Counting social interactions for discrete subsets of the plane

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Abstract

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We will use dynamical, geometric, and analytic techniques to study **translation surfaces**. A translation surface is, informally, a collection of polygons in the plane with parallel sides identified by translation to form a surface with a singular Euclidean structure. Understanding the geometry and behavior of flows on translation surfaces and their moduli spaces has lead to the development of many new and revolutionary techniques, including the work of award winning mathematicians like Mirzakhani, McMullen, Eskin, Yoccoz, and Zorich.

My work studies geodesic flows on translation surfaces by analyzing subsets of $\mathbb{R}^2$ corresponding to saddle connections, which are special trajectories of this flow. **Saddle connections** are straight line trajectories on a translation surface connecting two singular points, with none in their interior. Saddle connections are the driver for the parabolic behavior of geodesic flows on translation surfaces, as two nearby parallel lines behave differently under dynamics of the geodesic flow once a saddle connection comes between them.

This thesis will consider discrete subsets of the plane usually arising from saddle connections on translation surfaces. The following results are all partial answers to the following question: Given a surface, how are saddle connections distributed in the plane?

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DEDICATION

to my partner Max who has been with me for every step of this crazy math adventure

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Chapter 1

INTRODUCTION AND PRELIMINARIES

1.1 Background

1.1.1 Translation surfaces

A *translation surface* is a surface formed by taking a finite number of polygons in the plane and gluing opposite sides by translation, where surfaces are equivalent up to cutting and pasting of these polygons via translation. Equivalently a translation surface is a closed Riemann surface X with a nonzero holomorphic 1-form ω. For more background see [Mas06], [HS06], [Esk06].

Given $A \in SL(2,\mathbb{R})$ and (X,ω) a translation surface, we produce a new translation surface $A \cdot (X,\omega)$, which is the surface with charts of (X,ω) composed with A acting linearly on $\mathbb{R}^2$. The *Veech group* is the stabilizer subgroup of this action

$$SL(X,\omega) \stackrel{def}{=} \{A \in SL(2,\mathbb{R}) : A \cdot (X,\omega) = (X,\omega)\}.$$

The Veech group is always discrete and in fact trivial for almost every translation surface [GJ96]. Chapters 2 and give results for closed $SL(2,\mathbb{R})$ orbits where there is a large Veech group and natural invariant probability measure inherited from the Haar measure on $SL(2,\mathbb{R})$.

Chapters 3 and 4 will focus on generic surfaces with a trivial Veech group. The moduli space Ω_g of compact genus g area 1 translation surfaces (where $(X_1,\omega_1) \sim (X_2,\omega_2)$ if there is a biholomorphism $f : X_1 \to X_2$ with $f_*\omega_2 = \omega_1$) is stratified by integer partitions of $2g-2$ (fixing the orders of the zeros of ω). These strata have at most 3 connected components [KZ03], and each connected component $\mathcal{H}$ carries a natural Lebesgue probability measure $\mu = \mu_{\mathcal{H}}$ [Mas82, Vee82]. The measure $\mu_{\mathcal{H}}$ constructed by Masur and Veech is ergodic and invariant under the $SL(2,\mathbb{R})$-action, and can be thought of as Lebesgue measure in appropriate coordinates on $\mathcal{H}$.

1.1.2 Saddle connections and holonomy vectors

Saddle connections on a translation surface (X, ω) are geodesics in the flat metric determined by ω connecting two zeros of ω with no zeros in its interior. Let SC_ω be the set of saddle connections on (X, ω) (we will now abbreviate our translation surfaces by ω, as there is a unique Riemann surface structure X for which ω is holomorphic), and for $\gamma \in \mathrm{SC}_\omega$, let

$$z_\gamma = \int_\gamma \omega \in \mathbb{C}$$

its holonomy vector. Let

$$\Lambda_\omega = \{z_\gamma : \gamma \in \mathrm{SC}_\omega\}$$

denote the set of holonomy vectors of saddle connections. Λ_ω is a countable discrete subset of the plane $\mathbb{C}$. The *length* $\ell(\gamma)$ of a saddle connection γ is

$$\ell(\gamma) = |z_\gamma|.$$

For $R > 0$, let $\Lambda_\omega(R)$ be the collection of saddle connections with holonomy vector of length at most R. Note that the assignment

$$\omega \mapsto \Lambda_\omega$$

is $SL(2, \mathbb{R})$-equivariant, that is

$$\Lambda_{g\omega} = g\Lambda_\omega.$$

A central theme of these results is to understand how the points in Λ_ω are distributed in the plane. In particular each result considers the limit as $R \to \infty$ for functions counting the number of points in $\Lambda_\omega(R)^k$ for $k \geq 1$ with specific relationships between the points. The Siegel–Veech transform is a useful tool for connecting geometry of $(\mathbb{R}^2)^k$ to counting problems for subsets of $\Lambda_\omega(R)^k$.

1.1.3 Siegel–Veech transform

For $k \geq 1$, set $V_{\omega,k} = (\Lambda_\omega)^k$, and let $B_c((\mathbb{R}^2)^k)$ be the set of bounded measurable functions with compact support on $(\mathbb{R}^2)^k$.

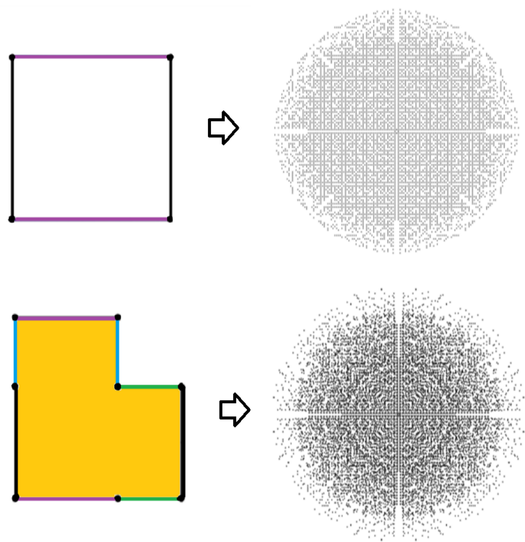

Figure 1.1: Top is the square torus, and bottom is the golden L, with side length the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. On the right are the corresponding holonomy vectors in $\Lambda_\omega(100)$. The saddle connections start and end at the black dots which all map to the same point under identification. In both of these cases, the Veech group is non-trivial given by $SL(2, \mathbb{Z}) = H_3$ and H_5 the third and fifth Hecke triangle groups described in Chapter 2

Definition 1.1.1. *For $f \in B_c((\mathbb{R}^2)^k)$ and a translation surface ω, we define the Siegel–Veech transform by*

$$\hat{f}(\omega) = \sum_{(v_1,\ldots,v_k)\in V_{\omega,k}} f(v_1,\ldots,v_k).$$

In the above definition $k = 1$ is the classical Siegel–Veech transform, and for particualar $f \in B_c((\mathbb{R}^2)^k)$ of the form $f(x_1,\cdots,x_k) = h(x_1)\cdots h(x_k)$ for $h \in B_c(\mathbb{R}^2)$, $\hat{f}$ corresponds to the kth power of the classical Siegel–Veech Transform of h on $\mathbb{R}^2$. Veech [Vee98] proved that the classical Siegel–Veech transform is integrable, that is $\hat{f} \in L^1(\mu)$ and moreover there exists a constant c called the *Siegel–Veech constant* so that

Theorem 1.1.2. *For $f \in B_c(\mathbb{R}^2)$,*

$$\int \hat{f}\, d\mu = c \int_{\mathbb{R}^2} f(x)\, dx,$$

where for the purpose of this thesis either

- *μ is the $SL(2,\mathbb{R})$ invariant probability measure on a closed $SL(2,\mathbb{R})$-orbit for non-trivial $SL(X,\omega)$.*

- *μ is the probability measure on a connected component of the stratum $\mathcal{H}$.*

On connected strata of translation surfaces, the Siegel–Veech transform is in L^2 with respect to the Masur–Veech measure [ACM19]. The transform has also been used on connected strata of translation surfaces to get asymptotic counting information for $\Lambda_\omega(R)$ ([EM01], which was latter effectivized by [NRW20]. Similar strategies were used in [AC12] to show that for generic surfaces, the smallest gap in angle between two holonomy vectors in $\Lambda_\omega(R)$ shrinks faster than $\frac{1}{R^2}$.

For closed $SL(2,\mathbb{R})$–orbits, understanding of the Siegel–Veech transform was focused on generalizing to $SL(n\mathbb{R})/SL(n,\mathbb{Z})$ via work of Siegel, Schmidt, and Rogers [Sie45, Sch60, Rog55], or more recently in [BNRW19] generalizing and effectivizing Veech's work from [Vee89] and [Vee98].

1.2 Outline

1.2.1 Higher moments for Hecke triangle groups

In Chapter 2, we compute higher moments of the Siegel–Veech transform over sets of surfaces with the Hecke triangle groups as their stabilizer group.

Definition 1.2.1. *The Hecke triangle group H_q for integers $q \geq 3$ is the discrete subgroup of $SL(2,\mathbb{R})$ generated by*

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & \lambda_q \\ 0 & 1 \end{bmatrix}, \ \text{where } \lambda_q = 2\cos\left(\frac{\pi}{q}\right).$$

Note $H_3 = SL(2,\mathbb{Z})$ and for all $q \geq 3$, H_q has finite co-volume in $SL(2,\mathbb{R})$. For more information on Hecke triangle groups see [LL16].

Let V_q be the discrete subset of $\mathbb{R}^2$ defined by

$$V_q = H_q \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which corresponds to a subset of saddle connections of a translation surface when q is odd (see section 2.2.2). Define $Y_q = SL(2,\mathbb{R})/H_q$ with corresponding Haar probability measure μ.

The main theorem of this section gives a higher moment formula for the Siegel–Veech transform for a single orbit of H_q.

Theorem 1.2.2. *Let $q \geq 3$ and $f \in B_c((\mathbb{R}^2)^k)$ for $k \in \mathbb{N}$. Let the corresponding Siegel–Veech transform (Definition 1.1.1) for $[g] \in SL(2,\mathbb{R})/H_q$ be given by*

$$\hat{f}(g) = \sum_{(v_1,\ldots,v_k)\in V_q^k} f(gv_1,\ldots,gv_k).$$

Then there is an exact formula for

$$\int_{SL(2,\mathbb{R})/H_q} \hat{f}(g)\, d\mu(g),$$

given when $k = 2$ in Theorem 2.1.3, and for all $k \geq 2$ in Theorem 2.1.4.

1.2.2 Pairs for generic surfaces

Continuing with applications of an L^2 formula, we now work with counting in a connected component of the stratum, $\mathcal{H}$. As an application of the fact that the Siegel–Veech transform is in L^2 ([ACM19]), we will outline a mostly complete proof towards the following conjecture about a function $N_A(\omega, R)$ that counts pairs of holonomy vectors in $\Lambda_\omega(R)$ with determinant at most A.

Conjecture 1.2.3. *There is a constant $c = c(A, \mathcal{H})$ such that for μ-almost every $\omega \in \mathcal{H}$,*

$$\lim_{R\to\infty} \frac{N_A(\omega, R)}{R^2} = c.$$

1.2.3 Gaps around 0 for generic surfaces

Finally, as we remain focusing on pairs of points, we will stay on a connected component of the stratum $\mathcal{H}$, but instead of counting pairs with a bounded determinant, we will look at the angle between two points in $\Lambda_\omega(R)$. So we define the smallest gap about zero out to radius R by

$$\zeta_\omega(R) = \min\{\phi \in \Theta_\omega(R) : \phi \geq 0\} - \max\{\phi \in \Theta_\omega(R) : \phi \leq 0\}.$$

We will then present an outline of proof and key generalization of the Borel-Cantelli Lemma in order to prove the main conjecture of this chapter, which fine tunes the convergence information obtained in [AC12].

Conjecture 1.2.4. *Let $\psi : [1, \infty) \to [1, \infty)$ be a nondecreasing continuous function.*

- *If $\int_1^\infty \frac{1}{\psi(t)^2}\, dt < \infty$, then for almost every ω (with respect to Masur–Veech measure)*

$$\liminf_{R\to\infty} \psi(R) R^2 \zeta_\omega(R) = \infty.$$

- *If $\int_1^\infty \frac{1}{\psi(t)^2}\, dt = \infty$, then for almost every ω,*

$$\liminf_{R\to\infty} \psi(R) R^2 \zeta_\omega(R) = 0.$$

Chapter 2

A HIGHER MOMENT FORMULA FOR THE SIEGEL–VEECH TRANSFORM OVER QUOTIENTS BY HECKE TRIANGLE GROUPS

Online first publication at Groups, Geometry and Dynamics, to appear in print ([Fai20]).

2.1 Introduction

Building on work of Siegel, Schmidt, and Rogers [Sie45, Sch60, Rog55] we compute higher moments of the Siegel–Veech transform over sets of surfaces with the Hecke triangle groups (See Definition 1.2.1) as their stabilizer group. In Theorem 1.1.2, the corresponding Siegel–Veech constant is given by $c = \frac{1}{c(q)}$ where

$$c(q) \overset{def}{=} \pi\left(\pi - \frac{\pi}{q} - \frac{\pi}{2}\right).$$

We will first prove the following theorem which computes the square of the classical Siegel–Veech transform on $B_c(\mathbb{R}^2)$. To state the theorem, we introduce the following two definitions:

Definition 2.1.1 (Set of non-vanishing determinants)**.** *Let*

$$\begin{aligned} N_q &\overset{def}{=} \{n \in \mathbb{Z}[\lambda_q] \setminus \{0\} : \textit{ there exists } \boldsymbol{v_1}, \boldsymbol{v_2} \in V_q \textit{ with } \det(\boldsymbol{v_1}\,\boldsymbol{v_2}) = n\} \\ &= \left\{ n \in \mathbb{Z}[\lambda_q] \setminus \{0\} : \textit{ there exists } 0 \le m < \lambda_q |n| \textit{ with } \begin{bmatrix} m \\ n \end{bmatrix} \in V_q \right\}, \end{aligned} \tag{2.1.1}$$

where Equation 2.1.1 will be proved in Lemma 2.3.5.

Definition 2.1.2 (q-geometric Euler totient function)**.** *For $b \in \mathbb{Z}[\lambda_q]$ define*

$$\varphi_q(b) = \#\left\{ 1 \le a \le \lambda_q\,|b| : \begin{bmatrix} a \\ b \end{bmatrix} \in V_q \right\}.$$

Note that $\lambda_3 = 1$ and $V_3 = SL(2,\mathbb{Z}) \cdot e_1$ so φ_3 reduces to the standard Euler totient function.

Theorem 2.1.3. *Let $f \in B_c(\mathbb{R}^2 \times \mathbb{R}^2)$, N_q be the set of non-vanishing determinants, and φ_q the q-geometric Euler totient function. Then,*

$$\int_{Y_q} \hat{f}\, d\mu(g) = \sum_{n \in N_q} \frac{\varphi_q(n)}{c(q)} \int_{SL(2,\mathbb{R})} f\left(gJ_n\right)\, d\eta + \frac{1}{c(q)} \int_{\mathbb{R}^2} \left(f(x,-x) + f(x,x)\right)\, dx \tag{2.1.2}$$

where $J_n = \begin{bmatrix} 1 & 1 \\ 0 & n \end{bmatrix}$*,* μ *is the Haar probability measure on* Y_q*,* η *is Haar measure on* $SL(2,\mathbb{R})$ *normalized so* $\eta(Y_3) = \frac{\pi^2}{6}$*, and* dx *is the Lebesgue measure on* $\mathbb{R}^2$ *normalized so the area of the unit square is 1.*

Note $\hat{f}$ is uniformly bounded by Lemma 16.10 of [Vee98], so both sides of Equation 2.1.2 are finite. The proof of Theorem 2.1.3 will use Schmidt's outline of proof (see section 2.2.1). It is a useful exercise to consider this proof in the case of Schmidt with $q = 3$. That is where $N_3 = \mathbb{Z} \setminus \{0\}$ and the constant $c(3) = \frac{\pi^2}{6} = \zeta(2)$. In section 2.5 we will see how the formula in Theorem 2.1.3 allows us to understand the asymptotic densities of saddle connections of translation surfaces with Veech group H_q for q odd. Theorem 2.1.3 is in fact a special case of the main theorem, which calculates the kth moment of the classical Siegel–Veech transform.

Theorem 2.1.4. *Let* $f \in B_c((\mathbb{R}^2)^k)$ *and define*

$$J_{n,m} = \begin{bmatrix} 1 & m \\ 0 & n \end{bmatrix}.$$

Then

$$\begin{aligned} &\int_{Y_q} \hat{f}\, d\mu(g) \\ &= \sum_{\substack{\lambda \in \mathbb{R}^k \\ \lambda = (1,\pm 1,\ldots,\pm 1)}} \frac{1}{c(q)} \int_{\mathbb{R}^2} f(\lambda x)\, dx \\ &+ \sum_{n \in N_q} \sum_{\substack{0 \le m < \lambda_q |n| \\ (m,n)^T \in V_q}} \sum_{1 \le j < k} \sum_{\lambda,\alpha,\beta} \frac{1}{c(q)} \int_{SL(2,\mathbb{R})} f\left(\lambda g \begin{bmatrix} 1 \\ 0 \end{bmatrix}, g J_{n,m} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right) d\eta(g). \end{aligned}$$

where for each $1 \le j < k$ *we have* $\lambda \in \mathbb{R}^j$ *is of the form* $(1, \pm 1, \ldots, \pm 1)$ *and* $\alpha = (0, \alpha_2, \ldots, \alpha_{k-j})$ *and* $\beta = (1, \beta_2, \ldots, \beta_{k-j})$ *where for each* $2 \le i \le k - j$ *we have*

$$\begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \in J_{n,m}^{-1} V_q. \tag{2.1.3}$$

2.1.1 Outline

In Section 2.2 we give an overview of the history of the problem, followed by the necessary background Veech groups, and the geometric Euler totient function. In Section 2.3 we prove Theorem 2.1.3, followed by Section 2.4 where we prove Theorem 2.1.4. Finally in Section 2.5 we explain how we found numerical evidence for the result.

2.1.2 Acknowledgments

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2.2 Background and history

We first give a summary of previous related results in the geometry of numbers, followed by background on translation surfaces, Veech groups, and the q-geometric Euler totient function.

2.2.1 Geometry of numbers

We will first focus on the mean and variance of the primitive Siegel transform, which is a special case of the Siegel–Veech transform defined in the previous section. First we set up some notation and definitions, then state the theorems of Siegel, Rogers, and Schmidt computing the mean and variance of the primitive Siegel transform.

Consider $f \in B_c(\mathbb{R}^d)$. We aim to understand f evaluated on *visible* lattice points in $\mathbb{R}^d$, where a point $\boldsymbol{v} = (v_1, \ldots, v_d)^T \in \mathbb{Z}^d$ is *primitive* or *visible* if $\gcd(v_1, \ldots, v_d) = 1$. We denote the set of primitive vector points by $\mathbb{Z}^d_{prim}$, which one can show $\mathbb{Z}^d_{prim} = SL(d, \mathbb{Z}) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Define

$X_d = SL(d,\mathbb{R})/SL(d,\mathbb{Z})$. By abuse of notation, for an equivalence class $g = [g] \in X_d$, we define the *primitive Siegel transform* by

$$\hat{f}(g) = \sum_{\boldsymbol{v}\in\mathbb{Z}^d_{prim}} f(gv).$$

In 1945, Siegel [Sie45], sections 5-6 showed

$$\int_{X_d} \hat{f}(g)\, d\mu(g) = \frac{1}{\zeta(d)} \int_{\mathbb{R}^d} f(x)\, dx \tag{2.2.1}$$

where the standard Lebesgue measure on $\mathbb{R}^d$ is dx, ζ is the Riemann zeta function, and μ is probability Haar measure on X_d.

In order to understand higher moments of $\hat{f}$, we split into the cases where $d = 2$ and $d > 2$. We address the latter case first.

For understanding higher moments of $\hat{f}$, C. A. Rogers' 1955 paper [Rog55], Theorem 5 solved the case for $\hat{f}^k$ with $d > 2$ and $k < d$. For simplicity, we will only consider the case $k = 2$ of Rogers' result. Recall for $f \in B_c(\mathbb{R}^d)$, and defining $h \in B_c(\mathbb{R}^2 \times \mathbb{R}^2)$ by $h(x,y) = f(x)f(y)$ we have

$$\sum_{v_1,v_2\in\mathbb{Z}^d_{prim}} h(gv_1, gv_2) = \sum_{v_1,v_2\in\mathbb{Z}^d_{prim}} f(gv_1)f(gv_2) = \left[\sum_{v\in\mathbb{Z}^d_{prim}} f(gv)\right]^2 = (\hat{f})^2.$$

Rogers showed that for $f \in B_c(\mathbb{R}^d)$, and $h(x,y) = f(x)f(y)$, the second moment of f is given by

$$\begin{aligned}\int_{X_d} (\hat{f})^2(g)\, d\mu(g) &= \int_{X_d} \sum_{v_1,v_2\in\mathbb{Z}^d_{prim}} h(gv_1, gv_2)\, d\mu(g) \\ &= \frac{1}{\zeta(d)^2} \int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,y)\, dx\, dy + \frac{1}{\zeta(d)} \int_{\mathbb{R}^d} \left[h(x,x) + h(x,-x)\right]\, dx.\end{aligned} \tag{2.2.2}$$

For a modern proof of Equation 2.2.2, see section 4 of [AM09].

For $k \geq d > 2$ and $f \in B_c(\mathbb{R}^d)$, the function $\hat{f}^k$ is not integrable (Proposition 7.1 of [KM99]). However when $d = 2$ we have $\hat{f}$ is bounded on X_2, and thus $\hat{f}^k$ integrable for any $k \geq 1$. So we now exclusively study the case $d = 2$. Rogers had a mistake in his paper claiming Equation 2.2.2 held for $d = 2$, which we can see does not work by setting h_0 to be the characteristic function of the set given by

$$\{(v_1, v_2) : \max(|v_1|, |v_2|) \leq R\,,\, \det(v_1 v_2) \notin \mathbb{Z}\,\}.$$

Applying Equation 2.2.2 to h_0, the left hand side of Equation 2.2.2 will be identically zero as for any $v_1, v_2 \in \mathbb{Z}^2_{prim}$

$$\det(gv_1\, gv_2) = \det(g)\det(v_1 v_2) \in \mathbb{Z},$$

and the right hand side will be nonzero as the vectors with integer determinant are a Lebesgue measure zero subset of $\mathbb{R}^2 \times \mathbb{R}^2$.

In correction to Rogers, Schmidt addressed the case where $d = 2$ (see [Sch60] Section 6).

Theorem 2.2.1. *Let $f \in B_c(\mathbb{R}^2 \times \mathbb{R}^2)$. Then*

$$\int_{Y_3} \hat{f}\, d\mu(g) = \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{\varphi(n)}{\zeta(2)} \int_{SL(2,\mathbb{R})} f\,(gJ_n)\; d\eta + \frac{1}{\zeta(2)} \int_{\mathbb{R}^2} f(x, -x) + f(x, x)\, dx \qquad (2.2.3)$$

where φ is the standard Euler totient function.

Note this formula does not look exactly like the formula in Schmidt [Sch60] as we have a different normalization of the Haar measure η. Note also that Theorem 2.2.1 is a special case of Theorem 2.1.3.

2.2.2 Hecke triangle groups as Veech groups

We will consider surfaces whose Veech group is given by $SL(X, \omega) = H_q$ for $q \geq 3$. When $q = 3$, $H_3 = SL(2, \mathbb{Z})$ which is the Veech group for the square torus. In general given a translation surface (X, ω) where we glue two regular $(2n + 1)$-gons and then identify opposite sides, Veech showed in [Vee89] that $SL(X, \omega) = H_{2n+1}$. For even Hecke triangle groups, Bouw and Möller [BM10] followed by a constructive proof of Hooper [Hoo13] were able to show that there exists a translation surface (X, ω) with $SL(X, \omega)$ conjugate to an index 2 subgroup of H_{2n}, but there is no translation surface with Veech group containing H_{2n}.

Notice the set of holonomy vectors for the square torus are

$$\mathbb{Z}^2_{prim} = SL(2,\mathbb{Z}) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = H_3 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The characterization is not as clean for other surfaces, but if (X,ω) is a translation surface with $SL(X,\omega)$ a lattice, then the set of holonomy vectors will always be given as a finite union of $SL(X,\omega)$-orbits [Vee89], 5th paragraph section 3. By studying the Siegel–Veech transform over V_q we will be able to understand asymptotic density of saddle connections for a class of translation surfaces [Vee98].

