| \le \frac{c_{13}}{t^{3/2}} e^{-\frac{\beta(|x|_\rho^2+|y|_\rho^2)}{t}} \le \frac{c_{14}}{\sqrt{t}}\, p^0_\beta(t,x,y), \qquad (7.2.13)
\end{aligned}$$

where in the second " $\le$ " it is used that $|y|_\rho > 1$. By combining (7.2.11), (7.2.12) and (7.2.13) we conclude

$$\left|\nabla_x \overline{p}_{D_0}(t,x,y)\right| \le \frac{c_{15}}{\sqrt{t}} p^0_\beta(t,x,y), \qquad (7.2.14)$$

which together with (7.2.9) shows that

$$|\nabla_x p(t,x,y)| \le \frac{c_{16}}{\sqrt{t}}\, p^0_\beta(t,x,y), \quad (x,y)\in D_0\times D_0. \qquad (7.2.15)$$

□

Now based on the above proposition, we are ready to show the following theorem which is called Duhamel's formula.

Theorem 7.2.2. *Let b be a measurable function on E which is in the family of $L^\infty(E) + L^{p_1,p_2}(E)$ with $p_1\in(1,\infty]$ and $p_2\in(2,\infty]$. Given vectors (α_1,α_2) and (β_1,β_2) satisfying $0<\alpha_1<\beta_1$ and $0<\alpha_2<\beta_2$. Set $\alpha=\alpha_1$ and $\beta=\beta_1$. The following relationship holds.*

$$\int_0^t \int_{z\in E} p^0_\alpha(t-s,x,z)|b(z)|\left|\nabla_z p^0_\beta(s,z,y)\right| dzds \le C_4(t) p^0_\alpha(t,x,y), \quad 0<s<t\le 1, \qquad (7.2.16)$$

where $C_4(t)$ is non-decreasing in t, $C_4(t)\to 0$ as $t\to 0$, and $C_4(t)/t\to 0$ as $t\to\infty$.

Proof. Without loss of generality, we assume b is non-negative. By Proposition 7.2.1, it suffices to show the following inequality.

$$\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D} p^0_\alpha(t-s,x,z)b(z)p^0_\beta(s,z,y)dzds \le c(t)p^0_\alpha(t,x,y). \qquad (7.2.17)$$

We prove (7.2.17) by considering different cases depending on the regions of x and y.

Case 1. $x, y \in \mathbb{R}_+$. In this case we consider the Chapman-Kolmogorov equation depending on the position of z as follows. When $x \in \mathbb{R}_+$, by Hölder's inequality we have for $p, q > 0$ satisfying $1/p + 1/q = 1$,

$$\begin{aligned}
&\int_{z\in\mathbb{R}_+} p^0_\alpha(t-s,x,z)b(z)p^0_\beta(s,z,y)dz \\
&\le \int_0^t \frac{1}{\sqrt{s}}\int_{\mathbb{R}_+}\frac{1}{\sqrt{t-s}}e^{-\frac{\alpha|x-z|^2}{t-s}}b(z)\frac{1}{\sqrt{s}}e^{-\frac{\beta|z-y|^2}{s}}dzds \\
&\le e^{-\frac{\alpha|x-y|^2}{t}}\int_0^t s^{-1+\frac{1}{2q}}\frac{1}{\sqrt{t-s}}\int_{z\in\mathbb{R}_+} b(z)s^{-\frac{1}{2q}}e^{-\frac{(\beta-\alpha)|z-y|^2}{s}}dzds \\
&\le e^{-\frac{\alpha|x-y|^2}{t}}\int_0^t s^{-1+\frac{1}{2q}}\frac{1}{\sqrt{t-s}}\|b\|_q\left(\int_{z\in\mathbb{R}_+}\left(s^{-\frac{1}{2q}}e^{-\frac{(\beta-\alpha)|z-y|^2}{s}}\right)^q dz\right)^{1/q}ds \\
&\le c\|b\|_p e^{-\frac{\alpha|x-y|^2}{t}}\cdot\frac{t^{\frac{1}{2q}}}{\sqrt{t}}.
\end{aligned}$$

Thus we have shown that the desired inequality holds in this case for $b \in L^p(\mathbb{R}_+)$ with $p \in (1,\infty]$. The same computation as above applies to the case that $z \in D_0$ with $|z|_\rho < 1$. When $z \in D_0$ with $|z|_\rho > 1$, again by Hölder's inequality we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}}\int_{z\in D_0,|z|_\rho>1}\frac{1}{t-s}e^{-\frac{\alpha\rho(x,z)^2}{t-s}}b(z)\frac{1}{s}e^{-\frac{\beta\rho(y,z)^2}{s}}dzds \\
&\le e^{-\frac{\alpha\rho(x,y)^2}{t}}\int_0^t\int_{z\in D_0,|z|_\rho>1}\frac{1}{t-s}e^{-\frac{\alpha|z|_\rho^2}{t-s}}b(z)dzds,
\end{aligned}$$

from which one can tell that for $b \in L^p(D_0\backslash B(a^*,1))$ with $p \in [1,\infty]$, the desired inequality holds with some $c(t)$ satisfying the desired restriction. Therefore the inequality has been proved for this case.

Case 2. $x, y \in D_0$, $|x|_\rho + |y|_\rho \le 9$. We first consider the case when $z \in D_0$, $|z|_\rho < 10$. In this case, by the heat kernel estimate we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}}\int_{z\in D_0,|z|_\rho<10} p^0_\alpha(t-s,x,z)b(z)p^0_\beta(s,z,y)dzds \\
&\le \int_0^t \frac{1}{\sqrt{s}}\int_{z\in D_0,|z|_\rho<10}\left[\frac{c}{\sqrt{t-s}}e^{-\frac{\alpha_1\rho(x,z)^2}{t-s}}+\frac{c}{t-s}\left(1\wedge\frac{|x|_\rho}{\sqrt{t-s}}\right)\left(1\wedge\frac{|z|_\rho}{\sqrt{t-s}}\right)e^{-\frac{\alpha_2|x-z|_e^2}{t-s}}\right] \\
&\cdot b(z)\left[\frac{c}{\sqrt{s}}e^{-\frac{\beta_1\rho(z,y)^2}{s}}+\frac{c}{s}\left(1\wedge\frac{|y|_\rho}{\sqrt{s}}\right)\left(1\wedge\frac{|z|_\rho}{\sqrt{s}}\right)e^{-\frac{\beta_2|y-z|_e^2}{s}}\right]dzds. \qquad (7.2.18)
\end{aligned}$$

The right hand side of (7.2.18) can be expanded into the sum of four terms. We now estimate the right hand side of (7.2.18) term by term. First of all for $p > 0$ we have by Hölder's inequality,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<10} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1\rho(x,z)^2}{t-s}} b(z) \frac{1}{\sqrt{s}} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \\
&\le \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-\frac{1}{2}+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho<10} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha_1\rho(x,z)^2}{t-s}} b(z) s^{-\frac{1}{2p}} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \\
&\le \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-\frac{1}{2}+\frac{1}{2p}} ds \|b\|_q \left(\int_{z\in D_0, |z|_\rho<10} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1 p\rho(x,z)^2}{t-s}} \frac{1}{\sqrt{s}} e^{-\frac{\beta p\rho(z,y)^2}{s}} dz \right)^{1/p} \\
&\le c\|b\|_q \cdot \frac{t^{\frac{1}{2p}}}{t^{\frac{1}{2}-\frac{1}{2p}}} \left(\int_{r=0}^{20} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1 p r^2}{t-s}} \frac{1}{\sqrt{s}} e^{-\frac{\beta_1 p(r-\rho(x,y))^2}{s}} dr \right)^{1/p} \\
&\le \frac{c}{\sqrt{t}} e^{-\frac{\alpha_1\rho(x,y)^2}{t}} \cdot t^{\frac{1}{2p}} \|b\|_q, \qquad (7.2.19)
\end{aligned}$$

which implies that $b \in L^q(D_0 \cap B(a^*, 1))$ with $q \in (1, \infty]$ will yield the desired inequality. Secondly, for the right hand side of (7.2.18), for $p, q, r > 0$ such that $1/p + 1/q + 1/r = 1$, by Hölder's inequality we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<10} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1\rho(x,z)^2}{t-s}} b(z) \frac{1}{s} \left(1 \wedge \frac{|y|_\rho}{\sqrt{s}}\right) \left(1 \wedge \frac{|z|_\rho}{\sqrt{s}}\right) e^{-\frac{\beta_2|z-y|_e^2}{s}} dzds \\
&\le \int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<10} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1\rho(x,z)^2}{t-s}} \frac{1}{s} e^{-\frac{\beta_2|z-y|_e^2}{s}} b(z) dzds \\
&= \int_0^t s^{-\frac{3}{2}+\frac{1}{2p}+\frac{1}{q}} (t-s)^{-\frac{1}{2}+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho<4} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} s^{-\frac{1}{2p}} e^{-\frac{\alpha|y-z|_e^2}{s}} s^{-\frac{1}{q}} e^{-\frac{(\beta-\alpha)|y-z|_e^2}{s}} b(z) dzds \\
&\le \int_0^t s^{-\frac{3}{2}+\frac{1}{2p}+\frac{1}{q}} (t-s)^{-\frac{1}{2}+\frac{1}{2p}} \left(\frac{1}{\sqrt{t}} e^{-\frac{p\alpha\rho(x,y)^2}{t}} \right)^{1/p} ds \cdot \|b\|_r \cdot \text{const} \\
&= c\|b\|_r e^{-\frac{\alpha\rho(x,y)^2}{t}} \frac{1}{t^{1-\frac{1}{2p}-\frac{1}{q}}}, \qquad (7.2.20)
\end{aligned}$$

