ON THE ROBIN PROBLEM IN FRACTAL DOMAINS

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Abstract. We study the solution to the Robin boundary problem for the Laplacian in a Euclidean domain. We present some families of fractal domains where the infimum is greater than 0, and some other families of domains were it is equal to 0. We also give a new result on "trap domains" defined in [BCM], i.e., domains where reflecting Brownian motion takes a long time to reach the center of the domain.

1. Introduction.

The Robin problem (also known as the "third" boundary problem) for a Euclidean domain $D \subset \mathbb{R}^d$ is to find a function u such that

$$\Delta u(x) = 0, \qquad x \in D, \tag{1.1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = cu, \qquad x \in \partial D, \tag{1.2}$$

with one or more side conditions, where $\mathbf{n}$ is the unit inward normal vector field on ∂D, $\partial u / \partial \mathbf{n}$ is the normal derivative of u in the distributional sense and $c > 0$ is a constant. See Gustafson and Abe [GA] for the history of this problem.

Our interest in the Robin problem stems from some recent applications in physics, electrochemistry, heterogeneous catalysis and physiology; see [FSF], [FS], [GFS], [Sa] and the references therein. Consider the mixed Dirichlet-Robin problem

$$\Delta u(x) = 0, \qquad x \in D \setminus B_*, \tag{1.3}$$

$$\frac{\partial u}{\partial \mathbf{n}} = cu, \qquad x \in \partial D, \tag{1.4}$$

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together with the side condition

$$u(x) = 1, \qquad x \in \partial B_*, \tag{1.5}$$

where $B_* \subset D$ is a fixed closed ball with non-zero radius. The solution to (1.3)-(1.5) represents the steady state of a system in which some particles move randomly in $D \setminus B_*$ and cross a semi-permeable membrane ∂D. The other part of the boundary, ∂B_*, is a source of particles and can be controlled so that we can assume a condition of type (1.5). The constant c in (1.4) is a physical characteristic of the membrane ∂D. One could consider a model with c dependent on $x \in \partial D$ but we will not do that in the present article. The constant c will play no role in our theorems so we will take $c = 1$ in the rest of the article.

In some applied situations, it is desirable to have as much flux through the boundary as possible. The points of a man-made or natural membrane ∂D where there is no flux can be considered an inefficient use of material. Hence, it is interesting to know when the flux is non-negligible through all points of the membrane. In other words, we would like to know whether $\inf_{x \in \partial D} \partial u / \partial \mathbf{n}(x) > 0$. In view of the relation (1.4) between the flux $\partial u / \partial \mathbf{n}$ and the density u of particles and the maximum principle for the harmonic function u, this condition is equivalent to $\inf_{x \in D \setminus B_*} u(x) > 0$. (By Lemma 2.4 below, we know u is non-negative.)

Definition 1.1. *We say that the whole surface of D is active if*

$$\inf_{x \in D \setminus B_*} u(x) > 0. \tag{1.6}$$

If it is not the case that the whole surface is active, we say part of the surface is nearly inactive.

In this paper we investigate the following problem.

Problem 1.2. *Give necessary and sufficient conditions of a geometric nature for the whole surface of D to be active.*

It is not difficult to show that the whole surface of a bounded Lipschitz domain is always active (see Remark 2.5(ii) below). We have posed Problem 1.2 in terms of u rather than $\partial u/\partial \mathbf{n}$ because we are interested in non-Lipschitz domains D; so there are some boundary points where $\mathbf{n}$ is not well-defined while the solution u is always well-defined, and, in fact, is smooth in $D \setminus B_*$. We do not have a complete solution to Problem 1.2, but we give a fairly explicit answer for some natural families of domains with fractal boundary.

We will approach Problem 1.2 using probabilistic methods. This agrees well with the motivating physical models. Suppose that X is reflecting Brownian motion in D, L is its local time on ∂D, and T_{B_*} is the hitting time of B_* by X. When D is a bounded C^3-smooth domain, it is known that (see [MS] and [Pa])

$$u(x) = \mathbf{E}_x \left[\exp\left(-\frac{1}{2} L_{T_{B_*}} \right) \right]. \tag{1.7}$$

This formula indicates that the third boundary problem (1.4) is more difficult to study from the probabilistic point of view than the corresponding Dirichlet and Neumann problems. This is because the Dirichlet problem corresponds to killed Brownian motion and killing on the boundary presents no technical problems. The Neumann boundary problem corresponds to reflecting Brownian motion. The construction of reflecting Brownian motion in an arbitrary domain D is a major technical challenge. Although this feat has been accomplished long time ago by Fukushima [Fu] on an abstract compactification, called the Martin-Kumarochi compactification, of D, many questions about the construction of reflecting Brownian motion on the Euclidean closure of a domain remain open (see [BBC]). Formula (1.7) shows that the Robin boundary problem (1.3)-(1.5) requires the construction and understanding of the local time. This is harder than constructing reflecting Brownian motion itself, because it is known that reflecting Brownian motion does not have a semimartingale decomposition in some domains. For some results in this area, see, e.g., DeBlassie and Toby [DT]. For information on the eigenvalue problem for the Laplacian with Robin boundary conditions, see Smits [Sm1], [Sm2].

The following are the main contributions of this paper. The list includes some technical results that may have independent interest.

(i) The solution of Problem 1.2 for a class of domains with fractal boundaries (Theorems 3.2 and 4.3).

(ii) A characterization of a class of "trap domains" in dimensions 3 and higher, improving a result in [BCM] (Theorem 5.1).

(iii) Clarification of the rigorous meaning of solution to the differential equation (1.3)-(1.5), its existence, uniqueness, and probabilistic representation for non-smooth domains ((2.3) and Lemma 2.4). In particular, we show that the solution to (1.3)-(1.5) is non-negative.

(iv) A semimartingale decomposition of reflecting Brownian motion in a class of fractal domains (Theorem 2.2).

(v) A sharp estimate for the Green function with Neumann boundary conditions in long and thin domains (Lemma 4.4).

(vi) A new version of the Neumann boundary Harnack principle, stronger than the one in [BH] (Lemma 2.8).

(vii) The proof that reflecting Brownian motion starting from the cusp point is not a semimartingale, for some cusps (Remark 4.14). This complements a result of Fukushima and Tomisaki [FT].

A simple example illustrating our main theorems is a cusp domain, defined for a fixed $\alpha > 1$ by

$$D = \left\{ x = (x_1, x_2, \dots, x_d) : 0 < x_1 < 1 \text{ and } x_1^\alpha > (x_2^2 + \dots + x_d^2)^{1/2} \right\}.$$

Applying the main results (Theorems 3.2 and 4.3) of this paper, we show in Example 3.4 (for $d = 2$) and Example 4.13 (for $d \geq 3$) that the whole boundary of D is active if $\alpha \in (1, 2)$, and part of ∂D is nearly inactive if $\alpha \geq 2$. There are more examples given in Sections 3 and 4.

The paper is organized as follows. Section 2 contains some technical preliminaries, many of which may have independent interest. Section 3 presents the solution to Problem 1.2 for a class of 2-dimensional domains, using techniques developed in [BCM]. Section 4 is devoted to Problem 1.2 in dimensions 3 and higher. Finally, Section 5 presents an application of the techniques developed in Section 4 to "trap domains" in dimensions 3 and higher.

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2. Reflecting Brownian motion in domains with fractal boundaries and the Neumann boundary Harnack principle.

This section is devoted to two important technical aspects of this paper. First, we will show that reflecting Brownian motion has a semimartingale decomposition for a class of fractal domains that contains some natural examples. The second technical result is a boundary Harnack principle for harmonic functions satisfying Neumann boundary conditions.

We will let $|\cdot|$ stand for the Euclidean norm in $\mathbb{R}^d$ (for any dimension $d \geq 1$), for the volume (d-dimensional Lebesgue measure) of a set $A \subset \mathbb{R}^d$, and for the $(d-1)$-dimensional surface area of the boundary ∂A of a set $A \subset \mathbb{R}^d$. The meaning will be obvious from the context so this notation should not lead to any confusion. For an open set D of $\mathbb{R}^d$, $C_c(D)$ and $C_c^\infty(D)$ denote the space of continuous functions with compact support in D and the space of smooth functions with compact support in D, respectively.

A ball with center x and radius r will be denoted $B(x,r)$. The notation will refer to an open ball, unless noted otherwise.

The harmonic measure of a set $A \subset \partial D$ in the domain D, relative to z, will be denoted $\omega(z, A, D)$.

The distribution of Brownian motion in $D \setminus B_*$ starting from $x \in \overline{D \setminus B_*}$, reflected on ∂D, and killed on ∂B_* will be denoted $\mathbf{P}_x$. The corresponding expectation will be denoted $\mathbf{E}_x$. The hitting time of a set A will be denoted T_A, i.e., $T_A = \inf\{t > 0 : X_t \in A\}$. We will sometimes write T_A^X or T_A^Y to show the dependence of the hitting time on the process.

We will use elements of excursion theory and Doob's h-processes. See [D] for the discussion of h-transforms in the case of (non-reflecting) Brownian motion, and [Sh] for conditioning of general Markov processes. Elements of excursion theory can be found in [Mv], [Bl], [Bu] and [Sh].

A real-valued function f defined on $A \subset \mathbb{R}^d$ is called Lipschitz with constant $\lambda < \infty$ if $|f(x) - f(y)| \leq \lambda |x - y|$ for all $x, y \in A$. A domain D is called Lipschitz if there exist

$r > 0$ and $\lambda < \infty$ such that for every $x \in \partial D$, the set $\partial D \cap B(x, r)$ is the graph of a Lipschitz function with constant λ in some orthonormal coordinate system. We call (λ, r) the Lipschitz characteristics of D.

Definition 2.1. We will say that a domain D belongs to class $\mathcal{D}$ if there exists an increasing sequence of domains $D_n \subset D$ with the following properties.

(i) Each D_n is a Lipschitz domain with characteristics (λ, r_n) (all the λ's are the same but the r_n's may differ) and $\bigcup_{n=1}^{\infty} D_n = D$.

(ii) For every $n \geq 1$, the set $\partial D_n \cap \partial D$ is a subset of the relative interior of $\partial D_{n+1} \cap \partial D$.

(iii) $\sup_{n \geq 1} |\partial D_n| < \infty$ and $\lim_{n \to \infty} |\partial D_n \setminus \partial D| = 0$.

The set $\partial_L D \stackrel{\text{df}}{=} \bigcup_n \partial D_n \cap \partial D$ will be called the Lipschitz part of ∂D.

Every bounded Lipschitz domain is in $\mathcal{D}$. See Examples 3.4, 3.6, 4.13 and 4.14 below for domains $D \in \mathcal{D}$ which are not Lipschitz.

Constructing a reflecting Brownian motion on a non-smooth domain D is a delicate problem. Let

$$W^{1,2}(D) \stackrel{\text{df}}{=} \{f \in L^2(D, dx) : \nabla f \in L^2(D, dx)\}$$

be the Sobolev space on D of order $(1, 2)$. Fukushima [Fu] used the Martin-Kuramochi compactification D^* of D to construct a continuous diffusion process X^* on D^* with transition semigroup denoted P_t, such that

$$\{f \in L^2(D, dx) : \sup_{t>0} \frac{1}{t} \int_D f(x)(f(x) - P_t f(x))dx < \infty\} = W^{1,2}(D)$$

and for $f \in W^{1,2}(D)$,

$$\mathcal{E}(f, f) \stackrel{\text{df}}{=} \lim_{t \to 0} \frac{1}{t} \int_D f(x)(f(x) - P_t f(x))dx = \frac{1}{2} \int_D |\nabla f(x)|^2 dx.$$

The pair $(\mathcal{E}, W^{1,2}(D))$ is called the Dirichlet space of X^* in $L^2(D^*, m)$, where m is Lebesgue measure on D extended to D^* by setting $m(D^* \setminus D) = 0$. See [FOT] for definitions and properties of Dirichlet spaces, including the notions of quasi-everywhere, quasi-continuous, etc. The process X^* could be called reflecting Brownian motion in D but it lives on an abstract space D^* that contains D as a dense open set. Chen [C1] proposed referring to the quasi-continuous projection X of X^* from D^* into the Euclidean closure $\overline{D}$ as reflecting Brownian motion in D. The projection process X is a continuous process on $\overline{D}$, but in general X is not a strong Markov process on $\overline{D}$ (for example this is the case when D is the unit disk with a slit removed). However when D is a Lipschitz domain, it is shown that X is the usual reflecting Brownian motion in D as constructed in [BH]. It was proved in [C1] that, roughly speaking, if ∂D has "finite surface measure," then X is a semimartingale and has a Skorokhod decomposition. This result was further sharpened in [CFW]. See the introductions of [C1] and [CFW] for the history of constructing reflecting Brownian motion on non-smooth domains.

Theorem 2.2. *If $D \in \mathcal{D}$, then reflecting Brownian motion X in D starting from $x \in D \cup \partial_L D$ has a semimartingale decomposition $X_t = x + W_t + N_t$, where W_t is a d-dimensional Brownian motion,*

$$N_t = \int_0^t \mathbf{n}(X_s) dL_s,$$

and L, the local time, is a non-decreasing continuous process that does not increase when X is not in $\partial_L D$, i.e., $\int_0^\infty \mathbf{1}_{(\partial_L D)^c}(X_t) dL_t = 0$. The Revuz measure of L for the process X^* is surface measure on $\partial_L D$.

Note that the local time L in our theorem satisfies the condition $\int_0^\infty \mathbf{1}_{(\partial_L D)^c}(X_t) dL_t = 0$, which is stronger than the usual condition $\int_0^\infty \mathbf{1}_D(X_t) dL_t = 0$.

Proof. Let $\{D_n, n \geq 1\}$ be the increasing sequence of Lipschitz domains in the definition of $D \in \mathcal{D}$. Let D^* be the Martin-Kuramochi compactification of D used in [Fu]. To be precise, for every $\alpha > 0$, let $\mathcal{H}_\alpha$ denote the space of all h in D such that $(\alpha - \frac{1}{2}\Delta)h = 0$ in D and having

$$\mathcal{E}_\alpha(h,h) \stackrel{\text{df}}{=} \frac{1}{2} \int_D |\Delta h(x)|^2 \, dx + \int_D u(x)^2 \, dx < \infty.$$

For $y \in D$, let $x \mapsto H_\alpha(x,y)$ be the unique α-harmonic function in $\mathcal{H}_\alpha$ such that

$$\mathcal{E}_\alpha(H_\alpha(\,\cdot\,, y),\, v(\,\cdot\,)) = v(y) \qquad \text{for every } v \in \mathcal{H}_\alpha.$$

Let $G^0_\alpha(x,y)$ be the α-resolvent density function for Brownian motion in D killed upon exiting D. Define

$$G_\alpha(x,y) \stackrel{\text{df}}{=} G^0_\alpha(x,y) + H_\alpha(x,y).$$

It is shown in [Fu] that $x \mapsto G_\alpha(x,y)$ is continuous on $D \setminus \{y\}$ and $G_\alpha(x,y) = G_\alpha(y,x)$. Define a metric δ on D by

$$\delta(x,y) = \int_D (|G_1(x,z) - G_1(y,z)| \wedge 1)\, dz$$

and let D^* be the completion of D under the metric δ. Fukushima [Fu] showed that there is a conservative continuous Hunt process X^* on $D^* \setminus N$ associated with the Dirichlet space $(\mathcal{E}, W^{1,2}(D))$ on $L^2(D^*, m)$, where N is a set that has zero capacity with respect to $(\mathcal{E}, W^{1,2}(D))$ and m is Lebesgue measure on D extended to D^* by defining $m(D^* \setminus D) = 0$. Since each coordinate function $x_i \in W^{1,2}(D)$, then each coordinate function admits a quasi-continuous version on D^*, which will be denoted as f_i. Note that $(f_1, \cdots, f_d)$ is defined quasi-everywhere on D^* and is a quasi-continuous map from D^* into $\overline{D}$. Define

$$X = (f_1(X^*), \cdots, f_d(X^*)).$$

Then X is a conservative continuous process on $\overline{D}$, which is called reflecting Brownian motion on $\overline{D}$ in [C1]. It coincides with the usual reflecting Brownian motion when D is a bounded Lipschitz domain.

Let X^n be reflecting Brownian motion on $\overline{D}_n$. It is known from [BH] that X^n has a Hölder continuous transition density function $p^n(t,x,y)$ on $(0,\infty) \times \overline{D}_n \times \overline{D}_n$. Its α-resolvent density function will be denoted as $G^n(x,y)$. Define

$$\tau_n = \inf\{t \geq 0 : X_t^n \in \partial D_n \setminus \partial D\}$$

and

$$G_\alpha^{n,0}(x,y_0) = G_\alpha(x,y_0) - \mathbf{E}_x^n \left[e^{-\alpha\tau_n} G_\alpha(X_{\tau_n}^n, y_0) \right].$$

It is easy to verify that $G_\alpha^{n,0}$ is the α-resolvent for reflecting Brownian motion in $\overline{D}_n$ killed upon hitting $\partial D_n \setminus \partial D$. Thus we have

$$G_\alpha(x,y_0) = G_\alpha^n(x,y) + \mathbf{E}_x^n \left[e^{-\alpha\tau_n} \left(G_\alpha(X_{\tau_n}^n, y_0) - G_\alpha^n(X_{\tau_n}^n, y_0) \right) \right].$$

By [BH], $x \mapsto G_\alpha^n(x,y_0)$ is continuous on $\overline{D}_n \setminus \{y_0\}$ and

$$x \mapsto \mathbf{E}_x^n \left[e^{-\alpha\tau_n} \left(G_\alpha(X_{\tau_n}^n, y_0) - G_\alpha^n(X_{\tau_n}^n, y_0) \right) \right]$$

is continuous on $\partial D_n \cap \partial D$ since it is harmonic in $D_n \setminus \{y_0\}$ with zero Neumann boundary conditions on $\partial D_n \cap \partial D$ and zero Dirichlet boundary conditions on $\partial D_n \setminus \partial D$. Hence we conclude that $x \mapsto G(x, y_0)$ extends continuously to $\partial D_n \cap \partial D$ under the Euclidean topology for every $n \geq 1$ and hence to $\partial_L D = \bigcup_{n=1}^{\infty} (\partial D_n \cap \partial D)$. This implies that

$$\partial_L D \subset (D^* \setminus D) \cap \partial D.$$

Note that on $\partial_L D$, $(f_1, \cdots, f_d)$ is the identity map and so $X_t^* = X_t$ when $X_t^* \in \partial_L D$.

Let σ_k denote surface measure on ∂D_k and let $\mathbf{n}_k(x)$ be the unit inward normal vector field on ∂D_k which is defined almost everywhere with respect to σ_k. By the definition of $D \in \mathcal{D}$,

$$k \mapsto \sigma_k(\partial D_k \cap \partial D)$$

is an increasing function and

$$\lim_{k\to\infty} \sigma_k(\partial D_k) = \lim_{k\to\infty} \sigma_k(\partial D_k \cap \partial D) = \sigma(\partial_L D),$$

since $\partial D_k \cap \partial D \subset \partial D_{k+1} \cap \partial D$ and $\lim_{k\to\infty} \sigma_k(\partial D_k \setminus \partial D) = 0$. Here σ is surface measure on $\partial_L D$. Since $\sup_{k\geq 1} \sigma_k(\partial D_k) < \infty$, there exist a subsequence $\{k_j, j \geq 1\}$ and finite signed measures $(\nu_1, \cdots, \nu_d)$ on D^* such that $\mathbf{n}_{k_j} \sigma_{k_j}$ converges weakly on D^* to $(\nu_1, \cdots, \nu_d)$; that is,

$$\lim_{j\to\infty} \int_{D^*} (g_1(x), \cdots, g_d(x)) \cdot \mathbf{n}_{k_j}(x)\, \sigma_{k_j}(dx) = \sum_{i=1}^{d} \int_{D^*} g_i(x)\, \nu_i(dx), \tag{2.1}$$

for all bounded continuous functions $\{g_1, \cdots, g_d\}$ on D^*. For every $1 \leq i \leq d$ and $k \geq 1$,

$$\begin{aligned} |\nu_i|(D^* \setminus (\partial D_k \cap \partial D)) &\leq \lim_{j\to\infty} \sigma_{k_j}(D^* \setminus (\partial D_k \cap \partial D)) \\ &= \lim_{j\to\infty} \sigma_{k_j}(\partial D_{k_j} \setminus (\partial D_k \cap \partial D)) \\ &= \sigma(\partial_L D \setminus \partial D_k). \end{aligned}$$

Thus for $1 \leq i \leq d$,

$$|\nu_i|(D^* \setminus \partial_L D) = \lim_{k\to\infty} |\nu_i|(D^* \setminus (\partial D_k \cap \partial D)) \leq \lim_{k\to\infty} \sigma(\partial_L D \setminus \partial D_k) = 0. \tag{2.2}$$

On the other hand, by the definition of $D \in \mathcal{D}$, $\mathbf{n}_k\, \sigma_k$ converges weakly on $\overline{D}$ to $\mathbf{n}\, \sigma$, where $\mathbf{n}$ is the unit inward normal vector field of D on $\partial_L D$ in the following sense:

$$\lim_{k\to\infty} \int_{\overline{D}} (g_1(x), \cdots, g_d(x)) \cdot \mathbf{n}_k(x)\, \sigma_k(dx) = \int_{\overline{D}} (g_1(x), \cdots, g_d(x)) \cdot \mathbf{n}(x)\, \sigma(dx)$$

for all bounded continuous functions $\{g_1, \cdots, g_d\}$ on $\overline{D}$ that vanish on $\partial D_n \setminus \partial D$ for some $n \geq 1$.