2.2.3 Geometric Euler totient function

Recall we define the *q-geometric Euler totient function* by

$$\varphi_q(b) = \# \left\{ 1 \le a \le \lambda_q |b| : \begin{bmatrix} a \\ b \end{bmatrix} \in V_q \right\},$$

where $\varphi_3 = \varphi$ is the standard Euler totient function. Since V_q is discrete and thus φ_q is finite and well defined. Though φ_q generalizes the standard Euler totient function, φ_q does *not* agree with the more standard Euler totient function defined for the ring of integers over a number field in terms of the product formula over prime ideals.

Following [LL16], we can define a *greatest common q-divisor* denoted $(a,b)_q$ for $a,b \in \mathbb{Z}[\lambda_q]$ using a Euclidean pseudo-algorithm. This greatest common q-divisor has many similar properties to the gcd function, including for any $t \neq 0$,

$$(ta,tb)_q = t \cdot (a,b)_q. \tag{2.2.4}$$

With this definition we also have the following useful characterization of elements of H_q as proved in Proposition 3.7 of [LL16].

Proposition 2.2.2. *A matrix* $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in SL(2, \mathbb{Z}[\lambda_q])$ *is in* H_q *if and only if* $(a, b)_q = (c, d)_q = 1$. *In fact if* $(a, b)_q < 1$ *or* $|b| < 1$, *then* $(a, b)^T$ *cannot be a column of a matrix in* H_q.

2.3 Orbits and integrals

The goal of this section is to prove Theorem 2.1.3.

Let $f \in B_c(\mathbb{R}^2 \times \mathbb{R}^2)$, and define $\hat{f}$ as in Definition 1.1.1. Consider the map

$$f \mapsto \int_{Y_q} \hat{f} d\mu.$$

This mapping is a positive linear functional which is $SL(2, \mathbb{R})$- invariant, where $SL(2, \mathbb{R})$ acts diagonally by $g \cdot (v_1, v_2) = (gv_1, gv_2)$ for $(v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2$. Hence by the Riesz representation theorem, there exists a measure ν so that

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} f \, d\nu = \int_{Y_q} \hat{f} \, d\mu \text{ for any } f \in B_c(\mathbb{R}^2 \times \mathbb{R}^2).$$

Since ν is $SL(2, \mathbb{R})$-invariant, we can write ν as a combination of measures on $SL(2, \mathbb{R})$ orbits of $\mathbb{R}^2 \times \mathbb{R}^2$. So to understand ν we need to understand our integral over $SL(2, \mathbb{R})$ orbits.

The outline of the proof is as follows. In section 2.3.1 we split $\mathbb{R}^2 \times \mathbb{R}^2$ into $SL(2, \mathbb{R})$ orbits under the diagonal action and find the possible $SL(2, \mathbb{R})$-invariant measures on these subsets. In section 2.3.2 we will reduce the uncountable number of orbits which occur in our setting to two linearly dependent orbits, and a countable number of linearly independent orbits. After setting up notation in section 2.3.3, in section 2.3.4 we reduce the linearly dependent case to Theorem 1.1.2, finally addressing the linearly independent case in section 2.3.5.

2.3.1 Decomposition into orbits

Let $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{\mathbf{0}\}$, similarly $\mathbb{Z}_0^n = \mathbb{Z}^n \setminus \{0\}$, and $\mathbb{Z}_0[\lambda_q] = \mathbb{Z}[\lambda_q] \setminus \{0\}$.

Lemma 2.3.1. *The following decomposes* $\mathbb{R}^2 \times \mathbb{R}^2$ *into disjoint* $SL(2, \mathbb{R})$ *orbits:*

$$\mathbb{R}^2 \times \mathbb{R}^2 = \left(\bigsqcup_{n \in \mathbb{R}_0} D_n \right) \sqcup \left(\bigsqcup_{t \in \mathbb{R}_0} LD_t \right) \sqcup H \sqcup V \sqcup \{0\},$$

where we have the linearly independent determinants,

$$D_n = \{(\boldsymbol{v}, \boldsymbol{w}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \det(\boldsymbol{v}\ \boldsymbol{w}) = n\},$$

the linearly dependent subsets

$$LD_t = \{(\boldsymbol{v}, t\boldsymbol{v}) : \boldsymbol{v} \in \mathbb{R}^2_0\},$$

and two special cases of linearly dependent vectors: horizontal and vertical

$$H = \{(\boldsymbol{v}, 0) : \boldsymbol{v} \in \mathbb{R}^2_0\} \quad V = \{(0, \boldsymbol{v}) : \boldsymbol{v} \in \mathbb{R}^2_0\}.$$

Proof. We will realize each subset as an orbit of $SL(2, \mathbb{R})$ under the diagonal action on $\mathbb{R}^2 \times \mathbb{R}^2$. Since $g \cdot \{0\} = 0$ for all $g \in SL(2, \mathbb{R})$, the point $\{0\}$ is an entire orbit.

Now notice that $g \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbb{R}^2_0$. Using this fact, for any $t \in \mathbb{R}_0$,

$$SL(2, \mathbb{R}) \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} t \\ 0 \end{bmatrix} \right) = \{(\boldsymbol{v}, t\boldsymbol{v}) : \boldsymbol{v} \in \mathbb{R}^2_0\} = LD_t.$$

Similarly for H and V, it suffices to see that they are both given by

$$H = SL(2, \mathbb{R}) \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \quad V = SL(2, \mathbb{R}) \cdot \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Finally, for $n \neq 0$, since $n\mathbb{R}^2_0 = \mathbb{R}^2_0$,

$$SL(2, \mathbb{R}) \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ n \end{bmatrix} \right) = \{(\boldsymbol{v}, n\boldsymbol{u}) : \det(\boldsymbol{v}\ \boldsymbol{u}) = 1\} = D_n.$$

Thus we have shown each of these subsets is an $SL(2, \mathbb{R})$ orbit. Finally, since every pair of elements in $\mathbb{R}^2$ is either linearly independent and thus have a nonzero determinant or linearly dependent and thus are scalar multiples we conclude every element $(v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ is contained in one of the given sets. Thus we have a decomposition of $\mathbb{R}^2 \times \mathbb{R}^2$ into $SL(2, \mathbb{R})$ orbits. □

The last task of this subsection is to determine the possible measures on each of our subsets. We will freely use the fact that Haar measure is unique up to scaling. In this section we will fix a particular scaling of Haar measure for each measure, and then by taking a linear combination of these different measures we can obtain ν.

On $\{0\}$, there is only one probability measure given by δ_0, which is trivially $SL(2,\mathbb{R})$ invariant.

On H, V, and LD_t for $t \in \mathbb{R}_0$, we have a copy of $\mathbb{R}_0^2$. Notice Lebesgue measure m_2 on $\mathbb{R}^2$ is $SL(2,\mathbb{R})$-invariant. So we will fix the standard Lebesgue measure giving the unit square $[0,1]^2$ volume 1 on each of the subsets H, V, and LD_t for $t \in \mathbb{R}_0$. Since $\{(0,0)\}$ is a measure zero subset, without loss of generality we can write integrals with respect to m_2 over all of $\mathbb{R}^2$. To see this measure is the unique $SL(2,\mathbb{R})$-invariant measure (up to scaling), consider the induced Haar measure under the quotient of $SL(2,\mathbb{R})/N \cong \mathbb{R}_0^2$ where $N = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in \mathbb{R} \right\}$.

To find a Haar measure on D_n, we will first find a Haar measure on $SL(2,\mathbb{R})$, then we will show how this can be viewed as a Haar measure on D_n. To construct a Haar measure on $SL(2,\mathbb{R})$, consider $SL(2,\mathbb{R})$ as a subset of $(\mathbb{R}^4, m_4)$ where m_k is Lebesgue measure on $\mathbb{R}^k$. As a result, for measurable $A \subseteq SL(2,\mathbb{R})$, we can define the cone measure

$$\eta(A) = m_4(C(A)) \text{ where } C(A) = \{\alpha g : \alpha \in (0,1], g \in A\}.$$

Under matrix multiplication, m_4 is $SL(2,\mathbb{R})$ invariant. Hence η is an $SL(2,\mathbb{R})$ invariant measure on $SL(2,\mathbb{R})$. Under this measure, the set of matrices with a zero in the top left corner is a null set. Thus we can write the measure $d\eta = da\, db\, ds$ under the coordinates

$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}.$$

With this normalization, in the quotient by $SL(2,\mathbb{Z})$, we can compute the pushforward defined in terms of the projection map π and fundamental domain F [AC14]

$$(\eta)_*(Y_3) = \eta(\pi^{-1}(Y_3) \cap F) = \frac{\pi^2}{6} = \zeta(2).$$

With this fixed normalization, η gives the Poincaré volume. This means that we in fact have

$$(\eta)_*(Y_q) = c(q).$$

Now having fixed Haar measure on D_1, for D_n with $n \neq 0$, we identify D_n with $D_1 = SL(2,\mathbb{R})$ as $D_n = D_1 J_n$. Since we can write $D_n = D_1 J_n$, we choose the coordinates on D_n to be the same as those on D_1. In this manner, we have η is the Haar measure we will choose as our normalization of Haar measure on D_n.

We've now decomposed $\mathbb{R}^2 \times \mathbb{R}^2$ into $SL(2,\mathbb{R})$ orbits, and fixed a normalization of Haar measure on each of these orbits.

Since Haar measure is unique up to scaling, we can now write our $SL(2,\mathbb{R})$ invariant measure on $\mathbb{R}^4$ as

$$\nu = a\delta_0 + \sum_{t\in\mathbb{R}\cup\{\infty\}} b_t\, m_2 + \sum_{n\in\mathbb{R}_0} c_n \eta$$

for some constants a, b_t, c_n. where b_∞ corresponds to V and b_0 corresponds to H.

2.3.2 Reduction to visible determinants and removal of zero term

We have shown

$$\begin{aligned} \int_{Y_q} \hat{f}\, d\mu &= \int_{\mathbb{R}^2\times\mathbb{R}^2} f\, d\nu \qquad (2.3.1)\\ &= af(0,0) + \int_{t\in\mathbb{R}\cup\{\infty\}} b_t \int_{\mathbb{R}^2} f(x,tx)\, dx + \int_{n\in\mathbb{R}_0} c_n \int_{SL(2,\mathbb{R})} f(gJ_n)\, d\eta, \end{aligned}$$

(where we define $(x,\infty x) = (0,x)$.)

The purpose of this section is to prove the following.

Lemma 2.3.2. *In Equation (2.3.1), $a = 0$, $t \in \{\pm 1\}$, and $n \in N_q$.*

Proof. To see that $t \in \{\pm 1\}$, consider the function f supported on LD_t for $t \in \mathbb{R}\cup\{\infty\}$ where $LD_0 = H$ and $LD_\infty = V$. That is, for some large R and $B(0,R)$ denoting the Euclidean ball in $\mathbb{R}^4$, let

$$f_{R,t}(x,y) = \chi_{B(0,R)\setminus\{0\}}(x,y)\chi_{LD_t}(x,y) \quad \text{for } x,y \in \mathbb{R}^2.$$

On the left hand side of Equation 2.3.1, notice

$$\begin{aligned}\hat{f}_{R,t}([g]) &= \sum_{v_1,v_2\in V_q} f_{R,t}(gv_1, gv_2)\\ &= \#\{v_1, v_2 \in V_q \cap B(0,R) : gv_1 = tgv_2\}\\ &= \#\{v_1, v_2 \in V_q \cap B(0,R) : v_1 = tv_2\}.\end{aligned}$$

If $\begin{bmatrix} a \\ c \end{bmatrix} \in V_q$ by Proposition 2.2.2, we have $(a,c)_q = 1$, and thus by Equation (2.2.4), $(ta,tc)_q = t$. So by Proposition 2.2.2 $\begin{bmatrix} ta \\ tc \end{bmatrix}$ cannot be an element of V_q unless $t = \pm 1$. Or more geometrically since V_q are the set of vectors visible from the origin, tv is never visible from the origin unless $t = \pm 1$. Hence we've shown

$$\hat{f}_{R,t} = \begin{cases} 0 & t \neq \pm 1 \\ \#\{v \in V_q \cap B(0,R)\} & t = \pm 1 \end{cases}.$$

On the right hand side of Equation (2.3.1), the only nonzero term will be the coefficient of b_t for if $x \in B\left(0, \frac{R}{t}\right)$, then $tx \in B(0,R)$. Thus $\int_{\mathbb{R}^2} f_{R,t}(x,tx)\,dx \geq m_2\left(B_{\mathbb{R}^2}(0,\frac{R}{t})\right) > 0$. But when $t \neq \pm 1$, the left hand side of Equation 2.3.1 is zero since $\hat{f}_R = 0$. Hence $b_t = 0$ for $t \neq \pm 1$.

We now want to show that the set of possible determinants is N_q. For the determinant n loci ($n \neq 0$), we similarly define

$$f_{R,n}(x,y) = \chi_{B(0,R)}(x,y)\chi_{D_n}(x,y) \quad \text{for } x, y \in \mathbb{R}^2.$$

We compute

$$\begin{aligned}\hat{f}_{R,n}([g]) &= \sum_{v_1,v_1\in V_q} f_{R,n}(gv_1, gv_1)\\ &= \#\{v_1, v_2 \in V_q \cap B(0,R) : \det(v_1\, v_2) = \det(gv_1, gv_2) = n\}.\end{aligned}$$

Since N_q is the set of determinants that can arise as the determinant of two elements in V_q, we can write

$$\hat{f}_{R,n} = \begin{cases} \#\{v_1, v_2 \in V_q \cap B(0,R) : \det(v_1 v_2) = n\} & n \in N_q \\ 0 & n \notin N_q. \end{cases}$$

On the right hand side of Equation (2.3.1), the only nonzero term corresponds to c_n, and

$$\int_{SL(2,\mathbb{R})} f_R\left(gJ_n\right)\, d\eta > 0$$

since $D_n \cap B(0,R)$ has positive cone measure. In order to match the left hand side of Equation (2.3.1) for $\hat{f}_{R,n}$, we conclude $c_n = 0$ for all $n \notin N_q$.

We conclude this proof by showing $a = 0$. To see this, consider the characteristic function over the set $\{(0,0)\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$. That is set $f_0(x,y) = \chi_{\{(0,0)\}}(x,y)$. Then on the right hand side of Equation (2.3.1), we have $f_0(0,0) = 1$, all other integrals are zero since $\{(0,0)\}$ is a measure zero subset of $\mathbb{R}^2$, and cannot show up in $SL(2,\mathbb{R})J_n$ for any n. Thus the right hand side of Equation (2.3.1) for f_0 is a. On the left hand side of Equation (2.3.1), $(0,0)$ is not a pair of visible vectors since $(0,0)$ cannot be the first column of a matrix in H_q, so the left hand side is zero. Thus we conclude $a = 0$. □

To summarize, in this section we reduced our Equation (2.3.1) to

Corollary 2.3.3.

$$\int_{Y_q} \hat{f}\, d\mu = \int_{\mathbb{R}^2} b_1 f(x,x) + b_{-1} f(x,-x)\, dx + \sum_{n \in N_q} c_n \int_{SL(2,\mathbb{R})} f\left(gJ_n\right)\, d\eta.$$

2.3.3 Notation and division into smaller lemmas

In the proceeding sections, we will compute the values for b_1, b_{-1}, and c_n for $n \in N_q$. In order to do this, we introduce the following notation: for D a discrete subset of $(\mathbb{R}^2)^k$ which is V_q-invariant under the diagonal action, define $f_D : Y_q \to \mathbb{R}$ by

$$f_D([g]) = \sum_{v \in D^k} f(gv)$$

In a similar manner define the functional $T_D : B_c((\mathbb{R}^2)^k) \to \mathbb{R}$ by

$$T_D(f) = \int_{Y_q} f_D([g])\, d\mu([g]).$$

We now define the following sets:

$$D_n^V = D_n \cap (V_q \times V_q) = \{(v, w) \in V_q \times V_q : \det(v\, w) = n\},$$

$$LD_{\pm 1}^V = \{(v, \pm v) : v \in V_q\}.$$

Then we can rewrite the left hand side of Corollary 2.3.3 as

$$\begin{aligned} &\int_{Y_q} \left[f_{LD_1^V} + f_{LD_{-1}^V} + \sum_{n \in N_q} f_{D_n^V} \right] d\mu \\ &= T_{LD_1^V}(f) + T_{LD_{-1}^V}(f) + \sum_{n \in N_q} T_{D_n^V}(f). \end{aligned}$$

Thus finding the coefficients in Corollary 2.3.3 is reduced to finding coefficients individually in each of these equations:

$$T_{LD_{\pm 1}}(f) = b_{\pm 1} \int_{\mathbb{R}^2} f(x, \pm x)\, dx, \tag{2.3.2}$$

and for each $n \in N_q$

$$T_{D_n^V}(f) = c_n \int_{SL(2,\mathbb{R})} f\left(g J_n\right)\, d\eta. \tag{2.3.3}$$

2.3.4 Reducing to Siegel–Veech formula in linearly dependent case

In this section, we will prove that the coefficients b_1 and b_{-1} in Equation (2.3.2) are given by $b_1 = b_{-1} = \frac{1}{c(q)}$ by reducing to the Siegel–Veech Primitive Integral Formula (Theorem 1.1.2). That is, we will prove the following:

Lemma 2.3.4. *For any $f \in B_c(\mathbb{R}^2 \times \mathbb{R}^2)$,*

$$T_{LD_{\pm 1}^V}(f) = \frac{1}{c(q)} \int_{\mathbb{R}^2} f(v, \pm v)\, dv$$

where $c(q)$ is the Poincaré volume of the unit tangent bundle over $\mathbb{H}^2/H_q$.

Proof. Given $f \in B_c(\mathbb{R}^2 \times \mathbb{R}^2)$, define $\overline{f} \in B_c(\mathbb{R}^2)$ by

$$\overline{f}_\pm(u) = f(u, \pm u).$$

So we now compute

$$\begin{aligned} T_{LD^V_{\pm 1}}(f) &= \int_{Y_q} \sum_{v \in V_q} f(gu, \pm gu)\, d\mu([g]) \\ &= \int_{Y_q} \sum_{u \in V_q} \overline{f}_\pm(gu)\, d\mu([g]) \\ &= \frac{1}{c(q)} \int_{\mathbb{R}^2} \overline{f}_\pm(x)\, dx && \text{(by Theorem 1.1.2)} \\ &= \frac{1}{c(q)} \int_{\mathbb{R}^2} f(x, \pm x)\, dx. \end{aligned}$$

This concludes the proof of the lemma. □

We've now shown $b_{\pm 1} = c(q)^{-1}$, in the next section, we address the coefficients c_n for $n \in N_q$.

2.3.5 Coefficients on loci with fixed determinant

The goal of this section is to prove that each $c_n = c(q)^{-1}\varphi_q(n)$ for $n \in N_q$. We will first decompose D^V_n into H_q orbits under the diagonal action, showing there are $\varphi_q(n)$ orbits which each contribute equally to $T_{D^V_n}$. After showing this, we will find the value over a single orbit.

Lemma 2.3.5. *Let $n \in \mathbb{Z}_0[\lambda_q]$. There exists $v_1, v_2 \in V_q$ with $\det(v_1 v_2) = n$ if and only if there exists $m \in \mathbb{Z}[\lambda_q]$ with $0 \leq m < \lambda_q|n|$ and $\begin{bmatrix} m \\ n \end{bmatrix} \in V_q$.*

In particular, the equality in Equation (2.1.1) for N_q holds.

Proof. First, suppose there exists $m \in \mathbb{Z}[\lambda_q]$ with $0 \leq m < \lambda_q|n|$ and $\begin{bmatrix} m \\ n \end{bmatrix} \in V_q$. Set $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which is in V_q since $Tv_1 = v_1$, and set $v_2 = \begin{bmatrix} m \\ n \end{bmatrix}$. Then $v_1, v_2 \in V_q$ with determinant n, so $n \in N_q$.

Conversely suppose $v_1, v_2 \in V_q$ with $\det(v_1 v_2) = n$. Let $g \in H_q$ so that $ge_1 = v_1$. Then,

$$g^{-1}[v_1\, v_2] = [e_1\, g^{-1}v_2].$$

Since the determinant is n,

$$g^{-1}v_2 = \begin{bmatrix} \ell \\ n \end{bmatrix}$$

for some $\ell \in \mathbb{Z}[\lambda_q]$. Applying the matrix T to the left j times for some $j \in \mathbb{Z}$ gives

$$T^j g^{-1}[v_1\, v_2] = [e_1\, T^j g^{-1}v_2]$$

where

$$T^j g^{-1}v_2 = \begin{bmatrix} \ell + jn\lambda_q \\ n \end{bmatrix}.$$

Thus we can find $j \in \mathbb{Z}$ so that $m = \ell + jn\lambda_q$ satisfies

$$0 \leq m < \lambda_q |n|,$$

and the proof is complete. □

Lemma 2.3.6. *For $n \in N_q$ The subset D_n^V is the union of $\varphi_q(n)$ different orbits*

$$D_n^V = \bigsqcup_{\substack{0 \leq m < |n|\lambda_q \\ (m,n)^T \in V_q}} E_n^{(m)},$$

where

$$E_n^{(m)} = \left\{ \gamma \cdot \begin{bmatrix} 1 & m \\ 0 & n \end{bmatrix} : \gamma \in H_q \right\}.$$

Proof. We will first show that the decomposition of every element in D_n^V can be written as an element $E_n^{(m)}$ for some m.

Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_n^V$$

From the proof of Lemma 2.3.5, there exists a matrix $h \in H_q$ with

$$h \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & m \\ 0 & n \end{bmatrix} \quad \text{for } 0 \leq m < \lambda_q |n|.$$

Since $\begin{bmatrix} b \\ d \end{bmatrix} \in V_q$, we also have $\begin{bmatrix} m \\ n \end{bmatrix} = h \begin{bmatrix} b \\ d \end{bmatrix} \in V_q$. We have now shown every element in D_n^V is in $E_n^{(m)}$ for some m with $0 \leq m < \lambda_q |n|$ and $\begin{bmatrix} m \\ n \end{bmatrix} \in V_q$.

