Detailing the Work of Leading a Productive Mathematics Discussion:
A Study of a Practice-Based Pedagogy of Elementary Teacher Education

Adrian Foster Cunard

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington
2014

Reading Committee:
Elham Kazemi, Chair
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Abstract

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In mathematics teacher education, we seek to support teachers to do increasingly complex kinds of teaching—teaching that aims to develop in all students a conceptual understanding of key mathematical ideas, powerful reasoning skills, and a view of themselves as people who do mathematics. But developing pedagogies of teacher preparation that support the enactment of such teaching by novices remains a central challenge for the field. I report in this dissertation on my study of a mathematics methods course in an elementary teacher preparation program that was organized around novice enactment of the complex practice of leading a productive mathematical discussion in the classroom. In the course, novices and the teacher educator worked on the practice through investigations, coached public rehearsals and enactments with children of a set of discussion-based mathematics lessons or *instructional activities*. I investigated the affordances of the use of a particular instructional activity, the String activity, for working on leading mathematical discussion with children. I sought to understand how pedagogies of enactment in the course scaffolded novice engagement with the practice.

I found that the String activity created opportunities to engage with the work of keeping the target mathematics at center stage throughout a discussion, or the practice of *orienting the children to the target mathematics*. I argue that coached public rehearsals of the String activity offered opportunities for novices and the teacher educator to *detail* that practice: to decompose what it would entail, to plan together for how it would look and sound. And I argue that enactments with children offered opportunities for novices to do what was rehearsed with children and see how they would respond; to encounter unanticipated responses from students; to manage the reality of pacing a lesson while engaging children's thinking; and to experience modeling from the teacher educator of high quality practice.

I also considered whether and how this work intersects with equity aims in mathematics education. The study raised questions about how to articulate the link between work in teacher preparation on orchestrating productive classroom discourse and the broader aims of improving learning opportunities for children in schools.

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Introduction

In mathematics teacher education, we seek to support teachers to do increasingly complex kinds of teaching—teaching that aims to develop in all students a conceptual understanding of key mathematical ideas, powerful reasoning skills, and a view of themselves as people who do mathematics. Supporting this kind of teaching has taken on increasing urgency in recent decades as the gaps in the outcomes of mathematics education along racial, cultural, and socioeconomic lines have come into sharp focus. Historically, in the field, we have been successful in helping beginning teachers to take up new understandings of mathematical ideas and to articulate new visions of mathematics teaching and learning (Clift & Brady, 2005). But it has been persistently challenging to help novice and veteran teachers alike to learn to *enact* the kinds of teaching practices that would realize such visions (Borko, Eisenhart, Brown, Underhill, Jones, Agard, 1992; Clift & Brady, 2005; Ensor, 2001; Kennedy, 1999). Kennedy (1999) calls this the "problem of enactment" in teacher education.

As McDonald, Kazemi & Kavanagh (2013) note, the field is currently turning substantive attention to this problem. They point to work by teacher educators on the development of more systematic pedagogies for preparing novice teachers for the *doing* of ambitious kinds of teaching. They highlight, in particular, work aimed at the identification of a set of core practices of high quality teaching and the organization of clinical preparation for teachers around novice learning of such practices (Ball, Sleep, Boerst, & Bass, 2009; Ball & Forzani, 2011; Grossman, Hammerness & McDonald, 2009; Lampert, Beasley, Ghousseini, Kazemi & Franke, 2010; Lampert, Franke, Kazemi, Ghousseini, Turrou, Beasley, Cunard & Crowe, 2013; NCATE, 2010). They and others (Ball & Forzani, 2011) suggest that this work will require the identification and specification of 1) core practices in the content areas; and 2) pedagogies of teacher education that support novice enactment of such practices.

In this dissertation, I will report on my study of a mathematics methods course in an elementary teacher preparation program that was organized around novice enactment of the core practice of leading a productive mathematical discussion in the classroom. In the course, novices and the teacher educator worked on the practice through *investigations*, *coached public rehearsals* and *enactments with children* of a set of discussion-based mathematics lessons or *instructional activities*. I sought to better understand the affordances of these pedagogies for supporting novice work on leading productive discussion. In the study, I investigated one of the instructional activities in particular, the String activity, to understand whether and how an instructional activity might provide an entry point for novices into a complex practice such as leading a productive discussion. And I analyzed how the pedagogies of the *coached public rehearsal* and *enactment with children* in the context of the course enabled work on the practice.

I will argue that the structure of the String activity enabled work in the course on key aspects of the practice of leading a productive mathematical discussion with children. In particular it created opportunities to engage with the work of keeping the target mathematics at center stage throughout a discussion, the practice of *orienting the children to the target mathematics*. I will argue that coached public rehearsals of the String activity offered opportunities for novices and the teacher educator to detail that practice: to decompose what it would entail, to plan together for how it would look and sound. And I will argue that the enactments with children offered opportunities for novices to do what was rehearsed with children and see how they would respond; to encounter unanticipated responses from students; to manage the reality of pacing a lesson while engaging children's thinking; and to experience modeling from the teacher educator of high quality practice.

In the following sections, I will offer background and describe the study. In Chapter 1, I will consider the aims of the improvement of both mathematics instruction and teacher education and will describe current thinking in the field about practice-based teacher preparation. In Chapter 2, I will describe the setting and the methods of the study.

CHAPTER 1: IMPROVING MATHEMATICS INSTRUCTION AND TEACHER EDUCATION

The improvement of mathematics instruction

The reform efforts of the last 25 years in mathematics education have centered on broadening the aims of school mathematics from the development of procedural fluency alone to include the development of competence in mathematical reasoning, of conceptual understanding of mathematical ideas, and of a productive disciplinary disposition in students (Kilpatrick, Swafford, and Findell, 2001; NCTM, 2000). These efforts have sought to replace a vision of mathematics as a fixed body of knowledge with a vision of the discipline as a social and cultural practice through which a dynamic body of knowledge has developed (Bishop, 1988; Ernest, 2009; Stigler & Baranes, 1988). And they have sought to replace the attendant transmission models of teaching and learning with more participatory practices (Sfard, 1998).

While the field of mathematics education is in general agreement about the aims of improvement, procedurally-based and transmission-oriented instruction remains dominant in elementary mathematics classrooms in the U.S. (Hiebert & Stigler, 2000; Kilpatrick, Swafford & Findell, 2001). Further, it appears that children from marginalized groups are more likely to be in classrooms organized around such pedagogy than children from mainstream groups (Anyon, 1980, Haberman, 1991, Ladson-Billings, 1997; Oakes, 1990). Researchers note that a model of teaching and learning focused on transmission of decontextualized knowledge, memorization and compliance is unproductive for elementary students in general, but it is theorized to be particularly deleterious for students from non-dominant groups (McDairmid, 1991; Oakes, 1990; Villegas & Lucas, 2001). The negative effect of transmission models of mathematics teaching and learning on diverse groups in this country is well documented. Tests such as the NAEP have shown yawning gaps between the mathematical achievement of 4th graders from dominant groups and that of children from non-dominant groups for more than 40 years (NCES, 2011; NAGB, 2011). Children from non-dominant groups are more likely to be placed in remedial mathematics classes, to stop taking mathematics as soon as possible in high school, and are radically underrepresented in STEM fields in this country (Darling-Hammond, 2007; Martin, 2009).

Villegas & Lucas (2002) articulate the critical link between current visions of mathematics teaching and learning and equity in mathematics classrooms. They say,

> …New ways of thinking acknowledge the critical role that student diversity plays in the teaching and learning process and they assume that differences—whether individual or cultural—are strengths to be built upon, not problems to be managed. All students regardless of background are depicted as capable thinkers who continuously strive to make sense of new information and who bring rich experiences and ideas to this sense-making process. This represents a significant departure from the conventional mind-set in which students who do not conform to the ways of speaking, thinking and acting that schools have traditionally been organized to expect are automatically seen as deficient and in need of fixing. Implicit in this shift in perspective is also a recognition that educators have a responsibility to adjust their practice in order to build on the diverse backgrounds of the students." (p. 76)

The underlying aim of this paper is to understand and articulate whether and how work in a practice-based pedagogy of teacher preparation might intersect with the work of developing equitable elementary mathematics classrooms. Drawing on visions from the mathematics education literature for the improvement of mathematics instruction and the urgency to improve learning opportunities in mathematics for students from non-dominant groups, I will consider how the mathematics methods course supported work on three broad aims:

1. Broad-based participation in classroom mathematics;
2. The attainment of fluency and conceptual understanding of key mathematical ideas by a broad range of students;
3. The development of productive relationship to the discipline of mathematics in a broad range of students

Current visions of high quality mathematics instruction place emphasis on classroom mathematical discourse as a locus for supporting broad participation, the attainment of important mathematics and the development of productive identity in students (Franke, Kazemi, & Battey, 2007; Hiebert & Grouws, 2006; Kilpatrick, Swafford, & Findell, 2001; NCTM, 2000). When discourse is productive, participants reason aloud together about mathematical ideas, conjecture, justify, agree and disagree in order to develop both collective and individual mathematical thinking. But research on novice and veteran teachers alike tells us that the practice of leading a productive mathematics discussion with children is among the hardest to enact in the work of high quality mathematics instruction (Ball, 1993; Heaton, 2000; Kazemi & Stipek, 2001; Nathan & Knuth, 2003; Sherin, 2002; Williams & Baxter, 1999).

The details of enactment

It is worth considering what is meant by *enactment*, what we know about it and why it matters. Literature on high quality mathematics teaching and learning makes a distinction between high quality *tasks* and *the way those tasks are enacted* in the classroom (Doyle, 1993; Stein, Grover & Henningsen, 1996). But both emerge as critical to the quality of student learning. *Tasks* are the activities in which students and teachers engage. Examples of mathematical tasks in an elementary classroom might involve solving a word problem, constructing a shape with particular features, playing a game, completing a practice worksheet, collecting and interpreting data and such. The *enactment of a task* refers to what the teacher does with it: how the task gets introduced to students, how she elicits and responds to student questions and ideas about it, how she supports students' engagement with it and with one another inside it. It is the discourse around the task (Franke, Kazemi & Battey, 2005) and that discourse shapes the nature and quality of the interactions among the teacher, the students, and the mathematics content (Cohen, Raudenbusch & Ball, 2003; Lampert 2001) as the task is being worked on in the classroom.

The literature on instructional tasks (Doyle, 1983) suggests that the structure and nature of the tasks in which students engage shapes student learning in two ways. First it serves to focus the learner's attention on particular aspects of content-- to foreground some ideas and leave others in the background. Second, it structures student engagement in particular learning processes such as developing opinions, reasoning, agreeing or disagreeing, critiquing and so forth. The mathematics education literature suggests that tasks that draw attention to mathematical relationships, that place a high level of cognitive demand on students, that require the justification of thinking, and that involve collective discussion of mathematical ideas can support the development of procedural and conceptual fluency in students (Franke, Kazemi & Battey, 2007; Hiebert & Grouws, 2007; Kilpatrick, Swafford, Findell, 2001; Lampert, 2001; NCTM, 2000). And tasks that build on what children know, have multiple entry points for engagement, emphasize reasoning over speed, and are organized around collaboration can support participation and the development of positive disciplinary relationships in children (Anderson, 1989; Boaler, 2002b; Boaler & Staples, 2008; Kilpatrick, Swafford & Findell, 2001; Marshall & Weinstein, 1984; NCTM, 2000; Warren & Roseberry, 1995).

The nature of the tasks used in the mathematics classroom also has important consequences for equity. The literature on supporting the rich cultural diversity of students in mathematics learning points to mathematics tasks that are collaborative (Boaler & Staples, 2008; Au, 2009); that place a high level of intellectual demand on students (Anyon, 1980; Haberman, 1991); that build on children's varied ways of knowing (Boaler, 2002a&b; Cobb & Hodge, 2002; Gutierrez & Rogoff, 2003; Howard, 2010; Gay, 2010; Warren & Roseberry, 1995); and that have multiple entry points for participation (Boaler & Staples, 2008). Table 1 summarizes some of what the literature suggests about the features of high quality tasks relative to supporting: broad participation in classroom mathematics, the attainment of important mathematical ideas by a broad range of students, and the development of productive disciplinary relationships in a broad range of students.

But there is also a substantive body of literature that suggests that how rich instructional tasks are *enacted* in the classroom shapes the learning to which students have access in important ways (Boaler, 2002a; Doyle, 1983; Gresalfi, Martin, Hand & Greeno, 2009; Hiebert & Grouws, 2007; Hiebert & Wearne, 1993; Kazemi & Stipek, 2001; Cohen, 1990; O'Connor & Michaels, 1993; Stein, Grover & Henningsen, 1996; Sleep, 2012). Scholars point to the idea that the kinds of interactions that take place between teachers, students, and content, the *details of enactment* in the mathematics classroom, can have a critical impact on equity. Boaler (2002a) and Gutierrez (2002) avenue for work towards equitable mathematics classrooms. The literature

Table 1: Features of mathematics tasks that can support broad participation, the attainment of important mathematics, and the development of productive disciplinary relationships in a broad range of students

	Support broad participation in mathematics work	**Support attainment of important mathematics**	**Support the development of productive mathematical identity**
Task features that…	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that are organized around collaboration/ collective activity • Tasks that have multiple solution paths • Tasks that have multiple entry points (NCTM, 2000; Kilpatrick, Swafford & Findell, 2001; Boaler, 2002; Boaler & Staples, 2008; Gay, 2010; Howard, 2012; Gutierrez, 2002; Gutierrez & Rogoff, 2003; Cobb & Hodge, 2006; Warren & Roseberry, 1995)	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that focus attention on meaning and mathematical relationships • Tasks that place a high level of cognitive demand on students • Tasks that involve multiple models of a mathematical idea • Tasks that require the justification of ideas (Boaler, 1997; Bransford, Brown & Cocking, 2010; Franke, Kazemi & Battey, 2007; Hiebert & Groews, 2007; NCTM, 2000; Kilpatrick, Swafford & Findell, 2001; Silver & Stein, 1996; Stein, Gover & Henningsen, 1996; Villegas & Lucas, 2002)	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that are organized around collaboration/ collective activity • Tasks that are non-competitive • Tasks that do not foreground speed (Marshall & Weinstein, 1984; Boaler & Greeno, 2001; Boaler & Staples, 2008; Anderson, 1989)

suggests that the nature and quality of the interactions in a mathematics classroom impacts not only who participates in mathematics and how (Boaler, 2002b; Delpit, 2012; Nasir, Roseberry, Warren & Lee, 2006; Gay, 2010; Lubienski, 2002; Morrison, Robbins & Rose, 2008; Parks, 2012), but students' access to high-level intellectual activity (Anyon, 1980, Haberman, 1991; Ladson-Billings, 1997; Henningsen & Stein, 1997), the degree to which mathematical goals are reached during the lesson (Ball, 1993; Sleep, 2012), and the way in which students get positioned relative to the discipline and their peers (Boaler, 2002a; Boaler & Staples, 2008; Ernest, 2009; Gutierrez, 2002; Gresalfi, Martin, Hand & Greeno, 2009; Ladson-Billings, 1997; Martin, 2009). Table 2 summarizes some of what the literature suggests about the features of high quality enactment.

Table 2: Features of enactment that can support broad participation, the attainment of important mathematics, and the development of productive disciplinary relationships in a broad range of students

	Support Broad Participation in Classroom Mathematics	Support attainment of key mathematics	Support development of productive mathematical identity
Enactment features that...	• Making explicit/teaching social and socio-mathematical norms [*Supporting the development of shared social norms in the mathematics classroom and giving children access to disciplinary norms.*] • Making explicit for children how to participate and contribute ideas • Introducing task in a way that supports access [*Attending from the outset to the variety of ways of knowing that children will bring to the task*] • Using multiple participation structures in the same activity [*Supporting the various ways of participating that children bring to the mathematics classroom*] [*Broadening what counts as participation*] • Eliciting multiple responses and solution paths [*Supporting the variety of ways of knowing that children bring to the classroom*] • Pacing to allow students time to think (Au, 2009; Boaler, 2002; Boaler & Staples, 2008; Cobb & Hodge, 2006; Franke, Kazemi & Battey, 2007; Gutierrez & Rogoff, 2003; Lampert, 2001; Lubienski, 2002; Parks, 2012; Yackel & Cobb, 1996)	• Pressing for justification • Maintaining focus on mathematics of the task [*Teaching towards instructional goals*] • Pressing children to do the intellectual work • Highlighting key mathematical ideas [*Teaching towards instructional goals*] • Foregrounding meaning making • Providing scaffolding [*Assisting so that children can complete a task, but not so much that the challenge or complexity of the task is reduced*] • Modeling mathematical ideas in multiple ways (Boaler & Staples, 2008; Doyle, 1983; Franke, Kazemi & Battey, 2007; Henningsen & Stein, 1997; Sleep, 2012; Kazemi & Stipek, 2001; Yackel & Cobb, 1996; Hiebert & Groews, 2007)	• Eliciting multiple responses and solution paths [*Communicating the valuing of a variety of ways of knowing*] • Communicating the valuing of reasoning [*Communicating the valuing of what students do know; positioning children as sensemakers*] • Evaluating ideas based on disciplinary norms [*rather than on the status of the speaker*] • Orienting students to one another's ideas [*Positioning students as authors of knowledge and ideas*] (Boaler, 2002; Boaler & Greeno, 2001; Boaler & Staples, 2008; Franke, Kazemi & Battey, 2007; Lampert, 2001; Yackel & Cobb, 1996)

The details of the enactment of high quality tasks appear both to matter and to present a robust challenge in the work of teaching. In much of the literature on how teachers have attempted to take up reform oriented teaching, teachers have good math tasks in their hands; the curricula that have been adopted by many schools and districts reflect the aims of mathematics reform and are oriented around building both procedural fluency and conceptual understanding in students. But an ample literature shows the challenge of detailing enactment so that it supports productive aims (Ball, 1993; Boaler, 2002; Borko, Eisenhart, Brown, Underhill, Jones, Agard,1992; Cohen, 1990; Heaton, 2000; Henningsen & Stein, 1997; Kazemi & Stipek, 2001; Sleep, 2012; Sherin, 2002; Williams & Baxter, 1999). Henningsen & Stein (1997) for example offered an empirical look at how the details of the enactment of a high quality task can dramatically lower the cognitive demand placed on children as they work. They found that the details of the task set-up, of the teacher's responses to student challenges, and the pacing of the work impacted the degree to which the demand of the task was maintained or lowered. Kazemi & Stipek (2001) likewise pointed to the use of high quality tasks and classroom structures by the two teachers they studied. Although both teachers had given children challenging problems and were attempting to engage with students' thinking, differences in the questions they asked dramatically changed the press on students to engage deeply with mathematical ideas. The work of using a mathematics task so that a broad range of children engage meaningfully with mathematical ideas and come to see themselves as doers of mathematics is complex, interactional, interpretive, and improvisational. It entails fluency with the mathematics of the task as well as the capacity to elicit students' understandings as the task unfolds and to build on those understandings towards curricular goals. It entails attention to the details of enactment.

Core practices and practice-based pedagogies of teacher education

University coursework in teacher preparation has not until recently attended to the work of the enactment of teaching practice. Teacher education research and program design, in recent decades, have been informed by cognitive perspectives on learning. Such perspectives have focused attention on the nature of expert teacher knowledge and on the kinds of teacher education practices that would develop such knowledge. The emphasis in this agenda has been on the things that teachers know and how they come to know them as well as on teacher reasoning and processes of decision-making (Berliner, 1986, 2001; Clift & Brady, 2005; Grossman, 2005; Leinhardt, 1989; Shulman, 1987; Sternberg & Horvath, 1995). 'Methods' courses, therefore, have attended to developing specialized content knowledge for teaching and 'foundations' courses have attended to developing theory, philosophies, and purposes for education that might underpin productive disposition and decision-making (Clift & Brady, 2005; Grossman, Hammerness & McDonald, 2009).

This agenda has been enormously fruitful in mathematics teacher education. It has given us a robust representation of the nature of specialized content knowledge necessary for teaching mathematics (Ball, Thames & Phelps, 2009) and key insights into children's learning trajectories in elementary mathematics content (Carpenter, Fennema, Franke, Levi, Empson, 1999) among other things. But the central limitation of this perspective as a frame for teacher education is that knowing and thinking about teaching are in essence equated with the *doing* of teaching. It suggests that understanding what good teachers know and how they think will enable us to engage in good teaching practice. So clinical preparation, preparation for the deployment of developed knowledge and the enactment of high quality teaching practices, has generally been left to the idiosyncrasies of novices' student teaching experiences. It has been assumed that novice teachers would carry with them into schools the stuff of their university courses and, with the help of a mentor teacher or university supervisor, turn it into good teaching practice. But as has been made clear from studies of the impact of teacher education on beginning teacher practice, knowing things is not the same as deploying that knowledge (Borko, Eisenhart, Brown, Underhill, Jones, Agard, 1992; Clift & Brady, 2005; Ensor, 2001).

More recently, social perspectives have come to the fore in shaping our thinking about teacher learning. From a situative perspective (Greeno, 1998, 2006), knowledge is understood to be developed and used differently in different situations (Boaler, 2002a; Carraher, Carraher, & Schliemann, 1985; Greeno, 1998; Reed & Lave, 1981). Such a perspective on learning to teach offers a way to foreground the contextual and interactive aspects of the work and to join ideas about knowledge and its deployment. This perspective focuses our attention on the practices in which novice teachers engage in learning to teach. If we want to attend to what novice teachers *know* and are able to *do*, we need to attend to both *knowing* and *doing* in the learning practices in which novice teachers engage.

McDonald, Kazemi & Kavanagh (2013) have pointed to a turn in teacher education towards the development of pedagogies of teacher preparation focused on both knowing and doing in teaching. Ball & Cohen (1999) called for the development of *practice-based pedagogies* of teacher education and Grossman, Hammerness & McDonald (2009) proposed the notion of *pedagogies of enactment* for the field organized around a set of core practices of high quality teaching. A rich research agenda has grown to advance these ideas (Ball, Sleep, Boerst, & Bass, 2009; Ball & Forzani, 2009; Grossman, Hammerness & McDonald, 2009; Kennedy, 1999; Lampert, Beasley, Ghousseini, Kazemi & Franke, 2010; Lampert, Franke, Kazemi, Ghousseini, Turrou, Beasley, Cunard & Crowe, 2013; McDonald, Kazemi, & Kavanagh, 2013;

NCATE, 2010; Zeichner, 2012).