Since $\partial_L D \subset D^* \cap \overline{D}$, we conclude from (2.1) and (2.2) that

$$(\nu_1, \cdots, \nu_d) = \mathbf{n}\,\sigma \qquad \text{on } D^*.$$

By Theorem 4.4 of [C1], σ is a smooth measure of X^* and thus it determines a positive continuous additive function L of X^*. Moreover,

$$X_t = X_0 + W_t + \int_0^t \mathbf{n}(X_s) dL_s \qquad \text{for } t \geq 0,$$

where W is a d-dimensional Brownian motion. The above Skorokhod decomposition holds for quasi-every starting point X_0^* in D^* with $X_0 = f(X_0)$. However, since the α-resolvent density function $x \mapsto G(x, y_0)$ is continuous on $\partial_L D \cup (D \setminus \{y_0\})$, reflecting Brownian motion X^* can be defined to start from every point $x \in D \cap \partial_L D$ (cf. [FOT]). Hence the above Skorokhod decomposition holds for every starting point $X_0 \in D \cup \partial_L D$. Clearly, since σ is carried on $\partial_L D$,

$$\int_0^\infty 1_{\{X_s \notin \partial_L D\}} dL_s = \int_0^\infty 1_{\{X_s^* \notin \partial_L D\}} dL_s = 0.$$

This proves the theorem. □

Remark 2.3. Let $\tau_{D \cup \partial_L D}$ be the first exit time from $D \cup \partial_L D$ by reflecting Brownian motion on D. Starting from $x \in D \cup \partial_L D$, $\{X_t, t < \tau_{D \cup \partial_L D}\}$ is a strong Markov process on $D \cup \partial_L D$, since it coincides with $\{X_t^*, t < \tau_{D \cup \partial_L D}\}$. Here

$$\tau_{D \cup \partial_L D} \stackrel{\text{df}}{=} \inf\{t > 0 : X_t \notin D \cup \partial_L D\} = \inf\{t > 0 : X_t^* \notin D \cup \partial_L D\}.$$

However even under the conditions of Theorem 2.2, reflecting Brownian motion on D may not be a strong Markov process. For example, let D be the union of $\{(x, y) \in \mathbb{R}^2 : |y| > |x| \text{ and } |y| \leq 1\}$ and $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$. Then clearly reflecting Brownian motion X on $\overline{D}$ can not have the strong Markov property since when X_t is at the origin $\mathbf{0}$, one can not tell how it will be reflected unless one knows where it came from.

Of course, in this example, for starting points in $\overline{D}$ other than the origin $\mathbf{0}$, reflecting Brownian motion will not visit $\mathbf{0}$. But one can modify this example so that the set of such non-Markovian points has positive capacity so it will be visited by the reflecting Brownian motion. Here is such an example. Let K be the standard Cantor set in $[0,1]$. Let $C = \{(x,y) \in \mathbb{R}^2 : |x| < |y| < 1\}$. Define $D \subset \mathbb{R}^2$ by

$$D = \{(x,y) : 1 < x^2 + y^2 < 9\} \cup \bigcup_{x \in K} ((x,0) + C)).$$

Clearly D satisfies the assumptions of Theorem 2.2. Note $\widetilde{K} \stackrel{\text{df}}{=} K \times \{0\}$ is the set of non-Markovian points and $\widetilde{K}$ has positive capacity (see [C2]) and so will be visited by reflecting Brownian motion in D. □

Now we make precise the meaning of solution to the partial differential equation with Robin and Dirichlet boundary conditions (1.3)-(1.5) for $D \in \mathcal{D}$. Define

$$W^{1,2}(D; B_*) \stackrel{\text{df}}{=} \{u \in W^{1,2}(D) : u = 0 \text{ q.e. on the closed ball } B_*\}.$$

We say u is a (weak) solution of (1.3)-(1.5) if the following two conditions are satisfied.

(i) u and its distributional derivative ∇u are in $L^2(D \setminus B_*)$ and for any bounded $g \in W^{1,2}(D; B_*)$,

$$\int_{D \setminus B_*} \nabla g(x) \cdot \nabla u(x)\, dx = -c \int_{\partial_L D} g(x) u(x) \sigma(dx). \tag{2.3}$$

(ii) u is continuous in a neighborhood of ∂B_* and $u = 1$ on ∂B_*.

Note that any $f \in W^{1,2}(D; B_*)$ admits a quasi-continuous version on $D^* \setminus B_*$. Throughout this paper, we will always represent such f by its quasi-continuous version, which will still be denoted as f. In particular, f is well defined q.e. on $D^* \setminus D$. Since $\partial_L D \subset D^* \setminus D$ and σ is a smooth measure of X^* according to Theorem 2.2, f is well defined σ-a.e. on $\partial_L D$ for every $f \in W^{1,2}(D; B_*)$. Hence the right hand side of (2.3) is well defined.

Lemma 2.4. *The partial differential equation with Robin and Dirichlet boundary conditions (1.3)-(1.5), where $c > 0$ is a constant, has a unique solution $u(x)$ given by $u(x) = \mathbf{E}_x\left[\exp(-\frac{c}{2}L_{T_{B_*}})\right]$. In particular, u is non-negative.*

Proof. We first establish existence. Note that $u(x) \overset{\text{df}}{=} \mathbf{E}_x[\exp(-\frac{c}{2}L_{T_{B_*}})]$ is well defined for every $x \in D \cup \partial_L D$ and for q.e. $x \in D^*$. By the Markov property of X^*, for $x \in D \cup \partial_L D$ (as well as for q.e. other $x \in D^*$),

$$v(x) \overset{\text{df}}{=} 1 - \mathbf{E}_x\left[\exp(-\frac{c}{2}L_{T_{B_*}})\right] = \frac{c}{2}\mathbf{E}_x\left[\int_0^{T_{B_*}} u(X_s^*)dL_s\right].$$

Let $X^{*,0}$ be reflecting Brownian motion X^* killed upon hitting B_* and let the transition semigroup be denoted by $\{P_t^0, t \geq 0\}$. It is known (cf. [FOT]) that the Dirichlet form of $X^{*,0}$ is $(\mathcal{E}, W^{1,2}(D; B_*))$ on $L^2(D^* \setminus B_*, m)$. For q.e. $x \in D^* \setminus B_*$,

$$v(x) - \mathbf{P}_t^0 v(x) = \frac{c}{2}\mathbf{E}_x\left[\int_0^t u(X_s^{*,0})dL_s\right].$$

Hence

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\int_{D\setminus B_*} v(x)(v(x) - P_t^0 v(x))\,dx &= \frac{c}{2}\lim_{t\to 0}\int_{D\setminus B_*} v(x)\mathbf{E}_x\left[\int_0^t u(X_s^{*,0})dL_s\right]dx \\ &= \frac{c}{2}\int_{\partial_L D} v(x)u(x)\sigma(dx) < \infty.\end{aligned}$$

Thus $v \in W^{1,2}(D; B_*)$, and a similar calculation to the above yields that for any bounded $g \in W^{1,2}(D; B_*)$

$$\begin{aligned}\frac{1}{2}\int_{D\setminus B_*} \nabla g(x)\cdot\nabla v(x)\,dx &= \mathcal{E}(g, v) \\ &= \lim_{t\to 0}\frac{1}{t}\int_{D\setminus B_*} g(x)(v(x) - P_t^0 v(x))\,dx \\ &= \frac{c}{2}\int_{\partial_L D} g(x)u(x)\sigma(dx).\end{aligned}$$

This shows that v is harmonic in $D \setminus B_*$ and $\frac{\partial v}{\partial \mathbf{n}} = -cu$. In particular, v is continuous in $D \setminus B_*$. Since every point of ∂B_* is regular, we see that v vanishes continuously on ∂B_*. Translating these properties to the function $u = 1 - v$ shows that u is a solution to (1.3)-(1.5).

Now we show the uniqueness. Suppose that u_1 and u_2 are two solutions for (1.3)-(1.5). Define $w \stackrel{\text{df}}{=} u_1 - u_2$. Then $w \in W^{1,2}(D; B_*)$ and it follows from (2.3) that

$$\int_{D\setminus B_*} \nabla g(x) \cdot \nabla w(x)\, dx = -c \int_{\partial_L D} g(x) w(x) \sigma(dx).$$

Letting $g = ((-n) \vee w) \wedge n$ in the above and then letting $n \to \infty$, we have

$$\int_{D\setminus B_*} |\nabla w(x)|^2\, dx = -c \int_{\partial_L D} |w(x)|^2 \sigma(dx).$$

Since $c > 0$, we must have

$$\int_{D\setminus B_*} |\nabla w(x)|^2\, dx = \int_{\partial_L D} |w(x)|^2 \sigma(dx) = 0. \tag{2.4}$$

Since $D \setminus B_*$ is connected, w has to be constant in $D \setminus B_*$, while the second equality in (2.4) implies that $w = 0$ σ-a.e. on $\partial_L D$. Therefore $w = 0$ in $D \setminus B_*$ and hence $u_1 = u_2$. This establishes the uniqueness and completes the proof of this Lemma. □

Remark 2.5. (i) A simple modification of the above argument establishes the existence and uniqueness for solutions to the Robin problem (1.3)-(1.5) with c being a bounded non-negative function. The solution in this case can be represented as

$$u(x) = \mathbf{E}_x \left[\exp\left(-\frac{1}{2} \int_0^{T_{B_*}} c(X_s) dL_s \right) \right] \qquad \text{for } x \in D \setminus B_*.$$

(ii) Suppose that D is a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \geq 3$ and u is the solution to (1.3)-(1.5). By Jensen's inequality, we have

$$u(x) \geq \exp\left(-\frac{c}{2} \mathbf{E}_x L_{T_{B_*}} \right) \qquad \text{for } x \in D \setminus B_*.$$

Let $G_{D\setminus B_*}$ be the Green function of the reflecting Brownian motion killed upon hitting B_*. It is known from [BH] that

$$G_{D\setminus B_*}(x, y) \leq \frac{c_1}{|x-y|^{d-2}} \qquad \text{for } x, y \in \overline{D} \setminus B_*.$$

It follows then

$$\sup_{x\in D\setminus B_*} \mathbf{E}_x L_{T_{B_*}} = \sup_{x\in D\setminus B_*} \int_{\partial D} G_{D\setminus B_*}(x,y)\sigma(dy) \le \sup_{x\in D\setminus B_*} \int_{\partial D} \frac{c_1}{|x-y|^{d-2}}\sigma(dy) < \infty.$$

Hence $\inf_{x\in D\setminus B_*} u(x) > 0$. In other words, the whole surface of a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \ge 3$ is always active. □

Let $D \subset \mathbb{R}^d$ be a Lipschitz domain and let O be a connected open set in $\mathbb{R}^d$. The following definition of "Neumann boundary conditions" for a harmonic function is standard in analysis and PDE (cf. [K]).

Definition 2.6. A function h defined on $D \cap O$ is said to be harmonic in $D \cap O$ with zero Neumann boundary conditions on $\partial D \cap O$ if $h \in W^{1,2}(O_1 \cap D)$ for every relatively compact open subset O_1 of O and

$$\int_{O\cap D} \nabla h(x) \cdot \nabla \psi(x) dx = 0 \tag{2.5}$$

for every $\psi \in C_c^\infty(O_1)$ and consequently for every continuous $\psi \in W^{1,2}(O_1)$ that vanishes on ∂O_1.

The following lemma says that functions expressed in terms of the hitting distribution of reflecting Brownian motion in D are harmonic functions with zero Neumann boundary conditions in the sense of Definition 2.6.

Lemma 2.7. *Let X be reflecting Brownian motion in the Lipschitz domain D, and O a connected open subset of $\mathbb{R}^d$. Define $\tau \stackrel{\text{df}}{=} \inf\{t > 0 : X_t \notin \overline{D} \cap O\}$. Then for any bounded measurable function ψ on $\partial O \cap \overline{D}$,*

$$h(x) \stackrel{\text{df}}{=} \mathbf{E}_x \left[\psi(X_\tau)\right], \qquad x \in \overline{D} \cap O,$$

is a harmonic function in $D \cap O$ with zero Neumann boundary conditions at $\partial D \cap O$.

Proof. Without loss of generality, we may assume that $\psi \ge 0$. Define

$$X_t^0 \stackrel{\text{df}}{=} \begin{cases} X_t & \text{if } t < \tau \\ \partial & \text{if } t \ge \tau, \end{cases}$$

which is reflecting Brownian motion in D killed upon leaving O. It is well-known that X^0 is a symmetric Markov process on $\overline{D}\cap O$ with Dirichlet form $(\mathcal{E}, W^{1,2}(D;O^c))$, where

$$W^{1,2}(D;O^c) \stackrel{\text{df}}{=} \{u \in W^{1,2}(D):\, u=0 \text{ q.e. on } O^c\}.$$

The transition semigroup for X^0 will be denoted by $\{P_t^0, t\geq 0\}$.

Let O_1 be a relatively compact open subset of O and let $f\geq 0$ be C_c^1 with $\text{supp}[f]\subset O$ and $f=1$ on O_1. Define $u(x)\stackrel{\text{df}}{=} f(x)h(x)$. Then for $x\in D\cap O$,

$$u(x)-P_t^0u(x)=\mathbf{E}_x\left[(f(X_0)-f(X_t))h(X_t);\, t<\tau\right]+\mathbf{E}_x\left[f(X_0)h(X_\tau);\, t\geq\tau\right].$$

Note that by time-reversal,

$$\begin{aligned}
&\int_{D\cap O} u(x)\mathbf{E}_x\left[(f(X_0)-f(X_t))h(X_t);\, t<\tau\right]dx\\
&\quad=\int_{D\cap O}\mathbf{E}_x\left[f(X_0)h(X_0)(f(X_0)-f(X_t))h(X_t);\, t<\tau\right]dx\\
&\quad=\int_{D\cap O}\mathbf{E}_x\left[f(X_t)h(X_t)(f(X_t)-f(X_0))h(X_0);\, t<\tau\right]dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{D\cap O} u(x)\mathbf{E}_x\left[(f(X_0)-f(X_t))h(X_t);\, t<\tau\right]dx\\
&\quad=\frac{1}{2}\int_{D\cap O}\mathbf{E}_x\left[(f(X_0)-f(X_t))^2h(X_0)h(X_t);\, t<\tau\right]dx\\
&\quad\leq\frac{\|h\|_\infty^2}{2}\int_{D\cap O}\mathbf{E}_x\left[(f(X_t)-f(X_0))^2;\, t<\tau\right]dx.
\end{aligned}$$

Thus

$$\begin{aligned}
&\limsup_{t\to 0}\frac{1}{t}\int_{D\cap O}u(x)(u(x)-P_t^0u(x))dx\\
&\quad\leq\limsup_{t\to 0}\Big(\frac{\|h\|_\infty^2}{2t}\int_{D\cap O}\mathbf{E}_x\left[(f(X_t)-f(X_0))^2;\, t<\tau\right]dx\\
&\qquad+\frac{\|h\|_\infty^2}{t}\int_{D\cap O}f(x)^2\mathbf{P}_x(t\geq\tau)dx\Big)\\
&\quad\leq\|h\|_\infty^2\int_{D\cap O}|\nabla f(x)|^2dx<\infty,
\end{aligned}$$

by Lemma 4.5.2(i) and (4.5.7) of [FOT]. This implies, by Lemma 1.3.4 of [FOT], that $u\in W^{1,2}(D;O^c)$ and so $h\in W^{1,2}(O_1)$.

$$u(x)=\mathbf{E}_x\left[u(X^0_{\tau_{O_1}})\right]\qquad\text{for } x\in O\cap\overline{D},$$

where $\tau_{O_1} \stackrel{\rm df}{=} \inf\{t > 0 : X_t^0 \notin O_1 \cap \overline{D}\}$. Hence by Theorem 4.3.2 of [FOT], u is $\mathcal{E}$-orthogonal to $W^{1,2}(D; O_1^c)$; that is,

$$\frac{1}{2}\int_{D\cap O} \nabla u(x)\cdot\nabla\phi(x)\,dx = 0 \qquad \text{for every } \phi \in W^{1,2}(D; O_1^c).$$

This shows that u and therefore h is harmonic in $D \cap O_1$ with zero Neumann boundary conditions on $\partial D \cap O_1$. Since O_1 is an arbitrary relatively compact open subset of O, we conclude that u is harmonic in $D\cap O$ with zero Neumann boundary conditions on $\partial D \cap O$. □

The following version of the Neumann boundary Harnack principle is similar to (but slightly more general) than Theorem 3.9 of [BH]. The result in [BH] is limited to smooth domains whose boundaries are locally graphs of Lipschitz functions (although the constant in that theorem depends only on the Lipschitz constant λ) and to harmonic functions h as in our Lemma 2.7, with non-negative ψ.

Lemma 2.8 (Neumann boundary Harnack principle). *Suppose that $\Phi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function with constant $\lambda < \infty$, i.e., $|\Phi(x)-\Phi(y)| \le \lambda|x-y|$ for all $x, y \in \mathbb{R}^{d-1}$. Assume that $\Phi(0) = 0$ and let $D = \{x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_d > \Phi((x_1, \dots, x_{d-1}))\}$. If $r > 0$, $c_1 > 1$, and $h : B(0, c_1 r) \cap D \to [0,\infty)$ is harmonic with zero Neumann boundary conditions on $B(0, c_1 r) \cap \partial D$ then*

$$h(x) \ge c_2 h(y) \qquad \text{for all } x, y \in B(0, r) \cap D, \tag{2.6}$$

where $c_2 > 0$ depends only on λ and c_1.

Proof. For $(y_1, \cdots, y_d) \in \mathbb{R}^d$, denote $\widetilde{y} \stackrel{\rm df}{=} (y_1, \cdots, y_{d-1})$. Define a one-to-one map ϕ : $\phi(\widetilde{y}, y_d) = (\widetilde{y}, y_d - \Phi(\widetilde{y}))$. As Φ is Lipschitz, the Jacobians of ϕ and its inverse ϕ^{-1} are bounded, with the bound depending only on the Lipschitz constant λ. Under ϕ, $\frac{1}{2}\Delta$ is mapped into a uniformly elliptic divergence form operator L with coefficient matrix $A(x)$ (see Remark 2.1.4 of [K]). Let $U \stackrel{\rm df}{=} \phi(B(0, c_1 r) \cap \overline{D})$ and $u(x) \stackrel{\rm df}{=} h(\phi^{-1}(x))$ for $x \in U$.

Using the change of variable formula, we conclude from (2.5) that for every continuous $\psi \in W^{1,2}(U)$ that vanishes on $\partial U \cap \{y \in R^d : y_d > 0\}$,

$$\int_U A(x)\nabla u(x) \cdot \nabla \psi(x) dx = 0. \tag{2.7}$$

Let U^- be the "mirror" reflection of U with respect to the hyperplane $\{(\widetilde{y}, y_d) : y_d = 0\}$, that is, $U^- = \{y = (\widetilde{y}, y_d) : (\widetilde{y}, -y_d) \in U\}$. For $y = (\widetilde{y}, y_d) \in U_-$, define $A(y) = A((\widetilde{y}, -y_d))$ and $u(y) = u((\widetilde{y}, -y_d))$. Then $L \overset{\text{df}}{=} \nabla(A\nabla)$ is the uniformly elliptic divergence form operator defined on the domain $U \cup U^-$. It now follows from (2.7) and its corresponding version for U^- that

$$\int_{U\cup U^-} A(x)\nabla u(x) \cdot \nabla \psi(x) dx = 0 \qquad \text{for every } \psi \in C_c^\infty(U \cup U^-).$$

Hence u is a non-negative L-harmonic function on $U \cup U^-$. The desired Harnack inequality for h now follows from the Harnack inequality for the L-harmonic function u. □

Remarks 2.9. (i) Some regularity conditions for a harmonic function with zero Neumann boundary conditions have to be assumed (such as those formulated in Definition 2.6) in order for the Neumann boundary Harnack principle to hold, even if D has a C^∞ boundary. The Neumann boundary Harnack principle does not need to hold for a harmonic function in $D \cap B(x_0, r)$ which satisfies zero Neumann boundary conditions only *almost everywhere* on $\partial D \cap B(x_0, r)$. For example, let D be a half-space in $\mathbb{R}^d$, $d \geq 3$, with ∂D passing through the origin, and let $h(x) = |x|^{2-d}$. Then h satisfies the Neumann boundary conditions everywhere except at the origin. The Neumann boundary Harnack principle does not hold for this function h in $D \cap B(0, 1)$.

(ii) We will apply Lemma 2.8 to two classes of functions. One of these families consists of harmonic functions defined in a probabilistic way, as in Lemma 2.7. That lemma shows that Lemma 2.8 is applicable to harmonic functions in this family.

We will also apply Lemma 2.8 to the Green function $x \to G(x, y)$, where $y \in D \setminus B_*$, and $G(\,\cdot\,, y)$ is the density of the expected occupation measure for the reflecting Brownian motion in a Lipschitz domain D killed upon hitting B_*, starting from y. To see that

Lemma 2.8 can be applied, consider any $y \in D \setminus B_*$ and let U be any relatively compact subdomain of $D \setminus (B_* \cup \{y\})$. Then for $x \in U$, $G(x,y) = \mathbf{E}_x[G(X_{\tau_U}, y)]$. So by Lemma 2.7, $x \to G(x,y)$ is "locally" in $W^{1,2}(D)$ and is harmonic with zero Neumann boundary conditions on ∂D.

3. Simply connected planar domains.

This section will present some results based on ideas developed in [BCM], a paper on "trap" domains. We will present a new result on trap domains in Section 5. In this section, we will review only as much of the material from [BCM] as is relevant to Problem 1.2. We will use complex analytic notation and concepts. Consult [Po] for the definitions of prime ends, harmonic measure, etc.

