Essays on Omnichannel Sale

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Abstract

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In this dissertation, we study three problems related to omnichannel sale. In the first two chapters, we study omnichannel fulfillment where a retailer can use either the online or the offline channel to back up the other one by fulfilling its demand, because of stockout or low inventory availability. In the last chapter, we study an incentive problem where sales agents compete for a common customer base. This problem is loosely related to similar problems in omnichannel sales – when an online order is filled by an offline store, or vice versa, how should sales credit be shared?

In the first chapter, we study *store fulfillment* strategy of an omnichannel retailer that would like to leverage its established offline retail channel infrastructure to capture online sales after stockout. We consider a single newsvendor-type product that is sold in both online and offline channels to non-overlapping markets with independent Poisson demand. The offline store can

fulfill online demand at an additional handling and fulfillment cost, *k*, but not vice versa. We characterize the optimal rationing policy which determines whether online demand should be fulfilled or not given the remaining time and inventory. Due to the challenges of implementing the optimal policy, we further propose two simple and effective heuristics. We also show that integrating the rationing policy into retailer's higher-level inventory stocking and supply chain design decisions can have a significant impact on the retailer's inventory level and profitability. As a result, we propose an integrated policy, where the retailer builds separate inventory stocks for each channel but can use the offline inventory to back up online sales, subject to a rationing heuristic.

In the second chapter, we study discounted home delivery strategy of an omnichannel retailer that would like to leverage its online channel to help with offline sales when offline store has limited inventory. We consider a single newsvendor-type product that is sold in both online and offline channels to non-overlapping markets with independent Poisson demand. Store has option to offer customers discount *d* to incentivize them to accept home delivery of item rather than taking an inventory unit from the store. We assume customers are heterogeneous in their discount sensitivity, ranging from price sensitive to leadtime sensitive. We characterize the optimal dynamic discounting policy which determines at any point in season whether the offline store should offer discount or not, and if yes, the optimal discount level. Again, due to the challenges of implementing the optimal policy, we propose two simple and effective heuristics. We conduct an extensive numerical study and find that retailers can considerably benefit from discounted home delivery policy.

In the last chapter, we study sales agent's competition for a common customer base when sales agents are heavily paid by commission. We assume agents can affect customers perceived service quality through their service time. We build a simple model that captures agents time and quality trade off while competing for a common customer base. We assume that the agent's service time can increase sales probability, but the agents incur cost for their effort. We show that agent's speed up in the decentralized setting, in comparison to the central solution, which results in lower expected sale for the system. We suggest financial, operational, and informational tools that can align agent's incentive with that of the service provider's.

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DEDICATION

I want to also thank my parents for their patience and support through all these years. I can't imagine getting here without their support. They held my hand whenever I fell and motivated me to move each step ahead. They told me just taking ***one step at a time*** will get you there. It did! Their wisdom and open view of the world allowed me to expand my ambitions beyond not only geographical boundaries but my mental boundaries.

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Getting to the ***finish line*** is a cheerful and happy experience when you have your friends and your loved ones on your side.

To passing many more finish lines in the future!

INTRODUCTION

With the rapid advance of technology, the interaction of customers with retailers has evolved significantly. Customers used to interact with retailers through one channel (for example, online or offline). However, nowadays they may follow different paths to make a purchase. Some might start by researching online, but end up purchasing in a store, or vice versa. In response to shifts in customer behavior, retailers have also been integrating their back-end operations to make efficient use of their inventory and resources in terms of matching demand with supply. When a customer places an order online, a retailer might choose to fulfill a customer's demand from a store nearby rather than a warehouse, either because of stockout, or some other reason. On the other hand, when a customer in the store finds an item to be out-of-stock, a sales associate can order the item for home delivery either using inventory in the warehouse or at other stores. These are just two strategies retailers use to create a seamless experience for customer's regardless of the channel they use to make a purchase. This is called omnichannel fulfillment.

The ultimate idea of omnichannel fulfillment is to have multiple channels perform as a single channel by back each other up. For example, when an online channel runs out-of-stock, demand will be routed to physical stores, either the ones nearby or those with low expectation of running out of inventory. This strategy is called "*store fulfillment.*". Successful implementation of this strategy requires stores to have a rationing policy that can decide whether to use a unit of inventory to satisfy an online demand or to protect it for future demand in a store. In chapter 2, we study store fulfillment strategy in-depth and characterize the optimal rationing policy and suggest heuristics that are practical and effective.

While store fulfillment strategy is used to backup an online channel using store inventory, the reverse also exists. For example, when a store runs out of an item in a specific color or size, they might offer customers home delivery with the shipping cost discounted or free of charge. Currently in the industry a discounted home delivery offer is only made in the event of a stockout, however there might be benefits to making this offer even before stockout happens. In chapter 3, we study this problem in-depth and characterize the optimal discounted home delivery policy. Given the remaining time and remaining inventory, optimal policy determines if home delivery should be offered and at what discount level. We also suggest heuristics that are easy to implement and efficient.

In omnichannel, the boundary between online shopping and in-store shopping behavior disappears; therefore, all sales channels are serving a common customer pool. Seeing customers walking out of their store and losing a sale or commission would be of great concern to sales agents who are heavily paid by commission. However, this situation is not specific to omnichannel. Agents even within the same channel serving a common customer base might have the same concern which can create competition between them. Such competition might have quality implications for the system and therefore the experience of customers. In chapter 4, we study sales agent competition for sales within the common customer base and how that affects service quality. When agents are independent decision makers, we show that they provide lower service quality than system optimal. We then suggest different mechanisms; contract, allocation scheme, and customer choice to incentivize agents to increase their service time to the system's optimal level.

CHAPTER 1.STORE FULFILLMENT STRATEGY

1.1 INTRODUCTION AND MOTIVATION

Online shopping appeals to consumers for its convenience, information abundance, and possible lower price. Spurred by rapid development and spread of Internet and mobile technologies, online shopping has expanded exponentially. Whereas total retail sales in the US grew at an average annual rate of 5.46% from 2009 to 2013, online sales grew at an average annual rate of 15.65% in the same period (U.S. Census Bureau 2015). Online-only retailers, such as Amazon.com, led the charge with an average annual growth of 25.29% during that time (InternetRetailer 2014), but traditional retailers, such as Macy's and Walmart, did not stand still, and also invested heavily in their ecommerce expansion. Traditional retailers' online sales also grew an average of 15.56% annually during the same period (InternetRetailer 2014).

Online orders require different capabilities from the fulfillment center than offline ones. For example, the fulfillment center needs to be able to pick many items quickly in small batches and combine them for shipment, whereas traditional offline warehouses are set up to move products in large quantities to a smaller number of destinations. To manage online demand, many traditional retailers simply built more warehouses configured specifically for online orders. With an entry price of at least $100 million dollar per warehouse (Banjo *et al.* 2014), this infrastructure investment imposes a huge financial burden. Innovative retailers, such as Nordstrom and Macy's, started to explore alternative ways to cope with the surge in online orders. They realized that they already had plenty of inventory/storage capability all around the country – in the form of department stores – that could be used to meet the growing online sales (Mattioli 2012). When used properly, these established offline stores present the retailer with a great opportunity to integrate online and offline fulfillments. For example, 30% of Target's online orders were fulfilled from the stores (Chao 2016). This particular type of "omnichannel" approach allows the retailer to strategically position and use some of its inventory for both online and offline sales.

One possible implementation is to allow in-store pickup of online orders. For customers who want instant gratification, this is a great option. It also increases traffic to offline store, which may lead to extra sales (Gallino 2014). This chapter, however, analyzes a complementary approach called "store fulfillment". Using this approach, when online stock runs out, an online order can be routed to an offline store where a clerk (*e.g.*, Macy's) or a logistics supplier employee (*e.g.*, USPS's) will take the order and pick items from shop floor, pack them up in a backroom, and then ship to the customer. Naturally, the system should be judicious in filling online orders from the offline store. Some online orders should be rejected if the offline store is low on inventory or if it anticipates sufficient demand in its own store for the remaining inventory units.

There are several benefits to using the store fulfillment approach: 1) the retailer needs no special, extra effort to track and offer additional incentive to capture some online demand that would otherwise be lost, 2) it increases inventory turnover by reducing total inventory and results in a more efficient use of offline inventory, and 3) it helps to maintain offline margin by preemptively shipping out inventory units that could be left over at the end of the season. For the most part, customer experience is unaffected – the retailer does not need to inform customers when its online store runs out. Many customers are simply not aware, or do not care, that the shipment they receive come directly from an offline store, rather than an online warehouse.

The store fulfillment approach also has downsides. At present, many offline stores are not set up/organized for online order picking, packing, and shipping (Baird & Kilcourse 2011). The logistics costs and inefficiency of stores versus warehouses in handling online orders may result in margin erosion (Manhattan Associates 2011, Weedfald 2014). Moreover, it creates more work and inconvenience for store clerks who must now fulfill online orders and help in-store customers (Banjo *et al.* 2014). Store fulfillment can cost three to four times more when compared with that in an online warehouse (Banjo 2012). According to a PwC survey of CEOs (PwC 2014), only 16% of all 410 respondents say they can fulfill omnichannel demand profitably. Moreover, 67% ranked fulfillment cost as the highest cost for fulfilling orders. Thus, a successful omnichannel fulfillment strategy must seek balance between satisfying online demand and curbing fulfillment cost. It is our aim in this research to derive efficient and profitable fulfillment strategies and provide insights about managing store fulfillment.

While our research is motivated by the store fulfillment strategy adopted by large retailers, our model is equally applicable to smaller, offline retailers that are moving to become omnichannel. The initial heavy capital requirement for opening an online channel, including building a website with all its associated e-commerce functions (billing, fulfillment, processing returns, etc.), poses a big challenge to small retailers. Seizing this opportunity, a number of e-commerce platforms provide fulfillment services. For example, Fulfillment by Amazon (FBA) offers to manage inventory and fulfillment for independent sellers, who would retain ownership of their inventory, but let FBA handle the physical stocking, handling, and shipping of the products. Facing such choices, sellers must decide whether to use such services and, if so, how to coordinate the management of inventory with their existing offline store. From private communications, we know some Amazon sellers will completely rely on FBA to manage all of their inventory, yet others will divide up inventory between FBA and their own warehouse, and use the latter to satisfy both demand generated by their own website and Amazon demand that exceeds the inventory placed with FBA.

For products with a short sales season, the retailer may also have to make real-time inventory rationing decisions, when both online and offline demand chase after a limited quantity of offline inventory. The Yeti Rambler was such a highly sought after product for the 2015 Christmas season. Some retailers, such as Illinois-based Ace Hardware Corp. had to cut off online sales during the last few weeks of the holiday season, in order to prioritize sales to local, offline customers who they believe are more profitable.

This research aims to tackle the fulfillment problem at all three levels described above. At the strategic level, the retailer must decide whether to stock channel-specific inventory or rely on just the offline inventory to fulfill both demand streams. Then, at the tactical level once the fulfillment structure is determined, the retailer must decide the amount of inventory to stock. Finally, at the operational level, the retailer must be able to ration the remaining offline inventory, in real time, between the two demand streams to maximize profit. We refer to these three decisions as the fulfillment structure, stocking, and rationing decisions, respectively.

Our contribution to the academic literature and business practice is four-fold: First, we build an integrated model to tackle all three of the problems described above. Second, our model is set in a realistic continuous-time framework and, we can characterize the structure of the optimal rationing policy through which we develop two simple yet effective heuristics. Third, we are able to provide concrete insights and guidance to the omnichannel retailer regarding its fulfillment structure and stocking decisions; namely, the retailer should shift some of its online inventory to the offline channel and use a judicious rationing policy to achieve profit maximization. Fourth, using an extensive numerical study, we demonstrate the value of integrating all these decisions, and show that our proposed approach is both profit-efficient and robust.

The rest of this chapter is organized as follows. In Section 1.2, we review several related literature streams. In Section 1.3, we build analytical models to study the retailer's problems. In Section 1.4, we use an extensive numerical study to further explore the results developed in Section 1.3, and gain managerial insights into the retailer's decisions. Finally, we conclude in Section 1.5 by summarizing the contribution of this research and directions for future work.

1.2 Literature Review

Our study of the retailer's fulfillment strategy is closely related to the literature on e-fulfillment and multi-channel distribution (see Agatz *et al.* 2008, de Koster 2002, Ricker and Kalakota 1999). Both Bretthauer *et al.* (2010) and Alptekinoglu and Tang (2005) study static allocation followed by Mahar *et al.* (2009) who consider the dynamic allocation of online sales across supply chain locations. More recently, Mahar *et al.* (2010) and Mahar *et al.* (2012) explore store configuration when in-store pickups and returns are allowed. The paper that comes closest to our research is Bendoly *et al.* (2007) who study whether online orders should be handled in a centralized or decentralized fashion. In this chapter, not only do we compare these fulfillment structures, we also integrate this decision with the stocking and rationing decisions. This makes our approach more practical and closer to the omnichannel ideal.

Another stream of literature that's closely related to the fulfillment structure and stocking aspects of our research is that on inventory pooling, which started with Eppen (1979) who showed the benefit of warehouse consolidation in a single-period setting. This seminal work has since been extended to include the examinations of correlated and general demand distributions (Corbett and Rajaram 2006), demand variability (Gerchak and Mossman 1992, Ridder *et al.* 1998, Gerchak and He 2003, Berman *et al.* 2011, Bimpikis and Markakis 2014), and holding and penalty costs (Chen and Lin 1989, Mehrez and Stulman 1984, Jönsson and Silver 1987). Some researchers have identified conditions under which pooling may not be beneficial, such as service levels less than 0.5 (Wee and Dada 2005) and right skewed demand distribution under product substitution (Yang and Scharge 2009). When the demand streams are non-identical, Eynan (1999) shows numerically that if the margins are different, lower margin customers serve as a secondary outlet of leftovers. Ben-Zvi and Gerchak (2012) model demand pooling with different shortage cost, and show that retailers are better off if they pool their inventory and give priority to customers with higher underage cost when allocating inventory after demand is realized.

Our model differs in two aspects. First, unlike the above models where demand streams are different in only one dimension, our demand streams are different in several dimensions: not only do they vary in margin and leftover cost, the online orders also incur an extra handling and fulfillment cost if they are filled from the offline store. Second, our inventory rationing is performed as demand unfurls in real time, not after all the demand is realized as is the case in many previous works. Similar to the aforementioned papers, we develop our model in a single-period setting. Readers interested in periodic-review inventory pooling are referred to Erkip *et al.* (1990), Benjaafar et al (2005), and Song (1994).

The rationing of inventory between online and offline demand in our model is related to three separate but overlapping streams of research: inventory rationing, transshipment, and substitution.

INVENTORY RATIONING The inventory rationing literature is concerned with how to use pooled inventory to satisfy several classes of demand. Kleijn and Dekker (1998) give a review of early papers in the literature. In the periodic-review setting, Veinott (1965) first proves the optimality of threshold based rationing policy. His work is extended by Topkis (1968), Evans (1968), and Kaplan (1969). In the single-period setting, Nahmias and Demmy (1981) present a model for two demand classes and Moon and Kang (1998) extend it to multiple classes.

Our model differs from the existing literature (*e.g.,* Nahmias & Demmy 1981, Atkins & Katircioglu 1995, Frank *et al.* 2003, Deshpande *et al.* 2003, Melchiors *et al.* 2000) in that demand arrivals and decision epochs are continuous within a single, finite period setting. Chen *et al.* (2011) is the only other paper with a similar setting but they approximate the continuous arrivals by discretizing time. Another distinguishing feature of our model is that demand margins are endogenized by the retailer's rationing decision, because the margin on an online demand is lower if it's satisfied by a unit of offline inventory.

LATERAL TRANSSHIPMENT Under lateral transshipment, if one retail store is out of stock, another store can supply it at a cost. Lee (1987) studies a two-echelon model with one depot and *n* identical stores, and evaluates three rules on choosing which store should be the origin of transshipment. Wee and Dada (2005) find the optimal transshipment origin in a similar two-echelon model with one warehouse and *n* identical stores. Unlike these two papers which assume inventory is monitored in continuous time, the majority of works in the literature study the rationing problem in periodic-review inventory models. Moreover, to simplify analysis, they assume that transshipment occurs either at the end of the period after demand is realized (Krishnan and Rao 1965, Tagaras 1989, Tagaras and Cohen 1992, Robinson 1990; Rudi *et al.* 2001), or at the beginning of each period in anticipation of stockout (Allen 1958, Gross 1963, Karmakar and Patel 1977, Herer and Rashit 1999). In contrast, although we study a single-period inventory model, we allow rationing decisions to be made continuously throughout the period, *as demand arrives*. Only a few other papers in the literature allow transshipment decisions *within* a period in the periodic-inventory setting. Archibald *et al.* (1997) use a finite-horizon continuous-time Markov decision process to study whether to use transshipment or place an emergency order. Axsäter (2003) studies a store that uses a (R,Q) policy to replenish from the supplier, supplemented by lateral transshipment. Due to the complexity of the model, he derives a myopic rationing heuristic, which is still too complicated to be incorporated into the stocking problem. In our research, not only we are able to characterize the optimal rationing policy, we also develop a simple, effective heuristic that could be used in future modeling work. For a more detailed review on lateral transshipments, please see Paterson *et al.* (2011).

SUBSTITUTION This research has similarity to those on firm-driven product substitution, because when online inventory runs out, offline inventory can be used as a perfect substitute, at an extra cost. Pasternack and Drezner (1991) consider two substitutable products with stochastic demand within a single period. They show that total order quantity under substitution may increase or decreases depending on the substitution revenue. Bassok *et al.* (1999) show concavity and submodularity of the expected profit function under various assumptions in a single-period setting with downward product substitution. Deflem and Van Nieuwenhuyse (2013) examine the benefits of downward substitution between two products in a single-period setting. Again, all these papers make the simplifying assumption that substitution occurs at the end of the period after demand is realized. In contrast, in this research, substitution decisions are made in real time. For a review on the substitution literature, please see Shin *et al.* (2015).

1.3 MODEL SETUP

We consider the inventory management of a single product in a newsvendor-type setting, for an omnichannel retailer with one online and one offline store. The product under study has a long replenishment leadtime and short sales season, so there is only one chance before the season to stock.

Some retailers match online and offline sales prices, but many don't. We assume the retailer has fixed online and offline prices, but impose no restriction on the relationship between the two. Let p_n denote the product's overall unit profit margin at store $n \in \{0,1\}$ (throughout the chapter, we use subscript 0 for the online store, and 1 for the offline store). Furthermore, the two stores have independent, non-overlapping Poisson customer arrivals during the season with mean rates of λ_0 and λ_1.

Let T be the length of the sales season and S_n be the initial stocking level at store n. Any unsatisfied demand will be lost. When excess online demand is satisfied by a unit of offline inventory, there is an extra cost of k (similar to Axsäter 2003), representing the higher handling, overhead, and shipment costs in the offline store compared to warehouses. Therefore, a unit of offline inventory can fetch a profit of p_1 if used to satisfy an offline demand, but only $p_0 - k$ if used to satisfy an online demand. We make the reasonable assumption that $p_1 > p_0 - k$; this implies that the offline store prefers satisfying its own customer to an online one (which is clearly the case in the Yeti Rambler example, Nassauer 2015). The product has a life cycle of one season, so any leftover at the end of the sales season will be cleared. Let h_n be the cost of having a unit leftover of the product at store $n \in \{0,1\}$ at the end of the season. Furthermore, we assume $h_n \leq p_n$. This assumption applies to many products such as those in garment or fashion industry, and is commonly used in the literature (see, for example, Yang and Schrage 2009).

As described earlier, in an omnichannel setting the retailer has option to fulfil online orders using offline demand. Adding this new option to the model of separate channels has other implications. In such setting, retailer needs to make decision at three levels of strategical, tactical, and operational. Retailer's decisions are described below.

- ***Fulfillment Channel Design Decision:*** Since offline stores are now capable of handling online orders, the retailer has an option of either assigning all online orders to offline store or keeping both channel inventory and using offline store in the case of stock-out. Therefore, at the strategic level, the retailer must decide whether to stock channel-specific inventory (Non-Pooling) or rely on just the offline inventory to fulfill both demand streams (Pooling).
- ***Stocking Decision:*** Since offline inventory can be substituted for online inventory, it is expected that stocking decision will be different depending on channel design. Therefore, once the fulfillment channel design is determined, at the tactical level retailer must decide on the amount of inventory to stock (S_1 only or S_0 and S_1).

- ***Rationing Decision:*** At the operational level, the retailer must ration offline inventory, in real time, between the two demand streams with margin of p_1 and $p_0 - k$ to maximize profit.

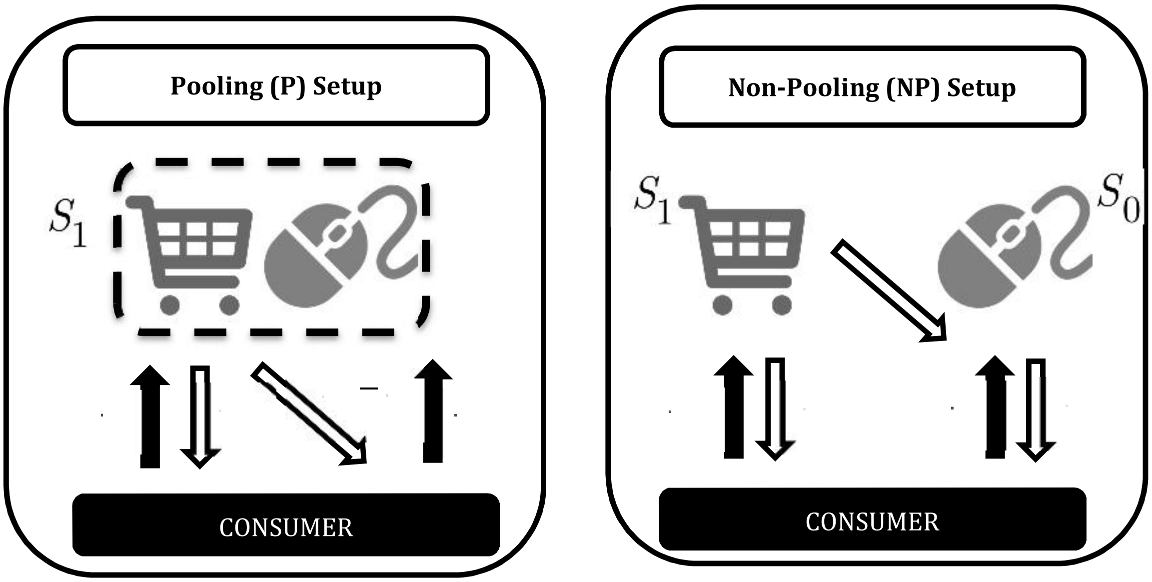

Figure 1.1 Channel Design of P and NP Fulfillment

While these decisions can be analyzed separately, we find the value of integrating these three levels of decision in our model to create more practical insight. In Section 1.3.1, we focus on the fulfillment channel design and stocking decisions assuming no inventory rationing. Then in Section 1.3.2, we derive the optimal rationing policy and develop two practical, effective heuristics for given inventory level(s). In Section 1.3.3, we integrate all the decisions.

1.3.1 Fulfillment Structure and Inventory Stocking Problems

When the retailer lets another party (*i.e.*, FBA) handle its online order fulfillment, or decides to have a dedicated online warehouse himself, each of the online and offline stores will have a dedicated pile of inventory. We call this the NP ("no pooling") structure, and the retailer must decide on the stocking level for each store, S_0 and S_1. On the other hand, retailer could also keep just one pile of inventory at the offline store, and use it to satisfy both the online and offline demand. We call this the P ("pooling") structure, and the retailer only needs to decide on the offline stocking level S_1 for both stores. There are pros and cons to each structure. In the P structure, the offline store reaps the inventory pooling benefit. In the NP structure, online demand can be satisfied using online inventory, thus avoiding the additional cross-channel handling and fulfillment cost k. In this section, we study when each structure should be adopted, assuming first-come first-served among all demand arrivals (i.e. there is no rationing).

Let $\Pi^{NP}(S_0, S_1)$ be the retailer's expected profit in the NP structure, given the stocking levels S_0 and S_1, and $\Pi^{P}\left(S_1\right)$ be the retailer's expected profit in the P structure, given the total stocking level S_1. We assume that the retailer is risk-neutral and seeks to maximize his expected profit.

In the NP structure, the two stores operate as separate newsvendor systems with independent Poisson demand with average rate λ_n, margin p_n, and leftover cost h_n. In the P structure, both stores are operated centrally as a single newsvendor system with Poisson demand with average rate $\lambda_{01} = \lambda_0 + \lambda_1$, leftover cost h_1, and a weighted product margin of

$$\omega = \frac{\lambda_1}{\lambda_0 + \lambda_1} p_1 + \frac{\lambda_0}{\lambda_0 + \lambda_1}\left(p_0 - k\right). \tag{1.1}$$

Clearly, the cost of using offline inventory to satisfy online demand, k, reduces the overall expected product margin. When k is high enough such that $\omega \leq 0$ then answer to fulfillment channel design becomes trivial as retailer will lose money under P structure, and thus NP will be the best choice. Therefore, here we are only considering when $\omega \geq 0$.

The newsvendor model results are summarized in Proposition 1. Following Hadley and Within (1963), we denote the PDF and the complementary CDF of a Poisson random variable with a rate of λ by $p(j,\lambda)$ and $P(j,\lambda)$, respectively.

Proposition 1 (Newsvendor stocking level and profit function)

1. In the NP structure, the optimal inventory level for store $n \in \{0,1\}$, S_n^{NP}, is the largest *S* such that:

$$P(S,\lambda_n T) \geq \frac{h_n}{h_n + p_n} \tag{1.2}$$

Furthermore, the retailer's optimal total profit function is:

$$\Pi^{NP}\left(S_0^{NP}, S_1^{NP}\right) = \sum_{n=0}^{1}\left(p_n S_n^{NP} - \left(p_n + h_n\right)\sum_{y=0}^{S_n^{NP}}\left(S_n^{NP} - y\right)p\left(y,\lambda_n T\right)\right). \tag{1.3}$$

2. In the P structure, the retailer's optimal inventory level at the offline store, S_1^P, is the largest *S* such that:

$$P(S,\lambda_{0,1} T) \geq \frac{h_1}{h_1 + \omega} \tag{1.4}$$

Furthermore, the retailer's optimal total profit function is:

$$\Pi^{P}\left(S_1^{P}\right) = \omega S_1^P - \left(\omega + h_1\right)\sum_{y=0}^{S_1^{P}}\left(S_1^{P} - y\right)p\left(y,\lambda_{0,1} T\right). \tag{1.5}$$

Proposition 1 shows the optimal stocking levels in the NP and P structures. The next proposition compares these quantities. In the P structure, the offline inventory is used to satisfy both online and offline demand; thus, the offline inventory must increase accordingly; hence, the first result in Proposition 2. This result is expected, but its proof is non-trivial due to the presence of extra handling and fulfillment cost k.

The second part of Proposition 2 means the total inventory is lower in the P structure, which is consistent with standard pooling results. Whereas the first result holds for any desired service level, the second result depends on our assumption of service level higher than 0.5. Wee and Dada (2005) show a similar result when stores are identical, demand is normally distributed, and service levels are above 0.5. We extend their results to the case with non-identical stores with the additional fulfillment cost, *k*.

Proposition 2 $S_1^{NP} \le S_1^{P} \le S_0^{NP} + S_1^{NP}$

Next, we compare the retailer's expected profit in these two settings.

Proposition 3

a) $\Pi^P\left(S_1^P\right) - \Pi^{NP}\left(S_0^{NP}, S_1^{NP}\right)$ is decreasing in k.

b) There exists a finite $\bar{k}$ such that $\Pi^P\left(S_1^P\right) - \Pi^{NP}\left(S_0^{NP}, S_1^{NP}\right) > 0$ if and only if $k < \bar{k}$.

Proposition 3 states that the preference of one structure to the other has a threshold form: smaller values of *k* favor P and larger values of *k* favor NP, with the threshold being $\bar{k}$. (When the NP structure is always preferred, $\bar{k}$ is set to be zero.) A numerical example is presented in Figure 1.2. This result is intuitive as large values of k impose heavy penalty for every fulfillment of online demand by offline inventory, pushing the retailer to carry online-specific inventory.