Drawing on Grossman and colleagues, McDonald, Kazemi, & Kavanagh (2013) suggest that core practices are, among other things, those which occur with high frequency in teaching and that have the potential to move student disciplinary thinking forward in substantive ways (Grossman, Hammerness & McDonald, 2008). Building on the work of Lampert and colleagues (Lampert & Graziani, 2009; Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2009; Lampert, Kazemi, Franke, et al. 2013), they offer a framework, shown in Figure 1, for organizing teacher preparation in the content areas around supporting novices to learn to enact a set of identified core practices.

In this work, core practices are embedded in a high quality disciplinary lesson, or *instructional activity* (Lampert & Graziani, 2009), that will get worked on by novices in repeated cycles of investigation and enactment. As Lampert and colleagues note, instructional activities can offer a way of structuring the interactions between teacher, students and content so that aspects of the work become routine, and novices can focus attention on the complexity of responding to student ideas (Lampert & Graziani, 2009; Lampert et al. 2010). Novices engage with the practice at different levels and through different pedagogies at each stage.

The top half of the diagram includes pedagogies often used in teacher education in recent years, those that Grossman and colleagues refer to as pedagogies of investigation (Grossman, Hammerness & McDonald, 2009). These pedagogies *represent* teaching practice as opposed to being practice itself (Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson, 2009). They include activities such as watching video of teachers and students, having the teacher educator model teaching practice, investigating protocols and lesson plans, and reading classroom cases.

(McDonald, Kazemi, & Kavanagh, 2013)

Figure 1: Cycle of investigation and enactment of core practices and pedagogies that support each phase

They also include post hoc debriefs of lessons, examination of student work, and reflections on teaching. The pedagogies on the bottom half of the diagram are those that engage novices and the teacher educator in the *doing of* practice, or are what Grossman and colleagues (2009) referred to as pedagogies of enactment. Work such as ‘preparing for and rehearsing an instructional activity’ and ‘enacting the activity with students’ offer novices the opportunity to *approximate* authentic, high quality teaching practice. Approximations are sheltered, low-stakes opportunities to try out practice. Grossman, Compton et al. (2009) suggest that novices preparing for teaching traditionally have fewer opportunities to engage in approximations of practice. They suggest that approximations can offer opportunities for “deliberate practice” (Ericsson, Krampe & Tesch-Romer, 1993) of challenging aspects of the work and sheltered opportunity to make errors. McDonald, Kazemi, & Kavanagh (2012) theorize that lacing together pedagogies of investigation and enactment can offer novices the opportunity to develop both knowledge of core practices and the capacity to enact them.

One of the suppositions of the design of a practice-based pedagogy of teacher education is that it allows teacher educators to scaffold (Greenfield, 1984; Wood, Bruner & Ross, 1976) novice work on the details of enacting a complex task such as leading a classroom discussion. In theory, each of the elements of the pedagogy provides slightly different opportunities to extend novice capacity for productive enactment of worthwhile mathematics tasks. But we have only begun to develop images of how novice teachers and teacher educators operate inside these opportunities (Lampert et al., 2013). Scholars in the field suggest that the work will require the identification and detailing of: core practices of high quality teaching in the content areas; instructional activities in which such practices can be embedded; and pedagogies through which the enactment of such practices can be worked on in teacher preparation (McDonald, Kazemi, & Kavanagh, 2013; Ball & Forzani, 2011).

In this study, I aimed to contribute to efforts to specify the core practice of leading a productive discussion with young children in the mathematics classroom and to understand how the pedagogies of rehearsal and enactment of an instructional activity might enable novice work on enacting such a complex practice. I also sought to contribute to thinking in the field about whether and how this work intersects with equity aims in mathematics education. In Chapter 2, I will explain the study, the setting and my analytic methods.

CHAPTER 2: STUDYING A PRACTICE-BASED PEDAGOGY OF MATHEMATICS TEACHER EDUCATION

The Study

In this study, I analyzed video recordings of the work that took place in the mathematics methods course in the elementary teacher education program at the University of Washington in the fall quarters of 2010 and 2011. The course was the product of a design study entitled *Learning Teaching in, from, and for Practice* (Lampert et al., 2013) undertaken between 2008 and 2012 to develop a clinically-based pedagogy for preparing high-quality elementary mathematics teachers. I was involved in the course as an observer or a teaching assistant in three of the four years of the project. The course was organized around the core practice of leading mathematical discussion with children. The practice was embedded in a set of classroom mathematics lessons referred to as *instructional activities* (Lampert & Graziani, 2009; Lampert, Beasley, Ghousseini, Kazemi & Franke, 2010) which were used in repeated cycles of investigation, coached rehearsal, enactment with children and reflection (Lampert et al., 2013; McDonald, Kazemi, and Kavanagh, 2013). All of the instructional activities used in the course involved engaging children in whole group discussion of numeric and computational relationships. I investigated one of these activities in particular, *the String activity*, in order to home in on how a core practice such as leading mathematical discussion might be nested in a classroom activity. And I investigated two of the pedagogies of the course, *coached public rehearsals* and *enactments with children* that took place in the context of course, in order to better understand the affordances of these pedagogies of enactment. I investigated the following question:

> How did the pedagogies of the mathematics methods course enable work on the practice of leading a productive mathematical discussion?

> 1. What aspects of orchestrating a productive mathematical discussion can be worked on through the String activity?
>
> 2. How was the practice of leading a productive mathematical discussion worked on in coached public rehearsals and in enactments with children in the context of the course?

In the next section I will describe the study that will be reported here. I will describe the mathematics methods course, the data set, the questions that guided my analysis and my analytic methods.

Study Setting: The Mathematics Methods course

The Mathematics Methods course took place over the fall and winter quarters of a five-quarter masters in teaching program for beginning elementary teachers at the University of Washington. In the fall, the course took place at an elementary school once a week for four hours during each of the ten weeks of the quarter. In the winter, the course took place on the university campus. It is the work that happened in the fall quarter that was the subject of this study. Videos of the course were collected over the four years as part of the larger study.

In the fall, the course was organized around a set of five designed mathematics lessons or *instructional activities*. Novice teachers taught the activities with small groups of children in a fifth grade class with whom they had a relationship during the course sessions each week. The pedagogy of the course involved collective investigation of and planning for leading an activity, coached public rehearsal, enactment of the lesson with children, and then collective reflection on the teaching episode (Lampert et al., 2013; McDonald, Kazemi & Kavanagh, 2013). The pedagogy was cyclical in nature and would begin anew with each instructional activity.

Instructional Activities

Instructional activities included Choral Counting, Quick Images, Problem Solving, Math

Games, and Strings (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2009). Each of the activities is structured around conversation about mathematical ideas. The activities offer engagement with mathematical problems, patterns and relationships to support computational fluency and to prompt conversation. They were designed to provide entry points for novice teachers into the practice of orchestrating productive mathematical discourse with children.

During the course, novice teachers familiarized themselves with the structure and mathematical aims of the activities by engaging in them as learners, by watching video of both novice and veteran teachers enacting the activity with children, and through engagement with a protocol prepared by teacher educators that outlined the structure of the activity.

The String Activity

In this study, I investigated videos of novice teachers and the teacher educator working on one of the five instructional activities: *the String activity*. The activity, as it was enacted in the mathematics methods course under investigation here, was drawn from the work of Fosnot and colleagues (Fosnot & Dolk, 2001; Fosnot & Uittenbogaard, 2007) and Parrish (2010). It involves posing a set of 3 to 6 computation problems one after another for students to solve. The problems are purposefully sequenced to highlight a mathematical relationship, operational property or meaning, or a computational strategy. In this course, the set of problems and the relationships among them were used as a prompt for conversation. Leading the activity involves posing the problems one by one, eliciting student strategies for solving them and orchestrating collective discussion of the relationships among them. Often, a visual model of the problems is used to further engage children's understanding of the target relationship. The mathematical task for students is to notice, use, and articulate the relationship among the problems. The work for the teacher is to use the problems, the visual model and the knowledge of the group to prompt and support a collaborative discussion of the target ideas.

I selected the String activity for this study because it is mathematically rich and, in my observation of the course, the most challenging pedagogically for novice teachers. It provides opportunities for novices to work with children on developing conceptual understanding of and fluency with whole number computation across the four operations. And it is designed to engage children with mathematical meaning making and pattern seeking. But the structure of offering up problems that are related to one another and asking children to describe and use the relationship is challenging pedagogically and by its nature, sets novices in unfamiliar territory.

As a sampling decision, looking at the String activity it offered me a way to investigate how rehearsals and enactments might support novice work on interacting with children and mathematics in very new ways. And a close investigation of a single activity allowed me to dig more deeply into what it means to work on a complex practice such as leading a discussion through a particular instructional activity. This sampling choice constrained my investigation in the sense that the String activity offers a very particular structure for leading a discussion. The mathematical goal of the activity is narrow in that a given set of problems is designed to prompt conversation about a single mathematical relationship. There is, in a way, a right answer as children investigate the relationship among the problems. This is different from prompting a conversation in which the goal might be to elicit a broad range of ideas, to explore multiple relationships, or in which there are multiple right answers to a prompt. The work of orchestrating a conversation about a String centers on getting ideas to converge on a single relationship or relationships and this, by its nature, constrained the kinds of teaching practices that I saw in play in both rehearsals and enactments of the String activity.

Coached public rehearsals and enactments with children

In the iterations of the course that I investigated, coached public rehearsals for leading an instructional activity took place immediately preceding its enactment by novices with children. Video of the work done in rehearsal and in enactments with children was collected weekly. It is these videos, and in particular, videos of rehearsal and enactment of the String activity, which were the subject of this investigation.

In rehearsals, one to four novices would practice leading an instructional activity that they had planned for together outside class. Their plans were based on a protocol for the activity and on analysis of the activity done in the context of course. Other novices would play the roles of students in the class. The teacher educator would stand nearby offering coaching, suggestions, guidance, and opportunities for discussion during the rehearsal.

Recent analysis of rehearsals (Lampert et al., 2013) shows that they were typically 12-15 minutes in length. Novices leading the activity spent on average about half of the time in the flow of the lesson, engaging "students" (played by other novices) in the activity, posing questions, and responding to student ideas; and about half of the time in exchanges with the teacher educator about their work. In these exchanges, the flow of the lesson would be stopped and the teacher educator or other novices would make suggestions, ask questions, or model teaching moves. The rehearsals had, on average, 14 such exchanges during the 12-15 minutes.

Immediately following the rehearsal, novices and the teacher educator would enter a classroom of children, with whom they had an ongoing relationship, to enact the lessons. The students were in the first half of their fifth grade year at culturally and socioeconomically diverse schools in the Seattle Public School District. The class of children was split into small groups that rotated through a set of stations each manned by a group of novices. Novices took turns leading lessons individually or in pairs as the groups of children rotated. Enactments lasted from 15-30 minutes. The teacher educator intervened in enactments to insert questions for students, to interact with students, or to offer suggestions for novice teachers in some cases and not in others.

Data

I analyzed a subset of the videos that were collected in the course during the 2010 and 2011 school years. I selected these two years because by this point in the larger study, the structure of the course and the way in which the instructional activities were used was stable allowing for investigation of the pedagogy itself. The fact that the videos are from later rather than earlier years of the pedagogy allowed me to investigate teacher educator practices that were informed by experience.

The cohort of novice teachers each year was divided into four groups. Four rehearsals, each led by one of three teacher educators, took place during each class session. Two of the teacher educators were graduate teaching assistants; one was a faculty member. The faculty teacher educator led rehearsals and enactments of two of the instructional activities: Choral Counting and Strings. Only the rehearsals and enactments that were led by the faculty member were recorded.

In total over the two years, 44 videos of rehearsals and enactments of Choral Counting and Strings were collected. In this study, I analyzed a subset of those: the 24 videos of the String activity that were collected, or 12 pairs of rehearsals and the enactments that immediately followed them. Table 3 shows the 24 videos that were analyzed. It shows the date of the rehearsal and enactment, which of the four groups of novices led the activity, the set of problems that was included in the String, the mathematical idea that the String was designed to support and the visual model that was offered. (In some class sessions, more than one rehearsal and enactment of a String took place. These are indicated with a, b, or c at the end of the date.) In Appendix 1, I have included further detail about the mathematics of some of these sets of problems and the visual models that were used to highlight the target mathematics.

Analysis

I sought to understand how and whether the String activity itself structured opportunities to work on the practice of leading a productive mathematical discussion in the elementary classroom and how the pedagogies of the course enabled novice work on the practice. Because the literature on high quality teaching pointed to the importance of both the nature of the *instructional tasks* that are used in the classroom and *how they are enacted*, my analysis needed two major components. The first was to understand the String activity itself as an instructional task. I sought to describe its features as a high quality task and to understand the entailments of enacting it productively. This would help me better understand the opportunities that engagement with it can offer for novice learning. The following questions guided my analysis:

1. What aspects of orchestrating a productive mathematics discussion can be worked on through the String activity?

    - What are the key features of the String activity?
    - What are the teaching practices entailed in a productive enactment of the String activity?

The second component of my analysis built on the first and related to understanding how the pedagogy of the course supported work on the practices entailed in the productive enactment of a String. I analyzed two of the pedagogies of the course that were used to support high quality enactment: coached public rehearsals and the enactments with children that took place weekly in the course. I sought to better understand the work for novices of learning to lead high quality mathematical talk and the scaffolding provided by these pedagogies.

Table 3: Videos of the String activity

	String Rehearsal	String Enactment	String	Mathematical Idea	Visual Model
Year 1: 2010-2011 School Year	November 2, 2010, Group 2	November 2, 2010, Group 2,	10x11 9x11 10x14 9x14	Compensating Strategy: Distributive property of multiplication over subtraction	Arrays
	November 9, 2010, Group 3a	November 9, 2010, Group 3a	10x15 9x15 11x15	Compensating Strategy: Distributive property of multiplication over subtraction	Arrays
	November 9, 2010, Group 3b	November 9, 2010, Group 3b	6x8 12x4 24x4	Proportional Reasoning	Arrays
	November 16, 2010, Group 4a	November 16, 2010, Group 4a	3x4 3x8 6x8 12x4	Proportional Reasoning	Arrays
	November 16, 2010, Group 4b	November 16, 2010, Group 4b	3x4 3x8 6x8 12x4	Proportional Reasoning	Arrays
	November 30, 2010, Group 1	November 30, 2010, Group 1	36÷6 72÷6 72÷12 144÷24	Proportional Reasoning	Number Line
Year 2: 2011-2012 School Year	November 8, 2011, Group 2	November 8, 2011, Group 2	6x20 6x100 6x120 6x119	Distributive property of multiplication over addition and over subtraction	Set Model
	November 15, 2011, Group 1	November 15, 2011, Group 1	4x100 8x50 16x25	Proportional Reasoning	Set Model
	November 29, 2011, Group 4	November 29, 2011, Group 4	57-20 57-26 53-30 53-34	Subtracting in increments of tens and ones	Number Line
	December 6, 2011, Group 3a	December 6, 2011, Group 3a	91-60 91-63 94-50	Subtracting in increments of tens and ones	Number Line
	December 6, 2011, Group 3b	December 6, 2011, Group 3b	3x50 3x100 3x149	Distributive property of multiplication over addition and over subtraction	Arrays
	December 6, 2011, Group 3c	December 6, 2011, Group 3c	35x8 70x4 140x2	Proportional reasoning	Arrays

The following questions guided my analysis:

2. How was the practice of leading a productive mathematical discussion worked on in coached public rehearsals and in enactments with children in the context of the course?
    - What aspects of leading a String were worked on in rehearsals?

- What aspects of leading a String were worked on in enactments with children?
- Did novices take up work that was done in rehearsal as they enacted Strings with children?
- What aspects of leading mathematical discussion were challenging for novices in enactments?

Analyzing the String activity

I first sought to understand the features of the String activity as a high quality task for use with children. I used what I learned from the literature on high quality tasks as a set of features against which I could examine the String activity. Table 4 represents a summary of the task features suggested by the literature (referenced in Chapter 1 as well) as supporting participation, the attainment of important mathematics, and the development of productive identity. This set of features is by no means exhaustive. But it provided me with a lens through which to look at the academic task at hand. I mapped each of the features listed in the table onto the String activity. This brought into focus the activity's most important features, and helped me consider how and whether Strings might provide a structure for work on high quality mathematics instruction. I will offer my findings from this analysis in Chapter 3.

Next I sought to understand what is involved in a productive enactment of the String activity. I began again with the descriptions from the literature about the features of enactment that can support participation, attainment of mathematics and productive identity development. Table 1 (referenced in Chapter 1) summarizes what I called the "features of productive enactment" that were highlighted by the literature.

In my analysis, I considered each general phase of the String activity in relationship to the features in Table 2. For each phase of the activity, I worked to specify the kinds of teaching practices and moves that could support productive aims. I generated a protocol for the activity based on this analysis. I then used the protocol to guide my analysis of the work that happened in

Table 1: Features of high quality mathematics tasks

	Support broad participation in mathematics work	Support attainment of important mathematics	Support the development of productive mathematical identity
Task features that...	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that are organized around collaboration/ collective activity • Tasks that have multiple solution paths • Tasks that have multiple entry points (NCTM, 2000; Kilpatrick, Swafford & Findell, 2001; Boaler, 2002; Boaler & Staples, 2008; Gay, 2010; Howard, 2012; Gutierrez, 2002; Gutierrez & Rogoff, 2003; Cobb & Hodge, 2006; Warren & Roseberry, 1995)	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that focus attention on meaning and mathematical relationships • Tasks that place a high level of cognitive demand on students • Tasks that involve multiple models of a mathematical idea • Tasks that require the justification of ideas (Boaler, 1997; Bransford, Brown & Cocking, 2010; Franke, Kazemi & Battey, 2007; Hiebert & Groews, 2007; NCTM, 2000; Kilpatrick, Swafford & Findell, 2001; Silver & Stein, 1996; Stein, Gover & Henningsen, 1996; Villegas & Lucas, 2002)	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that are organized around collaboration/ collective activity • Tasks that are non-competitive • Tasks that do not foreground speed (Marshall & Weinstein, 1984; Boaler & Greeno, 2001; Boaler & Staples, 2008; Anderson, 1989)

rehearsals and enactments. The protocol was refined further during successive passes through the video data as I watched novice teachers and the teacher educator try, work on or leave out what would turn out to be key moves and teaching practices. I offer this protocol along with rest of my analysis of the String activity and its entailments in Chapter 3.

Analyzing rehearsals and enactments

To answer my second question about the opportunities to work on the enactment of high quality teaching practices in rehearsals and enactments, I analyzed pairs of videos of novice teachers rehearsing for and then enacting the String activity with children during the course. I used Studiocode© software to generate coded timelines of the videos. I coded for four things: the teaching practices on which novices and the teacher educator were working in both rehearsals and enactments, the interventions made by the teacher educator in both settings, the problems of practice that emerged in enactments, and the uptake by novices in enactments of the work that had happened in rehearsal. Codes were refined through successive passes through the video data.

To code for the teaching practices that were being worked on, I drew on the protocol for the activity that I generated in the previous analysis focusing again on the aims of supporting 1) broad-based participation, 2) the attainment of important mathematics, 3) the development of a productive disciplinary relationship. I created a set of codes that would help me to identify whether and how high-quality teaching practices were being worked on in these settings.

Table 2: Features of productive enactment

	Support Broad Participation in Classroom Mathematics	Support attainment of key mathematics	Support development of productive mathematical identity
Enactment features that...	• Making explicit/teaching social and socio-mathematical norms [*Supporting the development of shared social norms in the mathematics classroom and giving children access to disciplinary norms.*] • Making explicit for children how to participate and contribute ideas • Introducing task in a way that supports access [*Attending from the outset to the variety of ways of knowing that children will bring to the task*] • Using multiple participation structures in the same activity [*Supporting the various ways of participating that children bring to the mathematics classroom*] [*Broadening what counts as participation*] • Eliciting multiple responses and solution paths [*Supporting the variety of ways of knowing that children bring to the classroom*] • Pacing to allow students time to think (Au, 2009; Boaler, 2002; Boaler & Staples, 2008; Cobb & Hodge, 2006; Franke, Kazemi & Battey, 2007; Gutierrez & Rogoff, 2003; Lampert, 2001; Lubienski, 2002; Parks, 2012; Yackel & Cobb, 1996)	• Pressing for justification • Maintaining focus on mathematics of the task [*Teaching towards instructional goals*] • Pressing children to do the intellectual work • Highlighting key mathematical ideas [*Teaching towards instructional goals*] • Foregrounding meaning making • Providing scaffolding [*Assisting so that children can complete a task, but not so much that the challenge or complexity of the task is reduced*] • Modeling mathematical ideas in multiple ways (Boaler & Staples, 2008; Doyle, 1983; Franke, Kazemi & Battey, 2007; Henningsen & Stein, 1997; Sleep, 2012; Kazemi & Stipek, 2001; Yackel & Cobb, 1996; Hiebert & Groews, 2007)	• Eliciting multiple responses and solution paths [*Communicating the valuing of a variety of ways of knowing*] • Communicating the valuing of reasoning [*Communicating the valuing of what students do know; positioning children as sensemakers*] • Evaluating ideas based on disciplinary norms [*rather than on the status of the speaker*] • Orienting students to one another's ideas [*Positioning students as authors of knowledge and ideas*] (Boaler, 2002; Boaler & Greeno, 2001; Boaler & Staples, 2008; Franke, Kazemi & Battey, 2007; Lampert, 2001; Yackel & Cobb, 1996)

To code the teacher educator's interventions in both rehearsal and enactments. I coded for where she intervened, what the interventions were about and how she did it (for example if she stopped the rehearsal to make a suggestion to the novices, or stopped the enactment to model asking a particular question to children). This would help me further understand the scaffolding offered by these pedagogies as well as the challenge areas that arose for novices in working on enacting Strings. To code for problems of practice that emerged as novices enacted lessons with children, I noted episodes in the enactments where novices looked perplexed, where the lesson had stalled, where children offered ideas that novices had trouble managing, and places where the lesson was not unfolding as anticipated. This would help me refine my understanding of the entailments of productive enactment and the challenges that it posed for novices. And lastly, I coded for the degree to which the work that happened in rehearsal shaped what the novices did in enactments with children, or the *uptake* by novices of what was rehearsed. This would help me better understand the affordances of lacing these two pedagogies together to support novice work on complex practice. For each segment of the String (the introduction, the posing of the first problem and so forth), I kept track of what was worked on in rehearsal and then I considered the degree to which that work was reflected in the enactment. I coded uptake in enactments as *high, medium,* or *low* for each segment of the lesson. These codes applied to aspects of the lesson that could be planned for, and not to aspects of the lesson that were contingent on student responses. For a given segment of a String, there was often lengthy discussion among the group of novices and the teacher educator and numerous decisions made about what would be done with children. I counted the uptake as *high* if the novices stayed very close to the plan they had developed in the rehearsal. I coded uptake for a given segment as *medium* where novices left out even one or two parts of what had been rehearsed. And I coded uptake as *low* for a given segment where novices skipped over altogether or nearly altogether what had been rehearsed.

