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Essays on Dynamic Markets

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Abstract

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This dissertation studies three markets. In chapter 1, we study a dynamic two-sided many-to-one matching model. We provide sufficient conditions for the existence of a dynamically stable matching and show that some but not all results for the college admissions problem can be generalized to our dynamic model. Without the sufficient conditions, we define regret-free dynamic stability, which allows agents to endogenously give up blocking powers. A regret-free dynamic stable matching always exists. In Chapter 2, we study airport slot allocation problems during severe weather. We assume airlines have lexicographic preferences and introduce a new mechanism, Multiple Trading Cycles (MTC), to allocate landing slots. In contrast to the currently used mechanism, MTC is individually rational, Pareto efficient, strategy-proof, non-manipulable by postponing a flight cancelation, and respects property rights over slots. In chapter 3, we develop a simple model of two intermediates competing for N suppliers, which is motivated by an observation from the fishing industry. Each intermediate receives a privately observed, i.i.d profit shock in each period. Intermediates use public observable, retroactive payments to entice suppliers to sell to them in the upcoming period. The competition can capture suppliers' responsiveness to the price difference and sensitivity to the sizes of price differences. When intermediates are not so patient, we prove that this model has a symmetric monotone pure strategy stationary Markov perfect Bayesian Equilibrium, in which an intermediate pays less when having a larger number of suppliers.

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Chapter 1

DYNAMIC COLLEGE ADMISSIONS PROBLEM

1.1 Introduction

Students and universities are matched once and for all in the college admissions problem. But this is not always the case in reality. A student might want to transfer from a university to another for various reasons. For example, a student might want to go to a community college because it is closer to home or more affordable.¹ In a report from the National Center of Education Statistics by Radford et al. [2010], an estimate of 26.6% of the students started college in 2003-2004 and obtained a Bachelor's degree by spring 2009 are transfer students.

In this chapter, we study a two-period many-to-one matching model where students can transfer from one school to another. Students' preferences are induced by strict rankings over two-period plans. Each university has a quota in each period. Modeling university preferences is more complicated, and there are many ways to do so. To keep university preferences relatively simple in the two-period environment, we take a stance in here: We assume a university cares not only about its graduating class but also not wasting resources on students who will transfer out. This assumption has several implications: First, the preference induced choice function of universities might fail irrelevance of rejected contracts and path independence.² Second, when dynamically stable matchings exist, transfer students in these matchings must have attended community colleges in the first period. Third, universities have preferences over matchings instead of over matches. Forth,

¹A community college offers two-year college education with almost no barrier to entry. Students might go to a community college and then transfer to a university. There are a few community colleges that offer four-year degrees in a few majors.

²Irrelevance of rejected contracts is also known as consistency.

it gives rise to one type of intertemporal externalities. In our model, although there is no externality within each period, there are several types of intertemporal externalities, which students and universities might impose on themselves and others.

A dynamically stable matching, which is individually rational and unblocked in each period, does not generally exist. We provide sufficient conditions that guarantee the existence of a dynamically stable matching. We now highlight some of the results under these sufficient conditions. A modified deferred acceptance algorithm identifies a dynamically stable matching. There exists a student-optimal, a student-pessimal, a university-optimal and a university-pessimal matchings; the set of dynamically stable matchings forms a lattice with respect to a partial order based on the students' common preferences but not with respect to a partial order based on the universities' common preferences. The rural hospital theorem continues to hold but becomes weaker than it is in the college admissions problem. We also show that many of the results for the college admissions problem cannot be generalized even under a condition that makes the two-period model a stack of two college admissions problems

Without the aforementioned sufficient conditions, we propose a novel solution concept, regret-free dynamic stability, which weakens the unblockness under the original preference profile. We make an assumption on market evolution to illustrate how agents give up their blocking powers endogenously. Preferences will be updated on the paths to final outcomes and are endogenously determined by some realized matchings. A regret-free dynamic stable matching is a dynamically stable matching under the final preference profile. There might be different regret-free dynamically stable matchings in the same final preference profile or in different final preference profiles. We use a deferred acceptance algorithm that mimics the market evolution to identify a regret-free dynamically stable matching, so a regret-free dynamic stable matching always exists.

This chapter makes several contributions to the literature. First, to the best of our knowledge, this is the first paper that studies dynamic *two-sided* many-to-one matchings. Second, we found

that many results for the college admissions problem can be generalized to our model under mild preference restrictions, while some cannot be generalized even under some strong preference restriction. In that sense, we provide a robustness check to results in the literature. Third, in a related one-to-one market of our model, the existence results are not special cases of those in the literature, so we provide new existence results to dynamic two-sided one-to-one matching problems.

Though we present our model using terminologies in college admissions, our study provides insights to other labor markets (such as the market between hospitals and doctors, the market between universities and faculties, and so on).

1.2 Related Literature

This chapter relates to several strands of the existing literature. The first strand is the rapidly growing dynamic matching literature. There are several papers study various dynamic two-sided one-to-one matching models. [Damiano and Lam \[2005\]](#) introduce self-sustaining stability and strict self-sustaining stability, where both solution sets contain the recursive core [[Becker and Chakrabarti, 1995](#)]. [Kurino \[2009\]](#) considers a matching is a contingency plan and proposes credible group stability, which requires immunity to any “defensible” group deviations. [Bando \[2012\]](#) shows that a pairwise stable matching exists if the choice functions satisfy substitutability or history independence. [Kotowski \[2015\]](#) proposes a solution concept called robust prescient stability, which assumes agents are farsighted. [Doval \[2017\]](#) is another paper that considers a matching as a contingent plan; the author proposes a stability concept, dynamic stability, which accommodates the trade-off between matching now and waiting for a better match; dynamically stable matchings may fail to exist in two-sided economies. The closest works to ours are [Kadam and Kotowski \[Forthcoming\]](#) and [Zhang and Zheng \[2016\]](#). [Kadam and Kotowski \[Forthcoming\]](#) consider a model with no commitment (in other words, partnerships are unbinding), their dynamic stability is indeed the same as the one we use in the related one-to-one market. [Zhang and Zheng \[2016\]](#) study matching

markets with three types of commitment — full, one-sided, and two-sided; they propose various stability concepts in these environments. Similar to a dynamic school choice problem, there is one-sided commitment in our model. We will discuss these two papers in greater detail in Section 1.3.3.1.

There are several papers that study dynamic *one-sided* many-to-one matching models. [Pereyra \[2013\]](#), [Kurino \[2014\]](#), [Kennes et al. \[2014\]](#) study dynamic object allocation problems with overlapping generations in teacher assignments, house assignments, and daycare assignments, respectively. [Dur \[2012\]](#) studies a slightly different school choice model, which does not have overlapping generations and agents with siblings are in two but not necessarily consecutive periods.

The second strand is the classic matching literature. Since the original many-to-one matching problem has been introduced by [Gale and Shapley \[1962\]](#), it has been studied extensively in the literature [[Gale and Shapley, 1962](#), [Roth, 1984](#), [1985](#), [1986](#), [Roth and Sotomayor, 1989](#), [1992](#)]. We generalize many results to our model under some preference restrictions and also provide counterexamples for many results that could not be generalized under a stronger preference restriction.

The third strand is the literature on matching with externalities. There are several types of externalities in our model. Matching models with externalities [[Sasaki and Toda, 1996](#), [Pycia and Yenmez, 2016](#)] normally define agents' preferences over matchings instead of matches. Follow this literature, agents in our model also have their preferences over matchings instead of matches.

This chapter also relates to [Antler \[2015\]](#), which studies two-sided matching with endogenous preferences. In that paper, agents' preferences depend on the matching mechanism and are therefore endogenous. By contrast, when we study regret-free dynamic stability, agents' preferences are endogenous because they can endogenously give up their blocking powers and thus update their preferences on the paths to regret-free dynamically stable matchings.

1.3 The Model

1.3.1 The Benchmark: College Admissions Problem

We briefly review the original college admissions problem in [Gale and Shapley \[1962\]](#). There is a finite set of students S and a finite set of universities U . Each student can match with at most one university or remain unmatched (being unmatched is denoted by \emptyset). There is a capacity vectors $q = (q_{u_1}, q_{u_2}, \dots, q_{|U|})$, where q_u is the quota of university u .

A *matching* μ is a mapping that satisfies

1. $\forall s \in S, \mu(s) \in U \cup \{\emptyset\}$;
2. $\forall u \in U, \mu(u) \subseteq S$ and $|\mu(u)| \leq q_u$;
3. $\mu(s) = u$ if and only if s is in $\mu(u)$.

Each student s has a strict preference relation P_s over $U \cup \{\emptyset\}$, and a university is *acceptable* to s if $uP_s\emptyset$. Each university u has a strict preference relation P_u over $S \cup \{\emptyset\}$, and a student is *acceptable* to u if $sP_u\emptyset$. Let P_u^* be u 's preference over sets of students induced by P_u .³ Let R_i be the weak preference relation induced by P_i for all $i \in S \cup U$. Let $R_S = (R_s)_{s \in S}$ and $R_U^* = (R_u^*)_{u \in U}$ be the preference profile of students and universities, respectively. (S, U, q, R_S, R_U^*) is a *college admissions problem*. If $q_u = 1$ for all $u \in U$, the model degenerates to the marriage problem.

A matching μ is *individually rational* if for each student s $\mu(s)R_s\emptyset$ and for each university u $sP_u\emptyset$ for all $s \in \mu(u)$. A matching μ is *stable* if it is individually rational and there does not exist a university u and a student s such that $uP_s\mu(s)$ and $sP_u\sigma$ for some $\sigma \in \mu(u)$.

³ P_u^* is responsive to P_u . See the next subsection for the definition of responsiveness.

1.3.2 A Two-period Model

There is a finite set of students S and a finite set of universities U . Each student can match with at most one university in each period or remain unmatched. There are two exogenous capacity vectors $q^1 = (q_{u_1}^1, q_{u_2}^1, \dots, q_{|U|}^1)$ and $q^2 = (q_{u_1}^2, q_{u_2}^2, \dots, q_{|U|}^2)$, where q_u^1 is the quota of university u for freshmen in the first period and q_u^2 is the quota for transfer students in the second period.⁴ We often write (q_u^1, q_u^2) to describe u 's capacity constraints, where one of them can be zero. There is an anonymous community college c that only exists in the first period and has $|S|$ seats to enroll all students, i.e., $q_c = |S|$. In the second period, a university cannot drop the students it admitted in the first period, but a student can leave the university he attended freely.

A **period-1 matching** μ^1 is a mapping satisfies

1. $\forall s \in S, \mu^1(s) \in U \cup \{c\} \cup \{\emptyset\}$;
2. $\forall u \in U, \mu^1(u) \subseteq S$ and $|\mu^1(u)| \leq q_u^1$;
3. $\mu^1(s) = u$ if and only if s is in $\mu^1(u)$.

A **period-2 matching** μ^2 is a mapping satisfies

1. $\forall s \in S, \mu^2(s) \in U \cup \{\emptyset\}$;

⁴A report from University of California (UC) shows that the proportions of transfer students in member institutions of Association of American Universities (AAU) are quite stable from 2009-2013: about 30% for UC, about 24% for AAU public institutions, and about 11% for AAU private institutions (see http://www.ucop.edu/institutional-research-academic-planning/_files/Transfer_Student_Brief.pdf).

On the UW website, it is stated explicitly: "The University of Washington Seattle generally dedicates thirty percent of its new undergraduate spaces annually to Washington community college transfer students" (see <https://admit.washington.edu/apply/transfer/policies/holistic-review>).

When quotas are endogenous, even in the benchmark model, there exists no stable mechanism that is immune to manipulation via quotas [Sönmez, 1997]. Konishi and Ünver [2006b] show that if university preferences are strongly monotonic, then the student optimal stable mechanism is immune to manipulation via quotas; indeed, strongly monotone preferences is the maximal domain of university preferences that guarantees such result [Kojima, 2007]. Strong monotonicity is quite restricted, and our model does not assume university preferences satisfy that.

2. $\forall u \in U, \mu^2(u) \subseteq S$ and $|\mu^2(u) \setminus \mu^1(u)| \leq q_u^2$;
3. $\mu^2(s) = u$ if and only if s is in $\mu^2(u)$.

A **matching** μ is a mapping such that $\forall i \in S \cup U, \mu(i) = (\mu^1(i), \mu^2(i))$ and $\mu^1(u) \cap (\mu^2(u) \setminus \mu^1(u)) = \emptyset$, where the latter means a student cannot transfer to his own university. Let \mathcal{M}^t be the set of period- t matchings, and let \mathcal{M} be the set of matchings.

A student may match with two different schools in period-1 and period-2. We call such sequence of matches a plan; (x, y) is a *plan* for a student where he is matched with x in the first period and with y in the second period. The complete set of plans is $(U \cup \{c\} \cup \emptyset) \times (U \cup \emptyset)$. The set of plans is rich, to simplify our definitions, we make two assumptions.⁵ (S1) Students plan to graduate from some universities at the end. (S2) Students who do not go to school in the first period cannot go to any university in the second period.⁶ The reduced set of plans is $\{(U \cup \{c\}) \times U\} \cup (\emptyset, \emptyset)$. We sometimes write $xy \equiv (x, y)$ and \emptyset in place of (\emptyset, \emptyset) if no confusion arises. Each student $s \in S$ has a *strict* ranking P_s over the reduced set of plans. Let R_s be weak ranking induced by P_s . A plan xy is *acceptable* to student s if $xy R_s \emptyset$. We denote the strict preference relation over the set of matchings for agent s by \succ_s and the indifference relation by \sim_s . Since students only care about their own plans, $\mu \succ_s \mu'$ if and only if $\mu(s) R_s \mu'(s)$. Let $\succ_S = (\succ_s)_{s \in S}$ be the preference profile of students.

For each u , let P_u be a strict ranking over $S \cup \{\emptyset\}$. Let $q_u = q_u^1 + q_u^2$, where q_u is the maximum number of students that u can possibly admit. Let P_u^* be the ranking over sets of students 2^S that is *responsive* (with capacity q_u) to P_u [Roth, 1985], that is, $\forall S' \subseteq S, \forall s' \in S'$ and $\forall s \notin S'$

1. $s P_u s' \iff S' \setminus s' \cup s P_u^* S'$;

⁵Results of this chapter can still be obtained without these assumptions. See footnote 10, 14, 27, 29, 34 and 30 regarding S1. See footnote 13 and 33 regarding S2.

⁶We consider the first period schooling as necessary general education or experience in other labor markets.

$$2. |S'| < q_u \text{ and } sP_u \emptyset \iff S' \cup sP_u^* S';$$

$$3. |S'| > q_u \implies \emptyset P_u^* S'.^7$$

Let R_u^* be the weak ranking induced by P_u^* . To compare $\mu^1(u)$'s and $\mu^2(u) \setminus \mu^1(u)$'s, we use P_u^t , which is the ranking over sets of students 2^S that is responsive (with capacity q_u^t) to P_u . Let R_u^t be the weak ranking induced by P_u^t .

Educating a student or training an employee can be very costly. If an institution believes someone is going to use its position as a stepping stone, then it might not be its best interest to recruit this person. Therefore, a university cares not only about its graduating class but also not wasting resources on the students who will transfer out. In some universities, more than 90% of their transfer students are from community colleges.⁸ In other words, in these universities, only a very small portion of transfer students are from other universities. One possible explanation is that universities do not want to waste resources on students who will transfer out.⁹ We incorporate this idea when we define acceptable matchings. Student s is *acceptable* to university u if $sP_u \emptyset$. Let N_u^μ be the number of students that will transfer out from u given μ .¹⁰ A matching μ is **acceptable** to u , if $\forall s \in \mu^2(u)$, $sP_u \emptyset$ and $N_u^\mu = 0$.¹¹ In other words, a matching is unacceptable to u if any student is unacceptable or will transfer out given μ . We write $\mu \succsim_u \emptyset$ if a matching is acceptable. For

⁷We use s to denote the singleton set $\{s\}$ if no confusion arises. While there may exist more than one P_u^* that is responsive to P_u , P_u is the unique ranking over singleton sets in these P_u^* 's.

⁸Over the years, in the University of California system, the proportion of admitted transfer students that are coming from community colleges is about 90% (see the first link in footnote 4). In particular, this number varies from 93-94% for UC Berkeley from 2014-2016 (see <http://news.berkeley.edu/topics/admissions>. Relevant articles released on May 15, 2014, July 2, 2015, and July 6, 2016) and 92-93% for UCLA from 2010-2016 (see http://www.admission.ucla.edu/prospect/Adm_tr/Tr_Prof.htm). For the University of Washington, Seattle, the number is 86% in 2016 (see <https://admit.washington.edu/why-uw/about#transfer-admission-class-profile>).

⁹ Another possible explanation is Assumption 1 below, which is not assumed throughout the chapter.

¹⁰Without S1, this term should also include the number of students that will drop out from u given μ .

¹¹A similar assumption to $N_u^\mu = 0$ can be found in Dur et al. [2017], which is called "Negative Net-Balance Aversion."

any acceptable matchings $\mu, \mu', \mu^2(u)R_u^*\mu^2(u)$ if and only if $\mu \succsim_u \mu'$.¹² Let $\succsim_U = (\succsim_u)_{u \in U}$ be the preference profile of universities.

Agents have preferences over matchings, and P 's are just rankings in our model, while they are preferences in the college admissions problem. By an abuse of terminology, we often call P_s the preference of s (over plans) and P_u the preference of u (over students), so for xP_iy , we say i prefers x to y instead of i ranks x higher than y . Also, for convenience, we often only write the acceptable objects to denote an agent's ranking or preference. For example,

$$P_s : u_1u_1, u_2u_1$$

means to student s , u_1u_1 is the most preferred plan, u_2u_1 follows the second, and no other plan is acceptable. The community college has no preference. $(S, U, q^1, q^2, \succsim_s, \succsim_U)$ is a *dynamic college admissions problem*.

A matching μ is **individually rational** if $\forall s \in S, \mu \succsim_s \emptyset$ and $\forall u \in U, \mu \succsim_u \emptyset$. Equivalently, a matching μ is *individually rational* if $\forall s \in S, \mu(s)R_s\emptyset$ and $\forall u \in U, \forall s \in \mu^2(u), sP_u\emptyset$ and $N_u^\mu = \emptyset$.

A matching μ is **period-1 blocked** if $\exists (s, u) \in S \times U$ such that

(Type A) $(u, u)P_s(\mu^1(s), \mu^2(s))$ and $sP_u\sigma$ for some $\sigma \in \mu^1(u)$, or

(Type B) $(c, u)P_s(\mu^1(s), \mu^2(s))$ and $sP_u\sigma$ for some $\sigma \in \mu^2(u) \setminus \mu^1(u)$.¹³

A matching μ is **period-2 blocked** if $\exists (s, u) \in S \times U$ such that $(\mu^1(s), u)P_s(\mu^1(s), \mu^2(s))$ and $sP_u\sigma$ for some $\sigma \in \mu^2(u) \setminus \mu^1(u)$.¹⁴ A matching is **unblocked** if it is not period-1 blocked or period-2

¹²Note that even $\mu(u) = (\mu^1(u), \mu^2(u)) \in 2^S \times 2^S$, universities' preferences are not defined over a subset of $2^S \times 2^S$ because N_u^μ depends on others' matchings. See the example in Appendix 1.7.4.

¹³Without S2, " (s, u) or (c, u) " should be in place of " (c, u) ," where (s, u) here is a plan where student s matches with himself in the first period and then matches with u in the second period. Note that the formal statement of S2 is: $\forall s \in S, \emptyset P_s s u$.

¹⁴Without S1, " $(\mu^1(s), s)$ or $(\mu^1(s), u)$ " should be in place of " $(\mu^1(s), u)$."

blocked. A matching is **dynamically stable** if it is individually rational and unblocked.

1.3.3 Discussions on Modeling Issues

1.3.3.1 Relationships with Other Models

We now relate our model to models in [Kadam and Kotowski \[Forthcoming\]](#) and [Zhang and Zheng \[2016\]](#). Since their models are on dynamic one-to-one matching problems, we briefly describe the relevant details of the related one-to-one market of a dynamic college admissions problem (henceforth, the related one-to-one market).¹⁵ This market is the counterpart of the related marriage market of a college admissions problem [[Roth and Sotomayor, 1989](#)]. Firstly, we divide each $u \in U$ into q_u^1 period-1 positions and q_u^2 period-2 positions. Each position is then an independent agent in the related one-to-one market. Secondly, we construct preferences for these newly created agents and also modify the preferences for students. Each position's preference over matchings is induced by a strict ranking over two-period plans.

In dynamic matching markets, the feature that once enrolled at a “school,” a “student” can stay enrolled in the next period without being replaced by a new student has different representations: It is modeled as part of schools' priority structures [[Pereyra, 2013](#), [Kennes et al., 2014](#)], part of schools' preferences [[Kadam and Kotowski, Forthcoming](#)], property rights [Kurino \[2014\]](#), and commitment [Diamantoudi et al. \[2015\]](#), [Zhang and Zheng \[2016\]](#). Our model belongs to the last category. The corresponding preference restriction in [Kadam and Kotowski \[Forthcoming\]](#) is *strong inertia*.¹⁶

We say P_i satisfies *perfect inertia* if $x \neq y$, then $\emptyset P_i xy$. Perfect inertia is slightly stronger than strong inertia. Let P_{u^1} be the strict ranking of a period-1 position of u (in other words, P_{u^1} is the “preference of u^1 over plans”). To embed the related one-to-one market into the model of [Kadam](#)

¹⁵The complete description of a related one-to-one market is relegated to Appendix 1.7.2.

¹⁶ P_i satisfies *strong inertia* if $x \neq y$ implies $xxP_i xy$ and $yyP_i xy$.

and Kotowski [Forthcoming], we need the following condition: For each $u \in U$, P_u satisfies perfect inertia.¹⁷ In their model, to guarantee the existence of a dynamically stable matching, two sets of preference restrictions have been proposed, where both sets include a restriction called *sequential improvement complementarity* (SIC).¹⁸ We do not assume preferences in our model satisfy SIC; for instance, the preferences of students that only want to match with universities in the second period do not satisfy SIC.¹⁹

The aforementioned students can be viewed as students that will “arrive” in the second period. The related one-to-one market and the one-sided commitment with arrivals of students model of Zhang and Zheng [2016] are different in solution concepts.²⁰ They study two stability concepts that restrict agents blocking powers. One puts restrictions on both period-1 blocking and period-2 blocking, and stable matchings are guaranteed to exist. Another puts restrictions only on period-2 blocking, and stable matchings are guaranteed to exist under two sets of preference restrictions. Stability concepts in this chapter do not restrict agents blocking powers.

Sum up, our existence results for dynamic two-sided one-to-one matching problems are not special cases of those in their papers. We propose different preference restrictions that guarantee the existence of dynamically stable matchings. Without these preference restrictions, we propose another stability concept.

¹⁷Strong inertia is not enough because quotas are exogenous. There are already q_u^2 period-2 positions of u in the related one-to-one market. Strong inertia does not rule out us plans, so if a period-1 position of u is matched with a us plan when all q_u^2 period-2 positions of u are matched, then the quota restriction in period-2 is violated. Acceptability rules out ss' plans (and su plans when S1 is not assumed). Therefore, only ss plans are preferred to \emptyset .

¹⁸ P_i satisfies *sequential improvement complementarity* if $xyP_i xxR_{ij}$ implies $yyP_i xx$.

¹⁹If student s has a preference $P_s : cu$. By SIC, $uuP_s \emptyset$, a contradiction. Note that SIC is a restriction for preferences over plans, so it is not applicable to universities' preferences over students. However, in the related one-to-one market, period-2 positions' preferences over plans do not satisfy SIC.

²⁰S1 is not assumed in their model, but as mentioned in footnote 5, S1 can be removed from our model as well.

1.3.3.2 Externalities

Although there is no externality within each period, there are several types of intertemporal externalities in our problem: A student's matchings in the first period and in the second period impose externalities on his future self and past self, respectively, because his preference may be history-dependent.²¹ A university and a student's period-2 matching might impose externalities on universities (via N_u^u 's) and students in the first period.²² A university itself might be able to impose an externality on some student.²³

If one takes plans as contracts, then it is tempting to map this model into a two-sided matching with contracts and externalities model. [Pycia and Yenmez \[2016\]](#) introduce such a model in a general many-to-many setting. In their model, a contract x specifies a buyer $s(x)$, a seller $u(x)$, and terms (freshman admission and transfer admission in this context). But a contract (s, uu') specifies two sellers and one buyer, and thus it is not a two-sided contract but a multilateral contract. Even one tolerates this violation, an essential condition for choice functions to guarantee the existence of a stable allocation, irrelevance of rejected contracts, might fail.²⁴ One might also consider (s, uu') to be two contracts $((s, uu)$ and (s, cu')) and allow a student to sign both of them, but when s transfers to u' in the second period, he breaches the contract (s, uu) and such feature is absent in matching with contracts.

²¹Consider an example where a student has a preference $P_s : cu$. Suppose he cannot period-1 block with u , then he might choose the stay out of the market, so being unmatched in the second period impose an externality to his past self. Now suppose at the beginning of the second period, there is an empty seat at u . But su is not an acceptable plan for s , so being unmatched in the first period imposes an externality to his future self.

²²See Example 2 below and Example D.1 in Appendix 1.7.4.

²³See Example A.2 in Appendix 1.7.1.2.

²⁴See Proposition 4. Also, the stability mentioned here is more general than pairwise stability.

1.3.3.3 *Pairwise Stabilities*

Dynamic stability and regret-free dynamic stability are all pairwise. We prefer pairwise stabilities for the following five reasons. First, since our analysis is motivated by the college admissions problem, we want our model to be closer to its origin. Second, many results in the literature are for stability. When our model reduces to the college admissions problem, both of our pairwise stabilities reduce to stability. We are interested in what results continue to hold in our model. Third, coordination among a large group of agents is unlikely in admission markets. Forth, more restricted assumptions are needed to ensure positive results for stability concepts that allow a large coalition to block. Fifth, pairwise stabilities are relatively more trackable.

Agents in our model are myopic (in the sense that deviations can be countered by subsequent deviations and agents do not know the terminal consequences of their deviations), and our pairwise stabilities are also myopic (in the sense that they assume agents are myopic).²⁵ This is consistent with the college admissions problem because stability is myopic, and the use of stability indicates agents are implicitly assumed to be myopic in that model. When a farsighted agent is contemplating a deviation, he might want to conjecture all possible subsequent market evolutions induced by his deviation.²⁶ This type of elaboration will be burdensome once the size of the market gets large, and it is even more so in a dynamic market.

The following two characters of stability are inherited by our pairwise stabilities. First, as

²⁵Xue [1998] clarifies that it is not the notion of stability that reflects foresight or myopia but the situation that formalizes how individuals view and use their alternatives.

When agents are farsighted, solution concepts that are farsighted should be used. For instance, see Mauleon et al. [2011], Kotowski [2015], Herings et al. [2017] (deviations are pairwise in the last two).

²⁶Farsightedness might be perfect and imperfect. Even with imperfect farsightedness, agents might or might not fully understand the final outcomes induced by their deviations (the latter is less common. Kotowski [2015], Doval [2017] are examples. A blocking coalition in Doval [2017] is farsighted only in the sense that it understands how the block affects the continuation through a contingent matching, but it does not consider the possible reactions of other coalitions to their block. Agents in Kotowski [2015] know what continuation plans can occur in the future but do not know which plan will occur). Given a type of farsightedness, to define stability concepts, agents are often assumed to be optimistic or conservative when they compare their alternatives. See Xue [1998] for a detailed discussion.

mentioned, stability is myopic, which we consider it is one reason why there may exist matchings that all colleges strictly prefer to the college-optimal stable matching in the college admissions problem [Roth, 1985]. Second, when joins with responsiveness, stability assumes a university “maximizes” its outcome seat-by-seat. We consider this is another reason that contributes to the aforementioned result in [Roth, 1985].

Our pairwise stabilities assume universities do not utilize the externalities that they can impose on others. This is because some sort of farsightedness is necessary: When universities do utilize these externalities, they must be able to anticipate future developments of the market and enforce some desirable outcomes.

In Appendix 1.7.1, we also discuss two solution concepts that do not restrict to pairwise deviations — dynamic core and dynamic group stability, where the former coincides with the set of dynamically stable matchings under some preference restrictions, while the latter is equivalent to dynamic stability under a stronger preference restriction.

1.4 Dynamically Stable Matchings

Proposition 1: The set of dynamically stable matchings might be empty.

Proof: We prove it by an example.

Example 1. Consider the following market. $S = \{s_1\}$, $U = \{u_1, u_2\}$,

$$P_{u_1} : s_1 q_{u_1} = (1, 0),$$

$$P_{u_2} : s_1 q_{u_2} = (0, 1),$$

$$P_{s_1} : u_1 u_2, u_1 u_1.$$

Let

$$\mu = \left(\begin{array}{cc} u_1 & u_2 \\ s_1 & \emptyset \\ s_1 & \emptyset \end{array} \right) \begin{array}{l} \} \mu^1 \\ \} \mu^2 \end{array} \text{ and } \mu' = \left(\begin{array}{cc} u_1 & u_2 \\ s_1 & \emptyset \\ \emptyset & s_1 \end{array} \right).$$

The matrix μ is read as $\mu(u_1) = (\{s_1\}, \{s_1\})$, $\mu(u_2) = (\{\emptyset\}, \{\emptyset\})$, and $\mu(s_1) = u_1u_1$. The set of unblock matchings is $\{\mu'\}$, and the set of individually rational matchings is $\{\emptyset\}$, where \emptyset here is the empty matching. In Example 1, unblockness is incompatible with individual rationality, so there does not exist a dynamically stable matching. \square

Since a dynamically stable matching does not always exist, to continue our analysis, we need to either make assumptions on agent's preferences or modify the stability concept. We will go down both routes in this chapter.

1.4.1 Existence and Properties Under Preference Restrictions

We provide two preference restrictions that can help us to circumvent the requirement $N_u^\mu = 0$ for each u in the individual rationality.

Assumption 1 (A1): $\forall s \in S, \emptyset P_s uu'$ where $u' \neq u$.²⁷

In words, all uu' plans are unacceptable to students. Under Assumption 1, no student *wants to match with a different university in the second period*.²⁸ A1 is another possible explanation of the fact that more than 90% of the transfer students are coming from community colleges in some

²⁷S1 is required to exclude us plans. Note that the formal statement of S1 is: $\forall s \in S, \emptyset P_s us$. So if S1 is not assumed, " us and uu' " should be in place of " uu' " in A1.

²⁸One might interpret this assumption as students are committed to the universities once they are admitted. Recall universities are also committed. The related one-to-one market under Assumption 1 is indeed very close to the full commitment model of [Zhang and Zheng \[2016\]](#). Here are several observations. First, the commitments of students are coming from special preferences here, while they are coming from limited blocking powers in that model. Second, solution concepts in that model require an extra condition which dynamically stable matchings in our model might violate because of the presence of community college. Third, the related one-to-one market is just an auxiliary market, and even under Assumption 1, some results for the related one-to-one market could not be generalized to the dynamic college admissions problem.

universities.

Assumption 2 (A2): $\forall u \in U$, if $uu'P_suu$ where $u' \neq u$ with $q_{u'}^2 > 0$ and $sP_{u'}\emptyset$ for some s , then $\emptyset P_{u's}$.²⁹

In words, a student that prefers uu' to uu in which u' has a positive period-2 quota and this student who is acceptable at u' is unacceptable to university u . Under Assumption 2, no student *can* match with a different university in the second period. It relates to our second approach where we modify the stability concept. For preference restriction, we consider A1 is more realistic than A2 and will focus on it as all results in this chapter that hold under A1 will also hold under A2.

Theorem 1: Under A1, the set of dynamically stable matchings is nonempty.

The theorem is proved by using a simple generalization of deferred acceptance algorithm [Gale and Shapley, 1962]. The following algorithm identifies a dynamically stable matching under A1 or A2.

Student-proposing plan deferred acceptance algorithm (S-PDA): Start with the empty matching μ , where $\mu(i) = (\emptyset, \emptyset)$ for all $i \in S \cup U$.

Step $n \geq 1$: Each student proposes his most preferred plan that has not been rejected. Each university keeps the most preferred students subject to its quotas and rejects all other students.

When a uu' plan is a student's most preferred plan: if A1 is assumed, leave the student unassigned; if A2 is assumed, skip to the next plan.³⁰ Terminates when no more proposal is made. Denote the final matching by μ_S .

This algorithm terminates in finite steps. When A1 is assumed and a student reports a preference that does not satisfy it, leaving the student alone ensures that no student has an incentive to report such preference in the first place. When A2 is assumed, a uu' plan will always be rejected

²⁹Without S1, " usP_suu or" should be put after "if."

³⁰ If S1 is not assumed, as mentioned in 27 and 29, we would need to modify A1 and A2. Then " us or uu' " should be in place of " uu' " here.

by u , so it is innocuous to skip it.

Lemma 1: Under A1, μ_S is dynamically stable.

Theorem 1 is a direct result of Lemma 1. In the related one-to-one market, we can define the **university-proposing plan deferred acceptance algorithm (U-PDA)**, where positions propose to students (each student has a quota of 1). Let μ_U denote the matching that its corresponding matching is the final matching of U-PDA.

A dynamically stable matching is **student-optimal (student-pessimal)** if each student gets the best (worst) outcome within the set of dynamically stable matchings. A **university-optimal (university-pessimal)** stable matching is defined analogously.

Theorem 2: Under A1, μ_S is student-optimal and μ_U is university-optimal.

Let $\mu >_S \mu'$ denote that all students like μ at least as well as μ' with at least one student strictly prefers, that is, $\mu \succeq_s \mu'$ for all s , and $\mu \succ_s \mu'$ for at least one s . $>_S$ is a partial order on the set of matchings that represents students' common preferences. $>_U$ is defined analogously.

Theorem 3: Under A1, the set of dynamically stable matchings forms a lattice under $>_S$.

However, the set of dynamically stable matchings does not form a lattice under $>_U$. This is in contrast to the college admissions problem, where the lattice under $>_S$ is the dual lattice of the one under $>_U$.

Assumption 3 (A3): $\forall s \in S$, either $\nexists xy \neq uu$ such that $xyP_s\emptyset$ or $\nexists xy \neq cu$ such that $xyP_s\emptyset$.

In other words, only uu plans or only cu plans are preferred to \emptyset for each student. It is obvious that A3 is stronger than A1. We use both A1 and A3 to emphasize the gap between our negative results and positive results. Under A3, the two-period model can be viewed as a stack of two college admissions problems and all externalities are eliminated.

Proposition 2: Under A3, the set of dynamically stable matchings does not form a lattice under $>_U$.

Proposition 3: Under A1, μ_U is student-pessimal and μ_S is university-pessimal.

Proposition 3, although seemingly, is not a direct result of Theorem 2. Theorem 2, 3 and Proposition 2, 3 imply that under A1, students always have common preferences between two dynamically stable matchings, while universities can agree on which matchings are university optimal and university pessimal, they might not have common preferences between some dynamically stable matchings.

Theorem 4 (Weak Pareto Optimality for the Students): Under A1, there is no individually rational matching μ (dynamically stable or not) such that $\mu \succ_s \mu_S$ for all $s \in S$.

The counterpart for universities does not hold [Roth, 1985].³¹

Theorem 5: Under A1, the set of students and positions that are matched is the same for all dynamically stable matchings.

Theorem 6 (Rural Hospitals Theorem): Under A1, any university that does not fill its quota in some period at some dynamically stable matching will match with the same set of students at every dynamically stable matching for that period.

Theorem 6 is weaker than its original version, which says any university that does not fill its quota at some stable matching gets the same sets of students at every stable matching. Here a university does not fill one of its quotas might still get different sets of students at different dynamically stable matchings given another quota is filled. The proof of Theorem 6 relies on Lemma C.2 in Appendix 1.7.3. This lemma is weaker than its original version in the college admissions problem. Indeed, the main results of Roth and Sotomayor [1989] also rely on the original version of this lemma; nevertheless, in Roth and Sotomayor [1992], they indicate that some results of the college admissions problem also depend critically on it (for example, the opposite result of Proposition 2).³² Although Lemma C.2 holds in dynamic college admissions problem, it is not strong enough to support those results in the two-period environment. In Appendix 1.7.1, we

³¹Though it holds for positions in the related one-to-one market.

³²These results are in 5.6.3 of Roth and Sotomayor [1992].

provide additional results to illustrate the failures of the aforementioned results.

Theorem 7: Under A1, if μ and μ' are dynamically stable matchings, then $\mu^2(u) = \mu'^2(u)$ if and only if $\mu(u) = \mu'(u)$.

Theorem 7 has no parallel in the college admissions problem. It says there does not exist two period-1 matchings that would give u the same final outcome at two different dynamically stable matchings. The intuition behind this theorem is simple. Consider a university u with 1 position in each period and $P_u : s_1$. If $\mu^2(u) = \{s_1\}$ and $\mu'^2(u) = \{s_1\}$ but $\mu^1(u_1) = \emptyset \neq s_1 = \mu'^1(u_1)$, then they cannot be both dynamically stable because s_1 has strict preference between cu_1 and u_1u_1 .

Theorem 8: Under A1, suppose μ and μ' are dynamically stable matchings.

- (a) If $\mu(s)P_s\mu'(s)$ for some $s \in \mu^1(u)$ and $\mu(s')P_{s'}\mu'(s')$ for some $s' \in \mu^2(u) \setminus \mu^1(u)$, then $\mu'(u)P_u^*\mu(u)$.
- (b) If $\mu(s)P_s\mu'(s)$ for some $s \in \mu^1(u)$ and $\mu^2(u) \setminus \mu^1(u) = \mu'^2(u) \setminus \mu'^1(u)$, or $\mu(s)P_s\mu'(s)$ for some $s \in \mu^2(u) \setminus \mu^1(u)$ and $\mu^1(u) = \mu'^1(u)$, then $\mu'(u)P_u^*\mu(u)$.
- (c) If $\mu(u)P_u^*\mu'(u)$, then $\exists s \in \mu^1(u)$ such that $\mu'(s)P_s\mu(s)$ or $s' \in \mu^2(u) \setminus \mu^1(u)$ such that $\mu'(s')P_{s'}\mu(s')$.

There might exist two dynamically stable matchings μ and μ' such that $\mu(s)P_s\mu'(s)$ for some $s \in \mu^1(u)$ and $\mu'(s')P_{s'}\mu(s')$ for some $s' \in \mu^2(u) \setminus \mu^1(u)$ (or the opposite), while $\mu'(u)P_u^*\mu(u)$ (see footnote 43 in Appendix 1.7.1).

Let $\succsim = (\succsim_i)_{i \in S \cup U}$ be the preference profile of all agents. A mechanism φ is a function that maps a preference profile to a matching. Let $\varphi(\succsim)$ denotes the matching outcome of φ given the preference profile \succsim . A mechanism φ is said to be *stable* if it yields a dynamically stable matching for each preference profile. A mechanism φ is said to be **strategy-proof** if there does not exist a preference profile \succsim , an agent $i \in S \cup U$ with \succsim'_i such that $\varphi(\succsim'_i, \succsim_{-i}) \succsim_i \varphi(\succsim)$ (although we put \succsim_i here, agent i only reports P_i to the mechanisms in this chapter).

A well-known result of the marriage problem says there does not exist a stable matching mechanism that is strategy-proof [Roth, 1982]. Since our model subsumes the marriage problem and the college admissions problem, it inherits *all* of their negative results.

A mechanism φ is said to be **group strategy-proof for students** if there does not exist a preference profile \succsim , a subset of students $S' \subseteq S$ with $(\succsim'_s)_{s \in S'}$ such that $\varphi((\succsim'_s)_{s \in S'}, (\succsim_s)_{s \in U \cup S \setminus S'}) \succ_s \varphi(\succsim)$ for all $s \in S'$. We define the **S-PDA mechanism** to be a mechanism that uses the S-PDA algorithm to produce a matching for each input. By Theorem 1, the S-PDA mechanism is stable.

Theorem 9: Under A1, the S-PDA mechanism is group strategy-proof for students.

1.4.2 Some Observations

1.4.2.1 The Connection between Period-1 and Period-2 Matchings in Dynamically Stable Matchings

In a subgame perfect equilibrium of a finitely repeated non-cooperative game, a stage game Nash equilibrium must be played in the last period. In the same spirit, in a dynamically stable matching μ , once agents arrive in the second period, the period-2 matching given μ^1 should be stable.

Let $P_s^{\mu^1}$ be the conditional preference (over universities) of student s given μ^1 , that is, $u' P_s^{\mu^1} u \iff (\mu^1(s), u') P_s(\mu^1(s), u)$. Given μ^1 , we say a student is *active* given μ^1 in period-2 if $\exists u' \in U$, $u' \neq \mu^1(s)$ such that $u' P_s^{\mu^1} \mu^1(s)$ for some u' with $q_{u'}^2 > 0$ and $s P_{u'} \emptyset$. Let $S_a^{\mu^1} \subseteq S$ denote the set of active students given μ^1 . We can partition S into two sets, $S_a^{\mu^1}$ and $S_{inactive}^{\mu^1} \equiv S \setminus S_a^{\mu^1}$. Let $R_S^{\mu^1} = (R_s^{\mu^1})_{s \in S_a^{\mu^1}}$ denote the preference profile that consists of the conditional preferences of all students in $S_a^{\mu^1}$. We say a university u is active in period-2 if $q_u^2 > 0$. Let $U_a \subseteq U$ denote the set of active universities. Let $P_u^{\mu^1}$ be the conditional preference of u given μ^1 such that $P_u^{\mu^1}$ agrees with P_u^2 for $s \notin \mu^1(u)$ and for $s \in \mu^1(u)$, $\emptyset P_u^{\mu^1} s$ (recall a student cannot transfer to his own university). Let $R_u^{\mu^1}$ be the weak preference induced by $P_u^{\mu^1}$. Let $R_U^{\mu^1} = (R_u^{\mu^1})_{u \in U_a}$ be the preference profile of all universities in U_a . Then $A^{\mu^1} \equiv (S_a^{\mu^1}, U_a, q^2, R_S^{\mu^1}, R_U^{\mu^1})$ is a college admissions problem and we call it the period-2 market given μ^1 .

