OMITTABLE LINES

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1. Introduction. A family $\mathcal{L}$ of $n = n(\mathcal{L})$ (straight) lines in the plane, not all through one point, determines an *aggregate* $\mathcal{A}(\mathcal{L})$. An aggregate is the geometric structure consisting of all the lines of $\mathcal{L}$ together with all the *vertices* $\mathcal{V}(\mathcal{L})$, that is, intersections of two or more of the lines of $\mathcal{L}$. By "plane" we mean the real projective plane, which we envisage as modeled by the *extended Euclidean plane* — the ordinary Euclidean plane augmented by the *points at infinity* (each such point corresponds to a family of all mutually parallel Euclidean lines) and the *line-at-infinity* (which is formed by the totality of the points at infinity).

A vertex $V \in \mathcal{V}(\mathcal{L})$ of an aggregate $\mathcal{A}(\mathcal{L})$ is said to be k-*fold* if it lies on precisely k lines of $\mathcal{L}$. A 2-fold vertex is called *ordinary*. A well-known but nontrivial result of combinatorial geometry is that every aggregate contains at least one ordinary vertex. This was conjectured by J. J. Sylvester [13] in 1893, but forgotten and independently asked by P. Erdös some forty years later. Detailed references and proofs may be found, among others, in [1], [3], [2], [11].

We are interested in the following question: Given an aggregate $\mathcal{A}(\mathcal{L})$, are there any lines of $\mathcal{L}$ that contain no ordinary points? Each line with this property is said to be an *omittable line* of $\mathcal{L}$, since its omission from $\mathcal{L}$ does not decrease the number of vertices of the aggregate. We denote the totality of omittable lines of $\mathcal{L}$ by $\mathcal{O}(\mathcal{L})$, and the number of lines in $\mathcal{O}(\mathcal{L})$ by $g(\mathcal{L})$. We are interested in the relations between $g(\mathcal{L})$ and $n(\mathcal{L})$. In particular, we shall discuss various types of families of lines that can serve as $\mathcal{O}(\mathcal{L})$ for some $\mathcal{L}$. The topic is equivalent (by duality in the projective plane) to the questions on omittable points considered in [8]. The aim of the present note is to extend the results of [8], and to correct an omission and an error of that paper (see Section 2 below). We shall freely use both the "omittable lines" and the dual "omittable points" approaches.

2. History of the topic. The concept of omittable points was introduced by K. Koutsky and V. Polák [10] in 1960; however, it seems that for a long time this paper elicited no additional investigations. Independently, the concept was introduced by W. F. Smyth [12] in 1989. Working in the omittable points version, Koutsky and Polák proved several results, which we can formulate as follows.

Theorem 1. If an aggregate $\mathcal{A}(\mathcal{L})$ of $n = n(\mathcal{L})$ lines, not all in a pencil, contains $g = g(\mathcal{L}) \geq 3$ omittable lines that form a pencil, then $n \geq 3g$. Moreover, for every $g \geq 3$ there exists such a set $\mathcal{L}$ of 3g points with g omittable lines in a pencil.

Theorem 2. Any set $\mathcal{O}$ of lines forming a pencil is the set $\mathcal{O}(\mathcal{L})$ of omittable lines for some aggregate $\mathcal{A}(\mathcal{L})$ of lines $\mathcal{L}$.

In a series of talks on arrangements of lines, which I gave in 1974 at the University of Washington, I presented the results of Koutsky and Polák with simplified proofs and with some additional examples.

A quarter century later, in 1999, I published a short note [8] describing the work of Koutsky and Polák; the note contained the above Theorem 1 and the following strengthening of Theorem 2.

Theorem 3. Any set $\mathcal{O}$ of g lines forming a pencil is the set $\mathcal{O}(\mathcal{L})$ of omittable lines for some aggregate $\mathcal{A}(\mathcal{L})$ of lines $\mathcal{L}$ with $n(\mathcal{L}) \leq g + 2^g$.

I am indebted to Dr. Jonathan Lenchner for pointing out (in a private communication) two errors of the note [8]. In Theorem 1 of [8] the first part should have been "If a set of n points, not all collinear, contains $k \geq 3$ *collinear* omittable points then $n \geq 3k$". Unfortunately, the word "collinear" italicized here was omitted from the formulation in [8], although it was used in the proof, and the assertion is clearly not valid without it. (This error was also mentioned by Richard Koch, in his review of [8] in the "Zentralblatt" v. 941 #51019; unfortunately, I noticed the review only recently.) The second error concerns the claim that the example in Figure 2 of [8] shows 13 points of which 6 are omittable. However, the example has only 4 omittable points. As indicated in Figure 5 below, the smallest number of lines with 6 omittable points that I know is 14.