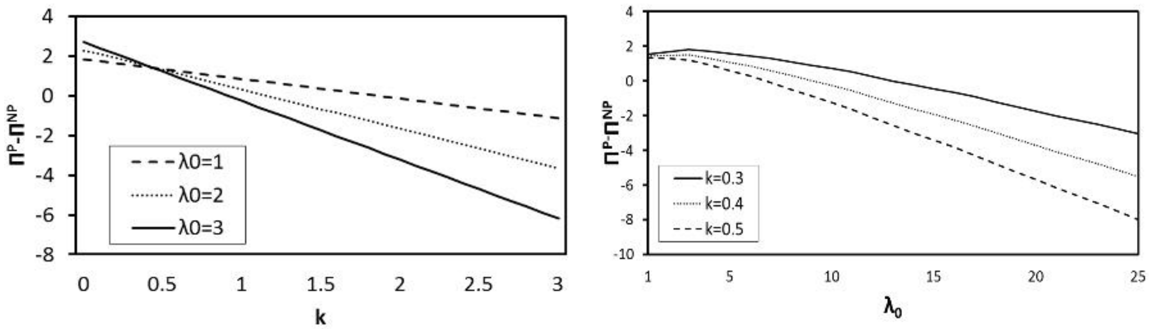

Figure 1.2 Difference between Optimal P Profit and Optimal NP Profit by k and λ_0

$(\lambda_1 = 10, p_0 = 10, p_1 = 10, h_0 = 1, h_1 = 1)$

From Figure 1.2, we further observe that the threshold $\bar{k}$ is decreasing in λ_0. That is, when online demand is large, it makes more sense to have online-specific inventory in order to avoid the cross-channel handling and fulfillment cost k. The next proposition gives theoretical support to this observation.

Proposition 4 $\Pi^P\left(S_1^P\right) - \Pi^{NP}\left(S_0^{NP}, S_1^{NP}\right)$ is submodular in k and λ_0.

Submodularity means the threshold on *k* that we found in Proposition 3 is decreasing in λ_0. Figure 1.3 depicts a typical dominance map of the P and NP structures. There is a monotonically decreasing switching curve in the k-λ_0 space. Below the curve, inventory pooling (P structure) is preferred and above it channel-specific inventory (NP structure) is preferred. A similar threshold on λ_0 is numerically observed by Bendoly *et al.* (2007).

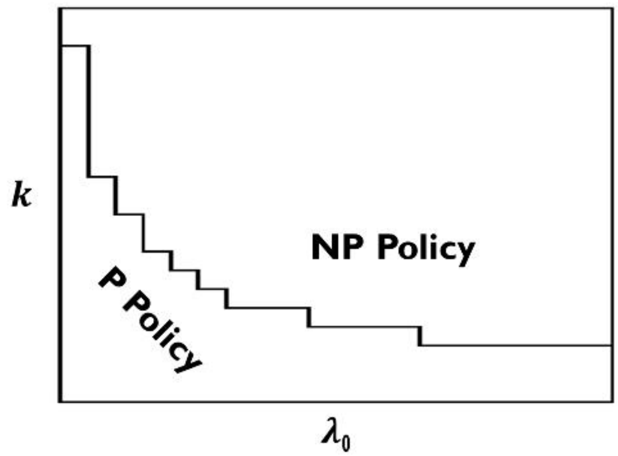

Figure 1.3 Decision Map for P versus NP Fulfillment

$(\lambda_1 = 10, p_0 = 10, p_1 = 10, h_0 = 1, h_1 = 1)$

The P structure always leads to lower inventory (Proposition 2), but not necessarily higher profit (Proposition 4), due to the cross-channel fulfillment cost *k* which is a novel, and important, feature of our model. This result adds to the existing literature on inventory pooling, which has explored the benefits and effect of pooling in regards to demand distribution, demand correlation, and cost parameters asymmetry (Yang and Schrage 2009, Gerchak and Mossman 1992, Pasternack and Drezner 1991).

1.3.2 Inventory Rationing Problem

Our analysis in Section 1.3.1 does not incorporate an inventory rationing strategy for allocating inventory to demand from the two channels. However, since a unit of offline inventory gets a higher margin when it is used to satisfy an offline demand ($p_1 > p_0 - k$), it may be sensible to protect some offline inventory for possible future offline customers, rather than using them to satisfy immediate online customers. Thus, a carefully chosen inventory rationing policy should improve profit for the retailer. Let θ $(0 \le \theta < T)$ denote the time in the season when the online store runs out of stock and thus, its demands are routed to the offline store. In the P structure, $\theta = 0$; rationing starts at the beginning of the season, and is applied throughout the season. In the NP structure, θ represents the instant online store runs out of inventory. From θ onward, all online demand is routed to the offline store but due to the rationing policy, it is not always filled. Any offline demand, however, is always satisfied as long as there is inventory.

Our inventory rationing problem is similar to the multi-class revenue management (RM) capacity allocation problem (Talluri and Van Ryzin 2004). A common solution approach is to approximate the problem using discrete-time setting dynamic program, where each time interval has a demand of only 0 or 1. However we choose to keep the continuous-time framework instead so that optimal policy determines the decision at any point in time rather than discrete times.

Let $t \in [0, T]$ denote the elapsed time from the beginning of the sales season. For the rationing problem, because of the memory-less property of Poisson arrival, the state variable for decision making is (i, t) where *i* is the level of offline inventory at time *t*. Following Liang (1999), the Bellman equation can be written as follows:

$$V(i,t) = \max_{u=0,1} \int_t^T \left(\alpha_0 \left(u\left(p_0 - k\right) + V\left(i-u,\tau\right)\right) + \alpha_1 \left(p_1 + V\left(i-1,\tau\right)\right)\right) \lambda_{0,1} p\left(0, \lambda_{0,1}\left(\tau - t\right)\right) d\tau \quad (1.6)$$

where u represents the decision to accept ($u = 1$) or reject online demand ($u = 0$). If an online demand arrives at time $t \geq \theta$, it is routed to the offline store, and the retailer must decide whether to sell to this customer and make the sure $p_0 - k$, or to reject the online demand and keep the unit for possible future offline use. Therefore, the retailer should optimally reject the online demand (i.e., protect this unit of inventory for offline use) if and only if its expected future value is higher than $p_0 - k$.

Intuitively, the marginal value of an extra unit of inventory should be higher when there is more time left in the sales season to sell it. Also at any given time, the marginal value of an extra unit of inventory should be decreasing in the existing inventory level: when the inventory level is low, an extra unit is highly likely to result in additional sales; this benefit diminishes as there is more inventory as the likelihood of being a left over starts to dominate. Therefore, we expect $V\left(i,t\right)$ to be concave in i, and submodular in $\left(i,t\right)$. Lemma 1 parallels a similar result in Liang (1999).

Lemma 1 $V\left(i,t\right)$ has the following properties:

a) $V\left(i,t\right)$ is concave in i,

b) $V\left(i,t\right)$ is submodular in $\left(i,t\right)$.

The concavity and sub-modularity properties are useful for deriving the optimal rationing policy for the retailer. When an online demand occurs, the retailer can realize a fixed profit of $p_0 - k$ if he accepts the demand. However, that comes at the expense of losing one unit of inventory, which could be used to fill a future higher-margin offline demand. The concavity in i shows decreasing marginal value in i, therefore the optimal rationing decision should have a threshold structure: if it is optimal to accept an online demand when on hand inventory is *i*, then it will also be optimal to accept online demand at inventory levels larger than *i*. We formally define a threshold-based rationing policy as follows:

There exists an inventory threshold $\tau\left(t\right)$ *for all* $t \in [0,T]$ *where an online demand at time* $t > 0$ *is accepted if and only if the offline inventory at time t is above* $\tau\left(t\right)$.

Submodularity of $V\left(i,t\right)$ further suggests that the marginal value of an additional unit of inventory is also decreasing in time. This means that when there is less time left, the retailer should protect less by lowering the protection level. Therefore, the inventory-based threshold is also decreasing in time. The following theorem is a key result in our study of the rationing problem.

Theorem 1 There exist a positive integer l and points in time $\left\{t_j\right\}_{j=0}^{l}$ where $0 = t_l < t_{l-1} \ldots < t_1 < t_0 < T$, such that a threshold-based rationing policy is optimal for the offline inventory, with the thresholds defined as follows: $\tau\left(t\right) = 0$ on $t \in [t_0, T]$ and $\tau\left(t\right) = j$ on $t \in [t_j, t_{j-1})$, $\forall 1 \leq j \leq l$.

To understand the specific pattern of the optimal threshold policy, we first consider *t* just before the end of the season (i.e. $t \approx T$). Any offline inventory at that point is almost certain to end up as leftover. Therefore, if an online demand occurs, the offline store should accept it and take the sure profit $p_0 - k$. As we move backward from *T*, more time is left in the sales season, and the probability of the marginal unit being leftover decreases. There comes a time, denoted by t_0, when the marginal value of protecting an offline inventory equals to $p_0 - k$. Then t_0 is the first indifference point when retailer becomes indifferent between protecting and not protecting one unit (last unit) of offline inventory. Moving further away from t_0, the retailer now strictly prefers to protect the last unit, so the threshold jumps to one. This procedure can be applied recursively to find the other *indifference times*, t_j, $\left(1 \le j \le l\right)$. At those time points, the retailer is indifferent between protecting *j* units and *j*+1 units. That is, it is equally optimal for the retailer to protect *j* units and *j*+1 units. Therefore, we also call $t_j \left(0 \le j \le l\right)$ the indifference points. For expositional purpose, we say the retailer prefers to protect j units at t_j.

Theorem 1 simplifies the computation of the optimal policy. It now suffices to calculate all the indifference points. For $t \in [t_j, t_{j-1})$ where $\tau(t) = j$, the maximum profit function can be written as:

$$\begin{aligned} V\left(i,t\right) &= \sum_{m=0}^{i-j-1}\left(\omega m + V\left(i-m, t_{j-1}\right)\right) p\left(m, \lambda_{0,1}\left(t_{j-1}-t\right)\right) \\ &+ \int_{\xi=t}^{t_{j-1}}\left(\omega\left(i-j\right) + \sum_{n=0}^{\infty}\left(p_1\left(n \wedge j\right) + V\left(\left(j-n\right)^{+}, t_{j-1}\right)\right) p\left(n, \lambda_1\left(t_{j-1}-\xi\right)\right)\right) \lambda_{0,1} p\left(i-j-1, \lambda_{0,1}\left(\xi - t\right)\right) d\xi. \end{aligned} \tag{1.7}$$

We can then use (6) and the indifference points property to recursively find t_0 t_1, t_2, … backwards. Figure 1.4 illustrates the structure of the optimal rationing policy via a numerical example, for different values of λ_0. As expected, when λ_0 increases, retailer stocks more inventory at time 0, and the optimal rationing policy employs more threshold levels, making it computationally expensive to find. Apart from computation time, deploying multiple frequently changing thresholds is not practical in industry. So it is worthwhile to study if such complicated threshold policy is required or we are able to capture most benefit, just by deploying a couple of these thresholds. This makes the case for developing simple yet effective heuristics for practical use.

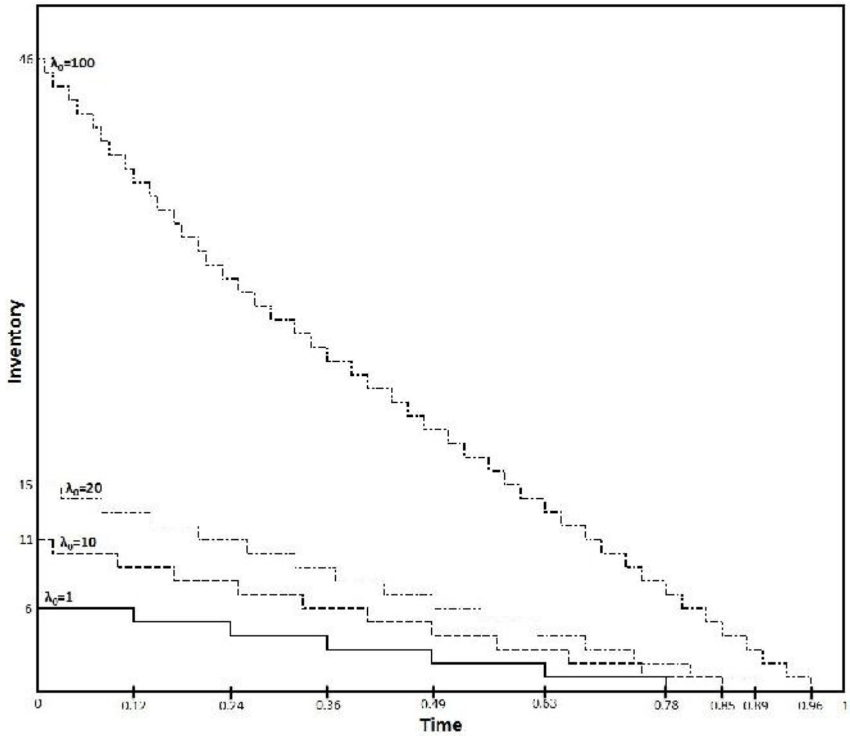

Figure 1.4 Optimal Rationing Policy as λ_0 increases

$(\lambda_1 = 10, p_0 = 10, p_1 = 10, h_1 = 1, k = 1)$

To do so, we take two different approaches. In the first, we limit the number of thresholds that can be used. In the second, we retain the multi-threshold structure of the optimal policy, but use a simple function to approximate $V\left(i,t\right)$ in (6). This function does not need to be evaluated recursively, so the indifference points are easier to find. We study these two heuristics in Sections 1.3.2.1 and 1.3.2.2 respectively.

1.3.2.1 Single Threshold (ST) Heuristic

In the optimal rationing policy, the threshold varies over time; therefore, a natural simplification is to use a fixed threshold throughout the season. This replaces the staircase shape of the optimal policy by a fixed horizontal line. Once this single threshold is set at time θ, it is used for the rest of the season. Because of this, the threshold should reflect the level of inventory, i, at time θ. For example, at time θ, if the inventory level is high, there isn't much need to protect inventory and the retailer should be more concerned about having leftover at the end of the season; therefore, a low value of single threshold should be set, and vice versa. We define this single threshold (ST) heuristic as follows:

If at time θ there are i units of inventory remaining in offline store, then there exists an inventory threshold $\tau^{ST}\left(i,\theta\right) \le i$. An online demand at time $t > \theta$ is accepted if and only if the offline inventory at time t is above $\tau^{ST}\left(i,\theta\right)$.

For ST, the offline store's maximum expected profit for $[\theta, T]$ can be written as $\Pi^{ST}\left(i,\theta\right) = \max_{\tau} H^{ST}\left(\tau \middle| i,\theta\right)$ where

$$\begin{aligned} H^{ST}\left(\tau \middle| i,\theta\right) &= \sum_{m=0}^{i-\tau-1}\left(\omega m - h_1\left(i-m\right)\right)p\left(m,\lambda_{0,1}(T-\theta)\right) \\ &+\int_{\xi=\theta}^{T}\left(\omega\left(i-\tau\right)+p_1\tau-\left(p_1+h_1\right)\sum_{n=0}^{\tau}\left(\tau-n\right)p\left(n,\lambda_1\left(T-\xi\right)\right)\right)\lambda_{0,1}p\left(i-\tau-1,\lambda_{0,1}(\xi-\theta)\right)d\xi. \end{aligned} \tag{1.8}$$

Even though the optimal ST threshold $\tau^{ST}\left(i,\theta\right)$ appears more complicated, it is much easier to use than the optimal policy because the threshold needs to be calculated only once and used for the rest of the season. Furthermore, we note that unlike the profit function of the optimal policy, $V\left(i,t\right)$ in (1.7), evaluation of $H^{ST}\left(\tau \middle| i,\theta\right)$ in (1.8) is not recruvie. Thus, the ST heuristic is much simpler to compute than the optimal rationing policy.

Lemma 2 $H^{ST}\left(\tau \middle| i,\theta\right)$ has the following properties:

a) $H^{ST}\left(\tau \middle| i,\theta\right)$ is concave in τ and i.

b) $H^{ST}\left(\tau \middle| i,\theta\right)$ is submodular in $\left(i,\theta\right)$.

The properties in Lemma 2 allow us to find the ST threshold for any given i and θ with the first order condition: $\tau^{ST}\left(i,\theta\right)$ is the largest τ such that $\Delta_{\tau} H^{ST}\left(\tau \middle| i,\theta\right) \geq 0$. The concavity and submodularity also imply that the optimal threshold $\tau^{ST}\left(i,\theta\right)$ is decreasing in both i and θ.

1.3.2.2 Newsvendor Thresholds (NT) Heuristic

We now take a different heuristic approach to keep the structure of the optimal policy but simplify how the thresholds are calculated. To do so, we approximate all the $V\left(i,t\right)$ terms on the right hand side of (6) by the newsvendor profit function for the offline store, $G_1\left(i,t\right) = p_1 i - \left(p_1 + h_1\right)\sum_{j=0}^{i}\left(i-j\right)p\left(i,\lambda_1\left(T-t\right)\right)$. This way, the calculation of indifference point simplifies to:

$$
\begin{aligned}
H^{NT}\left(i,t\right) = &\sum_{m=0}^{i-j-1}\left(\omega m + G_1\left(i-m,t_{j-1}\right)\right)p\left(m,\lambda_{0,1}\left(t_{j-1}-t\right)\right) \\
&+\int_{\xi=t}^{t_{j-1}}\left(\omega\left(i-j\right)+\sum_{n=0}^{\infty}\left(p_1\left(n\wedge j\right)+G_1\left(\left(j-n\right)^{+},t_{j-1}\right)\right)p\left(n,\lambda_1\left(t_{j-1}-\xi\right)\right)\right)\lambda_{0,1}p\left(i-j-1,\lambda_{0,1}\left(\xi-t\right)\right)d\xi.
\end{aligned}
\tag{1.9}
$$

We call the heuristic based on optimizing $H^{NT}\left(i,t\right)$ the newsvendor thresholds (NT) heuristic.

Lemma 3 $H^{NT}\left(i,t\right)$ has the following properties:

a) $H^{NT}\left(i,t\right)$ is submodular in $\left(i,t\right)$,

b) $H^{NT}\left(i,t\right)$ is concave in i.

The structural properties of $H^{NT}\left(i,t\right)$ in Lemma 3 allows us to show that the NT heuristic, just like the optimal policy, has the threshold structure.

There exists an inventory threshold $\tau^{NT}\left(t\right) = \max\left\{i : \Delta_i H^{NT}\left(i,t\right) \geq p_0 - k\right\}$ *for all* $t \in [0,T]$ *such that an online demand at time* $t > \theta$ *is accepted if and only if the offline inventory at time* t *is above* $\tau^{NT}\left(t\right)$.

Proposition 5 There exists a positive integer *n* and points in time $\{t_j'\}_{j=0}^{n}$, where $0 = t_n^{'} < ... < t_1^{'} < t_0^{'} < T$, such that the optimal NT threshold is $\tau^{NT}\left(t\right) = j$ for $t \in [t_j^{'}, t_{j-1}^{'})$ and $\tau^{NT}\left(t\right) = 0$ for $t \in [t_0, T]$.

Proposition 5 shows that the NT thresholds behave in a similar fashion to the optimal ones and decrease in unit steps over time. Clearly, the t_j's in Proposition 5 represents the indifference points, and are key to the complete characterization of the NT heuristic. Because the profit function evaluation of the NT heuristic is non-recursive, we are able to compute the indifference points in closed-form solutions, which require a straightforward inversion of a Poisson complementary CDF:

Proposition 6 The NT heuristic's indifference points are solutions to the following equation:

$$P\left(j, \lambda_1\left(T - t_{j-1}^{'}\right)\right) = \frac{p_0 - k + h_1}{p_1 + h_1}. \tag{1.10}$$

In the following Proposition, we show that the NT heuristic thresholds are always smaller or equal to the optimal policy thresholds. This means the NT heuristic will have fewer thresholds to compute (for clear delineation, we use superscript OPT to indicate the optimal rationing policy):

Proposition 7 $\tau^{NT}\left(t\right) \leq \tau^{OPT}\left(t\right)$ for all $t \in \left[\theta, T\right]$.

Both ST and NT heuristics are simple to understand, analyze, and implement. We will examine and verify their effectiveness in Section 1.4.

1.3.3 Integrated Fulfillment Structure, Stocking, and Rationing Policy

In Sections 1.3.1 and 1.3.2 we analyzed the fulfillment structure, stocking, and rationing decisions separately. Next, we integrate them into a coherent inventory policy. To that end, we use superscript ^{X,Y} to indicate an integrated policy (X,Y) where $X \in \left\{P, NP\right\}$ represents the fulfillment structure and $Y \in \left\{\varnothing, OPT\right\}$ represents the no rationing and optimal rationing policies. In the numerical studies in Section 1.4, we will extend Y to include all the rationing policies, therefore $Y \in \left\{\varnothing, ST, NT, OPT\right\}$.

Clearly, once the rationing policy is incorporated, the retailer should adjust his initial stocking level(s) accordingly. The following lemma provides needed technical properties of the profit function that are useful in computing the optimal stock levels.

Lemma 4

a) $\Pi^{P,OPT}(S_1)$ is concave in S_1 .

b) $\Pi^{NP,OPT}(S_0, S_1)$ is concave in S_1 and submodular in S_0 and S_1 .

We are also able to characterize how the integration of the optimal rationing policy affects the retailer's stocking levels and expected profits.

Proposition 8

a) $\Pi^{P,\varnothing}(S_1) \leq \Pi^{P,OPT}(S_1)$ and $\Pi^{NP,\varnothing}(S_0, S_1) \leq \Pi^{NP,OPT}(S_0, S_1)$.

b) $S_1^{NP,\varnothing} \leq S_1^{NP,OPT}$ and $S_0^{NP,\varnothing} \geq S_0^{NP,OPT}$.

Although the profit comparison in Proposition 8a) are for the same fixed S_0 and S_1, maximizing over them would preserve the direction of profit improvement. The profit increase is expected: Since no rationing is always a feasible action for the optimal rationing policy, profit should increase from the use of optimal rationing. We study the magnitude of such a profit improvement numerically in Section 4, and show that it can be substantial, especially in the NP structure.

The Proposition 8b) is intuitive but not straightforward to show. Within the NP structure, the use of optimal rationing helps the retailer to shift inventory from the online store to the offline store where its use is more flexible and, thus, more valuable. This shift in inventory allocation matches the trend in practice where large retailers are putting a higher emphasis on using offline stores to satisfy online demand, and moving inventory in that direction.

Proposition 8 demonstrate the effects of using the optimal rationing policy on the retailer's inventory and profit, within a fulfillment structure. Next, we discuss the impact of the rationing policy on the retailer's fulfillment structure itself.

Broadly speaking, rationing offers benefits at both the operational level and the strategic level. On the operational level, it helps the retailer to better utilize its limited offline inventory, in much the same way the expected marginal seat revenue (EMSR) model helps airlines to sell tickets in the RM literature. By differentiating the two types of customers and rejecting the lower-margin online customers at the appropriate time, the retailer can achieve a higher margin and better inventory utilization. This is how rationing helps in the P structure.

On the strategic level, rationing allows the retailer to combine the benefits of both P and NP as it can now place some inventory in the online store, in order to minimize cross-channel fulfillment costs caused by k. The retailer can choose this amount to be suitably low to minimize leftovers, knowing that it can always use offline inventory and rationing to handle any excess online demand. This is how rationing helps in the NP structure.

For any fixed stocking level(s), either benefit (operational or strategic) could dominate. However, when the retailer sets its inventory level(s) optimally, the operational benefit is minimized, and the strategic benefit will dominate. Therefore, we expect rationing to offer bigger profit improvement to a retailer that uses the NP fulfillment structure than to one that uses the P fulfillment structure. This will be numerically confirmed in Section 1.4.

1.4. Numerical Studies

In this section, we use a set of numerical examples to study the retailer's fulfillment structure, stocking, and rationing decisions. The three subsections here mirror those in Section 1.3. In Section 1.4.1 we compare the P and NP structures without the use of any rationing ($Y = \varnothing$). In Section 1.4.2 we compare the rationing policies and study their impact on the retailer's profit, margin, and inventory. Then in Section 1.4.3 we examine the integrated policies.

Without loss of generality, we normalize $p_0 = p_1 = 10$, and $T = 1$. Then we set λ_0 and k values at five different levels ranging from low to high.

- $\lambda_1 \in \{10,\ 20\}$ and $\lambda_0 \in \{0.2,\ 0.5,\ 0.8,\ 1,\ 2\} * \lambda_1$
- $k \in \{0.02,\ 0.05,\ 0.1,\ 0.2,\ 0.5\} * p_0$

Let λ_α represent ratio of online to offline demand rate, therefore $\lambda_\alpha \in \{0.2,\ 0.5,\ 0.8,\ 1,\ 2\}$. Because p_0 is fixed at 10, setting the online leftover cost h_0 is equivalent to setting service level $SL_0 = \dfrac{p_0}{p_0 + h_0}$. The latter is easier to understand in practice, so we set

- $SL_0 \in \{0.65,\ 0.75,\ 0.85,\ 0.95\}$, which implies $h_0 \in \left\{\dfrac{7}{13},\ \dfrac{1}{3},\dfrac{3}{17},\dfrac{1}{19}\right\}$.

The offline leftover cost is generally higher than the online one, so h_1 are set relative to h_0:

- $h_1 \in \{1,\ 1.25,\ 1.6\} * h_0$, which leads to the following offline store service levels for values of online service level:

SL_0	0.65			0.75			0.85			0.95		
SL_1	0.54	0.60	0.65	0.65	0.71	0.75	0.78	0.82	0.85	0.92	0.94	0.95

There are 600 cases in total.

1.4.1. Inventory Stocking Decisions

In Section 1.3.1, we prove that there is a switching curve on the λ_0-k space that demarcates whether the retailer should hold store-specific inventory, or pool all of his inventory at the offline store, in order to maximize total expected profit. Using the 600 examples, we study the switching curve and performance differences between the two structures on profit, product margin, and inventory levels. To streamline presentation, we show all the results as (NP, $\varnothing$)'s performance deviation from (P, $\varnothing$), defined as:

$$Performance\,Dev = \frac{Performance^{NP,\varnothing} - Performance^{P,\varnothing}}{Performance^{P,\varnothing}}, \tag{9}$$

where $Performance \in \left\{Profit, Margin, Inventory\right\}$. Margin is defined as profit per unit of inventory. All the tables present average performance deviations across the given set of cases.

Table 1.1 Policy (NP, $\varnothing$)'s Profit Deviation from Policy (P, $\varnothing$) by k and λ_α

Demand Rate (λ_α) \ Handling Cost (k)	0.2	0.5	1	2	5
2	-0.61%	1.54%	5.32%	13.79%	49.75%
1	-1.76%	-0.17%	2.59%	8.60%	31.67%
0.8	-2.08%	-0.67%	1.76%	7.01%	26.54%
0.5	-2.60%	-1.55%	0.25%	4.05%	17.39%
0.2	-2.86%	-2.34%	-1.45%	0.37%	6.24%

Table 1.1 clearly illustrates a switching curve pattern. Each cell corresponds to a possible combination of λ_α and k values, and, because there are 25 such combinations, each cell represents the average performance deviation for 12 cases. It is clear that P is better for small values of λ_α and k, and the switching curve is monotonically decreasing in both values. This confirms the analytical results in Proposition 3.

It should be noted that throughout our analysis we have ignored the fixed cost of building online warehouse, if the retailer decides to use the NP structure. This is a reasonable assumption if NP corresponds to using fulfillment services provided by an online platform, since the vast majority of the cost charged by such services (*i.e.*, Amazon Fulfillment) is variable based on volume of fulfillment; the fixed monthly fee is nominal. In the case when the retailer needs to build and operate the online warehouse by itself, a fixed cost would only affect the threshold of determining whether the P or NP structure is preferred. The switching curve would be shifted upward but the qualitative results would hold. Finally, our analysis here is on one SKU. The retailer may have already built the online warehouse for other purposes or other items. In such a case, sunk costs should not be considered in the fulfillment structure decision.

The 600 cases can clearly be divided into two groups: 385 cases in which (NP, $\varnothing$) is preferred and 215 in which (P, $\varnothing$) is preferred. Table 1.2 presents the performance comparisons between the two groups. The profit numbers in Table 1.1 are aggregated from those in Table 1.2, and the other performance measures can be aggregated similarly.

Table 1.2 Policy (NP, $\varnothing$)'s Performance Deviation from Policy (P, $\varnothing$)

	Policy (NP, $\varnothing$) **Is Preferred (385 cases)**	**Policy** (P, $\varnothing$) **Is Preferred (215 cases)**
Profit Dev	11.22%	-2.14%
Margin Dev	2.55%	-7.28%
Inventory Dev	8.48%	5.62%

When (NP, $\varnothing$) is preferred, the profit difference is significant (11.22% on average). This is accompanied by a big jump in inventory (8.48%), which is understandable given the lack of inventory pooling. Margin is modestly higher (2.55%). In the remaining cases (215 of 600) where (P,ϕ) is preferred, when either λ_0 or k is low, (P, $\varnothing$) outperforms (NP, $\varnothing$) by a modest amount

(2.14%) but it is able to achieve a significant inventory pooling benefit (inventory is lower by 5.62% on average), leading to a 7.28% increase in product margin.

These observations are in line with our expectation: when the NP structure is preferred, it uses a dedicated online stock to reduce the fulfillment cost of online orders, but that comes at the expense of increased inventory. Thus, whereas profit is higher significantly, the product margin enhancement is modest. Conversely, when the P structure is preferred, the profit improvement is limited, but it does so with lower inventory and a much improved product margin.