We start with some definitions that apply to domains in any number of dimensions. Let X be normally reflecting Brownian motion on $\overline{D} \subset \mathbb{R}^d$, $d \geq 2$, starting from $x \in D$ and killed upon hitting a closed ball B_*. As is mentioned in the previous section, X is obtained as the projection of reflecting Brownian motion X^* on the Martin-Kuramochi compactification of D into $\overline{D}$. The distributions of both X and X^* will be denoted $\mathbf{P}_x$ and the corresponding expectations will be denoted $\mathbf{E}_x$. Let $G(x,y)$ be defined on $(D \setminus B_*) \times (D \setminus B_*)$ by

$$\int_{(D\setminus B_*)\cap A} G(x,y)dy = \mathbf{E}_x \int_0^{T_{B_*}} \mathbf{1}_{\{X_t \in A\}} dt, \qquad A \subset \overline{D},$$

where dy denotes d-dimensional Lebesgue measure. Clearly $G(x,y)$ is a symmetric function on $(D \setminus B_*) \times (D \setminus B_*)$. It follows from Lemma 3.2 of [CFW] that the function $G(x,y)$ can be extended continuously to $(D^* \setminus B_*) \times (D \setminus B_*)$, where D^* is the Martin-Kuramochi compactification of D as mentioned in the proof of Theorem 2.2 in the previous section. Note that (cf. Section 2.1 of [BCM]) if $x, y \in D \setminus B_*$ and $x \neq y$ then $t \mapsto G(X^*_t, y)$ is a continuous local martingale. It is easy to see that $G(x,y)$ is the Green function for the domain $D \setminus B_*$ with (zero) Neumann boundary conditions on ∂D (in the distributional sense) and (zero) Dirichlet boundary conditions on ∂B_*.

For the rest of this section, suppose that D is a simply connected open subset of the complex plane $\mathbf{C}$, z_* is the center of B_*, and z_0 is a prime end in D. Consider a collection

$\{\gamma_n\}_{n\geq 1}$ of non-intersecting cross cuts of D that do not intersect B_* and such that γ_{n+1} separates γ_n from z_0 and the γ_n's tend to z_0. Suppose further that σ is a curve in D connecting z_* to z_0 such that $\sigma \cap \gamma_n$ is a single point z_n, for each n. This system of curves divides D into subregions: let Ω_n denote the component of $D \setminus \gamma_n$ which does not contain z_*. Thus $D_n = \Omega_n \setminus \Omega_{n+1}$ is the region between γ_n and γ_{n+1}. Write $\Omega_1 \setminus \sigma = \Omega^+ \cup \Omega^-$, where each set Ω^+ and Ω^- is connected, and set $D_n^+ = \Omega^+ \cap D_n$ and $D_n^- = \Omega^- \cap D_n$. See Figure 3.1.

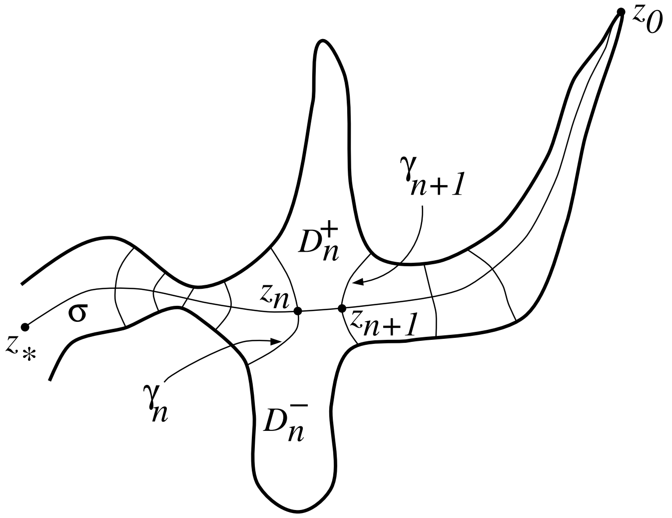

Figure 3.1. Hyperbolic blocks.

Recall that the harmonic measure of a set $A \subset \partial D$ in the domain D, relative to z, is denoted $\omega(z, A, D)$.

Definition 3.1 We will say that the system of curves $\{\gamma_n\} \cup \sigma$ divide D into *hyperbolic blocks* tending to the prime end z_0 if for some $c_* > 0$ and all $n \geq 1$, the following conditions hold:

(i) $c_* \leq \omega(z_*, \partial\Omega^+ \cap \partial D, D) \leq 1/2$ and $c_* \leq \omega(z_*, \partial\Omega^- \cap \partial D, D) \leq 1/2$,

(ii) for all $n \geq 1$ and for all $z \in \partial D_n^+ \cup \{z_{n-1}\}$, we have $\omega(z, \partial D_n^+ \cap \partial D, D) \geq c_*$,

(iii) for all $n \geq 1$ and for all $z \in \partial D_n^- \cup \{z_{n-1}\}$, we have $\omega(z, \partial D_n^- \cap \partial D, D) \geq c_*$.

We will call a system of hyperbolic blocks *regular* if it satisfies in addition the following condition,

(iv) for every $n \geq 1$ there exists $z \in \partial D_n^+ \cap \partial D_n^-$ such that $\omega(z, \partial D_n^- \cap \partial D, D_n) \geq c_*$ and $\omega(z, \partial D_n^+ \cap \partial D, D_n) \geq c_*$.

For every simply connected domain and any prime end z_0, there exists a family of regular hyperbolic blocks. Here is one way to construct $\{\gamma_n\}_{n\geq 1}$ and σ. Suppose that φ is a conformal map of the upper half plane $\mathbb{H}$ onto D, such that $\varphi(0) = z_0$ and $\varphi(i) = z_*$. Then we can take $\gamma_n = \varphi(\mathbb{H} \cap \{|z| = 2^{-n}\})$, $n \geq 1$, and $\sigma = \{\varphi(iy) : 0 < y \leq 1\}$. The conformal invariance of harmonic measure makes it is easy to verify that the ensemble $\{\gamma_n\} \cup \sigma$ divides D into hyperbolic blocks tending to z_0. Condition (iv) is satisfied by $z = \varphi(i(3/4)2^{-n})$. Hyperbolic blocks are useful because they can be constructed geometrically, without knowledge of any properties of the mapping φ; see [BCM] for examples of hyperbolic blocks.

In typical examples, verifying conditions (i)-(iv) is not harder than verifying just (i)-(iii). We did not include (iv) in the definition of hyperbolic blocks in order to keep the same nomenclature as that in [BCM].

Theorem 3.2. *Let $D \in \mathcal{D}$ be a simply connected planar domain.*

(i) If there exist constants $c_*, c' \in (0, \infty)$ such that for each prime end $z_0 \in \partial D$ there is a system of curves $\{\gamma_n\} \cup \sigma$ dividing D into regular hyperbolic blocks with parameter c_* and

$$\sup_{z_0} \sum_n n|\partial D_n \cap \partial D| \leq c', \tag{3.1}$$

then the whole surface of D is active.

(ii) Let r_n denote the distance between γ_n and γ_{n+1}. If for some prime end $z_0 \in \partial D$, there is a system of curves $\{\gamma_n\} \cup \sigma$ dividing D into regular hyperbolic blocks with

$$\sum_{n=1}^{\infty} n r_n = \infty, \tag{3.2}$$

then part of the surface of D is nearly inactive.

Example 3.4 and especially Example 3.6 show that the gap between parts (i) and (ii) of Theorem 3.2 is not large.

We need a lemma to prove Theorem 3.2.

Lemma 3.3 *Suppose that $D \in \mathcal{D}$ and the $\{\gamma_k\}$ divide D into regular hyperbolic blocks. For $n \geq 1$ and $y \in \gamma_{n-1}$, let $x \mapsto h_y(x)$ be the Poisson kernel with pole at y for reflecting Brownian motion in Ω_{n-1} killed upon hitting γ_{n-1}. Let $\mathbf{P}_x^y$ denote the distribution of Doob's h_y-transform of reflecting Brownian motion on D^* killed upon hitting γ_{n-1}, starting from $x \in \Omega_{n-1}$. Note that $\mathbf{P}_x^y$-a.s., the process will stay in the closure of Ω_{n-1} in D^* until its lifetime. Let r_n denote the distance between γ_{n-1} and γ_n. There exist $c_1, p_1 > 0$, depending only on D and c_*, such that $\mathbf{P}_x^y(L_{T_{\gamma_{n-1}}} > c_1 r_n) > p_1$ for any $x \in \gamma_n$.*

Proof. Let φ be a one-to-one conformal map of D_{n-1} onto the unit disc $S = \{z \in \mathbf{C} : |z| < 1\}$, such that $\varphi(\partial D_{n-1}^+ \cap \partial D) = I_1 \stackrel{\text{df}}{=} \{z = e^{i\theta} : \theta_1 \leq \theta \leq \pi - \theta_1\}$ and $\varphi(\partial D_{n-1}^- \cap \partial D) = I_2 \stackrel{\text{df}}{=} \{z = e^{-i\theta} : \theta_1 \leq \theta \leq \pi - \theta_1\}$, for some $0 < \theta_1 < \pi/2$. By condition (iv) in Definition 3.1 and conformal invariance, there exists a point $x_1 \in S$ such that $\omega(x_1, I_1, S) \geq c_*$ and $\omega(x_1, I_2, S) \geq c_*$. This easily implies that there exists $\theta_2 = \theta_2(c_*) \in (\pi/4, \pi/2)$ such that $\theta_1 < \theta_2$. Let $\theta_3 = (\pi/2 + \theta_2)/2$, $J_1 = \{z = e^{i\theta} : \theta_3 \leq \theta \leq \pi - \theta_3\}$ and $J_2 = \{z = e^{-i\theta} : \theta_3 \leq \theta \leq \pi - \theta_3\}$. For some $c_2 = c_2(c_*) > 0$ and every $z \in S$ on the imaginary axis, $\omega(z, J_1 \cup J_2, S) \geq c_2$. Let J^r and J^ℓ be the right and left connected components of $\partial S \setminus (I_1 \cup I_2)$. Let X be Brownian motion in S with normal reflection on $I_1 \cup I_2$ killed upon hitting $J^r \cup J^\ell$. It is easy to see that for some $c_3 = c_3(c_*) > 0$ and every $z \in J_1 \cup J_2$, if reflecting Brownian motion in S starts from z, then it hits J^ℓ before hitting J^r with probability greater than c_3 but less than $1 - c_3$.

Let $\gamma_{n-1/2} = \varphi^{-1}(\{z = a + bi \in S : a = 0\})$, $K_1 = \varphi^{-1}(J_1)$ and $K_2 = \varphi^{-1}(J_2)$. By conformal invariance, for every $x \in \gamma_{n-1/2}$, we have $\omega(z, K_1 \cup K_2, D_{n-1}) \geq c_2$, and for every point $x \in K_1 \cup K_2$, the probability that reflecting Brownian motion in D starting from x hits γ_{n-1} before hitting γ_n is in the range $(c_3, 1 - c_3)$.

Find $c_4 \in (0, 1/8)$ so small that a Brownian motion W starting from x will make a double loop in an annulus $B(x, r) \setminus B(x, 3r/4)$ for some $r \in (c_4 r_n, r_n/8)$, and then will make a crossing from $B(x + ir/2, r/16)$ to the ball $B(x + i2r, r/16)$ within the convex hull of the two balls, all before leaving $B(x, r_n/3)$, with probability greater than $1 - c_3/2$. A "double loop" in $B(x, r) \setminus B(x, 3r/4)$ means that there exist $t_1 < t_2$ such that $W_t \in$

$B(x,r) \setminus B(x,3r/4)$ for all $t \in (t_1,t_2)$, and a continuous version of $t \to \arg(W_t - x)$ increases by 4π over the interval $[t_1,t_2]$. Note that c_4 may be chosen independently of r_n, by Brownian scaling.

Consider a reflecting Brownian motion $X_t = x + W_t + N_t$ on D, starting from a point $x \in K_1 \cup K_2$. Suppose that $x + W_t$ makes a double loop in an annulus $B(x,r) \setminus B(x,3r/4)$ for some $r \in (c_4 r_n, r_n/8)$, and then it makes a crossing from $B(x+ir/2,r/16)$ to the ball $B(x+i2r,r/16)$ within the convex hull of the two balls, before leaving $B(x,\, r_n/3)$, during a time interval $[t_1,t_2]$. Suppose moreover, that $L_{t_2} - L_{t_1} \le r/16$. We will show that the two assumptions taken together yield a contradiction. The second assumption implies that $|N_t - N_{t_1}| \le r/16$ for all $t \in [t_1,t_2]$. This implies that X will make more than one loop in $B(x,17r/16) \setminus B(x,11r/16)$ and then it will make a crossing from $B(x+ir/2,r/8)$ to the ball $B(x+i2r,r/8)$ within the convex hull of the two balls, before leaving $B(x,\, r_n/2)$. This is impossible because then X would make a closed loop around x within $B(x,r_n/2)$, and hence it would have to cross the boundary of D. We conclude that if the first assumption holds, then $L_{t_2} - L_{t_1} \ge r/16 \ge c_4 r_n/16 = c_5 r_n$. Since the first event has probability greater than $1 - c_3/2$ and the process X starting from $x \in K_1 \cup K_2$ can hit γ_{n-1} before γ_n with probability greater than c_3, the event that X hits γ_{n-1} before γ_n and $L_{t_2} - L_{t_1} \ge c_5 r_n$ has probability greater than $c_3/2$. This implies that reflecting Brownian motion in D starting from $x \in \gamma_{n-1/2}$ will hit γ_{n-1} before γ_n and $L_{T_{\gamma_{n-1}}} \ge c_5 r_n$ with probability greater than $c_6 > 0$. Hence, reflecting Brownian motion in D conditioned to hit γ_{n-1} before γ_n and starting from $x \in \gamma_{n-1/2}$ will accumulate more than $c_5 r_n$ units of local time on ∂D_{n-1} before hitting γ_{n-1} with probability greater than c_6.

Let A be the interior of $\overline{D_{n-2} \cup D_{n-1}}$ and let ψ be a one-to-one conformal mapping of A onto a rectangle $R = \{a + ib : a_1 < a < a_2, 0 < b < 1\}$, such that γ_{n-2} is mapped onto the left side of R and γ_n is mapped onto the right side of R. Lemma 3.4 of [BCM] and a simple argument show that $a_2 - a_1$ is bounded above by a constant. Since the hyperbolic blocks are regular, there exists a point $x \in \psi(D_{n-2})$ such that the harmonic measure of the upper part of R in $\psi(D_{n-2})$ is greater than c_*, and the same is true for the lower part of the boundary. An analogous statement is true for $\psi(D_{n-1})$. All this easily implies that the distance of $\psi(\gamma_{n-1})$ from the left and right sides of R is bounded below by $c_7 = c_7(c_*) > 0$. Let $R_1 = \{a + ib : a_1 + c_7/2 < a < a_2 - c_7/2, 0 < b < 1\}$. By the Neumann boundary Harnack principle (Lemma 2.8), for any positive harmonic function h in R_1 with Neumann boundary conditions on the upper and lower sides of R_1, $h(x) \le c_8 h(z)$ for all $x, z \in \psi(\gamma_{n-1})$. This applies, in particular, to $h_y \circ \psi^{-1}$. By conformal invariance, $h_y(x) \le c_8 h_y(z)$ for all $x, z \in \gamma_{n-1}$.

Let $g(x)$ be the harmonic function in D_{n-1} with Neumann boundary conditions on $\partial D_{n-1} \cap \partial D$, equal to 1 on γ_{n-1} and equal to 0 on γ_n. Reflecting Brownian motion in D_{n-1} conditioned to hit γ_{n-1} before γ_n is a g-transform of the unconditioned process. We have already proved that the g-process starting from $x \in \gamma_{n-1/2}$ will accumulate more than $c_5 r_n$ units of local time on ∂D_{n-1} before hitting γ_{n-1} with probability greater than c_6. By the strong Markov property applied at the hitting time of $\gamma_{n-1/2}$, the same holds if the starting point belongs to γ_n. Without loss of generality, we may and do assume that $h_y(x_0) = 1$ for some $x_0 \in \gamma_{n-1}$. Since $0 < c_9 < g(x)/h_y(x) < c_{10} < \infty$ for $x \in \gamma_{n-1}$, an elementary argument shows that the h_y-process starting from $x \in \gamma_n$ will accumulate more than $c_5 r_n$ units of local time on ∂D_{n-1} before hitting γ_{n-1} with probability greater than c_6. $\square$

Proof of Theorem 3.2. (i) Let $d_D(x) \stackrel{\text{df}}{=} \text{dist}(x, \partial D)$. Consider $x_0 \in D$. It is not hard to see that there exists $z_0 \in \partial D$ and a corresponding family of γ_n's such that $x_0 \in D_{n_0}$ for some n_0 and $\text{dist}(x_0, \partial D_n) \ge c_1 d_D(x_0)$, where $c_1 \in (0,1)$ is a constant depending only on D. By the proof of Theorem 2.2 (see especially Lemmas 3.4 and 3.5) in [BCM], $G(x_0, \cdot)$ is bounded by $c_2 k$ on D_k for $k \le n_0 - 1$. Hence $G(x_0, \cdot)$ is bounded by $c_2 n_0$ on D_{n_0-1}. By the Harnack principle, it is bounded by $c_3 n_0$ on $\partial B(x_0, c_1 d_D(x_0)/2)$, and since

$$G(x_0, x) = \mathbf{E}_x \left[G(x_0, X_{T_{B(x_0, c_1 d_D(x_0)/2)}}) \right] \qquad \text{for } x \in D \setminus B(x_0, c_1 d_D(x_0)/2),$$

the same bound holds on $D \setminus B(x_0, c_1 d_D(x_0)/2)$. We obtain,

$$\begin{aligned}
\mathbf{E}_{x_0} L_{T_{B_*}} &= \sum_{n=1}^{\infty} \mathbf{E}_{x_0} \int_0^{T_{B_*}} \mathbf{1}_{\partial D_n \cap \partial D}(X_t) dL_t \\
&= \sum_{n=1}^{n_0-1} \mathbf{E}_{x_0} \int_0^{T_{B_*}} \mathbf{1}_{\partial D_n \cap \partial D}(X_t) dL_t + \sum_{n=n_0}^{\infty} \mathbf{E}_{x_0} \int_0^{T_{B_*}} \mathbf{1}_{\partial D_n \cap \partial D}(X_t) dL_t \\
&= \sum_{n=1}^{n_0-1} \int_{\partial D_n \cap \partial D} G(x_0, x) \sigma(dx) + \sum_{n=n_0}^{\infty} \int_{\partial D_n \cap \partial D} G(x_0, x) \sigma(dx) \\
&\le \sum_{n=1}^{n_0-1} |\partial D_n \cap \partial D| \sup_{x \in \partial D_n \cap \partial D} G(x_0, x) + \sum_{n=n_0}^{\infty} |\partial D_n \cap \partial D| \sup_{x \in \partial D_n \cap \partial D} G(x_0, x) \\
&\le \sum_{n=1}^{n_0-1} |\partial D_n \cap \partial D| \, c_2 n + \sum_{n=n_0}^{\infty} |\partial D_n \cap \partial D| \, c_3 n_0 \\
&\le \sum_{n=1}^{\infty} c_4 \, n \, |\partial D_n \cap \partial D|.
\end{aligned}$$

This is bounded by a constant independent of x_0, by assumption (3.1). Hence we obtain $\sup_{x \in D} \mathbf{E}_x L_{T_{B_*}} < \infty$ and, therefore, $\inf_{x \in D} u(x) = \inf_{x \in D} \mathbf{E}_x \exp(-L_{T_{B_*}}) > 0$. This means that the whole surface of D is active.

(ii) Find a prime end $z_0 \in \partial D$ and a family of γ_n's such that (3.2) holds, that is,

$$\sum_{n=1}^{\infty} n r_n = \infty.$$

Note that in the case of a simply connected domain D in $\mathbb{R}^2$, the Martin-Kuramochi boundary D^* of D and the corresponding reflecting Brownian motion X^* on D^* can be realized as follows. Let φ be a conformal map from $\mathbb{H} \stackrel{\text{df}}{=} \{a + bi : b > 0\}$ to D and define D^* to be the union of D and its prime ends. The map φ extends continuously to a one-to-one map from $\overline{\mathbb{H}}$ to D^*. Let Y be reflecting Brownian motion on $\overline{\mathbb{H}}$. Then $\varphi(Y)$ is a time change of reflecting Brownian motion X^* on D^*. We will use this constructed reflecting Brownian motion X^* in this proof. Recall that $G(z_0, x)$ is well defined for $x \in D^* \setminus \{z_0\}$ by the second paragraph of this section. For $a \ge 0$, define

$$\eta_a \stackrel{\text{df}}{=} \{x \in D^* \setminus (B_* \cup \{z_0\}) : G(z_0, x) = a\}.$$

First, we claim that there exist positive integers m_0, m_1 and a positive constant a_0 such that there is at least one D_n, but at most m_1 such sets, between η_a and η_{a+m_0}, for every $a > a_0$.

Recall that z_* is the center of B_*. Let $\varphi : \mathbb{H} \to D$ be a one-to-one conformal mapping, such that $\varphi(0) = z_0$ and $\varphi(i) = z_*$. Define $h : D \to \mathbb{R}$ by $h(z) = -\log|\varphi^{-1}(z)|$. Then h is harmonic in D with Neumann boundary conditions and a pole at z_0. Let $\eta^*_a = \{x \in D : h(x) = a\}$. Lemma 3.4 of [BCM] and conformal invariance easily imply that there exists an integer $\widetilde{m}_0$ such that for any $a \in \mathbb{R}$ there is at least one D_n between η^*_a and $\eta^*_{a+\widetilde{m}_0}$. It follows from the conformal invariance of the Green function that $h_1(z) \stackrel{\mathrm{df}}{=} G(z_0, \varphi(z))$ is the Green function for reflecting Brownian motion in $\mathbb{H}$ starting from 0 and killed upon hitting $\varphi(B_*)$. It is easy to see that $h_1(z)$ and $-\log|z|$ are comparable on $\mathbb{H} \cap \{z : |z| < r\}$, for some $r > 0$. This implies the existence of a positive integer m_0 and a constant $a_0 > 0$ such that there is at least one D_n between η_a and η_{a+m_0} for every $a > a_0$. From (3.1), the inequalities preceding (3.2) and (3.3) in [BCM] as well as Lemma 3.5 of [BCM], we see that there exists $m_1 < \infty$ such that there are at most m_1 sets D_n between any η_a and η_{a+m_0}.