Lenchner's remarks led me to investigate again the topic of omittable lines, and the results of this activity are the main contents of this note.

3. The main results. In trying to describe the results on omittable lines, it seems most useful to start by distinguishing various possibilities regarding the sets of omittable lines, rather than with aggregates of lines as such. In particular, we are interested in finding families $\mathcal{O}$ of lines that can serve as $\mathcal{O}(\mathcal{L})$ for appropriate $\mathcal{L}$. For these families $\mathcal{O}$ we would like to decide whether every member is $\mathcal{O}(\mathcal{L})$ for some $\mathcal{L}$, as well as find special members for which it is possible to find families of lines $\mathcal{L}$ with small $n(\mathcal{L})$. We find it convenient to consider separately the following cases:

(a) $\mathcal{O}(\mathcal{L})$ is a pencil;

(b) $\mathcal{O}(\mathcal{L})$ is a near-pencil;

(c) $\mathcal{O}(\mathcal{L})$ consists of a pencil together with two lines not in the pencil.

(d) Other possibilities for $\mathcal{O}(\mathcal{L})$; we call these "sporadic" families of lines.

For a given $\mathcal{L}$, the smallest known $n(\mathcal{L}^*)$ for $\mathcal{L}^*$ isomorphic with $\mathcal{L}$, will be denoted $n^*(\mathcal{L})$.

The case (a) is covered by Theorems 1 and 2 formulated above, while Theorem 3 supplies additional information concerning the relation between the size g of any pencil and an upper bound on $n^*(g)$. This information can succinctly be stated as follows:

Theorem 4. *Every* pencil of size g is $\mathcal{O}(\mathcal{L})$ for some $\mathcal{L}$ with $n(\mathcal{L}) \leq g + 2^g$, and $n(\mathcal{L}) \geq 3g$ for every g and $\mathcal{L}$. For each $g \geq 3$ there *exist* pencils of g lines for which $n^*(\mathcal{L}) = 3g$.

Proof. For completeness, and also since the proof requires only minor changes in order to establish Theorem 5, we repeat the arguments from [8]. The first part of the theorem is most intuitively argued in the dual formulation, hence we assume that we a given a family $\mathcal{P}$ of g collinear points. Without loss of generality we may take the line containing $\mathcal{P}$ to be the line-at-infinity. Consider the convex hull C of the points of $\mathcal{L}$ that lie in the finite part of the plane. Obviously, C is 2-dimensional, and every point of $\mathcal{P}$ determines two support parallel lines of C; since the point is omittable, each of the support lines contains two vertices of C. Hence each point of $\mathcal{P}$ is associated in this way with four vertices of C, and each vertex is associated with two points of $\mathcal{P}$. Therefore, by convexity, C has 2g vertices and so $n(\mathcal{L}) \geq 2g + g = 3g$.

To get the upper bound, let the g points $p_1, p_2, \dots, p_g$ of $\mathcal{P}$ be on the line-at-infinity. We start with any points q_1, q_2 not on the line-at-infinity, collinear with p_1, then construct translates q_3, q_4 of q_1, q_2 by a suitable vector v_2 in the direction of p_2; the four points q_j are translated by a vector v_3 in the direction of p_3; the length of the vectors being chosen at each step are such that no three of the resulting points q_j are collinear, nor are pairs of the q_j's accidentally collinear with any of the p_j. Repeating the same procedure for the remaining p_i's we arrive at a set $\mathcal{L}$ consisting of $g + 2^g$ points and having precisely the set $\mathcal{P}$ of omittable points.

To establish the existence part we revert to the "omittable lines" formulation. It is sufficient to start with a regular 2g-gon $\mathcal{R}$, and take as $\mathcal{L}$ the 2g lines determined by the sides of $\mathcal{R}$, together with the g mirrors (lines of symmetry) of $\mathcal{R}$ that pass through pairs of its antipodal vertices. These g mirrors form $\mathcal{O}(\mathcal{L})$. This is illustrated in Figure 1. In it, as well as in other diagrams, the lines in $\mathcal{O}(\mathcal{L})$ are shown in red, and the lines of $\mathcal{L}$ that are not omittable are shown in black. ♦