Overall, we learned from this set of tests that, when no rationing policy is in place, selecting a proper fulfillment structure could lead to significant improvement in profit, product margin, and/or inventory.

1.4.2. Inventory Rationing Decision

Base tests

To isolate the effect of the rationing policy, in this part we will fix the P/NP structure, and study how the various rationing policies compare with the baseline case of no-rationing. We define the performance deviation to be:

$$Performance\,Dev = \frac{Performance^{X,Y} - Performance^{X,\varnothing}}{Performance^{X,\varnothing}}, \tag{10}$$

where $Performance \in \left\{Profit, Margin, Inventory\right\}$, $X \in \left\{P, NP\right\}$, and $Y \in \left\{ST, NT, OPT\right\}$.

Table 1.3 provides a summary of the results when the retailer sets the stocking level(s) optimally for each rationing policy.

Table 1.3 Policy (X,Y)'s Performance Dev. from Policy (X, ∅), High Service Level (Using Optimal Stocking)

	X = P			**X = NP**		
	Y = ST	**Y = NT**	**Y = OPT**	**Y = ST**	**Y = NT**	**Y = OPT**
Profit Dev	0.01%	0.09%	0.12%	2.13%	2.15%	2.16%
Margin Dev	0.08%	0.34%	0.54%	6.31%	6.38%	6.42%
Inventory Dev	-0.07%	-0.24%	-0.40%	-3.85%	-3.90%	-3.93%

Two notable observations can be made from Table 1.3. First, the impact of rationing in the NP fulfillment structure is significant, for the optimal policy as well as the ST, NT heuristics. The retailer can reduce inventory by about 4% on average and improve profit by more than 2%. In contrast, although rationing is still beneficial, its impact is minimal in the P fulfillment structure. This is consistent with our assessment in Section 1.3.3 that the benefit of rationing is more significant in the NP structure than in the P structure.

Second, we note that both heuristics perform well, especially in the NP structure when the two heuristics are very close to the optimal policy. This has to do with the timing of θ, the start of rationing in the season. In the NP structure, when the retailer sets inventory levels optimally, θ should and usually does happen towards the end of the sales season. In such a case, only a few of the optimal policy's multiple thresholds will take effect. Thus, the simplification to a single threshold (in the case of ST) and the approximation of the value function (in the case of NT) will not deviate too much from the optimal policy. The two heuristics perform very well.

In practice, the two heuristics offer different advantages: ST is simpler and has analytical properties that make it more suitable to be used in analytical modeling; NT performs better and is more effective when θ happens early – due to either the P fulfillment structure, or an early occurrence of online stockout.

ADDITIONAL TESTS – LOW SERVICE LEVELS

Intuitively, rationing is most valuable when there are high demands for a limited number of inventory units. Therefore, we extend our tests to low service levels in both channels. We keep all the other parameters the same and change the service levels as follows:

SL_0	0.25			0.35			0.45			0.55		
SL_1	0.17	0.21	0.25	0.25	0.30	0.35	0.34	0.40	0.45	0.43	0.49	0.55

This gives us another 600 test cases, and the profit deviations are provided in Table 1.4:

Table 1.4 Policy (X,Y)'s Performance Dev. from Policy (X, $\varnothing$), Low Service Level (Using Optimal Stocking)

	X = P				**X = NP**		
	Y = ST	**Y = NT**	**Y = OPT**		**Y = ST**	**Y = NT**	**Y = OPT**
Profit Dev	0.01%	0.36%	0.54%		7.70%	7.81%	7.88%

We notice that the profit improvements in Table 4 are significantly higher than those in Table 3, especially for the NP structure. This confirms our intuition that rationing is more valuable for lower service levels. This is applicable to products with higher leftover cost *h* or lower margin *p*. It should be noted that the comparison between P and NP structures remain valid. That is, rationing has a much higher impact in the NP structure than in the P structure.

ADDITIONAL TESTS – UNDERSTOCKING

To further test how much rationing can help when inventory is low, we perform additional tests. Table 3 assumes that the retailer sets his inventory level(s) optimally. However, in practice the retailer may not be able to achieve the optimal stocking levels, either because of miscalculation of parameters or stocking levels, or because of limited supply of the product due to scarcity. In the next set of tests, we do not use the optimal stocking levels for the two stores. Instead, we fix their inventory level at 10%, 30%, and 50% below the optimal stocking level using optimal rationing. Then we apply the ST, NT, and OPT rationing policies to these fixed levels of inventory and compare the results.

Results for P structure are provided in Table 5**,** and they confirm our intuition. First, as understocking gets more severe, the benefit of rationing gets more pronounced. At the 30% understocking, the NT and optimal policies can get almost about 1% profit increase which is non-trivial for retailers who usually get single digit margins in practice. At 50% level, the profit improvement is substantial. Second, in the P structure, $\theta = 0$ and rationing starts immediately at the beginning of the season. Therefore, the ST heuristic, which uses a single threshold throughout the season, doesn't do as well, while the NT heuristic, by keeping the optimal policy's flexibility to adjust threshold over time, outperforms ST. The optimal policy, of course, does the best and can contribute a $4\%^+$ profit improvement.

Table 1.5 Policy (P,Y)'s Profit Deviation from Policy (P, $\varnothing$) when understocked (with retailer understocking below the (P,OPT) level)

	Y = ST	**Y = NT**	**Y = OPT**
10% understock	0.04%	0.22%	0.33%
30% understock	0.48%	0.93%	1.50%
50% understock	2.55%	2.66%	4.32%

Understocking in the NP structure is a bit more complicated as there are two stocking levels to consider. Below, we take the optimal (NP,OPT) stocking levels as the base and test the combination of 10%, 30%, and 50% below the base at each store, resulting in 9 combinations, each of which is presented by a cell in Table 1.6.

Table 1.6 Policy (NP,ST)/(NP,NT)/(NP,OPT)'s Profit Deviation from Policy (NP, $\varnothing$) (with retailer understocking from the optimal (NP,OPT) stocking levels)

(NP,ST) / (NP,NT) / (NP,OPT)		**Offline Understocking Levels**		
		10%	**30%**	**50%**
Online Understocking Levels	**10%**	5.83 / 5.86 / 5.88	2.22 / 2.25 / 2.28	0.52 / 0.53 / 0.56
	30%	9.42 / 9.49 / 9.58	3.40 / 3.46 / 3.57	0.73 / 0.75 / 0.82
	50%	12.95 / 13.08 / 13.32	4.46 / 4.55 / 4.82	0.90 / 0.92 / 1.09

It is notable that as the offline understocking increases the rationing policies' profit improvement suffers, but as the online understocking increases the rationing policies' profit improvement rises. This makes sense because rationing depends on the offline inventory. The more understocked the offline store, the fewer units of inventory can be used for rationing, and the less impactful rationing becomes. On the contrary, as the online store gets more understocked, θ gets smaller and there is a longer time period for rationing. That's when the benefit of rationing shines through. The message is that rationing makes the biggest impact when the more profitable channel has scarce resource.

Regardless of the combination of understocking, however, the rationing policies consistently outperform no rationing in the NP structure by a large margin.

SUMMARY

All the test results consistently suggest that rationing could have a significant impact on the retailer's inventory levels and profit, especially in the NP fulfillment structure and/or when the inventory level is low (due to insufficient supply or low service level target). Moreover, both heuristics perform extremely well in the NP structure. In the P structure, because there is a longer rationing time period, the multi-threshold NT heuristic tends to perform better than the inflexible, single-threshold ST heuristic, but overall the two heuristics are effective.

1.4.3. Impact of Inventory Rationing on Fulfillment Structure and Stocking Decisions

In the previous section, we study the impact of rationing within fixed P and NP structures. Here, we will integrate all the decisions, and see how rationing impacts the fulfillment structure as well. Table 7 compares the performance between (P,OPT) and (NP,OPT). This parallels Table 1.2 where $(P, \varnothing)$ and $(NP, \varnothing)$ are compared.

Table 1.7 Policy (NP,OPT)'s Performance Deviation From (P,OPT)

	Policy (NP,OPT) Is Preferred (595 Cases)	**Policy (P,OPT) Is Preferred (5 Cases)**
Profit Dev	8.56%	-0.13%
Margin Dev	4.59%	-0.13%
Inventory Dev	3.64%	0.00%

As predicted in Section 3.3, the use of rationing increases the advantage of the NP structure. This is clearly illustrated in Table 1.7, where 595 cases out of 600 prefer (NP,OPT) over (P,OPT). In contrast, the (NP, $\varnothing$) policy is preferred in only 385 cases in Table 1.2. Moreover, in the remaining 5 cases where (P,OPT) is preferred, the performance of the (NP,OPT) policy is nearly as good.

The robustness of the (NP,OPT) policy has practical implications. First, parameter estimates may not be accurate in practice. More importantly, our analysis is on a single product, but the retailer handles a large number of such single-season products. It's unrealistic to expect the retailer to tailor its fulfillment structure for each individual product. Therefore, retailers who prefer a uniform fulfillment structure for all of its products can choose the NP structure with the confidence that, together with an appropriate rationing policy, this approach would be either optimal or near-optimal for all the products.

Like Table 1.1, Table 1.8 below breaks down the profit deviation by the k and λ_0 parameter values. Comparing the two tables, we can see clearly that the use of rationing moves the switching curve closer to the lower left corner, just as discussed in section 3.3. The dominance of the (NP,OPT) policy is also apparent: it is optimal in almost all the cases. Depending on the k and λ_0 parameters, the profit advantage of the (NP,OPT) policy also could be very substantial.

Table 1.8 Policy (NP,OPT)'s Profit Deviation from Policy (P,OPT) by *k* and λ_α

$\lambda_\alpha = 2$	1.41%	3.40%	6.96%	14.99%	48.89%
$\lambda_\alpha = 1$	0.94%	2.35%	4.86%	10.45%	31.96%
$\lambda_\alpha = 0.8$	0.77%	1.95%	4.08%	8.82%	26.53%
$\lambda_\alpha = 0.5$	0.45%	1.23%	2.66%	5.84%	17.25%
$\lambda_\alpha = 0.2$	0.14%	0.56%	1.32%	3.02%	9.14%
	k = 0.2	*k* = 0.5	*k* = 1	*k* = 2	*k* = 5

We already know from Table 1.3 that both (NP,ST) and (NP,NT) are effective policies within the NP fulfillment structure. Results of a further analysis, presented in Table 1.9 below, shows that (NP,ST) and (NP,NT) have nearly as good a performance against (P,OPT). Therefore, we conclude that the retailer should consider using either (NP,ST) or (NP,NT) as its stocking-rationing policy. Relating back to the TRUA case, online demand may be too small initially, but once it starts to increase, it would be better for TRUA to establish an online warehouse to handle a base level of online demand, and then use offline stores as backup. Similarly, smaller retailers should seriously consider using the fulfillment services provided by online platforms.

Table 1.9 Policy (NP,ST)/(NP,OPT)'s Performance Deviation from Policy (P,OPT)

	Policy (NP,ST)/ (NP,NT) Is Preferred (595 Cases)	**Policy (P,OPT) Is Preferred (5 Cases)**
Profit Dev	8.53% / 8.55%	-0.13% / -0.13
Margin Dev	4.47% / 4.55%	-0.13% / -0.13
Inventory Dev	3.73% / 3.67%	0.00% / 0.00

1.5. Summary and Future Research Directions

In this research, we study the inventory management problem of an omnichannel retailer who already has an established offline store and is looking to leverage it to help with online sales. Our model incorporates relevant decision factors at three different levels: fulfillment structure (strategic), inventory (tactical), and rationing (operational). We derive the optimal rationing policy structure and develop two simple heuristics that we demonstrate, through an extensive numerical test, to be very effective. Integrating the rationing policy into higher-level decisions, we showed that it can have significant impact on the retailer's stocking and fulfillment structure decisions. The integrated (NP, ST) policy – where the retailer has an inventory stock dedicated to online sales, but can also use offline inventory as backup when needed, subject to the ST rationing heuristic – is proved to be simple, effective, and robust.

Being a first model in our attempt to analyze the omnichannel strategies, this research also points to several directions for future research. The current research focuses on the store fulfillment online-to-offline approach, where inventory backup is uni-directional; it would be interesting and important to extend the study to bi-directional backup and rationing. This requires the retailer to invest in store staffing to actively capture potential lost sale in the store. The retailer could also invest in technology (*i.e.*, QR codes, apps, online portal in the store) so that customers can order items online directly when shopping in the offline store (*e.g.*, Athletha). This trend is slowly taking hold in practice so a rigorous analytical study would offer guidance.

Even within the online-to-offline framework it would be important to incorporate the option of in-store pickup into the analytical model. We can extend our model by having three classes of demand to the offline store. A nested threshold policy may be optimal but a rigorous study is needed to find out its impact on the stocking decisions as well as the overall cross-channel fulfillment structure for the retailer.

Another important topic is cross-channel product return. Many retailers now allow customers to return items bought online to offline stores. To the extent most such returns are resalable, this makes the inventory decisions much more complicated, and interesting.

Finally, we consider a retailer that owns both channels, so the decisions are centralized to maximize total profit. Practically, however, each channel may have its own profit target and consideration, even within the same retailer. In such a case, one must consider incentive issues: for example, when an offline store fills an online order who gets the credit and how much credit? Getting a good handle on this is essential to the success of an omnichannel approach.

APPENDIX

Proof of Proposition 1

Retailer's expected profit when demands are pooled (P structure) can be calculated using ***demand realization***. However, analysis of that model becomes complicated, so we use ***weighted margin*** based on mean demands similar to Ben-Zvi and Gerchak, 2012. In this case, we have a newsvendor with margin of ω and leftover cost of h_1 and Poisson demand with rate of $\lambda_{0,1}$. Therefore, optimal stocking level of retailer can be calculated using critical ratio. Please refer to Hadley and Within (1963) on page 298 for proof of optimal stocking level for newsvendor model.

Proof of Proposition 2

We provide the proof in two parts as follows:

A. $S_1^P \geq S_1^{NP}$

B. $S_1^P \leq S_0^{NP} + S_1^{NP}$

We need the following two lemmas on the Poisson distribution function first. As a reminder, k is bounded such that $\omega \geq 0$.

Lemma A1 $p(n,\lambda) < \dfrac{1}{2}$ for all n and λ.

Proof: For any given $n > 0$, we have $p(n-1,n) = p(n,n)$ and $p(n-1,n) + p(n,n) < 1$. So $p(n,n) < 1/2$. Next, $\dfrac{\partial p(n,\lambda)}{\partial \lambda} = p(n-1,\lambda) - p(n,\lambda) = p(n-1,\lambda)\left(1 - \dfrac{\lambda}{n}\right)$. So $p(n,\lambda)$ is nondecreasing in λ for $\lambda \le n$ and non-increasing in λ for $\lambda \ge n$. Therefore, $p(n,\lambda) \le p(n,n) < 1/2$ for all n and λ.

Lemma A2 $P(n,\lambda)[1 - P(n,\lambda)] \le \lambda p(n-1,\lambda)$ for all n and λ.

Proof: Define $L(\mathrm{n},\lambda) \triangleq -2P(n,\lambda) - (n-1) + \lambda$. Then $\dfrac{\partial L(n,\lambda)}{\partial \lambda} = -2p(n-1,\lambda) + 1 > 0$ due to Lemma A1. Moreover, $L(n,0) \le 0$ and $L(n,\infty) = \infty$. This means there exists a finite A such that $L(n,\lambda) < 0$ for $\lambda < A$, $L(n,A) = 0$, and $L(n,\lambda) > 0$ for $\lambda > A$.

Next, define $G(n,\lambda) \triangleq P(n,\lambda)[1 - P(n,\lambda)] - np(n,\lambda)$. Since $\dfrac{\partial G(n,\lambda)}{\partial \lambda} = L(n,\lambda)p(n-1,\lambda)$, $G(n,\lambda)$ is decreasing for $\lambda < A$, and increasing for $\lambda > A$. Noting that $G(n,0) = G(n,\infty) = 0$, we conclude that $G(n,\lambda) \le 0$ for n and λ. Since $\lambda p(n-1,\lambda) = np(n,\lambda)$, we complete the proof.

Part A) Now, we go back to Proposition 2. Define $J(\lambda_0) \triangleq \dfrac{\omega P\left(S_1^{NP}, \lambda_{0,1}\right)}{1 - P\left(S_1^{NP}, \lambda_{0,1}\right)}$ where

$\lambda_{0,1} = \lambda_0 + \lambda_1$. We know $\lim\limits_{\lambda_0 \to 0^+} J(\lambda_0) = \dfrac{p_1 P\left(S_1^{NP}, \lambda_1 T\right)}{\sum_{y=0}^{S_1^P - 1} p\left(y, \lambda_{0,1} T\right)} \ge \dfrac{p_1 \dfrac{h_1}{p_1 + h_1}}{\sum_{y=0}^{S_1^P - 1} p\left(y, \lambda_{0,1} T\right)} \ge \dfrac{p_1 \dfrac{h_1}{p_1 + h_1}}{\dfrac{p_1}{p_1 + h_1}} = h_1$ and

$\lim\limits_{\lambda_0 \to \infty} J(\lambda_0) = \infty$. Moreover,

$$\frac{\partial J(\lambda_0)}{\partial \lambda_0} = \frac{-\left(\omega - p_0 + k\right) P\left(S_1^{NP}, \lambda_{0,1}\right)\left[1 - P\left(S_1^{NP}, \lambda_{0,1}\right)\right] + \omega \lambda_{0,1} p\left(S_1^{NP} - 1, \lambda_{0,1}\right)}{\lambda_{0,1}\left[1 - P\left(S_1^{NP}, \lambda_{0,1}\right)\right]^2}$$
$$= \frac{\omega\left\{\lambda_{0,1} p\left(S_1^{NP} - 1, \lambda_{0,1}\right) - P\left(S_1^{NP}, \lambda_{0,1}\right)\left[1 - P\left(S_1^{NP}, \lambda_{0,1}\right)\right]\right\}}{\lambda_{0,1}\left[1 - P\left(S_1^{NP}, \lambda_{0,1}\right)\right]^2} \geq 0 \quad \text{due to Lemma A2 and } p_0 > k.$$

Therefore, $J(\lambda_0) > h_1$ for all S_1^{NP} and λ_0, which means $P\left(S_1^{NP}, \lambda_{0,1}\right) < \frac{h_1}{h_1 + \omega}$. Next, $\omega \leq p_1$ implies $\frac{h_1}{h_1 + \omega} \geq \frac{h_1}{h_1 + p_1}$. These two facts together lead to $S_1^P < S_1^{NP}$.

Part B) Now, we prove $S_1^P \leq S_0^{NP} + S_1^{NP}$. For expositional ease, we assume equation (4) holds as equality (that is, stock levels can be fractional). Let $f(p) = \frac{h_1}{h_1 + p}$. It is a convex function, so we have $f(\omega) \geq \alpha_1 f(p_1) + \alpha_0 f(p_0 - k)$. Moreover, $f(p_0 - k) \geq f(p_0) = \frac{h_1}{h_1 + p_0} \geq \frac{h_0}{h_0 + p_0}$, where the first inequality is due to monotonicity of f and the second is due to $h_1 \geq h_0$. Therefore, we have shown that

$$\frac{h_1}{h_1 + \omega} \geq \alpha_1 \frac{h_1}{h_1 + p_1} + \alpha_0 \frac{h_0}{h_0 + p_0}. \tag{1.11}$$

Applying mean value theorem, we know that there exists $(u, v) = c\left(S_0^{NP}, \lambda_0\right) + (1 - c)\left(S_0^{NP} + S_1^{NP}, \lambda_{0,1}\right)$ for some $0 \leq c \leq 1$ such that

$$P\left(S_0^{NP} + S_1^{NP}, \lambda_{0,1}\right) = P\left(S_0^{NP}, \lambda_0\right) + \left.\frac{\partial P}{\partial S}\right|_{(u,v)}\left(S_1^{NP}\right) + \left.\frac{\partial P}{\partial \lambda}\right|_{(u,v)}\left(\lambda_1\right). \tag{1.12}$$

Using a continuous $P(u,v)$, we know $\frac{\partial P(u,v)}{\partial u} = -p(u,v)$ and $\frac{\partial P(u,v)}{\partial v} = p(u,v)$. So (1.12) becomes

$$P\left(S_0^{NP} + S_1^{NP}, \lambda_{0,1}\right) = P\left(S_0^{NP}, \lambda_0\right) - p(u,v)\left(S_1^{NP} - \lambda_1\right). \tag{1.13}$$

Similarly, there exists $(x,y) = d\left(S_1^{NP}, \lambda_1\right) + (1-d)\left(S_0^{NP} + S_1^{NP}, \lambda_{0,1}\right)$ for some $0 \le d \le 1$ such that

$$P\left(S_0^{NP} + S_1^{NP}, \lambda_{0,1}\right) = P\left(S_1^{NP}, \lambda_1\right) - p(x,y)\left(S_0^{NP} - \lambda_0\right). \tag{1.14}$$

A weighted sum of (1.13) and (1.14) gives us

$$P\left(S_0^{NP} + S_1^{NP}, \lambda_{0,1}\right) = \alpha_0 P\left(S_0^{NP}, \lambda_0\right) + \alpha_1 P\left(S_1^{NP}, \lambda_1\right) - \alpha_0 p\left(u,v\right)\left(S_1^{NP} - \lambda_1\right) - \alpha_1 p\left(x,y\right)\left(S_0^{NP}\lambda_0\right).$$

Because $S_n^{NP} \ge \lambda_n$ for $n = 0,1$, this means

$$P\left(S_0^{NP} + S_1^{NP}, \lambda_{0,1}\right) \le \alpha_0 P\left(S_0^{NP}, \lambda_0\right) + \alpha_1 P\left(S_1^{NP}, \lambda_1\right) + \alpha_0 \frac{h_0}{h_0 + p_0} + \alpha_1 \frac{h_1}{h_1 + p_1} \tag{1.15}$$

Combine (1.11) and (1.15), $P\left(S_0^{NP} + S_1^{NP}, \lambda_{0,1}\right) \le \frac{h_1}{h_1 + \omega}$ and $S_1^P \le S_0^{NP} + S_1^{NP}$ follows.

Proof of Proposition 3

Please note $\Pi^P\left(S\right)$ can be also written as:

$$\Pi^P\left(S\right) = \omega\left(k\right)\left[D\left(\lambda_{0,1}T\right) \wedge S\right] - h_1\left(S - D\left(\lambda_{0,1}T\right)\right)^+ \tag{1.16}$$

Since k is a variable now, $\Pi\left(k\right)=\max_{S}\Pi^{P}\left(S,k\right)$.

$$\frac{\partial\Pi\left(k\right)}{\partial k}=\frac{\partial\Pi^{P}\left(S,k\right)}{\partial S}\frac{\partial S_{k}^{P}}{\partial k}+\frac{\partial\Pi^{P}\left(S,k\right)}{\partial k}. \tag{1.17}$$

Please note $\Pi^{P}\left(S,k\right)$ is strictly concave in S. Considering that $\left.\frac{\partial\Pi^{P}\left(S,k\right)}{\partial S}\right|_{S=S_{k}^{P}}=0$ we can simplify (1.17) as follows:

$$\frac{\partial\Pi\left(k\right)}{\partial k}=\frac{\partial\Pi^{P}\left(S_{k}^{P},k\right)}{\partial k}=\frac{\partial\omega}{\partial k}\left[D\left(\lambda_{0,1}T\right)\wedge S_{k}^{P}\right]<0 \tag{1.18}$$

Because under structure NP there is no handling cost *k*, we have $\frac{\partial\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)}{\partial k}=0$. Therefore, the profit difference $\Pi^{P}\left(S_{1}^{P}\right)-\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)$ is decreasing in *k*.

Moreover, it is clear that as $k\rightarrow\infty$, we have $\omega\rightarrow-\infty$, hence $\Pi^{P}\left(S_{1}^{P}\right)\rightarrow 0$. So $\lim_{k\rightarrow\infty}\Pi^{P}\left(S_{1}^{P}\right)-\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)=-\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)<0$. At $k=0$, there are two possible cases:

a) $\Pi^{P}\left(S_{1}^{P}\right)>\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)$. In this case, we define $\bar{k}$ to be the unique solution to $\Pi^{P}\left(S_{1}^{P}\right)=\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)$. Due to strict monotonicity in k , we must have $\Pi^{P}\left(S_{1}^{P}\right)>\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)$ for all $k<\bar{k}$ and $\Pi^{P}\left(S_{1}^{P}\right)<\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)$ for all $k>\bar{k}$.

b) $\Pi^{P}\left(S_{1}^{P}\right)\leq\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)$. In this case we simply define $\bar{k}=0$ and automatically get, due to strict monotonicity in k, that $\Pi^{P}\left(S_{1}^{P}\right)<\Pi^{NP}\left(S_{0}^{NP},S_{1}^{NP}\right)$ for all $k>0$.

Proof of Proposition 4

Both k and λ_0 are variables in this proposition, so we include them as additional function arguments whenever necessary. For example: we denote $\Pi\left(\lambda_0, k\right) = \max_S \Pi^P\left(S, \lambda_0, k\right)$ and let its optimizer be denoted as S^P_{k,λ_0}; thus, $\Pi\left(\lambda_0, k\right) = \Pi^P\left(S^P_{k,\lambda_0}, \lambda_0, k\right)$.

$$\frac{\partial \Pi\left(\lambda_0, k\right)}{\partial k} = \frac{\partial \Pi^P\left(S^P_{k,\lambda_0}, \lambda_0, k\right)}{\partial k} = \frac{\partial \omega}{\partial k}\left(S^P_{k,\lambda_0} - \int_{x=0}^{S^P_{k,\lambda_0}} \left(S^P_{k,\lambda_0} - x\right) p\left(x, \lambda_{0,1} T\right)\right) \tag{1.19}$$

Then:

$$\frac{\partial^2 \Pi\left(\lambda_0, k\right)}{\partial \lambda_0 \partial k} = \left.\frac{\partial^2 \Pi^P\left(S, k, \lambda_0\right)}{\partial k \partial S}\right|_{S=S^P_{k,\lambda_0}} \frac{\partial S^P_{k,\lambda_0}}{\partial k} + \left.\frac{\partial^2 \Pi^P\left(S, k, \lambda_0\right)}{\partial \lambda_0 \partial k}\right|_{S=S^P_{k,\lambda_0}} \tag{1.20}$$

which using (1.19) can be written as follows:

$$\frac{\partial^2 \Pi\left(\lambda_0, k\right)}{\partial \lambda_0 \partial k} = \frac{\partial^2 \omega}{\partial \lambda_0 \partial k}\left(S^P_{k,\lambda_0} - \int_{x=0}^{S^P_{k,\lambda_0}} \left(S^P_{k,\lambda_0} - x\right) p\left(x, \lambda_{0,1} T\right)\right) + \frac{\partial \omega}{\partial k}\left(\frac{\partial S^P_{k,\lambda_0}}{\partial \lambda_0} \int_{x=S^P_{k,\lambda_0}}^{\infty} p\left(x, \lambda_{0,1} T\right)\right) \tag{1.21}$$

Since $\frac{\partial S^P_{k,\lambda_0}}{\partial \lambda_0} \geq 0$, $\frac{\partial^2 \omega}{\partial \lambda_0 \partial k} \leq 0$, and $\frac{\partial \omega}{\partial k} \leq 0$, the proof is complete.

Proof Lemma 1

Please refer to Liang (1999) on pages 119-120.

Proof of Theorem 1

Based on Lemma 1, protection thresholds are non-increasing unit step function throughout the season. At the indifference points we have:

$$\Delta_i V\left(i, t_{i-1}\right) = p_0 - k \tag{1.22}$$

It is intuitive as it says a unit of inventory should be open to share with online when $\Delta_i V\left(i,t\right) < p_0 - k$ and protected when $\Delta_i V\left(i,t\right) \geq p_0 - k$. Since Lemma 1 shows that $\Delta_i V\left(i,t\right)$ is decreasing in t and i, therefore $t_l < t_{l-1} \ldots < t_1 < t_0$. So it remains to show that there exist a t_0 close enough to end of season such that retailer would prefer to protect no unit after that. Because $\Delta_i V\left(i,t\right)$ is decreasing in i, if it is optimal to not protect last unit of inventory close enough to the end of season, it will also be optimal to not protect if we have more inventory available ($i > 1$). Therefore, it suffices to show t_0 exist when only one unit of inventory left, *i.e.* $\Delta_i V\left(i,t\right)$ is at its maximum for a given t.

Now consider an online demand that arrives at time $t > t_0$ when the offline store has one unit of inventory. If the retailer accepts the online demand, then he receives a sure $p_0 - k$ from that unit of inventory. If the retailer rejects the online demand, however, he may expect to sell it later at a higher profit of p_1. This happens with a probability of at most $P\left(1, \lambda_1\left(T-t\right)\right)$ (“at most” because policy may reject other online demands that arrive after t). On the other hand, the unit may be left over at the end of the season and incur a cost of $-h_1$. This happens with a probability of *at least* $p\left(0, \lambda_1\left(T-t\right)\right)$.