Let α_j be the sum of nr_n restricted to integers n such that D_{n-1} lies between η_a and η_{a+m_0+1}, where a is of the form $km_0 + j$. Every set D_{n-1} lies between η_a and η_{a+m_0+1} for some integer a, namely for the largest integer a such that $\eta_a \cap \Omega_{n-1} = \emptyset$. This and (3.2) imply that $\sum_{j=0}^{m_0} \alpha_j = \infty$. We will assume without loss of generality that $\alpha_0 = \infty$.

We define $k(n)$ to be the integer k which maximizes kr_k among all k's such that D_{k-1} lies between $\eta_{(n-1)m_0}$ and η_{nm_0} (we take the largest of the k's with these properties if the above definition does not uniquely identify $k(n)$). If we restrict the sum in (3.2) to $k(n)$'s, its value will be infinite, because there are at most m_1 sets D_{n-1} between any η_a and η_{a+m_0}. By Lemma 3.5 of [BCM] and the comparability of $-\log|\varphi(z)|$ and $G(z_0, z)$ for z in a neighborhood of z_0, $c_1 n \leq k(n) \leq c_2 n$.

By (3.1), the inequalities preceding (3.2) and (3.3) in [BCM] as well as Lemma 3.5 of [BCM], $c_1 k \leq G(z_0, x) \leq c_2 k$ for $x \in D_k$ for $k \geq 1$. Let β be the smallest integer multiple of m_0 greater than $\max\{2, (c_2/c_1)\}$. We have

$$\sum_{j=1}^{\infty} \sum_{n=\beta^{2j-1+m}}^{\beta^{2j+m}} n r_{k(n)} = \infty$$

for $m = 0$ or 1 and we will assume without loss of generality that we can take $m = 0$, i.e.,

$$\sum_{j=1}^{\infty} \sum_{n=\beta^{2j-1}}^{\beta^{2j}} n r_{k(n)} = \infty. \tag{3.3}$$

Let X^* be reflecting Brownian motion on D^* starting from some $x_0 \in D_{n_0}$, where n_0 is large. Define

$$\begin{aligned}
S_j &\stackrel{\text{df}}{=} \inf\{t > 0 : X_t^* \in \eta_{\beta^{2j}}\}, \quad j \geq 1, \\
T_1^{j,n} &\stackrel{\text{df}}{=} \inf\{t > S_j : X_t^* \in \eta_{nm_0}\}, \quad n \geq 1, \\
U_k^{j,n} &\stackrel{\text{df}}{=} \inf\{t > T_k^{j,n} : X_t^* \in \eta_{(n-1)m_0}\}, \quad n, k \geq 1, \\
T_k^{j,n} &\stackrel{\text{df}}{=} \inf\{t > U_{k-1}^{j,n} : X_t^* \in \eta_{nm_0}\}, \quad n, k \geq 2, \\
N_n^j &\stackrel{\text{df}}{=} \max\{k : U_k^{j,n} \leq S_{j-1}\}, \quad n \geq 1.
\end{aligned}$$

In other words, N_n^j is the number of downcrossings of $[(n-1)m_0, nm_0]$ by $M_t \stackrel{\text{df}}{=} G(z_0, X_t^*)$ between times S_j and S_{j-1}. This is of interest to us only for n such that $[(n-1)m_0, nm_0] \subset [\beta^{2j-2}, \beta^{2j}]$. The process M is a continuous local martingale so it is a time-change of Brownian motion, until it hits 0.

Consider a one-dimensional Brownian motion W starting from β^{2j} and killed at the hitting time T of β^{2j-2}. It follows easily from the Ray-Knight Theorem that there is an event A with probability greater than $p_1 > 0$, such that on A, the local time L_T^x accumulated by W at the level x before time T is greater than $c_4\beta^{2j}$ for all $x \in (\beta^{2j-1}, \beta^{2j})$. We will apply excursion theory to excursions of W from the set $\{nm_0 : \beta^{2j-1} \leq nm_0 \leq \beta^{2j}\}$. Given the local time $\{L_T^x, x = nm_0 \in [\beta^{2j-1}, \beta^{2j}]\}$ and assuming the event A occurs, the distribution of the number of excursions going from nm_0 to $(n-1)m_0$ is minorized by a Poisson random variable with expectation $K_n \geq c_5\beta^{2j}/m_0 \stackrel{\text{df}}{=} c_6\beta^{2j}$. Conditional on $\{L_T^x, x = nm_0 \in [\beta^{2j-1}, \beta^{2j}]\}$, these random variables are independent. Let $T^M(b) = \inf\{t > 0 : M_t = b\}$ and $M_t^j = \{M_{t+T^M(\beta^{2j})}, t \in [0, T^M(\beta^{2j-2}) - T^M(\beta^{2j})]\}$. Since M_t^j is a time-change of W_t, there exists an event A' with $\mathbf{P}_{x_0}(A') > p_1$, such that on A', conditional on the local time of M^j, the numbers of excursions of M^j between consecutive points of $\{nm_0, \beta^{2j-1} \le nm_0 \le \beta^{2j}\}$ are independent random variables minorized by independent Poisson random variables with means $K_n \ge c_6 \beta^{2j}$.

Note that the processes M^j are independent. We will now condition the process X^* on the local times of M^j's and the endpoints of excursions of X from $\{\eta_{nm_0}, \beta^{2j-1} \le nm_0 \le \beta^{2j}\}$.

Recall that $k(n)$ is an integer such that D_{n-1} lies between $\eta_{(n-1)m_0}$ and η_{nm_0}. An easy argument based on Lemma 3.3 shows that given endpoints of an excursion of X^* going from η_{nm_0} to $\eta_{(n-1)m_0}$, the amount of local time accumulated by the excursion on ∂D is greater than $c_7 r_{k(n)}$ with probability greater than $p_2 > 0$.

Let J_n be the distribution of the local time accumulated by X^* on the part of ∂D between $\eta_{(n-1)m_0}$ and η_{nm_0}, during the time interval $(T^M(\beta^{2j}), T^M(\beta^{2j-2}))$. We have shown that on an event A_j of probability greater than p_1, J_n is stochastically minorized by a random variable I_n whose distribution is Poisson with mean greater than $p_2 c_6 \beta^{2j} \cdot c_7 r_{k(n)}$. Hence J_n is minorized by a random variable I_n which has mean λ_n greater than $c_8 r_{k(n)} \beta^{2j}$ and variance λ_n. Moreover, we can assume that the I_n's are independent given A_j. Hence, the local time accumulated by X between hitting of $\eta_{\beta^{2j}}$ and $\eta_{\beta^{2j-2}}$, on the part of ∂D between these curves, is stochastically minorized by a random variable H_j such that on the event A_j, its mean is bounded below by $\sum_{j:\, \beta^{2j-1} \le nm_0 \le \beta^{2j}} c_8 r_{k(n)} \beta^{2j} \ge \sum_{j:\, \beta^{2j-1} \le nm_0 \le \beta^{2j}} c_9 n r_{k(n)}$ and the variance is equal to its mean. It follows that H_j takes a value larger than $b_j \stackrel{\mathrm{df}}{=} \frac{1}{2} \sum_{j:\, \beta^{2j-1} \le nm_0 \le \beta^{2j}} c_9 n r_{k(n)}$ with probability greater than $p_2 > 0$. Since the M^j's are independent, we can assume that the H_j's are independent. Let Λ_j be independent random variables with $P(\Lambda_j = b_j) = 1 - P(\Lambda_j = 0) = p_2$. Since the reflecting Brownian motion X^* starting from $x_0 \in D_{n_0}$ has to go through γ_j for $j = n_0, n_0 - 1, \cdots, 1$ before reaching $\gamma_0 \stackrel{\mathrm{df}}{=} \partial B_*$, the distribution of the local time accumulated by X^* before hitting B_* is minorized by the distribution of $\sum_{j=1}^{n_0} \Lambda_j$. In view of (3.3), $\sum_{j=1}^{\infty} b_j = \infty$, and this easily implies that $\sum_{j=1}^{\infty} \Lambda_j = \infty$, a.s. Hence, for any $b \in (0, \infty)$, there is some n_0 such that $\mathbf{P}\left(\sum_{j=1}^{n_0} \Lambda_j > b\right) > 1 - 1/b$. This implies that for any $x_0 \in D_{n_0}$, $\mathbf{P}_{x_0}(L_{T_{B_*}} > b) > 1 - 1/b$. Therefore, $\inf_{x \in D} \mathbf{E}_x \exp(-L_{T_{B_*}}) = 0$ and we see that part of the surface of D is nearly inactive. $\square$

Example 3.4. Our first example is very simple. Suppose that for some $\alpha > 1$,

$$D = \{x = (x_1, x_2) : |x_2| \leq x_1^\alpha \text{ and } 0 < x_1 < 1\}.$$

The interesting range of the parameter is $\alpha > 1$. We will show that if $\alpha \in (1, 2)$ then the whole surface of D is active and when $\alpha \geq 2$ then it is not.

It is easy to see that it is sufficient to analyze only one boundary point, namely, $(0, 0)$. We generate a corresponding system of hyperbolic blocks by letting γ_n's be vertical cuts of the domain at distance $2^{-k} + j2^{-k\alpha}$ from 0, for all $j \geq 0$ such that $2^{-k} + j2^{-k\alpha} \leq 2^{-k+1} - 2^{-k\alpha}$, for all $k \geq 2$.

The number of hyperbolic blocks whose distance from 0 is between 2^{-k} and 2^{-k+1} is of order $2^{-k(1-\alpha)}$. Hence the blocks in this family have indices n of order $\sum_{j \leq k} 2^{-j(1-\alpha)} \approx 2^{-k(1-\alpha)}$. The perimeter of each of these blocks is of order $2^{-k\alpha}$, so the contribution from these blocks to the sum in (3.1) is of order $2^{-k(1-\alpha)} \cdot 2^{-k(1-\alpha)} \cdot 2^{-k\alpha} = 2^{-k(2-\alpha)}$. If $\alpha < 2$ then $\sum_{k \geq 1} 2^{-k(2-\alpha)} < \infty$, so part (i) of Theorem 3.2 implies that the whole surface of D is active.

The distance between γ_n and γ_{n+1} is comparable to the perimeter of D_n, so the same calculation as above shows that the sum in (3.2) is comparable to $\sum_{k \geq 1} 2^{-k(2-\alpha)}$ and this is infinite for $\alpha \geq 2$. Therefore part of ∂D is nearly inactive when $\alpha \geq 2$.

The multidimensional version of this example will be discussed in Example 4.13.

It is interesting to compare the above result with the semimartingale property of reflecting Brownian motion X in D starting from the tip $\mathbf{0} \stackrel{\text{df}}{=} (0, 0)$. It is shown in DeBlassie and Toby [DT] that X starting from $\mathbf{0}$ is a semimartingale if and only if $\alpha < 2$. See also Theorem 3.1(i) of Burdzy and Toby [BT] for a similar result. Fukushima and Tomisaki [FT] proved for domains in the shape of multidimensional cusps that reflecting Brownian motion starting from the cusp point is a semimartingale if $\alpha < 2$. We will show in Remark 4.14 below that it is not a semimartingale when $\alpha \geq 2$. $\square$

Remark 3.5. In the definition of $\mathcal{D}$, it is required that $|\partial_L D|$ be finite. One can of course relax this condition using localization. However if $|\partial_L D| = \infty$ (under whatever generalization one uses) and if u is a weak solution to (1.3)-(1.5) in the sense of (2.3) with $0 \leq u \leq c$, then $\inf_{x\in\partial_L D} u(x) = 0$. For suppose otherwise, that is, there exists $c_0 > 0$ such that $u(x) \geq c_0$ for every $x \in \partial_L D$. Let g be a smooth function with compact support in $\mathbb{R}^d$ such that $g = 0$ on B_* and $g = 1$ on ∂D. Then by (2.3) we have

$$\int_{D\setminus B_*} \nabla g(x) \cdot \nabla u(x) dx = -c \int_{\partial_L D} g(x)u(x)\sigma(dx) = -\infty.$$

This is impossible since the left hand side should be finite by the Cauchy-Schwarz inequality. □

Example 3.6. We will analyze a fractal domain which contains channels that become thinner at the same rate as the single channel in Example 3.4. In the present example, the distance between γ_n and γ_{n+1} is much smaller than the perimeter of D_n for some n. Nevertheless, there is no gap between conditions (3.1) and (3.2) for this family of domains.

Suppose that $\alpha > 0$, $\beta > 1$ and let $a_k = \sum_{j=1}^k 2^{-(j-1)\alpha}$. Let $\mathcal{S}_n$ be the family of all binary (zero-one) sequences of length n. We will write $\mathbf{s} = (s_1, s_2, \dots, s_n)$ for $\mathbf{s} \in \mathcal{S}_n$. For integer $k \geq 1$ and $\mathbf{s} \in \mathcal{S}_k$, we set $b_\mathbf{s} = \sum_{j=1}^k s_j 2^{-j}$. Let $A_* = [0,1]^2$, for $k \geq 1$ and $\mathbf{s} \in \mathcal{S}_k$ let

$$A_\mathbf{s} = \{(x_1, x_2) \in \mathbb{R}^2 : a_k \leq x_1 \leq a_{k+1}, b_\mathbf{s} \leq x_2 \leq b_\mathbf{s} + 2^{-k\beta}\},$$

and let D be the connected component of the interior of $A_* \cup \bigcup_{k\geq 1} \bigcup_{\mathbf{s}\in\mathcal{S}_k} A_\mathbf{s}$ that contains the open square $(0,1)^2$ (see Figure 3.2). (Note that when $\beta \geq 2$, the interior of $A_* \cup \bigcup_{k\geq 1} \bigcup_{\mathbf{s}\in\mathcal{S}_k} A_\mathbf{s}$ is disconnected.)

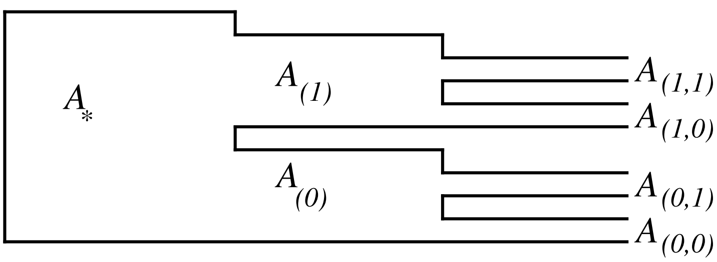

Figure 3.2.

The interesting range of parameters is $\beta > \alpha > 0$. If $\alpha \le 1$ then $|\partial D| = \infty$, so part of the surface of D is nearly inactive by Remark 3.5. We will show that if $1 < \alpha < \beta < 2\alpha$, then the whole surface of D is active and when $\beta \ge 2\alpha > 2$, then part of the surface is nearly inactive.

As in the case of Example 3.4, we will analyze only the family of hyperbolic blocks corresponding to a boundary point at the end of a channel. The analysis of other boundary points is straightforward but tedious so it is omitted. Fix a boundary point z_0 at the end of a channel, i.e., a point whose first coordinate is $\sum_{j=1}^{\infty} 2^{-(j-1)\alpha}$. Let $\mathcal{A}_k$ be the family of vertical lines $K_{k,n} = \{(x, y) : x = a_k + n2^{-k\beta}\}$, with $n \ge 1$ such that $a_k + n2^{-k\beta} \le a_{k+1}$. Let $\mathcal{C}_k$ be the family of these line segments in $K_{k,n} \cap D$ for $K_{k,n} \in \mathcal{A}_k$ that separate z_0 from A_*. Let $\{\gamma_n\}$ be the relabelled family $\bigcup_k \mathcal{C}_k$.

Let $\{\mathbf{s}_k \in \mathcal{S}_k, k \ge 1\}$ be the sequence such that the channel formed by the $A_{\mathbf{s}_k}$'s approaches z_0. The number of hyperbolic blocks D_n defined by the γ_n's needed to reach $A_{\mathbf{s}_k}$ is of order $\sum_{j \le k} 2^{-j\alpha}/2^{-j\beta} \approx 2^{k(\beta-\alpha)}$. Consider a hyperbolic block D_n which intersects $A_{\mathbf{s}_k}$ The set D_n may be either a square or it may contain a "tree" of thin channels. Consider first D_n's that are squares. There are about $2^{k(\beta-\alpha)}$ such hyperbolic blocks, so they correspond to n in (3.1) of order $\sum_{j \le k} 2^{j(\beta-\alpha)}$, which is within a constant multiple of $2^{k(\beta-\alpha)}$. The perimeter of any such D_n is of order $2^{-k\beta}$, so the total contribution of such D_n's to (3.1) is of order $2^{k(\beta-\alpha)} 2^{k(\beta-\alpha)} 2^{-k\beta} \approx 2^{k(\beta-2\alpha)}$. The series $\sum_k 2^{k(\beta-2\alpha)}$ is summable if and only if $\beta < 2\alpha$.

Next consider a D_n which intersects $A_{\mathbf{s}}$ with $\mathbf{s} \in \mathcal{S}_k$ and contains a side "tree" of thin channels. The length of its boundary is of order $\sum_{j \ge k} 2^{(j-k)} 2^{-j\alpha} \approx 2^{-k\alpha}$. It corresponds to n in (3.1) of order $2^{k(\beta-\alpha)}$. There are at most two such D_n's for each $A_{\mathbf{s}}$, so their contribution to (3.1) is of order $2^{k(\beta-\alpha)} 2^{-k\alpha} \approx 2^{k(\beta-2\alpha)}$. Hence, the contribution of D_n's with side channels is of the same order as the contribution of D_n that have the square shape. We conclude that (3.1) holds if $\beta < 2\alpha$.

If $\beta \ge 2\alpha$, then the contribution of the square D_n's is enough to make the left hand side of (3.2) infinite, due to the estimates presented above. □

4. Higher dimensional domains.

This section is devoted to a family of multidimensional domains. The family may seem small, but it contains many examples that arise naturally in the context of the present paper and that of [BCM]. Before presenting the main result of this section, Theorem 4.3, we will state a definition and a technical assumption.

Recall the family $\mathcal{D}$ from Definition 2.1.

Definition 4.1. We will say that a domain $D \subset \mathbb{R}^d$, $d \geq 3$, belongs to the family $\mathcal{D}_1$ if $|D| < \infty$, $D \in \mathcal{D}$, and it satisfies the properties listed below. Recall the meaning of the Lipschitz constant λ from Definition 2.1. Let $\gamma_0 \stackrel{\text{df}}{=} \partial B_*$. For every boundary point $z \in \partial D$ there exists a family of disjoint smooth $(d-1)$-dimensional surfaces $\{\gamma_n\}_{n\geq 1}$, such that $\gamma_n \subset D$, and the set $D \setminus \gamma_n$ consists of two open connected components, Ω_n and Ω'_n. For every n, we have $z \in \overline{\Omega}_n$, $B_* \subset \Omega'_n$, and $\Omega_{n+1} \subset \Omega_n$. Let r_n be the distance between γ_n and γ_{n+1}. There exist $k_0 < \infty$ and $0 < \alpha_1, \alpha_2, \dots, \alpha_7 < \infty$, depending only on D, satisfying the following conditions.

(i) For $n \geq 0$, $\alpha_1 < r_n/r_{n+1} < \alpha_2$ and γ_n can be covered by at most k_0 balls of radius $\alpha_3 r_n$.

(ii) For every $n \geq 0$, there exists a curve $\Gamma \subset D$ of length less than $\alpha_4 r_n$, connecting γ_n and γ_{n+1}, whose distance from ∂D is greater than $\alpha_5 r_n$.

(iii) For every $n \geq 0$ and $x \in \overline{\gamma}_n \cap \partial D$, there exists an orthonormal coordinate system CS with the property that $\partial D \cap B(x, \alpha_6 r_n)$ is the graph of a Lipschitz function with constant λ in CS.

(iv) For every $n \geq 0$ and $x \in \overline{\gamma}_n \cap \partial D$, there exists an orthonormal coordinate system CS with the property that $\partial \Omega_n \cap B(x, \alpha_6 r_n)$ is the graph of a Lipschitz function with Lipschitz constant α_7 in CS, and the analogous statement is true for Ω'_n in place of Ω_n.

We will write $D_n \stackrel{\text{df}}{=} \Omega_n \setminus \overline{\Omega_{n+1}}$.

Note that it follows from part (ii) of Definition 4.1 that there is a constant $\alpha_8 > 0$ depending only on $D \in \mathcal{D}_1$ such that $|D_n| \geq \alpha_8 r_n^d$ for every $n \geq 0$.

Our proof of the second part of our main result in this section, Theorem 4.3, requires the following technical assumption, Condition 4.2. We will discuss ways of verifying this assumption after the proof of Theorem 4.3.

Condition 4.2. *There exist* $0 < m_0 \le m_1 < \infty$ *such that for any* $z \in \partial D$ *and* γ_n*'s as in Definition 4.1, if* $n > m_1$ *and* $x_0 \in \Omega_n$*, then* $\sup_{x \in \gamma_{n-m_0}} G(x_0, x) \le \inf_{x \in \gamma_n} G(x_0, x)$*. Here* $G(x, y)$ *is the Green function of reflecting Brownian motion in* D *killed upon hitting* B_**.*

Recall that we say that the whole surface of D is active if (1.6) holds.