Two key changes occurred in my thinking as I made successive passes through the data. It became clear immediately that I would not be able to speak directly to the *attainment of important mathematics*. That would require data about children's learning over time, which was not in the domain of this study. But I could speak to the work that the novice teachers were doing in single lessons to move student thinking *towards* mathematical goals, to the places in the lessons where teacher moves were (or should have been) tightly laced with the mathematical goal. Drawing on the work of Sleep (2012), I substituted *steering instruction towards mathematical goals* for the *attainment of important mathematics* in my framework.

The next thing I noticed was that pointing to the work of supporting productive identity development in a lesson protocol was a bit of a cipher because again, this data would not support claims about changes in children's identity relative to mathematics. I could only code for the presence of practices that we *think* support productive identity development. I found that as I did this, I was coding for places where attention to student ideas would be given; where student ideas would be elicited, highlighted, or built on in some way. So rather than coding these elements of the String activity as *supporting productive mathematics identity development,* I coded them more directly as *building on student thinking*. Table 4 provides a list of the codes and their explanations.

In the following chapters, I will offer my findings on both the practices entailed in leading a String productively and the ways that rehearsals and enactments offered opportunities to work on such practices.

Table 4: String rehearsal and enactment analysis codes and descriptions

SUPPORTING BROAD PARTICIPATION		SET EXPECTATIONS	Telling children how to sit, how to show they are attending, general expectations for behavior
		PACING	Attending to the pacing of the lesson. Moving efficiently when necessary and slowing down when necessary
		HOW TO JOIN	Telling children how to let the group know when they have something to say: "Give me a thumbs-up when you are ready," e.g.
		ELICIT MULTIPLE ANSWERS	Asking multiple children to share their answers to a problem
		WHAT TO TALK ABOUT	Cueing children explicitly about what to discuss. "Turn and talk to your partner about where Sarah got the 20 and the 100 when she solved this."
		LISTEN IN	Listening in as children turn and talk to a partner or as they work independently to solve problems
		MATERIALS	Offering children materials that might support their access to the mathematics: paper/pencil, whiteboards, counters
		MULTIPLE PARTICIPATION STRUCTURES	Offering multiple ways for children to share ideas in a single lesson: talking with a partner, listening to the teacher, working independently, talking with a group
		PRESS FOR PARTICIPATION	Explicitly engaging children who have participated less in the lesson
		PHYSICAL	Attending to where to stand, where children are seated, how the board should be set up to give students access to learning
STEERING INSTRUCTION TOWARDS GOALS	ORIENTING Ss TO THE MATHEMATICS	ARTICULATE AIMS-BEG	Telling children at the beginning of the lesson what they will be doing in the lesson and why
		ARTICULATE AIMS-END	Telling children or supporting them to articulate the target of the lesson
		PROMPT TO CONNECT PROBLEMS	Prompting children to investigate the connections among the problems
		CONNECT MODEL TO MATH	Making a connection between a visual model and the operation that it represents
		CONNECT MODEL TO TARGET	Making a connection between a visual model and the goal of the lesson
		HIGHLIGHT/DWELL	Highlighting or calling attention to a student idea or strategy that supports work towards the goal
		PACING TO REACH AIMS	Attending to the pacing in order to get to the target problem(s)
		OFFER WELL DEVELOPED MODEL	Offering a model that will support student understanding of the target idea
BUILDING ON STUDENT THINKING		ELICIT THINKING	Asking students to contribute ideas
		ELICIT JUSTIFICATION	Asking students to explain how and why their solutions work
		PROBE	Asking follow-up questions to deepen student thinking
		FOLLOW UP	Asking follow-up questions for any other purpose
		ORIENTING TO OTHER Ss	Directing students attention to the thinking of another student
		ELICITING MULTIPLE IDEAS	Asking multiple students for solution strategies or ideas
		PURPOSEFUL SELECTION	Selecting who will contribute ideas purposefully rather than at random
		SUPPORT Ss TO ARTICULATE TARGET	Helping students to use and articulate the mathematical relationship that is the goals of the lesson
		USE STUDENT ERROR	Making use of a student error to move mathematical thinking forward
		Ss DO INTELLECTUAL WORK	Allowing the students to do the mathematical and reasoning work of the lesson rather than the teacher doing it

PART II: LEARNING FROM ANALYSIS OF REHEARSALS AND ENACTMENTS OF THE STRING ACTIVITY

In the next four chapters, I will describe my findings as I investigated whether and how nesting the String activity inside a practice-based pedagogy in the mathematics methods course offered opportunities to work on the practices involved in leading a productive mathematical discussion in the elementary classroom. I sought to understand how the String activity offers opportunities to work on the practice of leading discussion and how the practice was worked on through the pedagogies of rehearsal and enactment.

In Chapter 3, I offer my analysis of the String activity itself. I analyze the features of the activity as a high quality task and argue that these features create entailments for productive enactment. I offer a protocol for the activity that includes key aspects of the work of orienting children to the mathematical ideas that are the goal of the string.

In Chapter 4, I offer my analysis of how practice was worked on in coached rehearsals. I found that the kind of work that was done with the String activity before the rehearsal shaped the level at which practice could get worked on in rehearsal. I found that rehearsals emerged as an opportunity for novices and the teacher educator to *detail* complex practice. I found that rehearsals of Strings offered the opportunity for novice teachers and the teacher educator to work on the details of *orienting children to the target mathematics* in orchestrating productive discussion and, in particular, to detail the practices of 1) *developing a visual model which can support understanding of a mathematical idea*; 2) *orienting children to the meaning of a visual model through talk*; 3) *prompting students to investigate mathematical relationships*; and 4) *using a visual model to highlight a mathematical relationship*.

In Chapter 5, I offer my analysis of the enactments of the String activity that occurred with children immediately following each rehearsal. I investigated problems of practice that emerged in enactments to better understand the opportunities for work on high quality teaching practice that are unique to enactments. I found that the central problem of practice that novices encountered was that of managing unanticipated responses from students and that enactments highlighted several key practices that can support work on unanticipated responses. The first was the practice of *selecting call-on routines purposefully*. The second was the practice of asking children to *add on* to one another's ideas in order to develop complete responses. And the third was the practice of *orienting students to the mathematics of a visual model* to support the development of conceptual understanding. I will also share the interventions that the teacher educator made in enactments to scaffold novice work *on orienting children to the mathematics of the String*.

In Chapter 6, I offer what I found as I looked across rehearsals and enactments. I found that uptake of the work that was done in rehearsal was high in enactments. But I also found that enactments rarely played out as rehearsed. I noted that where adults in rehearsal almost offered correct and complete responses when they played the roles of children, the actual children offered a wide variety of responses in enactment. I found that although uptake of rehearsal work was high, sometimes novices lost track of what had been rehearsed and did something else. And I found that keeping lessons paced well with children was a challenge.

CHAPTER 3: WORKING ON ORCHESTRATING PRODUCTIVE MATHEMATICAL DISCUSSION THROUGH THE STRING ACTIVITY

One of the aims of this study is to contribute to the identification and specification of core practices of high quality mathematics teaching and to consider how such practices might be embedded in activities that can support novice learning of such teaching. In this chapter, I will consider the nature of the String activity as a mathematical task for use in the classroom and argue that the String activity has key features of a high quality task. I will argue that the features of an instructional activity drive what can get worked on when it is used in teacher preparation. I will offer a protocol for the activity that includes key practices of a successful enactment that emerged from my analysis of the both the activity and the work that was done on the activity in the context of the course.

Working on mathematical relationships with Strings of related problems

The String activity, as it was enacted in course under study here, was drawn from the work of Fosnot and colleagues (Fosnot & Dolk, 2001; Fosnot & Uittenbogaard 2007) and Parrish (2010). A set of 3-6 computations problems are posed to children one at a time. The problems are sequenced purposefully to highlight a mathematical relationship, operational property or meaning, or a computational strategy. The mathematics task for children is to work with their peers to identify the relationship among the problems in the String, and to use that relationship to support fluent computation. Very frequently (and in all of the work on Strings analyzed here) the work of identifying the relationship among the problems is scaffolded by a picture or a visual model of what is happening in each problem. The work for the teacher is to support the collective activity of the children as they investigate the mathematics.

Below is an example of a String of related problems that can support upper elementary children's use of the distributive property of multiplication over subtraction (Fosnot & Uittenbogaard, 2007). This String is structured so that the first problems are easy to solve mentally. They are offered as problems that can be used to solve the later, harder problems and can be thought of as 'helper' problems. It is a key aspect of work on supporting broad participation, the attainment of important mathematics, and the development of productive disciplinary relationships that children engage very explicitly with the relationship between what they already know and new mathematical ideas.

6x20
6x100
6x120
6x119

We want children to consider on their own that the first two known problems are inside the third, more challenging problem and can be combined to solve it. This offers children a view of the distributive property of multiplication over addition. Once the third problem is solved, it becomes the 'helper problem' for the fourth one which can be though of as the 'target problem' in the set. We want children to consider the relationship between the third and fourth problems: 6x119 is one group of 6 less than 6x120. The relationship highlights both the meaning of multiplication as 'groups of groups' and the distributive property of multiplication over subtraction. Below are the problems with a visual model that might be used to support children's thinking about these relationships. Each problem is represented as an array and the arrays are stacked one under the other to allow the relationships among them to stand out.

Analysis of novice work on the String activity revealed how complex the work is of prompting children to investigate and articulate the relationships among the problems in a String, and of using a visual model like the one in Figure 2 to support their investigation.

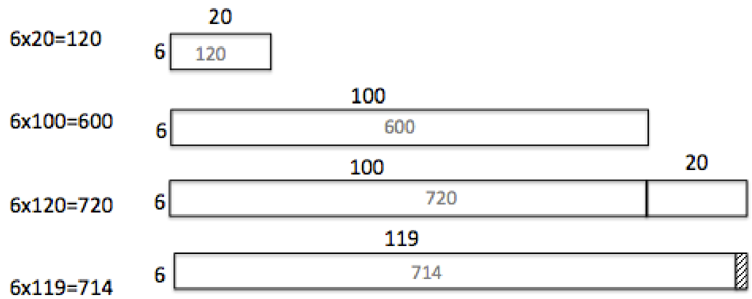


Figure 2: Modeling the problems in a String to highlight the distributive property of multiplication over addition and subtraction

Another set of problems was also designed to highlight the meaning of multiplication as 'groups of groups' and to support children's use of proportional reasoning (Fosnot & Uittenbogaard, 2007) but the structure of the problem set is a bit different. In the previous String, some of the problems were included as helper problems for others to highlight an operational property. In this set of problems the mathematical idea that doubling a factor will double the product is nested in the change that happens from problem to problem.

6x4
6x8
12x8
12x16

The first problem will be in many children's repertoire of known problems because, again, it is a key premise of high quality mathematics instruction that children are able to link new mathematical ideas to known ones. In the second problem we want children to consider the fact that one of the factors is double that of the first problem and that therefore the product will be doubled. In the third problem a factor is doubled again, and so the product will be doubled again. And the same thing happens with the fourth problem. Below is a visual model that highlights this doubling relationship. The arrays show the doubling of the side lengths and the doubling of the area of the rectangle each time.

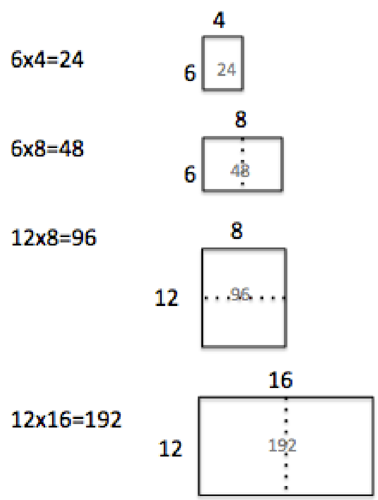


Figure 3: Modeling the problems in String to show the relationship between the factors and the product in multiplication

These examples show two ways that a String of problems might be set up to highlight mathematical relationships, operational properties and/or meanings. Both Strings emphasize the meaning of multiplication as 'groups of groups'. But one highlights the distributive property and the other highlights the relationship of the factors to the product in multiplication. One String emphasizes using easier problems to solve harder problems and the other directs attention to a particular change from problem to problem.

A String of related problems as a prompt for a goal oriented, collaborative classroom discussion

Although the problem sets and the mathematical goals differ from String to String, the general structure of the activity remains the same. The phases of the lesson are as follows. The teacher:

1. Introduces the lesson communicating to children that they will be working on solving computation problems and investigating how the problems are related to one another.
2. Poses the first problem and elicits answers to it from students.
3. Offers a visual model of the problem and talks with children about how the model shows the operation.
4. Poses the next problem prompting children to consider the relationship between the problems and to consider how this relationship might be helpful.
5. Elicits from students solutions and strategies for solving.

6. Offers a visual model of the next problem and talks with children about whether and how it shows the relationship to the previous problem.

The problem posing, prompting of connections, eliciting of solutions and strategies and visual modeling continues for each problem in the String until the target problem(s) have been solved.

Although the general structure of the String activity remains the same across sets of problems, the different structures of the sets and their differing goals necessitate differing kinds of prompting and questioning on the part of the teacher. In Table 5 below, I have offered examples of the ways in which the aims of the prompting that a teacher might offer differ between these two Strings. The table points to the way in which the details of enactment might need to differ given differing Strings. Other Strings that were worked on in by novices in this study are included in Appendix 1.

Table 5: How the structures and goals of a String necessitate different kinds of prompts

String	**Structure and Goals**	**Kinds and purpose of prompts from teacher** ***Prompts aim to support students to:***
6x20 6x100 6x120 6x119	• Easier problems are used to solve harder problems • Distributive property of multiplication over addition and subtraction	• See and articulate multiplication as groups of groups in an array • See and articulate the relationship between the first two problems and the third problem • Use and articulate the idea that the fourth problem can be solved by taking a group of 6 away from the product of the third problem • See and articulate the removal of the groups using the visual model
6x4 6x8 12x8 12x16	• Problems double from one to the next • Proportional reasoning	• Help students see and articulate multiplication as groups of groups in an array • Notice the doubling relationship between the factors in each problem • Noticing the relationship of the change in the products given the changes in the factors between problems • Seeing and articulating the change in the product and factors using visual representation

The String activity as a high quality mathematics task for children

I sought to understand how and whether the String activity itself enabled work on high quality teaching practices. I argue that the features of the tasks that are selected as instructional activities for use in teacher education will drive the teaching practices that can get worked on. Table 1 (also shown earlier) shows some of task features suggested by the literature as supporting: 1) broad participation in mathematical work; 2) the attainment of important mathematics; and 3) the development of a productive disciplinary relationship in children.

Table 1: Features of mathematics tasks that support broad participation, the attainment of important mathematics, and the development of a productive disciplinary relationship in a broad range of students

	Support broad participation in mathematics work	**Support attainment of important mathematics**	**Support the development of productive mathematical identity**
Task features that…	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that are organized around collaboration/ collective activity • Tasks that have multiple solution paths • Tasks that have multiple entry points (NCTM, 2000; Kilpatrick, Swafford & Findell, 2001; Boaler, 2002; Boaler & Staples, 2008; Gay, 2010; Howard, 2012; Gutierrez, 2002; Gutierrez & Rogoff, 2003; Cobb & Hodge, 2006; Warren & Roseberry, 1995)	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that focus attention on meaning and mathematical relationships • Tasks that place a high level of cognitive demand on students • Tasks that involve multiple models of a mathematical idea • Tasks that require the justification of ideas (Boaler, 1997; Bransford, Brown & Cocking, 2010; Franke, Kazemi & Battey, 2007; Hiebert & Groews, 2007; NCTM, 2000; Kilpatrick, Swafford & Findell, 2001; Silver & Stein, 1996; Stein, Gover & Henningsen, 1996; Villegas & Lucas, 2002)	• Tasks that are in the zone of proximal development for students; that build on what students already know • Tasks that are organized around collaboration/ collective activity • Tasks that are non-competitive • Tasks that do not foreground speed (Marshall & Weinstein, 1984; Boaler & Greeno, 2001; Boaler & Staples, 2008; Anderson, 1989)

As it was conceived of for the purposes of the mathematics methods course, the String activity has all of the above features except for having multiple entry points (in the left-hand column). Below, I will briefly consider a few of the features above relative to the String activity to highlight it as a high-quality task. But I will also point to the relationship between key task features and the practices entailed productive enactment. I argue that productive enactment needs to engage the key features of the task. This list is not exhaustive and does not fully describe the task. It is meant to highlight a just a few of the most critical features of the String activity and to offer a way to consider how task features open up opportunities for work on practice.

A String is designed to build on what students know

A String begins with a problem or problems that students are likely to have memorized. The problems that follow are related to the known problem(s) in key ways. The String activity is designed specifically to help students learn to leverage what they already understand about operations (such as how to solve 6x20 and 6x100) to develop new and more sophisticated understandings. Both the equity literature and the mathematics education literature point to the importance of the use of tasks that build on what students know. These literatures suggest that building on what students know can support participation by a broad range of students, can support the attainment of important mathematics and can support the development of a positive relationship to mathematics in children. The work for the teacher in using a task that builds on what children know involves the practice of focusing children's attention on what they know and supporting children to make the connection between the goal of the lesson and what they know. It entails substantive prompting and questioning. It also entails attention to pacing that highlights some knowledge as tacit for the group.

Another practice that this feature raises is that of task selection. While the task is designed to build on what students know, the selection of the String is critical to engaging this feature. If the initial problems in the String are not in the realm of what students already know then this feature is not relevant. Task selection takes a deep understanding of children's developmental trajectories in mathematics and is often somewhat outside novice capacity. In the math methods course under investigation here, the practice of selecting the String was generally done by or guided by the teacher educator.

A String is designed to draw attention to operational meanings and mathematical relationships.

A String is designed to highlight a particular mathematical relationship, property, or operational meaning. It is designed to prompt an investigation of how computational problems relate to one another and of how those relationships can be leveraged to support efficient computation. The work for teachers in leading talk that centers on the relationships is in keeping the focus on mathematical meanings, in continually orienting children's attention to the target mathematics by prompting investigation and articulation of meanings, posing questions and using the visual model to scaffold their work.

A String involves multiple models of a mathematical idea

The structure of the String activity offers the opportunity to connect computation problems, which are numeric models of mathematical ideas, to visual models of the same ideas in order to build conceptual understanding. Visual models include number lines, drawings, and arrays among others. Different representations of mathematical ideas highlight different aspects of those ideas. Connecting multiple models of the same idea offers children the opportunity to deepening their thinking about that idea. Offering a visual model and using it to support student understanding generates substantive entailments for productive enactment. A teacher needs a well-developed model, a set of questions that supports students to make sense of the model and a way of scaffolding the building of conceptual understanding using the model.

A String is designed as a cognitively demanding task

Stein, Grover & Henningsen (1996) suggest the features of tasks that place a high level of demand on students. Drawing on their categorizations, the structure of the String activity offers children opportunities to work on mathematical procedures and connect them to key mathematical concepts and ideas. Because it focuses student attention on computational procedures while connecting those procedures to mathematical meanings and properties, they call it a *procedures with connections* task. It suggests computational pathways that make underlying ideas explicit; represents mathematical ideas in multiple ways; and requires cognitive effort to make the target connections. But while the structure of the task offers opportunities to engage with mathematical ideas in these ways, the maintenance of this level of demand is contingent upon how the task is enacted by teachers, on the degree to which they allow children to do the intellectual work inside a task. The work for teachers involves posing questions responding to children's ideas in such a way as to maintain demand, and orienting them to one another's ideas so to allow children to be the ones to do the mathematical work.

A String as a prompt for discussion is designed to allow students to collaborate/engage in collective activity

Because it was used in the mathematics methods course as a prompt for group conversation, the String activity became a collective and collaborative task. In this setting and with the support of the teacher, individual children's fragmentary understandings can be pooled to build new ideas. This is intended to broaden the ways in which children can participate and to broaden what counts as productive participation. The literature suggests that collaborative tasks can both support participation in classroom mathematics and support the development of productive disciplinary dispositions in children. The collective nature of the String activity, the notion of building new, shared understandings from children's ideas raises the most challenging set of entailments for productive enactment. It requires teaching practices that support children to make ideas about the target mathematics public, and practices that orient children towards understanding and building on the ideas of others towards a mathematical goal.

The key teaching practices mentioned in this section that go along with engaging the important features of the String activity are:

- Orienting children to the mathematics of the String throughout the lesson
- Using a visual model to support student thinking
- Allowing children to do the intellectual work
- Orienting children to one another's thinking
- Supporting children to make thinking public
- Supporting children to build on one another's ideas

These are offered to illustrate the way in which the features of an instructional task create opportunities for novice work on important practices of high quality teaching.

Table 6 offers a protocol for the String activity which takes into account these key practices. It specifies for example some of the work of orienting children to the mathematics of the String throughout the lesson, and supporting their engagement with a visual model. These emerged as both critical to the lesson and challenging for novices. On the left-hand side of the table, I offer an image of how that phase of the activity might sound. And on the right-hand side I offer notes connecting that phase to ideas raised here so for.

But what also emerged from this analysis was that although practice can be represented in this way, and specified to a certain level, ample work remains to support enactment. The level of detail that is possible in a written representation of practice like this protocol is inadequate to inform novice practice fully. There are decisions that need to get made at every step about what to say, how and when to say it, where to stand, and so forth. But also, much of what happens in a discussion-based lesson is contingent upon student ideas and inputs. In the following chapters, I will offer images of the ways in which the pedagogies of rehearsal and enactment in the mathematics methods course offered opportunities to detail practice further and to engage novices with the contingent interactive aspects of productive practice.