To see that we have no duplicate representatives of our orbits, let $0 < m_1, m_2 \leq \lambda_q |n|$ with $m_1 \neq m_2$ and $\begin{bmatrix} m_1 \\ n \end{bmatrix}, \begin{bmatrix} m_2 \\ n \end{bmatrix} \in V_q$. Without loss of generality suppose $m_2 > m_1$. If the representatives

$$\begin{bmatrix} 1 & m_1 \\ 0 & n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & m_2 \\ 0 & n \end{bmatrix}$$

were in the same H_q orbit, there would exist an element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H_q$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & m_1 \\ 0 & n \end{bmatrix} = \begin{bmatrix} 1 & m_2 \\ 0 & n \end{bmatrix}.$$

This implies $a = 1, c = 0, d = 1$, and $b = \frac{m_2 - m_1}{n}$. Since $0 < m_2 - m_1 < \lambda_q n$, we have $0 < b < \lambda_q$. This is a parabolic element with upper right entry smaller than the generating matrix T, and thus not in H_q. Therefore we conclude that $E_n^{(m)}$ are $\varphi_q(n)$ distinct H_q orbits whose union is all of D_n^V. □

Lemma 2.3.7. *For a fixed m with $0 \leq m < \lambda_q |n|$ and* $\begin{bmatrix} m \\ n \end{bmatrix} \in V_q$,

$$T_{E_n^{(m)}}(f) = \frac{1}{c(q)} \int_{SL(2,\mathbb{R})} f\left(gJ_n\right)\, d\eta_2.$$

Proof. Let $\pi : SL(2,\mathbb{R}) \to Y_q$ be the projection map $g \mapsto [g]$. Recall we normalize η so that $\pi_*(\eta)(Y_q) = c(q)$. Hence $\pi_*\left(c(q)^{-1}\eta\right) = \mu$. Moreover, to push a function from $SL(2,\mathbb{R})$ to a function on X_2, we have to sum over the orbits H_q. Thus,

$$\frac{1}{c(q)}\int_{SL(2,\mathbb{R})} f\left(g\begin{bmatrix}1 & m\\ 0 & n\end{bmatrix}\right)\,d\eta = \int_{Y_q}\sum_{\gamma\in H_q} f\left(g\cdot\gamma\cdot\begin{bmatrix}1 & m\\ 0 & n\end{bmatrix}\right)\,d\mu = T_{E_n^{(m)}}(f).$$

For the last part of the lemma, we compute the following

$$\begin{aligned}
&\frac{1}{c(q)}\int_{SL(2,\mathbb{R})} f\left(g\begin{bmatrix}1 & m\\ 0 & n\end{bmatrix}\right)\,d\eta\\
&= \frac{1}{c(q)}\int_{SL(2,\mathbb{R})} f\left(g\begin{bmatrix}1 & \frac{1-m}{n}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & m\\ 0 & n\end{bmatrix}\right)\,d\eta\left(g\begin{bmatrix}1 & \frac{1-m}{n}\\ 0 & 1\end{bmatrix}\right)\\
&= \frac{1}{c(q)}\int_{SL(2,\mathbb{R})} f\left(g\begin{bmatrix}1 & 1\\ 0 & n\end{bmatrix}\right)\,d\eta(g)
\end{aligned}$$

where the last equality follows from the fact that $SL(2,\mathbb{R})$ is a unimodular group, so the Haar measure η is both left and right invariant under the action of $SL(2,\mathbb{R})$. □

Lemma 2.3.7 shows that $T_{E_n^{(m)}}(f)$ is constant for with respect to m. Hence we conclude

$$T_{D_n^V} = \sum_{\substack{0\le m<\lambda_q|n|\\ (m,n)^T\in V_q}} T_{E_n^{(m)}} = \frac{\varphi_q(n)}{c(q)}\int_{SL(2,\mathbb{R})} f\left(gJ_n\right)\,d\eta(g).$$

In conclusion, we've now shown that

$$T_{D_n^V} = \frac{\varphi_q(n)}{c(q)}\int_{SL(2,\mathbb{R})} f\left(gJ_n\right)\,d\eta\left(g\right)$$

As well as

$$T_{LD_{\pm1}} = \frac{1}{c(q)}\int_{\mathbb{R}^2} f(x,\pm x)\,dx.$$

Putting these results together with Corollary 2.3.3, we have now shown Theorem 2.1.3 holds.

2.4 Higher moments

We will prove Theorem 2.1.4 which is the generalization of Theorem 2.1.3 which corresponds to higher moments of the classical Siegel–Veech Transform on $\mathbb{R}^2$.

2.4.1 Decomposition into orbits

We first decompose $(\mathbb{R}^2)^k$ into $SL(2,\mathbb{R})$ orbits. Given a point in $(\mathbb{R}^2)^k$, either all the terms are linearly dependent, or there exist two terms in the k-tuples which are linearly dependent.

Lemma 2.4.1. *The following decomposes $(\mathbb{R}^2)^k$ into disjoint $SL(2,\mathbb{R})$ orbits:*

$$(\mathbb{R}^2)^k = \left(\bigsqcup_{\lambda} LD_\lambda\right) \sqcup \left(\bigsqcup_{n,\lambda,\alpha,\beta} D_{n,\lambda,\alpha,\beta}\right).$$

In the linearly dependent case,

$$LD_\lambda = \{\lambda x : x \in \mathbb{R}^2\}$$

for $\lambda \in \mathbb{R}^k$ with first nonzero entry (if it exists) given by 1.

In the linearly independent case,

$$D_{n,\lambda,\alpha,\beta} \overset{def}{=} \{(\lambda x, \alpha x + \beta y) : \det(x\, y) = n\},$$

where $n \in \mathbb{R}_0$ is the determinant of the first nonzero vector with the first linearly independent vector. For $0 \leq j < k$ we have $\lambda \in \mathbb{R}^j$ where the first nonzero entry is 1 and $\alpha, \beta \in \mathbb{R}^{k-j}$ where $\alpha = (0, \alpha_2, \ldots, \alpha_{k-j})$ and $\beta = (1, \beta_2, \ldots, \beta_{k-j})$.

Proof. We first claim that LD_λ and $D_{n,\lambda,\alpha,\beta}$ can be written as $SL(2,\mathbb{R})$ orbits. Indeed since $SL(2,\mathbb{R})$ acts transitively and linearly by matrix multiplication on $\mathbb{R}^2 \setminus \{0\}$ we can write

$$SL(2,\mathbb{R}) \cdot \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} = LD_\lambda.$$

Similarly since $SL(2,\mathbb{R})$ acts transitively on determinant n subsets as proved in Lemma 2.3.1 and linearly on $\mathbb{R}^2$, we can write

$$SL(2,\mathbb{R})\cdot\left(\lambda\begin{bmatrix}1\\0\end{bmatrix},\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\n\end{bmatrix}\right)=D_{n,\lambda,\alpha,\beta}.$$

Next we show that the union of the orbits in fact covers all of $(\mathbb{R}^2)^k$. To do this consider a vector $v=(v_1,v_2,\ldots,v_k)\in(\mathbb{R}^2)^k$. If all $v_i=0$ then $v\in LD_{\mathbf{0}}$. Otherwise there is some first nonzero v_i which we will call x. If $\dim(\operatorname{span}(v_1,\ldots,v_k))=1$, then every other element will be a linear multiple of x. Hence $v\in LD_\lambda$ where λ has first nonzero entry is 1 and all remaining entries are real numbers.

If however $\dim(\operatorname{span}(v_1,\ldots,v_k))=2$, then set y to be the first vector after x which is linearly independent of x. For all v_i which occur after y, v_i can be written as a linear combination of x and y, thus written as $\alpha_i x+\beta_i y$ for some real numbers. Since LD_λ and $D_{n,\lambda,\alpha,\beta}$ are subsets of $(\mathbb{R}^2)^k$ we conclude that

$$(\mathbb{R}^2)^k=\left(\bigcup_\lambda LD_\lambda\right)\cup\left(\bigcup_{n,\lambda,\alpha,\beta}D_{n,\lambda,\alpha,\beta}\right).$$

Finally we finish the proof by proving each of these orbits is distinct. Since all pairs of entries in LD_λ have determinant 0 and $SL(2,\mathbb{R})$ preserves determinants, we know that the LD_λ and $D_{n,\lambda,\alpha,\beta}$ must be disjoint.

Now suppose that $LD_\lambda=LD_{\lambda'}$. Since the first nonzero vector must have a coefficient of 1, if $\lambda_j=1$ is the first nonzero element, then $\lambda'_j=1$ as well. Now every vector after the first vector is linearly dependent on the first nonzero vector, so there is a unique coefficient and $\lambda=\lambda'$.

Similarly, since we choose a coefficient of 1 for the first nonzero vector, and a coefficient of 1 for the first vector which is linearly independent, we have a unique representation of the linear combinations $\alpha_i x+\beta_i y$. Hence the $D_{n,\lambda,\alpha,\beta}$ are also all disjoint. This completes the proof. □

2.4.2 Reduction to smaller lemmas

Using the notation of section 2.3.3 we will rewrite $\int_{Y_q} \hat{f}\, d\mu$, reducing the proof of Theorem 2.1.4 to smaller lemmas.

Define $D^V_{n,\lambda,\alpha,\beta} = D_{n,\lambda,\alpha,\beta} \cap (V_q)^k$ and $LD^V_\lambda = LD_\lambda \cap (V_q)^k$. Moreover for $1 \leq m \leq |n|$ with $(m,n)^T \in V_q$ define

$$E^{(m)}_{n,\lambda,\alpha,\beta} = H_q \cdot \left(\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}, J_{m,n} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right)$$

where $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is a $2 \times (k-j)$ matrix and

$$J_{n,m} = \begin{bmatrix} 1 & m \\ 0 & n \end{bmatrix}$$

corresponds to $J_n = J_{n,1}$ in the $k = 2$ case. By Lemma 2.3.6, we can write

$$D_{n,\lambda,\alpha,\beta} = \bigsqcup_{\substack{0 \leq m < \lambda_q |n| \\ (m,n)^T \in V_q}} E^{(m)}_{n,\lambda,\alpha,\beta}.$$

Lemma 2.4.2. *We have*

$$\int_{Y_q} \hat{f}\, d\mu = \sum_\lambda T_{LD^V_\lambda} + \sum_{n \in N_q} \sum_{\substack{0 \leq m < \lambda_q |n| \\ (m,n)^T \in V_q}} \sum_{\lambda,\alpha,\beta} T_{E^{(m)}_{n,\lambda,\alpha,\beta}}$$

where in the linearly dependent case $\lambda \in \mathbb{R}^k$, where the first element of λ must be 1, and any remaining elements of λ must be ± 1.

In the linearly independent case, given $n \in N_q$, there exists a unique $0 \leq m < \lambda_q |n|$ so that the two vectors lie in the H_q orbit of $J_{m,n}$ Given n and m, we have $\lambda \in \mathbb{R}^j$ with first entry 1 and all remaining elements ± 1. Moreover $\alpha = (0, \alpha_2, \ldots, \alpha_{k-j})$ and $\beta = (1, \beta_2, \ldots, \beta_{k-j})$ where α and β satisfy Equation 2.1.3 for each $2 \leq i \leq k-j$.

Proof. By Lemma 2.4.1, given $v \in (V_q)^k$, we must have $v \in LD^V_\lambda$ or $D^V_{n,\lambda,\alpha,\beta}$ for some λ or $(n, \lambda, \alpha, \beta)$.

If $v \in LD^V_\lambda$, first note that the zero vector is not in V_q. Hence the first entry in λ must be 1. Since the vectors are in V_q and must be constant multiples of the first vector, all other vectors must be $\pm v_1$. Thus λ must have the specified form.

Moving onto the linearly independent case, if $v \in D^V_{n,\lambda,\alpha,\beta}$, then we can write $v = (\lambda v_1, \alpha v_1 + \beta v_j)$ for some $1 < j \leq k$ where v_j is the first vector after v_1 which is not co-linear with v_1. Since all the vectors in λv_1 must be in V_q, λ must have first entry 1 and all other entries ± 1.

Setting $\det(v_1\, v_j) = n$ by the definition of N_q we have $n \in N_q$.

Finally we need to determine the criterion for $\alpha = (0, \alpha_2, \ldots, \alpha_{k-j})$, $\beta = (1, \beta_2, \ldots, \beta_{k-j})$. So that for all $1 \leq i \leq k - j$ we have $\alpha_i v_1 + \beta_i v_j \in V_q$.

By Lemma 2.3.6, there exists $\gamma \in H_q$ and $0 \leq m < \lambda_q |n|$ with $(m, n)^T \in V_q$ so that

$$\gamma \cdot (v_1 | v_j) = \begin{bmatrix} 1 & m \\ 0 & n \end{bmatrix}.$$

So we have

$$\gamma \cdot (\alpha_i v_1 + \beta_i v_j) = \begin{bmatrix} \alpha_i + m\beta_i \\ n\beta_i \end{bmatrix}.$$

Since H_q acts transitively on V_q, $\alpha_i v_1 + \beta_i v_j \in V_q$ is equivalent to

$$\begin{bmatrix} 1 & m \\ 0 & n \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} = \begin{bmatrix} \alpha_i + m\beta_i \\ n\beta_i \end{bmatrix} \in V_q.$$

Multiplying by the inverse of $J_{n,m}$ we see α and β must satisfy equation 2.1.3.

So for each n, there exists m with $0 \leq m < \lambda_q |n|$ so that $(m, n)^T \in V_q$. Given this m, we already have the requirement for λ, and the α and β must satisfy Equation 2.1.3. This concludes the proof. □

Now that we have decomposed $\int_{Y_q} \hat{f}\, d\mu$, the higher moments case is complete once we prove the following two lemmas.

Lemma 2.4.3. *Given the restrictions of λ in Lemma 2.4.2,*

$$T_{LD^V_\lambda} = \frac{1}{c(q)} \int_{\mathbb{R}^2} f(\lambda v)\, dv.$$

Proof. The proof strategy is identical to the strategy in Lemma 2.3.4 □

Lemma 2.4.4. *Given the restrictions of $n, m, \lambda, \alpha, \beta$ in Lemma 2.4.2*

$$T_{E^{(m)}_{n,\lambda,\alpha,\beta}} = \frac{1}{c(q)} \int_{SL(2,\mathbb{R})} f\left(\lambda g \begin{bmatrix} 1 \\ 0 \end{bmatrix}, g J_{n,m} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right) d\eta(g).$$

Proof. This proof strategy is identical to the proof in Lemma 2.3.7. Note we cannot use the change of variables to get equal contribution for each $T_{E^{(m)}_n}$ as in the case of Lemma 2.3.7 because the criterion for which α and β can occur depends on m. □

Combining Lemma 2.4.2 with Lemma 2.4.3 and Lemma 2.4.4, we have now concluded the proof of Theorem 2.1.4.

2.5 Numerical evidence

This section discusses how to interpret Theorem 2.1.4 in terms of a counting problem. We will focus on the case $k = 2$, that is Theorem 2.1.3. The following proposition is from section 16 of [Vee98].

Proposition 2.5.1. *For $B(0, R)$ the ball of radius R in $\mathbb{R}^2$,*

$$\lim_{R\to\infty} \frac{\#\{V_q \cap B(0,R)\}}{\pi R^2} = \frac{1}{c(q)}.$$

We will use the notation $\#\{V_q \cap B(0,R)\} \sim \frac{\pi R^2}{c(q)}$ where $f(R) \sim g(R)$ if and only if $\lim_{R\to\infty} f(R)/g(R) = 1$.

In the case $q = 3$ this can be interpreted as the probability a randomly chosen integer vector is primitive is $\frac{1}{\zeta(2)}$. See Figure 2.1 and Figure 2.2 for visualization of the points of V_q for $q = 3, 4, 5$.

To construct the set V_q, we used a Farey tree construction in the first quadrant and then used the 4-fold symmetry of V_q, that is $(a, b) \in V_q$ implies $(a, -b), (-a, b), (-a, -b) \in V_q$. The generalization of the Farey tree construction as found in [LL16] begins with the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V_q \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V_q.$$

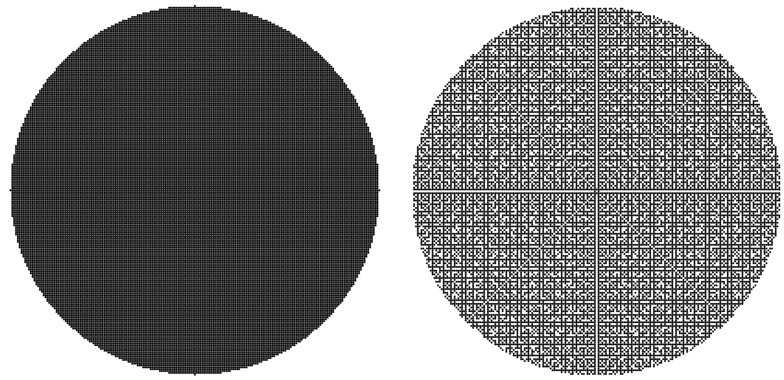


Figure 2.1: For $R = 100$, the left hand plot includes all integer pairs (a, b) and the right hand plot includes all primitive integers pairs (a, b) with $gcd(a, b) = 1$ where $a^2 + b^2 \leq 100^2$. This demonstrated the expectations that for large R, the number of points on the left should be approximately $\pi 100^2$, but on the right the number of points should be $\frac{\pi 100^2}{\zeta(2)}$.

Then for $i = 2, \ldots, q-1$ we add the vectors

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} \lambda_q a_{i-1} - a_{i-2} \\ \lambda_q b_{i-1} - b_{i-2} \end{bmatrix}$$

where

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Iterating this step between each pair of adjacent vectors we obtain the elements of V_q.

Now that we've generated plots for V_q, we can now count pairs of elements in V_q corresponding to the square of the Siegel–Veech transform. Specifically for $f = \chi_{B_{\mathbb{R}^4}(0,R)}$ the characteristic function of the Euclidean ball in $\mathbb{R}^4$, we want to understand $T_{D_n^V}(f)$ which will asymptotically grow like the function

$$\text{Count}_q(R, n) \overset{def}{=} \# \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_n^V : a^2 + b^2 + c^2 + d^2 \leq R^2 \right\}.$$

Theorem 2.1.3 states that

$$\frac{\text{Count}_q(R, n)}{\int_{SL(2,\mathbb{R})} f(gJ_n)\, d\eta} \sim \frac{\varphi_q(n)}{c(q)},$$

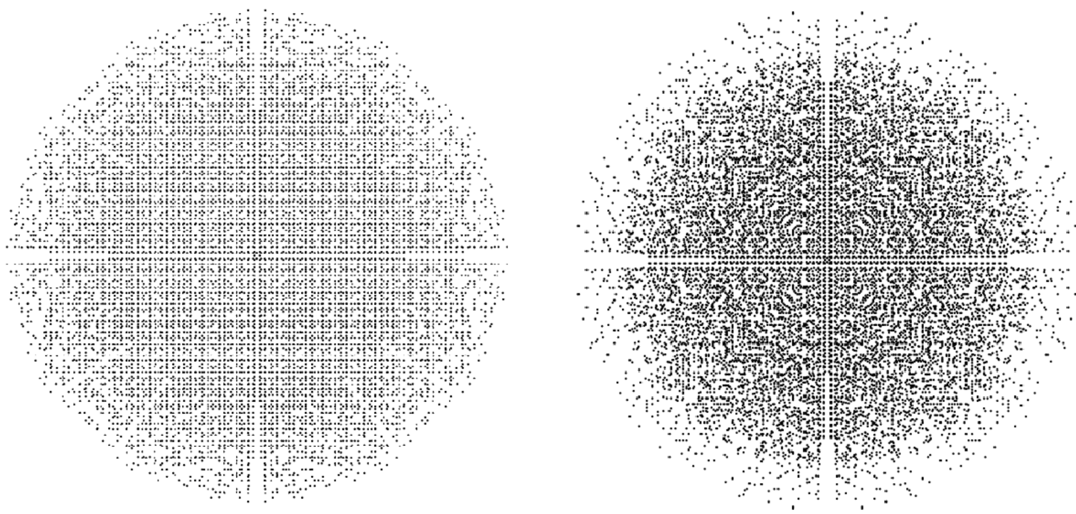


Figure 2.2: On the left is a plot of the vectors V_4, and on the right is a plot of the vectors V_5. These plots were generated using the Farey Tree construction.

which is not useful for understanding data without more knowledge about $\int_{SL(2,\mathbb{R})} f(gJ_n)\, d\eta$.

Newman [New88] showed that $\mathrm{Count}_3(R, 1) \sim 6R^2$. In particular combining with Theorem 2.1.3, we obtain

$$\int_{SL(2,\mathbb{R})} f(gJ_1)\, d\eta \sim \pi^2 R^2.$$

Next using the result of Schmidt [Sch60], we can extend this result to the fact that when $q = 3$,

$$\int_{SL(2,\mathbb{R})} f\left(gJ_n\right)\, d\eta \sim \frac{\pi^2}{n} R^2.$$

Thus we deduce that for any $q \geq 3$,

$$\frac{\mathrm{Count}_q(R, n)}{R^2} \sim \frac{1}{c(q)} \cdot \varphi_q(n) \cdot \frac{\pi^2}{n} = \frac{\varphi_q(n) \cdot \pi^2}{n \cdot c(q)}.$$

Indeed in our numerical experiments we obtained the desired results. In Figure 2.3, we show the convergence for $k = 1, 2, 3, 4$. Recall D_n^V can be decomposed into $\varphi_q(n)$ orbits $E_n^{(m)}$ where $0 \leq m < \lambda_q|n|$, and on each orbit we were able to verify we had density asymptotic

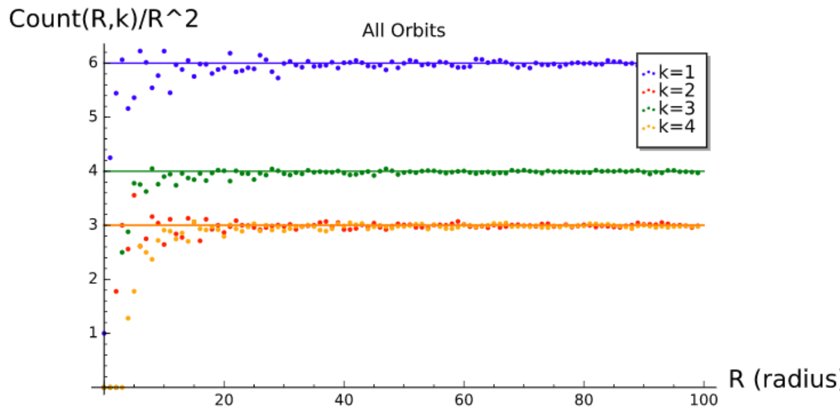


Figure 2.3: For $q = 3$ this is a plot of R on the horizontal axis, and a plot of $\frac{\text{Count}(R,k)}{R^2}$ on the vertical axis. Notice that $k = 2$ and $k = 4$ both converge to 3 since $\frac{6\varphi(4)}{4} = \frac{6\varphi(2)}{2} = 3$.

to $\frac{\pi^2}{n \cdot c(q)}$ as desired. Finally in Figure 2.5, we provide a visualization for pairs of elements in V_q for $q = 3$ and $q = 5$.

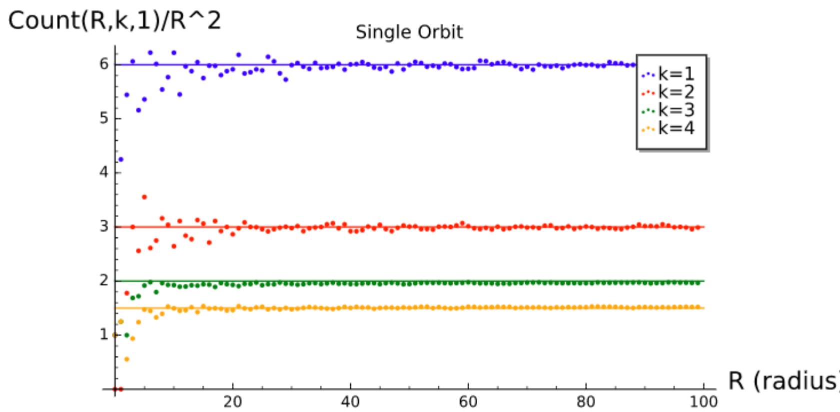


Figure 2.4: For $q = 3$, a plot of R on the horizontal axis, and a plot of $\frac{\text{Count}(R,n,1)}{R^2}$ on the vertical axis, where $\text{Count}(R, n, 1)$ is number of elements within the ball of radius R, which are in the orbit $SL(2, \mathbb{Z}) \cdot J_n$.

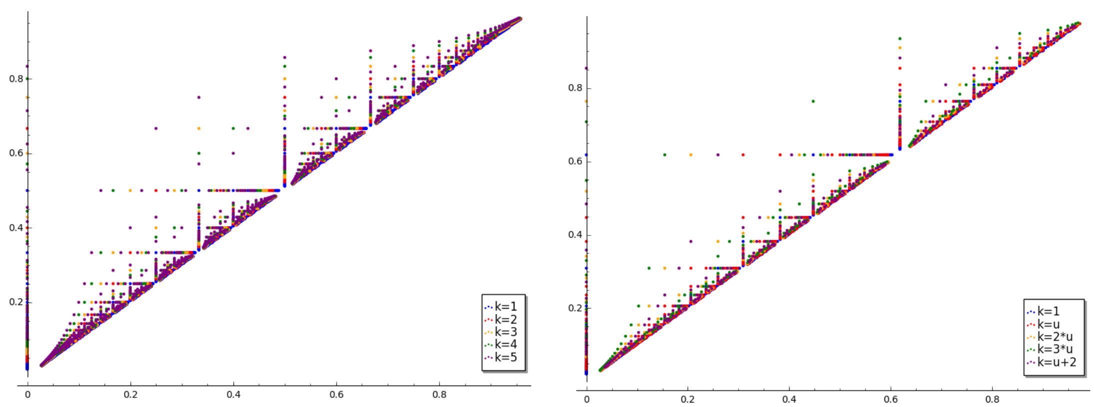


Figure 2.5: For each element (a, b) in V_q, we can visualize these points by considering a/b, which for simplicity we will consider the pairs with $a/b \in [0, 1]$. These pictures plot the points in D_k^V by placing a point if the point a/b on the x-axis and the point c/d on the y-axis have the corresponding pairs (a, b) and (b, c) in D_k^V. On the left are pairs of primitive vectors, and on the right are pairs of vectors in V_5 where in the legend $u = \phi$, the golden ratio.