provided that $1/q + 1/2p > 1/2$. Therefore by choosing an appropriate pair of (p, q), the desired inequality holds for $b \in L^r(D_0 \cap B(a^*, 1))$ with $r \in (2, \infty]$. Thirdly for the right hand side of (7.2.18) we have for any $p, q, r > 0$ satisfying $1/p + 1/q + 1/r = 1$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<10} \frac{1}{t-s}\left(1\wedge \frac{|x|_\rho}{\sqrt{t-s}}\right)\left(1\wedge \frac{|z|_\rho}{\sqrt{t-s}}\right) b(z) e^{-\frac{\alpha_2|x-z|_e^2}{t-s}} \frac{1}{\sqrt{s}} e^{-\frac{\beta_1\rho(z,y)^2}{s}} dz ds \\
&\le \int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<10} \frac{1}{t-s} e^{-\frac{\alpha_2|x-z|_e^2}{t-s}} \frac{1}{\sqrt{s}} e^{-\frac{\beta_1\rho(z,y)^2}{s}} b(z) dz ds \\
&= \int_0^t s^{-1+\frac{1}{p}} \int_{z\in D_0, |z|_\rho<10} s^{-\frac{1}{p}} e^{-\frac{\beta\rho(z,y)^2}{s}} (t-s)^{-\frac{1}{p}} e^{-\frac{(\alpha_1+\alpha_2)|x-z|_e^2}{2(t-s)}} (t-s)^{-\frac{1}{q}} e^{-\frac{(\alpha_2-\alpha_1)|x-z|_e^2}{2(t-s)}} t^{-\frac{1}{r}} b(z) dz ds \\
&\le \int_0^t s^{-1+\frac{1}{p}} \left(\int_{z\in D_0, |z|_\rho<10} \frac{1}{s} e^{-\frac{p\beta\rho(z,y)^2}{s}} \frac{1}{t-s} e^{-\frac{p(\alpha_1+\alpha_2)|x-z|_e^2}{2(t-s)}} dz\right)^{1/p} \\
&\quad \times \left(\int_{z\in D_0, |z|_\rho<10} \frac{1}{t-s} e^{-\frac{q(\alpha_2-\alpha_1)|z-x|_e^2}{2(t-s)}} dz\right)^{1/q} \|b\|_r \frac{1}{t^{1/r}} ds \\
&\le \int_0^t \frac{1}{s^{1-1/p}} ds \left(\frac{1}{\sqrt{s}} \cdot \frac{1}{\sqrt{t-s}} \cdot \frac{1}{\sqrt{t}} e^{-\frac{p\alpha\rho(x,y)^2}{t}}\right)^{1/p} \cdot \|b\|_r \frac{c}{t^{1/r}} ds \\
&\le c\|b\|_r \frac{1}{t^{1/(2p)+1/r}} e^{-\frac{\alpha\rho(x,y)^2}{t}}. \qquad (7.2.21)
\end{aligned}$$

It has been assumed in the above computation that $1/(2p)+1/r < 1/2$. Therefore we would like $b \in L^r(D_0 \cap B(a^*, 1))$ with $r \in (2, \infty]$. Finally to check the last term of (7.2.18), it suffices to notice that

$$\frac{c}{\sqrt{t}} e^{-\frac{\gamma\rho(x,y)^2}{t}} + p_{D_0}(t,x,y) \le p(t,x,y) \le \frac{c}{\sqrt{t}} e^{-\frac{\alpha\rho(x,y)^2}{t}} + p_{D_0}(t,x,y),$$

which implies that the desired inequality holds for $b \in L^p(D_0 \cap B(a^*, 10))$ with $p \in (2, \infty]$. When $z \in \mathbb{R}_+$, we have for $p, q > 0$ satisfying $1/p + 1/q = 1$,

$$\begin{aligned}
&\int_0^t \int_{z\in\mathbb{R}_+} p_\alpha^0(t-s,x,z) b(z) \frac{1}{\sqrt{s}} p_\beta^0(s,z,y) dz ds \\
&\le \int_0^t \frac{1}{\sqrt{s}} \int_{z\in\mathbb{R}_+} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha\rho(x,z)}{t-s}} b(z) \frac{1}{\sqrt{s}} e^{-\frac{\beta\rho(z,y)^2}{s}} dz ds \\
&\le \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-\frac{1}{2}+\frac{1}{2p}} \int_{z\in\mathbb{R}_+} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} s^{-\frac{1}{2p}} e^{-\frac{\beta\rho(z,y)^2}{s}} b(z) dz ds \\
&\le \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-\frac{1}{2}+\frac{1}{2p}}\, ds \cdot c t^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,y)^2}{t}} \|b\|_q \\
&\le \frac{c}{\sqrt{t}} e^{-\frac{\alpha\rho(x,y)^2}{t}} \|b\|_q \cdot t^{\frac{1}{2p}}, \qquad (7.2.22)
\end{aligned}$$

where $p > 0$. Thus the inequality holds for $b \in L^q(\mathbb{R}_+)$ with $q \in (1, \infty]$. To complete the proof to this case, we need to prove the inequality for the case that $z \in D_0$, $|z|_\rho \ge 10$. For

$p, q > 0$ satisfying $1/p + 1/q = 1$, it holds

$$\begin{aligned}
&\int_0^t \int_{z\in D_0, |z|_\rho \geq 10} \frac{1}{\sqrt{s}} p_\alpha^0(t-s,x,z)b(z)p_\beta^0(s,z,y)dzds \\
&\leq \int_0^t cs^{-1+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho \geq 10} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} s^{-\frac{1}{p}-\frac{1}{2}} e^{-\frac{\beta\rho(z,y)^2}{s}} b(z)dzds \\
&\leq \int_0^t cs^{-1+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho \geq 10} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} s^{-\frac{1}{p}} e^{-\frac{\beta\rho(z,y)^2}{s}} b(z)dzds \\
&\leq \int_0^t cs^{-1+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \left(\int_{z\in D_0, |z|_\rho \geq 10} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha\rho(x,z)^2}{t-s}} \frac{1}{s} e^{-\frac{p\beta\rho(z,y)^2}{s}} \right)^{1/p} \|b\|_q ds \\
&\leq \int_0^t cs^{-1+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \left(\int_{r\geq 9} \frac{r}{s} e^{-\frac{p\beta r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} \|b\|_q ds \\
&\leq \int_0^t cs^{-1+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \left(\int_{r\geq 9} \frac{1}{\sqrt{s}} e^{-\frac{p\alpha r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} \|b\|_q ds \\
&\leq \frac{t^{1/p}}{t} e^{-\frac{\alpha\rho(x,y)^2}{t}} \|b\|_q,
\end{aligned} \tag{7.2.23}$$

thus the desired inequality holds for $b \in L^q(D_0 \backslash B(a^*, 10))$ with $q \in (1, \infty]$. The proof to this case is thus complete.

Case 3. $x, y \in D_0$, $|x|_\rho \geq 6$, $|y|_\rho < 3$. Again we prove the inequality for different regions of z. When $z \in \mathbb{R}_+$, we have for $p, q > 0$ satisfying $1/p + 1/q = 1$,

$$\begin{aligned}
&\int_0^t \int_{z\in\mathbb{R}_+} p_\alpha^0(t-s,x,z)b(z)p_\beta^0(s,z,y)dzds \\
&\leq ce^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t s^{-1+\frac{1}{2p}}(t-s)^{-1+\frac{1}{2p}} \int_{z\in\mathbb{R}_+} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha|z|_\rho^2}{t-s}} s^{-\frac{1}{2p}} e^{-\frac{\beta|z|_\rho^2}{s}} b(z)dzds \\
&\leq ce^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t s^{-1+\frac{1}{2p}}(t-s)^{-1+\frac{1}{2p}} \int_{z\in\mathbb{R}_+} \frac{1}{(t-s)^{\frac{1}{2p}}} e^{-\frac{\alpha|z|_\rho^2}{t-s}} \frac{1}{s^{\frac{1}{2p}}} e^{-\frac{\beta|z|_\rho^2}{s}} b(z)dzds \\
&\leq ce^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t s^{-1+\frac{1}{2p}}(t-s)^{-1+\frac{1}{2p}} ds \cdot \frac{\|b\|_q}{t^{\frac{1}{2p}}} \leq \frac{c\|b\|_q}{t^{1-\frac{1}{2p}}} e^{-\frac{\alpha\rho(x,y)^2}{t}}.
\end{aligned} \tag{7.2.24}$$

Thus the desired inequality holds for $b \in L^q(\mathbb{R}_+)$ with $q \in (1, \infty]$.