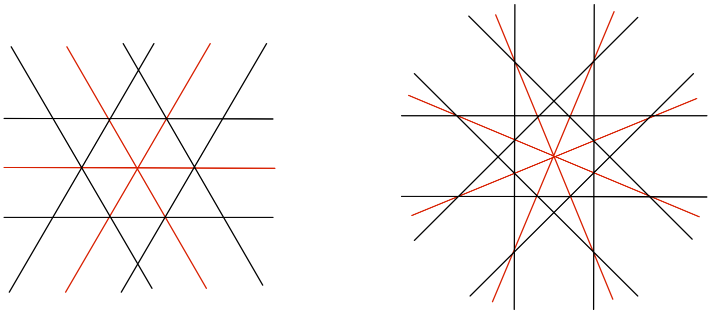


Figure 1. Examples of aggregates in which $\mathcal{O}(\mathcal{L})$ is a pencil and $n(\mathcal{L}) = 3g$. Here $g = 3, 4$.

These results can be extended to case (b).

Theorem 5. *Every* near-pencil of g lines is $\mathcal{O}(\mathcal{L})$ for some family $\mathcal{L}$ with $3g - 2 \leq n(\mathcal{L}) = 2^g + g - 2$; for each *even* $g \geq 4$ there *exist* near-pencils of g lines for which $n^*(\mathcal{L}) = 3g - 2$.

Proof. To prove the lower bound on $n(\mathcal{L})$ we only need to repeat the argument from the proof of Theorem 2, noticing that now we need only $2(g - 1)$ support lines, yielding $n(\mathcal{L}) > 2(g - 1) + g = 3g - 2$. For the upper bound we repeat the construction in the previous proof; now the line at infinity carries $q - 1$ points $p_1, p_2, \dots, p_{g-1}$, and the point p_g is not on that line. We modify the construction by first taking q_1 to coincide with p_g. This yields a set $\mathcal{S}$ of 2^{g-1} finite points, which includes p_g. All the other points of $\mathcal{P}$ are omittable with respect to this set $\mathcal{S}$, but we need to take care of p_g. To do this we construct a set $\mathcal{S}^*$ by mapping $\mathcal{S}\backslash\{p_g\}$ homothetically with center p_g and a ratio chosen not to generate any extraneous collinearities. Thus we can take as $\mathcal{L}$ the union of $\mathcal{S}$ with $\mathcal{S}^*$ and with the $g - 1$ points at infinity, for a total of $2^{g-1} + 2^{g-1} - 1 + g - 1 = 2^g + g - 2 = n(\mathcal{L})$ points.

On the other hand, again in the omittable line version, start with the same construction as above, with $\mathcal{R}$ a regular $(2g-2)$-gon; if g is even, then adding to the resulting aggregate the line-at-infinity yields a family $\mathcal{L}$ with $n(\mathcal{L}) = 3g - 2$. Examples are shown in Figure 2, while the illustration at right in Figure 1 makes clear that the construction does not work for odd q. ◆

For families of type (c) we have only the analog of the last part of Theorems 4 and 5:

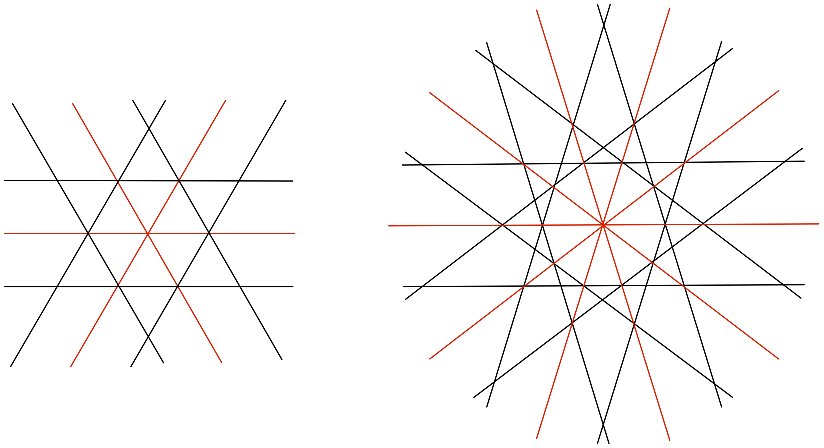


Figure 2. Examples of aggregates in which $\mathcal{O}(\mathcal{L})$ is a near-pencil; each aggregate includes the line at infinity, which is omittable. Shown are examples with $g = 4, 6$.

Theorem 6. For suitable families $\mathcal{F}$ of $g \geq 4$ lines, consisting of a pencil of $g - 2$ lines and two lines not in the pencil, there exist families $\mathcal{L}$ such that $\mathcal{F} = \mathcal{O}(\mathcal{L})$ and $n(\mathcal{L}) = 4g - 9$.