Therefore, if we define $\overline{t} = T + \frac{1}{\lambda_1}\ln\left(\frac{p_1 - \left(p_0 - k\right)}{p_1 + h_1}\right)$, then for any $t > \overline{t}$, we have

$$p_0 - k \geq p_1 P\left(1, \lambda_1\left(T - t\right)\right) - h_1 p\left(0, \lambda_1\left(T - t\right)\right).$$

Thus, the retailer would make less expected profit if he rejects the online demand at $t > \overline{t}$. Defining $t_0 = \min\{\overline{t} \mid \text{it is optimal to accept an online demand for all } t \in [\overline{t}, T]\}$, then we know it's optimal not to protect any offline inventory after t_0.

Proof of Lemma 3

Part A) The first order difference of $H^{ST}\left(\tau \middle| i, \theta\right)$ with respect to τ is as follow:

$$\begin{aligned}\Delta_\tau H^{ST}\left(\tau \middle| i, \theta\right) = &-h_1 + \left(p_1 + h_1\right) P\left(i - \tau, \lambda_{0,1}\left(T - \theta\right)\right) \\ &-\left(p_1 + h_1\right)\left(\alpha_1\right)^{-\left(i-\tau\right)} \sum_{n=0}^{\tau-1} p\left(n + i - \tau, \lambda_1\left(T - \theta\right)\right) \ {}_1F_1\left(i - \tau, n + i - \tau + 1, -\lambda_0\left(T - \theta\right)\right)\end{aligned}$$

where by convention ${}_1F_1\left(a; b; z\right)$ indicates a confluent hypergeometric function. Then

$$\Delta_\tau^2 H^{ST}\left(\tau \middle| i, \theta\right) = -\left(p_1 + h_1\right)\left(\alpha_1\right)^{-\left(i-\tau\right)} p\left(i - 1, \lambda_1\left(T - \theta\right)\right) \ {}_1F_1\left(i - \tau, i - 1 + 1, -\lambda_0\left(T - \theta\right)\right) \leq 0.$$

Since $\Delta_i \Delta_\tau H^{ST}\left(\tau \middle| i, \theta\right) = -\left(p_1 + h_1\right)\left(\alpha_1\right)^{-\left(i-\tau-1\right)} p\left(i - 1, \lambda_1\left(T - \theta\right)\right) \ {}_1F_1\left(i - \tau, i, -\lambda_0\left(T - \theta\right)\right) \leq 0$, then is also concave in i.

Part B) $\partial \frac{\Delta_\tau H^{ST}\left(\tau \middle| i, \theta\right)}{\partial \theta} = -\lambda_{0,1}\left(p_1 + h_1\right)\left(\alpha_1\right)^{-\left(i-r-1\right)} p\left(i - 1, \lambda_1\left(T - \theta\right)\right) \ {}_1F_1\left(i - r, i, -\lambda_0\left(T - \theta\right)\right) \leq 0.$

***Proof of Lemma** 4*

Part A) To prove $H^{NT}(i,t)$ is submodular in (i,t), we start from the end of the season and work our way backward. Let's look at $[t,T]$ where $T-t\le\Delta t$. Assume Δt is small enough such that there is only a single protection level active in this period ($\tau^{NT}(t)$). When $0\le i\le\tau^{NT}(t)$, we can write:

$$\partial\frac{\Delta_i H^{NT}(i,t)}{\partial t}=-\lambda_1(p_1+h_1)p(i-1,\lambda_1(T-t)) \tag{1.23}$$

When $i>\tau^{NT}(t)$, we can write:

$$\begin{aligned}\partial\frac{\Delta_i H^{NT}(i,t)}{\partial t}&=-\lambda_0(p_0-k+h_1)p\left(i-\tau^{NT}(t)-1,\lambda_{0,1}(T-t)\right)\\&\quad-\lambda_1(p_1+h_1)(\alpha_1)^{\tau^{NT}(t)}p(i-1,\lambda_{0,1}(T-t))\ {}_1F_1\left(\tau^{NT}(t),i,\lambda_0(T-t)\right)\end{aligned} \tag{1.24}$$

(1.23) and (1.24), so $\partial\frac{\Delta_i H^{NT}(i,t)}{\partial t}\le 0$ is decreasing on $\Delta_i H^{NT}(i,t)$ where $[t,T]$. $T-t\le\Delta t$.

Next we extend the analysis to show that $\Delta_i H^{NT}(i,t)$ is decreasing in t on $[t,T]$ where $T-t\le 2\Delta t$. When $0\le i\le\tau^{NT}(t)$, we can write:

$$\partial\frac{\Delta_i H^{NT}(i,t)}{\partial t}=-\lambda_1 p_1 p(i,\lambda_1(T-\Delta t-t))\lambda_1\sum_{n=0}^{i}\Delta_i^2 H^{NT}(i-n,T-\Delta t)p(n,\lambda_1(T-\Delta t-t))\le 0$$

When $i>\tau^{NT}(t)$, we can write:

$$\partial\frac{\Delta_i H^{NT}\left(i,t\right)}{\partial t}=-wp\left(i-\tau^*-1,\lambda_{0,1}\left(T-\Delta t-t\right)\right)+\sum_{m=0}^{i-\tau^*-1}\Delta_i^2 G_1\left(i-m,T-\Delta t\right)p\left(m,\lambda_{0,1}\left(T-\Delta t-t\right)\right)$$
$$-\lambda_1\alpha_1 p_1\left(\alpha_1\right)^{-\left(i-\tau^*\right)}p\left(i-1,\lambda_1\left(T-\Delta t-t\right)\right)\ {}_1F_1\left(i-\tau^*,i,-\lambda_0\left(T-\Delta t-t\right)\right)$$
$$+\lambda_1\alpha_1\left(\alpha_1\right)^{-\left(i-\tau^*\right)}\sum_{n=i-r}^{i-1}\Delta_i^2 G_1\left(i-n,T-\Delta t\right)p\left(n,\lambda_1\left(T-\Delta t-t\right)\right)\ {}_1F_1\left(i-\tau^*,n+1,-\lambda_0\left(T-\Delta t-t\right)\right)$$

Therefore, since $\Delta_i^2 G_1\left(i,t\right)\le 0$, we showed $\Delta_i H^{NT}\left(i,t\right)$ is decreasing in t on $\left[t,T\right]$ where $T-t\le 2\Delta t.$ We can repeat this process for $3\Delta t$, $4\Delta t$, … and eventually cover the whole season. Therefore $H^{NT}\left(i,t\right)$is submodular in i and t.

Part B) To show $H^{NT}\left(i,t\right)$ is concave in i, we follow a similar analysis to part A. First, we show that $\Delta_i H^{NT}\left(i,t\right)$ is decreasing in i where $T-t\le\Delta t$. Then we extend this to the whole season. When $0\le i\le\tau^{NT}\left(t\right)$, second order difference with respect to i is as follow:

$$\Delta_i^2 H^{NT}\left(i,t\right)=-\left(p_1+h_1\right)p\left(i-1,\lambda_1\left(T-t\right)\right)\le 0 \tag{1.25}$$

When $i>\tau^{NT}\left(t\right)$, second order difference with respect to i is also negative as follow:

$$\Delta_i^2 H^{NT}\left(i,t\right)=-\alpha_0\left(p_0-k+h_1\right)p\left(i-\tau^{NT}\left(t\right)-1,\lambda_{0,1}\left(T-t\right)\right)$$
$$-\left(p_1+h_1\right)\left(\alpha_1\right)^{-i+\tau^{NT}\left(t\right)+2}p\left(i-1,\lambda_1\left(T-t\right)\right)\ {}_1F_1\left(i-\tau^{NT}\left(t\right),i,-\lambda_0\left(T-t\right)\right)$$

Therefore $H^{NT}\left(i,t\right)$ is concave on$\left[t,T\right]$ where $T-t\le\Delta t$.

Now, we step back Δt and check $H^{NT}\left(i,t\right)$on $\left[t,T\right]$ where $T-t\le 2\Delta t$. When $0\le i\le\tau^{NT}\left(t\right)$, second order difference with respect to i is as follow:

$$\Delta_i^2 H^{NT}\left(i,t\right)=-p_1 p\left(i-1,\lambda_1\left(T-\Delta t-t\right)\right)+\sum_{n=0}^{i-1}\Delta_i^2 G_1\left(i-n,T-\Delta t\right)p\left(n,\lambda_1\left(T-\Delta t-t\right)\right)$$

Since $\Delta_i^2 G_1(i,t) \le 0$, then $\Delta_i^2 H^{NT}(i,t)$ is negative. When $i > \tau^{NT}(t)$, second order difference with respect to i is as follow:

$$\begin{aligned}
\Delta_i^2 H^{NT}(i,t) = & -p_1(\alpha_1)^{\tau^{NT}(t)+1} p\left(i-1,\lambda_{0,1}(T-\Delta t-t)\right) F\left(\tau^{NT}(t), i, \lambda_0(T-\Delta t-t)\right) \\
& +\alpha_0\left[\Delta_{\tau^*} G_1\left(\tau^{NT}(t), T-\Delta t\right)-\left(p_0-k\right)\right] p\left(i-\tau^{NT}(t)-1, \lambda_{0,1}(T-\Delta t-t)\right) \\
& +\sum_{m=0}^{i-\tau^{NT}(t)-1} \Delta_i^2 G_1(i-m, T-\Delta t) p\left(m, \lambda_{0,1}(T-\Delta t-t)\right) \\
& +(\alpha_1)^{-\left(i-\tau^{NT}(t)-2\right)} \sum_{n=i-\tau^{NT}(t)}^{i-1}\left[\Delta_i^2 G_1(i-n, T-\Delta t)\right] p\left(n, \lambda_1(T-\Delta t-t)\right) F\left(i-\tau^{NT}(t), n+1, -\lambda_0(T-\Delta t-t)\right).
\end{aligned}$$

Because $\Delta_i^2 G_1(i,t) \le 0$, it suffices to show $\Delta_{i=\tau} G_1(i, T-\Delta t) - (p_0 - k) \le 0$. At $i = \tau^{NT}(t)$, we know that $\Delta_i H^{NT}(i,t) \ge p_0 - k$. Because $\Delta_i H^{NT}(i,t)$ is submodular, it then follows that $H^{NT}(i,t)$ is concave in i on $[t,T]$ where $T - t \le 2\Delta t$. In a similar fashion, we can repeat this process for $3\Delta t$, $4\Delta t$, … and eventually cover the whole season. Therefore $H^{NT}(i,t)$ is concave in i on $[0,T]$.

Proof of Proposition 5

Please recall that solution to indifference points of optimal rationing policy is identified using $\Delta_i V(i, t_{i-1}) = p_0 - k$, however by approximating end of period value using $G_1(i,t)$ in the optimal value function, $\Delta_i H^{NT}(i, t'_{i-1}) = p_0 - k$ will be used to calculate indifference points. $\Delta_i H^{NT}(i,t)$ on $t \in [0,T]$ is simplified as follow, just as in the classic newsvendor model:

$$\Delta_i H^{NT}(i,t) = p_1 - (p_1 + h_1)\sum_{n=0}^{i-1} p\left(n, \lambda_1(T-t)\right) \tag{1.26}$$

By Lemma 3, $\Delta_i H^{NT}\left(i,t\right)$ is decreasing in t. Therefore, at a given i either $\Delta_i H^{NT}\left(i,t\right)$ is always below $p_0 - k$ or there exists a time (t'_{i-1}) after which it goes below $p_0 - k$. When the former happens $t'_{i-1} = 0$, while in the latter case $t'_{i-1} > 0$. We know at the limits (1.26) is:

$$\lim_{t \to T} \Delta_i H^{NT}\left(i,t\right) = -h_1$$

$$\lim_{T \to \infty} \lim_{t \to 0} \Delta_i H^{NT}\left(i,t\right) = \lim_{T \to \infty} \left(p_1 - \left(p_1 + h_1\right) \sum_{n=0}^{i-1} p\left(n, \lambda_1 \left(T - t\right)\right) \right) = p_1$$

since $\Delta_i H^{NT}\left(i,t\right)$ is monotonically decreasing in i as proved earlier. Therefore, starting from $i = 1$ there exit a t'_0 such that $\Delta_i H^{NT}\left(1,t'_0\right) = p_0 - k$. We know $\Delta_i H^{NT}\left(i,t\right)$ is also decreasing in i (*i.e.* concavity) therefore following holds:

$$\Delta_i H^{NT}\left(2,t'_0\right) \leq \Delta_i H^{NT}\left(1,t'_0\right) = p_0 - k$$

This means that there exist a $t'_1 \leq t'_0$ such that $\Delta_i H^{NT}\left(2,t'_1\right) = p_0 - k$. This procedure continuous until we reach a level of inventory ($i = n + 2$) where $t'_{n+1} \leq 0$. Then all the indifference points have been identified as follows: $0 = t'_n < ... < t'_1 < t'_0 < T$

Proof of Proposition 6

Equation (1.26) using $\Delta_i H^{NT}\left(i,t'_{i-1}\right) = p_0 - k$ can be further simplified as $\left(p_1 + h_1\right) \sum_{n=0}^{i-1} p\left(n, \lambda_1 \left(T - t\right)\right) = p_1 - p_0 + k.$ In the last step, it can be written as $P\left(i, \lambda_1 \left(T - \bar{t}_i\right)\right) = \dfrac{p_0 - k + h_1}{p_1 + h_1}$ to complete the proof.

A Lemma Used in the Proofs of Proposition 7, Lemma 4, and Proposition 8

Lemma A3 $\Delta_i V(i,\theta) \geq \Delta_i G_1(i,\theta)$ for any θ and $i \geq 0$.

Proof: Let $\delta(i,\theta) = V(i,\theta) - G_1(i,\theta)$. Since $V(i,\theta)$ is the optimal value function on $[\theta,T]$, we know that $\delta(i,\theta) \geq 0$ for all θ and i. It remains to be show that $\Delta_i \delta(i,\theta) \geq 0$ for all θ and $i \geq 0$.

- When $i = 1$, $\Delta_i V(i,\theta) \geq \Delta_i G_1(i,\theta)$ holds easily because $V(0,\theta) = G_1(0,\theta) = 0$.
- When $i > 1$, the proof is more complicated. We follow a backward induction and start from end of the season where $\theta \in [t_0,T]$. In this period, first order and second order differences with respect to i can be written as follow:

$$\Delta_i \delta(i,\theta) = (\omega + h_1) P(i,\lambda_{0,1}(T-\theta)) - (p_1 + h_1) P(i,\lambda_1(T-\theta)) \quad \forall i \geq 0$$

$$\Delta_i^2 \delta(i,\theta) = -(\omega + h_1) p(i-1,\lambda_{0,1}(T-\theta)) + (p_1 + h_1) p(i-1,\lambda_1(T-\theta)) \quad \forall i \geq 1.$$

Now define a ratio $R_i = \dfrac{(\omega + h_1) p(i-1,\lambda_{0,1}(T-\theta))}{(p_1 + h_1) p(i-1,\lambda_1(T-\theta))} = \dfrac{\omega + h_1}{p_1 + h_1} e^{-\lambda_0(T-\theta)} \left(1 + \dfrac{\lambda_0}{\lambda_1}\right)^{i-1}$. It is easy to see that $\Delta_i^2 \delta(i,\theta) \leq 0 \Leftrightarrow R_i \geq 1$. It is also easy to see that R_i is positive and strictly increasing in i. Depending on the initial value of R_i, there are two possible cases:

Case I- $R_1 \geq 1$. In this case $R_i \geq 1, \forall i$, because it's increasing in *i*. Therefore,

$$(\omega_1 + h_1) p(n,\lambda_{0,1}(T-\theta)) \geq (p_1 + h_1) p(n,\lambda_1(T-\theta)) \quad for\ \forall n \geq 0$$

Summing over *n* we get:

$$\left(\omega_1 + h_1\right)\sum_{n=i}^{\infty} p\left(n, \lambda_{0,1}\left(T-\theta\right)\right) \geq \left(p_1 + h_1\right)\sum_{n=i}^{\infty} p\left(n, \lambda_1\left(T-\theta\right)\right) \quad \text{for } \forall \text{i} \geq 0$$

That is, $\left(\omega_1 + h_1\right)P\left(i, \lambda_{0,1}\left(T-\theta\right)\right) \geq \left(p_1 + h_1\right)P\left(i, \lambda_1\left(T-\theta\right)\right)$. Thus, $\Delta_i \delta\left(i,\theta\right) \geq 0$.

Case II- $R_1 < 1$. Because R_i is increasing and $\lim_{i\to\infty} R_i = \infty$, there must exist an i' such that $R_i < 1$ for $0 \leq i < i'$ and $R_i \geq 1$ for $i \geq i'$.

Case II(1): For $i \geq i'$, $R_i \geq 1$. As in Case I, we immediately have $\Delta_i \delta\left(i,\theta\right) \geq 0$.

Case II(2): For $0 \leq i < i'$, $R_i < 1$ and $\delta\left(i,\theta\right)$ is convex in *i*. Because we know $\delta\left(0,\theta\right) = 0$, and $\delta\left(i,\theta\right) \geq 0$, it must be the case that $\delta\left(i,\theta\right)$ is increasing in i for all $0 \leq i < i'$. (That is, the decreasing part of a convex function is not possible here.)

Therefore, combining both cases we get $\Delta_i \delta\left(i,\theta\right) \geq 0$ for all *i* and $\theta \in \left[t_0, T\right]$.

We now extend the analysis to $\theta \in \left[t_1, t_0\right]$. Since $\tau\left(t\right) = 1$ on $\theta \in \left[t_1, t_0\right]$, we know $\Delta_i V\left(1,\theta\right) = \Delta_i G_1\left(1,\theta\right)$. Similar to what we did earlier, we can show that $\delta\left(i,\theta\right) = V\left(i,\theta\right) - G_1\left(i,\theta\right)$ is always increasing in *i* for $i \geq 2$. Knowing $\Delta_i \vartheta\left(1,\theta\right) = 0$, we conclude that $\Delta_i \vartheta\left(i,\theta\right) \geq 0$ for all i on $\theta \in \left[t_1, t_0\right]$.

This proof process can be repeated for $\theta \in \left[t_2, t_1\right]$ … to show that $\Delta_i \vartheta\left(i,\theta\right) \geq 0$ for all i and θ.

Proof of Proposition 7

First, as a recall, we describe how the indifference points are calculated under the optimal rationing policy. Then we compare that with NT heuristic.

For the optimal rationing policy, indifference points are calculated using $\Delta_i V(i, t_i) = p_0 - k$. To find the first indifference point, t_0, we will use following:

$$\Delta_i V(i = 1, t_0) = p_1 - (p_1 + h_1) p(0, \lambda_1 (T - t_0)) = p_0 - k \tag{1.27}$$

Then we calculate $V(i, t_0)$ for $\forall i \geq 1$. The second indifference point, t_1, is calculated using following:

$$\Delta_i V(i = 2, t_1) = p_1 + \sum_{n=0}^{1} [\Delta_i V(2 - n, t_0) - p_1] p(n, \lambda_1 (t_0 - t_1)) = p_0 - k \tag{1.28}$$

Then we calculate $V(i, t_1)$ for $\forall i \geq 1$. The procedure continuous till $V(i, 0)$ for $\forall i \geq 1$ is calculated.

Now, we compare the procedure with that of NT heuristic. While equation (1.27) stays the same, equation (1.28) changes to:

$$\Delta_i H^{NT}(i = 2, t_1') = p_1 + \sum_{n=0}^{1} [\Delta_i G_1(2 - n, t_0') - p_1] p(n, \lambda_1 (t_1' - t_0')) = p_0 - k \tag{1.29}$$

Therefore, both policy match up until first indifference point ($t_0 = t_0'$), which means:

$$\tau^{NT}(t) = \tau^{OPT}(t) = 0 \ \ \forall t \in [t_0', T]$$

In the next step to find t_1' using(1.29), we need marginal value at $t_0 = t_0'$ which is calculated using $\Delta_i V(i, t_0)$ and $\Delta_i G_1(i, t_0')$ respectively in optimal and NT heuristics. Comparing (1.28), we find (1.29), we find $\sum_{n=0}^{1} [\Delta_i G_1(2 - n, t_0') - p_1] p(n, \lambda_1 (t_1' - t_0')) = \sum_{n=0}^{1} [\Delta_i V(2 - n, t_0) - p_1] p(n, \lambda_1 (t_0 - t_1))$.

Because of Lemma A3 and $t_0 = t_0'$, it is clear that $t_1' \geq t_1$. This means under NT heuristic, protection level of one is optimal action for a longer period compared to that of optimal policy. Similarly, this analysis can be extended to show that $\overline{t}_n \geq t_n$. Therefore, there are fewer protection levels under the NT heuristic than under the optimal policy.

Proof of Lemma 4

Part A) We note $\Pi^{P,OPT}(S_1) = V\left(S_1, 0\right)$.

Therefore, based on Lemma 1, we conclude that $\Pi^{P,OPT}(S_1)$ is concave in S_1.

Part B1) Recall that we define G_1 to be the newsvendor profit function for the offline store:

$$G_1\left(i,t\right) = p_1 i - \left(p_1 + h_1\right)\sum_{j=0}^{i}\left(i-j\right)p\left(i,\lambda_1\left(T-t\right)\right).$$

We now similarly define G_0 to be the newsvendor profit function for the online store:

$$G_0\left(i,t\right) = p_0 i - \left(p_0 + h_0\right)\sum_{j=0}^{i}\left(i-j\right)p\left(i,\lambda_0\left(T-t\right)\right).$$

We can write the (NP,OPT) profit function as follows:

$$\begin{aligned}\Pi^{NP,OPT}(S_0,S_1) = {} & G_0(S_0,0) + G_1(S_1,0)\left(1 - P\left(S_0,\lambda_0 T\right)\right) + \int_{\theta=0}^{T} p_1 E\left[D_1(\theta)\wedge S\right]\lambda_0 p\left(S_0 - 1,\lambda_0\theta\right)d\theta \\ & + \int_{\theta=0}^{T}\left\{\sum_{u=0}^{S_1} V\left(S_1 - u,\theta\right)p(u,\lambda_1\theta)\right\}\lambda_0 p(S_0 - 1,\lambda_0\theta)d\theta\end{aligned}$$

Its first order difference with respect to S_1 is as follow:

$$\begin{aligned}\Delta_{S_1}\Pi^{NT,OPT}(S_0,S_1) = {} & \Delta_{S_1} G_1(S_1,0)\left(1 - P\left(S_0,\lambda_0 T\right)\right) \\ & + \int_{\theta=0}^{T}\left\{p_1 P(S_1,\lambda_1\theta) + \sum_{u=0}^{S_1-1}\Delta_{S_1} V\left(S_1 - u,\theta\right)p(u,\lambda_1\theta)\right\}\lambda_0 p(S_0 - 1,\lambda_0\theta)d\theta\end{aligned}$$

And the second order difference with respect to S_1 is as follow:

$$\Delta_{S_1}^2 \Pi^{NT,OPT}(S_0,S_1) = \Delta_{S_1}^2 G_1(S_1,0)\big(1-P\left(S_0,\lambda_0 T\right)\big) \\ +\int_{\theta=0}^{T}\left\{-p_1 p(S_1-1,\lambda_1\theta)+\sum_{u=0}^{S_1-1}\Delta_{S_1}^2 V\left(S_1-u,\theta\right)p(u,\lambda_1\theta)\right\}\lambda_0 p(S_0-1,\lambda_0\theta)d\theta$$

Since both $G_1(S_1,0)$ and $V\left(S_1-u,\theta\right)$ are concave (Propositions 1 and 5), we conclude that $\Pi^{NT,\mathrm{OPT}}(S_0,S_1)$ is concave in S_1.

Part B2) The (NP,OPT) profit function can be also written as follow:

$$\Pi^{NP,OPT}(S_0,S_1) = \Pi^{NP,\varnothing}(S_0,S_1) \\ +\int_{\theta=0}^{T}\left\{\sum_{u=0}^{S_1}\Big(V\left(S_1-u,\theta\right)-G_1\left(S_1-u,\theta\right)\Big)p(u,\lambda_1\theta)\right\}\lambda_0 p(S_0-1,\lambda_0\theta)d\theta$$

We first show that the following two inequalities hold:

$$p(S_0-1,\lambda_0\theta)-p(S_0-2,\lambda_0\theta)\le 0\,, \tag{1.30}$$

$$\Delta_{S_1}V\left(S_1-u,\theta\right)-\Delta_{S_1}G_1\left(S_1-u,\theta\right)\ge 0\,. \tag{1.31}$$

Considering our assumption that the service levels are more than 0.5, we have $S_n^{NP,\varnothing}\ge\lceil\lambda_n T\rceil\ge\lceil\lambda_0\theta\rceil$, and (1.30) holds. (1.31) follows from Lemma A3.

Next, the profit function's cross difference with respect S_0 and S_1 can be written as follow:

$$\Delta_{S_0}\Delta_{S_1}\Pi^{NP,OPT}(S_0,S_1) = \int_{\theta=0}^{T}\left\{\sum_{u=0}^{S_1-1}\Big(\Delta_{S_1}V\left(S_1-u,\theta\right)-\Delta_{S_1}G_1\left(S_1-u,\theta\right)\Big)p(u,\lambda_1\theta)\right\}\lambda_0\Big(p(S_0-1,\lambda_0\theta)-p(S_0-2,\lambda_0\theta)\Big)d\theta.$$

Thus $\Delta_{S_0}\Delta_{S_1}\Pi^{NP,OPT}(S_0,S_1)\leq 0$ because of (1.30) and (1.31). That is, the profit function is submodular.

Proof of Proposition 8

Part A) Because ϕ is a feasible rationing policy, its performance by definition is worse than the optimal rationing policy. Therefore, we have $\Pi^{P,\phi}(S_1)\leq \Pi^{P,OPT}(S_1)$ and $\Pi^{NP,\varnothing}(S_0,S_1)\leq \Pi^{NP,OPT}(S_0,S_1)$.

Part B1) Please note following:

$$\begin{aligned}\Delta_{S_1}\Pi^{NP,OPT}(S_0,S_1) &= \Delta_{S_1}G_1\left(S_1,0\right)\\ &+\int_{\theta=0}^{T}\left\{\sum_{u=0}^{S_1-1}\left(\Delta_{S_1}V\left(S_1-u,\theta\right)-\Delta_{S_1}G_1\left(S_1-u,\theta\right)\right)p(u,\lambda_1\theta)\right\}\lambda_0 p(S_0-1,\lambda_0\theta)d\theta\end{aligned}$$

Plugging in $S_1=S_1^{NP,\varnothing}$, we note $\Delta_{S_1}G_1(S_1^{NP,\varnothing},0)=0$ and Lemma A3 imply that the LHS is also positive: $\Delta_{S_1}\Pi^{NP,OPT}(S_0,S_1^{NP,\varnothing})\geq 0$. Then, because $\Delta_{S_1}\Pi^{NP,OPT}(S_0,S_1^{NP,OPT})\geq 0$, the concavity of $\Pi^{NP,OPT}(S_0,S_1)$ in S_1 means $S_1^{NP,OPT}\geq S_1^{NP,\varnothing}$.

Part B2) Please note following:

$$\begin{aligned}\Delta_{S_0}\Pi^{NP,OPT}(S_0,S_1) &= \Delta_{S_0}G_0(S_0,0)\\ &+\int_{\theta=0}^{T}\left\{\sum_{u=0}^{S_1}\left[V\left(S_1-u,\theta\right)-G_1\left(S_1-u,\theta\right)\right]p(u,\lambda_1\theta)\right\}\lambda_0\left[p(S_0-1,\lambda_0\theta)-p(S_0-2,\lambda_0\theta)\right]d\theta\end{aligned}$$

For any $S_0\geq S_0^{NP,\varnothing}$, the assumption of $S\geq 0.5$ means $S_0\geq S_0^{NP,\varnothing}\geq\lceil\lambda_n T\rceil\geq\lceil\lambda_0\theta\rceil$. So $p(S_0-1,\lambda_0\theta)\leq p(S_0-2,\lambda_0\theta)$. Plugging in $S_0=S_0^{NP,\varnothing}$, we get $\Delta_{S_0}\Pi^{OPT}(S_0^{NP,\varnothing},S_1)\leq 0$ for all

$S_0 \geq S_0^{NP,\varnothing}$, because $\Delta_{S_0} G_0(S_0^{NP,\varnothing}, 0) = 0$ and $V\left(S_1 - u, \theta\right) \geq G_1\left(S_1 - u, \theta\right)$. Since $\Pi^{NP,OPT}(S_0, S_1)$ is decreasing in S_0 on $S_0 \geq S_0^{NP,\varnothing}$, we must have $S_0^{NP,OPT} \leq S_0^{NP,\varnothing}$.