When $z \in D_0$, $|z|_\rho < 4$ by the heat kernel estimate for BMVD we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<4} p^0_\alpha(t-s,x,z)b(z)p^0_\beta(s,z,y)dzds \\
\le &\int_0^t \frac{c}{\sqrt{s}} \int_{|z|_\rho<4} \frac{1}{t-s}e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z)\left(\frac{1}{\sqrt{s}}e^{-\frac{\beta\rho(z,y)^2}{s}} + \frac{1}{s}e^{-\frac{\beta|z-y|_e^2}{s}}\right)dzds \\
= &\int_0^t \frac{c}{\sqrt{s}} \int_{z\in D_0|z|_\rho<4} \frac{1}{t-s}e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z)\frac{1}{\sqrt{s}}e^{-\frac{\beta_1\rho(z,y)^2}{s}}dzds+ \\
&\int_0^t \frac{c}{\sqrt{s}} \int_{z\in D_0|z|_\rho<4} \frac{1}{t-s}e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z)\frac{1}{s}e^{-\frac{\beta_2|z-y|_e^2}{s}}dzds. \qquad (7.2.25)
\end{aligned}$$

For the first term on the right hand side of (7.2.25) we have for $p,q>0$ satisfying $1/p+1/q = 1$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0|z|_\rho<4} \frac{1}{t-s}e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z)\frac{1}{\sqrt{s}}e^{-\frac{\beta_1\rho(z,y)^2}{s}}dzds \\
= &\int_0^t s^{-1+\frac{1}{2p}}(t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0|z|_\rho<4} (t-s)^{-\frac{1}{2p}}e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z)s^{-\frac{1}{2p}}e^{-\frac{\beta_1\rho(z,y)^2}{s}}dzds \\
\le &\int_0^t s^{-1+\frac{1}{2p}}(t-s)^{-1+\frac{1}{2p}}ds\left(\int_{z\in D_0|z|_\rho<4} \frac{1}{\sqrt{t-s}}e^{-\frac{p\alpha\rho(x,z)^2}{t-s}} \frac{1}{\sqrt{s}}e^{-\frac{p\beta_1\rho(z,y)^2}{s}}dz\right)^{1/p} \|b\|_q \\
\le &\int_0^t s^{-1+\frac{1}{2p}}(t-s)^{-1+\frac{1}{2p}}ds\left(\int_{r=0}^{10} \frac{c}{\sqrt{s}}e^{-\frac{p\beta r^2}{s}} \frac{1}{\sqrt{t-s}}e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}}dr\right)^{1/p} \|b\|_q \\
\le &e^{-\frac{\alpha\rho(x,y)^2}{t}}\frac{c\|b\|_q}{t^{1-\frac{1}{2p}}}. \qquad (7.2.26)
\end{aligned}$$

Therefore the desired inequality holds for $b \in L^q(D_0 \cap B(a^*,4))$ with $q \in (1,\infty]$. Now for the second term on the right hand side of (7.2.25) we first have the following computation result.

$$\begin{aligned}
&\int_{z\in D_0,|z|_\rho<4} \frac{1}{\sqrt{t-s}}e^{-\frac{\alpha\rho(x,z)^2}{t-s}} \frac{1}{\sqrt{s}}e^{-\frac{\alpha|y-z|_e^2}{s}} \frac{1}{\sqrt{s}}e^{-\frac{(\beta-\alpha)|y-z|_e^2}{s}}dz \\
\le &\left(\int_{z\in D_0,|z|_\rho<4} \frac{1}{t-s}e^{-\frac{2\alpha\rho(x,z)^2}{t-s}} \frac{1}{s}e^{-\frac{2\alpha\rho(y,z)^2}{s}}dz\right)^{1/2} \left(\int_{z\in D_0,|z|_\rho<4} \frac{1}{s}e^{-\frac{2(\beta-\alpha)|y-z|_e^2}{s}}dz\right)^{1/2} \\
\le &\left(\int_{r=|x|_\rho-4}^{|x|_\rho+4+2\epsilon} \int_{\theta=-\arctan\frac{4+\epsilon}{|x|_\rho+\epsilon}}^{\arctan\frac{4+\epsilon}{|x|_\rho+\epsilon}} \frac{r}{t-s}e^{-\frac{2\alpha r^2}{t-s}} \frac{1}{s}e^{-\frac{2\alpha(r-\rho(x,y))^2}{s}}d\theta dr\right)^{1/2} \times \text{const} \\
\le &c\cdot\left(\int_{r=|x|_\rho-4}^{|x|_\rho+4+2\epsilon} \frac{1}{t-s}e^{-\frac{2\alpha r^2}{t-s}} \frac{1}{s}e^{-\frac{2\alpha(r-\rho(x,y))^2}{s}}dr\right)^{1/2} \le c\cdot\left(\frac{1}{\sqrt{s(t-s)}}\cdot\frac{1}{\sqrt{t}}e^{-\frac{2\alpha\rho(x,y)^2}{t}}\right)^{1/2}. \\
&\qquad (7.2.27)
\end{aligned}$$

It is used in the fourth " $\leq$ " that $\lim_{x\to\infty} x \arctan \frac{1}{x} = 1$. Now we are ready to estimate the second term on the right hand side of (7.2.25). Below we assume $1/p + 1/q + 1/r = 1$.

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<4} \frac{1}{s(t-s)} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} e^{-\frac{\beta|y-z|_e^2}{s}} b(z)dzds \\
&= \int_0^t s^{-\frac{3}{2}+\frac{1}{2p}+\frac{1}{q}} (t-s)^{-1+\frac{1}{2p}} \\
&\qquad \int_{z\in D_0, |z|_\rho<4} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} s^{-\frac{1}{2p}} e^{-\frac{\alpha|y-z|_e^2}{s}} s^{-\frac{1}{q}} e^{-\frac{(\beta-\alpha)|y-z|_e^2}{s}} b(z)dzds \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{2p}+\frac{1}{q}} (t-s)^{-1+\frac{1}{2p}} \left(\frac{1}{\sqrt{s(t-s)}} \cdot \frac{1}{\sqrt{t}} e^{-\frac{p\alpha\rho(x,y)^2}{t}} \right)^{1/2p} ds \|b\|_r \\
&= e^{-\frac{\alpha\rho(x,y)^2}{t}} \frac{c\|b\|_r}{t^{\frac{3}{2}-\frac{1}{4p}-\frac{1}{q}}}. \qquad (7.2.28)
\end{aligned}$$

In the computation above, as long as $1/r < 1/2$, we can choose p, q such that $1/(4p) + 1/q > 1/2$, which implies that the desired inequality holds for $b \in L^r(D_0 \cap B(a^*, 4))$ with $r > 2$. Now we prove the same inequality for the subcase when $z \in D_0$, $|z|_\rho \geq 4$. Now we have for $p, q > 0$ satisfying $1/p + 1/q = 1$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho\geq 4, z\in D_0} \frac{1}{t-s} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \\
&= \int_0^t cs^{-\frac{3}{2}+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho\geq 4} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) s^{-\frac{1}{p}} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \\
&\leq \int_0^t cs^{-\frac{3}{2}+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{z\in D_0, |z|_\rho\geq 4} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha\rho(x,z)^2}{t-s}} \frac{1}{s} e^{-\frac{p\beta\rho(z,y)^2}{s}} dz \right)^{1/p} \|b\|_q ds \\
&\leq \int_0^t cs^{-\frac{3}{2}+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{r\geq 1} \frac{r}{s} e^{-\frac{p\beta r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} \|b\|_q ds \\
&\leq \int_0^t cs^{-\frac{3}{2}+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{r\geq 1} \frac{1}{\sqrt{s}} e^{-\frac{p\alpha r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} \|b\|_q ds \\
&\leq \|b\|_q \frac{c}{t^{\frac{3}{2}-\frac{1}{p}}} e^{-\frac{\alpha\rho(x,y)^2}{t}}, \qquad (7.2.29)
\end{aligned}$$

provided that $1/p > 1/2$. Therefore the desired inequality holds for $b \in L^q(D_0 \backslash B(a^*, 4))$ with $q \in (2, \infty]$. This completes the proof of Case 3.

Case 4. $x, y \in D_0$, $|x|_\rho \le 1$, $|y|_\rho > 3$. Again we first consider $z \in \mathbb{R}_+$.