Proof. The examples needed to establish Theorem 6 can be describes as follows. For a given $g \geq 4$ we start with a regular $(2g - 4)$-gon. The family $\mathcal{L}$ consists of the $2g - 4$ lines determined by the sides of the polygon, and $2g - 5$ of the mirrors of the polygon — that is, all mirrors except one that bisects a pair of opposite sides of the polygon. Then $\mathcal{F} = \mathcal{O}(\mathcal{L})$ consists of the $g - 2$ mirrors that pass through the vertices of the polygon, together with the two lines determined by the sides that are bisected by the mirror that was not included in $\mathcal{L}$. These families illustrate $n(\mathcal{L}) = 4g - 9$. Examples are shown in Figure 3. ♦

Theorem 7. Sporadic families $\mathcal{L}$ are known for $(g(\mathcal{L}), n(\mathcal{L})) = (1, 5)$, $(2, 7)$, $(5, 13)$, $(6, 14)$, $(7, 19)$, $(8, 20)$, $(9, 21)$, $(8, 32)$, $(9, 33)$, $(10, 34)$, $(11, 35)$, $(12, 36)$, $(13, 37)$.

Proof. Examples establishing this are shown in Figure 4. These sporadic aggregates are noteworthy because the sets $\mathcal{O}(\mathcal{L})$ are different from the ones covered by Theorems 4, 5 and 6. Several examples not explicitly illustrated can be derived by adding the line at infinity, or deleting one or several lines, as appropriate. ■

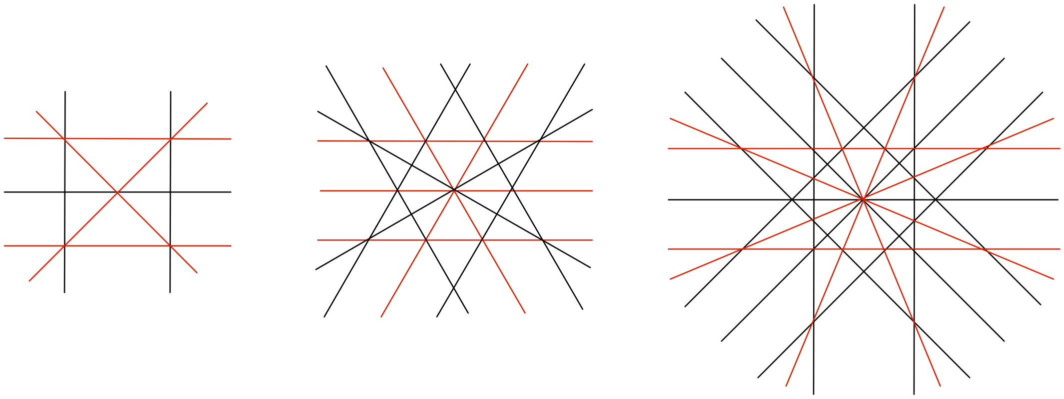
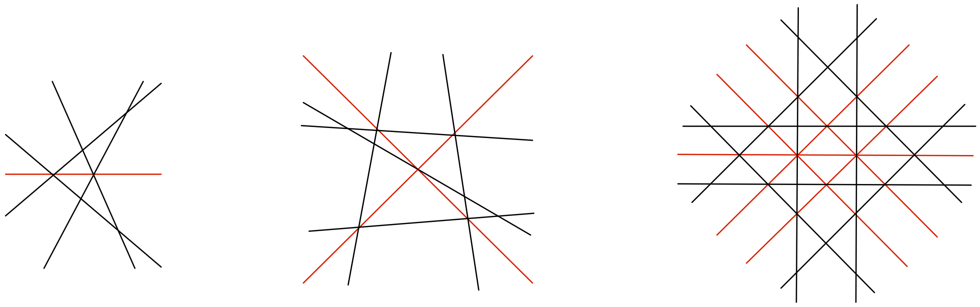