Table 6: Proposed protocol for enacting the String Activity emphasizing the work of orienting children to the target mathematics

Introducing the lesson and its purpose	
"We are going to do some mental math with [division, multiplication etc]. I am going to give you some problems to solve mentally and we are going to think about how you can use problems that you do know to solve problems that you don't know." *OR* "We are going to do some mental math with [division, multiplication etc.]. I am going to give you some problems to try to solve mentally and I want you to think about how these problems are related to one another. I think we can notice something interesting about [multiplication, division etc.] if we think together about how these problems are related."	• This begins the work of orienting children to the target mathematics.
Setting expectations for participation	
"I am going to give you a problem to solve. When you have an answer, I want you to give me a thumbs-up on your chest rather than calling out your idea. While we are working on these problems, I want you to listen carefully to one another's ideas. Ready?"	• Make sure the children are seated where they can see. This and setting expectations supports access to the mathematics and participation.
Posing the helper problem(s)	
[Write the problem on the board] "Give me a thumbs up when you have an answer."	• Write problem clearly and with room at the side for the visual model. • Give students enough time to think. Move to next step when nearly all students are ready. • This supports participation.
Eliciting multiple answers from group	
Either elicit chorally: "What answers did we get? Everyone call out what you got together." *OR* *Elicit answers from multiple children one at a time:* "What answers did we get? What did you get? What did you get? Does anyone have another answer?"	• Taking multiple responses supports participation. It broadens who participated and communicates to children that all reasoning will be considered.
Offering a visual model of the problem and asking students to make sense of the model	
" I am going to show this problem with [an array, a number line etc.]." [*Offer model on board*.]	• Draw model next to problem

Eliciting from students the meaning of the model and key features that will help with the goal	
"How does this [array, number line etc.] show [this problem]."	• Consider what features of the model students need to see/be able to talk about in order to support the goal. • Elicit thinking about the model until key features (features that are needed to support the goal) are clear • This is part of orienting students to the mathematics and helping them to use the model to support their thinking
Repeating for each helper problem in the set	
Posing target problem (s) and prompting students to investigate its relationship to earlier problems.	
"I am going to give you another problem and I want you to think about whether the problem(s) that you just solved could help you with this next one." *OR* " I am going to give you another problem and I want you to think about how our [array, number line etc.] will change for this problem." *[Write problem on the board.]* "Give me a thumbs up when you are ready."	• This is part of orienting children to the mathematics and prompting them to look for connections that will highlight target ideas.
"Turn and talk to the person next to you about what you got and how you got it."	• Listen in on turn and talk to select target strategies and others • This is part of supporting participation—using multiple participation structures. It gives children who might not want to talk to the whole group an opportunity to talk to a partner. It also offers children the chance to hear one another's ideas or to have their thinking scaffolded by a peer if they the problem is challenging for them. • It also gives the teacher an opportunity to select who will share. This is part of steering instruction towards curricular goals.
Eliciting answers and conversation about target relationship	
"What answers did we get? What did you get? What did you get?"	• Eliciting multiple answers places emphasis on reasoning rather than on having the right answer. It broadens who can participate.
"_________ [student selected during turn and talk], can you share with us how these [earlier] problems helped you solve this one?" *OR* "__________ [student selected during turn and talk], can you share with us your idea about how the [array, number line etc] will look different for this problem?"	• This is the work of steering instruction towards mathematical goals.
Offering model of target problem and asking students to make sense of the model in a way that gets out the target relationship	

"Here is [an array, a number line etc.] that shows your thinking. Where is [are] this [these] problems in this [array, number line]?"	• This is the work of orienting children to the mathematics. It is supporting children to use the visual model to support thinking about the target ideas.
Helping students to articulate the target strategy or relationship	
"At the beginning, I said that we were going to think about how easier problems can help us solve harder problems. How did these easier [multiplication, division etc.] problems help us solve the harder one?" *OR* "At the beginning, I said that we were going to think about how these problems were related. How were they related? Can someone try to make an idea about [multiplication, division etc.] that these problems help us see?"	• This is orienting children to the target mathematics. It is supporting them to articulate the target relationship.

CHAPTER 4: WORKING ON ORCHESTRATING PRODUCTIVE MATHEMATICAL DISCUSSION THROUGH COACHED PUBLIC REHEARSALS OF THE STRING ACTIVITY

Each week in the mathematics methods course, 2-4 novices came to class prepared to lead a String. They had seen the teacher educator model a String, had engaged with a protocol for the activity and had planned the lesson with a group of peers. A group of novices played the role of students and gathered on the rug near a whiteboard. Individuals or pairs of novices rehearsed leading the 'students' through the String, posing questions, eliciting and responding to student ideas and offering a visual model. While novices rehearsed, the teacher educator offered thoughts and suggestions, posed questions, and prompted discussion of the work throughout.

I sought to understand what aspects of the practice of leading a String productively were worked on in these rehearsals each week. I will offer two key findings about the 12 rehearsals of the String activity that I analyzed. First, it emerged that the *representations of the practice* of leading a String (Grossman et al, 2009) with which novices engaged in preparation shaped what got worked on in the rehearsals in important ways. Second, the teaching practice that was worked on most intensely in rehearsals of the String activity was that of *steering instruction towards mathematical goals*. Rehearsing this practice attended specifically to detailing the work of *orienting children's attention to the target mathematics*. Rehearsals supported work on and highlighted the importance of four key aspects of supporting children to attend to the target mathematics. They were:

1. Developing a visual model which supports understanding of a mathematical idea
2. Orienting children to the meaning of the model

3. Prompting students to investigate relationships among the problems in the String
4. Using the model to highlight a mathematical relationship

Representations of practice shape rehearsal

Before public rehearsals of the String activity took place in the course, novices had engaged with some representations of the practice of leading a String both in class and on their own in both years that I studied. In both years, the teacher educator had modeled leading a String; novices had engaged with a protocol of some type for the activity; and they had been given some common planning time with their peers to prepare for leading the activity. But the specifics of those representations differed in the two years and the differences affected the level at which practice was worked on in rehearsals. The rehearsals of the String activity from both years in my data set were, for all of the novices, a first attempt at leading a String. Noting what the novices were able to do independently (that is, without consultation with one another or with the teacher educator in the moment) in the rehearsals indicated the ways in which the representations of practice with which novices had previously engaged prepared them for enactment.

In the 2010-2011 school year, the representations of practice with which novices engaged before rehearsals had prepared them somewhat less for the logistics of leading a String. Therefore, that year, more time was spent in rehearsals on supporting novice understanding of the basic structure of the activity, the wording of an appropriate launch, when to introduce a visual model, how to pose the problems and so forth. In the 2011-2012 school year, novices had worked with a slightly different set of representations of practice before the rehearsals, and so they began the rehearsals prepared to introduce the lesson, pose problems one at a time, elicit responses from students, and attempt to represent the math. The work in which they then engaged in rehearsal attended more to the details of enactment than to the basic structure of the activity.

Novices in 2010-2011 had experienced one String being modeled by the teacher educator in the context of the course and had analyzed the lesson. They had also watched a commercially available video of a teacher enacting a String (Fosnot & Uittenbogaard, 2007). To prepare for their own teaching, they had been given a set of problems, one or two paragraphs about the construction of the String, and a written example of some dialogue with children about the problems drawn from a commercially available professional development resource entitled *Contexts for Learning Mathematics* (Fosnot & Uiteenbogaard, 2007) (Appendix 1). Also, just before the rehearsal the group of teachers who would be teaching the String that day had 30 minutes or so to talk about their plans.

Three of the six String rehearsals I analyzed from 2010-2011, involved work on the basic structure and logistics of the lesson. Novices and the teacher educator constructed an introduction during the rehearsal, planned for the basics of posing the problems and eliciting student ideas, and/or for how and when to offer a visual model of the mathematics. So for example, in a rehearsal that took place in November of 2010, the teacher educator (TE) began the rehearsal preparing the novices to introduce the String offering an example of an introduction.

> TE: The problems are supposed to connect with one another and we need to prompt that connection. And you want to make your introduction short. So you want to say something like, 'Today in this rotation we are going to work with some mental math problems and we are going to work on using what you know to figure out problems that are harder, or using facts you know to figure out facts that are harder.' OK? Something short, 10 seconds…"
> [November 16, 2010, Group 4a, String Rehearsal 1]

In another rehearsal a novice teacher (NT) began the lesson with an invented introduction in which he attempted to address student understanding of the array he would later use to represent the problems.

> NT: OK guys, today we are going to work on multiplication and we are going to use arrays. Does anybody remember what an array is?

He spent the next two minutes eliciting student ideas about arrays and showing them the array that he would offer to model the first problem in the String. The teacher educator (TE) intervened to help him focus and shorten his introduction, again modeling an example.

> TE: OK can I stop you? What is happening to this intro?
> NT: We kind of went off on arrays.
> TE: OK so we want them to know what an array is [inaudible]. So save the question for when you put 10x15 up. Say, 'I am going to put up an array for this. Why am I calling this an array?'…So try to think, 'OK in this rotation, we are going to work on some multiplication problems. We are going to think about how one problem can help us with the next. I am going to put up a problem, I want you to think about what it is and give me a thumb.' That is a 30 second intro, which is what you need to maximize instructional time.
> [November 9, 2010, Group 3a, String Rehearsal]

The work in this rehearsal on the introduction took about three and a half minutes out of the 11 minute rehearsal. The first problem of the String was not posed until nearly 4 minutes into the rehearsal.

Novices in 2011-2012 had also experienced and analyzed a String in the context of the class. They had also watched video. But they had an additional resource to support their planning. They had Parrish's (2010) *Number Talks*, a book that offers examples of problem sets to be used with various grade levels along with visual models that can support the mathematics. They had been prompted to select a String of a particular difficulty level from the book. They had also been given a planning template that offered a flow for the lesson and some prompts to support their planning (Appendix

2). And like novices in the previous year, they had spent about 30 minutes solidifying plans with the TE and their teaching partners just prior to the lesson. In the 6 rehearsals that I analyzed from this year, none of them involved work on the introduction or the basic structure of the lesson. It was typical for novices in this cohort to stand up, offer an introduction that would be adequate to get children started and then to pose the first problem with very little support from the teacher educator. An example in November of 2011 showed a novice (NT1), her teaching partner (NT2) and the teacher educator (TE) detailing an introduction in just 23 seconds. The novice began:

> NT1: I don't know if you guys remember, but a couple of weeks ago, we did some problems…
> NT2: [Waves his arm hello to remind her to begin by saying hello to the group]
> NT1: Oh, hello everybody.
> TE: 'Everybody' is better than 'guys.'
> NT1: OK so we talked about some problems and how they related to one another and how sometimes we can use problems that are easier to solve problems with more difficult numbers. So first I am going to give you a multiplication problem that you probably already know and you can use your papers if you want. The first problem is…
> [November 8, 2011, Group 2, String Rehearsal]

In this rehearsal, an introduction was given, the first problem was posed and 'students' were working on it 33 seconds into the rehearsal.

The point of making this comparison is not to suggest that the rehearsals from one of these years were better or more effective than from the other. The data I analyzed would not support this kind of conclusion. Nor is it an attempt to comment on exactly what kinds of representations of practice novices might engage with before rehearsal. What seems relevant is, first, that how novices engage with representations of practice before working on an approximation such as a coached rehearsal can shape what gets worked on in the rehearsal. When novices work ahead of time with representations that support them in understanding the logistics of the lesson, then work in rehearsal might be on other aspects of practice. And second, I would argue that these rehearsals show that representations of practice such as video, modeling, and protocols can, on their own, support novices to be able to enact certain key aspects of practice. These rehearsals offer evidence of novices offering productive introductions, moving through the steps of a lesson, eliciting student ideas and offering beginning visual models of target mathematical ideas based only on the representations of practice with which they have engaged.

However, in next section, I will show that while representations of practice can support novices in taking up key structural aspects of the work of teaching, robust work on the details of practice are necessary to support productive novice enactment. In the following section, I will offer examples of the way in which rehearsal offered opportunities to put the meat on the bones of complex practice.

Rehearsing the work of orienting children's attention to target mathematics

Episodes of rehearsal emerged as opportunities to work on the details of the practice of leading a String. While the goal of a String is for children to investigate and use connections among problems to understand operations, supporting this investigation is complex work for teachers. The work involves making sure that the target mathematics stays center stage. It requires substantive and purposeful prompting and questioning and skillful use of a visual model. It entails continuous work on directing children's attention to what is important. Much of the time in rehearsal was spent in detailing this work of *orienting children's attention to the target mathematics* throughout the lesson. Three key aspects of this practice were highlighted and detailed in these rehearsals. They were:

1. Developing a visual model which supports understanding of a mathematical idea
2. Orienting children to the meaning of the model
3. Prompting students to investigate relationships among the problems in the String
4. Using the model to highlight a mathematical relationship

In this section, I will offer examples of how the details of each of these practices were worked on in rehearsal.

Developing a mathematical model in rehearsal

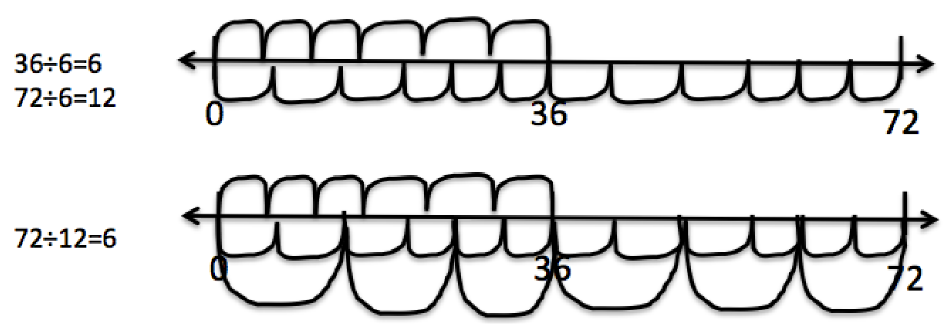
The first aspect of the practice of orienting children to the mathematics of a String that was worked in rehearsal was that of offering a well-developed visual model of the mathematical idea. The way in which Fosnot & Uittenbogaard (2007) and Parrish (2010) conceived of the String activity entails linking each problem in the String to a visual model such as an array, a number line or a picture that highlights the meaning of the operation and the target relationship. For a given String, an appropriate model needs to be selected and then fleshed out for the given problems. In some cases in these teaching episodes, novices had developed a model on their own or in conjunction with the teacher educator. In others, they had drawn the model from a commercial resource. But in all of the String rehearsals across both years, work on the details of the visual model was predominant. Decisions involved the nature of the model, how to lay it out on the board, what numbers to include in it, how big it should be, and even what colors should be used to draw it.

An example of work on the development of a visual model during a String rehearsal occurred in November of 2010. Students were preparing to pose the following set of problems designed to support students to engage with division as a ratio between the dividend and the divisor:

36÷6
72÷6
72÷12
144÷24

Students had elected to model the problems in this String using a number line as suggested by Fosnot & Uittenbogaard (2007, p. 59). Below is the representation of the model that was given to them for planning purposes and then the conversation that ensued in the rehearsal in order to detail the use of this complex model.

Figure 4: Model novices used to illustrate proportional reasoning



Novice Teacher 1 (NT1) began by posing the first problem in the set to the group, eliciting a choral response of "6!" and then beginning to draw a number line on the board below the problem. She started with 36 on the right hand end of the number line and made 6 arcs backwards towards 0 to indicate 6 groups of 6. Work on the model began immediately in a conversation that involved the novice teacher (NT1), her teaching partner (NT2), three novices in the audience (NTaud1,2,3) and the teacher educator (TE).

NTaud1: Do we want—sorry, not to interrupt but—but do we want to put the number line to the side so that we can put 72 [from the next problem] right under so they can see that relationship?
TE: Yeah-- the use of board space. It would be good because you want to have the numbers right underneath each other [marking in the air with her hand what she means].
…
TE: So you decided to go backwards I notice [by making jumps of 6 back from 36 to 0].

NT1: Yeah, I just feel like division is…it makes more sense to go backwards.
TE: So also what else needs to be drawn on there for them to see that it is 6 groups of 6?
NT1: [Begins to number each of the arcs from left to right 1,2,3,4….Then erases and labels them from right to left 6,5,4,3,2…]
NT2: Or should we write 6,12,18..?
NT1: [Looks at TE for advice.]
TE: What do you need to mark?
NT1: [Labels the end point of each arc with 6,12,18…]
NTaud2: In the example [from the pages of Fosnot & Uittenbogaard (2007)], they don't have the numbers on the number line. They just have the 0 and the 36. They don't have the jumps marked.
TE: They don't have the jumps marked.
NTaud2: It makes sense to mark the jumps.
NT1: Yeah, I actually did it too [showing her lesson plan].
TE: So that is part of thinking about what is going to make sense to your audience here. And the meaning, the reason why we are doing this is because there are six sixes in 36.
NT2: It's kind of a lot of numbers though in one small space [circling his pencil around the representation on the board]. Do you think we could have just the jumps marked 1 through 6 to 36 or…
NTaud3: Or you could write "Six jumps."

The conversation continued briefly and in the end, they decided not to write any numbers at all on the number line other than 0 and 36. A bit later in the rehearsal, work on the model got taken up again as NT2 noticed that, as the numbers in the problems get bigger, their number line needed to get bigger.

NT2: So anticipating [the bigger numbers in the next problems] that though, we should think about our number line. If we are going to get all the way to 144, are we going to come all the way down here [running his pen in a line along the board] and just keep jumping? Or should we start a new number line?
TE: You could start a new number line. You could say, 'Let's shrink this down now. What's going to happen?' Because the number line is doubling, right?
NT2: [Takes notes on his lesson plan.]
NT1: But the next problem is 72 divided by 12. So if we took the 12 [spreads her fingers over two arcs of six].
TE: You have to show that every two 6s is a 12.
NT2: Then we could convert our number line to 12s.
[November 30, 2010, Group 1, String Rehearsal]

This conversation continued briefly and the rehearsal went on. In this rehearsal novice teachers and the teacher educator spent about a third of the rehearsal time (or about 5 minutes of the 14 minute 41 seconds) developing this number line to show the relationship between divisors and dividends.

This focus on visual modeling was common in rehearsals in both years, with substantive time spent developing a model in 7 of the 12 rehearsals I analyzed. This work exemplifies the way in which representations of the practice of using a visual model (such a images of a model in a protocol) are not quite adequate to support novice enactment. The pedagogy of rehearsal offered opportunities to flesh out the details of using the model to support student learning. The double number line was described and illustrated for novices in the text that went with the String (Fosnot & Uittenbogaard, 2007) and the representation was clearly enough get the novices to a rehearsal with a beginning visual model in hand. Watching the videos, it seems clear that the novices themselves understood the models and their connection to the mathematics of the String. But what is also clear is that the representations of this practice were inadequate to support a beginner in presenting a well-developed visual model. Mediated work detailing the use of each model was clearly necessary to support enactment.

Rehearsing to help students connect a visual model to an operation

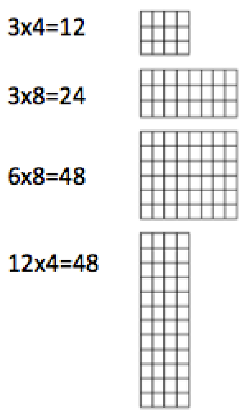
A second aspect of the practice of orienting children to the mathematics of a String that was detailed in rehearsals was that of helping children *connect the visual model to the target operation*—helping them understand the meaning of the model. Visual models can offer illustration and proof of mathematical relationships and meanings, but students need to engage with the way relationships appear in a given model. So for example, a model can highlight multiplication as 'groups of groups' but this feature needs to be made explicit to students. Novice teachers and the teacher educator rehearsed the work of orienting students to the mathematics of the model.

For example, in a rehearsal that occurred on November 2010, the novices worked on leading a string designed to highlight multiplication as groups of groups and to support students in using proportional reasoning:

> 3x4
> 3x8
> 6x8
> 12x4

They planned to model each problem with an array to illustrate how the doubling and halving of the factors from problem to problem would affect the product.

Figure 5: Modeling proportional reasoning



They began the rehearsal with the arrays for each problem cut out of ¾ inch graph paper. Novices posed the first problem, 3x4, and had the group call out the answer. They posted the first array and then the following exchange ensued:

> NT1: [To NT2] Then you are going to do the next [problem], right? No?
> NTaud1: You need to talk about the array as 3x4.
> NT2: So can someone come up and show me where they see 3x4 on this array? Sarah, do you want to come show us?
> NTaud2: [Comes to the board and runs her finger down the first column of 3 then across the top row of 4 to show the dimensions of the array.] Three and four.
> NT1: [Writes the dimensions on the sides of the array.]
> NT2: Good, thank you. OK, I am going to give you another one that you probably already know.
> NTaud3: Should we slow down here?
> NTaud4: The first couple are supposed to be easy.
> TE: The first one you don't expect to have much conversation about because you start with one they know. The thing that you are skipping over a bit that you probably want to do… First of all when we are in the room you will have some markers. Feel free to write on the arrays….1,2,3 and 1,2,3,4 [pointing to the outer row and column]. But the important thing about the array is that it's not just labeling the outside. The reason why there are 12 is that there are either

[running her finger down the columns] four 3s or [running her finger across the rows] three 4s.

NT2: Ohhh.

TE: And 'How many squares are there in here?' Twelve. You want to make the connection between the groups and the total.

NT2: [inaudible]

TE: Just to say, 'You have worked with arrays. Let's just quickly go over […] How does this show 3x4?' And you want to establish, sure, there's three this way and four this way but that is just seven. The point is that there are [running her finger across the array] three groups. And you want to have that established.

[November 16, 2010, Group 4a, String Rehearsal]

Novices and the teacher educator rehearsed helping students make sense of the array and its relationship to multiplication in a way that will support the goal of the String. In this case what surfaces is the way that arrays are commonly labeled and referred to by dimensions of length and width. So when NTaud2 came up to the board to show how the array represents 3x4, she ran her finger along the sides to show that one has a dimension of 3 units and the other of 4 units. But in order to understand the effect that doubling one of the factors in a multiplication problem will have, the problem needs to be understood as representing groups of groups. To make this explicit, the dimensions of the array need to be articulated as referring to the number of rows and columns of a certain length, as 3 rows of 4 units and 4 columns of 3 units, rather than simply as side lengths. The conversation is about how to push children beyond articulating the dimensions to articulating the presence of groups in the array.