Theorem 4.3. *Suppose that* $D \subset \mathbb{R}^d$*,* $d \ge 3$*, is such that* $D \in \mathcal{D}_1$*.*

(i) If for each boundary point $z \in \partial D$*, there exists a system of surfaces* $\{\gamma_n\}$ *as in Definition 4.1 such that*

$$\sup_{z \in \partial D} \sum_{n \ge 1} |\partial D_n \cap \partial D| \sum_{k=1}^{n} r_k^{2-d} < \infty, \tag{4.1}$$

then the whole surface of D *is active. Here* $|\partial D_n \cap \partial D|$ *denotes the* $(d-1)$*-dimensional surface measure of* $\partial D_n \cap \partial D$*.*

(ii) Suppose now that Condition 4.2 holds. If there exists a boundary point $z \in \partial D$ *and a family of surfaces* $\{\gamma_n\}$ *as in Definition 4.1, such that*

$$\sum_{n \ge 1} r_n^{d-1} \sum_{k=1}^{n} r_k^{2-d} = \infty, \tag{4.2}$$

then part of the surface of D *is nearly inactive.*

The proof of the above theorem will be preceded by a few lemmas. Recall from Section 1 that $B_* \subset D$ is a fixed reference ball. In our proofs, c_j, k_j, m_j and p_j, $j = 0, 1, \ldots$, will denote strictly positive and finite constants depending only on D.

Lemma 4.4. *Let* $D \in \mathcal{D}_1$*,* $z_0 \in \partial D$*, and let* $\{\gamma_n\}$ *and* $\{r_n\}$ *be defined relative to* z_0 *as in Definition 4.1. There exist* $c_1, c_2 \in (0, \infty)$*, depending only on* D*, such that*

$$c_1 \sum_{k=1}^{n} r_k^{2-d} \le G(x, y) \le c_2 \sum_{k=1}^{n} r_k^{2-d},$$

for all $n \ge 1$*,* $x \in \gamma_n$ *and* $y \in \gamma_{n+1}$*.*

Proof. Let $\{\gamma_n, n \geq 0\}$ and $\{\Omega_n, n \geq 0\}$ be as in Definition 4.1, and recall that $G(x,y)$ is the Green function for reflecting Brownian motion X^* on D^* killed upon hitting B_*. As we observed in Section 2, $X^* = X$ on $D \cup \partial_L D$. For $k \geq 0$, let

$$G_{\Omega_k}(x,y) \stackrel{\text{df}}{=} G(x,y) - \mathbf{E}_x \left[G(X_{T_{\gamma_k}}, y) \right], \qquad x, y \in \Omega_k,$$

and note that $G_{\Omega_k}(x,y)$ is the Green function for reflecting Brownian motion in Ω_k killed upon hitting γ_k. Note that since $\gamma_0 \stackrel{\text{df}}{=} \partial B_*$, $G_{\Omega_0}(x,y) = G(x,y)$.

It follows easily from Definition 4.1 that we can find points $z_k \in D_k$, $k \geq 0$, and finite positive constants $c_0 < c_1$ depending only on D such that

$$\operatorname{dist}(z_k, \partial D_k) > c_0 r_k \quad \text{and} \quad \max\{\operatorname{dist}(z_k, \gamma_k),\ \operatorname{dist}(z_k, \gamma_{k+1})\} < c_1 r_k$$
$$\text{for every } k \geq 0.$$

Let $B_k \stackrel{\text{df}}{=} B(z_k, c_0 r_k/2)$. Starting at any point in $\partial B(z_k, c_0 r_k/4)$, the expected time that Brownian motion spends in B_k before hitting ∂D_k is larger than $c_2 r_k^2$. By the support theorem for standard d-dimensional Brownian motion, starting from any point in $\partial B(z_k, 3c_0 r_k/4)$, there is probability at least $p_1 > 0$ (not depending on k) that the Brownian motion will hit the ball $B(z_k, c_0 r_k/4)$ before hitting ∂D_k. So starting at such a point the expected time spent in B_k before hitting ∂D_k is at least $p_1 c_2 r_k^2$. This implies that

$$\int_{B_k} G_{\Omega_k}(x,y)\,dx \geq p_1 c_2 r_k^2 \qquad \text{for every } y \in \partial B(z_k, 3c_0 r_k/4).$$

Using the Harnack inequality and the fact that $|B_k| = c_3 r_k^d$, it follows that

$$G_{\Omega_k}(z_k, y) \geq c_4 r_k^{2-d} \qquad \text{for every } y \in \partial B(z_k, 3c_0 r_k/4).$$

By the Harnack and the Neumann boundary Harnack principle (Lemma 2.8), we have

$$G_{\Omega_k}(x,y) \geq c_5 r_k^{2-d}, \qquad x \in \gamma_{k+1} \text{ and } y \in \gamma_{k+2}. \tag{4.3}$$

On the other hand, starting in B_k the expected amount of time reflecting Brownian motion X in D spends in B_k before exiting the ball $B(z_k, 3c_0 r_k/4)$ is bounded by $c_6 r_k^2$. By the support theorem for standard Brownian motion, there exists $p_2 > 0$ such that starting at any point in $\partial B(z_k, 3c_0 r_k/4)$, there is probability at least p_2 of hitting γ_k before hitting B_k. So the number of crossing from $B(z_k, 3c_0 r_k/4)$ to B_k by reflecting Brownian motion $X^{(k)}$ in Ω_k killed upon hitting γ_k is majorized by a geometric random variable with mean $1/p_2$. This implies that the expected amount of time spent in B_k by $X^{(k)}$ starting at any point in γ_{k+2} is at most $c_7 r_k^2$; that is

$$\int_{B_k} G_{\Omega_k}(x, y)dx \le c_7 r_k^2 \qquad \text{for } y \in \gamma_{k+2}.$$

Since $|B_k| = c_3 r_k^d$, the Harnack inequality implies that

$$G_{\Omega_k}(z_k, y) \le c_8 r_k^{2-d} \qquad \text{for every } y \in \gamma_{k+2}.$$

Again by the Harnack and the Neumann boundary Harnack inequality, we have

$$G_{\Omega_k}(x, y) \le c_9 r_k^{2-d} \qquad \text{for every } x \in \gamma_{k+1} \text{ and } y \in \gamma_{k+2}. \tag{4.4}$$

For $k \ge 0$, it follows from the strong Markov property that

$$G_{\Omega_k}(x, y) = G_{\Omega_{k+1}}(x, y) + \mathbf{E}_x \left[G_{\Omega_k}(X_{T_{\gamma_{k+1}}}, y) \right] \qquad \text{for } x, y \in \Omega_{k+1}.$$

Consequently, for every $x \in \Omega_{k+1}$ and $y \in \overline{\Omega}_{k+2}$.

$$G_{\Omega_k}(x, y) = G_{\Omega_{k+1}}(x, y) + \mathbf{E}_x \left[\mathbf{E}_y \left[G_{\Omega_k}(X_{T_{\gamma_{k+1}}}, Y_{T^Y_{\gamma_{k+2}}}) \right] \right], \tag{4.5}$$

where Y is a reflecting Brownian motion in D independent of X and $T^Y_{\gamma_{k+2}}$ is the first hitting time of γ_{k+2} by Y. Let $a_k(x, y) = \mathbf{E}_x \left[\mathbf{E}_y \left[G_{\Omega_k}(X_{T_{\gamma_{k+1}}}, Y_{T^Y_{\gamma_{k+2}}}) \right] \right]$ and note that by (4.3) and (4.4), for $x \in \Omega_{k+1}$ and $y \in \overline{\Omega}_{k+2}$,

$$c_5 r_k^{2-d} \le a_k(x, y) \le c_9 r_k^{2-d}. \tag{4.6}$$

Fix $n \ge 1$. By (4.5) and (4.6), for every $x \in \gamma_n$ and $y \in \gamma_{n+1}$ and $0 \le k \le n-2$,

$$G_{\Omega_k}(x, y) - G_{\Omega_{k+1}}(x, y) = a_k(x, y).$$

Adding these equations for $0 \le k \le n-2$, we obtain

$$G_{\Omega_0}(x, y) - G_{\Omega_{n-1}}(x, y) = \sum_{k=0}^{n-2} a_k(x, y),$$

or

$$G(x,y) = G_{\Omega_0}(x,y) = G_{\Omega_{n-1}}(x,y) + \sum_{k=0}^{n-2} a_k(x,y). \tag{4.7}$$

By (4.3) and (4.4), $c_5 r_{n-1}^{2-d} \leq G_{\Omega_{n-1}}(x,y) \leq c_9 r_{n-1}^{2-d}$. This, (4.6) and (4.7) imply that

$$c_5 \sum_{k=0}^{n-1} r_k^{2-d} \leq G(x,y) \leq c_9 \sum_{k=0}^{n-1} r_k^{2-d},$$

for $x \in \gamma_n$ and $y \in \gamma_{n+1}$. By Definition 4.1, $\alpha_1 < r_{n-1}/r_n < \alpha_2$ and $\alpha_1 < r_0/r_1 < \alpha_2$, so

$$c_{10} \sum_{k=1}^{n} r_k^{2-d} \leq G(x,y) \leq c_9 \sum_{k=1}^{n} r_k^{2-d},$$

for $x \in \gamma_n$ and $y \in \gamma_{n+1}$. □

Lemma 4.5. *For $n \geq 3$ and $y \in \gamma_{n-3}$, let $x \mapsto h_y(x)$ be the Poisson kernel with pole at y for reflecting Brownian motion in Ω_{n-3} killed upon hitting γ_{n-3}. Let $\mathbf{P}_x^y$ denote the distribution of the h_y-transform of reflecting Brownian motion in D^* killed upon hitting γ_{n-3}, starting from $x \in \Omega_{n-3}$. There exist $c_1, p_1 > 0$, depending only on D, such that $\mathbf{P}_x^y(L_{T_{\gamma_{n-3}}} > c_1 r_n) > p_1$ for any collection of γ_k's as in Definition 4.1, any $n \geq 3$, $x \in \gamma_n$ and $y \in \gamma_{n-3}$.*

Proof. All constants c_j that appear in this proof depend only on D. The conditions listed in Definition 4.1 imply existence of $c_2 > 0$ and a point $x_0 \in \partial D_{n-2} \cap \partial D$ such that (i) the distance from x_0 to $\gamma_{n-1} \cup \gamma_{n-2}$ is greater than $2c_2 r_n$, and (ii) there exists an orthonormal coordinate system CS_{x_0} such that $B(x_0, c_2 r_n) \cap \partial D$ is the graph of a Lipschitz function with the Lipschitz constant λ. Recall that λ is the constant in the definition of $\mathcal{D}$ and, hence, in the definition of $\mathcal{D}_1$. We will assume that $x_0 = 0$ in CS_{x_0} and the positive part of the d-th coordinate axis intersects $B(x_0, c_2 r_n) \cap D$.

Let $h(x) = \mathbf{P}_x(T_{\gamma_{n-3}} < T_{\gamma_n})$ and let x_1 be the intersection point of $\partial B(x_0, c_2 r_n/2)$ and the positive part of the d-th coordinate axis in CS_{x_0}. It is easy to show, using Definition 4.1 and a "Harnack chain of balls" argument, that $h(x_1) > c_3 > 0$. By the boundary Harnack principle (Lemma 2.8), we have $h(x) > c_4 > 0$ for all $x \in B(x_0, 3c_2 r_n/4) \cap D$.

Recall that X^* is reflecting Brownian motion on the Martin-Kuramochi compactification D^* of D, and X is the quasi-continuous projection of X^* into $\overline{D}$. As we noted in Section 2, $X = X^*$ on $D \cup \partial_L D$. Let X^* start from a point $x \in B(x_0, c_2 r_n/2) \cap \partial D \subset \partial_L D$. It follows from Theorem 2.2 that $X_t = x + W_t + N_t$, where W_t is a d-dimensional Brownian motion starting from 0 and $N_t = \int_0^t \mathbf{n}(X_s) dL_s$ is the singular push on the boundary $\partial_L D$. Assume without loss of generality that $\lambda > 1$. Let $x_2 = (0, 0, \dots, 0, -c_2 r_n/10)$, $B_1 = B(x_2, c_2 r_n/(100\lambda))$, $B_2 = B(0, c_2 r_n/(100\lambda))$, and let C_1 be the convex hull of $B_1 \cup B_2$. Consider the event A that the Brownian motion W hits B_1 before leaving C_1 in less than $c_5 r_n^2$ units of time. By the support theorem and Brownian scaling, the probability of A is greater than $p_2 > 0$. Let $T_* = T^X_{\partial B(x_0, 3c_2 r_n/4)} \wedge c_5 r_n^2$. We will argue that if A occurs, then $|N_{T_*}| \geq c_6 r_n$. To see this, first suppose that $T^X_{\partial B(x_0, 3c_2 r_n/4)} < c_5 r_n^2$. Since A holds, W stays in C_1, and it follows that $|W_{T_*}| \leq c_2 r_n/5$. Since $|X_{T_*} - x| \geq c_2 r_n/4$, we have $|N_{T_*}| \geq c_2 r_n/20$. If $T^X_{\partial B(x_0, 3c_2 r_n/4)} \geq c_5 r_n^2$ and A holds, let $t_0 \leq T_*$ be a time when $W_{t_0} \in B_1$. Since $x + B_1$ is at a distance greater than $c_2 r_n/(100\lambda)$ from D and $X_{t_0} \in \overline{D}$, we must have $|N_{T_*}| \geq c_2 r_n/(100\lambda)$. We see that with probability p_2 or greater, X accumulates at least $c_6 r_n$ units of local time before leaving $B(x_0, 3c_2 r_n/4)$. Since $h(x) > c_4$ for all $x \in B(x_0, c_2 r_n) \cap D$, X starting from any point $x \in B(x_0, c_2 r_n/2) \cap \partial D$ has a chance $p_3 > 0$ (depending only on D) of accumulating at least $c_6 r_n$ units of local time and hitting γ_{n-2} before hitting γ_n, by the strong Markov property applied at the time of hitting of $\partial B(x_0, 3c_2 r_n/4)$.

Suppose that $x_1 \in \gamma_{n-1}$ lies at least $c_7 r_n$ units away from ∂D. By the support theorem for Brownian motion, the chance that reflecting Brownian motion X starting from x_1 will hit $B(x_0, c_2 r_n/2) \cap \partial D$ before hitting any other part of $\partial D \cup \gamma_n \cup \gamma_{n-2}$ is greater than $p_4 > 0$, depending only on D. By the strong Markov property applied at the hitting time of $B(x_0, c_2 r_n) \cap \partial D$, reflecting Brownian motion starting from x_1 has a chance greater than $p_5 > 0$ of accumulating at least $c_6 r_n$ units of local time inside $B(x_0, 3c_2 r_n/4)$ and hitting γ_{n-2} before hitting γ_n. Hence, the h-transform of X starting from x_1 has a chance greater than $p_5 > 0$ of accumulating at least $c_6 r_n$ units of local time inside $B(x_0, 3c_2 r_n/4)$ before its lifetime. The boundary Harnack principle shows that the same is true for any $x \in \gamma_{n-1}$, except that the bound for the probability has to be replaced with a new value

$p_6 > 0$.

Now consider the h_y-transform of X starting from a point of γ_n. It must hit γ_{n-1} and then γ_{n-2} on its way to γ_{n-3}. The strong Markov property applied at the hitting times of γ_{n-1} and γ_{n-2} and the claims proved so far show that an h-process starting from any point in γ_n has a chance $p_6 > 0$ of accumulating at least $c_6 r_n$ units of local time inside $B(x_0, 3c_2 r_n/4)$ before its lifetime. By the Neumann boundary Harnack principle (Lemma 2.8), there are positive constants $c_7 < c_8$ such that

$$c_7 \frac{h(x)}{h(z)} \le \frac{h_y(x)}{h_y(z)} \le c_8 \frac{h(x)}{h(z)} \qquad \text{for } x, z \in \gamma_{n-2} \cup \gamma_{n-1}.$$

A routine argument based on this observation allows us to extend the claim to the h_y-process. $\square$

Proof of Theorem 4.3. The main idea of this proof is the same as that of the proof of Theorem 3.2 but some details are different.

(i) Consider $x_0 \in D$. It is not hard to see that there exists $z_0 \in \partial D$ and a corresponding family of γ_n's, as in Definition 4.1, such that $x_0 \in D_{n_0+1}$ for some $n_0 \ge 1$ and $\text{dist}(x_0, \partial D) \ge c_1 r_{n_0+1}$. For $y \in \Omega'_{n_0-1}$, by the strong Markov property of X,

$$G(x_0, y) = \mathbf{E}_{x_0}\left[G(X_{T_{\gamma_{n_0}}}, y)\right] = \mathbf{E}_{x_0}\left[\mathbf{E}_y\left[G\Big(X_{T_{\gamma_{n_0}}}, Y_{T^Y_{\gamma_{n_0-1}}}\Big); T^Y_{\gamma_{n_0-1}} < T^Y_{B_*}\right]\right],$$

where Y is a reflecting Brownian motion in D^* killed upon hitting ∂B_* starting from y and independent of X, and $T^Y_{\gamma_k}$ is the first hitting time of γ_k by Y. Hence by Lemma 4.4, $G(x_0, y)$ is bounded by $c_2 \sum_{k=1}^{n_0} r_k^{2-d}$ for $y \in \Omega'_{n_0-1}$. By the Harnack principle, $y \mapsto G(x_0, y)$ is bounded by $c_3 \sum_{k=1}^{n_0} r_k^{2-d}$ on $\partial B(x_0, c_1 r_{n_0+1}/2)$, and the maximum principle implies that the same bound holds on $D \setminus B(x_0, c_1 r_{n_0+1}/2)$. We obtain, with σ denoting the surface measure on $\partial_L D = \bigcup_{n=1}^{\infty}(\partial D_n \cap \partial D)$,

$$\begin{aligned}
\mathbf{E}_{x_0} L_{T_{B_*}} &= \mathbf{E}_{x_0}\left[\int_0^{T_{B_*}} \mathbf{1}_{\partial_L D}(X_t)dL_t\right] \\
&= \int_{\partial_L D} G(x_0,x)\sigma(dx) \\
&= \sum_{n=1}^{n_0-1} \int_{\partial D_n \cap \partial D} G(x_0,x)\sigma(dx) + \sum_{n=n_0}^{\infty} \int_{\partial D_n \cap \partial D} G(x_0,x)\sigma(dx) \\
&\le \sum_{n=1}^{n_0-1} |\partial D_n \cap \partial D| \sup_{x\in\partial D_n \cap \partial D} G(x_0,x) + \sum_{n=n_0}^{\infty} |\partial D_n \cap \partial D| \sup_{x\in\partial D_n \cap \partial D} G(x_0,x) \\
&\le \sum_{n=1}^{n_0-1} |\partial D_n \cap \partial D|\, c_4 \sum_{k=1}^{n} r_k^{2-d} + \sum_{n=n_0}^{\infty} |\partial D_n \cap \partial D|\, c_3 \sum_{k=1}^{n_0} r_k^{2-d} \\
&\le \sum_{n=1}^{\infty} c_5 |\partial D_n \cap \partial D| \sum_{k=1}^{n} r_k^{2-d}.
\end{aligned}$$

This is bounded by a constant independent of x_0, by assumption (4.1). Hence we obtain $\sup_{x\in D} \mathbf{E}_x L_{T_{B_*}} < \infty$ and, therefore, $\inf_{x\in D} u(x) = \inf_{x\in D} \mathbf{E}_x \exp(-L_{T_{B_*}}) > 0$. This means that the whole surface of D is active.

(ii) Consider a point $z_0 \in \partial D$ and a corresponding family of γ_n's satisfying (4.2). Let $\{x_n, n \ge 1\}$ be a sequence in D that converges to z_0. There is a subsequence $\{x_{n_j}, j \ge 1\}$ that converges to some z_0^* in D^*, the Martin-Kuramochi compactification of D. Note that $x \mapsto G(z_0^*, x)$ is well defined on $D^* \setminus \{z_0^*\}$ by the second paragraph of Section 3. In particular,

$$G(z_0^*, x) = \lim_{j\to\infty} G(x_{n_j}, x) \qquad \text{for } x \in D^* \setminus \{z_0^*\}.$$

For $a \ge 0$, define

$$\eta_a = \{x \in D^* \setminus (B_* \cup \{z_0^*\}) : G(z_0^*, x) = a\}.$$

Note that $\eta_0 = \partial B_*$. Recall the definition of the integer $m_0 > 1$ from Condition 4.2. Define for $n \ge 1$,

$$a_n = \inf_{x\in\gamma_{3nm_0}} G(z_0^*, x).$$

By the Neumann boundary Harnack principle applied to the function $x \mapsto G(z_0^*, x)$ on γ_{3nm_0}, there is a constant $c_0 \in (1,\infty)$ depending only on D such that $\inf_{x\in\gamma_{3nm_0}} G(z_0^*, x) \ge$

$c_0\mathbf{E}_x[G(z_0^*, X_{T_{\gamma_{3nm_0}}})]$ for every $x \in \gamma_{3(n+1)m_0}$. For $n \geq 0$,

$$
\begin{aligned}
a_{n+1} - a_n &= \inf_{x \in \gamma_{3(n+1)m_0}} G(z_0^*, x) - \inf_{x \in \gamma_{3nm_0}} G(z_0^*, x) \\
&\leq \inf_{x \in \gamma_{3(n+1)m_0}} \Big(G(z_0^*, x) - c_0 \mathbf{E}_x \Big[G(z_0^*, X_{T_{\gamma_{3nm_0}}}) \Big] \Big) \\
&\leq \inf_{x \in \gamma_{3(n+1)m_0}} \Big(c_0 G(z_0^*, x) - c_0 \mathbf{E}_x \Big[G(z_0^*, X_{T_{\gamma_{3nm_0}}}) \Big] \Big) \\
&= c_0 \inf_{x \in \gamma_{3(n+1)m_0}} G_{\Omega_{3nm_0}}(z_0^*, x) \\
&= c_0 \inf_{x \in \gamma_{3(n+1)m_0}} \mathbf{E}_{z_0^*} \Big[G_{\Omega_{3nm_0}}(X_{T_{\gamma_{3(n+2)m_0}}}, x) \Big] \\
&\leq c_1 r_{3nm_0}^{2-d}, \qquad (4.8)
\end{aligned}
$$

where the last inequality is due to (4.4) and $c_1 > 0$ is a constant depending only on D.