Chapter 3

COUNTING PAIRS OF SADDLE CONNECTIONS FOR TYPICAL TRANSLATION SURFACES.

This chapter is joint work with Jaydev Athreya and Howard Masur.

3.1 Introduction

We are interested in the distribution of *pairs* of saddle connections, in particular the growth rate of the following counting function, the count of pairs of bounded *virtual area*. Fix $A > 0$, and define

$$N_A(\omega, R) = \#\{(z_1, z_2) \in \Lambda_\omega(R)^2 : 0 < |z_1 \wedge z_2| \leq A, |z_2| \leq |z_1|\},$$

where for $z = x + iy, w = u + iv$,

$$z \wedge w = xv - yu = \Im(\overline{z}w)$$

is the (signed) area of the parallelogram spanned by z and w. Our main results are on the $R \to \infty$ behavior of these functions. We fix $\mathcal{H}$ to be a connected component of a stratum. Our main result is an almost sure asymptotic growth result for the set of pairs of saddle connections with bounded virtual area.

Conjecture 3.1.1. *There is a constant $c = c(A, \mathcal{H})$ such that for μ-almost every $\omega \in \mathcal{H}$, so that*

$$\lim_{R\to\infty} \frac{N_A(\omega, R)}{R^2} = c.$$

3.1.1 History and prior results

The study of counting problems for saddle connections is very active, and connected to many different areas of mathematics, from low-dimensional dynamical systems to algebraic geometry. Motivated by problems in counting special trajectories for billiards in rational polygons, Masur [Mas90] proved that the counting function

$$N_\omega(R) = \#\left(\Lambda_\omega \cap B(0, R)\right)$$

has quadratic upper and lower bounds for all ω, that is, there are $0 < c_1 = c_1(\omega) < c_2 = c_2(\omega)$ so that for all R,

$$c_1 R^2 \leq N(\omega, R) \leq c_2 R^2.$$

Subsequently, Veech [Vee98] showed there is a constant $c = c(\mathcal{H})$ such that

$$\lim_{R\to\infty} \int_{\mathcal{H}} \left| \frac{N(\omega, R)}{R^2} - c\pi \right| d\mu_{\mathcal{H}}(\omega) = 0,$$

an L^1-quadratic asympotic result. Inspired by Veech's approach, Eskin–Masur [EM01] adapted ideas from homogeneous dynamics (specifically, the work of Eskin–Margulis–Mozes [EMM95, EMM98] on quantitative versions of Oppenheim's conjecture) and an ergodic theorem of Nevo [Nev17] to improve this to a pointwise asymptotic result, showing that for $\mu_{\mathcal{H}}$ almost every $\omega \in \mathcal{H}$,

$$\lim_{R\to\infty} \frac{N(\omega, R)}{R^2} = c\pi.$$

More recently, Nevo–Rühr–Weiss [NRW20], using error term estimates in Nevo's ergodic theorem coming from mixing properties of the *Teichmüller geodesic flow*, showed that there is an $\alpha < 2$ such that for almost every $\omega \in \mathcal{H}$,

$$N(\omega, R) = c\pi R^2 + o(R^\alpha).$$

Our approach uses crucially ideas from all of these results: we will, using ideas similar to Eskin-Masur [EM01], set up our counting problem as an integral over a piece of an $SL(2, \mathbb{R})$-orbit on $\mathcal{H}$, and then apply the ergodic theorem of Nevo [Nev17]. To implement our strategy, we will need upper bounds in the spirit of [Mas90], and approximation ideas carefully implemented in [NRW20].

The Siegel-Veech transform

Given a bounded, compactly supported function $f \in B_c(\mathbb{C})$, the Siegel–Veech transform for $k = 1$ (Definition 1.1.1) $\hat{f} \in L^1(\mathcal{H}, \mu_{\mathcal{H}})$. In fact a crucial ingredient in Eskin-Masur's asymptotic result is that $\widehat{f} \in L^{1+\beta}$ for some $\beta > 0$. We will need similar results for the generalized Siegel-Veech transform. Given a bounded compactly supported function $h \in B_c(\mathbb{C}^2)$, we define the Siegel–Veech transform by

$$\widehat{h}(\omega) = \sum_{z_1, z_2 \in \Lambda_\omega} h(z_1, z_2).$$

For example, if h is the indicator function of the set

$$D_{R,A} = \{(z,w) \in \mathbb{C}^2 : |w| \leq |z| \leq R, 0 < |z \wedge w| < A\},$$

$$\widehat{h}(\omega) = N_A(\omega, R).$$

In our proof of our asymptotic counting result for N_A, we will use a result of Athreya-Cheung-Masur [ACM19] which shows that $h \in L^{1+\beta}$ for $h \in B_c(\mathbb{C}^2)$ (which is essentially equivalent to showing that for $f \in B_c(\mathbb{C})$, $\widehat{f} \in L^{2+\beta}(\mathcal{H})$.)

3.1.2 Strategy of proof

We now outline the strategy of proof of Conjecture 3.1.1. First, we recall the strategy of Eskin-Masur for understanding the counting function $N(\omega, R)$: they construct a function $f \in B_c(\mathbb{C})$ (essentially the indicator function of a trapezoid), which satisfied

$$\frac{1}{2\pi}\int_0^{2\pi} f(g_t r_\theta z) \approx e^{-2t}\chi_{A(e^t/2, e^t)}(z),$$

where the matrices

$$g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \quad r_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{3.1.1}$$

act $\mathbb{R}$-linearly on $\mathbb{C}$, and $A(R_1, R_2), 0 < R_1 < R_2$ is the annulus

$$A(R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z| < R_2\}.$$

Putting $e^t = R$, and adding the above expression over all $z \in \Lambda_\omega$, we obtain

$$\frac{1}{R^2}\left(N(\omega, R) - N(\omega, R/2)\right) \approx \frac{1}{2\pi}\int_0^{2\pi} \widehat{f}(g_t r_\theta \omega) d\theta.$$

This reduces the counting problem to a problem of understanding the sequence of integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \widehat{f}(g_t r_\theta \omega) d\theta.$$

Nevo's ergodic theorem deals precisely with integrals of this form, but with some compactness and smoothness assumptions on the integrand. A series of approximation arguments is required to implement Nevo's theorem, and the final convergence relies on Masur's upper bounds as well as the fact that $\widehat{f} \in L^{1+\beta}$. We will construct a function $h_A \in B_c(\mathbb{C}^2)$ so that

$$\frac{1}{2\pi}\int_0^{2\pi} \widehat{h_A}(g_t r_\theta(z,w)) \approx e^{-2t}\chi_{D(e^t,A)}(z,w), \tag{3.1.2}$$

where the action of $SL(2,\mathbb{R})$ on $\mathbb{C}^2$ is by $\mathbb{R}$-linear transformations in each coordinate, that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x+iy) = (ax+by) + i(cx+dy).$$

Adding (3.1.2) over all $(z,w) \in \Lambda^2_\omega$, we obtain

$$\frac{1}{R^2} N_A(\omega, R) \approx \frac{1}{2\pi}\int_0^{2\pi} \widehat{h_A}(g_t r_\theta \omega) d\theta. \tag{3.1.3}$$

Once again, we will need a series of approximation arguments to implement Nevo's theorem, and the final convergence will rely on the result of Athreya-Cheung-Masur that $\widehat{h} \in L^{1+\beta}$, and an upper bound result similar to Masur's that we prove in §3.4.

3.1.3 Organization of Preliminary Results

In §3.2, we carefully construct our function h_A and show it is very close to having the desired properties. In §3.3, we will address the error terms not accounted for in §3.2, and state a sufficient condition on the error terms to obtain Conjecture 3.3.1. Finally in §3.4 we show quadratic upper bounds, using a proof strategy that in future work will be modified to complete Conjecture 3.3.1 by proving Lemma 3.3.2.

Acknowledgments. We'd like to thank the Mathematical Sciences Research Institute where this work originated in the Fall 2019 program on Holomorphic Differentials in Mathematics and Physics. We'd like to thank the Fields Institute where we had preliminary discussion in Fall 2018 during the program on Teichmüller Theory and its Connections to Geometry, Topology and Dynamics. We'd like to thank David Aulicino, Yair Minsky, and John Smillie for useful discussions.

3.2 Approximation function and properties

In this section, we construct the function h_A and show that for a large set h_A capture the stated goal of (3.1.2). Fix $A > 0$, and define the following sets:

$$\mathcal{T} = \left\{\zeta \in \mathbb{C} : \frac{1}{2} \leq \operatorname{Im}(\zeta) \leq 1, \, |\operatorname{Re}(\zeta)| \leq \operatorname{Im}(\zeta)\right\}$$

and for each $\zeta \in \mathbb{C}$ let

$$\mathcal{R}_{A,\zeta} = \{\eta \in \mathbb{C} : 0 < |\zeta \wedge \eta| \leq A, \, |\operatorname{Im}(\eta)| \leq \operatorname{Im}(\zeta)\}.$$

Finally for a set $\mathcal{S} \subseteq \mathbb{C}$ define the fibered set

$$\mathcal{S}_A^{\mathcal{R}} = \{(\zeta, \eta) \in \mathbb{C}^2 : \zeta \in \mathcal{S}, \, \eta \in \mathcal{R}_{A,\zeta}\}.$$

Now we set $h_A = \chi_{\mathcal{T}_A^{\mathcal{R}}}$.

Lemma 3.2.1. *The fibered set $\mathcal{T}_A^{\mathcal{R}}$ has the following properties*

1. *Equivariance under geodesic flow. That is for $t > 0$ if g_t as defined in Equation* (3.1.1) *acts diagonally on $\mathbb{C}^2$,*

$$g_t(\mathcal{T}_A^{\mathcal{R}}) = (g_t\mathcal{T})_A^{\mathcal{R}}.$$

2. *Computation of endpoints under geodesic flow. Set $t > 0$. Then the trapezoid $g_{-t}\mathcal{T}$ has endpoints*

$$\pm\frac{e^{-t}}{2} + i\frac{e^{t}}{2}, \quad \pm e^{-t} + ie^{t}.$$

Moreover for each $\zeta \in g_{-t}\mathcal{T}$ the parallelogram $\mathcal{R}_{A,\zeta}$ has endpoints

$$\frac{\pm A + \operatorname{Im}(\zeta)\operatorname{Re}(\zeta)}{\operatorname{Im}(\zeta)} + i\operatorname{Im}(\zeta), \quad \frac{\pm A - \operatorname{Im}(\zeta)\operatorname{Re}(\zeta)}{\operatorname{Im}(\zeta)} - i\operatorname{Im}(\zeta).$$

Proof. 1. This follows from the fact that $g_t \in SL(2, \mathbb{R})$ so $|g_{-t}\zeta \wedge g_{-t}\eta| = |\zeta \wedge \eta|$. Moreover $\operatorname{Im}(g_{-t}\zeta) = e^t \operatorname{Im}(\zeta)$.

2. To find the endpoints of $\mathcal{R}_{A,\zeta}$ for $\zeta \in g_{-t}\mathcal{T}$ we will compute the intersection of the lines $|\zeta \wedge \eta| = A$ and $|\operatorname{Im}(\eta)| = \operatorname{Im}(\zeta)$.

   Thus the possible solutions for $\operatorname{Re}(\eta)$ are given by

$$\operatorname{Re}(\eta) = \frac{\pm A \pm \operatorname{Im}(\zeta)\operatorname{Re}(\zeta)}{\operatorname{Im}(\zeta)}.$$

□

Next we will define a quantity to capture a magnitude so that whenever we consider pairs where the first is rotated by θ to be in $g_{-t}\mathcal{T}$, the second will have small enough magnitude to be in the fiber. That is given $(\zeta, \eta) \in \mathbb{C}^2$ and $t > 0$, we want $|\eta|$ small enough so that $\eta \in R_{A,r_\theta\zeta}$ whenever $\zeta \in g_{-t}\mathcal{T}$. To this end, define

$$L^{\min}_{t,\zeta} = \min_{\substack{\theta\in[0,2\pi)\\ r_\theta\zeta\in g_{-t}\mathcal{T}}} \min_{\substack{\operatorname{Im}(\xi)=\operatorname{Im}(r_\theta\zeta)\\ |\xi\wedge r_\theta\zeta|\le A}} |\xi|.$$

Lemma 3.2.2. *Fix A. Then there exists $c_A > 0$ and there exists $T > 0$ so that for any $\zeta \in \mathbb{C}$ and $t > T$,*

$$L^{\min}_{t,\zeta} \geq |\zeta|\sqrt{1 - c_A e^{-4t}}$$

Proof. Fix θ so that $r - \theta\zeta \in g_{-t}\mathcal{T}$. Choose ξ_θ so that

$$|\xi_\theta| = \min_{\substack{\operatorname{Im}(\xi)=\operatorname{Im}(r_\theta\zeta)\\ |\xi\wedge r_\theta\zeta|\le A}} |\xi|.$$

Notice that part (2) of Lemma 3.2.1 gives bounds for $\operatorname{Re}(\xi_\theta)$ by taking the top side of the parallelogram $R_{A,r_\theta\zeta}$, so

$$\begin{aligned} |\xi_\theta| &= \sqrt{\operatorname{Re}(\xi_\theta)^2 + \operatorname{Im}(r_\theta\zeta)^2} \\ &= \begin{cases} \operatorname{Im}(r_\theta\zeta) & 0 \in \left[\frac{-|A|}{\operatorname{Im}(r_\theta\zeta)} + \operatorname{Re}(r_\theta\zeta), \frac{|A|}{\operatorname{Im}(r_\theta\zeta)} + \operatorname{Re}(r_\theta\zeta)\right] \\ \sqrt{\left|\frac{|A|}{\operatorname{Im}(r_\theta\zeta)} - |\operatorname{Re}(r_\theta\zeta)|\right|^2 + \operatorname{Im}(r_\theta\zeta)^2} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2.1}$$

Now to minimize Equation 3.2.1 over $\theta \in [0, 2\pi)$ such that $r_\theta \zeta \in g_{-t}\mathcal{T}$.

In the first case, whenever $r_\theta\zeta \in g_{-t}\mathcal{T}$, the smallest possible imaginary part must occur when $r_\theta\zeta$ makes angle θ_t with the vertical axis. Thus

$$|\zeta| \geq \mathrm{Im}(r_\theta\zeta) \geq |\zeta|\cos(\theta_t) = \frac{|\zeta|}{\sqrt{1+e^{-4t}}} \geq |\zeta|\sqrt{1-e^{-4t}}.$$

Notice the condition on the first case is equivalent to

$$|\mathrm{Re}(r_\theta\zeta)|\,\mathrm{Im}(r_\theta\zeta) \leq |A|.$$

So using the fact that $|\mathrm{Re}(r_\theta\zeta)|\,\mathrm{Im}(r_\theta\zeta) \leq |\zeta|^2\sin(\theta_t) \leq 1$, we know that we are in the first case whenever $|A| \geq 1$. In the second case, we have

$$|\xi_\theta| = |\zeta|\sqrt{1 + \frac{|A|}{\mathrm{Im}(r_\theta\zeta)^2|\zeta|^2}(|A| - 2|\mathrm{Re}(r_\theta\zeta)|\,\mathrm{Im}(r_\theta\zeta))}.$$

Using the estimates from the first case, and the fact that $|\mathrm{Re}(r_\theta\zeta)| \leq |\zeta|\sin(\theta_t) = \frac{|\zeta|e^{-2t}}{\sqrt{1+e^{-4t}}}$,

$$|\xi_\theta| \geq |\zeta|\sqrt{1 + \frac{|A|}{|\zeta|^4}\left(|A| - 2\frac{|\zeta|^2e^{-2t}}{\sqrt{1+e^{-4t}}}\right)} \geq |\zeta|\sqrt{1+e^{-4t}|A|\left(|A|-2\right)}$$

Note the estimate above requires t large enough so that $1 + e^{-4t}|A|(|A|-2) \geq 0$. Moreover since we are only concerned with $|A| \leq 1$, this lower bound satisfies

$$|\zeta|\sqrt{1+e^{-4t}|A|(|A|-2)} \leq |\zeta|.$$

Thus $c_A = 1$ if $|A| \geq 1$ and $|A|(2-|A|)$ if $|A| \leq 1$. □

We now work to understand the average of h_A over expanding circles.

Lemma 3.2.3. *For $t > 0$ and $(z, w) \in \mathbb{R}^2 \times \mathbb{R}^2$,*

$$\frac{1}{2\pi}\int_0^{2\pi} h_A(g_t r_\theta z, g_t r_\theta w)\, d\theta \leq e^{-2t}.$$

Proof. We note that $g_{-t}\mathcal{T}$ is contained in the sector $\{(r,\theta) : \theta \in [\frac{\pi}{2} - \theta_t, \frac{\pi}{2} + \theta_t]\}$ where $\tan(\theta_t) = e^{-2t}$. Thus for any z,

$$|\{\theta : r_\theta z \in g_{-t}\mathcal{T}\}| \leq 2\theta_t.$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} h_A(g_t r_\theta z, g_t r_\theta w)\, d\theta \leq \frac{\arctan(e^{-2t})}{\pi} \leq e^{-2t}.$$

□

This next lemma captures the fact that for a set close the the set we care about h_A captures the right information. The next section will account for the errors between Equation 3.2.2 and the Heuristic argument of section 3.1.2.

Lemma 3.2.4. *For $t > 0$ if*

$$\frac{e^t}{2}\sqrt{1+e^{-4t}} \leq |z| \leq e^t \text{ and } |w| \leq L_{t,z}^{\min}, \text{ with } |z \wedge w| \leq A \tag{3.2.2}$$

then

$$\frac{1}{2\pi}\int_0^{2\pi} h_A(g_t r_\theta(z,w))\, d\theta = \frac{\arctan(e^{-2t})}{\pi}.$$

Proof. By Lemma 3.2.3 we have one inequality. So to obtain equality we want to show

$$|\{\theta : r_\theta z \in g_{-t}\mathcal{T}\}| = 2\theta_t.$$

Note in the first component, since rotations preserve length, the set $K(z) \stackrel{def}{=} \{r_\theta z : \theta \in [0, 2\pi)\}$ will intersect $g_{-t}\mathcal{T}$ for the full angle $2\theta_t$.

We now consider the second component. For each θ so that $r_\theta z \in g_{-t}\mathcal{T}$, consider $r_\theta w$. Note $|z \wedge w| \leq A$ and moreover the condition on $|w| \leq L_{t,z}^{\min}$ guarantees that $r_\theta w \in \mathcal{R}_{A, r_\theta z}$. Thus $r_\theta(z,w) \in (g_{-t}\mathcal{T})_A^{\mathcal{R}} = g_{-t}\mathcal{T}_A^{\mathcal{R}}$. Hence the full angle $2\theta_t = \arctan(e^{-2t})\pi^{-1}$ is the resulting integral. □

3.3 Error terms

Our eventual goal will be to prove

Conjecture 3.3.1.

$$\left|\big(N_A(\omega, e^t) - N_A(\omega, e^t/2)\big) - \pi e^{2t}\frac{1}{2\pi}\int_0^{2\pi} \hat{h}_A(g_t r_\theta \omega)\, d\theta\right| = o(e^{2t}), \tag{3.3.1}$$

In particular,

$$\lim_{t\to\infty} \frac{(N_A(\omega, e^t) - N_A(\omega, e^t/2))}{\pi e^{2t}} = \lim_{t\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \hat{h}_A(g_t r_\theta \omega)\, d\theta.$$

Proof. To prove this lemma, we first rewrite the LHS of 3.3.1 as

$$\left|\sum_{j=0}^{4} \sum_{(z,w)\in M_t^j} \left[(\chi_{D_{e^t,A}}(1-\chi_{D_{e^t/2,A}}))(z,w) - \frac{e^{2t}}{2\pi}\int_0^{2\pi} h_A(g_t r_\theta z, g_t r_\theta w)\, d\theta\right]\right|, \tag{3.3.2}$$

where

$$M_t^0 = \{(z,w) \in \Lambda_\omega(e^t)^2 : (z,w) \text{ satisfy Equation (3.2.2)}\},$$

$$M_t^1 = \left\{(z,w) \in \Lambda_\omega(e^t)^2 : (z,w) \in D_{e^t,A} \setminus D_{e^t/2,A},\ \text{with } \frac{e^t}{2} \le |z| < \frac{e^t}{2}\sqrt{1+e^{-4t}}\right\},$$

$$M_t^2 =$$
$$\left\{(z,w) \in \Lambda_\omega(e^t)^2 : (z,w) \in D_{e^t,A} \setminus D_{e^t/2,A},\ \frac{e^t}{2}\sqrt{1+e^{-4t}} \le |z| \le e^t,\ L_{t,z}^{\min} < |w| \le |z|\right\},$$

$$M_t^3 = \{(z,w) \in \Lambda_\omega(e^t)^2 : \exists\, \theta \text{ with } (g_t r_\theta z, g_t r_\theta w) \in \mathcal{T}_A^{\mathcal{R}} \text{ and } |z| > e^t \text{ or } |z| < \frac{e^t}{2}\},$$

and

$$M_t^4 = \{(z,w) \in \Lambda_\omega(e^t)^2 : \exists\, \theta \text{ with } (g_t r_\theta z, g_t r_\theta w) \in \mathcal{T}_A^{\mathcal{R}} \text{ and } e^t/2 \le |z| \le e^t \text{ but } |w| > |z|\}.$$

The idea is that M_t^0 is the main set which should be large with $|M_t^0|$ on the order of e^{2t}, but in which the difference is strictly smaller than e^{2t}. All other sets are "small" in that the possible area in $\mathbb{R}^2 \times \mathbb{R}^2$ in which these points could live is $O(e^{-2t})$, and $|M_t^j|$ is strictly less than e^{2t}.

Claim 1: Equation (3.3.2) **is equal to the left hand side of Equation** (3.3.1)**.**

First define

$$f_t^+(z,w) = (\chi_{D_{e^t,A}}(1-\chi_{D_{e^t/2,A}}))(z,w), \quad f_t^-(z,w) = \frac{\pi e^{2t}}{2\pi}\int_0^{2\pi} h_A(g_t r_\theta z, g_t r_\theta w)\, d\theta.$$

Then notice

$$\left|\big(N_A(\omega,e^t) - N_A(\omega, e^t/2)\big) - \frac{\pi e^{2t}}{2\pi}\int_0^{2\pi} \hat{h}_A(g_t r_\theta \omega)\, d\theta\right| = \left|\sum_{(z,w)\in\Lambda_\omega(e^t)^2} f_t^+(z,w) - f_t^-(z,w)\right|.$$

For $(z,w) \in \Lambda_\omega(e^t)^2$, we claim $f_t^+(z,w) = f_t^-(z,w) = 0$ unless $(z,w) \in M_t^j$ for some $j = 0,1,2,3,4$.

Indeed by definition we must have $(z,w) \in D_{e^t,A} \setminus D_{e^t/2,A}$ in order for $f_t^+(z,w) > 0$. However M_t^0 and M_t^2 capture a large subset of the possible $|z|$ with restrictions on $|w|$. Then M_t^1 captures all other possible $|z|$ and all the corresponding possible w.

So now we only need to show that when $(z,w) \notin D_{e^t,A} \setminus D_{e^t/2,A}$, then $f_t^-(z,w) = 0$ unless $(z,w) \in M_t^3$ or $(z,w) \in M_t^4$. But as in the definition of M_t^3 and M_t^4 we can only have $f_t^-(z,w) > 0$ if there exists some θ with $(g_t r_\theta z, g_t r_\theta w) \in \mathcal{T}_A^{\mathcal{R}}$. Then the only possibilities since $(z,w) \notin D_{e^t,A} \setminus D_{e^t/2,A}$ is either $|z| > e^t$ or $|z| < e^t/2$ as in M_t^3, or as in M_t^4 we have $e^t/2 < |z| < e^t$ but $|w| > |z|$.

Lastly by definition it is quick to see that M_t^j are all pairwise disjoint for $j = 0,1,2,3,4$.