CHAPTER 2. DISCOUNTED HOME DELIVERY STRATEGY

2.1 INTRODUCTION AND MOTIVATION

Stockout is an experience most customers have had at some point in their lives. While customers who purchase online may have a higher tolerance for such events, it is a completely different matter for customers who shop in a store and expect to find the item they seek in stock. A study by Sterling Commerce Inc. (2009) showed at least 25% of the time customers left stores without the item they were looking for. A similar study by Deloitte Inc. (2007) showed that seventy-two percent of consumers say that finding their favorite items to be out-of-stock decreases their willingness to shop with that retailer. Customers who choose to shop in a store spend time and effort to get to a store because they expect retailers to have what they want. If a retailer fails to meet this expectation, the customer's disappointment might reduce the chances of them coming back. Therefore, it is not surprising that retailers are very concerned with not wanting customers to experience stockout.

While retailers have become more experienced in dealing with stockouts, there is still considerable room for improvement. A traditional solution has been transshipment, which requires the physical transfer of items between stores to rebalance the inventory. A more recent solution is to create inventory visibility across the system so that sales associates are able to locate out-of-stock items in other stores for customers to pick up in-person. However, in-person pick-up is not always feasible as the store might be far to reach for customer either from distance or time perspective. In this case sales associates can place an order for home delivery from online warehouse or directly from supplier. Customers who are not in need of an item immediately may accept the option of home delivery. There will be some customers who will just walk away and possibly substitute the item.

Retailers want to capture a sale before a customer leaves the store, so retailers have been mostly making the home delivery option more attractive compared with the shopping charge on their website. For example, Macy's offers $50 free shipping threshold for home delivery purchase made in store while the threshold online is $99. In some cases, a sales associate may choose to override the threshold and offer free shipping. Ann Taylor is another retailer that provides free shipping for home delivery if an item is not available in the store. While most retailers such as Macy's and Ann Taylor handle home delivery through sales associates, others such as Kohl use kiosks in the store. While retailers are incentivizing customers by subsidizing the shipping cost partially or completely, the incentive can take other forms such as an immediate price reduction or a discount coupon for future purchase. We call this the *discounted home delivery* strategy.

Customers react to the discounted home delivery option different due to their heterogeneity on many dimensions. In this regard we focus on sensitivity to price and leadtime. Customers who purchase in store can be placed on a continuum from highly price sensitive to leadtime sensitive. Discounted home delivery after stockout helps retailers to not only capture sales that might be otherwise lost; but also keep price sensitive customers happy. However, there is no reason to offer this option only after stockout, it can also be offered when there are still some inventory units left. By doing so, the retailer can induce some price sensitive customers to take the home delivery option while preserving the inventory for leadtime sensitive customers who would otherwise be lost if the home delivery option is not made available until stockout.

The rest of this chapter is organized as follows. In section 2.2 we review the related literature. In section 2.3 we introduce the model setup. In Section 2.4 we study the retailers' optimal discounted home delivery strategy. In section 2.5 we build simple and practical heuristics that address challenges with regard to implementation of the optimal policy. In Section 2.6 we perform numerical analysis to evaluate a retailer's profit and optimal stocking using a discounted home delivery strategy. In Section 2.7 we conclude this chapter with a summary and a discussion about directions for future research.

2.2 LITERATURE REVIEW

Our study of retailers' discounted home-delivery strategy is related to dynamic pricing, and customer retention. In this section, we focus on the literature review of the most relevant papers.

Dynamic Pricing The relevant research stream in dynamic pricing is the literature on multi-product dynamic pricing without replenishment in the retail industry. For comprehensive surveys on pricing in the revenue management literature, see Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003), and Chen and Simchi-Levi (2012).

There are a few other studies that consider multiple products with shared resources (such as Maglaras and Meissner 2006; Gallego and van Ryzin 1997). Gallego and van Ryzin (1997) studied the dynamic pricing problem of multiple products in a network revenue management context with limited inventory and price-sensitive Poisson demand. Even though the paper considered airline yield management applications, the model is extendible to retail settings.

Ding *et al.* (2006) study dynamic discounting for single product with multiple customer classes. They consider stochastic demand within a single period setting. However, they divide a period into several stages and assume that the firm observes the demand in the beginning of each stage and then the firm makes allocation decision based on that. They assume the probability of unfulfilled demand waiting for delayed delivery is *linear* in the discount offered. In another paper, Ding *et al.* (2007) investigate dynamic pricing for multiple customer classes but with deterministic demand in an EOQ type environment. They assume the retailer knows the customer classes therefore retailer can make a decision, to not serve some customer class from inventory but rather offer them discount to accept delayed delivery.

Customer Retention Another related stream of work in the inventory management literature looks at the economic incentives to retain customers in the presence of stockouts. Most of this literature considers durable products with multiple ordering opportunities. Customers will be offered a discount to wait till future periods, therefore partial backlogging is possible. Although these problem are similar in nature to ours, they deal with a different problem because offering such discount might result in lowering higher-class customer's satisfaction in the future. Therefore, the optimal policy in these problems considers a different set of tradeoffs than ours.

Cheung (1998) considers a continuous-review model where a discount can be offered to customers who are willing to accept backorders or substitutable units even before the inventory is depleted, but the proportion of backordering customers is not a function of discount as is the case in our work. Bhargava *et al.* (2006) study a model where customers wait for demand fulfillment if the effective price, a discounted price plus a waiting cost, exceeds their valuation. DeCroix and Arreola- Risa (1998) consider backordering incentives for a simple inventory system, but their analysis does not exploit different customer classes or dynamic discount adjustments. Deshpande (2000) considers the design of a pricing mechanism and rationing policy for serving customers with differing shortage costs. Gale and Holmes (1993) discuss the use of advance-purchase discounted prices to divert demand from a peak period flight to an off-peak period flight. Chen (2001) studies optimal pricing and replenishment strategies that balance discounts with the benefits of advance demand information. Chen finds optimal static prices as opposed to the dynamic discounts we use. Wang *et al.* (2002) study a problem of meeting demand from two demand classes with different lead time requirements. Their paper focuses on studying required inventory levels in a two-echelon supply chain, where each location follows a base stock policy with no inventory rationing, whereas we specifically focus on dynamic pricing for a single location.

2.3. MODEL SETUP

In our model setup, we consider the inventory management of a single seasonal product such as fashion items with finite sale horizon, for an omnichannel retailer that has one physical store and one virtual warehouse. We only consider the operations of the physical store and assume the retailer can use store inventory as well as virtual warehouse inventory to satisfy store demand. The *virtual warehouse* represents a virtual pool of available inventory outside the store which may include other stores, online warehouses, and suppliers. We refer to inventory in the virtual warehouse as backup inventory which for tractability purposes we assume to be infinite. In the case of big retailers such as Macy's and Nordstrom, if inventory outside one physical store can be pooled virtually then this assumption is reasonable.

Let p_n denote the product's unit profit margin at store $n \in \{0,1\}$ (throughout the chapter, we use subscript 0 for the warehouse, and 1 for the store). Unit sales price is a fixed; however, difference in cost of inventory can result in different margins for store inventory and backup inventory. When backup inventory is used, the retailer incurs additional shipping and handling cost, therefore it is reasonable to assume that $p_1 > p_0$. The product has a life cycle of one season, so any leftover at the end of the sales season need to be salvaged. Let h be the cost of having a unit leftover in the store at the end of the season.

When customers walk into a store and find the item they seek available, they will make the purchase and take the item home unless they are offered an option to receive discount of d for home delivery of the item. Customers can be viewed being on a continuum from price sensitive to leadtime sensitive. We also assume customers are myopic; therefore, they will not delay their purchase in anticipation of a higher discount in the future. When offered home delivery option, they will accept the offer if the discount is enough to offset the disutility of a longer delivery leadtime, otherwise they will walk away. We assume in this case demand is lost; there is no backlogging.

Let $\alpha(d)$ be the proportion of customers accepting home delivery at the discount level of d. We assume $\alpha(d)$ is increasing and concave in d. We assume that at zero discount no customer will accept home delivery. Therefore, a unit of store inventory can fetch a profit of p_1 and backup inventory can yield a profit of $\alpha(d)(p_0 - d)$. For notation simplicity, we will use α_d to denote $\alpha(d)$. Let T be the length of the sales season. We assume store demand follows the Poisson process with mean rates of λ. Therefore, when discounted home delivery is offered, rate of home delivery is $\lambda\alpha_d$ and rate of in store shopping is $\lambda(1-\alpha_d)$. Let $\lambda_0 = \lambda\alpha_d$ and $\lambda_1 = \lambda(1-\alpha_d)$.

In practice, α_d can depend on time as well. As time gets closer to the end of the season, customers' discount sensitivity decreases, which means that higher discount will be needed in order to capture the same α_d. While we assume α_d is independent of time, results of our study can be extended to this situation. In our model, d is the decision variable, however since α_d is monotone in d, we can solve the problem in terms of α_d as the decision variable instead. The optimal α_d is found throughout the season, it can be inverted to fine the discount value d.

2.4. Optimal Discounted Home Delivery Policy

The current industry practice to offer discounted home delivery after stockout is an effective strategy. However, as we discussed earlier this might not be the best strategy. It is possible that by offering this option even before stockout, retailers can delay the possible stockout and increase their profit. In this section, we study retailer's optimal discounting policy (OPT). The retailer has two decisions to make at any time in the season. First, the retailer should decide whether to offer discounted home delivery or not. The second decision is how much discount to offer. Once stockout occurs, the answer to first decision is positive. However, when a store has inventory available, the answer depends on the amount of inventory available. When lots of inventory is available, the option should not be offered. Recall from model setup that offering a zero discount results in no acceptance, which is equivalent to not providing the option. Therefore, we only study retailers discount decision (d), where $d = 0$ means *no-discount* and $d > 0$ means *discount*.

We use dynamic programming (DP) to formulate the retailer's profit. Continuous time setting is complicated to solve, therefore we follow a common practice in the literature to use discrete approximation to solve the problem and find the optimal policy. However later in the numerical analysis, we evaluate the optimal policy and suggested heuristics which are derived in the continuous time setting to check how well the approximation works. The methods for solving discrete DP's have been well developed in the operations research literature (Bertsekas 1987; Porteus 1980).

We divide the season to T time periods and use t to index time period numbers. Time indices run forward, so $t = 1$ is the first period and $t = T$ is the last period. We assume that the time period length, δ, is so small that in each period, there will be at most one arrival. Therefore, demand in each period follows the Bernoulli process where probability of one arrival is $P_1 = \lambda\delta$. The state variable for decision making is (i,t) where i is the level of offline inventory at time t. Let $V(i,t)$ be the retailer's optimal profit at the beginning of period t with i units of inventory. The corresponding Bellman equation can be written as:

$$V(i,t) = \max_{0 \le d \le p_0} \left\{ P_0 V(i,t+1) + P_1 \alpha_d \left(p_0 - d + V(i,t+1) \right) + P_1 (1-\alpha_d)\left(p_1 + V(i-1,t+1) \right) \right\} \tag{2.1}$$

with boundary condition $V(i,T+1) = -hi$ for all i. The first term in equation (2.1) represents no arrival and the second and third terms represent one arrival when home delivery is accepted and rejected respectively. Let $d_{i,t}^*$ be the optimal discount with i units of inventory at time t. We start by characterizing $d_{i,t}^*$ when inventory is zero and proceed to the positive inventory case. The Bellman equation for $i = 0$ simplifies as follows:

$$V(0,t) = V(0,t+1) + \max_{0 \le d \le p_0} P_1 \alpha_d \left(p_0 - d \right) \tag{2.2}$$

Proposition 9 For $i = 0$ and all t, the optimal discount level $d_{0,t}^*$ satisfies $\alpha'_{d_{0,t}^*}\left(p_0 - d_{0,t}^* \right) = \alpha_{d_{0,t}^*}$ and $d_{0,t}^* \in (0, p_0)$.

As we assume backup inventory is infinite, $d_{0,t}^*$ is an independent decision for every incoming demand as it is independent of t as shown in Proposition 9. Therefore, we simplify $d_{0,t}^*$ into d_0^*.

Next, we look at the optimal discount at t with positive inventory. Let's define first order difference of value function as $\Delta_i V(i,t) = V(i,t) - V(i-1,t)$ where $\Delta_i V(i,t)$represents marginal value of the i's unit of inventory at time t. The Bellman equation in (2.1) can be simplified as follows by rearranging terms:

$$V(i,t) = V(i,t+1) + P_1\left(p_1 - \Delta_i V(i,t+1)\right) + \max_{0 \le d \le p_0} P_1 W^{OPT}(d,i,t). \tag{2.3}$$

where $W^{OPT}(d,i,t) = \alpha_d\left(p_0 - d - p_1 + \Delta_i V(i,t+1)\right)$ is the expected gain from offering discount d with i units at time period t. Note that the superscript Y in $W^Y(d,i,t)$ represents the discounting policy. In the next sections, definition of Y will be extended to include others policies. Unlike optimal discount at zero inventory level, optimal discount for positive inventory depends on $\Delta_i V(i,t+1)$, marginal value of inventory. Therefore, to in order to characterize the behavior of $d_{i,t}^*$, we need to first study $\Delta_i V(i,t)$. When $\Delta_i V(i,t)$is high, the retailer is more willing to protect the unit and therefore offer discount for home delivery. The opposite holds when $\Delta_i V(i,t)$ is low. The following lemma shows characteristics of $\Delta_i V(i,t)$ with respect to i and t.

Lemma 5 For any i and t, $\Delta_i V(i+1,t) \le \Delta_i V(i,t)$ and $\Delta_i V(i,t) \ge \Delta_i V(i,t+1)$.

Lemma 5 shows that marginal value of inventory is decreasing in time and inventory level. Intuitively, the marginal value of an extra unit of inventory should be higher when there is more time left in the sales season to sell it. Also at any given time, the marginal value of an extra unit of inventory should be decreasing in the existing inventory level: when the inventory level is low, an extra unit is highly likely to result in additional sales; this benefit diminishes as there is more inventory as the likelihood of being a left over starts to dominate. Therefore, we expect that optimal discount to be higher in the early season for low value of inventory as marginal value of inventory will be high.

The following theorem shows that $d^*_{i,t}$ can be uniquely identified at any point in the season.

Theorem 2 For any i and t, there exists a unique $d^*_{i,t}$ such that

a) $d^*_{i,t} = 0$ if $p_0 - p_1 + \Delta_i V\left(i, t+1\right) < 0$,

b) otherwise $d^*_{i,t}$ is the unique solution to $\alpha'_{d^*_{i,t}} \left(p_0 - d^*_{i,t} - p_1 + \Delta_i V\left(i, t+1\right)\right) = \alpha_{d^*_{i,t}}$ and $d^*_{i,t} \in \left(0, p_0\right)$.

The proof is provided in the appendix and the idea is that $W^{OPT}\left(d, i, t\right)$ is a unimodal function, where it is either increasing at $d = 0$ or decreasing for all d. If it is decreasing at $d = 0$, then $d^*_{i,t} = 0$. Otherwise since it is a unimodal function, its first order condition can be used to calculate $d^*_{i,t}$. Next, we characterize $d^*_{i,t}$ with respect to i and t.

Proposition 10 For any i and t, $d^*_{i+1,t} \leq d^*_{i,t}$ and $d^*_{i,t} \geq d^*_{i,t+1}$.

Proposition 10 shows that optimal discount is decreasing in i and t, which is a result of Lemma 5. The result can also be explained considering the direct relationship between optimal discount and stockout. For example, at a given i, chance of stockout is decreasing as time gets closer to the end of the season, hence the discount level also decreases.

The next proposition further shows that there exists a point in the season after which no discount should be given whenever there is inventory.

Proposition 11 There exists t_0^* such that $d_{i,t}^* = 0$ for all $i > 0$ and $t \geq t_0^*$.

Intuitively speaking, near the end of the season, the expected value of any inventory unit is negative. Therefore, the retailer is better off offering no discount at all.

To completely characterize the optimal policy, we also need to calculate the optimal discount before t_0 is reached. Unfortunately, the calculation of $d_{i,t}^* = 0$ in part b) of Theorem 2 is quite involved and must be done for all $i > 0$ and $t < t_0^*$. Considering the scale of problems in the industry, *calculation* poses a practical challenge for implementation. Also the *dynamically* changing discount is hard to implement in industry. Perhaps a store's policy is to inform customers about discounts through putting signs near checkout areas. Therefore, either discount signs need to be digital and linked to the central system for continuous updating or, if the signs are physical, sales associates would be required to change them very frequently. Therefore, we develop simple and effective heuristics in the next section.

2.5. HEURISTICS

In this section, we present three heuristics. Let $\theta = \min\left\{t \mid 1 < t \leq T,\ i = 0 \text{ at time } t\right\}$ represent the random time when the first stockout happens. Similar to Proposition 9, we can see that it is optimal to offer discount d_0^* after θ for *any policy*. The calculation $d_0^* = \left(p_0 - \alpha_{d_0^*}\right) \Big/ \alpha'_{d_0^*}$ is fixed and simple enough, so we will let all of our heuristics use d_0^* after stockout. Our heuristics differ in how they discount before θ.

The first heuristic is called the *base* (B) policy and represents current practice in the industry, offering no discount before θ. We will use the base policy as the benchmark in our numerical analysis to show how much benefit retailers can expect if they switch to a more comprehensive policy. The second heuristic is called the *two discount level* (TD) policy, which offers a fixed discount level or zero, before θ. The third heuristic is called the *newsvendor* (NV) policy, where optimal discount can vary by i and t, but its calculation is simplified – the value function for the OPT policy is approximated by a static newsvendor profit function.

We now give the specifics of each heuristic in the next three sections.

2.5.1. Base Policy

The current industry practice is to *only* offer discounted home delivery when the store inventory is depleted. To represent this strategy, we study base policy which offers discount d_0^* on $\theta \le t \le T$ and zero discount on $1 \le t < \theta$. Since demand in each period follows a Bernoulli distribution, the total demand of season follows Binomial distribution. Let $\Pi^B\left(S\right)$ be the expected profit of retailer under B policy with initial stocking level of S. Let $p\left(j;T,P_1\right)$ be the probability distribution function of Binomial distribution and $P\left(j;T,P_1\right)=\sum_{i=j}^{\infty}p\left(i;T,P_1\right)$ be the tail of its cumulative distribution. Then $\Pi^B\left(S\right)$ can be written as:

$$\max_S \Pi^B\left(S\right)=\sum_{j=0}^{S}\left(p_1 j-h\left(S-j\right)\right)p\left(j;T,P_1\right)+\sum_{j=S+1}^{\infty}\left(p_1 S+\left(j-S\right)\alpha_d\left(p_0-d\right)\right)p\left(j;T,P_1\right) \quad (2.4)$$

Proposition 12 The following properties hold for $\Pi^B\left(S\right)$:

a) $\Pi^B\left(S\right)$ is strictly concave in S,

b) Optimal stocking level, S^B, is the largest S such that $P\left(S;T,P_1\right) \geq \dfrac{h}{\left(p_1+h\right)-\alpha_{d_0^*}\left(p_0-d_0^*\right)}$.

When there is no backup inventory, optimal stocking is the largest S such that $P\left(S;T,P_1\right) \geq \dfrac{h}{p_1+h}$. Comparing with

Proposition **12**, we can see that optimal stocking level is lower when backup inventory is available.

Proposition **12** shows that the possibility of using home delivery to satisfy customers after stockout can increase the store profit and reduce inventory at the same time.

2.5.2. TWO DISCOUNT LEVEL POLICY

In the two discount level (TD) policy, as the name suggests either discount will be zero or a fixed level $\widehat{d}$ before stockout. With only two discount level store operations will become more manageable and sale associates can use their time to further enhance the quality of customer in-store experience. The TD policy will also be easier to characterize as search paces of d is limited to zero or $\widehat{d}$. To find the optimal TD policy, a two stage optimization should be solved. In the second stage, $\widehat{d}$ is fixed, and the TD policy finds the optimal stock and profit given $\widehat{d}$. Then in the first stage, profit is optimized over $\widehat{d}$.

Let $H^{TD}\left(i,t \middle| \widehat{d}\right)$ be the expected maximum profit under TD policy with i units of inventory at time period t given the fixed $\widehat{d}$. For notation simplicity, we use $H^{TD}\left(i,t\right)$ to represent $H^{TD}\left(i,t \middle| \widehat{d}\right)$. The Bellman equation is:

$$H^{TD}\left(i,t\right) = H^{TD}\left(i,t+1\right) + P_1\left(p_1 - \Delta_i H^{TD}\left(i,t+1\right)\right) + \max_{d=0,\widehat{d}} P_1 W^{TD}\left(d,i,t\right) \tag{2.5}$$

where $W^{TD}\left(d,i,t\right) = \alpha_d\left(p_0 - d - p_1 + \Delta_i H^{TD}\left(i,t+1\right)\right)$ and the boundary condition is $H^{TD}\left(i,T+1\right) = -hi$. Next, we show that the TD value function has similar properties to those of the OPT function. The following result is parallel to Lemma 5.

Lemma 6 For any i and t , $\Delta_i H^{TD}\left(i+1,t\right) \le \Delta_i H^{TD}\left(i,t\right)$ and $\Delta_i H^{TD}\left(i,t\right) \ge \Delta_i H^{TD}\left(i,t+1\right)$.

Lemma 6 shows that the marginal value of inventory under TD policy is also decreasing in inventory and time, just as under the OPT policy. Similarly, we expect that TD discount has similar behaviors with respect to inventory and time to those of the OPT policy. Let $d_{i,t}^{TD}$ represent optimal discount under TD policy with i units at time period t . Next, we characterize $d_{i,t}^{TD}$ with positive inventory.

Proposition 13 If $p_0 - \widehat{d} - p_1 + \Delta_i H^{TD}\left(i,t+1\right) \ge 0$, $d_{i,t}^{TD} = \widehat{d}$, otherwise $d_{i,t}^{TD} = 0$.

The reason for Proposition 15 is that to find TD discount, we just need to compare $W^{TD}\left(\widehat{d},i,t\right)$ and $W^{TD}\left(0,i,t\right)$. Since $W^{TD}\left(0,i,t\right) = 0$, when $W^{TD}\left(\widehat{d},i,t\right) \ge 0$ then it is optimal for the retailer to offer $d_{i,t}^{TD} = \widehat{d}$. Based on Lemma 6 and Proposition 13, we can see that the TD policy is threshold based as characterized in Proposition 14.

Proposition 14 The TD discount policy is threshold based:

a) For a given $i > 0$, there exists a threshold on time period, t_i^{TD}, before which $d_{i,t}^{TD} = \hat{d}$ and after which $d_{i,t}^{TD} = 0$.

b) The threshold, t_i^{TD}, is decreasing in i.

Therefore, to characterize the TD policy for a given $\hat{d}$, we need to find t_i^{TD} for all i. Please note that Part b) of Proposition 14 is equivalent to inventory based threshold, i_t^{TD}, above which $d_{i,t}^{TD} = 0$ and below which $d_{i,t}^{TD} = \hat{d}$. Once DP is solved for a given $\hat{d}$, the optimal stocking and the optimal profit for $\hat{d}$ is found. Next step is to optimization over $\hat{d}$ as follows:

$$\max_{0 \le \hat{d} \le p_0} \Pi^{TD}\left(\hat{d}\right) = H^{TD}\left(S_{\hat{d}}, 1 \middle| \hat{d}\right) \tag{2.6}$$

where $S_{\hat{d}}$ is the optimal stocking given the fixed $\hat{d}$. Next, we show that to we can search for the optimal $\hat{d}$ on $\left[0, p_0\right]$.

Lemma 7 $H^{TD}\left(i, t \middle| \hat{d}\right)$ is concave in $\hat{d}$ for all i, t.

When $\hat{d} = 0$ and $\hat{d} = p_0$, the TD policy is the same as the B policy. However, for any $\hat{d} \in \left(0, p_0\right)$, the TD policy will perform better than B policy.

2.5.3. NEWSVENDOR POLICY

We now take a different approach that keeps the structure of optimal policy but simplifies how the optimal discount is calculated. Recall that to find the optimal discount under OPT policy, we optimize $W^{OPT}\left(d, i, t\right) = \alpha_d\left(p_0 - d - p_1 + \Delta_i V\left(i, t+1\right)\right)$ over d. To simplify calculation, we can approximate $\Delta_i V(i,t)$ by the first order of newsvendor profit. Therefore, we call this the newsvendor (NV) heuristic. The newsvendor profit is defined as follow:

$$G(d,i,t) = p_1 i - (p_1 + h)\sum_{n=0}^{i}(i-n)\,p\left(n;T-t+1,P_1(1-\alpha_d)\right). \tag{2.7}$$

Therefore, the first order difference of (2.7) is as follows:

$$\Delta_i G(d,i,t) = p_1 - (p_1 + h)\sum_{n=0}^{i-1} p\left(n;T-t+1,P_1(1-\alpha_d)\right). \tag{2.8}$$

This approximation assumes that a constant discount will be offered for the rest of the season at all positive inventory levels. The NV heuristics is myopic and will be updated in every time period, therefore the NV discount can vary. We begin by characterizing how NV discount ($d_{i,t}^{NV}$) is calculated and then show how its behavior changes with respect to i and t.

Proposition 15 There exists a unique $d_{i,t}^{NV}$ such that

a) $d_{i,t}^{NV} = 0$ if $p_0 - (p_1 + h)\sum_{n=0}^{i-1} p\left(n;T-t+1,P_1(1-\alpha_d)\right) < 0$,

b) otherwise, $d_{i,t}^{NV}$ is the unique solution to $\alpha'_{d_{i,t}^*}\left[p_0 - d_{i,t}^* - p_1 + \Delta_i G(d,i,t)\right] = \alpha_{d_{i,t}^*}$ and $d_{i,t}^{NV} \in (0, p_0)$.

Similar to the OPT policy, we show that $W^{NV}(d,i,t)$ is a unimodal function in d for all i and t. When it is decreasing for all d, then $d_{i,t}^* = 0$. Otherwise since it is a unimodal the first order condition can be used to calculate $d_{i,t}^*$. Next, we characterize how $d_{i,t}^{NV}$ changes with respect to i and t.

Proposition 16 For any i and t, $d_{i+1,t}^{NV} \leq d_{i,t}^{NV}$ and $d_{i,t+1}^{NV} \geq d_{i,t}^{NV}$.

Proposition 16 shows that the NV heuristic, just like the OPT policy, has decreasing discounts both with respect to time and inventory. Next, we characterize NV discount close to the end of the season which will help to map the complete NV policy.

Proposition 17 There exists a period t_0^{NV}, after which $d_{i,t}^{NV} = 0$ for all $i > 0$.

Using all the information we obtained on NV policy, we suggest the following procedure to calculate NV discount over the season. Start from $t = 1$ and $i = 1$ to calculate the NV discount and then move to the next period, until the NV discount is zero. At this point, go to the next inventory level and set $t = 1$ and $i = 2$, Follow this procedure until the NV discount at $t = 1$ and i is zero. Once the NV policy is calculated, we can calculate the retailer's expected profit using a dynamic program similar to the OPT policy.

2.6. NUMERICAL STUDIES

In this section, we use a set of numerical examples to study the retailer's profit and stocking level under the optimal policy and the suggested heuristics. Recall that the optimal policy and the heuristics are all derived in the discrete time setting which serves as an approximation to the underlying continuous time customer arrival and discounting processes. In the numerical tests we will evaluate all of them and see how well they perform in the real, continuous-time setting.

Without loss of generality, we normalize $p_0 = p_1 = 10$, $T = 1$, and $\delta = 0.001$. We set λ at five different levels ranging from low to high. Because p is fixed at 10, setting the leftover cost h is equivalent to setting a service level (SL), $SL = \dfrac{p}{p+h}$.

The parameter set is:

- o $\lambda \in \{5,10,20,50,75\}$.
- o $SL \in \{0.65, 0.75, 0.85, 0.95, 0.99\}$, which implies $h_0 \in \left\{\frac{7}{13}, \frac{1}{3}, \frac{3}{17}, \frac{1}{19}\right\}$.

We assume function form of $\alpha(d) = 1 - e^{-ad}$. We set a at five different levels ranging from low to high. The parameter set is:

- o $a \in \{0.05, 0.1, 0.2, 0.5, 1, 2, 5\}$.

Note that parameter a represents the upper bound on *discount sensitivity* of customers, showing how much home delivery rate increases if discount is slightly above zero discount.

There are 198 cases in total. In this section, we are interested in analyzing how much optimal profit and inventory change a retailer could expect from implementing the optimal policy and various suggested heuristics from the base policy. All the tables present average performance deviations across a given set of cases. To streamline presentation, we show all the results as performance deviation of OPT/NV/TD from policy B, defined as:

$$Performance\,Dev = \frac{Performance^{OPT/NV/TD} - Performance^{B}}{Performance^{B}}, \tag{9}$$

where $Performance \in \{Profit, Inventory\}$.