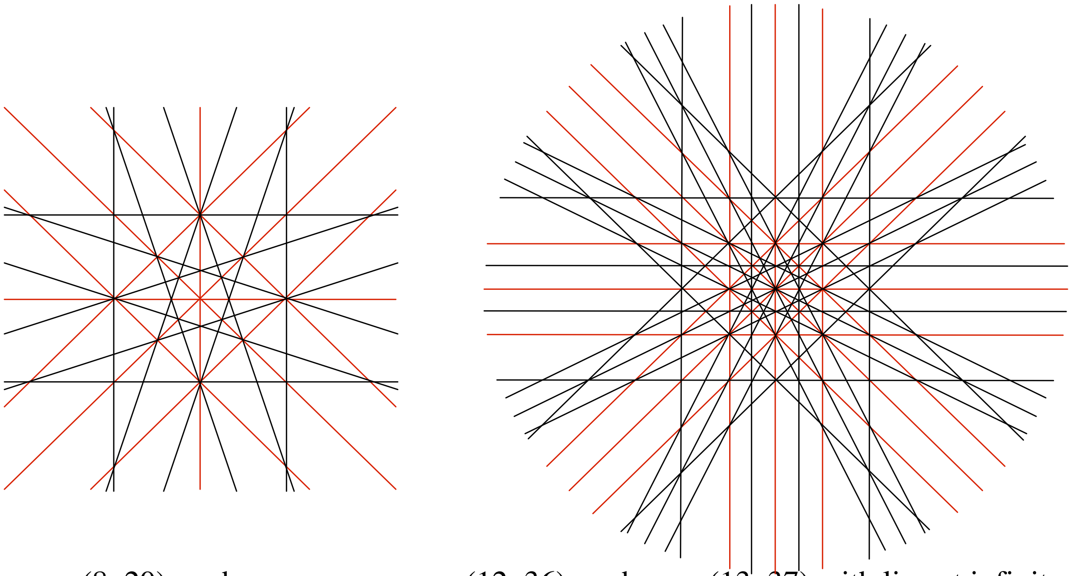
Figure 3. Examples of aggregates covered by Theorem 6, for g = 4, 5, 6.



g = 1, n = 5

g = 2, n = 7

g = 5, n = 13;
with line at infinity g = 6,   n = 14



(8, 20) as shown,
(9, 21) with line at infinity,
(7,19) without the middle vertical line.

(12, 36) as shown, (13, 37) with line at infinity.
Up to four of the middle omittable
lines can be deleted.

Figure 4. Representative members of the sporadic family. The values of (g, n($\mathcal{L}$)) are shown near each.