Considering M_t^0. By Lemma 3.2.4, and by the Taylor series expansion of $\arctan(x)$ we have the LHS of 3.3.1 when restricting to each $(z,w) \in M_t^0 \subset D_{e^t,A} \setminus D_{e^t/2,A}$ is

$$1 - \pi e^{2t} \cdot \frac{\arctan(e^{-2t})}{\pi} = 1 - 1 + e^{2t} \cdot O(e^{-6t}).$$

Thus using Proposition 3.4.1, $|M_t^0| \leq |D_{e^t,A} \setminus D_{e^t/2,A}| = N_A(\omega, e^t) - N_A(\omega, e^t/2) = O(e^{2t})$, so

$$\sum_{(z,w) \in M_t^0} \left| \left[(\chi_{D_{e^t,A}} - \chi_{D_{e^t/2,A}}))(z,w) - \frac{e^{2t}}{2\pi} \int_0^{2\pi} h_A(g_t r_\theta z, g_t r_\theta w)\, d\theta \right] \right| \leq |M_t^0| O(e^{-4t}) = o(e^{2t}). \tag{3.3.3}$$

Considering M_t^j for $j = 1,2,3,4$. We notice that

$$\lim_{t \to \infty} e^{-2t} \left| \sum_{j=1}^{4} \sum_{(z,w) \in M_t^j} f_t^+(z,w) - f_t^-(z,w) \right| \leq \lim_{t \to \infty} e^{-2t} \sum_{j=1}^{4} |M_t^j|,$$

since by Lemma 3.2.3 we have $0 < f_t^- \leq 1$, and the fact that $f_t^+(z,w) \in \{0,1\}$,

$$\max_{(z,w) \in M_t^j} |f_t^+(z,w) - f_t^-(z,w)| \leq 1.$$

Thus to conclude the proof of this Lemma, we require the following Lemma which in future work will be proved generalizing the proof in Section 3.4.

Lemma 3.3.2. *Let $j \in \{1, 2, 3, 4\}$. For all $\epsilon > 0$, there is a $T > 0$ so that for all $t > T$,*

$$|M_t^j| < \epsilon e^{2t}.$$

So for any $\epsilon > 0$, by Lemma 3.3.2 with $\frac{\epsilon}{5}$ and Equation (3.3.3), there is a $T > 0$ so that for $t > T$,

$$\left| \big(N_A(\omega, e^t) - N_A(\omega, e^t/2)\big) - e^{2t} \frac{1}{2\pi} \int_0^{2\pi} \hat{h}_A(g_t r_\theta \omega)\, d\theta \right| \leq \frac{\epsilon}{5} e^{2t} + \sum_{j=1}^{4} |M_t^j| \leq \epsilon e^{2t}.$$

Thus, we can conclude that for all $t > T$,

$$\left| \frac{(N_A(\omega, e^t) - N_A(\omega, e^t/2))}{\pi e^{2t}} - \frac{1}{2\pi} \int_0^{2\pi} \hat{h}_A(g_t r_\theta \omega)\, d\theta \right| < \epsilon.$$

Therefore, we have

$$\lim_{t \to \infty} \frac{(N_A(\omega, e^t) - N_A(\omega, e^t/2))}{\pi e^{2t}} = \lim_{t \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \hat{h}_A(g_t r_\theta \omega)\, d\theta.$$

and Theorem 3.3.1 is proved. □

3.4 Upper bounds

We want to show that in general we would expect at most quadratic upper bounds for pairs of saddle connections with bounded virtual area (Proposition 3.4.1). The techniques used in the proof will be extended in future work to prove Lemma 3.3.2.

Proposition 3.4.1. *Given $A > 0$, for a.e (X, ω) there exists C and $T > 0$ such that for all $t > T$,*

$$N_A(\omega, e^t) - N_A(\omega, e^t/2) \leq C e^{2t}.$$

3.4.1 Notation

For the remainder of this section we adopt the following notation. On a base surface (X, ω) we refer to a holonomy vector of a saddle connection z without subscripts. On the surface

$g_t r_\theta(X,\omega)$ the image holonomy vector $g_t r_\theta(z)$ will be denoted $z_{\theta,t}$. We denote θ_z to be angle so that $r_{\theta_z} z$ is vertical, that is $r_{\theta_z} = |z|i$.

We fix a stratum and let N be the maximum number of edges in any triangulation by saddle connections of any surface in the stratum. The number N only depends on the stratum. Choose

$$\delta < \frac{1}{2N}.$$

Fix $0 < \sigma < 1$.

Define the set

$$P_A(\omega, e^t) = \{(z,w) \in \Lambda_\omega(e^t) : e^t/2 \le |z|, |w| \le |z|, 0 < |z \wedge w| \le A\}.$$

So $|P_A(\omega,e^t)| = N_A(\omega,e^t) - N_A(\omega, e^t/2)$.

3.4.2 Collection of referenced theorems

In this section we state the theorems that we use from [Nev17], [MS91], and [ACM19].

Theorem 3.4.2. *For almost every $\omega \in \mathcal{H}$, for $\epsilon > 0$, and a smooth $\phi : \mathbb{R} \to \mathbb{R}$ so that*

$$\phi\big|_{\mathbb{R}\setminus[-(\log(2)+\epsilon),\epsilon]} = 0, \quad \phi\big|_{[-\log(2),0]} = 1, \quad \int \phi = \log(2) + \epsilon.$$

If $h = \chi_E$ for some set E in the stratum, then

$$\lim_{\tau\to\infty} \int_{-\infty}^{\infty} \phi(\tau - s)\left(\frac{1}{2\pi}\int_0^{2\pi} h(g_s r_\theta \omega)\, d\theta\right) ds = \int_{-\infty}^{\infty} \phi(s)\, ds \int_{\mathcal{H}} h\, \mu = (\log(2)+\epsilon)\mu(E).$$

Next we state a result originally due to Masur–Smillie [MS91], which we recall from [EMZ03].

Lemma 3.4.3. *For all $\epsilon, \kappa > 0$, the subset of $\mathcal{H}$ which have a saddle connection of lenght at most ϵ, and a non-homologous saddle connection with length at most κ is $O(\epsilon^2\kappa^2)$.*

Finally we state a counting lemma which is taken from sections 3.6.2 and 3.6.3 of [ACM19].

Lemma 3.4.4. *1. Fix a surface (X,ω) with shortest saddle connection v and second shortest saddle connection v'. Consider pairs of saddle connections $(z,w)\in\Lambda_\omega(\epsilon_0)^2$. If the shortest saddle connection v does not bound a cylinder or the shortest saddle connection v does bound a cylinder, but the cylinder has width at least ϵ_0, then the number of pairs saddle connections depends on the length of the second-shortest vector $|v'|$. Namely the number of pairs is*

$$O\left(\left(\frac{1}{|v'|^N}\right)^2\right) = O(|v'|^{-2N}).$$

2. Let $1>\sigma>0$, $j\in\mathbb{N}$, $\epsilon_0>0$. Define $H(\epsilon_0,\sigma^j)$ to be the set of surfaces where the shortest saddle connection has length in (σ^{j+1},σ^j) and bounds a cylinder of width at most ϵ_0, and the second shortest saddle connection is longer σ^{pj} for some $p>0$. Then

$$\mu(H(\epsilon_0,\sigma^j)) = O(\sigma^{3j}),$$

where the the implied constant depends on ϵ_0.

3.4.3 Counting bounds with Systoles

The following is a slight modification of Theorem 5.1 of Eskin Masur.

Lemma 3.4.5. *For any $L_0>0$ and $\delta>0$ there exists $C=C(\delta,L_0)$ such that for any $L<L_0$ and any surface (X,ω) in the stratum*

$$|\Lambda_{(X,\omega)}\cap B(0,L)| \le C\left(\frac{L}{\ell(X,\omega))}\right)^{1+\delta} \tag{3.4.1}$$

where $\Lambda_{(X,\omega)}$ is the set of holonomy vectors of saddle connections, $C=C(\delta,P)$ and $\ell(X,\omega)$ is the length of the shortest saddle connection of (X,ω).

Proof. Fix $L_0>0$, $\delta>0$, let $L<L_0$, and let (X,ω) be a surface in the stratum $\mathcal{H}$.

By Theorem 5.1 of [EM01] there is some $\kappa>0$ depending only on $\mathcal{H}$ and some constant $C(\delta,\mathcal{H})$ so that whenever $L<\kappa$,

$$|\Lambda_{(X,\omega)}\cap B(0,L)| \le C\left(\frac{L}{\ell(X,\omega)}\right)^{1+\delta}.$$

So we must now consider what happens if $L_0 > \kappa$. In this case, divide $[0, 2\pi)$ into intervals J_i of radius $\frac{\kappa^2}{4L_0^2}$ and center ϕ_i. Note that there are $O(L_0^2)$ such intervals.

Now for any $z \in \Lambda_{(X,\omega)} \cap B(0, L)$ choose ϕ_i with angle $\theta_z \in J_i$. We rotate z to almost vertical via $r_{-\phi_i}$, and then shrink $r_{-\phi_i}z$ to have length less than κ by applying $g_t r_{-\phi_i}$ where $e^t = \frac{2L_0}{\kappa}$. After rotating by $-\phi_i$, $r_{-\phi_i}z$ must lie in $B(0, L)$ with angle in $(-\frac{\kappa^2}{4L_0^2}, \frac{\kappa^2}{4L_0^2})$. Hence the largest possible imaginary component for $r_{-\phi_i}z$ is bounded above by $|z| < L < L_0$, so

$$\mathrm{Im}(g_t r_{-\phi_i} z) = \frac{\kappa}{2L_0} \mathrm{Im}(r_{-\phi_i} z) \leq \frac{\kappa}{2}.$$

Moreover the real component of $r_{-\phi_i}z$ must satisfy

$$\mathrm{Re}(g_t r_{-\phi_i} z) = \frac{2L_0}{\kappa} \mathrm{Re}(r_{-\phi_i} z) \leq \frac{2L_0}{\kappa} \mathrm{Im}(r_{-\phi_i} z) \tan\left(\frac{\kappa^2}{4L_0^2}\right) \leq \frac{2L_0^2}{\kappa}\frac{\kappa^2}{4L_0^2} = \frac{\kappa}{2},$$

where we used $\tan(x) < x$ for $0 < x < \frac{1}{4}$. Thus

$$|g_t r_{-\phi_i} z| \leq \kappa.$$

Moreover, for each i the systoles satisfy

$$\ell(X, \omega) \leq \ell(g_t r_{\phi_i}(X, \omega)) \cdot \frac{2L_0}{\kappa}.$$

So

$$|\Lambda_{(X,\omega)} \cap B(0, L)| = \sum_{i=1}^{O(L_0^2)} C \left(\frac{L}{\ell(g_t r_{-\phi_i}(X, \omega))}\right)^{1+\delta} \leq O(L_0^2) \left(\frac{2L_0}{\kappa}\right)^{1+\delta} C \left(\frac{L}{\ell((X, \omega))}\right)^{1+\delta}.$$

This completes the proof where we note the new constant C must depend on L_0 as well. ∎

3.4.4 Reducing to finitely many surfaces

This next Lemma we will use to only consider finitely many surfaces. Indeed the lemma shows that within a fixed range of angles for θ and times for t, $|g_t r_\theta z|$ cannot change too much.

Lemma 3.4.6. *Let $t > 0$ and consider an interval $I(\theta_0) = [\theta_0 - \frac{\pi}{e^{2t}}, \theta_0 + \frac{\pi}{e^{2t}})$. For any holonomy vector z of a saddle connection, and any $\theta \in I(\theta_0)$, and $e^t \leq e^s \leq 2e^t$ we have*

$$\frac{1}{\sqrt{8}\pi} \leq \frac{|g_s r_\theta z|}{|g_t r_{\theta_0} z|} \leq \sqrt{8}\pi.$$

Proof. The ratio achieves its maximum when $r_{\theta_0} z$ is vertical and $|\theta - \theta_0| = \frac{\pi}{e^{2t}}$. Without loss of generality, suppose $\theta = \theta_0 - \frac{\pi}{e^{2t}}$.

So we have $r_{\theta_0} z = |z|i$ and $|g_t r_{\theta_0} z| = e^{-t}|z|$. So we compute $r_\theta z = r_{-\frac{\pi}{e^{2t}}}|z|i$, and thus

$$|g_s r_\theta z| = |z|\sqrt{e^{2s}\sin^2\left(\frac{\pi}{e^{2t}}\right) + e^{-2s}\cos^2\left(\frac{\pi}{e^{2t}}\right)}.$$

Since $e^t \leq e^s \leq 2e^t$, $\sin(x) \leq x$ and $\cos(x) \leq 1$, we have

$$\frac{|g_s r_\theta z|}{|g_t r_{\theta_0} z|} \leq \sqrt{e^{2(t+s)}\left(\frac{\pi^2}{e^{4t}}\right) + e^{2(t-s)}} \leq \sqrt{4\pi^2 + 1} \leq \sqrt{8}\pi.$$

The argument for the minimum is similar.

□

3.4.5 Geodesics flow length bounds for pairs

In this section we collect bounds that hold under certain assumptions for pairs of holonomy vectors.

Lemma 3.4.7. *Given $(z, w) \in P_A(\omega, R)$, set θ_z to be the angle so that $r_{\theta_z} z$ is vertical. Then*

- $\frac{1}{2} \leq |g_t r_{\theta_z} z| \leq 1$,
- $|g_t r_{\theta_z} w| \leq |g_t r_{\theta_z} z| + 2A$,

Proof. Since $\frac{e^t}{2} \leq |z| \leq e^t$ the first conclusion is clear. Since $|w| \leq |z|$, the vertical component of $r_{\theta_z} w$ is at most $|z|$, so $\operatorname{Im}(g_t r_{\theta_z} w) \leq |g_t r_{\theta_z} z|$. Since $SL(2, \mathbb{R})$ preserves determinants, $|\operatorname{Re}(g_t r_{\theta_z} w)| \cdot |g_t r_{\theta_z} z| \leq A$. Hence

$$|g_t r_{\theta_z} w| \leq |\operatorname{Im}(g_t r_{\theta_z} w)| + |\operatorname{Re}(g_t r_{\theta_z} w)| \leq |g_t r_{\theta_z} z| + \frac{A}{|g_t r_{\theta_z} z|} \leq |g_t r_{\theta_z} z| + 2A.$$

□

3.4.6 Proof of Proposition 3.4.1

Proof. For each t so that $e^{2t} \in \mathbb{Z}$ partition $[0, 2\pi)$ into e^{2t} intervals $I(\theta_i)$ of radius $\frac{\pi}{e^{2t}}$ centered at points θ_i for $i = 1, \dots, e^{2t}$. If $e^{2t} \notin \mathbb{Z}$, take $\lfloor e^{2t} \rfloor$ intervals instead, but WLOG we will work with e^{2t}. We will do our counting on this finite set of surfaces $g_t r_{\theta_i}(X, \omega)$, and the Lemma 3.4.6 ensures error terms when computing lengths in a fixed interval change by at most a multiplicative constant which will be absorbed in our estimates.

Next for each i consider the holonomy vectors v_i, v_i' on (X, ω) so that $g_t r_{\theta_i} v_i$ and $g_t r_{\theta_i} v_i'$ are the shortest and second shortest saddle connections on $g_t r_{\theta_i}(X, \omega)$. Define $j = j(i)$ so that

$$\sigma^{j+1} \leq |g_t r_{\theta_i} v_i| \leq \sigma^j.$$

Now for each j we will break up the set of θ_i into two subsets depending on $|g_t r_{\theta_i} v_i'|$ the length of the second shortest saddle connection on the image surface $g_t r_{\theta_i}(X, \omega)$. In the first case, $|g_t r_{\theta_i} v_i'|$ will have a certain small prescribed upper bound, and we will use Nevo's ergodic theorem 3.4.2 to show that the set of θ_i is a small set. We then use Lemma 3.4.5 to count pairs of saddle connections for this small set of θ_i. In the second case, $|g_t r_{\theta_i} v_i'|$ will not satisfy the upper bound of the first case. Here we will count pairs using a bound in [ACM19] and then use some standard quadratic estimates to count the number of possible such θ_i.

Case I For each j define

$$N_j^I = \# \left\{ i \in \{1, \dots, e^{2t}\} : \sigma^{j+1} \leq |g_t r_{\theta_i} v_i| \leq \sigma^j, \text{ and } |g_t r_{\theta_i} v_i'| \leq \sigma^{\frac{j}{2N}} \right\}.$$

We first bound N_j^I using Theorem 3.4.2. Set $E\left(j, \frac{j}{2N}\right)$ to be the set of (Y, ω) with a saddle connection of length at most $\sqrt{8}\pi\sigma^j$, and a non homologous saddle connection of length at most $\sqrt{8}\pi\sigma^{\frac{j}{2N}}$. Similarly let $F\left(j, \frac{j}{2N}\right)$ be the set where there is a saddle connection of length at most σ^j and non homologous saddle connection of length at most $\sigma^{\frac{j}{2N}}$.

By Lemma 3.4.3,

$$\mu\left(E\left(j, \frac{j}{2N}\right)\right) = O\left(\sigma^{2j + 2\frac{j}{2N}}\right) = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

By setting h to be the characteristic function of $E\left(j, \frac{j}{2N}\right)$, then by Theorem 3.4.2, there is some τ_0 large enough so that for $t > \tau_0$,

$$\int_{-\infty}^{\infty} \phi(t-s)\left(\frac{1}{2\pi} h(g_s r_\theta \omega)\, d\theta\right) ds \le 2 \int h\, d\mu = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

Bounding the integral of ϕ below we have $\phi = 1$ for$-\log(2) \le t - s \le 0$. Hence we get switching the order of integration

$$\sum_{i=1}^{e^{2t}} \int_{I(\theta_i)} \int_t^{t+\log(2)} h(g_s r_\theta \omega)\, ds\, d\theta = \int_t^{t+\log(2)} \left(\int_0^{2\pi} h(g_s r_\theta \omega) d\theta\right) ds = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right). \quad (3.4.2)$$

Now if there is some $i \in \{1, \dots, e^{2t}\}$ so that $g_t r_{\theta_i}(X, \omega) \in F\left(j, \frac{j}{2N}\right)$, so i contributes to N_j^I, then Lemma 3.4.6 guarantees that $g_s r_\theta \omega \in E(j, \frac{j}{2N})$ whenever

$$(\theta, s) \in I(\theta_i) \times [t, t + \log(2)].$$

Since for each i contributing to N_j^I we have a full Annulus of integration, Equation (3.4.2) gives an upper bound

$$N_j^I \frac{2\pi}{e^{2t}} \cdot [t + \log(2) - t] = O\left(\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

Hence

$$N_j^I = O\left(e^{2t}\sigma^{2j}\sigma^{\frac{j}{N}}\right).$$

Now suppose we are given $(z, w) \in P_A(\omega, e^t)$. Choose i so that $\theta_z \in I(\theta_i)$. Lemma 3.4.7 and Lemma 3.4.6 show that $|g_t r_{\theta_i} z| \le \sqrt{8}\pi$ and $|g_t r_{\theta_i} w| \le \sqrt{8}\pi(1 + 2A)$. We will now apply Lemma 3.4.5 with $L_0 = \sqrt{8}\pi(1 + 2A)$.

Thus for each j and i, if $g_t r_{\theta_i}(X, \omega) \in F\left(j, \frac{j}{2N}\right)$, then

$$\begin{aligned}\#\left\{(z, w) \in P_A(\omega, e^t) : \theta_z \in I(\theta_i)\right\} &\le |\Lambda_{g_t r_{\theta_i}(X,\omega)} \cap B(0, L_0)|^2 \\ &= O\left(\left(\frac{1}{|g_t r_{\theta_i} v_i|}\right)^{2(1+\delta)}\right) = O\left(\sigma^{-j(2+2\delta)}\right).\end{aligned}$$

The number of possible i is given by N_j^I, so for t large enough and any j

$$\begin{aligned}&\#\left\{(z, w) \in P_A(\omega, e^t) : \exists\, i \text{ so that } \theta_z \in I(\theta_i), g_t r_{\theta_i}(X, \omega) \in F\left(j, \frac{j}{2N}\right)\right\} \\ &\le N_j O\left(\sigma^{-j(2+2\delta)}\right) = O\left(e^{2t}\sigma^{j(\frac{1}{N} - 2\delta)}\right).\end{aligned}$$

Summing over all possible j, since $\frac{1}{N} - 2\delta > 0$ and $\sigma < 1$,

$$\#\left\{(z,w) \in P_A(\omega, e^t) : \exists i, \text{ and } j(i) \text{ so that } \theta_z \in I(\theta_i),\ g_t r_{\theta_i}(X,\omega) \in F\left(j, \frac{j}{2N}\right)\right\}$$
$$= O\left(e^{2t}\sum_{j=1}^{\infty} \sigma^{j(\frac{1}{N}-2\delta)}\right) = O(e^{2t}). \tag{3.4.3}$$

Case II Now we consider the case where the second-shortest vector is bigger than $\sigma^{\frac{j}{2N}}$. For each j define

$$N_j^{II} = \#\{i \in \{1,\ldots,e^{2t}\} : \sigma^{j+1} \leq |g_t r_{\theta_i} v_i| \leq \sigma^j, \text{ and } |g_t r_{\theta_i} v_i'| > \sigma^{\frac{j}{2N}}\}.$$

If i contributes to N_j^{II}, then we first get an estimate on N_j^{II} by deducing the fact that $r_{\theta_i} v_i$ must be close to vertical. Specifically $|r_{\theta_i} v_i| = |v_i|$ satsifies

$$\frac{|r_{\theta_i} v_i|}{e^t |g_t r_{\theta_i} v_i|} e^t \sigma^{j+1} \leq |v_i| \leq \frac{|r_{\theta_i} v_i|}{e^t |g_t r_{\theta_i} v_i|} e^t \sigma^j.$$

Notice $\frac{|r_{\theta_i} v_i|}{e^t |g_t r_{\theta_i} v_i|} \leq 1$, so we can choose $k_i \geq j$ so that

$$\sigma^{k_i+1} e^t \leq |v_i| \leq \sigma^{k_i} e^t.$$

Notice the horizontal component $(r_{\theta_i} v_i)_1$ satisfies

$$e^t (r_{\theta_i} v_i)_1 \leq \sqrt{e^{2t}(r_{\theta_i} v_i)_1^2 + e^{-2t}(r_{\theta_i} v_i)_2^2} = |g_t r_{\theta_i} v_i| \leq \sigma^j.$$

So if ϕ_i is the angle which makes $r_{\theta_i} v_i$ vertical, then

$$|\sin|\phi_i - \theta_i|| = \frac{|(r_{\theta_i} v_i)_1|}{|r_{\theta_i} v_i|} \leq \frac{\sigma^j}{e^t} \cdot \frac{1}{\sigma^{k_i+1} e^t} = e^{-2t} \sigma^{j-k_i-1}. \tag{3.4.4}$$

By symmetry,

$$|\{\theta \in [0,2\pi) : \sin|\phi_i - \theta| \leq e^{-2t}\sigma^{j-k_i-1}\}| = 4|\{\theta \in [\phi_i, \pi/2+\phi_i) : \sin|\phi_i - \theta| \leq e^{-2t}\sigma^{j-k_i-1}\}|.$$

Then when $|\phi_i - \theta| \leq \frac{\pi}{2}$ we have $\sin|\phi_i - \theta| > \frac{2}{3}|\phi_i - \theta|$. Since the θ_i are spread evenly over $[0, 2\pi)$, we can estimate

$$\begin{aligned}
&\#\{i \in \{1, \ldots, e^{2t}\} : \sigma^{k_i+1}e^t \\
&\leq |v_i| \leq \sigma^{k_i}e^t, |g_t r_{\theta_i} v_i'| > \sigma^{\frac{j}{2N}}\} \qquad = e^{2t}\frac{|\{\theta \in [0, 2\pi) : \sin|\phi_i - \theta| \leq e^{-2t}\sigma^{j-k_i-1}\}|}{2\pi} \\
&\leq \frac{4e^{2t}}{2\pi}\left|\{\theta : |\phi_i - \theta| \leq \frac{3}{2}e^{-2t}\sigma^{j-k_i-1}\}\right| \\
&= O(e^{2t} \cdot e^{-2t}\sigma^{j-k_i}) = O(\sigma^{j-k_i}).
\end{aligned}$$

Next, we want a bound on the number of v_i with this specific k_i. Using quadratic growth from [Mas90], the number of possible v_i with length at most $\sigma^{k_i}e^t$ is $O(\sigma^{2k_i}e^{2t})$. Since $k_i \geq j$, we have $\sigma^{k_i} \leq \sigma^j$, so

$$M_j \leq O(\sigma^{2k_i}e^{2t}) \cdot O(\sigma^{j-k_i}) \leq O(\sigma^{2j}e^{2t}).$$

Now for each i in N_j^{II} we want to count the number of pairs of saddle connections of length at most $\sqrt{8}(1 + 2A)\pi$. Indeed this is sufficient to understand pairs $(z, w) \in P_A(z, w)$ by Lemma 3.4.7 and Lemma 3.4.6.