In this exchange, novices were working on making explicit the way in which a model represents a mathematical idea so that the model can later be used to illustrate a more complex relationship. They are working on helping students to articulate the mathematics in the model to support understanding of the next problems in the String. So while saying that the side lengths of the array are 3 units and 4 units is totally mathematically true, this is not the information about the model that is important for this

String. Novices need to consider as they work on a String the aspects of the model are most important for illustrating the target relationship. And then they need to practice supporting students to articulate these aspects of the model.

Connecting the model to the mathematics is the work of helping children to connect a visual model to an interpretation of an operation as soon as the model is introduced in order to support the building of later ideas. How to do this and what features of the array were important needed to get worked out in rehearsal. This practice turned out to be important because, as I will show in the next section, it emerged that the degree to which novices helped children connect a model to the mathematics early in the String would have an effect their success in supporting kids to articulate a target relationship at the end of the lesson.

Rehearsing to prompt students to look for the relationship among problems

Another important aspect of orienting children's attention to the target mathematics that emerged in rehearsal was the work of *prompting children to look for and use connections among the problems* in the String. Interestingly, novices in both years entered the rehearsals offering these kinds of prompts independently. For example, in the rehearsal from November 2011, novices worked on leading the following string designed to support student to use both partial products and compensating strategies to solve multi-digit problems:

6x20
6x100
6x120
6x119

Novices posed the first two problems and then as they got to the third problem, which can be solved by combining the products of the first two, NT1 said to the group,

> NT1: OK now I am going write a little bit harder one on the board. I want you to use your strategies. And also, think about the two problems that we have already figured out. See if that can help you solve this next one.

The group solved the problem and strategies for doing so were elicited from the group. And then NT2 posed the final problem, which can be solved by subtracting a group of 6 from the previous product. She offered a similar prompt to look for connections among the problems:

> NT2: OK now it's time for [inaudible]. And as you are thinking about this problem, I want you to think about the problem you just solved and how you can use that problem to help you solve this problem. And when you have an idea give me a thumbs-up.
>
> [November 8, 2011, Group 2, String rehearsal]

These prompts were commonly offered without intervention by the teacher educator in the rehearsal. This indicates that such prompts had been, in both years, well enough included in the representations of practice that preceded rehearsal for students to take them up.

Where the prompting needed refinement in rehearsal was when the prompt that would best support student work was something other than, 'Think about how this problem can help you solve this problem.' This prompt supports Strings in which the goal is to find and use easier problems to solve harder problems. But this is not the goal of all Strings. For example the following set of problems was rehearsed in November of 2010:

3x4
3x8
6x8
12x4

As mentioned earlier, the goal of this String is to support students to understand multiplication as groups of groups and to begin to use proportional reasoning. We want to give students the opportunity to consider the behavior of multiplication as an operation. If one factor in a multiplication problem is doubled, the product is doubled; if one factor is halved the product is halved; and if one factor is halved and the other is doubled, the product will remain the same. Children will often have single-digit problems, such as the ones in this String, memorized. The String is not designed to highlight a strategy that will make solving these problems easier. The small numbers in the String allow for children to use already familiar problems to investigate relationships—in this case the relationship between 3x4 and 3x8. Novices posed the first problem posted the array, and then moved on to the next problem. As they posted the array for the second problem the following conversation ensued:

> NT2: So should we have talked about this [problem] being helpful to this [problem] before we posted this [array]?
> TE: Not necessarily.
> NT2: OK. So this is an array for 3x8…[to TE] And should I go over having someone come and show what 3x8 is?
> TE: You know at this point, this is where you want to start getting them to make the connections, right? [Comes to the board][…] And you want to line these up [moving the arrays on the board so they are one under the other with left edges aligned] so that you can make the comparison. 'What happens to the array as we move from 3x4 to 3x8.'
> NT2: [Trying the prompt] So do you notice anything once we went from 3x4 down to 3x8? Do you notice anything?
> [November 16, 2010, Group 4, String Rehearsal 1]

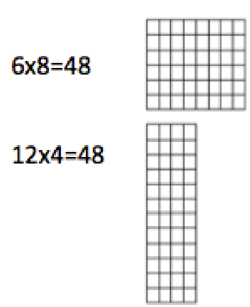
The novices here work with the TE to develop the prompt that will direct children's attention to the change in the array from problem to problem. This is a different kind of prompt than those used earlier because the goal of the String is a bit different. A similar conversation took place in each of the rehearsals in which the String highlighted this doubling and halving relationship. Novices needed time in rehearsal to consider how to phrase the prompts that would direct children to investigate the relationships among the problems.

Rehearsing to use the model to represent target relationships

The third aspect of orienting children's attention to the target mathematics that was worked on rehearsal was *linking the model to the target mathematics*. This is the work of supporting students to see and articulate how the relationship among problems is illustrated in the model. It emerged as worthwhile to separate this from the work of prompting an initial understanding of the representation because it required a different and seemingly more complex set of prompts.

For example, as the rehearsal described above progressed, novices were ready to pose the final problem, 12x4. One factor from the previous problem has been is doubled and one factor has been halved, so the product will remain the same. Novices posed the problem, elicited responses, and then posted a 12x4 array next to the problem. Then a discussion ensued among the novice teachers and the teacher educator about how to use the arrays to help students visualize why the product remains the same when a proportional change has happened in the factors. They aimed to prompt children in such a way that they would notice that the 6x8 array could be cut in half and rearranged to make the 12x4 array. But they needed to work out together how to do this.

Figure 6: Modeling the relationship between 6x8 and 12x4



NT2: So we look at the array of 4 rows of 12 or 12 rows of 4. So its 12x4=48. Can anyone tell me how they see a relationship between these two [arrays] [indicating the previous 6x8 array]?
TE: [Suggesting a prompt] 'What happened to the array? How can we… since they both equal 48…'
NT2: Can we make them both look the same?
TE: 'How can we make the arrays look the same?'
NT2: They have the same answer, any ideas?
TE: This probably is a turn and talk kind of a place because it takes a little thinking […]
NT2: So I want you to turn and talk with your neighbor about how we could make the array for 6x8 look like the array for 12x4 since they have the same answer.
NTaud1: [To TE] So we are trying to make [the 6x8 array] turn in to the 12x4?
NT1: I think this [12x4 array] turns into this [6x8] array. Not this [6x8 array] to this [12x4 array].
NTaud1: Yeah, which one do you want us to change?
NT2: [Shrugs as if to say to a student, 'I am not going to make that decision for you. That is your job.']
NT1: Is it this one to this one, because the other group did the opposite.
[….]
TE: [….] So you are trying to transform [this earlier array of 6x8] into this [array for 12x4] because that is the order of the problems…
NT1 & 2: Oh, OK.
NT1: But it could transform back the other way.
TE: It could transform the other way, but because of the sequence, we want to do that first…
NT2: [To the group] OK so we want to know how you can turn 6x8 into 12x4, the array for 6x8 into the array for 12x4. So talk with your partner and think if you can find a way to do that.
TE: […] And this is where, while they are [talking] you want to listen in because you want to use one of their ideas. 'Maria, I liked how you were describing that. Can you come up and show us?' And this is where you can actually cut [the array] so we'll have scissors…OK?
[November 16, 2010, Group 4a, String Rehearsal]

The teacher educator intervened first to help NT2 refine her first prompt so that she was actually directing children to fully articulate and show how the arrays are related. And then they needed to move on to work out the specifics of prompting children to turn one array into the other. This kind of work on how to use the visual model to deepen children's understanding of the goal of the String appeared in a similar form in many of the String rehearsals. But as I will discuss in the next section, having rehearsed this set of moves will not always be enough to support children to articulate the target mathematics when they engage in the work of the String.

In this section I offered examples of the way in which the representation of practice with which novices worked before rehearsal shaped the level of detail that could be engaged in rehearsal. The more representations of practices prepared novices for the structure and logistics of the lesson, the more attention could be paid in rehearsal to detailing practices of leading the String. Also, I offered examples of how rehearsals of the String activity enabled novices and the teacher educator to put the meat on the bones of the practice of orienting children to the target mathematics. Specifically, I offered examples of novice teachers and the teacher educator detailing the key practices of developing a mathematical model to support student understanding, helping children to connect a model to operational meanings, prompting students to connect problems, and using a model to highlight target mathematics. In the next chapter, I will offer my analysis of the aspects of orchestrating productive mathematical discussion that were worked on in enactments of the rehearsed lessons that novices did with children.

CHAPTER 5: WORKING ON ORCHESTRATING PRODUCTIVE MATHEMATICAL DISCUSSION IN ENACTMENTS OF THE STRING ACTIVITY WITH CHILDREN

Immediately following each of the rehearsals of the String activity analyzed previously, novice teachers led the String with a small group of 5th grade students from a class with which they had been working weekly since the mid-October. The lessons were done in 15-minute rotations with groups of 6-7 students. The teacher educator sat with the children and participated in the lessons making comments and offering prompts for children in addition to those that the novices offered in the course of the lesson. I analyzed these enactments on their own in order to understand the way the use of enactments in the context of the course offered opportunities for work on the complex practice of leading productive discussion. I gave particular consideration to the problems of practice that novices encountered as one way of understanding these opportunities. I also attended to the interventions and insertions that the teacher educator made in the lessons to support novice work as another way of understanding these opportunities. In this chapter, I will offer examples of the problems of practice that emerged, the teacher educator's interventions and the aspects of orchestrating productive talk that they highlight.

The problems of practice that emerged in enactments, almost without exception, involved a novice teacher asking a question that did not yield the intended response from students (Leinhardt & Steele, 2005). Consideration of the kinds of responses that novices encountered and their attempts to navigate them, highlighted several key practices that are related to orchestrating discussion in the classroom: the purposeful use of call-on routines, asking children to add-on to incomplete responses, and orienting children to the mathematics of a visual model. The majority of the interventions that the teacher educator made were intended to support novice work in *orienting students to the target mathematics*. She posed questions or made comments that prompted students to attend to key ideas or to consider key relationships. This theme of the work involved in orienting kids to mathematical ideas in the orchestration of productive talk emerged in all aspects of my analysis.

It seems important to say that these enactments represent novice teachers' very first attempts at leading a String with children. They are, for many of the novices in fact, among their first experiences teaching children at all. My analyses of the problems of practice and the interventions are not attempts to speak to what novices could have done better, to how these lessons could have been more successful, or to what novices need to learn next. It goes without saying that these lessons will be flawed in all kinds of ways and parsing the flaws is not especially useful. I was instead interested in thinking about the kinds of challenges that emerge repeatedly, how they might inform our thinking as teacher educators about leading productive discussion, and whether they might suggest ways to scaffold novice work on leading such discussion.

Problem of practice in enactments: Managing unanticipated responses

The problems of practice that emerged as novices enacted the rehearsed String activity with children consistently related to the work of managing unanticipated responses. There were instances in which children offered responses that were completely incorrect; instances in which they offered responses that were partially correct and partially incorrect; and instances in which they offered a partial correct but incomplete response. These instances presented novice teachers with challenges. They needed to decide in the moment how to respond in a way that would support student learning and keep the conversation moving towards the target idea. Considering the kinds of unanticipated responses and novices' attempts to manage them shed light on the work of leading productive mathematical talk in the classroom. Three practices emerged as particularly salient in managing unanticipated responses:

1. *Purposefully selecting call-on routines* (Leinhardt & Steele, 2005) emerged as an important tool for managing what enters a conversation.
2. Asking children to *add on to one another's ideas* surfaced as a worthwhile move when children offer partial responses in conversation.
3. The work of *orienting students to the mathematics of a visual model* from the outset of the String emerged as critical for supporting students' capacity to articulate and use the target idea at the end.

Unanticipated responses and call-on routines.

One kind of unanticipated response that students offered in these enactments was an error— an answer that was completely wrong and that novices had not considered in rehearsal. In a String enactment in November of 2010, a novice teacher gathered a group of five Fifth Graders in a row of chairs by a whiteboard. His fellow novice teachers sat ranged around behind the children. He had rehearsed this String just before joining the children and planned to work with the group to solve a multi-digit multiplication problem using a compensating strategy. He planned offer two related problems to help the children see how the first might be used to solve the second:

10x15
9x15

As he got ready to put the first problem up, he said:

NT: I am going to write a problem up and when I write it up there, I want you to think in your head as quick as you can, but quietly. And when you have an answer, I want you to put your thumb up. Here we go guys. [Writes 10x15 on

> the board] Alright, I see some of us have it. I am going to wait until I see everybody's thumbs. You should be able to get this one pretty quick. [Pauses]

And then he called on a student who had her thumb up and who offered the nonsensical answer of 10 (to the problem 10x15). This left the NT with the job of responding to and untangling her thinking.

> NT: K., could you tell us the answer to 10x15?
> K: I got…. 10.
> NT: What was that?
> K: 10.
> NT: [Squats down near her. Pauses.] You got 10?
> TE: Try 10x5 first. What's 10x5?
> [November 9, 2010, Group 3a, String Enactment]

The teacher educator intervened and together she and the NT spent about two minutes helping the student solve the problem. K.'s answer might suggest to us that she is only beginning to develop an understanding of multi-digit multiplication. And the NT and TE did successfully support her in getting to an answer of 150 by breaking down the problem into 10x5 and 10x10.

But thinking about this episode from the perspective of leading a String to support broad participation, to reach mathematical goals, and to build on what students know, offers a more nuanced view. It brings into focus the practice of choosing a method for eliciting student responses, or in Leinhardt & Steele's (2005) terms, choosing a "call-on routine," that will best support the aims of the lesson. In other rehearsals and enactments of Strings, novices were encouraged to pose the first problem, give the students a moment to think and a way to communicate that they were ready, and to have them *elicit multiple responses* either by calling on multiple children one at a time or by asking children to call out the answer chorally. This example above, in which the novice calls on a single student for an answer, highlights the benefits of the latter routine. From the perspective of supporting broad participation, the latter routine can get all students involved in the String from the outset. It also may avoid singling out a student for whom the mathematics is challenging, as happened here. Given the goal of the String, which involves using 10x15 as a benchmark for solving 9x15, the bulk of the time wants to be spent on investigating the relationship between the two problems. So while supporting an individual student's beginning understanding of multiplication is always important, it is not the goal of this group work. What is needed is a call-on routine that positions 10x15 as tacit knowledge in the group and moves them towards consideration of how such knowledge is helpful.

Another call-on routine related to managing unanticipated responses that is highlighted in these enactments is that of posing a question to students, asking them to turn and talk to their neighbors, listening in on their conversations and then *purposefully selecting* the responses that will get shared with the whole group (Lampert, 2001; Stein, Engel, Smith & Hughes, 2008). In many of the enactments, novice teachers posed a question and had students talk about their answers, but then called on a student whose response they had not already heard with an elicitation such as, "Who has an idea they would like to share?" So for example, an enactment that took place later in November 2010 novices led a String that supported children to investigate what happens to the product when factors are doubled and/or halved from problem to problem. They posed the first two problems and as they posed the third, they asked children to consider how it was related to the others. They asked children to turn and talk and they listened in. But as they gathered the group back, they called on two students to whom they had not listened in the turn and talk. The students both offered correct answers, but had used strategies that were unrelated to the target and left the novices having to navigate making the next step towards the goal [November 30, 2010, Group 1, String Enactment].

The aim here is not to avoid errors or to elicit only perfect responses, nor is it to say that the NT did one thing but he should have done another thing. Rather, it is to point out the way in which the details of the call-on routine in any given moment matter in important ways to the work of maintaining the aims of the String, to the work of leading productive discussion. There are two aims that can be served by using purposeful selection in this context. The first is that working towards a mathematical goal in a conversation with children requires active steering on the part of a teacher. Purposeful selection of student ideas is one powerful mechanism for steering. It allows the teacher to at least partly choose the pathway that the conversation takes, to drive the bus, as it were. The second is that the purposeful selection of ideas to be shared can offer novices support in reducing the unanticipated responses that land on table. For novices, this modicum of control is highly useful. Accomplished teachers are able to manage the wide range of well developed, partially developed, and incorrect ideas that children will bring to a conversation (cf. Lampert, 2001). They are able to assess in the moment how a child's idea intersects with a goal and how to help the group use it. But novices are still developing an understanding of the range of children's ideas relative to a topic and a repertoire for responding to that range. Moves that allow them to pre-screen ideas can support them in attaining mathematical goals as they develop.

Unanticipated responses and building ideas with the group.

It was common across enactments of Strings, as it is in most classroom discussions, for students to offer answers and ideas that are only partially developed or only partially correct. This situation came up a number of times in enactments and novices responded similarly each time: they tried earnestly to get that one student to give a more complete response by asking follow up questions that almost always stalled the conversation. In orchestrating mathematical discussion, we are often aiming to help multiple students make public the fragments of understanding and partially developed ideas that they possess and to combine them to build new, shared ideas. Watching enactments brings to the surface the consistent challenge for beginners of drawing out multiple partial understandings that are distributed among the group to build new ideas.

For example the enactment that took place on November 16, 2010, novices worked on a doubling and halving String with children. They had posed the first two problems, elicited answers, and posted arrays.

3x4=12
3x8=24

NT2 made an attempt at prompting students to connect the two problems:

> NT2: Did anyone use this problem [3x4] to help them solve this problem [3x8]
> Ss: No.
> NT2: No? You just knew this problem…

At this point she was hard pressed to help them make the connection between the problems. The teacher educator offered a prompt to move the conversation along:

> TE: How are they related though? What happens to the array?
> NT2: Do you see any relationship between these two?

Now, there was an audible gasp from a student and 3 of the 5 students visible in the video raised their hands, with one lunging forward from the edge of his seat. NT2 called on a student who offered a partial answer:

NT2: K.?
S1: 12 is half of 24.

Student 1 was looking at the products and noticed that product of the first problem was half of the product of the second problem. This is of course a correct and important observation, and it represents an important piece of the relationship between the problems. A full response, and the goal of the String, would involve articulating that because one of the factors is doubled from the first problem to the second, the product had doubled. Trying to figure out what to do with this partial response, novices try pressing her to offer a more complete response:

NT2: 12 is half of 24… Why do you think that happens?
Ss: [Silence.]
NT1: Do you see any relationship?
Ss: [Silence.]
NT2: Well, we'll keep going and we'll see if we can figure out a way that they are related.
[November 16, 2010, Group 4a, String Enactment]

What this stumbling spot brings to the surface is the rich potential of the collective knowledge of the group and the challenge for beginners of drawing on it. While these novices pressed on K, they had three other students waving their arms to share what they noticed about the two problems and no doubt these ideas would expand on Student 1's noticing about the change in the product. Rather than trying to get S1 on her own to see more than she has, they can take her partial idea and have the group expand on it. Even if those raising their hands each only have a piece of it, together the children will have articulated the bulk of the relationship.

Again, the point here is not to point out what the novices did as opposed to what they could have done. Noticing places that are consistently challenging for novices help us think about the work of orchestrating productive talk. Learning to take the fragments that children bring and use them to build shared ideas among the group is of the essence in learning to orchestrate talk productively. The consistency with which novices respond to partial understandings by homing in on one student points out the shift that we are asking novices to make in understanding the collective nature of orchestrating talk.

Unanticipated responses and connecting the model to the mathematics.

The third place in which unanticipated responses arose in enactments occurred at the end of the lessons. As novices posed the target problem(s) in the Strings, they often found that children were not quite able to use and articulate the target relationship. Children would offer alternative strategies, partial articulations of the target, or occasionally be unable to solve the target problem at all. Novices clearly puzzled over these moments and tried a variety of strategies to backfill the ideas that the String was designed to develop. In any given enactment, one could point a number of moments in which novices might have made a different choice in order to help children work on target mathematics. But a theme that appeared across the enactments was the importance of the work on engaging children early in the lesson with mathematics of the visual model offered.   The nature of this work appeared directly related to children's capacity to use and articulate the target strategy at the end.

The work that novices did with visual models in their work with students varied. They had almost always developed a model in rehearsal that they offered to support the String. But how they helped students connect the model to the mathematics of the String varied widely. In some cases, they supported students to articulate the key meanings of the model. In some case novices articulated the meaning of the model for students, and in some cases they seemed to take for granted that the picture alone would support student understanding and missed the opportunity to surface the mathematics that it represented. Looking at multiple enactments, it appeared that the more work that NTs did in the beginning to support children to articulate the meaning of the visual model, the more likely children were to be able to use the target strategy at the end and vice versa. As noted earlier, work on connecting visual models to the mathematics was a heavy focus in rehearsal. The importance of this work played out clearly in enactments.

I will offer two examples that highlight the relationship that I saw between work on the model and children's engagement with the target at the end of the String. I will offer one example in which the brief conversation that the novices have with students about the model during work on the first two problems set the children up for engagement with the target mathematics. Then I will offer an example in which the NTs did not quite involve the children with the meaning of the complex (but potentially very helpful) model they had developed. As a result, the children were much harder pressed to make use of it to support their work on the target mathematics.

In the second enactment that took place on November 16, 2010, novice teachers gathered a group of 7 children by a whiteboard to work on a doubling and halving String:

3x4
3x8
6x8
12x8

They posed the first problem, elicited solutions, posted a 3x4 array and asked children to consider how the array represented 3x4.

NT2: So how does this equal three times four? [Calls on student.]
S1: It's…it's [coming to board] three down and four across [running his hand down and across the array].
NT2: [Labeling the dimensions of the array] So three down and four across.
S1: [Inaudible]..so it is three times four inside. And so that is 12.

NT2: [Marking groups] So we know there are three groups of four and so we know there are 12 squares here.
S1: Yeah.