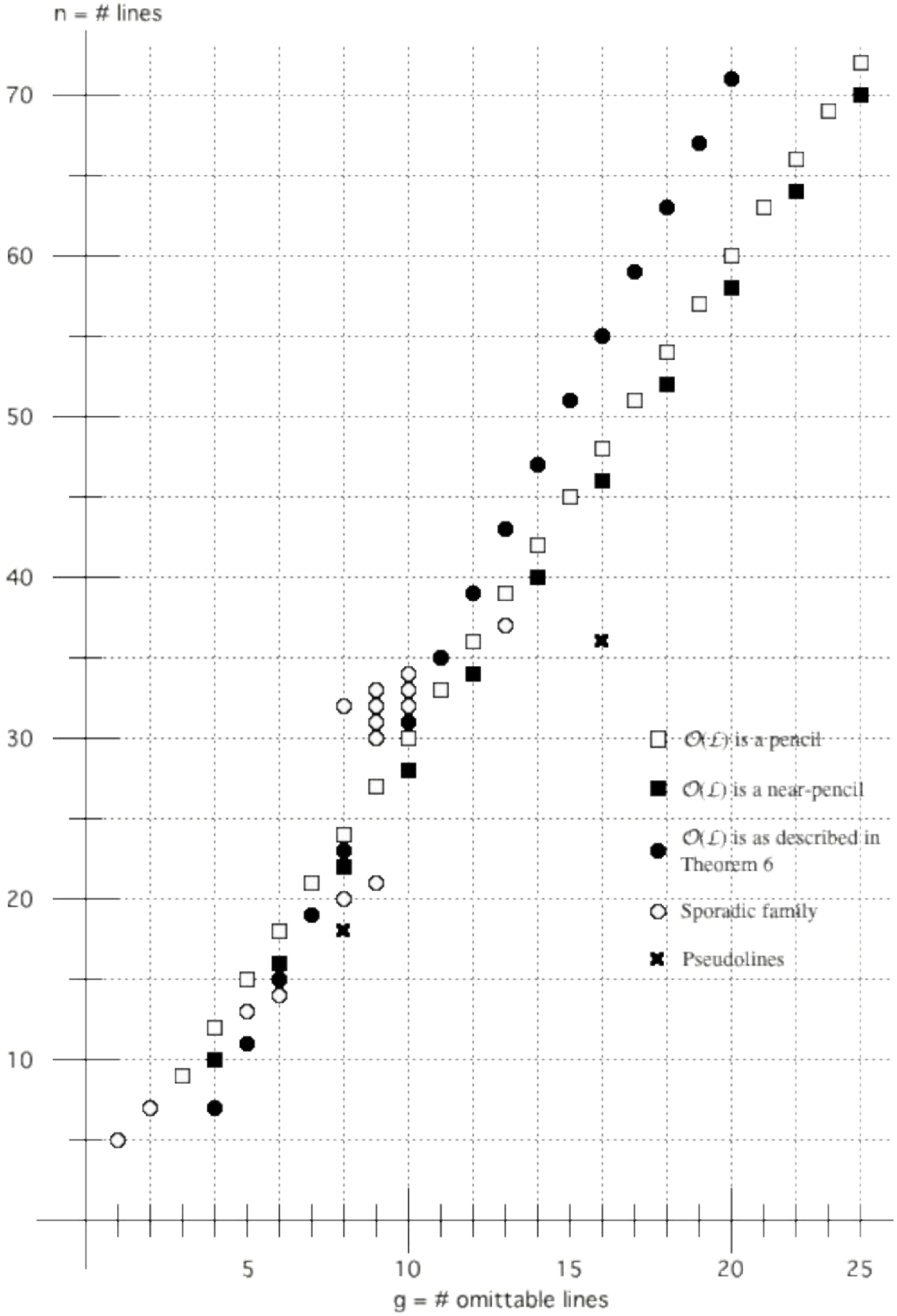


Figure 5. The explicitly known pairs (g, $n(\mathcal{L})$) of numbers of omittable lines and all lines in an aggregate. To avoid clutter, points corresponding to aggregates with $\mathcal{O}(\mathcal{L})$ a pencil that are to the left of (g, n) = (g, 3g) are not shown. The four families and the aggregates of pseudolines are explained in the text.

4. Comments.

(i) The term "aggregate" is introduced here for want of any word in the literature that describes the family consisting of a set of lines and their points of intersection. The topic discussed here is often framed as pertaining to arrangements of lines — but arrangements include the edges and faces determined by the set of lines, and these are not relevant in the present context. If a better term can be suggested, I would be happy to replace "aggregate", which was the best I could come up. It may be noted that in German the situation is even worse, sine there is no accepted translation for "arrangement", and the term "configuration" is used instead (see, for example, [B]) despite its well-known designation for a special class of families of points and lines.

(ii) It is clear that projective transformations applied to any family $\mathcal{L}$ of lines apply the same transformations to the set $\mathcal{O}(\mathcal{L})$ of omittable lines. However, in many cases, the same pair (g($\mathcal{L}$) , n($\mathcal{L}$)) occurs in projectively inequivalent aggregates, and even in aggregates that are not isomorphic. For example, the aggregate in Figure 6 is isomorphic to the aggregate in Figure 1, but is not projectively equivalent to it. (It should be noted that considered as *arrangements* these two examples are not isomorphic. This illustrates the need for a separate term for aggregates.) On the other hand, the aggregate with (g, n) = (12, 36) shown in the last part of Figure 4 is clearly not isomorphic with the example with the same (g, n) = (12, 36) constructed in the proof of Theorem 4.

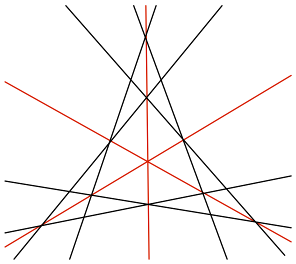


Figure 6. An aggregate with (g, n) = (3, 9) which is isomorphic with the first example in Figure 1, but not projectively equivalent to it.

(iii) The upper bounds on n($\mathcal{L}$) in Theorems 4 and 5 seem excessive. While it may be very hard to find exact upper bounds, deciding whether the growth is exponential would seem interesting.

(iv) The examples with (g, n) = (g, 3g) seem exceptional in several respects. To mention just two: An arbitrary number k ≤ g – 3 of the omittable lines may be deleted from the family while the remaining g – k lines remain omittable. In the opposite direction, there is no limit to the number of lines that can be added to the family while keeping those same g omittable lines. The illustration in Figure 7 is an example that admits many variations.

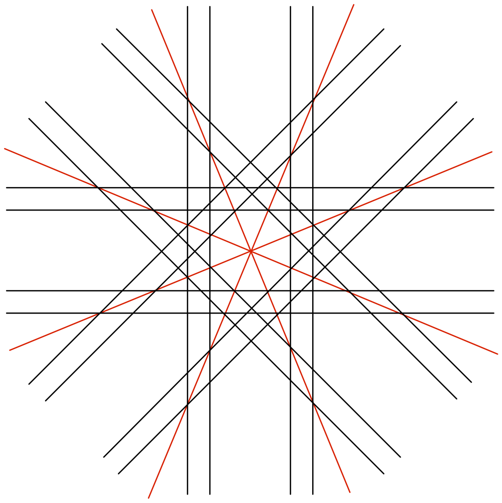


Figure 7. An example of an aggregate with $\mathcal{O}(\mathcal{L})$ a pencil but $n > 3g$.