On the other hand, for $x \in \gamma_n$,

$$
G(z_0^*, x) = \lim_{j \to \infty} G(x_{n_j}, x) = \lim_{j \to \infty} \mathbf{E}_{x_{n_j}} \Big[G\big(X_{T_{\gamma_{3(n+1)m_0}}}, x\big) \Big].
$$

So it follows from Lemma 4.4 that there are positive constants $c_2 < c_3$ depending only on D such that

$$
c_2 \sum_{k=1}^{3nm_0} r_k^{2-d} \leq a_n = \inf_{x \in \gamma_{3nm_0}} G(z_0^*, x) \leq c_3 \sum_{k=1}^{3nm_0} r_k^{2-d} \qquad \text{for every } n \geq 1. \qquad (4.9)
$$

It follows from Definition 4.1 that for some $c_4 < \infty$, $r_n^{2-d}/r_{n-1}^{2-d} \leq c_4$ for every $n \geq 1$. Hence,

$$
\begin{aligned}
\frac{a_n}{a_{n-1}} &\leq \frac{c_3 \sum_{k=1}^{3nm_0} r_k^{2-d}}{c_2 \sum_{k=1}^{3(n-1)m_0} r_k^{2-d}} \leq \frac{c_3}{c_2} + \frac{c_3 \sum_{k=3(n-1)m_0+1}^{3nm_0} r_k^{2-d}}{c_2 r_{3(n-1)m_0}^{2-d}} \qquad (4.10) \\
&\leq \frac{3c_3 m_0}{c_2}(1 + c_4^{3m_0}) < \infty.
\end{aligned}
$$

Let $\beta \overset{\text{df}}{=} \max\{2,\, (3c_3 m_0/c_2)(1 + c_4^{3m_0})\}$ and $n_j = \inf\{n : a_n \geq \beta^j\}$. By (4.10),

$$
\beta^j \leq a_{n_j} \leq a_{n_j - 1}\beta \leq \beta^{j+1}. \qquad (4.11)
$$

Since by (4.2),

$$
\sum_{m=0}^{3m_0-1} \sum_{i=0}^{1} \sum_{j=1}^{\infty} \sum_{\{n : 3nm_0 + m \in (n_{4j-4+2i}, n_{4j-2+2i}]\}} r_{3nm_0+m}^{d-1} \sum_{k=1}^{3nm_0+m} r_k^{2-d} = \infty,
$$

without loss of generality, we may and do assume the sum is infinite for $m = 0$ and $i = 1$, i.e.,

$$\sum_{j=1}^{\infty} \sum_{\{n: 3nm_0 \in (n_{4j-2}, n_{4j}]\}} r_{3nm_0}^{d-1} \sum_{k=1}^{3nm_0} r_k^{2-d} = \infty. \tag{4.12}$$

Let X^* be reflecting Brownian motion on D^* starting from some $x_0 \in D_{n_0}$, where n_0 is large. Define

$$\begin{aligned} S_j &= \inf\{t > 0 : X_t^* \in \eta_{a_{n_j}}\}, \quad j \ge 1, \\ T_1^{j,n} &= \inf\{t > S_j : X_t^* \in \eta_{a_n}\}, \quad n \ge 1, \\ U_k^{j,n} &= \inf\{t > T_k^{j,n} : X_t^* \in \eta_{a_{n-1}}\}, \quad n, k \ge 1, \\ T_k^{j,n} &= \inf\{t > U_{k-1}^{j,n} : X_t^* \in \eta_{a_n}\}, \quad n, k \ge 2, \\ N_n^j &= \max\{k : U_k^{j,n} \le S_{j-1}\}, \quad n \ge 1. \end{aligned}$$

In words, N_n^j is the number of downcrossings of $[a_{n-1}, a_n]$ by $M_t \stackrel{\text{df}}{=} G(z_0^*, X_t^*)$ between times S_j and S_{j-1}. This is of interest to us only for n such that $[a_{n-1}, a_n] \subset [a_{n_{j-1}}, a_{n_j}]$.

The process M is a continuous local martingale so it is a time-change of Brownian motion, until it hits 0.

Consider a one-dimensional Brownian motion W_t starting from $a_{n_{4j}}$ and killed at the hitting time T of $a_{n_{4j-4}}$. Note that by (4.11), $a_{n_{4j}} \ge \beta^{4j}$, $a_{n_{4j-4}} \le \beta^{4j-3}$, and $a_{n_{4j-2}} \ge \beta^{4j-2}$. It follows from the Ray-Knight theorem that there is an event A with probability greater than $p_1 > 0$, such that on A, the local time L_T^x accumulated by W at level x before time T is greater than $c_5 \beta^j$ for all $x \in (a_{n_{4j-2}}, a_{n_{4j}})$. We will apply excursion theory to excursions of W from the set $\{a_n : n_{4j-2} \le n \le n_{4j}\}$. Given the local time $\{L_T^x : x = a_n \in \{a_{n_{4j-2}}, a_{n_{4j}}\}\}$ and assuming the event A occurs, the distribution of the number of excursions going from a_{n-1} to a_n is minorized by a Poisson random variable with expectation $K_j \ge c_5 \frac{\beta^{4j}}{a_n - a_{n-1}}$. By (4.8), we have

$$K_j \ge \frac{c_5}{c_1} \frac{\beta^{4j}}{r_{3nm_0}^{2-d}}.$$

Conditional on $\{L_T^x, x = a_n \in [a_{n_{4j-2}}, a_{n_{4j}}]\}$, these random variables are independent.

Let $T^M(b) = \inf\{t > 0 : M_t = b\}$ and $M_t^j = \{M_{t+T^M(a_{n_{4j}})}, t \in [0, T^M(a_{n_{4j-4}}) - T^M(a_{n_{4j}})]\}$. Since M_t^j is a time-change of W_t, there exists an event A' with $\mathbf{P}_{x_0}(A') > p_1$, such that on A', conditional on the local time of M^j at $\{a_n, n_{4j-1} \le n \le n_{4j}\}$, the numbers of excursions of M^j between consecutive points of $\{a_n, n_{4j-1} \le n \le n_{4j}\}$ are independent random variables minorized by independent Poisson random variables with means $K_n \ge \frac{c_5}{c_1} \frac{\beta^{4j}}{r_{3nm_0}^{2-d}}$. By (4.9) and (4.11), there is a constant $c_6 > 0$ depending only on D such that

$$K_j \ge c_6 \frac{\sum_{k=1}^{n_{4j}} r_k^{2-d}}{r_{3nm_0}^{2-d}}$$

Note that the processes M^j are independent. We will now consider the process X conditioned on the values of the following random elements: local times of the M^j's accumulated at levels $\{a_n, n_{4j-1} \le n \le n_{4j}\}$, and the endpoints of excursions of X from $\{\eta_{a_n}, n_{4j-1} \le n \le n_{4j}\}$.

It follows from the definition of a_n and Condition 4.2 that the following condition is satisfied

$$\textit{There are at least three consecutive } D_k\textit{'s between } \eta_{a_{n-1}} \textit{ and } \eta_{a_n}. \tag{4.13}$$

This and an easy argument based on Lemma 4.5 show that given endpoints of an excursion of X going from η_{a_n} to $\eta_{a_{n-1}}$, the amount of local time accumulated by the excursion on ∂D is greater than $c_7 r_{3nm_0}$ with probability greater than $p_2 > 0$.

Let J_n be the distribution of the local time accumulated by X on the part of ∂D between $\eta_{a_{n-1}}$ and η_{a_n} during the time interval $(T^M(a_{n_{4j}}), T^M(a_{n_{4j-4}}))$. We have shown that on an event A_j of probability greater than p_1, J_n is stochastically minorized by a random variable I_n whose distribution is Poisson with mean greater than

$$p_2\, c_6 \frac{\sum_{k=1}^{n_{4j}} r_k^{2-d}}{r_{3nm_0}^{2-d}} c_7 r_{3nm_0} = p_2\, c_6\, c_7\, r_{3nm_0}^{d-1} \sum_{k=1}^{n_{4j}} r_k^{2-d}.$$

Hence J_n is minorized by a random variable with mean $\lambda_n \ge c_8 r_{3nm_0}^{d-1} \sum_{k=1}^{n_{4j}} r_k^{2-d}$ and variance λ_n. Moreover, we can assume that the I_n's are independent given A_j. Hence, the local time accumulated by X between the hitting of $\eta_{a_{n_{4j}}}$ and $\eta_{a_{n_{4j-2}}}$, on the part of ∂D between these surfaces, is stochastically minorized by a random variable H_j such that on the event A_j, its mean is bounded below by

$$\sum_{\{n:3nm_0\in[n_{4j-2},n_{4j}]\}} c_8 r_{3nm_0}^{d-1}\sum_{k=1}^{n_{4j}} r_k^{2-d}$$

and the variance equals its mean. It follows that H_j takes a value no less than

$$b_j \stackrel{\text{df}}{=} \frac{1}{2}\sum_{\{n:3nm_0\in[n_{4j-2},n_{4j}]\}} c_8 r_{3nm_0}^{d-1}\sum_{k=1}^{n_{4j}} r_k^{2-d}$$

with probability greater than $p_3 > 0$.

Since the M^j's are independent, we can assume that the H_j's are independent. Let Λ_j be independent random variables with $P(\Lambda_j = b_j) = 1 - P(\Lambda_j = 0) = p_3$. The distribution of the local time accumulated by reflecting Brownian motion starting from $x_0 \in D_{n_{4j_0}}$ before hitting B_* is minorized by the distribution of $\sum_{j=1}^{j_0}\Lambda_j$. In view of (4.12), $\sum_{j\geq 1} b_j = \infty$, and this easily implies that $\sum_{j\geq 1}\Lambda_j = \infty$, a.s. Hence, for any $b < \infty$, there is some j_0 such that $\mathbf{P}(\sum_{j\leq j_0}\Lambda_j > b) > 1-1/b$. This implies that $\mathbf{P}_{x_0}(L_{T_{B_*}} > b) > 1-1/b$ for every $x_0 \in D_{n_{4j_0}}$. Therefore, $\inf_{x\in D}\mathbf{E}_x\left[\exp(-L_{T_{B_*}})\right] = 0$ and we see that part of the surface of D is nearly inactive. □

We note that the only place where Condition 4.2 is used in the proof of Theorem 4.3(ii) is to prove (4.13). We will next discuss Condition 4.2 but first we need a lemma.

Lemma 4.6. *Let $D \in \mathcal{D}_1$, $z \in \partial D$ and let $\{D_n, n \geq 0\}$ and $\{\gamma_n, n \geq 0\}$ be as in Definition 4.1. There exist $\alpha_0, p_0 > 0$ depending only on D such that the following holds.*

(i) For every $n \geq 1$ and every positive harmonic function h on the interior of $\overline{D_{n-1}\cup D_n}$, with Neumann boundary conditions on $\partial D \cap \overline{D_{n-1}\cup D_n}$,

$$h(x) \leq \alpha_0 h(y) \qquad \text{for every } x, y \in \gamma_n.$$

(ii) For every $n \geq 1$,

$$\mathbf{P}_x(T_{\gamma_{n+1}} < T_{\gamma_{n-1}}) \geq p_0 \qquad \text{for every } x \in \gamma_n.$$

Proof. (i) Let h be a positive harmonic function on the interior of $\overline{D_{n-1} \cup D_n}$, with Neumann boundary conditions on $\partial D \cap \overline{D_{n-1} \cup D_n}$. It follows from parts (i) and (iv) of Definition 4.1 that there are $k_1 < \infty$ and $c_1 > 0$, depending only on D, such that γ_n can be covered by at most k_1 balls $B(x_k, c_1 r_n)$. Moreover, for each one of these balls, either $B(x_k, 2c_1 r_n) \subset D$ or $B(x_k, 2c_1 r_n) \cap \partial D$ is the graph of a Lipschitz function. If $x, y \in \gamma_n$ and both points belong to one of balls $B(x_k, c_1 r_n)$, then there is a constant $c_2 > 0$ depending only on D such that $h(x) \leq c_2 h(y)$, either by the usual Harnack principle (if the ball is inside D) or by the Neumann boundary Harnack principle proved in Lemma 2.8. It follows by a Harnack chain argument that $h(x) \leq \alpha_0 h(y)$ for any $x, y \in \gamma_n$, where $\alpha_0 = c_2^{k_1}$.

(ii) According to the definition of $D \in \mathcal{D}_1$, there is $k_2 < \infty$ such that there exists a "Harnack chain of balls" connecting γ_n and γ_{n+1}, that is, we can find a sequence $B(x_1, r), B(x_2, r), \ldots, B(x_k, r)$ in Ω_{n-1} with $k \leq k_2$, $x_1 \in \gamma_n$, $x_k \in \gamma_{n+1}$ and $x_j \in B(x_{j-1}, r/2)$ for $j = 2, \ldots, k$. The existence of this "Harnack chain of balls" and the Harnack inequality easily imply that for some $p_1 > 0$ depending only on D and some $x \in \gamma_n$,

$$\mathbf{P}_x(T_{\gamma_{n+1}} < T_{\gamma_{n-1}}) \geq p_1.$$

Applying part (i) of this lemma to harmonic function $x \mapsto \mathbf{P}_x(T_{\gamma_{n+1}} < T_{\gamma_{n-1}})$, we conclude that there is some $p_0 > 0$ depending only on D such that $\mathbf{P}_x(T_{\gamma_{n+1}} < T_{\gamma_{n-1}}) \geq p_0$ for every $x \in \gamma_n$. □

Part (ii) of Theorem 4.3 has been proved under the assumption that Condition 4.2 holds. Condition 4.2 seems to be difficult to verify in a direct way. We will state two other conditions, Conditions 4.7 and 4.8, that are easier to verify in examples. We will show that Condition 4.7 implies Condition 4.8 and Condition 4.8 implies Condition 4.2. In some examples, Condition 4.7 is the easiest condition to verify, but in some other examples Condition 4.8 holds even though Condition 4.7 does not. Lemma 4.11 below shows how one can verify Condition 4.7 in some examples.

Condition 4.7. *Let α_0 and p_0 be the constants in Lemma 4.6 and let T_{γ_n} be the first hitting time of γ_n by reflecting Brownian motion in D. There exist $0 < m_0 < m_1 \leq \infty$*

such that for any $z \in \partial D$ and the γ_n's as in Definition 4.1 corresponding to z, if $n > m_1$,

$$\mathbf{P}_x(T_{\gamma_{n-m_0-1}} < T_{\gamma_{n+1}}) \leq \alpha_0^{-2} p_0 \qquad \text{for every } x \in \gamma_n,$$

and

$$\mathbf{P}_x(T_{\gamma_{n+1}} < T_{\gamma_{n-m_0-1}}) \leq \alpha_0^{-2} p_0 \qquad \text{for every } x \in \gamma_{n-m_0}.$$

Condition 4.8. *There exist $0 < m_0 \leq m_1 < \infty$ such that for any $z \in \partial D$ and the γ_n's as in Definition 4.1, the following is true for $n > m_1$. Let A be the interior of $\overline{\bigcup_{n-m_0-1 \leq k \leq n} D_k}$ and $\mu_x(dy) = \mathbf{P}_x(T_{\gamma_{n-m_0-1} \cup \gamma_{n+1}} \in dy)$, for $x \in A$. In other words, μ_x is harmonic measure on the set $\gamma_{n-m_0-1} \cup \gamma_{n+1}$ inside A for Brownian motion reflected on ∂D. Then the Radon-Nikodym derivative $d\mu_z/d\mu_y \leq 1$ on γ_{n+1} and $d\mu_z/d\mu_y \geq 1$ on γ_{n-m_0-1}, for $z \in \gamma_{n-m_0}$ and $y \in \gamma_n$.*

Lemma 4.9. *Condition 4.7 implies Condition 4.8.*

Proof. Recall the constants α_0 and p_0 from Lemma 4.6.

Let A_n be the interior of $\overline{\bigcup_{n-m_0-1 \leq k \leq n} D_k}$ and $\mu_x(dy) = \mathbf{P}_x(X_{T_{\gamma_{n-m_0-1} \cup \gamma_{n+1}}} \in dy)$, for $x \in A_n$. In other words, μ_x is harmonic measure in A_n for Brownian motion reflected on ∂D. Fix a set $C \subset \gamma_{n+1}$. By Lemma 2.7, $x \mapsto \mu_x(C)$ is a non-negative harmonic function of $x \in A_n$. By Lemma 4.6,

$$\mu_x(C) \leq \alpha_0 \mu_y(C) \qquad \text{and} \qquad \mu_y(\gamma_{n+1}) \leq \alpha_0 \mu_x(\gamma_{n+1})$$

for $x, y \in \gamma_n$, and so

$$\mu_x(C) \leq \alpha_0^2 \, \frac{\mu_y(C)}{\mu_y(\gamma_{n+1})} \, \mu_x(\gamma_{n+1}) \qquad \text{for } x, y \in \gamma_n. \tag{4.14}$$

By Lemma 4.6 and Condition 4.7, for $n > m_1$, $\mu_x(\gamma_{n+1}) \geq p_0$ for all $x \in \gamma_n$ and $\mu_z(\gamma_{n+1}) \leq \alpha_0^{-2} p_0$ for all $z \in \gamma_{n-m_0}$. Hence,

$$\mu_z(\gamma_{n+1})/\mu_x(\gamma_{n+1}) \leq \alpha_0^{-2}, \tag{4.15}$$

for all $z \in \gamma_{n-m_0}$, $x \in \gamma_n$ and $n > m_1$. If reflecting Brownian motion in D starts from a point in γ_{n-m_0}, it has to hit γ_n before hitting γ_{n+1}. Hence, by the strong Markov property, (4.14), and (4.15), we obtain for $n > m_1$, $C \subset \gamma_{n+1}$, $z \in \gamma_{n-m_0}$ and $y \in \gamma_n$,

$$\begin{aligned}
\mu_z(C) &= \int_{\gamma_n} \mu_x(C)\mathbf{P}_z(X_{T_{\gamma_{n-m_0-1}\cup\gamma_n}} \in dx) \\
&\leq \alpha_0^2 \frac{\mu_y(C)}{\mu_y(\gamma_{n+1})} \int_{\gamma_n} \mu_x(\gamma_{n+1})\mathbf{P}_z(X_{T_{\gamma_{n-m_0-1}\cup\gamma_n}} \in dx) \\
&= \alpha_0^2 \frac{\mu_y(C)}{\mu_y(\gamma_{n+1})}\mu_z(\gamma_{n+1}) \\
&\leq \mu_y(C).
\end{aligned}$$

Since C is an arbitrary subset of γ_{n+1}, the Radon-Nikodym derivative $d\mu_z/d\mu_y \leq 1$ on γ_{n+1} for $z \in \gamma_{n-m_0}$ and $y \in \gamma_n$ with $n > m_1$. Similarly, $d\mu_z/d\mu_y \geq 1$ on γ_{n-m_0-1} for $z \in \gamma_{n-m_0}$ and $y \in \gamma_n$ with $n > m_1$. Therefore Condition 4.8 is satisfied. □

Lemma 4.10. *Condition 4.8 implies Condition 4.2.*

Proof. We will consider $n \geq m_1 + 1$. Suppose that $x_0 \in \Omega_{n+1}$ and choose c_1 so large that $K \stackrel{\text{df}}{=} \{x \in D \setminus B_* : G(x_0, x) \geq c_1\} \subset \Omega_{n+1}$. We will prove that $\sup_{x\in\gamma_{n-m_0}} G(x_0, x) \leq \inf_{x\in\gamma_n} G(x_0, x)$. It will suffice to show that $\mathbf{P}_x(T_K < T_{B_*}) \leq \mathbf{P}_y(T_K < T_{B_*})$ for $x \in \gamma_{n-m_0}$ and $y \in \gamma_n$, because $G(x_0, x) = c_1\mathbf{P}_x(T_K < T_{B_*})$ for $x \in D \setminus (B_* \cup K)$.

Our proof will use the technique of coupling. We will construct two reflecting Brownian motions in D on a common probability space, X starting from $x \in \gamma_{n-m_0}$ and Y starting from $y \in \gamma_n$, such that $\{T^X_K < T^X_{B_*}\} \subset \{T^Y_K < T^Y_{B_*}\}$ almost surely, that is,

$$\mathbf{P}\left(T^X_K < T^X_{B_*} \text{ and } T^Y_K \geq T^Y_{B_*}\right) = 0.$$

Let A_n be the interior of $\overline{\bigcup_{n-m_0-1\leq k\leq n} D_k}$ and define

$$\mu_x(dy) \stackrel{\text{df}}{=} \mathbf{P}_x(X_{T_{\gamma_{n-m_0-1}\cup\gamma_{n+1}}} \in dy) \qquad \text{for } x \in A_n.$$

Given $x \in \gamma_{n-m_0}$ and $y \in \gamma_n$, we will define some random variables on a common probability space. Let $\eta_{x,n+1}$ be a random variable taking values in γ_{n+1} and having distribution $\mu_x(dz)/\mu_x(\gamma_{n+1})$. Let $\eta_{y,x,n+1}$ take values in γ_{n+1} with distribution $(\mu_y(dz) -$

$\mu_x(dz))/(\mu_y(\gamma_{n+1}) - \mu_x(\gamma_{n+1}))$, and let $I_{x,n+1}$ take values 0 or 1, with $P(I_{x,n+1} = 1) = \mu_x(\gamma_{n+1})$. Note that $\eta_{y,x,n+1}$ is well defined because, under Condition 4.8, $\mu_y(dz) \geq \mu_x(dz)$ on γ_{n+1}. Similarly, let $\eta_{y,n-m_0-1}$ be a random variable taking values in γ_{n-m_0-1} and having distribution $\mu_y(dz)/\mu_y(\gamma_{n-m_0-1})$ on γ_{n-m_0-1}. Let $\eta_{x,y,n-m_0-1}$ take values in γ_{n-m_0-1} and have distribution $(\mu_x(dz)-\mu_y(dz))/(\mu_x(\gamma_{n-m_0-1})-\mu_y(\gamma_{n-m_0-1}))$. Let $I_{y,n-m_0-1}$ take values 0 or 1, and assume that $P(I_{y,n-m_0-1} = 1) = \mu_y(\gamma_{n-m_0-1})$. Due to Condition 4.8, we may and do assume that the I's are constructed so that $I_{x,n+1} + I_{y,n-m_0-1} \leq 1$, a.s., and we let $I_{x,y} = 1 - I_{x,n+1} - I_{y,n-m_0-1}$. Moreover, the η's are constructed so that they are independent, and independent of the I's.