Let $m^{\mu^1} \in M^{\mu^1}$ be a matching, where M^{μ^1} is the set of matchings in A^{μ^1} . If an active student is not matched with a university at m^{μ^1} , then he either continues to match with his original university

or becomes unmatched if he was matched with the community college. Formally, $\forall s \in S_a^{\mu^1}$,

1. if $m^{\mu^1}(s) \neq \emptyset$ then $\mu^2(s) = m^{\mu^1}(s)$;
2. if $m^{\mu^1}(s) = \emptyset$ and $\mu^1(s) = u$, then $\mu^2(s) = u$;
3. if $m^{\mu^1}(s) = \emptyset$ and $\mu^1(s) = c$, then $\mu^2(s) = \emptyset$.³³

Then, if $s \in S_{inactive}^{\mu^1}$, then $\mu^2(s) = \mu^1(s)$. These four cases fully specify $\mu^2(s)$ for all students given any μ^1 . Let $S_u^{m^{\mu^1}}$ be the set of students who are active in u and are matched with some university at m^{μ^1} , that is, $\forall s \in S_u^{m^{\mu^1}}$, $s \in \mu^1(u)$ and $m^{\mu^1}(s) \neq \emptyset$.³⁴ For each university u , $\mu^2(u) = \mu^1(u) \setminus S_u^{m^{\mu^1}} \cup m^{\mu^1}(u)$. It is easy to see that if m^{μ^1} is not stable, then $\mu = (\mu^1, \mu^2)$ is not dynamically stable. In particular, if m^{μ^1} is not individually rational, then μ is not individually rational; if m^{μ^1} is blocked, then μ is period-2 blocked. We call m^{μ^1} the period-2 matching given μ^1 . So in a dynamically stable matching μ , once agents arrive in the second period, the period-2 matching given μ^1 should be stable.

It is well-known that the set of stable matchings is nonempty in a college admissions problem [Gale and Shapley, 1962]. Let \mathfrak{M}^{μ^1} be the set of stable matchings in A^{μ^1} . Let $m^{\mu^1} \in \mathfrak{M}^{\mu^1}$ be a stable matching and let $\mu_{\mu^1}^2$ be the corresponding unblocked period-2 matching. Fixing any μ^1 , there will be a college admissions problem A^{μ^1} and thus a set of stable matchings \mathfrak{M}^{μ^1} ; we can construct an $\mu_{\mu^1}^2$ for each $m^{\mu^1} \in \mathfrak{M}^{\mu^1}$ and obtain a set of unblocked period-2 matchings $\mathcal{M}_{\mu^1}^2 \subseteq \mathcal{M}^2$. Thus, dynamic stability selects a matching μ of the form $(\mu^1, \mu_{\mu^1}^2)$ such that it is individually rational, period-1 unblocked and $S_u^{m^{\mu^1}} = \emptyset$ for each u (the latter is equivalent to $N_u^{\mu} = 0$ for each u).

³³Without S2, “ c or s ” should be in place of “ c ” in 3.

³⁴Without S1, $S_u^{m^{\mu^1}}$ should also include students that will drop out from u given μ .

1.4.2.2 Relationship with the College Admissions Problem

Under the following conditions, the model reduces to the college admissions problem (in other words, we can construct a college admissions problem that is isomorphic to the dynamic college admissions problem).

Condition 1: $\forall s \in S, xyP_s\emptyset \implies xy = uu$ for some u .

Condition 2: $\forall s \in S, xyP_s\emptyset \implies xy = cu$ for some u .

Condition 3: $q_u^1 = 0$ for each $u \in U$.

Condition 4: $q_u^2 = 0$ for each $u \in U$.

Condition 1 implies that only uu plans are preferred to \emptyset , and Condition 2 implies that only cu plans are preferred to \emptyset . Both of them are stronger than A3.

1.4.3 Preference Induced Choice Functions

Preferences are primitives of this model. Here we show that the preference induced choice function of universities has some interesting properties. We borrow some notations from [Pycia and Yenmez \[2016\]](#) and call plans “contracts.” Let C_u be the preference induced choice function of u , where $C_u(X|\mu) = \underset{\sim_u}{\max}\{Y \subseteq X|\mu\}$ is the set of contracts chosen by u from X given μ is the set of contracts currently held. Let $C_U(X|\mu) = \cup_{u \in U} C_u(X|\mu)$. A choice function C satisfies the *irrelevance of rejected contracts* if for all $Y, Z, \mu \subseteq X$, we have $C(Z|\mu) \subseteq Y \subseteq Z \implies C(Y|\mu) = C(Z|\mu)$.

Proposition 4: C_U might not satisfy irrelevance of rejected contracts.

Proof: Let $Y = \{(s_1, u_1 u_1)\}$ and $Z = Y \cup (s, u_1 u_2)$. Then in Example 1, $\mu = Y = C_U(Y|\emptyset) \neq C_U(Z|\emptyset) = \emptyset$.³⁵ □

A choice function C is *path-independent* if for every μ, Y and Z , $C(C(Y|\mu) \cup C(Z|\mu)|\mu) = C(Y \cup Z|\mu)$. [Aizerman and Malishevski \[1981\]](#) show that a choice function is path independent if

³⁵A result in [Hatfield et al. \[2012\]](#) says that a choice function induced by a strict preference relation satisfies irrelevance of rejected contracts. There is no contradiction here because C_{u_1} ($= C_U$) depends on both $P_{u_1}^*$ and $N_{u_1}^{\mu_1}$.

and only if it satisfies substitutability and irrelevance of rejected contracts.

Corollary 1: C_U might not be path-independent.

In Example 1, $\mu = Y = C_U(C_U(Y|\emptyset) \cup C_U(Z|\emptyset)|\emptyset) \neq C_U(Y \cup Z|\emptyset) = \emptyset$. In [Yenmez \[2015\]](#), universities' choice functions might not be path-independent, but it is because they might fail to satisfy substitutability.

1.5 Regret-Free Dynamic Stability

We now modify the stability concept. A student cannot commit to a uu plan if there are better alternatives available to him in the second period. Dynamic stability does not take this lack of commitment into account. Without preference restrictions, dynamic stability must be relaxed. We motivate our modification by Example 2 below.

We except a final market outcome is acceptable to each u . Therefore, only cu and uu plans would be matched. A *reduced* students' preference profile $\underline{\succ}_S$ is obtained by eliminating all uu' plans from \succ_S . We say μ is a *proposed* matching if it is dynamically stable under $(\underline{\succ}_S, \underline{\succ}_U)$. In any market $(S, U, q^1, q^2, \underline{\succ}_S, \underline{\succ}_U)$, the set of proposed matchings is a superset of the set of dynamically stable matchings, and if a proposed matching is not period-2 blocked, then it is dynamically stable.

Example 2. Consider the following market. $S = \{s_1, s_2, s_3\}$, $U = \{u_1, u_2\}$,

$$P_{u_1} : s_1, s_2, s_3 \quad q_{u_1} = (1, 1),$$

$$P_{u_2} : s_3, s_1, s_2 \quad q_{u_2} = (0, 1),$$

$$P_{s_1} : u_1 u_2, u_1 u_1,$$

$$P_{s_2} : cu_2, cu_1,$$

$$P_{s_3} : cu_1, cu_2.$$

In this market,

$$\mu^{(0)} = \begin{pmatrix} u_1 & u_2 & c \\ s_1 & \emptyset & s_2, s_3 \\ s_1, s_3 & s_2 & \emptyset \end{pmatrix}$$

is dynamically stable under $(\underline{\succ}_S, \underline{\succ}_U)$, so it is a proposed matching. Under $(\underline{\succ}_S, \underline{\succ}_U)$, it is not period-1 blocked, but it is period-2 blocked by u_2 and s_1 . u_1 knows if it does not take action, s_1 would transfer out in the second period.³⁶ As a myopic agent who cannot anticipate future development of the market, u_1 cannot foresee this when it admits s_1 , and now it “regrets.” Here, u_2 and s_2 ’s period-2 matching impose an externality on u_1 and s_1 in the first period via $N_u^{\mu^{(0)}}$.

The regret-free property of blocking does not hold in our model. Indeed, in models with externalities, such a property often does not hold.³⁷ For DA algorithms to perform well, the regret-free property is in general necessary. Therefore, preference restrictions are often made to restore this property, as we did before.

In [Zhang and Zheng \[2016\]](#) and [Doval \[2017\]](#), different stability concepts are defined after restricting agents’ blocking powers in different ways (some feasible blocks under one restriction become infeasible under another restriction). Restricting agents’ blocking powers exogenously determine what blocks can occur. Instead, we assume agents would give up blocking powers endogenously, so what blocks can occur are also endogenous determined.

In Example 2, since s_1 is acceptable to u_1 , we need to come up with an alternative way to let u_1 drop s_1 in this situation. We discuss 3 possible approaches that are reminiscent of the “dropping strategy” in [Kojima and Pathak \[2009\]](#), where universities list some acceptable students as unacceptable. First, if $u u' P_s u u$ where $u' \neq u$ with $q_{u'}^2 > 0$ and $s P_{u'} \emptyset$, then u list this student as unacceptable. This has the same effect of A2. But it is too exclusive because this student might want to

³⁶ u_1 knows s_1 would transfer out is based on the current market condition, and it does not mean u_1 is farsighted.

³⁷See [Bando et al. \[2016\]](#) for a survey on two-sided matching with externalities.

match with u with cu , which is totally fine. This leads us to the second approach: List this student as unacceptable in the first period but acceptable in the second period. However, we cannot express this preference in our model.³⁸ Fortunately, we can express it in the related one-to-one market. (u^1 is some period-1 position of u , u^2 is some period-2 position of u' , and \bar{U} is the set of positions.)

Assumption 4 (A4): $\forall u^1 \in \bar{U}$, if $u^1 u^2 P_s u^1 u^1$, $u^2 \in \bar{U}$ and $u' s P_{u^2} \emptyset$ for some s , then $\emptyset P_{u^1} s s$.

$u^2 \in \bar{U}$ is equivalent to $q_{u'}^2 > 0$. A4 is weaker than A2 but it also ensures no student can match with a different university in the second period. Therefore, all results in this chapter that hold under A1 will also hold under A4. We say a type A block (s, uu) is **nullified** if u no longer accepts s in the first period. The second approach, which simply nullifies a type A block when s satisfy conditions in A4, is still too exclusive (see μ' in Example 2 below, which would be excluded under the second approach). The third approach, which we focus on, only nullifies type A blocks that matter on the paths to final outcomes.

We now make an assumption on market evolution to illustrate how agents give up blocking powers endogenously. There are two possible ways to describe nullifications in preference profiles. One is similar to A4, which makes student *cannot* match with the nullified plan (by modifying positions' preferences). Another is similar to A1, which makes students *do not want to* match with the nullified plan (by modifying students' preferences). They are equivalent to each other. The first one is more natural but it invokes the related one-to-one market. We employ the second one in the main text (see the first one in Appendix 1.7.2.1, where the solution concept and algorithm below are also adjusted accordingly).

An assumption on market evolution: Students propose plans in the reduced students' preference profile \succ_{S^0} , and the market evolves until a proposed matching arises, denote this matching by $\mu^{(0)}$.³⁹ If there exists a set of student $S^{(0)}$ such that for each $s \in S^{(0)}$, $\mu^{(0)}$ is period-2 blocked

³⁸Hatfield and Kominers [2017] show that expressiveness can impact the set of stable matchings.

³⁹Note that in a dynamic college admissions problem with a reduced students' preference profile, a matching that is period-2 blocked by a pair can also be period-1 by the same pair, so all blocks can be viewed as period-1 blocks.

by s and some u' under (\succsim_S, \succsim_U) , then the universities that they are currently matched nullify the type A blocks that involve those students. For $s \in S^{(0)}$, update \succsim_s to $\succsim_s^{(1)}$ such that $\emptyset P_s^{(1)} u u$ and otherwise preserves P_s , where u is the university s is currently matched. Let $\succsim_S^{(1)}$ denote the new students' preference profile. Students propose plans in $\underline{\succsim_S^{(1)}}$, and the market evolves until a proposed matching arises, denote this matching by $\mu^{(1)} \dots$. Market evolution stops as soon as a proposed matching $\mu^{(t)}$ is not period-2 blocked under $(\succsim_S^{(t)}, \succsim_U)$, which implies $\mu^{(t)}$ is dynamically stable under $(\succsim_S^{(t)}, \succsim_U)$. Denote such t by N .

Since the number of students and the number of possible plans are finite, there must exist such $N < \infty$. In another word, there exists a finite sequence $\{(\succsim_S, \mu^{(0)}), (\succsim_S^{(1)}, \mu^{(1)}), \dots, (\succsim_S^{(N)}, \mu^{(N)})\}$ such that $\mu^{(N)}$ is a dynamically stable matching under $(\succsim_S^{(N)}, \succsim_U)$.

A matching μ is **regret-free dynamically stable** if it satisfies

1. Individual rationality — $\forall s \in S, \mu(s) R_s \emptyset; \forall u \in U, s P_u \emptyset$ and $N_u^\mu = 0$; and
2. Unblockness under $(\succsim_S^{(N)}, \succsim_U)$.

Therefore, $\mu^{(N)}$ is a regret-free dynamically stable matching. Recall in Example 1, μ a proposed matching because it is dynamically stable under $(\underline{\succsim_S}, \succsim_U)$, but it is period-2 blocked under $(\underline{\succsim_S}, \succsim_U)$, so $\underline{\succsim_S}$ is updated to $\underline{\succsim_S}^{(1)}$, where $(s_1, u_1 u_1)$ is nullified. The empty matching \emptyset is then the unique proposed matching, and it is not not period-2 blocked under $(\underline{\succsim_S}^{(1)}, \succsim_U)$, so it is a regret-free dynamically stable matching. (In the proof of Proposition 4, we saw $C_U(Z \cup Y | \emptyset) = \emptyset$, but the empty matching is not a satisfactory outcome under $(\underline{\succsim_S}, \succsim_U)$ as it is period-1 blocked there.)

Agents' preferences are now endogenous because they depend on the realized proposed matchings. There might be another regret-free dynamically stable matching $\mu'^{(N)}$ or $\mu^{(N')}$ with $N \neq N'$ (see the continuation of Example 2 below). In other words, different regret-free dynamically stable

Therefore, for each such problem, we can construct a marriage market that is isomorphic to its related one-to-one market. Then we can apply a result in [Roth and Vate \[1990\]](#), which says starting from an arbitrary matching, the process of randomly choosing blocking pairs to match will converge to a stable matching with probability one.

matchings can arise in the same final preference profile or in different final preference profiles. Obviously, regret-free dynamic stability inherits all negative results of dynamic stability.

Remark 1: Nullifying *all* those type A blocks has two implications. First, it allows us to make no assumption on who to drop and whom to be dropped. Second, it implicitly assumes universities are conservative and do not expect others to help. In other words, universities believe the market itself will not evolve to conditions that favor them in any way. This prevents gridlocks. Consider a university chooses not to drop any students because it expects other universities would drop and in hope that the market would evolve to conditions where it can keep some of these students. Such argument could be applied to any university, as a result, there might be a gridlock where all of them abstain from dropping students.

Remark 2: Nullification implicitly assumes universities never look back, i.e., once a type A block is nullified, it would not be resumed. This prevents cycles; otherwise, if universities have short memories and the market somehow evolves back to the condition where the student was dropped, then the university would need to drop him again, and thus the matching might never converge to a dynamically stable matching. For instance, in Example 1, the market might go from μ to \emptyset as u_1 drops s_1 but then goes from \emptyset to μ as u_1 forgets the past and admit s_1 back again.

Remark 3: The set of dynamically stable matchings is the intersection of the set of unblocked matchings and the set of individually rational matchings, and this intersection might be empty. Essentially, regret-free dynamic stability weakens unblockness by searching individually rational matchings that are unblocked in the updated preference profiles. Since the set of individually rational matchings is always non-empty (the empty matching is always inside), the set of regret-free dynamically stable matchings is also nonempty. In Example 1, the empty matching is the only individually rational matching and so it is the unique regret-free dynamically stable matching.

To show the existence of a regret-free dynamically stable matching, we use the **regret-free plan deferred acceptance algorithm (RF-PDA):**

Start with the empty matching μ , where $\mu(i) = (\emptyset, \emptyset)$ for all $i \in S \cup U$.

Round $r = 1$:

Step 0: Construct the *reduced* students' preference profile $\underline{\succ}_S$.

Step $n \geq 1$: Each student proposes his most preferred plan that has not been rejected according to $\underline{\succ}_S$. Each university keeps the most preferred students subject to its quotas and rejects all other students. Pauses when no more proposal is made. Denote this tentative matching by $\mu^{(0)}$.

If $\nexists (s, u') \in S \times U$ such that $(\mu^{(0),1}(s), u') P_s \mu^{(0)}(s)$ and $s P_{u'} \sigma$ for some $\sigma \in \mu^{(0),2}(u') \setminus \mu^{(0),1}(u')$ under $(\underline{\succ}_S, \underline{\succ}_U)$, terminates. Otherwise, for each of these students s with $u = \mu^{(0),1}(s)$, update $\underline{\succ}_S$ to $\underline{\succ}_S^{(1)}$ such that $\emptyset P_s^{(1)} uu$ and otherwise preserves P_s . Let $\underline{\succ}_S^{(1)}$ denote the new students' preference profile, proceed to the next round.

Round $r \geq 2$:

Step 0: Construct the *reduced* students' preference profile $\underline{\succ}_S^{(r)}$.

Step $n \geq 1$: Each student proposes his most preferred plan that has not been rejected according to $\underline{\succ}_S^{(r)}$. Each university keeps the most preferred students subject to its quotas and rejects all other students. Pauses when no more proposal is made. Denote this tentative matching by $\mu^{(r)}$.

If $\nexists (s, u') \in S \times U$ such that $(\mu^{(r),1}(s), u') P_s \mu^{(r)}(s)$ and $s P_{u'} \sigma$ for some $\sigma \in \mu^{(r),2}(u') \setminus \mu^{(r),1}(u')$ under $(\underline{\succ}_S^{(r)}, \underline{\succ}_U)$, terminates. Otherwise, for each of these students s with $u = \mu^{(r),1}(s)$, update $\underline{\succ}_S^{(r)}$ to $\underline{\succ}_S^{(r+1)}$ such that $\emptyset P_s^{(r+1)} uu$ and otherwise preserves $P_s^{(r)}$. Let $\underline{\succ}_S^{(r+1)}$ denote the new students' preference profile, proceed to the next round.

Denote the final matching by μ_{RF} . This algorithm terminates in finite rounds and finite steps.

Theorem 10: The set of regret-free dynamically stable matchings is nonempty.

By Lemma 1, μ_{RF} is dynamically stable in $(S, U, q^1, q^2, \underline{\succ}_S^{(N)}, \underline{\succ}_U)$. It is not period-2 blocked under $(\underline{\succ}_S^{(N)}, \underline{\succ}_U)$ because otherwise, RF-PDA, which mimics market evolutions, would not terminate and μ_{RF} would not be the final matching. Therefore, μ_{RF} is a regret-free dynamically stable matching.

Back to Example 2, $\mu^{(0)}$ is a proposed matching under \succsim_S , but it is period-2 blocked by u_2 and $s_1 \in S^{(0)}$. u_1 will nullify the type A block $(s_1, u_1 u_1)$, so P_{s_1} is updated to $P_{s_1}^{(1)} : u_1 u_2, \emptyset$. (Students in the \emptyset column are unmatched.)

$$\mu^{(1)} = \begin{pmatrix} u_1 & u_2 & c & \emptyset \\ \emptyset & \emptyset & s_2, s_3 & s_1 \\ s_3 & s_2 & \emptyset & s_1 \end{pmatrix}$$

then is a regret-free dynamically stable matching as it is dynamically stable under $(\succsim_S^{(1)}, \succsim_U)$.

Consider another regret-free dynamically stable matching

$$\mu' = \begin{pmatrix} u_1 & u_2 & c \\ s_1 & \emptyset & s_2, s_3 \\ s_1, s_2 & s_3 & \emptyset \end{pmatrix}.$$

It is a dynamically stable matching under (\succsim_S, \succsim_U) . Note that $(s_1, u_1 u_1)$ is nullified on the path to $\mu^{(1)}$, but it is not nullified on the path to μ' . If we run RF-PDA in this market, $\mu^{(0)}$ will be obtained in round 1, and $\mu^{(1)} = \mu_{RF}$ will be obtained in round 2.

We extend the definition of student-optimality to regret-free dynamically stable matchings. The following proposition follows from observing $\mu^{(1)}$ and μ' are the only regret-free dynamically stable matchings in the market of Example 2, and s_1 prefers the latter, while other students prefer the former.

Proposition 5: There might not exist a student-optimal regret-free dynamically stable matching.

A similar result can be found in [Abizada \[2016\]](#) and [Abizada and Dur \[2017\]](#), which study college admissions with budget constraints and college admissions with complementarities, respectively. They show that a student-optimal stable allocation might not exist in their models.⁴⁰

⁴⁰In [Kominers and Sönmez \[2016\]](#), they found that an agent-optimal matching might not exist in their matching

Corollary 2: The set of regret-free dynamically stable matchings does not form a lattice under $>_s$.

Corollary 1 is a direct result of Proposition 4. We extend the definition of stability to mechanisms that produce regret-free dynamically stable matchings. We define the RF-PDA mechanism to be a mechanism that uses the RF-PDA algorithm to produce a matching for each input. By Theorem 10, the RF-PDA mechanism is stable.

Proposition 6: The RF-PDA mechanism is not strategy-proof for students.

Proof: In Example 1, the dominant strategy for s_1 is to state $P'_{s_1} : u_2u_2$ instead of his true preference $P_{s_1} : u_2u_1$. The only regret-free dynamically stable matching under $(\succ'_{s_1}, \succ_{-s_1})$ matches s_1 to u_2u_2 , but s_1 can then transfer to u_1 . \square

1.6 Conclusion

We have studied a dynamic two-sided many-to-one matching model that generalizes the college admissions problem. When there is no preference restriction, a dynamically stable matching fails to exist because individual rationality might be incompatible with unblockness. We propose conditions that restrict the preference profile of one side of the market to resolve this conflict (A1, A3 for students and A2, A4 for universities). Thus, under any of these conditions, a dynamically stable matching always exists.

Without the above conditions, we resolve this conflict endogenously. We assume a university can endogenously choose not to admit some acceptable student in the first period. Agents' preferences thus become endogenous. The final matching depends on the realized proposed matchings and is individually rational under the original preference profile and unblocked under the final preference profile. A regret-free dynamically stable matching always exists and there might be other regret-free dynamic stable matchings in the same final preference profile or in different final preference profiles.

with slot-specific priorities model.

1.7 Appendix

1.7.1 Additional Results

1.7.1.1 Negative Results

Theorem A.1: In a college admissions problem, if universities and students have strict preference over individuals, then universities have strict preferences over those groups of students that they may be assigned at stable matchings.⁴¹

This implies if μ and μ' are two different stable matchings, then there does not exist a responsive preference that is indifferent between $\mu(u)$ and $\mu'(u)$. This is Theorem 3 of [Roth and Sotomayor \[1989\]](#). They use the following example to illustrate the result: consider a university u with a quota of 2 and $P_u : s_1, s_2, s_3, s_4$. Let $\mu(u) = \{s_1, s_4\}$ and $\mu'(u) = \{s_2, s_3\}$. Then they cannot both be stable because $s_1 P_u s_2$ but $s_4 R_u s_3$ and $s_3 P_u s_4$ but $s_2 R_u s_1$ (we cannot compare these two sets of students by using responsiveness).

Proposition A.1: Under A3, the result of Theorem A.1 does not hold in the dynamic college admissions problem.

Proof: We prove it by an example.

Example A.1. Consider the following market. $S = \{s_1, s_2, s_3, s_4\}$, $U = \{u_1, u_2\}$,

$$P_{u_1} : s_1, s_2, s_3, s_4 \quad q_{u_1} = (1, 1),$$

$$P_{u_2} : s_4, s_3, s_2, s_1 \quad q_{u_2} = (1, 1),$$

$$P_{s_1} : u_2 u_2, u_1 u_1,$$

$$P_{s_2} : u_1 u_1, u_2 u_2,$$

$$P_{s_3} : c u_2, c u_1,$$

⁴¹Alternative statement of Theorem A.1 [[Roth and Sotomayor, 1992](#)]: In a college admissions problem, if μ and μ' are two stable matchings, and u is a university with $q_u = k$ such that $\mu(u) \neq \mu'(u)$, $\mu(u) = \{s_1, s_2, \dots, s_k\}$ and $\mu'(u) = \{s'_1, s'_2, \dots, s'_k\}$, where the students are listed in order of P_u , that is, $\forall i, s_i P_u s_{i+1}$ and $s'_i P_u s'_{i+1}$. If i is any index such that $s_i P_u s'_i$, then $s_\gamma R_u s'_\gamma$ for all $\gamma \in \{1, 2, \dots, k\}$ and $\mu(u) R_u^* \mu'(u)$.

$P_{s_4} : cu_1, cu_2.$

In this market,

$$\mu = \begin{pmatrix} u_1 & u_2 & c \\ s_1 & s_2 & s_3, s_4 \\ s_1, s_4 & s_2, s_3 & \emptyset \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} u_1 & u_2 & c \\ s_2 & s_1 & s_3, s_4 \\ s_2, s_3 & s_1, s_4 & \emptyset \end{pmatrix},$$

so both $\mu^2(u_1) = \{s_1, s_4\}$ and $\mu'^2(u_1) = \{s_2, s_3\}$ are now dynamically stable outcomes for u_1 . However, $s_1 P_u s_2$ but $s_4 \not R_u s_3$ and $s_3 P_u s_4$ but $s_2 \not R_u s_1$, which contradicts the conclusion of Theorem 1*. \square

Theorem A.2: In a college admissions problem, if μ and μ' are both stable matchings and $\mu(u) R_u^* \mu'(u)$ for some u , then $s P_u s'$ for all $s \in \mu(u)$ and $s' \in \mu'(u) \setminus \mu(u)$.

In words, u prefers every student in $\mu(u)$ to every student in $\mu'(u)$ but not in $\mu(u)$. This is Theorem 4 of [Roth and Sotomayor \[1989\]](#). They use the following example to illustrate the result: consider a university u with a quota of 2 and $P_u : s_1, s_2, s_3, s_4$. Let $\mu(u) = \{s_1, s_3\}$ and $\mu'(u) = \{s_2, s_4\}$. Note that we cannot use Theorem A.1 to determine whether they are both stable or not. Obviously, $\mu(u) R_u^* \mu'(u)$ by responsiveness and transitivity. By Theorem A.2, they cannot both be stable since $s_3 \not P_u s_2$. Corollary A.1 follows immediately from Theorem A.2 and responsiveness.

Corollary A.1: In a college admissions problem, if R_u^* and $R_u^{*'}$ are responsive to P_u , then for every pair of stable matchings μ and μ' , $\mu(u) R_u^* \mu'(u)$ if and only if $\mu(u) R_u^{*'} \mu'(u)$.

Proposition A.2: Under A3, the results of Theorem A.2 and Corollary A.1 do not hold in the dynamic college admissions problem.

Proof: Consider Example A.1 again. In that market,

$$\mu_S = \begin{pmatrix} u_1 & u_2 & c \\ s_2 & s_1 & s_3, s_4 \\ s_2, s_4 & s_1, s_3 & \emptyset \end{pmatrix} \text{ and } \mu_U = \begin{pmatrix} u_1 & u_2 & c \\ s_1 & s_2 & s_3, s_4 \\ s_1, s_3 & s_2, s_4 & \emptyset \end{pmatrix},$$

so both $\mu_S^2(u_1) = \{s_2, s_4\}$ and $\mu_U^2(u_1) = \{s_1, s_3\}$ are now dynamically stable outcomes for u_1 . However, $s_3 \not P_u s_2$, which contradicts the conclusion of Theorem A.2.

Now consider

$$R_{u_1}^* : \{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4$$

and

$$R_{u_1}' : \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4.$$

Both of them are responsive to P_{u_1} , and we have $\mu^2(u_1) = \{s_1, s_4\} R_{u_1}^* \{s_2, s_3\} = \mu'^2(u_1)$ and $\mu^{2'}(u_1) = \{s_2, s_3\} R_{u_1}' \{s_1, s_4\} = \mu^2(u_1)$, which contradicts the conclusion of Corollary A.1. \square

The following three theorems can be found in 5.6.3 of [Roth and Sotomayor \[1992\]](#). Theorem A.3 is parallel to Theorem B.2 in the related one-to-one market.

Theorem A.3: In a college admissions problem, if μ and μ' are stable matchings, then $\mu >_S \mu'$ if and only if $\mu' >_U \mu$.

Proposition A.3: Under A3, the result of Theorem A.3 does not hold in the dynamic college admissions problem.

Proof: Consider μ and μ' in Example A.1 again with

$$R_{u_1}^* : \{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4$$

and

$$R_{u_2}^* : \{s_3, s_4\}, \{s_2, s_4\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_1, s_3\}, \{s_1, s_2\}, s_4, s_3, s_2, s_1.$$

Now $\mu >_U \mu'$ but $\mu' \not>_S \mu$ since s_3 and s_4 prefer μ over μ' , which contradicts the conclusion of Theorem A.3. \square

If μ and μ' are matchings, we define

$$\lambda(u) = \mu \vee_U \mu' = \begin{cases} \mu(u) & \text{if } \mu^2(u) R_u^* \mu'^2(u) \\ \mu'(u) & \text{otherwise} \end{cases}, \lambda(s) = \mu \wedge_S \mu' = \begin{cases} \mu(s) & \text{if } \mu'(s) P_s \mu(s) \\ \mu'(s) & \text{otherwise} \end{cases};$$

similarly,

$$v(u) = \mu \wedge_U \mu' = \begin{cases} \mu(u) & \text{if } \mu'^2(u) R_u^* \mu^2(u) \\ \mu'(u) & \text{otherwise} \end{cases}, v(s) = \mu \vee_S \mu' = \begin{cases} \mu(s) & \text{if } \mu(s) P_s \mu'(s) \\ \mu'(s) & \text{otherwise} \end{cases}.$$

If λ is an acceptable matching, then λ is the least upper bound for $\{\mu, \mu'\}$ under $>_U$ and the greatest lower bound for $\{\mu, \mu'\}$ under $>_S$.⁴² Theorem A.4 is parallel to Theorem C.3 in the related one-to-one market.

Theorem A.4: In a college admissions problem, if μ and μ' are stable matchings, then λ and v are stable matchings.

Proposition A.4: Under A3, the result of Theorem A.4 does not hold in the dynamic college admissions problem.

Proof: Consider μ and μ' in Example A.1 again with

$$R_{u_1}^* : \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4,$$

⁴²Note that we put acceptability in the statement, which is not needed in the college admissions problem.

and

$$R_{u_2}^* : \{s_3, s_4\}, \{s_2, s_4\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_1, s_3\}, \{s_1, s_2\}, s_4, s_3, s_2, s_1.$$

$\lambda = \mu \vee_U \mu' (\neq \mu_U, \text{ but } \mu_U = \mu \wedge_S \mu')$ is not a matching since both of them want $\{s_2, s_3\}$. This contradicts the conclusions of Theorem A.4. \square

Theorem A.5: In a college admissions problem, if μ and μ' are stable matchings and $u = \mu(s)$ or $u = \mu'(s)$, then if $\mu(u)P_u^* \mu'(u)$ then $\mu'(s)R_s \mu(s)$ (and if $\mu'(s)P_s \mu(s)$ then $\mu(u)R_u^* \mu'(u)$).

Theorem A.5 is an analogue of Lemma C.1 in the related one-to-one market, and Theorem 8 is the corresponding result in our model.

Proposition A.5: Under A3, the result of Theorem A.5 does not hold in the dynamic college admissions problem.

Proof: Consider μ and μ' in Example A.1 again with

$$R_{u_1}^* : \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}, s_1, s_2, s_3, s_4.$$

$\{s_2, s_3\} = \mu'^2(u_1)R_{u_1}^* \mu^2(u_1) = \{s_1, s_4\}$ and $\mu'(s_2) = u_1 u_1 P_{s_2} \mu(s_2) = u_2 u_2$, which contradicts the conclusion of Theorem A.5.⁴³ \square

The above results fail because when considering two dynamically stable outcomes for a university, $\mu^2(u)$ and $\mu'^2(u)$, one outcome can be from a matching consists of a superior period-1 matching and an inferior period-2 matching, while another outcome comes from a matching consists of an inferior period-1 matching and a superior period-2 matching.

Corollary A.2: In a college admissions problem, the set of stable matchings forms a lattice under $>_U$ and $>_S$, and the lattice under $>_U$ is the dual to the lattice under $>_S$.

⁴³ $\mu'(s_3) = cu_2 P_{s_3} \mu(s_3) = cu_1$, so there might exist two dynamically stable matchings μ and μ' such that $\mu(s)P_s \mu'(s)$ for some $s \in \mu^1(u)$ and $\mu'(s')P_{s'} \mu(s')$ for some $s' \in \mu^2(u) \setminus \mu^1(u)$, while $\mu'(u)P_u^* \mu(u)$. For the opposite, consider s_1, s_4 , and u_2 with $\{s_1, s_4\}R_{u_2}^* \{s_2, s_3\}$. Both u_2 and $s_4 \in \mu^2(u_2) \setminus \mu^1(u_2)$ prefer μ , while $s_1 \in \mu^1(u_2)$ prefers μ' .

Corollary A.2 follows from Theorem A.3 and Theorem A.4, and it is parallel to Corollary C.2 in the related one-to-one market.

Proposition A.6: Under A3, the result of Corollary A.2 does not hold in the dynamic college admissions problem.

Proof: We prove it by an example.

Example A.2. Consider the following market. $S = \{s_1, s_2, s_3, s_4, s_5\}$, $U = \{u_1, u_2, u_3\}$,

$$P_{u_1} : s_1, s_2, s_3, s_4, s_5 \quad q_{u_1} = (1, 1),$$

$$P_{u_2} : s_2, s_1, s_4, s_5, s_3 \quad q_{u_2} = (1, 1),$$

$$P_{u_3} : s_5, s_3, s_4 \quad q_{u_3} = (0, 1),$$

$$P_{s_1} : u_2 u_2, u_1 u_1,$$

$$P_{s_2} : u_1 u_1, u_2 u_2,$$

$$P_{s_3} : c u_2, c u_3, c u_1,$$

$$P_{s_4} : c u_3, c u_1, c u_2,$$

$$P_{s_5} : c u_1, c u_2, c u_3.$$

In this market, there are six dynamically stable matchings:

$$\mu_S = \begin{pmatrix} u_1 & u_2 & u_3 & c \\ s_2 & s_1 & \emptyset & s_3, s_4, s_5 \\ s_2, s_5 & s_1, s_3 & s_4 & \emptyset \end{pmatrix}, \mu_1 = \begin{pmatrix} u_1 & u_2 & u_3 & c \\ s_1 & s_2 & \emptyset & s_3, s_4, s_5 \\ s_1, s_5 & s_2, s_3 & s_4 & \emptyset \end{pmatrix},$$

$$\mu_2 = \begin{pmatrix} u_1 & u_2 & u_3 & c \\ s_2 & s_1 & \emptyset & s_3, s_4, s_5 \\ s_2, s_4 & s_1, s_5 & s_3 & \emptyset \end{pmatrix}, \mu_3 = \begin{pmatrix} u_1 & u_2 & u_3 & c \\ s_2 & s_1 & \emptyset & s_3, s_4, s_5 \\ s_1, s_4 & s_2, s_5 & s_3 & \emptyset \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} u_1 & u_2 & u_3 & c \\ s_2 & s_1 & \emptyset & s_3, s_4, s_5 \\ s_2, s_3 & s_1, s_4 & s_5 & \emptyset \end{pmatrix}, \text{ and } \mu_U = \begin{pmatrix} u_1 & u_2 & u_3 & c \\ s_2 & s_1 & \emptyset & s_3, s_4, s_5 \\ s_1, s_3 & s_2, s_4 & s_5 & \emptyset \end{pmatrix}.$$

Consider these matchings with

$$R_{u_1}^* : \dots, \{s_1, s_4\}, \dots, \{s_2, s_3\}, \dots, \{s_2, s_4\}, \dots, \{s_1, s_5\}, \dots, s_1, s_2, s_3, s_4, s_5$$

and

$$R_{u_2}^* : \dots, \{s_1, s_4\}, \dots, \{s_2, s_3\}, \dots, s_2, s_1, s_4, s_5, s_3$$

(we put more restrictions on $R_{u_1}^*$ and leave $R_{u_2}^*$ more flexible. Many $R_{u_1}^*, R_{u_2}^*$ pairs can work here).

Then the set of dynamically stable matchings under $>_U$ is as shown in Figure 1.1a. $\mu_4 >_U \mu_1$ because $\{s_2, s_3\}R_{u_1}^* \{s_1, s_5\}$, $\{s_1, s_4\}R_{u_2}^* \{s_2, s_3\}$, and $s_5P_{u_3}s_4$. μ_1 and μ_2 is not comparable under $>_U$ because $\{s_2, s_4\}R_{u_1}^* \{s_1, s_5\}$ but $s_3P_{u_3}s_4$. Similarly, μ_3 and μ_4 is not comparable under $>_U$ because $\{s_1, s_4\}R_{u_1}^* \{s_2, s_3\}$ but $s_5P_{u_3}s_4$. The readers can verified the rest by using responsiveness and transitivity.

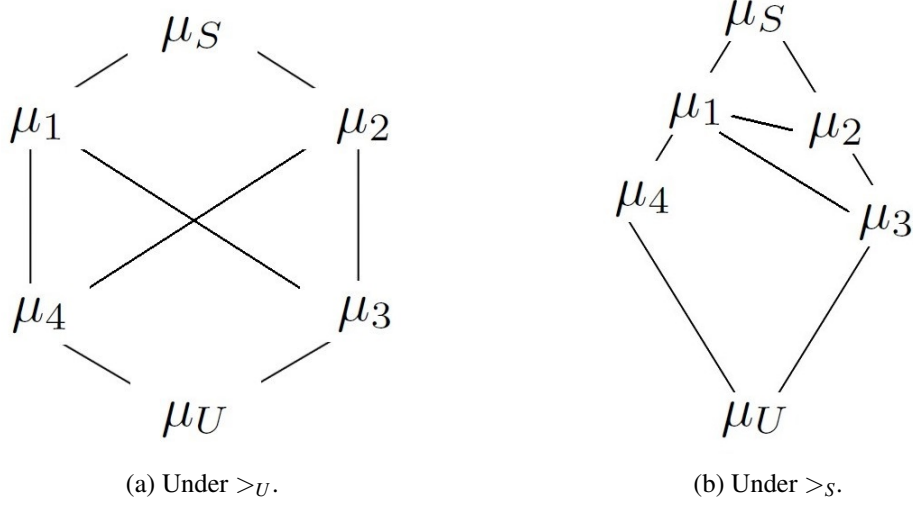


Figure 1.1: The set of dynamically stable matchings.

Therefore, μ_1 and μ_2 have common upper bounds μ_3 , μ_4 , and μ_U , but none of them is the least upper bound, so the set of dynamically stable matchings does not form a lattice under $>_U$. \square

By checking the meets and joins under $>_S$, the readers can also verify the set of dynamically stable matchings indeed forms a lattice under $>_S$ as shown in Figure 1.1b.

1.7.1.2 Non-Pairwise Solution Concepts

A matching μ is **weakly period-1 dominated** by another matching $\mu' = (\mu_B^1, \mu_B^2)$ via a coalition $B \subseteq S \cup U$ if (i) $\forall l \in B$, $i \in \mu_B^1(l)$ implies $l \in \mu_B^1(i)$ and $i \in B$, (ii) $\forall l \in B$, $\mu' \succ_l \mu$, and (iii) $\exists l \in B$ such that $\mu' \succ_l \mu$. A matching μ is **weakly period-2 dominated** by another matching $\mu' = (\mu^1, \mu_B^2)$ via a coalition $B \subseteq S \cup U$ if (i) $\forall s \in B$, $\mu_B^2(s) = u$ implies $s \in \mu_B^2(u) \setminus \mu^1(u)$ and $u \in B$, (ii) $\forall u \in B$, $s \in \mu_B^2(u) \setminus \mu^1(u)$ implies $\mu_B^2(s) = u$ and $s \in B$, (iii) $\forall l \in B$, $\mu' \succ_l \mu$, and (iv) $\exists l \in B$ such that $\mu' \succ_l \mu$. The set of matchings that are not weakly period- t dominated by any other matching is the

dynamic core (defined by weak domination).⁴⁴ It is easy to see the dynamic core is inside the Pareto set.⁴⁵ Theorem A.6 generalizes a result which says a matching is in the core if and only if it is stable in a college admissions problem.⁴⁶

Theorem A.6: Under A1, a matching is in the dynamic core if and only if it is dynamically stable.

Proof of Theorem A.6: (\implies) If μ is not dynamically stable via a single agent because μ is not individually rational, then it is clearly not in the dynamic weak core since it is weakly period-1 dominated by any matching μ' with $\mu'(s) = \emptyset$ via a coalition $B = \{s\}$ or by any matching μ' with $\mu'^2(u) = \mu^2(u) \setminus S'$ via a coalition $B = u \cup \mu^2(u) \setminus S'$ for some S' such that $\emptyset P_u s$ for all $s \in S'$.

If μ is not dynamically stable via a pair of student and university (s, u) with $s P_u \sigma$ for some $\sigma \in \mu^1(u)$ or $\mu^2(u) \setminus \mu^1(u)$. Suppose μ is period-1 blocked, then it is weakly period-1 dominated by any matching μ' with $\mu'(s) = uu$, $\mu'^1(u) = s \cup \mu^1(u) \setminus \sigma$, and $\mu'^2(u) \setminus \mu'^1(u) = \mu^2(u) \setminus \mu^1(u)$ via a coalition $B = u \cup s \cup \mu^2(u) \setminus \sigma$. Similarly, suppose μ is period-2 blocked, then it is weakly period-2 dominated by any matching μ' with $\mu'^2(s) = u$ and $\mu'^2(u) \setminus \mu^1(u) = s \cup \mu^2(u) \setminus \mu^1(u) \setminus \sigma$.

(\impliedby) Suppose μ is not in the dynamic weak core. Then μ is weakly period-1 or period-2 dominated by some matching μ' via some coalition B , and hence some student or university prefer μ' to μ . If μ is not individually rational, then it is not dynamically stable. Now, suppose μ satisfies individual rationality.

(A) Suppose μ is weakly period-2 dominated by some matching $\mu' = (\mu^1, \mu_B^2)$ via some coalition B .

⁴⁴In the dynamic matching literature, the core equivalent is sometimes called “recursive core” [Damiano and Lam, 2005].