Set $\epsilon_0 = \sqrt{8}(1 + 2A)\pi$ to match notation of Lemma 3.4.4. In the first case, the number of pairs is bounded by the length of the second shortest saddle connection. In our case the second shortest saddle connection satisfies $|g_t r_{\theta_i} v_i'| > \sigma^{\frac{j}{2N}}$, so

$$\#\{(z, w) \in \Lambda_\omega(\epsilon_0)\} = O\left(\left(\sigma^{\frac{j}{2N}}\right)^{-2N}\right) = O(\sigma^{-j}).$$

Hence for each j the number of pairs of length at most $\sqrt{8}(1 + 2A)\pi$ is $N_j^{II} \cdot \#\{(z, w) \in \Lambda_\omega(\epsilon_0)\} = O(\sigma^j e^{2t})$.

Set $G\left(j, \frac{j}{2N}\right)$ to be be the set where there is a saddle connection of length at most σ^j and non homologous saddle connection of length at least $\sigma^{\frac{j}{2N}}$, and the shortest vector does not bound a cylinder or bounds a cylinder of width at least $\sqrt{8}(1 + 2A)\pi$. Then we have

$$\begin{aligned}
&\#\left\{(z, w) \in P_A(\omega, e^t) : \exists i, j(i) \text{ so that } \theta_z \in I(\theta_i), g_t r_{\theta_i}(X, \omega) \in G\left(j, \frac{j}{2N}\right)\right\} \\
&\leq O\left(\sum_{j=0}^{\infty} O(\sigma^j e^{2t})\right) = O(e^{2t}).
\end{aligned}$$

To finish the proof we need to cover the last case from Lemma 3.4.4. In this case we don't have a count for the number of pairs, but we have a bound on the measure. Namely,

$$\mu(H(\epsilon_0, \sigma^j)) = O\left(\sigma^{3j}\right) = O\left(\sigma^{2j+\frac{j}{N}}\right).$$

In this case we can use the estimates from Case I. Namely the set $H_j = H(\epsilon_0, \sigma^j)$ is the set of surfaces with shortest saddle connection at most σ^j that bounds a cylinder with distance across at most $\sqrt{8}(1+2A)\pi$, and the length of the second shortest saddle connection is at least σ^{pj} where $p = \frac{1}{2N}$. Then $\mu(H_j) = O(\sigma^{2j+\frac{j}{N}})$, and we follow the exact method as in Case I to get a bound of $O(e^{2t})$ on the number of pairs in this case.

This concludes the proof of the theorem, where we note T in the statement must be large enough to apply Theorem 3.4.5 in both Case I and Case II. Moreover when applying Lemma 3.4.5 and Lemma 3.4.4, the constant C depends on A. □

Chapter 4

DECAY RATE FOR THE GAPS IN ANGLES BETWEEN SADDLE CONNECTIONS

This chapter is joint work with Jonathan Chaika.

4.1 *Problem Statement and Conjectured Theorem Statement*

Let $\Lambda_\omega(R) \subseteq \mathbb{C}$ be the set of holonomy vectors of saddle connections on a translation surface ω which have length at most R. Masur ([Mas90]) showed that $|\Lambda_\omega(R)|$ grew quadratically like R^2.

Let $\Theta_\omega(R) \subseteq [-\pi, \pi)$ be the set of directions of holonomy vectors in $\Lambda_\omega(R)$. If the angles of the holonomy vectors were all equally distributed throughout $[-\pi, \pi)$, then all of the gaps between points in $\Theta_\omega(R)$ would be on the order of $\frac{1}{R^2}$. Indeed [AC12] showed that for almost every surface the smallest gap in $\Theta_\omega(R)$ shrinks faster than $\frac{1}{R^2}$. Specifically for almost every surface,

$$\lim_{R\to\infty} R^2 \min_{\theta_i,\theta_j\in\Theta_\omega(R)} |\theta_i - \theta_j| = 0.$$

We want to better understand the decay rate of the smallest gap for generic surfaces, which would give us information about how the holonomy vectors are distributed in the plane. In order to understand the *smallest gap*, we will first work to understand the smallest gap *about a point.* Specifically in this chapter we will study the smallest gap in directions of holonomy vectors about zero. In other words define the smallest gap about zero out to radius R by

$$\zeta_\omega(R) = \min\{\phi \in \Theta_\omega(R) : \phi \geq 0\} - \max\{\phi \in \Theta_\omega(R) : \phi \leq 0\}.$$

We want to understand the rate at which $\zeta_\omega(R)$ shrinks. To that end, we want to create a $0-1$ law which gives precise conditions on the shrinking rate of $\zeta_\omega(R)$.

Conjecture 4.1.1. *Fix $\mathcal{H}$ a connected component of the stratum with probability measure μ, the Masur–Veech measure. Let $\psi : [1,\infty) \to [1,\infty)$ be a nondecreasing continuous function.*

- *If $\int_1^\infty \frac{1}{\psi(t)^2}\,dt < \infty$, then for almost every ω (with respect to Masur–Veech measure)*

$$\liminf_{R\to\infty} \psi(R)R^2\zeta_\omega(R) = \infty.$$

- *If $\int_1^\infty \frac{1}{\psi(t)^2}\,dt = \infty$, then for almost every ω,*

$$\liminf_{R\to\infty} \psi(R)R^2\zeta_\omega(R) = 0.$$

4.1.1 Outline

In Section 4.2, we will state Theorem 4.2.2 which gives a partial result towards Conjecture 4.1.1. A key component of Theorem 4.2.2 is appplying a generalized version of the Borel–Cantelli lemma. In 4.3 we will state the version of the Borel–Cantelli Lemma that we will use, and provide sets which satsify the assumptions. In Section 4.4 we will prove the last remaining assumption to apply the Borel-Cantelli generalization. The proof in 4.4 depends on Corollary 4.5.9 which we will prove in Section 4.5. Finally in Section 4.6 we provide a proof of the Borel–Cantelli generalization.

4.2 Theorem statement and connection to Conjecture

We have a strong indication that Conjecture 4.1.1 should be true. We will first define the sets in order to state the main theorem, which captures a $0-1$ law with limsup sets. The main theorem has one very technical assumption that would need modifying in order to achieve the conjecture. After stating the main theorem, we will give a heuristic argument for connecting the main theorem to the conjecture.

4.2.1 Defintions and Main Theorem Statement

Fix a stratum $\mathcal{H}$, with complex dimension $2g+s-1$ with s the number of distinct singularities. For $\omega \in \mathcal{H}$ there exists an injectivity radius r so that we can write $\omega = (x_1, x_2, x_3) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{2g+s-3}$. Let ℓ_0 be as in Corollary 4.5.9, for $0 < \delta < 1$, and $I \stackrel{def}{=} (-\frac{\pi}{12}, \frac{\pi}{12})$. Define $b = e^{\ell_0}$.

Definition 4.2.1 (Definition of the A's). *Fix $0 < \sigma < 1$. Let $\omega_0 = (x_1^0, x_2^0, x_3^0)$ have two horizontal holonomy vectors of length 1, and let r be the injectivity radius. Choose $0 < c < 1$ so that the trapezoids $T^{\pm}_{c,\sigma,j}$ are contained inside the ball of radius r in $\mathbb{C}^{2g+s-1}$, where $T^{\pm}_{c,\sigma,j}$ are the trapezoids with corners $(c, 0), (1, 0), (1, \frac{\pm\sigma}{\psi(2^j)}), (c, c\frac{\pm\sigma}{\psi(2^j)})$. Define the set in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^{2g+s-3}$ by*

$$H_{c,\sigma,j} = T^{+}_{c,\sigma,j} \times T^{-}_{c,\sigma,j} \times B.$$

where $B = B_{\mathbb{C}^{2g+s-3}}(x_3^0, r)$.

Finally define for $k \in \mathbb{N}$,

$$A_k = g_{\log(b^k)} H_{c,\sigma,k}.$$

Where

$$g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Theorem 4.2.2. *Let* $\psi : [1, \infty) \to \mathbb{R}_{\geq 0}$ *be a nondecreasing continuous function. For a connected component* $\mathcal{H}$ *of a stratum, with* μ *Masur–Veech measure,*

- *If* $\sum_{i=1}^{\infty} \frac{1}{\psi(2^i)^2}\, dt < \infty$*, then* $\mu(\limsup A_k) = 0$.
- *If* $\sum_{i=1}^{\infty} \frac{1}{\psi(2^i)^2}\, dt = \infty$*, so that*

$$\forall i\, \forall j \text{ so that } j > i + \frac{4}{\delta} \log\left(\frac{1}{\mu(A_i)}\right), \text{ we in fact have } e^{-\frac{\delta}{4}|i-j|} \leq \frac{\sigma^4}{7^4} \frac{1}{\psi(2^j)^4} \quad (4.2.1)$$

  then $\mu(\limsup A_k) = 1$.

The technical assumption of Equation (4.2.1) says that for each i, we want the tail end to not be too small. The intuition is that the divergence of the series $\frac{1}{\psi(2^j)}$, the fact that $e^{-\frac{\delta}{4}|i-j|}$ is summable for each fixed i, and monotonicity of ψ should allow for a large family of ψ where Theorem 4.2.2 is true. The key to this proof will be discussed in Section 4.3 where we state and verify assumptions of a generalization of the Borel–Cantelli Lemma.

Proof. By Lemma 4.3.5, and the first part of Proposition 4.3.1, the first part is true.

For the second direction, applying Proposition 4.3.1 to the sets with verified properties in Section 4.3.4, we know $\mu(\limsup_{j\to\infty} A_j) > 0$.

We claim that $\limsup_{j\to\infty} A_j$ is invariant under $g_{\log(b^{-t})}$ for $t > 0$. Suppose

$$x \in g_{\log(b^{-t})} \limsup_{j\to\infty} g_{\log(b^j)} H_{c,\sigma,j},$$

and let $n \in \mathbb{N}$. So given $n + t$, there exists some $j \geq n + t$ so that

$$x \in g_{\log(b^{-t+j})} H_{c,\sigma,j} \subseteq g_{\log(b^{-t+j})} H_{c,\sigma,-t+j}$$

So for each n we have $j - t \geq n$ so that $x \in g_{\log(b^{-t+j})} H_{c,\sigma,-t+j}$. Hence $x \in \limsup_{j\to\infty} g_j H_j$. Since $\limsup A_j$ is a positive measure set invariant under $g_{-\log(b^t)}$, by ergodicity we conclude $\mu(\limsup_{j\to\infty} A_j) = 1$. □

4.2.2 Heuristic of connection between Conjecture 4.1.1 and Theorem 4.2.2.

To obtain an idea for Conjecture 4.1.1, suppose $\zeta_\omega(R) \leq \frac{\sigma}{f(R)}$ for some constant $\sigma > 0$. This means that there are at least two holonomy vectors in the circular wedge where $0 \leq |z| \leq R$ and $\arg(z) \in (-\frac{\sigma}{f(R)}, \frac{\sigma}{f(R)})$. Since $\zeta_\omega(R) \to 0$ we want $f(R) \to \infty$ to get asymptotic information. Recall for small angles $\tan(\phi) \sim \phi$. And thus we can approximate our wedge by the triangle with vertices $(0,0)$, $(R, \frac{R\sigma}{f(R)})$. So up to this triangle-to-wedge approximation, if we can show that almost every surface has two holonomy vectors in the triangle infinitely often, then for every $m \in \mathbb{N}$ there exists $\rho > m$ so that $\zeta_\omega(\rho) \leq \frac{\sigma}{f(R)}$. And thus $\limsup_{R\to\infty} f(R)\zeta_\omega(\rho) \leq \sigma$. Letting $\sigma \to 0$ we conclude $\limsup_{R\to\infty} f(R)\zeta_\omega(R) = 0$. In order for the triangle to be a good approximation of the wedge,the height of the triangle needs to be small relative to R. In particular it is known from [AC12] that $f(R) = R^2$ is necessary. To get finer results we set $f(R) = R^2\psi(R)$ where $\psi(R)$ is some function non-decreasing in R. In summary we've given a heuristic argument for why small gaps about zero corresponds to having two holonomy vectors in a triangle.

Now that we've made a connection between gaps and trapezoids used to define the A_k sets, notice that the gap about zero would have width at most the angle of A_k. So if $\omega_0 \in \limsup A_k$ would give a subsequence R_k where

$$\zeta_{\omega_0}(R_k) \leq \frac{\sigma}{R_k^2 \psi(R_k)}.$$

Since the $\liminf$ is the smallest possible cluster point of a sequence, we conclude

$$\liminf_{R\to\infty} R^2\psi(R)\zeta_{\omega_0}(R) \leq \sigma.$$

Then taking $\sigma \to 0$ we obtain the second part of Conjecture 4.1.1, and the first part is similar in strategy.

4.3 Properties of Borel–Cantelli Sets

In this section, we will define sets which give us geometric information about gaps, and show they have the properties desired to apply Proposition 4.3.1.

4.3.1 Statement of Borel–Cantelli Generalization

The following captures the Borel-Cantelli lemma, and a version a converse statement.

Proposition 4.3.1 (Exponential Decay Borel-Cantelli)**.** *Let $C \geq 1$, $0 < \delta < 1$, and $(A_k)_{k=1}^{\infty}$ be measurable sets.*

If $\sum_{k=1}^{\infty} \mu(A_k) < \infty$, then $\mu\left(\cap_{N=1}^{\infty} \cup_{i=N}^{\infty} A_i\right) = \mu(\limsup A_i) = 0$.

Conversely, suppose the following hold:

1. $\sum_{k=1}^{\infty} \mu(A_k) = \infty$
2. *For all $i \leq j$, $\mu(A_i) \geq \mu(A_j)$.*
3. *For all i, for all j so that $j > i + C\log(\frac{1}{\mu(A_i)})$ we have*

$$\mu(A_i \cap A_j) \leq C\mu(A_i)[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}]$$

4. *For all i, for all j so that $i < j \leq i + C\log(\frac{1}{\mu(A_i)})$ we have that there exists measurable sets B_i, C_j so that*
   - *(a)* $B_i \subset A_i$ *and* $A_j \subset C_j$
   - *(b)* $\mu(B_i) > \frac{1}{C}\mu(A_i)$
   - *(c)* $\mu(C_j) < C\mu(A_j)^{\frac{1}{2}}$
   - *(d)* $\mu(B_i \cap C_j) < C\mu(B_i)\left(2^{-(j-i)(1-\delta)} + \mu(C_j)^{\frac{1+\delta}{2}}\right)$.

Then $\cap_{N=1}^{\infty} \cup_{i=N}^{\infty} A_i = \limsup A_i$ has positive measure.

4.3.2 Definitions

Definition 4.3.2 (Definition of the B's)**.** *For $k \in \mathbb{N}$ define*

$$B_k = g_{\log(b^k)} g_{\log\left(\sqrt{\frac{\psi(2^k)}{\sigma}}\right)} \bigcup_{\theta \in I} r_\theta W_k$$

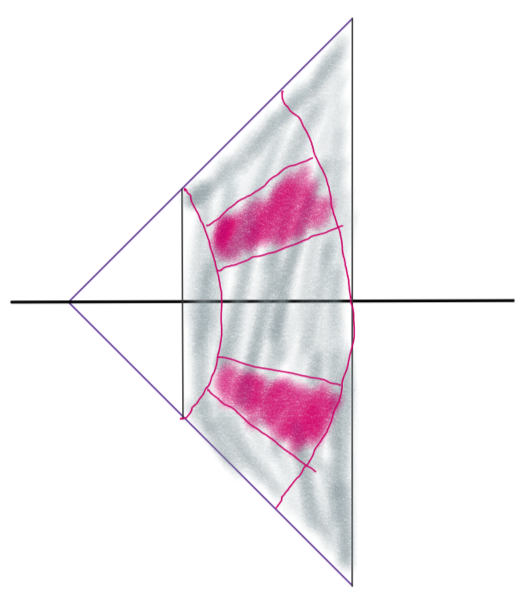
where we have the following definitions. We first pull back the set A_k so that trapezoids in $H_{c,\sigma,k}$ makes a 45 degree angle so

$$\widetilde{W_k} = g_{log\left(\sqrt{\frac{\sigma}{\psi(2^k)}}\right)} g_{-log(b^k)} A_k.$$

Then we restrict to a smaller subset of $\widetilde{W_k}$ which is rotation invariant up to a certain set of angles, $I \stackrel{def}{=} (-\frac{\pi}{12}, \frac{\pi}{12})$, which involves restricting angles as well as lengths. That is W_k is the set of ω with two holonomy vectors v_1 and v_2 satisfying

$$c\sqrt{\frac{2\sigma}{\psi(2^k)}} \leq |v_1|, |v_2| \leq \sqrt{\frac{\sigma}{\psi(2^k)}} \quad \arg(v_1) \in \left(\frac{\pi}{12}, \frac{\pi}{6}\right) \text{ and } \arg(v_2) \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right).$$

In the graphic below, the gray shaded region corresponds to the region $\tilde{W}_k$ and the pink region corresponds to W_k.



Definition 4.3.3 (Definition of the C's)**.** *Define*

$$C_k = g_{\log(b^k)} g_{\log\left(\sqrt{\frac{\psi(2^k)}{\sigma}}\right)} S_1(c, \sigma, 2^k)$$

where

$$S_j(c, \sigma, t) = \left\{ \omega : \omega \text{ has } j \text{ h.v. length in } \left(c\sqrt{\frac{\sigma}{\psi(t)}}, \sqrt{\frac{2\sigma}{\psi(t)}} \right) \right\}.$$

4.3.3 Measure bounds

In this section we will state two measure bounds for the measure of the set $H_{c,\sigma,j}$. We will make use of the following lemma due to [MS91] , which we quote from [AC12].

Lemma 4.3.4 (H.Masur, J. Smillie)**.** *There is a constant M such that for all $\epsilon, \kappa > 0$, the subset of $\mathcal{H}$ consisting of those flat surfaces, which have saddle connection of length at most ϵ, has volume at most $M\epsilon^2$. The volume of the set of flat surfaces with a saddle connection of length at most ϵ and a nonhomologous saddle connection with length at most κ is at most $M\epsilon^2\kappa^2$.*

Lemma 4.3.5. *Given a connected component of the stratum $\mathcal{H}$ containing ω_0, there is a constant $m = m(\alpha, r)$ where r is an injectivity radius for ω_0 so that*

$$\mu(H_{c,\sigma,j}) = \frac{m\sigma^2}{\psi(2^j)^2}.$$

Proof. Since the injectivity radius of ω_0 is r and c is chosen small enough so that $H_{c,\sigma,j}$ is within the injectivity radius, the measure μ on $\mathcal{H}$ is locally lebesgue. Hence if m_j is Lebesgue measure on $\mathbb{C}^j$, by symmetry of $T^{\pm}_{c,\sigma,j}$,

$$\mu(H_{c,\sigma,j}) = m_1(T^+_{c,\sigma,j})^2 m_{2g+s-3}(B).$$

Note that

$$m_1(T^+_{c,\sigma,j}) = \frac{1}{2}\left(\frac{\sigma}{\psi(2^j)} + c\frac{\sigma}{\psi(2^j)}\right)(1-c) = \frac{\sigma}{\psi(2^j)}\frac{(1-c^2)}{2}.$$

Hence using the fact that $c = c(r)$, $m = \frac{1-c^2}{2} m_{2g+s-3}(B)$ gives desired measure. ☐

4.3.4 Properties

In this section we verify that the sets satisfy the assumptions of Proposition 4.3.1 under the assumption that $\psi : [1, \infty) \to \mathbb{R}$ is nondecreasing, satisfies Equation (4.2.1) and

$$\sum_{k=1}^{\infty} \frac{1}{\psi(2^k)^2} = \infty,$$

Notice the assumption of Equation 4.2.1 is a techinical assumption that is necessary only for Lemma 4.3.6.

1. We claim $\sum_{k=1}^{\infty} \mu(A_k) = \infty$.

   Proof. By the measure bounds of Lemma 4.3.5 and the fact that g_t preserves measure,

   $$\sum_{k=1}^{\infty} \mu(A_k) \geq \sum_{k=1}^{\infty} m \frac{\sigma^2}{\psi(2^k)^2} = \infty.$$

   □

2. We claim for all $i \leq j$, $\mu(A_i) \geq \mu(A_j)$.

   Proof. By Lemma 4.3.5, $i \leq j$ implies $\psi(2^i) \leq \psi(2^j)$, so

   $$\mu(A_i) = \frac{m\sigma^2}{\psi(2^i)^2} \geq \frac{m\sigma^2}{\psi(2^j)^2} = \mu(A_j).$$

   □

3. Exponential Decay for far away pairs.

   Lemma 4.3.6. *Fix $0 < \delta < 1$. There exists a constant C so that for all $j > m_i$ where $m_i = i + \frac{4}{\delta} \log \left(\frac{1}{\mu(A_i)} \right)$,*

   $$\mu(A_i \cap A_j) \leq C\mu(A_i) \left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|} \right].$$

   This will be proved in Section 4.4.

4. Fix i. Now let's consider j so that $i < j \leq i + C \log\left(\frac{1}{\mu(A_i)}\right)$.

   (a) As constructed $B_i \subseteq A_i$ and $A_j \subseteq C_j$.