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in\mathbb{R}_+} p^0_\alpha(t-s,x,z)b(z)p^0_\beta(s,z,y)dzds \\
&\le c\int_0^t \int_{z\in\mathbb{R}_+} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha\rho(x,y)^2}{t}} b(z) e^{-\frac{(\beta-\alpha)\rho(z,y)^2}{s}} dzds \\
&\le ce^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t \frac{1}{\sqrt{t-s}} ds \int_{z\in\mathbb{R}_+} b(z) e^{-\frac{(\beta-\alpha)|z|^2}{t}} dz,
\end{aligned} \tag{7.2.30}$$

thus taking $b \in L^p(\mathbb{R}_+)$ with $p \in [1,\infty]$ will satisfy the desired inequality.

When $z \in D_0$, $|z|_\rho < 2$, we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<2} p^0_\alpha(t-s,x,z)b(z)p^0_\beta(s,z,y)dzds \\
\le &\int_0^t \frac{c}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<2} \left[\frac{1}{\sqrt{t-s}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} + \frac{1}{t-s} e^{-\frac{\alpha_2|x-z|_e^2}{t-s}} \right] b(z)\, \frac{1}{s} e^{-\frac{(\alpha+\beta)|z-y|_e^2}{2s}} dzds \\
= &\int_0^t \frac{c}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<2} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{(\alpha+\beta)|z-y|_e^2}{2s}} dzds \\
+ &\int_0^t \frac{c}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<2} \frac{1}{t-s} e^{-\frac{\alpha|x-z|_e^2}{t-s}} b(z)\, \frac{1}{s} e^{-\frac{(\alpha+\beta)|z-y|_e^2}{2s}} dzds.
\end{aligned} \tag{7.2.31}$$

The first term of the (7.2.31) can be computed as follows.

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<2} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{(\alpha+\beta)|z-y|_e^2}{2s}} dzds \\
&\le c\int_0^t \int_{z\in D_0, |z|_\rho<2} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha\rho(x,y)^2}{t}} b(z) e^{-\frac{(\beta-\alpha)\rho(z,y)^2}{2s}} dzds \\
&\le ce^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t \frac{1}{\sqrt{t-s}} ds \int_{z\in D_0, |z|_\rho<2} b(z) e^{-\frac{(\beta-\alpha)}{t}} dz,
\end{aligned} \tag{7.2.32}$$

thus the desired inequality will be satisfied for $b \in L^q(D_0 \cap B(a^*, 2))$ for $p \in [1,\infty]$. Now for the second term of (7.2.31) we have for $p, q > 0$ satisfying $1/p + 1/q = 1$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<2} \frac{1}{t-s} e^{-\frac{\alpha_2|x-z|_e^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \\
&\le \int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<2} \frac{1}{t-s} e^{-\frac{\alpha_2|x-z|_e^2}{t-s}} b(z) e^{-\frac{(\beta+\alpha)\rho(z,y)^2}{2s}} dzds \\
&\le \int_0^t (t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho<2} \frac{1}{s^{1/p}} e^{-\frac{(\beta+\alpha)\rho(z,y)^2}{2s}} \frac{1}{(t-s)^{\frac{1}{2p}}} e^{-\frac{\alpha_2|x-z|_e^2}{t-s}} b(z) dzds \\
&\le \int_0^t (t-s)^{-1+\frac{1}{2p}} \left(\int_{z\in D_0, |z|_\rho<2} \frac{1}{s} e^{-\frac{p(\beta+\alpha)\rho(z,y)^2}{2s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha_2|x-z|_e^2}{t-s}} dz \right)^{1/p} ds \cdot \|b\|_q \\
&\le \int_0^t (t-s)^{-1+\frac{1}{2p}} \left(\int_{r>1} \frac{r}{s} e^{-\frac{p(\alpha+\beta)r^2}{2s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha_2(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} ds \cdot \|b\|_q \\
&\le \int_0^t (t-s)^{-1+\frac{1}{2p}} \left(\int_{r>1} \frac{1}{\sqrt{s}} e^{-\frac{p\alpha r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha_2(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} ds \cdot \|b\|_q \\
&\le \frac{c\|b\|_q}{t^{\frac{1}{2p}}} e^{-\frac{\alpha\rho(x,y)^2}{t}} \cdot t^{\frac{1}{2p}} = c\|b\|_q e^{-\frac{\alpha\rho(x,y)^2}{t}}, \qquad (7.2.33)
\end{aligned}$$

assuming $p > 0$. Therefore the desired inequality holds for $b \in L^q(D_0 \cap B(a^*, 2))$ with $q \in (1, \infty]$. Now for $D_0 \backslash B(a^*, 2)$, we have for $p, q > 0$ satisfying $1/p + 1/q = 1$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho\ge 2} \frac{1}{t-s} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \\
&\le \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho\ge 2} s^{-\frac{1}{2p}-\frac{1}{2}} e^{-\frac{\beta\rho(z,y)^2}{s}} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) dzds \\
&\le \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-1+\frac{1}{2p}} \int_{r>0} r s^{-\frac{1}{2p}-\frac{1}{2}} e^{-\frac{\beta r^2}{s}} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha(r-\rho(x,y))^2}{t-s}} b(z) dzds \\
&\le \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{r>0} \frac{c}{\sqrt{s}} e^{-\frac{\alpha r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} ds \cdot \|b\|_q \\
&\le \frac{c\|b\|_q}{t^{1-\frac{1}{2p}}} e^{-\frac{\alpha\rho(x,y)^2}{t}}, \qquad (7.2.34)
\end{aligned}$$

assuming that $1/(2p) > 0$. Thus the desired inequality holds for $b \in L^q(D_0 \backslash B(a^*, 2))$ with $q \in (1, \infty]$. The proof to this case is complete by combining (7.2.30), (7.2.31) and (7.2.34),
Case 5. $x, y \in D_0$, $|x|_\rho \ge 1$, $|y|_\rho \ge 3$. Again first of all when $z \in \mathbb{R}_+$ we have for $p, q > 0$

satisfying $1/p + 1/q = 1$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in\mathbb{R}_+} \frac{1}{t-s} e^{-\frac{\alpha(|x|_\rho^2+|z|_\rho^2)}{t-s}} b(z) \frac{1}{s} e^{-\frac{\beta(|y|_\rho^2+|z|_\rho^2)}{s}} dzds \\
&\leq \int_0^t \int_{z\in\mathbb{R}_+} \frac{c}{t-s} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) e^{-\frac{\alpha\rho(z,y)^2}{s}} dzds \\
&\leq c \cdot e^{-\frac{\alpha(|x|_\rho^2+|y|_\rho^2)}{t}} \int_0^t \int_{z\in\mathbb{R}_+} \frac{1}{t-s} e^{-\frac{\alpha|z|^2}{t-s}} b(z) e^{-\frac{\alpha|z|^2}{s}} dzds \\
&\leq c \cdot e^{-\frac{\alpha(|x|_\rho^2+|y|_\rho^2)}{t}} \int_0^t (t-s)^{-1+\frac{1}{2p}} \int_{z\in\mathbb{R}_+} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha|z|^2}{t-s}} s^{-\frac{1}{2p}} e^{-\frac{\beta|z|^2}{s}} b(z) dzds \\
&\leq \frac{c\|b\|_q}{t^{\frac{1}{2p}}} e^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t (t-s)^{-1+\frac{1}{2p}} ds \leq c\|b\|_q e^{-\frac{\alpha\rho(x,y)^2}{t}}, \qquad (7.2.35)
\end{aligned}$$

assuming that $p > 0$. Therefore the inequality holds for $b \in L^q(\mathbb{R}_+)$ with $q \in (1, \infty]$. Also when $z \in D_0$ it holds

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0} \frac{1}{t-s} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{\beta\rho(y,z)^2}{s}} dzds \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} s^{-\frac{1}{p}} e^{-\frac{\beta\rho(y,z)^2}{s}} b(z) dzds \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{z\in D_0} \frac{1}{s} e^{-\frac{p\beta\rho(z,y)^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha\rho(x,z)^2}{t-s}} \right)^{1/p} \|b\|_q ds \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{r>0} \frac{r}{s} e^{-\frac{p\beta r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} ds \cdot \|b\|_q \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{r>0} \frac{1}{\sqrt{s}} e^{-\frac{p\alpha r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} ds \cdot \|b\|_q \\
&\leq \frac{c\|b\|_q t^{1/p-1/2}}{t} e^{-\frac{\alpha\rho(x,y)^2}{t}}, \qquad (7.2.36)
\end{aligned}$$

assuming $1/p > 1/2$, which implies that the inequality holds for $b \in L^q(D_0)$ with $q \in (2, \infty]$. (7.2.36) together with (7.2.35) proves Case 5.