⁴⁵A matching μ is **Pareto dominated** by μ' if (i) $\forall l \in S \cup U$, $\mu' \succ_l \mu$, and (ii) $\exists l \in S \cup U$ such that $\mu' \succ_l \mu$. A matching μ is **Pareto efficient** if it is not Pareto dominated by any other matching, and we call the set of matchings that are not Pareto dominated by any other matching the **Pareto set**.

⁴⁶This is Proposition 5.36 of Roth and Sotomayor [1992]. A matching μ is **weakly dominated** by another matching μ' via a coalition $B \subseteq S \cup U$ if (i) $\forall l \in B$, $i \in \mu'(l)$ implies $i \in B$, (ii) $\forall l \in B$, $\mu' \succ_l \mu$, and (iii) $\exists l \in B$ such that $\mu' \succ_l \mu$. The set of matchings that are not weakly dominated by any other matching is the **core (defined by weak domination)**. We do not define dynamic core defined by *strong* domination because it is a generalization of the core defined by strong domination, which might contain unstable outcomes in the college admissions problem (see 5.7 of Roth and Sotomayor [1992]).

(A.1) Suppose $\mu_B^2(u) \setminus \mu^1(u) P_u^2 \mu^2(u) \setminus \mu^1(u)$ for some $u \in B$. There must exist $s \in \mu_B^2(u) \setminus \mu^1(u) \setminus \mu^2(u)$ and $\sigma \in \mu^2(u) \setminus \mu^1(u) \setminus \mu_B^2(u)$ such that $s P_u \sigma$; otherwise, $\sigma R_u s \forall s \in \mu_B^2(u) \setminus \mu^1(u) \setminus \mu^2(u)$ implies $\mu^2(u) \setminus \mu^1(u) P_u^2 \mu_B^2(u) \setminus \mu^1(u)$ by responsiveness. By definition, $s \in \mu_B^2(u) \setminus \mu^1(u)$ implies $s \in B$, so $u = \mu'(s) R_s \mu(s)$; furthermore, $s \notin \mu^2(u) \setminus \mu^1(u)$ and $s \in \mu_B^2(u) \setminus \mu^1(u)$, so $\mu'(s) = (\mu^1(s), \mu_B^2(s)) \neq (\mu^1(s), \mu^2(s)) = \mu(s)$. Therefore, $\mu'(s) P_s \mu(s)$. Hence, μ it is period-2 blocked by (s, u) .

(A.2) Suppose $\mu'(s) P_s \mu(s)$ for some $s \in B$ with $\mu_B^2(s) = u$, this implies $u \in B$, so $\mu_B^2(u) \setminus \mu^1(u) R_u^2 \mu^2(u) \setminus \mu^1(u)$. $\mu_B^2(s) = u$ with $\mu'(s) P_s \mu(s)$ also implies $\mu^2(s) \neq u$ (a student cannot switch between cu and uu given μ^1), so $\mu_B^2(u) \setminus \mu^1(u) \neq \mu^2(u) \setminus \mu^1(u)$. Hence, $\mu_B^2(u) \setminus \mu^1(u) P_u^2 \mu^2(u) \setminus \mu^1(u)$. This implies that there is a student $s' \in \mu_B^2(u) \setminus \mu^1(u) \setminus \mu^2(u)$ (possibly different from s) and $\sigma \in \mu^2(u) \setminus \mu^1(u) \setminus \mu_B^2(u)$ such that $s' P_u \sigma$. Then μ is period-2 blocked by (s', u) .

(B) Suppose μ instead is weakly period-1 dominated by some matching $\mu' = (\mu_B^1, \mu_B^2)$ via some coalition B .

(B.1) Suppose $\mu_B^2(u) P_u^* \mu^2(u)$ for some $u \in B$. There must exist $s \in \mu_B^2(u) \setminus \mu^2(u)$ and $\sigma \in \mu^2(u) \setminus \mu_B^2(u)$ such that $s P_u \sigma$; otherwise, $\sigma R_u s \forall s \in \mu_B^2(u) \setminus \mu^2(u)$ implies $\mu^2(u) P_u^* \mu_B^2(u)$ by responsiveness. By definition, $s \in \mu_B^2(u)$ implies $s \in B$, so $\mu'(s) R_s \mu(s)$; furthermore, $s \in \mu_B^2(u) \setminus \mu^2(u)$ implies $\mu'(s) \neq \mu(s)$. Therefore, $\mu'(s) P_s \mu(s)$. Hence, μ it is period-1 or period-2 blocked by (s, u) .

(B.2) Suppose $\mu'(s) P_s \mu(s)$ for some $s \in B$ with $\mu_B^2(s) = u$, this implies $u \in B$, so $\mu_B^2(u) R_u^* \mu^2(u)$.

(B.2.1) If $\mu_B^2(u) \neq \mu^2(u)$, then $\mu_B^2(u) R_u^* \mu^2(u)$. This in turn implies there is a student $s' \in \mu_B^2(u) \setminus \mu^2(u)$ (possibly different from s) and $\sigma \in \mu^2(u) \setminus \mu_B^2(u)$ such that $s' P_u \sigma$. Then μ is period-1 or period-2 blocked by (s', u) .

(B.2.2) If $\mu_B^2(u) = \mu^2(u)$, then it means s switches between cu and uu . Without loss of generality, assume $uu P_s cu$. So $\mu'(s) = uu$, then either there is an unmatched period-1 position under μ or there is no unmatched period-1 position under μ but there exists some student s' such that

$\mu'(s') = cuP_{s'}uu = \mu(s')$ and he switches his seat with s (note that $s' \in B$, so $\mu'(s')R_{s'}\mu(s')$, since there is no more period-1 seat for s under μ , given $\mu_B^2(u) = \mu^2(u)$, s' has to move to cu under μ' , this implies $\mu'(s') = cuP_{s'}uu = \mu(s')$). In the first case, μ is period-1 blocked by (s, u) , and in the second case, μ is period-1 blocked by (s, u) or (s', u) . \square

[Roth and Sotomayor \[1989\]](#) define another notion of multilateral blocking, group stability, that does not require the coalition members to match only with other members of the coalition after a deviation. We extend this definition to dynamic two-sided many-to-one matching problems. A **period-1 group deviation** from μ is a group $B \subseteq S \cup U$ and some matching $\mu' = (\mu_B^1, \mu_B^2)$ such that (i) $\forall s \in B, \mu_B^2(s) \in B$, (ii) $\forall u \in B, \sigma \in \mu_B^2(u) \implies \sigma \in B \cup \mu^2(u)$, and (iii) $\forall l \in B, \mu' \succ_l \mu$. A **period-2 group deviation** from μ is a group $B \subseteq S \cup U$ and a matching $\mu' = (\mu^1, \mu_B^2)$ such that (i) $\forall s \in B, \mu_B^2(s) \in B$, (ii) $\forall u \in B, \sigma \in \mu_B^2(u) \implies \sigma \in B \cup \mu^2(u) \setminus \mu^1(u)$, and (iii) $\forall l \in B, \mu' \succ_l \mu$. A matching μ is **dynamically group stable** if it is immune to any period- t group deviation from μ .⁴⁷

We provide two results for this solution concept — one under A3 and another under A1. Theorem A.7 below generalizes a result which says a matching is group stable if and only if it is stable in a college admissions problem.⁴⁸

Theorem A.7: Under A3, a matching is dynamically group stable if and only if it is dynamically stable.

Proof of Theorem A.7: Suppose μ is not dynamically stable. If μ is not individually rational via a student or university or it is blocked by a student-university pair, then it is not dynamically group stable via the group consisting of the agent or pair.

Suppose μ is not dynamically group stable. If there is a (period-1 or period-2) group deviation

⁴⁷This is a different concept to the dynamic group stability in [Kurino \[2009\]](#), which is defined in a dynamic marriage market.

⁴⁸This result is Proposition 1 of [Roth and Sotomayor \[1989\]](#). A **group deviation** from μ is a group $B \subseteq S \cup U$ and a matching μ' such that (i) $\forall s \in B, \mu'(s) \in B$, (ii) $\forall u \in B, \sigma \in \mu'(u) \implies \sigma \in B \cup \mu(u)$, and (iii) $\forall i \in B, \mu' \succ_i \mu$. A matching μ is **group stable** if it is immune to any group deviation from μ .

[Konishi and Ünver \[2006a\]](#) defines a more general definition for many-to-many matching problems.

from μ via group B . Let u be in B . Under A3, $\mu'(u)R_u^*\mu(u)$ implies that there exists in $s \in \mu'(u) \setminus \mu(u)$ and $\sigma \in \mu(u) \setminus \mu'(u)$ such that $sP_u\sigma$. So s is in B and prefers to match with u instead of $\mu(s)$, so (s, u) (period-1 or period-2) blocks μ . \square

Corollary A.3: Under A3, the set of dynamically stable matchings, the dynamic core and the set of the set of dynamically stable matchings coincide with each other.⁴⁹

Corollary A.3 follows from Theorem A.6 and Theorem A.7. Indeed, Theorem A.7 holds because A3 eliminates all externalities. Recall dynamic stability assumes agents are myopic, universities maximize their outcome seat-by-seat (when joins with responsiveness) and do not utilize the externalities that they can impose on others. Dynamic group stability assumes none of these.

Proposition A.8: Under A1, the set of dynamically group stable matchings might not coincide with the set of dynamically stable matchings.

Proof: We prove it by an example.

Example A.3. $S = \{s_1, s_2\}$, $U = \{u_1\}$,

$$P_{u_1} : s_1, s_2 \ q_{u_1} = (1, 1),$$

$$P_{s_1} : u_1 u_1, c u_1,$$

$$P_{s_2} : u_1 u_1.$$

It is easy to check that the unique dynamically stable matching and the unique element in the dynamic core is:

$$\mu = \begin{pmatrix} u_1 & c & \emptyset \\ s_1 & \emptyset & s_2 \\ s_1 & \emptyset & s_2 \end{pmatrix}.$$

⁴⁹In a college admissions problem, by the results in footnote 46 and 48, the set of stable matchings, the core and the set of group stable matchings coincide with each other.

Yet, the unique dynamically group stable matching is

$$\mu' = \begin{pmatrix} u_1 & c \\ s_2 & s_1 \\ s_1, s_2 & \emptyset \end{pmatrix}.$$

There is a period-1 group deviation from μ with $B = \{u_1, s_2\}$ and μ' . □

Note that μ'^2 is an unblocked period-2 matching given μ'^1 , but μ'^1 is not period-1 unblocked.⁵⁰

Example A.2 shows that when a university is not myopic (u_1 anticipates μ' is the outcome if it deviates with s_2 from μ), does not maximize its outcome seat-by-seat (u_1 chooses not to period-1 block with s_1) and utilize the externalities that it can impose on others (u_1 forces s_1 to go to community college), it can improve its outcome. However, it is easy to see that any mechanism that produces μ' is not strategy-proof as s_1 can use a truncation strategy [Roth and Rothblum, 1999] — reports $\widehat{P}_{s_1} : u_1 u_1$.

Indeed, μ' might be also a predicted outcome of some farsighted solution concepts based on indirect dominance because μ' indirectly dominates μ , where μ is a predicted outcome of myopic solution concepts based on direct dominance — dynamic stability and dynamic core.⁵¹

⁵⁰Unstable matchings might arise even when agents are myopic. See [Diamantoudi et al. \[2015\]](#).

⁵¹The indirect dominance relation is introduced by [Harsanyi \[1974\]](#), and it has been formalized by [Chwe \[1994\]](#). It captures the idea that Farsighted coalition would consider the final outcomes induced by their deviations.

A matching μ is indirectly dominated by μ' if there exists a sequence of matching $\mu^0, \mu^1, \dots, \mu^Z$ (where $\mu^0 = \mu$ and $\mu^Z = \mu'$) and a sequence of coalitions S^0, S^1, \dots, S^{Z-1} such that for any $z \in \{1, \dots, Z\}$, the following conditions hold: (i) $\forall i \in S^{z-1}, \mu^z \succ_i \mu^{z-1}$; (ii) Coalition S^{z-1} can enforce the matching μ^z over μ^{z-1} . Direct dominance is obtained by setting $Z = 1$.

There are different notions of enforceability (see [Mauleon et al. \[2011\]](#)). We take “enforce” literally here. In Example A.2, μ' indirectly dominates μ , with $S^0 = \{u_1\}$, $\mu^1 = \emptyset$, $S^1 = \{u_1, s_2\}$, μ^2 where $\mu^2(s_2) = u_1 u_1$, $S^2 = \{u_1, s_1\}$, and $\mu^3 = \mu'$.

1.7.2 The Related one-to-one Market

Consider a dynamic college admissions problem $(S, U, q^1, q^2, \succ_S, \succ_U)$. We divide each $u \in U$ into q_u^1 period-1 positions and q_u^2 period-2 positions; denote them $u_1^1, u_2^1, \dots, u_{q_1}^1$ and $u_1^2, u_2^2, \dots, u_{q_2}^2$, respectively. Let $\beta \leq q^1$ and $\gamma \leq q^2$. Denote the set of period-1 positions by U^1 , the set of period-2 positions by U^2 , and the set of positions by \bar{U} . A generic element of \bar{U} is denoted by p . Since each position has a quota of 1, we do not need to consider rankings over sets of students but only rankings over students. Let $x \in S \cup \{u\}$. $\forall u \in U$, u in a plan represents self-matching. For each $u^1 \in U^1$, $sP_u x \implies ssP_{u^1} x$.⁵² For each $u^2 \in U^2$, $x \in S$, $sP_u x \implies usP_{u^2} ux$. Each student's ranking over plans is modified by replacing any uu plans by $u_1^1 u_1^1, u_2^1 u_2^1, \dots, u_{q_1}^1 u_{q_1}^1$, then any uy plans by $u_1^1 y, u_2^1 y, \dots, u_{q_1}^1 y$, and finally any yu plans by $yu_1^2, yu_2^2, \dots, yu_{q_2}^2$. We use bar to indicate an object is in the related one-to-one market. By construction, a student s 's rankings in the original problem and the related one-to-one market are "different." We use \bar{S} to denote the set of students here. For $i \in \bar{S} \cup \bar{U}$, $\bar{\mu} \succ_i \mu'$ if and only if $\bar{\mu}(i)R_i \mu'(i)$. By an abuse of terminology, we call P_p the preference of p over plans.

The related one-to-one market of $(S, U, q^1, q^2, \succ_S, \succ_U)$ is described by $(\bar{S}, \bar{U}, \succ_{\bar{S}}, \succ_{\bar{U}})$. Let $\bar{\mu}$, with $\bar{\mu}^1(u_\beta^1) \neq \bar{\mu}^2(u_\gamma^2) \forall \beta, \gamma$, be a matching that matches student in $\mu^t(u)$ to the ordered positions of a university according to the ranking of s in P_u ; that is, if s is u 's most preferred student in $\mu^1(u)$, then $\bar{\mu}^1(s) = u_1^1$, and if s is u 's second most preferred student in $\mu^2(u)$, then $\bar{\mu}^2(s) = u_2^2$, etc.⁵³ Let $\bar{\mathcal{M}}$ denote the set of $\bar{\mu}$'s.⁵⁴ Since P_u 's are strict, by defining $\bar{\mu}$ as above, there is a natural bijection between \mathcal{M} and $\bar{\mathcal{M}}$. We call $\bar{\mu}$ the *corresponding matching* of μ .

A matching $\bar{\mu}$ is **individually rational** if $\forall i \in \bar{S} \cup \bar{U}$, $\bar{\mu}(i)R_i \emptyset$. A matching $\bar{\mu}$ is **period-1 blocked** if $\exists (s, p) \in \bar{S} \times \bar{U}$ such that

⁵²See footnote 17 for more discussions.

⁵³ $\bar{\mu}^1(u_\beta^1) \neq \bar{\mu}^2(u_\gamma^2) \forall \beta, \gamma$ here means a student cannot transfer to his own university.

⁵⁴ $\bar{\mathcal{M}}$ is a subset of the set of matchings in $(\bar{S}, \bar{U}, \succ_{\bar{S}}, \succ_{\bar{U}})$.

(**Type A**) $u^1 u^1 P_s \bar{\mu}(s)$ and $ss P_{u^1} \bar{\mu}(u^1)$; or

(**Type B**) $cu^2 P_s \bar{\mu}(s)$ and $us P_{u^2} \bar{\mu}(u^2)$.

A matching $\bar{\mu}$ is **period-2 blocked** if $\exists (s, p) \in \bar{S} \times \bar{U}$ such that $(\bar{\mu}^1(s), u^2) P_s \bar{\mu}(s)$ and $(\bar{\mu}^1(u^2), s) P_{u^2} \bar{\mu}(u^2)$.

Unblockness and dynamic stability are the same as before.

Lemma B.1: A matching μ is dynamically stable if and only if its corresponding matching $\bar{\mu}$ is dynamically stable.

The proof is a straightforward verification, so it is omitted.

1.7.2.1 Alternative Assumption on Market Evolution

Let $(\bar{S}, \bar{U}, \succ_{\bar{S}}, \succ_{\bar{U}})$ be the related one-to-one market of $(S, U, q^1, q^2, \succ_S, \succ_U)$.

Assumption on Market Evolution-2: Students propose plans in the reduced students' preference profile $\succ_{\bar{S}}$, and the market evolves until a proposed matching arises, denote this matching by $\bar{\mu}^{(0)}$.

If there exists a set of student $S^{(0)}$ such that for each $s \in S^{(0)}$, $\bar{\mu}^{(0)}$ is period-2 blocked by s and some u^2 , then for all $u^1 \in \bar{U}$ where some $\bar{\mu}(u^1) = ss$ with $s \in S^{(0)}$, update \succ_{u^1} to $\succ_{u^1}^{(1)}$ such that $\emptyset P_{u^1}^{(1)} ss$ and otherwise preserves P_{u^1} . Let $\succ_{\bar{U}}^{(1)}$ denote the new positions' preference profile.

Students propose plans in $\succ_{\bar{S}}$, and the market evolves until a proposed matching arises, denote this matching by $\bar{\mu}^{(1)}$ Market evolution stops as soon as a proposed matching $\bar{\mu}^{(t)}$ is not period-2 blocked under $(\succ_{\bar{S}}, \succ_{\bar{U}}^{(t)})$, which implies $\bar{\mu}^{(t)}$ is dynamically stable under $(\succ_{\bar{S}}, \succ_{\bar{U}}^{(t)})$. Denote such t by N .

We now redefine the regret-free dynamic stability if this assumption is used. A matching $\bar{\mu}$ is **regret-free dynamically stable** if it satisfies

1. Individual rationality — $\forall i \in \bar{S} \cup \bar{U}, \bar{\mu}(i) R_i \emptyset$; and
2. Unblockness under $(\succ_{\bar{S}}, \succ_{\bar{U}}^{(N)})$.

We now modify RF-PDA. The following algorithm, RF-PDA-2, will find the corresponding matching of μ_{RF} because at each round, the set of nullified plans is the same as the one in RF-PDA.

RF-PDA-2:

Start with the empty matching μ , where $\bar{\mu}(i) = (\emptyset, \emptyset)$ for all $i \in \bar{S} \cup \bar{U}$.

Round $r = 1$:

Step 0: Construct the *reduced* students' preference profile $\underline{\succ}_{\bar{S}}$.

Step $n \geq 1$: Each student proposes his most preferred plan that has not been rejected according to $\underline{\succ}_{\bar{S}}$. Each position keeps the most preferred plan and rejects all others. Pauses when no more proposal is made. Denote this tentative matching by $\bar{\mu}^{(0)}$.

If $\nexists (s, u'^2) \in \bar{S} \times \bar{U}$ such that $(\bar{\mu}^{(0),1}(s), u') P_s \bar{\mu}^{(0)}(s)$ and $(\bar{\mu}^{(0),1}(u'^2), s) P_{u'^2} \bar{\mu}^{(0)}(u'^2)$ under $(\underline{\succ}_{\bar{S}}, \underline{\succ}_{\bar{U}})$, terminates. Otherwise, for all $u^1 \in \bar{U}$ such that some $\bar{\mu}^{(0)}(u^1_\beta) = ss$ where s is one of these students, update $\underline{\succ}_{u^1}$ to $\underline{\succ}_{u^1}^{(1)}$ such that $\emptyset P_{u^1}^{(1)} ss$ and otherwise preserves P_{u^1} . Let $\underline{\succ}_{\bar{U}}^{(1)}$ denote the new positions' preference profile, proceed to the next round.

Round $r \geq 2$:

Step $n \geq 1$: Each student proposes his most preferred plan that has not been rejected according to $\underline{\succ}_{\bar{S}}$. Each position keeps the most preferred plan according to $\underline{\succ}_{\bar{U}}^{(r)}$ and rejects all others. Pauses when no more proposal is made. Denote this tentative matching by $\bar{\mu}^{(r)}$.

If $\nexists (s, u'^2) \in \bar{S} \times \bar{U}$ such that $(\bar{\mu}^{(r),1}(s), u') P_s \bar{\mu}^{(r)}(s)$ and $(\bar{\mu}^{(r),1}(u'^2), s) P_{u'^2} \bar{\mu}^{(r)}(u'^2)$ under $(\underline{\succ}_{\bar{S}}, \underline{\succ}_{\bar{U}}^{(r)})$, terminates. Otherwise, for all $u^1 \in \bar{U}$ such that some $\bar{\mu}^{(r)}(u^1_\beta) = ss$ where s is one of these students, update $\underline{\succ}_{u^1}^{(r)}$ to $\underline{\succ}_{u^1}^{(r+1)}$ such that $\emptyset P_{u^1}^{(r+1)} ss$ and otherwise preserves $P_{u^1}^{(r)}$. Let $\underline{\succ}_{\bar{U}}^{(r+1)}$ denote the new positions' preference profile, proceed to the next round.

Denote the final matching by $\bar{\mu}_{RF}$. This algorithm terminates in finite rounds and finite steps.

1.7.3 Omitted Proofs and Additional Results

Proof of Lemma 1: μ_S is individually rational since, at each step of the algorithm, no university keeps an unacceptable plan, no student proposes an unacceptable plan, and by construction $N^\mu = 0$ for each u .

μ_S is unblocked. Suppose $uuP_s\mu_S(s)$ or $cuP_s\mu_S(s)$ for some s . This means s has applied to u and was rejected at some step n of S-PDA. Denote a tentative matching of u at step n by $\mu_S^n(u)$. $\mu_S^n(u)$ only improves weakly as n goes up, i.e., $\mu_S^{n+1}(u)R_u^*\mu_S^n(u)$, so u will not block with s . \square

Let $\bar{\mu}_S$ be the final matching of S-PDA in the related one-to-one market (by construction, it is the corresponding matching of μ_S). Let $\bar{\mu}_U$ be the final matching of U-PDA.

A dynamically stable matching is **position-optimal (position-pessimal)** if each position gets the best (worst) outcome within the set of dynamically stable matchings. A student and a position are said to be *achievable* to each other if they are matched in some dynamically stable matching.

Theorem C.1: Under A1, $\bar{\mu}_S$ is student-optimal and $\bar{\mu}_U$ is position-optimal.

Proof of Theorem C.1: We prove $\bar{\mu}_S$ is student-optimal by showing that no student is rejected by an achievable position. The proof is by induction. Assume that up to a given step in the algorithm no student has been rejected by a position who is achievable for him so far. At this step, suppose p rejects s . If s is unacceptable, then p is unachievable for s .

If p rejects s because a preferable student s' has proposed to it. We must show p is unachievable for s . We know s' prefers p to any plan except those who have previously rejected him (so those plans by assumption are unachievable for him). For a contradiction, consider a hypothetical matching $\bar{\mu}$ that matches s to p and everyone else to an achievable match such that if s is achievable for p , then $\bar{\mu}$ is one of the dynamically stable matchings. But s' prefers p to his match at $\bar{\mu}$, so s' and p will form a blocking pair. Hence, there is no dynamically stable matching that matches s and p , so p is unachievable for s .

By a symmetry argument, $\bar{\mu}_U$ is position-optimal. \square

Proof of Theorem 2: From Theorem C.1, it is obvious that μ_S is student-optimal (otherwise, there exists some student who can match with an unachievable position (from another university or the same university but in a different period) in the related one-to-one market).

Let μ be some dynamically stable matching other than μ_U . By Theorem C.1, $\bar{\mu}_{\bar{U}}$ is position-optimal, that is, $\forall p \in \bar{U}$, $\bar{\mu}_{\bar{U}}(p)P_p\bar{\mu}(p)$ and $\exists p \in \bar{U}$ such that $\bar{\mu}_{\bar{U}}(p)P_p\bar{\mu}(p)$. By responsiveness, $\forall u \in U$, $\mu_U^1(u)R_u^1\mu^1(u)$ and $\mu_U^2(u) \setminus \mu_U^1(u)R_u^2\mu^2(u) \setminus \mu^1(u)$, and $\exists u \in U$ such that $\mu_U^1(u)P_u^1\mu^1(u)$ or $\mu_U^2(u) \setminus \mu_U^1(u)P_u^2\mu^2(u) \setminus \mu^1(u)$. By construction, $\mu_U^1(u)$ and $\mu_U^2(u) \setminus \mu_U^1(u)$ are disjoint subsets of S , so are $\mu^1(u)$ and $\mu^2(u) \setminus \mu^1(u)$. Then by responsiveness again, $\forall u \in U$, $\mu_U(u)R_u^*\mu(u)$ and $\exists u \in U$, $\mu_U(u)P_u^*\mu(u)$. Under A1, this implies $\mu_U \succsim_u \mu$ for all u , and $\mu_U \succ_u \mu$ for at least one u . So $\mu_U >_U \mu$. \square

Let $\bar{\mu} >_{\bar{U}} \bar{\mu}'$ denote that all positions like $\bar{\mu}$ at least as well as $\bar{\mu}'$ with at least one position strictly prefers, that is, $\bar{\mu} \succ_p \bar{\mu}'$ for all p , and $\bar{\mu} \succ_p \bar{\mu}'$ for at least one p .

Theorem C.2: Under A1, if $\bar{\mu}$ and $\bar{\mu}'$ are dynamically stable matchings, then $\bar{\mu} >_{\bar{S}} \bar{\mu}'$ if and only if $\bar{\mu}' >_{\bar{U}} \bar{\mu}$.

Proof of Theorem C.2: Suppose it is not true such that $\bar{\mu} >_{\bar{S}} \bar{\mu}'$ and $\bar{\mu} >_{\bar{U}} \bar{\mu}'$. Then $\exists s \in \bar{S}$ such that $\bar{\mu}(s)P_s\bar{\mu}'(s)$, where $\bar{\mu}(s)P_s\emptyset$ by strict preferences. Then the position $p = \bar{\mu}(s)$ must match with someone else. Since p also has a strict preference and $\bar{\mu} >_{\bar{U}} \bar{\mu}'$, p and s will form a blocking pair at $\bar{\mu}'$. So $\bar{\mu}'$ is not dynamically stable, a contradiction. \square

Theorem C.2 says that any dynamically stable matching that is weakly better for all student is weakly worse for all positions, and vice versa. Theorem C.1 and C.2 together imply the following corollary.

Corollary C.1: Under A1, $\bar{\mu}_{\bar{U}}$ is student-pessimal and $\bar{\mu}_{\bar{S}}$ is university-pessimal.

Proof of Proposition 2: From Corollary C.1, it is obvious that μ_U is student-pessimal (otherwise, there exists some position who can match with an unachievable student in the related one-to-one market).

Let μ be some dynamically stable matching other than μ_S . By Corollary C.1, $\bar{\mu}_{\bar{S}}$ is position-pessimal, that is, $\forall p \in \bar{U}$, $\bar{\mu}(p)P_p\bar{\mu}_{\bar{S}}(p)$ and $\exists p \in \bar{U}$ such that $\bar{\mu}(p)P_p\bar{\mu}_{\bar{S}}(p)$. By responsiveness, $\forall u \in U$, $\mu^1(u)R_u^1\mu_S^1(u)$ and $\mu^2(u) \setminus \mu^1(u)R_u^2\mu_S^2(u) \setminus \mu_S^2(u)$, and $\exists u \in U$ such that $\mu^1(u)P_u^1\mu_S^1(u)$ or $\mu^2(u) \setminus \mu^1(u)P_u^2\mu_S^2(u) \setminus \mu_S^1(u)$. By construction, $\mu^1(u)$ and $\mu^2(u) \setminus \mu^1(u)$ are disjoint subsets of S , so are $\mu_S^1(u)$ and $\mu_S^2(u) \setminus \mu_S^1(u)$. Then by responsiveness again, $\forall u \in U$, $\mu(u)R_u^*\mu_S(u)$ and $\exists u \in U$, $\mu(u)P_u^*\mu_S(u)$. Under A1, this implies $\mu \succsim_u \mu_S$ for all u , and $\mu \succ_u \mu_S$ for at least one u . So $\mu >_U \mu_S$. \square

Lemma C.1 (Decomposition Lemma): Under A1, let $\bar{\mu}$ and $\bar{\mu}'$ be dynamically stable matchings. Let $S^{\bar{\mu}}$ ($\bar{U}^{\bar{\mu}}$) be the sets of students (positions) who prefer $\bar{\mu}$ to $\bar{\mu}'$, and let $S^{\bar{\mu}'}$ ($\bar{U}^{\bar{\mu}'}$) be those who prefer $\bar{\mu}'$. Then $\bar{\mu}$ and $\bar{\mu}'$ map $S^{\bar{\mu}}$ onto $\bar{U}^{\bar{\mu}'}$ and $S^{\bar{\mu}'}$ onto $\bar{U}^{\bar{\mu}}$. (That is, both $\bar{\mu}$ and $\bar{\mu}'$ match any student who prefers $\bar{\mu}'$ to a position who prefers $\bar{\mu}$, and vice versa.)

Proof of Lemma C.1: Suppose for some s , $p = \bar{\mu}(s)P_s\bar{\mu}'(s) = p'$. Then since $\bar{\mu}'$ is dynamically stable, this implies $\bar{\mu}'(p)P_p\bar{\mu}(p) = s$. Hence, $\bar{\mu}(S^{\bar{\mu}'})$ is contained in $\bar{U}^{\bar{\mu}'}$, so $|S^{\bar{\mu}}| \leq |\bar{U}^{\bar{\mu}'}|$. Symmetrically, $\bar{\mu}(\bar{U}^{\bar{\mu}'})$ is contained in $S^{\bar{\mu}}$, so $|S^{\bar{\mu}}| \geq |\bar{U}^{\bar{\mu}'}|$. Since $\bar{\mu}$ and $\bar{\mu}'$ are one-to-one and $S^{\bar{\mu}}$ and $\bar{U}^{\bar{\mu}'}$ are finite, both $\bar{\mu}$ and $\bar{\mu}'$ are onto. \square

$$\bar{\lambda}(p) = \bar{\mu} \vee_{\bar{U}} \bar{\mu}' = \begin{cases} \bar{\mu}(p) & \text{if } \bar{\mu}(p)P_p\bar{\mu}'(p) \\ \bar{\mu}'(p) & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{v}(p) = \bar{\mu} \wedge_{\bar{U}} \bar{\mu}' = \begin{cases} \bar{\mu}(p) & \text{if } \bar{\mu}'(p)P_p\bar{\mu}(p) \\ \bar{\mu}'(p) & \text{otherwise} \end{cases}.$$

$\bar{\lambda}(s) = \bar{\mu} \wedge_S \bar{\mu}'$ and $\bar{v}(s) = \bar{\mu} \vee_S \bar{\mu}'$ are defined as in Appendix 1.7.1. If $\bar{\lambda}$ is an acceptable matching, then $\bar{\lambda}$ will be the least upper bound for $\{\bar{\mu}, \bar{\mu}'\}$ under $>_{\bar{U}}$ and the greatest lower bound for $\{\bar{\mu}, \bar{\mu}'\}$ under $>_{\bar{S}}$.

Theorem C.3 (Lattice Theorem): Under A1, if $\bar{\mu}$ and $\bar{\mu}'$ are dynamically stable matchings, then $\bar{\lambda}$ and \bar{v} are dynamically stable matchings.

Proof of Theorem C.3: By definition, $\bar{\lambda}(i) = \bar{\mu}'(i) \forall i \in S^{\bar{\mu}'} \cup \bar{U}^{\bar{\mu}}$ and $\bar{v}(i) = \bar{\mu}(i) \forall i \in S^{\bar{\mu}} \cup \bar{U}^{\bar{\mu}'}$. By Lemma C.1, $\bar{\lambda}$ is therefore a matching. Note that $\bar{\lambda}$ is acceptable because A1 is assumed ($N_u^{\bar{\mu}} = 0$ is satisfied for all u) and no one is matched to an unacceptable plan under $\bar{\mu}$ and $\bar{\mu}'$. Suppose (s, p)

blocks $\bar{\lambda}$. Then $sP_p\bar{\lambda}(p)$ and $pP_s\bar{\lambda}(s)$. Hence (s, p) blocks $\bar{\mu}$ if $\bar{\lambda}(p) = \bar{\mu}(p)$ or $\bar{\mu}'$ if $\bar{\lambda}(p) = \bar{\mu}'(p)$. In either case we have a contradiction.

By a symmetric argument, \bar{v} is also a dynamically stable matching. \square

Corollary C.2 below follows from Theorem C.2 together with Theorem C.3.

Corollary C.2: Under A1, the set of dynamically stable matchings forms a lattice under $>_{\bar{U}}$ and $>_{\bar{S}}$, and the lattice under $>_{\bar{U}}$ is the dual to the lattice under $>_{\bar{S}}$.

Proof of Theorem 3: The result follows from Lemma B.1 and Corollary C.2. \square

Proof of Proposition 2: See the proof of Proposition A.6. \square

For a set of agents $L \subseteq \bar{S} \cup \bar{U}$, let $\bar{\mu}(L)$ be the set of agents that match with some member of L in $\bar{\mu}$.

Proof of Theorem 4: We show the result by using the S-PDA in the related one-to-one market. Let $\bar{\mu}$ be the corresponding matching of μ . The matching $\bar{\mu}$ would match every student s to some position that had rejected him in the algorithm (because a preferable student s' proposed to this position). So all those positions $\bar{\mu}(\bar{S})$ have been matched under $\bar{\mu}_{\bar{S}}$, this implies $\bar{\mu}_{\bar{S}}(\bar{\mu}(\bar{S})) = \bar{S}$. Hence, all students would have been matched under $\bar{\mu}_{\bar{S}}$ and $\bar{\mu}_{\bar{S}}(\bar{S}) = \bar{\mu}(\bar{S})$. But since all students are matched under $\bar{\mu}_{\bar{S}}$, any position which gets a proposal in the last step of the algorithm at which proposals were issued has not rejected any acceptable student (otherwise it is not in the last step given all students are matched), i.e., the algorithm stops as soon as every position in $\bar{\mu}_{\bar{S}}(\bar{S})$ has an acceptable proposal. So such a position must be unmatched at $\bar{\mu}$ since every student prefers $\bar{\mu}$ to $\bar{\mu}_{\bar{S}}$, which contradicts $\bar{\mu}_{\bar{S}}(\bar{S}) = \bar{\mu}(\bar{S})$. \square

Proof of Theorem 5: $|\bar{\mu}_{\bar{S}}(\bar{U})| = |\bar{\mu}_{\bar{S}}(\bar{S})| \geq |\bar{\mu}_{\bar{U}}(\bar{S})| = |\bar{\mu}_{\bar{U}}(\bar{U})| \geq |\bar{\mu}_{\bar{S}}(\bar{U})|$. \square

Lemma C.2: Under A1, let μ and μ' be dynamically stable matchings and let $\bar{\mu}$ and $\bar{\mu}'$ be their corresponding matchings, respectively. If $\mu^1(u) \neq \mu'^1(u)$ for some u such that $\bar{\mu}(u_i^1)P_{u^1}\bar{\mu}'(u_i^1)$ for some u_i^1 , then $\bar{\mu}(u_\beta^1)R_{u^1}\bar{\mu}'(u_\beta^1)$ for all period-1 positions u_β^1 of u . If $\mu^2(u) \setminus \mu^1(u) \neq \mu'^2(u) \setminus \mu'^1(u)$ for some u such that $\bar{\mu}(u_\gamma^2)P_{u^2}\bar{\mu}'(u_\gamma^2)$ for some u_γ^2 , then $\bar{\mu}(u_\gamma^2)R_{u^2}\bar{\mu}'(u_\gamma^2)$ for all period-2 positions

u_γ^2 of u .

Proof of Lemma C.2: We show that $\bar{\mu}(u_\beta^1)P_{u^1}\bar{\mu}'(u_\beta^1)$ for all $i < \beta$. Suppose not. $\exists \beta$ such that $\bar{\mu}(u_\beta^1)P_{u^1}\bar{\mu}'(u_\beta^1)$ and $\bar{\mu}'(u_{\beta+1}^1)R_{u^1}\bar{\mu}(u_{\beta+1}^1)$. Note that u_β^1 is matched under $\bar{\mu}$ (we cannot have $\emptyset = \bar{\mu}(u_\beta^1)P_{u^1}\bar{\mu}'(u_\beta^1)$), so Theorem 5 implies $\bar{\mu}'(u_\beta^1) \in \bar{S}$. Let $s \equiv \bar{\mu}'(u_\beta^1)$. By Lemma C.1, $u_\beta^1 \equiv \bar{\mu}'(s)P_s\bar{\mu}(s)$, and since $s \equiv \bar{\mu}'(u_\beta^1)P_{u^1}\bar{\mu}'(u_{\beta+1}^1)R_{u^1}\bar{\mu}(u_{\beta+1}^1)$, $s \neq \bar{\mu}(u_{\beta+1}^1)$. Since by construction $u_{\beta+1}^1$ is the next most preferred plan in the preference of s , $u_{\beta+1}^1 P_s \bar{\mu}(s)$. So s and $u_{\beta+1}^1$ will block $\bar{\mu}$, which contradicts the dynamic stability of $\bar{\mu}$. Since i is arbitrary, the first part follows.

The second part follows from the same arguments. \square

Proof of Theorem 6: Suppose not. Then $\exists u_i^1$ such that $\bar{\mu}(u_i^1)P_{u^1}\bar{\mu}'(u_i^1)$, then from the proof of Lemma C.2 we know that $\bar{\mu}(u_\beta^1)P_{u^1}\bar{\mu}'(u_\beta^1)$ for all $i < \beta$. But we know by Theorem 5, if a period-1 position is unmatched in some dynamically stable matching, then it is unmatched under all dynamically stable matchings. This means that $\bar{\mu}(u_{q_u}^1) = \bar{\mu}'(u_{q_u}^1) = \emptyset$ for the last period-1 position in u , and this contradicts $\bar{\mu}(u_{q_u}^1)P_{u^1}\bar{\mu}'(u_{q_u}^1)$. So such u_i^1 does not exist, that is, $\bar{\mu}(u_\beta^1) = \bar{\mu}'(u_\beta^1)$ for all β .

The same result can be shown for the second period using the same arguments. \square

Proof of Theorem 7: Suppose u fills all its quota under μ and μ' in both periods; otherwise by Theorem 5, $\mu^1(u) = \mu'^1(u)$ or $\mu^2(u) \setminus \mu^1(u) = \mu'^2(u) \setminus \mu'^1(u)$.

(\Leftarrow) This direction is by the definition of μ .

(\Rightarrow) If $\mu(u) \neq \mu'(u)$ but $\mu^2(u) = \mu'^2(u)$. Without loss of generality, suppose $\mu^1(u)P_u^1\mu'^1(u)$ and $\mu^2(u) \setminus \mu^1(u)P_u^2\mu'^2(u) \setminus \mu'^1(u)$.

Let $\bar{\mu}$ and $\bar{\mu}'$ be the corresponding matching of μ and μ' , respectively. There must exist s and s' such that $s = \bar{\mu}(u_\beta^1)$, $s = \bar{\mu}'(u_\gamma^2)$, $s' = \bar{\mu}'(u_\beta^1)$ and $s' = \bar{\mu}(u_\gamma^2)$. Since preferences are strict, without loss of generality, suppose sP_us' and cuP_suu . Then (s, u_γ^2) will block $\bar{\mu}$, a contradiction. \square

Proof of Theorem 8: Let $\bar{\mu}$ and $\bar{\mu}'$ be the corresponding matchings of μ and μ' , respectively.

(a) Suppose u fills all its quota under μ and μ' in both periods; otherwise, by Theorem 5,

$\mu^1(u) = \mu'^1(u)$ or $\mu^2(u) \setminus \mu^1(u) = \mu'^2(u) \setminus \mu'^1(u)$. This will imply $\mu(s) = \mu'(s)$ or $\mu(s') = \mu'(s')$.

Suppose $\mu(s)P_s\mu'(s)$ and $\mu(s')P_{s'}\mu'(s')$. $\exists u_\beta^1, u_\gamma^2$ such that $\bar{\mu}(u_\beta^1) = s$ and $\bar{\mu}'(u_\gamma^2) = s'$. Since μ' is dynamically stable, $\exists s'' = \bar{\mu}'(u_\beta^1)P_{u^1}s$ and $\exists s''' = \bar{\mu}(u_\gamma^2)P_{u^2}s'$. By Lemma C.2, $\bar{\mu}'(u_i^1)R_{u^1}\bar{\mu}(u_i^1)$ for all period-1 positions u_i^1 of u with u_β^1 strictly prefers, and $\bar{\mu}'(u_i^2)R_{u^2}\bar{\mu}(u_i^2)$ for all period-2 positions u_i^2 of u with u_γ^2 strictly prefers. So by responsiveness, $\mu^1(u)P_u^1\mu^1(u)$ and $\mu^2(u) \setminus \mu^1(u)P_u^2\mu^2(u) \setminus \mu^1(u)$. By responsiveness again, $\mu'(u)P_u^*\mu(u)$.

(b) We prove the first part here, the second part follows from a symmetric argument. Suppose u fills all its period-1 positions under μ and μ' , otherwise, by Theorem 5, $\mu(u) = \mu'(u)$. $\exists u_\beta^1$ such that $\bar{\mu}(u_\beta^1) = s$. Since μ' is dynamically stable, $\exists s'' = \bar{\mu}'(u_\beta^1)P_{u^1}s$. By Lemma C.2, $\bar{\mu}'(u_i^1)R_{u^1}\bar{\mu}(u_i^1)$ for all period-1 positions u_i^1 of u with u_β^1 strictly prefers. So by responsiveness, $\mu^1(u)P_u^1\mu^1(u)$ and so $\mu'(u)P_u^*\mu(u)$.

(c) Suppose u fills all its period-1 positions or period-2 positions under μ and μ' ; otherwise, by Theorem 5, $\mu(u) = \mu'(u)$.