Table 2.1 Policy OPT/NV/TD's Profit Deviation from Policy B

	NV Policy	TD Policy	OPT Policy
Average	0.81%	0.83%	0.99%
MAX	2.24%	2.30%	2.68%

In Table 2.1, profit deviation results show that OPT policy can result in ~1% profit increases. Maximum profit increase is 2.68%, which is significant. While the TD policy slightly does better than the NV policy, comparing individual cases shows that in 52.8% of the cases NV policy is outperforming (0.64% vs. 0.58%) TD policy.

Table 2.2 Policy OPT/NV/TD's Inventory Deviation from Policy B

	NV Policy	TD Policy	OPT Policy
Average	-6.57%	-4.71%	-6.43%

In Table 2.2 we observe optimal inventory reduction which is due to using backup inventory. The OPT policy on average reduces stocking level around 6.43%, which is less reduction compared to NV policy. So while expected profit from NV policy is not as much as OPT and TD policy, the inventory reduction is higher. Numerically what we can see is that NV policy discounts heavier compared to OPT policy; as a result, inventory reduction is more.

In Table 2.1 and Table 2.2, we analyzed how a retailer's performance in terms of profit and inventory changes with discounted home delivery policy. It is also important to understand a retailer's performance trend with respect to demand and service level.

Table 2.3 Policy OPT's Profit Deviation from Policy B by λ and SL

Service Level	5	10	20	50	75
0.65	1.91%	1.98%	2.02%	1.81%	1.66%
0.75	1.44%	1.59%	1.59%	1.38%	1.26%
0.85	1.06%	1.07%	1.03%	0.90%	0.82%
0.95	0.44%	0.42%	0.39%	0.33%	0.30%
0.99	0.09%	0.08%	0.08%	0.06%	0.06%

λ (Demand Rate)

In Table 2.3 we can see that retailers with a service level below 0 benefit significantly from implementing OPT policy compared to B policy. When service level is high, unit values are high while holding cost is low, therefore retailers with high service level tend to carry lots of inventory. Therefore, they will not need to use backup inventory as much. Next, we analyze profit and inventory deviation with respect to customer's discount sensitivity.

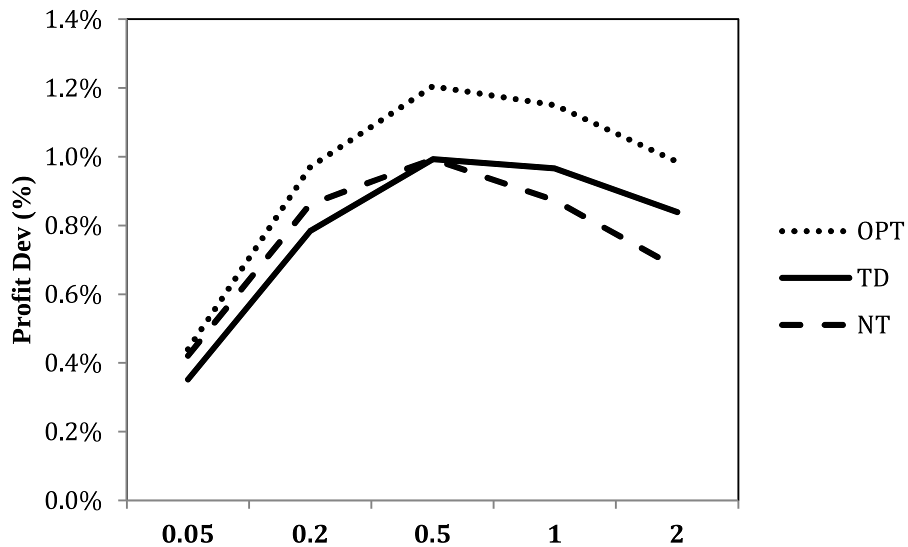

Figure 2.1 Profit Deviation by Discount Sensitivity

Profit deviation is concave in discount sensitivity. When discount sensitivity is low, it means that customers are mostly leadtime sensitive. Therefore, increasing discount will not have much benefit. On the other hand, when discount sensitivity is high, customers are mostly price sensitive. Therefore, retailer can capture them just by offering discount after stockout. However, with middle range of price sensitivity, there is more heterogeneity among customers. In that case retailer can discounted home delivery to successfully segment the customers before stockout. Therefore, for middle ranges of price sensitivity, OPT, NT, and ST outperform B policy.

2.7. SUMMARY AND FUTURE RESEARCH DIRECTIONS

In this chapter, we have studied the discounted home delivery strategy. This strategy helps retailers to delay stock and to save the sale after stockout.

We characterized the optimal action of retailers under stochastic demand for any given time and inventory in the season. We noted the challenges of implementing the optimal policy and suggest two heuristics, *two discount level* and *newsvendor*, to address these challenges. In the numerical section we analyzed retailer profit and inventory decisions with discounted home delivery strategies. We showed that profit increase as well as inventory reduction is significant with OPT, NV, or TD policy compared to B policy.

Much work remains to be done. There are three extensions that we suggest here. First, we assumed customers are myopic; however, in real life customers can be strategic. Therefore, one extension to this model is to consider strategic customers who might visit a store often in order to determine the best time to make a purchase. Second, we made the assumption that there is an ample amount of backup inventory. This is a reasonable assumption when considering big retailers with many stores, warehouses, and suppliers. However, it is worthwhile to consider finite backup inventory. Third, we assumed that $\alpha(d)$ is independent of inventory level for tractability purposes. However, $\alpha(d)$ could be different when we have positive inventory available compared to no inventory. When there is no inventory available, customers discount sensitivity will be higher.

APPENDIX

Proof of Proposition 9

Please note:

$$V\left(0,t\right)=V\left(0,t+1\right)+\max_{0\le d\le p_0}P_1\alpha_d\left(p_0-d\right) \tag{2.9}$$

For this proof, let's define $f\left(d\right)=\alpha_d\left(p_0-d\right)$. The first and second order derivatives with respect to d are as follows:

$$\begin{aligned} f'\left(d\right)&=\alpha_d'\left(p_0-d\right)-\alpha \\ f''\left(d\right)&=\alpha_d''\left(p_0-d\right)-2\alpha_d' \end{aligned} \tag{2.10}$$

Since $\alpha_d'>0$ and $\alpha_d''<0$ then $f''\left(d\right)<0$. Therefore $f\left(d\right)$ is a strictly concave function, as a result of which $d_{0,t}^*$ is unique the solution to $f'\left(d\right)=0$. Since $f'\left(0\right)=\alpha_0'\cdot p_0>0$ and $f'\left(p_0\right)=-\alpha_{p_0}<0$, then $0<d_{0,t}^*<p_0$.

Proof of Lemma 5

We prove Lemma 5 in two parts as follows:

A. For any i and t, $\Delta_i V\left(i+1,t\right)\le\Delta_i V\left(i,t\right)$

B. For any i and t, $\Delta_i V\left(i,t\right)\ge\Delta_i V\left(i,t+1\right)$

Part A) The proof is by induction on t. For all i, we know $\Delta_i V\left(i,T+1\right)=-h$, therefore it holds at the boundary. Now we assume that it holds at period $t+1$ and show that it holds at period t as well. For all $i>0$, at period t, we can write:

$$\begin{aligned}\Delta_i V\left(i+2,t\right)-\Delta_i V\left(i+1,t\right)=&\left(1-P_1\right)\left(\Delta_i V\left(i+2,t+1\right)-\Delta_i V\left(i+1,t+1\right)\right)\\&+P_1\left(\Delta_i V\left(i+1,t+1\right)-\Delta_i V\left(i,t+1\right)\right)\\&+P_1\alpha_{d^*_{i+2,t}}\left(p_0-d^*_{i+2,t}+p_1+\Delta_i V\left(i+2,t+1\right)\right)\\&-P_1\alpha_{d^*_{i+1,t}}\left(p_0-d^*_{i+1,t}+p_1+\Delta_i V\left(i+1,t+1\right)\right)\\&-P_1\alpha_{d^*_{i+1,t}}\left(p_0-d^*_{i+1,t}+p_1+\Delta_i V\left(i+1,t+1\right)\right)\\&+P_1\alpha_{d^*_{i,t}}\left(p_0-d^*_{i,t}+p_1+\Delta_i V\left(i,t+1\right)\right)\end{aligned}\tag{2.11}$$

From the optimality of $d^*_{i+1,t}$, the following inequalities hold:

$$\begin{aligned}\alpha_{d^*_{i+1,t}}\left(p_0-d^*_{i+1,t}+p_1+\Delta_i V\left(i+1,t+1\right)\right)\geq\alpha_{d^*_{i+2,t}}\left(p_0-d^*_{i+2,t}+p_1+\Delta_i V\left(i+1,t+1\right)\right)\\\alpha_{d^*_{i+1,t}}\left(p_0-d^*_{i+1,t}+p_1+\Delta_i V\left(i+1,t+1\right)\right)\geq\alpha_{d^*_{i,t}}\left(p_0-d^*_{i,t}+p_1+\Delta_i V\left(i+1,t+1\right)\right)\end{aligned}\tag{2.12}$$

Substituting (2.12) into (2.11), we obtain:

$$\begin{aligned}\Delta_i V\left(i+2,t\right)-\Delta_i V\left(i+1,t\right)\leq&\left(1-P_1\right)\left(\Delta_i V\left(i+2,t+1\right)-\Delta_i V\left(i+1,t+1\right)\right)\\&+P_1\left(\Delta_i V\left(i+1,t+1\right)-\Delta_i V\left(i,t+1\right)\right)\\&+P_1\alpha_{d^*_{i+2,t}}\left(p_0-d^*_{i+2,t}+p_1+\Delta_i V\left(i+2,t+1\right)\right)\\&-P_1\alpha_{d^*_{i+2,t}}\left(p_0-d^*_{i+2,t}+p_1+\Delta_i V\left(i+1,t+1\right)\right)\\&-P_1\alpha_{d^*_{i,t}}\left(p_0-d^*_{i,t}+p_1+\Delta_i V\left(i+1,t+1\right)\right)\\&+P_1\alpha_{d^*_{i,t}}\left(p_0-d^*_{i,t}+p_1+\Delta_i V\left(i,t+1\right)\right)\end{aligned}\tag{2.13}$$

Rearranging and canceling terms yield the following:

$$\begin{aligned}\Delta_i V(i+2,t)-\Delta_i V(i+1,t) \le &\left(1-P_1\left(1-\alpha_{d^*_{i+2,t}}\right)\right)\left(\Delta_i V(i+2,t+1)-\Delta_i V(i+1,t+1)\right)\\ &+P_1\left(1-\alpha_{d^*_{i,t}}\right)\left(\Delta_i V(i+1,t+1)-\Delta_i V(i,t+1)\right)\end{aligned} \tag{2.14}$$

From the induction assumption, we know $\Delta_i V(i+1,t+1) \le \Delta_i V(i,t+1)$ for all i. By definition of α_d and P_1, we also know that $0 \le \alpha_d,\ P_1 \le 1$. Therefore we have showed $\Delta_i V(i+2,t)-\Delta_i V(i+1,t) \le 0$ holds for all i at period t as well. As a result, it holds for all i and t.

Part B) Please note:

$$\begin{aligned}\Delta_i V(i,t)-\Delta_i V(i,t+1) = &-P_1\left(\Delta_i V(i,t+1)-\Delta_i V(i-1,t+1)\right)\\ &+P_1\alpha_{d^*_{i,t}}\left(p_0-d^*_{i,t}-p_1+\Delta_i V(i,t+1)\right)\\ &-P_1\alpha_{d^*_{i-1,t}}\left(p_0-d^*_{i-1,t}-p_1+\Delta_i V(i-1,t+1)\right)\end{aligned} \tag{2.15}$$

From the optimality of $d^*_{i,t}$, the following inequality holds:

$$\alpha_{d^*_{i,t}}\left(p_0-d^*_{i,t}-p_1+\Delta_i V(i,t+1)\right) \ge \alpha_{d^*_{i-1,t}}\left(p_0-d^*_{i-1,t}-p_1+\Delta_i V(i,t+1)\right) \tag{2.16}$$

Substituting (2.16) into (2.15), we obtain:

$$\begin{aligned}\Delta_i V(i+1,t)-\Delta_i V(i+1,t+1) \ge &-P_1\left(\Delta_i V(i+1,t+1)-\Delta_i V(i,t+1)\right)\\ &+P_1\alpha_{d^*_{i,t}}\left(p_0-d^*_{i,t}-p_1+\Delta_i V(i+1,t+1)\right)\\ &-P_1\alpha_{d^*_{i,t}}\left(p_0-d^*_{i,t}-p_1+\Delta_i V(i,t+1)\right)\end{aligned} \tag{2.17}$$

Rearranging and canceling terms yield

$$\Delta_i V(i+1,t) - \Delta_i V(i+1,t+1) \geq -\left(1-\alpha_{d^*_{i,t}}\right) P_1 \left(\Delta_i V(i+1,t+1) - \Delta_i V(i,t+1)\right) \quad (2.18)$$

We showed in part A) that $\Delta_i V(i+1,t+1) \leq \Delta_i V(i,t+1)$, therefore $\Delta_i V(i+1,t) \geq \Delta_i V(i+1,t+1)$.

Proof of Theorem 2

Recall from (2.3):

$$W^{OPT}(d,i,t) = \alpha_d \left(p_0 - d - p_1 + \Delta_i V(i,t+1)\right) \quad (2.19)$$

The first and second order derivatives with respect to d are as follows:

$$\frac{\partial W^{OPT}(d,i,t)}{\partial d} = \alpha'_d \left(p_0 - d - p_1 + \Delta_i V(i,t+1)\right) - \alpha_d \quad (2.20)$$

$$\frac{\partial^2 W^{OPT}(d,i,t)}{\partial d^2} = \alpha''_d \left(p_0 - d - p_1 + \Delta_i V(i,t+1)\right) - 2\alpha'_d \quad (2.21)$$

For brevity, we will shorten $\frac{\partial W^{OPT}(d,i,t)}{\partial d}$ and $\frac{\partial^2 W^{OPT}(d,i,t)}{\partial d^2}$ as $\frac{\partial W}{\partial d}$ and $\frac{\partial^2 W}{\partial d^2}$ respectively.

For this proof, let's define $f(d) = p_0 - d - p_1 + \Delta_i V(i,t+1)$. We know $f'(d) = -1 < 0$ and $f(p_0) = -p_1 + \Delta_i V(i,t+1) < 0$. Therefore, either $f(d) \leq 0$ for all d or there exist $\overline{d}$ such that $f(\overline{d}) = 0$. Therefore, at any (i,t) two cases are possible:

I. When $f\left(d\right) < 0$ for all d, because $\alpha_d' > 0$ and $\alpha_d > 0$, we know $\dfrac{\partial W}{\partial d} < 0$

II. Similar to I, when $f\left(d\right) \leq 0$ for $d \in \left[\overline{d}, p_0\right]$, we know $\dfrac{\partial W}{\partial d} < 0$. On the other hand, since $f\left(d\right) > 0$ on $d \in \left[0, \overline{d}\right)$, $\alpha_d'' < 0$, and $\alpha_d' > 0$, we know $\dfrac{\partial^2 W}{\partial d^2} < 0$. Please note $\left.\dfrac{\partial W}{\partial d}\right|_{d=0} > 0$ and $\left.\dfrac{\partial W}{\partial d}\right|_{d=\overline{d}} < 0$.

Therefore $W^{OPT}\left(d,i,t\right)$ is a unimodal function. In case I, $d_{i,t}^* = 0$ and in case II, $d_{i,t}^*$ is the unique solution to the first order condition and $d_{i,t}^* \in \left(0, \overline{d}\right)$.

Proof of Proposition 10

We prove Proposition 10 in two parts as follows:

A. For any i and t, $d_{i+1,t}^* \leq d_{i,t}^*$

B. For any i and t, $d_{i,t}^* \geq d_{i,t+1}^*$

Part A) The first order difference of (2.20) with respect to i is as follows:

$$\Delta_{i+1} \frac{\partial W^{OPT}\left(d,i,t\right)}{\partial d} = \alpha_d'\left(\Delta_i V\left(i+1,t+1\right) - \Delta_i V\left(i,t+1\right)\right). \tag{2.22}$$

Based on Lemma 5, $\Delta_i V\left(i+1,t+1\right) \leq \Delta_i V\left(i,t+1\right)$, therefore $\Delta_{i+1} \dfrac{\partial W^{OPT}\left(d,i,t\right)}{\partial d} \leq 0$ which means $d_{i+1,t}^* \leq d_{i,t}^*$.

Part B) Let's define $\Delta_t \dfrac{\partial W^{OPT}\left(d,i,t\right)}{\partial d} = \dfrac{\partial W^{OPT}\left(d,i,t\right)}{\partial d} - \dfrac{\partial W^{OPT}\left(d,i,t+1\right)}{\partial d}$. The first order difference of (2.20) with respect to t is as follows:

$$\Delta_{t+1} \frac{\partial W^{OPT}\left(d,i,t\right)}{\partial d} = \alpha_d'\left(\Delta_i V\left(i,t+1\right) - \Delta_i V\left(i,t+2\right)\right) \tag{2.23}$$

Based on Lemma 5, $\Delta_i V\left(i,t+1\right) \geq \Delta_i V\left(i,t+2\right)$, therefore $\Delta_{t+1} \dfrac{\partial W^{OPT}\left(d,i,t\right)}{\partial d} \geq 0$ which means $d_{i,t}^* \geq d_{i,t+1}^*$.

Proof of Proposition 11

At period $t = T$:

$$\frac{\partial W^{OPT}\left(d,i,T\right)}{\partial d} = \alpha_d'\left(p_0 - d - p_1 - h\right) - \alpha_d \tag{2.24}$$

Since $p_0 < p_1$ and $\alpha_d' > 0$, then $\dfrac{\partial W^{OPT}\left(d,i,T\right)}{\partial d} < 0$ for all d which means $d_{i,T}^* = 0$. We know $d_{i,t}^* \geq d_{i,t+1}^*$, then if we move backward in time, either TD discount is always zero or there might be a period at which $d_{i,t}^* > 0$. Since $d_{i+1,t}^* \leq d_{i,t}^*$, $d_{1,t}^*$ will change sign sooner than any other i at a given t. Therefore t_0^* is the largest t such that the followings hold:

$$\left(p_0 - p_1 + \Delta_i V\left(1,t+1\right)\right) \geq 0 \quad and \quad \left(p_0 - p_1 + \Delta_i V\left(1,t+2\right)\right) < 0. \tag{2.25}$$

Proof of

***Proposition** 12*

The first order difference of $\Pi^B\left(S\right)$ with respect to S is:

$$\Delta_S \Pi^B\left(S\right) = p_1 - \alpha_d\left(p_0 - d\right) - \left(p_1 - \alpha_d\left(p_0 - d\right) + h\right)\sum_{j=0}^{S-1} p\left(j;T,P_1\right) \tag{2.26}$$

The second order difference of $\Pi^B\left(S\right)$ with respect to S is:

$$\Delta_S^2 \Pi^B\left(S\right) = -\left(p_1 - \alpha_d\left(p_0 - d\right)\right) p\left(S-1;T,P_1\right) \tag{2.27}$$

Therefore $\Pi^B\left(S\right)$ is strictly concave in S.

Proof of Lemma 6

Proof is parallel to that of Lemma 5.

Proof of Proposition 13

We want to optimize $W^{TD}\left(d,i,t\right) = \alpha_d\left(p_0 - d - p_1 + \Delta_i H^{TD}\left(i,t+1\right)\right)$ over $d \in \left\{0,\widehat{d}\right\}$. We know $W^{TD}\left(0,i,t\right) = 0$. Therefore, if $W^{TD}\left(\widehat{d},i,t\right) \geq 0$, $d_{i,t}^{TD} = \widehat{d}$, otherwise $d_{i,t}^{TD} = 0$. For $W^{TD}\left(\widehat{d},i,t\right) \geq 0$, it is sufficient if $p_0 - \widehat{d} - p_1 + \Delta_i H^{TD}\left(i,t+1\right) \geq 0$.

Proof of Proposition 14

For this proof, we define $f\left(\hat{d},i,t\right)=p_0-\hat{d}-p_1+\Delta_i H^{TD}\left(i,t+1\right)$. By Proposition 13, if $f\left(\hat{d},i,t\right)\geq 0$, then $d_{i,t}^{TD}=\hat{d}$, otherwise $d_{i,t}^{TD}=0$.

Part A) We know $f\left(\hat{d},i,T\right)=p_0-\hat{d}-p_1-h<0$. Therefore, when $t=T$, $d_{i,t}^{TD}=0$ for all $i>0$. By Lemma 6, $\Delta_i H^{TD}\left(i,t\right)\geq\Delta_i H^{TD}\left(i,t+1\right)$, therefore $f\left(\hat{d},i,t\right)\geq f\left(\hat{d},i,t+1\right)$ which means $d_{i,t}^{TD}\geq d_{i,t+1}^{TD}$. This means going backward either TD discount is always zero for a given i or there is a period at which, called t_i^{TD}, TD discount changes to $\hat{d}$ and remains fixed for the rest of the season. Therefore t_i^{TD} is such that:

$$\left(p_0-\hat{d}-p_1+\Delta_i H^{TD}\left(i,t_i^{TD}\right)\geq 0\right) \quad and \quad \left(p_0-\hat{d}-p_1+\Delta_i H^{TD}\left(i,t_i^{TD}+1\right)<0\right) \quad (2.28)$$

Part B) By Lemma 6, $\Delta_i H^{TD}\left(i+1,t\right)\leq\Delta_i H^{TD}\left(i,t\right)$, therefore $f\left(\hat{d},i+1,t\right)\leq f\left(\hat{d},i,t\right)$ which means $d_{i+1,t}^{TD}\leq d_{i,t}^{TD}$. As a result, t_i^{TD} is decreasing i.

Proof of Lemma 7

We prove concavity in $\hat{d}$ using induction on t. At the last period, $t=T$, we know that $d_{i,t}^{TD}=0$ holds for all $i>0$. That is, there will be no discounting regardless of the inventory. Thus the value function is independent of the $\hat{d}$. It's trivially concave in $\hat{d}$ as a result. We assume that it is concave in $\hat{d}$ at period $t+1$ and show that it is concave at period t as well.

For all $i\geq i_{t+1}^{TD}$, we know:

$$H^{TD}\left(i,t\right)=P_0 H^{TD}\left(i,t+1\right)+P_1\left(p_1+H^{TD}\left(i-1,t+1\right)\right) \tag{2.29}$$

The second order of (2.29) with respect to $\widehat{d}$ is as follows:

$$\frac{\partial^2 H^{TD}\left(i,t\right)}{\partial \widehat{d}^2}=P_0\frac{\partial^2 H^{TD}\left(i,t+1\right)}{\partial \widehat{d}^2}+P_1\frac{\partial^2 H^{TD}\left(i-1,t+1\right)}{\partial \widehat{d}^2} \tag{2.30}$$

By induction assumption, $H^{TD}\left(i,t+1\right)$ is concave for all $i \ge i_{t+1}^{TD}$, then $\frac{\partial^2 H^{TD}\left(i,t\right)}{\partial \widehat{d}^2}<0$ for all $i \ge i_{t+1}^{TD}$ as well.

For $i < i_{t+1}^{TD}$, we know:

$$\begin{aligned}H^{TD}\left(i,t\right)&=P_0 H^{TD}\left(i,t+1\right)+P_1\left(p_1+H^{TD}\left(i-1,t+1\right)\right)\\&\quad+P_1\alpha_{\widehat{d}}\left(p_0-\widehat{d}+p_1+\Delta_i H^{TD}\left(i,t+1\right)\right)\end{aligned} \tag{2.31}$$

Therefore, second orders of (2.31) with respect to $\widehat{d}$ is:

$$\begin{aligned}\frac{\partial^2 H^{TD}\left(i,t\right)}{\partial \widehat{d}^2}&=P_0\frac{\partial^2 H^{TD}\left(i,t+1\right)}{\partial \widehat{d}^2}+P_1\frac{\partial^2 H^{TD}\left(i-1,t+1\right)}{\partial \widehat{d}^2}\\&\quad+P_1\alpha''_{\widehat{d}}\left(p_0-\widehat{d}+p_1+\Delta_i H^{TD}\left(i,t+1\right)\right)\\&\quad+2P_1\alpha'_{d}\left(-1+\frac{\partial \Delta_i H^{TD}\left(i,t+1\right)}{\partial \widehat{d}}\right)+P_1\alpha_{\widehat{d}}\frac{\partial^2 \Delta_i H^{TD}\left(i,t+1\right)}{\partial \widehat{d}^2}\end{aligned} \tag{2.32}$$

We can show that $\frac{\partial \Delta_i H^{TD}\left(i,t+1\right)}{\partial \widehat{d}}=P_1\alpha'_{\widehat{d}}\Delta_i^2 H^{TD}\left(i,t+2\right)$ and $\frac{\partial^2 \Delta_i H^{TD}\left(i,\tau+1\right)}{\partial \dot{d}^2}=P_1\alpha''_{d}\Delta_i^2 H^{TD}\left(i,\tau+2\right)$. Therefore (2.32) simplifies to:

$$\begin{aligned}\frac{\partial^2 H^{TD}\left(i,t\right)}{\partial \hat{d}^2} &= P_0 \frac{\partial^2 H^{TD}\left(i,t+1\right)}{\partial \hat{d}^2} + P_1 \frac{\partial^2 H^{TD}\left(i-1,t+1\right)}{\partial \hat{d}^2} \\ &+P_1 \alpha''_{\hat{d}}\left(p_0 - \hat{d} + p_1 + \Delta_i H^{TD}\left(i,t+1\right)\right) \\ &+2\left(P_1 \alpha'_{\hat{d}}\right)^2 \Delta_i^2 H^{TD}\left(i,t+2\right) - \left(P_1\right)^2 \alpha_{\hat{d}} \alpha''_{\hat{d}} \Delta_i^2 H^{TD}\left(i,t+2\right) - 2P_1 \alpha'_{\hat{d}}\end{aligned} \tag{2.33}$$

Therefore, since (2.33) is also negative, concavity proof is complete.

Proof of Proposition 15

The first and second order derivatives with respect to d are as follows:

$$\frac{\partial W^{NT}\left(d,i,t\right)}{\partial d} = \alpha'_d \left(p_0 - d - \left(p_1 + h\right)\sum_{n=0}^{i-1} p\left(n;T-t+1,P_1\left(1-\alpha_d\right)\right)\right) - \alpha_d \tag{2.34}$$

$$\begin{aligned}\frac{\partial^2 W^{NV}\left(d,i,t\right)}{\partial d^2} &= \alpha''_d \left(p_0 - d - \left(p_1 + h\right)\sum_{n=0}^{i-1} p\left(n;T-t+1,P_1\left(1-\alpha_d\right)\right)\right) \\ &-\alpha'_d \left(2 + \left(p_1 + h\right)\alpha'_d P_1 \sum_{n=0}^{i-1} \frac{T-t+1}{T-t+1-n} p\left(n;T-t,P_1\left(1-\alpha_d\right)\right)\right)\end{aligned} \tag{2.35}$$

For brevity of this proof, we will shorten $\frac{\partial W^{NV}\left(d,i,t\right)}{\partial d}$ and $\frac{\partial^2 W^{NV}\left(d,i,t\right)}{\partial d^2}$ as $\frac{\partial W}{\partial d}$ and $\frac{\partial^2 W}{\partial d^2}$ respectively. For this proof, let's define $f\left(d\right) = p_0 - d - \left(p_1 + h\right)\sum_{n=0}^{i-1} p\left(n;T-t+1,P_1\left(1-\alpha_d\right)\right)$.

We know $f'\left(d\right) < 0$ and $f\left(p_0\right) < 0$. Therefore, either $f\left(d\right) < 0$ for all d or there exist $\overline{d}$ such that $f\left(\overline{d}\right) = 0$. Therefore, at any $\left(i,t\right)$ two cases are possible:

I. When $f\left(d\right) < 0$ for all d, because $\alpha'_d > 0$ and $\alpha_d > 0$, we know $\frac{\partial W}{\partial d} < 0$

II. Similar to I, when $f\left(d\right) \le 0$ for $d \in \left[\overline{d}, p_0\right]$, we know $\frac{\partial W}{\partial d} < 0$. On the other hand, since $f\left(d\right) > 0$ on $d \in \left[0, \overline{d}\right)$, $\alpha_d'' < 0$, and $\alpha_d' > 0$, we know $\frac{\partial^2 W}{\partial d^2} < 0$. Please note $\left.\frac{\partial W}{\partial d}\right|_{d=0} > 0$ and $\left.\frac{\partial W}{\partial d}\right|_{d=\overline{d}} < 0$.

Therefore $W^{NV}\left(d,i,t\right)$ is a unimodal function. In case I, $d_{i,t}^* = 0$ and in case II, $d_{i,t}^*$ is the unique solution to the first order condition and $d_{i,t}^* \in \left(0, \overline{d}\right)$.