Case 6. $x \in \mathbb{R}_+$, $y \in D_0$, $|y|_\rho \geq 2$. When $z \in \mathbb{R}_+ \cup (D_0 \cap B(a^*, 1))$ we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in\mathbb{R}_+\cup(D_0\cap B(a^*,1))} p_\alpha^0(t-s,x,z)b(z)p_\beta^0(s,z,y)dzds \\
&\leq c\int_0^t \frac{1}{\sqrt{s}} \int_{z\in\mathbb{R}_+\cup(D_0\cap B(a^*,1))} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha|x-z|^2}{t-s}} b(z)\frac{1}{s}e^{-\frac{\beta\rho(z,y)^2}{s}}dzds \\
&\leq ce^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t \frac{1}{\sqrt{s}} \int_{z\in\mathbb{R}_+\cup(D_0\cap B(a^*,1))} \frac{1}{\sqrt{t-s}} b(z)\frac{1}{s}e^{-\frac{(\beta-\alpha)\rho(z,y)^2}{s}}dzds \\
&\leq ce^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t \int_{z\in\mathbb{R}_+\cup(D_0\cap B(a^*,1))} \frac{1}{\sqrt{t-s}} b(z)e^{-\frac{(\beta-\alpha)\rho(z,y)^2}{2s}}dzds \leq c\|b\|_q e^{-\frac{\alpha\rho(x,y)^2}{t}},
\end{aligned} \tag{7.2.37}$$

provided that $b \in L^q(\mathbb{R}_+ \cup (D_0 \cap B(a^*, 1)))$ with $q \in [1, \infty]$.

Finally for Case 6, when $z \in D_0$, $|z|_\rho > 1$ we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0,|z|_\rho>1} \frac{1}{t-s} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z)\frac{1}{s}e^{-\frac{\beta\rho(y,z)^2}{s}}dzds \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0,|z|_\rho>1} (t-s)^{-\frac{1}{2p}}e^{-\frac{\alpha\rho(x,z)^2}{t-s}} s^{-\frac{1}{p}}e^{-\frac{\beta\rho(y,z)^2}{s}}b(z)dzds \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \left(\int_{z\in D_0,|z|_\rho>1} \frac{1}{s}e^{-\frac{p\beta\rho(z,y)^2}{s}} \frac{1}{\sqrt{t-s}}e^{-\frac{\alpha\rho(x,z)^2}{t-s}}\right)^{1/p} \|b\|_q ds \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \left(\int_{r>0} \frac{r}{s}e^{-\frac{\beta r^2}{s}} \frac{1}{\sqrt{t-s}}e^{-\frac{\alpha(r-\rho(x,y))^2}{t-s}}dr\right)^{1/p} ds\cdot\|b\|_q \\
&\leq \int_0^t s^{-\frac{3}{2}+\frac{1}{p}}(t-s)^{-1+\frac{1}{2p}} \left(\int_{r>0} \frac{1}{\sqrt{s}}e^{-\frac{\alpha r^2}{s}} \frac{1}{\sqrt{t-s}}e^{-\frac{\alpha(r-\rho(x,y))^2}{t-s}}dr\right)^{1/p} ds\cdot\|b\|_q \\
&\leq \frac{c\|b\|_q t^{\frac{1}{p}-\frac{1}{2}}}{t} e^{-\frac{\alpha\rho(x,y)^2}{t}}.
\end{aligned} \tag{7.2.38}$$

provided that $1/p > 1/2$. Thus again the inequality holds for $b \in L^q(D_0 \backslash B(a^*, 2))$ with $q \in (2, \infty]$. (7.2.38) together with (7.2.37) proves Case 6.

Case 7. $x \in \mathbb{R}_+$, $y \in D_0$, $|y|_\rho < 2$. Again we first consider the case when $z \in \mathbb{R}_+$.

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in\mathbb{R}_+} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha|x-z|_e^2}{t-s}} b(z)\frac{1}{\sqrt{s}}e^{-\frac{\beta\rho(z,y)^2}{s}}dzds \\
&\leq e^{-\frac{\alpha\rho(x,y)^2}{t}} \int_0^t s^{-1+\frac{1}{2p}} \frac{1}{\sqrt{t-s}} \int_{z\in\mathbb{R}_+} b(z)s^{-\frac{1}{2p}}e^{-\frac{(\beta-\alpha)\rho(z,y)^2}{s}}dzds \leq \frac{c\|b\|_q t^{\frac{1}{2p}}}{\sqrt{t}} e^{-\frac{\alpha\rho(x,y)^2}{t}},
\end{aligned} \tag{7.2.39}$$

provided that $p > 0$. So again the inequality holds for $b \in L^q(\mathbb{R}_+)$ with $q \in (1, \infty]$. When $z \in D_0$, $|z|_\rho < 3$, we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho < 3} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} b(z) \\
&\qquad \left(\frac{1}{\sqrt{s}} e^{-\frac{\beta_1 \rho(z,y)^2}{s}} + \frac{1}{s} \left(1 \wedge \frac{|y|_\rho}{\sqrt{s}}\right) \left(1 \wedge \frac{|z|_\rho}{\sqrt{s}}\right) e^{-\frac{\beta_2 |z-y|_e^2}{s}} \right) dz ds \\
&\le \int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho < 3} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} b(z) \frac{1}{\sqrt{s}} e^{-\frac{\beta_1 \rho(z,y)^2}{s}} dz ds \\
&+ \int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho < 3} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{\beta_2 |z-y|_e^2}{s}} dz ds. \qquad (7.2.40)
\end{aligned}$$

For the first term on the right hand side of (7.2.41), we can apply the same computation as (7.2.39) to conclude that the inequality holds for $b \in L^q(D_0 \cap B(a^*, 3))$ with $q \in (1, \infty]$. For the second term on the right hand side of (7.2.41), we have for p, q, r such that $1/p + 1/q + 1/r = 1$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho < 3} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1 \rho(x,z)^2}{t-s}} b(z) \frac{1}{s} \left(1 \wedge \frac{|y|_\rho}{\sqrt{s}}\right) \left(1 \wedge \frac{|z|_\rho}{\sqrt{s}}\right) e^{-\frac{\beta_2 |z-y|_e^2}{s}} dz ds \\
&\le \int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho < 3} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1 \rho(x,z)^2}{t-s}} \frac{1}{s} e^{-\frac{\beta_2 |z-y|_e^2}{s}} b(z) dz ds \\
&\le \int_0^t \frac{1}{\sqrt{s}} \int_{z \in D_0, |z|_\rho < 3} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1 \rho(x,z)^2}{t-s}} \frac{1}{s} e^{-\frac{\beta_2 |z-y|_e^2}{s}} b(z) dz ds \\
&= \int_0^t s^{-\frac{3}{2} + \frac{1}{2p} + \frac{1}{q}} (t-s)^{-\frac{1}{2} + \frac{1}{2p}} \\
&\qquad \int_{z \in D_0, |z|_\rho < 3} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} s^{-\frac{1}{2p}} e^{-\frac{\alpha |y-z|_e^2}{s}} s^{-\frac{1}{q}} e^{-\frac{(\beta-\alpha)|y-z|_e^2}{s}} b(z) dz ds \\
&\le \int_0^t s^{-\frac{3}{2} + \frac{1}{2p} + \frac{1}{q}} (t-s)^{-\frac{1}{2} + \frac{1}{2p}} \left(\frac{1}{\sqrt{t}} e^{-\frac{p\alpha \rho(x,y)^2}{t}} \right)^{1/p} ds \cdot \|b\|_r \times \text{const} \\
&= c\|b\|_r e^{-\frac{\alpha \rho(x,y)^2}{t}} t^{-1 + \frac{1}{2p} + \frac{1}{q}}, \qquad (7.2.41)
\end{aligned}$$

provided that $1/(2p) + 1/q > 1/2$. Therefore by choosing an appropriate pair of (p, q), the inequality is satisfied if $b \in L^r(D_0 \cap B(a^*, 3))$ with $r \in (2, \infty]$. Finally when $z \in D_0$, $|z|_\rho \ge 3$

we have for $p > 1$

$$
\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho \geq 3} \frac{1}{t-s} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \\
&\leq \int_0^t \frac{c}{\sqrt{t-s}} \int_{z\in D_0, |z|_\rho \geq 3} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{(\alpha+\beta)\rho(z,y)^2}{2s}} dzdr \\
&\leq \int_0^t c(t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho \geq 3} \frac{1}{(t-s)^{\frac{1}{2p}}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{s^{1/p}} e^{-\frac{(\alpha+\beta)\rho(z,y)^2}{2s}} dzds \\
&\leq \int_0^t c(t-s)^{-1+\frac{1}{2p}} \left(\int_{z\in D_0, |z|_\rho \geq 3} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha\rho(x,z)^2}{t-s}} \frac{1}{s} e^{-\frac{p(\alpha+\beta)\rho(z,y)^2}{2s}} dz \right)^{1/p} \|b\|_q ds \\
&\leq \|b\|_q \int_0^t c(t-s)^{-1+\frac{1}{2p}} \left(\int_{r>1} \frac{r}{s} e^{-\frac{p(\alpha+\beta)r^2}{2s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dz \right)^{1/p} ds \\
&\leq \|b\|_q \int_0^t c(t-s)^{-1+\frac{1}{2p}} \left(\int_{r>1} \frac{1}{\sqrt{s}} e^{-\frac{p\alpha r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dz \right)^{1/p} ds \\
&\leq c\|b\|_q e^{-\frac{\alpha\rho(x,y)^2}{t}}, \qquad (7.2.42)
\end{aligned}
$$

by assuming that $1/p < 1$, therefore the inequality holds for $b \in L^q(D_0 \backslash B(a^*, 3))$ with $q \in (1, \infty]$. (7.2.42) together with (7.2.39) and (7.2.41) proves Case 7.