Suppose $\mu(u)P_u^*\mu'(u)$, then either $\mu^1(u)P_u^1\mu^1(u)$ or $\mu^2(u) \setminus \mu^1(u)P_u^2\mu^2(u) \setminus \mu^1(u)$. Recall $\bar{U}^{\bar{\mu}} = \{p \in \bar{U} \mid \bar{\mu}(p)P_p\bar{\mu}'(p)\}$ and $S^{\bar{\mu}'} = \{s \in S \mid \bar{\mu}'(s)P_s\bar{\mu}(s)\}$.

If $\mu^1(u)P_u^1\mu^1(u)$, then by Lemma C.2, $\bar{\mu}(u_i^1)R_{u^1}\bar{\mu}'(u_i^1)$ for all period-1 position of u with $u_{q_u^1}^1$ strictly prefers. Let $u_j^1, \dots, u_{q_u^1}^1$ be the positions that strictly prefer $\bar{\mu}$ to $\bar{\mu}'$. That is, $u_j^1, \dots, u_{q_u^1}^1 \in \bar{U}^{\bar{\mu}}$, where u_j^1 is the one with the lowest index among these positions. Let $s = \bar{\mu}(u_j^1)$. By Lemma C.1, $s \in S^{\bar{\mu}'}$. Also, $s \neq \bar{\mu}'(u_i^1) \forall i$ (suppose not. If $i \geq j$, then u_j^1 does not strictly prefer $\bar{\mu}$. If $i < j$, then it contradicts the way we pick u_j^1). So $\mu'(s)P_s\mu(s)$.

If $\mu^2(u) \setminus \mu^1(u)P_u^2\mu^2(u) \setminus \mu^1(u)$, by the same arguments, we can find s' such that $\mu'(s')P_{s'}\mu(s')$. \square

Lemma C.3 (Blocking Lemma): Under A1, let $\bar{\mu}$ be any individually rational matching and let S' be the set of students who prefer $\bar{\mu}$ to $\bar{\mu}_{S'}$. If S' is nonempty, there is a pair (s, p) that blocks $\bar{\mu}$ such that $s \in \bar{S} \setminus S'$ and $p \in \bar{\mu}(S')$.

This result also holds for $\bar{\mu}_{\bar{U}}$ in the related one-to-one market, but it does not hold for μ_U since

$U \setminus U'$ might be empty.⁵⁵

Proof of Lemma C.3:

Case 1: $\bar{\mu}(S') \neq \bar{\mu}_{\bar{S}}(S')$. Pick a p in $\bar{\mu}(S') \setminus \bar{\mu}_{\bar{S}}(S')$, that is, $\bar{\mu}(p) = s' \in S'$ and $\bar{\mu}_{\bar{S}}(p) = s \notin S'$. For $\bar{\mu}_{\bar{S}}$ to be dynamically stable, $sP_p s'$. But $s \notin S'$, so he does not prefer $\bar{\mu}$ to $\bar{\mu}_{\bar{S}}$ and by strictness of preferences, we have $\bar{\mu}_{\bar{S}}(s) = pP_s \bar{\mu}(s)$. So (s, p) blocks $\bar{\mu}$.

Case 2: $\bar{\mu}(S') = \bar{\mu}_{\bar{S}}(S') = \bar{U}'$. Let p be the last position in \bar{U}' to receives a proposal from an acceptable student in S' in the S-PDA. Under $\bar{\mu}$, since $p \in \bar{U}'$, it matches with some student in S' . Let us denote him s' . We show that s' is rejected before p gets this last proposal. Since $pP_{s'} \bar{\mu}_{\bar{S}}(s')$, s' must has proposed to p and get rejected in S-PDA and then ends up matching with $\bar{\mu}_{\bar{S}}(s')$. But if p only rejects s' when it gets its last proposal from an acceptable student in S' , then the proposal of s' to $\bar{\mu}_{\bar{S}}(s')$ would have came later, which contradicts how we pick p . So p rejects s' before it got its last proposal from an acceptable student in S' , that implies it holds a proposal from some other student. Let us denote him s . So this s would prefer p to $\bar{\mu}_{\bar{S}}(s)$ since he proposed to $\bar{\mu}_{\bar{S}}(s)$ after p rejected him. Since s later proposed to $\bar{\mu}_{\bar{S}}(s)$ and p is the last position in \bar{U}' to receives a proposal from an acceptable student in S' , $\bar{\mu}_{\bar{S}}(s) \notin \bar{\mu}_{\bar{S}}(S') = \bar{\mu}(S')$. Thus, $s \notin S'$. That implies $pP_s \bar{\mu}_{\bar{S}}(s)P_s \bar{\mu}(s)$. p rejects s' because of s . So (s, p) block $\bar{\mu}$. \square

Proof of Theorem 9: We use language in the related one-to-one market in this proof. Under A1, by design, a student does not have an incentive to report a uu' plan in his reported preference.

Suppose for contradiction that some set of students $S' \subseteq \bar{S}$ misreport their preferences such that the reported preference profile is $\hat{\succ} = ((\hat{\succ}'_s)_{s \in S'}, (\hat{\succ}_s)_{s \in \bar{U} \cup \bar{S} \setminus S'})$ and $\bar{\mu}'_{\bar{S}}(s)P_s \bar{\mu}_{\bar{S}}(s)$ for all $s \in S'$, where $\bar{\mu}'_{\bar{S}}$ is the S-PDA outcome under the reported preference profile. Since no one has an incentive to report an unacceptable plan, $\bar{\mu}'_{\bar{S}}$ is individually rational under the true preference profile. Note that $\bar{\mu}'_{\bar{S}}(s)P_s \bar{\mu}_{\bar{S}}(s)$ for all $s \in S'$. We can apply the Lemma C.3 to the true preference profile $\hat{\succ}$. So $\exists (s', p)$, where $s' \in \bar{S} \setminus S'$ and $p \in \bar{\mu}_{\bar{S}}(S')$, that blocks $\bar{\mu}'_{\bar{S}}$ under the true preference profile.

⁵⁵See the proof of proposition 1 in Roth [1985].

$s' \in \bar{S} \setminus S'$ and $p \in \bar{\mu}_{\bar{S}}(S')$ also imply $\widehat{\succ}_{s'} = \widetilde{\succ}_{s'}$ and $\widehat{\succ}_p = \widetilde{\succ}_p$, so s' and p also block $\bar{\mu}_{\bar{S}}$ under $\widehat{\succ}$. But by Lemma 1, $\bar{\mu}_{\bar{S}}$ is dynamically stable under $\widehat{\succ}$. A contradiction. \square

1.7.4 An Additional Example

The following example shows even though $\mu(u) = (\mu^1(u), \mu^2(u)) \in 2^S \times 2^S$, universities' preferences are not over a subset of $2^S \times 2^S$ because N_u^μ depends on others' matchings.

Example D.1. Consider the following market. $S = \{s_1, s_2, s_3\}$, $U = \{u_1, u_2, u_3\}$,

$$P_{u_1} : s_1 \quad q_{u_1} = (1, 0),$$

$$P_{u_2} : s_2, s_1, s_3 \quad q_{u_2} = (0, 1),$$

$$P_{u_3} : s_3, s_2 \quad q_{u_3} = (1, 0),$$

$$P_{s_1} : u_1 u_2, u_1 u_1,$$

$$P_{s_2} : u_3 u_3, c u_2,$$

$$P_{s_3} : c u_2, u_3 u_3.$$

In this market,

$$\mu = \begin{pmatrix} u_1 & u_2 & u_3 & c \\ s_1 & \emptyset & s_3 & s_2 \\ s_1 & s_2 & s_3 & \emptyset \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} u_1 & u_2 & u_3 & c \\ s_1 & \emptyset & s_2 & s_3 \\ s_1 & s_3 & s_2 & \emptyset \end{pmatrix}$$

Even though $\mu(u_1) = \mu'(u_1)$, μ is acceptable to u_1 , but μ' is not because $N_u^{\mu'} \neq \emptyset$ as s_1 will transfer out given μ' .

Chapter 2

AIRPORT SLOT ALLOCATION PROBLEMS

2.1 Introduction

In this chapter¹, we study airport slot allocation problems during severe weather.² We assume airlines have lexicographic preferences. Our model is built on [Schummer and Vohra \[2013\]](#), [Schummer and Abizada \[2017\]](#), where they also study this topic with different assumptions on airlines' preferences.³

Landing schedules are made in advance, but unpredictable conditions, such as severe weather, may lead to reallocations of landing slots as the arrival capacity at the affected airport is often reduced below the number of initially scheduled flights.⁴ In the United States, during severe weather, the Federal Aviation Administration (FAA) implements Ground Delay Programs to reallocate landing slots. When severe weather is forecasted, the FAA initiates a ground delay program (GDP) a few hours ahead. The duration of a GDP is usually several hours. In a GDP, the FAA assigns new arrival times (landing slots) to aircraft departing airports in the contiguous U.S. and Canada to the affected airport. These aircraft are also assigned new (and usually delayed) departure times at their

¹This chapter is a joint work with Alexander Rodivilov.

²A slot at an airport is essentially a time interval that allows an airline to land a plane.

³This is an important topic: In the United States, the total cost of all flight delays in 2007 was estimated at 31.2 billion dollars, in which 8.3 billion dollars were costs to airlines [[Ball et al., 2010a](#)]. In this particular year, the weather's share of total delay minutes is 43.6% (it varies from 32.8%-49.7% in the period of 2004–2016. See <https://www.rita.dot.gov/bts/help/aviation/html/understanding.html>).

The costs of weather-caused flight delays are nontrivial to airlines as their profits were just 5 billion dollars [[Air Transport Association, 2008](#)] in the same year.

⁴Unpredictable conditions are not limited to severe weather conditions. For instance, runway closures caused by aircraft incidents are also included. Our model can also be applied in this situation, see the last part of footnote [20](#) and the last part of footnote [27](#).

origin airports (while the aircraft are still on the ground).⁵ In a current GDP, the FAA uses a 2-step procedure to carry out the reassignment of landing slots. The first step is to assign slots to flights. The current algorithm is called Ration-by-Schedule (RBS). It orders flights by increasing original scheduled time of arrival and then assigns slots sequentially. That is, the first flight is assigned to the first available GDP slot, the second flight is assigned to the second available GDP slot, etc.⁶ RBS may assign slots to flights that have been canceled or delayed by their airlines and so cannot feasibly use their assigned slots. The airlines can adjust their schedules by substitutions and cancellations, but flight cancellations and delays may consequently create vacant slots in the landing schedule. The second step is to reassign these newly created vacant slots to airlines that can use them. The mechanism in place now is called Compression.⁷ Compression utilizes information reported by the airlines (cancellations and earliest feasible arrival times of flights) to exchange slots among them to produce a new landing schedule. Essentially, when an airline cannot use a slot, Compression exchanges it with a later slot that is owned by some airline that can use the original slot.⁸ In the current GDP, the first assignment is created by running RBS and possible Compression. But as airlines update their information, Compression might be run multiple times.

We propose a new mechanism to replace the currently used mechanism. Now we describe some deficiencies of RBS and Compression that motivate us. We show that RBS does not respect a form of property rights *before* a GDP starts. Note that slots of different lengths are different objects. A GDP converts initial slots into GDP slots, but such conversion is just a re-division of time intervals. Under RBS, owning an early initial slot gives the airline an early GDP slot, which is not the same

⁵These new departure times are calculated based on the new arrival times. For more details, see Section 17-9-1 of the *Facility Operation and Administration*. This document is available at https://www.faa.gov/air_traffic/publications.

⁶The OAG schedule is considered to be the initial schedule in the industry, where OAG is an air travel intelligence company.

⁷So the currently used mechanism is a combination of RBS and Compression.

⁸Compression moves flights up in the schedule to fill those vacant slots. For more details, see [Schummer and Vohra \[2013\]](#) or [Vossen and Ball \[2006a\]](#).

object it had at the beginning, while such time interval (of the GDP slot) might be entirely owned by another airline before the start of a GDP. A mechanism that respects property rights before a GDP starts would endow a GDP slot to the airline that owns the entire time interval of this GDP slot before a GDP starts. RBS might lead to outcomes that are not individually rational and thus outside the core in our model. We show that Compression does not respect a form of property rights *after* a GDP starts (in reassignments).⁹ Also, Compression produces outcomes that might be outside the core and is manipulable by postponing a flight cancelation [Schummer and Vohra, 2013].¹⁰ Moreover, it is not strategy-proof [Schummer and Abizada, 2017].¹¹ We show that these negative results for Compression also hold in our preference domain. In addition, we show that Compression is not individually rational and not Pareto efficient.¹²

Our mechanism, Multiple Trading Cycles (MTC), will overcome the aforementioned problems. In our model, airlines have lexicographic preferences, in which each airline has an importance ranking over its flights. MTC solicits importance rankings and earliest feasible arrival times of flights from airlines. If the time interval of a slot is entirely owned by some airline, then this slot is considered to be owned by the airline in MTC. There are three stages in MTC. In the first stage, each slot that is demanded by only 1 airline (in a sense we make precise below) will be identified and assigned. In the second stage, all slots being assigned are demanded by more than 1 airline. According to an ordering determined by MTC, an airline picks a slot for its most important remaining flight or picks a slot pro forma (if it has no remaining flight). If the airline picks a slot that is owned by another airline, then the latter can pick a slot for its most important remaining flight or pick a slot pro forma. If a cycle forms, each airline in the cycle picks a slot

⁹Schummer and Vohra [2013] define property rights based on core allocations from an initial endowment of landing slots and claim that Compression does not respect such a form of property rights.

¹⁰Indeed, they show Compression fails a condition that is weaker than non-manipulable by postponing a flight cancelation.

¹¹Schummer and Vohra [2013] show that Compression is strategy-proof in their preference domain.

¹²Schummer and Vohra [2013] show that Compression is Pareto efficient in their preference domain.

for its most important remaining flight or pick a slot pro forma. This stage finishes when each non-canceled flight has been assigned a slot. In the last stage, each airline that has canceled flights will get the same number of slots. (The last stage is consistent with RBS. MTC can also be used to perform reassignments, and the slots assigned in the last stage might be valuable in a subsequent reassignment.)

We now highlight some desirable properties of MTC.¹³ In contrast to RBS and Compression, MTC respects a form of property rights before and after a GDP starts, produces outcomes that are individually rational, Pareto efficient and in the core (in this problem, the core might not be a subset of the Pareto set), is strategy-proof and non-manipulable by postponing a flight cancelation. By truth-telling, MTC minimizes the *expected* delays for each airline lexicographically. That is, for each airline, it minimizes the expected delay for the most important flight then minimizes the expected delay for the second most important flight, and so on.¹⁴

We provide two algorithms to find the outcome of MTC. We also provide a modified version of MTC called Multiple Trading Cycles-2. MTC-2 has all the desirable properties of MTC while it possibly favors some airlines. Finally, we extend our model to allow indifferences in preferences. In that extended model, a modified version of MTC with tiebreaking inherits most of the desirable properties while it possibly produces outcomes outside the core as agents might be in more than one cycle under their true preferences, where one of the cycles might lead to a better outcome for the agents (this does not happen in models where agents have unit demand and non-strict preferences).

¹³Before the currently used mechanism was adopted, the FAA used a mechanism called Grover-Jack, which assigns slots based on feasible departure times reported by the airlines. This mechanism provides incentives for airlines to manipulate its information (see [Schummer and Vohra \[2013\]](#) for more discussions on this “double penalty” problem). The currently used mechanism avoids this problem by using the originally scheduled times of arrival instead of the reported feasible departure times to allocate slots. MTC also avoids this problem as it does not use reported feasible departure times to allocate slots.

¹⁴RBS minimizes the delay for each flight in a lexicographic order. It is easy to see it minimizes the delay for the first flight then minimizes the delay for the second flight, and so on. [Vossen and Ball \[2006a\]](#) show RBS lexicographically minimize the maximum delay with respect to the original schedule. Their formulation is different, but the intuition is similar.

Our mechanism might be useful in other applications. For example, when a set of objects (or tasks) is being distributed to several teams and team members have heterogeneous preferences, given each team has an internal ranking over its members, a slight modification of MTC can be used in this environment.¹⁵

2.2 *Related Literature*

The two papers most related to ours are [Schummer and Vohra \[2013\]](#) and [Schummer and Abizada \[2017\]](#). Importantly, both papers take RBS outcomes as initial endowments and focus on the reassignment step. In [Schummer and Vohra \[2013\]](#), preferences are incomplete as not every pair of feasible landing schedules is comparable; they propose a mechanism called Trading Cycle (TC).¹⁶ In [Schummer and Abizada \[2017\]](#), preferences are induced by linear delay costs. They separately consider airlines' incentives to report flights' feasible arrival times, relative delay costs (weights), and cancelations. They propose a mechanism called Deferred Acceptance with Self Optimization (DASO). DASO is not Pareto efficient (this is because DASO does not use weights, which are necessary to determine Pareto efficient outcomes). DASO is non-manipulable via weights and non-manipulable by postponing a flight cancelation, though it is still manipulable by intentional flight delay. By contrast, our mechanism achieves full strategy-proofness (non-manipulable via feasible arrival times and rankings), Pareto efficiency and is non-manipulable by postponing a flight cancelation in the lexicographic preference domain. Lexicographic preferences assume that the weight of a flight is infinitesimal compared to the weight of a more important flight, and this feature is absent from the preferences in [Schummer and Abizada \[2017\]](#).

¹⁵See [Schummer and Vohra \[2013\]](#) for more potential applications.

¹⁶They assume an airline is made better off only if it moves a flight up in the schedule while no others move down, so airlines' preferences are induced by feasible arrival times and the current landing schedule. The current landing schedule is preference-incomparable with another landing schedule that puts some of airline a 's flights in earlier positions and some in later positions.

In this chapter and [Schummer and Abizada \[2017\]](#), an airline might be made better off even if a flight is moved down.

Another related paper is [Abdulkadiroğlu and Sönmez \[1999\]](#). Indeed, when (i) no airline owns a canceled flight, (ii) each airline owns exactly one non-canceled flight, and (iii) each airline owns at most one GDP slot, our model degenerates to the housing allocation with existing tenants model, and MTC reduces to YGMH-IGYT (with random ordering).¹⁷ Given (ii) and (iv) no airline owns a GDP slot, our model reduces to a house allocation problem [[Hylland and Zeckhauser, 1979](#)], and MTC reduces to random serial dictatorship. Given (i), (ii), (v) each airline owns exactly one GDP slot, and (vi) each slot is owned by some airline, our model reduces to a housing market [[Shapley and Scarf, 1974](#)], and MTC reduces to the core mechanism.¹⁸

[Kurino \[2014\]](#), [Kennes et al. \[2014\]](#), [Pereyra \[2013\]](#) study dynamic object allocation problems with overlapping generations in house assignments, daycare assignments, and teacher assignments, respectively. They also propose mechanisms that respect the property rights over the objects induced by the allocation in the previous period.

[Konishi et al. \[2001\]](#) also generalize the housing market. In their model, multiple types of indivisible goods are traded. They show that the core may be empty and there is no Pareto efficient, individually rational, and strategy-proof (deterministic) mechanism. In our context, we obtain positive results on these properties for MTC (which is stochastic). [Chun and Park \[2017\]](#) study a slot allocation problem assuming monetary transfers are feasible. By contrast, we assume monetary transfers are infeasible.

In the transportation literature on GDP, optimization is the main focus.¹⁹ [Vossen \[2002\]](#) pro-

¹⁷With unit demand, agents' preferences are trivially lexicographic.

(i) is non-trivial when some airline owns a GDP slot. Without (i), when some airline owns a GDP slot, the airline has the ability to remove it from the set of available GDP slots (see Section 2.6.1 for more details), and such feature is absent in the housing allocation with existing tenants model.

This mechanism is called Top Trading Cycle mechanism in [Abdulkadiroğlu and Sönmez \[1999\]](#), [Sönmez and Ünver \[2005\]](#) and YGMH-IGYT mechanism in [Sönmez and Ünver \[2010\]](#).

¹⁸In a housing market with strict preferences, there is a unique matching in the core [[Roth and Postlewaite, 1977](#)], and Gale's top trading cycles algorithm (attributed to David Gale by [Shapley and Scarf, 1974](#)) can be used to find the outcome of the core mechanism.

(i), (ii), (v) and (vi) imply that the number of flights equals the number of slots.

¹⁹See [Vossen and Ball \[2006a,b\]](#), [Bard and Mohan \[2008\]](#), [Ball et al. \[2009\]](#), [Glover and Ball \[2013\]](#).

poses a proportional random assignment method, in which each flight is entitled to an equal share of each slot it can use. Balakrishnan [2007] uses the housing market model by treating flights as agents. These two papers do not take airlines' incentives into account. Ball et al. [2010b] propose an algorithm called Ration-by-Distance that assigns slots to flights based on distance. They show that Ration-by-Distance minimizes total expected delay, while MTC minimizes the expected delays for each airline lexicographically.

2.3 Model

There is a finite set of airlines A and a finite set of flights $F^o = \cup_{a \in A} F_a^o$, where F_a^o is the set of flights owned by airline a . Some flights might be canceled during the GDP or before the GDP starts; we use $F \subseteq F^o$ to denote the set of non-canceled flights and $F_a \subseteq F_a^o$ to denote the set of flights owned by airline a that are not canceled. There is a set of initial slots $S^o = \{s_1^o, s_2^o, \dots, s_{|L|}^o\}$, where the length of each slot is normalized to one unit of time. Note that $|F^o|$ of the $|L|$ initial slots were owned by some airlines. Let the set of available GDP slots be $S = \{s_1, s_2, \dots\}$, where the length of each slot is l units of time ($l > 1$).²⁰

There is an earliest feasible arrival time $e_f \in S$ for each flight $f \in F$, and f can be feasibly assigned to a slot s_n only if $e_f \leq s_n$.²¹ With a slight abuse of notation, we use n instead of s_n to express e_f in our examples. Let $e = (e_f)_{f \in F}$ be the vector of all earliest feasible arrival times and

²⁰Note that available GDP slots do not have to be adjacent since there are exempted flights in a GDP (for example, international and airborne flights are exempted). There will always be feasible slots for flights, yet they might be late and outside the GDP time window. The number of arrivals a runway can accept (per 60 minutes) is called maximum runway arrival capacity. If 1 unit of time is 1 minute and the maximum runway arrival capacity (MRAC) during a GDP is 30, then $l = \frac{60}{MRAC \times 1 \text{ unit of time}} = \frac{60}{30 \times 1} = 2$.

There is a single runway in our model. See Section 2.6.2 for an extended model with multiple runways, in which there are multiple slots available at a time.

When a GDP is implemented because of runway closures caused by aircraft incidents but not severe weather, $l = 1$ is possible. However, all results in this chapter hold for $l = 1$.

²¹For $n = 1, 2, \dots$, slot s_n is the time interval $[1 + (n - 1)l, 1 + nl]$, where "1" represents the starting time of s_1^o . We use s_n to represent $1 + (n - 1)l$ on the timeline. If some slot is not available, then we can omit it from S ; for example, if s_2 is not available, then $S = \{s_1, s_3, \dots\}$. Following the literature, we call e_f the earliest feasible arrival *time* of f , but strictly speaking, e_f is the earliest feasible arrival *slot* of f .

$e_a = (e_f)_{f \in F_a}$ be the vector of airline a 's earliest feasible arrival times. A **landing schedule** is an injective function $\Pi : F \rightarrow S$ that assigns each flight to a landing slot. Let \mathcal{M} be the set of all landing schedules. A landing schedule Π is **feasible** if $\Pi(f) \geq e_f$ for all $f \in F$. A landing schedule Π is **non-wasteful** if $\nexists f \in F$ and $s \in S$ such that $\Pi^{-1}(s) = \emptyset$ and $e_f \leq s < \Pi(f)$.

We denote a subset of GDP slots owned by some airline a by $\Phi(a)$, where $\Phi : A \rightarrow 2^S$ is a **slot ownership function** such that $a \neq a' \implies \Phi(a) \cap \Phi(a') = \emptyset$. For $A' \subseteq A$, let $\Phi(A') \equiv \cup_{a \in A'} \Phi(a)$. Φ is *consistent* with Π if $\forall a \in A, \forall f \in F_a, \Pi(f) \in \Phi(a)$. An **assignment** is a pair (Π, Φ) that satisfies consistency. An *initial landing schedule* is an injective function $\Pi^o : F \rightarrow S^o$. Given some initial landing schedule Π^o , one can infer slot $\Pi^o(f)$ is initially endowed to airline a if $f \in F_a^o$. $\Phi^o : A \rightarrow 2^{S^o}$ is an *initial slot ownership function* such that $a \neq a' \implies \Phi^o(a) \cap \Phi^o(a') = \emptyset$. Given an initial assignment (Π^o, Φ^o) , the set of initial slots owned by airline a is $\Phi^o(a)$.

2.3.1 Preferences

An airline's preference over landing schedules might be very complex.²² To simplify the problem, we assume the airlines have lexicographic preferences.²³

Each airline has an importance ranking over its flights. $\forall a \in A$, let R_a be a strict total order over F_a . If $f \in F_a$ is more important than $f' \in F_a$, we write $f R_a f'$. Let $R = (R_a)_{a \in A}$ be the importance ranking profile. All else being equal, airline a prefers flight $f \in F_a$ to land as early as possible (but not earlier than e_f). Given a landing schedule $\Pi \in \mathcal{M}$, we define the **delay** for each flight f by $d_f(\Pi) = \Pi(f) - e_f$, where $\Pi(f)$ is the slot assigned to f in Π .²⁴

²²See Schummer and Abizada [2017] for more discussions.

²³In the two-sided many-to-one matching literature, Dutta and Massó [1997] study a model where the one side agents have lexicographic preferences, and Abizada and Dur [2017] study a model with complementarities where the many side agents have lexicographic preferences. Schulman and Vazirani [2012] and Saban and Sethuraman [2014] study allocation of divisible goods under lexicographic preferences. Fujita et al. [2015] study exchange with multiple indivisible goods under lexicographic preferences. Ehlers [2002] and Ehlers [2003] study locating multiple public goods when agents have lexicographic preferences.

²⁴Note that the delay of a flight f is with respect to e_f but not its original slot $\Pi^o(f)$.

Airline a 's preference over landing schedules is induced by R_a and e_a (recall that $e_a = (e_f)_{f \in F_a}$ and $d_f(\Pi) = \Pi(f) - e_f$). For any two landing schedules Π and Π' , airline a (lexicographically) prefers Π to Π' if and only if the first non-zero coordinate of $x_a = (x_1, x_2, \dots, x_{|F_a|})$ is positive, where $x_i = d_{f_{a,i}}(\Pi') - d_{f_{a,i}}(\Pi)$ for $i \in \{1, \dots, |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$, and we write $\Pi \succ_a \Pi'$.²⁵ Conversely, if the first non-zero coordinate of x_a is negative, Π' is preferred to Π . If airline a is indifferent between Π and Π' , we write $\Pi \sim_a \Pi'$; this happens when all coordinates of x_a equal to 0. Let $\Pi_a : F_a \rightarrow \Phi(a)$ denote a **landing schedule for a** . $\Pi \sim_a \Pi'$ implies $\Pi_a = \Pi'_a$.²⁶ Since airlines only care about their own flights, we will also use \succsim_a to compare landing schedules for a .

A **schedule lottery** is a probability distribution over the set of all landing schedules \mathcal{M} . Let $\Delta \mathcal{M}$ denote the set of all schedule lotteries. We denote a schedule lottery by $\mathcal{L} = \sum p_\Pi \cdot \Pi$ where $p_\Pi \in [0, 1]$ is the probability weight of landing schedule Π and $\sum_\Pi p_\Pi = 1$. We now extend an airline's preference to allow it to compare schedule lotteries. Given a schedule lottery $\mathcal{L} \in \Delta \mathcal{M}$, the **expected delay** for f is $d_f(\mathcal{L}) = \sum_\Pi p_\Pi \cdot (\Pi(f) - e_f)$. For any schedule lotteries \mathcal{L} and \mathcal{L}' , $\mathcal{L} \succ_a \mathcal{L}'$ if and only if the first non-zero coordinate of $x_a = (x_1, x_2, \dots, x_{|F_a|})$ is positive, where $x_i = d_{f_{a,i}}(\mathcal{L}') - d_{f_{a,i}}(\mathcal{L})$ for $i \in \{1, \dots, |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$; other cases are the same as above.

Let $\succsim = (\succsim_a)_{a \in A}$ be the preference profile of all airlines. An **instance** of an Airport Slot Allocation Problem is a tuple $I = (S, A, F^o, R, e, \Phi)$. An instance is not equivalent to an airport slot allocation problem for two reasons. Firstly, there can be multiple instances in an airport slot allocation problem (Section 2.4.3). Secondly, an airport slot allocation problem includes contents that are outside an instance (Section 2.6.1). A GDP is an airport slot allocation problem but the opposite is not true because airport slot allocation problems subsume problems that are different from GDPs (as discussed in Section 2.2).

²⁵Lexicographic preference does not rule out the possibility that an airline prefers an infeasible landing schedule to a feasible landing schedule.

²⁶When there are multiple runways (as in Section 2.6.2), $\Pi \sim_a \Pi'$ implies Π_a is effectively the same as Π'_a .

2.3.2 Mechanisms and Their Properties

In an instance I , parameters other than R and e are fixed. R and e will be reported by the airlines. A (direct) schedule mechanism $\varphi : (R, e) \rightarrow \mathcal{M}$ is a mapping that selects a landing schedule for every strategy profile (R, e) . Let $\varphi_f(R, e)$ be the slot that is assigned to f in $\varphi(R, e)$, and $\varphi_a(R, e)$ be the landing schedule for a in $\varphi(R, e)$. A (direct) lottery mechanism $\phi : (R, e) \rightarrow \Delta \mathcal{M}$ is a mapping that selects a schedule lottery for every strategy profile (R, e) . $\phi_f(R, e)$ and $\phi_a(R, e)$ are defined analogously. The strategy space for airline a is $\mathcal{R}_a \times S^{|F_a|}$, where \mathcal{R}_a is the set of strict total orders over F_a and $S^{|F_a|}$ is the vector space of airline a 's earliest feasible arrival times.

A schedule mechanism φ is *regular* if for any strategy profile (R, e) , the induced ownership function $\Phi^{\varphi(R, e)}$ is consistent with $\varphi(R, e)$. Let $\underline{\phi(R, e)}$ be a realized landing schedule of the schedule lottery $\phi(R, e)$. A lottery mechanism ϕ is regular if for any strategy profile (R, e) and any realization $\underline{\phi(R, e)}$, the induced ownership function $\Phi^{\underline{\phi(R, e)}}$ is consistent with $\underline{\phi(R, e)}$. A schedule mechanism φ is *feasible (non-wasteful)* if for any strategy profile (R, e) , $\varphi(R, e)$ is feasible (non-wasteful).

A schedule mechanism or lottery mechanism is **strategy-proof** if truth-telling is a dominant strategy in its induced preference revelation game. A landing schedule Π is **Pareto efficient** if $\nexists \Pi'$ such that (i) $\forall a \in A, \Pi' \succ_a \Pi$, and (ii) $\exists a \in A, \Pi' \succ_a \Pi$. The set of Pareto efficient landing schedules is the **Pareto set**. A schedule mechanism φ is *Pareto efficient* if for any strategy profile (R, e) , $\varphi(R, e)$ is Pareto efficient.

Let $\Phi^{exante}(a)$ be the set of available GDP slots that their time intervals are entirely owned by airline a before the GDP starts.²⁷ For $A' \subseteq A$, let $S_{A'} \in \{\Phi^{exante}(A'), \Phi(A')\}$ (if $A' = \{a\}$, we write S_a).

²⁷Formally, $\Phi^{exante}(a) = \{s_n \in S[[1 + (n-1)l, 1 + nl] \subseteq \cup_{s_n^g \in \Phi^o(a)} [n, n+1]]\}$. See footnote 21 for the definition of s_n .

In the case that there are runway closures caused by aircraft incidents and only one runway left, the construction of $\Phi^{exante}(a)$ should follow the description in Section 2.6.2.2 rather than the description here.

An airline has the right to swap its own flights within its own set of slots.²⁸ A landing schedule for a $\Pi_a^{S_a}$ is **self-optimized** (with respect to S_a) if $\forall f \in F_a, \Pi_a^{S_a}(f), \Pi'_a(f) \in S_a, \Pi_a^{S_a} \succsim_a \Pi'_a$. Since the preference of airline $a \in A$ is strict, $\Pi_a^{S_a}$ is necessarily unique. To construct $\Pi_a^{S_a}$, order slots in S_a in ascending order, assign a 's most important flight to the earliest slot that it can feasibly use, then assign a 's second most important flight to the earliest slot (among those remaining) that it can feasibly use, and so on until there is no more slot or no more flight.

A landing schedule Π is **individually rational** if $\forall a \in A, \Pi_a \succsim_a \Pi_a^{S_a}$. A schedule mechanism φ is *individually rational* if for any strategy profile (R, e) , $\varphi(R, e)$ is individually rational. A landing schedule Π is in the core if no subgroup of airlines could reallocate their slots to each other and make themselves better off than in Π . Formally, a landing schedule Π is in the **core** if $\nexists \Pi'$ and $A' \subseteq A$ such that (i) $\forall f \in \cup_{a \in A'} F_a, \Pi'(f) \in S_{A'}$, and (ii) $\forall a \in A', \Pi' \succ_a \Pi$.²⁹ A schedule mechanism φ is **core-selecting** if for any strategy profile (R, e) , $\varphi(R, e)$ is in the core. If S_A is empty, then any feasible mechanism is individually rational and core-selecting.³⁰

Schummer and Vohra [2013] consider that Compression does not respect property rights because it might produce outcomes outside the core. We propose the following explicit definition for mechanism instead.³¹ A schedule mechanism or lottery mechanism **respects property rights** over S_A if $\forall a \in A, \forall s \in S_a, a$ can use the slot by itself or trade it for a better slot if there is any. A slot s' is *better* than s for a if (i) s' can be used by a flight $f \in F_a$ that has not been assigned a slot or is currently assigned a slot later than s' , (ii) s cannot be used by a flight $f' \in F_a$ that is more important than f , or s can be used by f' but f' is currently assigned a feasible slot earlier

²⁸See Section 17-9-5 of the *Facility Operation and Administration*. The link to download this document is in footnote 5.

²⁹The core defined by weak domination might be empty. For example, when an airline with a GDP slot cancels all of its flights and such GDP slot is demanded by more than 1 airline, the core defined by weak domination is empty.

³⁰Note that this is the counterpart of any acceptable mechanism is individually rational and core-selecting in a house allocation problem.

³¹When there are indifferences in preferences, a mechanism that respects property rights might produce outcomes outside the core. More details can be found in Section 2.6.2.

than s . We say a mechanism respects property rights *before* a GDP starts if it respects property rights over $\Phi^{exante}(A)$, and we say a mechanism respects property rights *after* a GDP starts if it respects property rights over $\Phi(A)$.³² So a mechanism respects property rights before and after a GDP starts if it respects property rights over S_A .

A lottery mechanism is *ex post individually rational* if it only gives positive probability to landing schedules that are individually rational. A lottery mechanism is *ex post Pareto efficient* if it only gives positive probability to landing schedules that are Pareto efficient. Other *ex post properties* for a lottery mechanism are defined analogously.

2.4 The Mechanism

Given a set \mathcal{X} , a **(priority) ordering** (of its elements) is a bijective function $z(\mathcal{X}) : \{1, 2, \dots, |\mathcal{X}|\} \rightarrow \mathcal{X}$. Let $Z(\mathcal{X})$ be the set of orderings ($|Z(\mathcal{X})| = |\mathcal{X}|!$).

We define the **Multiple Trading Cycles** mechanism to be a mechanism that produces a landing schedule for each input using the following algorithm.

Algorithm 1:

Pre-competition Stage (Identification and Allocation of Non-scarce Resources):

- (a) Order flights in F in increasing order of e_f (break ties arbitrarily).
- (b) Assign flights sequentially to the earliest slot in S that each flight can feasibly use (there might be gaps between occupied slots). Denote the tentative landing schedule by $\hat{\Pi}$ and the set of occupied slots by S^{0-0} .
- (c) Let $F^{0-0} = F$. Find the earliest $s \in S^{0-0}$ such that
 - (c-i) $s = \hat{\Pi}^{-1}(f)$ for some $f \in F_a^{0-0}$ and $e_f > s(-1)$, where $s(-1)$ is the last slot before s in S^{0-0} (f occupies s in $\hat{\Pi}$ and will not compete for slots earlier than s . Since in $\hat{\Pi}$, all flights that arrive

³²TC in Schummer and Vohra [2013] and DASO in Schummer and Abizada [2017] indeed respect property rights after a GDP starts.

strictly earlier than f will get a slot strictly earlier than s , this condition also implies that $\forall f' \in F^{0-0}$ with $e_{f'} \leq s(-1)$, $\hat{\Pi}^{-1}(f') \leq s(-1)$. Therefore, there are sufficient slots to accommodate flights that arrive earlier than f , so they will not compete for s);

(c-ii) If f' has $e_f \leq e_{f'} \leq s$, then $f' \in F_a^{0-0}$ (all flights that want s belong to a).

(We say s is *demanded* only by airline a (not can be used only by a) if (c-i) and (c-ii) hold simultaneously (note that (c-i) is trivially satisfied if s is the earliest occupied slot) and s is demanded by more than 1 airline if either (c-i) or (c-ii) fails (such a slot might be assigned to different airlines in different feasible and non-wasteful mechanisms). For each slot in S^{0-t} (see below, $t = 0, 1, \dots$), it is a *non-scarce* resource if it is demanded only by 1 airline, and it is a *scarce* resource if it is demanded by more than 1 airline. We can further categorize the slots in S^{0-t} into four groups: type 1 slots satisfy (c-i) but not (c-ii), type 2 slots do not satisfy both, type 3 slots satisfy (c-ii) but not (c-i), and type 4 slots satisfy both. Only type 4 slots are non-scarce resources.

Example 1:

S	s_1	s_2	s_3	s_4
$\hat{\Pi}$	$f_{a,1}$	$f_{b,1}$	$f_{c,1}$	$f_{d,1}$
e_f	1	1	2	4

The number below each flight is its earliest feasible arrival time (we use n to mean s_n for e_f as mentioned, so $e_{f_{a,1}} = 1$ means $e_{f_{a,1}} = s_1$). In this example, $S^{0-0} = \{s_1, s_2, s_3, s_4\}$ and $\hat{\Pi}$ is shown in the table above. s_1 is a type 1 slot since (c-i) is trivially satisfied as s_1 is the earliest occupied slot but (c-ii) fails as $f_{b,1}$ also wants s_1 . s_2 is a type 2 slot since (c-i) fails as $f_{b,1}$ wants s_1 and (c-ii) fails as $e_{f_{a,1}} \leq e_{f_{b,1}} \leq 2$ but $f_{b,1} \notin F_a^{0-t}$. s_3 is a type 3 slot since (c-i) fails as $f_{c,1}$ wants s_2 but (c-ii) is trivially satisfied (there is no such f'). s_4 is a type 4 slot as (c-i) and (c-ii) hold simultaneously. It is easy to see that $f_{d,1}$ will not compete for slots earlier than s_4 and there are sufficient slots to accommodate flights that arrive earlier than $f_{d,1}$, so those flights will not compete for s_4 . Also, there is no other flight arrives at the same time or later. Therefore, any feasible and non-wasteful

mechanism will assign s_4 to $f_{d,1}$. The pre-competition stage identifies and allocates type 4 slots to avoid strategic issues that we will discuss later.)

Assign s to $f_{a,i}$, where $f_{a,i} \in F_a^{0-0}$ is the most important flight with $e_f \leq e_{f_{a,i}} \leq s$. Remove $f_{a,i}$ from F^{0-0} and s from S^{0-0} . If $f \neq f_{a,i}$, modify $\hat{\Pi}$ in the following way: Start from f , move each flight to the next slot in S^{0-0} until $\hat{\Pi}(f_{a,i})$ is filled. For $t = 0, 1, \dots$, update F^{0-t} to $F^{0-(t+1)}$ and S^{0-t} to $S^{0-(t+1)}$ (a type 3 slot in S^{0-t} can become a type 4 slot in $S^{0-(t+1)}$).

(d) Repeat (c) until all slots are demanded by more than 1 airline.

Denote the resulting sets by S^{main} and F^{main} .

Main Stage (Allocation of Scarce Resources):

If this is the first assignment in a GDP, for each $a \in A$, construct $S_a = \Phi^{exante}(a)$ according to the initial slot ownership function Φ^o (see footnote 27). Otherwise, let $S_a = \Phi(a)$ (from the last assignment). Create $|F_a^o|$ surrogates of a for each $a \in A$, name them $a(1), a(2), \dots, a(|F_a^o|)$. Denote the set of surrogates by \mathcal{A} ($|\mathcal{A}| = |F^o|$). Randomly select an ordering $z(\mathcal{A})$ with uniform distribution over $Z(\mathcal{A})$. For each a , rearrange its surrogates to their positions in the ordering such that they are in the order of $a(1), a(2), \dots, a(|F_a^o|)$ (alternatively, one can draw a surrogate at a time (without replacement) and let the first surrogate of $a \in A$ be $a(1)$, the second be $a(2)$, and so on). Denote the resulting ordering by z .

Let $a(i) \in \{a(1), \dots, a(|F_a^{main}|)\}$ represent the i -th important flight in F_a^{main} (according to R_a); we call it a remaining flight of a . Let $a(i) \in \{a(|F_a^{main}| + 1), \dots, a(|F_a|)\}$ represent the $i - |F_a^{main}|$ -th important flight in $F_a \setminus F_a^{main}$ (according to R_a); we call it a duplicate flight of a (each of these flights will be *assigned pro forma* the slot it was assigned in the pre-competition stage. They are here to reward airlines that gave them slots and so property rights over those slots are respected). Let $a(i) \in \{a(|F_a| + 1), \dots, a(|F_a^o|)\}$ represent a canceled flight of a ; we call it a dummy flight of a (to be consistent with RBS, airlines with canceled flights will get the same number of slots).

Let $S^1 = S^{0-0}$ and $F^1 = F$.

Step 1 - Without loss of generality, let $a(1)$ be the first flight in z .

(i) If $a(1)$ is a dummy flight, remove it and skip to the next flight in line.

If $a(1)$ is a remaining flight, let $a(1)$ pick the earliest feasible slot in $S^1 \cap S^{main}$. If $a(1)$ is a duplicate flight, let $a(1)$ pick the earliest feasible slot in $S^1 \setminus S^{main}$.