Then NT2 posed the next problem, 3x8, elicited responses and posted the array. Then NT1 prompted the students to attend to the relationship between the arrays. This simple prompting of attention to the way in which the array modeled the mathematics supported the students to engage with the key idea of the String right away:

NT1: Ms. A. is going to put up the array for 3x8 and I want you take a look at these arrays and think about how they are similar. Think about how they are similar. [Four of the 7 children raise their hands.]
S2: If you cut […] the 3x8 array in half you get the 3x4.
NT1: If you cut the 3x8 array in half it's going to be how many 3x4s?
S2: Two.
NT1: Two 3x4s fit inside the 3x8.
NT2: [Takes the 3x4 array and lays it over the 3x8 array showing that it fits twice.]
S3: [….] I had one more thing about what they have in common.
NT1: OK, go ahead.
S3: Um…both of them are 'three by' something.
NT1: Both of them are 'three by' something.
NT2: [Marking on the array] Three by…
S3: 'Three by' different numbers.
NT1: Three rows of something.
NT2: So what goes here [pointing to the row of 8 in the 3x8 array]?
S3: It is the doubled number of 4 [referring to the doubling of the factor from the first problem to the second].
NT1: What is the 'doubled number of 4?'
S3: Eight.
NT1: And where do you see the 'doubled number of 4' in these numbers here [gesturing to the two problems on the board]? Does anybody see that?
S4: [Coming up to the board and pointing to 8 in 3x8] Here.
NT1: [Drawing an arrow from 4 in the first problem to 8 in the second problem and marking it with 'x2' to show the doubling] So 4x2 is 8. OK see if you can use this information to solve the next problem. Use what we are doing here to solve this next problem.

This set of prompts and children's engagement with them set the group up from the outset to see the way in which the arrays modeled the key mathematics (that doubling one of the factors will double the product). And because the work with the meaning of the arrays and their relationship to the target mathematics was so clear, as the novice started to pose the last problem, one of the students correctly predicted that its answer would be

96. He had seen the doubling relationship highlighted by the String clearly enough to predict the answer before the problem was posed. The novices took up this prediction [in a lovely invention on the activity that they developed in the moment] saying:

> S1: So you are thinking of a problem that would equal 96. Who could think of a problem that would equal 96? Would you turn and talk to your neighbor […] Can we think of a problem that might equal 96?
> [November 16, 2010, Group 4b, String Enactment]

And because the children have engaged from the outset with the meaning of the arrays and their relationship to the target mathematics, all 4 of the children who volunteer ideas for the what the fourth problem might be offer problems that highlight doubling one of the factors. The problems that are already there are:

3x4
3x8
6x8

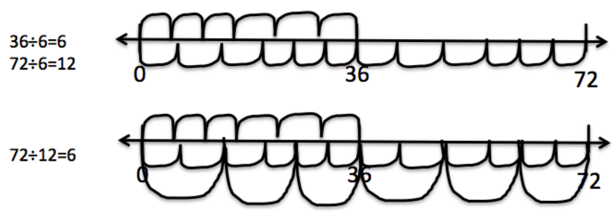
And the children suggest that the fourth problem might be:

6x16
12x8
3x32

This work is an example of the way in which supporting students to see the key mathematics in the representation from the outset can support attainment of the goal towards the end of the String. The following is an example in which the novices did less work around engaging the students with the meaning of a model and in the end have a trickier time getting students to the goal. On November 30, 2010, novices began a complex String designed to highlight ideas about proportional reasoning relative to division. They have planned to pose the following problems and to represent them on a number line as shown below:

36÷6
72÷6
72÷12
144÷24

Figure 7: Modeling proportional reasoning



NTs posed the first problem, elicited answers from the group, and then posted the number line saying:

> NT1: So we have six groups of six in thirty six. So we are going to draw that on our number line. [Draws 6 arcs between 0 and 36.] So we want you guys to look at this one and use the numbers that we learned in this one for the one that Mr. P is going to give you.
> [November 30, 2010, Group 1, String Enactment]

NT1 did not stop after posting the number line to help students make sense of the meaning of the representation, agree on what each of the arcs mean, agree that you can show division using a number line like this and so forth. NT2 just went on to the next problem. This was the case throughout the String. In the next problem, 72÷6, a student noted that the quotient would be double because the dividend was double. But again, NT2 posted the continuation of the number line without quite pausing to help students articulate how the model showed the student's idea. He just moved on to the next problem, 72÷12. In this problem, student use of the picture could be really helpful for bringing ideas about proportional reasoning to the surface. The picture offered the possibility of showing groups of 12 as 2 groups of 6. But because the students were not quite linked to the picture, they solved the problem using a range of strategies that were not related to the goal. NTs elicited them all, listening and recording carefully, and then called a student to show groups of 12 on the number line (although no one has spoken about groups of 12). But again, they did not quite pause to help the students link the relationship among the problems. As a result, when they got to the final problem, 144÷24, students were not yet able to use ideas about proportional reasoning and could not yet draw on the picture to help them. (This final problem in the set also makes a leap over an intermediate problem, 144÷12, and so raises the burden on students to apply what they have learned from the early problems.)

The point here, again, is not to parse this enactment line by line for missteps or to speak to what these NTs should or should not have done. It is instead to illustrate the notion that how we engage children in connecting visual models to the target mathematics seems to matter in reaching instructional goals. The work of using visual models to support conceptual understanding is a key practice of ambitious mathematics teaching. These enactments offered a window into novices' first attempts at engaging in this practice and suggested a decomposition that makes what is important more explicit. It appeared in these enactments that when we offer children a visual model to support conceptual understanding, we want to:

1. Consider what aspects of the model need to be brought to the surface in order to support the goal
2. Elicit children's thinking about the model as soon as it is offered and make sure that key aspects are articulated among the group
3. Prompt children to connect the model to the mathematics as problems and models are elaborated
4. Prompt children to articulate how the model illustrates the goal

In the next section, I will describe how the teacher educator intervened in enactments and how those interventions shed light on the work of orchestrating productive mathematical discussion.

Teacher educator interventions in enactments: Helping to orient students to the mathematics of the String

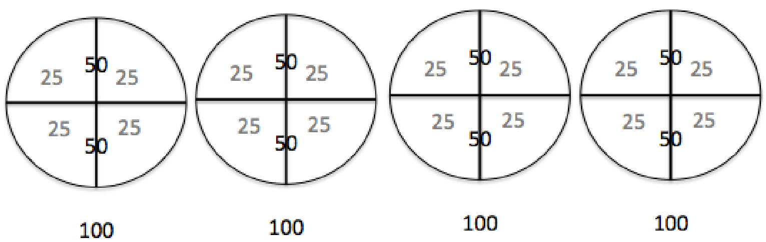
As enactments of Strings lessons unfolded with small groups of children, the teacher educator made occasional insertions into the conversation. She would comment on what was happening, interact with children as they talked with partners, suggest questions that NTs might ask, or pose questions to the children herself. These insertions serve as attempts on the part of the TE to model key aspects of practice for the NTs as well as to support the success of the lesson with children. Analysis of the interventions revealed that, in large part, they were aimed at supporting the novices in *orienting the children to the mathematics* of the String. The work of orienting children to the mathematics entails directing their attention in some way to key mathematical ideas as they emerge. This can be done by posing questions, by prompting students to attend to particular things, or by highlighting particular inputs, among other things. Work on orienting had been developed in rehearsal, but what emerged in the enactment is how nuanced the work is and how constant it needs to be when leading a conversation towards an instructional goal. Watching the TEs attempts to support the orienting work in enactments highlighted its importance, the challenge it presents novices and some of the kinds of moves entailed in doing it.

I will offer a set of examples of teacher educator interventions from a single enactment that were typical of the kind and nature of the interventions made in other enactments. In this case the teacher educator inserted questions, prompts or comments at ten different points in the 13-minute lesson with children. And as was typical of the insertions made in enactments, all of these were attempts to make sure that the children's intellectual attention was on the key mathematics that would support reaching the goal. The novices had planned for a String to support the development of ideas about proportional reasoning:

100x4
50x8
25x16

They developed a representation that involved 4 circles—each representing a set of 100. They planned to represent all three problems on the same picture by cutting each circle in half to show 8 groups of 50 and then in half again to show 16 groups of 25:

Figure 8: Modeling proportional reasoning



NT1 posed the first problem, 100x4, elicited the answer and student strategies for solving it, and then began to draw the four circles. The TE intervened:

TE:	OK, but 100 times 4 means what? What does it mean?
S1:	One hundred 4 times.
S2:	Four hundreds.
TE:	It means four hundreds.

This quick insertion was designed to highlight a key aspect of the picture that would be drawn. The TE was modeling probing student responses in order to elicit mathematical ideas that would be important to the String. Although all the students were readily able to solve 4x100, no one had said that they had added 100 four times. The TE was directing student attention to the 'groupness' of multiplication in order to make the representation more useful.

Novices posed the next question, 50x8, and elicited responses from the group and the TE intervened again:

NT1:	OK so all together, what is 50x8?
Ss:	400.
NT1:	400. Excellent.
TE:	It's the same as 100x4? Hmmm…

The fact that all three of the problems have the same product is key to the mathematics of the String. The TE was modeling using a brief comment to direct students' attention to this relationship between the problems.

Later in the lesson, students worked to solve 25x16 on their whiteboards. The TE engaged with a student, T., who had begun to use an algorithm to solve, and then the TE addressed the group:

TE:	So, T., before you think about it just with the numbers, everybody look at that picture and think about: How do you change the picture?
NT2:	How is the picture going to change?
TE:	How is that picture going to change? Think about that. That is what we want you to do right now. Work on how is it going to change when we get to 25x16?

The TE was modeling the idea of pressing children to engage with the target mathematics. As 5^{th} graders, many of the children will be able to solve a 2- digit by 2-digit multiplication problem using the standard algorithm. But learning to solve this problem is not the goal of the string. The goal of the String is to support students in seeing the relationship among the problems, in noticing that if one factor is double and the other is halved, the product will remain the same. The picture offers a way to prove that this is true. So as the children began to solve the problem, the TE modeled making sure that their attention was on the picture. About 45 seconds later, as some children finished their work on this problem, the TE modeled pressing again.

TE: If you think you did it with numbers, try to do it with the picture now. We had 8 groups of 50. How do we make 16 groups of 25?

She was modeling the importance of being a bit unrelenting in this orienting work. Even though they have been prompted once, some of the children did not take up the prompt. So as they finished, she prompted them again.

When all of the children finished, the NTs invited a student, A., to share his thinking about the problem. The TE offered another interjection designed to heighten the children's attention to the target idea.

NT2: I notice that A. had an interesting way to solve this problem. Would you like to explain to us what you did?
TE: Why don't you go up there [to the chart] and change the picture for us. And let's really look at A. and see if you had a similar idea about how to change the picture […]. Let's watch A. and see how he is going to change the picture.
A: It is dividing by 2 [pointing to the numbers in the equations]. It is 100 divided by 2 is 50. And then 50 divided by 2 is 25. And right here are doubling, so 4 times 2 is 8 and 8 times 2 is 16. And down here [gesturing to the picture] it is 50 divided by 2. So you just put [inaudible] [drawing a line through the model to divide each group of 100 into fourths].
TE: So why is he drawing that line? What is he doing when he draws that line? C.?
C: Well he's dividing the circles into four equal groups.
[November 15, 2011, Group 1, String Enactment]

Here again, the TE modeled keeping student thinking oriented to the target mathematics. She invited A. to the board to represent his idea publicly and then modeled orienting the children to his work by asking them to articulate why he did what he did.

In this enactment with children, the TE made ten brief insertions into the flow of the lesson and eight of them were related to the work of orienting children's attention to the mathematical goal of the String. She modeled: probing student answers to surface the idea that multiplication means 'groups of groups' at the beginning of the String, highlighting the fact that the product stays the same in each problem in spite of the change in the factors, pressing children to use the target relationship to solve a problem, and orienting children to one another's work as the target strategy is being used. These moves were all fairly small, but they point to the way in which the work of orienting children to the mathematics needs to be steady and constant in orchestrating productive discussion.

The point of offering these examples is not to suggest that this is the way in which or the topic on which teacher educators should intervene in enactments with students, nor is it to suggest that these kinds of insertions are the best vehicle for supporting novice teacher learning. I offer them because seeing where insertions seem necessary offers insight into the work of leading productive mathematics discussion with children. The fact each time the TE feels compelled to make an insertion into the lesson, she does so with the intention of orienting the kids to the math, reminds us of how important this orienting work is in expert practice. And it highlights that enactments offer teacher educators opportunities to further model expert practice. Watching these enactments made clear that in productive talk, most of the moves that an expert teacher makes will, in one way or another, draw attention to key mathematics. It also highlighted the challenge of this focusing work for novices and the opportunity that the String activity offers to engage with this skill.

In this chapter, I have offered examples of the some of the ways in which the practice of leading productive mathematics talk was worked on in the enactments of the String activity that took place with children in the mathematics methods course. Interactions with children around the problems in the String allowed for novices to encounter the work of managing unanticipated responses from children. Children offered errors, partially correct ideas and incomplete ideas in addition to their correct and complete ones. Novice's efforts to respond to these inputs highlighted three practices related to unanticipated responses. The first was the purposeful selection of call-on routines that can support maintaining some control for novices over what enters the conversation. The second was the practice of helping children pool partial understandings (rather than trying to question an individual child towards understanding) by asking them to build on one another's thinking. And the third was supporting children's understanding and capacity to articulate the goal through the use of the model throughout the String. The enactments with children also offered the teacher educator opportunities to model the necessary constancy of the work of orienting children's attention to the target mathematical ideas throughout the String.

In the next chapter, I will offer my findings as I looked across the pedagogies of rehearsal and enactment together.

CHAPTER 6: WORKING ON ORCHESTRATING PRODUCTIVE MATHEMATICAL DISCUSSION ACROSS REHEARSALS AND ENACTMENTS OF THE STRING ACTIVITY

In the previous two chapters, I have described findings from separate analyses of rehearsals and enactments. In this chapter, I will offer briefly what I learned by setting rehearsals and enactments side by side. I sought to understand how rehearsals supported the work of enactment and what the work across these settings can tell us about the work of learning to lead productive mathematical discussion.

I will offer two findings from looking across rehearsals and enactments:

1) Uptake in enactments of the work done in rehearsal is high. Enactments reflect plans, procedures, and practices that were engaged in rehearsal.

2) In spite of the high uptake of rehearsal work, enactments with children rarely played out as rehearsed. There were three main causes: 1) Adults in rehearsal offered on-target answers and ideas almost exclusively where children in enactments did not; 2) Although novices took up most of what was rehearsed, sometimes they did not; 3) Although the end of the lesson was often rehearsed, novices frequently ran out of time before getting there in enactments.

I will offer examples that show the uptake of rehearsal work. And then I will offer examples of the ways in which rehearsals and enactments differed.

Uptake of rehearsal work in enactment

Looking across any pair of rehearsals and enactments shows that work done in rehearsal shaped what the novices did in enactments. There were a total of 41 String activity segments that I could code for the degree to which novices took up what had been rehearsed as they enacted the lessons. (It occurred in a number of enactments that novices ran out of time before posing all of the problems that they had rehearsed. So these segments could not be coded for uptake.) I had coded the uptake of rehearsal work for each segment of the lesson as high, medium or low, only counting uptake as high in segments where the novices stuck very close to what they had planned for in rehearsal. I was coding in these segments for aspects of the work that could be planned for, and not for aspects which would be contingent upon student inputs. In 35 of the 41 lesson segments, uptake of the work that was done in rehearsal was high. What is reflected in this uptake in particular, is the detailing work that was done on the structure of the lesson in rehearsal: how exactly to phrase the introduction, how to phrase questions, how to set up the board, when to try turn-and-talk, how to represent the mathematical idea visually and so forth.

The tables below show examples of what I coded as high uptake drawn from three pairs of rehearsals and enactments. These examples typify the ways in which rehearsal work was reflected in enactments. On the left is a transcript from of the rehearsal of a segment of a String (such as ‘Introducing the String,’ or ‘Posing the helper problems’) and on the right is a transcript of that same segment as it played out in enactment. Places in each setting that show uptake are italicized.

In the first example, the teacher educator and the novice worked in rehearsal on how to get the lesson started, and on how to phrase a question that would begin to engage children with the visual model. The teacher educator suggested the wording for a brief introduction to the String that would give children access to the work and suggested a way of prompting orientation to the array. In the enactment, the novice teacher took up both of these suggestions closely.

Introducing the String: Rehearsal	**Introducing the String**: Enactment
NT: OK guys, today we are going to work on multiplication and we are going to use arrays. Does anybody remember what an array is? [A two minute conversation about arrays ensues.] TE: OK can I stop you? What is happening to this intro? NT: We kind of went off on arrays. TE: OK so we want them to know what an array is [inaudible]. *So save the question for when you put 10x15 up. Say, 'I am going to put up an array for this. Why am I calling this an array?' ...So try to think, 'OK in this rotation, we are going to work on some multiplication problems. We are going to think about how one problem can help us with the next. I am going to put up a problem, I want you to think about what it is and give me a thumb.' That is a 30 second intro, which is what you need to maximize instructional time.* [November 9, 2010, Group 3, String Rehearsal]	NT: *Today we are going to work on some multiplication problems and we are going to use easier multiplication problems to solve harder multiplication problems. So I am going to write a problem on the board and when I write it up there, I want you to think in your head. Try to come up with an answer as quickly as you can but quietly. And when you have an answer I want you to put your thumb up. All right here we go.* [Poses the first problem and elicits responses.] *NT: OK I am going to put up an array for 10x15. Can anyone tell me why this is an array for this fact.* [November 9, 2010, Group 3, String Enactment]

In the next example, the TE and the NTs talked together about the purpose for the first problems in the String and about how to pace the beginning of the lesson to reflect

Posing helper problems: Rehearsal	**Posing helper problems**: Enactment
NT1: [Having posed the first problem in the string] R. I see that you have your thumb up. Can you share your strategy for 6x20? R: Yeah. I did 6x2 and then since 20 is 2 tens I did 6 times 2 tens and that is 12 tens so it is 120. [A nearly 2 minute conversation ensues about her strategy and representing it.] TE: *One thing that you can think about in a String is that you don't necessarily need to ask for strategies on the first [problem.] Because—* NT1: *The first one we can just kind of go—* TE: *Because the idea is that you are giving them problems that are familiar to them. So you can just say, 'What are your ideas? Shout them out.' And we can all say 120 so you can hear that we all got the same one. If you heard lots of different numbers then you can say, 'OK I am hearing 120 and 140, let's check and see which one it is.' Because you really want to move—* NT1: You want to get to the meat— TE: You want to get to the meat. NT2: *So would we do that for the other problem in the string too—for 6x100, too?* TE: Yeah. NT1: Same thing for that, kind of go through it quickly? TE: Yeah because it is really about how those two help you with the other ones. NT1: OK that makes sense, so I can just say, 'Put your thumb up. *OK all together now...*' [Nov 8, 2011, Group 2, String Rehearsal]	NT1: [Having posed the first problem in the String]. *So what is our answer?* Ss: *120.* NT1: *120.* [Writes the product on the board.] TE: You know what, A. drew a really nice picture for us too. She drew six groups of 20. NT1: *Oh yeah, how many groups of 20 do we have her? 1, 2, 3, 4, 5, 6. Hmm. Nicely done.* NT2: *OK here is another problem. Once you figure this problem out, show a thumbs-up. [Writes the next problem and waits.] All right, so let's say this all together. What is 6x100?* Ss: *600.* NT2: *And I see A. drew groups of 100. So we have how many groups of 100?* Ss: *6.* [Nov 8, 2011, Group 2, String Enactment]

that purpose. The teacher educator suggested that the novices not stop after the first couple of problems to elicit strategies for solving them because they are problems with which children are already fluent. In the enactment, the NTs took this up and moved quickly through the first couple of problems stopping only to highlight a representation of multiplication as groups of groups that a student had offered and that would support the goal of the String. Where in the rehearsal, novices spent several minutes getting the first problem out, in the enactment, they got the first two problems out in a minute or so.

In the next example, the teacher educator in rehearsal offered the novice a way to orient students to the mathematics of the String: by directing their attention to how the visual model changes from problem to problem in addition how the numbers change from problem to problem. Novices took up the orienting work suggested by the teacher

Connecting Problems: Rehearsal	**Connecting Problems**: Enactment
NT1: I want you to look at this first problem and think about how it could help you with the second problem [Poses the second problem.] TE: And they might just know this problem […] But the point here is to say, 'This might be a number that we know, but let's try also to think about…is there a way to change the array from six times eight to twelve times four? And what is happening to the array?' Even if this is a fact they know, you can still work on the relationships between the problems. [NT1 elicits solutions and strategies including the doubling and halving relationship.] NT1: OK let's put up an array to see if we can see that halving and doubling. TE: Yeah, so you could put this [array] up and say, 'Huh, I wonder. What did we do to this one to make it into this one?' [….] This might be a place for a turn and talk and then it would be a place to cut that first [array] and making it fit [over the second one.] [November 9, 2010, Group 3, String Rehearsal 2]	NT1: So let's look at the second problem. And I want you guys to think, 'How could I use the first problem to answer the second problem?' NT2: So we are going to have you […] turn and talk to each other about how you would use this problem to help you solve this one. Everyone is going to get a board and you are going to work together […] [Poses the second problem. Students solve problem in pairs as NTs move around.] NT2: […] So I am going to put up an array now of twelve times four, and I want you to look at this for a moment. Think about it. How did you use this problem to help you solve this problem? […] H: Well six is half of 12 and 4 is half of 8 [referring to the doubling and halving relationship between the problems.] [Other children share this same observation.] NT2: OK so we have heard three people share that strategy. How would you use this array of twelve times four to do what H. and M. and C. said? Would you think about that and then when you have an idea put your thumb up. How could you change this array to look like this array? [November 9, 2010, Group 3 String Enactment 2]

educator, probing the students in enactment for how the problems were related to one another and then for how the array changed from one problem to the next. These three examples were typical of the way that consultations between novices and the teacher educator in rehearsal were reflected in enactment. What is important to note is that these examples all come from places in the lesson that were the least contingent upon student responses. They were aspects of the lesson that generally did not need as much adjusting in the moment and so they could be done as rehearsed. An introduction, the posing of an initial prompt, or a planned moment for a turn-and-talk can be executed as rehearsed when children are there because they are not dependent on what students say or have said. Looking across rehearsal and enactment, these aspects of the lessons almost always fully reflected the work that was done in rehearsal. And at the same time, the lessons usually went very differently from the way they were rehearsed.

Differences between rehearsals and enactments

There were three common things that happened in enactments that caused them to play out differently from the way rehearsals had. The first was that adults offered complete and on-target responses when they played the roles of students in rehearsal. But, as noted in Chapter 5, children in enactments did not always do this. Second, occasionally, in the stress of enactment, novices forgot what had been rehearsed and did something different. And third, although the end of the lessons were usually rehearsed, and is usually where the target mathematics is addressed in a String, frequently in enactments novices ran out of time before getting to the target problems in a set.