Case 8. $x \in D_0$, $|x|_\rho \geq 3$, $y \in \mathbb{R}_+$. First of all when $z \in \mathbb{R}_+$ it holds

$$
\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z\in\mathbb{R}_+} \frac{1}{t-s} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{\sqrt{s}} e^{-\frac{\beta|z-y|^2}{s}} dzds \\
&= \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-1+\frac{1}{2p}} \int_{z\in\mathbb{R}_+} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) s^{-\frac{1}{2p}} e^{-\frac{\beta|z-y|^2}{s}} dzds \leq c\|b\|_q t^{\frac{1}{2p}-1} e^{-\frac{\alpha\rho(x,y)^2}{t}},
\end{aligned}
\qquad (7.2.43)
$$

for $b \in L^q(\mathbb{R}_+)$ with $q \in (1, \infty]$. When $z \in D_0$, $|z|_\rho \leq 3$, it holds

$$
\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho \leq 3, z\in D_0} \frac{1}{t-s} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) \frac{1}{\sqrt{s}} e^{-\frac{\beta|z-y|_e^2}{s}} dzds \\
&\leq \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-1+\frac{1}{2p}} \int_{z\in D_0, |z|_\rho \leq 3} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha\rho(x,z)^2}{t-s}} b(z) s^{-\frac{1}{2p}} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \\
&\leq \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{z\in D_0, |z|_\rho \leq 3} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha\rho(x,z)^2}{t-s}} \frac{1}{\sqrt{s}} e^{-\frac{p\beta\rho(z,y)^2}{s}} dz \right)^{1/p} \|b\|_q ds \\
&\leq \int_0^t s^{-1+\frac{1}{2p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{r>0} \frac{c}{\sqrt{s}} e^{-\frac{p\beta r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha(r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} \|b\|_q ds \\
&\leq c\|b\|_q t^{\frac{1}{2p}-1} e^{-\frac{\alpha\rho(x,y)^2}{t}}, \qquad (7.2.44)
\end{aligned}
$$

provided that $p > 0$. Thus the inequality holds for $b \in L^q(D_0 \cap B(a^*, 3))$ with $q \in (1, \infty]$. When $z \in D_0$, $|z|_\rho > 3$ it holds

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho > 3, z \in D_0} \frac{1}{t-s} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} b(z) \frac{1}{s} e^{-\frac{\beta \rho(z,y)^2}{s}} dz ds \\
&\le \int_0^t c s^{-1+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \int_{z \in D_0, |z|_\rho > 3} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} b(z) s^{-\frac{1}{p}} e^{-\frac{\beta \rho(z,y)^2}{s}} dz ds \\
&\le \int_0^t c s^{-1+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{z \in D_0, |z|_\rho > 3} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha \rho(x,z)^2}{t-s}} \frac{1}{s} e^{-\frac{p\beta \rho(z,y)^2}{s}} dz \right)^{1/p} \|b\|_q ds \\
&\le \int_0^t c s^{-1+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{r>3} \frac{r}{s} e^{-\frac{p\beta r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha (r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} \|b\|_q ds \\
&\le \int_0^t c s^{-1+\frac{1}{p}} (t-s)^{-1+\frac{1}{2p}} \left(\int_{r>3} \frac{1}{\sqrt{s}} e^{-\frac{p\alpha r^2}{s}} \frac{1}{\sqrt{t-s}} e^{-\frac{p\alpha (r-\rho(x,y))^2}{t-s}} dr \right)^{1/p} \|b\|_q ds \\
&\le c\|b\|_q t^{\frac{1}{p}-1} e^{-\frac{\alpha \rho(x,y)^2}{t}},
\end{aligned} \tag{7.2.45}$$

provided that $p > 0$. Therefore the desired inequality holds for $b \in L^q(D_0 \backslash B(a^*, 3))$ with $q \in (1, \infty]$. (7.2.45) together with (7.2.44) and (7.2.43) proves Case 8.

Case 9. $x \in D_0$, $|x|_\rho \le 3$, $y \in \mathbb{R}_+$. First of all if $z \in \mathbb{R}_+$,

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{z \in \mathbb{R}_+} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha(|x|_\rho^2 + |z|_\rho^2)}{t-s}} \frac{1}{\sqrt{s}} e^{-\frac{\beta |z-y|^2}{s}} dz ds \\
&\le \int_0^t s^{-1+\frac{1}{2p}} \int_{z \in \mathbb{R}_+} (t-s)^{-\frac{1}{2p}} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} b(z) s^{-\frac{1}{2p}} e^{-\frac{\beta \rho(z,y)^2}{s}} dz ds \\
&\le c\|b\|_q t^{\frac{1}{2p}-\frac{1}{2}} e^{-\frac{\alpha \rho(x,y)^2}{t}},
\end{aligned} \tag{7.2.46}$$

provided that $p > 0$, therefore the desired inequality holds for $b \in L^q(\mathbb{R}_+)$ with $q \in (1, \infty]$. When $z \in D_0$, $|z|_\rho \ge 3$ we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{s}} \int_{|z|_\rho \ge 3, z \in D_0} \frac{1}{t-s} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} \frac{1}{s} e^{-\frac{\rho(z,y)^2}{s}} dz ds \\
&\le c \int_0^t \int_{|z|_\rho \ge 3, z \in D_0} \frac{1}{t-s} e^{-\frac{\alpha \rho(x,z)^2}{t-s}} b(z) e^{-\frac{\alpha \rho(z,y)^2}{s}} dz ds. \\
&\le c\|b\|_q e^{-\frac{\alpha \rho(x,y)^2}{t}},
\end{aligned} \tag{7.2.47}$$

for $b \in L^q(D_0 \backslash B(a^*, 3))$ with $q \in (1, \infty]$. To complete the proof to Case 9, the remaining part is when $z \in D_0$, $|z|_\rho < 3$, in which case we have

$$\int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<3} \left[\frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1 \rho(x,z)^2}{t-s}} + \frac{1}{t-s}\left(1 \wedge \frac{|x|_\rho}{\sqrt{t-s}}\right)\left(1 \wedge \frac{|z|_\rho}{\sqrt{t-s}}\right) e^{-\frac{\alpha_2 |x-z|_e^2}{t-s}} \right]$$

$$\times\, b(z) \frac{1}{\sqrt{s}} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds$$

$$\leq \int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<3} \frac{1}{\sqrt{t-s}} e^{-\frac{\alpha_1 \rho(x,z)^2}{t-s}} b(z) \frac{1}{\sqrt{s}} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds$$

$$+ \int_0^t \frac{1}{\sqrt{s}} \int_{z\in D_0, |z|_\rho<3} \frac{1}{t-s} e^{-\frac{\alpha_2 |x-z|_e^2}{t-s}} b(z) \frac{1}{\sqrt{s}} e^{-\frac{\beta\rho(z,y)^2}{s}} dzds \tag{7.2.48}$$

The first term on the right hand side of (7.2.48) can be computated in the same way as (7.2.46) provided that $b \in L^q(D_0 \cap B(a^*, 3))$ with $q \in (1, \infty]$. The second term on the right hand side of (7.2.48) can be computated in the same way as (7.2.21) provided that $b \in L^r(D_0 \cap B(a^*, 3))$ with $r \in (2, \infty]$. Now we have proved Case 9, which proves this proposition. Since these nine cases cover all the possibilities of the regions of x, y and z, we have proved the inequality for $L^{p_1,p_2}(E)$ with $p_1 \in (1, \infty]$ and $p_2 \in (2, \infty]$, which immediate implies the conclusion of the theorem. □

Remark 7.2.3. *For $\alpha-$stable processes, the following "3P" property holds regarding the transition density:*

$$p(t,x,z) \wedge p(s,z,y) \leq C \cdot p(t+s,x,y), \forall x,y,z \in E,\ t>0.$$

We point out that such a property does not hold for BMVD. A counterexample is as follows. Let x be on $\mathbb{R}$, $|x| = \sqrt{t}$. $y = z \in \mathbb{R}^2$. $|y|_\rho = |z|_\rho = \sqrt{t}$. Choose $s = t^{3/4} > t$. Hence by the small time heat kernel estimate of BMVD,

$$p(t,x,z) \wedge p(s,z,y) \asymp \frac{1}{\sqrt{t}} \wedge \left[\frac{1}{s}\left(1 \wedge \frac{\sqrt{t}}{\sqrt{s}}\right)^2 + \frac{1}{\sqrt{s}}\right] = \frac{1}{\sqrt{t}} \wedge \left(\frac{t}{s^2} + \frac{1}{\sqrt{s}}\right)$$
$$= \frac{1}{\sqrt{t}} \wedge \left(\frac{1}{t^{1/2}} + \frac{1}{t^{3/8}}\right) = \frac{1}{\sqrt{t}},$$

while

$$p(t+s,x,y) \asymp \frac{1}{\sqrt{t+s}} \asymp \frac{1}{\sqrt{s}} = \frac{1}{t^{3/8}}.$$

Therefore

$$\frac{p(t,x,z) \wedge p(s,z,y)}{p(t+s,x,y)} \asymp \frac{t^{-1/2}}{t^{-3/8}} = t^{-1/8} \to \infty \text{ as } t \downarrow 0,$$

which contradicts the "3P" property.