(ii) If $a(1)$ picks a slot in (a) $S^1 \cap S_a$, (b) $S^1 \setminus S_A$ or $S^1 \cap S_b$ but b has no remaining/duplicate flight in F^1 , assign this slot to $a(1)$, go to the next step.

(iii) If $a(1)$ picks a slot $s \in S^1 \cap S_b$ and b has a remaining/duplicate flight in F^1 , modify z by inserting $b(1)$ in front of $a(1)$.

(iii-i) If $b(1)$ picks a slot in $S^1 \setminus S_A$ or a slot in $S^1 \cap S_c$ but c has no remaining/duplicate flight in F^1 , assign this slot to $b(1)$ and assign s to $a(1)$, go to the next step.

Let $F^{1-0} = F^1$ and $S^{1-0} = S^1$.

(iii-ii) If $b(1)$ picks s , assign s to $b(1)$. Let $S^{1-1} = S^{1-0} \setminus \{s\}$ and $F^{1-1} = F^{1-0} \setminus \{b(1)\}$. Let $a(1)$ pick the next available slot in S^{1-1} .

When (iii-ii), (iii-iii) or (iii-iv)-(a) are repeated, for $t = 1, \dots$, update S^{1-t} to $S^{1-(t+1)}$ and F^{1-t} to $F^{1-(t+1)}$.

(iii-iii) If $b(1)$ picks a slot s' in $S^1 \cap S_b \setminus \{s\}$, assign this slot to $b(1)$, modify z by inserting $b(2)$ behind $b(1)$. If there is no $b(2)$ or $b(2)$ is a dummy flight, then it will be in case (ii)-(b); otherwise, apply (iii) to $b(2)$ with $S^{1-1} = S^{1-0} \setminus \{s'\}$ in place of S^1 and $F^{1-1} = F^{1-0} \setminus \{b(1)\}$ in place of F^1 .

(iii-iv) If $b(1)$ picks a slot $s'' \in S^1 \cap S_c$ and c has a remaining/duplicate flight in F^1 , modify z by inserting $c(1)$ in front of $b(1)$, apply (iii)' to $c(1)$ with $b(1)$ in place of $a(1)$, where (iii)' is a generalization of (iii): For (iii-i), replace “assign this slot to...” by “then it will be in case (iii-iv)-(b).” For (iii-iii), replace “then it will be in case (ii)-(b); otherwise, apply (iii)” by “then it will be in case (iii-iv)-(b); otherwise, apply (iii)'.”

For each airline $a \in A$, let s_a be some slot in $S^{1-t} \cap S_a$ for some $t \in \mathbb{N}$. If there is (a) a cycle $(x(k), s_y, y(\cdot), \dots, s_z, z(\cdot), s_x)$ of slots and most important remaining/duplicate flights such that $x(\cdot)$

picks $s_y, \dots, z(\cdot)$ picks s_x ($a(1)$ is not in the cycle), remove all flights in the cycle by assigning them the slots they pick. Let $F^{1-(t+1)} = F^{1-t} \setminus \{x(k), y(\cdot), \dots, z(\cdot)\}$ and $S^{1-(t+1)} = S^{1-t} \setminus \{s_y, \dots, s_z, s_x\}$.

If s_x is demanded by the flight that inserted $x(k)$, let it pick the next available slot (if $x(k)$ is inserted by $x(k-1)$, then check if s_x is demanded by the flight that inserted $x(k-1)$, and so on. Denote the flight of x that was inserted by another airline by $x(\cdot)$); otherwise, modify z by inserting $x(k+1)$ behind $x(k)$. If there is no $x(k+1)$ or $x(k+1)$ is a dummy flight, then it will be in case (iii-iv)-(b). Otherwise, apply (iii)' to $x(k+1)$ with $S^{1-(t+1)}$ in place of S^1 , $F^{1-(t+1)}$ in place of F^1 , and the flight that inserted $x(\cdot)$ in place of $a(1)$.

After possible repetitions of (iii-ii), (iii-iii) and (iii-iv)-(a), at the end, there must be (b) a chain $(a(1), s_d, d(\cdot), \dots, s_g, g(\cdot), s''')$ with s''' in $S^{1-T} \setminus S_A$ or $S^{1-T} \cap S_w$ but w has no remaining/duplicate flight in F^{1-T} , or (c) a cycle $(a(1), s_d, d(\cdot), \dots, s_g, g(\cdot), s_a)$. Remove all flights in the chain/cycle by assigning them the slots they pick. Go to the next step. (In both cases, d does not have to be b , and even $d = b$, s_d does not have to be s . $a(1)$ would pick the next available slot if b uses s or there is a cycle that contains s but not $a(1)$, in which b trades s for a slot owned by another airline.)

Denote the resulting sets by S^2 and F^2 .

Step $n \geq 2$ - Without loss of generality, let $a(i)$ be the next flight in line.

(i) If $a(i)$ is a dummy flight, remove it and skip to the next flight in line.

If $a(i)$ is a remaining flight, let $a(i)$ pick the earliest feasible slot in $S^n \cap S^{main}$. If $a(i)$ is a duplicate flight, let $a(i)$ pick the earliest feasible slot in $S^n \setminus S^{main}$.

(ii) If $a(i)$ picks a slot in (a) $S^n \cap S_a$, (b) $S^n \setminus S_A$ or $S^n \cap S_b$ but b has no remaining/duplicate flight in F^n , assign this slot to $a(i)$, go to the next step.

(iii) If $a(i)$ picks a slot $s \in S^n \cap S_b$ and b has a remaining/duplicate flight in F^n , modify z by inserting $b(j)$ in front of $a(i)$, where $b(j)$ is b 's most important remaining/duplicate flight in F^n .

(iii-i) If $b(j)$ picks a slot in $S^n \setminus S_A$ or a slot in $S^n \cap S_c$ but c has no remaining/duplicate flight in F^n , assign this slot to $b(j)$ and assign s to $a(i)$, go to the next step.

Let $F^{n-0} = F^n$ and $S^{n-0} = S^n$.

(iii-ii) If $b(j)$ picks s , assign s to $b(j)$. Let $S^{n-1} = S^{n-0} \setminus \{s\}$ and $F^{n-1} = F^{n-0} \setminus \{b(j)\}$. Let $a(i)$ pick the next available slot in S^{n-1} .

When (iii-ii), (iii-iii) or (iii-iv)-(a) are repeated, for $t = 1, \dots$, update S^{n-t} to $S^{n-(t+1)}$ and F^{n-t} to $F^{n-(t+1)}$.

(iii-iii) If $b(j)$ picks a slot s' in $S^n \cap S_b \setminus \{s\}$, assign this slot to $b(j)$, modify z by inserting $b(j+1)$ behind $b(j)$. If there is no $b(j+1)$ or $b(j+1)$ is a dummy flight, then it will be in case (ii)-(b); otherwise, apply (iii) to $b(j+1)$ with $S^{n-1} = S^{n-0} \setminus \{s'\}$ in place of S^n and $F^{n-1} = F^{n-0} \setminus \{b(j)\}$ in place of F^n .

(iii-iv) If $b(j)$ picks a slot $s'' \in S^n \cap S_c$ and c has a remaining/duplicate flight in F^n , modify z by inserting $c(\cdot)$ in front of $b(j)$, where $c(\cdot)$ is c 's most important remaining/duplicate flight in F^n . Apply (iii)' to $c(\cdot)$ with $b(j)$ in place of $a(i)$ (see the description for (iii)' in Step 1).

For each airline $a \in A$, let s_a be some slot in $S^{n-t} \cap S_a$ for some $t \in \mathbb{N}$. If there is (a) a cycle $(x(k), s_y, y(\cdot), \dots, s_z, z(\cdot), s_x)$, remove all flights in the cycle by assigning them the slots they pick. Let $F^{n-(t+1)} = F^{n-t} \setminus \{x(k), y(\cdot), \dots, z(\cdot)\}$ and $S^{n-(t+1)} = S^{n-t} \setminus \{s_y, \dots, s_z, s_x\}$.

If s_x is demanded by the flight that inserted $x(k)$, let it pick the next available slot (if $x(k)$ is inserted by $x(k-1)$, then check if s_x is demanded by the flight that inserted $x(k-1)$, and so on. Denote the flight of x that was inserted by another airline by $x(\cdot)$); otherwise, modify z by inserting $x(k+1)$ behind $x(k)$. If there is no $x(k+1)$ or $x(k+1)$ is a dummy flight, then it will be in case (iii-iv)-(b). Otherwise, apply (iii)' to $x(k+1)$ with $S^{n-(t+1)}$ in place of S^n and $F^{n-(t+1)}$ in place of F^n , and the flight that inserted $x(\cdot)$ in place of $a(i)$.

After possible repetitions of (iii-ii), (iii-iii) and (iii-iv)-(a), at the end, there must be (b) a chain $(a(i), s_d, d(\cdot), \dots, s_g, g(\cdot), s''')$ with s''' in $S^{n-T} \setminus S_A$ or $S^{n-T} \cap S_w$ but w has no remaining/duplicate flight in F^{n-T} , or (c) a cycle $(a(i), s_d, d(\cdot), \dots, s_g, g(\cdot), s_a)$. Remove all flights in the chain/cycle by assigning them the slots they pick. Go to the next step.

Denote the resulting sets by S^{n+1} and F^{n+1} .

The main stage stops when $F^k = \emptyset$ for some $k \geq 1$.

Supplemental Stage:

Let $V = S \setminus S^{0-0}$ be the set of remaining vacant slots. Start from the earliest slot in $V \cap S_A$, if a slot is in some S_a and a has a dummy flight, assign it to a and remove a dummy flight of a . Repeat until there is no more slot can be assigned by the above procedure. Denote the resulting set by V^1 . Assign the earliest slot in V^1 to the dummy flight with the highest order in z . Repeat until there is no more dummy flight.

Remark 1: (iii-i) to (iii-iv), which provide a complete picture of all cycles that are triggered by $a(i)$, are not explicitly described in the YGMH-IGYT algorithm — if an existing tenant j is inserted to the top by i , then the step can simply start over. This is because when j picks a vacant house or a house of another existing tenant who is already assigned another house, i is next in line to pick the vacated house of i (in the language of our model, this vacated house is a slot in $S^n \cap S_i$ but i has no remaining/duplicate flight in F^n). By contrast, in Algorithm 1, when a flight $b(j)$ is inserted to the top by $a(i)$, the step cannot simply start over because when $b(j)$ picks a slot in $S^n \setminus S_A$ or a slot in $S^n \cap S_c$ but c has no remaining/duplicate flight in F^n , even though $a(i)$ is next in line, b might still have remaining/duplicate flights and yet we do not want to insert $b(j+1)$ in this case. Therefore, we need (iii-i) to (iii-iv) to exhaust all possibilities after a flight is being inserted.

2.4.1 An Alternative Algorithm

[Abdulkadiroğlu and Sönmez \[1999\]](#) provide 2 algorithms to find the outcome of the YGMH-IGYT mechanism. One is the YGMH-IGYT algorithm, and the other is the top trading cycles algorithm. Algorithm 1 generalizes the YGMH-IGYT algorithm, and Algorithm 2 below generalizes the top trading cycles algorithm. Comparing these two algorithms, Algorithm 1 is more transparent on how cycles form, while Algorithm 2 is more transparent on how it works. (We change the main

stage from “Step 1....”)

Algorithm 2:

In general, at Step h :

- Each remaining flight in F^h points to the earliest feasible slot in $S^h \cap S^{main}$;
- Each duplicate flight in F^h points to the earliest feasible slot in $S^h \setminus S^{main}$;
- Each slot in $S^h \setminus S_A$ or $S^h \cap S_a$ where a has no remaining/duplicate flight in F^h points to the remaining/duplicate flight in F^h with the highest priority in z (if such flight is a dummy flight, remove it and skip to the next flight in line); and
- Each slot in $S^h \cap S_a$ where a has a remaining/duplicate flight in F^h points to the most important flight in $F^h \cap F_a$.

Since $|F|$ and $|S^{0-0}|$ are finite, there is at least one cycle. Each airline can be in at most 1 cycle in each step. Every flight in a cycle is assigned (or assigned pro forma) the slot that it points to and removed with such slot. Whenever there is a slot in $S^h \setminus S_A$ or $S^h \cap S_a$ where a has no remaining/duplicate flight in F^h in a cycle, the remaining/duplicate flight in F^h with the highest priority in z is also in the cycle. The set of slots that are not removed at the end of Step h is denoted by S^{h+1} . The set of flights that are not removed at the end of Step h is denoted by F^{h+1} . The main stage stops when $F^h = \emptyset$ for some $h \geq 1$.

Theorem 1: For a given ordering z , Algorithm 1 and Algorithm 2 produce the same outcome.

Note that in Algorithm 2, a cycle that is not removed at any step remains a cycle at the next step (as the earliest feasible slots for the flights in a cycle still remain). Therefore, removing one cycle (instead of multiple) at a time will not change its outcome. Algorithm 1 removes cycles in Algorithm 2 one at a time but possibly multiple at a step. Recall (iii-ii) in Algorithm 1: If $b(j)$

picks $s \in S^n \cap S_b$ (the slot demanded by $a(i)$), assign s to $b(j)$, and let $a(i)$ pick the next available slot. In this case, s leaves earlier than $a(i)$ in Algorithm 2, so $a(i)$ cannot pick s . Recall (iii-iii) in Algorithm 1: If $b(j)$ picks a slot s' in $S^n \cap S_b \setminus \{s\}$, assign this slot to $b(j)$, modify z by inserting $b(j+1)$ behind $b(j)$. In this case, $b(j)$ leaves earlier than $a(i)$ in Algorithm 2, so $b(j+1)$ should be in place of $b(j)$. In general, for any given ordering z , any flights that are inserted in front of $a(i)$ but not in a cycle/chain that contains $a(i)$ in Algorithm 1 leave earlier than $a(i)$ in Algorithm 2, and any flights that are inserted in front of $a(i)$ and in a cycle/chain that contains $a(i)$ in Algorithm 1 are in the same cycle in Algorithm 2.

2.4.2 Some Observations

Both algorithms stop in at most $|F^o|$ steps. The following claim implies that S^{0-0} is assigned in any feasible and non-wasteful mechanism.

Claim 1: S^{0-0} is assigned in any feasible and non-wasteful landing schedule Π .

There is a pattern for scarce resources. Scarce resources are sequences of adjacent slots in S^{main} such that each **sequence** starts with a type 1 slot and ends with a type 3 slot. Each sequence contains one type 1 slot, some type 3 slots, and possibly some type 2 slots. In any feasible and non-wasteful landing schedule, a flight that gets a slot in a sequence in $\hat{\Pi}$ will *always* get a slot in the same sequence (it is infeasible to get a slot earlier than the sequence as (c-i) holds for the first slot in the sequence. It is wasteful if the flight gets a slot later than the sequence—by feasibility, the number of flights that can feasibly use the slots in the sequence is fixed, the flight gets a slot later than the sequence implies there exists some slot in S^{0-0} that is empty); a flight that gets a slot outside a sequence in $\hat{\Pi}$ will *never* get a slot in that sequence (it is infeasible to get a slot in a sequence earlier than the slot it gets in $\hat{\Pi}$ as (c-i) holds for the type 1 or type 4 slot that locates right after the sequence in S^{main} . It is wasteful if the flight gets a slot in a sequence later than the slot it gets in $\hat{\Pi}$ —by feasibility, the number of flights that can feasibly use the slots earlier than this

sequence is fixed, the flight gets a slot in this sequence implies there exists some slot earlier than this sequence in S^{0-0} is empty). A type 1 slot can be followed by a type 3 slot. Also, there might be more than one type 3 slot.

Example 2:

S	s_1	s_2	s_3
$\hat{\Pi}$	-	$f_{a,1}$	$f_{b,1}$
e_f	-	1	2

S	s_1	s_2	s_3	s_4
$\hat{\Pi}$	-	$f_{a,1}$	$f_{b,1}$	$f_{b,2}$
e_f	-	1	2	2

In the left table above, $s_1 \notin S^{0-t}$. It is easy to see that s_1 is a type 1 slot and s_3 is a type 3 slot in S^{0-t} . In the right table above, s_4 is also a type 3 slot.

Airline a 's probability of getting the first position in z is $\frac{|F_a^o|}{|F^o|}$. Given it gets the first position, the probability of getting the second position declines to $\frac{|F_a^o|-1}{|F^o|-1}$. Airline a "pays" positions $a(|F_a^1|+1), \dots, a(|F_a|)$ (possibly to other airlines) for the slots it obtains in the pre-competition stage. These positions have zero values to a (with respect to \succ_a), but they might be valuable to other airlines.

Recall that when (i) no airline owns a canceled flight, (ii) each airline owns exactly one non-canceled flight, and (iii) each airline owns at most one GDP slot, MTC reduces to YGMH-IGYT (with random ordering) as the pre-competition stage and the supplemental stage become redundant. However, one can modify YGMH-IGYT by using the pre-competition stage: Fix an ordering. If a house is acceptable to only one agent i and the house is his top choice, assign this house to i . If the house is owned by some agent j , let j take i 's position if it is earlier than j 's position (in the ordering) and i 's house if i owns a house. Repeat this procedure until there is no more such house. Then run YGMH-IGYT with the reduced ordering (with these i 's eliminated). The outcome does not change because j would be inserted in front of i when it is i 's turn or i has been inserted as some agent demands i 's house. Each agent that is assigned a house here always obtains the same house (in any acceptable and non-wasteful mechanism). In Section 2.6.1, we discuss another modification that is also meaningful to MTC but insignificant to YGMH-IGYT.

2.4.3 Subsequent Reassignments

In the current GDP, the first assignment is created by running RBS and possible Compression. But as airlines update their information, Compression might be run multiple times. MTC can also be used to perform reassignments.

A new instance is a tuple $\tilde{I} = (\tilde{S}, \tilde{A}, \tilde{F}^o, \tilde{R}, \tilde{e}, \tilde{\Phi})$, where $\tilde{\Phi}$ is the slot ownership function from the last assignment restricted to \tilde{A} and \tilde{S} .³³ If an airline freezes a flight in a slot $s \in S_a$ (“airlines will also have the capability to freeze flights they don’t want moved up through the submission of an earliest time of arrival” [Wambsganss, 1996]), then $s \notin \tilde{S}$. If $\tilde{F}_a = \emptyset$ for some airline a (all flights in F_a^o are canceled or frozen in some slots), then $a \notin \tilde{A}$ and $F_a^o \not\subseteq \tilde{F}^o$. For $a \in \tilde{A}$, now its preference over landing schedules, $\tilde{\succ}_a$, is induced by \tilde{R}_a and \tilde{e}_a . Vacant slots obtained from the last supplemental stage might become valuable (in the sense that airlines can use them or trade them for better slots) in this new instance.³⁴

2.4.4 Examples

Since MTC is a rather complex mechanism, a giant example that shows all of its features would be quite involved. Thus, we use two examples to demonstrate MTC.

Example 3:

³³In this chapter, we assume l is constant across instances. But if l changes, to construct $\tilde{\Phi}$, one can either modify the formula in footnote 27 or ignore the change in l .

³⁴The supplemental stage of MTC assigns the earliest slots in $V \cap S_a$ to a if a has dummy flights but not some random slots in V . Suppose when a has a dummy flight, we assign the earliest slot in $V \cap S_a$, s , to $b \in A$ and assign a random slot s' to a . Then if a wants s but not s' in the next instance, it has to trade with b (this potentially hurts some less important flights of a). Here, s cannot be compared with s' in the former instance but s is better than s' in the latter instance. Since a 's preference might change, it seems that assigning s to a in the former instance is more appropriate.

S^o	s_1^o	s_2^o	s_3^o	s_4^o	s_5^o	s_6^o	s_7^o	s_8^o	s_9^o	s_{10}^o	s_{11}^o	s_{12}^o	s_{13}^o	s_{14}^o
F^o	$f_{c,3}$	$f_{b,(1)}$	$f_{a,1}$	$f_{a,2}$	$f_{a,3}$	$f_{b,2}$	$f_{c,2}$	$f_{c,1}$	$f_{b,1}$	$f_{a,4}$	$f_{a,5}$	$f_{a,6}$	$f_{b,3}$	$f_{a,7}$
Φ^o	c	b	a	a	a	b	c	c	b	a	a	a	b	a
e_f	1	-	2	3	4	2	5	6	8	7	10	10	12	10
R	3	-	1	2	3	2	2	1	1	4	5	6	3	7

$S (l=2)$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}
$\hat{\Pi}$		$f_{c,3}$	$f_{a,1}$	$f_{b,2}$	$f_{a,2}$	$f_{a,3}$	$f_{c,2}$	$f_{c,1}$	$f_{a,4}$	$f_{b,1}$	$f_{a,6}$	$f_{a,5}$	$f_{a,7}$	$f_{b,3}$
Φ^{exante}			a		c		a							
RBS+Comp	$f_{c,3}$	$f_{b,2}$	$f_{a,1}$	$f_{a,2}$	$f_{a,3}$	$f_{c,2}$	$f_{c,1}$	$f_{b,1}$	$f_{a,4}$	$f_{a,5}$	$f_{a,6}$	$f_{a,7}$	$f_{b,3}$	b
MTC	$f_{c,3}$	$f_{a,1}$	$f_{a,2}$	$f_{a,3}$	$f_{b,2}$	$f_{c,1}$	$f_{a,4}$	$f_{b,1}$	$f_{c,2}$	$f_{a,5}$	$f_{a,6}$	$f_{b,3}$	$f_{a,7}$	b

Flight $f_{b,(1)}$ is a canceled flight, where the subscript $b, (1)$ indicates it is a canceled flight of b with index (1) (we use this notation occasionally and might simply use “-” to indicate a flight is canceled). R is the importance ranking profile. In this example, $s_2 \in \Phi^{exante}(a)$ (s_3^o and $s_4^o \in \Phi^o(a)$ cover the entire time interval of s_2) but a 's most important flight $f_{a,1}$ with $e_{f_{a,1}} = 2$ cannot use it because of RBS. This shows RBS does not respect property rights before a GDP starts. Since s_2 would be assigned to $f_{a,1}$ in any individually rational landing schedule, RBS might lead to outcome that is not individually rational and thus not in the core (any individually rational outcome dominates the RBS+Comp outcome via subgroup $\{a\}$). Also, the RBS+Comp outcome is not Pareto efficient: $f_{c,1}R_c f_{c,2}$ but $e_{f_{c,1}} \leq \Pi(f_{c,2}) < \Pi(f_{c,1})$.

Putting e with $\Phi^{RBS+Comp}$ (the induced ownership function of RBS+Comp outcome) back into Compression will give the same outcome, and that implies Compression is not Pareto efficient, not individually rational and thus not core-selecting.³⁵ The violation of individual rationality is not because s_2 has not been assigned to $f_{a,1}$ (RBS+Compression produces an assignment with $s_2 \in \Phi^{RBS+Comp}(b)$) but because the landing schedule is not self-optimized yet (recall we define individual rationality through self-optimization).

³⁵Example 5 below shows that Compression is not core-selecting but not because of violations of individual rationality.

Airline a 's preference over landing schedules in this instance is the following:

$$\succsim_a: (s_2, s_3, s_4, s_7, s_{10}, s_{10}, s_{10}), (s_2, s_3, s_4, s_7, s_{10}, s_{10}, s_{11}), \dots, \\ (s_2, s_3, s_4, s_7, s_{10}, s_{11}, s_{12}), \dots, (s_3, s_3, s_4, s_7, s_{10}, s_{10}, s_{10}), \dots,$$

where each element is of the form $(\Pi(f_{a,1}), \Pi(f_{a,2}), \dots, \Pi(f_{a,7}))$. Airline b and c 's preferences can be expressed similarly. Now we run MTC.

Pre-competition stage: A tentative landing schedule $\hat{\Pi}$ is created as shown in the second table above and the set of occupied slot S^{0-0} is $\{s_1, s_2, \dots, s_{13}\}$.

The first type 4 slot is s_1 as it is demanded only by c with $f_{c,3}$, so $f_{c,3}$ is assigned this slot. The resulting sets are S^{0-1} and F^{0-1} .

The second type 4 slot is s_{10} as it is demanded only by a with $f_{a,5}, f_{a,6}, f_{a,7}$, so the most important flight among these three, $f_{a,5}$, is assigned this slot. Since $\hat{\Pi}(f_{a,5}) = s_{11}$, s_{11} is empty now. Update $\hat{\Pi}$ by moving $f_{a,6}$ into s_{11} . The resulting sets are S^{0-2} and F^{0-2} .

The third type 4 slot is s_{11} as it is demanded only by a with $f_{a,6}, f_{a,7}$, so the most important flight among these two, $f_{a,6}$, is assigned this slot. The resulting sets are S^{0-3} and F^{0-3} . (Note that $f_{a,5} \notin F^{0-2}$ and $s_{10} \notin S^{0-2}$. Without the previous iteration, (c-i) does not hold for s_{11} . In S^{0-2} , $s(-1)$ of s_{11} is s_9 .)

Now all slots are demanded by more than 1 airline (s_{12} is demanded by a with $f_{a,6}$ and b with $f_{b,3}$). $S^{main} = S^{0-3} = \{s_2, s_3, \dots, s_9, s_{12}, s_{13}\}$ and

$$F^{main} = F^{0-3} = \{f_{a,1}, f_{a,2}, f_{a,3}, f_{a,4}, f_{a,7}, f_{b,1}, f_{b,2}, f_{b,3}, f_{c,2}, f_{c,3}\}.$$

Main stage: For each a , $S_a = \Phi^{exante}(a)$ is constructed as shown in the second table above.

$$z = (a(1), a(2), a(3), b(1), c(1), a(4), b(2), b(3), b(4), a(5), a(6), a(7), c(2), c(3)).$$

$a(1), \dots, a(5)$ represent $f_{a,1}, f_{a,2}, f_{a,3}, f_{a,4}, f_{a,7}$ (remaining flights), respectively. $a(6)$ and $a(7)$ represent $f_{a,5}$ and $f_{a,6}$ (duplicate flights), respectively. $b(1), \dots, b(3)$ represent $f_{b,1}, \dots, f_{b,3}$, respectively. $b(4)$ represents $f_{b,(1)}$ (dummy flight). $c(1)$ and $c(2)$ represent $f_{c,1}$ and $f_{c,2}$, respectively. Lastly, $c(3)$ represents $f_{c,3}$. $S^1 = S^{0-0}$ and $F^1 = F$.

Step 1: $a(1)$ picks $s_2 \in S^1 \cap S_a$ for $f_{a,1}$. $f_{a,1}$ is assigned this slot. The resulting sets are S^2 and F^2 . ((ii)-(a))

Step 2: $a(2)$ picks $s_3 \in S^2 \setminus S_A$ for $f_{a,2}$. $f_{a,2}$ is assigned this slot. The resulting sets are S^3 and F^3 . ((ii)-(b))

Step 3: $a(3)$ picks $s_4 \in S^3 \cap S_c$ for $f_{a,3}$ and c has a remaining/duplicate fight in F^3 , modify z by inserting $c(1)$, c 's most important remaining/duplicate fight in F^3 , in front of $a(3)$. $c(1)$ picks $s_6 \in S^3 \cap S_a$ for $f_{c,1}$. $a(3)$ and $c(1)$ form a cycle. $f_{c,1}$ is assigned s_6 and $f_{a,3}$ is assigned s_4 . The resulting sets are S^4 and F^4 . ((iii-vi)-(c))

Step 4: $b(1)$ picks $s_8 \in S^4 \setminus S_A$ for $f_{b,1}$. $f_{b,1}$ is assigned this slot. The resulting sets are S^5 and F^5 .

Step 5: Note that $c(1)$ has been inserted in front. Now $a(4)$ picks $s_7 \in S^5 \setminus S_A$. $f_{a,4}$ is assigned this slot. The resulting sets are S^6 and F^6 .

Step 6: $b(2)$ picks $s_5 \in S^6 \setminus S_A$ for $f_{b,2}$ (s_2, s_3 and s_4 have already been assigned). $f_{b,2}$ is assigned this slot. The resulting sets are S^7 and F^7 .

Step 7: $b(3)$ picks $s_{12} \in S^7 \setminus S_A$ for $f_{b,3}$. $f_{b,3}$ is assigned this slot. The resulting sets are S^8 and F^8 .

Step 8: $b(4)$ is a dummy flight, remove it and skip to $a(5)$. ((i))

$a(5)$ picks $s_{13} \in S^8 \setminus S_A$ for $f_{a,7}$ ($a(5)$ is a remaining flight, so it picks the earliest feasible slot in $S^8 \cap S^{main}$. s_{10} and s_{11} are in $S^8 \setminus S^{main}$, and s_{12} has been assigned before). $f_{a,7}$ is assigned this slot. The resulting sets are S^9 and F^9 .

Step 9: $a(6)$ picks $s_{10} \in S^9 \setminus S_A$ for $f_{a,5}$ ($a(6)$ is a duplicate flight, so it picks the earliest feasible

slot in $S^9 \setminus S^{main}$. Note that s_{10} is the slot $f_{a,5}$ was assigned in the pre-competition stage). $f_{a,5}$ is assigned this slot pro forma. The resulting sets are S^{10} and F^{10} .

Step 10: $a(7)$ picks $s_{11} \in S^{10} \setminus S_A$ for $f_{a,6}$. $f_{a,6}$ is assigned this slot pro forma. The resulting sets are S^{11} and F^{11} .

Step 11: $c(2)$ picks $s_9 \in S^{11} \setminus S_A$ for $f_{c,2}$ (s_5, s_6, s_7 and s_8 have already been assigned). $f_{c,2}$ is assigned this slot. The resulting sets are S^{12} and F^{12} .

Step 12: $c(3)$ picks $s_1 \in S^{12} \setminus S_A$ for $f_{c,3}$. $f_{c,3}$ is assigned this slot pro forma. The resulting sets are S^{13} and F^{13} . Note that $F^{13} = \emptyset$. The main stage stops here.

Supplemental stage:

$V = S \setminus S^{0-0} = \{s_{14}, s_{15}, \dots\}$. There is no slot in $V \cap S_a$ for some a , so $V^1 = V$. Assign the earliest slot s_{14} to b .

Example 4:

S	s_1	s_2	s_3	s_4	s_5	s_6
F^o	$f_{c,1}$	$f_{a,1}$	$f_{a,2}$	$f_{a,3}$	$f_{a,(1)}$	$f_{b,1}$
Φ	c	a	a	a	a	b
\tilde{e}_f	5	2	4	1	-	2
R	1	1	2	3	-	1
$\hat{\Pi}$	$f_{a,3}$	$f_{b,1}$	$f_{a,1}$	$f_{a,2}$	$f_{c,1}$	
MTC	$f_{a,3}$	$f_{a,1}$	$f_{b,1}$	$f_{a,2}$	$f_{c,1}$	a

In this example, $f_{c,1}$ has been delayed and cannot use the slot it was assigned in the last assignment. The same for $f_{a,2}$.

Pre-competition stage:

A tentative landing schedule $\hat{\Pi}$ is created as shown in the table above, and the set of occupied slot S^{0-0} is $\{s_1, s_2, \dots, s_5\}$.

The first type 4 slot is s_1 as it is demanded only by a with $f_{a,3}$, so $f_{a,3}$ is assigned this slot. The resulting sets are S^{0-1} and F^{0-1} .

The second type 4 slot is s_4 as it is demanded only by a with $f_{a,2}$, so $f_{a,2}$ is assigned this slot. The resulting sets are S^{0-2} and F^{0-2} .

The third type 4 slot is s_5 as it is demanded only by c with $f_{c,1}$, so $f_{c,1}$ is assigned this slot. The resulting sets are S^{0-3} and F^{0-3} .

Now all slots are demanded by more than 1 airline. $S^{main} = S^{0-3} = \{s_2, s_3\}$ and $F^{main} = F^{0-3} = \{f_{a,1}, f_{b,1}\}$.

Main stage:

For each a , let $S_a = \Phi(a)$ from the last assignment (in the table above).

$$z = (b(1), a(1), a(2), c(1), a(3), a(4)).$$

$a(1)$ represents $f_{a,1}$. $a(2)$ and $a(3)$ represent $f_{a,2}$ and $f_{a,3}$, respectively. $a(4)$ represents $f_{a,(1)}$. $b(1)$ represents $f_{b,1}$. $c(1)$ represents $f_{c,1}$. $S^1 = S^{0-0}$ and $F^1 = F$.

Step 1: $b(1)$ picks $s_2 \in S^1 \cap S_a$ for $f_{b,1}$ and a has a remaining/duplicate fight in F^1 , modify z by inserting $a(1)$ in front of $b(1)$. $a(1)$ picks $s_2 \in S^1 \cap S_a$ for $f_{a,1}$. Assign s_2 to $f_{a,1}$. The resulting sets are S^{1-1} and F^{1-1} . ((iii)-(ii))

$b(1)$ picks the next available slot $s_3 \in S^{1-1} \cap S_a$ and a has a remaining/duplicate fight in F^{1-1} , modify z by inserting $a(2)$ in front of $b(1)$.

$a(2)$ picks $s_4 \in S^{1-1} \cap S_a$. Assign s_4 pro forma to $f_{a,2}$ and modify z by inserting $a(3)$ behind $a(2)$. The resulting sets are S^{1-2} and F^{1-2} . ((iii)-(iii))

$a(3)$ picks $s_1 \in S^{1-2} \cap S_c$ for $f_{a,3}$ and c has a remaining/duplicate fight in F^{1-2} , modify z by inserting $c(1)$ in front of $a(3)$.

$c(1)$ picks $s_5 \in S^{1-2} \cap S_a$ for $f_{c,1}$. $a(3)$ and $c(1)$ form a cycle. $f_{c,1}$ is assigned pro forma s_5 and $f_{a,3}$ is assigned pro forma s_1 . The resulting sets are S^{1-3} and F^{1-3} . ((iii-vi)-(a))

s_5 is not demanded by $b(1)$ (s_5 is some s_x in Algorithm 1, and $b(1)$ is the flight that inserted

$x(k-1)$, which is $a(2)$ here), so $a(4)$ should be inserted behind $a(3)$. But $a(4)$ is a dummy flight, which means a has no remaining/duplicate flight in F^{1-3} . So $f_{b,1}$ is assigned s_3 . The resulting sets are S^{1-4} and F^{1-4} . ((iii-vi)-(b))

Note that $F^{1-4} = \emptyset$. The main stage stops here.

Supplemental stage:

$V = S \setminus S^{0-0} = \{s_6, \dots\}$. $s_6 \in V \cap S_b$ but b has no dummy flight. So $V^1 = V$. Assign the earliest slot s_6 to a .

2.5 Properties of the Mechanism

Proposition 1: The multiple trading cycles mechanism ϕ is regular, ex post feasible, ex post non-wasteful, and respects property rights over S_A .

Example 5:

S	s_1	s_2	s_3	s_4	s_5
F^o	-	-	$f_{a,1}$	$f_{c,1}$	$f_{b,1}$
Φ	a	b	a	c	b
e_f	-	-	2	1	1
Compression	$f_{c,1}$	$f_{b,1}$	$f_{a,1}$	b	a
MTC	$f_{b,1}$	$f_{a,1}$	$f_{c,1}$	a	b

We have shown that RBS does not respect property rights before a GDP starts by Example 3. Example 5 shows that Compression does not respect property rights after a GDP starts and is not core-selecting but not because of violations of individual rationality. For a , s_2 is better than s_1 and s_3 , while s_4 is not better than s_1 or s_3 . Yet in the first step of Compression, c obtains s_1 for $f_{c,1}$ while a obtains s_4 .³⁶ By contrast, in MTC, a trades s_1 for s_2 . The Compression outcome is not in the core because it is dominated by the MTC outcome via subgroup $\{a, b\}$.

Proposition 2: The multiple trading cycles mechanism ϕ is ex post individually rational.

³⁶Then b obtains s_4 for $f_{b,1}$ while a obtains s_5 . Finally, $f_{b,1}$ is moved to s_2 .

If an airline a only uses its own slots in S_a under any ordering z , $\phi_a(R, e) = \Pi_a^{S_a}$ with probability 1. But if it uses some other slots under some ordering z , $\phi_a(R, e)$ would be preferred to $\Pi_a^{S_a}$.

Proposition 3: The multiple trading cycles mechanism ϕ is ex post Pareto efficient.

Suppose $\varphi^z(R, e)$ is a realized landing schedule of MTC. $\Phi^{\varphi^z(R, e)}$ is its induced ownership function. Proposition 2 and 3 imply that putting (R, e) with $\Phi^{\varphi^z(R, e)}$ back into MTC would not change the outcome because there will be no trading cycle for any ordering z' . If $a(i)$ picks the slot of $b(j)$, then $b(k)$ will be inserted in front of $a(i)$ where $b(k)$ is the most important remaining flight of b ($b(k)$ and $b(j)$ will not be duplicate flights as $a(i)$ demands the slot of $b(j)$). The procedure continues in a similar way until some flight f picks its top choice. Then the flight that inserted f picks its own slot. The same for the next flight that inserted it, and so on. At the end, $a(i)$ picks the next slot and the same argument applies until it gets the slot it was assigned in $\varphi^z(R, e)$.

In a housing market with strict preferences, the core is a subset of the Pareto set.³⁷ But in an airport slot allocation problem with strict preferences, the core might not be a subset of the Pareto set since $S_A \neq S$.³⁸ A landing schedule in the core is not necessary Pareto efficient if some airline can benefit by having a slot that is not owned by some airline. The following example shows a Pareto efficient landing schedule might not be in the core.

Example 6:

F^o	$f_{a,1}$	$f_{a,2}$	$f_{b,1}$	S	s_1	s_2	s_3
e_f	1	1	1	Φ	a		a
R	1	2	1	Π	$f_{b,1}$	$f_{a,1}$	$f_{a,2}$
				Π'	$f_{a,1}$	$f_{b,1}$	$f_{a,2}$

³⁷In that problem, the core is equivalent to the core defined by weak domination, and the core defined by weak domination is a subset of the Pareto set.

³⁸Indeed, this is also true for a house allocation with existing tenants problem with strict preferences. A similar but distinct result can be found in [Roth and Sotomayor \[1992\]](#): In a two-sided college admissions problem, the college-optimal stable matching does not need to be Pareto efficient for the colleges (Theorem 5.10), but this matching is in the core defined by weak domination (Theorem 5.36).

Π is Pareto efficient, but it is not in the core since a can use slots only in $\Phi(a)$ and be better off (as in Π'). Proposition 3 and Theorem 2 below imply that MTC selects landing schedules from the intersection of the core and the Pareto Set.

Theorem 2: The multiple trading cycles mechanism ϕ is ex post core-selecting.

Theorem 3: The multiple trading cycles mechanism ϕ is strategy-proof.

There are two sources of strategy-proofness. The first one is the randomness introduced in the main stage together with a feature of lexicographic preference that it does not sacrifice the benefit of a flight for the benefit of a less important flight. By truth-telling, MTC minimizes the expected delays for each airline lexicographically. But if an airline deviates, it might be able to reduce the expected delays for some of its flights, but the expected delay for a more important flight will increase.³⁹ The next example will show the importance of randomness.

Recall that when (ii) each airline owns exactly one non-canceled flight and (iv) no airline owns a GDP slot, MTC reduces to random serial dictatorship; if we fix an ordering z , it further reduces to serial dictatorship. However, when (iv) holds but not (ii), MTC with fixed ordering is different from serial dictatorship. In this context, airlines are agents, so serial dictatorship would allow airline a to pick all slots it wants, then allow airline b to pick all slots it wants (among those remaining), etc. It is well-known that serial dictatorship is strategy-proof, and the reason is that an airline does not need to manipulate its report to get the best set of available slots, but MTC with fixed ordering does not have this feature. We illustrate this point by the following example.

Example 7: (MTC with fixed ordering is not strategy-proof)

Consider a case where (iv) holds but not (ii). There is no non-scarce resource in this example, so we can skip the pre-competition stage. Fix an ordering z , where $z = (a(1), b(1), a(2), b(2))$. $a(1)$ represents the most important flight according to R_a and $a(2)$ represents the other. $b(1)$ represents the most important flight according to R_b and $b(2)$ represents the other.

³⁹If the lexicographic preference assumption is relaxed, the same result might still be obtained if the size of the market goes to infinite.

F^o	$f_{a,1}$	$f_{a,2}$	$f_{b,1}$	$f_{b,2}$
e_f	1	2	2	1
R	1	2	1	2
\widehat{e}_f	2	1	2	1
\widehat{R}	2	1	1	2

S	s_1	s_2	s_3	s_4
$\varphi^z(R, e)$	$f_{a,1}$	$f_{b,1}$	$f_{a,2}$	$f_{b,2}$
$\varphi^z(R_a, \widehat{e}_a, (R, e)_{-a})$	$f_{a,2}$	$f_{a,1}$	$f_{b,1}$	$f_{b,2}$
$\varphi^z(\widehat{R}_a, e_a, (R, e)_{-a})$	$f_{a,1}$	$f_{a,2}$	$f_{b,1}$	$f_{b,2}$

In this example, by either misreporting its importance ranking or earliest feasible arrival times (as in the left table above), a can gain by having s_2 (see the right table above. In the middle case, it can swap slots for $f_{a,1}$ and $f_{a,2}$).

The second source of strategy-proofness is the design of the pre-competition stage. If an airline knows one of the slots will be used by one of its flights only, say the most important one, then it will have the incentive to misreport its ranking such that this flight is the least important one; alternatively, it can misreport its earliest feasible arrival times such that each of its flights picks a slot for the next most important flight and the least important flight picks a slot for the most important flight. By doing either of these, all of its remaining flights would be weakly better off if one runs MTC without the pre-competition stage.

Example 8: (MTC without the pre-competition stage is not strategy-proof)

F^o	$f_{a,1}$	$f_{a,2}$	$f_{b,1}$
e_f	3	1	1
R	1	2	1
\widehat{e}_f	1	3	1
\widehat{R}	2	1	1

S	s_1	s_2	s_3
φ^{z^1} or $\widehat{\varphi}^{z^1}$ or $\widehat{\varphi}^{z^2}$	$f_{a,2}$	$f_{b,1}$	$f_{a,1}$
φ^{z^2} or φ^{z^3} or $\widehat{\varphi}^{z^3}$	$f_{b,1}$	$f_{a,2}$	$f_{a,1}$

We drop the arguments for $\varphi^z(R, e)$ and write φ^z . In this example, given a can swap slots for $f_{a,1}$ and $f_{a,2}$ whenever necessary, reporting either \widehat{e}_a or \widehat{R}_a will give a the same outcome, but reporting both of them together will give a the outcome of $\varphi^z(R, e)$. We use $\widehat{\varphi}^z$ to represent $\varphi^z(\widehat{R}_a, e_a, (R, e)_{-a})$ and $\varphi^z(R_a, \widehat{e}_a, (R, e)_{-a})$ after necessary self-optimization. There are three possible orderings: $z^1 = (a(1), a(2), b(1))$, $z^2 = (a(1), b(1), a(2))$, and $z^3 = (b(1), a(1), a(2))$. Let $a(i)$

represent $f_{a,i}$ for $i \in \{1, 2\}$ and $b(1)$ represent $f_{b,1}$. By reporting either \widehat{e}_a or \widehat{R}_a , a can strictly gain if z^2 is realized and lose nothing under z^1 and z^3 .