Proof of Proposition 16

Part A) The first order difference of (2.34) with respect to i is:

$$\Delta_{i+1} \frac{\partial W^{NV}\left(d,i,t\right)}{\partial d} = -\left(p_1 + h\right)\alpha_d' p\left(i-1; T-t+1, P_1\left(1-\alpha_d\right)\right) \tag{2.36}$$

Since $\Delta_{i+1} \frac{\partial W^{NV}\left(d,i,t\right)}{\partial d} < 0$, we know $d_{i+1,t}^{NV} \le d_{i,t}^{NV}$.

Part B) The cross order derivative of (2.34) with respect to t is:

$$\Delta_{t+1} \frac{\partial W^{NV}\left(d,i,t\right)}{\partial d} = \left(p_1 + h\right)\alpha_d' \sum_{n=0}^{i-1} p\left(n; T-t+1, P_1\left(1-\alpha_d\right)\right) \sum_{j=0}^{n-1} \frac{1}{T-t+1-j} \tag{2.37}$$

Since $\Delta_{t+1} \frac{\partial W^{NV}\left(d,i,t\right)}{\partial d} > 0$, we know $d_{i,t}^{NV} \ge d_{i,t+1}^{NV}$.

Proof of Proposition 17

Let's look at the last period $t = T$. The first order condition at $t = T$ for all $i > 0$ simplifies to:

$$\frac{\partial W^{NV}\left(d,i,T\right)}{\partial d} = \lambda\left\{\alpha_d'\left(p_0 - d - p_1 - h\right) - \alpha_d\right\} < 0 \tag{2.38}$$

Therefore $d_{i,T}^{NT} = 0$ for all $i > 0$. However since $d_{i,t}^{NT}$ is decreasing in t, if we move backward there might be a period at which $d_{i,t}^{NT}$ becomes greater than zero. Since $d_{i,t}^{NT}$ is decreasing in i, when $i = 1$ sign of $\frac{\partial W^{NV}\left(d,i,T\right)}{\partial d}$ will change sooner than any other i going backward. We can use (2.34) to characterize this point as follow:

$$p_0 - \left(p_1 + h\right) p\left(0; T - t_0^{NV} + 1, P_1\left(1 - \alpha_d\right)\right) = 0 \tag{2.39}$$

CHAPTER 3. COMPETITION FOR COMMON CUSTOMER BASE

3.1 Introduction and Motivation

In this chapter, we study an incentive problem in a service setting where the service provider has multiple channels to serve the same homogeneous customer base. We use a very broad definition of channel so that it can include different formats of service or simply different service agents who all can serve the customer base. Because the agents serve the same pool of potential customers, there exists an inherent competition among agents for customers. For example, when one agent serves the only remaining customer in a queue, the other agent(s) after completing service will be idle as they wait for more customers. Therefore, an agent's share of the customer base is the result of competition. Intuitively, we know that this will affect an agent's decision on how much effort to put forth or how fast to work, both of which affect service quality. We want to know how to coordinate such a system through various financial, operational, and informational means available to the system provider. We hope that the results and insights developed in this research, while obtained in a service setting, could have some relevance to similar incentive problems often faced by omnichannel retailers.

Sales associates are paid through a combination of salary, commission, and bonuses. A combination of base salary and commissions are more prevalent in the industry as they promote a more sales-focused culture. Commissions are easy for a company to set as they will never overpay. Commission is popular not only among big companies such as Nordstrom, and IBM, but also among small companies. Fishbowl (Williams and Scott 2013) is a small company with around 100 employees. They put every employee on their commission system and believe that it helped them increase their sales by about 60% between 2007 and 2013.

Commission promotes aggregate sale, but not the sales process. Such behavior will likely decrease the quality of customer service and therefore customer satisfaction. Even though customer satisfaction as a criterion of salesforce compensation is growing, only 47% of US contact centers reward their agents based on customer satisfaction ratings (ContactBabel 2012).

This chapter is organized as follows. In section 3.2 we review the related literature. In section 3.3 we introduce the model setup. In Section 3.4, we find system's optimal service quality in the centralized setting. In section 3.5, we find agent's service quality when system is decentralized. In section 3.6, we suggest three solution mechanisms as contracts, allocation, and customer choice. Finally, in section 3.7 we summarize our findings and talk about future research direction for agent's competition.

3.2 Literature Review

This research is related to salesforce compensation, quality and speed trade-off, as well as resource allocation. Speed and quality trade-off literature has a rather short research history. Anand *et al.* (2011) when they studied customer intense services. They assumed that time spent with customer's increases service quality with diminishing marginal value; this became a common assumption in the literature. They captured quality through endogenous demands of strategic customers. They considered single servers as well as multiple competing servers and show that service value offered through competition is higher and that servers will slow down. Tong and Rajagopalan (2014) studied pricing decisions of firms when customers have different quality valuations. The dynamic aspect of the quality–speed trade-off is analyzed in Kostami and Rajagopalan (2014). There is a branch in this stream of literature that considers the speed–quality trade-off in a health care setting (Alizamir *et al.* 2013; Akan *et al.* 2011). There is another branch that considers the speed–quality trade-off in a call center setting, where quality is defined as percentage of calls that are resolved during the first call (de Vericourt & Zhou 2005; Zhan &Ward 2014). A separate branch of the literature looked at judgment/diagnostic accuracy under congestion where the value of an additional test or cue is weighted against delaying service to other customers and creating congestion (de Vericourt & Sun 2009; Alizamir *et al.* 2013).

Our research is relevant to the extensively studied resource allocation literature. In the literature, with limited resources available, agents compete based on different factors such as service level (Benjaafar *et al.* 2007; Jin and Ryan 2012; Elahi 2013), delivery frequency (Ha *et al.* 2003; Cachon and Zhang 2007), and price (Jin and Ryan 2012). Gilbert and Weng (1998) modeled a principal that allocates limited demand between two competing agents with costly service rates. They characterized the condition under which consolidated or non-consolidate queues would be performing better. Cachon and Lariviere (1999) studied supply chain performance when a supplier allocates its fixed supply across retailers using different allocation mechanism. They specifically analyzed a *turn-and-earn allocation* mechanism based on past sales. Lu and Lariviere (2012) have studied performance of turn-and-earn allocation mechanisms in the long-run using a dynamic setting. Interested readers are referred to Zhou and Ren (2010) for a review of service-based outsourcing.

Salesforce compensation has been extensively studied in the marketing literature; see Coughlan (1993) and Albers and Mantrala (2008) for a comprehensive review. Many of these studies considered asymmetrical information cases where moral-hazard or adverse selection or a combination of the two exists (started with Basu *et al.* 1985). Mainly this literature has studied the effectiveness of different combinations of compensation packages such as salary, commission, and bonuses (Holmstrom 1979, Basu *et al.* 1985, Raju and Srinivasan 1996, Oyer 2000). In operations management literature there are few studies on salesforce compensation with regard to operation issues such as inventory (Chen 2000, Chen 2005, Dai and Jerath 2013, Saghafian and Chao 2014).

3.3 MODEL SETUP

Consider a service center with a service provider and two identical agents. We assume the center sells a product that has a fixed price of p. We also assume the arrival of customers follows a Poisson process with a constant rate λ. Customers arrive to a queue that is common to both agents. When an agent becomes available, they will immediately serve the next in line on a first-come, first-served (FCFS) basis. Customers who find both agents idle are assigned randomly. We assume customer purchase decisions depend not only on the price of the product but also on the quality of the interaction they perceive to have received from the agents. We assume agents are identical in that they have the same effort cost function and receive the same commission c from each sale.

Each customer brings a random amount of work to the system. We assume customers are homogeneous and their work amount follow *i.i.d.* exponential distribution.

, which means they bring *i.i.d.* work amount. Moreover, we assume that the work amount all follows the same exponential distribution. Agents exert effort which affects the speed of service. The random service time for each customer simply equals the amount of work divided by the service speed. Therefore, customer service time follows the *i.i.d.* exponential distribution, and the average service time and the agent speed have a reciprocal-type relationship. We can equivalently view the agents' decision as determining their average service time. Let $t_i\left(i=1,2\right)$ represent the average time agent i spends with customers, and $R\left(t\right)$ the resultant service quality given t. Since higher service quality leads to higher sales probability, we will normalize $R\left(t\right)$ to be the probability of making a sale to each customer if the agent puts in an effort that leads to average service time of t. For the rest of the chapter we will simply use $R\left(t\right)$to represent both service quality and sales probability.

It is common to assume that service quality is an increasing function of t. After all, the more service time a customer receives, the better treated they feel. However, that is only true up to a point, since time itself is valuable to customers. If a service takes too long, any additional marginal benefit from longer service time may be offset by the customer's loss of time. Therefore, we assume that $R\left(t\right)$ is increasing in t up to a certain point, denoted as t_p, with diminishing marginal value. For t above t_p, $R\left(t\right)$ is decreasing and reaches zero before $t=2/\lambda$. We assume $R\left(0\right)=R\left(2/\lambda\right)=0$.

When two agents choose t_1 and t_2 respectively, the system can be modeled as a FCFS 2-server heterogeneous queuing system under Poisson arrival with rate λ, and two exponential servers with service rates of $1/t_1$ and $1/t_2$. It is clear that the proportion of customers served by each agent must depends on the average service time of both agents. Let $\phi_i\left(t_1,t_2\right)$ denote the share of customers served by agent i. Based on Kalai *et al.* (1992), the steady-state market share of agent i is:

$$\phi_i(t_i,t_j) = \frac{\lambda t_j^2 + t_i + t_j}{\lambda(t_i^2 + t_j^2) + 2(t_i + t_j)}. \tag{3.1}$$

Then an agent i's ($i = 1,2$) sale $S_i\left(t_1,t_2\right)$ can be calculated as $\lambda\phi_i\left(t_1,t_2\right)R\left(t_i\right)$. Agent i's utilization ρ_i can be calculated as $\lambda\phi_i\left(t_1,t_2\right)t_i$. We model the agents' cost of effort as an increasing and convex function of their utilization ρ_i, and denote it by $D\left(\rho_i\right)$. We assume $D\left(0\right) = 0$ and $D'\left(0\right) = 0$. At the same time, for every sale, agents earn a commission of c while the service provider earns $p - c$. Thus agent's *i*'s income is $c \cdot S_i\left(t_1,t_2\right)$ and its profit function is $c \cdot S_i\left(t_1,t_2\right) - D\left(\rho_i\right)$.

Agents face the following trade-off with regard to service time. On the one hand, spending more time with each customer can lead to higher $R\left(t\right)$, up to a point. On the other hand, longer service time also means a potentially smaller $\phi_i\left(t_1,t_2\right)$, and a higher effort cost. Therefore, when determining the optimal average service time, agents must trade off potential higher sales with potential higher effort cost, and optimize the total profit function in a game theoretical setting.

3.4 Centralized Model

We first present the centralized optimal solution for the service center where both the agents' and the service providers' profit has been taken into account. Later we will compare this solution with that in the decentralized setting. Let $\Pi^f_{sys}(t_1, t_2)$ be system profit under centralized setting where f stands for *system optimal*, a.k.a. *first best.* The centralized problem is as follows:

$$\begin{aligned} \max\ \Pi^f_{sys}(t_1, t_2) &= p\sum_{i=1}^{2} S_i\left(t_1, t_2\right) - \sum_{i=1}^{2} D\left(\rho_i\right) \\ s.t. \qquad \rho_i &\leq 1. \end{aligned} \tag{3.2}$$

Since both agents are identical in terms of their cost of effort and service quality function, it is not optimal for a system to decide different service times for each agent. Therefore, we study the symmetrical equilibrium where agents make similar decisions.

Proposition 18 There exists a unique symmetrical equilibrium $\left(t_f, t_f\right)$ for all c, where t_f is the unique solution to $p\dfrac{dR\left(t_f\right)}{dt} = \left.\dfrac{dD}{d\rho}\right|_{\rho=\frac{\lambda t_f}{2}}$, and $t_f \in \left(0, t_p\right)$.

The result shows that quality experienced by customers at systems optimal (t_f) is lower than maximum possible $\left(t_p\right)$, due to agent's cost of effort. Next we look at the decentralized model where agents decide on their t_i given the commission offered by the service provider.

3.5 DECENTRALIZED MODEL

In the decentralized setting, the game between the service provider and the agents is a Stackelberg game. The service provider offers commission c per sale and, given that, agents decide on t_i as a result of game between them. Let $\Pi^s_{sp}(c)$ be the service provider's expected profit under the decentralized setting where s superscript stands for *second best*. Agent i then maximizes his/her profit as:

$$\begin{aligned} \max_{t_i} \quad & U_i\left(t_1,t_2\right) = c \cdot S_i\left(t_1,t_2\right) - D\left(\rho_i\right) \\ s.t. \quad & \rho_i \le 1. \end{aligned}$$

Therefore, the service provider's problem under decentralized setting is as follows:

$$\begin{aligned} \max \ \Pi_{sp}^{s}(c) \ & = \left(p-c\right)\sum_{i=1}^{2} S_i\left(t_1^*,t_2^*\right) \\ s.t. \quad t_i^* & = \arg\max_{t_i}\left\{S_i\left(t_1,t_2\right)c - D\left(\rho_i\right)\middle| t_i : \rho_i \le 1\right\} \end{aligned} \tag{3.3}$$

We look for a symmetrical equilibrium, as we did earlier.

Proposition 19 There exist a symmetrical equilibrium $\left(t_s,t_s\right)$ for all c, where t_s is the unique solution to $c\frac{\partial S_i\left(t,t\right)}{\partial t_i} - \frac{\partial \rho_i\left(t,t\right)}{\partial t_i}\frac{dD}{d\rho}\bigg|_{\rho=\frac{\lambda t}{2}} = 0$, and $t_c \in \left(0,t_f\right)$.

The result shows that there is no level of commission that can coordinate this system. Agents will always spend less time on average with customers compared to the systems optimal. Therefore, *agent's speedup* results in lower customer service quality, sales probability, and therefore system profitability.

Agent's speedup is not an artifact of system double marginalization, because even if agents get full commission $c = p$, they will still exert effort lower than the centralized optimum. We believe that this result is strongly driven by the competition for customers from the same queue, as evidenced by the function $\phi_i\left(t_1,t_2\right)$ in equation (3.1).

Therefore, we have shown a simple commission contract is not able to coordinate the system. Next, we look at the effect of operational and financial solution mechanisms in aligning agent incentive with that of the service provider. It should be noted that the speedup result is in contrast to Krishnan *et al.* (2011), who showed that the agents' strategy under competition is to slow down. This is expected as they do not model competition between agents. They assume customers will join the queue if the expected service value considering wait time and price is positive. They also assume that once customers have joined a queue that they will make a purchase. However, in our model a customer's purchase decision depends on the quality of interaction with an agent.

3.6 SOLUTION MECHANISMS

3.6.1 CONTRACTS

3.6.1.1 Revenue and Profit Sharing Contracts

We start by considering revenue sharing contracts which are extensively studied in the literature and applied in the industry. We have three players in the model, one service provider and two agents. Therefore, we use two parameters to show how revenue is shared among the players. Of the total system revenue, let the service provider receive a θ proportion, and agent i receive a $\gamma_i\left(i=1,2\right)$ proportion. Obviously, $\theta+\gamma_1+\gamma_2=1$. Agents i then maximize their own profit as:

$$\begin{aligned} \max_{t_i} \quad & U_i\left(t_1,t_2\right)=\gamma_i\left(1-\theta\right)pS_i\left(t_1,t_2\right)-D\left(\rho_i\right) \\ s.t. \quad & \rho_i \le 1. \end{aligned} \tag{3.4}$$

Then the system provider solves the following maximization problem:

$$\begin{aligned} &\max \Pi_{sp}^{rs}(\theta, \gamma_1, \gamma_2) = \theta p \sum_{i=1}^{2} S_i\left(t_1^*, t_2^*\right) \\ &\qquad s.t. \quad t_i^* = \arg\max_{t_i} U_i\left(t_1, t_2\right) \qquad i = 1,2 \end{aligned} \tag{3.5}$$

Proposition 20 There does not exist θ and $\gamma_i\left(i = 1,2\right)$ such that $t_{rs} = t_f$.

Therefore Proposition 20 means a revenue sharing contract that coordinates the system doesn't exist. However, a contract that satisfies the following conditions (3.6) and (3.7) will always coordinate the system if it's feasible:

$$\sum_{i=1}^{2} U_i\left(t_1, t_2, C^{pr}\left(t_1, t_2, \theta, \gamma_1, \gamma_2\right)\right) = \left(1-\theta\right)\ \Pi_{sys}\left(t_1, t_2\right) \tag{3.6}$$

$$U_i\left(t_i, C^{pr}\left(t_1, t_2, \theta, \gamma_1, \gamma_2\right)\right) = \gamma_i\left(1-\theta\right)\ \Pi_{sys}\left(t_1, t_2\right) \tag{3.7}$$

where $C^{pr}\left(t_1, t_2, \theta, \gamma_1, \gamma_2\right)$ is the contract payment, and $U_i\left(t_i, C^{pr}\left(t_1, t_2, \theta, \gamma_1, \gamma_2\right)\right)$ is agent i 's utility. With slight abuse of the notation, θ and $\gamma_i\left(i = 1,2\right)$ here represents how profit, not revenue, is shared among the parties. This contract works because it shares system profit among players, and therefore each player's goal will be aligned with the system goal. We will refer to this as the "profit sharing" contract, hence the superscript pr. Note that designing a profit sharing contract requires knowledge of agent cost, which is their private information, in addition to system revenue. We can solve (3.6) and (3.7) to find θ and $\gamma_i\left(i = 1,2\right)$ that coordinate the system:

Proposition 21 Choose $\gamma_1 = \gamma_2 = \dfrac{1-\theta}{2}$ for any θ , the following profit sharing contract coordinates the system: $C_i^{pr}\left(t_1, t_2, \theta, \gamma_1, \gamma_2\right) = \left(1-\theta\right)p + \theta \dfrac{D\left(\rho_i\right)}{S_i\left(t_1, t_2\right)}$.

The result shows that in a coordinating profit sharing contract the two agents should receive an equal share of the system profit. Moreover, θ can be chosen to flexibly determine how system profit can be shared between the service provider and the agents so that each party can be better off.

3.6.2 ALLOCATION

3.6.2.1 Proportional Allocation

Operational tools, such as routing, are as effective as financial tools in aligning incentives. One such tool is to allocate customers to agents based on their previous sales resolution record (i.e. $R\left(t_i\right)$), similar to the "earn and turn" strategy employed in the auto industry (Cachon and Lariviere 1999; Lu and Lariviere 2012). We call this method "*proportional allocation*", and note that it uses sales probability rather than actual sales records. Again with slight abuse of the notation, θ and $\gamma_i\left(i=1,2\right)$ here represent how profit, not revenue, is shared among the parties. Let $\alpha\left(t_1,t_2\right)$ be agent i's share of the customer base under proportional allocation. This can be written as:

$$\alpha\left(t_1,t_2\right)=\frac{R\left(t_i\right)^{\gamma_i}}{R\left(t_i\right)^{\gamma_i}+R\left(t_j\right)^{\gamma_j}} \tag{3.8}$$

Proposition 22 Choose $\gamma_1=\gamma_2$, for any θ, there exists a symmetrical equilibrium $\left(t_\alpha,t_\alpha\right)$ for $c=\widehat{c}_\alpha$, where $\widehat{c}_\alpha = p\dfrac{\gamma_i\dfrac{dR\left(t_f\right)}{dt}t_f+2R\left(t_f\right)}{R\left(t_f\right)\left(\gamma_i+2\right)}<p$ such that $t_\alpha=t_f$.

Therefore Proposition 22 shows that under proportional allocation and a simple commission contract when $c = \widehat{c}_{\alpha}$, agents put in effort at the system optimal level. While it coordinates and using γ_i can arbitrarily share profit between parties, designing such routing policy and keeping fairness to customers (*i.e.* follow FCFS) is a challenging issue.

3.6.2.2 Customer's Choice

The customer choice model allows customers to choose the agent based on disclosed information such as customer satisfaction. If a service provider creates a platform that provides information about each agent and allows customers to leave reviews for each agent on the service quality received, then future customers may select and agent based on this information. We will use a logit choice model to represent how customers choose an agent based on disclosed service quality information. Again with slight abuse of the notation, θ and $\gamma_i \left(i = 1,2\right)$ here represent how profit, not revenue, is shared among the parties. Let $\beta\left(t_1, t_2\right)$ represent the share of the customer base for agent i in a logit customer choice model. This can be written as follows:

$$\beta\left(t_1, t_2\right) = \frac{e^{R\left(t_i\right)^{\gamma_i}}}{e^{R\left(t_i\right)^{\gamma_i}} + e^{R\left(t_j\right)^{\gamma_j}}} \tag{3.9}$$

Proposition 23 Choose $\gamma_1 = \gamma_2$, for any θ, there exists a symmetrical equilibrium $\left(t_\beta, t_\beta\right)$ for $c = \widehat{c}_\beta$, where $\widehat{c}_\beta = \dfrac{p\left(\gamma_i \dfrac{dR}{dt_i} t_f + 2\right)}{\gamma_i R\left(t_f\right) + 2} < p$, such that $t_\beta = t_f$.

Therefore, result shows that when customers choose agents based on disclosed service quality information using a logit model, the simple commission contract can align an agent's incentive with that of the system. Proposition 23 also shows that using γ_i profit can be arbitrarily shared between parties to make them better off.

3.6.3 Comparison of Solution Mechanism

We have suggested two categories of solution mechanisms. In the first category (*i.e.* financial contracts), we provide incentive to agents to make the right decision, while in the second category (*i.e.* allocation) the service provider manipulates input (market share) to force agents to make the right decision.

Contracts are suitable mechanisms to use when an agent's pay can be adjusted to reflect their effort, cost, and so on. Profit sharing contracts require knowledge of an agent's cost of effort, and while this information is private, there are industry norms that can be used to estimate their cost.

Allocation mechanisms are more suitable when a service provider can allocate customers either directly by proportional allocation or indirectly through disclosing information for customers to choose. However, designing a customer routing algorithm for proportional allocation that can fairly allocate customers to satisfy first-come, first-served is a challenging task. The customer choice model can be easier to implement but may result higher commission costs.

3.7 Summary and Future Research Directions

In this chapter we studied agent competition for a common customer base within a decentralized setting. Comparing the result with that of centralized solution, we showed that agent speedup and service quality is lower than system optimal. This is not a desirable outcome for the system as expected system profit will decrease.

To solve agent suboptimal behavior caused by competition, we have examined contracts and operational tools to coordinate the system. We showed that profit sharing contract can coordinate the system. However, this contract requires precise knowledge of the optimal solution and the private information of agents, and are hence not very robust. We also looked at proportional allocation policy as a means to coordinate. We showed that an allocation policy based on the past sale resolution record of agents can coordinate the system under a simple commission contract. However, implementing such a policy might violate FCFS rule. Lastly, we showed that customers provided with agent's service quality (i.e. service time), they can self-select which eventually coordinates with a more practical approach.

Much work remains to be done. There are three extensions that we suggest here. First, we have analyzed agent static decision making by looking at average service time, while in real life decisions are made dynamically. Second, we have assumed customers are homogenous, but they can be heterogeneous in how they value their time. Third, we have assumed agents have identical, effort cost, and resolution probability. It would be worthwhile to extend this model to consider the case of heterogeneous agents.

APPENDIX

Notation Definition

For simplicity of notation in this appendix, we use:

$\frac{\partial \phi_i\left(t,t\right)}{\partial t_i} = \frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i}\Big\vert_{t_1=t_2=t}$	$\frac{\partial^2 \phi_i\left(t,t\right)}{\partial t_i^{\ 2}} = \frac{\partial^2 \phi_i\left(t_1,t_2\right)}{\partial t_i^{\ 2}}\Big\vert_{t_1=t_2=t}$
$\frac{dD}{d\rho}\Big\vert_{\rho=\lambda\phi_i\left(t_1,t_2\right)t_i} = \frac{dD\left(\rho_i\right)}{d\rho}$	$\frac{d^2 D}{d\rho^2}\Big\vert_{\rho=\lambda\phi_i\left(t_1,t_2\right)t_i} = \frac{d^2 D\left(\rho_i\right)}{d\rho^2}$
$\frac{dR}{dt}\Big\vert_{t=t_i} = \frac{dR\left(t_i\right)}{dt}$	$\frac{dR}{dt}\Big\vert_{t=t_i} = \frac{dR\left(t_i\right)}{dt}$

Frequently Used Relationships in Proofs

Recall $\phi_i(t_1,t_2) = \frac{\lambda t_j^2 + t_1 + t_2}{\lambda(t_1^2 + t_2^2) + 2(t_1 + t_2)}$, based on which following can be shown:

$$\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i} = \frac{-\lambda(t_1 + t_2)^2 - 2\lambda^2 t_j^2 t_i}{\left(\lambda(t_1^2 + t_2^2) + 2(t_1 + t_2)\right)^2} < 0, \tag{3.10}$$

$$\frac{\partial^2 \phi_i\left(t_1,t_2\right)}{\partial t_i^2} = -2\frac{\lambda\phi_i\left(t_1,t_2\right) + 2\left(\lambda t_i + 1\right)\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i}}{\lambda(t_1^2 + t_2^2) + 2(t_1 + t_2)}. \tag{3.11}$$

Simplifying (3.10) and (3.11) for a symmetrical equilibrium where $t_1 = t_2 = t$, we obtain:

$$\frac{\partial \phi_i\left(t,t\right)}{\partial t_i} = -\frac{\lambda}{4 + 2\lambda t} < 0, \tag{3.12}$$

$$\frac{\partial^2 \phi_i\left(t_1,t_2\right)}{\partial t_i^2} = \frac{\lambda^2}{2(2+\lambda t)^2} > 0, \tag{3.13}$$

$$\lambda \frac{\partial \phi_i\left(t,t\right)}{\partial t_i} t + \lambda \phi_i\left(t,t\right) = \frac{\lambda}{2+\lambda t} > 0, \tag{3.14}$$

$$\lambda \frac{\partial^2 \phi_i\left(t,t\right)}{\partial t_i^{\,2}} t + 2\lambda \frac{\partial \phi_i\left(t,t\right)}{\partial t_i} = -\frac{4\lambda + \lambda^2 t}{2(2+\lambda t)^2} < 0. \tag{3.15}$$

Since $R\left(t_i\right)$ is concave and differentiable, then using Taylor approximation we know:

$$R\left(t_i\right) - t_i \frac{dR\left(t_i\right)}{dt} > 0. \tag{3.16}$$

Proof of Proposition 18

The first order condition of (3.2) is as follows:

$$\begin{aligned} \frac{\partial \Pi_{sys}\left(t_1,t_2\right)}{\partial t_i} = {} & \lambda p\left(\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i} R\left(t_i\right) + \phi_i\left(t_1,t_2\right)\frac{dR\left(t_i\right)}{dt}\right) - \lambda\left(\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i} t_i + \phi_i\left(t_1,t_2\right)\right)\frac{dD\left(\rho_i\right)}{d\rho} \\ & + \lambda p\left(\frac{\partial \phi_j\left(t_1,t_2\right)}{\partial t_i} R\left(t_j\right)\right) - \lambda\left(\frac{\partial \phi_j\left(t_1,t_2\right)}{\partial t_i} t_j\right)\frac{dD\left(\rho_i\right)}{d\rho}. \end{aligned} \tag{3.17}$$

We assume everyone will be served therefore:

$$\phi_1\left(t_1,t_2\right) + \phi_2\left(t_1,t_2\right) = 1 \rightarrow \begin{cases} \dfrac{\partial \phi_1\left(t_1,t_2\right)}{\partial t_i} = -\dfrac{\partial \phi_2\left(t_1,t_2\right)}{\partial t_i} \\ \dfrac{\partial^2 \phi_1\left(t_1,t_2\right)}{\partial t_i^2} = -\dfrac{\partial^2 \phi_2\left(t_1,t_2\right)}{\partial t_i^2} \end{cases}. \tag{3.18}$$

We can then simplify (3.17) using (3.18) as follows:

$$\frac{\partial \Pi_{sys}\left(t_1,t_2\right)}{\partial t_i} = \lambda p \frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i}\left(R\left(t_i\right)-R\left(t_j\right)\right) -\lambda \frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i}\left(t_i \frac{dD\left(\rho_j\right)}{d\rho} - t_j \frac{dD\left(\rho_j\right)}{d\rho}\right) + \lambda \phi_i\left(t_1,t_2\right)\left(p\frac{dR\left(t_i\right)}{dt} - \frac{dD\left(\rho_i\right)}{d\rho}\right). \tag{3.19}$$

Simplifying (3.19) using $t_i = t_j$ to find the symmetrical equilibrium, we obtain:

$$\left.\frac{\partial \Pi_{sys}\left(t_1,t_2\right)}{\partial t_i}\right|_{t_1=t_2=t} = \frac{\lambda}{2}\left(p\frac{dR}{dt} - \left.\frac{dD}{d\rho}\right|_{\rho=\frac{\lambda t}{2}}\right). \tag{3.20}$$

Following a similar procedure, the second order derivate of (3.17) for symmetrical equilibrium is as follows:

$$\left.\frac{\partial^2 \Pi_{sys}\left(t_1,t_2\right)}{\partial t_i^2}\right|_{t_1=t_2=t} = 2\lambda \frac{\partial \phi_i\left(t,t\right)}{\partial t_i}\left(p\frac{dR}{dt} - \left.\frac{dD}{d\rho}\right|_{\rho=\frac{\lambda t}{2}}\right) + \frac{\lambda p}{2}\frac{d^2R}{dt^2} - \lambda^2\left(\left(\frac{\partial \phi_i\left(t,t\right)}{\partial t_i}t + \frac{1}{2}\right)^2 + \left(\frac{\partial \phi_i\left(t,t\right)}{\partial t_i}t\right)^2\right)\left.\frac{d^2D}{d\rho^2}\right|_{\rho=\frac{\lambda t}{2}}. \tag{3.21}$$

For this proof, let's define $f\left(t\right) = p\frac{dR}{dt} - \left.\frac{dD}{d\rho}\right|_{\rho=\frac{\lambda t}{2}}$. Since $\frac{d^2R}{dt^2} < 0$ and $\frac{d^2D}{d\rho^2} > 0$, then $f'\left(t\right) = p\frac{d^2R}{dt^2} - \frac{\lambda}{2}\left.\frac{d^2D}{d\rho^2}\right|_{\rho=\frac{\lambda t}{2}} < 0$. Therefore $f\left(t\right)$ is decreasing for all t. Since $\lim_{t\to 0} f\left(t\right) = p\frac{dR}{dt} > 0$ and $\lim_{t\to \frac{2}{\lambda}} f\left(t\right) = -\left.\frac{dD}{d\rho}\right|_{\rho=\frac{\lambda t}{2}} < 0$, therefore there is unique solution to $f\left(t\right) = 0$. Let's call it $\overline{t}$ where $f\left(\overline{t}\right) = 0$.