7.3 Small Time Heat Kernel Estimates for Drifted BMVD

Now our goal is to get the heat kernel estimate for BMVD with drift by making use of Theorem 7.2.2. We will inductively define functions $|k|_n : \mathbb{R} \times E \times E \to \mathbb{R}$. Namely, for $t > 0$ and $x, y \in E$, we let $|k|_0(t,x,y) := p(t,x,y)$, where $p(t,x,y)$ is the transition density of BMVD, and then we define

$$|k|_n(t,x,y) := \int_0^t \int_E |k|_{n-1}(t-s,x,z)|b(z)||\nabla_z p(s,z,y)|dzds,$$

for $n \geq 1$. When $t \leq 0$, we let $|k|_n(t,x,y) = 0$ for all $n \geq 0, x, y \in E$.

Let $C_1 = C_1(b,t)$ be the constant in Theorem 7.2.2. If we suppose that $|k|_{n-1}(t,x,y) \leq C_1^{n-1} p(t,x,y)$, which is satisfied for $n = 1$, then by Theorem 7.2.2,

$$|k|_n(t,x,y) \leq C_1^{n-1} \int_E \int_0^t p(t-s,x,z)|b(z)||\nabla_z p(s,z,y)|dsdz \leq C_1^n p(t,x,y) < \infty,$$

which holds for all $n \geq 0$, $t \in \mathbb{R}$, and $x, y \in E$.

We will now define functions $k_n : \mathbb{R} \times E \times E \to \mathbb{R}$. For $t > 0$ and $x, y \in E$, we let $k_0(t,x,y) := p(t,x,y)$, and we inductively define

$$k_n(t,x,y) := \int_0^t \int_E k_{n-1}(t-s,x,z)b(z) \cdot \nabla_z p(s,z,y)dzds,$$

for $n \geq 1$. When $t \leq 0$ we let $k_n(t,x,y) = 0$ for all $n \geq 0, x, y \in E$. By induction and the convergence of $|k|_n$, each $k_n(t,x,y)$ is well defined and

$$|k_n(t,x,y)| \leq |k|_n(t,x,y) \leq C_1^n p(t,x,y) < \infty. \tag{7.3.1}$$

Lemma 7.3.1. *Fix $T > 0$ that is sufficiently small. For $0 < t < T$, there is $C_2 = C_2(b,T)$ such that*

$$C_2^{-1} p(t,x,y) \leq \sum_{n=0}^{\infty} k_n(t,x,y) \leq \sum_{n=0}^{\infty} |k_n(t,x,y)| \leq C_2 p(t,x,y),$$

for all $x, y \in E$, C_2 is nondecreasing in t, and $C_2 \to 1$ as $t \to 0$.

Proof. By Theorem 7.2.2, we can pick $T > 0$ sufficiently small so that $C_1 < 1/2$,

$$\sum_{n=1}^{\infty} |k|_n(t,x,y) \leq \frac{C_1}{1-C_1} p(t,x,y),$$

and

$$\sum_{n=0}^{\infty} |k_n(t,x,y)| \le \frac{1}{1-C_1} p(t,x,y),$$

$$\sum_{n=0}^{\infty} k_n(t,x,y) \ge p(t,x,y) - \sum_{n=1}^{\infty} |k_n(t,x,y)| \ge \frac{1-2C_1}{1-C_1} p(t,x,y).$$

We see that the desired inequalities hold with $C_2 = (1-C_1)/(1-2C_1)$. Theorem 7.2.2 yields that C_2 is nondecreasing in t, and $C_2 \to 1$ as $t \to 0$. □

In particular, the series

$$p^b(t,x,y) := \sum_{n=0}^{\infty} k_n(t,x,y) \tag{7.3.2}$$

absolutely converges almost uniformly on $(0,T] \times E \times E$, where T is fixed in Lemma 7.3.1

Now we need to identify the relationship between the family of $\{p^b(t,x,y)\}$ and $\{G^b_\alpha\}$ defined by (7.1.1). That is, we want to claim $p^b(t,x,y)$ is exactly the transition density of BMVD with drift which has been defined in terms of Girsanov transform by(7.1.1).

For $\alpha > 0$, $n = 0, 1, \ldots$, and $x, y \in E$ we define

$$u^{(n)}_\alpha(x,y) := \int_0^\infty e^{-\alpha t} k_n(t,x,y)dt \le C_1^n \int_0^\infty e^{-\alpha t} p(t,x,y)dt \le C_1^n u_\alpha(x,y). \tag{7.3.3}$$

Therefore we can define

$$u^b_\alpha(x,y) = \sum_{n=0}^{\infty} u^{(n)}_\alpha(x,y) = \sum_{n=0}^{\infty} \int_0^\infty e^{-\alpha t} k_n(t,x,y)dt. \tag{7.3.4}$$

We now claim G^b_α defined in Theorem 7.1.1 is actually the resolvent of $p^b(t,x,y)$.

Lemma 7.3.2. *$G_\alpha(b\cdot\nabla G^0_\alpha)^n$ has kernel $\{u^{(n)}_\alpha(x,y)\}$, and the family of $\{G^b_\alpha\}$ is the resolvent of $\{p^b(t,x,y)\}$ defined by* (7.3.2).

Proof. By Fubini's theorem, for $n \ge 1$ and $x \ne y$ we have

$$u^{(n)}_\alpha(x,y) = \int_E u^{(n-1)}_\alpha(x,y) b(z) \cdot \nabla_z u_\alpha(z,y)dz. \tag{7.3.5}$$

Thus the statement is true for $n = 0$. By induction, for every bounded f

$$\begin{aligned} G^0_\alpha(b \cdot G^0_\alpha)^n f(x) &= \int_E u^{(n-1)}_\alpha(x,y) b(y) \nabla G^0_\alpha f(y)dy \\ &= \int_E u^{(n-1)}_\alpha(x,y) \int_E b(y) \cdot \nabla_y u_\alpha(y,z) f(z)dzdy \\ &= \int_E \left(\int_E u^{(n-1)}_\alpha(x,y) b(y) \cdot \nabla_y u_\alpha(y,z)dy \right) f(z)dz = \int_E u^{(n)}_\alpha(x,z) f(z)dz, \end{aligned}$$

for all $n \geq 1$.

Since $G^b_\alpha = \sum_{n=0}^{\infty} G^0_\alpha (b \cdot G^0_\alpha)^n$, we have actually showed that G^b_α has kernel $\sum_{n=0}^{\infty} u^{(n)}_\alpha (x,y)$ which equals $u^b_\alpha(x,y)$. Again in view of (7.3.4) $\sum_{n=0}^{\infty} u^{(n)}_\alpha(x,y)$ is the resolvent kernel of $p^b(t,x,y)$ defined by (7.3.2), we see that the family of $\{G^b_\alpha\}$ is indeed the resolvents of $\{p^b(t,x,y)\}$. □

The following corollary is an immediately consequence of the lemma 7.3.2.

Corollary 7.3.3. *The family of $\{p^b(t,x,y)\}$ is indeed the transition density of BMVD with drift defined in Theorem 7.1.1.*

In view of Lemma 7.3.1 and (7.3.2), we immediately have the following theorem, which gives the small time heat kernel estimate for BMVD with drift.

Theorem 7.3.4. *Let $T > 0$ be fixed. Let b be in the family of $L^\infty(E) + L^{p_1,p_2}(E)$ with $p_1 \in (1,\infty]$ and $p_2 \in (2,\infty]$. Let $p^b(t,x,y)$ be the transition density of BMVD with drift b. It holds*

$$p^b(t,x,y) \asymp p(t,x,y), \quad \textit{for } (t,x,y) \in E \times E \times (0,T],$$

where $p(t,x,y)$ is the transition density of BMVD.