(v) No examples of aggregates $\mathcal{A}(\mathcal{L})$ are known for which $\mathcal{O}(\mathcal{L})$ is a near-pencil with $g(\mathcal{L})$ odd.

(vi) The four families described above reflect the character of the set $\mathcal{O}(\mathcal{L})$ of omittable lines in examples known at present — a pencil, a near-pencil, a pencil and two lines not in the pencil, and other possibilities. The small number of examples in the sporadic family hints at the difficulty of constructing aggregates with the set $\mathcal{O}(\mathcal{L})$ far from being a pencil. We propose:

Conjecture 1. There are only finitely many sporadic families $\mathcal{O}(\mathcal{L})$.

In fact, there probably are no such families with $n \geq 37$.

(vii) Theorem 5 provides a counterexample to conjecture (A) of [12], that for $g \geq 6$ all omittable points are collinear. However, no counterexample is known for Conjecture (E) of [12], which can be formulated as follows: Let $\mathcal{M}$ and $\mathcal{N}$ be sets of $m \geq 1$ omittable and $k \geq 1$ non-omittable lines, respectively, of any aggregate $\mathcal{A}(\mathcal{L})$, such that any intersection point of $M \in \mathcal{M}$ and $N \in \mathcal{N}$ belongs to no other line in $\mathcal{M} \cup \mathcal{N}$. Then there are at least $m + k - 2$ lines of $\mathcal{L}$ that do not belong to $\mathcal{M} \cup \mathcal{N}$. Moreover, if either $m = 1$ or $k = 1$ then there are at least $m + k - 1$ such lines.

(viii) Theorem 6 provides a counterexample to Conjecture 2 of [8], which essentially asserted that pencils and near-pencils are the only infinite families that can occur as

$\mathcal{O}(\mathcal{L})$. If true, the above Conjecture 1 would imply that the sets $\mathcal{O}(\mathcal{L})$ of Theorems 4, 5, and 6 are the only infinite families. A related conjecture, close to Conjecture 1 of [8], is:

Conjecture 2. $\limsup_{n \to \infty} g/n = 1/3$.

(viii) Two problems arise in connection with the family (c) of aggregates. One is whether the bound of Theorem 6 is a lower bound for all families of lines of type (c); most likely this is true, but a proof seems elusive. The other question concerns the upper bound on $n(\mathcal{L})$ in terms of $g(\mathcal{L})$ for families of this kind — if such a bound exists at all. I conjecture that there is no bound, and in fact that the following family of points (in the Euclidean plane) does not coincide with $\mathcal{O}(\mathcal{L})$ for any family of points $\mathcal{L}$ belonging to (the dual version) of (c). Here are six points, in Cartesian coordinates: (0,0), (1, 0), (2, 0), (0, 2), (b, 1-b), (c, 1-c), for suitable reals b and c, which I believe are not the set of omittable points of any aggregate.

(ix) The various aggregates with relatively large g are either simplicial, or close to simplicial. (Here an aggregate is said to be *simplicial* provided the arrangement generates by its lines has only triangles as faces. More detail about simplicial aggregates can be found in [6] and [9].) This is the reason why the present paper deals with omittable *lines* instead of the omittable *points* considered in [8]. And just as with simplicial arrangements it is frequently impossible to add or delete lines without changing the character of the aggregate, so here as well, concerning aggregates with relatively large g. In most cases, the addition or deletion of a line decreases g by more than one.

(x) An aggregate of *pseudolines* in the projective plane consists of family of lines that have been modified in a finite part in such a way that any two have just a single point in common, at which they cross each other. See [7], [4] or [5] for an introduction to this topic. For such aggregates the concept of omittable (pseudo)lines can be defined with no change, and it is not surprising that new examples are possible. Two interesting cases are shown in Figures 8 and 9. For these families (g, n) equals (8, 18) or (16, 36), respectively. This seems to indicate that one should not expect Conjecture 2 to be valid for aggregates of pseudolines.

(xi) Many other directions of investigation of omittable lines are completely open. For example, how many lines (or pseudolines) can be omitted simultaneously? In the 3-dimensional space, are there any omittable planes in aggregates of finite families of planes? Here a plane is said to be omittable if its deletion does not decrease the number of vertices of the aggregate.