   (b) Following the strategy of Lemma 4.3.5 where we compute the area of the sectors instead of trapezoids,

$$\mu(W_i)\frac{\sigma^2}{\psi(2^i)^2}\left(\frac{\pi}{24}(1-2c)\right)^2 m_{2g+s-3}(B).$$

   Thus by Lemma 4.3.5,

$$\mu(B_i) \geq \mu(W_i) = \mu(A_i)\frac{1}{m}\left(\frac{\pi}{24}(1-2c)\right)^2 m_{2g+s-3}(B).$$

   (c) To see this inequality holds, since the measure is invariant under geodesic flow and by Lemma 4.3.5,

$$\mu(A_j) = \mu(H_{c,\sigma,j}) = m\frac{\sigma^2}{\psi(2^j)^2}.$$

   By Masur–Smillie Lemma 4.3.4,

$$\mu(C_j) = \mu(S_1(c, \sigma, 2^j)) \leq M\frac{\sigma}{\psi(2^j)}.$$

   Thus

$$\mu(C_j) \leq \frac{M}{\sqrt{m}}\sqrt{\frac{m\sigma^2}{\psi(2^j)^2}} = \frac{M}{\sqrt{m}}\mu(A_j)^{\frac{1}{2}}.$$

   (d) To obtain an upper bound for $\mu(B_i \cap C_j)$, by g_t-invariance of μ, it suffices to find an upper bound for $\mu(\tilde{B}_i \cap \tilde{C}_j)$ where $\tilde{B}_i = \bigcup_{\theta \in I} r_\theta W_i = I \cdot W_i$ and $\tilde{C}_j = g_{f(i,j)} S_1(c, \sigma, 2^j)$ where

$$f(i,j) = \log\left(b^{j-i}\sqrt{\frac{\psi(2^j)}{\psi(2^i)}}\right).$$

   We first use the fact that in $S_1(c, \sigma, 2^j)$, the shortest possible saddle connection has length $\ell(\omega) \in \left(c\sqrt{\frac{\sigma}{\psi(2^j)}}, \sqrt{\frac{2\sigma}{\psi(2^j)}}\right)$. Choose τ large enough to satisfy the assumption of Corollary 4.5.9. By (4.5.2),

$$C_{\delta,\tau}^{-1}\left(\frac{\psi(2^j)}{2\sigma}\right)^{\frac{1+\delta}{2}} \leq V_\delta^{(\tau)}(\omega) \leq \frac{C_{\delta,\tau}}{c}\left(\frac{\psi(2^j)}{\sigma}\right)^{\frac{1+\delta}{2}}.$$

Thus

$$
\begin{aligned}
&\mu(\tilde{B}_i \cap \tilde{C}_j) \\
&\leq \mu\left(\left\{\omega \in \tilde{B}_i : V_\delta^{(\tau)}(g_{-f(i,j)}\omega) \geq C_{\delta,\tau}^{-1}\left(\frac{\psi(2^j)}{2\sigma}\right)^{\frac{1+\delta}{2}}\right\}\right) \\
&\leq C_{\delta,\tau}\left(\frac{\psi(2^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \int_{I\cdot W_i} V_\delta^{(\tau)}(g_{-f(i,j)}\omega)\, d\mu(\omega) \qquad \text{(by Markov's inequality)} \\
&\leq C_{\delta,\tau}\left(\frac{\psi(2^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \int_{SO(2)\cdot W_i/SO(2)} \int_I V_\delta^{(\tau)}(g_{-f(i,j)} r_\theta \tilde{\omega})\, d\theta\, d\tilde{\mu}(\tilde{\omega})
\end{aligned}
$$

(Disintegrating the measure $\mu = d\theta\, d\tilde{\mu}$ on $\mathcal{H}/SO(2) \times SO(2)$ and increasing to a full $SO(2)$ orbit)

$$
\leq C_{\delta,\tau}\left(\frac{\psi(2^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \int_{SO(2)\cdot W_i/SO(2)} ce^{-(1-\delta)(f(i,j))} V_\delta^{(\tau)}(\tilde{\omega}) + b_\tau |I|\, d\tilde{\mu}(\tilde{\omega})
$$

(By Corollary 4.5.9 and monotonicity of ψ, $f(i,j) \geq \log(b^{j-i}) = \ell_0(j-i) > \ell_0$.)

$$
= C_{\delta,\tau}\left(\frac{\psi(2^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \left(ce^{-(1-\delta)(f(i,j))} \int_{W_i} V_\delta^{(\tau)}(\omega)\, d\mu(\omega) + b_\tau |I| \mu(W_i)\right)
$$

(Since $SO(2)W_i$ is $SO(2)$-invariant)

$$
\leq C_{\delta,\tau}\left(\frac{\psi(2^j)}{2\sigma}\right)^{-\frac{1+\delta}{2}} \mu(W_i)\left(ce^{-(1-\delta)(f(i,j))} \frac{C_{\delta,\tau}}{c}\left(\frac{\psi(2^j)}{\sigma}\right)^{\frac{1+\delta}{2}} + b_\tau |I|\right)
$$

(Since all holonomy vectors in W_i are contained in the circle the upper bound for $V_\delta^{(\tau)}$ applies)

$$
\leq \mu(\tilde{B}_i)\left(ce^{-(1-\delta)f(i,j)} C_{\delta,\tau}^2 2^{\frac{1+\delta}{2}} + b_\tau |I| C_{\delta,\tau}\left(\frac{2\sigma}{\psi(2^j)}\right)^{\frac{1+\delta}{2}}\right) \qquad \text{(Since } W_i \subseteq \tilde{B}_i.\text{)}
$$

Our goal is to compare the equation on the right hand side to the volume of C_j. Note by the construction of the Masur–Veech measure, there is a constant m so that

$$
\mu(C_j) = \mu(S_1(c,\sigma,2^j)) \geq m\left(\sqrt{\frac{2\sigma}{\psi(2^j)}}\right)^2 = m\frac{2\sigma}{\psi(2^j)}.
$$

Thus we have

$$\begin{aligned}
\mu(B_j \cap C_j) &= \mu(\tilde{B}_j \cap \tilde{C}_j) \\
&\leq \mu(B_i)\left(cC_{\delta,\tau}^2 2^{\frac{1+\delta}{2}} e^{-(1-\delta)f(i,j)} + b_\tau |I| C_{\delta,\tau}\left(\frac{\mu(C_j)}{m}\right)^{\frac{1+\delta}{2}}\right) \\
&= \mu(B_i)\left(cC_{\delta,\tau}^2 2^{\frac{1+\delta}{2}} 2^{-(j-i)(1-\delta)}\left(\frac{\psi(2^i)}{\psi(2^j)}\right)^{\frac{1-\delta}{2}} + b_\tau |I| C_{\delta,\tau}\left(\frac{\mu(C_j)}{m}\right)^{\frac{1+\delta}{2}}\right) \\
&\qquad \text{(By the definition of } f(i,j)\text{)} \\
&\leq \mu(B_i)\left(cC_{\delta,\tau}^2 2^{\frac{1+\delta}{2}} 2^{-(j-i)(1-\delta)} + \frac{b_\tau |I| C_{\delta,\tau}}{m^{\frac{1+\delta}{2}}}\mu(C_j)^{\frac{1+\delta}{2}}\right) \\
&\qquad \text{(Since } \psi(R) \text{ is a non-decreasing sequence and } i<j,\ \psi(2^i)\leq\psi(2^j)\text{)} \\
&\leq \mu(B_i)\left(c_1 2^{-(j-i)(1-\delta)} + c_2\mu(C_j)^{\frac{1+\delta}{2}}\right). \\
&\qquad \text{(defining } c_1 = cC_{\delta,\tau}^2 2^{\frac{1+\delta}{2}} \text{ and } c_2 = \tfrac{b_\tau|I|C_{\delta,\tau}}{m^{\frac{1+\delta}{2}}}\text{)}
\end{aligned}$$

Thus picking $C > \max\{c_1, c_2\}$ we obtain the desired inequality.

4.4 *Exponential decay of correlations for far away pairs*

The goal of this section is to prove Lemma 4.3.6, so for the reminder of this section we will fix i and suppose $j > m_i$. To do this, we want to use exponential mixing of the geodesic flow.

Theorem 4.4.1 (Stated from [AEZ12] Theorem C.4, see [AGY06]). *Fix $\mathcal{H}$ a connected component of the stratum and μ an $SL(2,\mathbb{R})$-invariant probability measure. There exists $C > 0$ and $\delta > 0$ so that for all h_1, h_2 Lipschitz and compactly supported, there exists a C_K depending only on the shortest sytole of a surface in the compact set so that for all $t \geq 0$*

$$\left|\int h_1(h_2 \circ g_t)\, d\mu - \int h_1\, d\mu \int h_2\, d\mu\right| \leq C(C_K + \|h_1\|_\infty + \|h_2\|_{Lip})(C_K + \|h_2\|_\infty + \|h_2\|_{Lip})e^{-\delta t}.$$

In order to apply Theorem 4.4.1, we need to use bump functions to approximate the sets A_i andd A_j. By g_t-invariance of μ, $\mu(A_i \cap A_j) = \mu(H_{c,\sigma,i} \cap g_{j-i}H_{c,\sigma,j})$. So it will suffice to define our bump function to approximate $H_{c,\sigma,i}$ and $H_{c,\sigma,j}$.

Definition 4.4.2. *For each i and each $j > i + \frac{4}{\delta}\log\left(\frac{1}{\mu(A_i)}\right)$, define*

$$\epsilon_{i,j} = e^{-\frac{\delta}{4}|i-j|}.$$

Then for $\ell \in \{i, j\}$, define

$$\rho_{i,j}^{\ell}(x_1, x_2, x_3) = f_1^{\ell}(x_1) f_2^{\ell}(x_2) f_3(x_3)$$

where

$$f_1^{\ell}(x_1) = \min\left\{1, \frac{1}{\epsilon_{i,j}} dist(x_1, \partial T_{c,\sigma,\psi(2^{\ell})})\right\} \cdot \chi_{T_{c,\sigma,\psi(2^{\ell})}},$$
$$f_2^{\ell}(x_2) = \min\left\{1, \frac{1}{\epsilon_{i,j}} dist(x_2, \partial T_{c,\sigma,\psi(2^{\ell})})\right\} \chi_{T_{c,\sigma,\psi(2^{\ell})}},$$

and

$$f_3^{\ell}(x_3) = \min\left\{1, \frac{1}{\epsilon_{i,j}} dist(x_3, \partial B(0,1))\right\} \chi_{B(0,1)}.$$

Lemma 4.4.3. *The functions $\rho_{i,j}^{\ell}$ are $\frac{1}{\epsilon_{i,j}}$-Lipschitz.*

Proof. Note that f_k^{ℓ} for $k = 1, 2, 3$ are all $\frac{1}{\epsilon_{i,j}}$- Lipschitz as

$$|f_k^{\ell}(x_k) - f_k^{\ell}(y_k)| \le \frac{1}{\epsilon_{i,j}} |\text{dist}(x_k, \partial T_{c,\sigma,\psi(2^{\ell})}) - \text{dist}(y_k, \partial T_{c,\sigma,\psi(2^{\ell})})| \le \frac{1}{\epsilon_{i,j}} |x_k - y_k|$$

since distance as a function is always 1-Lipschitz.

Now we claim that the function $\rho_{i,j}^{\ell}$ is $\frac{1}{\epsilon_{i,j}}$-Lipschitz with respect to the distance on $\mathcal{H}$ given by

$$d_{\mathcal{H}}((x_1, x_2, x_3) - (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|.$$

To see this, let (x_1, x_2, x_3) and (y_1, y_2, y_3) be fixed. We compute

$$\begin{aligned}
&|\rho(x_1, x_2, x_3) - \rho(y_1, y_2, y_3)| \\
&\le |f_1(x_1) f_2(x_2) f_3(x_3) - f_1(y_1) f_2(x_2) f_3(x_3)| \\
&\quad + |f_1(y_1) f_2(x_2) f_3(x_3) - f_1(y_1) f_2(y_2) f_3(x_3)| \\
&\quad + |f_1(y_1) f_2(y_2) f_3(x_3) - f_1(y_1) f_2(y_2) f_3(y_3)| \\
&\le \frac{1}{\epsilon_{i,j}} \left(f_2(x_2) f_3(x_3) |x_1 - y_1| + f_1(y_1) f_3(x_3) |x_2 - y_2| + f_1(y_1) f_2(y_2) |x_3 - y_3| \right) \\
&\le \frac{1}{\epsilon_{i,j}} d_{\mathcal{H}}((x_1, x_2, x_3) - (y_1, y_2, y_3)). \qquad \text{(Since } f_i \le 1\text{)}
\end{aligned}$$

□

We can now use the definition of the $\rho^{\ell}_{i,j}$ to state a corollary of Theorem 4.4.1.

Corollary 4.4.4. *Fix $0 < \delta < 1$. Then there exists a constant C so that for all $j > m_i$,*

$$\int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i}) \leq C\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right].$$

Proof. We know from definition $\left\|\rho^{\ell}_{i,j}\right\|_\infty = 1$, and from Lemma 4.4.3 that $\left\|\rho^{\ell}_{i,j}\right\| = \frac{1}{\epsilon_{i,j}} = e^{\frac{\delta}{4}|i-j|}$. By 4.4.1, writing $C_K + 1 = C_K^+$,

$$\begin{aligned}\left|\int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i}) - \int \rho^i_{i,j} \int \rho^j_{i,j}\right| &\leq C(C_K^+ + e^{\frac{\delta}{4}|i-j|})^2 e^{-\delta|i-j|} \\ &= C(C_K^+)^2 e^{-\delta|i-j|} + 2CC_K^+ e^{-\frac{3\delta}{4}|i-j|} + Ce^{-\frac{\delta}{2}|i-j|} \\ &\quad (\text{Since } e^{-\frac{\delta}{2}|i-j|} \geq e^{-\frac{3}{4}\delta|i-j|} \geq e^{-\delta|i-j|} \text{ as } e^{-\delta|i-j|} < 1) \\ &\leq \tilde{C}e^{-\frac{\delta}{2}|i-j|}.\end{aligned}$$

So we have by construction of $\rho^{\ell}_{i,j}$, $\int \rho^{\ell}_{i,j} = \mu(H_{c,\sigma,\ell}) = \mu(A_\ell)$, so

$$\int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i}) \leq \int \rho^i_{i,j} \int \rho^j_{i,j} + \tilde{C}e^{-\frac{\delta}{2}|i-j|} \leq \mu(A_i)\mu(A_j) + \tilde{C}e^{-\frac{\delta}{2}|i-j|}.$$

Now since $j > m_i$, we have

$$e^{-\frac{\delta}{4}|i-j|} < \mu(A_i).$$

Since $\tilde{C} > 1$,

$$\int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i}) \leq \tilde{C}\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right].$$

□

Next our goal is to get a relationship between $\mu(A_i \cap A_j)$ and $\int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i})$.

Lemma 4.4.5. *For all $j > m_i$,*

$$\mu(A_i \cap A_j) \leq \int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i}) + \mu(E_j) + \mu(E_i)$$

where $E_\ell = \{\omega : \rho^{\ell}_{i,j} \in (0,1)\}$.

Proof. We first make a general claim. **Claim** If $0 \leq g \leq f \leq 1$ then

$$\int f \leq \int g + \mu\{f \neq g\}.$$

To see this is true, we write

$$\int f = \int g + f - g = \int g + \int_{\{f \neq g\}} f - g \leq \int g + \mu\{f \neq g\}$$

where the last inequality follows since $f - g \leq 1$.

The proof now follows from the claim where $f = \chi_{H_{c,\sigma,i} \cap g_{j-i} H_{c,\sigma,j}}$ and $g = \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i})$, combined with the fact that $f \neq g$ occurs when $\rho^i_{i,j} \in (0,1)$ or $\rho^j_{i,j} \in (0,1)$. So at worst,

$$\mu\{f \neq g\} \leq \mu(E_i) + \mu(E_j).$$

□

Finally to finish the proof of Theorem 4.3.6 we will relate $\int \rho^i_{i,j} \int \rho^j_{i,j}$ to $\mu(E_i) + \mu(E_j)$ from Lemma 4.4.5

Lemma 4.4.6. *There exists a constant $C > 1$ so that*

$$C \int \rho^i_{i,j} \int \rho^j_{i,j} > \mu(E_i) + \mu(E_j).$$

Before proving this lemma, we will first show that this Lemma is sufficient to get Lemma 4.3.6.

Proof of Lemma 4.3.6. Fix i and let $j > m_i$. Then

$$\begin{aligned}
\mu(A_i \cap A_j) &\leq \int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i}) + \mu(A_i) + \mu(A_j) && \text{(By Lemma 4.4.5)} \\
&\leq \int \rho^i_{i,j}(\rho^j_{i,j} \circ g_{j-i}) + C \int \rho^i_{i,j} \int \rho^j_{i,j} && \text{(By Lemma 4.4.6)} \\
&\leq C\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right] + C\mu(A_i)\mu(A_j) \\
&\qquad \text{(by Corollary 4.4.4, and since } \textstyle\int \rho^\ell_{i,j} \leq \mu(H_{c,\sigma,\ell}) = \mu(A_\ell)) \\
&\leq \tilde{C}\mu(A_i)\left[\mu(A_j) + e^{-\frac{\delta}{4}|i-j|}\right] && \text{(Setting } \tilde{C} = 2C.)
\end{aligned}$$

□

Proof of Lemma 4.4.6. Since $\rho^j_{i,j} \leq \rho^j_{i,j}$, we have

$$\int \rho^i_{i,j} \int \rho^j_{i,j} \geq \left(\int \rho^j_{i,j} \right)^2 \geq \mu\{\rho^j_{i,j} = 1\}^2 \tag{4.4.1}$$

Now we want to get a lower bound for the measure of the set where $\rho^j_{i,j} = 1$. To do this, we need to find the area of the subset of the trapezoid $T^+_{c,\sigma,j}$ where the $\rho^j_{i,j} = 1$. To do this, note the horizontal line $y = \epsilon_{i,j}$ from $c + \epsilon$ to $1 - \epsilon$ give the height of the inner trapezoid. The line which is length $\epsilon_{i,j}$ away from the line $y = \frac{\sigma}{\psi(2^j)} x$ is given by

$$y = \frac{\sigma}{\psi(2^j)} x - \epsilon \sqrt{1 + \frac{\sigma^2}{\psi(2^j)^2}}.$$

Thus the four corners of the trapezoid where $\rho^j_{i,j} = 1$ are given by $(c + \epsilon, \epsilon), (1 - \epsilon, \epsilon)$,

$$\left(1 - \epsilon, \frac{\sigma}{\psi(2^j)}(1 - \epsilon) - \epsilon \sqrt{1 + \frac{\sigma^2}{\psi(2^j)^2}}\right), \left(c + \epsilon, \frac{\sigma}{\psi(2^j)}(c + \epsilon) - \epsilon \sqrt{1 + \frac{\sigma^2}{\psi(2^j)^2}}\right).$$

Hence the area of subset of $T^+_{c,\sigma,j}$ where $\rho^j_{i,j} = 1$ is given by

$$\frac{(1 - c - 2\epsilon_{i,j})}{2} \left(\frac{\sigma}{\psi(2^j)}(1 + c) - 2\epsilon_{i,j} \sqrt{1 + \frac{\sigma^2}{\psi(2^j)^2}} - 2\epsilon_{i,j} \right)$$

By symmetry, the area of $T^-_{c,\sigma,j}$ where $\rho^j_{i,j} = 1$ is the same as the area for $T^+_{c,\sigma,j}$. Thus the total area is the product of the two trapezoids with the product of the ball where $n+4 = 2g+s-1$

and $\sigma(n)$ is gives the volume of the n-ball. That is

$$\mu(\{\rho^j_{i,j}=1\}) = \frac{(1-c-2\epsilon_{i,j})^2}{4}\left(\frac{\sigma}{\psi(2^j)}(1+c)-2\epsilon_{i,j}\sqrt{1+\frac{\sigma^2}{\psi(2^j)^2}}-2\epsilon_{i,j}\right)^2\cdot\sigma(n)(r-\epsilon_{i,j})^n$$

$$\geq d_n(1-c-2\epsilon_{i,j})^2\left(\frac{\sigma}{\psi(2^j)}(1+c)-2\epsilon_{i,j}\sqrt{1+\frac{\sigma^2}{\psi(2^j)^2}}-2\epsilon_{i,j}\right)^2$$

(Assuming $r-\epsilon\geq\frac{r}{2}$ so d_n is some constant depending only on n)

$$\geq d_n\left(\frac{1}{2}-2\epsilon_{i,j}\right)^2\left(\frac{\sigma}{\psi(2^j)}-2\epsilon_{i,j}\sqrt{1+\frac{\sigma^2}{\psi(2^j)^2}}-2\epsilon_{i,j}\right)^2$$

(Assuming $1-c\geq\frac{1}{2}$ which we can do by making c larger if neccessary)

$$\geq \frac{d_n}{16}\left[\frac{\sigma}{\psi(2^j)}-6\epsilon_{i,j}\right]^2 \qquad (\epsilon_{i,j}<\tfrac{1}{8}\text{ implies }\tfrac{1}{2}-2\epsilon_{i,j}>\tfrac{1}{4})$$

$$\geq \frac{d_n}{16}\epsilon_{i,j}^{\frac{1}{2}}$$

(Since $6\epsilon_{i,j}+\epsilon_{i,j}^{\frac{1}{4}}\leq 7\epsilon_{i,j}^{\frac{1}{4}}\leq\frac{\sigma}{\psi(2^j)}$ by assumption on ψ Equation (4.2.1).)

Combining this fact with equation 4.4.1, we obtain

$$\int\rho^i_{i,j}\int\rho^j_{i,j}\geq\frac{d_n^2}{16^2}\epsilon_{i,j}. \tag{4.4.2}$$

Now from the other end we want an upper bound for $\mu\{\rho^i\in(0,1)\}+\mu\{\rho^j\in(0,1)\}\leq 2C\epsilon_{i,j}$.

Given $\ell=i$ or $\ell=j$, we have the area of the $\epsilon_{i,j}$-boundary of one of the trapezoids $T^{\pm}_{c,\sigma,2^\ell}$

is given by

$$\begin{aligned}
&\mu(\partial_{\epsilon_{i,j}} T^{\pm}_{c,\sigma,2^\ell}) \\
&= \frac{\sigma}{2\psi(2^\ell)}(1-c^2) - \left(\frac{1-c}{2} - \epsilon_{i,j}\right)\left(\frac{\sigma}{\psi(2^\ell)}(1+c) - 2\epsilon_{i,j}\sqrt{1+\frac{\sigma^2}{\psi(2^\ell)^2}}\right) \\
&\hspace{5cm}\text{(Taking area of } T^{\pm}_{c,\sigma,\ell} \text{ less the area where } \rho^{\ell}_{i,j} = 1) \\
&= \epsilon_{i,j}\left[\frac{\sigma}{\psi(2^\ell)}(1+c) + (1-c-2\epsilon_{i,j})\left[1+\sqrt{1+\frac{\sigma^2}{\psi(2^\ell)^2}}\right]\right] \\
&\le \epsilon_{i,j}\left[1+c+(1-c)(1+\sqrt{2})\right] \\
&\text{(Assuming } \sigma < \psi(2^\ell) \text{ which is easy since } \sigma < 1 \text{ is fixed and } \psi \ge 1 \text{ is non-decreasing)} \\
&\le \epsilon_{i,j} C. \hspace{5cm} \text{(Where } C \text{ depends on } c)
\end{aligned}$$

Thus

$$\mu(\{\rho^i_{i,j} \in (0,1)\}) + \mu(\{\rho^j_{i,j} \in (0,1)\}) \le 2C\epsilon_{i,j} \tag{4.4.3}$$

Combining Equation 4.4.3 with Equation 4.4.2,

$$\int \rho^i_{i,j} \int \rho^j_{i,j} \ge \frac{d_n^2}{16^2}\epsilon_{i,j} \ge \frac{d_n^2}{16^2(2C)}\left[\mu\{\rho^i_{i,j} \in (0,1)\} + \mu\{\rho^j_{i,j} \in (0,1)\}\right] \tag{4.4.4}$$

So setting $\tilde{C} = \frac{16^2(2C)}{d_n^2}$, we can assume $\tilde{C} > 1$ since d_n is bounded above by a fixed constant, and we can make C larger if necessary. □

4.5 Construction and Properties for averages of lengths of complexes

The main goal of this section is to prove Corollary 4.5.9, which requires combining[Ath06] and [Doz19] with a few modifications to fit the goals of this paper.

Definition 4.5.1. *A complex K in ω is a closed subset of X whose boundary ∂K consists of a union of disjoint (in the interior) saddle connections such that if ∂K contains three saddle connections bounding a triangle, then the interior of that triangle is in K. Given a complex K the complexity of K is the number of saddle connections needed to triangulate K. For any*

$\delta > 0$ and $k \in \mathbb{N}$, if M is the complexity of ω,

$$\alpha_k(\omega) = \max_{\substack{K \text{ complexity } k \\ area(K) < 2^{k-M-1}}} \frac{1}{|\partial K|^{1+\delta}}.$$

If the set over which we take the maximum is empty, then we set $\alpha_k(\omega) = 0$.

Definition 4.5.2. *Given a function f on $\mathcal{H}$ and a point $\omega \in \mathcal{H}$, we let*

$$Ave_t(f)(\omega) = \frac{1}{2\pi} \int_0^{2\pi} f(g_t r_\theta \omega)\, d\theta.$$

Note that $\alpha_1(\omega) = \frac{1}{\ell(\omega)^{1+\delta}}$ where $\ell(\omega)$ is the length of the shortest saddle connection. Since M is finite for all k large enough, $\alpha_k(\omega) = 0$. For more information and intuition for the α_k, see Section 5.3 of [Doz19]. From Proposition 5.3 of [Doz19], we have

Proposition 4.5.3. *Fix a stratum $\mathcal{H}$, and $0 < \delta < \frac{1}{2}$. We can find a constant b such that for any interval $I \subseteq S^1$, there exists a constant c_I such that for all $\omega \in \mathcal{H}$ and $T \geq 0$,*

$$\int_I \alpha_k(g_T r_\theta \omega)\, d\theta < c_I e^{-(1-2\delta)T} \sum_{j \geq k} \alpha_j(\omega) + b|I|.$$

The strategy we will take is to extend this theorem to a function V_δ which is a weighted average of the α_k functions. We want to weight the average to have nice properties, so our goal is to recreate the following theorem for integrating over an interval I instead of $[0, 2\pi)$.

Lemma 4.5.4 (Lemma 6.2 [AG13], Proof in [Ath06])**.** *Let $\mathcal{V}$ be a neighborhood of the identity in $SL(2,\mathbb{R})$. Fix $\mathcal{H}$ a conncted stratum of $\mathcal{H}(\alpha)$. For every $0 < \delta < 1$ there exists $c_1 > 0$ so that for all $t > 0$ there exists a function $V_\delta^{(t)} : \mathcal{H} \to [1, \infty)$ and a scalar b_t satisfying the following properties. For all $\omega \in \mathcal{H}$,*

$$(Ave)_t(V_\delta^{(t)})(\omega) = \int_0^{2\pi} V_\delta^{(t)}(g_t r_\theta \omega)\, d\theta \leq c_1 e^{-(1-\delta)t} V_\delta^{(t)}(\omega) + b_t.$$

Moreover, V_δ is logsmooth. That is

$$V_\delta^{(t)}(g\omega) \leq c_3 V_\delta^{(t)}(\omega) \tag{4.5.1}$$

for all $\omega \in \mathcal{H}$ *and* $g \in \mathcal{V}$.