For $x \in A_n$ and $z \in \gamma_{n-m_0-1} \cup \gamma_{n+1}$, let $\mathbf{Q}_x^z$ denote the distribution of reflecting Brownian motion X in A_n starting from x, conditioned on leaving $\overline{A_n} \setminus \{\gamma_{n-m_0-1} \cup \gamma_{n+1}\}$ through z. Let $\mathbf{Q}_{x,y}^z$ denote the distribution of a pair of processes $(\widehat{X}, \widehat{Y})$, such that the distribution of $\widehat{X}$ is $\mathbf{Q}_x^z$ and the distribution of $\widehat{Y}$ is $\mathbf{Q}_y^z$. The processes $\widehat{X}$ and $\widehat{Y}$ are defined on the same probability space but no further relationship such as independence is assumed. In particular, the two processes do not necessarily reach z at the same time.

We will now define a distribution for a pair of processes $(\widetilde{X}, \widetilde{Y})$ starting from $x, y \in \gamma_{n-m_0} \cup \gamma_n$, such that either $x = y$ or $x \in \gamma_{n-m_0}$ and $y \in \gamma_n$. If $x = y \in \gamma_{n-m_0}$, we define $\widetilde{X}_t = \widetilde{Y}_t$ for all $t \geq 0$, and the distribution of $\widetilde{X}$ is that of reflecting Brownian motion in D, killed upon hitting γ_n. Similarly, if $x = y \in \gamma_n$, define $\widetilde{X}_t = \widetilde{Y}_t$ for all $t \geq 0$, and the distribution of $\widetilde{X}$ is that of reflecting Brownian motion in D, killed upon hitting γ_{n-m_0}, The most significant case is when $x \in \gamma_{n-m_0}$ and $y \in \gamma_n$. In this case we let

$$Z_x \stackrel{\text{df}}{=} \eta_{x,n+1} I_{x,n+1} + \eta_{y,n-m_0-1} I_{y,n-m_0-1} + \eta_{x,y,n-m_0-1} I_{x,y}$$

and

$$Z_y \stackrel{\text{df}}{=} \eta_{x,n+1} I_{x,n+1} + \eta_{y,n-m_0-1} I_{y,n-m_0-1} + \eta_{y,x,n+1} I_{x,y}.$$

Note that by our construction, Z_x and Z_y have distributions μ_x and μ_y, respectively. When $I_{y,n-m_0-1} = 1$ and $Z_x = Z_y = \eta_{y,n-m_0-1} = z \in \gamma_{n-m_0-1}$, we define the (conditional) distribution of $(\widetilde{X}, \widetilde{Y})$ to be $\mathbf{Q}_{x,y}^z$ until the processes hit γ_{n-m_0-1}, and then we "continue them as a single reflecting Brownian motion in D starting from z until it hits γ_{n-m_0}." In other words, if $\widetilde{X}$ hits γ_{n-m_0-1} at time t_0 and $\widetilde{Y}$ hits γ_{n-m_0-1} at time t_1 then $\widetilde{X}_{t_0+t} = \widetilde{Y}_{t_1+t}$

for $t \geq 0$. Similarly, when $I_{x,n+1} = 1$ and $Z_x = Z_y = \eta_{x,n+1} = z \in \gamma_{n+1}$, we define the (conditional) distribution of $(\widetilde{X}, \widetilde{Y})$ to be $\mathbf{Q}^z_{x,y}$ until the processes hit γ_{n+1}, and then we continue them as a single reflecting Brownian motion in D starting from z until it hits γ_n. When $I_{x,y} = 1$ and $Z_x = z_1 \in \gamma_{n-m_0-1}$ and $Z_y = z_2 \in \gamma_{n+1}$, we let $\widetilde{X}$ have (conditional) distribution $\mathbf{Q}^x_{z_1}$ and then we continue it as reflecting Brownian motion in D starting from z_1 until it hits γ_{n-m_0}, and we let $\widetilde{Y}$ be independent from $\widetilde{X}$ with conditional distribution $\mathbf{Q}^{z_2}_y$, and we continue it as reflecting Brownian motion in D starting from z_2 until it hits γ_n. We call the distribution of the processes constructed above $\mathbf{P}_{x,y}$. Note that under $\mathbf{P}_{x,y}$ each one of the processes $\widetilde{X}$ and $\widetilde{Y}$ is a reflecting Brownian motion in D. Under $\mathbf{P}_{x,y}$, the processes $\widetilde{X}$ and $\widetilde{Y}$ start from $x, y \in \gamma_{n-m_0} \cup \gamma_n$, i.e., $\widetilde{X}_0 = x$ and $\widetilde{Y}_0 = y$, they have random lifetimes ζ^X and ζ^Y, not necessarily equal, $\widetilde{X}_{\zeta^X-}, \widetilde{Y}_{\zeta^Y-} \in \gamma_{n-m_0} \cup \gamma_n$, and either $\widetilde{X}_{\zeta^X-} = \widetilde{Y}_{\zeta^Y-}$ or $\widetilde{X}_{\zeta^X-} \in \gamma_{n-m_0}$ and $\widetilde{Y}_{\zeta^Y-} \in \gamma_n$. The essential property of $\mathbf{P}_{x,y}$ is that if $\widetilde{X}$ enters Ω_{n+1} before it is killed, then the part of the trajectory of $\widetilde{X}$ after the hitting time of Ω_{n+1} is a time shift of the trajectory of $\widetilde{Y}$ after its hitting time of Ω_{n+1}. Similarly, under $\mathbf{P}_{x,y}$, if $\widetilde{Y}$ enters Ω'_{n-m_0-1} before it gets killed, then the part of the trajectory of $\widetilde{Y}$ after the hitting time of Ω_{n-m_0-1} is a time shift of the trajectory of $\widetilde{X}$ after its hitting time of Ω_{n-m_0-1}.

We will use the distributions $\mathbf{P}_{x,y}$ to construct processes X and Y which are defined on the whole time interval $[0, \infty)$. Suppose that $x \in \gamma_{n-m_0}$ and $y \in \gamma_n$ and let (X^1, Y^1) have distribution $\mathbf{P}_{x,y}$. Let (X^2, Y^2) have conditional distribution $\mathbf{P}_{x_2,y_2}$ given the event $\{X^1_{\zeta^{X^1}-} = x_2 \text{ and } Y^1_{\zeta^{Y^1}-} = y_2\}$. We continue by induction. Given (X^k, Y^k), we let (X^{k+1}, Y^{k+1}) have conditional distribution $\mathbf{P}_{x_{k+1},y_{k+1}}$ given the event $\{X^k_{\zeta^{X^k}-} = x_{k+1} \text{ and } Y^k_{\zeta^{Y^k}-} = y_{k+1}\}$. It is easy to see that $\sum_k \zeta^{X^k} = \infty$ and $\sum_k \zeta^{Y^k} = \infty$, a.s. Set $\zeta^{X^0} = \zeta^{Y^0} = 0$. For $k \geq 0$ and $t \in \big[\sum_{0\leq j\leq k} \zeta^{X^j}, \sum_{0\leq j\leq k+1} \zeta^{X^j}\big)$, define

$$X_t \stackrel{\text{df}}{=} X^{k+1}\Big(t - \sum_{0\leq j\leq k} \zeta^{X^j}\Big).$$

Similarly, for $t \in \big[\sum_{0\leq j\leq k} \zeta^{Y^j}, \sum_{0\leq j\leq k+1} \zeta^{Y^j}\big)$, define

$$Y_t \stackrel{\text{df}}{=} Y^{k+1}\Big(t - \sum_{0\leq j\leq k} \zeta^{Y^j}\Big).$$

It is straightforward to check that X and Y are reflecting Brownian motions in D and $\{T_K^X < T_{B_*}^X\} \subset \{T_K^Y < T_{B_*}^Y\}$. This proves that $\mathbf{P}_x(T_K < T_{B_*}) \le \mathbf{P}_y(T_K < T_{B_*})$ for $x \in \gamma_{n-m_0}$ and $y \in \gamma_n$ and, as we pointed out at the beginning of this proof, this implies that $\sup_{x\in\gamma_{n-m_0}} G(x_0, x) \le \inf_{x\in\gamma_n} G(x_0, x)$. ◻

Recall λ from Definition 2.1.

Lemma 4.11. *For any c_1 and λ there exists c_2 such that the following holds. Suppose that for some n and m_2 we have $|D_k| \le c_1 r_k^d$ for all $n - m_2 - 1 \le k \le n$. Then for all $x \in \gamma_n$,*

$$\mathbf{P}_x(T_{\gamma_{n-m_2-1}} < T_{\gamma_{n+1}}) \le c_2 \frac{r_n^{2-d}}{\sum_{i=n-m_2-1}^{n} r_i^{2-d}},$$

and for all $x \in \gamma_{n-m_2}$ we have

$$\mathbf{P}_x(T_{\gamma_{n+1}} < T_{\gamma_{n-m_2-1}}) \le c_2 \frac{r_{n-m_2-1}^{2-d}}{\sum_{i=n-m_2-1}^{n} r_i^{2-d}}.$$

Proof. We prove the second inequality, the first one being very similar. Write j for $n - m_2 - 1$. If a, b are integers with $j \le a \le b \le n$, set $U_{a,b} = \bigcup_{k=a}^{b} D_k$, define

$$C_{a,b} = \inf\Big\{ \int_{U_{a,b}} |\nabla f(x)|^2 dx : f = 0 \text{ on } \gamma_a \text{ and } f = 1 \text{ on } \gamma_{b+1} \Big\}, \tag{4.16}$$

and let $R_{a,b} = C_{a,b}^{-1}$. $C_{a,b}$ is called the conductance across $U_{a,b}$ and $R_{a,b}$ the resistance. Consider reflecting Brownian motion in $U_{a,b}$ killed on hitting γ_a and let $G_{a,b}(x, y)$ be the corresponding Green function. We use the fact that with respect to this process $C_{a,b}$ is equal to the capacity of γ_{b+1}; see [FOT].

Using Definition 4.1 we can find a constant c_3 independent of k and points $z_k \in D_k$ such that $\operatorname{dist}(z_k, \partial D_k) \ge c_3 r_k$. Let B_k be the ball of radius $c_3 r_k/2$ centered at z_k. Starting at any point that is a distance $c_3 r_k/4$ from z_k, the expected time that Brownian motion in D_k spends in B_k before hitting ∂D_k is larger than $c_4 r_k^2$. By the support theorem for standard d-dimensional Brownian motion, starting from any point that is a distance $3c_3 r_k/4$ from z_k, there is probability at least $p_1 > 0$ (not depending on k) that the Brownian motion will hit the ball of radius $c_3 r_k/4$ about z_k before hitting ∂D_k. So starting at such a point the expected time spent in B_k before hitting ∂D_k is at least $p_1 c_4 r_k^2$. Using the Harnack inequality and the fact that $|B_k| = c_5 r_k^d$, it follows that $G_{k,k}(z_k, y) \geq c_6 r_k^{2-d}$ if $|y - z_k| = 3c_3 r_k/4$. By the Neumann boundary Harnack principle,

$$G_{k,k}(z_k, y) \geq c_7 r_k^{2-d}, \qquad y \in \gamma_{k+1}. \tag{4.17}$$

Consider reflecting Brownian motion in D_k killed on hitting γ_k and let ν_k be the capacitary measure for γ_{k+1}. Then

$$\begin{aligned} 1 \geq \mathbf{P}_{z_k}(T_{\gamma_{k+1}} < T_{\gamma_k}) &= \int G_{k,k}(z_k, y)\nu_k(dy) \\ &\geq c_7 r_k^{2-d} \nu_k(\gamma_{k+1}) \\ &= c_7 r_k^{2-d} C_{k,k}. \end{aligned}$$

Therefore

$$C_{k,k} \leq c_7^{-1} r_k^{d-2}$$

and

$$R_{k,k} \geq c_7 r_k^{2-d}. \tag{4.18}$$

Next, if $a_1 \leq a_2 < a_2 + 1 \leq a_3$, let f_1 be the function on U_{a_1,a_2} at which the infimum in (4.16) is attained and similarly f_2 the function on U_{a_2+1,a_3}. Let $\beta = C_{a_2+1,a_3}/(C_{a_1,a_2} + C_{a_2+1,a_3})$ and define f on U_{a_1,a_3} by setting the restriction of f on U_{a_1,a_2} to be equal to βf_1 and the restriction of f on U_{a_2+1,a_3} to be equal to $\beta + (1-\beta) f_2$. Then

$$\begin{aligned} C_{a_1,a_3} &\leq \int_{U_{a_1,a_3}} |\nabla f(x)|^2 dx = \beta^2 \int_{U_{a_1,a_2}} |\nabla f_1|^2 + (1-\beta)^2 \int_{U_{a_2+1,a_3}} |\nabla f_2|^2 \\ &= \beta^2 C_{a_1,a_2} + (1-\beta)^2 C_{a_2+1,a_3} \\ &= \frac{C_{a_1,a_2} C_{a_2+1,a_3}}{C_{a_1,a_2} + C_{a_2+1,a_3}}. \end{aligned} \tag{4.19}$$

This is equivalent to

$$R_{a_1,a_3} \geq R_{a_1,a_2} + R_{a_2+1,a_3}. \tag{4.20}$$

By (4.18), (4.20), and induction, we obtain

$$R_{j,n} \geq \sum_{i=j}^{n} c_7 r_i^{2-d},$$

or

$$C_{j,n} \le \frac{1}{\sum_{i=j}^{n} c_7 r_i^{2-d}}. \tag{4.21}$$

Recall that B_j is the ball of radius $c_3 r_j/2$ about z_j. Starting in B_j the expected amount of time the process spends in B_j before exiting the ball of radius $3c_3 r_j/4$ about z_j is bounded by $c_8 r_j^2$. By the support theorem for standard Brownian motion, there exists $p_2 > 0$ such that starting at any point that is a distance $3c_3 r_j/4$ from z_j, there is probability at least p_2 of hitting γ_j before hitting $\partial D_j \setminus \gamma_j$. A standard argument allows us to conclude that the expected amount of time spent in B_j starting at any point of $U_{j,n}$ is at most $c_9 r_j^2$. Since $|B_j| = c_{10} r_j^d$, the Harnack inequality implies that $G_{j,n}(z_j, y) \le c_{11} r_j^{2-d}$ if $y \in \gamma_{n+1}$. The Neumann boundary Harnack inequality then implies that

$$G_{j,n}(x,y) \le c_{12} r_j^{2-d}, \qquad x \in \gamma_{j+1}, \quad y \in \gamma_{n+1}. \tag{4.22}$$

Let ν be the equilibrium measure for γ_{n+1} with respect to reflecting Brownian motion in $U_{j,n}$ killed on hitting γ_j. Combining (4.21) and (4.22),

$$\mathbf{P}_x(T_{\gamma_{n+1}} < T_{\gamma_j}) = \int G_{j,n}(x,y)\nu(dy) \le c_{12} r_j^{2-d} C_{j,n}, \qquad x \in \gamma_{j+1}.$$

This proves the lemma. □

If the r_n are comparable, then Lemma 4.11 implies Condition 4.7 for sufficiently large m_0.

Remark 4.12. If $D \in \mathcal{D}_1$, then a "typical point" $x \in \partial D$ has a neighborhood $U \subset \overline{D}$ such that $\partial D \cap U$ is the graph of a Lipschitz function. The Green function satisfies $G(x,y) \le c_1 |x-y|^{2-d}$, for $x, y \in U$, where c_1 depends only on the Lipschitz constant characterizing ∂D; to see this we flatten the boundary and reflect over a hyperplane as in the proof of Lemma 2.8, and then use the result of [LSW]. The upper estimate in Lemma 4.4 follows from this immediately.

Example 4.13. Our first example in this section is a multidimensional version of Example 3.4. Suppose that $d \ge 3$ and for some $\alpha > 1$,

$$D = \left\{ x = (x_1, x_2, \dots, x_d) : 0 < x_1 < 1 \text{ and } x_1^\alpha > (x_2^2 + \dots + x_d^2)^{1/2} \right\}.$$

We will restrict the parameter range to $\alpha > 1$. We will show that if $\alpha \in (1, 2)$ then the whole surface of D is active and when $\alpha \geq 2$ then part of the surface is nearly inactive.

We will analyze only one boundary point, the origin, in view of Remark 4.12. We let the γ_k's be intersections of D with $(d-1)$-dimensional hyperplanes perpendicular to the first axis, at distances $2^{-k} + j2^{-k\alpha}$ from 0, for all $j \geq 0$ such that $2^{-k} + j2^{-k\alpha} \leq 2^{-k+1} - 2^{-k\alpha}$, for all $k \geq 1$.

Note that for some c_1 and any m_0 there exists m_1 such that for any $n > m_1$ we have $1/c_1 \leq r_j/r_k \leq c_1$ for all $n \leq j, k \leq n + m_0$. This and Lemma 4.11 easily imply that Condition 4.7 holds.

The number of D_n's whose distance from 0 lies between 2^{-k} and 2^{-k+1} is of order $2^{-k(1-\alpha)}$. For D_n's in this range, $\sum_{m=1}^n r_m^{2-d} \approx \sum_{j \leq k} 2^{-j(1-\alpha)} 2^{-j\alpha(2-d)} \approx 2^{-k(1+\alpha(1-d))}$. The surface area, $|\partial D_n \cap \partial D|$, is of order $2^{-k\alpha(d-1)}$, so the contribution from these sets to the sum in (4.1) is of order $2^{-k(1-\alpha)} \cdot 2^{-k(1+\alpha(1-d))} \cdot 2^{-k\alpha(d-1)} = 2^{-k(2-\alpha)}$. If $\alpha < 2$ then $\sum_{k \geq 1} 2^{-k(2-\alpha)} < \infty$, so part (i) of Theorem 4.3 implies that the whole surface of D is active.

A similar calculation shows that the sum in (4.2) is comparable to $\sum_{k \geq 1} 2^{-k(2-\alpha)}$ and this is infinite for $\alpha \geq 2$. Hence, by Theorem 4.3 (ii), part of the surface of D is nearly inactive if $\alpha \geq 2$. $\square$

Remark 4.14. Fukushima and Tomisaki [FT] studied reflecting Brownian motion in unbounded cusps

$$\widetilde{D} \stackrel{\text{df}}{=} \left\{ x = (x_1, x_2, \dots, x_d) : x_1 > 0 \text{ and } x_1^\alpha > (x_2^2 + \dots + x_d^2)^{1/2} \right\}$$

and derived a Green function estimate (see Lemma 5.4 and 5.5 in [FT]). Their proof can be adapted to get the Green function upper bound estimate for reflecting Brownian motion in the truncated cusps D as defined in Example 4.13 and to show that for $1 < \alpha < 2$,

$$\int_{\partial D} G_{D \setminus B_*}(x, y) \sigma(dy) < \infty.$$

Thus this gives an alternative proof for the boundary of D to be active when $1 < \alpha < 2$. The main goal of the paper [FT] is to show that reflecting Brownian motion in $\widetilde{D}$ starting from the cusp point $\mathbf{0} \stackrel{\text{df}}{=} (0, \cdots, 0)$ is a semimartingale when $\alpha < 2$. Using Theorem 4.3(ii) (and its proof) in this paper, we can settle the remaining case by showing that reflecting Brownian motion in $\widetilde{D}$ starting from $\mathbf{0}$ is not a semimartingale when $\alpha \geq 2$. Clearly, this is equivalent to the fact that reflecting Brownian motion in D starting from $\mathbf{0}$ is not a semimartingale when $\alpha \geq 2$.

By Theorem 2.1 of [FT], reflecting Brownian motion X in D is a strong Feller process on $\overline{D}$ and thus can start from every point in $\overline{D}$. Let $\alpha \geq 2$. According to Example 4.13, part of ∂D is nearly inactive. By the proof of Theorem 4.3(ii),

$$\lim_{x \to \mathbf{0}} \mathbf{P}_x \left(L_{T_{B_*}} > b \right) > 1 - \frac{1}{b} \qquad \text{for every } b > 0.$$

Were X a semimartingale starting from $\mathbf{0}$, the Skorokhod decomposition for X

$$X_t = X_0 + W_t + \int_0^t \mathbf{n}(X_s) dL_s \qquad \text{for } t \geq 0$$

would hold under $\mathbf{P}_x$ for every $x \in \overline{D}$. It follows from weak convergence and the second to the last display that

$$\mathbf{P}_{\mathbf{0}}(L_{T_{B_*}} = \infty) = 1.$$

This is a contradiction since $\mathbf{P}_{\mathbf{0}}(T_{B_*} < \infty) > 0$. Therefore X starting from $\mathbf{0}$ cannot be a semimartingale. □

Example 4.15. This is a multidimensional analogue of Example 3.6. Suppose that $\alpha > 0$, $\beta > 1$ and let $a_k = \sum_{j=1}^k 2^{-(j-1)\alpha}$. Let $\mathcal{S}_n$ be the family of all binary (zero-one) sequences of length n. We will write $\mathbf{s} = (s_1, s_2, \dots, s_n)$ for $\mathbf{s} \in \mathcal{S}_n$. For integer $k \geq 1$ and $\mathbf{s} \in \mathcal{S}_k$, we set $b_{\mathbf{s}} = \sum_{j=1}^k s_j 2^{-j}$. Let $A_* = [0,1]^d$. For $k \geq 1$ and $\mathbf{s} \in \mathcal{S}_k$ let

$$A_{\mathbf{s}} = \{(x_1, \dots, x_d) : a_k \leq x_1 \leq a_{k+1}, ((x_2 - b_{\mathbf{s}})^2 + x_3^2 + \dots + x_d^2)^{1/2} \leq f_{\mathbf{s}}(x_1)\},$$

where $c_1 2^{-k\beta} \leq f_{\mathbf{s}}(x_1) \leq 2^{-k\beta}$ and $c_1 > 0$ does not depend on $\mathbf{s}$. Assume that all functions $f_{\mathbf{s}}$ are Lipschitz with the same Lipschitz constant. Let D be the connected component of the interior of $A_* \cup \bigcup_{k \geq 1} \bigcup_{\mathbf{s} \in \mathcal{S}_k} A_{\mathbf{s}}$ that contains the open box $(0,1)^d$.