We know for all t, $\frac{\partial \phi_i(t,t)}{\partial t_i} < 0$. Also on $t \in [0, \overline{t}]$, we know $f(t) \geq 0$, therefore the second order, (3.21), is negative. We also know that the first order, (3.20), is strictly negative on $t \in \left(\overline{t}, \lambda/2\right]$. Therefore, function $\Pi_{sys}(t_1,t_2)\Big|_{t_1=t_2=t}$ is unimodal and the symmetrical equilibrium (t_f, t_f) is the unique solution to the first order condition, (3.20).

Proof of Proposition 19

We prove Proposition 19 in two parts as follows:

A. There exists a symmetrical equilibrium (t_s, t_s) for all c, where t_s is the unique solution to

$$c\frac{\partial S_i(t,t)}{\partial t_i} - \frac{\partial \rho_i(t,t)}{\partial t_i}\frac{dD}{d\rho}\bigg|_{\rho=\frac{\lambda t}{2}} = 0.$$

B. $t_c \in (0, t_f)$.

Part A) The first and second order derivatives of $U_i(t_1,t_2)$ are as follows:

$$\frac{\partial U_i(t_1,t_2)}{\partial t_i} = \lambda c\left[\frac{\partial \phi_i(t_1,t_2)}{\partial t_i}R(t_i) + \phi_i(t_1,t_2)\frac{dR(t_i)}{dt}\right] - \lambda\left[\frac{\partial \phi_i(t_1,t_2)}{\partial t_i}t_i + \phi_i(t_1,t_2)\right]\frac{dD}{d\rho}\bigg|_{\rho=\lambda\phi_i(t_1,t_2)t_i}, \tag{3.22}$$

$$\begin{aligned}\frac{\partial^2 U_i(t_1,t_2)}{\partial t_i^2} = {} & \lambda c\left[\frac{\partial^2 \phi_i(t_1,t_2)}{\partial t_i^2}R(t_i) + 2\frac{\partial \phi_i(t_1,t_2)}{\partial t_i}\frac{dR(t_i)}{dt} + \phi_i(t_1,t_2)\frac{d^2R(t_i)}{dt^2}\right] \\ & -\lambda\left[\frac{\partial^2 \phi_i(t_1,t_2)}{\partial t_i^2}t_i + 2\frac{\partial \phi_i(t_1,t_2)}{\partial t_i}\right]\frac{dD}{d\rho}\bigg|_{\rho=\lambda\phi_i(t_1,t_2)t_i} - \lambda\left[\frac{\partial \phi_i(t_1,t_2)}{\partial t_i}t_i + \phi_i(t_1,t_2)\right]^2\frac{d^2D}{d\rho^2}\bigg|_{\rho=\lambda\phi_i(t_1,t_2)t_i}.\end{aligned} \tag{3.23}$$

We know $R(t_i) - t_i\frac{dR(t_i)}{dt} > 0$ and $\frac{\partial \phi_i(t_1,t_2)}{\partial t_i} < 0$, therefore the following inequality holds:

$$\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i} R\left(t_i\right) + \phi_i\left(t_1,t_2\right)\frac{dR\left(t_i\right)}{dt} < \frac{dR\left(t_i\right)}{dt}\left(\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i} t_i + \phi_i\left(t_1,t_2\right)\right) \tag{3.24}$$

Using (3.24), we find an upper bound for the first order as follows:

$$\frac{\partial U_i\left(t_1,t_2\right)}{\partial t_i} < \lambda\left(\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i} t_i + \phi_i\left(t_1,t_2\right)\right)\left(c\frac{dR\left(t_i\right)}{dt} - \left.\frac{dD}{d\rho}\right|_{\rho=\lambda\phi_i\left(t_1,t_2\right)t_i}\right) \tag{3.25}$$

Next, we follow a similar procedure to find an upper bound on the second order derivative. We know $R\left(t_i\right) - t_i\frac{dR\left(t_i\right)}{dt} > 0$ and $\frac{\partial^2 \phi_i\left(t_1,t_2\right)}{\partial t_i^{\,2}} > 0$, therefore the following inequality holds:

$$\frac{\partial^2 \phi_i\left(t_1,t_2\right)}{\partial t_i^{\,2}} R\left(t_i\right) + 2\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i}\frac{dR\left(t_i\right)}{dt} < \frac{R\left(t_i\right)}{t_i}\left(\frac{\partial^2 \phi_i\left(t_1,t_2\right)}{\partial t_i^{\,2}} t_i + 2\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i}\right) \tag{3.26}$$

Using (3.26), an upper bound for the second order is as follows:

$$\begin{aligned}\frac{\partial^2 U_i\left(t_1,t_2\right)}{\partial t_i^{\,2}} < {} & \lambda\left(\frac{\partial^2 \phi_i\left(t_1,t_2\right)}{\partial t_i^{\,2}} t_i + 2\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i}\right)\left(c\frac{R\left(t_i\right)}{t_i} - \left.\frac{dD}{d\rho}\right|_{\rho=\lambda\phi_i\left(t_1,t_2\right)t_i}\right) \\ & +\lambda c\phi_i\left(t_1,t_2\right)\frac{d^2R\left(t_i\right)}{dt^2} - \lambda\left[\frac{\partial \phi_i\left(t_1,t_2\right)}{\partial t_i} t_i + \phi_i\left(t_1,t_2\right)\right]^2 \left.\frac{d^2D}{d\rho^2}\right|_{\rho=\lambda\phi_i\left(t_1,t_2\right)t_i}\end{aligned} \tag{3.27}$$

For this proof, Let's define $f_i\left(t_1,t_2\right) = c\left.\frac{dR}{dt}\right|_{t=t_i} - \left.\frac{dD}{d\rho}\right|_{\rho=\rho_i}$ and $f\left(t\right) = f_1\left(t,t\right) = f_2\left(t,t\right)$. Since $\frac{d^2R}{dt^2} < 0$ and $\frac{d^2D}{d\rho^2} > 0$, then $f'\left(t\right) = c\frac{d^2R}{dt^2} - \frac{\lambda}{2}\left.\frac{d^2D}{d\rho^2}\right|_{\rho=\frac{\lambda t}{2}} < 0$. Therefore $f\left(t\right)$ is decreasing for all t. Since $\lim_{t\to 0} f\left(t\right) = c\frac{dR}{dt} > 0$ and $\lim_{t\to\frac{2}{\lambda}} f\left(t\right) = -\left.\frac{dD}{d\rho}\right|_{\rho=\frac{\lambda t}{2}} < 0$, then there is unique solution to $f\left(t\right)=0$. Let's call it $\overline{t}$ where $f\left(\overline{t}\right)=0$.

Remark 1) We know $\frac{\partial\phi_i\left(t,t\right)}{\partial t_i}t + \phi_i\left(t,t\right) > 0$ and $f\left(t_i\right) \le 0$ for $t \in \left[\overline{t}, \lambda/2\right)$, therefore $\frac{\partial U_i\left(t_1,t_2\right)}{\partial t_i} \le 0$ on $t \in \left[\overline{t}, \lambda/2\right)$.

We know $R\left(t_i\right) - t_i\frac{dR\left(t_i\right)}{dt} > 0$. We also know that on $t \in \left[0,\overline{t}\right)$, $f\left(t_i\right) > 0$. Therefore $c\frac{R\left(t_i\right)}{t_i} > c\frac{dR\left(t_i\right)}{dt} > \left.\frac{dD}{d\rho}\right|_{\rho=\lambda\phi_i\left(t_1,t_2\right)t_i}$, which result in $c\frac{R\left(t_i\right)}{t_i} - \left.\frac{dD}{d\rho}\right|_{\rho=\lambda\phi_i\left(t_1,t_2\right)t_i} > 0$.

Remark 2) We know $\frac{\partial^2\phi_i\left(t_1,t_2\right)}{\partial t_i^2}t_i + 2\frac{\partial\phi_i\left(t_1,t_2\right)}{\partial t_i} < 0$ and $c\frac{R\left(t_i\right)}{t_i} - \left.\frac{dD}{d\rho}\right|_{\rho=\lambda\phi_i\left(t_1,t_2\right)t_i} > 0$ on $t \in \left[0,\overline{t}\right)$, therefore, $\frac{\partial^2 U_i\left(t_1,t_2\right)}{\partial t_i^2} < 0$ on $t \in \left[0,\overline{t}\right)$.

Considering remarks 1 and 2, $U_i\left(t_1,t_2\right)$ is a unimodal function and the equilibrium $t_1 = t_2 = t_c$ is the unique solution to the first order condition, (3.22).

Part B) If we evaluate the solution to equation (3.20), t_f, in equation (3.22), we obtain:

$$\left.\frac{\partial U_i\left(t_1,t_2\right)}{\partial t_i}\right|_{t_1=t_2=t_f} = \lambda c\left[\frac{\partial\phi_i\left(t_f,t_f\right)}{\partial t_i}R\left(t_f\right) + \phi_i\left(t_f,t_f\right)\frac{dR\left(t_f\right)}{dt}\right] - \lambda p\left[\frac{\partial\phi_i\left(t_f,t_f\right)}{\partial t_i}t_f + \phi_i\left(t_f,t_f\right)\right]\frac{dR\left(t_f\right)}{dt} \tag{3.28}$$

We know (3.16) and (3.14), therefore:

$$\frac{\partial \phi_i(t_1,t_2)}{\partial t_i} R(t_i) + \phi_i(t_1,t_2)\frac{dR(t_i)}{dt_i} < \frac{dR_i(t)}{dt_i}\left(\frac{\partial \phi_i(t_1,t_2)}{\partial t_i} t_i + \phi_i(t_1,t_2)\right) \tag{3.29}$$

Using (3.29), we obtain the following upper bound to (3.28):

$$\left.\frac{\partial U_i(t_1,t_2)}{\partial t_i}\right|_{t_1=t_2=t_f} < \lambda \frac{dR(t_f)}{dt}\left(\frac{\partial \phi_i(t_f,t_f)}{\partial t_i} t_f + \phi_i(t_f,t_f)\right)(c-p) \tag{3.30}$$

Therefore, for all $c \in [0, p_0]$, $\left.\frac{\partial U_i(t_1,t_2)}{\partial t_i}\right|_{t_1=t_2=t_f} < 0$. It means there is no level of commission that can coordinate the system.

Proof of Proposition 20

The first order of (3.4) is as follows:

$$\begin{aligned} \frac{\partial U_i(t_1,t_2)}{\partial t_i} &= \lambda\gamma_i p\left[\frac{\partial \phi_i(t_1,t_2)}{\partial t_i} R(t_i) + \phi_i(t_1,t_2)\frac{dR(t_i)}{dt}\right] + \lambda\gamma_i p\left[\frac{\partial \phi_j(t_1,t_2)}{\partial t_i} R(t_j)\right] \\ &\quad -\lambda\left[\frac{\partial \phi_i(t_1,t_2)}{\partial t_i} t_i + \phi_i(t_1,t_2)\right]\left.\frac{dD}{d\rho}\right|_{\rho=\lambda\phi_i(t_1,t_2)t_i} \end{aligned} \tag{3.31}$$

The second order of (3.31)is as follows:

$$\frac{\partial^2 U_i(t_1,t_2)}{\partial t_i^2} = \lambda\gamma_i p\left[2\frac{\partial \phi_i(t_1,t_2)}{\partial t_i}\frac{dR(t_i)}{dt} + \phi_i(t_1,t_2)\frac{d^2R(t_i)}{dt^2}\right] - \lambda\left[\frac{\partial \phi_i(t_1,t_2)}{\partial t_i} t_i + \phi_i(t_1,t_2)\right]^2 \left.\frac{dD}{d\rho}\right|_{\rho=\lambda\phi_i(t_1,t_2)t_i}$$

Since $\frac{\partial \phi_i(t_1,t_2)}{\partial t_i} < 0$, $\frac{d^2R}{dt^2} < 0$, and $\frac{dD}{d\rho} > 0$, the second order condition holds.

If we evaluate (3.31) at centralized solution(t_f, t_f), we obtain:

$$\frac{\partial U_i\left(t_f,t_f\right)}{\partial t_i} = \lambda p \frac{dR\left(t_f\right)}{dt}\left(\phi_i\left(t_f,t_f\right)\gamma_i - \left(\phi_i\left(t_f,t_f\right) + \frac{\partial \phi_i\left(t_f,t_f\right)}{\partial t_i} t_i\right)\right) \tag{3.32}$$

To show $t_{rs} = t_f$, it is equivalent to show:

$$\phi_i\left(t_f,t_f\right)\gamma_i = \left(\phi_i\left(t_f,t_f\right) + \frac{\partial \phi_i\left(t_f,t_f\right)}{\partial t_i} t_i\right) \tag{3.33}$$

We use (3.14) to simplify (3.33) as follows:

$$\gamma_i = \frac{2}{2+\lambda t} \tag{3.34}$$

We can drive an identical condition for the other agent. Therefore $\gamma_i = \gamma_j$. Since $\theta + \gamma_i + \gamma_j = 1$, then (3.34) can be written as:

$$\theta = \frac{-2+\lambda t}{2+\lambda t} \tag{3.35}$$

Constraint of agent's utility maximization is $\lambda t < 2$, which implies from (3.35) that $\theta < 0$. Therefore, there doesn't exist θ such that $t_{rs} = t_f$.

Proof of Proposition 21

Using (3.6) and (3.7), we can define a flexible contract that can both coordinate the system and divide the profit arbitrarily. Please note that (3.6) simplifies to:

$$C^{pr}\left(t_1,t_2,\theta,\gamma_1,\gamma_2\right) = \left(1-\theta\right)p + \frac{\theta}{\lambda}\frac{\sum_{i=1}^{2} D\left(\lambda\phi_i\left(t_1,t_2\right)t_i\right)}{\sum_{i=1}^{2}\phi_i\left(t_1,t_2\right)R\left(t_i\right)}, \tag{3.36}$$

and (3.7) simplifies to:

$$C^{pr}\left(t_1,t_2,\theta,\gamma_1,\gamma_2\right) = \frac{\gamma_i\left(\lambda p\sum_{i=1}^{2}\phi_i\left(t_1,t_2\right)R\left(t_i\right) - \sum_{i=1}^{2} D\left(\lambda\phi_i\left(t_1,t_2\right)t_i\right)\right)}{\lambda\phi_i\left(t_1,t_2\right)R\left(t_i\right)} + \frac{D\left(\lambda\phi_i\left(t_1,t_2\right)t_i\right)}{\lambda\phi_i\left(t_1,t_2\right)R\left(t_i\right)}. \tag{3.37}$$

Comparing (3.36) and (3.37), The *sufficient* conditions for existence of such a contract are:

$$\gamma_i = \frac{\phi_i\left(t_1,t_2\right)R\left(t_i\right)}{\sum_{i=1}^{2}\phi_i\left(t_1,t_2\right)R\left(t_i\right)} \tag{3.38}$$

$$\frac{\phi_j\left(t_1,t_2\right)R\left(t_i\right)}{D\left(\lambda\phi_j\left(t_1,t_2\right)t_j\right)} = \frac{\phi_i\left(t_1,t_2\right)R\left(t_i\right)}{D\left(\lambda\phi_i\left(t_1,t_2\right)t_i\right)} \tag{3.39}$$

Please note that for a symmetrical equilibrium (3.39) holds, while equation (3.38) means $\gamma_1 = \gamma_2$.

Therefore, the contract specified by (3.36) can be simplified to the following for any θ:

$$C^{pr}\left(t_1,t_2,\theta,\gamma_1,\gamma_2\right) = \left(1-\theta\right)p + \frac{\theta}{\lambda}\frac{D\left(\lambda\phi_i\left(t_1,t_2\right)t_i\right)}{\phi_i\left(t_1,t_2\right)R_i\left(t_i\right)} \tag{3.40}$$

where $\gamma_1 = \gamma_2 = \dfrac{1-\theta}{2}$.

Proof of Proposition 22

Agent i's optimization problem for the *proportional allocation* scheme is as follows:

$$\begin{aligned} \max_{t_i} \quad & U_i^{\alpha}\left(t_1,t_2\right) = \lambda\alpha_i\left(t_1,t_2\right)R\left(t_i\right) - D\left(\rho_i\right) \\ s.t. \quad & \rho_i \leq 1. \end{aligned}$$

where $\alpha_i\left(t_1,t_2\right) = \dfrac{R\left(t_i\right)^{\gamma_i}}{R\left(t_i\right)^{\gamma_i} + R\left(t_j\right)^{\gamma_j}}$. The first order derivative of $U_i^{\alpha}\left(t_1,t_2\right)$ is as follows:

$$\frac{\partial U_i^{\alpha}\left(t_1,t_2\right)}{\partial t_i} = \lambda c\left[\frac{\partial \alpha_i\left(t_1,t_2\right)}{\partial t_i}R\left(t_i\right) + \alpha_i\left(t_1,t_2\right)\frac{dR\left(t_f\right)}{dt_i}\right] - \lambda\left[\frac{\partial \alpha_i\left(t_1,t_2\right)}{\partial t_i}t_i + \alpha_i\left(t_1,t_2\right)\right]\frac{dD}{d\rho}\bigg|_{\rho=\lambda\alpha_i\left(t_1,t_2\right)t_i}$$

If we evaluate the first order at $\left(t_f,t_f\right)$, we obtain:

$$\frac{\partial U_i^{\alpha}\left(t_f,t_f\right)}{\partial t_i} = \lambda c\left[\frac{\partial \alpha_i\left(t_f,t_f\right)}{\partial t_i}R\left(t_f\right) + \alpha_i\left(t_f,t_f\right)\frac{dR\left(t_f\right)}{dt_i}\right] - \lambda p\left[\frac{\partial \alpha_i\left(t_f,t_f\right)}{\partial t_i}t_f + \alpha_i\left(t_f,t_f\right)\right]\frac{dR\left(t_f\right)}{dt} \tag{3.41}$$

When $c = 0$, we can simplify (3.41) as follows:

$$\frac{\partial U_i^{\alpha}\left(t_f,t_f\right)}{\partial t_i}\bigg|_{c=0} = -\lambda p\left[\frac{\partial \alpha_i\left(t_f,t_f\right)}{\partial t_i}t_f + \alpha_i\left(t_f,t_f\right)\right]\frac{dR\left(t_f\right)}{dt} < 0 \tag{3.42}$$

When $c = p$, we can simplify (3.41) as follows:

$$\frac{\partial U_i^{\alpha}\left(t_f,t_f\right)}{\partial t_i}\bigg|_{c=p} = \lambda p\frac{\partial \alpha_i\left(t_f,t_f\right)}{\partial t_i}\left[R\left(t_f\right) - t_f\frac{dR\left(t_f\right)}{dt}\right] > 0 \tag{3.43}$$

Any $t \geq t_p$ is suboptimal because the purchase probability goes down and the effort cost increases. Therefore, we only consider $t < t_p$, where $\frac{dR}{dt} > 0$ by assumption. We know

$$\frac{\partial \alpha_i\left(t_1,t_2\right)}{\partial t_i} = \theta_i R\left(t_i\right)^{\gamma_i - 1} \frac{R\left(t_j\right)^{\gamma_j} \frac{dR}{dt}}{\left(R\left(t_i\right)^{\gamma_i} + R\left(t_j\right)^{\gamma_j}\right)^2} > 0\text{; therefore } \frac{\partial \alpha_i\left(t_1,t_2\right)}{\partial t_i} t_i + \alpha_i\left(t_1,t_2\right) > 0.$$

Behavior of (3.41) with respect to c depends on $\frac{\partial \alpha_i\left(t_1,t_2\right)}{\partial t_i} R\left(t_i\right) + \alpha_i\left(t_1,t_2\right)\frac{dR\left(t_i\right)}{dt_i}$ which is positive because $R\left(t_i\right) - t_i \frac{dR\left(t_i\right)}{dt} > 0$ and $\frac{\partial \alpha_i\left(t_1,t_2\right)}{\partial t_i} > 0$:

$$\frac{\partial \alpha_i\left(t_1,t_2\right)}{\partial t_i} R\left(t_i\right) + \alpha_i\left(t_1,t_2\right)\frac{dR\left(t_i\right)}{dt_i} > \frac{dR\left(t_i\right)}{dt}\left(\frac{\partial \alpha_i\left(t_1,t_2\right)}{\partial t_i} t_i + \alpha_i\left(t_1,t_2\right)\right) > 0. \tag{3.44}$$

Therefore (3.41) is increasing in c which means there exist a level of commission, $\hat{c}_a$, such that $\left.\frac{\partial U_i\left(t_f,t_f\right)}{\partial t_i}\right|_{c=\hat{c}_a} = 0$. This means coordination under proportional allocation is possible with the simple commission contract. We can use (3.41) to find $\hat{c}_a$ as follow:

$$\hat{c}_\alpha = \frac{p\left(\frac{\partial \alpha_i\left(t_f,t_f\right)}{\partial t_i} t_f + \alpha_i\left(t_f,t_f\right)\right)\frac{dR\left(t_f\right)}{dt}}{\frac{\partial \alpha_i\left(t_f,t_f\right)}{\partial t_i} R\left(t_f\right) + \alpha_i\left(t_f,t_f\right)\frac{dR\left(t_f\right)}{dt_i}} = p\frac{\gamma_i \frac{dR\left(t_f\right)}{dt} t_f + 2R\left(t_f\right)}{R\left(t_f\right)\left(\gamma_i + 2\right)} \tag{3.45}$$

Proof of Proposition 23

Agent i 's optimization problem for the *customer choice* scheme is as follows:

$$\max_{t_i} \; U_i^\beta\left(t_1,t_2\right) = \lambda\beta_i\left(t_1,t_2\right)R\left(t_i\right) - D\left(\rho_i\right)$$
$$s.t. \;\; \rho_i \le 1$$

where $\beta\left(t_1,t_2\right) = \dfrac{e^{R\left(t_i\right)^{\gamma_i}}}{e^{R\left(t_i\right)^{\gamma_i}} + e^{R\left(t_j\right)^{\gamma_j}}}$. The first order derivative of $U_i^\beta\left(t_1,t_2\right)$ is as follows:

$$\frac{\partial U_i^\beta\left(t_1,t_2\right)}{\partial t_i} = \lambda c\left[\frac{\partial\beta_i\left(t_1,t_2\right)}{\partial t_i}R\left(t_i\right) + \beta_i\left(t_1,t_2\right)\frac{dR\left(t_i\right)}{dt}\right] - \lambda\left[\frac{\partial\beta_i\left(t_1,t_2\right)}{\partial t_i}t_i + \beta_i\left(t_1,t_2\right)\right]\frac{dD}{d\rho}\bigg|_{\rho=\lambda\beta_i\left(t_1,t_2\right)t_i}$$

Evaluating it at $\left(t_f,t_f\right)$, we obtain:

$$\frac{\partial U_i^\beta\left(t_1,t_2\right)}{\partial t_i}\bigg|_{t_1=t_2=t} = \lambda c\left[\frac{\partial\beta_i\left(t_f,t_f\right)}{\partial t_i}R\left(t_f\right) + \beta_i\left(t_f,t_f\right)\frac{dR\left(t_f\right)}{dt}\right] - \lambda p\left[\frac{\partial\beta_i\left(t_f,t_f\right)}{\partial t_i}t_f + \beta_i\left(t_f,t_f\right)\right]\frac{dR\left(t_f\right)}{dt} \qquad (3.46)$$

When $c = 0$, we can simplify (3.46) as follows:

$$\frac{\partial U_i^\beta\left(t_f,t_f\right)}{\partial t_i}\bigg|_{c=0} = -\lambda p\left[\frac{\partial\beta_i\left(t_f,t_f\right)}{\partial t_i}t_f + \beta_i\left(t_f,t_f\right)\right]\frac{dR\left(t_f\right)}{dt} < 0 \qquad (3.47)$$

When $c = p$, we can simplify (3.46) as follows:

$$\frac{\partial U_i^\beta\left(t_f,t_f\right)}{\partial t_i}\bigg|_{c=p} = \lambda p\frac{\partial\beta_i\left(t_f,t_f\right)}{\partial t_i}\left[R\left(t_f\right) - t_f\frac{dR\left(t_f\right)}{dt}\right] > 0 \qquad (3.48)$$

We only consider all $t < t_p$ where $\frac{dR}{dt} > 0$, as all $t \geq t_p$ are suboptimal since purchase probability goes down and effort cost increases. We know $\frac{\partial \beta_i\left(t_1,t_2\right)}{\partial t_i} = \frac{dR}{dt_i} \frac{\gamma_i e^{R\left(t_i\right)^{\gamma_i}} e^{R\left(t_j\right)^{\gamma_j}}}{\left(e^{R\left(t_i\right)^{\gamma_i}} + e^{R\left(t_j\right)^{\gamma_j}}\right)^2} > 0$ and therefore $\frac{\partial \beta_i\left(t_1,t_2\right)}{\partial t_i} t_i + \beta_i\left(t_1,t_2\right) > 0$.

Behavior of (3.46) with respect to c depends on $\frac{\partial \beta_i\left(t_1,t_2\right)}{\partial t_i} R\left(t_i\right) + \beta_i\left(t_1,t_2\right) \frac{dR\left(t_i\right)}{dt}$ which is positive because $R\left(t_i\right) - t_i \frac{dR\left(t_i\right)}{dt} > 0$ and $\frac{\partial \beta_i\left(t_1,t_2\right)}{\partial t_i} > 0$:

$$\frac{\partial \beta_i\left(t_1,t_2\right)}{\partial t_i} R\left(t_i\right) + \beta_i\left(t_1,t_2\right) \frac{dR\left(t_i\right)}{dt_i} > \frac{dR\left(t_i\right)}{dt} \left(\frac{\partial \beta_i\left(t_1,t_2\right)}{\partial t_i} t_i + \beta_i\left(t_1,t_2\right) \right) > 0 \tag{3.49}$$

Therefore (3.46) is increasing in c which means there exist a level of commission, $\hat{c}_\beta$, such that $\left.\frac{\partial U_i^\beta\left(t_f,t_f\right)}{\partial t_i}\right|_{c=\hat{c}_\beta} = 0$. This means coordination under proportional allocation is possible with the simple commission contract. We can use (3.41) to find $\hat{c}_\beta$ as follow:

$$\hat{c}_\beta = \frac{p\left(\frac{\partial \beta_i\left(t_f,t_f\right)}{\partial t_i} t_f + \beta_i\left(t_f,t_f\right) \right) \frac{dR\left(t_f\right)}{dt}}{\frac{\partial \beta_i\left(t_f,t_f\right)}{\partial t_i} R\left(t_f\right) + \beta_i\left(t_f,t_f\right) \frac{dR\left(t_f\right)}{dt_i}} = \frac{p\left(\gamma_i \frac{dR}{dt_i} t_f + 2 \right)}{\gamma_i R\left(t_f\right) + 2} \tag{3.50}$$

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