Finally, there exists a constant $C_{\delta,t}$ *so that*

$$\frac{V_\delta^{(t)}(\omega)}{V_\delta(\omega)} \in [C_{\delta,t}^{-1}, C_{\delta,t}] \tag{4.5.2}$$

where $V_\delta = \max\{1, \alpha_1(\omega)\} = \max\{1, \frac{1}{\ell(\omega)^{1+\delta}}\}$

We want to change Lemma 4.5.4 to restricting over an interval I.

We now explicitly construct V_δ using the following result.

Proposition 4.5.5 (Proposition 5.4 [Doz19])**.** *Fix* $\mathcal{H}$ *and* $0 < \delta < 1$*. There exists* $C > 0$ *so that for any* $t > 0$*, there exists constants* b_t *and* w_t *so that for any* k *and any* $\omega \in \mathcal{H}$*,*

$$Ave_t(\alpha_k)(\omega) \leq Ce^{-t(1-\delta)}\alpha_k(\omega) + w_t \sum_{j>k} \alpha_j(\omega) + b_t. \tag{4.5.3}$$

Definition 4.5.6. *Fix* δ *and* $t > 0$*. Define*

$$\lambda_k^{(t)} = \left(\frac{w_t}{C} + 1\right)^k$$

where w_t *and* C *are the constants of 4.5.3. Define*

$$V_\delta^{(t)}(\omega) = \sum_{k=0}^{M} \lambda_k^{(t)} \alpha_k(\omega)$$

where M *is the maximum complexity of* ω*.*

Proof of Lemma 4.5.4. We first claim

$$\lambda_k C e^{-(1-\delta)t} + w_t \sum_{j=0}^{k-1} \lambda_j \leq 2C\lambda_k e^{-(1-\delta)t}. \tag{4.5.4}$$

To see this holds, note that $e^{-(1-\delta)t} \geq 1$. Thus since $\lambda_k \geq 1$, we have

$$1 - \frac{1}{\lambda_k} \leq 1 \leq \left(\frac{w_t}{C} + 1 - 1\right)\frac{C}{w_t} e^{-(1-\delta)t}.$$

Simplifying and using the finite geometric series formula, this implies

$$\sum_{j=0}^{k-1} \lambda_j = \frac{\lambda_k - 1}{\lambda_1 - 1} \leq \lambda_k \frac{C}{w_t} e^{-(1-\delta)t}.$$

Multiplying by w_t and adding $\lambda_k C e^{-(1-\delta)t}$ to each side yields Equation 4.5.4.

We now want to prove that on average V_δ shrinks over circles of radius t. To see this, we compute

$$\begin{aligned}
\mathrm{Ave}_t(V_\delta)(\omega) &= \sum_{k=0}^{n} \lambda_k \mathrm{Ave}_t(\alpha_k)(\omega) \\
&\leq \sum_{k=0}^{n} \lambda_k \left(C e^{-t(1-\delta)} \alpha_k(\omega) + w_t \sum_{j>k} \alpha_j(\omega) + b_t \right) && \text{(By 4.5.3)} \\
&= \sum_{k=0}^{n} \lambda_k C e^{-t(1-\delta)} \alpha_k(\omega) + w_t \sum_{k=1}^{n} \alpha_k(\omega) \left(\sum j = 0^{k-1} \lambda_j \right) + b_t \\
& && \text{(Replacing } b_t = b_t \left(\textstyle\sum_{j=1}^{n} \lambda_j \right)) \\
&\leq 2C e^{-t(1-\delta)} \lambda_0 \alpha_0(\omega) + \sum_{k=1}^{n} \alpha_k(\omega) \left[\lambda_k C e^{-t(1-\delta)} + w_t \sum_{j=0}^{k-1} \lambda_j \right] + b_t \\
& && \text{(By Equation 4.5.4 and replacing } C \text{ with } 2C) \\
&\leq C e^{-t(1-\delta)} \sum_{k=0}^{n} \lambda_k \alpha_k(\omega) + b_t \\
&= C e^{-t(1-\delta)} V_\delta(\omega) + b_t.
\end{aligned}$$

The logsmoothness of the V_δ follows from [EM01]. □

Now that we have defined V_δ with the logsmooth property, we now proceed to extending the results of [Doz19] to include the V_δ function.

Lemma 4.5.7. *There exists a constant $c_2 > 0$ so that for any $\tau \geq 0$ and $I \subseteq S^1$ an interval, there exists $t_0(\tau, |I|) \geq 0$ so that for any $\omega \in \mathcal{H}$ and $t > t_0$, we have*

$$\int_I V_\delta^{(\tau)}(g_{t+\tau} r_\theta \omega)\, d\theta \leq c_2 \int_J Ave_\tau(V_\delta^{(\tau)})(g_t r_\theta \omega)\, d\theta$$

where $J \subseteq S^1$ is an interval (that could depend on all other parameters) with $|J| = |I|$.

Proof. Note that this result would follow directly from linearity combined with Lemma 5.2 of [Doz19], except as stated in Lemma 5.2 the interval J could depend on α_i. However following the proof exactly using linearity to replace each α_i with $V_\delta^{(\tau)}$, we take the interval $2I$ with the same center as I and twice the length. Then in the last 5 lines of the proof, we write $2I = J_1 \cup J_2$ as a union of two intervals with $|J_1| = |J_2| = |I|$. Then

$$\max_{j=1,2} \int_{J_j} \mathrm{Ave}_\tau(V_\delta^{(\tau)}(g_t r_\theta \omega) \geq \frac{1}{2} \int_{2I} \mathrm{Ave}_\tau V_\delta^{(\tau)}(g_t r_\theta \omega).$$

Now define J (which now depends on V_δ^τ instead of individual α_k to be the interval on which the maximum is achieved, and the proof follows by linearity as desired. □

Now we state Proposition 5.3 of [Doz19] for the V_δ functions.

Proposition 4.5.8. *Fix a stratum $\mathcal{H}$ and $0 < \delta < 1$. Let c_1 and c_2 be the constants of Lemma 4.5.4 and Lemma 4.5.7, respectively. Choose $\tau \geq 0$ large enough so that*

$$c_1 c_2 e^{-(1-\delta)\tau} < \frac{1}{2}.$$

Let $I \subseteq S^1$ be an interval and by Lemma 4.5.7 let m be the smallest possible integer so that $(m-1)\tau > t_0(\tau, |I|)$. That is $m = 1 + \left\lceil \frac{t_0(\tau,|I|)}{\tau} \right\rceil$. There are constants $c = c(\tau, \delta, |I|) > 0$ and $b_\tau = b(\tau, \delta)$ so that for all $n \geq m$ and for any $\omega \in \mathcal{H}$,

$$\int_I V_\delta^{(\tau)}(g_{n\tau} r_\theta \omega)\, d\theta < c e^{-(1-\delta)n\tau} V_\delta^{(\tau)}(\omega) + b_\tau |I|. \tag{4.5.5}$$

Proof. Let $n \geq m$ and $\omega \in \mathcal{H}$. Our goal is to construct the constants c and b_τ to that Equation 4.5.5 holds. Indeed applying Lemma 4.5.7 followed by Lemma 4.5.4, we have

$$\begin{aligned}
\int_I V_\delta^{(\tau)}(g_{n\tau} r_\theta \omega)\, d\theta &\leq c_2 \int_{J_{n-1}} \mathrm{Ave}_\tau(V_\delta^{(\tau)})(g_{(n-1)\tau} r_\theta \omega)\, d\theta \\
&\leq c_2 \left(\int_{J_{n-1}} c_1 e^{-(1-\delta)\tau} V_\delta^{(\tau)}(g_{(n-1)\tau} r_\theta \omega)\, d\theta + b_\tau \right) \\
&= c_2 c_1 e^{-(1-\delta)\tau} \int_{J_{n-1}} V_\delta^{(\tau)}(g_{(n-1)\tau} r_\theta \omega)\, d\theta + c_2 b_\tau |I|
\end{aligned}$$

where the last equality follows from the fact that $|J_{n-1}| = |I|$.

Now repeatedly applying this inequality for $n-1, n-2, \ldots, m$ with I replaced by J_{n-1}, J_{n-2} through J_m which all have length $|I|$, we obtain

$$\int_I V_\delta^{(\tau)}(g_{n\tau} r_\theta \omega) \leq \left(c_1 c_2 e^{-(1-\delta)\tau}\right)^{n-m+1} \int_{J_{m-1}} V_\delta^{(\tau)}(g_{(m-1)\tau} r_\theta \omega)\, d\theta$$
$$+ |I| b_\tau c_2 \sum_{j=0}^{n-m+1} \left(c_1 c_2 e^{-(1-\delta)\tau}\right)^j$$

(By our choice of τ the geometric sum is at most 2, so replacing b_τ with $2c_2 b_\tau$,)

$$\leq \left(c_1 c_2 e^{-(1-\delta)\tau}\right)^{n-m+1} \int_{J_{m-1}} V_\delta^{(\tau)}(g_{(m-1)\tau} r_\theta \omega)\, d\theta + |I| b_\tau$$

By the logsmooth property of V_δ from Lemma 4.5.4, splitting into small steps, there exists some $k(m, \tau)$ so that

$$V_\delta^{(\tau)}(g_{(m-1)\tau} r_\theta \omega) \leq c_3^{k(m,\tau)} V_\delta^{(\tau)}(\omega).$$

Thus we can write our constant c as

$$c = (c_1 c_2)^{n-m+1} \left(e^{-(-1\delta)\tau}\right)^{-m+1} c_3^{k(m,\tau)} |I|$$

where we note m depends on τ and $|I|$, so c depends only on δ, τ and $|I|$.

Thus we obtain

$$\int_I V_\delta^{(\tau)}(g_{n\tau} r_\theta \omega)\, d\theta \leq c e^{-\tau n(1-\delta)} V_\delta^{(\tau)}(\omega) + b_\tau |I|.$$

□

Corollary 4.5.9. *Fix a stratum $\mathcal{H}$ and $0 < \delta < 1$. There exists $\tau \geq 0$ so that for any interval $I \subseteq S^1$, there exists constants $c = c(\tau, \delta, |I|) > 0$ and $b_\tau = b(\tau, \delta)$ so that there exists an ℓ_0 so that for all $\ell \geq \ell_0$ and for any $\omega \in \mathcal{H}$,*

$$\int_I V_\delta^{(\tau)}(g_\ell r_\theta \omega)\, d\theta \leq c e^{-(1-\delta)\ell} V_\delta^{(\tau)}(\omega) + b_\tau |I|.$$

Proof. We choose τ to satisfy the assumption of Proposition 4.5.8. Choose $\ell \geq m\tau$ where m is defined in Proposition 4.5.8. Pick

$$n_0 = \min\{n \in \mathbb{N} : n\tau > \ell\},$$

and note $n_0 - 1 \geq m$. Let $r = n_0\tau - \ell$. Choose step sizes of r_0 so that $r = kr_0$ for some $k \in \mathbb{N}$ and $g_{-r_0} \in \mathcal{V}$ so we can apply (4.5.1).

Then from Proposition 4.5.8,

$$\begin{aligned}
\int_I V_\delta^{(\tau)}(g_\ell r_\theta \omega)\, d\theta &= \int_I V_\delta^{(\tau)}(g_{-kr_0} g_{n_0\tau} r_\theta \omega)\, d\theta \\
&\leq c_3^k \int_I V_\delta^{(\tau)}(g_{n_0\tau} r_\theta \omega)\, d\theta && \text{(By (4.5.1))} \\
&\leq c_3^k \left[ce^{-(1-\delta)n_0\tau} V_\delta^{(\tau)}(\omega) + b_\tau |I| \right] && \text{(By (4.5.5))} \\
&\leq c_3^k \left[ce^{-(1-\delta)\ell} V_\delta^{(\tau)}(\omega) + b_\tau |I| \right] && \text{(Since } n_0\tau \geq \ell\text{)} \\
&\leq c_3^{\frac{\tau}{r_0}} \left[ce^{-(1-\delta)\ell} V_\delta^{(\tau)}(\omega) + b_\tau |I| \right] && \text{(Since } r \leq \tau\text{)}
\end{aligned}$$

Thus the final constants only depend on τ and not ℓ and we obtain the desired result. □

4.6 Proof of Generalized Borel–Cantelli

To prove Proposition 4.3.1, we build from [Pet02], which invokes the Chung-Erdös Inequality.

Lemma 4.6.1 (Chung-Erdös Inequality). *If $(A_k)_{k=1}^\infty$ is a sequence of measurable sets with $\mu\left(\bigcup_{k=1}^N A_k\right) > 0$, then*

$$\mu\left(\bigcup_{k=1}^N A_k\right) \geq \frac{\left(\sum_{k=1}^N \mu(A_k)\right)^2}{\sum_{k,j=1}^n \mu(A_k \cap A_j)}.$$

Proof. Beginning with the numerator on the right hand side, multiply by a fancy 1 which is the characteristic function of the set where the sum of the characteristic functions is nonzero

$$\left(\sum_{k=1}^N \mu(A_k)\right)^2 = \left[\int \left(\sum_{j=1}^N \chi_{A_j}\right) \chi_{\{\sum \chi_{A_j} > 0\}}\, d\mu\right]^2.$$

Using Schwarz's inequality and the fact that

$$\int \chi^2_{\{\sum \chi_{A_j}>0\}}\, d\mu = \int \chi_{\{\sum \chi_{A_j}>0\}}\, d\mu = \mu\left(\bigcup_{j=1}^{N} A_j\right),$$

we have

$$\left(\sum_{k=1}^{N} \mu(A_k)\right)^2 \leq \int \left(\sum_{j=1}^{N} \chi_{A_j}\right)^2 d\mu\mu\left(\bigcup_{j=1}^{N} A_j\right) = \left[\sum_{j,i=1}^{N} \mu(A_i \cap A_j)\right] \mu\left(\bigcup_{k=1}^{N} A_k\right).$$

Rearranging we obtain the desired inequality. □

Proof of Proposition 4.3.1. Since for all i there are $B_i \subseteq A_i$, we have

$$\limsup B_i \subseteq \limsup A_i.$$

So it suffices to show that $\mu(\limsup B_i) > 0$.

We first claim there exists some $\tilde{C} \geq 1$ depending only on C large enough so that if $\tilde{m}_i = i + \tilde{C} + \tilde{C}\log\left(\frac{1}{\mu(B_i)}\right)$,

$\tilde{1}$. $\sum_{k=1}^{\infty} \mu(B_k) = \infty$.

Proof. Follows from 1 and 4b. □

$\tilde{2}$. For all $i \leq j$, $\mu(B_i) \geq \frac{1}{C}\mu(B_j)$.

Proof. Follows from 2 and 4b. □

$\tilde{3}$. For all i and for all j so that $j > \tilde{m}_i$,

$$\mu(B_i \cap B_j) \leq \tilde{C}\mu(B_i)\left[\mu(B_j) + e^{-\frac{\delta}{4}|i-j|}\right].$$

Proof. Combine assumption 3 with 4a and 4b. □

$\tilde{4}$. For all i and for all j with $i < j < \tilde{m}_i$,

$$\mu(B_i \cap B_j) < \tilde{C}\mu(B_i)\left[2^{-|i-j|(1-\delta)} + \mu(B_j)^{\frac{1+\delta}{4}}\right].$$

Proof. Combine conditions 4b and 4c with that fact that $\frac{1+\delta}{4} \in (\frac{1}{4}, \frac{1}{2})$ and $C \geq 1$ to obtain

$$\mu(C_j)^{\frac{1+\delta}{2}} \leq C\mu(A_j)^{\frac{1+\delta}{4}} \leq C(C\mu(B_j))^{\frac{1+\delta}{4}} \leq \tilde{C}^2\mu(B_j)^{\frac{1+\delta}{4}}.$$

□

We will now show that that the measure of $\limsup B_n$ has positive measure. To do this we will adapt the proof of Theorem 2.1 of [Pet02].

By the Chung-Erdös inequality,

$$\mu\left(\bigcup_{k=n}^{N} B_k\right) \geq \frac{\left(\sum_{k=n}^{N} \mu(B_k)\right)^2}{\sum_{i,j=n}^{N} \mu(B_i \cap B_j)}. \tag{4.6.1}$$

We want to find an upper bound for the denominator, and

$$\begin{aligned}
&\sum_{i,j=n}^{N} \mu(B_i \cap B_j) \\
&= 2\sum_{i=n}^{N}\sum_{j>i} \mu(B_i \cap B_j) + \sum_{k=n}^{N} \mu(B_k) \\
&\leq \tilde{C}\left[\sum_{k=n}^{N} \mu(B_k) + 2\sum_{i=n}^{N}\sum_{j>\tilde{m}_i}^{N} \mu(B_i)\mu(B_j) + \mu(B_i)e^{-\frac{\delta}{4}|i-j|}\right. \\
&\qquad \left. + 2\sum_{i=n}^{N}\sum_{i<j\leq\tilde{m}_i} \mu(B_i)2^{-|i-j|(1-\delta)} + \mu(B_i)\mu(B_j)^{\frac{1+\delta}{4}}\right]
\end{aligned}$$

(By 4.6 and 4.6 and since $\tilde{C} \geq 1$)

$$\begin{aligned}
&\leq \tilde{C}\left[\sum_{k=n}^{N} \mu(B_k) + \left(\sum_{k=n}^{N} \mu(B_k)\right)^2 + \frac{2}{1-e^{-\frac{\delta}{4}}}\sum_{k=n}^{N} \mu(B_k)\right. \\
&\qquad \left. + \frac{2}{1-2^{-(1-\delta)}}\sum_{k=n}^{N} \mu(B_k) + 2C\tilde{C}\sum_{k=n}^{N} \mu(B_k) + 2C\tilde{C}C' + 2C\tilde{C}\sum_{k=n}^{N} \mu(B_k)\right]
\end{aligned}$$

(By justifications 1,2,3,4 below.)

$$= \tilde{C}\left[D\sum_{k=n}^{N} \mu(B_k) + D' + \left(\sum_{k=n}^{N} \mu(B_k)\right)^2\right] \tag{4.6.2}$$

where $D' = 2C\tilde{C}C'$ and $D = 1 + \frac{2}{1-e^{-\frac{\delta}{4}}} + \frac{2}{1-2^{-(1-\delta)}} + 4C\tilde{C}$.

Justification 1 Since the measures are always non-negative,

$$\begin{aligned} 2\sum_{i=n}^{N}\sum_{j>m_i}^{N}\mu(B_i)\mu(B_j) &\leq 2\sum_{i=n}^{N}\sum_{j>i}^{N}\mu(B_i)\mu(B_j) \\ &= \left(\sum_{i=n}^{N}\mu(B_i)\right)^2 - \sum_{i=n}^{N}\mu(B_i)^2 \\ &\leq \left(\sum_{i=n}^{N}\mu(B_i)\right)^2. \end{aligned}$$

Justification 2 Making the change of variables $k = j - i$, by the geometric series formula

$$\sum_{j>\tilde{m}_i}^{N} e^{-\frac{\delta}{4}|i-j|} = \sum_{k=1+\tilde{m}_i-1}^{N-1} e^{-\frac{\delta}{4}k} \leq \sum_{k=0}^{\infty} e^{-\frac{\delta}{4}k} = \frac{1}{1-e^{-\frac{\delta}{4}}}.$$

Thus

$$2\sum_{i=n}^{N}\sum_{j>\tilde{m}_i}^{N}\mu(B_i)e^{-\frac{\delta}{4}|i-j|} \leq 2\sum_{i=n}^{N}\mu(B_i)\frac{1}{1-e^{-\frac{\delta}{4}}}.$$

Justification 3 Similar to Justification 2, by the geometric series formula,

$$2\sum_{i=n}^{N}\sum_{i<j\leq\tilde{m}_i}\mu(B_i)2^{-|i-j|(1-\delta)} \leq 2\sum_{i=n}^{N}\sum_{k=0}^{\infty}2^{-(1-\delta)k} \leq \frac{2}{1-2^{-(1-\delta)}}\sum_{i=n}^{N}\mu(B_i).$$

Justification 4 First note for any $j > i$ by assumptions 2, 4(a) and 4(b),

$$\mu(B_j) \leq \mu(A_j) \leq \mu(A_i) \leq C\mu(B_i).$$

Note also that $\frac{5+\delta}{4} \in \left(\frac{5}{4}, \frac{6}{4}\right)$ so $\frac{5+\delta}{4}$ is a power bigger than 1 with $\mu(B_i) \leq 1$, so

$$\mu(B_i)^{\frac{5+\delta}{4}} \leq \mu(B_i).$$

Thus

$$\begin{aligned} 2\sum_{i=n}^{N}\sum_{i<j\leq\tilde{m}_i}\mu(B_i)\mu(B_j)^{\frac{1+\delta}{4}} &\leq 2C\sum_{i=n}^{N}\mu(B_i)^{\frac{5+\delta}{4}}(\tilde{m}_i - i) \\ &\leq 2C\tilde{C}\sum_{i=n}^{N}\mu(B_i) + 2C\tilde{C}\sum_{i=n}^{N}\mu(B_i)^{\frac{5+\delta}{4}}\log\left(\frac{1}{\mu(B_i)}\right) \end{aligned}$$

So to finish this justification it suffices to show there is a constant C' so that

$$\sum_{i=n}^{N} \mu(B_i)^{\frac{5+\delta}{4}} \log\left(\frac{1}{\mu(B_i)}\right) \leq C' + \sum_{i=n}^{N} \mu(B_i),$$

Indeed to see this, choose n_0 large enough so that for all $i \geq n_0$,

$$\log\left(\frac{1}{\mu(B_i)}\right) \leq \mu(B_i)^{\frac{-1-\delta}{4}}.$$

Then if $n \geq n_0$,

$$\sum_{i=n}^{N} \mu(B_i)^{\frac{5+\delta}{4}} \log\left(\frac{1}{\mu(B_i)}\right) \leq \sum_{i=n}^{N} \mu(B_i).$$

Otherwise if $n \leq n_0$

$$\sum_{i=n}^{N} \mu(B_i)^{\frac{5+\delta}{4}} \log\left(\frac{1}{\mu(B_i)}\right) \leq \sum_{i=n}^{n_0-1} \mu(B_i) \log\left(\frac{1}{\mu(B_i)}\right) + \sum_{i=n_0}^{N} \mu(B_i) \leq C' + \sum_{i=n}^{N} \mu(B_i).$$

where $C' > 0$ is the bound for the finite sum.

Now that the 4 justifications are complete, we now combine Equation 4.6.1 and 4.6.2 to get

$$\begin{aligned}
\liminf_{N\to\infty} \mu\left(\bigcup_{k=n}^{N} B_k\right) &\geq \liminf_{N\to\infty} \frac{\left(\sum_{k=n}^{N} \mu(B_k)\right)^2}{\tilde{C}\left[D\sum_{k=n}^{N} \mu(B_k) + D' + \left(\sum_{k=n}^{N} \mu(B_k)\right)^2\right]} \\
&= \liminf_{N\to\infty} \frac{1}{\tilde{C}\left[\frac{D}{\sum_{k=n}^{N} \mu(B_k)} + \frac{D'}{\left(\sum_{k=n}^{N} \mu(B_k)\right)^2} + 1\right]} \\
&= \frac{1}{\tilde{C}} \qquad \text{(By assumption 1 and 4(b) we have } \textstyle\sum_{k=1}^{\infty} \mu(B_i) = \infty.\text{)}
\end{aligned}$$

Hence $\mu\left(\bigcup_{k=n}^{\infty} B_k\right) \geq \frac{1}{\tilde{C}}$.

Setting $D_n = \bigcup_{k=n}^{\infty} B_k$, we have a nested decreasing sequence of sets and thus

$$\mu(\limsup B_n) = \mu\left(\bigcap_{n=1}^{\infty} D_n\right) = \lim_{n\to\infty} \mu(D_n) \geq \frac{1}{\tilde{C}} > 0.$$

□

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