Note that s_3 would be assigned in the pre-competition stage. MTC avoids the aforementioned manipulations by assigning non-scarce resources (type 4 slots) in the pre-competition stage without asking airlines to give up anything meaningful to them in the main stage. For completeness, we provide an example in Appendix 2.8.3 to show Compression is not strategy-proof in our preference domain.

2.6 Extensions

2.6.1 Outside an Instance

An airline can freeze a flight $f \in F_a^o$ in a slot $s \in S_a$ and effectively remove s from an instance. This is not a strategy in the induced preference revelation game of a mechanism, but rather a way to change the game. Schummer and Abizada [2017] show that DASO is non-manipulable by postponing a flight cancelation (an airline cannot gain by freezing a canceled flight in a slot $s \in S_a$ and reusing it later), while Schummer and Vohra [2013] show that both Compression and TC fail an even weaker condition.⁴⁰ For completeness, we provide an example in Appendix 2.8.3 to show Compression is manipulable by postponing a flight cancelation in our preference domain.

The above non-manipulable condition is defined for deterministic mechanisms. We define the corresponding condition for stochastic mechanisms. A lottery mechanism ϕ is **manipulable by postponing a flight cancelation** if there are instances $I = (S, A, F^o, R, e, \Phi)$ and $I' = (S \setminus \{s\}, A, F^o \setminus \{f\}, R, e, \Phi)$, airline $a \in A$, $f \in F_a^o \setminus F_a$ and slot $s \in S_a$ such that $\exists \mathcal{L} \succ_a \phi^I(R, e)$, where $\mathcal{L} = \sum p_\Pi \cdot \Pi$ and each Π is some landing schedule that contains landing schedule for a $\Pi_a^{\phi_a^{I'}(R, e) \cup \{s\}}$. In words, a lottery mechanism is manipulable by postponing a flight cancelation if an airline can gain

⁴⁰Compression and TC fail a condition called non-manipulable via slot destruction—an airline cannot gain by freezing a canceled flight in a slot $s \in S_a$.

by freezing a canceled flight in a slot $s \in S_a$ and then self-optimize using the slots in $\phi_a^{I'}(R, e) \cup s$.

Theorem 4-1: The multiple trading cycles mechanism ϕ is non-manipulable by postponing a flight cancelation.

The proof is based on the following ideas. Suppose airline a freezes a canceled flight $f \in F_a^o \setminus F_a$ in a slot $s \in S_a$, so both s and f are removed from the instance. First, the probabilities of getting better positions (for remaining flights in the main stage) in z are higher if f is not removed. For instance, the probability of $a(1)$ being the first flight in z is $\frac{|F_a^o - 1|}{|F^o - 1|}$ if f is removed, and such probability increases to $\frac{|F_a^o|}{|F^o|}$ if f is not removed. Second, removing s means a does not use s to trade, but MTC is ex post individually rational, putting s back into the instance would only make a weakly better off.

It is easy to see that from Claim 1 and Proposition 1, if a slot $s \in S^{0-0}$ is removed from an instance, then each flight with $e_f \leq s$ would get a slot that is weakly later than otherwise, and if a flight $f \in F$ is removed from an instance, then each flight in $F \setminus \{f\}$ would get a weakly earlier slot in MTC (for any given z with f removed).

Can an airline gain by freezing a non-canceled flight $f_{a,i} \in F_a$ in a slot $s \in S_a$? The answer is maybe.

Theorem 4-2: Suppose $f_{a,i}$ is the most important flight of a in an instance I and MTC is used. If the earliest feasible available slot for $f_{a,i}$, s , is in S_a and s is a scarce resource, then a can weakly gain by freezing $f_{a,i}$ in s .

Putting s and $f_{a,i}$ into the instance would make a “pay” the position of $a(1)$ in any ordering z to get s even though s is in S_a (in this case, if a removes s and f , then the theorem applies to the next most important flight of a). However, if such s is a non-scarce resource, freezing $f_{a,i}$ in s makes a weakly worse off (this is by Theorem 4-1 because $f_{a,i}$ in this situation is effectively the same as a canceled flight to a in MTC). We illustrate these points by the following example.

Example 9:

F^o	$f_{a,1}$	$f_{a,2}$	$f_{b,1}$	S	s_1	s_2	s_3
e_f	1	1	1	Φ	a		
\underline{e}_f	3	1	1	φ^{z^1} or φ^{z^4}	$f_{a,1}$	$f_{a,2}$	$f_{b,2}$
\tilde{R}	1	2	1	φ^{z^2} or φ^{z^3} or φ^{z^5}	$f_{a,1}$	$f_{b,1}$	$f_{a,2}$
\tilde{e}_f	-	1	1	Φ			a
\tilde{R}	-	1	1	φ^{z^1} or φ^{z^2} or φ^{z^4}	$f_{a,2}$	$f_{b,1}$	$f_{a,1}$
				φ^{z^3} or φ^{z^5}	$f_{b,1}$	$f_{a,2}$	$f_{a,1}$

Let φ^z represent $\varphi^z(R, e)$ and $\underline{\varphi}^z$ represent $\varphi^z(R, \underline{e})$. Let $b(1)$ represent $f_{b,1}$.

Case 1: $s_1 \in \Phi(a)$ is a scarce resource ($e_{f_{a,1}} = 1$). There are three possible orderings: $z^1 = (a(1), a(2), b(1))$, $z^2 = (a(1), b(1), a(2))$, and $z^3 = (b(1), a(1), a(2))$. Let $a(i)$ represent $f_{a,i}$ for $i \in \{1, 2\}$. Each ordering realizes with probability $\frac{1}{3}$. While $f_{a,1}$ always get s_1 (under z^3 , $a(1)$ would be inserted in front of $b(1)$), $f_{a,2}$ gets s_2 with probability $\frac{1}{3}$ (only under z^1). Consider if a freezes $f_{a,1}$ in s_1 , so both $f_{a,1}$ and s_1 are removed from the instance. There are two possible orderings: $z^4 = (a(1), b(1))$ and $z^5 = (b(1), a(1))$. Let $a(1)$ represent $f_{a,2}$. Each ordering realizes with probability $\frac{1}{2}$. Therefore, $f_{a,2}$ gets s_2 with probability $\frac{1}{2}$, so its expected delay is lower.

Case 2: $s_3 \in \Phi(a)$ is a non-scarce resource ($e_{f_{a,1}} = 3$). s_3 is assigned to $f_{a,1}$ in the pre-competition stage. There are also three possible orderings z^1 , z^2 and z^3 as in Case 1. Let $a(1)$ represent $f_{a,2}$ and $a(2)$ represent $f_{a,1}$ ($f_{a,1}$ is a duplicate flight). $f_{a,2}$ gets s_2 with probability $\frac{2}{3}$ (under z^1 and z^2). Consider if a freezes $f_{a,1}$ in s_3 , so both $f_{a,1}$ and s_3 are removed from the instance. Again, there are also two possible orderings z^4 and z^5 as in Case 1. Let $a(1)$ represent $f_{a,2}$. Each ordering realizes with probability $\frac{1}{2}$. Therefore, $f_{a,2}$ gets s_1 with probability $\frac{1}{2}$, so its expected delay is higher.

Case 3: Consider when $f_{a,1}$ is canceled ((\tilde{R}, \tilde{e}) is reported). $s_3 \in \Phi(a)$ is not in S^{0-0} . $a(2)$ now represents a dummy flight $f_{a,1}$. Everything else is the same as in Case 2.

The following mechanism is inspired by Theorem 4-2. We call the MTC that uses the following modified main stage **Multiple Trading Cycles-2** (MTC-2). All results of MTC hold for MTC-2.

MTC and MTC-2 are not the same mechanism as MTC-2 possibly favors some airlines that fit the description of Theorem 4-2.

Modified Main Stage:

If this is the first assignment in a GDP, for each $a \in A$, construct $S_a = \Phi^{exante}(a)$ according to the initial slot ownership function Φ^o . Otherwise, let $S_a = \Phi(a)$ (from the last assignment). Start from the earliest slot in $S^{main} \cap S_a$, if a slot in S_a is the earliest feasible available slot in S^{main} to the most important flight of a in F^{main} , assign it to this flight and update S^{main} to $S^{(1)}$ and F^{main} to $F^{(1)}$ (in general, update $S^{(t)}$ to $S^{(t+1)}$ and $F^{(t)}$ to $F^{(t+1)}$). Then start from the earliest slot in $S^{(t)} \cap S_a$ for $t = 1, 2, \dots$, repeat the above procedure until there is no more such slot. Denote the set of slots being assigned here by S^{top} and the set of flights that are assigned a slot here by F^{top} .

Create $|F_a^o| - |F_a^{top}|$ surrogates of a for each $a \in A$, name them $a(1), a(2), \dots, a(|F_a^o| - |F_a^{top}|)$. Denote the set of surrogates by \mathcal{A} ($|\mathcal{A}| = |F^o| - |F^{top}|$). Randomly select an ordering $z(\mathcal{A})$ with uniform distribution over $Z(\mathcal{A})$. For each a , rearrange its surrogates to their positions in the ordering such that they are in the order of $a(1), a(2), \dots, a(|F_a^o| - |F_a^{top}|)$. Denote the resulting ordering by z .

Let $a(i) \in \{a(1), \dots, a(|F_a^{main}| - |F_a^{top}|)\}$ represent the i -th important flight in $F_a^{main} \setminus F_a^{top}$ (according to R_a); we call it a remaining flight of a . Let $a(i) \in \{a(|F_a^{main}| - |F_a^{top}| + 1), \dots, a(|F_a| - |F_a^{top}|)\}$ represent the $i - |F_a^{main}| - |F_a^{top}|$ -th important flight in $F_a \setminus F_a^{main}$ (according to R_a); we call it a duplicate flight of a . Let $a(i) \in \{a(|F_a| - |F_a^{top}| + 1), \dots, a(|F_a^o| - |F_a^{top}|)\}$ represent a canceled flight of a ; we call it a dummy flight of a .

Let $S^1 = S^{0-0} \setminus S^{top}$ and $F^1 = F \setminus F^{top}$. (We only change the procedure before Step 1.)

In Case 1 of Example 9, MTC-2 assigns s_1 to $f_{a,1}$ and then selects z^4 and z^5 randomly, while MTC selects z^1 , z^2 , and z^3 randomly. One can modify YGMH-IGYT by using the modified main stage: Fix an ordering. If a house is owned by an agent and the house is his top choice, assign this house to this agent. Repeat this procedure until there is no more such house. Then run YGMH-

IGYT with the reduced ordering (with these agents eliminated). The outcome does not change because the positions of these agents in the original ordering are independent to the final outcome. Each agent that is assigned a house here always obtains the same house (in any individually rational mechanism). This modification is compatible with the previous one.

2.6.2 Multiple Runways

2.6.2.1 An Extended Model

When there are multiple runways, there will be multiple slots available at a time. Let m be the number of runways, so the set of available GDP slots is $S^m = \{s_{1,1}, s_{1,2}, \dots, s_{1,m}, s_{2,1}, \dots\}$. We assume airlines are indifferent between slots of the same time.⁴¹ Since we need strict preferences in MTC, we can use tiebreaking rules to eliminate these indifferences.

Tiebreaking rule-1: Given S_A and a preference profile \succsim induced by (R, e) with $e_f \in S$ for each flight f , construct a strict preference profile \succ^{main} with $e_f \in S^m$ for each flight f as follows: For any airline a , given two slots of the same time,

(1) if both slots are in S_a or S_b (for some $b \in A$) or $S \setminus S_A$, then the slot with the lower index is strictly better,

(2) if one slot is in S_a and another is in S_b , then the one in S_a is strictly better,

(3) if one slot is in S_b and another is in S_c , then the one _____ is strictly better (see discussion below), and

(4) if one slot is in S_A and another is in $S \setminus S_A$, then the one in $S \setminus S_A$ is strictly better.

Anything (fixed or random) based on some exogenous parameters that do not create indifferences can be filled in the blank. [Abdulkadiroğlu and Sönmez \[1999\]](#) propose a tie-breaking rule

⁴¹The are many papers study allocation problems with non-strict preferences. See [Quint and Wako \[2004\]](#), [Ergin \[2008\]](#), [Abdulkadiroğlu et al. \[2009\]](#), [Alcalde-Unzu and Molis \[2011\]](#), [Jaramillo and Manjunath \[2012\]](#), [Ehlers \[2014\]](#), [Erdil and Ergin \[2017\]](#).

where in the situation of (3), a slot owned by a higher ranked owner in z is preferred.⁴² The intuition behind (2) and (4) is that trading with another airline (for a slot in S^{main}) is not free (potentially make some slots in S_a unavailable to itself), so an airline might want to avoid trading whenever possible. Under Tiebreaking rule-1, all else being equal, for any airline a , slots in $S \setminus S_a$ are the best, slots in S_a are in the middle, and slot in some S_b are the worst.

Tiebreaking rule-2: Given S_A and a preference profile \succsim induced by (R, e) with $e_f \in S$ for each flight f , construct a strict preference profile $\succ^{pre-competition}$ with $e_f \in S^m$ for each flight f as follows: For any airline a , given two slots of the same time,

- (1') if both slots are in S_a or $S \setminus S_a$, then the slot with the lower index is strictly better, and
- (2') if one slot is in S_a and another is in $S \setminus S_a$, then the one in $S \setminus S_a$ is strictly better.

The intuition behind (2') is that trading a slot $s \in S_a$ for a slot in $S \setminus S^{main}$ eliminates the potential gain from s in the main stage. Under Tiebreaking rule-2, all else being equal, for any airline a , slots in $S \setminus S_a$ are the best, slots in S_a are the worst.

2.6.2.2 Modified MTC

For the first assignment, we need to construct $\Phi^{exante}(a)$ for each $a \in A$. Let $S^{o,m} = \{s_{1,1}^o, s_{1,2}^o, \dots, s_{1,m}^o, s_{2,1}^o, \dots\}$. The indices of runways are not important in the construction of $\Phi^{exante}(a)$ as we treat $s_{1,r}^o$ for $r \in \{1, \dots, m\}$ as s_1^o . We now construct $\Phi^{exante}(a)$ according to the initial slot ownership function $\Phi^o(A)$: Select an arbitrary airline $a \in A$, if the time interval of a GDP slot s_n is entirely owned by airline a before the GDP starts, endow $s_{n,1}$ to a for each n .⁴³ Remove one copy of each initial slot that covers these s_n 's. If s_n is still being covered (by the remaining set of initial slots), endow $s_{n,2}$ to a for each n . Procedure continues in a similar way until no more s_n is covered. Then select an arbitrary airline $b \in A$...(if a is endowed $s_{n,r}$ for some n , then other airlines might be endowed

⁴²Under that tiebreaking rule, in situation (4), the one in S_A is strictly better.

⁴³Formally, $s_n = [1 + (n-1)l, 1 + nl] \subseteq \cup_{s_{n,r}^o \in \Phi^o(a)} [n, n+1]$.

$s_{n,r+1}$, and so on. If in a situation where t slots at time s_n are being endowed but only r slots are available ($r < t$, there can be exempted flights, crossing runways, etc.), then remove all s_n 's from $\Phi^{exante}(A)$.

Modified Pre-competition Stage:

(a) Order flights in F in increasing order of e_f (break ties arbitrarily).

(b) Assign flights sequentially to the earliest slot (start from the one with the lowest index) in S that each flight can feasibly use. Denote the tentative landing schedule by $\hat{\Pi}$ and the set of occupied slots by S^{0-0} .

(c) Let $F^{0-0} = F$. Find an earliest $s \in S^{0-0}$ such that

(c-i) $s = \hat{\Pi}^{-1}(f)$ for some $f \in F_a^{0-0}$ and $e_f > s(-1)$, where $s(-1)$ is some last slot before s in S^{0-0} (f occupies s in $\hat{\Pi}$ and will not compete for slots earlier than s . Since in $\hat{\Pi}$, all flights that arrive strictly earlier than f will get a slot *no later* (compare to “strictly earlier” in the single runway problem) than s , this condition also implies that $\forall f' \in F^{0-0}$ with $e_{f'} \leq s(-1)$, $\hat{\Pi}^{-1}(f') \leq s$. Therefore, there are sufficient slots to accommodate flights that arrive earlier than f , so they will not compete for s);

(c-ii) (a) If f' has $e_f \leq e_{f'} \leq s$, then $f' \in F_a^{0-0}$; or (b) $\nexists f'$ with $e_f \leq e_{f'} \leq s$ such that $s(+1) = \hat{\Pi}^{-1}(f')$, where $s(+1)$ is some next slot after s in S^{0-0} ((a) says all flights that want s belong to a , and (b) says each flight that wants s get a slot no later than s).

Assign s tentatively to $f_{a,i}$ if (c-ii)-(a) is satisfied, where $f_{a,i} \in F_a^{0-0}$ is the most important flight with $e_f \leq e_{f_{a,i}} \leq s$. Remove $f_{a,i}$ from F^{0-0} and s from S^{0-0} . If $f \neq f_{a,i}$, modify $\hat{\Pi}$ in the following way: Start from f , move each flight to the next slot in S^{0-0} until $\hat{\Pi}(f_{a,i})$ is filled.

Assign s tentatively to f if (c-ii)-(b) is satisfied (or both (c-ii)-(a) and (c-ii)-(b) are satisfied).

For $t = 0, 1, \dots$, update F^{0-t} to $F^{0-(t+1)}$ and S^{0-t} to $S^{0-(t+1)}$.

(d) Repeat (c) until all slots are demanded by more than 1 airline.

Denote the resulting set of flights by F^{main} .

(Since both main stage and modified main stage use S^{main} as an input. We need to construct S^{main} . Unlike the single runway problem, S^{main} might be different from the last S^{0-t} (see Example 13).)

Construct S_a for each $a \in A$ as in MTC. Create $|F_a \setminus F_a^{main}|$ surrogates of a for each $a \in A$, name them $a(1), a(2), \dots, a(|F_a \setminus F_a^{main}|)$. Denote the set of surrogates by \mathcal{A}^0 ($|\mathcal{A}^0| = |\cup_{a \in A} F_a \setminus F_a^{main}|$). Randomly select an ordering $z(\mathcal{A}^0)$ with uniform distribution over $Z(\mathcal{A}^0)$. For each a , rearrange its surrogates to their positions in the ordering such that they are in the order of $a(1), a(2), \dots, a(|F_a \setminus F_a^{main}|)$. Denote the resulting ordering by z^0 . Let $a(i) \in \{a(1), \dots, a(|F_a \setminus F_a^{main}|)\}$ represent the i -th important flight in $F_a \setminus F_a^{main}$ (according to R_a). According to Tiebreaking rule-2, tentatively assign the first flight in z^0 the slot it wants, then tentatively assign the second flight in z^0 the slot it wants (among those remaining), and so on.

Denote the resulting set of slots by S^{main} .

(Each slot that is tentatively assigned will be assigned to one of the flights that obtains a slot of the same time here in the main stage. Assigning slots tentatively allows us to apply Tiebreaking rule-1 for all flights in the main stage (Tiebreaking rule-2 has fulfilled its mission already); otherwise, we would need to apply Tiebreaking rule-2 for duplicate flights. The two other stages are the same except the S_a for each $a \in A$ is constructed earlier and a duplicate flight will be assigned pro forma the slot it was assigned in the pre-competition stage or a slot of the same time. The final assignment depends on the realized ordering in the main stage and Tiebreaking rule-1.

We now say s is demanded only by *some* airline a if (c-i) and (c-ii) hold simultaneously and s is demanded by more than 1 airline if either (c-i) or (c-ii) fails. Recall the 4 types of slots: Type 1 slots satisfy (c-i) but not (c-ii), type 2 slots do not satisfy both, type 3 slots satisfy (c-ii) but not (c-i), and type 4 slots satisfy both.)

Example 10:

F^o	$f_{a,1}$	$f_{b,1}$	$f_{c,1}$	$f_{d,1}$	$f_{e,1}$
e_f	1	1	2	4	4

S (Runway 1)	$s_{1,1}$	$s_{2,1}$	$s_{3,1}$	$s_{4,1}$
(Runway 2)				$s_{4,2}$
$\hat{\Pi}$	$f_{a,1}$	$f_{b,1}$	$f_{c,1}$	$f_{d,1}$
				$f_{e,1}$

This example is modified from Example 1. We only need to check the new condition (c-ii)-(b). $s_{1,1}$ is a type 1 slot since (c-i) is satisfied, (c-ii)-(a) fails, and (c-ii)-(b) fails as $\hat{\Pi}(f_{b,1}) = s(+1) = s_{2,1}$ with $e_{f_{a,1}} \leq e_{f_{b,1}} \leq s_{1,1}$. $s_{2,1}$ is a type 2 slot since (c-i) fails, (c-ii)-(a) fails, and (c-ii)-(b) fails as $\hat{\Pi}(f_{c,1}) = s(+1) = s_{3,1}$ with $e_{f_{b,1}} \leq e_{f_{c,1}} \leq s_{2,1}$. $s_{3,1}$ is a type 3 slot since (c-i) fails, (c-ii)-(a) is satisfied, and (c-ii)-(b) is satisfied (there is no such f'). $s_{4,1}$ is a type 4 slot as (c-i) and (c-ii)-(b) hold simultaneously ((c-ii)-(a) fails because $f_{d,1} \leq f_{e,1} \leq s_{4,1}$ but $f_{e,1} \notin F_d^{0-t}$). $s_{4,2}$ is also a type 4 slot by similar reasoning. Now consider the following modified Example.

Example 11:

F^o	$f_{a,1}$	$f_{b,1}$	$f_{c,1}$	$f_{d,1}$	$f_{d,2}$	$f_{d,3}$
e_f	1	1	2	4	4	4

S (Runway 1)	$s_{1,1}$	$s_{2,1}$	$s_{3,1}$	$s_{4,1}$	$s_{5,1}$
(Runway 2)				$s_{4,2}$	
$\hat{\Pi}$	$f_{a,1}$	$f_{b,1}$	$f_{c,1}$	$f_{d,1}$	$f_{d,3}$
				$f_{d,2}$	

$s_{4,1}$ and $s_{4,2}$ is a type 4 slot as (c-i) and (c-ii)-(a) hold simultaneously (for $s_{4,1}$, (c-ii)-(b) fails as $\hat{\Pi}(f_{d,3}) = s(+1) = s_{5,1}$ with $f_{d,1} \leq f_{d,3} \leq s_{4,1}$. The same for $s_{4,2}$). $s_{5,1}$ is also a type 4 slot in $S^{0-(t+2)}$ (after the removals of $s_{4,1}$ and $s_{4,2}$) as (c-i), (c-ii)-(a) and (c-ii)-(b) hold simultaneously.

Remark 2: If a slot $s_{n,1}$ does not satisfy (c-ii) (some flight of other airlines also wants this slot but has been assigned a later slot in $\hat{\Pi}$), then each $s_{n,r}$ also fails (c-ii). But if a slot $s_{n,1}$ does not satisfy (c-i), some $s_{n,r}$ might satisfy (c-i). Therefore, type 1 and type 2 slots may coexist in some $(s_{n,1}, s_{n,2}, \dots, s_{n,m})$ (by construction, type 2 slots have lower indices). Similarly, type 3 and type 4 slots may coexist in some $(s_{n,1}, s_{n,2}, \dots, s_{n,m})$ (a slot is type 3 or type 4 is determined in the pre-competition stage. Also, there can be slots in $S \setminus S^{0-0}$ here as well).

Example 12:

F^o	$f_{a,1}$	$f_{b,1}$	$f_{c,1}$	$f_{d,1}$
e_f	1	1	2	2

S (Runway 1)	$s_{1,1}$	$s_{2,1}$	$s_{3,1}$
(Runway 2)		$s_{2,2}$	
$\hat{\Pi}$	$f_{a,1}$	$f_{b,1}$	$f_{d,1}$
		$f_{c,1}$	

In Example 12, $s_{2,1}$ is a type 2 slots and $s_{2,2}$ is a type 1 slot. Note that $s_{1,1}$ is also a type 1 slot. $s_{3,1}$ is a type 3 slot since (c-i) fails but (c-ii)-(b) is satisfied.

Example 13:

F^o	$f_{a,1}$	$f_{b,1}$	$f_{c,1}$
e_f	1	1	2

S (Runway 1)	$s_{1,1}$	$s_{2,1}$
(Runway 2)		$s_{2,2}$
$\hat{\Pi}$	$f_{a,1}$	$f_{b,1}$
		$f_{c,1}$

In Example 13, $s_{2,1}$ is a type 3 slots and $s_{2,2}$ is a type 4 slot. Suppose now $s_{2,2} \in \Phi(c)$. In the modified pre-competition stage, $s_{2,2}$ is tentatively assigned to $f_{c,1}$, so $S^{0-1} = \{s_{1,1}, s_{2,1}\}$. However, according to Tiebreaking rule-2, $f_{c,1}$ will pick $s_{2,1}$. Therefore, $S^{main} = \{s_{1,1}, s_{2,2}\} \neq S^{0-1}$. Now $s_{2,2}$ is a type 3 slots and $s_{2,1}$ is a type 4 slot. Note that if there is a $s_{2,3}$, then it is in $S \setminus S^{0-0}$.

We call $(s_{n,1}, s_{n,2}, \dots, s_{n,m})$ a **slot group**. We say a slot group it a *type x slot group* if all slots in the slot group are type x slots. Scarce resources in the extended model are sequences of adjacent *slot groups* in S^{main} such that each **sequence** starts with a type 1 slot group and ends with type 3 slots. Note that $s_{2,2}$ in Example 13 (without the supposition) is not in any sequence as it is not a scarce resource. Suppose there is no gap in S^{main} and there are always m slots at time s_n , a sequence is of the form

$$((s_{n,1}, s_{n,2}, \dots, s_{n,m}), (s_{n+1,1}, s_{n+1,2}, \dots, s_{n+1,m}) \dots, (s_{n+t,1}, s_{n+t,2}, \dots, s_{n+t,m})),$$

where $(s_{n,1}, s_{n,2}, \dots, s_{n,m})$ is a type 1 slot group and $(s_{n+t,1}, s_{n+t,2}, \dots, s_{n+t,m})$ contains some type 3

slots. Each sequence contains one type 1 slot group and possibly more type 1 slots (as in Example 12), some type 3 slots, and possibly some type 2 slots. As before, in any feasible and non-wasteful landing schedule, a flight that gets a slot in a sequence in $\hat{\Pi}$ will *always* get a slot in the same sequence and a flight that gets a slot outside a sequence in $\hat{\Pi}$ will *never* get a slot in that sequence.

Since airlines are indifferent between slots of the same time, there might be multiple individually rational landing schedules for some airline a . But a is indifferent between any of these landing schedules as each of its flights would get a slot of the same time only. The next proposition is the main result for this extended model. We call the MTC that uses the modified pre-competition stage and Tiebreaking rule-1 in the main stage **modified MTC with tiebreaking** (by replacing MTC with MTC-2, we can define *modified MTC-2 with tiebreaking*. Again, all results are the same).

Proposition 4: The results of Claim 1, Proposition 1, 2, 3 and Theorem 1, 3, 4-1, 4-2 hold for the modified MTC with tie-breaking in the extended model.

This result is for the true preference profile \succsim .⁴⁴ Example 14 shows the modified MTC with tie-breaking might produce outcomes outside the core.

Example 14:

	S (Runway 1)	$s_{1,1}$	$s_{2,1}$	$s_{3,1}$	$s_{4,1}$	$s_{5,1}$	
	(Runway 2)	$s_{1,2}$					
	Φ	b	a	a			
		c					
F^o	$f_{a,1}$	$f_{a,2}$	$f_{b,1}$	$f_{c,1}$	$f_{d,1}$	$f_{e,1}$	
e_f	1	2	3	2	1	1	
	Case 1	$f_{a,1}$	$f_{a,2}$	$f_{b,1}$	$f_{c,1}$	$f_{e,1}$	
		$f_{d,1}$					
	Case 2	$f_{a,1}$	$f_{c,1}$	$f_{b,1}$	$f_{a,2}$	$f_{e,1}$	
		$f_{d,1}$					

In situation (3) of Tiebreaking rule-1, break ties based on alphabetical order. That is, slots in S_a are better than slots in S_b , etc. There is no non-scarce resource in this example, so we can skip the pre-competition stage.

⁴⁴Indeed, Proposition 4 also holds for \succsim^{main} .

$$z = (a(1), a(2), c(1), b(1), d(1), e(1)).$$

Let $a(i)$ represent $f_{a,i}$ for $i = \{1, 2\}$ and others represent their only flights.

Case 1: $a(1)$ picks $s_{1,1} \in S_b$ for $f_{a,1}$, then $b(1)$ will be inserted in front of $a(1)$. $b(1)$ picks $s_{3,1} \in S_a$ for $f_{b,1}$. $a(1)$ and $b(1)$ form a cycle. $f_{a,1}$ is assigned $s_{1,1}$ and $f_{b,1}$ is assigned $s_{3,1}$. Then $s_{2,1}$ goes to $a(2)$, $s_{4,1}$ goes to $c(1)$, $s_{1,2}$ goes to $d(1)$, and $s_{5,1}$ goes to $e(1)$.

Now suppose MTC breaks ties based on reverse alphabetical order.

Case 2: $a(1)$ picks $s_{1,2} \in S_c$ for $f_{a,1}$, then $c(1)$ will be inserted in front of $a(1)$. $c(1)$ picks $s_{2,1} \in S_a$ for $f_{c,1}$. $a(1)$ and $c(1)$ form a cycle. $f_{a,1}$ is assigned $s_{2,1}$ and $f_{c,1}$ is assigned $s_{2,1}$. Then $s_{3,1}$ goes to $a(2)$, $s_{4,1}$ goes to $b(1)$, $s_{1,2}$ goes to $d(1)$, and $s_{5,1}$ goes to $e(1)$.

Note that both landing schedules are Pareto efficient and individually rational but the Case 2 landing schedule is not in the core because it is dominated by the Case 1 landing schedule via subgroup $\{a, b\}$. Yet MTC respects property rights over S_A because the definition only requires that a trades a slot in S_a for a better slot and $s_{1,2}$ is better than $s_{2,1}$ and $s_{3,1}$. When $f_{a,1}$ is picking a slot, a is in two cycles under its true preference, where one of the cycles leads to a better outcome for a .⁴⁵ This does not happen in models where agents have unit demand and non-strict preferences.⁴⁶

2.7 Conclusion

This chapter studies airport slot allocation problems during severe weather. We assume airlines have lexicographic preferences. We argue that the currently used mechanism does not respect property rights before and after a GDP starts. The mechanism we proposed solicits information such as earliest feasible arrival times and importance rankings from the airlines. Based on this

⁴⁵Indeed, condition (iv) in the Proof of Theorem 2 no longer holds in the extended model because now $f_{a,j}$ does not have to pick in $S_{A'}$. In Example 14, $f_{a,1}$ can pick $s_{1,2} \notin S_{A'} = S_a \cup S_b$.

⁴⁶In such models, an agent could be in more than one cycle under his true preference, but none of the cycles would lead to a better outcome for the agent.

information, the mechanism produces outcomes that are individually rational, Pareto efficient and in the core. Our mechanism also respects property rights before and after a GDP starts, is strategy-proof and non-manipulable by postponing a flight cancelation.

In the extended model with multiple runways, a modified version of our mechanism with tiebreaking inherits most of the aforementioned properties but might produce outcomes outside the core. It remains an open question whether there exist some endogenous tie-breaking rules that can resolve this problem.

2.8 Appendix

2.8.1 Omitted Proofs

(Proofs are in the language of Algorithm 1 unless otherwise noted.)

Proof of Theorem 1: The two algorithms only differ in their main stages. For any set of slot $S' \subset S$ and set of flights $F' \subseteq F$, the main stage of Algorithm 1 assigns (or assigns pro forma) the next series of slots in one of two possible ways.

Case 1: There is a chain $(a(i), s_b, b(j), s_c, \dots, s_d, d(k), s)$ (which may consist of a single flight $a(i)$ and a single slot s), where $a(i), b(j), \dots, d(k)$ are most important remaining/duplicate flights, and $a(i)$ has the highest priority in z and demands s_b , $b(j)$ demands s_c, \dots , $d(k)$ demands s in $S' \setminus S_A$ or $S' \cap S_e$ but e has no remaining/duplicate flight in F' . Every flight in the chain is assigned (or assigned pro forma) the slot that it demands. Note that this is a cycle in Algorithm 2 for (S', F') . (Case 1 includes (ii)-(b), (iii-i), and (iii-iv)-(b).)

Case 2: There is a cycle $(a(i), s_b, b(j), \dots, s_d, d(k), s_a)$ (which may consist of a single flight $a(i)$ and a single slot s_a), and every flight in the cycle is assigned (or assigned pro forma) the slot that it demands. This is also a cycle in Algorithm 2 for (S', F') . (Case 2 includes (ii)-(a), (iii-ii), (iii-iii), (iii-iv)-(a), and (iii-iv)-(c).)

Hence Algorithm 1 finds a cycle in Algorithm 2 and implements the associated trades for any

set of slots and set of flights. In Algorithm 2, a cycle that is not removed at any step remains a cycle at the next step, so for any given ordering z , the main stages of Algorithm 1 and Algorithm 2 produce the same outcome. ■

Proof of Claim 1: Suppose not. $\exists s \in S^{0-0}$ such that $\Pi^{-1}(s) = \emptyset$, where Π is some feasible and non-wasteful landing schedule. By feasibility, the number of slots earlier than s that can be occupied is fixed. $\Pi^{-1}(s) = \emptyset$ but $s \in S^{0-0}$ then implies $\exists f \in F$ such that $e_f \leq s < \Pi(f)$. Contradicts to the non-wastefulness of Π . ■

Proof of Proposition 1: Regularity: This is by the construction of the mechanism. Ex post feasibility: For any ordering z , at each stage and each step, no flight gets an infeasible slot. Ex post non-wastefulness: This is also by the construction of the mechanism. For any ordering z , let φ^z be the induced schedule mechanism. If $\exists f \in F$ such that $s \in V$ with $e_f < s$, then it must be the case that $\varphi_f^z(R, e) < s$.

Respects property rights over S_A : Without loss of generality, we can focus on the main stage. At each step a slot in some S_a is being assigned (or assigned pro forma), there are three possibilities: (i) It is assigned to the most important remaining/duplicate flight of a ; (ii) a trades it for a better slot for its most important remaining/duplicate flight; (iii) a has no more remaining/duplicate flight and this slot is assigned to some airline in the main stage (or the supplemental stage). In (iii), there is no better slot for a . ■

Proof of Proposition 2: For any ordering z , let φ^z be the induced schedule mechanism. Let $x_i = d_{f_{a,i}}(\Pi_a^{S_a}) - d_{f_{a,i}}(\varphi_a^z(R, e))$ for $i \in \{1, \dots, |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$.

If $x_i = 0$ for all $i \in \{1, \dots, |F_a|\}$, then $\Pi_a^{S_a} = \varphi_a^z(R, e)$. Otherwise, let x_j be the first non-zero coordinate of $x_a = (x_1, x_2, \dots, x_{|F_a|})$. x_j will always be positive since airline a picks $\varphi_{f_{a,j}}^z(R, e)$ instead of $\Pi_a^{S_a}(f_{a,j})$, which means $e_{f_{a,i}} \leq \varphi_{f_{a,j}}^z(R, e) < \Pi_a^{S_a}(f_{a,j})$. Hence, $\forall a \in A$, $\varphi_a^z(R, e) \succeq_a \Pi_a^{S_a}$. ■

Proof of Proposition 3: (We use the language of Algorithm 2 for this proof.) Flights that leave

in the pre-competition stage are already getting the earliest slot they can get without hurting any flights that are more important than them within their airlines, and no slot in $S \setminus S^{main}$ can be used to make flights leave in the main stage better off.

Consider the main stage, for any ordering z , any flight that leaves at step 1 is assigned its top choice that is available and cannot be made better off. Any flight that leaves at Step 2 is assigned its top choice that is available (among those remaining) and cannot be made better off without hurting some flight who left at Step 1. Proceeding in a similar fashion, no flight can be made better off without hurting some flight that left at an earlier step.

Moreover, for an airline, a flight left at an earlier step is more important than a flight left later, so it cannot make itself better off as well. Therefore, ϕ is ex post Pareto efficient. ■

Proof of Theorem 2: For any ordering z , let ϕ^z be the induced schedule mechanism. Suppose $\exists \Pi'$ and $A' \subseteq A$ such that (i) $\forall f \in \cup_{a \in A'} F_a$, $\Pi'(f) \in S_{A'}$, and (ii) $\forall a \in A'$, $\Pi' \succ_a \phi^z(R, e)$. Therefore, $\forall a \in A'$, the first non-zero coordinate of $x_a = (x_1, x_2, \dots, x_{|F_a|})$ is positive where $x_i = d_{f_{a,i}}(\phi_a^z(R, e)) - d_{f_{a,i}}(\Pi')$ for $i \in \{1, \dots, |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$.

Consider $f_{a,i}$ where x_i is the first non-zero coordinate of x_a for $a \in A'$. Note that $\Pi'(f_{a,i}) \in S_{A'}$ is better than $\phi_{f_{a,i}}^z(R, e)$, and $\Pi'(f_{a,i})$ is not available when $f_{a,i}$ is picking a slot in $\phi^z(R, e)$, so $\Pi'(f_{a,i})$ is not assigned in the supplemental stage. There is a $\Pi'(f_{a,i})$ for each $a \in A'$; let S_T be the collection of $\Pi'(f_{a,i})$ for all $a \in A'$. S_T is the set of slots that makes airlines in A' prefer Π' .

(i) If $\Pi'(f_{a,i})$ is used by some $f_{a,j}$ in $\phi^z(R, e)$, then it must be $f_{a,j} R_a f_{a,i}$. Since x_i is the first non-zero coordinate, $x_j = 0$, i.e., $f_{a,j}$ is getting the same slot in Π' , a contradiction.

The same argument applies to all airlines in A' . Therefore, $\forall a \in A'$, $\Pi'(f_{a,i})$ is used by some other airline $a' \in A'$ in $\phi^z(R, e)$.

The fact that $\Pi'(f_{a,i})$ is not available when $f_{a,i}$ is picking a slot in $\phi^z(R, e)$ together with (i) implies $\Pi'(f_{a,i})$ is not assigned in the pre-competition stage, so it must be assigned in the main stage. Let $s_a \in S_T \cap S_a$ ($a \in A'$) be the first slot in S_T that is being assigned to some $f \in F$ in

$\varphi^z(R, e)$. a will pick a slot for its most important remaining flight $f_{a,j}$ before s_a is assigned (either it is $f_{a,j}$'s turn or $f_{a,j}$ has been inserted; if a has no remaining flight, then it will be in case (iii) below). At this point, all slots in S_T are available (otherwise it contradicts the way we pick s_a).

(ii) If $f_{a,i} = f_{a,j}$, a picks $\varphi_{f_{a,i}}^z(R, e)$ but not $\Pi'(f_{a,i})$, a contradiction.

(iii) If $f_{a,i} R_a f_{a,j}$, this means $\Pi'(f_{a,i})$ is still available after $f_{a,i}$ picked a slot in $\varphi^z(R, e)$, a contradiction.

(iv) If $f_{a,j} R_a f_{a,i}$, we have $\varphi_{f_{a,j}}^z(R, e) = \Pi'(f_{a,j}) \in S_{A'}$.

By (iv), $\varphi_{f_{a,j}}^z(R, e) \neq s_a$ otherwise it contradicts the supposition that s_a makes some airline in A' prefer Π' . Therefore, $f_{a,j}$ picks some slot other than s_a . That means airline a trades s_a for $\varphi_{f_{a,j}}^z(R, e) \in S_b$ and $b \in A'$.

(♠) Let $\varphi_{f_{b,j}}^z(R, e)$ be the slot obtained by b in this trade. Because all slots in S_T are still available, the flight $f_{b,j}$ is more important than $f_{b,i}$, so $\varphi_{f_{b,j}}^z(R, e) = \Pi'(f_{b,j}) \in S_{A'}$. If $\varphi_{f_{b,j}}^z(R, e) \in S_a$ we have a cycle.

If $\varphi_{f_{b,j}}^z(R, e) \in S_c$, $c \in A'$ will be the next airline in this trade, and the argument (♠) applies. Because none of the airlines in this trade gets a slot outside $S_{A'}$ and A' is finite, there must exist a cycle contains exclusively airlines in A' . Let $y \in A'$ be the airlines gets s_a for $f_{y,j}$. Recall $\Pi'(f_{y,i})$ is not available when $f_{y,i}$ is picking a slot in $\varphi^z(R, e)$. Since all slots in S_T are still available, $f_{y,j}$ is more important than $f_{y,i}$, and therefore $\varphi_{f_{y,j}}^z(R, e) = \Pi'(f_{y,j}) = s_a$. This contradicts the supposition that s_a makes some airline in A' prefer Π' . ■

Proof of Theorem 3: For any ordering z , let φ^z be the induced schedule mechanism. We inspect each stage to see if an airline a can be better off by misreporting R_a or e_a , that is, $\varphi^z(\widehat{R_a, e_a}, (R, e)_{-a}) \succ_a \varphi^z(R, e)$. Let $x_a = (x_1, x_2, \dots, x_{|F_a|})$ be a vector where $x_i = d_{f_{a,i}}(\varphi^z(R, e)) - d_{f_{a,i}}(\varphi^z(\widehat{R_a, e_a}, (R, e)_{-a}))$ for $i \in \{1, \dots, |F_a|\}$ and $f_{a,i} R_a f_{a,i+1}$. Let x_j be the first non-zero coordinate of x_a . If there is no such x_j , then $\varphi^z(\widehat{R_a, e_a}, (R, e)_{-a}) \sim_a \varphi^z(R, e)$, and we are done. Suppose not. $\varphi^z(\widehat{R_a, e_a}, (R, e)_{-a}) \succ_a \varphi^z(R, e)$ implies that x_j is positive, and so there exists some slot $s \in S$ such that $e_{f_{a,j}} \leq s =$

$\varphi_{f_{a,j}}^z(\widehat{R}_a, e_a, (R, e)_{-a}) < \varphi_{f_{a,j}}^z(R, e)$. Note that s is not used by a flight of a that is more important than $f_{a,j}$ since that would contradict x_j is the first non-zero coordinate of x_a .