We restrict the range of parameters to $\beta > \alpha$. Since we have assumed that $\alpha > 0$ and $\beta > 1$, the surface of D is finite and Remark 3.5 cannot be used to draw any conclusions.

We will show that if $\alpha < \beta < 2\alpha$ then the whole surface of D is active and when $\beta \geq 2\alpha$ then part of the surface is nearly inactive.

We will analyze only a family of D_n's corresponding to a boundary point at the end of a channel, in view of Remark 4.12. Fix a boundary point z_0 at the end of an infinite channel, i.e., a point whose first coordinate is $\sum_{j=1}^{\infty} 2^{-(j-1)\alpha}$. Let $\mathcal{A}_k$ be the family of hyperplanes $K_{k,n} = \{(x_1, \dots, x_d) : x_1 = a_k + n2^{-k\beta}\}$, with $n \geq 1$ such that $a_k + n2^{-k\beta} \leq a_{k+1}$. Let $\mathcal{C}_k$ be the family of connected components of $K_{k,n} \cap D$, for $K_{k,n} \in \mathcal{A}_k$, which separate z_0 from A_*. Let γ_n's be the relabelled family $\bigcup_k \mathcal{C}_k$.

This assumption on the magnitude of f and Lemma 4.11 imply that Condition 4.7 holds as long as the relevant D_n's belong to the same $A_{\mathbf{s}}$. Lemmas 4.9 and 4.10 then prove that Condition 4.2, i.e., $\sup_{x \in \gamma_{n-m_0}} G(x_0, x) \leq \inf_{x \in \gamma_n} G(x_0, x)$, holds if γ_{n-m_0} and γ_n belong to the same $A_{\mathbf{s}}$. Then clearly Condition 4.2 holds in full generality if we replace m_0 with $2m_0 + 1$.

The number of D_n's defined by the γ_n's, needed to reach $A_{\mathbf{s}}$ with $\mathbf{s} \in \mathcal{S}_k$ is of order $\sum_{j \leq k} 2^{-j\alpha}/2^{-j\beta} \approx 2^{k(\beta-\alpha)}$. Consider a D_n which intersects $A_{\mathbf{s}}$ with $\mathbf{s} \in \mathcal{S}_k$. The set D_n may either have diameter of order $2^{-k\beta}$ or it may contain a "tree" of thin channels. Consider first D_n's that have diameters of order $2^{-k\beta}$. There are $2^{k(\beta-\alpha)}$ such D_n's, up to a constant, so for n in this range, $\sum_{m=1}^{n} r_m^{2-d} \approx \sum_{j \leq k} 2^{j(\beta-\alpha)} 2^{-j\beta(2-d)} \approx 2^{k(\beta(d-1)-\alpha)}$. The surface area, $|\partial D_n \cap \partial D|$, is of order $2^{-k\beta(d-1)}$, so the total contribution of such D_n's to (4.1) is of order $2^{k(\beta-\alpha)} 2^{k(\beta(d-1)-\alpha)} 2^{-k\beta(d-1)} \approx 2^{k(\beta-2\alpha)}$. The series $\sum_k 2^{k(\beta-2\alpha)}$ is summable if and only if $\beta < 2\alpha$.

Next consider a D_n which intersects $A_{\mathbf{s}}$ with $\mathbf{s} \in \mathcal{S}_k$ and contains a side "tree" of thin channels. Its surface area, $|\partial D_n \cap \partial D|$, is of order $\sum_{j \geq k} 2^{j-k} 2^{-j\alpha} 2^{-j\beta(d-2)} \approx 2^{-k(\alpha+\beta(d-2))}$. There are at most two such D_n's for each $A_{\mathbf{s}}$, so their contribution to (4.1) is of order $2^{k(\beta(d-1)-\alpha)} 2^{-k(\alpha+\beta(d-2))} \approx 2^{k(\beta-2\alpha)}$. Hence, the contribution of D_n's with side channels is of the same order as the contribution of D_n that have diameter of order $2^{-k\beta}$. We conclude that (4.1) holds if $\beta < 2\alpha$.

If $\beta \geq 2\alpha$ then the contribution of D_n's with diameter of order $2^{-k\beta}$ is enough to make the left hand side of (4.2) infinite, due to the estimates presented above. □

Example 4.16. We will analyze a multidimensional fractal domain vaguely resembling the von Koch snowflake, except that we will add barriers partly blocking the passage between the building blocks. Suppose that the dimension of the space is $d \geq 3$ and fix a parameter $\rho \in (0, 1/2)$. We will impose further restrictions on ρ below. For $k \geq 0$, let $\mathcal{A}_k$ be a finite family of open cubes with edge length ρ^k, with edges parallel to the axes, and satisfying the following properties. The family $\mathcal{A}_0$ consists of one cube A_0. The family $\mathcal{A}_1$ consists of $2d$ cubes which are disjoint from each other and are disjoint from A_0. One side of any cube in $\mathcal{A}_1$ lies on a side of A_0 and these two sides of the two cubes have the same center. Now suppose that we have defined families $\mathcal{A}_k$ for $k \leq n$. Let A_n be the union of all cubes in $\bigcup_{k \leq n} \mathcal{A}_k$. Then $\mathcal{A}_{n+1}$ is the maximal family of disjoint cubes that do not intersect A_n and such that one side of each of these cubes lies on a side of a cube from the family $\mathcal{A}_n$, and has the same center. Let D_* be the union of all cubes in $\bigcup_{k \geq 0} \mathcal{A}_k$ and note that this set is not connected because all cubes in this family are disjoint. We transform D_* into a connected open set by adding "passages" between cubes. Fix a parameter $\beta > 1$. For any pair of cubes which belong to $\mathcal{A}_{n-1}$ and $\mathcal{A}_n$, and whose sides intersect and have a common center x, we add to D_* the open ball $B(x, 2\rho^{(n-1)\beta})$. We let D be the union of D_* and all such balls. Parts of the boundary of D are adjacent to D on both sides, and this is forbidden by Definition 4.1, strictly speaking. We could modify the domain D or even Definition 4.1 to cover this case, but that would be an unnecessary embellishment.

We will determine for which values of ρ the surface area is finite because the example is not interesting if $|\partial D| = \infty$; in such a case a part of the surface is nearly inactive by Remark 3.5. The surface area of a cube with edge length ρ^k is of order $\rho^{k(d-1)}$. The number of cubes in $\mathcal{A}_k$ is of order $(2d-1)^k$. The total surface area of cubes in $\mathcal{A}_k$ is of order $\rho^{k(d-1)}(2d-1)^k$. The surface area of D is finite if $\sum_k \rho^{k(d-1)}(2d-1)^k < \infty$, that is if $\rho^{d-1}(2d-1) < 1$. Hence, we are interested only in ρ less than $(2d-1)^{-1/(d-1)}$. The function $f(d) = (2d-1)^{-1/(d-1)}$ is increasing for $d \geq 3$ because, when we treat d as a real argument,

$$f'(d) = \frac{1 + (2d-1)\left(\log(2d-1) - 1\right)}{(d-1)^2 (2d-1)^{d/(d-1)}} > 0$$

for $d \geq 3$. We have $f(3) = 1/\sqrt{5}$, and $\lim_{d\to\infty} f(d) = 1$. Hence, we can take $\rho \in (0, 1/2 \wedge 1/\sqrt{5})$ for any d. We will see that, as long as the surface area is finite, the value of ρ does not play any role in this example.

As usual in our examples, we will analyze only a point $z \in \partial D$ that lies at the end of an "infinite" channel, i.e., such that any continuous path in D from the center z_* of A_0 to z must pass through at least one cube in every family $\mathcal{A}_k$. Let Γ be a continuous path in D from z_* to z that passes through a side of any cube in $\bigcup_{k\geq 0} \mathcal{A}_k$ at most once, and if it does so, then it passes through the center of that side. Let z_k be the intersection point of Γ and the side of the cube in $\mathcal{A}_k$ that is a part of a side of a cube in $\mathcal{A}_{k-1}$. The curve Γ passes through all the z_k's on its way from z_* to z.

For every z_k, let $\mathcal{C}_k$ be the family of all sets $\partial B(z_k, 2^j) \cap D$, where j satisfies $4\rho^{(k-1)\beta} \leq 2^{j-1} \leq 2^{j+1} \leq \rho^{k-1}/2$. Note that each set $\partial B(z_k, 2^j) \cap D$ contributes two sets to $\mathcal{C}_k$ and each one of these sets is a spherical cap. Let $\mathcal{C} = \bigcup_k \mathcal{C}_k$ and rename the elements of $\mathcal{C}$ as γ_n, in the order in which they have to be passed on the way from z_* to z within D. It is elementary to check that this family of γ_n's satisfies the conditions listed in Definition 4.1.

In this example, Condition 4.7 does not hold. We will argue that Condition 4.8 holds directly. Consider spheres $\partial B(0, 2^j), \partial B(0, 2^{j+k_0}), \partial B(0, 2^{j+k_0+m_0})$ and $\partial B(0, 2^{j+2k_0+m_0})$ and call them S_1, S_2, S_3 and S_4. It is not very hard to prove that there exist large k_0 and m_0, such that $\mathbf{P}_x(T_{S_4} \in A, T_{S_4} < T_{S_1}) \leq \mathbf{P}_y(T_{S_4} \in A, T_{S_4} < T_{S_1})$, for $A \subset S_4$, $x \in S_2$ and $y \in S_3$. We also have, for sufficiently large k_0 and m_0, that $\mathbf{P}_x(T_{S_1} \in A, T_{S_1} < T_{S_4}) \geq \mathbf{P}_y(T_{S_1} \in A, T_{S_1} < T_{S_4})$, for $A \subset S_1$, $x \in S_2$ and $y \in S_3$, although the two claims are not symmetric and require somewhat different justification. By the reflection principle, for reflecting Brownian motion in D, $\mathbf{P}_x(T_{\gamma_{n+2k_0+m_0}} \in A, T_{\gamma_{n+2k_0+m_0}} < T_{\gamma_n}) \geq \mathbf{P}_y(T_{\gamma_{n+2k_0+m_0}} \in A, T_{\gamma_{n+2k_0+m_0}} < T_{\gamma_n})$ for $A \subset \gamma_{n+2k_0+m_0}$, $x \in \gamma_{n+k_0}$ and $y \in \gamma_{n+k_0+m_0}$, provided γ_n and $\gamma_{n+2k_0+m_0}$ belong to the same family $\mathcal{C}_k$ and lie on the same side of ∂D. We also have $\mathbf{P}_x(T_{\gamma_n} \in A, T_{\gamma_n} < T_{\gamma_{n+2k_0+m_0}}) \geq \mathbf{P}_y(T_{\gamma_n} \in A, T_{\gamma_n} < T_{\gamma_{n+2k_0+m_0}})$ for $A \subset \gamma_n$, $x \in \gamma_{n+k_0}$ and $y \in \gamma_{n+k_0+m_0}$. If we take only every k_0-th element of the family

γ_n, this proves Condition 4.7 for γ_n's which belong to the same family $\mathcal{C}_k$ and lie on the same side of ∂D. Hence, Lemma 4.10 proves Condition 4.2 for n restricted in such a way. However, this implies that Condition 4.2 holds for all n, with m_0 replaced by $2m_0$, for the same reason as in Example 4.13.

The number of γ_n's in $\mathcal{C}_k$ is of order $\log((\rho^{k-1}/2)/(4\rho^{(k-1)\beta})) \approx k$. If $z_1 \in \Omega_{n+1}$ with sufficiently large n then by Lemma 4.4, the Green function $G(z_1, \cdot\,)$ can be bounded by $c_1 \sum_{j\le k} j\rho^{j\beta(2-d)} \le c_2 k\rho^{k\beta(2-d)}$ for $x \in D$ that lie between γ_n's in $\mathcal{C}_k$. The surface area of $\partial D_n \cap \partial D$ corresponding to $\gamma_n \in \mathcal{C}_k$ is bounded by $c_3\rho^{k(d-1)}$, so the contribution of such D_n's to the sum in (4.1) is bounded by $c_4 k^2\rho^{k\beta(2-d)}\rho^{k(d-1)} = c_4 k^2 \rho^{k(\beta(2-d)+d-1)}$. If $\beta < (d-1)/(d-2)$ then $\sum_k k^2\rho^{k(\beta(2-d)+d-1)} < \infty$ and Theorem 4.3 (i) implies that the whole surface of D is active.

To find a lower bound for (4.2), we take into account only one D_n corresponding to each family $\mathcal{C}_k$, namely the one with the largest surface area. We obtain as a lower bound for (4.2) the quantity $c_5 k\rho^{k\beta(2-d)}\rho^{k(d-1)} = c_5 k\rho^{k(\beta(2-d)+d-1)}$. If $\beta \ge (d-1)/(d-2)$ then $\sum_k k\rho^{k(\beta(2-d)+d)} = \infty$ so by Theorem 4.3 (ii), part of the surface of D is nearly inactive.

5. Trap domains.

The ideas developed in Section 4 allow us to prove a new result on "trap" domains introduced in [BCM]. The new result applies only to the class of domains $\mathcal{D}_1$ presented in Definition 4.1 but that class contains some very natural examples of fractal domains, such as the multidimensional version of the von Koch snowflake presented in Example 4.16, that were not covered by theorems proved in [BCM] (see Example 5.2 below). There was a big gap between results on two-dimensional domains and higher dimensional domains in [BCM]. At first we thought the gap was purely technical in nature—complex analytic methods could not be used in higher dimensions. It turns out that the gap is in fact "real," in the sense that the multidimensional examples are considerably different from the two-dimensional examples—compare our Example 5.2 and Proposition 2.15 of [BCM].

Recall that $B_* \subset D$ is a closed ball with positive radius and $T_{B_*} = \inf\{t \ge 0 : X_t \in B_*\}$ is the first hitting time of B_* by X. We say that $D \subset \mathbb{R}^d$, $d \ge 2$, is a *trap domain* if

$$\sup_{x\in D} \mathbf{E}_x T_B = \infty, \tag{5.1}$$

and otherwise D is called a *non-trap domain.* One can express (5.1) in a purely analytic way, namely, by saying that D is a trap domain if and only if

$$\sup_{x \in D \setminus B} \int_{D \setminus B} G(x, y) dy = \infty. \tag{5.2}$$

See [BCM] for further discussion of basic properties of trap domains.

Theorem 5.1. *Consider a domain $D \in \mathcal{D}_1$, $D \subset \mathbb{R}^d$, $d \geq 3$, with a finite volume.*

(i) If there exists a constant $c < \infty$ such that for each point $z \in \partial D$, there is a system of surfaces $\{\gamma_n, n \geq 0\}$ as in Definition 4.1 satisfying

$$\sum_{n=1}^{\infty} |D_n| \sum_{k=1}^{n} r_k^{2-d} \leq c, \tag{5.3}$$

then D is not a trap domain.

(ii) If there exists a boundary point $z \in \partial D$ and a system of surfaces $\{\gamma_n\}$ as in Definition 4.1, such that

$$\sum_{n} |D_n| \sum_{k=1}^{n} r_k^{2-d} = \infty, \tag{5.4}$$

then D is a trap domain.

Proof. (i) The proof is very similar to the proof of Theorem 4.3. Consider $x_0 \in D$. It is not hard to see that there exists $z_0 \in \partial D$ and a corresponding family of γ_n's, as in Definition 4.1, such that $x_0 \in D_{n_0}$ for some n_0 and $\operatorname{dist}(x_0, \partial D_{n_0}) \geq c_1 r_{n_0}$. By Lemma 4.4, $G(x_0, \cdot)$ is bounded by $c_2 \sum_{k=1}^{n_0} r_k^{2-d}$ on $D_{n_0 - 1}$. By the Harnack principle, it is bounded by $c_3 \sum_{k=1}^{n_0} r_k^{2-d}$ on $\partial B(x_0, c_1 r_{n_0}/2)$, and the maximum principle implies that the same bound holds on $D \setminus B(x_0, c_1 r_{n_0}/2)$. For $x \in B(x_0, c_1 r_{n_0}/2)$ we have $G(x_0, x) \leq c_4 |x - x_0|^{2-d} \sum_{k=1}^{n_0} r_k^{2-d} / (r_{n_0}/2)^{2-d}$, by comparison with the Green function in $\mathbb{R}^d$. We obtain, using the upper bound in Lemma 4.4 for $n < n_0$,

$$\begin{aligned}
\int_{D\setminus B} G(x_0, y)dy &= \sum_{n\geq 1} \int_{D_n} G(x_0, y)dy \\
&= \sum_{1\leq n<n_0} \int_{D_n} G(x_0, y)dy + \sum_{n\geq n_0} \int_{D_n\setminus B(x_0, c_1 r_{n_0}/2)} G(x_0, y)dy \\
&\quad + \int_{B(x_0, c_1 r_{n_0}/2)} G(x_0, y)dy \\
&\leq \sum_{1\leq n<n_0} |D_n| c_2 \sum_{k=1}^{n} r_k^{2-d} + \sum_{n\geq n_0} |D_n| c_3 \sum_{k=1}^{n_0} r_k^{2-d} + c_5 r_{n_0}^d \sum_{k=1}^{n_0} r_k^{2-d} \\
&\leq \sum_{n\geq 1} c_6 |D_n| \sum_{k=1}^{n} r_k^{2-d}.
\end{aligned}$$

This is bounded by a constant independent of x_0, by assumption (5.3). The theorem follows in view of (5.2).

(ii) A calculation similar to that in part (i), based on the lower bound in Lemma 4.9, easily implies part (ii) of the theorem. □

We would like to emphasize that part (ii) of Theorem 5.1 is much easier to prove than part (ii) of Theorem 4.3. This is because all we have to show is that the function $x \mapsto \mathbf{E}_x [T_{B_*}]$ is unbounded. In the proof of Theorem 4.3 (ii) we had to prove that the random variable $L_{T_{B_*}}$ for reflecting Brownian motion X^* starting from x converges to infinity in distribution as x approaches a boundary point $z_0 \in \partial D$.

Example 5.2. Recall the domain D and notation from Example 4.15. Recall that the number of γ_n's in $\mathcal{C}_k$ is of order $\log((\rho^{k-1}/2)/(4\rho^{(k-1)\beta})) \approx k$. If $z_1 \in \Omega_{n+1}$ with sufficiently large n then by Lemma 4.4, the Green function $G(z_1, \cdot)$ can be bounded by $c_1 \sum_{j\leq k} j\rho^{j\beta(2-d)} \leq c_2 k\rho^{k\beta(2-d)}$ for $x \in D$ that lie between γ_n's in $\mathcal{C}_k$. The volume of a D_n corresponding to a $\gamma_n \in \mathcal{C}_k$ is bounded by $c_3\rho^{kd}$, so the contribution of such D_n's to the sum in (5.3) is bounded by $c_4 k^2 \rho^{k\beta(2-d)}\rho^{kd} = c_4 k^2 \rho^{k(\beta(2-d)+d)}$. If $\beta < d/(d-2)$ then $\sum_k k^2 \rho^{k(\beta(2-d)+d)} < \infty$ and Theorem 5.1 (i) implies that D is not a trap domain.

To find a lower bound for (5.4), we take into account only one D_n corresponding to each family $\mathcal{C}_k$, namely the one with the largest volume. We obtain as a lower bound for (5.4) the quantity $c_5 k\rho^{k\beta(2-d)}\rho^{kd} = c_5 k\rho^{k(\beta(2-d)+d)}$. If $\beta \geq d/(d-2)$ then $\sum_k k\rho^{k(\beta(2-d)+d)} = \infty$ so by Theorem 5.1 (ii), D is a trap domain. $\square$

We will use the above example to compare Theorem 5.1 to a result about multidimensional trap domains proved in [BCM]. The result in [BCM] was based on the notion of a J_α-domain, used by Maz'ja in his book on Sobolev spaces [Maz]. Here is an informal definition of a J_α-domain (see [Maz] or [BCM] for the rigorous definition). We say that D is a J_α domain if for every smooth $(d-1)$-dimensional surface Λ which divides D into two connected components D_1^Λ and D_2^Λ, we have $\min(|D_1^\Lambda|, |D_2^\Lambda|)^\alpha \leq c_1|\Lambda|$, where $c_1 < \infty$ depends only on D (here $|\Lambda|$ is the $(d-1)$-dimensional surface area). Theorem 2.4 of [BCM] implies that if D is a J_α domain with $\alpha < 1$ then D is not a trap domain, and there exists a trap domain $D \in J_1$.

Roughly speaking, one can determine whether the domain D of Example 5.2 belongs to J_α with a given α by comparing the surface area of the opening between cubes in $\mathcal{A}_k$ and $\mathcal{A}_{k-1}$ to the volume of the cubes in $\bigcup_{j\geq k}\mathcal{A}_j$. The surface area is of order $\rho^{k\beta(d-1)}$ and the volume is of order ρ^{kd}, so D is a J_α domain with $\alpha < 1$ if $\beta < d/(d-1)$. Hence, for the family of domains in Example 5.2, Theorem 2.4 of [BCM] shows that D is not a trap domain if $\beta < d/(d-1)$, while Theorem 5.1 of this paper shows that this holds for all $\beta < d/(d-2)$, and in addition it shows that this result is sharp. The gap between the power of the two approaches is not as striking in dimensions $d \geq 3$ as it is in the 2-dimensional case, discussed in Proposition 2.15 of [BCM].

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