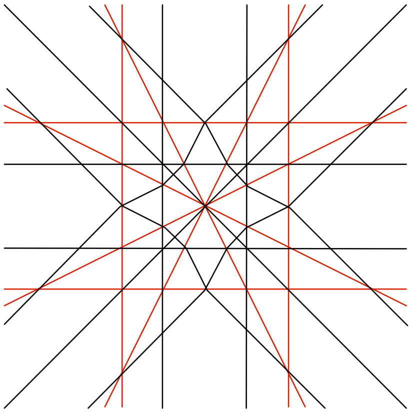


Figure 8. An aggregate of 18 pseudolines with eight omittable (pseudo)lines.

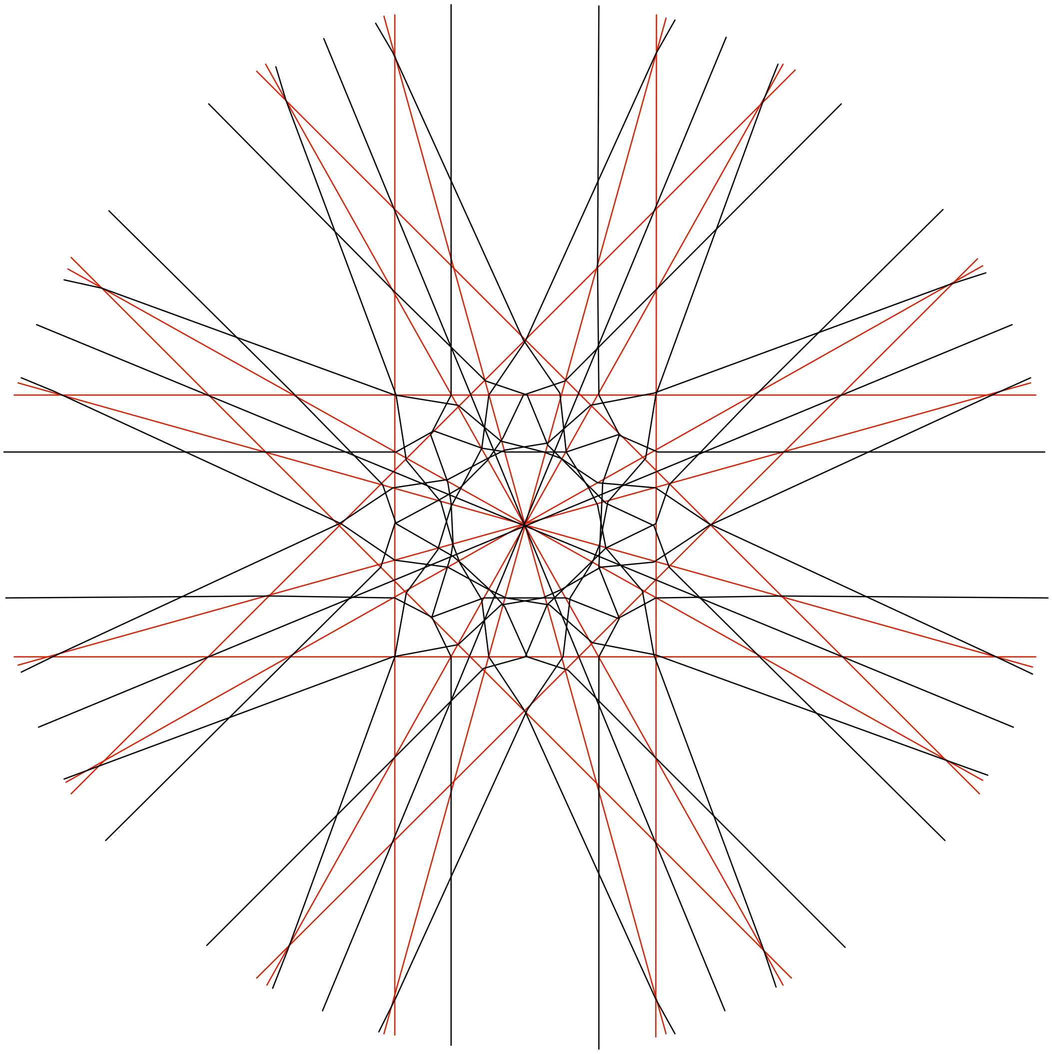


Figure 9. A pseudoline aggregate of 36 pseudolines, with 16 omittable.

Conjecture 3. There are no omittable planes in aggregates of planes in 3-dimensional space.

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