Adults offered target ideas in rehearsal where children did not

One common way in which enactments went differently from rehearsals was that adults in rehearsals almost always responded to prompts with correct answers, and in particular, answers that supported the goal of the String, when they were playing the roles of 'students'. Children of course do not always do this and so, as discussed in Chapter 5, novices were left managing unanticipated ideas.

This occurred in a rehearsal and enactment pair from November of 2010. The novices posed the first problem in the String, 6x8, elicited the answer from the group, posted a 6x8 array and prompted the group to talk about the array.

Rehearsal	Enactment with children
NT2: Who can show me how this is an array for 6x8? NTaud1: I think that there are 1,2,3,4,5,6 boxes in this column and there are 1,2,3,4,5,6,7,8 columns and that this 6 is represented 8 times [running her finger down each group of 6]. NT1: B., can you repeat what P. just said? NTaud2: She said that there are 6 on one side and 8 on the other side so there must be 8 sixes. [November 9, 2010, Group 3b, Rehearsal]	NT1: Who can tell me […] why is this an array for 6x8? R.? S1: Because it is six by eight. NT1: OK so you are seeing….[writing on the array] 1,2,3,4,5,6 and 1,2,3,4,5,6,7,8. Is that right? S1: Yeah. NT1: So how many squares are in there? S1: 48 NT1: OK [looking perplexed], does anyone see it a different way? Ss: [Silence.] NT1: So this is like 1 group of 6, right, 2 groups, 3 groups [running her finger along the groups]… Everyone see that?

The novices needed to get out the idea of groups in talking about the array and, in the rehearsal, the adults offered exactly the explanation of the array that was needed: that the array represented 8 groups of 6. But in the enactment, although they asked the question just as rehearsed, something different came up from students. When they asked students to talk about how the array connected to the problem, the student correctly offered that the problem 6x8 represented the dimensions of the array saying, "It is six by eight." The NT took this, marked it and then, looking perplexed because she realized she did not hear the idea that she needed, attempted to get someone to "add on" to his idea. But no one did. This left her showing the students the groups in the model herself.

In another example, novices posed the problems 6x20 and 6x100 and then posed 6x120 hoping students would use the first two to solve the third. They gave students time to think and then to talk to a partner about what they did. The NT had rehearsed listening in on students' talk and then selecting a student to share who had used the target strategy. She did so in the enactment as well. And she had rehearsed having a student repeat the target strategy to highlight it. In the rehearsal, an adult offered the target strategy, and then another adult readily repeated what had been said. In the enactment, a student shared the target strategy, but when the NT tried to have a student repeat it, he has not heard it. She had the student say the target strategy again. And again tried to follow up with another student.

Rehearsal	**Enactment with children**
NT1: S…I heard you and R talking about some great strategies. Do you mind sharing it with us? NTaud1: Sure, can I come up? [Comes to the board.] I knew that this [120] was actually [….] that I could separate this [6x120] into 6x100 and 6x20. And so I knew that I could use these answers, 6x100=600 and 6x20=120, if I added these two together, that would get the answer. NT1: […] This would be a good place for 'revoicing.' […] A. can you help us figure out how S got the 6x100 and the 6x20. NTaud2: S saw that the 6x120 could be broken up into two chunks: 6x100 and 6x20. [November 8, 2011, Group 2, String Rehearsal]	NT1: I heard you had something really interesting, how you figured that out. Will you share it with us? S1: Well, if we are doing 6x120, I split up 120 into 100 and 20 and decided that 6x20=120 and 6x100=600 and 120+600=720. NT1: D, did you hear how she did that? S2: No. NT1: OK I am going to have her share it one more time to you. Will you listen this time so you can hear? TE: She did something kind of cool. Let's see what she did to the 120. [S1 comes up to the board and shows again what she did.] NT1: D, where did she get 20 and 100? D: From right there where you just wrote it [referring to the numbers on the board]. NT1: Oh, so you saw 20 and 100 here. TE: It looks the same doesn't it? NT1: S, do you know where exactly down here, where did this 100 and this 20 come from? What number did this come from? S3: Up there [pointing to the numbers on the board]. NT1: So you are seeing it up there too. And what is 100+20? [Silence.] Does anyone know what 100+20 is? N? S4: 120. NT1: 120. Which I see right here. [November 8, 2011, Group 2, String Enactment]

When she probed a student to restate where S1 had gotten the 100 and the 20, he replied with "from right where you just wrote it." She tried three times to get someone to articulate that the 100 and the 20 are inside the 120 and can be used to solve it, but the students on whom she calls are not able to do so in that moment and so the NT asks a series of closed questions that led students there.

It was common across rehearsals and enactments for adults to have supplied ready-made, correct, and complete responses that lead straight to the goals in rehearsals and for children to have created a much more circuitous route to the goal for novices to navigate in enactment.

Novices did not always do what was rehearsed

Sometimes it occurred that an enactment went differently than rehearsed because the novice did not do what was rehearsed. This was relatively infrequent, and when it happened, the teacher educator usually intervened with a reminder about what had been rehearsed. An example that has been used in previous chapters can be used here again to point to the way that sometimes novices lost track in an enactment of what had been rehearsed. In November of 2010, novices and the teacher educator had rehearsed directing the students to attend to the change in the array rather than to how one problem might help to solve another. This prompt was more relevant to the particular String. But in the enactment she lost track of the change they had made. The TE stepped in quickly to remind her of the question they had planned for.

Rehearsal		**Enactment**	
NT2:	So should we have talked about this [problem] being helpful to this [problem] before we posted this [array]?	NT2:	Did anyone use this problem [3x4] to help them solve this problem [3x8]
TE:	Not necessarily.	Ss:	No.
NT2:	OK. So this is an array for 3x8…[To TE] And should I go over having someone come and show what 3x8 is?	NT2:	No? You just knew this problem…
		TE:	How are they related though? What happens to the array?

TE: You know at this point, this is where you want to start getting them to make the connections, right? [Comes to the board][…] And you want to line these up [moving the arrays on the board so they are one under the other with left edges aligned] so that you can make the comparison. 'What happens to the array as we move from 3x4 to 3x8.' NT2: [Trying the prompt] So do you notice anything once we went from 3x4 down to 3x8? Do you notice anything? [November 16, 2010, Group 4a, String Rehearsal]	NT2: Do you see any relationship between these two? [November 16, 2010, Group 4a String Enactment]

Pacing was a challenge

About half of the time, in enactments, things went more slowly than planned at some point in the String, and so the novices did not have time to work with students on the last problems or the goal of the String. Sometimes early problems were not quite as easy for students as planned. Sometimes the work of navigating an unanticipated response took longer than planned. Sometimes students took longer to solve target problems than planned, or took more prompting and directing than planned. This meant that novices did not always, in the enactments, get to engage with the work supporting students to use and articulate the goal of the String.

In this chapter, I have offered a few images of what emerged when I looked at the work that was done across rehearsals and enactments. I noted that novice uptake in enactments of the work that was done in rehearsal was high, particularly for aspects of the lessons that could be planned for, such as how to word things and when to use turn and talks. I also noted that enactments offered the teacher educator the opportunity to model the practice of orienting children to the mathematics of the lessons. This emerged as particularly salient in enacting a String, and particularly challenging for novices to do. And lastly, I noted that although uptake of rehearsal work was high, enactments rarely played out exactly as planned. Children offered ideas that had not been rehearsed for (noted in chapter 5 as well), novices sometimes lost track of what was rehearsed, and they frequently ran out of time to work on target ideas with students.

In the next chapter, I will offer a discussion of my findings.

PART III: DISCUSSION AND IMPLICATIONS

CHAPTER 7: DISCUSSION

The literature suggests that effective orchestration of classroom mathematical discussion can serve to broaden who participates in the mathematical work at hand, can support the development of conceptual understanding of and procedural fluency with important mathematics for a broad range of students, and can support a broad range of students to see themselves a people who do mathematics. But the literature also points to the substantive challenge of enacting the practice to meet such aims. I sought in this study to contribute to the specification of the practice of leading productive discussion and to the developing understanding of pedagogies of teacher preparation that might support novice enactment of such a complex practice.

Specifying the work of leading a productive mathematics discussion

Based on my analysis of the String activity, and of the work that novices and the teacher educator did on the activity in both coached rehearsals and in enactments with children, I offer a representation for thinking about the work of leading a String productively in Figure 9. It builds on my initial framework that focused leading productive mathematical discussion as a function of the work of supporting broad participation in classroom mathematics, the attainment of important mathematics, and the development in children of productive disciplinary relationships. I attend to these areas because they offered me a way to think about how the pedagogy of this course intersected with the aims of improving learning opportunities in mathematics for a broad range of children. Although in one sense, participation, attainment, and disciplinary engagement by children are aims of teaching, or intended outcomes, supporting them encompasses key teaching practices. It is teachers who enable children to participate. It is teachers who steer instruction to meet curricular goals. It is teachers who create environments in which children see mathematics as meaningful and themselves as doers of mathematics. These aims all involve things that teachers do. It seems worth decomposing this work a bit in order see if we can identify teaching practices which support these aims, to link practices with aims. A decomposition like this might offer us a way as teacher educators to help novices begin to link what they do with what happens for children.

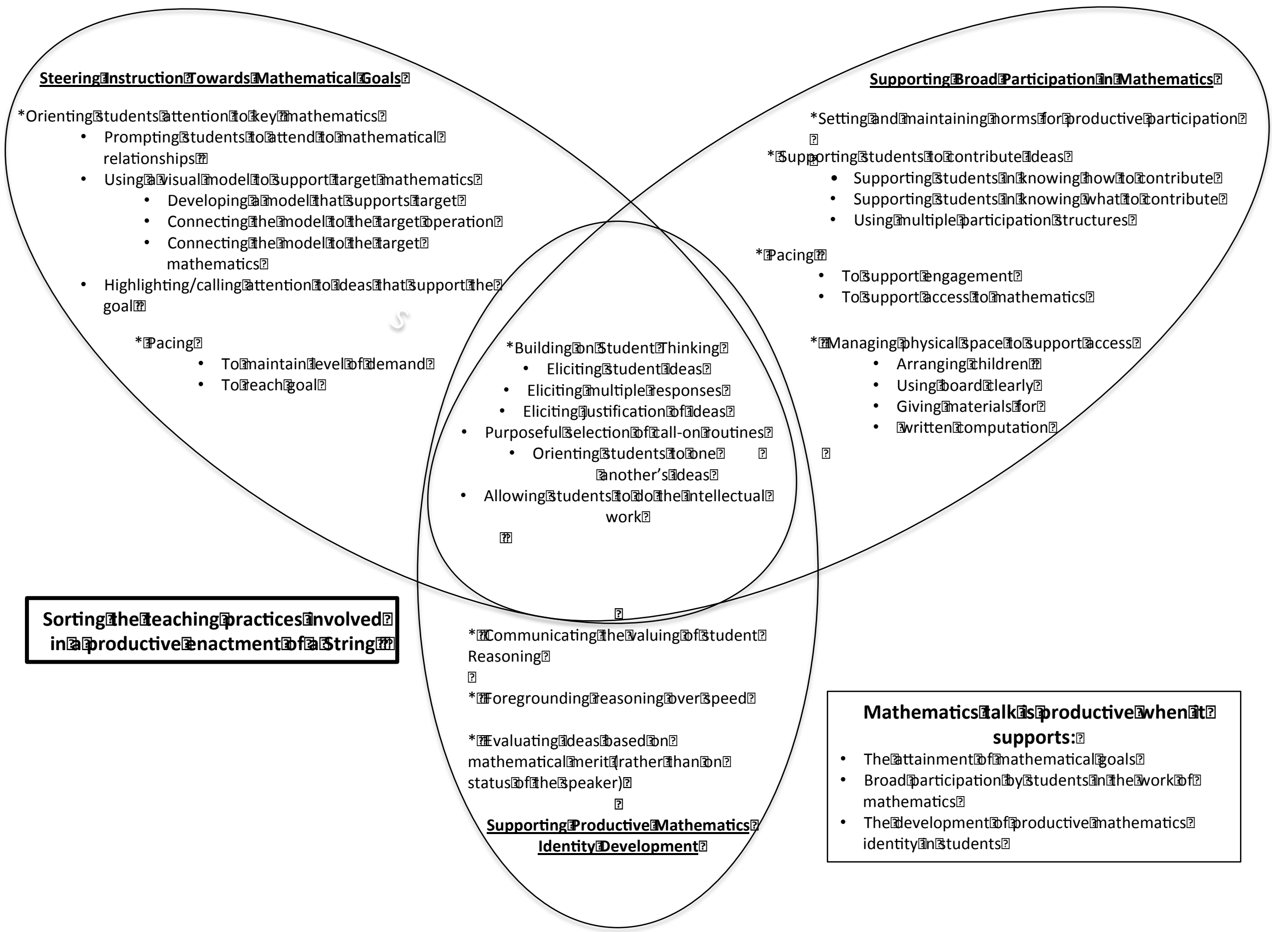
The representation includes four broad, overlapping practices that are encompassed in the work of leading a productive mathematical discussion. They are drawn from my observations of rehearsals and enactments of Strings, from my read of the literature on productive enactment, and from my definition of high quality teaching. They are: *steering instruction towards mathematical goals*, *supporting broad participation in mathematics*, *building on student thinking*, and *supporting productive identity development*.

The work of *building on student thinking* is at the center of the diagram. In considering the work of leading a String in particular and the work of leading a productive mathematical discussion in general, this practice emerged as the one that joins all of the others. It is our aim as mathematics educators to put the development of student thinking at the center of our work. Building on student thinking includes the practices of eliciting student ideas, highlighting key ideas, orienting students to one another's thinking, helping children to add on to one another's ideas, and making sure students are doing the mathematical work. These practices are inside of and support all of the other three aims and practices. We want to help teachers to steer instruction towards curricular goals *through* the work of building on student thinking. One way we have of supporting broad participation in classroom mathematics is placing student ideas at the center of classroom discourse. And one aspect of positioning children productively relative to the discipline is again placing their thinking at the center of attention to help them see themselves as people with important mathematical ideas.

The notion of *steering instruction towards mathematical goals* at the upper left of the diagram is drawn from the work of Sleep (2012). Her research focused on the work of steering instruction through much longer mathematics lessons that often involved multiple activities and materials. But the idea that keeping the target mathematics at the center of attention in nearly every move that is made during the lesson applies to lessons of any length and type. She raises some of the practices that are inside this work, two of which emerged as particularly salient in to the work of leading a String productively: "opening up and emphasizing key mathematical ideas," and "making sure students are doing the mathematical work." In leading a goal-oriented discussion like a String, a significant part of steering instruction is that of directing children's attention to key mathematics. And then all of the practices involved in eliciting and responding to children's ideas in order to build new ways of thinking and new understandings emerged as critical to this practice.

The practice of *supporting broad participation in mathematics* (at the upper right of the diagram) includes the work of setting norms with children for how to participate productively, communicating with children so that they know what and how to contribute to the discussion, pacing the lesson to ensure access to the work, using call-on routines purposefully, orienting children to one another's ideas, and highlighting children's ideas.

Figure 9: Visualizing the practices entailed in a productive enactment of a String



The literature points to the nature of classroom interactions around mathematical ideas as having a critical impact on the *development of productive mathematics identity.* What is communicated to children about how their assets are valued contributes to their developing relationship to the discipline. Evaluating ideas rather than speakers, foregrounding reasoning about ideas rather than quickness, and again placing their ideas at the center of the work all serve to position children productively and help them see themselves as doers of mathematics.

As with earlier ideas presented in this dissertation, this is not a complete or exhaustive representation of the work of leading a String. It is a proposal for a beginning way to specify practice in order to support thinking about the complexity of the work of leading a productive mathematical discussion with children and what it means to next such a practice inside a specific task of teaching.

I will use this representation to consider what I have learned about the affordances of the pedagogies of the mathematics methods course for supporting work on these practices. Using the representation helped me to see what kinds of practices got worked on through aspects of the pedagogy and those that were emphasized less.

Understanding the affordances of pedagogies of enactment

Below is the framework, offered by McDonald, Kazemi and Kavanagh (2013), for organizing teacher preparation around learning to enact a complex core practice such as leading a productive mathematical discussion (Figure 1). I investigated the ways in which aspects of this framework enabled work on the enactment of practice in the mathematics methods course. I investigated the affordances of nesting the core practice of leading productive discussion into the String activity (at the center of the diagram). And I investigated the pedagogies of the coached public rehearsal (at the bottom right of the diagram) and enactments with children (at the bottom left of the diagram) of the String activity in the context of the course. These investigations also raised an idea about the affordances of the work that happens in the upper right of the diagram as novices were introduced to and learned about the activity.

Figure 1: McDonald, Kazemi, & Kavanagh, 2013

The structure of the String activity enables work on the practices of steering instruction towards mathematical goals and supporting participation in mathematics.

I investigated, first, the way in which embedding the core practice of leading a productive mathematical discussion into the String activity enabled work on the practice. The nature and structure of the String activity creates a particular set of demands on the teacher. It is the work of navigating these demands that opens up opportunities for novice engagement with complex practice. In the String activity, the relationships among a set of computation problems becomes a prompt for a collaborative discussion among children and the teacher. The teacher needs to orchestrate the discussion in such a way as to build on student ideas to bring key mathematical relationships to the surface.

As a mathematics task for use with children, the String activity has key features suggested by the literature as supporting productive aims. It is designed to build on what students know, to highlight meanings and mathematical relationships, to place a high level of cognitive demand on students, to involve multiple models of mathematical ideas and to be collaborative. But it is the very features that make it a high quality task that make it incredibly complex to enact well. Orchestrating the discourse around the task in such a way as to engage these features is nuanced and detailed work because, as the literature suggests, it is possible to use a task that is designed to highlight mathematical meaning without doing so (Cohen, 1990; Kazemi & Stipek, 2001) and to enact a task that could place a high level of cognitive demand on students in such a way that the demand is lowered (Stein, Grover & Henningsen, 1997).

Because a String is a goal-oriented activity, it offers particular emphasis on the work of orienting children to the mathematical goals throughout the discussion. Making sure that the target mathematics comes to the surface as children offer a range of answers to or strategies for solving computation problems entails intentional work on the part of the teacher. Three practices emerged as particularly salient to orienting students' attention in a String. The first is the practice of prompting students to attend to mathematical relationships. One way to orient children's attention is by cueing them explicitly—by saying in essence, "Pay attention to this." In the String activity this involves prompts such as, "Consider how this problem might help you solve this one," or "Think about how this problem is related to this problem." Another practice is that of using a visual model to support student understanding. In leading a discussion about a String, this involves helping children connect the visual model (such as an array) to the operation being worked on in the String (such as multiplication) in such a way as to highlight the meaning of the operation. This involves engaging students in discussion of the model that will surface features of the model that are important to the goal of the String, for example, helping children notice that the array shows groups of groups. And it also involves helping children connect the model to the target relationship such as using the array to show the removal of one group in using a compensating strategy.

Because a String is designed as a collective discussion, it also opens opportunities for working on supporting broad participation. The structure of the activity is designed around the elicitation and use of children's ideas about how to solve the problems in the String and about the mathematical relationships they show. As such, it structures opportunities for novice work on the key practices of building on student's ideas. Engaging in the activity necessitates deciding when and how to eliciting children's ideas and when and how to respond to them.

But work with children on the String activity is a very particular instantiation of a mathematical discussion. Because the activity has such a narrow mathematical focus, it places a substantial constraint on the way in which children participate and on the range of ideas that they might bring to the conversation. Although in theory, multiple strategies and ideas would be considered when a String is enacted productively, the teacher will always be homing in on particular ideas in order to get at the mathematical goal. Strategies that emerge and ideas that may be quite valid and useful but that do not support reaching the goal will receive less attention that those that do. Also, the String that is selected will by definition constrain who can participate in the discussion. Where good discussion prompts often have multiple entry points in order to support participation by a broad range of learners, the String activity does not. If the opening problems in the String (the "helper problems") are not in the range of what some students are able to do, then they are unlikely to be able to participate fully in the discussion.

An instructional activity will always be a particular instantiation of the core practice is it intended to support. Understanding the affordances and constraints of an activity relative to a core practice seems important as these pedagogies are developed.

Work that happens in the process of introducing and learning about the activity can shape what happens in the next phase of the cycle.

It emerged in analysis of rehearsals that what happens in the upper right hand corner of the cycle of investigation enactment (McDonald, Kazemi, and Kavanagh, 2013) (Figure 1) can shape what happens in next phase in important ways. The ways in which novices were first introduced to and learned about the instructional activity of Strings shaped what novices were able to do and what could get worked on in the rehearsals of the activity that followed. Grossman and colleagues (2009) call the tools that get used in that upper right quadrant "representations of practice." In the case of this course, the activity was modeled by the teacher educator in this phase; novices got some form of protocol for the activity; and they engaged in planning discussions with their peers. These representations in the second year of the study were adequate to support novices to offer productive introductions to the String and to engage with much of the flow of the lesson independently. In the first year, in contrast, work in that upper quadrant had been somewhat less, and so the work of learning the flow of the lesson needed to happen in rehearsal.

Representations of practice have been the predominant pedagogies used in teacher education in recent decades. Mathematics education has focused on the use of written cases, videos of classrooms, engagement in teacher educator modeling of mathematics teaching, and novice engagement with instructional materials and mathematics activities. We have come to see more recently that while these activities do support changes in novice's ideas about mathematics teaching and learning and do support the development of their own mathematical understandings, these representations have not, on their own, effectively supported the enactment of practice. But this small finding gives us insight into how such representations of practice can be linked to other pedagogies to play an important role in novice work on the enactment of complex practice.

Coached public rehearsals enabled the work of detailing key practices of productive enactment

The rehearsals of the String activity in which the novices and the teacher educator engaged enabled work on what I called *detailing* practice. Detailing involved careful consideration and planning for exactly what to say, how and when to say it, where to stand, what to write, where to write it, and so forth. Watching the rehearsals highlights how critical this detailing work is to supporting novice enactment of practice. It is one thing to list "eliciting student thinking" as a key practice of high quality teaching. And it is another to specify exactly when in a particular lesson to elicit thinking, how to select from among numerous ways of eliciting the most appropriate for the moment, and then what to do with the range of possible responses. Novices begin often with a single kind of elicitation: asking a question, waiting for students to raise their hands, calling on one student, and then evaluating the response as right or wrong (Cazden, 2001). Rehearsals of Strings offered the opportunity for novices and the teacher educator to detail a variety of elicitation strategies.