Chapter 8

GREEN FUNCTION ESTIMATE

In this chapter, we estimate the Green function of BMVD with drift. We establish two-sided bounds for the Green function of BMVD with drift killed upon exiting a bounded $C^{1,1}$ open set $D \subset E$. Apparently, it is interesting only when $D \cap \mathbb{R}_+ \neq \emptyset$, $D \cap D_0 \neq \emptyset$.

8.1 Green Function Estimate

Theorem 8.1.1. *Let $g^b_D(x,y)$ be the Green function of BMVD with drift b killed upon exiting D, where D is a bounded $C^{1,1}$ domain of E and $b \in L^\infty(E) + L^{p_1,p_2}(E)$. It is also assumed that $D \cap \mathbb{R}_+ \neq \emptyset$, $D \cap D_0 \neq \emptyset$.*

$$g_D(x,y) \asymp \begin{cases} \delta_D(x) \wedge \delta_D(y), & x \in \mathbb{R}_+,\ y \in \mathbb{R}_+; \\ (\delta_D(y) \wedge 1)\,(\delta_D \wedge 1) + \ln\left(1 + \frac{\delta_{D,\epsilon}(x)\delta_{D,\epsilon}(y)}{|x-y|^2}\right), & x \in D_0,\ y \in D_0; \\ (\delta_D(y) \wedge 1) \cdot \delta_D(x), & x \in \mathbb{R}_+,\ y \in D_0, \end{cases}$$

where $\delta_D(\cdot) = dist(\cdot, \partial D)$, $\delta_{D,\epsilon}(\cdot) = dist\Big(\cdot, \partial\big(D \cap (D_0 \backslash D_\epsilon)\big)\Big)$.

Proof. There are three subcases depending on the locations of x and y.

Case 1. $x, y \in D \cap \mathbb{R}_+$. In order to prove the statement for this case, we claim

$$p^b_D(t,x,y) \leq c_1 e^{-c_2 t}\big(\delta_D(x) \wedge \delta_D(y)\big), \quad \text{on } (1,\infty) \times (D \cap \mathbb{R}_+) \cap (D \cap \mathbb{R}_+). \tag{8.1.1}$$

The proof to this claim is similar to the argument in [10]. We spell out all the details for the convenience of readers. Indeed, for some constant $a > 0$,

$$\begin{aligned}
\mathbb{P}^x(T_D^b < 1) &\geq \mathbb{P}^x(X_1^b \in E \backslash D) \\
&= \int_{E\backslash D} p^b(1,x,z)dz \\
&\geq \int_{z=a>0}^{\infty} e^{-|x-z|^2} dz + \int_{z\in\mathbb{R}^2, |z|>a>0} e^{-|x-z|^2} dz \geq c_1 e^{-\delta_D^2(x)}.
\end{aligned}$$

Thus

$$\sup_{x\in\mathbb{R}_+} \int_D p_D^b(1,x,y)dy = \sup_{x\in D} \mathbb{P}^x(T_D^b > 1) < 1.$$

By Markov property of BMVD with drift, there exist positive constants c_2 and c_3 such that

$$\int_D p_D^b(t,x,y) \leq c_2 e^{-c_3 t}, \qquad (t,x) \in (0,\infty) \times (D \cap \mathbb{R}_+).$$

Since

$$p_D^b(1,x,y) \leq p_{D_0\cup D}^b(1,x,y) \asymp (\delta_D(x)\wedge 1)(\delta_D(y)\wedge 1)p^b(1,x,y), \quad \text{for } x,y \in D\cap\mathbb{R}_+,$$

for any $(t,x,y) \in (1,\infty) \times (D \cap \mathbb{R}_+) \times (D \cap \mathbb{R}_+)$,

$$\begin{aligned}
p_D^b(t,x,y) &= \int_D p_D^b(t-1,x,z) p_D^b(1,z,y)dz \\
&\leq c_4\big(1\wedge \delta_D(y)\big) \int_D p_D^b(t-1,x,z)dz \leq c_4\big(1\wedge\delta_D(y)\big)c_2 e^{-c_3(t-1)}.
\end{aligned}$$

By switching x and y, we have

$$p_D^b(t,x,y) \leq c_4\big(1\wedge\delta_D(x)\big)c_2 e^{-c_3(t-1)}.$$

The claim that $p_D^b(t,x,y) \leq c_1 \delta_D(x) \wedge \delta_D(y) e^{-c_2 t}$ on $(1,\infty)\times(D\cap\mathbb{R}_+)\times(D\cap\mathbb{R}_+)$ is therefore proved. Now we need to show another claim that for some positive constant c_5

$$\int_0^1 p_D^b(t,x,y)dt \geq c_5 \int_1^\infty p_D^b(t,x,y)dt, \quad \text{for } (x,y) \in (D\cap\mathbb{R}_+)\times(D\cap\mathbb{R}_+).$$

To prove this claim, by invoking the heat kernel estimate of BMVD with drift, one has

$$\begin{aligned}
\int_0^1 p^b_{D}(t,x,y)dt &\asymp \int_0^1 \frac{c}{\sqrt{t}}\left(1\wedge\frac{\delta_D(x)}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)}{\sqrt{t}}\right)e^{-\frac{\rho(x,y)^2}{t}}dt \\
&\geq \int_0^1 \frac{c}{\sqrt{t}}\left(1\wedge\delta_D(x)\right)\left(1\wedge\delta_D(y)\right)e^{-\frac{\rho(x,y)^2}{t}}dt \\
&\geq \left(1\wedge\delta_D(x)\right)\left(1\wedge\delta_D(y)\right)\int_0^1 \frac{c}{\sqrt{t}}e^{-\frac{\rho(x,y)^2}{t}}dt \\
&\geq c\cdot\left(1\wedge\delta_D(x)\right)\left(1\wedge\delta_D(y)\right)\int_0^1 e^{-\frac{c}{t}}d\sqrt{t}dt \\
&\geq c_5\cdot\left(1\wedge\delta_D(x)\right)\left(1\wedge\delta_D(y)\right)\geq c_5\int_1^\infty p^b_{D}(t,x,y)dt,
\end{aligned}$$

which completes the proof to the claim. The fact that $\rho(x,y)\leq \mathrm{diam}\{D\}$ is used in the second last inequality. It now follows that

$$g^b_{D}(x,y)\asymp g_{D}(x,y), \qquad (x,y)\in(D\cap\mathbb{R}_+)\times(D\cap\mathbb{R}_+).$$

Case 2. $x\in\mathbb{R}_+$, $y\in D_0$. We have

$$\begin{aligned}
g^b_{D}(x,y) &= \int_{t=0}^\infty\int_{s=0}^t \mathbb{P}^{b,y}_D(T_{\{a^*\}}\in ds)p^b_{D}(t-s,a^*,x)dt \\
&= \int_{s=0}^\infty\int_{t=s}^\infty \mathbb{P}^{b,y}_D(T_{\{a^*\}}\in ds)p^b_{D}(t-s,a^*,x)dt \\
&= \int_{s=0}^\infty \mathbb{P}^{b,y}_D(T_{\{a^*\}}\in ds)\int_{t=s}^\infty p^b_{D}(t-s,a^*,x)dt \\
&= \mathbb{P}^{b,y}_D(T_{\{a^*\}}<\infty)g^b_{D}(a^*,x)=\mathbb{P}^{b,y}_D(T_{\{a^*\}}<\infty)\delta_D(x),
\end{aligned}$$

where the last equality is due to the result of the case when $x\in\mathbb{R}_+,\ y\in\mathbb{R}_+$. Since boundary Harnack Principle remains true for $\Delta+b\nabla$ on $\mathbb{R}^2$, it again holds

$$\mathbb{P}^{b,y}_D(T_{\{a^*\}}<\infty)\asymp(\delta_D\wedge 1)$$

It thus follows

$$g^b_{D}(x,y)=\mathbb{P}^{b,y}_D(T_{\{a^*\}}<\infty)\cdot g^b_{D}(a^*,x)\asymp(\delta_D(y)\wedge 1)\cdot\delta_D(x).$$

Case 3. $x\in D_0$, $y\in D_0$. Again by invoking the previous case, we have

$$\begin{aligned}
g^b_{D}(x,y) &= \mathbb{P}^{b,y}_D(T_{\{a^*\}}<\infty)g^b_{D}(a^*,x)+g^{b,2-BM}_{D,\epsilon}(x,y) \\
&\asymp(\delta_D(y)\wedge 1)(\delta_D(x)\wedge 1)+\ln\left(1+\frac{\delta_{D,\epsilon}(x)\delta_{D,\epsilon}(y)}{|x-y|^2}\right),
\end{aligned}$$

where $g^{b,2-BM}_{D,\epsilon}(x,y)$ is the Green function of 2-dimensional killed Brownian motion with drift upon exiting $D \cap D_0$ whose two-sided estimates can be found for example in [13]. The proof is therefore complete.

$\square$

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