In the pre-competition stage, the only possible way for airline a to get a slot that is feasible for $f_{a,j}$ but not being assigned in this stage is to make it a non-scarce resource (type 4 slot) such that it will be assigned to $f_{a,j}$. For contradiction, suppose there is a slot s' , which might be a type 1, type 2, or type 3 slot in the sequence that contains s , that can be converted to a type 4 slot and assigned to $f_{a,j}$ in the pre-competition stage. It is sufficient to show that $s \neq s'$ for all such s' .

Case 1: Suppose s' is the type 1 slot of the sequence. a can make s' a type 4 slot only when s' violates (c-ii) because of a (with $f_{a,x}, f_{a,x'}, \dots$) and at most one airline b . Then a can misreport $e_{f_{a,x}}, \widehat{e_{f_{a,x'}}}, \dots$ (some infeasible or later times) to give that slot to airline b . Since a cannot have such slot assigned to $f_{a,j}$, $s \neq s'$. (Note that this procedure (of misreporting $e_{f_{a,x}}, \widehat{e_{f_{a,x'}}}, \dots$) makes the type 1 slot satisfy (c-ii). It also makes the next slot a type 1 slot if it was a type 2 slot; moreover, it makes the next slot a type 4 slot if it was a type 3 slot. In other words, this procedure makes the next slot satisfy (c-i).)

Case 2: Suppose s' is some type 2 slot of the sequence. a can make s' a type 4 slot only by making it a type 1 slot first and then converting it to a type 4 slot (all by repeating the procedure in Case 1, if not impossible). But, again, a cannot have such slot assigned to $f_{a,j}$, so $s \neq s'$. (Observe that a can convert a type 2 slot to a type 3 slot only when the slot violates (c-ii) because of a (with $f_{a,x}, f_{a,x'}, \dots$) and at most one airline b . Then a can misreport $e_{f_{a,x}}, \widehat{e_{f_{a,x'}}}, \dots$ to make it a type 3 slot (with (c-ii) satisfied). But there is no way for a to convert it to a type 4 slot after that.)

Case 3: Suppose s' is some type 3 slot of the sequence. a can make s' a type 4 slot only when s' violates (c-i) because of a . Then a can misreport $e_{f_{a,j}}$ as s' ($> e_{f_{a,j}}$) and have this slot assigned to $f_{a,j}$ (if there is more than 1 such type 3 slots. Let flights that are assigned type 3 slots in $\hat{\Pi}$ be $f_{a,x}, f_{a,x'}, \dots$. a also needs to misreport $e_{f_{a,x}}, \widehat{e_{f_{a,x'}}}, \dots$. But this even requires $f_{a,j}$ to be the most important flight among $f_{a,j}, f_{a,x}, f_{a,x'}, \dots$ since a is only willing to sacrifice flights that are less

important than $f_{a,j}$ to help it). However, $f_{a,j}$ can always get a weakly earlier slot in the main stage. $s = s'$ would contradict x_j is positive ($\varphi_{f_{a,j}}^z(R, e) \leq \varphi_{f_{a,j}}^z(\widehat{R}_a, e_a, (R, e)_{-a}) = s' = s$), so $s \neq s'$.

Suppose a wants to manipulate the pre-competition stage to help $f_{a,j}$ to get a slot in the main stage. The only thing it might be able to change is S^{main} . By sacrificing some flights $f_{a,x}, f_{a,x'}, \dots$ that are less important than $f_{a,j}$, it might shrink the size of S^{main} (as described in Case 1 to 3 above), or it might enlarge the size of S^{main} by reporting $e_{f_{a,x}}, \widehat{e_{f_{a,x'}}}, \dots$ so that some slots $\varphi_{f_{b,x}}^z(R, e) = \widehat{e_{f_{a,x}}}, \varphi_{f_{g,x}}^z(R, e) = \widehat{e_{f_{a,x'}}}, \dots$ become scarce resources. But z is fixed, so those manipulations do not work: Suppose a wants to get $s = \varphi_{f_{c,x}}^z(R, e)$ for $f_{a,j}$, so it shrinks or enlarges S^{main} by misreporting earliest feasible arrival times of flights that are less important than $f_{a,j}$. Note that $f_{c,x}$ (with $e_{f_{c,x}} \leq s$) ranks higher than $f_{a,j}$ in z . If s is going to $f_{a,j}$, it means $f_{c,x}$ must be getting an earlier slot s' (shrinking S^{main} by giving (type 1 or type 2) slots away will only help but not hurt flights of other airlines. Shrinking S^{main} by removing type 3 slots would only make $f_{a,j}$ worse off (if a removes type 3 slots in some earlier sequences such that flights in earlier sequences are forced to pick slots in the sequence that contains s and $f_{c,x}$ gets a later slot because of that, then $f_{a,j}$ would get an even later slot. If a removes type 3 slots in the sequence that contains s , this would not help. $f_{a,j}$ wants $\varphi_{f_{c,x}}^z(R, e)$ instead of $\varphi_{f_{a,j}}^z(R, e)$ but it cannot get it under $\varphi^z(R, e)$. This implies $\varphi_{f_{c,x}}^z(R, e)$ is an earlier slot that cannot be removed by a . If a removes a type 3 slot in a later sequence, nothing will change for $f_{a,j}$). Enlarging S^{main} will not force $f_{c,x}$ to get a later slot too since $f_{a,x}, f_{a,x'}, \dots$ (with $e_{f_{a,x}}, \widehat{e_{f_{a,x'}}}, \dots$ reported) will not compete with $f_{b,x}, f_{g,x}, \dots$ before $f_{a,j}$ picks a slot); if s' is going to $f_{c,x}$, it means a flight $f_{d,x}$, which ranks higher than $f_{c,x}$ and picks s' in $\varphi^z(R, e)$, must be getting an earlier slot s'' , and so on. But this contradicts the finiteness of $|F|$.

In the main stage, if a wants to manipulate the outcome, it can misreport (i) R_a ; (ii) e_a ; or (iii) both.

There is no way to change the ranking of a single flight, and changing a flight's earliest feasible arrival time cannot help the flight itself (recall a wants to get s such that $e_{f_{a,j}} \leq s < \varphi_{f_{a,j}}^z(R, e)$,

but by misreporting $e_{f_{a,j}}, f_{a,j}$ either gets a slot strictly earlier than $e_{f_{a,j}}$, or a slot weakly later than $\varphi_{f_{a,j}}^z(R, e)$. Therefore, a must use a subset of its flights to help another subset.

Note that there is no way to use less important flights to help a_j as they always pick later than a_j . Eventually, there is only 1 channel to improve a 's outcome: use flights to help flights that are less important than them; that is, use $f_{a,i}, f_{a,i}, \dots \in F_{a,I}$ to help $f_{a,j}, f_{a,j}, \dots \in F_{a,J}$, where $f_{a,i} R_a f_{a,i'} \dots R_a f_{a,j} R_a f_{a,j'} \dots$

(i)' If a misreports R_a , in $\varphi^z(\widehat{R_a}, e_a, (R, e)_{-a})$, $f_{a,i}$ will pick later than itself in $\varphi^z(R, e)$. Let $f_{a,x}$ be the flight that takes $f_{a,i}$'s position. It is without loss of generality to assume $e_{f_{a,i}} \neq e_{f_{a,x}}$. Otherwise, we can exclude $f_{a,i}$ from $F_{a,I}$ as it effectively picks a slot for itself (a can always self-optimize at the end).

(ii)' If a misreports e_a , $f_{a,i}$ will pick some slot for a less important flight and some less important flight will pick a slot for it, then a can self-optimize given the slots it gets by misreporting.

(iii)' If a misreports both, one of the two cases above must happen.

Note that in all circumstances, $\varphi_{a,i}^z(R, e)$ is the best possible slot $f_{a,i}$ can get given z . In (i)', (ii)', and (iii)', a will pick a slot for $f_{a,i}$ strictly later than it would in $\varphi^z(R, e)$. In the main stage, each slot is demanded by more than 1 airline. Since z is arbitrary, that means there exists some realization such that $\varphi_{a,i}^z(R, e)$ would be picked by some other airline.

Sum up, the probability for $f_{a,i}$ to get an earlier slot is 0, while the probability of getting a later slot is bounded away from 0. Let $\mathcal{L}(\phi(R, e))$ be the schedule lottery induced by MTC if (R, e) is being reported. We have $d_{f_{a,i}}(\mathcal{L}(\phi(\widehat{R_a}, e_a, (R, e)_{-a}))) - d_{f_{a,i}}(\mathcal{L}(\phi(R, e))) > 0$, and this means a prefers to report R_a and e_a truthfully.

In the supplemental stage, if $f_{a,j}$ wants a slot that is being assigned in this stage, it will get that slot in a previous stage, a contradiction. ■

Proof of Theorem 4-1 and 4-2: (4-1) Suppose airline a freezes a canceled flight $f \in F_a^o \setminus F_a$ in a slot $s \in S_a$. $I = (S, A, F^o, R, e, \Phi)$ and $I' = (S \setminus \{s\}, A, F^o \setminus \{f\}, R, e, \Phi)$. First suppose s cannot

be used to trade (not demanded by another airline, so it can be in S^{0-0} only if a uses it).

Denote the probability of $a(i)$ in instance I is drawn before the t -th flight is being drawn by $q_{I,t}^i$. Given $a(i-1)$ is drawn before, when the t -th flight is being drawn but $a(i)$ has not been drawn yet, the probability for $a(i)$ to get this position t in z in instance I is denoted by $p_{I,t}^i$. We list $p_{I,t}^i$'s for $a(1)$'s and $a(2)$'s in the table below.

z	1	2	3	...	$ F^o - F_a^o $	$ F^o - F_a^o + 1$	$ F^o - F_a^o + 2$
$a(1)$ in I	$\frac{ F_a^o }{ F^o }$	$\frac{ F_a^o }{ F^o -1}$	$\frac{ F_a^o }{ F^o -2}$...	$\frac{ F_a^o }{ F_a^o +1}$	1	0
$a(1)$ in I'	$\frac{ F_a^o -1}{ F^o -1}$	$\frac{ F_a^o -1}{ F^o -2}$	$\frac{ F_a^o -1}{ F^o -3}$...	$\frac{ F_a^o -1}{ F_a^o }$	1	0
$a(2)$ in I	0	$\frac{ F_a^o -1}{ F^o -1}$	$\frac{ F_a^o -1}{ F^o -2}$...	$\frac{ F_a^o -1}{ F_a^o +1}$	$\frac{ F_a^o -1}{ F_a^o }$	1
$a(2)$ in I'	0	$\frac{ F_a^o -2}{ F^o -2}$	$\frac{ F_a^o -2}{ F^o -3}$...	$\frac{ F_a^o -2}{ F_a^o }$	$\frac{ F_a^o -2}{ F_a^o -1}$	1

Position $|F^o| - |F + a| + 1$ is the worst position $a(1)$ can get in *both* instances. Each initial ordering z in MTC with $a(1)$ in position $|F^o| - |F_a^o| + 1$ induces a landing schedule, and there are $(|F^o| - |F_a^o|)!$ of them. Let d_{max}^1 denote the expected delay for $a(1)$ in these landing schedules (in MTC, this is the average delay for $a(1)$ in these landing schedule because z has a uniform distribution). Similarly, let $d_{|F^o|-|F_a^o|}^1$ denote the expected delay for $a(1)$ in landing schedules that are induced by initial ordering z 's with $a(1)$ in position $|F^o| - |F_a^o|$. In general, let d_t^i denote the expected delay for flight $a(i)$ in landing schedules that are induced by initial ordering z 's with $a(i)$ in position t . Since getting an earlier position does not hurt $a(i)$, $d_t^i \leq d_{t+1}^i$ for all i and t (note that $i \leq t \leq |F^o| - |F + a| + i$). In general, $d_{max}^i \equiv d_{|F^o|-|F_a^o|+i}^i$. For brevity, we omit the domains of t 's below).

The expected delay for $a(1)$ in instance I when the $|F^o| - |F_a^o|$ -th flight is being drawn but $a(1)$ has not been drawn yet is

$$D_{|F^o|-|F_a^o|}^{1,I} = \frac{|F_a^o|}{|F_a^o|+1} \cdot d_{|F^o|-|F_a^o|}^1 + \left(1 - \frac{|F_a^o|}{|F_a^o|+1}\right) \cdot d_{max}^1,$$

and the expected delay for $a(1)$ in instance I when the $|F^o| - |F_a^o| - 1$ -th flight is being drawn but

$a(1)$ has not been drawn yet is

$$D_{|F^o|-|F_a^o|-1}^{1,I} = \frac{|F_a^o|}{|F_a^o|+2} \cdot d_{|F^o|-|F_a^o|-1}^1 + \left(1 - \frac{|F_a^o|}{|F_a^o|+2}\right) \cdot D_{|F^o|-|F_a^o|}^{1,I}.$$

In general, $D_t^{1,I} = p_{I,t}^1 \cdot d_t^1 + (1 - p_{I,t}^1) D_{t+1}^{1,I}$. We can calculate $D_t^{1,I}$'s recursively and eventually get

$$D_1^{1,I} = \frac{|F_a^o|}{|F^o|} \cdot d_1^1 + \left(1 - \frac{|F_a^o|}{|F^o|}\right) \cdot D_2^{1,I}.$$

For $a(1)$ in I' ,

$$D_{|F^o|-|F_a^o|}^{1,I'} = \frac{|F_a^o|-1}{|F_a^o|} \cdot d_{|F^o|-|F_a^o|}^1 + \left(1 - \frac{|F_a^o|-1}{|F_a^o|}\right) \cdot d_{max}^1.$$

Because $d_{|F^o|-|F_a^o|}^1 \leq d_{max}^1$ and $\frac{|F_a^o|}{|F_a^o|+1} > \frac{|F_a^o|-1}{|F_a^o|}$, we have $D_{|F^o|-|F_a^o|}^{1,I} \leq D_{|F^o|-|F_a^o|}^{1,I'}$. Since $p_{I,t}^1 > p_{I',t}^1$ for all t , we have $D_t^{1,I} \leq D_t^{1,I'}$ for all t . ($D_1^{1,I}$ is indeed the expected delay for $a(1)$ in $\phi^I(R, e)$, that is, $d_{a(1)}(\phi^I(R, e))$.)

In general, the expected delay for $a(i)$ in instance I when the t -th flight is being drawn but $a(i)$ has not been drawn yet is $D_t^{i,I}$ ($D_t^{i,I}$ for $t \leq i$ are the same as $D_i^{i,I}$, and $D_1^{i,I} = D_i^{i,I}$ is the expected delay for $a(i)$ in $\phi^I(R, e)$, that is, $d_{a(i)}(\phi^I(R, e))$). If $D_1^{1,I} = D_1^{1,I'}$, then we need to show $D_1^{2,I} \leq D_1^{2,I'}$.

Recall that $q_{I,t}^i$ is the probability of $a(i)$ in instance I is drawn before the t -th flight is being drawn. Observe that $q_{I,t}^1(p_{I,1}^1, p_{I,2}^1, \dots, p_{I,t-1}^1)$ is monotone in each argument. Since $p_{I,t}^1 > p_{I',t}^1$ for all t , we have $q_{I,t}^1 \geq q_{I',t}^1$ for all t (when the t -th flight is being drawn but $a(1)$ has not been drawn yet, the probability that $a(1)$ is not drawn is $(1 - p_{I,t}^1)$ for all t , and so $q_{I,t}^1 = 1 - \prod_{t-1} (1 - p_{I,t}^1)$). Observe that $q_{I,t}^2(q_{I,t-1}^1, p_{I,2}^2, \dots, p_{I,t-1}^2)$ is monotone in each argument. Since $p_{I,t}^2 > p_{I',t}^2$ for all t , we have $q_{I,t}^2 \geq q_{I',t}^2$ for all t (when the t -th flight is being drawn but $a(2)$ has not been drawn yet and $a(1)$ is drawn before (this has probability $q_{I,t}^1$), the probability that $a(2)$ is not drawn is $(1 - q_{I,t}^1 \cdot p_{I,t}^2)$ for all t ; therefore, $q_{I,t}^2 = 1 - \prod_{t-1} (1 - q_{I,t}^1 \cdot p_{I,t}^2)$). In general, $q_{I,t}^i(q_{I,t-1}^{i-1}, p_{I,i}^i, \dots, p_{I,t-1}^i)$ is monotone in each argument. Since $p_{I,t}^i > p_{I',t}^i$ for all i and t , we have $q_{I,t}^i \geq q_{I',t}^i$ for all i and t ($q_{I,t}^i =$

$1 - \prod_{t-1} (1 - q_{I,t}^{i-1} \cdot p_{I,t}^i)$). Therefore, we have $q_{I,t}^{i-1} \cdot p_{I,t}^i \geq q_{I',t}^{i-1} \cdot p_{I',t}^i$ for all i and t ($q_{I,t}^0 = 1$ for all t). Note that

$$D_t^{i,I} = q_t^{i-1} \cdot p_t^i \cdot d_t^i + (1 - q_t^{i-1} \cdot p_t^i) D_{t+1}^{i,I}$$

($D_t^{1,I} = p_t^1 \cdot d_t^1 + (1 - p_t^1) D_{t+1}^{1,I}$ because $q_{I,t}^0 = 1$ for all t). By the same recursive argument above, we have $D_1^{2,I} \leq D_1^{2,I'}$ and in general, $D_1^{i,I} \leq D_1^{i,I'}$ for all $a(i)$.

Self optimization after obtaining $\phi_a^{I'}(R, e)$: If s cannot be used by any flight of a , we are done. Otherwise, a can give s to some of its flights and obtain some self-optimized landing schedule for $a \Pi_a^{\phi_a^{I'}(R, e) \cup \{s\}}$. Let $a(j)$ be the most important flight that might be assigned s in all $\Pi_a^{\phi_a^{I'}(R, e) \cup \{s\}}$'s.

Case 1: If $a(j)$ obtains some scarce resource s' in MTC. The fact that s , which is not demanded by another airline, might be given to $a(j)$ implies that s is earlier than the sequence of slots that contains s' (suppose s is later than the sequence. This contradicts the way we pick $a(j)$ as $a(j)$ always gets an earlier slot in the sequence). The expected delay for $a(j)$ is therefore constant as it always gets s . Still, we have $D_1^{i,I} \leq D_1^{i,I'}$ for each $a(i)$ —in addition to all of its flights pick weakly later, each remaining flight that is less important than $a(j)$ picks strictly later because $a(j+1)$ can use $a(j)$'s position in z if s is in the instance, and so on. So a is not better off in this case (the best thing a can do is to let $a(j)$ pick a slot for $a(j+1)$, $a(j+1)$ pick a slot for $a(j+2)$, etc. But this is the same as having s put in the instance).

Case 2: If $a(j)$ obtains some non-scarce resource s' in MTC. The fact that s , which is not demanded by another airline, might be given to $a(j)$ implies that s is earlier than s' . The expected delay for $a(j)$ is therefore constant as it always gets s .

(♣) If s' cannot be used by other flights of a or s' can be used by some flights of a but each of these flights obtains a slot better than s' in MTC, we are done because of $D_1^{i,I} \leq D_1^{i,I'}$ for each $a(i)$. Otherwise, let $f \in F_a$ be the most important flight that might be assigned s' . If f obtains some scarce resource s'' in MTC, the expected delay for f is therefore constant as it always gets s' . Still, we have $D_1^{i,I} \leq D_1^{i,I'}$ for each $a(i)$ —in addition to all of its flights pick weakly later, each remaining

flight that is less important than f picks strictly later (as in Case 1). If f obtains some non-scarce resource s'' in MTC, repeat (♣) with s'' in place of s' . We will eventually have $D_1^{i,I} \leq D_1^{i,I'}$ for each $a(i)$ since a has finitely many flights. So a is not better off in this case.

Now suppose s might be used to trade. If s is used by a itself under all z 's, we are done as $D_1^{i,I} \leq D_1^{i,I'}$ for all $a(i)$. Suppose s is traded for some slot under some z . Let $a(k)$ be the most important flight that gets an earlier slot under all such z 's. If there is no flight that gets an earlier slot under all z 's, we are done as $D_1^{i,I} \leq D_1^{i,I'}$ for all $a(i)$. Otherwise, observe that $D_1^{k,I}$ is lowered as some d_i^i is reduced and thus $D_1^{k,I} < D_1^{k,I'}$ (in this case, it is possible that $D_1^{m,I} > D_1^{m,I'}$ for some $m > k$). So a is not better off in this case.

(4-2) Let $f_{a,1}$ be the most important flight of a . Suppose the earliest feasible available slot for $f_{a,1}$, s , is in S_a and s is a scarce resource. $a(1)$ in I represents $f_{a,1}$. $a(2)$ in I represents some flight of a . $D_1^{2,I}$ is the expected delay for this flight in instance I in MTC. Let $a(1)$ in I' represent this flight in I' . $D_1^{1,I'}$ is the expected delay for this flight in instance I' in MTC. We want to show $D_1^{2,I} \geq D_1^{1,I'}$.

Now d_i^i in I might be different from d_i^i in I' . Let $d_t^{i,I}$ denote the one in I . For $i \geq 2$, let $\tilde{d}_t^{i,I}$ be the expected delay for flight $a(i)$ in I in landing schedules that are induced by initial ordering z 's with $a(i)$ in position t conditioning on $a(1)$ gets the first position in z with probability 1 (that is, $q_{I,1}^1 = 1$). Observe that

$$D_t^{2,I} = q_{I,t}^1 \cdot p_{I,t}^2 \cdot d_t^{2,I} + (1 - q_{I,t}^1 \cdot p_{I,t}^2) D_{t+1}^{2,I} \geq p_{I,t}^2 \cdot \tilde{d}_t^{2,I} + (1 - p_{I,t}^2) D_{t+1}^{2,I} = p_{I',t}^1 \cdot d_{t-1}^{1,I'} + (1 - p_{I',t}^1) D_t^{1,I'} = D_t^{1,I'}.$$

For $t \geq i \geq 2$, each $q_{I,t}^i$ is monotone in $q_{I,t-i+1}^1$ and 1 is the optimum for all $q_{I,t-i+1}^1$ (as $q_{I,1}^1 = 1$), but in the second term, each $q_{I,t-i+1}^1$ is weakly less than 1, which would result in weakly larger expected delays for all other flights of a (this gives the weak inequality). $d_{t-1}^{1,I'}$ is effectively the same as $\tilde{d}_t^{2,I}$ for all $t \geq 2$ (this gives the second equality). If $D_1^{2,I} = D_1^{1,I'}$, then we need to show

$D_1^{3,I} \geq D_1^{2,I}$. But the argument is the same for all $a(i)$ in I with $i \geq 2$. ■

Proof of Proposition 4: (All proofs are the same except the one for Theorem 3. We provide the modification to the Proof of Theorem 3 for the modified pre-competition stage (all other parts are the same).)

...In the pre-competition stage, the only possible way for airline a to get a slot that is feasible for $f_{a,j}$ but not being assigned in this stage is to make it a non-scarce resource (type 4 slot) such that it will be assigned to $f_{a,j}$. We show that any slot that can be converted to a type 4 slot by a cannot be s .

Case 1: Suppose s is a type 1 slot (either in the type 1 slot group or a slot group with both type 1 and type 2 slots). a can give s to $f_{a,j}$ only when the type 1 slots at the same time are demanded by a with $f_{a,x}, \dots, f_{a,x'}, \dots$, where $f_{a,x}, \dots R_a f_{a,j} R_a f_{a,x'}, \dots$ and at most $n(s) - |f_{a,x}, \dots, f_{a,j}|$ flights that do not belong to a (call these flight $f_{b,x}, \dots$), where $n(s)$ is the number of these type 1 slots and $|f_{a,x}, \dots, f_{a,j}|$ is the number of a 's flights that are weakly more important than $f_{a,j}$. Then a can misreport $\widehat{e_{f_{a,x'}, \dots}}$ to give s to $f_{a,j}$ ($f_{a,x}, \dots$ and $f_{b,x}, \dots$ also get these slots). (Note that this procedure (of misreporting $\widehat{e_{f_{a,x'}, \dots}}$) makes the type 1 slots satisfy (c-ii). It also makes the next slot group a type 1 slot group if it was a type 2 slot group or a slot group with both type 1 and type 2 slots; moreover, it makes the next slot group a type 4 slot group if it was a type 3 slot group or a slot group with both type 3 and type 4 slots. In other words, this procedure makes the slots in next slot group satisfy (c-i).)

Case 2: Suppose s is a type 2 (either in a type 2 slot group or a slot group with both type 1 and type 2 slots). If not impossible, a can give s to $f_{a,j}$ only by repeating the procedure in Case 1. This requires at each previous iteration, type 1 slots at the same time are demanded by a with $f_{a,y}, \dots, f_{a,y'}, \dots$, where $f_{a,y}, \dots R_a f_{a,j} R_a f_{a,y'}, \dots$ and at most $n(s) - |f_{a,y}, \dots|$ flights that do not belong to a (call these flight $f_{c,y}, \dots$). Then a can misreport $\widehat{e_{f_{a,y'}, \dots}}$ so that $f_{a,y}, \dots$ and $f_{c,y}, \dots$ get these slots. At the end, it requires s to become a type 1 slot that can be given to $f_{a,j}$ as described in Case 1. (If not impossible, a can convert s to a type 3 slot by using the procedure in Case 1 to make s satisfy

(c-ii). But there is no way for a to convert it to a type 4 slot after that.)

Case 3: Suppose s is a type 3 slot (either in a type 3 slot group or a slot group with both type 3 and type 4 slots). a can give s to $f_{a,j}$ by misreport $e_{f_{a,j}}$ as s ($> e_{f_{a,j}}$) and have this slot assigned to $f_{a,j}$ (if there is more than 1 such type 3 slots. Let flights that are assigned type 3 slots in $\hat{\Pi}$ be $f_{a,x}, f_{a,x'}, \dots$. a also needs to misreport $e_{f_{a,x}}, \widehat{e_{f_{a,x'}}}, \dots$. But this even requires $f_{a,j}$ to be the most important flight among $f_{a,j}, f_{a,x}, f_{a,x'}, \dots$ since a is only willing to sacrifice flights that are less important than $f_{a,j}$ to help it).

In Case 1 and 2, there are sufficient slots to accommodate the only competitors $f_{b,x}, \dots, f_{c,y}, \dots$ and $f_{a,x}, \dots, f_{a,y}, \dots$ before s is being picked as $f_{b,x}, \dots, f_{c,y}, \dots$ and $f_{a,x}, \dots, f_{a,y}, \dots$ will always try to get a slot (weakly) earlier than s and flights $f_{a,x'}, \dots, f_{a,y'}, \dots$ that are less important than $f_{a,j}$ will not compete with them before $f_{a,j}$ gets a slot in the main stage. In Case 3, $f_{a,j}$ can get a weakly earlier slot in the main stage. In all cases, $f_{a,j}$ can always get a weakly earlier slot in the main stage, so $\varphi_{f_{a,j}}^z(R, e) \leq \varphi_{f_{a,j}}^z(\widehat{R}_a, e_a, (R, e)_{-a}) = s$, a contradiction... ■

2.8.2 Summary of properties

	Compression	TC	DASO	MTC
Preference domain*	-	$e, \Pi^{current}$	e, w^{**}	e, R
Individual rationality	No***	Yes	Yes	Yes
Pareto efficiency	No***	Yes	No	Yes
Core	No	Yes	No	Yes
Strategy-proofness	No***	Yes	No	Yes
Non-manipulable****	No	No	Yes	Yes
Ex post property rights	No	Yes	Yes	Yes
Ex ante property rights	No (because of RBS)			Yes

* Properties defined by preference are different in different preference domains.

** w are weights of flights.

*** Yes in the preference domain of [Schummer and Vohra \[2013\]](#).

**** Non-manipulable by postponing flight cancelation.

2.8.3 Examples for Compression

We use the following example to show Compression is not strategy-proof. This is the same example [Schummer and Abizada \[2017\]](#) used to show Compression is not strategy-proof. We convert weights (in that example) to rankings.

Example 15:

S	s_1	s_2	s_3	s_4
F^o	-	$f_{a,2}$	$f_{b,1}$	$f_{a,1}$
Φ	c	a	b	a
e_f	-	2	1	1
\hat{e}_f	-	1	1	2
R	-	2	1	1
Compression (e)	$f_{b,1}$	$f_{a,2}$	$f_{a,1}$	c
Compression (\hat{e})	$f_{a,2}$	$f_{a,1}$	$f_{b,1}$	c

If a slot is unusable to its owner, then the next flight in the schedule that can feasibly use it would be assigned this slot in Compression. If a reports e_a , then b obtains s_1 for $f_{b,1}$ and c obtains s_3 . Then a obtains s_3 for $f_{a,1}$ and c obtains s_4 . But if a reports \hat{e}_a , then a obtains s_1 for $f_{a,2}$ and c obtains s_2 . Then a obtains s_2 for $f_{a,1}$ and c obtains s_4 . We can see a strictly gain in this case (a can swap slots for $f_{a,1}$ and $f_{a,2}$).

We use the following example to show Compression is manipulable by postponing a flight cancelation. This is the same example [Schummer and Vohra \[2013\]](#) used to show Compression is manipulable via slot destruction. We put rankings on flights.

Example 16:

S	s_1	s_2	s_3	s_4	s_5	s_6	s_7
F^o	$f_{a,(1)}$	$f_{b,(1)}$	$f_{a,(2)}$	$f_{c,1}$	$f_{b,1}$	$f_{a,1}$	$f_{b,2}$
Φ	a	b	a	c	b	a	b
e_f	-	-	-	2	4	4	1
R	-	-	-	1	2	1	1
Compression	$f_{b,2}$	$f_{c,1}$	a	$f_{b,1}$	$f_{a,1}$	b	a
Compression'	$f_{a,(1)}$	$f_{b,2}$	$f_{c,1}$	$f_{a,1}$	$f_{b,1}$	a	b

$f_{a,(1)}$, $f_{b,(1)}$ and $f_{a,(2)}$ are canceled flights. In the first step of Compression, b obtains s_1 for $f_{b,2}$ while a obtains s_7 . Then b obtains s_4 for $f_{b,1}$ while c obtains s_2 for $f_{c,1}$. Finally, a obtains s_5 for $f_{a,1}$ while b obtains s_6 . But if a freezes $f_{a,(1)}$ in s_1 . In the first step of Compression', $f_{b,2}$ is moved to s_2 . Then c obtains s_3 for $f_{c,1}$ while a obtains s_4 for $f_{a,1}$. Finally, b obtains s_5 for $f_{b,1}$. It is easy to see a gains by having s_4 .

Chapter 3

DYNAMIC PRICE COMPETITION FOR SUPPLY

3.1 Introduction

Price competition models are mostly for demand. This chapter studies a dynamic price competition model for supply. The model is motivated by an observation from the fishing industry. The following quote from the IRS describes the situation:

It is common for fish processors to pay a retroactive bonus to fishermen (based on the ultimate sales price of the fish) in order to entice the fishermen to sell to the processor in the upcoming fishing season. ... Depending on the profitability of a particular fishing season, the bonus may or may not be paid. ... Each processor unilaterally decides to pay a bonus. The processor has no legal obligation to pay the bonus. ... Generally fishers do not enter contracts with processors that guarantee the fishers will receive a bonus ... written price guarantees to fishers by processors are largely unheard of.¹

One possible reason to explain why processors might abstain from specifying a written price is that they face uncertainties from both the supply side and the demand side. On the one hand, how much fish will be harvested by her fishermen is uncertain, and on the other hand, the final sales price she can get is also uncertain (this might be affected by the harvest of other places, which is uncertain).

The situation can be described as a game where the timing of each period is as follows: (1) fishermen choose one processor; (2) fishermen deliver fish and processors pay base prices; (3)

¹The quote is from the IRS's website, which is available at <https://www.irs.gov/businesses/small-businesses-self-employed/retroactive-bonus-payments-for-fishermen> (last accessed June 2018).

processors sell the fish to buyers and then pay bonuses to fishermen. There is a holdup problem in the stage game: the processors have all the ex-post bargaining powers, so there would be no trade as fishermen anticipate no payment. If this stage game is repeatedly played by a long-lived processor and a long-lived fisherman, then we can expect the processor to set the payment equal to the fisherman's cost to keep the fisherman in business and extract all rents in each period.

We develop a model that attempts to capture this situation. Though our model stems from the fishing industry, we think it might provide some insights to some oligopsony markets (e.g. markets in agricultural), so we will use intermediate (processor) and supplier (fisherman) instead hereafter. In our model, we slightly abstract from the above timeline: we assume each supplier delivers one unit of good and consolidate the base prices and bonuses into payments to be paid in the third stage. We show that there exists a symmetric stationary Markov Perfect Bayesian equilibrium in behavior strategies in general, and when intermediates are not so patient, we show that there exists a symmetric monotone pure strategy stationary Markov perfect Bayesian equilibrium. In the pure strategy equilibrium, an intermediate pays less when she has a larger number of suppliers, and vice versa; also, the equilibrium strategies are partially revealing in the sense that after an intermediate observes her opponent's payment, she can perfectly infer her opponent's profit shock if the payment is less than some threshold.

[Chen and Rosenthal \[1996\]](#), [Kováč and Schmidt \[2014\]](#) study dynamic duopoly price competition models with randomized pricing strategies. In [Chen and Rosenthal \[1996\]](#), they assume exactly one buyer will switch from the high-price seller to the low-price seller in the next period, while in [Kováč and Schmidt \[2014\]](#), they assume buyers exchange information via word-of-mouth communication such that the high-price seller loses certain percentages of her buyers in the current period. In these two models, the size of the price difference is irrelevant. By contrast, the size of the price difference plays an important role in our price competition; specifically, the larger the price difference, the larger the chance that the high-payment intermediate will get more than a threshold

number of suppliers in the next period. This means suppliers in our model are sensitive to the size of the price difference. The threshold number is determined by suppliers' responsiveness to the price difference, and this threshold number is higher if the suppliers are more responsive. In our language, Bertrand competition assumes consumers are extremely responsive to a price difference but completely insensitive to the size of the price difference because all consumers will switch to the low-price seller for an arbitrarily small price difference.

[Athey and Bagwell \[2001\]](#), [Athey et al. \[2004\]](#), [Athey and Bagwell \[2008\]](#) study repeated/dynamic Bertrand games in which prices are publicly observable and each firm receives a privately observed, i.i.d cost shock in each period. They study collusive equilibria in these games. In contrast, we focus on Markov perfect equilibria. One way to interpret Markov perfection is that it represents the absence of collusion.

3.2 The Model

The economy consists of 2 intermediates and N suppliers. Intermediates are the players of the game and suppliers are non-strategic. Time $t = 0, 1, 2, \dots$ is discrete and infinite. Intermediates are risk-neutral and share a common discount factor $\delta \in [0, 1)$. At time 0, intermediates design a contingency plan to maximize their expected payoffs. Each period consists of three stages:

1. Each supplier contracts with one intermediate $i \in \{\alpha, \beta\}$.
2. Each supplier delivers one unit of good to the intermediate.
3. Each intermediate, after observing her own (per unit) profit $\pi_{i,t}$, chooses to pay each of her supplier $w_{i,t}$.

Costs of the suppliers are constant and normalized to 0. $\pi_{i,t}$ captures the total effect of all shocks on i at time t and is only observed by i . The distribution of $\pi_{i,t}$ is common knowledge and has

a finite support $\Pi \equiv \{0, \pi^1, \dots, \bar{\pi}\}$. $\pi_{i,t}$ are i.i.d. across intermediates and across periods. Let $\Pi_t = (\pi_{i,t}, \pi_{-i,t})$. For each π in the support, let p_π be its probability mass.

Payments $w_{i,t} \geq 0$ are publicly observable and homogeneous across suppliers contracted with i at time t . Intermediates are liquidity-constrained and therefore cannot pay more than their profits in each period, so $w_{i,t} \leq \pi_{i,t}$. In periods with $\pi_{i,t} = \pi$, the set of feasible actions for intermediate i is a compact set $[0, \pi] = A^\pi \subseteq A^{\bar{\pi}} \subset \mathbb{R}$. Let $A = A^{\bar{\pi}} \times A^{\bar{\pi}}$. A generic element of A is denoted by a or (w_i, w_{-i}) . Let $n_{i,t}$ the number of suppliers that work for i at time t (and so $n_{-i,t} = N - n_{i,t}$). Let $n_t = (n_{i,t}, n_{-i,t}) \in \mathcal{N}$, where $\mathcal{N} \equiv \{(0, N), (1, N-1), \dots, (N, 0)\}$ with $|\mathcal{N}| = N + 1$. $u_{i,t} = n_{i,t}(\pi_{i,t} - w_{i,t})$ is the stage-game payoff for i in period t .

A game state $s_t = (n_t, \Pi_t)$. Let S_t be set of game states. In this game, time is not payoff-relevant. For brevity, we drop the subscript t whenever possible. Let $S \equiv \mathcal{N} \times \Pi^2$ be the set of Markov states with generic element $s = (n, \Pi)$; S is a finite state space with $(N + 1) \times |\Pi^2|$ elements. Let $\Delta(S)$ be the set of distribution over states. The evolution of the state is given by a function $q : A \times S \rightarrow \Delta(S)$, which maps each current action profile and state to a probability distribution of the next state. In this model, the transition probability only indirectly depends on the current state through actions. Let $q(s'|a)$ be the transition probability to state s' given a .

$$\begin{aligned} q(s'|a) &= q(n', \Pi'|a) \\ &= p(n'|a) \times p_{\pi'_i} \times p_{\pi'_{-i}}. \end{aligned} \tag{3.1}$$

Now we make assumptions on the conditional probability distribution, $P(n'|a)$, that is characterized by the probability mass function $p(n'|a)$. Let $d(w_i, w_{-i}) = |w_i - w_{-i}| \in [0, \bar{\pi}]$. For $k > 1$, let h be the number of n' 's such that $n' > \frac{N}{k}$ and l be the number of n' 's such that $n' < \frac{N}{k}$. Let $p(d)$ be a continuously differentiable function that satisfies the following conditions:

1. $p(0) = \frac{h}{N+1}$;

$$2. p(\bar{\pi}) = \frac{h+l}{N+1};$$

$$3. p'(d) > 0 \text{ and } p''(d) < 0;$$

$$4. \lim_{d \rightarrow 0} p'(d) = \infty;$$

$$5. \lim_{d \rightarrow \bar{\pi}} p'(d) = 0.$$

$\frac{h}{N+1}$ is the probability mass that is assigned to the h high states and $\frac{l}{N+1}$ is the probability mass that is assigned to the l low states when $d = 0$.

Now we set

$$p(n'_{(\cdot)} = n' | w_i, w_{-i}) = \begin{cases} \frac{p(d)}{h} & \text{for } n' > \frac{N}{k} \\ \frac{1-p(d)}{l} & \text{for } n' < \frac{N}{k} \\ \frac{1}{N+1} & \text{for } n' = \frac{N}{k} \end{cases}. \quad (3.2)$$

In any period, if $w_i \geq w_{-i}$, let

$$\begin{aligned} p(n'|a) &= p(n'_i = n' | w_i, w_{-i}) \\ &= p(n'_{-i} = N - n' | w_i, w_{-i}). \end{aligned}$$

This implies if $w_i < w_{-i}$, then

$$\begin{aligned} p(n'|a) &= p(n'_{-i} = n' | w_i, w_{-i}) \\ &= p(n'_i = N - n' | w_i, w_{-i}). \end{aligned}$$

We assume each intermediate has its own supplier, who serves as a device to compete when $n_{i,t} = 0$.

Observe that $0 \leq p(n'|a) \leq 1$ for all n' and $\sum_{n'} p(n'|a) = 1$, so $P(n'|a)$ is a probability distribution. When $w_i = w_{-i}$, $p(n'|a) = \frac{1}{N+1}$ for all n' . When $w_i > w_{-i}$, intermediate i that pays higher in this period will be more likely to have more than $\frac{N}{k}$ suppliers in the next period. k can be interpreted as a measurement of the suppliers' responsiveness to the price difference. Suppliers are more responsive as k becomes smaller. As k goes down, $\frac{N}{k}$ becomes bigger, so h becomes (weakly) smaller and l becomes (weakly) larger, which means there are less high states and more low states. This implies that there is a larger probability mass that can be shifted from the low states to the high states. When $k \leq 2$, suppliers are relatively responsive because if an intermediate pays higher than her opponent in this period, then she is more likely to have more suppliers in the next period. When k goes to 1, h reaches its minimum 1 and l reaches its maximum N , where $n' = N$ is the only high state. When k goes to ∞ , h reaches its maximum N and l reaches its minimum 1, where $n' = 0$ is the only low state. Figure 3.1a illustrates $p(n'|a)$ when there is no $n' = \frac{N}{k}$ and Figure 3.1b illustrates $p(n'|a)$ when there is an $n' = \frac{N}{k}$. The first line under the horizontal axis is n'_i given $w_i \geq w_{-i}$ and the second line is n'_{-i} , which is from N to 0.

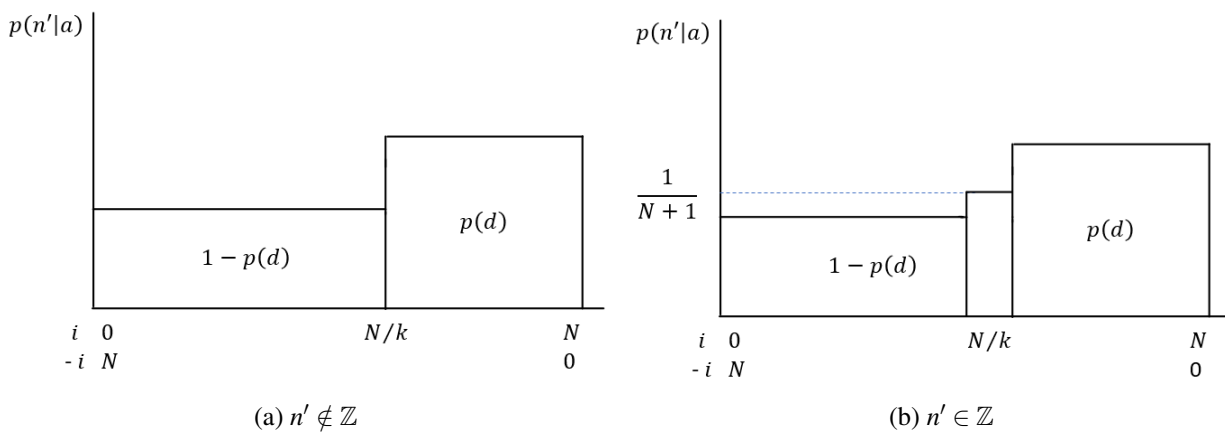


Figure 3.1: $p(n'|a)$

Consider the following family of functions:

$$p_\lambda(d) = \left(\frac{h}{N+1}\right) + \left(\frac{l}{N+1}\right) \left(1 - \left(1 - \frac{d}{\bar{\pi}}\right)^\lambda\right)^{\frac{1}{\lambda}}$$

for $1 < \lambda < \infty$.