They worked on having children call out answers chorally, calling on multiple students one at a time, or asking children to talk to a neighbor and then selecting purposefully who would share. Rehearsal time was also spent, in particular, detailing the work of steering instruction towards mathematical goals. Novices and the teacher educator worked on the details of the visual models that would be offered, the ways in which to help students make sense of and use models, and the ways in which to prompt students to investigate connections among the problems.

I would argue that work on detailing practice at this level is a key aspect of work on the enactment of practice with novices. We know that pedagogies of investigation can offer novices access to understanding mathematics better and to articulating productive visions of mathematics instruction. But we also know that on their own, they seem to leave novices unable to realize complex teaching practice in the classroom (Borko, Eisenhart, Brown, Underhill, Jones, Agard,1992; Ensor, 2001). The opportunity to put the meat on the bones of practice, as it were, to help novices think through exactly what will be involved in the *doing* of teaching is one of the central affordances of the pedagogy of rehearsal.

Enactments enabled novice engagement with the work of managing unanticipated responses, gave the teacher educator opportunities to model practice, and surfaced the challenge of pacing

Enactments of the activity that immediately followed each rehearsal gave novices the chance to try out with children what they had rehearsed. The uptake of the work that happened in rehearsal was universally high. Novices introduced the lessons as planned, offered the visual models they had developed, asked children to turn and talk in the places they planned to do so, phrased questions as suggested. This all suggests that rehearsals can shape proximal practice in critical ways. And it also emerged that enactments played out differently from rehearsals almost universally. Enactments gave the novices opportunities to encounter responses from children that had not been anticipated in rehearsal. Children offered errors, responses that were partially correct and partially incorrect, and responses that were correct but incomplete. Novices needed to make decisions in the moment about how to manage these responses. They were sometimes successful in using errors and partial understandings to move the conversation forward, and sometimes less so.

The work of orchestrating a String so that it uses children's ideas to build towards a goal is partly improvisational. While the structure of the set of problems in the String and the protocol for moving through them provides a scaffold for movement towards a goal, children bring a wide variety of ideas, understandings and confusions to the conversation. The work of managing unanticipated responses cuts to the core of this improvisation. It seems that encounters with this improvisational work is one of the affordances of the pedagogy that is unique to enactments.

Enactments also offered the teacher educator the opportunity to model practice. She offered insertions into the lesson posing questions to the students, making suggestions to the novices, prompting and probing student thinking as the lessons unfolded. It turned out that the vast majority of these insertions dealt with the work of orienting children to the mathematics of the String. This aspect of the work again emerged as critical to productive enactment of the lesson and challenging for novices.

Lastly, enactments surfaced the challenge for novices of pacing lessons to reach intended goals. In many enactments novices got waylaid early in the lesson and could not reach the end of the String before the time was up. Pacing is a common challenge for novices in all aspects of teaching, and it would seem that enactments in the context of a methods course offer a unique opportunity to engage with it.

What was in the foreground and what was in the background in the pedagogy

Looking again at Figure 9 in which I parsed some of the practices entailed in leading a String productively, what is notable is the amount of work that happened in rehearsals and enactments of Strings around *steering instruction towards mathematical goals*. I speculate that there are two reasons for this, both of which raise important ideas related to developing pedagogies of enactment. The first is that because the String activity is structured as goal oriented mathematical discussion, enacting it productively will by definition entail the work of directing children's attention to the target mathematics. This again raises the idea of the importance of the nature of the activities selected as containers for core practices in developing pedagogies of enactment in teacher education. The nature and structure of the activities selected will drive the teaching practices that can and will get worked on through the pedagogy. The second reason that I speculate that so much attention was given to the work of orienting children to the mathematics is that working towards enacting the lessons with children in the context of the course creates a certain urgency to make the lessons successful as lessons. In the brief time that novice teachers and the teacher educator have to rehearse, getting down the sequence of moves that will get the goal of the lesson out becomes paramount, and may, of necessity, dominate the work that happens. Learning to move a group of children towards a goal through discussion is a central aim of the course, and is critical in high quality mathematics instruction. But it means that the equally important work of supporting children's participation, and attending to children's identity development may be getting worked on less explicitly—at least in the rehearsal and enactment settings.

The work that happened in the upper left hand quadrant of McDonald, Kazemi, and Kavanagh's (2013) diagram in this course, the debriefing and reflections on the lessons that happened after enactments, were not the subject of this study. But it may be that work on who participates(ed) and how in the lessons and what could be done about that next, as well as how children are being positioned by the work, could be thought of as the central aim of the pedagogies that sit in that quadrant. It may be that in a course structure like this one, in which children will be present in the context of the methods course, the work that happens in rehearsals and enactments will focus most explicitly on getting a lesson up on its feet and on interacting with children's mathematical ideas in the moment. The work of considering and planning to support children's participation may well sit most fruitfully in the domain of the analysis of enactments that occurs afterwards.

Considering how the pedagogy intersected with equity aims

It was a central goal of mine in undertaking this study to consider whether and how the pedagogy of teacher preparation that was being developed in this course might intersect with the aims of the improvement of learning opportunities in mathematics for children from non-dominant groups. What I was able to do was to set both the String activity and the teaching practices that it entails against the aims of supporting broad participation in classroom mathematics, the attainment of important mathematical ideas for a broad range of students, and the development by a broad range of students of productive mathematical disposition. This helped me locate many of the teaching practices entailed in enacting a String in the context of their purpose relative to children and to notice the categories of practices that were foregrounded in in rehearsals and enactments.

But what I notice is that framing these brief episodes of novice teacher learning with the large aims of equitable improvement of mathematics was inadequate to make any substantive claims about work on equity in the course. I noticed that the bulk of both my attention and the attention of the participants in the course in rehearsals and enactments of Strings was on what we might think of as the nitty-gritty details of teaching. This is partly because the videos I analyzed represented first efforts at teaching for so many of the students and so, of necessity, dealt with very concrete needs for novices. Keeping the importance of these concrete needs in focus right along with the importance of much broader cultural reform is profoundly challenging. But also, analyzing very small slices of teaching—episodes of 15 minutes or so—is inadequate to speak to the work of equity in any substantive way in the absence of a framework which would help us locate these micro-interactions in a larger picture. It is my hunch that it is repeated patterns of interaction that occur over time between and among children, teachers and the discipline of mathematics that matter the most to developing equitable mathematics classrooms. It is children's repeated exposure to mathematics as a meaning-making discipline that matters to how they see mathematics rather than their participation once in an activity that highlights meaning. It is children's repeated experiences with having their thinking placed at the center of attention that matters rather than single episodes of teachers eliciting student ideas. It is repeated invitations and robust support for productive participation that broadens who participates, rather than one time exposure to open-ended prompts.

So what we are able to consider in a study such as this is whether and how learning to interact with children inside an activity like Strings might offer novices templates for patterns of interaction that can over time open up learning opportunities for a much broader range of students in elementary mathematics. But it would require a much richer framework to do so and data that might help us think about who these novices become as teachers over time.

Limitations of the study

This was a study of one instantiation of a pedagogy of enactment in mathematics teacher education. So it represents only one view of how such a pedagogy might come to life and what its various aspects afford. But it offered me the opportunity to look very closely at the details of this instantiation. It is also a teacher education practice to which I am very close, having been involved in the course from its inception. This makes it likely that I do not see in it what others might. But it also offered me fluency with and a deeper understanding of the data than I might have otherwise been able to attain. Because of the nature of the data, this study is not able to make claims about whether any of this elaborate work in teacher education helps novice teachers to enact high quality practices in their own classrooms as they become teachers of record. I am only able to use this data to offer images of what was worked on the course, and hope to contribute to theorizing in the field about the ways in which we can create opportunities for novice learning.

Directions for future research

It is not yet clear what novices carry with them from their work in this course. As a field of mathematics teacher educators, we do know that engaging children in mathematical discussion is not a common practice in elementary schools in this country.

So we know that when we organize mathematics teacher preparation around 'leading productive classroom discussion', we are preparing beginning teachers to do something that they are unlikely to see in the schools in which they will teach when they are done. This raises important and unavoidable questions about the impact of this work. It points clearly to how it needs to be laced together with other aspects of reform to support changes. Next steps for this research agenda would be to understand how what novices learn in this course dovetails with work that is being done in the settings in which they land. Following novices to understand whether, two or three years into their teaching, they are engaging children in meaningful mathematical discussions in their classrooms could inform next steps in the development of pedagogies of enactment in mathematics teacher education.

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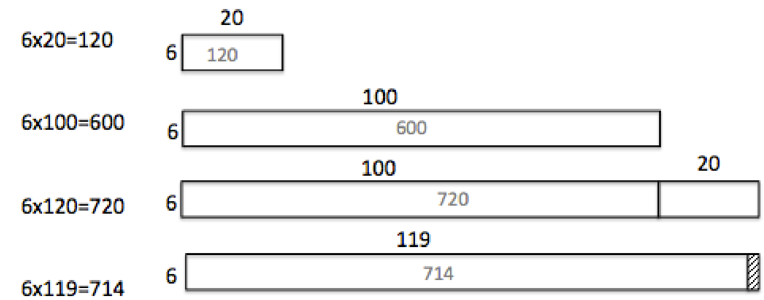
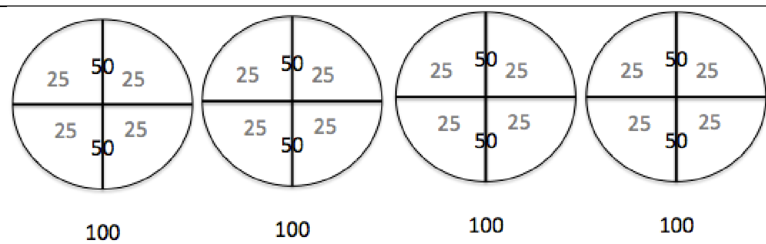
Appendix 1: The mathematics of the Strings used in each class session

String 1:

Problems	Visual Model
6x20 6x100 6x120 6x119	

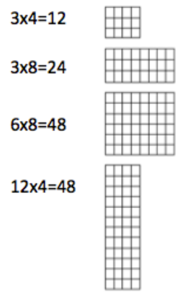
This set of problems is designed to engage children in work on the distributive property of multiplication over addition and then over subtraction. The first two problems can be combined to solve the third given the distributive property of multiplication over addition. The fourth problem highlights the distributive property of multiplication over subtraction. 6x119 can be thought of in relation to 6x120 as one less group of 6 and can be solved by subtracting a group of 6 from the product of 6x120.

String 2:

Problems	Model
100x4 50x8 25x16	

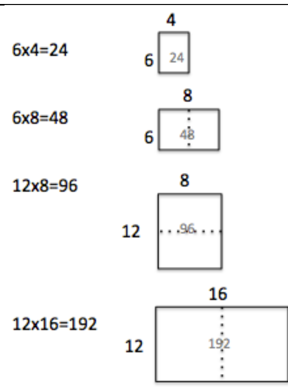
A second String was designed to engage children in work on the associative property of multiplication. It was represented using a set model. Given the associative property of multiplication, 100x4 can be thought of as 50x2x4 or 50x8. And 50x8 can be thought of as 25x2x8 or 25x16. The idea is that the factors in a multiplication problem can be multiplied in any order without changing the product. It also support students' proportional reasoning. As one of the factors is doubled the other is halved, leaving the product the same in each problem.

String 3:

Problems	
3x4 3x8 6x8 12x4	

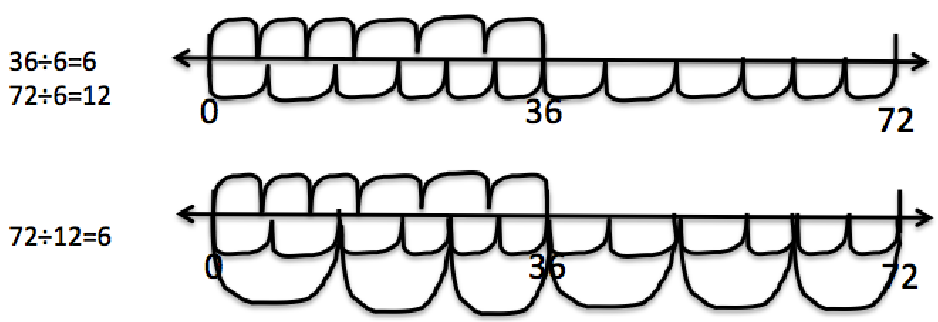
A third string I likewise highlights both the associative property and proportional reasoning. In the second problem one of the factors is double that of the first and so the product is doubled (3x8 can be thought of 3x4x2). In the fourth problem one of the factors is halved and one is doubled and so the product will remain the same.

String 4:

Problems	Model
6x4 6x8 12x8 12x16	

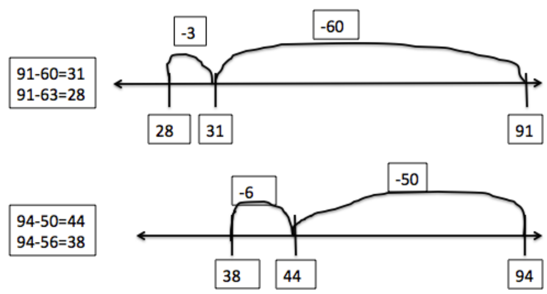
A fourth String was designed to highlight the meaning of multiplication as groups of groups and to help students develop proportional reasoning skills. Doubling one of the factors in a multiplication problem, as between the first two problems, means doubling the number of groups, therefore doubling the product.

String 5:

Problems	Model
36÷6 72÷6 72÷12 144÷24	

A fifth String highlights the distributive property of division over addition and the meaning of division as ratio of part to whole. It is designed to engage children in using proportional reasoning. The first problem should be a fact that children know. The dividend in the second problem is double that of the first. The representation should support students to note that 72÷6= (36÷6)+(36÷6). The third problem can be thought of as having the same ratio as that of the first as both the divisor and the dividend have doubled. The same thing happens in the 4th problem. Both the divisor and the dividend have doubled and so the ratio stays the same.

String 6:

Problems:	Model
91-60 91-63 94-50 94-56	

A sixth String highlights the interpretation of subtraction as removal, and supports children in removing first groups of 10s and then 1s from the minuend for ease of computation. These problems were represented using a number line to illustrate subtraction as the removal of one quantity from another.

Appendix 2: Representation of String Activity novice teachers worked with in 2010-2011 school year

Graph Paper Arrays · D10

Associative Property, Doubling and Halving

This string of related problems is designed to encourage students to use multiplication facts they know to find answers to other, more challenging ones. The string also supports use of the doubling and halving strategy. Do one problem at a time, giving enough think time before you start discussion. Use graph paper arrays and have scissors handy to cut them to match students' strategies. For example, if a student says, "I cut a 6 × 8 in half to make a 12 × 4," cut a 6 × 8 array into two 6 × 4 arrays and then place one above the other to make a new array, 12 × 4.

3 × 4
3 × 8
6 × 8
12 × 4 ← We could stop here.
24 × 2
48 × 1
3 × 16

If all goes well could give them a problem like 12×8 OR 12 x [illegible]6 to see if they can come up with a problem that will help them.

Behind the Numbers: How the String was Crafted

The first three problems are basic facts but they are presented one at a time and related in a way that supports the use of doubling (each one has a factor that is double one of the factors in the previous one). The next three problems in the string are all equivalent since one factor doubles while the other halves. Even if students do not use doubling and halving to produce the answers, the fact that the answers are the same will likely prompt a discussion on equivalence. The doubling and halving is not as easy to see in the last problem. Students must look back over the string to find the problem from which the last one was produced. And since the answer is the same as in the previous four problems, new relationships among the factors can be examined. The associative property underlies the doubling and halving strategy.

Inside One Classroom

A Portion of the Minilesson

Author's Notes

Miki (the teacher)**:** *(Writes on the chalkboard 6 × 8.)* Here is the next one...another multiplication fact. What is 6 times 8? C. J.?

Learning the basic multiplication facts is critical when moving on to more challenging multiplication problems.

C.J.: That's 48.

Miki: *(Writes 48 next to the problem and puts a 6 x 8 array up next to the problem.)* Great, and here's a 6 by 8 array I cut out. Can somebody show us how this is a 6 by 8 array? Megan?

The array provides students with a representation of their strategies. Eventually the array will become a tool with which to think.

Megan: There's 6 squares going down and 8 going across. *(Points to the rows and columns of the array.)*

Miki: Everybody agrees that we have 6 rows and 8 columns? *(Nods from the class.)* OK, here's another problem. *(Writes 12 × 4 directly underneath the first problem.)* Take a minute to think about how you would solve this problem. When you are ready to share, put a thumb up so you don't disturb other people who are still working on it. *(Waits several seconds before calling on students to share.)* Debbie?

Read this as example of teacher talk to make connections.

Debbie: Well, I just knew it. I knew 12 × 4 was 48. But I also noticed that it's the same answer as 6 × 8.

One of the reasons that each problem in a string is written directly underneath the previous one is so observations such as Debbie's can be made easily.

Miki: That's interesting, Debbie. Did anybody else notice that? I wonder why that happened.

Claire: I think you double and halve 6 by 8 so it's the same answer.

Miki: Can you say more about that, Claire?

Miki asks Claire to expand on her thinking in order to encourage conversation on this important strategy.

Claire: Well, 12 is double 6 and 4 is half of 8 so the answer is still 48.

Miki: *(Draws an arrow from the 6 to the 12 and from the 8 to the 4 and writes ×2 and ÷2 respectively next to the arrows.)* Claire, can you show us how the doubling and halving works using the 6 by 8 array?

Claire: You cut the 6 by 8 array in half. So you count 4 squares across and cut down the line there. *(Uses scissors to cut the 6 × 8 array in half vertically.)* And then you take one of the pieces and connect it to the bottom of the other piece so that you have 12 squares going down. *(Attaches one 6 x 4 piece to the bottom of the other 6 × 4 piece to make a 12 × 4.)* That's a 12 by 4 array now, and you still have 48.

The paper array is cut and moved to represent Claire's thinking.

Miki: This is interesting. Let's think about the next problem, and how the array can be doubled and halved the way that Claire just did. Here it is: 24 × 2. *(Writes 24 × 2 directly under 12 × 4.)*

Miki invites reflection on the strategy.

Appendix 3: Planning prompts offered to novice teachers in the 2011-2012 school year (taken from a homework sheet given to novices to support their preparation for teaching.)

Multiplication String. Pick a number talk from your number talk. Look at the strings for mulitiplication in your Number Talk book. You can choose a category 2 or 3 string: making landmark or friendly numbers (p. 270 - 271)

1. How do the problems build upon each other? What's the mathematical goal of posing these particular problems in this particular order? What mathematical idea do we want to engage students in thinking about by posing these problems in this order?
2. What connections do you see across the problems? How might you engage students in seeing these connections? What questions might you ask to support them in making these connections?
3. What representations might best support this string of related problems? Why? How do these visual representations help make sense of the mathematical connections?

Appendix 4: Protocol for String Activity Used in 2011-12 Methods Course

Time/Materials	Activity	Teaching Notes/Accomodations
	Choose a purposeful sequence of related problems **Be sure that you have thoughtfully chosen or created a string with a specific mathematical purpose. What is the string designed to highlight? What relationships or strategies do you want participants to notice?**	• Think about whether you are using doc camera, overhead, smart board, etc. • think about if you want to prepare any visuals ahead of time, like arrays
	Introduce task to students and anticipate the flow/pacing *"There are a few math problems we are going to solve today…"* *"Ready for the first one?"* *Remember not to call out your answer.* *You can give me a thumbs up when you are ready.*	• Keeping your purpose in mind will help you decide when to delve and when to gloss over particular problems and/or strategies. For example, you may want to tell participants you expect them to "just know" the first problem in a string (although this is not always the case!) Also, if someone shares a rather complicated strategy that does not match your goals, you may choose not to ask them a lot of probing questions. In contrast, if someone has shared the strategy you'd like people to focus on, slow the conversation down by asking someone else to restate. • Decide what management device you want to use for kids to signal that they have their answer. How does your management routine convey messages about competence, status, competition, speed, etc.?
	Pose the first problem **"I think this problem is one we can all do."** • Get answer(s) from students. Record. • Decide if you want to link answer to a particular representation.	• Start with a problem that you know the kids will find easy. • Decide if you want to be in charge of the representation or have the kids create or direct your representation. • Listen to response and decide if clarification, elaboration or explanation is needed. If a student shares a strategy you want to highlight, decide how much elaboration, re-voicing and rephrasing you want to do or request that other students do. • Decide if you want to request a different strategy or if you want to ask students to comment on or build upon current strategy. • Decide how to utilize other student voices to explain mathematical reasoning. • Decide how to record students' mathematical reasoning.
	Pose the Next Problem • Keep first problem visible	○ Think about how to keep the problems of the string visible to the students if you have also been recording their strategies. ○ Request answer(s) from student.

Time/Materials	Activity	Teaching Notes/Accomodations
	• Get answer(s) from students. Record.	○ Decide how you want a student to link their answer back to representation. Request student to describe how they got an answer(s). ○ Decide how you want students to treat different answers and strategies shared thus far. Do you want to comment or have students comment on their similarity or differences? Do you want to make an explicit link to how the strategies used on previous problems might support solving this problem?
	Pose Remaining Problems Explicitly indicate problems we just solved might help with next/last problem: "*Now I'm going to pose a new problem with different numbers. See if the work we've just done with ___ idea, helps you get the answer for this one.*"	• Pose each problem one at a time and consider all ideas from how you posed the previous two problems. NOTE: If the last problem in the string is an application of the ideas that the string is designed to focus attention to, explicitly tell students you are posing a new problem.
	Highlight the big ideas and close the task	• Discuss the specific strategy that this string was designed to address. Work with students to make connections among the problems that were posed to them within the string. Make the mathematical strategy/concept that this string highlighted explicit for students. • Decide if it is necessary to pose another similar problem where students might be able to use the strategy just discussed and highlighted in the string.

Challenges that might/will occur: *How will you handle these scenarios?*
Children offer incorrect responses

Many children seem to not be participating

You ask for any connections among the problems and get no response

Children are not seeing connections among problems: they are not using previous problems in the string to solve the harder problems that come nearer the end of the string