Lemma 1: (1) For all $1 < \lambda < \infty$, $p_\lambda(d)$ satisfies the 5 conditions. (2) For all $1 < \lambda < \infty$ and $d \in (0, \bar{\pi})$, $\frac{\partial p_\lambda(d)}{\partial \lambda} > 0$.² (3) If $\lambda = \infty$, then $p_\infty(\varepsilon) = \frac{h+l}{N+1}$ for all $\varepsilon > 0$.

(2) implies that for any fixed $d \in (0, \bar{\pi})$ and $\lambda < \lambda'$, $p_\lambda(d) < p_{\lambda'}(d)$ and so $p'_{\lambda'}(d)$ tends to 0 faster than $p'_\lambda(d)$, where the latter can be also observed from the expression of $p'_\lambda(d)$ directly (see Appendix 3.5). Figure 3.2 illustrates these facts. Figure 3.3 illustrates $p_\lambda(d)$ when $n' \in \mathbb{Z}$.

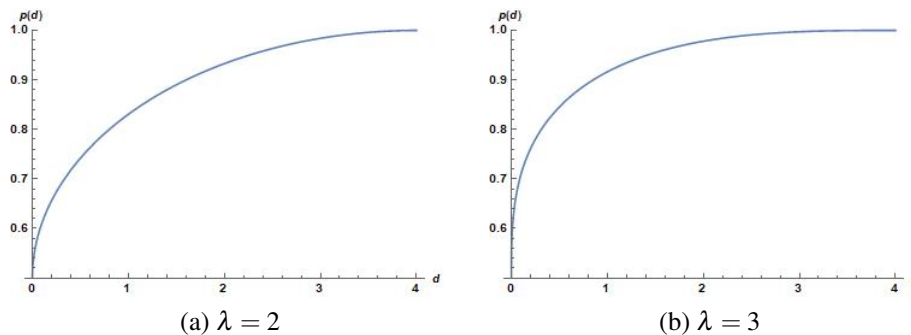


Figure 3.2: $p^\lambda(d)$ with $\frac{h}{N+1} = \frac{l}{N+1} = \frac{1}{2}$ ($n' \notin \mathbb{Z}$), $\bar{\pi} = 4$.

² $p_\lambda(0) = 0$ for all $\lambda > 1$, so $\frac{\partial p_\lambda(0)}{\partial \lambda} = 0$. $p_\lambda(\bar{\pi}) = 1$ for all $\lambda > 1$, and $\frac{\partial p_\lambda(\bar{\pi})}{\partial \lambda}$, which contain terms with $\ln(0)$, is not defined because the logarithmic function is not defined at 0 (see Appendix 3.5 for more details).

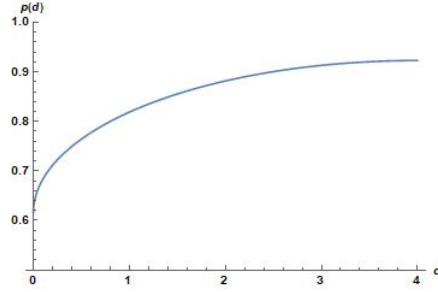


Figure 3.3: $p^\lambda(d)$ with $N = 12$, $k = 3$, $\frac{h}{N+1} = \frac{8}{13}$, $\frac{l}{N+1} = \frac{4}{13}$ ($n \in \mathbb{Z}$), $\bar{\pi} = 4$ and $\lambda = 2$.

λ can be interpreted as a measurement of the suppliers' sensitivity to the sizes of price differences. Suppliers are less sensitive if λ becomes larger. When $k \rightarrow 1$ and $\lambda \rightarrow \infty$, all suppliers will supply to an intermediate if the intermediate pays a slightly higher payment in this period. This is reminiscent of the consumers in the Bertrand price competition model, in which they will buy from the seller with a slightly lower price. In our language, Bertrand competition assumes consumers are extremely responsive to a price difference but completely insensitive to the size of the price difference.

Let $\mathcal{H}_i^t = (\mathcal{N} \times \Pi \times A)^t$ be i 's set of period t ex ante private histories ($\mathcal{H}_i^0 = \{\emptyset\}$) and $\tilde{\mathcal{H}}_i^t = (\mathcal{N} \times \Pi_i \times A)^t \times (\mathcal{N} \times \Pi_i)$ be i 's set of period t ex post private histories. Let $\tilde{\mathcal{H}}_i = \cup_{t=0}^{\infty} \tilde{\mathcal{H}}_i^t$ be i 's set of ex post private histories. Let $\mathcal{H}^t = (S \times A)^t$ be the set of period t ex ante histories ($\mathcal{H}^0 = \{\emptyset\}$) and \mathcal{H}^∞ be the set of outcomes.

A *pure strategy* for i is a mapping $\sigma_i : \tilde{\mathcal{H}}_i \rightarrow A^{\bar{\pi}}$. A *behavior strategy* for i is a mapping $\sigma_i : \tilde{\mathcal{H}}_i \rightarrow \mathcal{M}(A^{\bar{\pi}})$, where $\mathcal{M}(A^{\bar{\pi}})$ is the space of all (Borel) probability measures on $A^{\bar{\pi}}$. Without loss of generality, we restrict our attention to feasible strategies, i.e., strategies that specify a feasible action or assign probability 1 to the set of feasible actions A^π at every state with $\pi_i = \pi$.

Given the transition function q , a strategy profile σ induces a probability measure over \mathcal{H}^∞ . The discounted expected payoff of i at time 0 is

$$U_i^{s^0}(\sigma) = E_{s^0}^\sigma[(1 - \delta) \sum_{t=0}^{\infty} \delta^t n_{i,t}(\pi_{i,t} - w_{i,t})],$$

where the expectation is taken with respect to the probability measure over \mathcal{H}^∞ induced by σ and $s^0 \in S$ (s^0 is some initial state).

A *Markov strategy* for i is a mapping $\sigma_i : (\mathcal{N} \times \Pi_i)_t \rightarrow \mathcal{M}(A^{\bar{\pi}})$. In this chapter, we focus on a special and natural class of Markov strategies: stationary Markov strategies. A *stationary* Markov strategy for i is a mapping $\sigma_i : \mathcal{N} \times \Pi_i \rightarrow \mathcal{M}(A^{\bar{\pi}})$. A strategy profile σ is called a **stationary Markov perfect Bayesian equilibrium** if

$$U_i^{s^0}(\sigma_i, \sigma_{-i}) \geq U_i^{s^0}(\sigma'_i, \sigma_{-i})$$

for all i , all $s^0 \in S$ and all σ'_i . Note that this is a game of incomplete information and σ_i depends on π_i, n_i, n_{-i} and the distribution of π_{-i} . Because n_i pins down n_{-i} and the realization of π_{-i} is not observed by i , we sometimes write $\sigma_i(s)$ as $\sigma_i(n, \pi)$, where π and n are consistent with π_i and n_i in s .

Since the game has a stationary structure, we can use the standard dynamic programming approach to study this problem. Specifically, we can focus on a one-shot normal form game, where each player's payoff function is given by a convex combination of his stage-game payoff and expected continuation value. For s with $n_i = n$, and $\pi_i = \pi$, the solution of intermediate i 's optimization problem (of $U_i^{s^0}$) satisfies the following Bellman-equation:

$$V_i(n, \pi) = \max_{w \in A^{\bar{\pi}}} \left\{ \sum_{\pi_{-i}} p_{\pi_{-i}} \left[(1 - \delta)n(\pi - w) + \delta \sum_{s' \in S} V(s')q(s'|w, \sigma_{-i}(n_{-i}, \pi_{-i})) \right] \right\}. \quad (3.3)$$

A strategy profile σ is a **stationary Markov perfect Bayesian equilibrium** if

$$\sigma_i(n, \pi) \in \arg \max_{w \in A^\pi} \left\{ \sum_{\pi_{-i}} p_{\pi_{-i}} \left[(1 - \delta)n(\pi - w) + \delta \sum_{s' \in S} V(s')q(s'|w, \sigma_{-i}(n_{-i}, \pi_{-i})) \right] \right\}$$

for all i , all π and all n .

Each $s = (n_i, n_{-i}, \pi_i, \pi_{-i})$ induces a one-shot Bayesian normal form game with players' payoff functions given by

$$\tilde{u}_i^{(n_i, \pi_i)}(w_i(\cdot), w_{-i}(\cdot)) = \sum_{\pi_{-i}} p_{\pi_{-i}} \left[(1 - \delta)n_i(\pi_i - w_i(\cdot)) + \delta \sum_{s' \in S} V(s')q(s'|w_i(\cdot), w_{-i}(\cdot)) \right] \quad (3.4)$$

for all i , where $w_i : \{0, \dots, N\} \times \Pi \rightarrow \mathcal{M}(A^{\pi_i})$ is a (mixed) strategy for player i in this Bayesian game.³ Let W_i be the set of strategies for i . A Bayesian Nash equilibrium for this Bayesian game is a strategy profile $w(s) = (w_i(\cdot), w_{-i}(\cdot))$ that constitutes a Nash equilibrium of game $G(s) = [I, \{W_i\}, \{\tilde{u}_i(\cdot)\}]$; that is, $\tilde{u}_i^{(n_i, \pi_i)}(w_i(\cdot), w_{-i}(\cdot)) \geq \tilde{u}_i^{(n_i, \pi_i)}(w'_i(\cdot), w_{-i}(\cdot))$ for all $w'_i(\cdot) \in W_i$ and all i . Therefore, by construction, a strategy profile σ is a stationary Markov perfect Bayesian equilibrium if and only if for all states $s \in S$, $w(s) = \sigma(s)$ is a Nash equilibrium of the induced game $G(s)$.

3.3 Results

When $\delta = 0$, there is an obvious pure strategy equilibrium for the dynamic Bayesian game, where each intermediate pays zero in every period. Observe that $V_i(s)$ is always positive: Because 0 is in the support Π , the probability of $\pi_{-i} = 0$ is positive, which implies even if an intermediate always pays zero, in expectation, she will still get a positive number of suppliers in the next period. Because $V_i(\cdot)$ depends on the opponent's strategy, we do not know the shape of $V_i(\cdot)$. Therefore, we cannot conclude that the utility function (in $G(s)$) is quasi-concave in one's action, which is mostly

³If Π is public information, then each $s = (n_i, n_{-i}, \pi_i, \pi_{-i})$ induces a one-shot normal form game, where each player's payoff function is given by $u_i^{(n_i, \pi_i)}(w_i, w_{-i}) = (1 - \delta)n_i(\pi_i - w_i) + \delta \sum_{s' \in S} V(s')q(s'|w_i, w_{-i})$ for all i .

required for the existence of a pure-strategy Nash equilibrium in general. So for an arbitrary δ , the existence of a pure strategy equilibrium in the dynamic Bayesian game is not guaranteed.

Proposition 1: There exists a symmetric stationary Markov Perfect Bayesian equilibrium in behavior strategies in the dynamic Bayesian game.

Proof of Proposition 1: Construct an auxiliary game with $|\Pi|^2$ players for each $n \leq \frac{N+1}{2}$ ($N+1$ is not needed because of symmetry). The first $|\Pi|$ players are endowed with sets of feasible actions $A^0, A^{\pi^1}, \dots, A^{\bar{\pi}}$ and utility functions $\tilde{u}^{(n,k)} : A^k \times A^0 \times A^{\pi^1} \times \dots \times A^{\bar{\pi}} \rightarrow \mathbb{R}$ (as defined in equation 3.4) for $k \in \Pi$, respectively. The second $|\Pi|$ players are endowed with sets of feasible actions $A^0, A^{\pi^1}, \dots, A^{\bar{\pi}}$ and utility functions $\tilde{u}^{(N-n,k)} : A^k \times A^0 \times A^{\pi^1} \times \dots \times A^{\bar{\pi}} \rightarrow \mathbb{R}$ for $k \in \Pi$, respectively. Let $\mathcal{A} = \{A^0 \times A^{\pi^1} \times \dots \times A^{\bar{\pi}}\} \times \{A^0 \times A^{\pi^1} \times \dots \times A^{\bar{\pi}}\}$. Note that all these utility functions are continuous in \mathcal{A} and all sets of feasible actions are non-empty and compact subsets of \mathbb{R} . By [Glicksberg \[1952\]](#), there exists a mixed-strategy equilibrium for these auxiliary games.⁴ For stationarity, fix a mixed-strategy equilibrium in its corresponding auxiliary game for each n . For $s = (n_i, n_{-i}, \pi_i, \pi_{-i})$, find the auxiliary game with $n \in \{n_i, n_{-i}\}$ and let $w_i(n_i, \pi_i)$ be the mixed-strategy employed by the player with $n_i = n$, and $k = \pi_i$ in the fixed equilibrium; similarly, let $w_{-i}(n_{-i}, \pi_{-i})$ be the mixed-strategy employed by the player with $n_{-i} = N - n$, and $k = \pi_{-i}$ in this equilibrium. By construction, $w(s) = (w_i(n_i, \pi_i), w_{-i}(n_{-i}, \pi_{-i}))$ is a mixed-strategy Nash equilibrium of the game $G(s)$. Let $w(s) = \sigma(s)$, then σ is a symmetric stationary Markov Perfect Bayesian equilibrium in behavior strategies. \square

Now we restrict our attention to monotone pure strategies. We say a pure strategy σ_i is *mono-*

⁴Let $j \in \{n, N-n\}$. A mixed-strategy equilibrium of an auxiliary game is a $|\Pi|^2$ tuple of probability measure $(\mu_{n,0}^*, \mu_{n,\pi^1}^*, \dots, \mu_{n,\bar{\pi}}^*, \mu_{N-n,0}^*, \dots, \mu_{N-n,\bar{\pi}}^*)$, with $\mu_{(j,k)}^* \in \mathcal{M}(A^k)$, such that for each player (j,k)

$$\begin{aligned} & \int \tilde{u}^{(j,k)}(w_{(j,k)}, w_{(-j,k)}) d(\mu_{(j,k)}^*(w_{(j,k)}) \times \mu_{(-j,k)}^*(w_{(-j,k)})) \\ &= \max_{\mu_{(j,k)} \in \mathcal{M}(A^k)} \int \tilde{u}^{(j,k)}(w_{(j,k)}, w_{(-j,k)}) d(\mu_{(j,k)}(w_{(j,k)}) \times \mu_{(-j,k)}^*(w_{(-j,k)})) \end{aligned}$$

where $\mu_{(-j,k)}^*(w_{(-j,k)}) = \prod_k \mu_{(-j,k)}^*(w_{(-j,k)})$. See [Dasgupta and Maskin \[1986\]](#) for a more general definition.

tone if $\sigma_i(n, \pi) \leq \sigma_i(n-1, \pi)$ for all $\pi \in \Pi$ and all $n \geq 1$.

Main Theorem: There exists a $\bar{\delta} \in (0, 1)$ such that for $\delta \in (0, \bar{\delta})$, a symmetric monotone pure strategy stationary Markov perfect Bayesian equilibrium exists.

This result implies that if intermediates are not so patient, at the equilibrium of the Main Theorem, an intermediate would pay less when she has a large number of suppliers. This is rather intuitive: when an intermediate has a larger number of suppliers, paying them higher would be very expensive.

Proof of the Main Theorem: Suppose intermediate $-i$ uses some monotone pure strategy. We guess and verify. We guess $V_i(n-1) < V_i(n)$ for all n . Now we rewrite equation 3.3. It is obvious that

$$V_i(n, \pi) = \max_{w \in A^\pi} \left\{ (1 - \delta)n(\pi - w) + \delta \sum_{\pi_{-i}} p_{\pi_{-i}} \left[\sum_{s' \in S} V(s')q(s'|w, \sigma_{-i}(N-n, \pi_{-i})) \right] \right\}. \quad (3.5)$$

Let $V_i(n) = \sum_{\pi'} p_{\pi'} V_i(n, \pi')$ be the expected equilibrium value for i when $n_i = n$. By equation 3.1, we have

$$\begin{aligned} & \sum_{s' \in S} V(s')q(s'|w, \sigma_{-i}(N-n, \pi_{-i})) \\ &= \sum_{\pi'_{-i}} p_{\pi'_{-i}} \left[\sum_{\pi'} p_{\pi'} \left[\sum_{n'} V(n', \pi') p(n'|w, \sigma_{-i}(N-n, \pi_{-i})) \right] \right] \\ &= \sum_{\pi'_{-i}} p_{\pi'_{-i}} \left[\sum_{n'} [V_i(n') p(n'|w, \sigma_{-i}(N-n, \pi_{-i}))] \right]. \end{aligned} \quad (3.6)$$

Substitute 3.6 into equation 3.5, we have

$$V_i(n, \pi) = \max_{w \in A^\pi} \left\{ (1 - \delta)n(\pi - w) + \delta \sum_{\pi_{-i}} p_{\pi_{-i}} \left[\sum_{\pi'_{-i}} p_{\pi'_{-i}} \left[\sum_{n'} V_i(n') p(n'|w, \sigma_{-i}(N - n, \pi_{-i})) \right] \right] \right\}.$$

Observe that $V_i(n, \pi)$ is continuous in w . The following first-order conditions have to be satisfied whenever possible:

$$\underbrace{\frac{(1 - \delta)}{\delta} n}_{MC_n} = \underbrace{\sum_{\pi_{-i}} p_{\pi_{-i}} \left[\sum_{\pi'_{-i}} p_{\pi'_{-i}} \left[\sum_{n'} V_i(n') \frac{\partial p(n'|w, \sigma_{-i}(N - n, \pi_{-i}))}{\partial w} \right] \right]}_{MB_n}, \quad (3.7)$$

$$\frac{(1 - \delta)}{\delta} (n - 1) = \sum_{\pi_{-i}} p_{\pi_{-i}} \left[\sum_{\pi'_{-i}} p_{\pi'_{-i}} \left[\sum_{n'} V_i(n') \frac{\partial p(n'|w, \sigma_{-i}(N - n + 1, \pi_{-i}))}{\partial w} \right] \right], \quad (3.8)$$

where

$$\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w} = \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \frac{\partial p(d)}{\partial d} \frac{\partial d}{\partial w}.$$

For any fixed n , MC_n is the marginal cost and MB_n is the expected marginal benefit, which is also a function of w . Let w_n^* and w_{n-1}^* denote the solution of equation 3.7 and equation 3.8, respectively. Note that these first-order conditions are independent of the realized π . Observe that

$$\frac{\partial d}{\partial w} = \begin{cases} 1 & \text{for } w \geq \sigma_{-i}(\cdot) \\ -1 & \text{for } w < \sigma_{-i}(\cdot) \end{cases}.$$

Again, the probability of $\pi_{-i} = 0$ is positive. When $\pi_{-i} = 0$, $\frac{\partial d}{\partial w} = 1$. There is at least one high state n' with $\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} > 0$. Recall that $\frac{\partial p(d)}{\partial d} = p'(d)$ and $p'(0) = \infty$. Therefore, at least one term in $MB_n(0)$ equals ∞ , so $MB_n(0) = \infty$ for all $\sigma_{-i}(\cdot)$ and all n .

We take the derivative of the function 3.2 with respect to $p(d)$:

$$\frac{\partial p(n'|w_i, \sigma_{-i}(\cdot))}{\partial p(d)} = \frac{1}{h} > 0$$

for all $n' > \frac{N}{k}$ and

$$\frac{\partial p(n'|w_i, \sigma_{-i}(\cdot))}{\partial p(d)} = -\frac{1}{l} < 0$$

for all $n' < \frac{N}{k}$ and

$$\frac{\partial p(n'|w_i, \sigma_{-i}(\cdot))}{\partial p(d)} = 0$$

for $n' = \frac{N}{k}$. Note that $n'_i = n'$ when $w \geq \sigma_{-i}(\cdot)$ and $n'_i = N - n'$ when $w < \sigma_{-i}(\cdot)$. Therefore, when $w \geq \sigma_{-i}(\cdot)$, $\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w} > 0$ for all $n' > \frac{N}{k}$ and $\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w} < 0$ for all $n' < \frac{N}{k}$. Similarly, when $w < \sigma_{-i}(\cdot)$, $\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w} > 0$ for all $N - n' > N - \frac{N}{k}$ (see Figure 3.1a, these are the high states for the intermediate that is losing the price competition) and $\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w} < 0$ for all $N - n' < N - \frac{N}{k}$.

Note that $0 \leq p(n'|w_i, \sigma_{-i}(\cdot)) \leq 1$ and $\sum_{n'} p(n'|w_i, \sigma_{-i}(\cdot)) = 1$, so $\sum_{n'} \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w}$ must be 0 for any w . Recall that we assume $V_i(n-1) < V_i(n)$ for all n . Sum up, in $\sum_{n'} V_i(n') \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w}$, the coefficients of $V_i(n')$'s sum to zero and the coefficients of high (low) $V_i(n')$'s are positive (negative). $\sum_{n'} V_i(n') \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w}$ is therefore positive for $w \in (0, \bar{\pi})$, which in turn implies MB_n is positive for all $w \in (0, \bar{\pi})$ and all n .

$$\begin{aligned}
\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2} &= \frac{\partial}{\partial w} \left(\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \frac{\partial p(d)}{\partial d} \frac{\partial d}{\partial w} \right) \\
&= \underbrace{\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)^2}}_{=0} \frac{\partial p(d)}{\partial d} \frac{\partial d}{\partial w} \frac{\partial p(d)}{\partial d} \frac{\partial d}{\partial w} \\
&\quad + \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \frac{\partial^2 p(d)}{\partial d^2} \underbrace{\frac{\partial d}{\partial w} \frac{\partial d}{\partial w}}_{=1} \\
&\quad + \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \frac{\partial p(d)}{\partial d} \underbrace{\frac{\partial^2 d}{\partial w^2}}_{=0} \\
&= \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \underbrace{p''(d)}_{<0}.
\end{aligned}$$

Again $n'_i = n'$ when $w \geq \sigma_{-i}(\cdot)$ and $n'_i = N - n'$ when $w < \sigma_{-i}(\cdot)$. Therefore, when $w \geq \sigma_{-i}(\cdot)$, $\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2} < 0$ for all $n' > \frac{N}{k}$ and $\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2} > 0$ for all $n' < \frac{N}{k}$. Similarly, when $w < \sigma_{-i}(\cdot)$, $\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2} < 0$ for all $N - n' > N - \frac{N}{k}$ and $\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2} > 0$ for all $N - n' < N - \frac{N}{k}$. Since $\sum_{n'} \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w} = 0$ for any w , we have $\sum_{n'} \frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2} = 0$ for any w . Sum up, in $\sum_{n'} V_i(n') \frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2}$, the coefficients of $V_i(n')$'s sum to zero and the coefficients of high (low) $V_i(n')$'s are negative (positive). $\sum_{n'} V_i(n') \frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2}$ is therefore negative for all $w \in (0, \bar{\pi})$, which in turn implies $\frac{\partial MB_n}{\partial w}$ is negative for all $w \in (0, \bar{\pi})$. So MB_n is downward sloping for all n .

Observe that

$$\frac{\partial d}{\partial \sigma_{-i}(\cdot)} = \begin{cases} -1 & \text{for } w \geq \sigma_{-i}(\cdot) \\ 1 & \text{for } w < \sigma_{-i}(\cdot) \end{cases},$$

so $\frac{\partial d}{\partial \sigma_{-i}(\cdot)} \frac{\partial d}{\partial w} = -1$ for all $w \in (0, \bar{\pi})$.

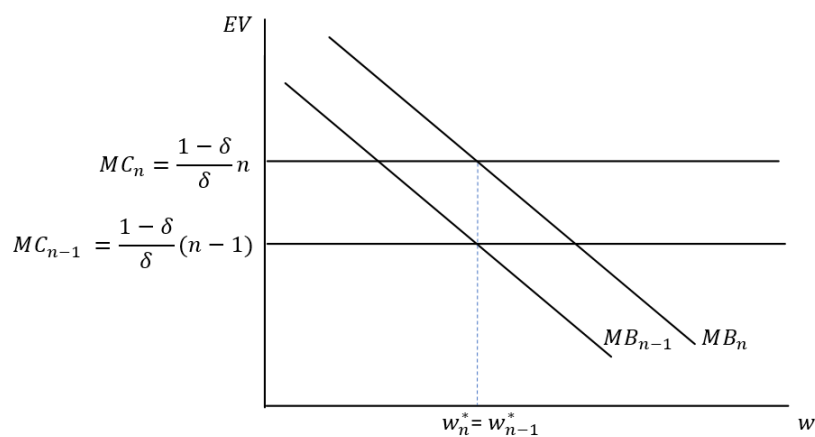
$$\begin{aligned}
\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w \partial \sigma_{-i}(\cdot)} &= \frac{\partial}{\partial \sigma_{-i}(\cdot)} \left(\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \frac{\partial p(d)}{\partial d} \frac{\partial d}{\partial w} \right) \\
&= \underbrace{\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)^2}}_{=0} \frac{\partial p(d)}{\partial d} \frac{\partial d}{\partial \sigma_{-i}(\cdot)} \frac{\partial p(d)}{\partial d} \frac{\partial d}{\partial w} \\
&\quad + \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \frac{\partial^2 p(d)}{\partial d^2} \underbrace{\frac{\partial d}{\partial \sigma_{-i}(\cdot)} \frac{\partial d}{\partial w}}_{=-1} \\
&\quad + \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \frac{\partial p(d)}{\partial d} \underbrace{\frac{\partial^2 d}{\partial w \partial \sigma_{-i}(\cdot)}}_{=0} \\
&= -\frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial p(d)} \underbrace{p''(d)}_{<0}.
\end{aligned}$$

Again $n'_i = n'$ when $w \geq \sigma_{-i}(\cdot)$ and $n'_i = N - n'$ when $w < \sigma_{-i}(\cdot)$. Therefore, when $w \geq \sigma_{-i}(\cdot)$, $\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w \partial \sigma_{-i}(\cdot)} > 0$ for all $n' > \frac{N}{k}$ and $\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w \partial \sigma_{-i}(\cdot)} < 0$ for all $n' < \frac{N}{k}$. Similarly, when $w < \sigma_{-i}(\cdot)$, $\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2} > 0$ for all $N - n' > N - \frac{N}{k}$ and $\frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w^2} < 0$ for all $N - n' < N - \frac{N}{k}$. Since $\sum_{n'} \frac{\partial p(n'|w, \sigma_{-i}(\cdot))}{\partial w} = 0$ for any w , we have $\sum_{n'} \frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w \partial \sigma_{-i}(\cdot)} = 0$ for any w . Sum up, in $\sum_{n'} V_i(n') \frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w \partial \sigma_{-i}(\cdot)}$, the coefficients of $V_i(n')$'s sum to zero and the coefficients of high (low) $V_i(n')$'s are positive (negative). $\sum_{n'} V_i(n') \frac{\partial^2 p(n'|w, \sigma_{-i}(\cdot))}{\partial w \partial \sigma_{-i}(\cdot)}$ is therefore positive for all $w \in (0, \bar{\pi})$, which in turn implies $\frac{\partial MB_n}{\partial \sigma_{-i}(\cdot)}$ is positive for all $w \in (0, \bar{\pi})$ for all n . Since $-i$ uses a monotone pure strategy, $\sigma_{-i}(N - n + 1, \pi_{-i}) \leq \sigma_{-i}(N - n, \pi_{-i})$ for all $\pi \in \Pi$ and all n . This implies $MB_{n-1} \leq MB_n$ for $w \in (0, \bar{\pi})$ and all $n \geq 1$.

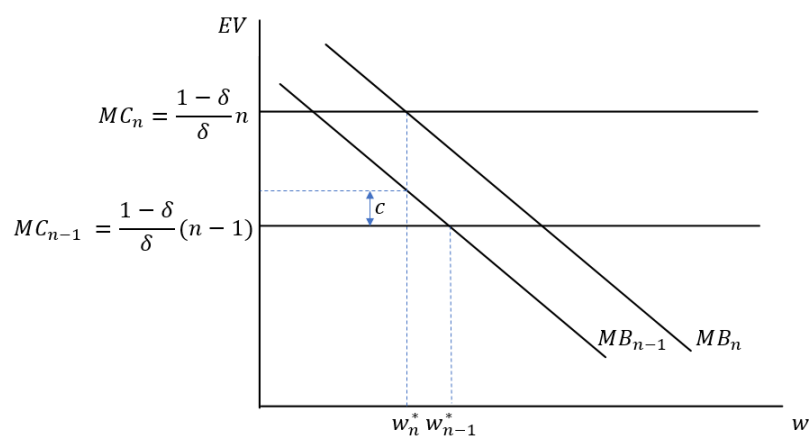
Note that MB_n is ∞ at 0, positive and downward sloping, so $0 < MB_n(\bar{\pi}) < \infty$ (only the MB induced by $\sigma_{-i}(s) = 0$ for all $s \in S$ has $MB(\bar{\pi}) = 0$). For sufficiently small δ , $\frac{1-\delta}{\delta}n = MC_n > MB_n(\bar{\pi})$ for all $n \geq 1$. This guarantees MC_n intercepts MB_n for all $n \geq 1$.

The difference between $MB_{n-1}(w)$ and $MB_n(w)$ is finite for all $w \in (0, \bar{\pi})$. For sufficiently small δ , $\frac{1-\delta}{\delta}n - \frac{1-\delta}{\delta}(n-1) = \frac{1-\delta}{\delta} > MB_n(w) - MB_{n-1}(w)$ for all $w \in (0, \bar{\pi})$. This implies for some small

δ we have $\frac{1-\delta}{\delta} \geq MB_n(w_n^*) - MB_{n-1}(w_n^*)$ for some w_n^* . Hence, $\frac{1-\delta}{\delta}n - \frac{1-\delta}{\delta}(n-1) = MB_n(w_n^*) - MB_{n-1}(w_n^*) + c$ for some $c \geq 0$. $MB_{n-1}(w_n^*) = \frac{1-\delta}{\delta}n$ by definition. So we have $MB_{n-1}(w_n^*) = \frac{1-\delta}{\delta}(n-1) + c$. Since MB_{n-1} is downward sloping and $MB_{n-1}(w_{n-1}^*) = \frac{1-\delta}{\delta}(n-1)$, we have $MB_{n-1}^{-1}(w_n^*) \leq MB_{n-1}^{-1}(w_{n-1}^*)$, i.e., $w_n^* \leq w_{n-1}^*$ (see Figure 3.4).



(a) $w_n^* = w_{n-1}^*, c = 0$.



(b) $w_n^* < w_{n-1}^*, c > 0$.

Figure 3.4: MB and MC

So for sufficiently small δ , the best response for a monotone pure strategy is therefore also a monotone pure strategy. Suppose $-i$ uses the same strategy as i , which gives symmetry. Again, MB_n is ∞ at 0, positive and downward sloping, which imply that when the first-order condition (equation 3.7) is not satisfied, the derivative of $V_i(n, \pi)$ is always positive. Note that the derivative of $V_i(0, \pi)$ is always positive because $0 = MC_0 < MB_0(\bar{\pi})$, so let $w_0^* = \bar{\pi}$. The equilibrium strategy profile can be specified by the following automaton: in state $(n, \pi_i \leq w_n^*)$, output $w_n^{\pi_i} = \pi_i$; in state $(n, \pi_i > w_n^*)$, output $w_n^{\pi_i} = w_n^*$ for all i .

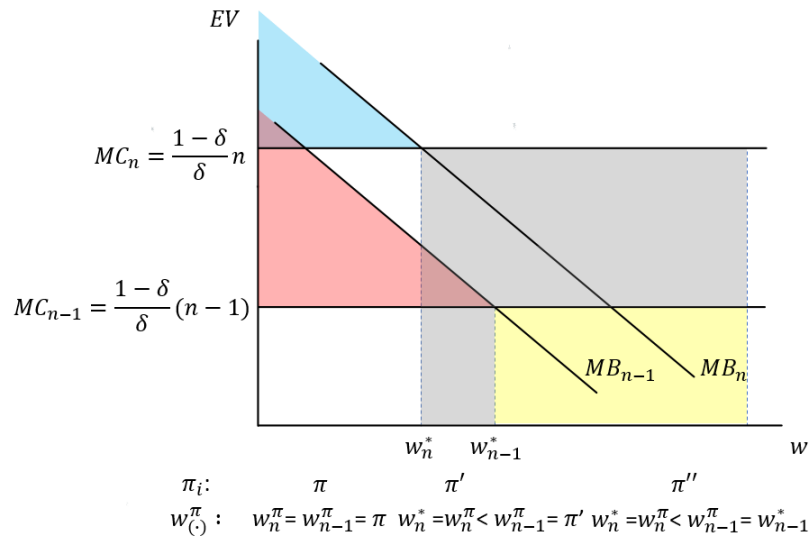


Figure 3.5: $V_i(n, \pi'')$

It remains to show $V_i(n - 1) < V_i(n)$ for all n . Note that

$$\begin{aligned}
V_i(n) &= \sum_{\pi} p_{\pi} \left[(1 - \delta)n(\pi - w_n^{\pi}) + \delta \sum_{\pi_{-i}} p_{\pi_{-i}} \left[\sum_{\pi'_{-i}} p_{\pi'_{-i}} \left[\sum_{n'} V_i(n') p(n' | w_n^{\pi}, \sigma_{-i}(N - n, \pi_{-i})) \right] \right] \right] \\
&= \sum_{\pi} p_{\pi} [(1 - \delta)n(\pi - w_n^{\pi})] + \delta \sum_{\pi} p_{\pi} \left[\sum_{\pi_{-i}} p_{\pi_{-i}} \left[\sum_{\pi'_{-i}} p_{\pi'_{-i}} \left[\sum_{n'} V_i(n') p(n' | w_n^{\pi}, \sigma_{-i}(N - n, \pi_{-i})) \right] \right] \right].
\end{aligned}$$

Observe that $\sum_{\pi} p_{\pi} [(1 - \delta)n(\pi - w_n^{\pi})] > \sum_{\pi} p_{\pi} [(1 - \delta)(n - 1)(\pi - w_{n-1}^{\pi})]$ because $n > n - 1$ and $w_n^{\pi} \leq w_{n-1}^{\pi}$ for all $\pi \in \Pi$ (see Figure 3.5, for $\pi \in [0, w_n^*]$, $w_n^{\pi} = w_{n-1}^{\pi} = \pi$, for $\pi' \in (w_n^*, w_{n-1}^*)$, $w_n^* = w_n^{\pi} < w_{n-1}^{\pi} = \pi'$, and for $\pi'' \in [w_{n-1}^*, \bar{\pi}]$, $w_n^* = w_n^{\pi} < w_{n-1}^{\pi} = w_{n-1}^*$.)

For sufficiently small δ , the difference between the second terms in $V_i(n)$ and $V_i(n - 1)$ will be dominated by the difference of the first terms. Lastly, let $\bar{\delta}$ be the largest δ such that $MC_n > MB_n(\bar{\pi})$, $w_n^* \leq w_{n-1}^*$ and $V_i(n - 1) < V_i(n)$ hold for all $n \geq 1$. \square

In Figure 3.5, at the equilibrium, when $\pi'' > w_{n-1}^*$, we can see the following: For i with $n - 1$ suppliers, the yellow and red areas represent her net benefit. Similarly, for i with n suppliers, the yellow, grey and blue areas represent her net benefit.

$MB_{n-1} \leq MB_n$ is somehow counterintuitive as one might think the expected marginal benefits should be higher when $-i$ behaves less aggressively. Indeed, when $-i$ uses some monotone pure strategy, we have $TB_{n-1} \geq TB_n$, where

$$TB_n = \sum_{\pi_{-i}} p_{\pi_{-i}} \left[\sum_{\pi'_{-i}} p_{\pi'_{-i}} \left[\sum_{n'} V_i(n') p(n' | w, \sigma_{-i}(N - n, \pi_{-i})) \right] \right]$$

is the expected total benefit for any n . This is because for any w , $P(n' | w, \sigma_{-i}(N - n + 1, \pi_{-i}))$ first order stochastically dominates $P(n' | w, \sigma_{-i}(N - n, \pi_{-i}))$ for all n' and $V_i(n - 1) < V_i(n)$ for all $n \geq 1$. We have shown that TB_n is strictly increasing and strictly concave. Note that the maximum expected total benefit can only be achieved by $TB(\bar{\pi})$, where TB is the expected total benefit

function induced by $\sigma_{-i}(s) = 0$ for all $s \in S$ (that is, $-i$ pays zero in all states). Figure 3.6 illustrates what we just discussed.

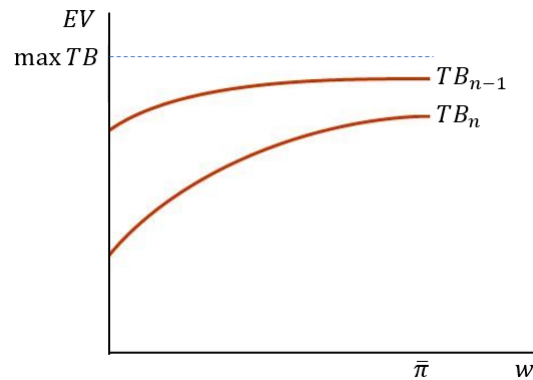


Figure 3.6: TB

Our interpretation of $MB_{n-1} \leq MB_n$ is that when $-i$ behaves less aggressively, the expected total benefit is higher for each w ; because the maximum expected total benefit is bounded, the marginal benefit of w has to be smaller. It is hard to see $\frac{\partial MB_n}{\partial \sigma_{-i}(\cdot)} > 0$ directly from the complex expression of MB_n . But we find that it is clear when $w = \bar{\pi}$: Suppose $\sigma_{-i}(N - n + 1, \pi_{-i}) < \sigma_{-i}(N - n, \pi_{-i})$. Let $d_{n-1} = \bar{\pi} - \sigma_{-i}(N - n + 1, \pi_{-i})$ and $d_n = \bar{\pi} - \sigma_{-i}(N - n, \pi_{-i})$. We have $d_{n-1} > d_n$, so $p(d_{n-1}) > p(d_n)$, which in turn implies $p'(d_{n-1}) < p'(d_n)$. Note that π_{-i} is arbitrary and there is a $p'(d)$ in every term of MB_n and MB_{n-1} , so $MB_n(\bar{\pi}) < MB_{n-1}(\bar{\pi})$.

This is a game with private monitoring since each intermediate privately observes its own profit shock and does not communicate before choosing a payment. However, the strategies in the Main Theorem are partially revealing in the sense that after an intermediate observes her opponent's payment, she can perfectly infer her opponent's profit shock if the payment is less than w_{n-i}^* but she cannot infer that if the payment is equal to w_{n-i}^* .

3.4 Conclusion

In this chapter, we introduce a dynamic price competition model for supply. Suppliers are responsive to the price difference and sensitive to the sizes of price differences. We found that a symmetric stationary Markov Perfect Bayesian equilibrium in behavior strategies always exists. We also found that a symmetric monotone pure strategy stationary Markov perfect Bayesian equilibrium exists when intermediates are not so patient. In this pure strategy equilibrium, an intermediate behaves more aggressively when she has a smaller number of suppliers and thus a smaller opportunity cost.

3.5 Appendix

Proof of Lemma 1: (1) The first two conditions are immediate.

$$p'_\lambda(d) = \frac{l}{(N+1)\bar{\pi}} \left(\left(1 - \left(1 - \frac{d}{\bar{\pi}} \right)^\lambda \right)^{\frac{1}{\lambda}-1} \left(1 - \frac{d}{\bar{\pi}} \right)^{\lambda-1} \right)$$

and it is positive.

$$p''_\lambda(d) = - \left(\frac{l}{(N+1)(d-\bar{\pi})^2} \left(1 - \left(1 - \frac{d}{\bar{\pi}} \right)^\lambda \right)^{\frac{1}{\lambda}-2} \left(1 - \frac{d}{\bar{\pi}} \right)^\lambda (\lambda-1) \right)$$

and it is negative. These give condition 3. For condition 4, we want to check the limit of $p'_\lambda(d)$ as d goes to 0, that is,

$$\lim_{d \rightarrow 0} \frac{l}{(N+1)\bar{\pi}} \left(\left(1 - \left(1 - \frac{d}{\bar{\pi}} \right)^\lambda \right)^{\frac{1}{\lambda}-1} \left(1 - \frac{d}{\bar{\pi}} \right)^{\lambda-1} \right).$$

$\left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda-1}$ goes to 1 as d goes to 0. But because $\lambda > 1$, $\frac{1}{\lambda} - 1$ is less than 0. Therefore,

$$\left(1 - \left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda}\right)^{\frac{1}{\lambda}-1} = \frac{1}{\left(1 - \left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda}\right)^{1-\frac{1}{\lambda}}}$$

and the denominator goes to 0 as d goes to 0. Hence we have $\lim_{d \rightarrow 0} p'_{\lambda}(d) = \infty$. For condition 5,

$\left(1 - \left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda}\right)^{\frac{1}{\lambda}-1}$ goes to 1 and $\left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda-1}$ goes to 0 as d goes to $\bar{\pi}$. Hence $\lim_{d \rightarrow \bar{\pi}} p'_{\lambda}(d) = 0$.

(2)

$$\frac{\partial p_{\lambda}(d)}{\partial \lambda} = \frac{l}{(N+1)\lambda^2} \left(1 - \left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \left(-\ln \left[1 - \left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda}\right] + \frac{\left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda} \lambda \ln \left[1 - \frac{d}{\bar{\pi}}\right]}{\left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda} - 1}\right).$$

Note that the logarithmic function is negative in $(0, 1)$ and $\left(1 - \frac{d}{\bar{\pi}}\right)^{\lambda} - 1$ is less than 0, so $\frac{\partial p_{\lambda}(d)}{\partial \lambda}$ is greater than 0 all $1 < \lambda < \infty$ and $d \in (0, \bar{\pi})$.

(3) Observe that $\left(1 - \frac{\varepsilon}{\bar{\pi}}\right)^{\infty}$ vanishes for all $\varepsilon > 0$, the result follows immediately. \square

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